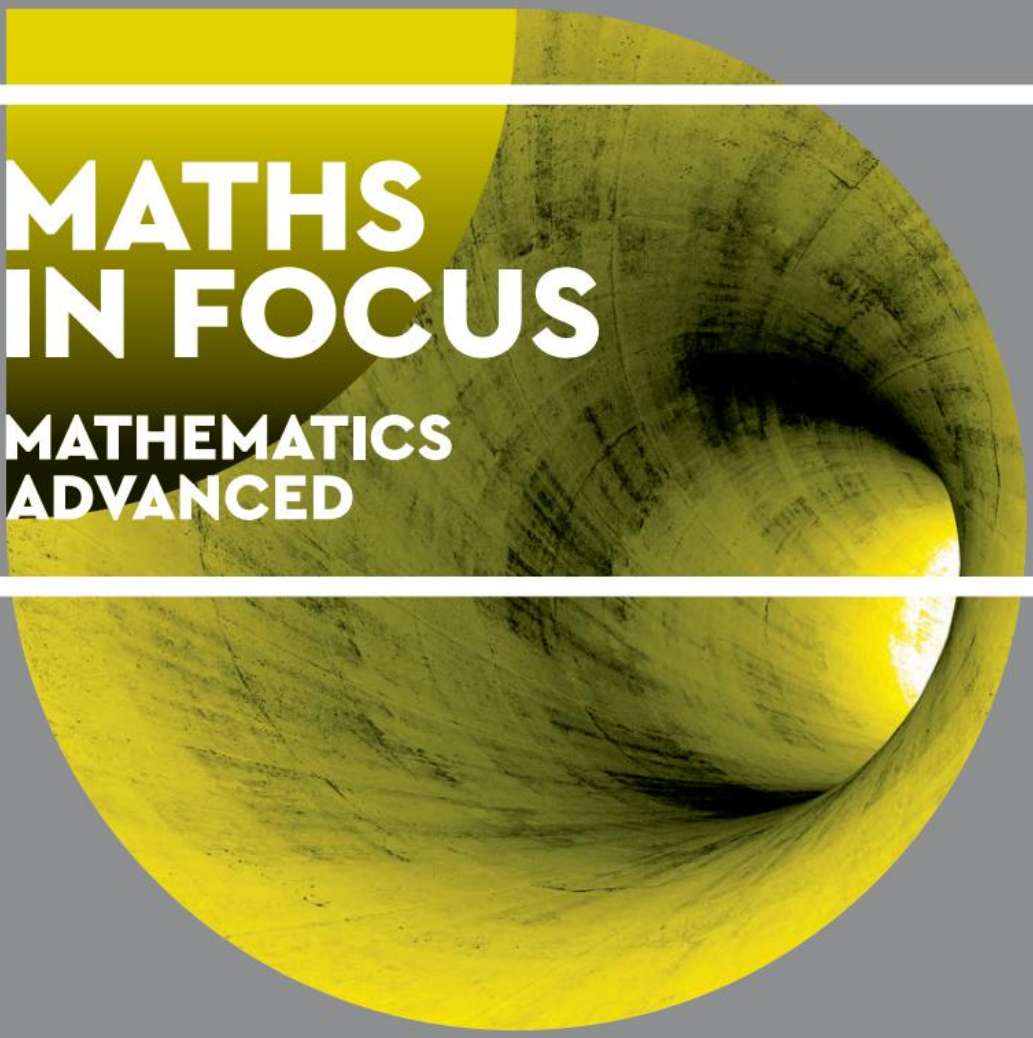
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MATHS IN FOCUS

MATHEMATICS ADVANCED

YEAR 11

Margaret Grove

3RD EDITION



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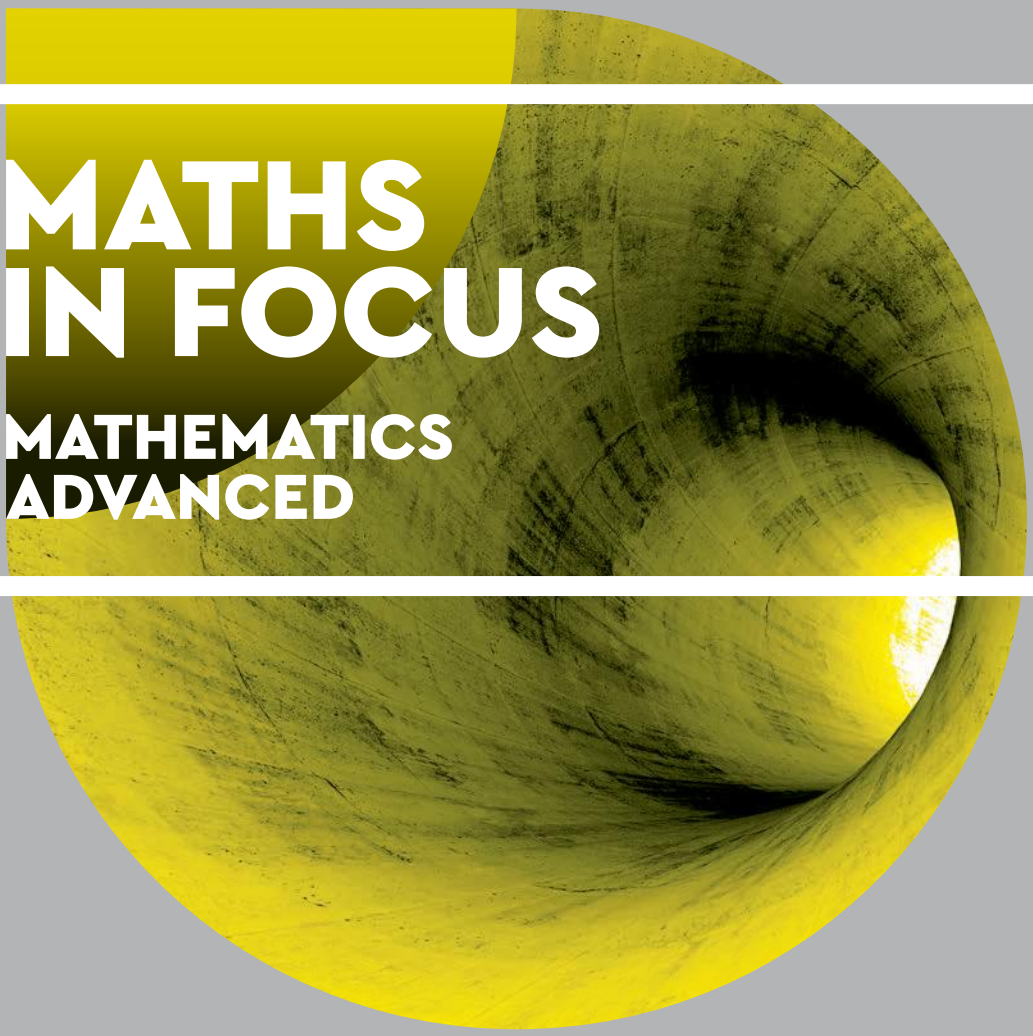
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MATHS IN FOCUS

MATHEMATICS ADVANCED

YEAR 11

Margaret Grove
Series editor: Robert Yen

3RD EDITION

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PREFACE

Maths in Focus 11 Mathematics Advanced has been rewritten for the new Mathematics Advanced syllabus (2017). In this 3rd edition of the book, teachers will find those familiar features that have made Maths in Focus a leading senior mathematics series, such as clear and abundant worked examples in plain English, comprehensive sets of graded exercises, chapter *Test Yourself* and *Challenge* exercises, Investigations, and practice sets of mixed revision and exam-style questions.

The Mathematics Advanced course is designed for students who intend to study at university, in a field that requires mathematics, especially calculus and statistics. This book covers the Year 11 Mathematics Advanced course. The theory follows a logical order, although some topics may be learned in any order. We have endeavoured to produce a practical text that captures the spirit of the course, providing relevant and meaningful applications of mathematics.

The *NelsonNet* student and teacher websites contain additional resources such as worksheets, video tutorials and topic tests. We wish all teachers and students using this book every success in embracing the new senior mathematics course.

ABOUT THE AUTHOR

Margaret Grove has spent over 30 years teaching HSC Mathematics, most recently at Bankstown TAFE College. She has written numerous senior mathematics texts and study guides over the past 25 years, including the bestselling *Maths in Focus* series for Mathematics and Mathematics Extension 1.

Margaret thanks her family, especially her husband Geoff, for their support in writing this book.

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Gaspare Carrozza and **Haroon Ha** from Homebush Boys High School wrote many of the *NelsonNet* worksheets.

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Tania Eastcott and **Elizabeth Nabhan** wrote the topic tests.

Roger Walter wrote the *ExamView* questions.

Shane Scott, **Brandon Pettis** and **George Dimitriadis** wrote the worked solutions to all exercise sets.

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CONTENTS

PREFACE iii
SYLLABUS REFERENCE GRID vii
ABOUT THIS BOOK ix
MATHEMATICAL VERBS xii
STUDY SKILLS xiii

* = Revision

1

ALGEBRAIC TECHNIQUES 2

F1.1 1.01 Index laws 4
F1.1 1.02 Zero and negative indices 7
F1.1 1.03 Fractional indices 9
* 1.04 Simplifying algebraic expressions ... 14
* 1.05 Expansion 16
* 1.06 Binomial products 17
* 1.07 Special products 18
* 1.08 Factorisation 19
* 1.09 Factorisation by grouping in pairs .. 20
* 1.10 Factorising trinomials 21
* 1.11 Further trinomials 23
* 1.12 Perfect squares 24
* 1.13 Difference of two squares 25
* 1.14 Mixed factorisation 26
F1.1 1.15 Simplifying algebraic fractions 27
F1.1 1.16 Operations with algebraic fractions 28
* 1.17 Substitution 30
F1.1 1.18 Simplifying surds 32
F1.1 1.19 Operations with surds 33
F1.1 1.20 Rationalising the denominator 37
Test yourself 1 41
Challenge exercise 1 45

2

EQUATIONS AND INEQUALITIES 46

* 2.01 Equations 48
* 2.02 Inequalities 51
F1.4 2.03 Absolute value 53
F1.4 2.04 Equations involving absolute values 56
E1.4 2.05 Exponential equations 58
* 2.06 Solving quadratic equations by factorisation 61
F1.1 2.07 Solving quadratic equations by completing the square 62
F1.1 2.08 Solving quadratic equations by quadratic formula 64
* 2.09 Formulas and equations 66
F1.3 2.10 Linear simultaneous equations 69
F1.3 2.11 Non-linear simultaneous equations .. 70
F1.3 2.12 Simultaneous equations with three unknown variables 72
Test yourself 2 74
Challenge exercise 2 76

PRACTICE SET 1 77

3

FUNCTIONS 82

F1.2 3.01 Functions 85
F1.2 3.02 Function notation 92
F1.2 3.03 Properties of functions 96
F1.3 3.04 Linear functions 103
C1.1 3.05 The gradient of a straight line 107
F1.3 3.06 Finding a linear equation 113

ISBN 9780170413152

F1.3 3.07 Parallel and perpendicular lines 117
F1.3 3.08 Quadratic functions 121
F1.3 3.09 Axis of symmetry 126
F1.3 3.10 The discriminant 130
F1.3 3.11 Finding a quadratic equation 135
F1.3 3.12 Cubic functions 139
F1.4 3.13 Polynomial functions 147
F1.3 3.14 Intersection of graphs 152
Test yourself 3 158
Challenge exercise 3 162

4

TRIGONOMETRY 164

T1.1 4.01 Trigonometric ratios 168
T1.1 4.02 Finding a side of a right-angled triangle 174
T1.1 4.03 Finding an angle in a right-angled triangle 178
T1.1 4.04 Applications of trigonometry 182
T1.1 4.05 The sine rule 190
T1.1 4.06 The cosine rule 198
T1.1 4.07 Area of a triangle 202
T1.1 4.08 Mixed problems 205
T1.2 4.09 Radians 211
T1.2 4.10 Length of an arc 216
T1.2 4.11 Area of a sector 219
Test yourself 4 222
Challenge exercise 4 225

PRACTICE SET 2 227

5

FURTHER FUNCTIONS 234

F1.4 5.01 The hyperbola 236
F1.4 5.02 Absolute value functions 243
F1.4 5.03 Circles and semicircles 248
F1.4 5.04 Reflections of functions 253
F1.2 5.05 Combined and composite functions 259
Test yourself 5 264
Challenge exercise 5 267

6

INTRODUCTION TO CALCULUS 268

C1.2, 1.3 6.01 Gradient of a curve 271
C1.1 6.02 Differentiability 279
C1.1–1.3 6.03 Differentiation from first principles 282
C1.3, 1.4 6.04 Short methods of differentiation 292
C1.4 6.05 Derivatives and indices 297
C1.4 6.06 Tangents and normals 299
C1.4 6.07 Chain rule 304
C1.4 6.08 Product rule 306
C1.4 6.09 Quotient rule 309
C1.4 6.10 Rates of change 311
Test yourself 6 317
Challenge exercise 6 320

7

PROBABILITY 322

S1.1 7.01 Set notation and Venn diagrams 324
S1.1 7.02 Relative frequency 328
S1.1 7.03 Theoretical probability 332
S1.1 7.04 Addition rule of probability 337
S1.1 7.05 Product rule of probability 340
S1.1 7.06 Probability trees 346
S1.1 7.07 Conditional probability 351
Test yourself 7 360
Challenge exercise 7 363

PRACTICE SET 3 364

ISBN 9780170413152

8

EXPONENTIAL AND LOGARITHMIC FUNCTIONS 370

E1.1, 1.4 8.01 Exponential functions..... 372
E1.3 8.02 Euler's number, *e* 377
E1.3 8.03 Differentiation of exponential functions..... 381
E1.1 8.04 Logarithms 385
E1.2 8.05 Logarithm laws..... 392
E1.1–1.4 8.06 Logarithmic functions..... 397
E1.4 8.07 Exponential equations 403
Test yourself 8 406
Challenge exercise 8 409

9

TRIGONOMETRIC FUNCTIONS 410

T1.2 9.01 Angles of any magnitude 412
T2 9.02 Trigonometric identities..... 419
T1.2, 2 9.03 Radians 425
T1.2, 2 9.04 Trigonometric functions..... 429
T2 9.05 Trigonometric equations 442
T2 9.06 Applications of trigonometric functions 446
Test yourself 9 449
Challenge exercise 9 451

10

DISCRETE PROBABILITY DISTRIBUTIONS 452

S1.2 10.01 Random variables..... 454
S1.2 10.02 Discrete probability distributions....456
S1.2 10.03 Mean or expected value..... 467
S1.2 10.04 Variance and standard deviation ... 475
Test yourself 10 481
Challenge exercise 10 485

PRACTICE SET 4 487

ANSWERS 492

INDEX 572

MATHS IN FOCUS AND NEW CENTURY MATHS 11–12



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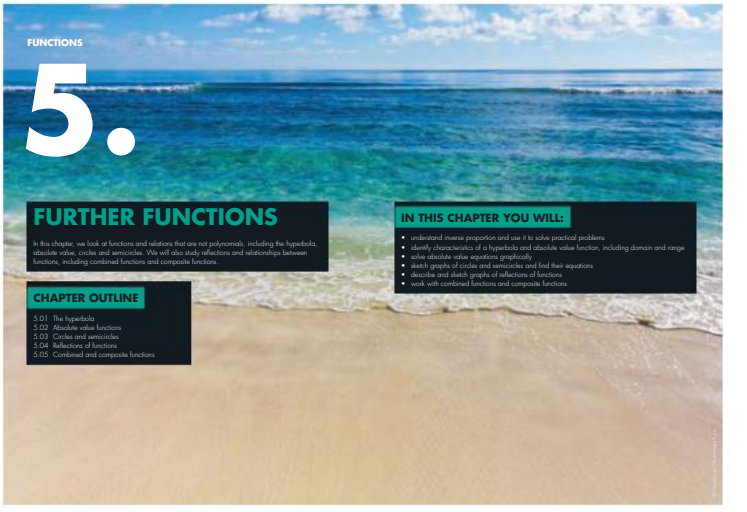
Topic and subtopic	*Maths in Focus 11 Mathematics Advanced* chapter
FUNCTIONS	
MA-F1 Working with functions	
F1.1 Algebraic techniques F1.2 Introduction to functions F1.3 Linear, quadratic and cubic functions F1.4 Further functions and relations	1 Algebraic techniques 2 Equations and inequalities 3 Functions 5 Further functions
TRIGONOMETRIC FUNCTIONS	
MA-T1 Trigonometry and measure of angles	
T1.1 Trigonometry T1.2 Radians	4 Trigonometry 9 Trigonometric functions
MA-T2 Trigonometric functions and identities	9 Trigonometric functions
CALCULUS	
MA-C1 Introduction to differentiation	
C1.1 Gradients of tangents C1.2 Difference quotients C1.3 The derivative function and its graph C1.4 Calculating with derivatives	6 Introduction to calculus
EXPONENTIAL AND LOGARITHMIC FUNCTIONS	
MA-E1 Logarithms and exponentials	
E1.1 Introducing logarithms E1.2 Logarithmic laws and applications E1.3 The exponential function and natural logarithms E1.4 Graphs and applications of exponential and logarithmic functions	8 Exponential and logarithmic functions
STATISTICAL ANALYSIS	
MA-S1 Probability and discrete probability distributions	
S1.1 Probability and Venn diagrams S1.2 Discrete probability distributions	7 Probability 10 Discrete probability distributions

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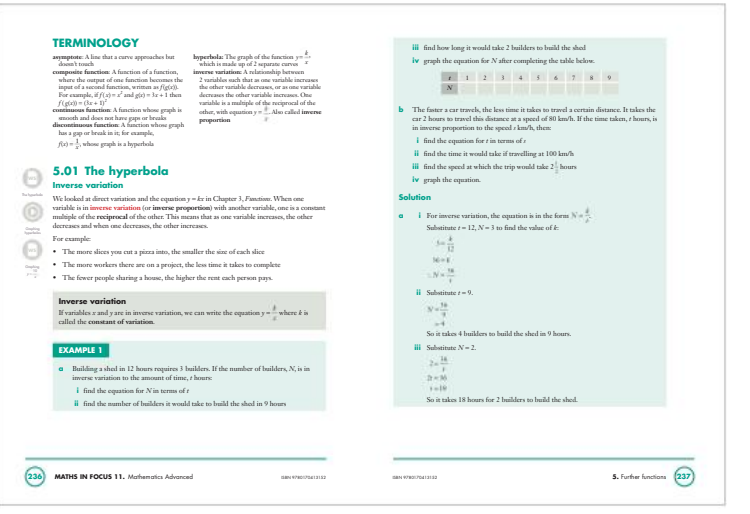
ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of chapter outcomes



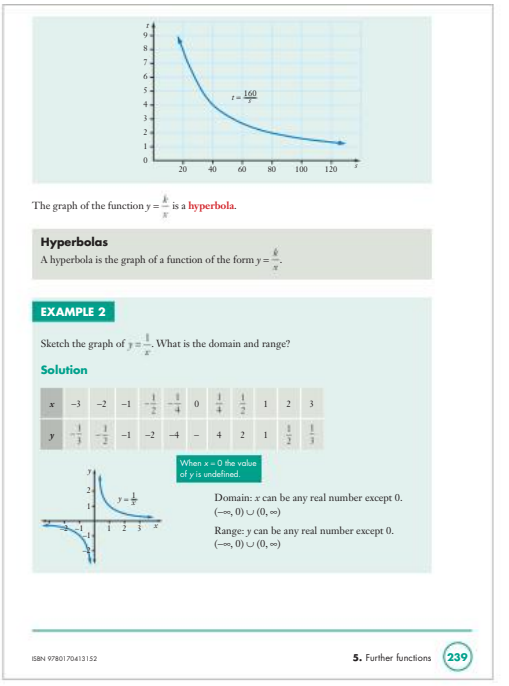
- **Terminology** is a chapter glossary that previews the key words and phrases from within the chapter

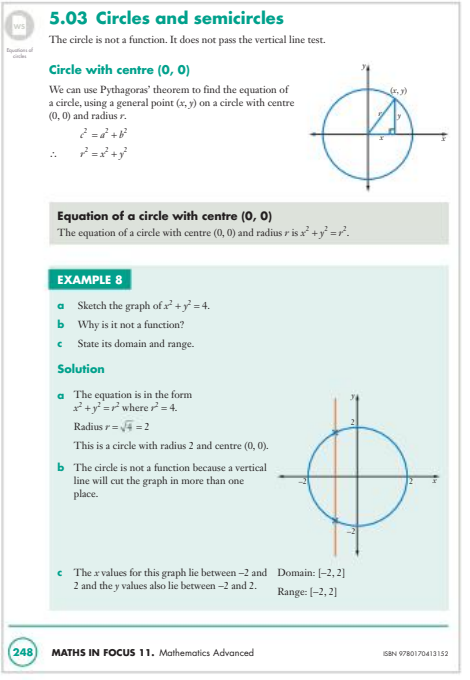


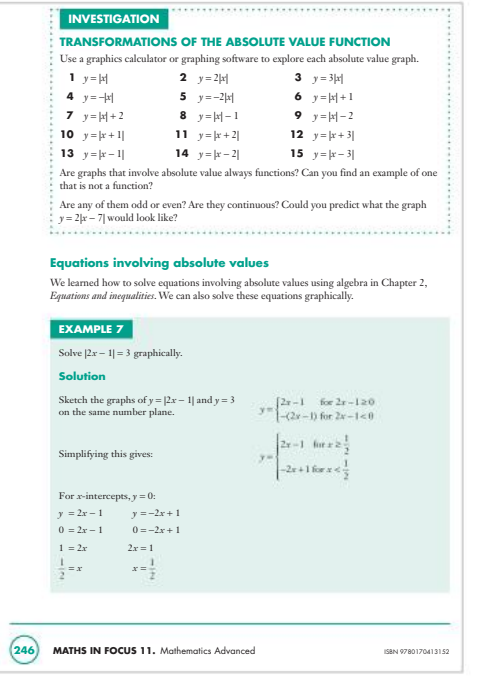
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IN EACH CHAPTER

- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the Terminology chapter glossary.
- Graded exercises include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website.
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- **Did you know?** contains interesting facts and applications of the mathematics learned in the chapter.

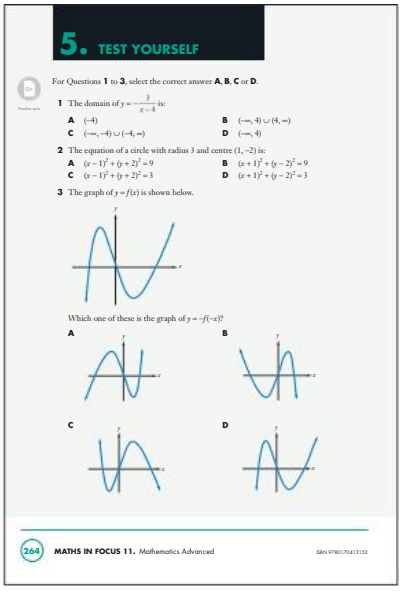






AT THE END OF EACH CHAPTER

- **Test Yourself** contains chapter revision exercises.
- If you have trouble completing the *Test Yourself* exercises, you need to go back and revise the chapter before trying the exercises again.
- **Challenge Exercise** contains chapter extension questions. Attempt these only after you are confident with the *Test Yourself* exercises, because these are more difficult and are designed for students who understand the topic really well.
- **Practice sets** (after several chapters) provide a comprehensive variety of mixed exam-style questions from various chapters, including short-answer, free-response and multiple-choice questions.


5. TEST YOURSELF

For Questions 1 to 3, select the correct answer A, B, C or D.

1 The domain of $y = \frac{3}{x-4}$ is:

A (-4) B $(-\infty, 4) \cup (4, \infty)$

C $(-\infty, -4) \cup (-4, \infty)$ D $(-\infty, 4)$

2 The equation of a circle with radius 3 and centre $(1, -2)$ is:

A $(x-1)^2 + (y+2)^2 = 9$ B $(x+1)^2 + (y-2)^2 = 9$

C $(x-1)^2 + (y+2)^2 = 3$ D $(x+1)^2 + (y-2)^2 = 3$

3 The graph of $y = f(x)$ is shown below.

Which one of these is the graph of $y = -f(-x)$?

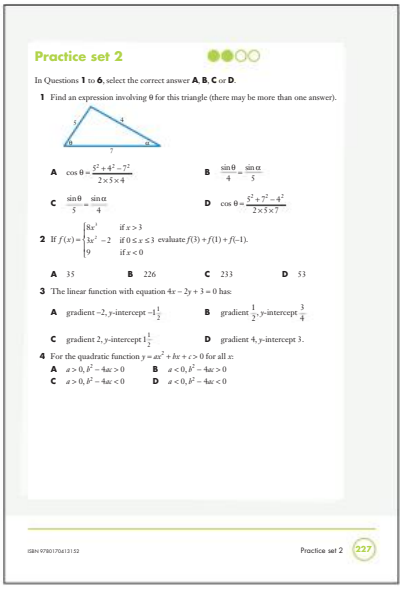
A B

C D

264 MATHS IN FOCUS 11. Mathematics Advanced

AT THE END OF THE BOOK

- **Answers** and **Index** (worked solutions on the teacher website).


Practice set 2

In Questions 1 to 6, select the correct answer A, B, C or D.

1 Find an expression involving θ for this triangle (there may be more than one answer).

A $\cos\theta = \frac{5^2 + 4^2 - 7^2}{2 \times 5 \times 4}$ B $\frac{\sin\theta}{4} = \frac{\sin\alpha}{5}$

C $\frac{\sin\theta}{5} = \frac{\sin\alpha}{4}$ D $\cos\theta = \frac{5^2 + 7^2 - 4^2}{2 \times 5 \times 7}$

2 If $f(x) = \begin{cases} 8x^3 & \text{if } x > 3 \\ 3x^2 - 2 & \text{if } 0 \le x \le 3 \\ 9 & \text{if } x < 0 \end{cases}$ evaluate $f(3) + f(1) + f(-1)$.

A 35 B 226 C 233 D 53

3 The linear function with equation $4x - 2y + 3 = 0$ has:

A gradient -2, y-intercept $-1\frac{1}{2}$ B gradient $\frac{1}{2}$, y-intercept $\frac{3}{4}$

C gradient 2, y-intercept $1\frac{1}{2}$ D gradient 4, y-intercept 3.

4 For the quadratic function $y = ax^2 + bx + c > 0$ for all x:

A $a > 0, b^2 - 4ac > 0$ B $a < 0, b^2 - 4ac > 0$

C $a > 0, b^2 - 4ac < 0$ D $a < 0, b^2 - 4ac < 0$

Practice set 2 227

NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, **www.nelsonnet.com.au**. These include:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Video tutorials:** worked examples explained by 'flipped classroom' teachers
- ***ExamView* quizzes:** interactive and self-marking

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NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website, also at **www.nelsonnet.com.au**, contains:

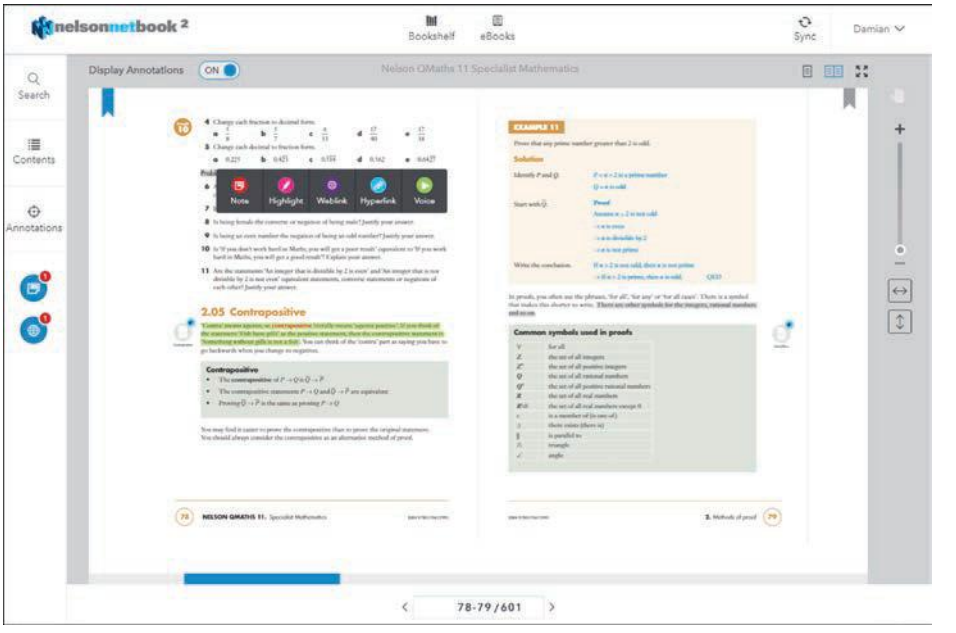
- A **teaching program,** in Microsoft Word and PDF formats
- **Topic tests,** in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- ***ExamView*** exam-writing software and questionbanks
- **Resource Finder:** search engine for *NelsonNet* resources

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- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students



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MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analyse: study in detail the parts of a situation

apply: use knowledge or a procedure in a given situation

classify, identify: state the type, name or feature of an item or situation

comment: express an observation or opinion about a result

compare: show how two or more things are similar or different

construct: draw an accurate diagram

describe: state the features of a situation

estimate: make an educated guess for a number, measurement or solution, to find roughly or approximately

evaluate, calculate: find the value of a numerical expression, for example 3×8^2 or $4x + 1$ when $x = 5$

expand: remove brackets in an algebraic expression, for example expanding $3(2y + 1)$ gives $6y + 3$

explain: describe why or how

factorise: opposite to **expand,** to insert brackets by taking out a common factor, for example factorising $6y + 3$ gives $3(2y + 1)$

give reasons: show the rules or thinking used when solving a problem. *See also* **justify.**

hence find/prove: find an answer or prove a result using previous answers or information supplied

interpret: find meaning in a mathematical result

justify: give reasons or evidence to support your argument or conclusion. *See also* **give reasons**

rationalise: make rational, remove surds

show that, prove: (in questions where the answer is given) use calculation, procedure or reasoning to prove that an answer or result is true

simplify: give a result in its most basic, shortest, neatest form, for example simplifying a ratio or algebraic expression

sketch: draw a rough diagram that shows the general shape or ideas, less accurate than **construct**

solve: find the value(s) of an unknown pronumeral in an equation or inequality

substitute: replace a variable by a number and evaluate

verify: check that a solution or result is correct, usually by substituting back into the equation or referring back to the problem

write, state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

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STUDY SKILLS

The Year 11 course introduces the basics of topics such as calculus that are then applied in the Year 12 course. You will struggle in the HSC if you don't set yourself up to revise the Year 11 topics as you learn new Year 12 topics. Your teachers will be able to help you build up and manage good study habits. Here are a few hints to get you started. There is no right or wrong way to learn. Different styles of learning suit different people. There is also no magical number of hours a week that you should study, because this will be different for every student. But just listening in class and taking notes is not enough, especially when you are learning material that is totally new.

If a skill is not practised within the first 24 hours, up to 50% can be forgotten. If it is not practised within 72 hours, up to 85–90% can be forgotten! So it is really important that, whatever your study timetable, new work must be looked at soon after it is presented to you.

With a continual succession of new work to learn and retain, this is a challenge. But the good news is that you don't have to study for hours on end!

IN THE CLASSROOM

In order to remember, first you need to focus on what is being said and done.

According to an ancient proverb:

I hear and I forget
I see and I remember
I do and I understand.

If you chat to friends and just take notes without really paying attention, you aren't giving yourself a chance to remember anything and will have to study harder at home.

If you are unsure of something that the teacher has said, the chances are that others are also not sure. Asking questions and clarifying things will ultimately help you gain better results, especially in a subject like mathematics where much of the knowledge and skills depend on being able to understand the basics.

Learning is all about knowing what you know and what you don't know. Many students feel like they don't know anything, but it's surprising just how much they know already. Picking up the main concepts in class and not worrying too much about other less important parts can really help. The teacher can guide you on this.

Here are some pointers to get the best out of classroom learning:

- Take control and be responsible for your own learning
- Clear your head of other issues in the classroom
- Active, not passive, learning is more memorable
- Ask questions if you don't understand something

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- Listen for cues from the teacher
- Look out for what are the main concepts.

Note-taking varies from class to class, but here are some general guidelines:

- Write legibly
- Use different colours to highlight important points or formulas
- Make notes in textbooks (using pencil if you don't own the textbook)
- Use highlighter pens to point out important points
- Summarise the main points
- If notes are scribbled, rewrite them at home.

AT HOME

You are responsible for your own learning and nobody else can tell you how best to study. Some people need more revision time than others, some study better in the mornings while others do better at night, and some can work at home while others prefer a library.

- Revise both new and older topics regularly
- Have a realistic timetable and be flexible
- Summarise the main points
- Revise when you are fresh and energetic
- Divide study time into smaller rather than longer chunks
- Study in a quiet environment
- Have a balanced life and don't forget to have fun!

If you are given exercises out of a textbook to do for homework, consider asking the teacher if you can leave some of them till later and use these for revision. It is not necessary to do every exercise at one sitting, and you learn better if you can spread these over time.

People use different learning styles to help them study. The more variety the better, and you will find some that help you more than others. Some people (around 35%) learn best visually, some (25%) learn best by hearing and others (40%) learn by doing.

- Summarise on cue cards or in a small notebook
- Use colourful posters
- Use mindmaps and diagrams
- Discuss work with a group of friends
- Read notes out aloud
- Make up songs and rhymes
- Exercise regularly
- Role-play teaching someone else

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ASSESSMENT TASKS AND EXAMS

You will cope better in exams if you have practised doing sample exams under exam conditions. Regular revision will give you confidence, and if you feel well prepared this will help get rid of nerves in the exam. You will also cope better if you have had a reasonable night's sleep before the exam.

One of the biggest problems students have with exams is in timing. Make sure you don't spend too much time on questions you're unsure about, but work through and find questions you can do first.

Divide the time up into smaller chunks for each question and allow some extra time to go back to questions you couldn't do or finish. For example, in a 2-hour exam with 6 questions, allow around 15 minutes for each question. This will give an extra half hour at the end to tidy up and finish off questions.

- Read through and ensure you know how many questions there are
- Divide your time between questions with extra time at the end
- Don't spend too much time on one question
- Read each question carefully, underlining key words
- Show all working out, including diagrams and formulas
- Cross out mistakes with a single line so it can still be read
- Write legibly

AND FINALLY...

Study involves knowing what you don't know, and putting in a lot of time into concentrating on these areas. This is a positive way to learn. Rather than just saying, 'I can't do this', say instead, 'I can't do this yet', and use your teachers, friends, textbooks and other ways of finding out.

With the parts of the course that you do know, make sure you can remember these easily under exam pressure by putting in lots of practice.

Remember to look at new work:

today, tomorrow, in a week, in a month.

Some people hardly ever find time to study while others give up their outside lives to devote their time to study. The ideal situation is to balance study with other aspects of your life, including going out with friends, working, and keeping up with sport and other activities that you enjoy.

Good luck with your studies!

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FUNCTIONS

1

ALGEBRAIC TECHNIQUES

This chapter revises and extends the algebraic techniques that you will need for this course. These include indices, algebraic expressions, expansion, factorisation, algebraic fractions and surds.

CHAPTER OUTLINE

1.01 Index laws
1.02 Zero and negative indices
1.03 Fractional indices
1.04 Simplifying algebraic expressions
1.05 Expansion
1.06 Binomial products
1.07 Special products
1.08 Factorisation
1.09 Factorisation by grouping in pairs
1.10 Factorising trinomials
1.11 Further trinomials
1.12 Perfect squares
1.13 Difference of two squares
1.14 Mixed factorisation
1.15 Simplifying algebraic fractions
1.16 Operations with algebraic fractions
1.17 Substitution
1.18 Simplifying surds
1.19 Operations with surds
1.20 Rationalising the denominator



IN THIS CHAPTER YOU WILL:

- identify and use index rules including fractional and negative indices
- simplify algebraic expressions
- remove grouping symbols including perfect squares and the difference of 2 squares
- factorise expressions including binomials and special factors
- simplify algebraic fractions
- use algebra to substitute into formulas
- simplify and use surds including rationalising the denominator

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TERMINOLOGY

binomial: A mathematical expression consisting of 2 terms; for example, $x + 3$ and $3x - 1$

binomial product: The product of binomial expressions; for example, $(x + 3)(2x - 1)$

expression: A mathematical statement involving numbers, pronumerals and symbols; for example, $2x - 3$

factor: A whole number that divides exactly into another number. For example, 4 is a factor of 28

factorise: To write an expression as a product of its factors; that is, take out the highest common factor in an expression and place the rest in brackets. For example, $2y - 8 = 2(y - 4)$

index: The power or exponent of a number. For example, 2^3 has a base number of 2 and an index of 3. The plural of index is **indices**

power: The index or exponent of a number. For example, 2^3 has a base number of 2 and a power of 3

root: A number that when multiplied by itself a given number of times equals another number. For example, $\sqrt{25} = 5$ because $5^2 = 25$

surd: A root that can't be simplified; for example, $\sqrt{3}$

term: A part of an expression containing pronumerals and/or numbers separated by an operation such as +, −, × or ÷. For example, in $2x - 3$ the terms are $2x$ and 3

trinomial: An expression with 3 terms; for example, $3x^2 - 2x + 1$

1.01 Index laws

An **index** (or **power** or **exponent**) of a number shows how many times a number is multiplied by itself. A **root** of a number is the inverse of the power.

For example:

- $4^3 = 4 \times 4 \times 4 = 64$
- $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$
- $\sqrt{36} = 6$ since $6^2 = 36$
- $\sqrt[3]{8} = 2$ since $2^3 = 8$
- $\sqrt[6]{64} = 2$ since $2^6 = 64$

Note: In 4^3 the 4 is called the base number and the 3 is called the index or power.

There are some general laws that simplify calculations with indices. These laws work for any m and n, including fractions and negative numbers.

Index laws

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

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EXAMPLE 1

Simplify:

a $m^9 \times m^7 \div m^2$ **b** $(2y^4)^3$ **c** $\dfrac{(y^6)^3 \times y^{-4}}{y^5}$

Solution

a $m^9 \times m^7 \div m^2 = m^{9+7-2}$
$= m^{14}$

b $(2y^4)^3 = 2^3(y^4)^3$
$= 2^3 y^{4\times 3}$
$= 8y^{12}$

c $\dfrac{(y^6)^3 \times y^{-4}}{y^5} = \dfrac{y^{18} \times y^{-4}}{y^5}$
$= \dfrac{y^{18+(-4)}}{y^5}$
$= \dfrac{y^{14}}{y^5}$
$= y^{14-5}$
$= y^9$

Exercise 1.01 Index laws

1 Evaluate without using a calculator:

a $5^3 \times 2^2$ **b** $3^4 + 8^2$ **c** $\left(\dfrac{1}{4}\right)^3$

d $\sqrt[3]{27}$ **e** $\sqrt[4]{16}$

2 Evaluate correct to 1 decimal place:

a 3.7^2 **b** $1.06^{1.5}$ **c** $2.3^{-0.2}$

d $\sqrt[3]{19}$ **e** $\sqrt[3]{34.8 - 1.2 \times 43.1}$ **f** $\dfrac{1}{\sqrt[3]{0.99 + 5.61}}$

3 Simplify:

a $a^6 \times a^9 \times a^2$ **b** $y^3 \times y^{-8} \times y^5$ **c** $a^{-1} \times a^{-3}$

d $w^{\frac{1}{2}} \times w^{\frac{1}{2}}$ **e** $x^6 \div x$ **f** $p^3 \div p^{-7}$

g $\dfrac{y^{11}}{y^5}$ **h** $(x^7)^3$ **i** $(2x^5)^2$

j $(3y^{-2})^4$ **k** $a^3 \times a^5 \div a^7$ **l** $\left(\dfrac{x^2}{y^9}\right)^5$

m $\dfrac{w^6 \times w^7}{w^3}$ **n** $\dfrac{p^2 \times (p^3)^4}{p^9}$ **o** $\dfrac{x^6 \div x^7}{x^2}$

p $\dfrac{a^2 \times (b^2)^6}{a^4 \times b^9}$ **q** $\dfrac{(x^2)^{-3} \times (y^3)^2}{x^{-1} \times y^4}$

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4 Simplify:

a $x^5 \times x^9$ **b** $a^{-1} \times a^{-6}$ **c** $\dfrac{m^7}{m^3}$

d $k^{13} \times k^6 \div k^9$ **e** $a^{-5} \times a^4 \times a^{-7}$ **f** $x^{\frac{2}{5}} \times x^{\frac{3}{5}}$

g $\dfrac{m^5 \times n^4}{m^4 \times n^2}$ **h** $\dfrac{p^{\frac{1}{2}} \times p^{\frac{1}{2}}}{p^2}$ **i** $(3x^{11})^2$

j $\dfrac{(x^4)^6}{x^3}$

5 Expand each expression and simplify where possible:

a $(pq^3)^5$ **b** $\left(\dfrac{a}{b}\right)^8$ **c** $\left(\dfrac{4a}{b^4}\right)^3$

d $(7a^5b)^2$ **e** $\dfrac{(2m^7)^3}{m^4}$ **f** $\dfrac{xy^3 \times (xy^2)^4}{xy}$

g $\dfrac{(2k^8)^4}{(6k^3)^3}$ **h** $(2y^5)^7 \times \dfrac{y^{12}}{8}$ **i** $\left(\dfrac{a^6 \times a^4}{a^{11}}\right)^{-3}$

j $\left(\dfrac{5xy^9}{x^8 \times y^3}\right)^3$

6 Evaluate a^3b^2 when $a = 2$ and $b = \dfrac{3}{4}$.

7 If $x = \dfrac{2}{3}$ and $y = \dfrac{1}{9}$, find the value of $\dfrac{x^3y^2}{xy^5}$.

8 If $a = \dfrac{1}{2}$, $b = \dfrac{1}{3}$ and $c = \dfrac{1}{4}$, evaluate $\dfrac{a^2b^3}{c^4}$ as a fraction.

9 a Simplify $\dfrac{a^{11}b^8}{a^8b^7}$.

b Hence evaluate $\dfrac{a^{11}b^8}{a^8b^7}$ as a fraction when $a = \dfrac{2}{5}$ and $b = \dfrac{5}{8}$.

10 a Simplify $\dfrac{p^5q^8r^4}{p^4q^6r^2}$.

b Hence evaluate $\dfrac{p^5q^8r^4}{p^4q^6r^2}$ as a fraction when $p = \dfrac{7}{8}$, $q = \dfrac{2}{3}$ and $r = \dfrac{3}{4}$.

11 Evaluate $(a^4)^3$ when $a = \left(\dfrac{2}{3}\right)^{\frac{1}{6}}$.

12 Evaluate $\dfrac{a^3b^6}{b^4}$ when $a = \dfrac{1}{2}$ and $b = \dfrac{2}{3}$.

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13 Evaluate $\frac{x^4 y^7}{x^5 y^5}$ when $x = \frac{1}{3}$ and $y = \frac{2}{9}$.

14 Evaluate $\frac{k^{-5}}{k^{-9}}$ when $k = \frac{1}{3}$.

15 Evaluate $\frac{a^4 b^6}{a^3 (b^2)^2}$ when $a = \frac{3}{4}$ and $b = \frac{1}{9}$.

16 Evaluate $\frac{a^6 \times b^3}{a^5 \times b^2}$ as a fraction when $a = \frac{1}{9}$ and $b = \frac{3}{4}$.

1.02 Zero and negative indices

Zero and negative indices

$$x^0 = 1$$
$$x^{-n} = \frac{1}{x^n}$$

WS Review of index laws

EXAMPLE 2

a Simplify $\left(\frac{ab^5c}{abc^4}\right)^0$.

b Evaluate 2^{-3}.

c Write in index form:

i $\frac{1}{x^2}$ **ii** $\frac{3}{x^5}$ **iii** $\frac{1}{5x}$ **iv** $\frac{1}{x+1}$

d Write a^{-3} without the negative index.

Solution

a $\left(\frac{ab^5c}{abc^4}\right)^0 = 1$

b $2^{-3} = \frac{1}{2^3}$
$= \frac{1}{8}$

c **i** $\frac{1}{x^2} = x^{-2}$

ii $\frac{3}{x^5} = 3 \times \frac{1}{x^5}$
$= 3x^{-5}$

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iii $\frac{1}{5x} = \frac{1}{5} \times \frac{1}{x}$

$= \frac{1}{5}x^{-1}$

iv $\frac{1}{x+1} = \frac{1}{(x+1)^1}$

$= (x+1)^{-1}$

d $a^{-3} = \frac{1}{a^3}$

Exercise 1.02 Zero and negative indices

1 Evaluate as a fraction or whole number:

a 3^{-3} **b** 4^{-1} **c** 7^{-3} **d** 10^{-4} **e** 2^{-8}

f 6^{0} **g** 2^{-5} **h** 3^{-4} **i** 7^{-1} **j** 9^{-2}

k 2^{-6} **l** 3^{-2} **m** 4^{0} **n** 6^{-2} **o** 5^{-3}

p 10^{-5} **q** 2^{-7} **r** 2^{0} **s** 8^{-2} **t** 4^{-3}

2 Evaluate:

a 2^0 **b** $\left(\frac{1}{2}\right)^{-4}$ **c** $\left(\frac{2}{3}\right)^{-1}$ **d** $\left(\frac{5}{6}\right)^{-2}$ **e** $\left(\frac{x+2y}{3x-y}\right)^{0}$

f $\left(\frac{1}{5}\right)^{-3}$ **g** $\left(\frac{3}{4}\right)^{-1}$ **h** $\left(\frac{1}{7}\right)^{-2}$ **i** $\left(\frac{2}{3}\right)^{-3}$ **j** $\left(\frac{1}{2}\right)^{-5}$

k $\left(\frac{3}{7}\right)^{-1}$ **l** $\left(\frac{8}{9}\right)^{0}$ **m** $\left(\frac{6}{7}\right)^{-2}$ **n** $\left(\frac{9}{10}\right)^{-2}$ **o** $\left(\frac{6}{11}\right)^{0}$

p $\left(-\frac{1}{4}\right)^{-2}$ **q** $\left(-\frac{2}{5}\right)^{-3}$ **r** $\left(-3\frac{2}{7}\right)^{-1}$ **s** $\left(-\frac{3}{8}\right)^{0}$ **t** $\left(-1\frac{1}{4}\right)^{-2}$

3 Change into index form:

a $\frac{1}{m^3}$ **b** $\frac{1}{x}$ **c** $\frac{1}{p^7}$ **d** $\frac{1}{d^9}$ **e** $\frac{1}{k^5}$

f $\frac{1}{x^2}$ **g** $\frac{2}{x^4}$ **h** $\frac{3}{y^2}$ **i** $\frac{1}{2z^6}$ **j** $\frac{3}{5t^8}$

k $\frac{2}{7x}$ **l** $\frac{5}{2m^6}$ **m** $\frac{2}{3y^7}$ **n** $\frac{1}{(3x+4)^2}$ **o** $\frac{1}{(a+b)^8}$

p $\frac{1}{x-2}$ **q** $\frac{1}{(5p+1)^3}$ **r** $\frac{2}{(4t-9)^5}$ **s** $\frac{1}{4(x+1)^{11}}$ **t** $\frac{5}{9(a+3b)^7}$

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4 Write without negative indices:

a	t^{-5}	**b**	x^{-6}	**c**	y^{-3}	**d**	n^{-8}	**e**	w^{-10}
f	$2x^{-1}$	**g**	$3m^{-4}$	**h**	$5x^{-7}$	**i**	$(2x)^{-3}$	**j**	$(4n)^{-1}$
k	$(x+1)^{-6}$	**l**	$(8y+z)^{-1}$	**m**	$(k-3)^{-2}$	**n**	$(3x+2y)^{-9}$	**o**	$\left(\frac{1}{x}\right)^{-5}$
p	$\left(\frac{1}{y}\right)^{-10}$	**q**	$\left(\frac{2}{p}\right)^{-1}$	**r**	$\left(\frac{1}{a+b}\right)^{-2}$	**s**	$\left(\frac{x+y}{x-y}\right)^{-1}$	**t**	$\left(\frac{2w-z}{3x+y}\right)^{-7}$

1.03 Fractional indices

INVESTIGATION

FRACTIONAL INDICES

Consider the following examples.

$\left(x^{\frac{1}{2}}\right)^2 = x^1$ (by index laws)

$= x$

$\left(\sqrt{x}\right)^2 = x$

So $\left(x^{\frac{1}{2}}\right)^2 = \left(\sqrt{x}\right)^2$

$= x$

$\therefore \quad x^{\frac{1}{2}} = \sqrt{x}$

Now simplify these expressions.

1	$\left(x^2\right)^{\frac{1}{2}}$	**2**	$\sqrt{x^2}$	**3**	$\left(x^{\frac{1}{3}}\right)^3$	**4**	$\left(x^3\right)^{\frac{1}{3}}$	**5**	$\left(\sqrt[3]{x}\right)^3$
6	$\sqrt[3]{x^3}$	**7**	$\left(x^{\frac{1}{4}}\right)^4$	**8**	$\left(x^4\right)^{\frac{1}{4}}$	**9**	$\left(\sqrt[4]{x}\right)^4$	**10**	$\sqrt[4]{x^4}$

Use your results to complete:

$x^{\frac{1}{n}} =$

WS Indices

WS Fractional indices and radicals

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Power of $\frac{1}{n}$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Proof

$\left(a^{\frac{1}{n}}\right)^n = a$ (by index laws)

$(\sqrt[n]{a})^n = a$

$\therefore a^{\frac{1}{n}} = \sqrt[n]{a}$

EXAMPLE 3

a Evaluate:

i $49^{\frac{1}{2}}$ **ii** $27^{\frac{1}{3}}$

b Write $\sqrt{3x-2}$ in index form.

c Write $(a+b)^{\frac{1}{7}}$ without fractional indices.

Solution

a **i** $49^{\frac{1}{2}} = \sqrt{49} = 7$ **ii** $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

b $\sqrt{3x-2} = (3x-2)^{\frac{1}{2}}$ **c** $(a+b)^{\frac{1}{7}} = \sqrt[7]{a+b}$

Further fractional indices

$$a^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{a}}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m$$

Proof

$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

$= \left(\sqrt[n]{a}\right)^m$

$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}}$

$= \sqrt[n]{a^m}$

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$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

Proof

$$\begin{aligned}\left(\frac{a}{b}\right)^{-n} &= \frac{1}{\left(\frac{a}{b}\right)^{n}} \\ &= \frac{1}{\frac{a^n}{b^n}} \\ &= 1 \div \frac{a^n}{b^n} \\ &= 1 \times \frac{b^n}{a^n} \\ &= \frac{b^n}{a^n} \\ &= \left(\frac{b}{a}\right)^{n}\end{aligned}$$

EXAMPLE 4

a Evaluate:

i $8^{\frac{4}{3}}$ **ii** $125^{-\frac{1}{3}}$ **iii** $\left(\frac{2}{3}\right)^{-3}$

b Write in index form:

i $\sqrt{x^5}$ **ii** $\frac{1}{\sqrt[3]{\left(4x^2-1\right)^2}}$

c Write $r^{-\frac{3}{5}}$ without the negative and fractional indices.

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Solution

a i $8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 \text{ (or } \sqrt[3]{8^4}\text{)}$

$= 2^4$

$= 16$

ii $125^{-\frac{1}{3}} = \frac{1}{125^{\frac{1}{3}}}$

$= \frac{1}{\sqrt[3]{125}}$

$= \frac{1}{5}$

iii $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3$

$= \frac{27}{8}$

$= 3\frac{3}{8}$

b i $\sqrt{x^5} = x^{\frac{5}{2}}$

ii $\frac{1}{\sqrt[3]{(4x^2-1)^2}} = \frac{1}{(4x^2-1)^{\frac{2}{3}}}$

$= (4x^2-1)^{-\frac{2}{3}}$

c $r^{-\frac{3}{5}} = \frac{1}{r^{\frac{3}{5}}} = \frac{1}{\sqrt[5]{r^3}}$

DID YOU KNOW?

Fractional indices

Nicole Oresme (1323–82) was the first mathematician to use fractional indices.

John Wallis (1616–1703) was the first person to explain the significance of zero, negative and fractional indices. He also introduced the symbol ∞ for infinity.

Research these mathematicians and find out more about their work and backgrounds. You could use keywords such as indices and infinity as well as their names to find this information.

Exercise 1.03 Fractional indices

1 Evaluate:

a $81^{\frac{1}{2}}$	**b** $27^{\frac{1}{3}}$	**c** $16^{\frac{1}{2}}$	**d** $8^{\frac{1}{3}}$	**e** $49^{\frac{1}{2}}$
f $1000^{\frac{1}{3}}$	**g** $16^{\frac{1}{4}}$	**h** $64^{\frac{1}{2}}$	**i** $64^{\frac{1}{3}}$	**j** $1^{\frac{1}{7}}$
k $81^{\frac{1}{4}}$	**l** $32^{\frac{1}{5}}$	**m** $0^{\frac{1}{8}}$	**n** $125^{\frac{1}{3}}$	**o** $343^{\frac{1}{3}}$
p $128^{\frac{1}{7}}$	**q** $256^{\frac{1}{4}}$	**r** $125^{\frac{2}{3}}$	**s** $4^{\frac{5}{2}}$	**t** $8^{\frac{2}{3}}$
u $9^{\frac{3}{2}}$	**v** $8^{-\frac{1}{3}}$	**w** $9^{-\frac{1}{2}}$	**x** $16^{-\frac{1}{4}}$	**y** $64^{-\frac{2}{3}}$

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2 Evaluate correct to 2 decimal places:

a $23^{\frac{1}{4}}$ **b** $\sqrt[4]{45.8}$ **c** $\sqrt[7]{1.24+4.3^2}$

d $\dfrac{1}{\sqrt[5]{12.9}}$ **e** $\sqrt[8]{\dfrac{3.6-1.4}{1.5+3.7}}$ **f** $\dfrac{\sqrt[4]{5.9\times 3.7}}{8.79-1.4}$

3 Write without fractional or negative indices:

a $y^{\frac{1}{3}}$ **b** $x^{\frac{1}{6}}$ **c** $a^{\frac{1}{2}}$ **d** $t^{\frac{1}{9}}$ **e** $y^{\frac{2}{3}}$

f $x^{\frac{3}{4}}$ **g** $b^{\frac{2}{5}}$ **h** $a^{\frac{4}{7}}$ **i** $x^{-\frac{1}{2}}$ **j** $d^{-\frac{1}{3}}$

k $x^{-\frac{1}{8}}$ **l** $y^{-\frac{1}{3}}$ **m** $a^{-\frac{1}{4}}$ **n** $z^{-\frac{3}{4}}$ **o** $y^{-\frac{3}{5}}$

p $(2x+5)^{\frac{1}{2}}$ **q** $(6q+r)^{\frac{1}{3}}$ **r** $(a+b)^{\frac{1}{9}}$ **s** $(3x-1)^{-\frac{1}{2}}$ **t** $(x+7)^{-\frac{2}{5}}$

4 Write in index form:

a $\sqrt{t}$ **b** $\sqrt[5]{y}$ **c** $\sqrt{x^3}$ **d** $\sqrt[3]{9-x}$ **e** $\sqrt{4s+1}$

f $\sqrt{(3x+1)^5}$ **g** $\dfrac{1}{\sqrt{2t+3}}$ **h** $\dfrac{1}{\sqrt{(5x-y)^3}}$ **i** $\dfrac{1}{\sqrt[3]{(x-2)^2}}$ **j** $\dfrac{1}{2\sqrt{y+7}}$

k $\dfrac{5}{\sqrt[3]{x+4}}$ **l** $\dfrac{1}{3\sqrt{y^2-1}}$ **m** $\dfrac{3}{5\sqrt[4]{(x^2+2)^3}}$

5 Write in index form and simplify:

a $x\sqrt{x}$ **b** $\dfrac{\sqrt{x}}{x}$ **c** $\dfrac{x}{\sqrt[3]{x}}$ **d** $\dfrac{x^2}{\sqrt[3]{x}}$ **e** $x\sqrt[4]{x}$

6 Write without fractional or negative indices:

a $(a-2b)^{-\frac{1}{3}}$ **b** $(y-3)^{-\frac{2}{3}}$ **c** $4(6a+1)^{-\frac{4}{7}}$ **d** $\dfrac{(x+y)^{-\frac{5}{4}}}{3}$ **e** $\dfrac{6(3x+8)^{-\frac{2}{9}}}{7}$

DID YOU KNOW?

The beginnings of algebra

One of the earliest mathematicians to use algebra was **Diophantus of Alexandria** in Greece. It is not known when he lived, but it is thought this may have been around 250 CE.

In Persia around 700–800 CE a mathematician named **Muhammad ibn Musa al-Khwarizmi** wrote books on algebra and Hindu numerals. One of his books was named *Al-Jabr wa'l Muqabala*, and the word **algebra** comes from the first word in this title.

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1.04 Simplifying algebraic expressions

EXAMPLE 5

Simplify:

a $4x^2 - 3x^2 + 6x^2$ **b** $x^3 - 3x - 5x + 4$ **c** $3a - 4b - 5a - b$

Solution

a $4x^2 - 3x^2 + 6x^2 = x^2 + 6x^2$
$= 7x^2$

b $x^3 - 3x - 5x + 4 = x^3 - 8x + 4$

Only 'like' terms can be added or subtracted.

c $3a - 4b - 5a - b = 3a - 5a - 4b - b$
$= -2a - 5b$

EXAMPLE 6

Simplify:

a $-5x \times 3y \times 2x$ **b** $\dfrac{5a^3b}{15ab^2}$

Solution

a $-5x \times 3y \times 2x = -30xyx$
$= -30x^2y$

b $\dfrac{5a^3b}{15ab^2} = \dfrac{1}{3}a^{3-1}b^{1-2}$
$= \dfrac{1}{3}a^2b^{-1}$
$= \dfrac{a^2}{3b}$

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Exercise 1.04 Simplifying algebraic expressions

1 Simplify:

a $9a - 6a$ **b** $5z - 4z$ **c** $4b - b$

d $2r - 5r$ **e** $-4y + 3y$ **f** $-2x - 3x$

g $2a - 2a$ **h** $-4k + 7k$ **i** $3t + 4t + 2t$

j $8w - w + 3w$ **k** $4m - 3m - 2m$ **l** $x + 3x - 5x$

m $8h - h - 7h$ **n** $3b - 5b + 4b + 9b$ **o** $-5x + 3x - x - 7x$

p $6x - 5y - y$ **q** $8a + b - 4b - 7a$ **r** $xy + 2y + 3xy$

s $2ab^2 - 5ab^2 - 3ab^2$ **t** $m^2 - 5m - m + 12$ **u** $p^2 - 7p + 5p - 6$

v $ab + 2b - 3ab + 8b$ **w** $ab + bc - ab - ac + bc$

x $a^5 - 7x^3 + a^5 - 2x^3 + 1$ **y** $x^3 - 3xy^2 + 4x^2y - x^2y + xy^2 + 2y^3$

2 Simplify:

a $5 \times 2b$ **b** $2x \times 4y$ **c** $5p \times 2p$

d $-3z \times 2w$ **e** $-5a \times -3b$ **f** $x \times 2y \times 7z$

g $8ab \times 6c$ **h** $4d \times 3d$ **i** $3a \times 4a \times a$

j $(-3y)^3$ **k** $(2x^2)^5$ **l** $2ab^3 \times 3a$

m $5a^2b \times -2ab$ **n** $7pq^2 \times 3p^2q^2$ **o** $5ab \times a^2b^2$

p $4h^3 \times -2h^7$ **q** $k^3p \times p^2$ **r** $(-3t^3)^4$

s $7m^6 \times -2m^5$ **t** $-2x^2 \times 3x^3y \times -4xy^2$

3 Simplify:

a $30x \div 5$ **b** $2y \div y$ **c** $\dfrac{8a^2}{2}$

d $\dfrac{8a^2}{a}$ **e** $\dfrac{8a^2}{2a}$ **f** $\dfrac{xy}{2x}$

g $12p^3 \div 4p^2$ **h** $\dfrac{3a^2b^2}{6ab}$ **i** $\dfrac{20x}{15xy}$

j $\dfrac{-9x^7}{3x^4}$ **k** $-15ab \div -5b$ **l** $\dfrac{2ab}{6a^2b^3}$

m $\dfrac{-8p}{4pqs}$ **n** $14cd^2 \div 21c^3d^3$ **o** $\dfrac{2xy^2z^3}{4x^3y^2z}$

p $\dfrac{42p^5q^4}{7pq^3}$ **q** $5a^9b^4c^{-2} \div 20a^5b^{-3}c^{-1}$ **r** $\dfrac{2(a^{-5})^2b^4}{4a^{-9}(b^2)^{-1}}$

s $-5x^4y^7z \div 15xy^8z^{-2}$ **t** $-9(a^4b^{-1})^3 \div -18a^{-1}b^3$

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WS

Expanding algebraic expressions

1.05 Expansion

When we remove grouping symbols we say that we are **expanding** an expression.

Expanding expressions

To expand an expression, use the distributive law:

$$a(b + c) = ab + ac$$

EXAMPLE 7

Expand and simplify:

a $5a^2(4 + 3ab - c)$ **b** $5 - 2(y + 3)$ **c** $2(b - 5) - (b + 1)$

Solution

a $5a^2(4 + 3ab - c) = 5a^2 \times 4 + 5a^2 \times 3ab - 5a^2 \times c$

$= 20a^2 + 15a^3b - 5a^2c$

b $5 - 2(y + 3) = 5 - 2 \times y - 2 \times 3$

$= 5 - 2y - 6$

$= -2y - 1$

c $2(b - 5) - (b + 1) = 2 \times b + 2 \times -5 - 1 \times b - 1 \times 1$

$= 2b - 10 - b - 1$

$= b - 11$

Exercise 1.05 Expansion

Expand and simplify each expression.

1 $2(x - 4)$ **2** $3(2h + 3)$ **3** $-5(a - 2)$

4 $x(2y + 3)$ **5** $x(x - 2)$ **6** $2a(3a - 8b)$

7 $ab(2a + b)$ **8** $5n(n - 4)$ **9** $3x^2y(xy + 2y^2)$

10 $3 + 4(k + 1)$ **11** $2(t - 7) - 3$ **12** $y(4y + 3) + 8y$

13 $9 - 5(b + 3)$ **14** $3 - (2x - 5)$ **15** $5(3 - 2m) + 7(m - 2)$

16 $2(h + 4) + 3(2h - 9)$ **17** $3(2d - 3) - (5d - 3)$ **18** $a(2a + 1) - (a^2 + 3a - 4)$

19 $x(3x - 4) - 5(x + 1)$ **20** $2ab(3 - a) - b(4a - 1)$ **21** $5x - (x - 2) - 3$

22 $8 - 4(2y + 1) + y$ **23** $(a + b) - (a - b)$ **24** $2(3t - 4) - (t + 1) + 3$

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1.06 Binomial products


Expanding binomial products

A **binomial expression** consists of 2 **terms**; for example, $x + 3$.

A set of 2 binomial expressions multiplied together is called a **binomial product**; for example, $(x + 3)(x - 2)$.

Each term in the first bracket is multiplied by each term in the second bracket.

Binomial product

$$(x + a)(x + b) = x^2 + bx + ax + ab$$

EXAMPLE 8

Expand and simplify:

a $(p + 3)(q - 4)$ **b** $(a + 5)^2$ **c** $(x + 4)(2x - 3y - 1)$

Solution

a $(p + 3)(q - 4) = pq - 4p + 3q - 12$

b $(a + 5)^2 = (a + 5)(a + 5)$
$= a^2 + 5a + 5a + 25$
$= a^2 + 10a + 25$

c $(x + 4)(2x - 3y - 1) = 2x^2 - 3xy - x + 8x - 12y - 4$
$= 2x^2 - 3xy + 7x - 12y - 4$

Exercise 1.06 Binomial products

Expand and simplify:

1 $(a + 5)(a + 2)$ **2** $(x + 3)(x - 1)$ **3** $(2y - 3)(y + 5)$ **4** $(m - 4)(m - 2)$

5 $(x + 4)(x + 3)$ **6** $(y + 2)(y - 5)$ **7** $(2x - 3)(x + 2)$ **8** $(h - 7)(h - 3)$

9 $(x + 5)(x - 5)$ **10** $(5a - 4)(3a - 1)$ **11** $(2y + 3)(4y - 3)$ **12** $(x - 4)(y + 7)$

13 $(x^2 + 3)(x - 2)$ **14** $(n + 2)(n - 2)$ **15** $(2x + 3)(2x - 3)$ **16** $(4 - 7y)(4 + 7y)$

17 $(a + 2b)(a - 2b)$ **18** $(3x - 4y)(3x + 4y)$ **19** $(x + 3)(x - 3)$ **20** $(y - 6)(y + 6)$

21 $(3a + 1)(3a - 1)$ **22** $(2z - 7)(2z + 7)$ **23** $(x + 9)(x - 2y + 2)$

24 $(b - 3)(2a + 2b - 1)$ **25** $(x + 2)(x^2 - 2x + 4)$ **26** $(a - 3)(a^2 + 3a + 9)$

27 $(a + 9)^2$ **28** $(k - 4)^2$ **29** $(x + 2)^2$ **30** $(y - 7)^2$

31 $(2x + 3)^2$ **32** $(2t - 1)^2$ **33** $(3a + 4b)^2$ **34** $(x - 5y)^2$

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35 $(2a+b)^2$ **36** $(a-b)(a+b)$ **37** $(a+b)^2$ **38** $(a-b)^2$

39 $(a+b)(a^2-ab+b^2)$ **40** $(a-b)(a^2+ab+b^2)$

WS Expanding expressions

1.07 Special products

Some binomial products have special results and can be simplified quickly using their special properties. Did you notice some of these in Exercise 1.06?

Special binomial products

Difference of two squares

$$(a+b)(a-b) = a^2 - b^2$$

Perfect squares

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

EXAMPLE 9

Expand and simplify:

a $(2x-3)^2$ **b** $(3y-4)(3y+4)$

Solution

a $(2x-3)^2 = (2x)^2 - 2(2x)3 + 3^2$
$= 4x^2 - 12x + 9$

b $(3y-4)(3y+4) = (3y)^2 - 4^2$
$= 9y^2 - 16$

Exercise 1.07 Special products

Expand and simplify:

1 $(t+4)^2$ **2** $(z-6)^2$ **3** $(x-1)^2$

4 $(y+8)^2$ **5** $(q+3)^2$ **6** $(k-7)^2$

7 $(n+1)^2$ **8** $(2b+5)^2$ **9** $(3-x)^2$

10 $(3y-1)^2$ **11** $(x+y)^2$ **12** $(3a-b)^2$

13 $(4d+5e)^2$ **14** $(t+4)(t-4)$ **15** $(x-3)(x+3)$

16 $(p+1)(p-1)$ **17** $(r+6)(r-6)$ **18** $(x-10)(x+10)$

19 $(2a+3)(2a-3)$ **20** $(x-5y)(x+5y)$ **21** $(4a+1)(4a-1)$

ISBN 9780170413152

22 $(7 - 3x)(7 + 3x)$ **23** $(x^2 + 2)(x^2 - 2)$ **24** $(x^2 + 5)^2$

25 $(3ab - 4c)(3ab + 4c)$ **26** $\left(x + \frac{2}{x}\right)^2$ **27** $\left(a - \frac{1}{a}\right)\left(a + \frac{1}{a}\right)$

28 $[x + (y - 2)][x - (y - 2)]$ **29** $[(a + b) + c]^2$ **30** $[(x + 1) - y]^2$

31 $(a + 3)^2 - (a - 3)^2$ **32** $16 - (z - 4)(z + 4)$ **33** $2x + (3x + 1)^2 - 4$

34 $(x + y)^2 - x(2 - y)$ **35** $(4n - 3)(4n + 3) - 2n^2 + 5$ **36** $(x - 4)^3$

37 $\left(x - \frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^2 + 2$ **38** $(x^2 + y^2)^2 - 4x^2y^2$ **39** $(2a + 5)^3$

1.08 Factorisation

WS
Factorising algebraic expressions

Factors divide exactly into an equal or larger number or term, without leaving a remainder.

Factorising

To **factorise** an expression, we use the distributive law in the opposite way from when we expand brackets.

$$ax + bx = x(a + b)$$

EXAMPLE 10

Factorise:

a $3x + 12$ **b** $y^2 - 2y$ **c** $x^3 - 2x^2$

d $5(x + 3) + 2y(x + 3)$ **e** $8a^3b^2 - 2ab^3$

Solution

a The highest common factor is 3. $3x + 12 = 3(x + 4)$

b The highest common factor is y. $y^2 - 2y = y(y - 2)$

c x and x^2 are both common factors. Take out the highest common factor, which is x^2. $x^3 - 2x^2 = x^2(x - 2)$

d The highest common factor is $x + 3$. $5(x + 3) + 2y(x + 3) = (x + 3)(5 + 2y)$

e The highest common factor is $2ab^2$. $8a^3b^2 - 2ab^3 = 2ab^2(4a^2 - b)$

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Exercise 1.08 Factorisation

Factorise:

1 $2y + 6$ **2** $5x - 10$ **3** $3m - 9$

4 $8x + 2$ **5** $24 - 18y$ **6** $x^2 + 2x$

7 $m^2 - 3m$ **8** $2y^2 + 4y$ **9** $15a - 3a^2$

10 $ab^2 + ab$ **11** $4x^2y - 2xy$ **12** $3mn^3 + 9mn$

13 $8x^2z - 2xz^2$ **14** $6ab + 3a - 2a^2$ **15** $5x^2 - 2x + xy$

16 $3q^5 - 2q^2$ **17** $5b^3 + 15b^2$ **18** $6a^2b^3 - 3a^3b^2$

19 $x(m + 5) + 7(m + 5)$ **20** $2(y - 1) - y(y - 1)$ **21** $4(7 + y) - 3x(7 + y)$

22 $6x(a - 2) + 5(a - 2)$ **23** $x(2t + 1) - y(2t + 1)$

24 $a(3x - 2) + 2b(3x - 2) - 3c(3x - 2)$ **25** $6x^3 + 9x^2$

26 $3pq^5 - 6q^3$ **27** $15a^4b^3 + 3ab$ **28** $4x^3 - 24x^2$

29 $35m^3n^4 - 25m^2n$ **30** $24a^2b^5 + 16ab^2$ **31** $2\pi r^2 + 2\pi rh$

32 $(x - 3)^2 + 5(x - 3)$ **33** $y^2(x + 4) + 2(x + 4)$ **34** $a(a + 1) - (a + 1)^2$

1.09 Factorisation by grouping in pairs

Factorising by grouping in pairs

If an expression has 4 terms, it can sometimes be factorised in pairs.

$$ax + bx + ay + by = x(a + b) + y(a + b)$$
$$= (a + b)(x + y)$$

EXAMPLE 11

Factorise:

a $x^2 - 2x + 3x - 6$ **b** $2x - 4 + 6y - 3xy$

Solution

a $x^2 - 2x + 3x - 6 = x(x - 2) + 3(x - 2)$
$= (x - 2)(x + 3)$

b $2x - 4 + 6y - 3xy = 2(x - 2) + 3y(2 - x)$
$= 2(x - 2) - 3y(x - 2)$
$= (x - 2)(2 - 3y)$

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Exercise 1.09 Factorisation by grouping in pairs

Factorise:

1 $2x + 8 + bx + 4b$ **2** $ay - 3a + by - 3b$ **3** $x^2 + 5x + 2x + 10$

4 $m^2 - 2m + 3m - 6$ **5** $ad - ac + bd - bc$ **6** $x^3 + x^2 + 3x + 3$

7 $5ab - 3b + 10a - 6$ **8** $2xy - x^2 + 2y^2 - xy$ **9** $ay + a + y + 1$

10 $x^2 + 5x - x - 5$ **11** $y + 3 + ay + 3a$ **12** $m - 2 + 4y - 2my$

13 $2x^2 + 10xy - 3xy - 15y^2$ **14** $a^2b + ab^3 - 4a - 4b^2$ **15** $5x - x^2 - 3x + 15$

16 $x^4 + 7x^3 - 4x - 28$ **17** $7x - 21 - xy + 3y$ **18** $4d + 12 - de - 3e$

19 $3x - 12 + xy - 4y$ **20** $2a + 6 - ab - 3b$ **21** $x^3 - 3x^2 + 6x - 18$

22 $pq - 3p + q^2 - 3q$ **23** $3x^3 - 6x^2 - 5x + 10$ **24** $4a - 12b + ac - 3bc$

25 $xy + 7x - 4y - 28$ **26** $x^4 - 4x^3 - 5x + 20$ **27** $4x^3 - 6x^2 + 8x - 12$

28 $3a^2 + 9a + 6ab + 18b$ **29** $5y - 15 + 10xy - 30x$ **30** $\pi r^2 + 2\pi r - 3r - 6$

1.10 Factorising trinomials



Factorising quadratic expressions

A **trinomial** is an expression with 3 terms; for example, $x^2 - 4x + 3$. Factorising a trinomial usually gives a **binomial product**.

We know that: $(x + a)(x + b) = x^2 + bx + ax + ab$

$= x^2 + (a + b)x + ab$

Factorising trinomials

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

Find values for a and b so that the sum $a + b$ is the middle term and the product ab is the last term.

EXAMPLE 12

Factorise:

a $m^2 - 5m + 6$

b $y^2 + y - 2$

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Solution

a $a + b = -5$ and $ab = 6$

To have $a + b = -5$, at least one number must be negative.

To have $ab = 6$, both numbers have the same sign. So both are negative.

For $ab = 6$: we could have $-6 \times (-1)$ or $-3 \times (-2)$

$-3 + (-2) = -5$ so $a = -3$ and $b = -2$.

So $m^2 - 5m + 6 = (m - 3)(m - 2)$

Check: $(m - 3)(m - 2) = m^2 - 2m - 3m + 6$

$= m^2 - 5m + 6$

b $a + b = 1$ and $ab = -2$

To have $ab = -2$, the numbers must have opposite signs. So one is positive and one is negative.

For $ab = -2$: we could have -2×1 or -1×2

$-1 + 2 = 1$ so $a = -1$ and $b = 2$.

So $y^2 + y - 2 = (y - 1)(y + 2)$

Check: $(y - 1)(y + 2) = y^2 + 2y - y - 2$

$= y^2 + y - 2$

Exercise 1.10 Factorising trinomials

Factorise:

1 $x^2 + 4x + 3$

2 $y^2 + 7y + 12$

3 $m^2 + 2m + 1$

4 $t^2 + 8t + 16$

5 $z^2 + z - 6$

6 $x^2 - 5x - 6$

7 $v^2 - 8v + 15$

8 $t^2 - 6t + 9$

9 $x^2 + 9x - 10$

10 $y^2 - 10y + 21$

11 $m^2 - 9m + 18$

12 $y^2 + 9y - 36$

13 $x^2 - 5x - 24$

14 $a^2 - 4a + 4$

15 $x^2 + 14x - 32$

16 $y^2 - 5y - 36$

17 $n^2 - 10n + 24$

18 $x^2 - 10x + 25$

19 $p^2 + 8p - 9$

20 $k^2 - 7k + 10$

21 $x^2 + x - 12$

22 $m^2 - 6m - 7$

23 $q^2 + 12q + 20$

24 $d^2 - 4d - 5$

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1.11 Further trinomials

When the coefficient of the first term is not 1, for example $5x^2 - 13x + 6$, we need to use a different method to factorise the trinomial.

The coefficient of the first term is the number in front of the x^2.

This method still involves finding 2 numbers that give a required sum and product but it also involves grouping in pairs.

Factorising quadratic expressions (Advanced)

Excel worksheet: Factorising trinomials

Excel spreadsheet: Factorising trinomials

EXAMPLE 13

Factorise:

a $5x^2 - 13x + 6$

b $4y^2 + 4y - 3$

Solution

a First, multiply the coefficient of the first term by the last term: $5 \times 6 = 30$.

Now $a + b = -13$ and $ab = 30$.

Since the sum is negative and the product is positive, a and b must be both negative.

2 numbers with product 30 and sum -13 are -10 and -3.

Now write the trinomial with the middle term split into 2 terms $-10x$ and $-3x$, and then factorise by grouping in pairs.

$$\begin{aligned} 5x^2 - 13x + 6 &= 5x^2 - 10x - 3x + 6 \\ &= 5x(x - 2) - 3(x - 2) \end{aligned}$$

If you factorise correctly, you should always find a common factor remaining, such as $(x - 2)$ here.

$$= (x - 2)(5x - 3)$$

b First, multiply the coefficient of the first term by the last term: $4(-3) = -12$

Now $a + b = 4$ and $ab = -12$.

Since the product is negative, a and b have opposite signs (one positive and one negative).

2 numbers with product -12 and sum 4 are 6 and -2.

Now write the trinomial with the middle term split into 2 terms $6y$ and $-2y$, and then factorise by grouping in pairs.

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$$4y^2 + 4y - 3 = 4y^2 + 6y - 2y - 3$$
$$= 2y(2y + 3) - 1(2y + 3)$$
$$= (2y + 3)(2y - 1)$$

There are other ways of factorising these trinomials. Your teacher may show you some of these.

Exercise 1.11 Further trinomials

Factorise:

1 $2a^2 + 11a + 5$ **2** $5y^2 + 7y + 2$ **3** $3x^2 + 10x + 7$

4 $3x^2 + 8x + 4$ **5** $2b^2 - 5b + 3$ **6** $7x^2 - 9x + 2$

7 $3y^2 + 5y - 2$ **8** $2x^2 + 11x + 12$ **9** $5p^2 + 13p - 6$

10 $6x^2 + 13x + 5$ **11** $2y^2 - 11y - 6$ **12** $10x^2 + 3x - 1$

13 $8t^2 - 14t + 3$ **14** $6x^2 - x - 12$ **15** $6y^2 + 47y - 8$

16 $4n^2 - 11n + 6$ **17** $8t^2 + 18t - 5$ **18** $12q^2 + 23q + 10$

19 $4r^2 + 11r - 3$ **20** $4x^2 - 4x - 15$ **21** $6y^2 - 13y + 2$

22 $6p^2 - 5p - 6$ **23** $8x^2 + 31x + 21$ **24** $12b^2 - 43b + 36$

25 $6x^2 - 53x - 9$ **26** $9x^2 + 30x + 25$ **27** $16y^2 + 24y + 9$

28 $25k^2 - 20k + 4$ **29** $36a^2 - 12a + 1$ **30** $49m^2 + 84m + 36$

1.12 Perfect squares

You have looked at expanding $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$.
These are called **perfect squares**.

When factorising, use these results the other way around.

EXAMPLE 14

Factorise:

a $x^2 - 8x + 16$ **b** $4a^2 + 20a + 25$

Solution

a $x^2 - 8x + 16 = x^2 - 2(4)x + 4^2$
$= (x - 4)^2$

b $4a^2 + 20a + 25 = (2a)^2 + 2(2a)(5) + 5^2$
$= (2a + 5)^2$

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Exercise 1.12 Perfect squares

Factorise:

1 $y^2 - 2y + 1$ **2** $x^2 + 6x + 9$ **3** $m^2 + 10m + 25$

4 $t^2 - 4t + 4$ **5** $x^2 - 12x + 36$ **6** $4x^2 + 12x + 9$

7 $16b^2 - 8b + 1$ **8** $9a^2 + 12a + 4$ **9** $25x^2 - 40x + 16$

10 $49y^2 + 14y + 1$ **11** $9y^2 - 30y + 25$ **12** $16k^2 - 24k + 9$

13 $25x^2 + 10x + 1$ **14** $81a^2 - 36a + 4$ **15** $49m^2 + 84m + 36$

16 $t^2 + t + \frac{1}{4}$ **17** $x^2 - \frac{4x}{3} + \frac{4}{9}$ **18** $9y^2 + \frac{6y}{5} + \frac{1}{25}$

19 $x^2 + 2 + \frac{1}{x^2}$ **20** $25k^2 - 20 + \frac{4}{k^2}$

1.13 Difference of two squares

Difference of two squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 15

Factorise:

a $d^2 - 36$ **b** $1 - 9b^2$ **c** $(a + 3)^2 - (b - 1)^2$

Solution

a $d^2 - 36 = d^2 - 6^2$

$= (d + 6)(d - 6)$

b $1 - 9b^2 = 1^2 - (3b)^2$

$= (1 + 3b)(1 - 3b)$

c $(a + 3)^2 - (b - 1)^2 = [(a + 3) + (b - 1)][(a + 3) - (b - 1)]$

$= (a + 3 + b - 1)(a + 3 - b + 1)$

$= (a + b + 2)(a - b + 4)$

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Exercise 1.13 Difference of two squares

Factorise:

1 $a^2 - 4$ **2** $x^2 - 9$ **3** $y^2 - 1$

4 $x^2 - 25$ **5** $4x^2 - 49$ **6** $16y^2 - 9$

7 $1 - 4z^2$ **8** $25t^2 - 1$ **9** $9t^2 - 4$

10 $9 - 16x^2$ **11** $x^2 - 4y^2$ **12** $36x^2 - y^2$

13 $4a^2 - 9b^2$ **14** $x^2 - 100y^2$ **15** $4a^2 - 81b^2$

16 $(x + 2)^2 - y^2$ **17** $(a - 1)^2 - (b - 2)^2$ **18** $z^2 - (1 + w)^2$

19 $x^2 - \frac{1}{4}$ **20** $\frac{y^2}{9} - 1$ **21** $(x + 2)^2 - (2y + 1)^2$

22 $x^4 - 1$ **23** $9x^6 - 4y^2$ **24** $x^4 - 16y^4$

1.14 Mixed factorisation

WS Factorising expressions

EXAMPLE 16

Factorise $5x^2 - 45$.

Solution

Using simple factors: $5x^2 - 45 = 5(x^2 - 9)$

The difference of 2 squares: $= 5(x + 3)(x - 3)$

Exercise 1.14 Mixed factorisation

Factorise:

1 $4a^3 - 36a$ **2** $2x^2 - 18$ **3** $3p^2 - 3p - 36$

4 $5y^2 - 5$ **5** $5a^2 - 10a + 5$ **6** $3z^3 + 27z^2 + 60z$

7 $9ab - 4a^3b^3$ **8** $x^3 - x$ **9** $6x^2 + 8x - 8$

10 $y^2(y + 5) - 16(y + 5)$ **11** $x^4 + 8x^3 - x^2 - 8x$ **12** $y^6 - 4$

13 $x^3 - 3x^2 - 10x$ **14** $x^3 - 3x^2 - 9x + 27$ **15** $4x^2y^3 - y$

16 $24 - 6b^2$ **17** $18x^2 + 33x - 30$ **18** $3x^2 - 6x + 3$

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19 $x^3 + 2x^2 - 25x - 50$ **20** $z^3 + 6z^2 + 9z$ **21** $3y^2 + 30y + 75$

22 $ab^2 - 9a$ **23** $4k^3 + 40k^2 + 100k$ **24** $3x^3 + 9x^2 - 3x - 9$

25 $4a^3b + 8a^2b^2 - 4ab^2 - 2a^2b$

1.15 Simplifying algebraic fractions

EXAMPLE 17

Simplify:

a $\dfrac{4x+2}{2}$ **b** $\dfrac{2x^2-3x-2}{x^2-4}$

Solution

a $\dfrac{4x+2}{2} = \dfrac{2(2x+1)}{2}$

$= 2x + 1$

b Factorise both top and bottom.

$\dfrac{2x^2-3x-2}{x^2-4} = \dfrac{(2x+1)(x-2)}{(x-2)(x+2)}$

$= \dfrac{2x+1}{x+2}$

Exercise 1.15 Simplifying algebraic fractions

Simplify:

1 $\dfrac{5a+10}{5}$ **2** $\dfrac{6t-3}{3}$ **3** $\dfrac{8y+2}{6}$

4 $\dfrac{8}{4d-2}$ **5** $\dfrac{x^2}{5x^2-2x}$ **6** $\dfrac{y-4}{y^2-8y+16}$

7 $\dfrac{2ab-4a^2}{a^2-3a}$ **8** $\dfrac{s^2+s-2}{s^2+5s+6}$ **9** $\dfrac{b^4-1}{b^2-1}$

10 $\dfrac{2p^2+7p-15}{6p-9}$ **11** $\dfrac{a^2-1}{a^2+2a-3}$ **12** $\dfrac{3(x-2)+y(x-2)}{x^2-4}$

13 $\dfrac{x^3+3x^2-9x-27}{x^2+6x+9}$ **14** $\dfrac{2p^2-3p-2}{2p^2+p}$ **15** $\dfrac{ay-ax+by-bx}{2ay-by-2ax+bx}$

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WS
Algebraic fractions

1.16 Operations with algebraic fractions

EXAMPLE 18

Simplify:

a $\frac{x-1}{5}-\frac{x+3}{4}$ **b** $\frac{2a^2b+10ab}{b^2-9}\div\frac{a^2-25}{4b+12}$ **c** $\frac{2}{x-5}+\frac{1}{x+2}$ **d** $\frac{2}{x+1}-\frac{1}{x^2-1}$

Solution

a
$$\begin{aligned}\frac{x-1}{5}-\frac{x+3}{4}&=\frac{4(x-1)-5(x+3)}{20}\\&=\frac{4x-4-5x-15}{20}\\&=\frac{-x-19}{20}\end{aligned}$$

b
$$\begin{aligned}\frac{2a^2b+10ab}{b^2-9}\div\frac{a^2-25}{4b+12}&=\frac{2a^2b+10ab}{b^2-9}\times\frac{4b+12}{a^2-25}\\&=\frac{2ab(a+5)}{(b+3)(b-3)}\times\frac{4(b+3)}{(a+5)(a-5)}\\&=\frac{8ab}{(a-5)(b-3)}\end{aligned}$$

c
$$\begin{aligned}\frac{2}{x-5}+\frac{1}{x+2}&=\frac{2(x+2)+1(x-5)}{(x-5)(x+2)}\\&=\frac{2x+4+x-5}{(x-5)(x+2)}\\&=\frac{3x-1}{(x-5)(x+2)}\end{aligned}$$

d
$$\begin{aligned}\frac{2}{x+1}-\frac{1}{x^2-1}&=\frac{2}{x+1}-\frac{1}{(x+1)(x-1)}\\&=\frac{2(x-1)}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2-1}{(x+1)(x-1)}\\&=\frac{2x-3}{(x+1)(x-1)}\end{aligned}$$

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Exercise 1.16 Operations with algebraic fractions

1 Simplify:

a $\frac{x}{2}+\frac{3x}{4}$ **b** $\frac{y+1}{5}+\frac{2y}{3}$ **c** $\frac{a+2}{3}-\frac{a}{4}$

d $\frac{p-3}{6}+\frac{p+2}{2}$ **e** $\frac{x-5}{2}-\frac{x-1}{3}$

2 Simplify:

a $\frac{3x+6}{5}\times\frac{10}{x+2}$ **b** $\frac{a^2-4}{3}\times\frac{5b}{a+2}$

c $\frac{t^2+3t-10}{xy^2}\div\frac{5t-10}{2xy}$ **d** $\frac{2a-6}{2x+4}\times\frac{5x+10}{4}$

e $\frac{5x+10-xy-2y}{15}\div\frac{7x+14}{3}$ **f** $\frac{3}{b+2}\times\frac{b^2+2b}{6a-3}$

g $\frac{3ab^2}{5xy}\div\frac{12ab-6a}{x^2y+2xy^2}$ **h** $\frac{ax-ay+bx-by}{x^2-y^2}\times\frac{x^2y+xy^2}{ab^2+a^2b}$

i $\frac{x^2-6x+9}{x^2-25}\div\frac{x^2-5x+6}{x^2+4x-5}$ **j** $\frac{p^2-4}{q^2+2q+1}\times\frac{5q+5}{3p+6}$

3 Simplify:

a $\frac{2}{x}+\frac{3}{x}$ **b** $\frac{1}{x-1}-\frac{2}{x}$ **c** $1+\frac{3}{a+b}$

d $x-\frac{x^2}{x+2}$ **e** $p-q+\frac{1}{p+q}$ **f** $\frac{1}{x+1}+\frac{1}{x-3}$

g $\frac{2}{x^2-4}-\frac{3}{x+2}$ **h** $\frac{1}{a^2+2a+1}+\frac{1}{a+1}$

4 Simplify:

a $\frac{a^2-5a}{y^2-4y+4}\div\frac{3a-15}{y^2-4}\times\frac{y^2-y-2}{5ay}$ **b** $\frac{3}{x-3}+\frac{2x+8}{x^2-9}\times\frac{x^2+3x}{4x-16}$

c $\frac{5b}{2b+6}\div\frac{b^2}{b^2+b-6}-\frac{b}{b+1}$ **d** $\frac{x^2-8x+15}{5x^2+10x}\div\frac{x^2-9}{10x^2}\times\frac{x^2+5x+6}{2x-10}$

5 Simplify:

a $\frac{5}{x^2-4}-\frac{3}{x-2}-\frac{2}{x+2}$ **b** $\frac{2}{p^2+pq}+\frac{3}{pq-q^2}$ **c** $\frac{a}{a+b}-\frac{b}{a-b}+\frac{1}{a^2-b^2}$

1.17 Substitution

Algebra is used for writing general formulas or rules, and we substitute numbers into these formulas to solve a problem.

EXAMPLE 19

a $V = \pi r^2 h$ is the formula for finding the volume of a cylinder with radius r and height h. Find V (correct to 1 decimal place) when $r = 2.1$ and $h = 8.7$.

b If $F = \frac{9C}{5} + 32$ is the formula for converting degrees Celsius (°C) into degrees Fahrenheit (°F), find F when $C = 25$.

Solution

a When $r = 2.1$, $h = 8.7$,

$$V = \pi r^2 h$$
$$= \pi(2.1)^2(8.7)$$
$$= 120.533\ldots$$
$$\approx 120.5$$

b When $C = 25$,

$$F = \frac{9C}{5} + 32$$
$$= \frac{9(25)}{5} + 32$$
$$= 77$$

This means that 25°C is the same as 77°F.

Exercise 1.17 Substitution

1 Given $a = 3.1$ and $b = -2.3$ find, correct to 1 decimal place:

a ab **b** $3b$ **c** $5a^2$ **d** ab^3

e $(a+b)^2$ **f** $\sqrt{a-b}$ **g** $-b^2$

2 For the formula $T = a + (n-1)d$, find T when $a = -4$, $n = 18$ and $d = 3$.

3 Given $y = mx + c$, the equation of a straight line, find y if $m = 3$, $x = -2$ and $c = -1$.

4 If $h = 100t - 5t^2$ is the height of a particle at time t, find h when $t = 5$.

5 Given vertical velocity $v = -gt$, find v when $g = 9.8$ and $t = 20$.

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6 If $y = 2^x + 3$ is the equation of a function, find y when $x = 1.3$, correct to 1 decimal place.

7 $S = 2\pi r(r + h)$ is the formula for the surface area of a cylinder. Find S when $r = 5$ and $h = 7$, correct to the nearest whole number.

8 $A = \pi r^2$ is the area of a circle with radius r. Find A when $r = 9.5$, correct to 3 significant figures.

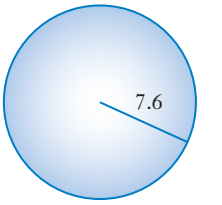
9 For the formula $u = ar^{n-1}$, find u if $a = 5$, $r = -2$ and $n = 4$.

10 Given $V = \frac{1}{3}lbh$ is the volume formula for a rectangular pyramid, find V if $l = 4.7$, $b = 5.1$ and $h = 6.5$.

11 The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Find m if $x_1 = 3$, $x_2 = -1$, $y_1 = -2$ and $y_2 = 5$.

12 If $A = \frac{1}{2}h(a + b)$ gives the area of a trapezium, find A when $h = 7$, $a = 2.5$ and $b = 3.9$.

13 $V = \frac{4}{3}\pi r^3$ is the volume formula for a sphere with radius r.
Find V to 1 decimal place for a sphere with radius $r = 7.6$.



14 The velocity of an object at time t is given by the formula $v = u + at$.
Find v when $u = \frac{1}{4}$, $a = \frac{3}{5}$ and $t = \frac{5}{6}$.

15 Given $S = \frac{a}{1 - r}$, find S if $a = 5$ and $r = \frac{2}{3}$. S is the sum to infinity of a geometric series.

16 $c = \sqrt{a^2 + b^2}$, according to Pythagoras' theorem. Find the value of c if $a = 6$ and $b = 8$.

17 Given $y = \sqrt{16 - x^2}$ is the equation of a semicircle, find the exact value of y when $x = 2$.

18 Find the value of E in the energy equation $E = mc^2$ if $m = 8.3$ and $c = 1.7$.

19 $A = P\left(1 + \frac{r}{100}\right)^n$ is the formula for finding compound interest. Find A correct to 2 decimal places when $P = 200$, $r = 12$ and $n = 5$.

20 If $S = \frac{a(r^n - 1)}{r - 1}$ is the sum of a geometric series, find S if $a = 3$, $r = 2$ and $n = 5$.

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1.18 Simplifying surds

An **irrational number** is a number that cannot be written as a ratio or fraction.

Surds such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are special types of irrational numbers.

If a question involving surds asks for an exact answer, then leave it as a surd.

Properties of surds

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\left(\sqrt{x}\right)^2 = \sqrt{x^2} = x \text{ for } x \geq 0$$

EXAMPLE 20

a Express $\sqrt{45}$ in simplest surd form.

b Simplify $3\sqrt{40}$.

c Write $5\sqrt{2}$ as a single surd.

Solution

a $\sqrt{45} = \sqrt{9 \times 5}$
$= \sqrt{9} \times \sqrt{5}$
$= 3 \times \sqrt{5}$
$= 3\sqrt{5}$

b $3\sqrt{40} = 3 \times \sqrt{4} \times \sqrt{10}$
$= 3 \times 2 \times \sqrt{10}$
$= 6\sqrt{10}$

c $5\sqrt{2} = \sqrt{25} \times \sqrt{2}$
$= \sqrt{50}$



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Exercise 1.18 Simplifying surds

1 Express these surds in simplest surd form:

a	$\sqrt{12}$	**b**	$\sqrt{63}$	**c**	$\sqrt{24}$	**d**	$\sqrt{50}$	**e**	$\sqrt{72}$
f	$\sqrt{200}$	**g**	$\sqrt{48}$	**h**	$\sqrt{75}$	**i**	$\sqrt{32}$	**j**	$\sqrt{54}$
k	$\sqrt{112}$	**l**	$\sqrt{300}$	**m**	$\sqrt{128}$	**n**	$\sqrt{243}$	**o**	$\sqrt{245}$
p	$\sqrt{108}$	**q**	$\sqrt{99}$	**r**	$\sqrt{125}$				

2 Simplify:

a	$2\sqrt{27}$	**b**	$5\sqrt{80}$	**c**	$4\sqrt{98}$	**d**	$2\sqrt{28}$	**e**	$8\sqrt{20}$
f	$4\sqrt{56}$	**g**	$8\sqrt{405}$	**h**	$15\sqrt{8}$	**i**	$7\sqrt{40}$	**j**	$8\sqrt{45}$

3 Write as a single surd:

a	$3\sqrt{2}$	**b**	$2\sqrt{5}$	**c**	$4\sqrt{11}$	**d**	$8\sqrt{2}$	**e**	$5\sqrt{3}$
f	$4\sqrt{10}$	**g**	$3\sqrt{13}$	**h**	$7\sqrt{2}$	**i**	$11\sqrt{3}$	**j**	$12\sqrt{7}$

4 Evaluate x if:

a	$\sqrt{x}=3\sqrt{5}$	**b**	$2\sqrt{3}=\sqrt{x}$	**c**	$3\sqrt{7}=\sqrt{x}$	**d**	$5\sqrt{2}=\sqrt{x}$	**e**	$2\sqrt{11}=\sqrt{x}$
f	$\sqrt{x}=7\sqrt{3}$	**g**	$4\sqrt{19}=\sqrt{x}$	**h**	$\sqrt{x}=6\sqrt{23}$	**i**	$5\sqrt{31}=\sqrt{x}$	**j**	$\sqrt{x}=8\sqrt{15}$

1.19 Operations with surds

EXAMPLE 21

Simplify $\sqrt{3}-\sqrt{12}$.

Solution

First, change into like surds.

$$\begin{aligned}\sqrt{3}-\sqrt{12}&=\sqrt{3}-\sqrt{4}\times\sqrt{3}\\&=\sqrt{3}-2\sqrt{3}\\&=-\sqrt{3}\end{aligned}$$

Multiplication and division, as in algebra, are easier to do than adding and subtracting.

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EXAMPLE 22

Simplify:

a $4\sqrt{2}\times 5\sqrt{18}$ **b** $\dfrac{2\sqrt{14}}{4\sqrt{2}}$ **c** $\left(\sqrt{\dfrac{10}{3}}\right)^2$

Solution

a $4\sqrt{2}\times 5\sqrt{18}=20\sqrt{36}$
$=20\times 6$
$=120$

b $\dfrac{2\sqrt{14}}{4\sqrt{2}}=\dfrac{2\times\sqrt{7}}{4}$
$=\dfrac{\sqrt{7}}{2}$

c $\left(\sqrt{\dfrac{10}{3}}\right)^2=\dfrac{10}{3}$
$=3\dfrac{1}{3}$

EXAMPLE 23

Expand and simplify:

a $3\sqrt{7}(2\sqrt{3}-3\sqrt{2})$ **b** $(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2})$ **c** $(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})$

Solution

a $3\sqrt{7}(2\sqrt{3}-3\sqrt{2})=3\sqrt{7}\times 2\sqrt{3}-3\sqrt{7}\times 3\sqrt{2}$
$=6\sqrt{21}-9\sqrt{14}$

b $(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2})=\sqrt{2}\times\sqrt{3}-\sqrt{2}\times\sqrt{2}+3\sqrt{5}\times\sqrt{3}-3\sqrt{5}\times\sqrt{2}$
$=\sqrt{6}-2+3\sqrt{15}-3\sqrt{10}$

c Using the difference of 2 squares: $(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})=\left(\sqrt{5}\right)^2-\left(2\sqrt{3}\right)^2$
$=5-4\times 3$
$=-7$

Exercise 1.19 Operations with surds

1 Simplify:

a $\sqrt{5}+2\sqrt{5}$ **b** $3\sqrt{2}-2\sqrt{2}$ **c** $\sqrt{3}+5\sqrt{3}$

d $7\sqrt{3}-4\sqrt{3}$ **e** $\sqrt{5}-4\sqrt{5}$ **f** $4\sqrt{6}-\sqrt{6}$

g $\sqrt{2}-8\sqrt{2}$ **h** $\sqrt{5}+4\sqrt{5}+3\sqrt{5}$ **i** $\sqrt{2}-2\sqrt{2}-3\sqrt{2}$

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j $\sqrt{5}+\sqrt{45}$ **k** $\sqrt{8}-\sqrt{2}$ **l** $\sqrt{3}+\sqrt{48}$

m $\sqrt{12}-\sqrt{27}$ **n** $\sqrt{50}-\sqrt{32}$ **o** $\sqrt{28}+\sqrt{63}$

p $2\sqrt{8}-\sqrt{18}$ **q** $3\sqrt{54}+2\sqrt{24}$ **r** $\sqrt{90}-5\sqrt{40}-2\sqrt{10}$

s $4\sqrt{48}+3\sqrt{147}+5\sqrt{12}$ **t** $3\sqrt{2}+\sqrt{8}-\sqrt{12}$ **u** $\sqrt{63}-\sqrt{28}-\sqrt{50}$

v $\sqrt{12}-\sqrt{45}-\sqrt{48}-\sqrt{5}$

2 Simplify:

a $\sqrt{7}\times\sqrt{3}$ **b** $\sqrt{3}\times\sqrt{5}$ **c** $\sqrt{2}\times3\sqrt{3}$

d $5\sqrt{7}\times2\sqrt{2}$ **e** $-3\sqrt{3}\times2\sqrt{2}$ **f** $5\sqrt{3}\times2\sqrt{3}$

g $-4\sqrt{5}\times3\sqrt{11}$ **h** $2\sqrt{7}\times\sqrt{7}$ **i** $2\sqrt{3}\times5\sqrt{12}$

j $\sqrt{6}\times\sqrt{2}$ **k** $\left(\sqrt{2}\right)^2$ **l** $\left(2\sqrt{7}\right)^2$

m $\sqrt{3}\times\sqrt{5}\times\sqrt{2}$ **n** $2\sqrt{3}\times\sqrt{7}\times-\sqrt{5}$ **o** $\sqrt{2}\times\sqrt{6}\times3\sqrt{3}$

3 Simplify:

a $\dfrac{4\sqrt{12}}{2\sqrt{2}}$ **b** $\dfrac{12\sqrt{18}}{3\sqrt{6}}$ **c** $\dfrac{5\sqrt{8}}{10\sqrt{2}}$ **d** $\dfrac{16\sqrt{2}}{2\sqrt{12}}$

e $\dfrac{10\sqrt{30}}{5\sqrt{10}}$ **f** $\dfrac{2\sqrt{2}}{6\sqrt{20}}$ **g** $\dfrac{4\sqrt{2}}{8\sqrt{10}}$ **h** $\dfrac{\sqrt{3}}{3\sqrt{15}}$

i $\dfrac{\sqrt{2}}{\sqrt{8}}$ **j** $\dfrac{3\sqrt{15}}{6\sqrt{10}}$ **k** $\dfrac{5\sqrt{12}}{5\sqrt{8}}$ **l** $\dfrac{15\sqrt{18}}{10\sqrt{10}}$

m $\dfrac{\sqrt{15}}{2\sqrt{6}}$ **n** $\left(\sqrt{\dfrac{2}{3}}\right)^2$ **o** $\left(\sqrt{\dfrac{5}{7}}\right)^2$

4 Expand and simplify:

a $\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)$ **b** $\sqrt{3}\left(2\sqrt{2}-\sqrt{5}\right)$ **c** $4\sqrt{3}\left(\sqrt{3}+2\sqrt{5}\right)$

d $\sqrt{7}\left(5\sqrt{2}-2\sqrt{3}\right)$ **e** $-\sqrt{3}\left(\sqrt{2}-4\sqrt{6}\right)$ **f** $\sqrt{3}\left(5\sqrt{11}+3\sqrt{7}\right)$

g $-3\sqrt{2}\left(\sqrt{2}+4\sqrt{3}\right)$ **h** $\sqrt{5}\left(\sqrt{5}-5\sqrt{3}\right)$ **i** $\sqrt{3}\left(\sqrt{12}+\sqrt{10}\right)$

j $2\sqrt{3}\left(\sqrt{18}+\sqrt{3}\right)$ **k** $-4\sqrt{2}\left(\sqrt{2}-3\sqrt{6}\right)$ **l** $-7\sqrt{5}\left(-3\sqrt{20}+2\sqrt{3}\right)$

m $10\sqrt{3}\left(\sqrt{2}-2\sqrt{12}\right)$ **n** $-\sqrt{2}\left(\sqrt{5}+2\right)$ **o** $2\sqrt{3}\left(2-\sqrt{12}\right)$

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5 Expand and simplify:

a $\left(\sqrt{2}+3\right)\left(\sqrt{5}+3\sqrt{3}\right)$ **b** $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{2}-\sqrt{7}\right)$ **c** $\left(\sqrt{2}+5\sqrt{3}\right)\left(2\sqrt{5}-3\sqrt{2}\right)$

d $\left(3\sqrt{10}-2\sqrt{5}\right)\left(4\sqrt{2}+6\sqrt{6}\right)$ **e** $\left(2\sqrt{5}-7\sqrt{2}\right)\left(\sqrt{5}-3\sqrt{2}\right)$ **f** $\left(\sqrt{5}+6\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{3}\right)$

g $\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)$ **h** $\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)$ **i** $\left(\sqrt{6}+3\sqrt{2}\right)\left(\sqrt{6}-3\sqrt{2}\right)$

j $\left(3\sqrt{5}+\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{2}\right)$ **k** $\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}+\sqrt{5}\right)$ **l** $\left(\sqrt{2}+9\sqrt{3}\right)\left(\sqrt{2}-9\sqrt{3}\right)$

m $\left(2\sqrt{11}+5\sqrt{2}\right)\left(2\sqrt{11}-5\sqrt{2}\right)$ **n** $\left(\sqrt{5}+\sqrt{2}\right)^2$

o $\left(2\sqrt{2}-\sqrt{3}\right)^2$ **p** $\left(3\sqrt{2}+\sqrt{7}\right)^2$ **q** $\left(2\sqrt{3}+3\sqrt{5}\right)^2$

r $\left(\sqrt{7}-2\sqrt{5}\right)^2$ **s** $\left(2\sqrt{8}-3\sqrt{5}\right)^2$ **t** $\left(3\sqrt{5}+2\sqrt{2}\right)^2$

6 If $a=3\sqrt{2}$, simplify:

a a^2 **b** $2a^3$ **c** $(2a)^3$

d $(a+1)^2$ **e** $(a+3)(a-3)$

7 Evaluate a and b if:

a $\left(2\sqrt{5}+1\right)^2=a+\sqrt{b}$ **b** $\left(2\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}-3\sqrt{5}\right)=a+b\sqrt{10}$

8 Expand and simplify:

a $\left(\sqrt{a+3}-2\right)\left(\sqrt{a+3}+2\right)$ **b** $\left(\sqrt{p-1}-\sqrt{p}\right)^2$

9 Evaluate $\left(2\sqrt{7}-\sqrt{3}\right)\left(2\sqrt{7}+\sqrt{3}\right)$.

10 Simplify $\left(2\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right)$.

11 If $\left(2\sqrt{3}-\sqrt{5}\right)^2=a-\sqrt{b}$, evaluate a and b.

12 Evaluate a and b if $\left(7\sqrt{2}-3\right)^2=a+b\sqrt{2}$.

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1.20 Rationalising the denominator



Rationalising the denominator



Surds

Rationalising the denominator of a fractional surd means writing it with a rational number (not a surd) in the denominator. For example, after rationalising the denominator $\frac{3}{\sqrt{5}}$ becomes $\frac{3\sqrt{5}}{5}$.

To rationalise the denominator, multiply top and bottom by the same surd as in the denominator:

Rationalising the denominator

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

EXAMPLE 24

Rationalise the denominator of $\frac{2}{5\sqrt{3}}$.

Solution

$$\frac{2}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{5\sqrt{9}}$$
$$= \frac{2\sqrt{3}}{5 \times 3}$$
$$= \frac{2\sqrt{3}}{15}$$

When there is a binomial denominator, we use the difference of 2 squares to rationalise it.

Rationalising a binomial denominator

To rationalise the denominator of $\frac{b}{\sqrt{c}+\sqrt{d}}$, multiply by $\frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}-\sqrt{d}}$.

To rationalise the denominator of $\frac{b}{\sqrt{c}-\sqrt{d}}$, multiply by $\frac{\sqrt{c}+\sqrt{d}}{\sqrt{c}+\sqrt{d}}$.

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EXAMPLE 25

a Write with a rational denominator:

i $\dfrac{\sqrt{5}}{\sqrt{2}-3}$ **ii** $\dfrac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}$

b Evaluate a and b if $\dfrac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}}=a+\sqrt{b}$.

c Evaluate $\dfrac{2}{\sqrt{3}+2}+\dfrac{\sqrt{5}}{\sqrt{3}-2}$ as a fraction with rational denominator.

Solution

a **i**
$$\frac{\sqrt{5}}{\sqrt{2}-3}\times\frac{\sqrt{2}+3}{\sqrt{2}+3}=\frac{\sqrt{5}\left(\sqrt{2}+3\right)}{\left(\sqrt{2}\right)^2-3^2}$$
$$=\frac{\sqrt{10}+3\sqrt{5}}{2-9}$$
$$=-\frac{\sqrt{10}+3\sqrt{5}}{7}$$

ii
$$\frac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}\times\frac{\sqrt{3}-4\sqrt{2}}{\sqrt{3}-4\sqrt{2}}=\frac{\left(2\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-4\sqrt{2}\right)}{\left(\sqrt{3}\right)^2-\left(4\sqrt{2}\right)^2}$$
$$=\frac{2\times3-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{3-16\times2}$$
$$=\frac{6-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{-29}$$
$$=\frac{-6+8\sqrt{6}-\sqrt{15}+4\sqrt{10}}{29}$$

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b $$\frac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{3\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}$$

$$=\frac{3\sqrt{9}+3\sqrt{6}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{3\times3+3\sqrt{6}}{3-2}$$

$$=\frac{9+3\sqrt{6}}{1}$$

$$=9+3\sqrt{6}$$

$$=9+\sqrt{9}\times\sqrt{6}$$

$$=9+\sqrt{54}$$

So $a = 9$ and $b = 54$.

c $$\frac{2}{\sqrt{3}+2}+\frac{\sqrt{5}}{\sqrt{3}-2}=\frac{2\left(\sqrt{3}-2\right)+\sqrt{5}\left(\sqrt{3}+2\right)}{\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right)}$$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{\left(\sqrt{3}\right)^2-2^2}$$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{3-4}$$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{-1}$$

$$=-2\sqrt{3}+4-\sqrt{15}-2\sqrt{5}$$

Exercise 1.20 Rationalising the denominator

1 Express with a rational denominator:

a $\frac{1}{\sqrt{7}}$ **b** $\frac{\sqrt{3}}{2\sqrt{2}}$ **c** $\frac{2\sqrt{3}}{\sqrt{5}}$ **d** $\frac{6\sqrt{7}}{5\sqrt{2}}$

e $\frac{1+\sqrt{2}}{\sqrt{3}}$ **f** $\frac{\sqrt{6}-5}{\sqrt{2}}$ **g** $\frac{\sqrt{5}+2\sqrt{2}}{\sqrt{5}}$ **h** $\frac{3\sqrt{2}-4}{2\sqrt{7}}$

i $\frac{8+3\sqrt{2}}{4\sqrt{5}}$ **j** $\frac{4\sqrt{3}-2\sqrt{2}}{7\sqrt{5}}$

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2 Express with a rational denominator:

a $\dfrac{4}{\sqrt{3}+\sqrt{2}}$ **b** $\dfrac{\sqrt{3}}{\sqrt{2}-7}$ **c** $\dfrac{2\sqrt{3}}{\sqrt{5}+2\sqrt{6}}$

d $\dfrac{\sqrt{3}-4}{\sqrt{3}+4}$ **e** $\dfrac{\sqrt{2}+5}{\sqrt{3}-\sqrt{2}}$ **f** $\dfrac{3\sqrt{3}+\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$

3 Express as a single fraction with a rational denominator:

a $\dfrac{1}{\sqrt{2}+1}+\dfrac{1}{\sqrt{2}-1}$ **b** $\dfrac{\sqrt{2}}{\sqrt{2}-\sqrt{3}}-\dfrac{3}{\sqrt{2}+\sqrt{3}}$

c $t+\dfrac{1}{t}$ where $t=\sqrt{3}-2$ **d** $z^2-\dfrac{1}{z^2}$ where $z=1+\sqrt{2}$

e $\dfrac{\sqrt{2}+3}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}$ **f** $\dfrac{\sqrt{3}}{\sqrt{2}+3}+\dfrac{\sqrt{2}}{\sqrt{3}}$

g $\dfrac{\sqrt{5}}{\sqrt{6}+2}-\dfrac{2}{5\sqrt{3}}$ **h** $\dfrac{\sqrt{2}+7}{4+\sqrt{3}}-\dfrac{\sqrt{2}}{4-\sqrt{3}}$

i $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\dfrac{2+\sqrt{3}}{\sqrt{3}+1}$

4 Find a and b if:

a $\dfrac{3}{2\sqrt{5}}=\dfrac{\sqrt{a}}{b}$ **b** $\dfrac{\sqrt{3}}{4\sqrt{2}}=\dfrac{a\sqrt{6}}{b}$ **c** $\dfrac{2}{\sqrt{5}+1}=a+b\sqrt{5}$

d $\dfrac{2\sqrt{7}}{\sqrt{7}-4}=a+b\sqrt{7}$ **e** $\dfrac{\sqrt{2}+3}{\sqrt{2}-1}=a+\sqrt{b}$

5 Show that $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}+\dfrac{4}{\sqrt{2}}$ is rational.

6 If $x=\sqrt{3}+2$, simplify:

a $x+\dfrac{1}{x}$ **b** $x^2+\dfrac{1}{x^2}$ **c** $\left(x+\dfrac{1}{x}\right)^2$

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1. TEST YOURSELF

For Questions **1** to **8**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 Rationalise the denominator of $\frac{\sqrt{3}}{2\sqrt{7}}$ (there may be more than one answer).

A $\frac{\sqrt{21}}{28}$ **B** $\frac{2\sqrt{21}}{28}$ **C** $\frac{\sqrt{21}}{14}$ **D** $\frac{\sqrt{21}}{7}$

2 Simplify $\frac{x-3}{5} - \frac{x+1}{4}$.

A $\frac{-(x+7)}{20}$ **B** $\frac{x+7}{20}$ **C** $\frac{x+17}{20}$ **D** $\frac{-(x+17)}{20}$

3 Factorise $x^3 - 4x^2 - x + 4$ (there may be more than one answer).

A $(x^2 - 1)(x - 4)$ **B** $(x^2 + 1)(x - 4)$

C $x^2(x - 4)$ **D** $(x - 4)(x + 1)(x - 1)$

4 Simplify $3\sqrt{2} + 2\sqrt{98}$.

A $5\sqrt{2}$ **B** $5\sqrt{10}$ **C** $17\sqrt{2}$ **D** $10\sqrt{2}$

5 Simplify $\frac{3}{x^2-4} + \frac{2}{x-2} - \frac{1}{x+2}$.

A $\frac{x+5}{(x+2)(x-2)}$ **B** $\frac{x+1}{(x+2)(x-2)}$ **C** $\frac{x+9}{(x+2)(x-2)}$ **D** $\frac{x-3}{(x+2)(x-2)}$

6 Simplify $5ab - 2a^2 - 7ab - 3a^2$.

A $2ab + a^2$ **B** $-2ab - 5a^2$ **C** $-13a^3b$ **D** $-2ab + 5a^2$

7 Simplify $\sqrt{\frac{80}{27}}$.

A $\frac{4\sqrt{5}}{3\sqrt{3}}$ **B** $\frac{4\sqrt{5}}{9\sqrt{3}}$ **C** $\frac{8\sqrt{5}}{9\sqrt{3}}$ **D** $\frac{8\sqrt{5}}{3\sqrt{3}}$

8 Expand and simplify $(3x - 2y)^2$.

A $3x^2 - 12xy - 2y^2$ **B** $9x^2 - 12xy - 4y^2$

C $3x^2 - 6xy + 2y^2$ **D** $9x^2 - 12xy + 4y^2$

9 Evaluate as a fraction:

a 7^{-2} **b** 5^{-1} **c** $9^{-\frac{1}{2}}$

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10 Simplify:

a $x^5 \times x^7 \div x^3$ **b** $(5y^3)^2$ **c** $\dfrac{(a^5)^4 b^7}{a^9 b}$ **d** $\left(\dfrac{2x^6}{3}\right)^3$ **e** $\left(\dfrac{ab^4}{a^5b^6}\right)^0$

11 Evaluate:

a $36^{\frac{1}{2}}$ **b** 4^{-3} as fraction **c** $8^{\frac{2}{3}}$

d $49^{-\frac{1}{2}}$ as a fraction **e** $16^{\frac{1}{4}}$ **f** $(-3)^0$

12 Simplify:

a $a^{14} \div a^9$ **b** $(x^5y^3)^6$ **c** $p^6 \times p^5 \div p^2$

d $(2b^9)^4$ **e** $\dfrac{(2x^7)^3 y^2}{x^{10} y}$

13 Write in index form:

a $\sqrt{n}$ **b** $\dfrac{1}{x^5}$ **c** $\dfrac{1}{x+y}$ **d** $\sqrt[4]{x+1}$ **e** $\sqrt[7]{a+b}$

f $\dfrac{2}{x}$ **g** $\dfrac{1}{2x^3}$ **h** $\sqrt[3]{x^4}$ **i** $\sqrt[7]{(5x+3)^9}$ **j** $\dfrac{1}{\sqrt[4]{m^3}}$

14 Write without fractional or negative indices:

a a^{-5} **b** $n^{\frac{1}{4}}$ **c** $(x+1)^{\frac{1}{2}}$ **d** $(x-y)^{-1}$ **e** $(4t-7)^{-4}$

f $(a+b)^{\frac{1}{5}}$ **g** $x^{-\frac{1}{3}}$ **h** $b^{\frac{3}{4}}$ **i** $(2x+3)^{\frac{4}{3}}$ **j** $x^{-\frac{3}{2}}$

15 Evaluate a^2b^4 when $a = \dfrac{9}{25}$ and $b = 1\dfrac{2}{3}$.

16 If $a = \left(\dfrac{1}{3}\right)^4$ and $b = \dfrac{3}{4}$, evaluate ab^3 as a fraction.

17 Write in index form:

a $\sqrt{x}$ **b** $\dfrac{1}{y}$ **c** $\sqrt[6]{x+3}$ **d** $\dfrac{1}{(2x-3)^{11}}$ **e** $\sqrt[3]{y^7}$

18 Write without the negative index:

a x^{-3} **b** $(2a+5)^{-1}$ **c** $\left(\dfrac{a}{b}\right)^{-5}$

19 Simplify:

a $5y - 7y$ **b** $\dfrac{3a+12}{3}$ **c** $-2k^3 \times 3k^2$ **d** $\dfrac{x}{3} + \dfrac{y}{5}$

e $4a - 3b - a - 5b$ **f** $\sqrt{8} + \sqrt{32}$ **g** $3\sqrt{5} - \sqrt{20} + \sqrt{45}$

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20 Factorise:

a $x^2 - 36$ **b** $a^2 + 2a - 3$ **c** $4ab^2 - 8ab$

d $5y - 15 + xy - 3x$ **e** $4n - 2p + 6$

21 Expand and simplify:

a $b + 3(b - 2)$ **b** $(2x - 1)(x + 3)$ **c** $5(m + 3) - (m - 2)$

d $(4x - 3)^2$ **e** $(p - 5)(p + 5)$ **f** $7 - 2(a + 4) - 5a$

g $\sqrt{3}\left(2\sqrt{2} - 5\right)$ **h** $\left(3 + \sqrt{7}\right)\left(\sqrt{3} - 2\right)$

22 Simplify:

a $\dfrac{4a - 12}{5b^3} \times \dfrac{10b}{a^2 - 9}$ **b** $\dfrac{5m + 10}{m^2 - m - 2} \div \dfrac{m^2 - 4}{3m + 3}$

23 The volume of a cube is $V = s^3$. Evaluate V when $s = 5.4$.

24 **a** Expand and simplify $\left(2\sqrt{5} + \sqrt{3}\right)\left(2\sqrt{5} - \sqrt{3}\right)$.

b Rationalise the denominator of $\dfrac{3\sqrt{3}}{2\sqrt{5} + \sqrt{3}}$.

25 Simplify $\dfrac{3}{x - 2} + \dfrac{1}{x + 3} - \dfrac{2}{x^2 + x - 6}$.

26 If $a = 4$, $b = -3$ and $c = -2$, find the value of:

a ab^2 **b** $a - bc$ **c** $\sqrt{a}$ **d** $(bc)^3$ **e** $c(2a + 3b)$

27 Simplify:

a $\dfrac{3\sqrt{12}}{6\sqrt{15}}$ **b** $\dfrac{4\sqrt{32}}{2\sqrt{2}}$

28 The formula for the distance an object falls is given by $d = 5t^2$. Find d when $t = 1.5$.

29 Rationalise the denominator of:

a $\dfrac{2}{5\sqrt{3}}$ **b** $\dfrac{1 + \sqrt{3}}{\sqrt{2}}$

30 Expand and simplify:

a $\left(3\sqrt{2} - 4\right)\left(\sqrt{3} - \sqrt{2}\right)$ **b** $\left(\sqrt{7} + 2\right)^2$

31 Factorise fully:

a $3x^2 - 27$ **b** $6x^2 - 12x - 18$ **c** $5y^2 - 30y + 45$

32 Simplify:

a $\dfrac{3x^4 y}{9xy^5}$ **b** $\dfrac{5}{15x - 5}$

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33 Simplify:

a $\left(3\sqrt{11}\right)^2$ **b** $\left(2\sqrt{3}\right)^3$

34 Expand and simplify:

a $(a+b)(a-b)$ **b** $(a+b)^2$

35 Factorise:

a $a^2-2ab+b^2$ **b** a^2-b^2

36 If $x=\sqrt{3}+1$, simplify $x+\frac{1}{x}$ and give your answer with a rational denominator.

37 Simplify:

a $\frac{4}{a}+\frac{3}{b}$ **b** $\frac{x-3}{2}-\frac{x-2}{5}$

38 Simplify $\frac{3}{\sqrt{5}+2}-\frac{\sqrt{2}}{2\sqrt{2}-1}$, writing your answer with a rational denominator.

39 Simplify:

a $3\sqrt{8}$ **b** $-2\sqrt{2}\times 4\sqrt{3}$ **c** $\sqrt{108}-\sqrt{48}$ **d** $\frac{8\sqrt{6}}{2\sqrt{18}}$

e $5a\times -3b\times -2a$ **f** $\frac{2m^3n}{6m^2n^5}$ **g** $3x-2y-x-y$

40 Expand and simplify:

a $2\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)$ **b** $\left(5\sqrt{7}-3\sqrt{5}\right)\left(2\sqrt{2}-\sqrt{3}\right)$ **c** $\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

d $\left(4\sqrt{3}-\sqrt{5}\right)\left(4\sqrt{3}+\sqrt{5}\right)$ **e** $\left(3\sqrt{7}-\sqrt{2}\right)^2$

41 Rationalise the denominator of:

a $\frac{3}{\sqrt{7}}$ **b** $\frac{\sqrt{2}}{5\sqrt{3}}$ **c** $\frac{2}{\sqrt{5}-1}$ **d** $\frac{2\sqrt{2}}{3\sqrt{2}+\sqrt{3}}$ **e** $\frac{\sqrt{5}+\sqrt{2}}{4\sqrt{5}-3\sqrt{3}}$

42 Simplify:

a $\frac{3x}{5}-\frac{x-2}{2}$ **b** $\frac{a+2}{7}+\frac{2a-3}{3}$ **c** $\frac{1}{x^2-1}-\frac{2}{x+1}$

d $\frac{4}{k^2+2k-3}+\frac{1}{k+3}$ **e** $\frac{\sqrt{3}}{\sqrt{2}+\sqrt{5}}-\frac{5}{\sqrt{3}-\sqrt{2}}$

43 Evaluate n if:

a $\sqrt{108}-\sqrt{12}=\sqrt{n}$ **b** $\sqrt{112}+\sqrt{7}=\sqrt{n}$ **c** $2\sqrt{8}+\sqrt{200}=\sqrt{n}$

d $4\sqrt{147}+3\sqrt{75}=\sqrt{n}$ **e** $2\sqrt{245}+\frac{\sqrt{180}}{2}=\sqrt{n}$

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1. CHALLENGE EXERCISE

1 Write $64^{-\frac{2}{3}}$ as a rational number.

2 Show that $2(2^k - 1) + 2^{k+1} = 2(2^{k+1} - 1)$.

3 Find the value of $\frac{a}{b^3c^2}$ in index form if $a = \left(\frac{2}{5}\right)^4$, $b = \left(-\frac{1}{3}\right)^3$ and $c = \left(\frac{3}{5}\right)^2$.

4 Expand and simplify:

a $4ab(a - 2b) - 2a^2(b - 3a)$ **b** $(y^2 - 2)(y^2 + 2)$ **c** $(2x - 5)^3$

5 Find the value of $x + y$ with rational denominator if $x = \sqrt{3} + 1$ and $y = \frac{1}{2\sqrt{5} - 3}$.

6 Simplify $\frac{2\sqrt{3}}{7\sqrt{6} - \sqrt{54}}$.

7 Factorise:

a $(x + 4)^2 + 5(x + 4)$ **b** $x^4 - x^2y - 6y^2$ **c** $a^2b - 2a^2 - 4b + 8$

8 Simplify $\frac{2xy + 2x - 6 - 6y}{4x^2 - 16x + 12}$.

9 Simplify $\frac{(a+1)^3}{a^2 - 1}$.

10 Factorise $\frac{4}{x^2} - \frac{a^2}{b^2}$.

11 **a** Expand $(2x - 1)^3$.

b Hence, or otherwise, simplify $\frac{6x^2 + 5x - 4}{8x^3 - 12x^2 + 6x - 1}$.

12 If $V = \pi r^2 h$ is the volume of a cylinder, find the exact value of r when $V = 9$ and $h = 16$.

13 If $s = u + \frac{1}{2}at^2$, find the exact value of s when $u = 2$, $a = \sqrt{3}$ and $t = 2\sqrt{3}$.

14 Expand and simplify, and write in index form:

a $\left(\sqrt{x} + x\right)^2$ **b** $\left(\sqrt[3]{a} + \sqrt[3]{b}\right)\left(\sqrt[3]{a} - \sqrt[3]{b}\right)$

c $\left(p + \frac{1}{\sqrt{p}}\right)^2$ **d** $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

15 Find the value of $\frac{a^3b^2}{c^2}$ if $a = \left(\frac{3}{4}\right)^2$, $b = \left(\frac{2}{3}\right)^3$ and $c = \left(\frac{1}{2}\right)^4$.

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2.

EQUATIONS AND INEQUALITIES

Equations are found in most branches of mathematics. They are also important in many other fields, such as science, economics, statistics and engineering. In this chapter you will revise basic equations and solve harder equations, including those involving absolute values, exponential equations, quadratic equations and simultaneous equations.

CHAPTER OUTLINE

2.01 Equations
2.02 Inequalities
2.03 Absolute value
2.04 Equations involving absolute values
2.05 Exponential equations
2.06 Solving quadratic equations by factorisation
2.07 Solving quadratic equations by completing the square
2.08 Solving quadratic equations by quadratic formula
2.09 Formulas and equations
2.10 Linear simultaneous equations
2.11 Non-linear simultaneous equations
2.12 Simultaneous equations with three unknown variables



IN THIS CHAPTER YOU WILL:

- solve equations and inequalities
- understand and use absolute values in equations
- solve simple exponential equations
- solve quadratic equations using 3 different methods
- understand how to substitute into and rearrange formulas
- solve linear and non-linear simultaneous equations

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TERMINOLOGY

absolute value $|x|$ is the absolute value of x, its size without sign or direction. Also the distance of x from 0 on the number line in either direction

equation A mathematical statement that has a pronumeral or unknown number and an equal sign. An equation can be solved to find the value of the unknown number, for example, $3x + 1 = 7$

exponential equation An equation where the unknown pronumeral is the power or index, for example, $2^x = 8$

inequality A mathematical statement involving an inequality sign with an unknown pronumeral, for example, $x - 7 \leq 12$

quadratic equation An equation involving x^2 in which the highest power of x is 2

simultaneous equations 2 or more equations that can be solved together to produce a solution that makes each equation true at the same time

PROBLEM

The age of Diophantus at his death can be calculated from his epitaph:

Diophantus passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh more as a bachelor; five years after his marriage a son was born who died four years before his father at half his father's final age. How old was Diophantus?

WS Equations

2.01 Equations

EXAMPLE 1

Solve each equation.

a $4y - 3 = 8y + 21$

b $2(3x + 7) = 6 - (x - 1)$

Solution

a
$$\begin{aligned} 4y - 3 &= 8y + 21 \\ 4y - 4y - 3 &= 8y - 4y + 21 \\ -3 &= 4y + 21 \\ -3 - 21 &= 4y + 21 - 21 \\ -24 &= 4y \\ \frac{-24}{4} &= \frac{4y}{4} \\ -6 &= y \\ y &= -6 \end{aligned}$$

b
$$\begin{aligned} 2(3x + 7) &= 6 - (x - 1) \\ 6x + 14 &= 6 - x + 1 \\ &= 7 - x \\ 6x + x + 14 &= 7 - x + x \\ 7x + 14 &= 7 \\ 7x + 14 - 14 &= 7 - 14 \\ 7x &= -7 \\ \frac{7x}{7} &= \frac{-7}{7} \\ x &= -1 \end{aligned}$$

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When an equation involves fractions, multiply both sides of the equation by the common denominator of the fractions.

EXAMPLE 2

Solve:

a $\frac{m}{3} - 4 = \frac{1}{2}$ **b** $\frac{x+1}{3} + \frac{x}{4} = 5$

Solution

a

$$\frac{m}{3} - 4 = \frac{1}{2}$$
$$6\left(\frac{m}{3}\right) - 6(4) = 6\left(\frac{1}{2}\right)$$
$$2m - 24 = 3$$
$$2m - 24 + 24 = 3 + 24$$
$$2m = 27$$
$$\frac{2m}{2} = \frac{27}{2}$$
$$m = \frac{27}{2}$$
$$= 13\tfrac{1}{2}$$

b

$$\frac{x+1}{3} + \frac{x}{4} = 5$$
$$12\left(\frac{x+1}{3}\right) + 12\left(\frac{x}{4}\right) = 12(5)$$
$$4(x+1) + 3x = 60$$
$$4x + 4 + 3x = 60$$
$$7x + 4 = 60$$
$$7x + 4 - 4 = 60 - 4$$
$$7x = 56$$
$$\frac{7x}{7} = \frac{56}{7}$$
$$x = 8$$

DID YOU KNOW?

History of algebra

Algebra was known in ancient civilisations. Many equations were known in Babylon, although general solutions were difficult because symbols were not used in those times.

Diophantus, around 250 CE, first used algebraic notation and symbols (e.g. the minus sign). He wrote a treatise on algebra in his *Arithmetica*, comprising 13 books. Only six of these books survived. About 400 CE, Hypatia of Alexandria wrote a commentary on them.

Hypatia was the first female mathematician on record, and was a philosopher and teacher. She was the daughter of Theon, who was also a mathematician and who ensured that she had the best education.

In 1799 **Carl Friedrich Gauss** proved the Fundamental Theorem of Algebra: that every algebraic equation involving a power of x has at least one solution, which may be a real number or a non-real number.

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Exercise 2.01 Equations

Solve each equation.

1 $t + 4 = -1$

2 $z + 1.7 = -3.9$

3 $y - 3 = -2$

4 $w - 2.6 = 4.1$

5 $5 = x - 7$

6 $1.5x = 6$

7 $5y = \frac{1}{3}$

8 $\frac{b}{7} = 5$

9 $-2 = \frac{n}{8}$

10 $\frac{r}{6} = \frac{2}{3}$

11 $2y + 1 = 19$

12 $33 = 4k + 9$

13 $7d - 2 = 12$

14 $-2 = 5x - 27$

15 $\frac{y}{3} + 4 = 9$

16 $\frac{x}{2} - 3 = 7$

17 $\frac{m}{5} + 7 = 11$

18 $3x + 5 = 17$

19 $4a + 7 = -21$

20 $7y - 1 = 20$

21 $3(x + 2) = 15$

22 $-2(3a + 1) = 8$

23 $7t + 4 = 3t - 12$

24 $x - 3 = 6x - 9$

25 $2(a - 2) = 4 - 3a$

26 $5b + 2 = -3(b - 1)$

27 $3(t + 7) = 2(2t - 9)$

28 $2 + 5(p - 1) = 5p - (p - 2)$

29 $3.7x + 1.2 = 5.4x - 6.3$

30 $\frac{b}{5} = \frac{2}{3}$

31 $\frac{5x}{4} = \frac{11}{7}$

32 $\frac{x}{3} - 4 = 8$

33 $\frac{5 + x}{7} = \frac{2}{7}$

34 $\frac{y}{2} = -\frac{3}{5}$

35 $\frac{x}{9} - \frac{2}{3} = 7$

36 $\frac{w - 3}{2} = 5$

37 $\frac{2t}{5} - \frac{t}{3} = 2$

38 $\frac{x}{4} + \frac{1}{2} = 4$

39 $\frac{x}{5} - \frac{x}{2} = \frac{3}{10}$

40 $\frac{x + 4}{3} + \frac{x}{2} = 1$

41 $\frac{p - 3}{2} + \frac{2p}{3} = 2$

42 $\frac{t + 3}{7} + \frac{t - 1}{3} = 4$

43 $\frac{x + 5}{9} - \frac{x + 2}{5} = 1$

44 $\frac{q - 1}{3} - \frac{q - 2}{4} = 2$

45 $\frac{x + 3}{5} + 2 = \frac{x + 7}{2}$

COULD THIS BE TRUE?

Half full = half empty

$\therefore$ full = empty



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2.02 Inequalities

> means greater than.

< means less than.

≥ means greater than or equal to.

≤ means less than or equal to.

Solving inequalities

The **inequality sign reverses** when:

- multiplying by a negative
- dividing by a negative
- taking the reciprocal of both sides.

On the number plane, we graph inequalities using arrows and circles (open for greater than and less than and closed in for greater than or equal to and less than or equal to).

Inequalities on a number line



WS
Inequalities on a number line

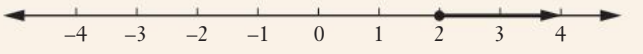
EXAMPLE 3

Solve each inequality and show its solution on a number line.

a $5x + 7 \geq 17$ **b** $3t - 2 > 5t + 4$ **c** $1 < 2z + 7 \leq 11$

Solution

a
$$5x + 7 \geq 17$$
$$5x + 7 - 7 \geq 17 - 7$$
$$5x \geq 10$$
$$\frac{5x}{5} \geq \frac{10}{5}$$
$$x \geq 2$$

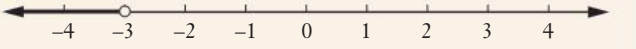


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b

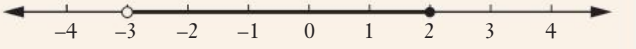
$$3t - 2 > 5t + 4$$
$$3t - 5t - 2 > 5t - 5t + 4$$
$$-2t - 2 > 4$$
$$-2t - 2 + 2 > 4 + 2$$
$$-2t > 6$$
$$\frac{-2t}{-2} < \frac{6}{-2}$$
$$t < -3$$

Remember to change the inequality sign when dividing by –2.



c

$$1 < 2z + 7 \leq 11$$
$$1 - 7 < 2z + 7 - 7 \leq 11 - 7$$
$$-6 < 2z \leq 4$$
$$-3 < z \leq 2$$



Exercise 2.02 Inequalities

1 Solve each equation and plot the solution on a number line.

a $x + 4 > 7$ **b** $y - 3 \leq 1$

2 Solve:

a $5t > 35$ **b** $3x - 7 \geq 2$ **c** $2(p + 5) > 8$

d $4 - (x - 1) \leq 7$ **e** $3y + 5 > 2y - 4$ **f** $2a - 6 \leq 5a - 3$

g $3 + 4y \geq -2(1 - y)$ **h** $2x + 9 < 1 - 4(x + 1)$ **i** $\frac{a}{2} \leq -3$

j $8 > \frac{2y}{3}$ **k** $\frac{b}{2} + 5 < -4$ **l** $\frac{x}{3} - 4 > 6$

3 Solve and plot each solution on a number line.

a $3 < x + 2 < 9$ **b** $-4 \leq 2p < 10$ **c** $2 < 3x - 1 < 11$

d $-6 \leq 5y + 9 \leq 34$ **e** $-2 < 3(2y - 1) < 7$

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2.03 Absolute value

The **absolute value** of a number is the size of the number without the sign or direction. So absolute value is always positive or zero.

We write the absolute value of x as $|x|$.

For example, $|4| = 4$ and $|-3| = 3$.

We can also define $|x|$ as the distance of x from 0 on the number line.

If x is positive, then its absolute value is itself.

If $x = 0$, then its absolute value is 0.

If x is negative, then its absolute value is its opposite, $-x$. Because x is already negative, the effect of the negative sign in front of it is to make it positive; for example, $-(-5) = 5$.

Absolute value

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

$|4| = 4$ since $4 \geq 0$.

$|-3| = -(-3)$ since $-3 < 0$

$= 3$

Properties of absolute value

Property	Example
$\|ab\| = \|a\| \times \|b\|$	$\|2 \times -3\| = \|2\| \times \|-3\| = 6$
$\|a\|^2 = a^2$	$\|-3\|^2 = (-3)^2 = 9$
$\sqrt{a^2} = \|a\|$	$\sqrt{(-5)^2} = \|-5\| = 5$
$\|-a\| = \|a\|$	$\|-7\| = \|7\| = 7$
$\|a - b\| = \|b - a\|$	$\|2 - 3\| = \|3 - 2\| = 1$
$\|a + b\| \leq \|a\| + \|b\|$	$\|2 + 3\| = \|2\| + \|3\|$ but $\|-3 + 4\| < \|-3\| + \|4\|$

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EXAMPLE 4

a Evaluate $|2| - |-1| + |-3|^2$.

b Show that $|a + b| \leq |a| + |b|$ when $a = -2$ and $b = 3$.

c Write expressions for $|2x - 4|$ without the absolute value signs.

Solution

a $|2| - |-1| + |-3|^2 = 2 - 1 + 3^2$

$= 10$

b

LHS	RHS
$= \|a + b\|$	$= \|a\| + \|b\|$
$= \|-2 + 3\|$	$= \|-2\| + \|3\|$
$= \|1\|$	$= 2 + 3$
$= 1$	$= 5$

Note: LHS means left-hand side and RHS means right-hand side.

Since $1 < 5$,

$|a + b| \leq |a| + |b|$

c $|2x - 4| = 2x - 4$ when $2x - 4 \geq 0$ i.e. when $2x \geq 4$

i.e. when $x \geq 2$

$|2x - 4| = -(2x - 4)$ when $2x - 4 < 0$ i.e. when $2x < 4$

$= -2x + 4$ i.e. when $x < 2$

CLASS DISCUSSION

ABSOLUTE VALUE

Are these statements true? If so, are there some values for which the expression is undefined (values of x or y that the expression cannot have)?

1 $\dfrac{x}{|x|} = 1$

2 $|2x| = 2x$

3 $|2x| = 2|x|$

4 $|x| + |y| = |x + y|$

5 $|x|^2 = x^2$

6 $|x|^3 = x^3$

7 $|x + 1| = |x| + 1$

8 $\dfrac{|3x - 2|}{3x - 2} = 1$

9 $\dfrac{|x|}{x^2} = 1$

10 $|x| \geq 0$

Discuss absolute value and its definition in relation to these statements.

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Exercise 2.03 Absolute value

1 Evaluate:

a $|7|$ **b** $|-5|$ **c** $|-6|$ **d** $|0|$

e $|2|$ **f** $|-11|$ **g** $|-2||3|$ **h** $3|-8|$

i $|-5|^2$ **j** $|-5|^3$

2 Evaluate:

a $|3| + |-2|$ **b** $|-3| - |4|$ **c** $|-5 + 3|$

d $|2 \times -7|$ **e** $|-3| + |-1|$ **f** $5 - |-2| \times |6|^2$

g $|-2 + 5 \times -1|$ **h** $3|-4|$ **i** $2|-3| - 3|-4|$

j $|5 - 7| + 4|-2|$

3 Evaluate $|a - b|$ if:

a $a = 5$ and $b = 2$ **b** $a = -1$ and $b = 2$ **c** $a = -2$ and $b = -3$

d $a = 4$ and $b = 7$ **e** $a = -1$ and $b = -2$

4 Write an expression for:

a $|a|$ when $a > 0$ **b** $|a|$ when $a < 0$ **c** $|a|$ when $a = 0$

d $|3a|$ when $a > 0$ **e** $|3a|$ when $a < 0$ **f** $|3a|$ when $a = 0$

g $|a + 1|$ when $a > -1$ **h** $|a + 1|$ when $a < -1$ **i** $|x - 2|$ when $x > 2$

5 Show that $|a + b| \leq |a| + |b|$ when:

a $a = 2$ and $b = 4$ **b** $a = -1$ and $b = -2$ **c** $a = -2$ and $b = 3$

d $a = -4$ and $b = 5$ **e** $a = -7$ and $b = -3$

6 Show that $\sqrt{x^2} = |x|$ when:

a $x = 5$ **b** $x = -2$ **c** $x = -3$

d $x = 4$ **e** $x = -9$

7 Use the definition of absolute value to write each expression without the absolute value signs.

a $|x + 5|$ **b** $|b - 3|$ **c** $|a + 4|$

d $|2y - 6|$ **e** $|3x + 9|$ **f** $|4 - x|$

g $|2k + 1|$ **h** $|5x - 2|$ **i** $|a + b|$

8 Find values of x for which $|x| = 3$.

9 Simplify $\dfrac{|n|}{n}$ where $n \neq 0$.

10 Simplify $\dfrac{x-2}{|x-2|}$ and state which value x cannot be.

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WS

Absolute value equations and inequalities

2.04 Equations involving absolute values

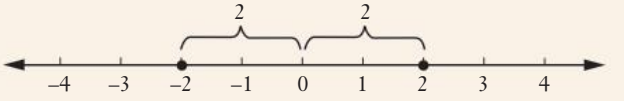
On a number line, $|x|$ means the distance of x from 0 in either direction.

EXAMPLE 5

Solve $|x| = 2$.

Solution

$|x| = 2$ means the distance of x from zero is 2 (in either direction).



$x = \pm 2$

CLASS DISCUSSION

ABSOLUTE VALUE AND THE NUMBER LINE

What does $|a - b|$ mean as a distance along the number line?

Select different values of a and b to help with this discussion.

EXAMPLE 6

Solve:

a $|x + 4| = 7$ **b** $|2x - 3| = 9$

Solution

a This means that the distance from $x + 4$ to 0 is 7 in either direction.

So $x + 4 = \pm 7$

$x + 4 = 7$ or $x + 4 = -7$

$x + 4 - 4 = 7 - 4$ $x + 4 - 4 = -7 - 4$

$x = 3$ $x = -11$

So $x = 3$ or -11.

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Checking your answer:

LHS $= |3 + 4|$
$= |7|$
$= 7$
$=$ RHS

LHS $= |-11 + 4|$
$= |-7|$
$= 7$
$=$ RHS

b $|2x - 3| = 9$

$2x - 3 = 9$
$2x = 12$
$x = 6$

or $2x - 3 = -9$
$2x = -6$
$x = -3$

So $x = 6$ or -3.

Checking your answer:

LHS $= |2 \times 6 - 3|$
$= |9|$
$= 9$
$=$ RHS

LHS $= |2 \times (-3) - 3|$
$= |-9|$
$= 9$
$=$ RHS

Exercise 2.04 Equations involving absolute values

1 Solve:

a $|x| = 5$ **b** $|y| = 8$ **c** $|x| = 0$

2 Solve:

a $|x + 2| = 7$ **b** $|n - 1| = 3$ **c** $9 = |2x + 3|$

d $|7x - 1| = 34$ **e** $\left|\frac{x}{3}\right| = 4$

3 Solve:

a $|8x - 5| = 11$ **b** $|5 - 3n| = 1$ **c** $16 = |5t + 4|$

d $21 = |9 - 2y|$ **e** $|3x + 2| - 7 = 0$

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WS
Exponential equations

2.05 Exponential equations

The word **exponent** means the power or index of a number.

So an **exponential equation** involves an unknown index or power; for example, $2^x = 8$.

EXAMPLE 7

Solve:

a $3^x = 81$ **b** $5^{2k-1} = 25$ **c** $8^n = 4$

Solution

a
$$3^x = 81$$
$$3^x = 3^4$$
$$\therefore x = 4$$

b
$$5^{2k-1} = 25$$
$$5^{2k-1} = 5^2$$
$$\therefore 2k - 1 = 2$$
$$2k = 3$$
$$\frac{2k}{2} = \frac{3}{2}$$
$$k = 1\frac{1}{2}$$

c It is hard to write 8 as a power of 4 or 4 as a power of 8, but both can be written as powers of 2.

$$8^n = 4$$
$$(2^3)^n = 2^2$$
$$2^{3n} = 2^2$$
$$\therefore 3n = 2$$
$$\frac{3n}{3} = \frac{2}{3}$$
$$n = \frac{2}{3}$$

To solve other equations involving indices, we do the opposite or inverse operation. For example, squares and square roots are inverse operations, and cubes and cube roots are inverse operations.

EXAMPLE 8

Solve:

a $x^2 = 9$ **b** $5n^3 = 40$

Solution

a There are two possible numbers whose square is 9.

$$x^2 = 9$$
$$x = \pm\sqrt{9}$$
$$\therefore x = \pm 3$$

b
$$5n^3 = 40$$
$$\frac{5n^3}{5} = \frac{40}{5}$$
$$n^3 = 8$$
$$n = \sqrt[3]{8}$$
$$n = 2$$

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INVESTIGATION

SOLUTIONS FOR EQUATIONS INVOLVING x^n

Investigate equations of the type $x^n = k$ where k is a constant; for example, $x^n = 9$.

Look at these questions.

1 What is the solution when $n = 0$?

2 What is the solution when $n = 1$?

3 How many solutions are there when $n = 2$?

4 How many solutions are there when $n = 3$?

5 How many solutions are there when n is even?

6 How many solutions are there when n is odd?

Exercise 2.05 Exponential equations

1 Solve:

a $2^n = 16$ **b** $3^y = 243$ **c** $2^m = 512$ **d** $10^x = 100\,000$

e $6^m = 1$ **f** $4^x = 64$ **g** $4^x + 3 = 19$ **h** $5(3^x) = 45$

i $4^x = 4$ **j** $\dfrac{6^k}{2} = 18$

2 Solve:

a $3^{2x} = 81$ **b** $2^{5x-1} = 16$ **c** $4^{x+3} = 4$ **d** $3^{n-2} = 1$

e $7^{2x+1} = 7$ **f** $3^{x-3} = 27$ **g** $5^{3y+2} = 125$ **h** $7^{3x-4} = 49$

i $2^{4x} = 256$ **j** $9^{3a+1} = 9$

3 Solve:

a $4^m = 2$ **b** $27^x = 3$ **c** $125^x = 5$ **d** $\left(\dfrac{1}{49}\right)^k = 7$

e $\left(\dfrac{1}{1000}\right)^k = 100$ **f** $16^n = 8$ **g** $25^x = 125$ **h** $64^n = 16$

i $\left(\dfrac{1}{4}\right)^{3k} = 2$ **j** $8^{x-1} = 4$

4 Solve:

a $2^{4x+1} = 8^x$ **b** $3^{5x} = 9^{x-2}$ **c** $7^{2k+3} = 7^{k-1}$

d $4^{3n} = 8^{n+3}$ **e** $6^{x-5} = 216^x$ **f** $16^{2x-1} = 4^{x-4}$

g $27^{x+3} = 3^x$ **h** $\left(\dfrac{1}{2}\right)^x = \left(\dfrac{1}{64}\right)^{2x+3}$ **i** $\left(\dfrac{3}{4}\right)^x = \left(\dfrac{27}{64}\right)^{2x-3}$

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5 Solve:

a $4^m = \sqrt{2}$ **b** $\left(\frac{9}{25}\right)^{k+3} = \sqrt{\frac{3}{5}}$ **c** $\frac{1}{\sqrt{2}} = 4^{2x-5}$

d $3^k = 3\sqrt{3}$ **e** $\left(\frac{1}{27}\right)^{3n+1} = \frac{\sqrt{3}}{81}$ **f** $\left(\frac{2}{5}\right)^{3n+1} = \left(\frac{5}{2}\right)^{-n}$

g $32^{-x} = \frac{1}{16}$ **h** $9^{2b+5} = 3^b\sqrt{3}$ **i** $81^{x+1} = \sqrt{3^x}$

6 Solve, giving exact answers:

a $x^3 = 27$ **b** $y^2 = 64$ **c** $n^4 = 16$

d $x^2 = 20$ **e** $p^3 = 1000$ **f** $2x^2 = 50$

g $6y^4 = 486$ **h** $w^3 + 7 = 15$ **i** $6n^2 - 4 = 92$

7 Solve and give the answer correct to 2 decimal places:

a $p^2 = 45$ **b** $x^3 = 100$ **c** $n^5 = 240$

d $2x^2 = 70$ **e** $4y^3 + 7 = 34$ **f** $\frac{d^4}{3} = 14$

g $\frac{k^2}{2} - 3 = 7$ **h** $\frac{x^3 - 1}{5} = 2$ **i** $2y^2 - 9 = 20$

8 Solve:

a $x^{-1} = 5$ **b** $a^{-3} = 8$ **c** $y^{-5} = 32$ **d** $x^{-2} + 1 = 50$

e $2n^{-1} = 3$ **f** $a^{-3} = \frac{1}{8}$ **g** $x^{-2} = \frac{1}{4}$ **h** $b^{-1} = \frac{1}{9}$

i $x^{-2} = 2\frac{1}{4}$ **j** $b^{-4} = \frac{16}{81}$

PUZZLE

Test your logical thinking and that of your friends.

1 How many months have 28 days?

2 If I have 128 sheep and take away all but 10, how many do I have left?

3 A bottle and its cork cost $1.10 to make. If the bottle costs $1 more than the cork, how much does each cost?

4 What do you get if you add 1 to 15 four times?

5 On what day of the week does Good Friday fall in 2030?

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2.06 Solving quadratic equations by factorisation



Quadratic equations by factorising

A **quadratic equation** is an equation involving a square. For example, $x^2 - 4 = 0$.

When solving quadratic equations by factorising, we use a property of zero.

For any real numbers a and b, if $ab = 0$ then $a = 0$ or $b = 0$.

EXAMPLE 9

Solve:

a $x^2 + x - 6 = 0$ **b** $y^2 - 7y = 0$ **c** $3a^2 - 14a = -8$

Solution

a
$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$\therefore x + 3 = 0$ or $x - 2 = 0$

$x = -3$ or $x = 2$

So the solution is $x = -3$ or 2.

b
$y^2 - 7y = 0$

$y(y - 7) = 0$

$\therefore y = 0$ or $y - 7 = 0$

$y = 7$

So the solution is $y = 0$ or 7.

c First we make the equation equal to zero so we can factorise and use the rule for zero.

$3a^2 - 14a = -8$

$3a^2 - 14a + 8 = -8 + 8$

$3a^2 - 14a + 8 = 0$

$(3a - 2)(a - 4) = 0$

$\therefore 3a - 2 = 0$ or $a - 4 = 0$

$3a = 2$ or $a = 4$

$\frac{3a}{3} = \frac{2}{3}$

$a = \frac{2}{3}$

So the solution is $a = \frac{2}{3}$ or 4.

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Exercise 2.06 Solving quadratic equations by factorisation

Solve each quadratic equation.

1 $y^2 + y = 0$
2 $b^2 - b - 2 = 0$
3 $p^2 + 2p - 15 = 0$
4 $t^2 - 5t = 0$
5 $x^2 + 9x + 14 = 0$
6 $q^2 - 9 = 0$
7 $x^2 - 1 = 0$
8 $a^2 + 3a = 0$
9 $2x^2 + 8x = 0$
10 $4x^2 - 1 = 0$
11 $3x^2 + 7x + 4 = 0$
12 $2y^2 + y - 3 = 0$
13 $8b^2 - 10b + 3 = 0$
14 $x^2 - 3x = 10$
15 $3x^2 = 2x$
16 $2x^2 = 7x - 5$
17 $5x - x^2 = 0$
18 $y^2 = y + 2$
19 $8n = n^2 + 15$
20 $12 = 7x - x^2$
21 $m^2 = 6 - 5m$
22 $x(x + 1)(x + 2) = 0$
23 $(y - 1)(y + 5)(y + 2) = 0$
24 $(x + 3)(x - 1) = 32$
25 $(m - 3)(m - 4) = 20$

WS
Completing the square

2.07 Solving quadratic equations by completing the square

Not all trinomials will factorise, so other methods need to be used to solve quadratic equations.

EXAMPLE 10

Solve:

a $(x + 3)^2 = 11$ **b** $(y - 2)^2 = 7$

Solution

a
$$(x+3)^2 = 11$$
$$x+3 = \pm\sqrt{11}$$
$$x+3-3 = \pm\sqrt{11}-3$$
$$x = \pm\sqrt{11}-3$$

b
$$(y-2)^2 = 7$$
$$y-2 = \pm\sqrt{7}$$
$$y-2+2 = \pm\sqrt{7}+2$$
$$y = \pm\sqrt{7}+2$$

To solve a quadratic equation such as $x^2 - 6x + 3 = 0$, which will not factorise, we can use the method of **completing the square**.

We use the perfect square:

$$a^2 + 2ab + b^2 = (a + b)^2$$

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EXAMPLE 11

Complete the square on $a^2 + 6a$.

Solution

Compare with $a^2 + 2ab + b^2$: $2ab = 6a$

$$b = 3$$

To complete the square: $a^2 + 2ab + b^2 = (a + b)^2$

$$a^2 + 2a(3) + 3^2 = (a + 3)^2$$

$$a^2 + 6x + 9 = (a + 3)^2$$

Completing the square

To complete the square on $a^2 \pm pa$, divide p by 2 and square it.

$$a^2 \pm pa + \left(\frac{p}{2}\right)^2 = \left(a \pm \frac{p}{2}\right)^2$$

EXAMPLE 12

Solve by completing the square:

a $x^2 - 6x + 3 = 0$

b $y^2 + 2y - 7 = 0$ (correct to 3 significant figures)

Solution

a $x^2 - 6x + 3 = 0$

$x^2 - 6x = -3$

$x^2 - 6x + 9 = -3 + 9$ $\left(\frac{6}{2}\right)^2 = 3^2 = 9$

$(x - 3)^2 = 6$

$\therefore x - 3 = \pm\sqrt{6}$

$x = \pm\sqrt{6} + 3$

The 3rd line shows the 'completing the square' step in both solutions.

b $y^2 + 2y - 7 = 0$

$y^2 + 2y = 7$

$y^2 + 2y + 1 = 7 + 1$ $\left(\frac{2}{2}\right)^2 = 1^2 = 1$

$(y + 1)^2 = 8$

$\therefore y + 1 = \pm\sqrt{8}$

$y = \pm\sqrt{8} - 1$

$y \approx 1.83$ or -3.83

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Exercise 2.07 Solving quadratic equations by completing the square

1 Solve and give exact solutions:

a $(x+1)^2 = 7$ **b** $(y+5)^2 = 5$ **c** $(a-3)^2 = 6$

d $(x-2)^2 = 13$ **e** $(2y+3)^2 = 2$

2 Solve and give solutions correct to one decimal place:

a $(h+2)^2 = 15$ **b** $(a-1)^2 = 8$ **c** $(x-4)^2 = 17$

d $(y+7)^2 = 21$ **e** $(3x-1)^2 = 12$

3 Solve by completing the square, giving exact solutions in simplest surd form:

a $x^2 + 4x - 1 = 0$ **b** $a^2 - 6a + 2 = 0$ **c** $y^2 - 8y - 7 = 0$

d $x^2 + 2x - 12 = 0$ **e** $p^2 + 14p + 5 = 0$ **f** $x^2 - 10x - 3 = 0$

g $y^2 + 20y + 12 = 0$ **h** $x^2 - 2x - 1 = 0$ **i** $n^2 + 24n + 7 = 0$

4 Solve by completing the square and writing answers correct to 3 significant figures:

a $x^2 - 2x - 5 = 0$ **b** $x^2 + 12x + 34 = 0$ **c** $q^2 + 18q - 1 = 0$

d $x^2 - 4x - 2 = 0$ **e** $b^2 + 16b + 50 = 0$ **f** $x^2 - 24x + 112 = 0$

g $r^2 - 22r - 7 = 0$ **h** $x^2 + 8x + 5 = 0$ **i** $a^2 + 6a - 1 = 0$

WS Quadratic formula

WS Quadratic equations

WS Problems involving quadratic equations

The quadratic formula

2.08 Solving quadratic equations by quadratic formula

Completing the square is difficult with harder quadratic equations such as $2x^2 - x - 5 = 0$. Completing the square on a general quadratic equation gives the following formula.

The quadratic formula

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof

Solve $ax^2 + bx + c = 0$ by completing the square.

$ax^2 + bx + c = 0$

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

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Completing the square:

$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$=\frac{-4ac+b^2}{4a^2}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{-4ac+b^2}{4a^2}}$$

$$=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$



Excel worksheet: The quadratic formula


Excel spreadsheet: The quadratic formula



Solving algebraic equations

EXAMPLE 13

a Solve $x^2 - x - 2 = 0$ by using the quadratic formula.

b Solve $2y^2 - 9y + 3 = 0$ by formula and give your answer correct to 2 decimal places.

Solution

a $a = 1, b = -1, c = -2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-2)}}{2(1)}$$

$$=\frac{1\pm\sqrt{1+8}}{2}$$

$$=\frac{1\pm\sqrt{9}}{2}$$

$$=\frac{1\pm3}{2}$$

$$=2 \text{ or } -1$$

b $a = 2, b = -9, c = 3$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-9)\pm\sqrt{(-9)^2-4(2)(3)}}{2(2)}$$

$$=\frac{9\pm\sqrt{81-24}}{4}$$

$$=\frac{9\pm\sqrt{57}}{4}$$

$$\approx 4.14 \text{ or } 0.36$$

Exercise 2.08 Solving quadratic equations by quadratic formula

1 Solve by formula, correct to 3 significant figures where necessary:

a $y^2 + 6y + 2 = 0$ **b** $2x^2 - 5x + 3 = 0$ **c** $b^2 - b - 9 = 0$

d $2x^2 - x - 1 = 0$ **e** $-8x^2 + x + 3 = 0$ **f** $n^2 + 8n - 2 = 0$

g $m^2 + 7m + 10 = 0$ **h** $x^2 - 7x = 0$ **i** $x^2 + 5x = 6$

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2 Solve by formula, leaving the answer in simplest surd form:

a	$x^2 + x - 4 = 0$	**b**	$3x^2 - 5x + 1 = 0$	**c**	$q^2 - 4q - 3 = 0$
d	$4h^2 + 12h + 1 = 0$	**e**	$3s^2 - 8s + 2 = 0$	**f**	$x^2 + 11x - 3 = 0$
g	$6d^2 + 5d - 2 = 0$	**h**	$x^2 - 2x = 7$	**i**	$t^2 = t + 1$

CLASS INVESTIGATION

FAULTY PROOF

Here is a proof that 1 = 2. Can you see the fault in the proof?

$x^2 - x^2 = x^2 - x^2$

$x(x - x) = (x + x)(x - x)$

$x = x + x$

$x = 2x$

$\therefore \quad 1 = 2$

2.09 Formulas and equations

Sometimes substituting values into a formula involves solving an equation.

EXAMPLE 14

a The formula for the surface area of a rectangular prism is given by $S = 2(lb + bh + lh)$. Find the value of b when $S = 180$, $l = 9$ and $h = 6$.

b The volume of a cylinder is given by $V = \pi r^2 h$. Evaluate the radius r, correct to 2 decimal places, when $V = 350$ and $h = 6.5$.

Solution

a

$S = 2(lb + bh + lh)$

$180 = 2(9b + 6b + 9 \times 6)$

$= 2\ (15b + 54)$

$= 30b + 108$

$72 = 30b$

$\frac{72}{30} = \frac{30b}{30}$

$2.4 = b$

b

$V = \pi r^2 h$

$350 = \pi r^2 (6.5)$

$\frac{350}{6.5\pi} = \frac{\pi r^2 (6.5)}{6.5\pi}$

$\frac{350}{6.5\pi} = r^2$

$\sqrt{\frac{350}{6.5\pi}} = r$

$4.14 = r$

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Exercise 2.09 Formulas and equations

1 Given that $v = u + at$ is the formula for the velocity of a particle at time t, find the value of t when $u = 17.3$, $v = 100.6$ and $a = 9.8$.

2 The sum of an arithmetic series is given by $S = \frac{n}{2}(a + l)$. Find l if $a = 3$, $n = 26$ and $S = 1625$.

3 The formula for finding the area of a triangle is $A = \frac{1}{2}bh$. Find b when $A = 36$ and $h = 9$.

4 The area of a trapezium is given by $A = \frac{1}{2}h(a + b)$. Find the value of a when $A = 120$, $h = 5$ and $b = 7$.

5 Find the value of y when $x = 3$, given the straight line equation $5x - 2y - 7 = 0$.

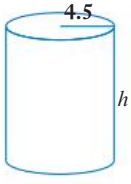
6 The area of a circle is given by $A = \pi r^2$. Find r correct to 3 significant figures if $A = 140$.

7 The area of a rhombus is given by the formula $A = \frac{1}{2}xy$ where x and y are its diagonals. Find the value of x correct to 2 decimal places when $y = 7.8$ and $A = 25.1$.

8 The simple interest formula is $I = Prn$. Find n if $r = 0.145$, $P = 150$ and $I = 326.25$.

9 The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Find y_1 when $m = -\frac{5}{6}$, $y_2 = 7$, $x_2 = -3$ and $x_1 = 1$.

10 The surface area of a cylinder is given by the formula $S = 2\pi r(r + h)$. Evaluate h correct to 1 decimal place if $S = 232$ and $r = 4.5$.



11 The formula for body mass index is $\text{BMI} = \frac{w}{h^2}$. Evaluate:

- **a** the BMI when $w = 65$ and $h = 1.6$
- **b** w when BMI $= 21.5$ and $h = 1.8$
- **c** h when BMI $= 19.7$ and $w = 73.8$.

12 A formula for depreciation is $D = P(1 - r)^n$. Find r if $D = 12\,000$, $P = 15\,000$ and $n = 3$.

13 The x value of the midpoint is given by $x = \frac{x_1 + x_2}{2}$. Find x_1 when $x = -2$ and $x_2 = 5$.

14 Given the height of a particle at time t is $h = 5t^2$, evaluate t when $h = 23$.

15 If $y = x^2 + 1$, evaluate x when $y = 5$.

16 If the surface area of a sphere is $S = 4\pi r^2$, evaluate r to 3 significant figures when $S = 56.3$.

17 The area of a sector of a circle is $A = \frac{1}{2}r^2\theta$. Evaluate r when $A = 24.6$ and $\theta = 0.45$.

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18 If $y = \dfrac{2}{x^3 - 1}$, find the value of x when $y = 3$.

19 Given $y = \sqrt{2x + 5}$, evaluate x when $y = 4$.

20 The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$. Evaluate r to 1 decimal place when $V = 150$.

INVESTIGATION

BODY MASS INDEX

Body mass index (BMI) is a formula that is used by health professionals to screen for weight categories that may lead to health problems.

The formula for BMI is $\text{BMI} = \dfrac{m}{h^2}$ where m is the mass of a person in kg and h is the height in metres.

For adults over 20, a BMI under 18.5 means that the person is underweight and over 25 is overweight. Over 30 is considered obese.



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The BMI may not always be a reliable measurement of body fat. Can you think of some reasons?

Is it important where the body fat is stored? Does it make a difference if it is on the hips or the stomach?

Research more about BMI generally.

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2.10 Linear simultaneous equations

You can solve two equations together to find one solution that satisfies both equations. Such equations are called **simultaneous equations** and there are two ways of solving them. The **elimination method** adds or subtracts the equations. The **substitution method** substitutes one equation into the other.

EXAMPLE 15

Solve simultaneously using the elimination method:

a $3a + 2b = 5$ and $2a - b = -6$

b $5x - 3y = 19$ and $2x - 4y = 16$

Solution

a

$3a + 2b = 5$ [1]

$2a - b = -6$ [2]

$[2] \times 2$: $4a - 2b = -12$ [3]

$[1] + [3]$: $3a + 2b = 5$ [1]

$7a = -7$

$a = -1$

Substitute $a = -1$ in [1]: $3(-1) + 2b = 5$

$-3 + 2b = 5$

$2b = 8$

$b = 4$

Check that the solution is correct by substituting back into *both* equations. → $\therefore$ Solution is $a = -1, b = 4$

b

$5x - 3y = 19$ [1]

$2x - 4y = 16$ [2]

$[1] \times 4$: $20x - 12y = 76$ [3]

$[2] \times 3$: $6x - 12y = 48$ [4]

$[3] - [4]$: $14x = 28$

$x = 2$

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Substitute $x = 2$ in [2]:

$$2(2) - 4y = 16$$
$$4 - 4y = 16$$
$$-4y = 12$$
$$y = -3$$

$\therefore$ Solution is $x = 2, y = -3$

Exercise 2.10 Linear simultaneous equations

Solve each pair of simultaneous equations.

1 $a - b = -2$ and $a + b = 4$
2 $5x + 2y = 12$ and $3x - 2y = 4$
3 $4p - 3q = 11$ and $5p + 3q = 7$
4 $y = 3x - 1$ and $y = 2x + 5$
5 $2x + 3y = -14$ and $x + 3y = -4$
6 $7t + v = 22$ and $4t + v = 13$
7 $4x + 5y + 2 = 0$ and $4x + y + 10 = 0$
8 $2x - 4y = 28$ and $2x - 3y = -11$
9 $5x - y = 19$ and $2x + 5y = -14$
10 $5m + 4n = 22$ and $m - 5n = -13$
11 $4w_1 + 3w_2 = 11$ and $3w_1 + w_2 = 2$
12 $3a - 4b = -16$ and $2a + 3b = 12$
13 $5p + 2q + 18 = 0$ and $2p - 3q + 11 = 0$
14 $7x_1 + 3x_2 = 4$ and $3x_1 + 5x_2 = -2$
15 $9x - 2y = -1$ and $7x - 4y = 9$
16 $5s - 3t - 13 = 0$ and $3s - 7t - 13 = 0$
17 $3a - 2b = -6$ and $a - 3b = -2$
18 $3k - 2h = -14$ and $2k - 5h = -13$

PROBLEM

A group of 39 people went to see a play. There were both adults and children in the group. The total cost of the tickets was $939, with children paying $17 each and adults paying $29 each. How many in the group were adults and how many were children? (Hint: let x be the number of adults and y the number of children.)

2.11 Non-linear simultaneous equations

In simultaneous equations involving **non-linear equations** there may be more than one set of solutions. When solving these, you need to use the substitution method.



Non-linear simultaneous equations

EXAMPLE 16

Solve each pair of equations simultaneously using the substitution method:

a $xy = 6$ and $x + y = 5$

b $x^2 + y^2 = 16$ and $3x - 4y - 20 = 0$

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Solution

a

$xy = 6$ [1]

$x + y = 5$ [2]

From [2]: $y = 5 - x$ [3]

Substitute [3] in [1]:

$x(5 - x) = 6$

$5x - x^2 = 6$

$0 = x^2 - 5x + 6$

$0 = (x - 2)(x - 3)$

$\therefore x = 2$ or $x = 3$

Substitute $x = 2$ in [3]: $y = 5 - 2 = 3$

Substitute $x = 3$ in [3]: $y = 5 - 3 = 2$

Solutions are $x = 2, y = 3$ and $x = 3, y = 2$

b

$x^2 + y^2 = 16$ [1]

$3x - 4y - 20 = 0$ [2]

From [2]:

$3x - 20 = 4y$

$\frac{3x - 20}{4} = y$ [3]

Substitute [3] into [1]:

$x^2 + \left(\frac{3x - 20}{4}\right)^2 = 16$

$x^2 + \left(\frac{9x^2 - 120x + 400}{16}\right) = 16$

$16x^2 + 9x^2 - 120x + 400 = 256$

$25x^2 - 120x + 144 = 0$

$(5x - 12)^2 = 0$

$\therefore 5x - 12 = 0$

$x = 2.4$

Substitute $x = 2.4$ into [3]:

$y = \frac{3(2.4) - 20}{4}$

$= -3.2$

So the solution is $x = 2.4, y = -3.2$

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Exercise 2.11 Non-linear simultaneous equations

Solve each pair of simultaneous equations.

1 $y = x^2$ and $y = x$
2 $y = x^2$ and $2x + y = 0$
3 $x^2 + y^2 = 9$ and $x + y = 3$
4 $x - y = 7$ and $xy = -12$
5 $y = x^2 + 4x$ and $2x - y - 1 = 0$
6 $y = x^2$ and $6x - y - 9 = 0$
7 $x = t^2$ and $x + t - 2 = 0$
8 $m^2 + n^2 = 16$ and $m + n + 4 = 0$
9 $xy = 2$ and $y = 2x$
10 $y = x^3$ and $y = x^2$
11 $y = x - 1$ and $y = x^2 - 3$
12 $y = x^2 + 1$ and $y = 1 - x^2$
13 $y = x^2 - 3x + 7$ and $y = 2x + 3$
14 $xy = 1$ and $4x - y + 3 = 0$
15 $h = t^2$ and $h = (t + 1)^2$
16 $x + y = 2$ and $2x^2 + xy - y^2 = 8$
17 $y = x^3$ and $y = x^2 + 6x$
18 $y = |x|$ and $y = x^2$
19 $y = x^2 - 7x + 6$ and $24x + 4y - 23 = 0$
20 $x^2 + y^2 = 1$ and $5x + 12y + 13 = 0$

WS
Simultaneous equations

2.12 Simultaneous equations with three unknown variables

Three equations can be solved simultaneously to find 3 unknown pronumerals.

EXAMPLE 17

Solve simultaneously: $a - b + c = 7$, $a + 2b - c = -4$ and $3a - b - c = 3$.

Solution

$a - b + c = 7$ [1]

$a + 2b - c = -4$ [2]

$3a - b - c = 3$ [3]

[1] + [2]:

$$\begin{array}{r} a - b + c = 7 \\ a + 2b - c = -4 \\ \hline 2a + b = 3 \end{array} \quad [4]$$

[1] + [3]:

$$\begin{array}{r} a - b + c = 7 \\ 3a - b - c = 3 \\ \hline 4a - 2b = 10 \end{array}$$

or $2a - b = 5$ [5]

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[4] + [5]:

$$\begin{array}{rl} 2a + b &= 3 \\ 2a - b &= 5 \\ \hline 4a &= 8 \\ a &= 2 \end{array}$$

Substitute $a = 2$ in [4]:

$$\begin{aligned} 2(2) + b &= 3 \\ 4 + b &= 3 \\ b &= -1 \end{aligned}$$

Substitute $a = 2$ and $b = -1$ in [1]:

$$\begin{aligned} 2 - (-1) + c &= 7 \\ 2 + 1 + c &= 7 \\ 3 + c &= 7 \\ c &= 4 \end{aligned}$$

$\therefore$ solution is $a = 2, b = -1, c = 4$

Exercise 2.12 Simultaneous equations with three unknown variables

Solve each set of simultaneous equations.

1 $x = -2$, $2x - y = 4$ and $x - y + 6z = 0$

2 $a = -2$, $2a - 3b = -1$ and $a - b + 5c = 9$

3 $2a + b + c = 1$, $a + b = -2$ and $c = 7$

4 $a + b + c = 0$, $a - b + c = -4$ and $2a - 3b - c = -1$

5 $x + y - z = 7$, $x + y + 2z = 1$ and $3x + y - 2z = 19$

6 $2p + 5q - r = 25$, $2p - 2q - r = -24$ and $3p - q + 5r = 4$

7 $2x - y + 3z = 9$, $3x + y - 2z = -2$ and $3x - y + 5z = 14$

8 $x - y - z = 1$, $2x + y - z = -9$ and $2x - 3y - 2z = 7$

9 $3h + j - k = -3$, $h + 2j + k = -3$ and $5h - 3j - 2k = -13$

10 $2a - 7b + 3c = 7$, $a + 3b + 2c = -4$ and $4a + 5b - c = 9$

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2. TEST YOURSELF

Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 Find the exact solution of $x^2 - 5x - 1 = 0$.

A $\frac{-5 \pm \sqrt{29}}{2}$ **B** $\frac{5 \pm \sqrt{21}}{2}$ **C** $\frac{5 \pm \sqrt{29}}{2}$ **D** $\frac{-5 \pm \sqrt{21}}{2}$

2 If $S = 4\pi r^2$, find the value of r when $S = 200$ (there may be more than one answer).

A $5\sqrt{\frac{2}{\pi}}$ **B** $\sqrt{\frac{200}{\pi}}$ **C** $10\sqrt{\frac{2}{\pi}}$ **D** $\sqrt{\frac{50}{\pi}}$

3 Solve the simultaneous equations $x - y = 7$ and $x + 2y = 1$.

A $x = 5, y = 2$ **B** $x = 5, y = -2$ **C** $x = -5, y = -2$ **D** $x = -5, y = 2$

4 Solve:

a $8 = 3b - 22$ **b** $\frac{a}{4} - \frac{a+2}{3} = 9$ **c** $4(3x + 1) = 11x - 3$

d $3p + 1 \leq p + 9$

5 The compound interest formula is $A = P(1 + r)^n$. Find, correct to 2 decimal places:

a A when $P = 1000$, $r = 0.06$ and $n = 4$

b P when $A = 12\,450$, $r = 0.055$ and $n = 7$.

6 Solve each pair of simultaneous equations.

a $x - y + 7 = 0$ and $3x - 4y + 26 = 0$ **b** $xy = 4$ and $2x - y - 7 = 0$

7 Solve:

a $3^{x+2} = 81$ **b** $16^y = 2$

8 Solve $|3b - 1| = 5$.

9 The area of a trapezium is given by $A = \frac{1}{2}h(a + b)$. Find:

a A when $h = 6$, $a = 5$ and $b = 7$

b b when $A = 40$, $h = 5$ and $a = 4$.

10 Solve $2x^2 - 3x + 1 = 0$.

11 Solve $-2 < 3y + 1 \leq 10$ and plot the solution on a number line.

12 Solve, correct to 3 significant figures:

a $x^2 + 7x + 2 = 0$ **b** $y^2 - 2y - 9 = 0$ **c** $3n^2 + 2n - 4 = 0$

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13 The surface area of a sphere is given by $A = 4\pi r^2$. Evaluate to 1 decimal place:

a A when $r = 7.8$ **b** r when $A = 102.9$

14 Solve the simultaneous equations $x^2 + y^2 = 16$ and $3x + 4y - 20 = 0$.

15 The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Evaluate to 2 significant figures:

a V when $r = 8$ **b** r when $V = 250$

16 For each equation, decide if it has:

A 2 solutions **B** 1 solution **C** no solutions.

a $x^2 - 6x + 9 = 0$ **b** $|2x - 3| = 7$ **c** $x^2 - x - 5 = 0$

d $2x^2 - x + 4 = 0$ **e** $3x + 2 = 7$

17 Solve simultaneously $a + b = 5$, $2a + b + c = 4$, $a - b - c = 5$.

18 Solve $9^{2x+1} = 27^x$.

19 Solve:

a $2(3y - 5) > y + 5$ **b** $3^{2x-1} = 27$ **c** $5x^3 - 1 = 39$

d $|5x - 4| = 11$ **e** $8^{x+1} = 4^x$ **f** $27^{2x-1} = 9$

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2. CHALLENGE EXERCISE

1 Find the value of y if $a^{3y-5} = \frac{1}{a^2}$.

2 The solutions of $x^2 - 6x - 3 = 0$ are in the form $a + b\sqrt{3}$. Find the values of a and b.

3 **a** Factorise $x^5 - 9x^3 - 8x^2 + 72$.

b Hence or otherwise solve $x^5 - 9x^3 - 8x^2 + 72 = 0$.

4 Solve the simultaneous equations $y = x^3 + x^2$ and $y = x + 1$.

5 Find the value of b if $x^2 - 8x + b$ is a perfect square. Hence solve $x^2 - 8x - 1 = 0$ by completing the square.

6 Considering the definition of absolute value, solve $\frac{|x-3|}{3-x} = x$, where $x \neq 3$.

7 Solve $x^{\frac{3}{2}} = \frac{1}{8}$.

8 Find the solutions of $x^2 - 2ax - b = 0$ by completing the square.

9 Solve $3x^2 = 8(2x - 1)$ and write the solution in the simplest surd form.

10 Solve $|2x - 1| = 5 - x$ and check solutions.

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Practice set 1

In Questions **1** to **6**, select the correct answer **A**, **B**, **C** or **D**.

1 Write $\dfrac{1}{3\sqrt{(x-2)^5}}$ in index form.

A $(x-2)^{-\frac{5}{3}}$ **B** $\dfrac{(x-2)^{-\frac{5}{2}}}{3}$ **C** $3(x-2)^{-\frac{5}{2}}$ **D** $\dfrac{1}{(x-2)^{\frac{5}{3}}}$

2 Simplify $\dfrac{(2a^3b)^3}{(ab)^2}$.

A $8a^7b$ **B** $8a^8b$ **C** $2a^7b$ **D** $2a^8b$

3 Evaluate $4^{-\frac{3}{2}}$.

A -8 **B** $\dfrac{1}{8}$ **C** $\dfrac{1}{6}$ **D** -6

4 Simplify $\dfrac{a^2-6a+9}{a^2-9}$.

A $\dfrac{1}{a+3}$ **B** $\dfrac{a-3}{a+3}$

C $\dfrac{a+3}{a-3}$ **D** $\dfrac{-6a+9}{a-9}$

5 Factorise $a^2-\dfrac{b^2}{4}$.

A $\left(a-\dfrac{b}{2}\right)^2$ **B** $\left(a+\dfrac{b}{4}\right)\left(a-\dfrac{b}{4}\right)$

C $\left(a+\dfrac{b}{2}\right)^2$ **D** $\left(a+\dfrac{b}{2}\right)\left(a-\dfrac{b}{2}\right)$

6 The solution to $x^2+2x-6=0$ is:

A $x=-1\pm 2\sqrt{7}$ **B** $x=\dfrac{2\pm\sqrt{28}}{2}$

C $x=\dfrac{-2\pm\sqrt{-20}}{2}$ **D** $x=-1\pm\sqrt{7}$

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7 Solve:

a $3x - 7 = 23$ **b** $5(b - 3) = 15$ **c** $\frac{x}{3} + 4 = 5$

d $4y - 7 = 3y + 9$ **e** $8z + 1 = 11z - 17$ **f** $2^x = 32$

g $9^{y-1} = 3$ **h** $x^2 - 3x = 0$ **i** $|x + 2| = 5$

j $|5a - 2| = 8$

8 Solve for p: $\frac{p-3}{2} - \frac{p+1}{5} = 1$.

9 Simplify $2\sqrt{12}$.

10 Factorise fully: $10x + 2xy - 10y - 2y^2$.

11 Write in index form:

a $\frac{1}{x}$ **b** $\sqrt[3]{x^4}$

12 Simplify the expression $8y - 2(y + 5)$.

13 Rationalise the denominator of $\frac{5}{5-\sqrt{2}}$.

14 Solve $2x^2 - 3x - 1 = 0$ correct to 3 significant figures.

15 Simplify $\frac{x+1}{5} \div \frac{x^2 - 2x - 3}{10}$.

16 Evaluate $(3.9)^4$ correct to 1 decimal place.

17 Simplify $2\sqrt{3} - \sqrt{27}$.

18 Expand and simplify $(x - 3)(x^2 + 5x - 1)$.

19 Expand and simplify $\sqrt{2}(3\sqrt{5} - 2\sqrt{2})$.

20 Simplify $\frac{2x+6}{2}$.

21 Solve $4a - 5 < 7a + 4$.

22 The radius r of a circle with area A is given by $r = \sqrt{\frac{A}{\pi}}$. Find r, correct to 2 decimal places, if $A = 7.59$.

23 Solve each set of simultaneous equations.

a $3a - b = 7$ and $2a + b = 8$

b $a + b - c = 8$, $b + c = 5$ and $a + 2c = 3$

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24 Solve $5 - 2x < 3$ and show the solution on a number line.

25 Solve the equation $x^2 - 4x + 1 = 0$, giving exact solutions in simplest surd form.

26 Write 7^{-2} as a rational number.

27 Solve the simultaneous equations $y = 3x - 1$ and $y = x^2 - 5$.

28 Find integers x and y such that $\dfrac{\sqrt{3}}{2\sqrt{3}+3} = x + y\sqrt{3}$.

29 Evaluate $|-2|^2 - |-1| + |4|$.

30 Factorise $8x^2 - 32$.

31 Rationalise the denominator of $\dfrac{2\sqrt{3}}{3\sqrt{5}-\sqrt{2}}$.

32 Simplify $2|-4| - |3| + |-2|$.

33 Rationalise the denominator of $\dfrac{\sqrt{5}+1}{2\sqrt{2}+3}$.

34 Simplify $\dfrac{(a^{-4})^3 \times b^6}{a^9 \times (b^{-1})^4}$.

35 Evaluate $4^{-\frac{3}{2}}$ as a rational number.

36 Simplify $2(x - 5) - 3(x - 1)$.

37 Solve $4^{2x+1} = 8$.

38 Write $\dfrac{1}{x+3}$ in index form.

39 Find the value of a^3b^{-2} in index form if $a = \left(\dfrac{1}{2}\right)^3$ and $b = \left(\dfrac{4}{5}\right)^2$.

40 Write $(3x+2)^{-\frac{1}{2}}$ without an index.

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41 Simplify:

a $8x - 7y - y + 4x$

b $\sqrt{124}$

c $\dfrac{x^2 - 9}{2x^2 + 5x - 3}$

d $\dfrac{1}{\sqrt{2}+1} + \dfrac{2}{\sqrt{2}-1}$

e $\dfrac{3}{x+1} + \dfrac{2}{x^2-1} - \dfrac{4}{x-1}$

f $x - \dfrac{1}{x}$ when $x = 2\sqrt{3}$

g $\dfrac{(x^{-2})^5 y^4 z^{-3}}{x^4 (y^3)^{-1} (z^{-4})^{-2}}$

h $\dfrac{a+b}{5a - 20ab^2} \div \dfrac{a^2 + 2ab + b^2}{3 - 6b}$

i $8\sqrt{5} - 3\sqrt{20} + 2\sqrt{45}$

j $\dfrac{a^3 b^2 (c^4)^2}{(a^2)^2 bc^5}$ if $a = \left(\dfrac{1}{2}\right)^2$, $b = \left(\dfrac{2}{3}\right)^3$ and $c = \left(\dfrac{4}{9}\right)^{-1}$.

42 The volume of a sphere is given by the formula $V = \dfrac{4}{3}\pi r^3$. Find the exact radius r if the volume V is $10\dfrac{2}{3}$ cm^3.

43 Find the value of k if $(2x + 5)^2 = 4x^2 + kx + 25$.

44 Simplify $\sqrt{81x^2 y^3}$.

45 Factorise:

a $5(a - 2)^2 + 40(a - 2)$

b $(2a - b + c)^2 - (a + 5b - c)^2$

46 Solve $-2 \leq \dfrac{8x - 1}{5} < 9$.

47 Simplify $\dfrac{x+1}{5} - \dfrac{x+2}{3}$.

48 Solve $x^2 - 5x = 0$.

49 Solve $x^2 - 5x - 1 = 0$ and write the solutions correct to 2 decimal places.

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50 Simplify $\sqrt{8}+\sqrt{98}$.

51 Write $\dfrac{3}{x^2+5x}-\dfrac{4}{x}+\dfrac{2}{x+5}$ as a single fraction.

52 Solve for x: $4^{2x-1}=\dfrac{1}{8}$.

53 Factorise:

a x^2-2x-8 **b** a^2-9 **c** y^2+6y+9

d $t^2+8t+16$ **e** $3x^2-11x+6$

54 Solve:

a $5x-4=2x+11$ **b** $y^2-2y-13=0$ (correct to 2 decimal places)

c $4^{2x}=8$ **d** $|2b+3|=7$

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3.



FUNCTIONS

Functions and their graphs are used in many areas such as mathematics, science and economics. In this chapter you will explore what functions are and how to sketch some types of graphs, including straight lines, parabolas and cubics.

CHAPTER OUTLINE

3.01 Functions
3.02 Function notation
3.03 Properties of functions
3.04 Linear functions
3.05 The gradient of a straight line
3.06 Finding a linear equation
3.07 Parallel and perpendicular lines
3.08 Quadratic functions
3.09 Axis of symmetry
3.10 The discriminant
3.11 Finding a quadratic equation
3.12 Cubic functions
3.13 Polynomial functions
3.14 Intersection of graphs



IN THIS CHAPTER YOU WILL:

- understand the definition of a function and use function notation
- test a function using the vertical line test
- identify a one-to-one function using the horizontal line test
- find the domain and range of functions including composite functions using interval notation
- identify even and odd functions
- understand a linear function, its graph and properties, including the gradient and axes intercepts
- graph situations involving direct linear variation
- find the equation of a line, including parallel and perpendicular lines
- identify a quadratic function, its graph and properties, including its axis of symmetry, turning point and axes intercepts
- solve quadratic equations and use the discriminant to identify the numbers and types of solutions
- find the quadratic equation of a parabola
- identify a cubic function, its graph and properties, including the shape, horizontal point of inflection and axes intercepts
- find a cubic equation
- identify a polynomial and its characteristics
- draw the graph of a polynomial showing intercepts
- solve simultaneous equations involving linear and quadratic equations, both algebraically and graphically, and solve problems involving intersection of graphs of functions (for example, break-even points)

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TERMINOLOGY

angle of inclination The angle a straight line makes with the positive x-axis measured anticlockwise

axis of symmetry A line that divides a shape into halves that are mirror-images of each other

break-even point The point at which a business' income equals its costs, making neither a profit nor a loss

coefficient A constant multiplied by a pronumeral in an algebraic term. For example, in ax^3 the a is the coefficient

constant term The term in a polynomial function that is independent of x

cubic function A function with x^3 as its highest power or degree

degree The highest power of x in a polynomial

dependent variable A variable whose value depends on another (independent) variable, such as y (depending on x)

direct variation A relationship between two variables such that as one variable increases so does the other, or as one variable decreases so does the other. One variable is a multiple of the other, with equation $y = kx$. Also called **direct proportion**

discriminant The expression $b^2 - 4ac$ that shows how many roots the quadratic equation $ax^2 + bx + c = 0$ has

domain The set of all possible values of x for a function or relation; the set of 'input' values

even function A function $f(x)$ that has the property $f(-x) = f(x)$; its graph is symmetrical about the y-axis

function A relation where every x value in the domain has a unique y value in the range

gradient The steepness of a graph at a point on the graph, measured by the ratio $\frac{\text{rise}}{\text{run}}$; or the change in y values as x values change

horizontal line test A test that checks if a function is one-to-one, whereby any horizontal line drawn on the graph of a function should cut the graph at most once. If the horizontal line cuts the graph more than once, it is not one-to-one

independent variable A variable whose value does not depend on another variable; for example, x in $y = f(x)$

intercepts The values where a graph cuts the x- and y- axes

interval notation A notation that represents an interval by writing its endpoints in square brackets [ ] when they are included and in parentheses ( ) when they are not included

leading coefficient The coefficient of the highest power of x. For example, $2x^4 - x^3 + 3x + 1$ has a leading coefficient of 2

leading term The term with the highest power of x. For example, $2x^4 - x^3 + 3x + 1$ has a leading term of $2x^4$

linear function A function with x as its highest power or degree

monic polynomial A polynomial whose leading coefficient is 1

odd function A function $f(x)$ that has the property $f(-x) = -f(x)$; its graph has point symmetry about the origin (0, 0)

one-to-one function A function in which every y value in the range corresponds to exactly one x value in the domain

parabola The graph of a quadratic function

piecewise function A function that has different functions defined on different intervals

point of inflection A point on a curve where the concavity changes, such as the turning point on the graph of a cubic function

polynomial An expression in the form $P(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$ where n is a positive integer or zero

quadratic function A function with x^2 as the highest power of x

range The set of all possible y values of a function or relation; the set of 'output' values

root A solution of an equation

turning point Where a graph changes from increasing to decreasing or vice versa; sometimes a turning point (horizontal inflection) where concavity changes

vertex A turning point

vertical line test A test that checks if a relation is a function, whereby any vertical line drawn on the graph of a relation should cut the graph at most once. If the vertical line cuts the graph more than once, it is not a function

zero An x value of a function or polynomial for which the y value is zero, that is, $f(x) = 0$

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3.01 Functions

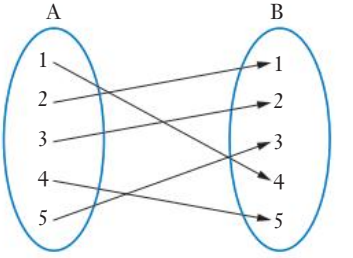
WS
Functions and relations

A **relation** is a set of **ordered pairs** (x, y) where the **variables** x and y are related according to some pattern or rule. The x is called the **independent variable** and the y is called the **dependent variable** because the value of y depends on the value of x. We usually choose a value of x and use it to find the corresponding value of y.

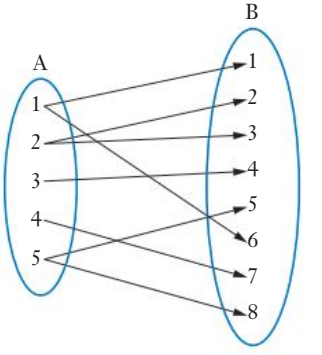
A relation can also be described as a mapping between 2 sets of numbers, with the set of x values, A, on the left and the set of y values B, on the right.

Types of relations

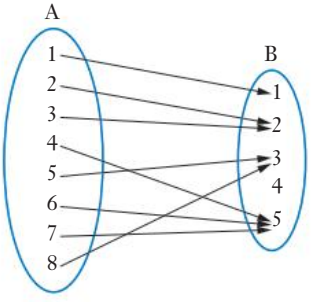
A **one-to-one** relation is a mapping where every element of A corresponds with exactly one element of B and every element of B corresponds with exactly one element of A. Each element has its own unique match.



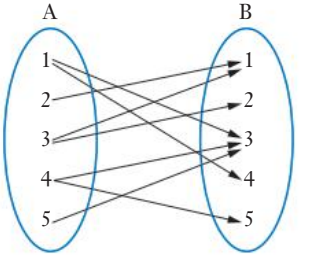
A **one-to-many** relation is a mapping where an element of A corresponds with 2 or more elements of B. For example, 5 in set A matches with 5 and 8 in set B.



A **many-to-one** relation is a mapping where 2 or more elements of A correspond with the same one element of B. For example, 4, 6 and 7 in set A match with 5 in set B.



A **many-to-many** relation is a mapping where 2 or more elements of A correspond with 2 or more elements of B. This is a combination of the one-to-many and many-to-one relations.



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Function

A **function** is a special type of relation where for every value of x there is a unique value of y.

The **domain** is the set of all values of x for which a function is defined.

The **range** is the set of all values of y as x varies.

A function could be a one-to-one or many-to-one relation.

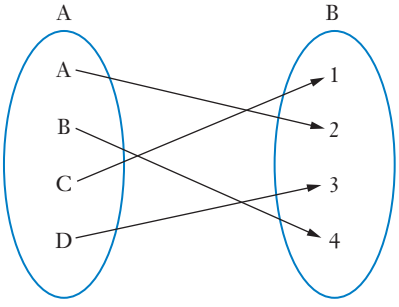
For example, this table matches a group of people with their eye colours.

Person	Anne	Jacquie	Donna	Hien	Marco	Russell	Trang
Colour	Blue	Brown	Grey	Brown	Green	Brown	Brown

The ordered pairs are (Anne, Blue), (Jacquie, Brown), (Donna, Grey), (Hien, Brown), (Marco, Green), (Russell, Brown) and (Trang, Brown).

This table represents a function, since for every person there is a unique eye colour. The domain is the set of people; the range is the set of eye colours. It is a many-to-one function since more than one person can correspond to one eye colour.

Here is a different function:



Set A is the domain, set B is the range.

The ordered pairs are (A, 2), (B, 4), (C, 1) and (D, 3).

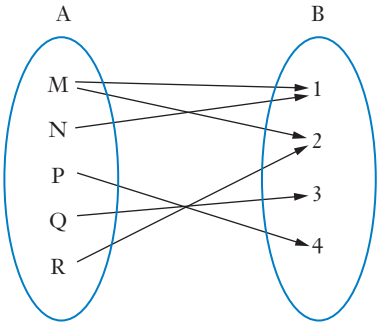
It is a function because every x value in A corresponds to exactly one x value in B.

It is a one-to-one function because every y value in B corresponds to exactly one x value in A.

Here is an example of a relation that is **not** a function. Can you see why?

In this example the ordered pairs are (M, 1), (M, 2), (N, 1), (P, 4), (Q, 3) and (R, 2).

Notice that M corresponds to 2 values in set B: 1 and 2. This means that it is **not** a function. Notice also that M and R both correspond with the same value 2. This is a many-to-many relation.

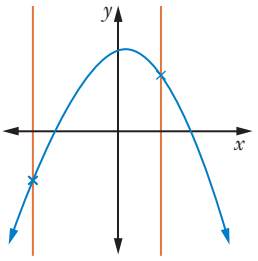


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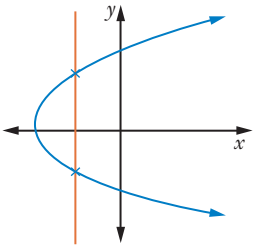
The vertical line test

Relations can also be described by algebraic rules or equations such as $y = x^2 + 1$ and $x^2 + y^2 = 4$, and hence graphed on a number plane. There is a very simple test called the **vertical line test** to test if a graph represents a function.

If any vertical line crosses a graph at only one point, the graph represents a function. This shows that, for every value of x, there is only one value of y.



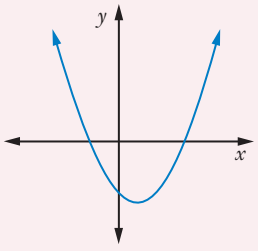
If any vertical line crosses a graph at more than one point, the graph does not represent a function. This shows that, for some value of x, there is more than one value of y.



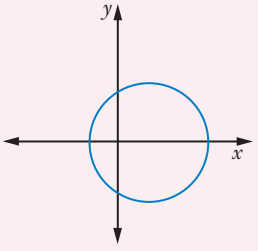
EXAMPLE 1

Does each graph or set of ordered pairs represent a function?

a

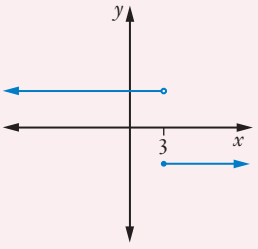


b



c $(-2, 3), (-1, 4), (0, 5), (1, 3), (2, 4)$

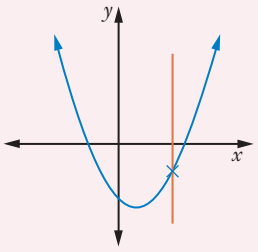
d



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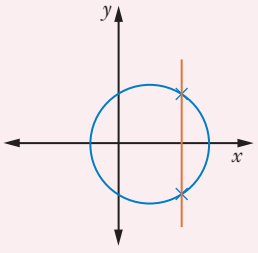
Solution

a



A vertical line only cuts the graph once. So the graph represents a function.

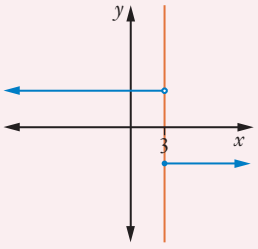
b



A vertical line can cut the curve in more than one place. So the circle does not represent a function.

c For each x value there is only one y value, so this set of ordered pairs is a function.

d



The open circle at $x = 3$ on the top line means that $x = 3$ is not included, while the closed circle on the bottom line means that $x = 3$ is included on this line.

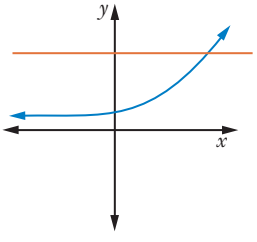
So a vertical line only touches the graph once at $x = 3$.

The graph represents a function.

The horizontal line test

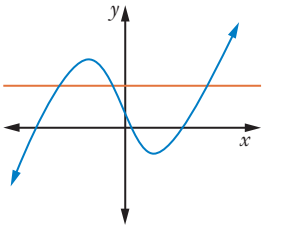
The **horizontal line test** is used on the graph of a function to test whether the function is **one-to-one**.

If any horizontal line crosses a graph at only one point, there is only one x value for every y value. The graph represents a one-to-one function.



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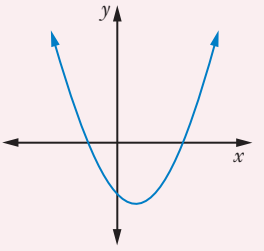
If any horizontal line crosses a graph at more than one point, this means that there are 2 or more x values that have the same y value. The graph does not represent a one-to-one function.



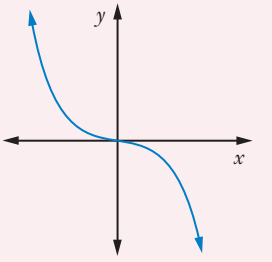
EXAMPLE 2

Does each graph represent a one-to-one function?

a

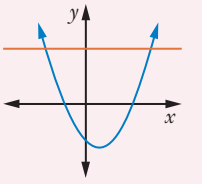


b

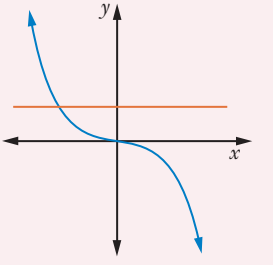


Solution

a A horizontal line cuts the curve in more than one place. The function is not one-to-one.



b A horizontal line cuts the curve in only one place. The function is one-to-one.



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DID YOU KNOW?

René Descartes

The number plane is called the **Cartesian plane** after René Descartes (1596–1650). Descartes used the number plane to develop analytical geometry. He discovered that any equation with two unknown variables can be represented by a line. The points in the number plane can be called Cartesian coordinates.

Descartes used letters at the beginning of the alphabet to stand for numbers that are known, and letters near the end of the alphabet for unknown numbers. This is why we still use x and y so often!

Research Descartes to find out more about his life and work.

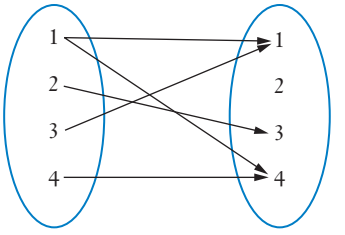
Exercise 3.01 Functions

1 List the ordered pairs for each relation, then state whether the relation is a one-to-one, one-to-many, many-to-one or many-to-many.

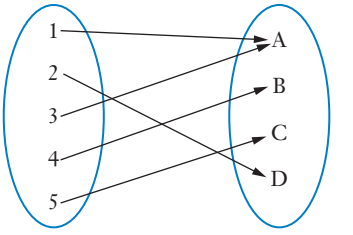
a

Name	Wade	Scott	Geoff	Deng	Mila	Stevie
Hair colour	Black	Blond	Grey	Black	Brown	Blond

b



c



d

x	3	5	8	9	5	8
y	5	±2	−7	3	6	0

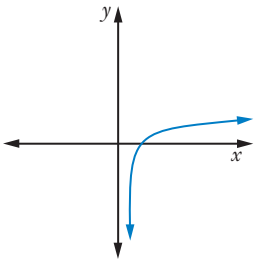
e

x	y
1	9
2	15
3	27
4	33
5	45

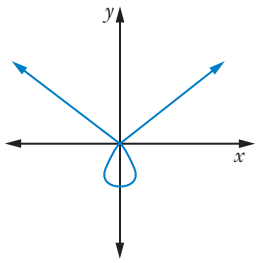
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2 Does each graph or set of ordered pairs represent a function? If it does, state whether it is one-to-one.

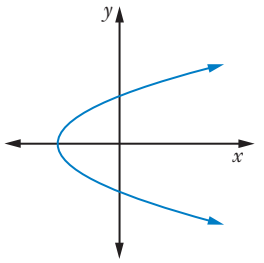
a



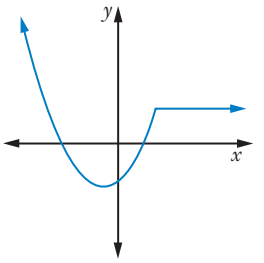
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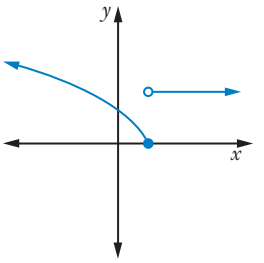
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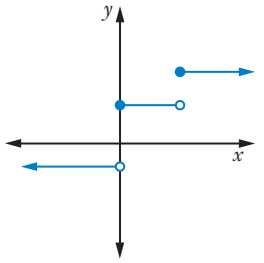
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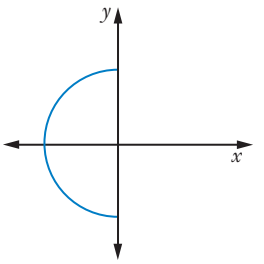
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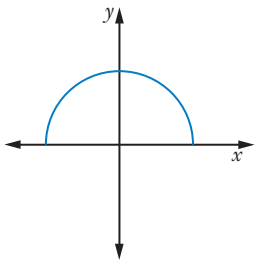
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g



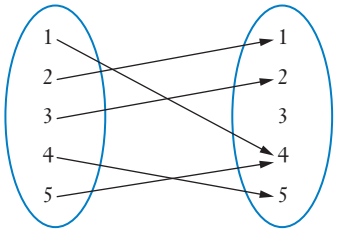
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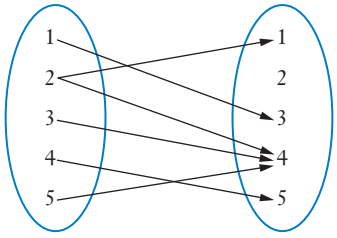
i (1, 3), (2, –1), (3, 3), (4, 0)

j (1, 3), (2, –1), (2, 7), (4, 0)

k



l



m (2, 5), (3, –1), (4, 0), (–1, 3), (–2, 7)

n

Person	Ben	Paula	Pierre	Hamish	Jacob	Leanne	Pierre	Lien
Sport	Tennis	Football	Tennis	Football	Football	Badminton	Football	Badminton

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o

A	3
B	4
C	7
D	3
E	5
F	7
G	4

3 A relation consists of the ordered pairs (–3, 4), (–1, 5), (0, –2), (1, 4) and (6, 8).

a Write the set of independent variables, x.

b Write the set of dependent variables, y.

c Describe the relation as one-to-one, one-to-many, many-to-one or many-to-many.

d Is the relation a function?

WS
Function notation

3.02 Function notation

Since the value of y depends on the value of x, we say that y is a function of x. We write this using **function notation** as $y = f(x)$.

EXAMPLE 3

a Find the value of y when $x = 3$ in the equation $y = 2x - 1$.

b Evaluate $f(3)$, given $f(x) = 2x - 1$.

Solution

a When $x = 3$:

$$\begin{aligned} y &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

b $f(x) = 2x - 1$

$$\begin{aligned} f(3) &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

Both questions in Example 3 are the same, but the second one looks different because it uses function notation.

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EXAMPLE 4

a If $f(x) = x^2 + 3x + 1$, find $f(-2)$.

b If $f(x) = x^3 - x^2$, find the value of $f(-1)$.

c Find the values of x for which $f(x) = 0$ given that $f(x) = x^2 + 3x - 10$.

Solution

a
$$f(x) = x^2 + 3x + 1$$
$$f(-2) = (-2)^2 + 3(-2) + 1$$
$$= 4 - 6 + 1$$
$$= -1$$

b
$$f(x) = x^3 - x^2$$
$$f(-1) = (-1)^3 - (-1)^2$$
$$= -1 - 1$$
$$= -2$$

c
$$f(x) = 0$$
$$x^2 + 3x - 10 = 0$$
$$(x + 5)(x - 2) = 0$$
$$x = -5, x = 2$$

A **piecewise function** is a function made up of 2 or more functions defined on different intervals.

EXAMPLE 5

a
$$f(x) = \begin{cases} 3x + 4 & \text{when } x \geq 2 \\ -2x & \text{when } x < 2 \end{cases}$$

Find $f(3), f(2), f(0)$ and $f(-4)$.

b
$$g(x) = \begin{cases} x^2 & \text{when } x > 2 \\ 2x - 1 & \text{when } -1 \leq x \leq 2 \\ 5 & \text{when } x < -1 \end{cases}$$

Find $g(1) + g(-2) - g(3)$.

Solution

a
$f(3) = 3(3) + 4$ since $3 \geq 2$
$= 13$
$f(2) = 3(2) + 4$ since $2 \geq 2$
$= 10$
$f(0) = -2(0)$ since $0 < 2$
$= 0$
$f(-4) = -2(-4)$ since $-4 < 2$
$= 8$

b
$g(1) = 2(1) - 1$ since $-1 \leq 1 \leq 2$
$= 1$
$g(-2) = 5$ since $-2 < -1$
$g(3) = 3^2$ since $3 > 2$
$= 9$
So $g(1) + g(-2) - g(3) = 1 + 5 - 9$
$= -3$

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You can also substitute pronumerals instead of numbers into functions.

EXAMPLE 6

Find $f(h + 1)$ given $f(x) = 5x + 4$.

Solution

Substitute $h + 1$ for x:

$$f(h + 1) = 5(h + 1) + 4$$
$$= 5h + 5 + 4$$
$$= 5h + 9$$

DID YOU KNOW?

Leonhard Euler

Leonhard Euler (1707–83), from Switzerland, studied functions and invented the function notation $f(x)$. He studied theology, astronomy, medicine, physics and oriental languages as well as mathematics, and wrote more than 500 books and articles on mathematics. He found time between books to marry and have 13 children, and even when he went blind he kept on having books published.

Exercise 3.02 Function notation

1 Given $f(x) = x + 3$, find $f(1)$ and $f(-3)$.

2 If $h(x) = x^2 - 2$, find $h(0)$, $h(2)$ and $h(-4)$.

3 If $f(x) = -x^2$, find $f(5)$, $f(-1)$, $f(3)$ and $f(-2)$.

4 Find the value of $f(0) + f(-2)$ if $f(x) = x^4 - x^2 + 1$.

5 Find $f(-3)$ if $f(x) = 2x^3 - 5x + 4$.

6 If $f(x) = 2x - 5$, find x when $f(x) = 13$.

7 Given $f(x) = x^2 + 3$, find any values of x for which $f(x) = 28$.

8 If $f(x) = 3^x$, find x when $f(x) = \frac{1}{27}$.

9 Find values of z for which $f(z) = 5$ given $f(z) = |2z + 3|$.

10 If $f(x) = 2x - 9$, find $f(p)$ and $f(x + h)$.

11 Find $g(x - 1)$ when $g(x) = x^2 + 2x + 3$.

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12 If $f(x) = x^2 - 1$, find $f(k)$ as a product of factors.

13 Given $f(t) = t^2 - 2t + 1$, find:

a t when $f(t) = 0$ **b** any values of t for which $f(t) = 9$.

14 Given $f(t) = t^4 + t^2 - 5$, find the value of $f(b) - f(-b)$.

15 $f(x) = \begin{cases} x^3 & \text{for } x > 1 \\ x & \text{for } x \leq 1 \end{cases}$

Find $f(5)$, $f(1)$ and $f(-1)$.

16 $f(x) = \begin{cases} 2x - 4 & \text{if } x > 1 \\ x + 3 & \text{if } -1 \leq x \leq 1 \\ x^2 & \text{if } x < -1 \end{cases}$

Find the value of $f(2) - f(-2) + f(-1)$.

17 Find $g(3) + g(0) + g(-2)$ if $g(x) = \begin{cases} x + 1 & \text{when } x \geq 0 \\ -2x + 1 & \text{when } x < 0 \end{cases}$

18 Find the value of $f(3) - f(2) + 2f(-3)$ when $f(x) = \begin{cases} x & \text{for } x > 2 \\ x^2 & \text{for } -2 \leq x \leq 2 \\ 4 & \text{for } x < -2 \end{cases}$

19 Find the value of $f(-1) - f(3)$ if $f(x) = \begin{cases} x^3 - 1 & \text{for } x \geq 2 \\ 2x^2 + 3x - 1 & \text{for } x < 2 \end{cases}$

20 If $f(x) = x^2 - 5x + 4$, find $f(x + h) - f(x)$ in its simplest form.

21 Simplify $\dfrac{f(x+h) - f(h)}{h}$ where $f(x) = 2x^2 + x$.

22 If $f(x) = 5x - 4$, find $f(x) - f(c)$ in its simplest form.

23 Find the value of $f(k^2)$ if $f(x) = \begin{cases} 3x + 5 & \text{for } x \geq 0 \\ x^2 & \text{for } x < 0 \end{cases}$

24 If $f(x) = \begin{cases} x^3 & \text{when } x \geq 3 \\ 5 & \text{when } 0 < x < 3 \\ x^2 - x + 2 & \text{when } x \leq 0 \end{cases}$

evaluate:

a $f(0)$ **b** $f(2) - f(1)$ **c** $f(-n^2)$

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25 If $f(x) = \dfrac{x^2 - 2x - 3}{x - 3}$:

- **a** evaluate $f(2)$.
- **b** explain why the function does not exist for $x = 3$.
- **c** by taking several x values close to 3, find the value of y that the function is moving towards as x moves towards 3.

WS
Function notation

3.03 Properties of functions

We can use the properties of functions, such as their **intercepts**, to draw their graphs.

Intercepts

The ***x*-intercept** of a graph is the value of x where the graph crosses the x-axis.

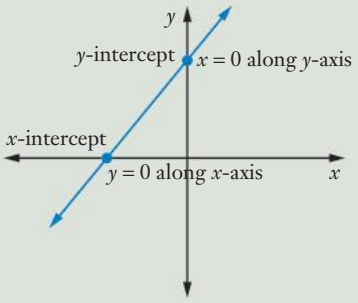
The ***y*-intercept** of a graph is the value of y where the graph crosses the y-axis.

Intercepts of the graph of a function

For x-intercept(s), substitute $y = 0$.

For y-intercept, substitute $x = 0$.

For the graph of $y = f(x)$, solving $f(x) = 0$ gives the x-intercepts and evaluating $f(0)$ gives the y-intercept.



EXAMPLE 7

Find the x- and y-intercepts of the function $f(x) = x^2 + 7x - 8$.

Solution

For x-intercepts, $y = f(x) = 0$:

$0 = x^2 + 7x - 8$

$= (x + 8)(x - 1)$

$x = -8, x = 1$

So x-intercepts are -8 and 1.

For y-intercept, $x = 0$:

$f(0) = 0^2 + 7(0) - 8$

$= -8$

So the y-intercept is -8.

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Domain and range

The **domain** of a function $y = f(x)$ is the set of all x values for which $f(x)$ is defined.

The **range** of a function $y = f(x)$ is the set of all y values for which $f(x)$ is defined.

Interval notation

- $[a, b]$ means the interval is between a and b, including a and b
- (a, b) means the interval is between a and b, excluding a and b
- $[a, b)$ means the interval is between a and b, including a but excluding b
- $(a, b]$ means the interval is between a and b, excluding a but including b
- $(-\infty, \infty)$ means that the interval includes the set of all real numbers R

EXAMPLE 8


Domain and range

Find the domain and range of each function.

a $f(x) = x^2$ **b** $y = \sqrt{x-1}$

Solution

a You can find the domain and range from the equation or the graph.

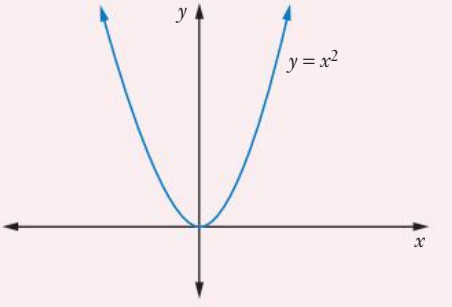
For $f(x) = x^2$, you can substitute any value for x. The y values will be 0 or positive.

So the domain is all real values of x and the range is all $y \geq 0$.

We can write this using interval notation:

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$



b The function $y = \sqrt{x-1}$ is only defined if $x - 1 \geq 0$ because we can only evaluate the square root of a positive number or 0.

For example, $x = 0$ gives $y = \sqrt{-1}$, which is undefined for real numbers.

So $x - 1 \geq 0$

$x \geq 1$

Domain: $[1, \infty)$

The value of $\sqrt{x-1}$ is always positive or zero. So $y \geq 0$.

Range: $[0, \infty)$

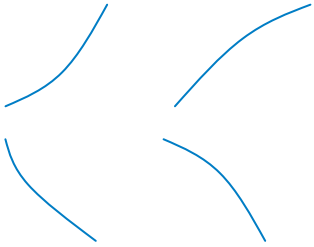
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Increasing and decreasing graphs

When you draw a graph, it helps to know whether the function is increasing or decreasing on an interval.

If a graph is **increasing**, y increases as x increases, and the graph is moving upwards.

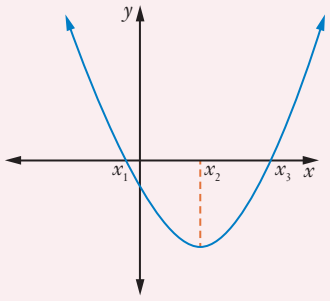
If a graph is **decreasing**, then y decreases as x increases, and the curve moves downwards.



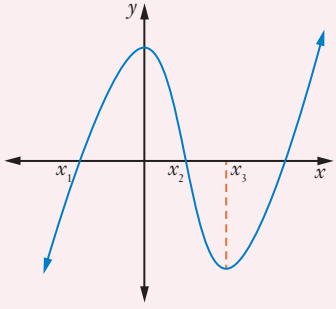
EXAMPLE 9

State the domain over which each curve is increasing.

a



b



Solution

a The curve is decreasing to the left of x_2 and increasing to the right of x_2, that is, when $x > x_2$.

So the domain over which the graph is increasing is (x_2, ∞).

b The curve is increasing on the left of the y-axis ($x = 0$), decreasing from $x = 0$ to $x = x_3$, then increasing again from $x = x_3$.

So the curve is increasing for $x < 0$, $x > x_3$.

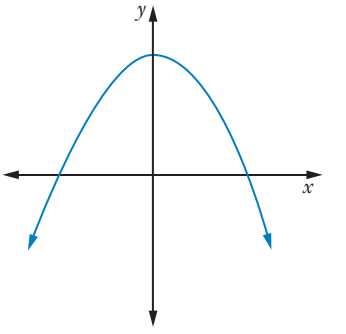
So the domain over which the graph is increasing is $(-\infty, 0) \cup (x_3, \infty)$.

The symbol $\cup$ is for 'union' and means 'and'. It stands for the union or joining of 2 separate parts. You will meet this symbol again in probability.

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Even and odd functions

Even functions have graphs that are symmetrical about the y-axis. The graph has line symmetry about the y-axis. The left and right halves are mirror-images of each other.



Even functions

A function is even if $f(x) = f(-x)$ for all values of x in the domain.

EXAMPLE 10

Show that $f(x) = x^2 + 3$ is an even function.

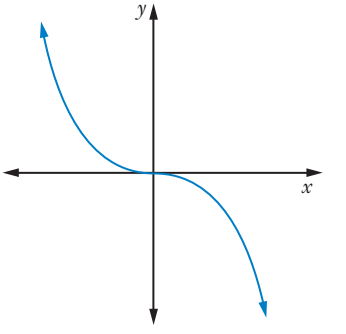
Solution

$$f(-x) = (-x)^2 + 3$$
$$= x^2 + 3$$
$$= f(x)$$

So $f(x) = x^2 + 3$ is an even function.

Odd and even functions

Odd functions have graphs that have point symmetry about the origin. A graph rotated 180° about the origin gives the original graph.



Odd functions

A function is odd if $f(-x) = -f(x)$ for all values of x in the domain.

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Odd and even functions

EXAMPLE 11

Show that $f(x) = x^3 - x$ is an odd function.

Solution

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$

So $f(x) = x^3 - x$ is an odd function.

INVESTIGATION

EVEN AND ODD FUNCTIONS

Explore the family of graphs of $f(x) = kx^n$, the **power functions**.

For what values of n is the function even?

For what values of n is the function odd?

Does the value of k change this?

Are these families of functions below even or odd? Does the value of k change this?

1 $f(x) = x^n + k$ **2** $f(x) = (x + k)^n$

Exercise 3.03 Properties of functions

1 Find the x- and y-intercepts of each function.

a $y = 3x - 2$ **b** $2x - 5y + 20 = 0$ **c** $x + 3y - 12 = 0$

d $f(x) = x^2 + 3x$ **e** $f(x) = x^2 - 4$ **f** $p(x) = x^2 + 5x + 6$

g $y = x^2 - 8x + 15$ **h** $p(x) = x^3 + 5$ **i** $y = \dfrac{x+3}{x}$

j $g(x) = 9 - x^2$

2 $f(x) = 3x - 6$

a Solve $f(x) = 0$.

b Find the x- and y-intercepts.

3 Show that $f(x) = f(-x)$ where $f(x) = x^2 - 2$. What type of function is it?

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4 $f(x) = x^3 + 1$

a Find $f(x^2)$. **b** Find $[f(x)]^2$. **c** Find $f(-x)$.

d Is $f(x) = x^3 + 1$ an even or odd function?

e Solve $f(x) = 0$.

f Find the intercepts of the function.

5 Show that $g(x) = x^8 + 3x^4 - 2x^2$ is an even function.

6 Show that $f(x)$ is odd, given $f(x) = x$.

7 Show that $f(x) = x^2 - 1$ is an even function.

8 Show that $f(x) = 4x - x^3$ is an odd function.

9 **a** Prove that $f(x) = x^4 + x^2$ is an even function.

b Find $f(x) - f(-x)$.

10 Are these functions even, odd or neither?

a $y = \dfrac{x^3}{x^4 - x^2}$ **b** $f(x) = \dfrac{1}{x^3 - 1}$ **c** $f(x) = \dfrac{3}{x^2 - 4}$

d $y = \dfrac{x - 3}{x + 3}$ **e** $f(x) = \dfrac{x^3}{x^5 - x^2}$

11 If n is a positive integer, for what values of n is the power function $f(x) = kx^n$:

a even? **b** odd?

12 Can the function $f(x) = x^n + x$ ever be:

a even? **b** odd?

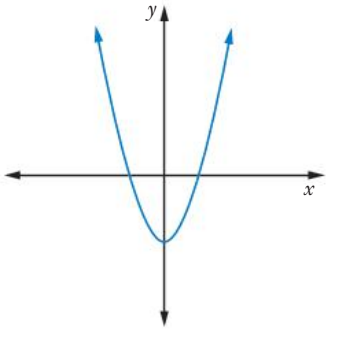
13 For the functions below, state:

i the domain over which the graph is increasing

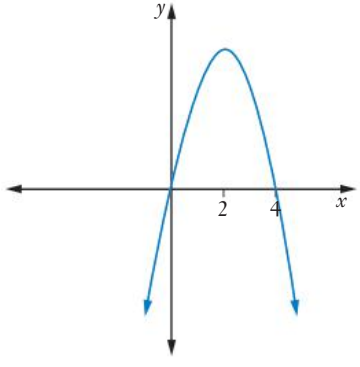
ii the domain over which the graph is decreasing

iii whether the graph is odd, even or neither.

a

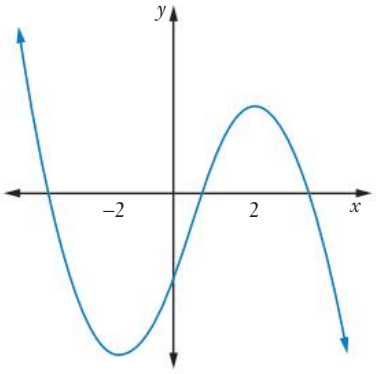


b

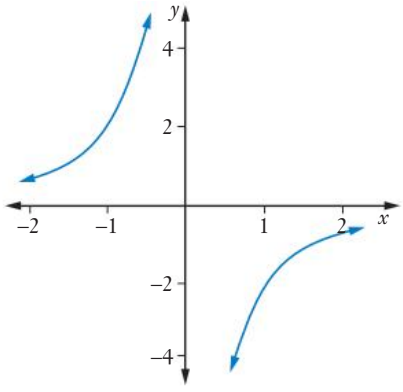


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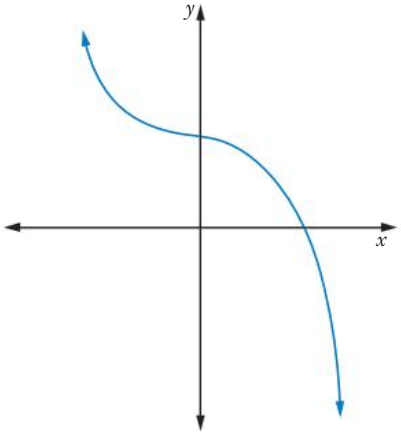
c



d



e



14 State the domain and range for each function.

a $f(x) = x^2 + 1$ **b** $y = x^3$ **c** $y = \sqrt{x}$

d $f(x) = \sqrt{x+5}$ **e** $y = -\sqrt{2x-6}$

15 $f(x) = (x-2)^2$

a Find $f(3)$.

b Find $f(-5)$.

c Solve $f(x) = 0$.

d Find the x- and y-intercepts.

e State the domain and range of $f(x)$.

f Find $f(-x)$.

g Is $f(x)$ even, odd or neither?

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3.04 Linear functions



A page of number planes



Graphing linear functions



x- and y-intercepts

Linear functions

A **linear function** has an equation of the form $y = mx + c$ or $ax + by + c = 0$.

Its graph is a straight line with one x-intercept and one y-intercept.

Direct variation

When one variable is in **direct variation** (or **direct proportion**) with another variable, one is a constant multiple of the other. This means that as one increases, so does the other.

Direct variation

If variables x and y are in direct proportion we can write the equation $y = kx$, where k is called the **proportionality constant**.

EXAMPLE 12

Huang earns \$20 an hour. Find an equation for Huang's income (I) for working x hours and draw its graph.

Solution

Income for 1 hour is \$20.

Income for 2 hours is \$20 × 2 or \$40.

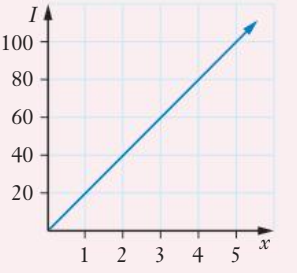
Income for 3 hours is \$20 × 3 or \$60.

Income for x hours is \$20 × x or \$20x.

We can write the equation as $I = 20x$.

We can graph the equation using a table of values.

x	1	2	3
I	20	40	60



$I = 20x$ is an example of direct variation. Direct variation graphs are always straight lines passing through the origin.

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EXAMPLE 13

a Find the x- and y-intercepts of the graph of $y = 2x - 4$ and draw its graph on the number plane.

b Find the x- and y-intercepts of the line with equation $x + 2y + 6 = 0$ and draw its graph.

Solution

a For x-intercept, $y = 0$:

$0 = 2x - 4$

$4 = 2x$

$2 = x$

So the x-intercept is 2.

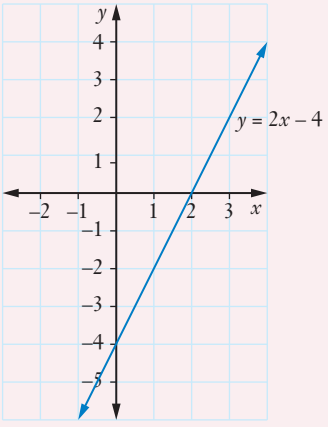
For y-intercept, $x = 0$:

$y = 2(0) - 4$

$= -4$

So the y-intercept is -4.

Use the intercepts to graph the line.



b For x-intercept, $y = 0$:

$x + 2(0) + 6 = 0$

$x + 6 = 0$

$x = -6$

So the x-intercept is -6.

For y-intercept, $x = 0$

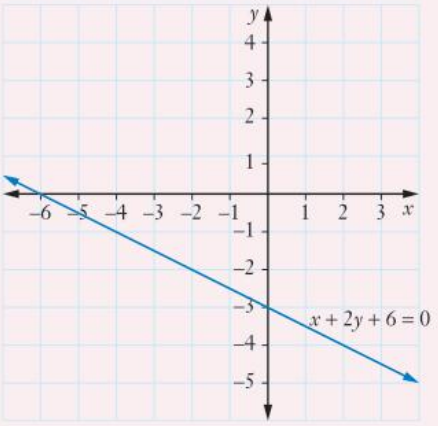
$0 + 2y + 6 = 0$

$2y = -6$

$y = -3$

So the y-intercept is -3.

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Domain and range of linear functions

The domain of a linear function is $(-\infty, \infty)$, all real numbers.

The range of a linear function is $(-\infty, \infty)$, all real numbers.

Horizontal and vertical lines

EXAMPLE 14

a Sketch the graph of $y = 2$ on a number plane. What is its domain and range?

b Sketch the graph of $x = -1$ on a number plane and state its domain and range.

Solution

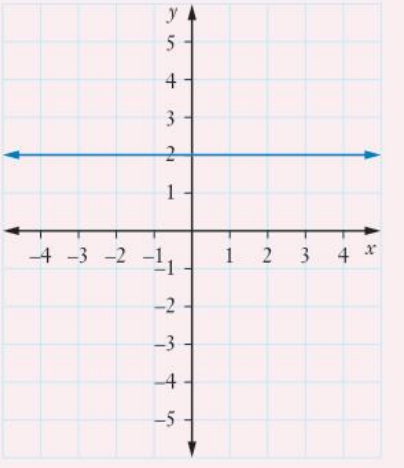
a x can have any value and y is always 2.

Some of the points on the line will be (0, 2), (1, 2) and (2, 2).

This gives a horizontal line with y-intercept 2.

The domain is all real x and the range is $y = 2$.

Domain $(-\infty, \infty)$, Range [2]



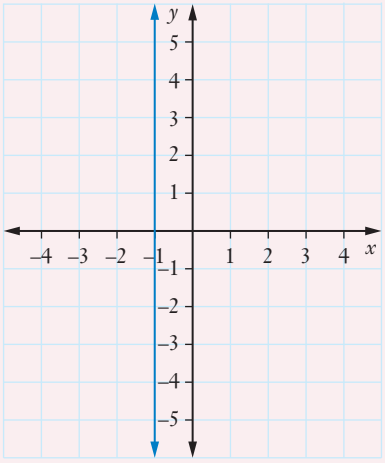
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b y can have any value and x is always -1.

Some of the points on the line will be $(-1, 0)$, $(-1, 1)$ and $(-1, 2)$.

This gives a vertical line with x-intercept -1.

Domain $[-1]$, Range $(-\infty, \infty)$.



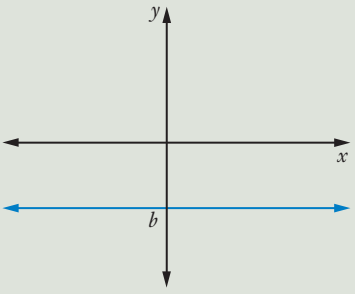
Horizontal lines

$y = b$ is a horizontal line with y-intercept b.

$y = b$ is a many-to-one function.

Domain $(-\infty, \infty)$

Range $[b]$



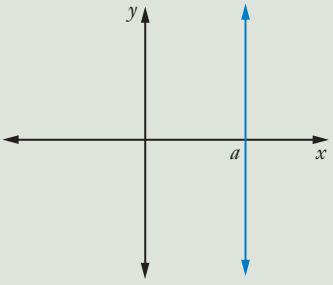
Vertical lines

$x = a$ is a vertical line with x-intercept a.

$x = a$ is not a function.

Domain $[a]$

Range $(-\infty, \infty)$



Exercise 3.04 Linear functions

1 Write an equation for:

- **a** the number of months (N) in x years
- **b** the amount of juice (A) in n lots of 2 litre bottles
- **c** the cost (c) of x litres of petrol at \$1.50 per litre
- **d** the number (y) of people in x debating teams if there are 4 people in each team
- **e** the weight (w) of x lots of 400 g cans of peaches.

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2 Find the equation and draw the graph of the cost (c) of x refrigerators if each refrigerator costs $850.

3 Find the x- and y-intercepts of the graph of each function.

a $y = x - 2$ **b** $y = 3x + 9$ **c** $y = 4 - 2x$

d $f(x) = 2x + 3$ **e** $f(x) = 5x - 4$ **f** $f(x) = 10x + 5$

g $x + y - 2 = 0$ **h** $2x - y + 4 = 0$ **i** $x - y + 3 = 0$

j $3x - 6y - 2 = 0$

4 Draw the graph of each linear function.

a $y = x + 4$ **b** $f(x) = 2x - 1$ **c** $f(x) = 3x + 2$

d $x + y = 3$ **e** $x - y - 1 = 0$

5 Find the domain and range of each equation.

a $3x - 2y + 7 = 0$ **b** $y = 2$ **c** $x = -4$

d $x - 2 = 0$ **e** $3 - y = 0$

6 Sketch each equation's graph and state its domain and range.

a $x = 4$ **b** $x - 3 = 0$

c $y = 5$ **d** $y + 1 = 0$

7 A supermarket has boxes containing cans of dog food. The number of cans of dog food is directly proportional to the number of boxes.

a If there are 144 cans in 4 boxes, find an equation for the number of cans (N) in x boxes.

b How many cans are in 28 boxes?

c How many boxes would be needed for 612 cans of dog food?

8 By sketching the graphs of $x - y - 4 = 0$ and $2x + 3y - 3 = 0$ on the same set of axes, find the point where they cross.

3.05 The gradient of a straight line

The **gradient** of a line measures its slope. It compares the vertical rise with the horizontal run.

Gradient and y-intercept of a line

The gradient of a line

$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$

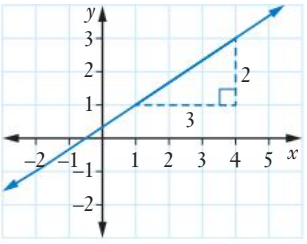


Positive gradient leans to the right.



Negative gradient leans to the left.

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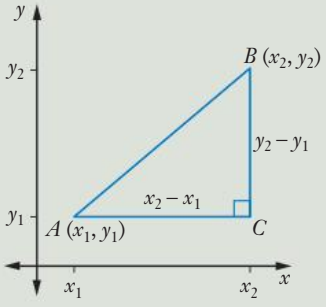
$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{2}{3}$$

On the number plane, gradient is a measure of the rate of change of y with respect to x.

Gradient formula

The gradient of the line joining points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



EXAMPLE 15

Find the gradient of the line joining points (2, 3) and (–3, 4).

Solution

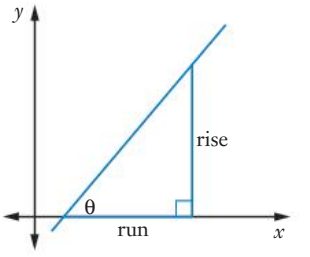
$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4-3}{-3-2}$$
$$= \frac{1}{-5}$$
$$= -\frac{1}{5}$$

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The angle of inclination of a line

The **angle of inclination**, θ, is the angle a straight line makes with the positive x-axis, measured anticlockwise.

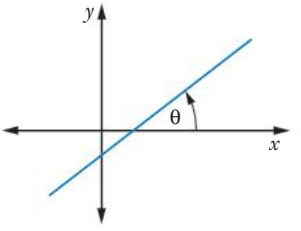
$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\text{opposite}}{\text{adjacent}}$$
$$= \tan\theta$$



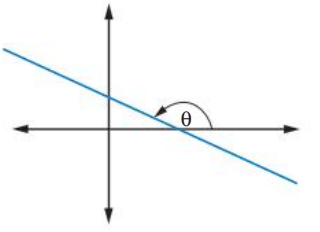
Gradient and angle of inclination of a line

$$m = \tan\theta$$

where m is the gradient and θ is the **angle of inclination**.



For an acute angle, $\tan\theta > 0$.



For an obtuse angle, $\tan\theta < 0$.

DISCUSSION

ANGLES AND GRADIENTS

1 What type of angles give a positive gradient?

2 What type of angles give a negative gradient? Why?

3 What is the gradient of a horizontal line? What angle does it make with the x-axis?

4 What angle does a vertical line make with the x-axis? Can you find its gradient?

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EXAMPLE 16

a Find the gradient of the line that makes an angle of inclination of 135°.

b Find correct to the nearest minute the angle of inclination of a straight line whose gradient is:

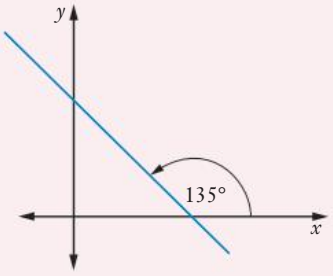
i 0.5 **ii** −3

Solution

a $m = \tan\theta$

$= \tan 135°$

$= -1$



b **i** $m = \tan\theta$

$\therefore \tan\theta = 0.5$

$\theta = \tan^{-1}(0.5)$

$= 26°33'54.18''$

$\approx 26°34'$

See page 172 of Chapter 4 if you need to revise rounding an angle to the nearest minute.

Operation	Casio scientific	Sharp scientific
Enter data.	SHIFT $\tan^{-1}$ 0.5 =	2ndF $\tan^{-1}$ 0.5 =
Change to degrees and minutes.	° ' "	2ndF D°M'S

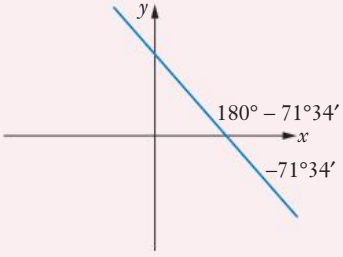
ii $m = \tan\theta$

$\therefore \tan\theta = -3$

$\theta = \tan^{-1}(-3)$

$= -71°33'54.18''$

$\approx -71°34'$



A negative gradient means that the angle of inclination is obtuse.

To find this angle, subtract the acute angle from 180°.

$\theta = 180° - 71°34'$

$= 108°26'$

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INVESTIGATION

GRAPHING $y = mx + c$

Graph each linear function using a graphics calculator or graphing software. Find the gradient of each function. What do you notice?

1	$y = x$	**2**	$y = 2x$	**3**	$y = 3x$	**4**	$y = 4x$
5	$y = -x$	**6**	$y = -2x$	**7**	$y = -3x$	**8**	$y = -4x$

Graph each function and find the y-intercept.

9	$y = x$	**10**	$y = x + 1$	**11**	$y = x + 2$	**12**	$y = x + 3$
13	$y = x - 1$	**14**	$y = x - 2$	**15**	$y = x - 3$		

The gradient–intercept equation of a straight line

The linear function with equation $y = mx + c$ has gradient m and y-intercept c.

WS
$y = mx + c$

EXAMPLE 17

a Find the gradient and y-intercept of the linear function $y = 7x - 5$.

b Find the gradient of the straight line with equation $2x + 3y - 6 = 0$.

Solution

a Gradient = 7, y-intercept = -5.

b First, change the equation into the form $y = mx + c$.

$$\begin{aligned} 2x + 3y - 6 &= 0 \\ 2x + 3y &= 6 \\ 3y &= 6 - 2x \\ &= -2x + 6 \end{aligned}$$

$$\begin{aligned} y &= \frac{-2x}{3} + \frac{6}{3} \\ &= -\frac{2}{3}x + 2 \end{aligned}$$

So the gradient is $-\frac{2}{3}$.

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Exercise 3.05 The gradient of a straight line

1 Find the gradient of the line joining the points:

a (3, 2) and (1, –2) **b** (0, 2) and (3, 6) **c** (–2, 3) and (4, –5)

d (2, –5) and (–3, 7) **e** (2, 3) and (–1, 1) **f** (–5, 1) and (3, 0)

g (–2, –3) and (–4, 6) **h** (–1, 3) and (–7, 7) **i** (1, –4) and (5, 5)

2 Find the gradient of the straight line, correct to 1 decimal place, whose angle of inclination is:

a 25° **b** 82° **c** 68°

d 100° **e** 130° **f** 164°

3 For each linear function, find:

i the gradient **ii** the y-intercept.

a $y = 3x + 5$ **b** $f(x) = 2x + 1$ **c** $y = 6x - 7$

d $y = -x$ **e** $y = -4x + 3$ **f** $y = x - 2$

g $f(x) = 6 - 2x$ **h** $y = 1 - x$ **i** $y = 9x$

4 Find the gradient of the linear function:

a with x-intercept 3 and y-intercept –1

b passing through (2, 4) and x-intercept 5

c passing through (1, 1) and (–2, 7)

d with x-intercept –3 and passing through (2, 3)

e passing through the origin and (–3, –1).

5 Find the angle of inclination, to the nearest minute, of a line with gradient:

a 2 **b** 1.7 **c** 6

d –5 **e** –0.85 **f** –1.2

6 For each linear function, find:

i the gradient **ii** the y-intercept.

a $2x + y - 3 = 0$ **b** $5x + y + 6 = 0$ **c** $6x - y - 1 = 0$

d $x - y + 4 = 0$ **e** $4x + 2y - 1 = 0$ **f** $6x - 2y + 3 = 0$

g $x + 3y + 6 = 0$ **h** $4x + 5y - 10 = 0$ **i** $7x - 2y - 1 = 0$

7 Find the gradient of each linear function.

a $y = -2x - 1$ **b** $y = 2$ **c** $x + y + 1 = 0$

d $3x + y = 8$ **e** $2x - y + 5 = 0$ **f** $x + 4y - 12 = 0$

g $3x - 2y + 4 = 0$ **h** $5x - 4y = 15$ **i** $y = \frac{2}{3}x + 3$

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j $y = \frac{x}{5} - 1$ **k** $y = \frac{2x}{7} + 5$ **l** $y = -\frac{3x}{5} - 2$

m $2y = -\frac{x}{7} + \frac{1}{3}$ **n** $3x - \frac{y}{5} = 8$ **o** $\frac{x}{2} + \frac{y}{3} = 1$

8 If the gradient of the line joining $(8, y_1)$ and $(-1, 3)$ is 2, find the value of y_1.

9 The gradient of the line through $(2, -1)$ and $(x, 0)$ is -5. Find the value of x.

10 The gradient of a line is -1 and the line passes through the points $(4, 2)$ and $(x, -3)$. Find the value of x.

11 The number of frequent flyer points that Mario earns on his credit card is directly proportional to the amount of money he spends on his card.

a If Mario earns 150 points when he spends \$450, find an equation for the number of points (P) he earns when spending d dollars.

b Find the number of points Mario earns when he spends \$840.

c If Mario earns 57 points, how much did he spend?

12 The points $A(-1, 2)$, $B(1, 5)$, $C(6, 5)$ and $D(4, 2)$ form a parallelogram. Find the gradients of all 4 sides of the parallelogram. What do you notice?

3.06 Finding a linear equation



Linear functions code puzzle


Linear modelling



Finding the equation of a line



Equations of lines

EXAMPLE 18

Find the equation of the line with gradient 3 and y-intercept -1.

Solution

The equation is $y = mx + c$ where m = gradient and $c = y$-intercept.

$m = 3$ and $c = -1$.

Equation is $y = 3x - 1$.

There is a formula you can use if you know the gradient and the coordinates of a point on the line.

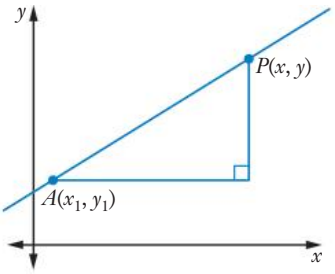
The point–gradient equation of a straight line

The linear function with equation $y - y_1 = m(x - x_1)$ has gradient m and the point (x_1, y_1) lies on the line.

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Proof

Let $P(x, y)$ be a general point on the line with gradient m that passes through $A(x_1, y_1)$.



Then line AP has gradient

$$m = \frac{y - y_1}{x - x_1}$$

$$m(x - x_1) = y - y_1$$

EXAMPLE 19

Find the equation of the line:

a with gradient -4 and x-intercept 1

b passing through $(2, 3)$ and $(-1, 4)$.

Solution

a The x-intercept of 1 means the line passes through the point $(1, 0)$.

Substituting $m = -4$, $x_1 = 1$ and $y_1 = 0$ into the formula:

$$y - y_1 = m(x - x_1)$$
$$y - 0 = -4(x - 1)$$
$$y = -4x + 4$$

b First find the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4 - 3}{-1 - 2}$$
$$= -\frac{1}{3}$$

Substitute the gradient and one of the points, say $(2, 3)$, into the formula.

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -\frac{1}{3}(x - 2)$$
$$3 \times (y - 3) = 3 \times -\frac{1}{3}(x - 2)$$
$$3y - 9 = -(x - 2)$$
$$= -x + 2$$
$$x + 3y - 9 = 2$$
$$x + 3y - 11 = 0$$

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Applications of linear functions

EXAMPLE 20

A solar panel company has fixed overhead costs of $3000 per day and earns $150 for each solar cell sold.

a Write the amount ($A) that the company earns on selling x solar cells each day.

b Find the amount the company earns on a day when it sells 54 solar cells.

c If the company earns $2850 on another day, how many solar cells did it sell that day?

d What is the **break-even point** for this company (where income and costs of production are equal)?

Solution

a The company earns $150 per cell, so it earns 150x$ for x cells.

Daily amount earned = value of solar cells sold – overhead costs.

So $A = 150x - 3000$

b Substitute $x = 54$:

$A = 150(54) - 3000 = 5100$

The company earns $5100 when it sells 54 solar cells.

c Substitute $A = 2850$:

$2850 = 150x - 3000$

$5850 = 150x$

$39 = x$

The company sold 39 solar cells that day.

d At the break-even point:

Income = overhead costs

$150x = 3000$ (or $A = 0$)

$x = 20$

So the break-even point is where the company sells 20 solar cells.

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Exercise 3.06 Finding a linear equation

1 Find the equation of the straight line:

a with gradient 4 and y-intercept -1

b with gradient -3 and passing through $(0, 4)$

c passing through the origin with gradient 5

d with gradient 4 and x-intercept -5

e with x-intercept 1 and y-intercept 3

f with x-intercept 3, y-intercept -4.

2 Find the equation of the straight line passing through the points:

a $(2, 5)$ and $(-1, 1)$ **b** $(0, 1)$ and $(-4, -2)$ **c** $(-2, 1)$ and $(3, 5)$

d $(3, 4)$ and $(-1, 7)$ **e** $(-4, -1)$ and $(-2, 0)$.

3 What is **a** the gradient and **b** the equation of the line with x-intercept 2 that passes through $(3, -4)$?

4 Find the equation of the line:

a parallel to the x-axis and passing through $(2, 3)$

b parallel to the y-axis and passing through $(-1, 2)$.

5 A straight line passing through the origin has a gradient of -2. Find:

a the y-intercept **b** its equation.

6 In a game, each person starts with 20 points, then earns 15 points for every level completed.

a Write an equation for the number of points earned (P) for x levels completed.

b Find the number of points earned for completing:

i 24 levels **ii** 55 levels **iii** 247 levels

c Find the number of levels completed if the number of points earned is:

i 2195 **ii** 7700 **iii** 12 665

7 A TV manufacturing business has fixed costs of \$1500 rental, \$3000 wages and other costs of \$2500 each week. It costs \$250 to produce each TV.

a Write an equation for the cost (c) of producing n TVs each week.

b From the equation, find the cost of producing:

i 100 TVs **ii** 270 TVs **iii** 1200 TVs

c From the equation find the number of TVs produced if the cost is:

i \$52 000 **ii** \$78 250 **iii** \$367 000

d If each TV sells for \$950, find the number of TVs needed to sell to break even.

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8 There are 450 litres of water in a pond, and 8 litres of water evaporate out of the pond every hour.

a Write an equation for the amount of water in the pond (A) after h hours.

b Find the amount of water in the pond after:

i 3 hours **ii** a day.

c After how many hours will the pond be empty?

9 Geordie has a $20 iTunes credit. He uses the credit to buy singles at $1.69 each.

a Write an equation for the amount of credit (C) left if Geordie buys x singles.

b How many songs can Geordie buy before his credit runs out?

10 Emily-Rose owes $20 000 and she pays back $320 a month.

a Write an equation for the amount of money she owes (A) after x months.

b How much does Emily-Rose owe after:

i 5 months? **ii** 1 year? **iii** 5 years?

c How long will it take for Emily-Rose to pay all the money back?

11 Acme Party Supplies earns $5 for every helium balloon it sells.

a If overhead costs are $100 each day, find an equation for the profit (P) of selling x balloons.

b How much profit does Acme make if it sells 300 balloons?

c How many balloons does it sell if it makes a profit of $1055?

d What is the break-even point for this business?

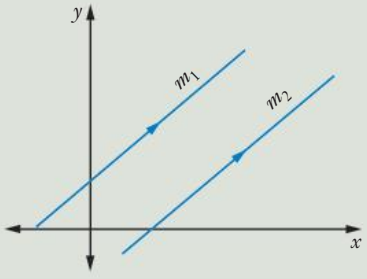
3.07 Parallel and perpendicular lines

WS Parallel and perpendicular lines

WS Linear functions

Gradients of parallel lines

If 2 lines are parallel, then they have the same gradient. That is, $m_1 = m_2$.



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EXAMPLE 21

a Prove that the straight lines with equations $5x - 2y - 1 = 0$ and $5x - 2y + 7 = 0$ are parallel.

b Find the equation of a straight line parallel to the line $2x - y - 3 = 0$ and passing through $(1, -5)$.

Solution

a First, change the equation into the form $y = mx + c$.

$5x - 2y - 1 = 0$

$5x - 1 = 2y$

$\frac{5}{2}x - \frac{1}{2} = y$

$\therefore \quad m_1 = \frac{5}{2}$

$5x - 2y + 7 = 0$

$5x + 7 = 2y$

$\frac{5}{2}x + \frac{7}{2} = y$

$\therefore \quad m_2 = \frac{5}{2}$

$m_1 = m_2 = \frac{5}{2}$

$\therefore$ the lines are parallel.

b $2x - y - 3 = 0$

$2x - 3 = y$

$\therefore m_1 = 2$

For parallel lines $m_1 = m_2$.

$\therefore m_2 = 2$

Substitute this and $(1, -5)$ into $y - y_1 = m(x - x_1)$:

$y - (-5) = 2(x - 1)$

$y + 5 = 2x - 2$

$y = 2x - 7$

CLASS INVESTIGATION

PERPENDICULAR LINES

Sketch each pair of straight lines on the same number plane.

1 $3x - 4y + 12 = 0$ and $4x + 3y - 8 = 0$

2 $2x + y + 4 = 0$ and $x - 2y + 2 = 0$

What do you notice about each pair of lines?

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Gradients of perpendicular lines

If 2 lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$,

that is, $m_2 = -\dfrac{1}{m_1}$.

EXAMPLE 22

a Show that the lines with equations $3x + y - 11 = 0$ and $x - 3y + 1 = 0$ are perpendicular.

b Find the equation of the straight line through (2, 3) that is perpendicular to the line passing through (–1, 7) and (3, 3).

Solution

a
$$3x + y - 11 = 0$$
$$y = -3x + 11$$
$$\therefore m_1 = -3$$
$$x - 3y + 1 = 0$$
$$x + 1 = 3y$$
$$\frac{1}{3}x + \frac{1}{3} = y$$
$$\therefore m_2 = \frac{1}{3}$$
$$m_1 m_2 = -3 \times \frac{1}{3} = -1$$
$\therefore$ the lines are perpendicular.

b Line through (–1, 7) and (3, 3):

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{7-3}{-1-3}$$
$$= \frac{4}{-4}$$
$$= -1$$

For perpendicular lines, $m_1 m_2 = -1$:

$$-1m_2 = -1$$
$$m_2 = 1$$

Substitute $m = 1$ and the point (2, 3) into $y - y_1 = m(x - x_1)$:

$$y - 3 = 1(x - 2)$$
$$= x - 2$$
$$y = x + 1$$

Exercise 3.07 Parallel and perpendicular lines

1 Find the gradient of the straight line:

- **a** parallel to the line $3x + y - 4 = 0$
- **b** perpendicular to the line $3x + y - 4 = 0$
- **c** parallel to the line joining $(3, 5)$ and $(-1, 2)$
- **d** perpendicular to the line with x-intercept 3 and y-intercept 2
- **e** perpendicular to the line that has an angle of inclination of $135°$
- **f** perpendicular to the line $6x - 5y - 4 = 0$
- **g** parallel to the line $x - 3y - 7 = 0$
- **h** perpendicular to the line passing through $(4, -2)$ and $(3, 3)$.

2 Find the equation of the straight line:

- **a** passing through $(2, 3)$ and parallel to the line $y = x + 6$
- **b** through $(-1, 5)$ and parallel to the line $x - 3y - 7 = 0$
- **c** with x-intercept 5 and parallel to the line $y = 4 - x$
- **d** through $(3, -4)$ and perpendicular to the line $y = 2x$
- **e** through $(-2, 1)$ and perpendicular to the line $2x + y + 3 = 0$
- **f** through $(7, -2)$ and perpendicular to the line $3x - y - 5 = 0$
- **g** through $(-3, -1)$ and perpendicular to the line $4x - 3y + 2 = 0$
- **h** passing through the origin and parallel to the line $x + y + 3 = 0$
- **i** through $(3, 7)$ and parallel to the line $5x - y - 2 = 0$
- **j** through $(0, -2)$ and perpendicular to the line $x - 2y = 9$
- **k** perpendicular to the line $3x + 2y - 1 = 0$ and passing through the point $(-2, 4)$.

3 Show that the lines with equations $y = 3x - 2$ and $6x - 2y - 9 = 0$ are parallel.

4 Show that lines $x + 5y = 0$ and $y = 5x + 3$ are perpendicular.

5 Show that lines $6x - 5y + 1 = 0$ and $6x - 5y - 3 = 0$ are parallel.

6 Show that lines $7x + 3y + 2 = 0$ and $3x - 7y = 0$ are perpendicular.

7 If the lines $3x - 2y + 5 = 0$ and $y = kx - 1$ are perpendicular, find the value of k.

8 Show that the line joining $(3, -1)$ and $(2, -5)$ is parallel to the line $8x - 2y - 3 = 0$.

9 Show that the points $A(-3, -2)$, $B(-1, 4)$, $C(7, -1)$ and $D(5, -7)$ are the vertices of a parallelogram.

10 The points $A(-2, 0)$, $B(1, 4)$, $C(6, 4)$ and $D(3, 0)$ form a rhombus. Show that the diagonals are perpendicular.

11 Find the equation of the straight line passing through $(6, -3)$ that is perpendicular to the line joining $(2, -1)$ and $(-5, -7)$.

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3.08 Quadratic functions

WS
Graphing quadratic functions
WS
Graphing quadratics

Quadratic functions

A **quadratic function** has an equation in the form $y = ax^2 + bx + c$, where the highest power of x is 2. The graph of a quadratic function is a **parabola**.

EXAMPLE 23

Graph the quadratic function $y = x^2 - x$.

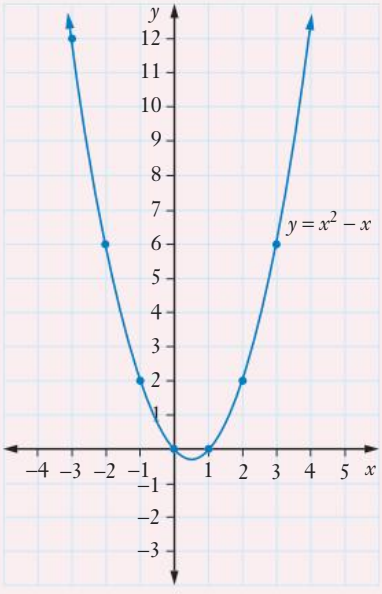
Solution

Draw up a table of values for $y = x^2 - x$.

x	–3	–2	–1	0	1	2	3
y	12	6	2	0	0	2	6

Plot (–3, 12), (–2, 6), (–1, 2), (0, 0), (1, 0), (2, 2) and (3, 6) and draw a parabola through them.

Label the graph with its equation.



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TECHNOLOGY

Transforming quadratic functions

Use a graphics calculator or graphing software to graph these quadratic functions. Look for any patterns.

$y = x^2$	$y = x^2 + 1$	$y = x^2 + 2$	$y = x^2 + 3$
$y = x^2 - 1$	$y = x^2 - 2$	$y = x^2 - 3$	$y = 2x^2$
$y = 3x^2$	$y = x^2 + x$	$y = x^2 + 2x$	$y = x^2 + 3x$
$y = x^2 - x$	$y = x^2 - 2x$	$y = x^2 - 3x$	$y = -x^2$
$y = -x^2 + 1$	$y = -x^2 + 2$	$y = -x^2 + 3$	$y = -x^2 - 1$
$y = -x^2 - 2$	$y = -x^2 - 3$	$y = -2x^2$	$y = -3x^2$
$y = -x^2 + x$	$y = -x^2 + 2x$	$y = -x^2 - x$	$y = -x^2 - 2x$

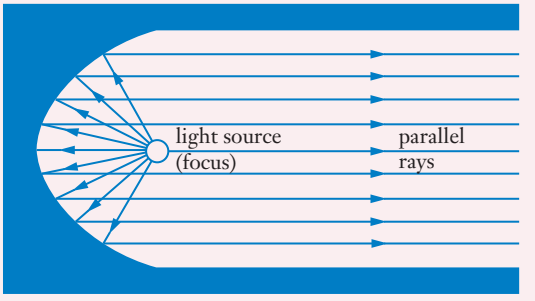
Could you predict where the graphs $y = x^2 + 9$, $y = 5x^2$ or $y = x^2 + 6x$ would lie?

Is the parabola always a function? Can you find an example of a parabola that is not a function?

DID YOU KNOW?

The parabola

The parabola shape has special properties that are very useful. For example, if a light is placed inside a parabolic mirror at a special place called the focus, then all light rays coming from this point and reflecting off the parabola shape will radiate out parallel to each other, giving a strong light. This is how car headlights work. The dishes of radio telescopes also use this property of the parabola, because radio signals coming in to the dish will reflect back to the focus.





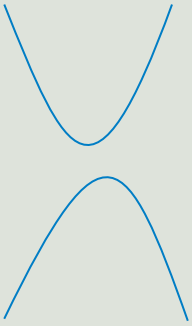
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Concavity and turning points

For the parabola $y = ax^2 + bx + c$:

- if $a > 0$ the parabola is **concave upwards** and has a **minimum turning point**.
- if $a < 0$ the parabola is **concave downwards** and has a **maximum turning point**.



The **turning point** is also called the **vertex** or **stationary point** of the parabola.

Notice also that the parabola is always symmetrical.

EXAMPLE 24

a **i** Sketch the graph of $y = x^2 - 1$, showing intercepts.

ii State the domain and range.

b **i** Find the x- and y-intercepts of the quadratic function $f(x) = -x^2 + 4x + 5$.

ii Sketch a graph of the function.

iii Find the maximum value of the function.

iv State the domain and range.

Solution

a **i** Since $a > 0$, the graph is concave upwards.

For x-intercepts, $y = 0$:

$0 = x^2 - 1$

$1 = x^2$

$x = \pm 1$

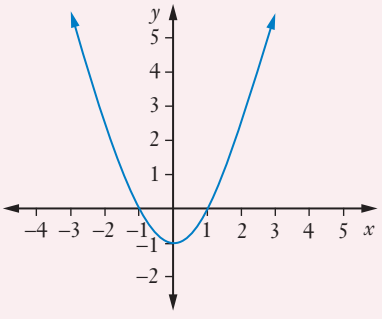
For y-intercept, $x = 0$:

$y = 0^2 - 1$

$= -1$

Since the parabola is symmetrical, the turning point is at $x = 0$, halfway between the x-intercepts -1 and 1.

When $x = 0$, $y = -1$: Vertex is $(0, -1)$.



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ii From the equation and the graph, x can have any value.

Domain $(-\infty, \infty)$

The values of y are greater than or equal to -1.

Range $[-1, \infty)$

b i For x-intercepts, $f(x) = 0$.

$0 = -x^2 + 4x + 5$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

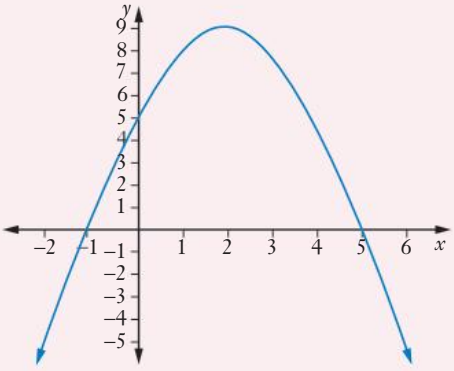
$x = 5, x = -1$

For y-intercept, $x = 0$.

$f(0) = -(0)^2 + 4(0) + 5$

$= 5$

ii Since $a < 0$, the quadratic function is concave downwards.



iii The turning point is halfway between $x = -1$ and $x = 5$.

$x = \frac{-1+5}{2}$

$= 2$

$f(2) = -(2)^2 + 4(2) + 5$

$= 9$

The maximum value of $f(x)$ is 9.

iv For the domain, the function can take on all real numbers for x.

Domain $(-\infty, \infty)$

For the range, $y \leq 9$.

Range $(-\infty, 9]$

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Exercise 3.08 Quadratic functions

1 Find the x- and y-intercepts of the graph of each quadratic function:

a $y = x^2 + 2x$ **b** $y = -x^2 + 3x$ **c** $f(x) = x^2 - 1$

d $y = x^2 - x - 2$ **e** $y = x^2 - 9x + 8$

2 Sketch each parabola and find its maximum or minimum value:

a $y = x^2 + 2$ **b** $y = -x^2 + 1$ **c** $f(x) = x^2 - 4$

d $y = x^2 + 2x$ **e** $y = -x^2 - x$ **f** $f(x) = (x - 3)^2$

g $f(x) = (x + 1)^2$ **h** $y = x^2 + 3x - 4$ **i** $y = 2x^2 - 5x + 3$

j $f(x) = -x^2 + 3x - 2$

3 For each parabola, find:

i the x- and y-intercepts **ii** the domain and range.

a $y = x^2 - 7x + 12$ **b** $f(x) = x^2 + 4x$ **c** $y = x^2 - 2x - 8$

d $y = x^2 - 6x + 9$ **e** $f(x) = 4 - x^2$

4 Find the domain and range of:

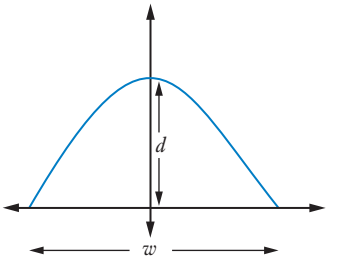
a $y = x^2 - 5$ **b** $f(x) = x^2 - 6x$ **c** $f(x) = x^2 - x - 2$

d $y = -x^2$ **e** $f(x) = (x - 7)^2$

5 A satellite dish is in the shape of a parabola with equation $y = -3x^2 + 6$, and all dimensions are in metres.

a Find d, the depth of the dish.

b Find w, the width of the dish, to 1 decimal place.



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WS Quadratic functions

WS Sketching quadratic functions

WS Features of a parabola

3.09 Axis of symmetry

Axis of symmetry of a parabola

The **axis of symmetry** of a parabola with the equation $y = ax^2 + bx + c$ is the vertical line with equation:

$$x = -\frac{b}{2a}.$$

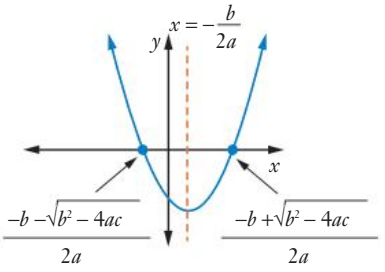
Proof

The axis of symmetry of a parabola lies halfway between the x-intercepts.

For the x-intercepts, $y = 0$

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



The x-coordinate of the axis of symmetry is the average of the x-intercepts.

$$x = \frac{\frac{-b-\sqrt{b^2-4ac}}{2a} + \frac{-b+\sqrt{b^2-4ac}}{2a}}{2} = \frac{\frac{-2b}{2a}}{2} = \frac{-2b}{4a} = -\frac{b}{2a}$$

Turning point of a parabola

The quadratic function $y = ax^2 + bx + c$ has a minimum value if $a > 0$ and a maximum value if $a < 0$.

The minimum or maximum value of the quadratic function is $f\left(-\frac{b}{2a}\right)$.

The turning point or vertex of a parabola is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

EXAMPLE 25

a Find the equation of the axis of symmetry and the minimum value of the quadratic function $y = x^2 - 5x + 1$.

b Find the equation of the axis of symmetry, the maximum value and the turning point of the quadratic function $y = -3x^2 + x - 5$.

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Solution

a Axis of symmetry:

$$x = -\frac{b}{2a}$$
$$= -\frac{(-5)}{2(1)}$$
$$= \frac{5}{2}$$
$$= 2\frac{1}{2}$$

$\therefore$ Axis of symmetry is the line $x = 2\frac{1}{2}$.

Minimum value $y = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1$

$$= \frac{25}{4} - \frac{25}{2} + 1$$
$$= -5\frac{1}{4}$$

b $x = -\frac{b}{2a}$

$$= -\frac{1}{2(-3)}$$
$$= \frac{1}{6}$$

$\therefore$ Axis of symmetry is the line $x = \frac{1}{6}$.

Maximum value $y = -3\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right) - 5$

$$= -\frac{1}{12} + \frac{1}{6} - 5$$
$$= -4\frac{11}{12}$$

The turning point is $\left(\frac{1}{6}, -4\frac{11}{12}\right)$.

EXAMPLE 26

Determine whether each function is even.

a $f(x) = x^2 + 3$ **b** $y = -x^2 + 3x$

Solution

a $f(x) = x^2 + 3$

$f(-x) = (-x)^2 + 3$

$= x^2 + 3$

$= f(x)$

So $f(x) = x^2 + 3$ is an even function.

b Let $f(x) = -x^2 + 3x$

$f(-x) = -(-x)^2 + 3(-x)$

$= -x^2 - 3x$

$\neq f(x)$

So $y = -x^2 + 3x$ is not an even function.

Exercise 3.09 Axis of symmetry

1 For the parabola $y = x^2 + 2x$, find the equation of its axis of symmetry and the minimum value.

2 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 - 4$.

3 Find the equation of the axis of symmetry and the minimum turning point of the parabola $y = 4x^2 - 3x + 1$.

4 Find the equation of the axis of symmetry and the maximum value of the parabola $y = -x^2 + 2x - 7$.

5 Find the equation of the axis of symmetry and the vertex of the parabola $y = -2x^2 - 4x + 5$.

6 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 + 3x + 2$.

7 Find the equation of the axis of symmetry and the coordinates of the vertex for each parabola:

a $y = x^2 + 6x - 3$ **b** $y = -x^2 - 8x + 1$ **c** $y = 3x^2 + 18x + 4$

d $y = -2x^2 + 5x$ **e** $y = 4x^2 + 10x - 7$

8 For each parabola, find:

i the equation of the axis of symmetry

ii the minimum or maximum value

iii the vertex.

a $y = x^2 + 2x - 2$ **b** $y = -2x^2 + 4x - 1$

9 Find the turning point of each function and state whether it is a maximum or minimum.

a $y = x^2 + 2x + 1$ **b** $y = x^2 - 8x - 7$ **c** $f(x) = x^2 + 4x - 3$

d $y = x^2 - 2x$ **e** $f(x) = x^2 - 4x - 7$ **f** $f(x) = 2x^2 + x - 3$

g $y = -x^2 - 2x + 5$ **h** $y = -2x^2 + 8x + 3$ **i** $f(x) = -3x^2 + 3x + 7$

10 For each quadratic function:

i find x-intercepts using the quadratic formula

ii state whether the function has a maximum or minimum value and find this value

iii sketch the graph of the function on a number plane

iv solve the quadratic equation $f(x) = 0$ graphically

a $f(x) = x^2 + 4x + 4$ **b** $f(x) = x^2 - 2x - 3$ **c** $y = x^2 - 6x + 1$

d $f(x) = -x^2 - 2x + 6$ **e** $f(x) = -x^2 - x + 3$

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11 **a** Find the minimum value of the parabola with equation $y = x^2 - 2x + 5$.

b How many solutions does the quadratic equation $x^2 - 2x + 5 = 0$ have?

c Sketch the parabola.

12 **a** Find the maximum value of the quadratic function $f(x) = -2x^2 + x - 4$.

b How many solutions are there to the quadratic equation $-2x^2 + x - 4 = 0$?

c Sketch the graph of the quadratic function.

13 Show that $f(x) = -x^2$ is an even function.

14 Determine which of these functions are even.

a $y = x^2 + 1$ **b** $f(x) = x^2 - 3$ **c** $y = -2x^2$

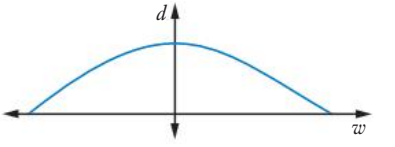
d $f(x) = x^2 - 3x$ **e** $f(x) = x^2 + x$ **f** $y = x^2 - 4$

g $y = x^2 - 2x - 3$ **h** $y = x^2 - 5x + 4$ **i** $p(x) = (x + 1)^2$

15 A bridge has a parabolic span as shown,

with equation $d = -\frac{w^2}{800} + 200$

where d is the depth of the arch in metres.



a Show that the quadratic function is even.

b Find the depth of the arch from the top of the span.

c Find the total width of the span.

d Find the depth of the arch at a point 10 m from its widest span.

e Find the width across the span at a depth of 100 m.



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WS
The discriminant

3.10 The discriminant

The solutions of an equation are also called the **roots** of the equation.

In the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression $b^2 - 4ac$ is called the **discriminant**.

It gives us information about the roots of the quadratic equation $ax^2 + bx + c = 0$.

EXAMPLE 27

Use the quadratic formula to find how many real roots each quadratic equation has.

a $x^2 + 5x - 3 = 0$ **b** $x^2 - x + 4 = 0$ **c** $x^2 - 2x + 1 = 0$

Solution

a
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$
$$= \frac{-5 \pm \sqrt{25 + 12}}{2}$$
$$= \frac{-5 \pm \sqrt{37}}{2}$$

There are 2 real roots:

$$x = \frac{-5 + \sqrt{37}}{2}, \frac{-5 - \sqrt{37}}{2}$$

b
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times 4}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{-15}}{2}$$

There are no real roots since $\sqrt{-15}$ has no real value.

c
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 1}}{2 \times 1}$$
$$= \frac{2 \pm \sqrt{0}}{2}$$
$$= 1$$

There are 2 real roots:

$x = 1, 1$

However, these are equal roots.

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The discriminant

The value of the **discriminant** $\Delta = b^2 - 4ac$ tells us information about the roots of the quadratic equation $ax^2 + bx + c = 0$.

When $\Delta \geq 0$, there are 2 real roots.

- If Δ is a perfect square, the roots are rational.
- If Δ is not a perfect square, the roots are irrational.

When $\Delta = 0$, there are 2 equal rational roots (or 1 rational root).

When $\Delta < 0$, there are no real roots.

EXAMPLE 28

a Show that the equation $2x^2 + x + 4 = 0$ has no real roots.

b Describe the roots of the equation:

i $2x^2 - 7x - 1 = 0$ **ii** $x^2 + 6x + 9 = 0$

c Find the values of k for which the quadratic equation $5x^2 - 2x + k = 0$ has real roots.

Solution

a $\Delta = b^2 - 4ac$

$= 1^2 - 4(2)(4)$

$= -31$

< 0

$\Delta < 0$, so the equation has no real roots.

b **i** $\Delta = b^2 - 4ac$

$= (-7)^2 - 4(2)(-1)$

$= 57$

> 0

$\Delta > 0$, so there are 2 real irrational roots.

Roots are irrational because 57 is not a perfect square.

ii $\Delta = b^2 - 4ac$

$= (6)^2 - 4(1)(9)$

$= 0$

$\Delta = 0$ so there are 2 real equal rational roots.

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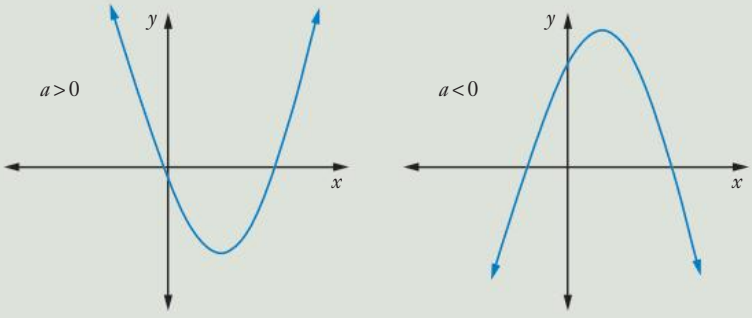
c For real roots, $\Delta \geq 0$.

$$b^2 - 4ac \geq 0$$
$$(-2)^2 - 4(5)(k) \geq 0$$
$$4 - 20k \geq 0$$
$$4 \geq 20k$$
$$k \leq \frac{1}{5}$$

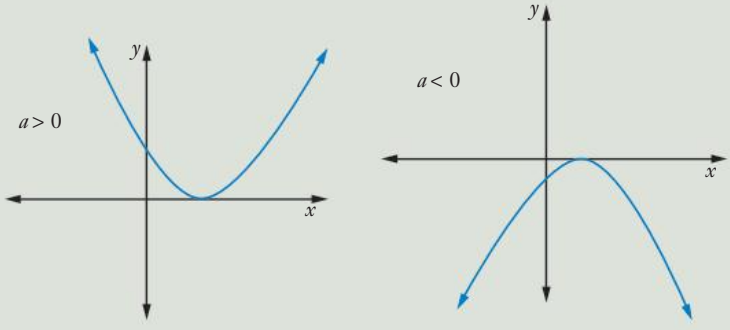
The discriminant and the parabola

The roots of the quadratic equation $ax^2 + bx + c = 0$ give the x-intercepts of the parabola $y = ax^2 + bx + c$.

If $\Delta > 0$, then the quadratic equation has 2 real roots and the parabola has 2 x-intercepts.

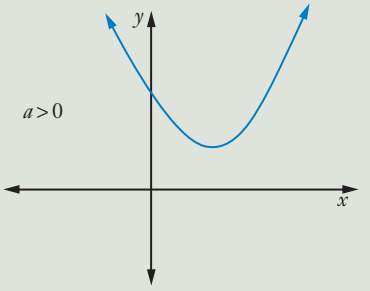


If $\Delta = 0$, then the quadratic equation has 1 real root or 2 equal roots and the parabola has one x-intercept.

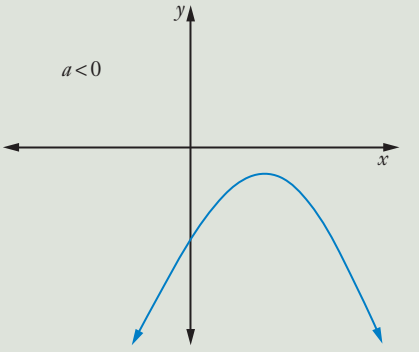


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If $\Delta < 0$, then the quadratic equation has no real roots and the parabola has no x-intercepts.



If $\Delta < 0$ and $a > 0$, then $ax^2 + bx + c > 0$ for all x.



If $\Delta < 0$ and $a < 0$, then $ax^2 + bx + c < 0$ for all x.

EXAMPLE 29

a Show that the parabola $f(x) = x^2 - x - 2$ has 2 x-intercepts.

b Show that $x^2 - 2x + 4 > 0$ for all x.

Solution

a $\Delta = b^2 - 4ac$

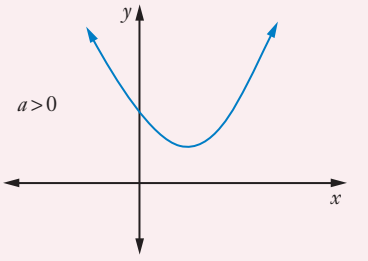
$= (-1)^2 - 4(1)(-2)$

$= 9$

> 0

So there are 2 real roots and the parabola has 2 x-intercepts.

b If $a > 0$ and $\Delta < 0$, then $ax^2 + bx + c > 0$ for all x.



$a = 1 > 0$

$\Delta = b^2 - 4ac$

$= (-2)^2 - 4(1)(4)$

$= -12$

< 0

Since $a > 0$ and $\Delta < 0$, $x^2 - 2x + 4 > 0$ for all x.

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Exercise 3.10 The discriminant

1 Find the discriminant of each quadratic equation.

a $x^2 - 4x - 1 = 0$ **b** $2x^2 + 3x + 7 = 0$ **c** $-4x^2 + 2x - 1 = 0$

d $6x^2 - x - 2 = 0$ **e** $-x^2 - 3x = 0$ **f** $x^2 + 4 = 0$

g $x^2 - 2x + 1 = 0$ **h** $-3x^2 - 2x + 5 = 0$ **i** $-2x^2 + x + 2 = 0$

2 Find the discriminant and state whether the roots of the quadratic equation are real or not real. If the roots are real, state whether they are equal or unequal, rational or irrational.

a $x^2 - x - 4 = 0$ **b** $2x^2 + 3x + 6 = 0$ **c** $x^2 - 9x + 20 = 0$

d $x^2 + 6x + 9 = 0$ **e** $2x^2 - 5x - 1 = 0$ **f** $-x^2 + 2x - 5 = 0$

g $-2x^2 - 5x + 3 = 0$ **h** $-5x^2 + 2x - 6 = 0$ **i** $-x^2 + x = 0$

3 Find the value of p for which the quadratic equation $x^2 + 2x + p = 0$ has equal roots.

4 Find any values of k for which the quadratic equation $x^2 + kx + 1 = 0$ has equal roots.

5 Find all the values of b for which $2x^2 + x + b + 1 = 0$ has real roots.

6 Evaluate p if $px^2 + 4x + 2 = 0$ has no real roots.

7 Find all values of k for which $(k + 2)x^2 + x - 3 = 0$ has 2 real unequal roots.

8 Prove that $3x^2 - x + 7 > 0$ for all real x.

9 Show that the line $y = 2x + 6$ cuts the parabola $y = x^2 + 3$ in 2 points.

10 Show that the line $3x + y - 4 = 0$ cuts the parabola $y = x^2 + 5x + 3$ in 2 points.

11 Show that the line $y = -x - 4$ does not touch the parabola $y = x^2$.

12 Show that the line $y = 5x - 2$ is a tangent to the parabola $y = x^2 + 3x - 1$.

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3.11 Finding a quadratic equation

WS
A page of parabolas

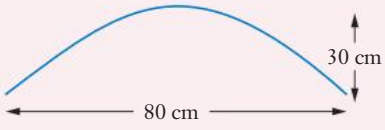
EXAMPLE 30

a Find the equation of the parabola that passes through the points $(-1, -3)$, $(0, 3)$ and $(2, 21)$.

b A parabolic satellite dish is built so it is 30 cm deep and 80 cm wide, as shown.

i Find an equation for the parabola.

ii Find the depth of the dish 10 cm out from the vertex.



Solution

a The parabola has equation in the form $y = ax^2 + bx + c$.

Substitute the points into the equation.

$(-1, -3)$:

$$-3 = a(-1)^2 + b(-1) + c$$
$$= a - b + c$$
$$\therefore a - b + c = -3 \quad [1]$$

$(0, 3)$:

$$3 = a(0)^2 + b(0) + c$$
$$= c$$
$$\therefore c = 3 \quad [2]$$

$(2, 21)$:

$$21 = a(2)^2 + b(2) + c$$
$$= 4a + 2b + c$$
$$\therefore 4a + 2b + c = 21 \quad [3]$$

Solve simultaneous equations to find a, b and c.

Substitute [2] into [1]:

$$a - b + 3 = -3$$
$$a - b = -6 \quad [4]$$

Substitute [2] into [3]:

$$4a + 2b + 3 = 21$$
$$4a + 2b = 18 \quad [5]$$

[4] $\times$ 2:

$$2a - 2b = -12 \quad [6]$$

[5] + [6]:

$$6a = 6$$
$$a = 1$$

Substitute $a = 1$ into [5]:

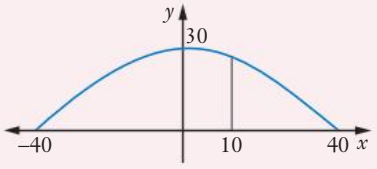
$$4(1) + 2b = 18$$
$$4 + 2b = 18$$
$$2b = 14$$
$$b = 7$$

$\therefore a = 1, b = 7, c = 3$

Thus the parabola has equation $y = x^2 + 7x + 3$.

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b i We can put the dish onto a number plane as shown. Since the parabola is symmetrical, the width of 80 cm means 40 cm either side of the y-axis.



The parabola passes through points (0, 30), (40, 0) and (–40, 0).

Substitute these points into $y = ax^2 + bx + c$.

(0, 30): $30 = a(0)^2 + b(0) + c \ = c$

So $y = ax^2 + bx + 30$

Substitute (40, 0) into $y = ax^2 + bx + 30$:

$0 = a(40)^2 + b(40) + 30$

$0 = 1600a + 40b + 30$ [1]

Substitute (–40, 0) into $y = ax^2 + bx + 30$:

$0 = a(-40)^2 + b(-40) + 30$

$0 = 1600a - 40b + 30$ [2]

[1] + [2]:

$$0 = 3200a + 60$$
$$a = \frac{-60}{3200}$$
$$= -\frac{3}{160}$$

Substitute a into [1]:

$$0 = 1600\left(-\frac{3}{160}\right) + 40b + 30$$
$$= -30 + 40b + 30$$
$$= 40b$$
$$0 = b$$

So $y = -\frac{3}{160}x^2 + 30$

ii Substitute $x = 10$:

$$y = -\frac{3}{160}(10)^2 + 30$$
$$= 28.125$$

So the depth of the dish at 10 cm is 28.125 cm.

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Exercise 3.11 Finding a quadratic equation

1 The braking distance of a car travelling at 100 km/h is 40 metres. The formula for braking distance (d) in metres is $d = kx^2$ where k is a constant and x is speed in km/h.

- **a** Find the value of k.
- **b** Find the braking distance at 80 km/h.
- **c** A dog runs out onto the road 15 m in front of a car travelling at 50 km/h. Will the car be able to stop in time without hitting the dog?
- **d** If the dog was 40 m in front of a car travelling at 110 km/h, would the car stop in time?

2 The area (A) of a figure is directly proportional to the square of its length (x). When $x = 5$ cm, its area is 125 cm^2.

- **a** Find the equation for the area.
- **b** Find the area when the length is 4.2 cm.
- **c** Find the length correct to 1 decimal place when the area is 250 cm^2.

3 The volume of a cylinder is given by $V = \pi r^2 h$ where r is the radius and h is height.

- **a** Find the equation for volume if the height is fixed at 8 cm.
- **b** Find the volume of a cylinder with radius 5 cm.
- **c** Find the radius if the volume is 100 cm^3.

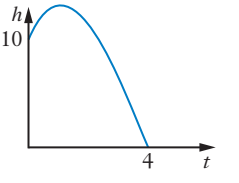
4 A rectangle has sides x and $3 - x$.

- **a** Write an equation for its area.
- **b** Draw the graph of the area.
- **c** Find the value of x that gives the maximum area.
- **d** Find the maximum area of the rectangle.

5 Find the equation of the parabola that passes through the points:

- **a** (0, –5), (2, –3) and (–3, 7)
- **b** (1, –2), (3, 0) and (–2, 10)
- **c** (–2, 21), (1, 6) and (–1, 12)
- **d** (2, 3), (1, –4) and (–1, –12)
- **e** (0, 1), (–2, 1) and (2, –7)

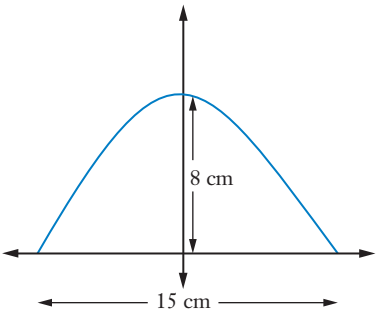
6 Grania throws a ball off a 10 m high cliff. After 1 s it is 22.5 m above ground and it reaches the ground after 4 s.



- **a** Find the equation for the height (h metres) of the ball after time t seconds.
- **b** Find the height of the ball after 2 seconds.
- **c** Find when the ball is in line with the cliff.

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7 A parabolic shaped headlight is 15 cm wide and 8 cm deep as shown.



a Find an equation for the parabola.

b Find the depth of the headlight at a point 3 cm out from its axis of symmetry.

c At what width from the axis of symmetry does the headlight have a depth of 5 cm?

8 **a** Find the equation of the parabola passing through (0, 0), (3, –3) and (–1, 5).

b Find the value of y when:

i $x = 5$ **ii** $x = -4$

c Find values of x when $y = -4$.

d Find exact values of x when $y = 2$.

9 **a** Find the equation of the quadratic function $f(x)$ that passes through points (1, 10), (0, 7) and (–1, 6).

b Evaluate $f(-5)$.

c Show that $f(x) > 0$ for all x.

10 Find the equation of a parabola with axis of symmetry $x = 1$, minimum value –2 and passing through (0, 0).

11 Find the equation of the quadratic function with axis $x = 3$, maximum value 13 and passing through (0, 4).



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3.12 Cubic functions

A **cubic function** has an equation where the highest power of x is 3, such as $f(x) = kx^3$, $f(x) = k(x - b)^3 + c$ and $f(x) = k(x - a)(x - b)(x - c)$ where a, b, c and k are constants.

WS Cubic functions

WS Graphing cubics

WS Graphing cubics 2

EXAMPLE 31

a Sketch the graph of the cubic function $f(x) = x^3 + 2$.

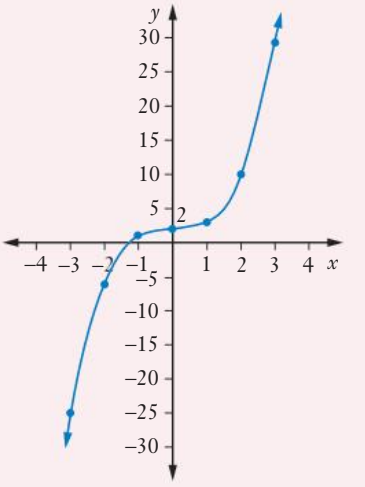
b State its domain and range.

c Solve the equation $x^3 + 2 = 0$ graphically.

Solution

a Draw up a table of values.

x	–3	–2	–1	0	1	2	3
y	–25	–6	1	2	3	10	29



b The function can have any real x or y value.

Domain $(-\infty, \infty)$

Range $(-\infty, \infty)$

c From the graph, the x-intercept is approximately –1.3.

So the root of $x^3 + 2 = 0$ is approximately $x = -1.3$.

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INVESTIGATION

TRANSFORMING CUBIC FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of some cubic functions, such as:

$y = x^3$	$y = x^3 + 1$	$y = x^3 + 3$	$y = x^3 - 1$
$y = x^3 - 2$	$y = 2x^3$	$y = 3x^3$	$y = -x^3$
$y = -2x^3$	$y = -3x^3$	$y = 2x^3 + 1$	$y = (x + 1)^3$
$y = (x + 2)^3$	$y = (x - 1)^3$	$y = 2(x - 2)^3$	$y = 3(x + 2)^3 + 1$

$y = (x - 1)(x - 2)(x - 3)$ $y = x(x + 1)(x + 4)$ $y = 2(x + 1)(x - 2)(x + 5)$

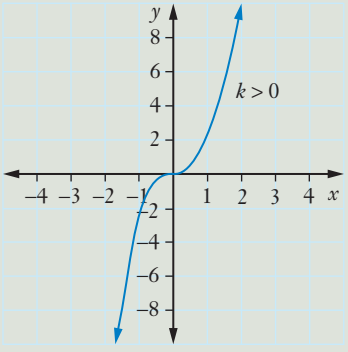
Can you see any patterns? Could you describe the shape of the cubic function? Could you predict where the graphs of different cubic functions would lie?

Is the cubic graph always a function? Can you find an example of a cubic that is not a function?

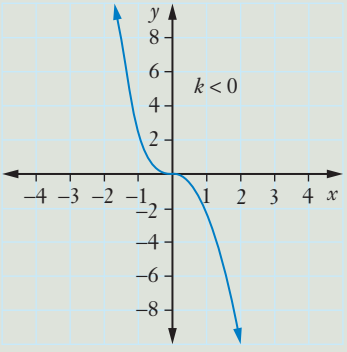
Point of inflection

The flat turning point of the cubic function $y = kx^3$ is called a **point of inflection**, which is where the concavity of the curve changes.

The graph of $y = kx^3$



This cubic curve is increasing and has a point of inflection at (0, 0) where the curve changes from concave downwards to concave upwards.



This cubic curve is decreasing and has a point of inflection at (0, 0) where the curve changes from concave upwards to concave downwards.

The graph of $y = k(x - b)^3 + c$

The graph of $y = k(x - b)^3 + c$ is the graph of $y = kx^3$ shifted so that its point of inflection is at (b, c).

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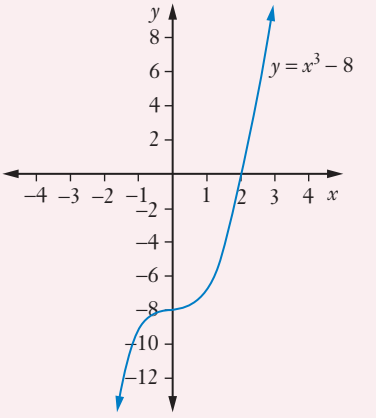
EXAMPLE 32

a Sketch the graph of $y = x^3 - 8$, showing intercepts.

b Sketch the graph of $f(x) = -2(x - 3)^3 + 2$.

Solution

a This is the graph of $y = x^3$ shifted downwards 8 units so that its point of inflection is at $(0, -8)$. Since $k > 0$, the function is increasing.



For x-intercepts, $y = 0$:

$0 = x^3 - 8$

$8 = x^3$

$x = 2$

For y-intercept, $x = 0$:

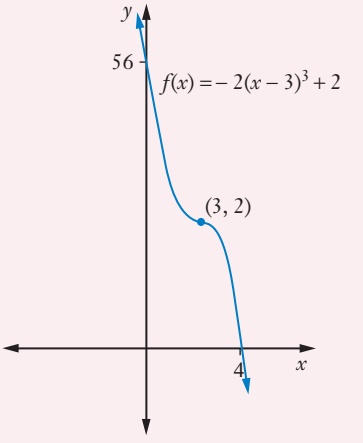
$y = 0^3 - 8$

$= -8$

The point of inflection is at $(0, -8)$, where the curve changes from concave downwards to concave upwards.

b Since $k < 0$, $f(x)$ is decreasing.

This is the graph of $y = -2x^3$ shifted upwards and to the right so that its point of inflection is at $(3, 2)$.



For x-intercepts, $f(x) = 0$:

$0 = -2(x - 3)^3 + 2$

$-2 = -2(x - 3)^3$

$1 = (x - 3)^3$

$1 = x - 3$

$x = 4$

For y-intercept, $x = 0$:

$y = -2(0 - 3)^3 + 2$

$= -2(-27) + 2$

$= 56$

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EXAMPLE 33

Show that $y = 2x^3$ is an odd function.

Solution

Let $f(x) = 2x^3$.

$$f(-x) = 2(-x)^3$$
$$= -2x^3$$
$$= -f(x)$$

So $y = 2x^3$ is an odd function.

A cubic function has one y-intercept and up to 3 x-intercepts. We can sketch the graph of a more general cubic function using intercepts. This will not give a very accurate graph but it will show the shape and important features.

The graph of $y = k(x - a)(x - b)(x - c)$

The graph of $y = k(x - a)(x - b)(x - c)$ has x-intercepts at a, b and c.

EXAMPLE 34

a **i** Sketch the graph of the cubic function $f(x) = x(x + 3)(x - 2)$.

ii Describe the shape of the graph and state its domain and range.

b Sketch the graph of the cubic function $f(x) = (x - 3)(x + 1)^2$ and describe its shape.

Solution

a **i** For x-intercepts, $f(x) = 0$:

$0 = x(x + 3)(x - 2)$

$x = 0, -3, 2$

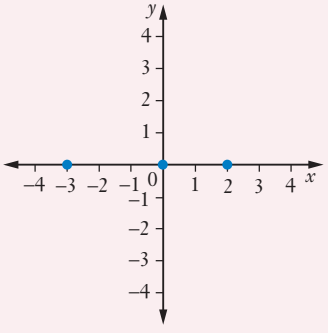
Plot x-intercepts on graph.

For y-intercept, $x = 0$:

$f(0) = 0(0 + 3)(0 - 2)$

$= 0$

So y-intercept is 0.



We look at which parts of the graph are above and which are below the x-axis between the x-intercepts.

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Test $x < -3$, say $x = -4$:

$f(-4) = -4(-4 + 3)(-4 - 2) = -24 < 0$

So here the curve is below the x-axis.

Test $-3 < x < 0$, say $x = -1$:

$f(-1) = -1(-1 + 3)(-1 - 2) = 6 > 0$

So here the curve is above the x-axis.

Test $0 < x < 2$, say $x = 1$:

$f(1) = 1(1 + 3)(1 - 2) = -4 < 0$

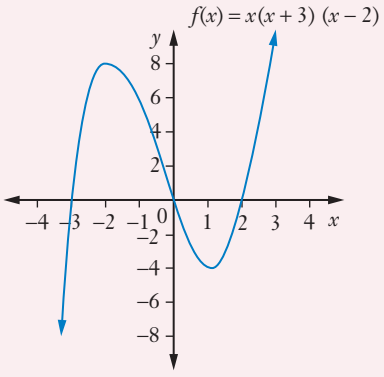
So here the curve is below the x-axis.

Test $x > 2$, say $x = 3$:

$f(3) = 3(3 + 3)(3 - 2) = 18 > 0$

So here the curve is above the x-axis.

We can sketch the cubic curve as shown.



ii The graph increases to a maximum turning point, then decreases to a minimum turning point. Then it increases again.

Domain $(-\infty, \infty)$

Range $(-\infty, \infty)$

b For x-intercepts, $f(x) = 0$:

$0 = (x - 3)(x + 1)^2$

$x = 3, x = -1$

So x-intercepts are -1 and 3.

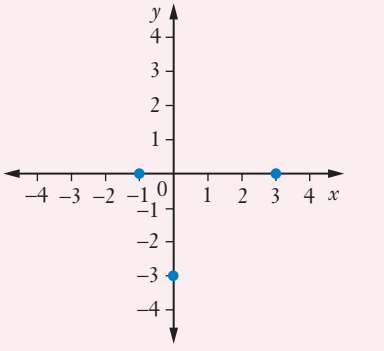
For y-intercept, $x = 0$:

$f(0) = (0 - 3)(0 + 1)^2$

$= (-3)(1)$

$= -3$

So y-intercept is -3.



We look at which parts of the graph are above and below the x-axis.

Test $x < -1$, say $x = -2$:

$f(-2) = (-2 - 3)(-2 + 1)^2 = -5 < 0$

So here the curve is below the x-axis.

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Test $-1 < x < 3$, say $x = 0$:

$f(0) = (0 - 3)(0 + 1)^2 = -3 < 0$

So here the curve is below the x-axis.

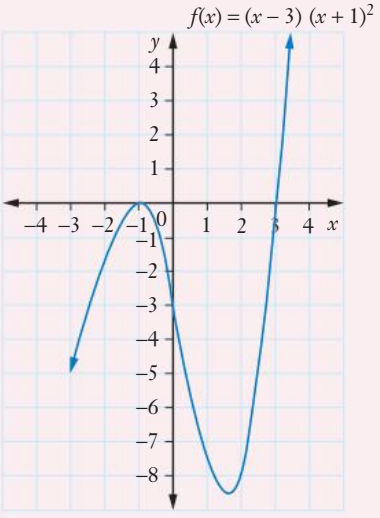
Test $x > 3$, say $x = 4$:

$f(4) = (4 - 3)(4 + 1)^2 = 25 > 0$

So here the curve is above the x-axis.

We can sketch the cubic curve as shown.

The graph increases to a maximum turning point, then decreases to a minimum turning point, then increases.



Finding a cubic equation

EXAMPLE 35

a Find the equation of the cubic function $y = kx^3 + c$ if it passes through (0, 16) and (4, 0).

b Find the equation of the cubic function $f(x) = k(x - a)(x - b)(x - c)$ if it has x-intercepts -1, 3 and 4 and passes through (1, 12).

Solution

a Substitute (0, 16) into $y = kx^3 + c$.

$16 = k(0)^3 + c$

$= c$

So $y = kx^3 + 16$.

Substitute (4, 0) into $y = kx^3 + 16$.

$0 = k(4)^3 + 16$

$= 64k + 16$

$-16 = 64k$

$k = -\frac{16}{64}$

$= -\frac{1}{4}$

So the equation is $y = -\frac{1}{4}x^3 + 16$.

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b $f(x) = k(x - a)(x - b)(x - c)$ has x-intercepts when $f(x) = 0$.

$0 = k(x - a)(x - b)(x - c)$

$x = a, b, c$

But we know x-intercepts are at -1, 3 and 4.

So $a = -1$, $b = 3$ and $c = 4$ (in any order).

So $f(x) = k(x - (-1))(x - 3)(x - 4)$

$= k(x + 1)(x - 3)(x - 4)$

To find k, substitute (1, 12):

$12 = k(1 + 1)(1 - 3)(1 - 4)$

$= k(2)(-2)(-3)$

$= 12k$

$1 = k$

So the cubic function is $f(x) = (x + 1)(x - 3)(x - 4)$.

Exercise 3.12 Cubic functions

1 Find the x- and y-intercept(s) of the graph of each cubic function.

a $y = x^3 - 1$ **b** $f(x) = -x^3 + 8$ **c** $y = (x + 5)^3$

d $f(x) = -(x - 4)^3$ **e** $f(x) = 3(x + 7)^3 - 3$ **f** $y = (x - 2)(x - 1)(x + 5)$

2 Draw each graph on a number plane.

a $y = -x^3$ **b** $p(x) = 2x^3$ **c** $g(x) = x^3 + 1$

d $y = (x + 2)^3$ **e** $y = -(x - 3)^3 + 1$ **f** $f(x) = -x(x + 2)(x - 4)$

g $y = (x + 2)(x - 3)(x + 6)$ **h** $y = x^2(x - 2)$ **i** $f(x) = (x - 1)(x + 3)^2$

3 Find the point of inflection of the graph of each cubic function by sketching each graph.

a $y = 8x^3 + 1$ **b** $y = -x^3 + 27$ **c** $f(x) = (x + 2)^3$

d $y = 2(x - 1)^3 - 16$ **e** $f(x) = -(x + 1)^3 + 1$

4 Find the x-intercept of the graph of each cubic function correct to one decimal place.

a $y = 2x^3 - 5$ **b** $f(x) = (x - 1)^3 + 2$ **c** $f(x) = -3x^3 + 1$

d $y = 2(x + 3)^3 - 3$ **e** $y = -3(2x - 1)^3 + 2$

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5 Describe the shape of each cubic function.

a $y = x^3 - 64$ **b** $f(x) = -(x - 3)^3$ **c** $y = x(x + 2)(x + 4)$

d $f(x) = -2(x + 3)(x + 1)(x - 4)$ **e** $y = x(x + 5)^2$

6 Solve graphically:

a $x^3 - 5 = 0$ **b** $x^3 + 2 = 0$ **c** $2x^3 - 9 = 0$

d $3x^3 + 4 = 0$ **e** $(x - 1)^3 + 6 = 0$ **f** $x(x + 2)(x - 1) = 0$

7 The volume of a certain solid has equation $V = kx^3$ where x is the length of its side in cm.

a Find the equation if $V = 120$ when $x = 3.5$.

b Find the volume when $x = 6$.

c Find x when $V = 250$.

8 The volume of a solid is directly proportional to the cube of its radius.

a If radius $r = 12$ mm when the volume V is 7238 mm^3, find an equation for the volume.

b Find the volume if the radius is 2.5 mm.

c Find the radius if the volume is 7000 mm^3.

9 Show that $f(x) = -x^3$ is an odd function.

10 Determine whether each function is odd.

a $y = 3x^3$ **b** $y = (x + 1)^3$ **c** $f(x) = -2x^3 - 1$

d $y = -5x^3$ **e** $y = (x - 2)^3 + 3$

11 A cubic function is in the form $y = kx^3 + c$. Find its equation if it passes through:

a (0, 0) and (1, 2) **b** (0, 5) and (2, –3) **c** (1, –4) and (–2, 23)

d (1, –2) and (2, 33) **e** (2, –29) and (–3, 111)

12 A cubic function is in the form $y = k(x - a)(x - b)(x - c)$. Find its equation if:

a it has x-intercepts 2, 3 and –5 and passes through the point (–2, –120)

b it has x-intercepts –1, 4 and 6 and passes through the point (3, 96)

c it has x-intercepts 1 and 3, y-intercept –27 and $k = -3$.

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3.13 Polynomial functions


Graphing power functions

A **polynomial** is a function defined for all real x involving powers of x in the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

where n is a positive integer or zero and $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

We generally write polynomials from the highest power down to the lowest, for example $P(x) = x^2 - 5x + 4$. We have already studied some polynomial functions, as linear, quadratic and cubic functions are all polynomials.

Polynomial terminology

$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$ is called a **polynomial expression**.

$P(x)$ has **degree** n (where n is the highest power of x).

$a_n, a_{n-1}, a_{n-2}, \ldots a_2, a_1$ and a_0 are called **coefficients**.

$a_n x^n$ is called the **leading term**.

a_n is the **leading coefficient**.

a_0 is called the **constant term**.

If $a_n = 1$, $P(x)$ is called a **monic polynomial**.

EXAMPLE 36

a Which of the following are polynomial expressions?

A $4 - x + 3x^2$ **B** $3x^4 - x^2 + 5x - 1$ **C** $x^2 - 3x + x^{-1}$

b $P(x) = x^6 - 2x^4 + 3x^3 + x^2 - 7x - 3$.

i Find the degree of $P(x)$.

ii Is the polynomial monic?

iii State the leading term.

iv What is the constant term?

v Find the coefficient of x^4.

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Solution

a **A** and **B** are polynomials but **C** is not, because it has a term of x^{-1} that is not a positive integer power of x.

b **i** Degree is 6 since x^6 is the highest power.

ii Yes, the polynomial is monic because the coefficient of x^6 is 1.

iii The leading term is x^6.

iv The constant term is -3.

v The coefficient of x^4 is -2.

Polynomial equations

$P(x) = 0$ is a **polynomial equation** of degree n.

The values of x that satisfy the equation are called the **roots** of the equation or the **zeros** of the polynomial $P(x)$.

EXAMPLE 37

a Find the zeros of the polynomial $P(x) = x^2 - 5x$.

b Show that the polynomial $p(x) = x^2 - x + 4$ has no real zeros.

Solution

a To find the zeros of the polynomial, solve $P(x) = 0$.

$$x^2 - 5x = 0$$

$$x(x - 5) = 0$$

$$x = 0, 5$$

So the zeros are 0, 5.

b Solve $p(x) = 0$.

$$x^2 - x + 4 = 0$$

The discriminant will show whether the polynomial has real zeros.

$$b^2 - 4ac = (-1)^2 - 4(1)(4)$$

$$= -15$$

$$< 0$$

So the polynomial has no real zeros.

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Graphing polynomials

EXAMPLE 38

a Write the polynomial $P(x) = x^4 + 2x^3 - 3x^2$ as a product of its factors.

b Sketch the graph of the polynomial.

Solution

a $P(x) = x^4 + 2x^3 - 3x^2$

$= x^2(x^2 + 2x - 3)$

$= x^2(x + 3)(x - 1)$

b For x-intercepts, $P(x) = 0$:

$0 = x^4 + 2x^3 - 3x^2$

$= x^2(x + 3)(x - 1)$

$x = 0, -3, 1$

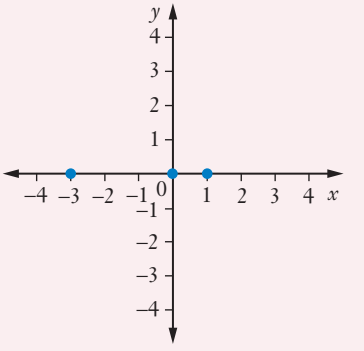
So the x-intercepts are -3, 0, 1.

For y-intercepts, $x = 0$:

$P(0) = 0^4 + 2(0)^3 - 3(0)^2$

$= 0$

So y-intercept is 0.



Test $x < -3$, say $x = -4$:

$P(-4) = (-4)^4 + 2(-4)^3 - 3(-4)^2 = 80 > 0$

So here the curve is above the x-axis.

Test $-3 < x < 0$, say $x = -1$:

$P(-1) = (-1)^4 + 2(-1)^3 - 3(-1)^2 = -4 < 0$

So here the curve is below the x-axis.

Test $0 < x < 1$, say $x = 0.5$:

$P(0.5) = (0.5)^4 + 2(0.5)^3 - 3(0.5)^2$

$= -0.4375 < 0$

So here the curve is below the x-axis.

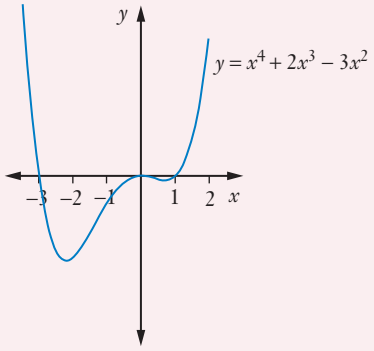
Test $x > 1$, say $x = 3$:

$P(3) = 3^4 + 2(3)^3 - 3(3)^2 = 108 > 0$

So here the curve is above the x-axis.

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We can sketch the graph of the polynomial as shown.



DID YOU KNOW?

'Poly' means many

The word 'polynomial' means an expression with many terms. (A binomial has 2 terms and a trinomial has 3 terms.) 'Poly' means 'many', and is used in many words, for example polygamy, polyglot, polygon, polyhedron, polymer, polyphonic, polypod and polytechnic. Do you know what all these words mean? Do you know any others with 'poly-'?

Exercise 3.13 Polynomial functions

1 Write down the degree of each polynomial:

a $5x^7 - 3x^5 + 2x^3 - 3x + 1$ **b** $3 + x + x^2 - x^3 + 2x^4$ **c** $3x + 5$

d $x^{11} - 5x^8 + 4$ **e** $2 - x - 5x^2 + 3x^3$ **f** 3

2 For the polynomial $P(x) = x^3 - 7x^2 + x - 1$, find:

a $P(2)$ **b** $P(-1)$ **c** $P(0)$

3 Given $P(x) = x + 5$ and $Q(x) = 2x - 1$, find:

a $P(-11)$ **b** $Q(3)$ **c** $P(2) + Q(-2)$

d the degree of $P(x) + Q(x)$ **e** the degree of $P(x)Q(x)$

4 For the polynomial $P(x) = x^5 - 3x^4 - 5x + 4$, find:

a the degree of $P(x)$ **b** the constant term

c the coefficient of x^4 **d** the coefficient of x^2.

5 Find the zeros of each polynomial.

a $P(x) = x^2 - 9$ **b** $p(x) = x + 5$ **c** $f(x) = x^2 + x - 2$

d $P(x) = x^2 - 8x + 16$ **e** $g(x) = x^3 - 2x^2 + 5x$

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6 Which of the following are not polynomials?

a $5x^4 - 3x^2 + x + \frac{1}{x}$ **b** $x^2 + 3^x$ **c** $x^2 + 3x - 7$

d $3x + 5$ **e** 0 **f** $4x^3 + 7x^{-2} + 5$

7 For the polynomial $P(x) = (a + 1)x^3 + (b - 7)x^2 + c + 5$, find values for a, b or c if:

a $P(x)$ is monic **b** the coefficient of x^2 is 3

c the constant term is -1 **d** $P(x)$ has degree 2

e the leading term has a coefficient of 5.

8 Given $P(x) = 2x + 5$, $Q(x) = x^2 - x - 2$ and $R(x) = x^3 + 9x$, find:

a any zeros of $P(x)$ **b** the roots of $Q(x) = 0$

c the degree of $P(x) + R(x)$ **d** the degree of $P(x)Q(x)$

e the leading term of $Q(x)R(x)$.

9 Given $f(x) = 3x^2 - 2x + 1$ and $g(x) = 3x - 3$:

a show $f(x)$ has no zeros **b** find the leading term of $f(x)g(x)$

c find the constant term of $f(x) + g(x)$ **d** find the coefficient of x in $f(x)g(x)$

e find the roots of $f(x) + g(x) = 0$.

10 State how many real roots there are for each polynomial equation $P(x) = 0$.

a $P(x) = x^2 - 9$ **b** $P(x) = x^2 + 4$

c $P(x) = x^2 - 3x - 7$ **d** $P(x) = 2x^2 + x + 3$

e $P(x) = 3x^2 - 5x - 2$ **f** $P(x) = x(x - 1)(x + 4)(x + 6)$

11 Sketch the graph of each polynomial by finding its zeros and showing the x- and y-intercepts.

a $f(x) = (x + 1)(x - 2)(x - 3)$ **b** $P(x) = x\,(x + 4)(x - 2)$

c $p(x) = -x(x - 1)(x - 3)$ **d** $f(x) = x(x + 2)^2$

e $g(x) = (5 - x)(x + 2)(x + 5)$

12 **i** Write each polynomial as a product of its factors.

ii Sketch the graph of the polynomial and describe its shape.

a $P(x) = x^3 - 2x^2 - 8x$ **b** $f(x) = -x^3 - 4x^2 + 5x$

c $P(x) = x^4 + 3x^3 + 2x^2$ **d** $A(x) = 2x^3 + x^2 - 15x$

e $P(x) = -x^4 + 2x^3 + 3x^2$

13 **a** Find the x-intercepts of the polynomial $P(x) = x(x - 1)(x + 2)^2$.

b Sketch the graph of the polynomial.

14 **a** Show that $(x - 3)(x - 2)(x + 2) = x^3 - 3x^2 - 4x + 12$.

b Sketch the graph of the polynomial $P(x) = x^3 - 3x^2 - 4x + 12$.

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3.14 Intersection of graphs

Solving equations graphically

EXAMPLE 39

a Sketch $y = x^2$ and $y = 1$ on the same set of axes, and hence solve $x^2 = 1$ graphically.

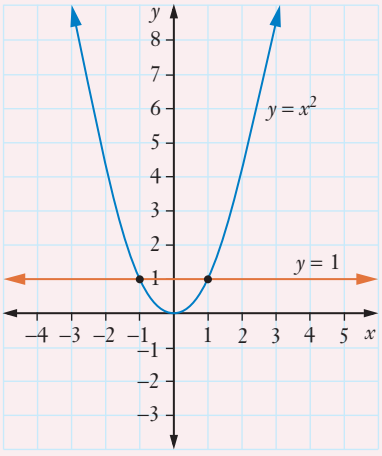
b Sketch $y = x^2 - x$ and $y = 2$ on the same set of axes, and hence solve $x^2 - x = 2$ graphically.

Solution

a $y = x^2$ is a parabola and $y = 1$ is a horizontal line, as shown.

To solve $x^2 = 1$ graphically, find the x values where the 2 graphs $y = x^2$ and $y = 1$ intersect.

The solution is $x = \pm 1$.

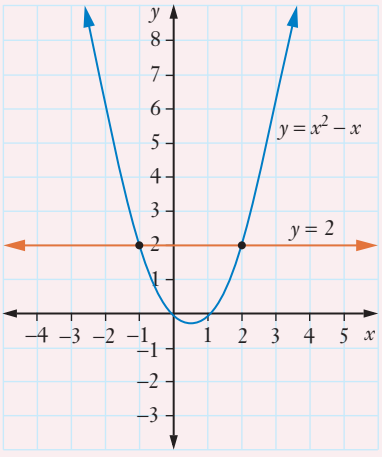


b $y = x^2 - x$ is a parabola with x-intercepts 0, 1 and y-intercept 0. Since $a > 0$, it is concave upwards.

$y = 2$ is a horizontal line.

The solutions of $x^2 - x = 2$ are the x values at the intersection of the 2 graphs.

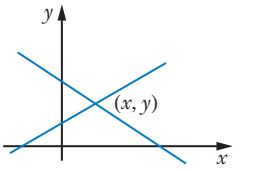
$x = -1, 2$.



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Intersecting lines

Two straight lines intersect at a single point (x, y).



The point of intersection can be found graphically or algebraically using simultaneous equations.

EXAMPLE 40

Find the point of intersection between lines $2x - 3y - 3 = 0$ and $5x - 2y - 13 = 0$.

Solution

Solve simultaneous equations.

$2x - 3y - 3 = 0$ [1]

$5x - 2y - 13 = 0$ [2]

[1] × 2:

$4x - 6y - 6 = 0$ [3]

[2] × 3:

$15x - 6y - 39 = 0$ [4]

[3] – [4]:

$-11x + 33 = 0$

$33 = 11x$

$3 = x$

Substitute $x = 3$ into [1]:

$2(3) - 3y - 3 = 0$

$-3y + 3 = 0$

$3 = 3y$

$1 = y$

So the point of intersection is (3, 1).

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Break-even points

Break-even points

EXAMPLE 41

A company that manufactures cables sells them for \$2 each. It costs 50 cents to produce each cable and the company has fixed costs of \$1500 per week.

a Find the equation for the income, \$$I$, on x cables per week.

b Find the equation for the costs, \$$C$, of manufacturing x cables per week.

c Find the break-even point (where income = costs).

d Find the profit on 1450 cables.

Solution

a $I = 2x$

b $C = 0.5x + 1500$

c Solving simultaneous equations:

$I = 2x$ [1]

$C = 0.5x + 1500$ [2]

Substitute [1] into [2]:

$2x = 0.5x + 1500$

$1.5x = 1500$

$x = 1000$

1000 cables is where income = costs.
Substitute $x = 1000$ into [1] (or [2]):

$I = 2(1000)$

$= 2000$

So the break-even point is (1000, 2000). 1000 cables gives an income and cost of \$2000.

d Profit = income − costs = $I - C$

Substitute $x = 1450$ into both equations.

$I = 2x$

$= 2(1450)$

$= 2900$

So income is \$2900.

$C = 0.5x + 1500$

$= 0.5(1450) + 1500$

$= 2225$

So costs are \$2225.

Profit = \$2900 − \$2225

= \$675

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Break-even point

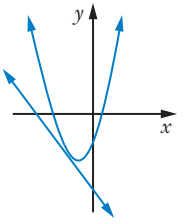
In business, the **break-even point** is the point where the **income** (or **revenue**) **equals costs.**

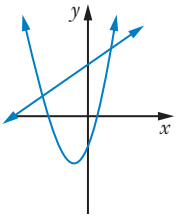
If income > costs, the business makes a **profit**.

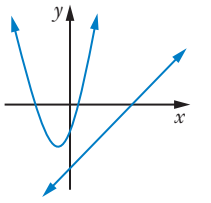
If income < costs, the business makes a **loss**.

Intersecting lines and parabolas

A line and a parabola can intersect at 1 or 2 points, or they may not intersect at all.







EXAMPLE 42

Find the points of intersection of the line $y = x - 1$ with the parabola $y = x^2 + 4x + 1$.

Solution

Solve simultaneous equations.

$y = x - 1$ [1]

$y = x^2 + 4x + 1$ [2]

Substitute [1] into [2]:

$x - 1 = x^2 + 4x + 1$

$0 = x^2 + 3x + 2$

$= (x + 2)(x + 1)$

$x = -2, -1$

Substitute $x = -2$ into [1]:

$y = -2 - 1$

$= -3$

Substitute $x = -1$ into [1]:

$y = -1 - 1$

$= -2$

So the 2 points of intersection are $(-2, -3)$ and $(-1, -2)$.

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Exercise 3.14 Intersection of graphs

1 **a** Given $f(x) = 2x - 4$, solve graphically:

i $f(x) = 0$

ii $f(x) = -2$

iii $f(x) = 4$

b By sketching the graph of $f(x) = x^2 - 2x$, solve graphically:

i $f(x) = 0$

ii $f(x) = 3$

c Use the sketch of $f(x) = x^3 - 1$ to solve graphically:

i $f(x) = 0$

ii $f(x) = 7$

iii $f(x) = -2$

2 Find the point of intersection between:

a $y = x + 3$ and $y = 2x + 2$

b $y = 3x - 1$ and $y = 5x + 1$

c $x + 2y - 4 = 0$ and $2x - y + 2 = 0$

d $3x + y - 2 = 0$ and $2x - 3y - 5 = 0$

e $4x - 3y - 5 = 0$ and $7x - 2y - 12 = 0$

3 Find points of intersection between:

a $y = x^2$ and $y = x$

b $y = x^2$ and $y = 4$

c $y = x^2$ and $y = x + 2$

d $y = x^2$ and $y = -2x + 3$

e $y = x^2 - 5$ and $y = 4x$

4 **a** Draw the graphs of $f(x) = x^2$ and $f(x) = (x - 2)^2$ on the same number plane.

b From the graph, find the number of points of intersection of the functions.

c From the graph or by using algebra, find any points of intersection.

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5 Find any points of intersection between the functions $f(x) = x^2$ and $f(x) = (x + 2)^2$.

6 Find any points of intersection between the curves $y = x^2 - 5$ and $y = 2x^2 + 5x + 1$.

7 Find any points of intersection between $y = 3x^2 - 4x - 4$ and $y = 5x^2 - 2$.

8 **a** If Paula's Posies' income on x roses is given by $y = 10x$ and the costs are $y = 3x + 980$, find the break-even point.

b Find the profit on 189 roses.

c Find the loss on 45 roses.

9 Find the number of calculators that a company needs to sell to break even each week if it costs \$3 to make each calculator and they are sold for \$15 each. Fixed overheads are \$852 a week.

10 Cupcakes Online sells cupcakes at \$5 each. The cost of making each cupcake is \$1 and the company has fixed overheads of \$264 a day.

a Find the equations for daily income and costs.

b Find how many cupcakes the company needs to sell daily to break even.

c What is the profit on 250 cupcakes?

d What is the loss on 50 cupcakes?

11 **a** The perimeter of a figure is in direct proportion to its side x. Find an equation for perimeter if the perimeter $y = 90$ cm when side $x = 5$ cm.

b The area of the figure is in direct proportion to the square of its side x. If the area of the figure is $y = 108$ cm^2 when $x = 3$ cm, find its equation.

c Find any x values for the side for which the perimeter and area will have the same y value.

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3. TEST YOURSELF

Practice quiz

For Questions **1** to **5**, select the correct answer **A**, **B**, **C** or **D**.

1 Which polynomial below is a monic polynomial with constant term 5 and degree 6?

A $P(x) = -x^6 + 5$ **B** $P(x) = 6x^5 - 3x^4 + 5$

C $P(x) = x^6 - 3x^4 + 5$ **D** $P(x) = 5x^6 - 3x^4 + 1$

2 The axis of symmetry and turning point of the quadratic function $f(x) = 1 + 2x - x^2$ are, respectively:

A $x = 1, (1, 2)$ **B** $x = -1, (-1, 4)$

C $x = 2, (2, 5)$ **D** $x = -2, (-2, 5)$

3 The linear function $2x - 3y - 6 = 0$ has x- and y-intercepts, respectively:

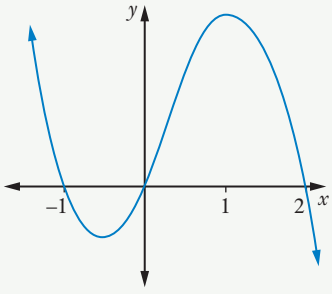
A -3 and 2 **B** 3 and -2 **C** -3 and -2 **D** 3 and 2

4 The domain and range of the straight line with equation $x = -2$ are:

A Domain $(-\infty, \infty)$, range $[-2]$ **B** Domain $[-2]$, range $(-\infty, \infty)$

C Domain $(-\infty, \infty)$, range $(-\infty, \infty)$ **D** Domain $[-2]$, range $[-2]$

5 Which cubic function has this graph?



A $y = x(x + 1)(x - 2)$

B $y = -x(x - 1)(x + 2)$

C $y = x(x - 1)(x + 2)$

D $y = -x(x + 1)(x - 2)$

6 If $f(x) = x^2 - 3x - 4$, find:

a $f(-2)$ **b** $f(a)$ **c** x when $f(x) = 0$

7 Sketch each graph and find its domain and range.

a $y = x^2 - 3x - 4$ **b** $f(x) = x^3$ **c** $2x - 5y + 10 = 0$

d $x = 3$ **e** $y = (x + 1)^3$ **f** $y = -2$

g $f(x) = -x^2 + x$ **h** $f(x) = x^2 + 4x + 4$

8 If $f(x) = 3x - 4$, find:

a $f(2)$ **b** x when $f(x) = 7$ **c** x when $f(x) = 0$

9 Sketch the graph of $P(x) = 2x^3 - 2x^2 - 4x$.

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10 Find the gradient of the straight line:

a passing through $(3, -1)$ and $(-2, 5)$ **b** with equation $2x - y + 1 = 0$

c perpendicular to the line $5x + 3y - 8 = 0$ **d** making an angle of inclination of $45°$.

11 For the parabola $y = x^2 - 4x + 1$, find:

a the equation of the axis of symmetry **b** the minimum value.

12 Sketch the graph of $f(x) = (x - 2)(x + 3)^2$, showing the intercepts.

13 For the polynomial $P(x) = x^3 + 2x^2 - 3x$, find:

a the degree **b** the coefficient of x

c the zeros **d** the leading term.

14 Find the x- and y-intercepts of:

a $2x - 5y + 20 = 0$ **b** $y = x^2 - 5x - 14$

c $y = (x + 2)^3$ **d** $2x - 5y - 10 = 0$

15 Find the point of intersection between lines $y = 2x + 3$ and $x - 5y + 6 = 0$.

16 For the quadratic function $y = -2x^2 - x + 6$, find:

a the equation of the axis of symmetry **b** the maximum value.

17 Find the domain and range of $y = -2x^2 - x + 6$.

18 For each quadratic equation, select the correct property of its roots **A**, **B**, **C** or **D**.

A real, different and rational **B** real, different and irrational

C equal **D** unreal.

a $2x^2 - x + 3 = 0$ **b** $x^2 - 10x - 25 = 0$ **c** $x^2 - 10x + 25 = 0$

d $3x^2 + 7x - 2 = 0$ **e** $6x^2 - x - 2 = 0$

19 Find the equation of the line:

a passing through $(2, 3)$ and with gradient 7

b parallel to the line $5x + y - 3 = 0$ and passing through $(1, 1)$

c through the origin, and perpendicular to the line $2x - 3y + 6 = 0$

d through $(3, 1)$ and $(-2, 4)$

e with x-intercept 3 and y-intercept -1.

20 The polynomial $f(x) = ax^2 + bx + c$ has zeros 4 and 5, and $f(-1) = 60$. Evaluate a, b and c.

21 Determine whether each function is even, odd or neither.

a $y = x^2 - 1$ **b** $y = x + 1$ **c** $y = x^3$

d $y = (x + 1)^2$ **e** $y = -5x^3$

22 Show that $f(x) = x^3 - x$ is odd.

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23 Prove that the line between (–1, 4) and (3, 3) is perpendicular to the line $4x - y - 6 = 0$.

24 Show that $-4 + 3x - x^2 < 0$ for all x.

25 For each pair of equations, state whether their graphs have 0, 1 or 2 points of intersection.

a $xy = 7$ and $3x - 5y - 1 = 0$ **b** $x^2 + y^2 = 9$ and $y = 3x - 3$

c $x^2 + y^2 = 1$ and $x - 2y - 3 = 0$ **d** $y = x^2$ and $y = 4x - 4$

26 Prove that the lines with equations $y = 5x - 7$ and $10x - 2y + 1 = 0$ are parallel.

27 Find the zeros of $g(x) = -x^2 + 9x - 20$.

28 Sketch the graph of $P(x) = 2x(x - 3)(x + 5)$, showing intercepts.

29 Solve $P(x) = 0$ when $P(x) = x^3 - 4x^2 + 4x$.

30 Find x if the gradient of the line through $(3, -4)$ and $(x, 2)$ is -5.

31 If $f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x^2 - 3 & \text{if } x < 1 \end{cases}$, find $f(5) - f(0) + f(1)$.

32 Given $f(x) = \begin{cases} 3 & \text{if } x > 3 \\ x^2 & \text{if } 1 \leq x \leq 3 \\ 2 - x & \text{if } x < 1 \end{cases}$

find:

a $f(2)$ **b** $f(-3)$ **c** $f(3)$

d $f(5)$ **e** $f(0)$

33 Find the equation of the parabola:

a that passes through the points $(-2, 18)$, $(3, -2)$ and $(1, 0)$

b with x-intercepts 3 and -2 and y-intercept 12.

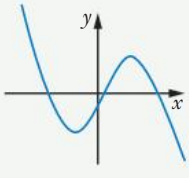
34 The area (A) of a certain shape is in direct proportion to the square of its length x. If the area is 448 cm^2 when $x = 8$, find:

a the equation for area **b** the area when $x = 10$

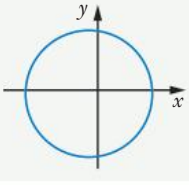
c x when the area is 1093.75 cm^2.

35 For each graph and set of ordered pairs, state whether it represents a function, and for those that do, whether it represents a one-to-one function.

a

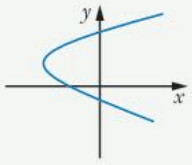


b

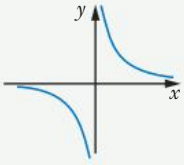


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c



d



e (1, 2), (2, 5), (–1, 4), (1, 3), (3, 4)

36 Find the equation of a cubic function $f(x) = kx^3 + c$ if it passes through the point (1, 2) and has y-intercept 5.

37 A company has costs given by $y = 7x + 15$ and income $y = 12x$. Find the break-even point.

38 **a** Find the equation of the straight line that is perpendicular to the line $y = \frac{1}{2}x - 3$ and passes through (1, –1).

b Find the x-intercept of this line.

39 Find values of m such that $mx^2 + 3x - 4 < 0$ for all x.

40 Find any points of intersection of the graphs of:

a $y = 3x - 4$ and $y = 1 - 2x$

b $y = x^2 - x$ and $y = 2x - 2$

c $y = x^2$ and $y = 2x^2 - 9$

41 Find the equation of the straight line passing through the origin and parallel to the line with equation $3x - 4y + 5 = 0$.

42 Find the equation of the line with y-intercept –2 and perpendicular to the line passing through (3, –2) and (0, 5).

43 The amount of petrol used in a car is directly proportional to the distance travelled.

a If the car uses 10.8 litres of petrol for an 87 km trip, find the equation for the amount of petrol used (A) over a distance of d km.

b Find the amount of petrol used for a 250 km trip.

c Find how far the car travelled if it used 35.5 L of petrol.

44 A function has equation $f(x) = x^3 - x^2 - 4x + 4$.

a Solve $f(x) = 0$.

b Find its x- and y-intercepts.

c Sketch the graph of the function.

d From the graph, state how many solutions there are for:

i $f(x) = 1$

ii $f(x) = -2$

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3. CHALLENGE EXERCISE

1 Find the values of b if $f(x) = 3x^2 - 7x + 1$ and $f(b) = 7$.

2 Sketch the graph of $y = (x + 2)^2 - 1$ in the domain $[-3, 0]$.

3 If points $(-3k, 1)$, $(k - 1, k - 3)$ and $(k - 4, k - 5)$ are collinear (lie on a straight line), find the value of k.

4 Find the equation of the line that passes through the point of intersection of the lines $2x + 5y + 19 = 0$ and $4x - 3y - 1 = 0$ and is perpendicular to the line $3x - 2y + 1 = 0$.

5 If $ax - y - 2 = 0$ and $bx - 5y + 11 = 0$ intersect at the point $(3, 4)$, find the values of a and b.

6 By writing each as a quadratic equation, solve:

a $(3x - 2)^2 - 2(3x - 2) - 3 = 0$

b $5^{2x} - 26(5^x) + 25 = 0$

c $2^{2x} - 10(2^x) + 16 = 0$

d $2^{2x+1} - 5(2^x) + 2 = 0$

7 Find the equation of the straight line through $(1, 3)$ that passes through the intersection of the lines $2x - y + 5 = 0$ and $x + 2y - 5 = 0$.

8 $f(x) = \begin{cases} 2x+3 & \text{when } x > 2 \\ 1 & \text{when } -2 \le x \le 2 \\ x^2 & \text{when } x < -2 \end{cases}$

Find $f(3)$, $f(-4)$, $f(0)$ and sketch the graph of the function.

9 If $h(t) = \begin{cases} 1-t^2 & \text{if } t > 1 \\ t^2-1 & \text{if } t \le 1 \end{cases}$, find the value of $h(2) + h(-1) - h(0)$ and sketch the curve.

10 If $f(x) = 2x^3 - 2x^2 - 12x$, find x when $f(x) = 0$.

11 Show that the quadratic equation $2x^2 - kx + k - 2 = 0$ has real rational roots.

12 Find the values of p for which $x^2 - x + 3p - 2 > 0$ for all x.

13 If $f(x) = 2x - 1$ show that $f(a^2) = f[(-a)^2]$ for all real a.

14 Find the equation of the straight line through $(3, -4)$ that is perpendicular to the line with x-intercept -2 and y-intercept 5.

15 Find any points of intersection between $y = x^2$ and $y = x^3$.

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16 Find the equation of a cubic function $y = ax^3 + bx^2 + cx + d$ if it passes through (0, 1), (1, 3), (–1, 3) and (2, 15).

17 Show that the quadratic equation $x^2 - 2px + p^2 = 0$ has equal roots.

18 A monic polynomial $P(x)$ of degree 3 has zeros –2, 1 and 6. Write down the equation of the polynomial.

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4.

TRIGONOMETRY

Trigonometry is used in many fields, such as building, surveying and navigating. It is the geometry and measurement of triangles.

This chapter covers the trigonometry of right-angled and non-right-angled triangles, and applies it to problems and real-life situations, including the use of angles of elevation and depression and bearings. This chapter also introduces radians, an alternative to degrees for measuring angle size. We will apply radians to circle measurement by finding the length of an arc and the area of a sector.

CHAPTER OUTLINE

4.01 Trigonometric ratios
4.02 Finding a side of a right-angled triangle
4.03 Finding an angle in a right-angled triangle
4.04 Applications of trigonometry
4.05 The sine rule
4.06 The cosine rule
4.07 Area of a triangle
4.08 Mixed problems
4.09 Radians
4.10 Length of an arc
4.11 Area of a sector



IN THIS CHAPTER YOU WILL:

- identify the trigonometric ratios
- solve right-angled triangle problems
- apply trigonometry to angles of elevation and depression and bearings
- understand and apply the sine and cosine rules
- find the area of a triangle given the length of two sides and the size of their included angle
- understand radians and convert between degrees and radians
- find the length of an arc and area of a sector of a circle

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TERMINOLOGY

ambiguous case: When using the sine rule to find an angle, there may be 2 possible angles – one acute and one obtuse

angle of depression: The angle between the horizontal and the line of sight when looking down to an object below

angle of elevation: The angle between the horizontal and the line of sight when looking up to an object above

bearing: A direction from one point on Earth's surface to another, measured in degrees. Bearings may be written as true bearings (clockwise from north) or as compass bearings (using N, S, E and W)

compass bearing: Angles specified as either side of north or south, for example N 20° W or S 67° E.

cosine rule: In any triangle $c^2 = a^2 + b^2 - 2ab\cos C$

radian: A unit of angle measurement equal to the size of the angle subtended at the centre of a unit circle by an arc of length 1 unit

sine rule: In any triangle $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

true bearing: True or three-figure bearings are measured from north and turning clockwise

DID YOU KNOW?

Ptolemy

Ptolemy (Claudius Ptolemaeus), in the second century, wrote *Hē mathē matikē syntaxis* (or *Almagest* as it is now known) on astronomy. This is considered to be the first treatise on trigonometry, but it was based on circles and spheres rather than on triangles. The notation 'chord of an angle' was used rather than sin, cos or tan.

Ptolemy constructed a table of sines from 0° to 90° in steps of a quarter of a degree. He also calculated a value of π to 5 decimal places, and established the relationship for $\sin(x \pm y)$ and $\cos(x \pm y)$.

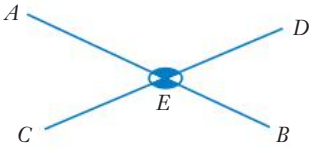
Geometry results

You will need to use some geometry when solving trigonometry problems. Here is a summary of the rules you may need.

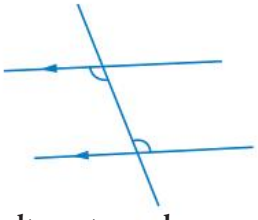
$\angle AEC$ and $\angle DEB$ are **vertically opposite angles**.

$\angle AED$ and $\angle CEB$ are also vertically opposite.

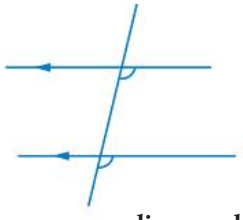
Vertically opposite angles are equal.



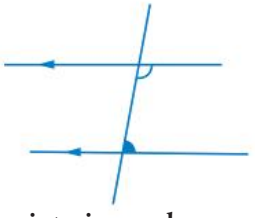
If lines are parallel, then:



alternate angles are equal.

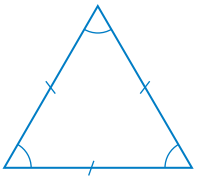


corresponding angles are equal.

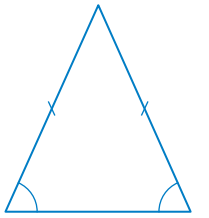


cointerior angles are **supplementary** (their sum is 180°).

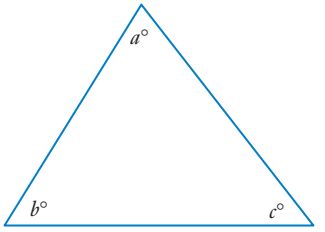
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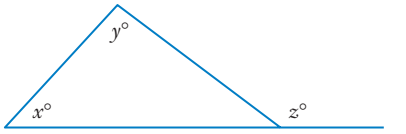
An **equilateral triangle** has 3 equal sides and 3 equal angles of size 60°.



An **isosceles triangle** has 2 equal sides and 2 equal angles.



The **sum of the interior angles** in any triangle is 180°, that is, $a + b + c = 180$.



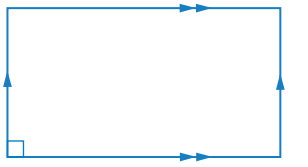
The **exterior angle** in any triangle is equal to the sum of the 2 opposite interior angles. That is, $x + y = z$.

A **parallelogram** is a quadrilateral with opposite sides parallel.



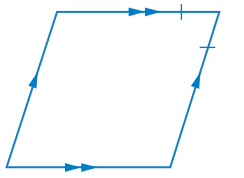
- Opposite sides are equal.
- Opposite angles are equal.
- Diagonals bisect each other.

A **rectangle** is a parallelogram with one angle a right angle.



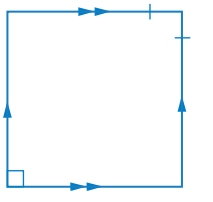
- Opposite sides are equal.
- All angles are right angles.
- Diagonals are equal and bisect each other.

A **rhombus** is a parallelogram with a pair of adjacent sides equal.



- All sides are equal.
- Opposite angles are equal.
- Diagonals bisect each other at right angles.

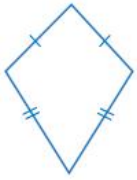
A **square** is a rectangle with a pair of adjacent sides equal.



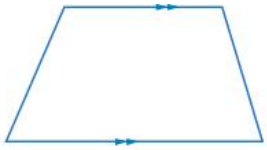
- All sides are equal.
- All angles are right angles.
- Diagonals are equal and bisect each other at right angles.
- Diagonals make angles of 45° with the sides.

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A **kite** is a quadrilateral with 2 pairs of adjacent sides equal.



A **trapezium** is a quadrilateral with one pair of sides parallel.



The **sum of the interior angles** in any quadrilateral is 360°, that is, $a + b + c + d = 360$.





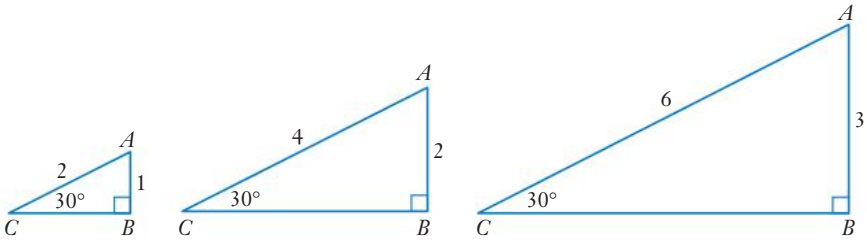
Trigonometric ratios



Trigonometry calculations

4.01 Trigonometric ratios

In similar triangles, pairs of corresponding angles are equal and sides are in proportion. For example:

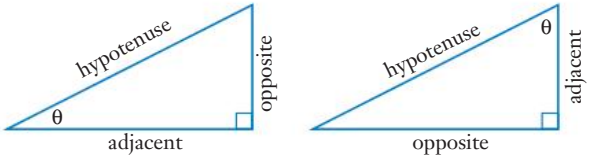


In any triangle containing an angle of 30°, the ratio $AB : AC = 1 : 2$. Similarly, the ratios of other corresponding sides will be equal. These ratios of sides form the basis of the trigonometric ratios.

The sides of a right-angled triangle

- The **hypotenuse** is the longest side, and is always opposite the right angle.
- The **opposite** side is opposite the angle marked in the triangle.
- The **adjacent** side is next to the angle marked.

The opposite and adjacent sides vary according to where the angle is marked. For example:



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The trigonometric ratios

Sine $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

Cosine $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

Tangent $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

DID YOU KNOW?

The origins of trigonometry

Trigonometry, or **triangle measurement**, progressed from the study of geometry in ancient Greece. Trigonometry was seen as applied mathematics. It gave a tool for the measurement of planets and their motion. It was also used extensively in navigation, surveying and mapping, and it is still used in these fields today.

Trigonometry was crucial in setting up an accurate calendar, since this involved measuring the distances between the Earth, Sun and Moon.

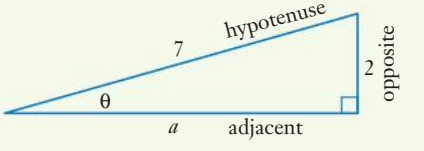
EXAMPLE 1

If $\sin\theta = \dfrac{2}{7}$, find the exact ratios of $\cos\theta$ and $\tan\theta$.

Solution

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{7}.$$

First draw a triangle with opposite side 2 and hypotenuse 7, then use Pythagoras' theorem to find the adjacent side.



$$c^2 = a^2 + b^2$$
$$7^2 = a^2 + 2^2$$
$$49 = a^2 + 4$$
$$45 = a^2$$
$$a = \sqrt{45}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{45}}{7}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{\sqrt{45}}$$

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Degrees, minutes, seconds

Angles are measured in degrees, minutes and seconds.

60 minutes = 1 degree ($60' = 1°$)

60 seconds = 1 minute ($60'' = 1'$)

When rounding numbers, you round up if the digit to the right is 5 or more. However, with angles, you round up to the next degree if there are 30 minutes or more.

Similarly, round angles up to the nearest minute if there are 30 seconds or more.

EXAMPLE 2

a Round to the nearest degree:

i $54°17'45''$ **ii** $29°32'52''$

b Round to the nearest minute:

i $23°12'22''$ **ii** $57°34'41''$ **iii** $84°19'30''$

Solution

a **i** $17'$ is less than $30'$ so rounding gives $54°$.

ii $32'$ is more than $30'$ so rounding gives $30°$.

b **i** $22''$ is less than $30''$ so rounding gives $23°12'$.

ii $41''$ is more than $30''$ so rounding gives $57°35'$.

iii $30''$ is exactly halfway so round up to $84°20'$.

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Decimal degrees and degrees-minutes-seconds

Scientific calculators have a [°'"] or [D°M'S] key for converting between decimal degrees and degrees, minutes, seconds.

EXAMPLE 3

a Change 58°19′ into a decimal.

b Change 45.236° into degrees and minutes.

Solution

a

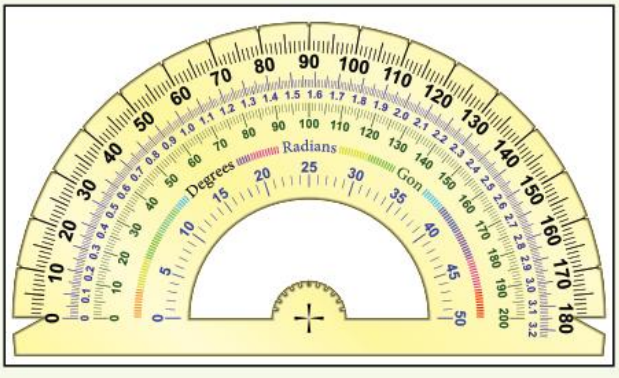
Operation	Casio scientific	Sharp scientific
Make sure the calculator is in degrees.	SHIFT SET UP deg	Press DRG until deg is on the screen
Enter data.	58 [°'"] 19 [°'"] =	58 [D°M'S] 19 [D°M'S]
Change to a decimal.	[°'"]	2ndF [D°M'S]

So 58°19′ = 58.31666 … ≈ 58.32

b

Operation	Casio scientific	Sharp scientific
Enter data.	45.236 =	45.236 =
Change to degrees and minutes.	[°'"]	2ndF [D°M'S]

So 45.236° = 45°14′9.6″ ≈ 45°14′



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EXAMPLE 4

a Find cos 58°19′ correct to 3 decimal places.

b If tan θ = 0.348, find θ in degrees and minutes.

Solution

a

Operation	Casio scientific	Sharp scientific
Enter data.	cos 58 ° ′ ″ 19 ° ′ ″ =	cos 58 D°M′S 19 D°M′S =

So cos 58°19′ = 0.52522 ... ≈ 0.525.

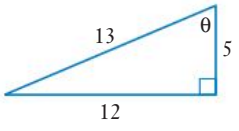
b To find the angle given the ratio, use the inverse key ($\tan^{-1}$).

Operation	Casio scientific	Sharp scientific
Enter data.	SHIFT $\tan^{-1}$ 0.348 =	2ndF $\tan^{-1}$ 0.348 =
Change to degrees and minutes.	° ′ ″	2ndF D°M′S

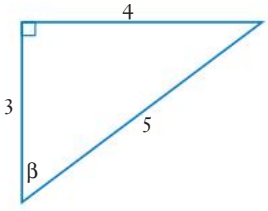
So θ = 19°11′16.43″ ≈ 19°11′.

Exercise 4.01 Trigonometric ratios

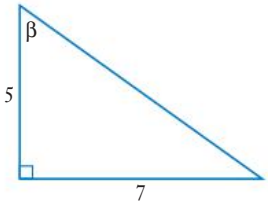
1 Write down the ratios of cos θ, sin θ and tan θ.



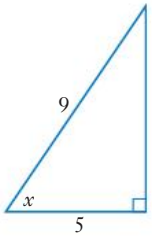
2 Find sin β, tan β and cos β.



3 Find the exact ratios of sin β, tan β and cos β.



4 Find exact values for cos x, tan x and sin x.



5 If $\tan\theta = \frac{4}{3}$, find cos θ and sin θ.

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6 If $\cos\theta = \frac{2}{3}$, find exact values for $\tan\theta$ and $\sin\theta$.

7 If $\sin\theta = \frac{1}{6}$, find the exact ratios of $\cos\theta$ and $\tan\theta$.

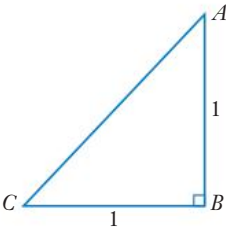
8 If $\cos\theta = 0.7$, find exact values for $\tan\theta$ and $\sin\theta$.

9 $\triangle ABC$ is a right-angled isosceles triangle with $\angle ABC = 90°$ and $AB = BC = 1$.

a Find the exact length of AC.

b Find $\angle BAC$.

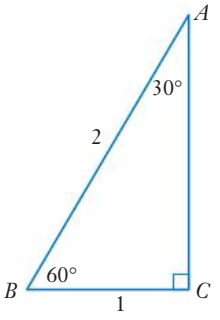
c From the triangle, write down the exact ratios of $\sin 45°$, $\cos 45°$ and $\tan 45°$.



10 **a** Using Pythagoras' theorem, find the exact length of AC.

b Write down the exact ratios of $\sin 30°$, $\cos 30°$ and $\tan 30°$.

c Write down the exact ratios of $\sin 60°$, $\cos 60°$ and $\tan 60°$.



11 Round each angle to the nearest degree.

a 47°13′12″ **b** 81°45′43″ **c** 19°25′34″ **d** 76°37′19″ **e** 52°29′54″

12 Round each angle to the nearest minute.

a 47°13′12″ **b** 81°45′43″ **c** 19°25′34″ **d** 76°37′19″ **e** 52°29′54″

13 Change to a decimal:

a 77°45′ **b** 65°30′ **c** 24°51′ **d** 68°21′ **e** 82°31′

14 Change into degrees and minutes:

a 59.53° **b** 72.231° **c** 85.887° **d** 46.9° **e** 73.213°

15 Find correct to 3 decimal places:

a $\sin 39°25'$ **b** $\cos 45°51'$ **c** $\tan 18°43'$ **d** $\sin 68°06'$ **e** $\tan 54°20'$

16 Find θ in degrees and minutes if:

a $\sin\theta = 0.298$ **b** $\tan\theta = 0.683$ **c** $\cos\theta = 0.827$

d $\tan\theta = 1.056$ **e** $\cos\theta = 0.188$

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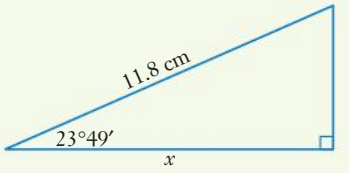
Finding an unknown side

4.02 Finding a side of a right-angled triangle

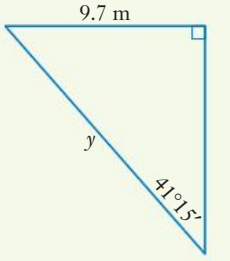
We can use trigonometry to find an unknown side of a triangle.

EXAMPLE 5

a Find the value of x, correct to 1 decimal place.



b Find the value of y, correct to 3 significant figures.



Solution

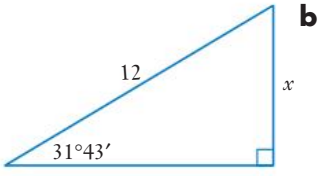
a
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos 23°49' = \frac{x}{11.8}$$
$$11.8 \cos 23°49' = x$$
$$x \approx 10.8 \text{ cm}$$

b
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin 41°15' = \frac{9.7}{y}$$
$$y \sin 41°15' = 9.7$$
$$y = \frac{9.7}{\sin 41°15'}$$
$$\approx 14.7 \text{ m}$$

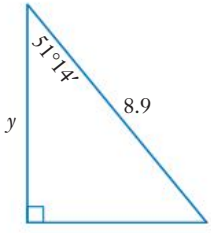
Exercise 4.02 Finding a side of a right-angled triangle

1 Find the values of all pronumerals, correct to 1 decimal place:

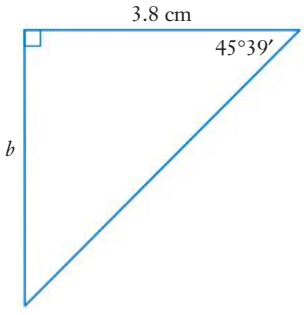
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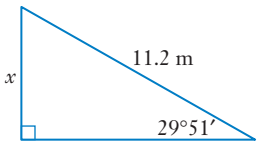


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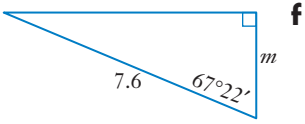


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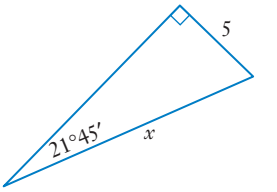
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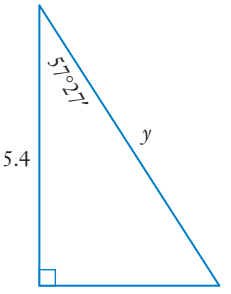
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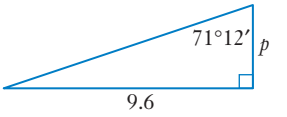
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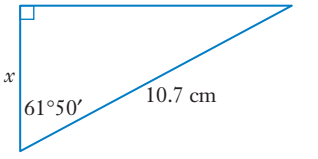
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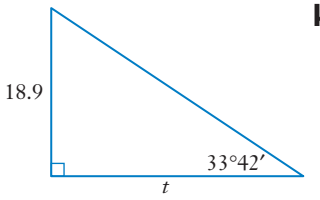
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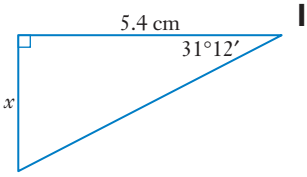
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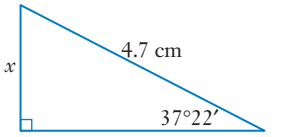
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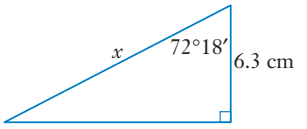
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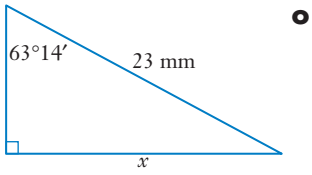
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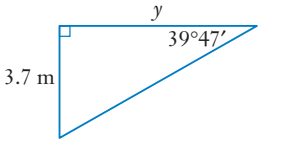
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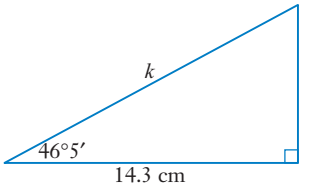
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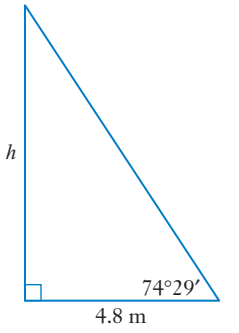
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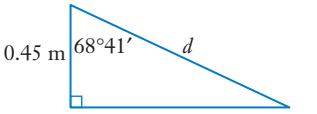
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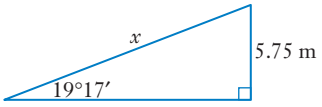


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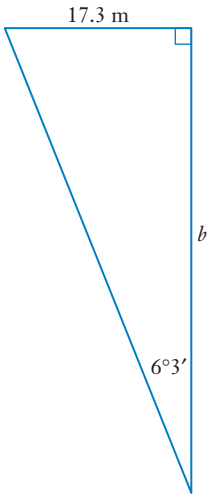


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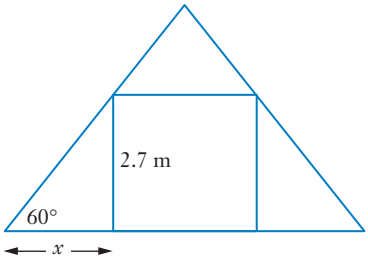
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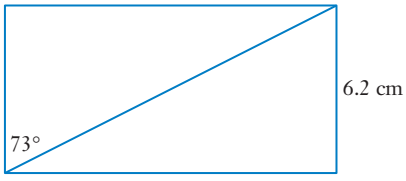
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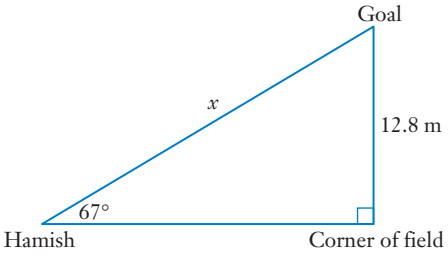
2 A roof is pitched at 60°. A room built inside the roof space is to have a 2.7 m high ceiling. How far in from the side of the roof will the wall for the room go?



3 A diagonal in a rectangle with width 6.2 cm makes an angle of 73° with the vertex as shown. Find the length of the rectangle correct to 1 decimal place.



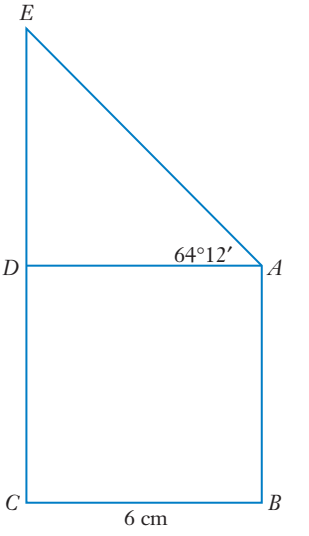
4 Hamish is standing on the sideline of a soccer field, and the goal is at an angle of 67° from his position as shown. The goal is 12.8 m from the corner of the field. How far does he need to kick a ball for it to reach the goal?



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5 Square $ABCD$ with side 6 cm has line CD produced to E as shown so that $\angle EAD = 64°12'$. Evaluate the length, correct to 1 decimal place, of:

a CE **b** AE



6 A right-angled triangle with hypotenuse 14.5 cm long has one interior angle of 43°36′. Find the lengths of the other two sides of the triangle.

7 A right-angled triangle ABC with the right angle at A has $\angle B = 56°44'$ and $AB = 26$ mm. Find the length of the hypotenuse.

8 A triangular fence is made for a garden inside a park. Three holes A, B and C for fence posts are made at the corners so that A and B are 10.2 m apart, AB and CB are perpendicular, and angle CAB is 59°54′. How far apart are A and C?

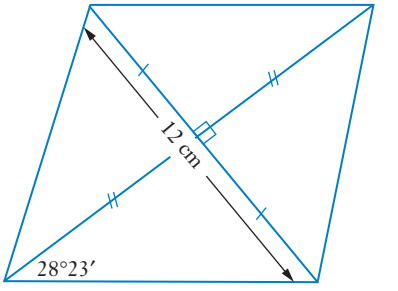
9 Triangle ABC has $\angle BAC = 46°$and $\angle ABC = 54°$. An altitude (perpendicular line) is drawn from C to meet AB at point D. If the altitude is 5.3 cm long, find, correct to 1 decimal place, the length of:

a AC **b** BC **c** AB

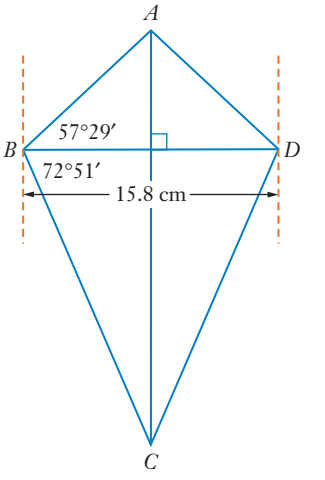
10 A rhombus has one diagonal 12 cm long and the other diagonal makes an angle of 28°23′ with the side of the rhombus.

a Find the length of the side of the rhombus.

b Find the length of the other diagonal.



11 Kite $ABCD$ has diagonal $BD = 15.8$ cm as shown. If $\angle ABD = 57°29'$ and $\angle DBC = 72°51'$, find the length of the other diagonal AC.



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Finding an unknown angle

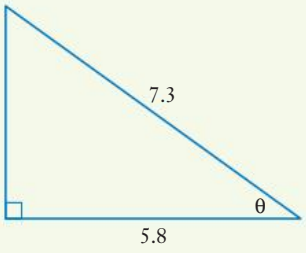
4.03 Finding an angle in a right-angled triangle

We can use trigonometry to find an unknown angle in a triangle.

EXAMPLE 6

Find the value of the pronumeral, in degrees and minutes.

a



b



Solution

a $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$= \dfrac{5.8}{7.3}$

$\therefore \theta = \cos^{-1}\left(\dfrac{5.8}{7.3}\right)$

$\approx 37°23'$

b $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

$= \dfrac{4.9}{2.1}$

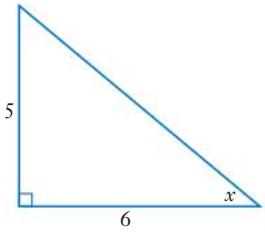
$\therefore \alpha = \tan^{-1}\left(\dfrac{4.9}{2.1}\right)$

$= 66°48'$

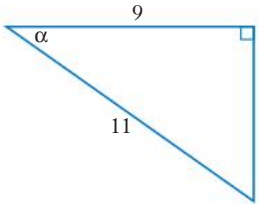
Exercise 4.03 Finding an angle in a right-angled triangle

1 Find the value of each pronumeral, in degrees and minutes:

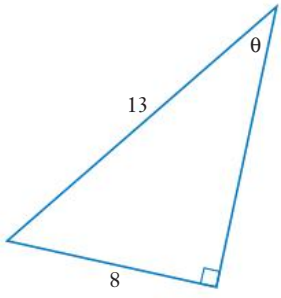
a



b

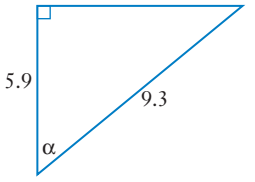


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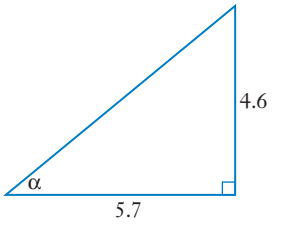


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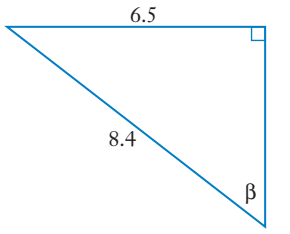
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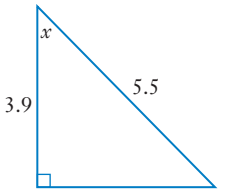
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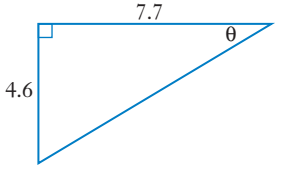
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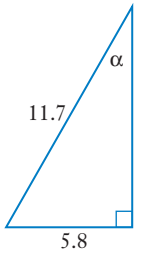
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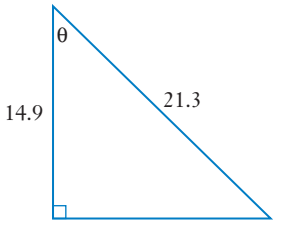
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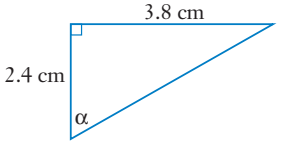
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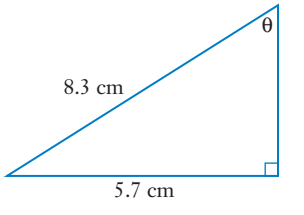
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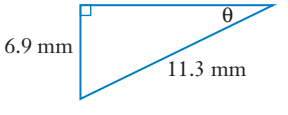
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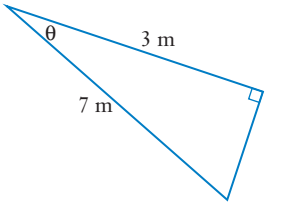
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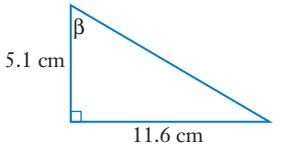
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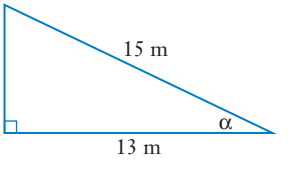
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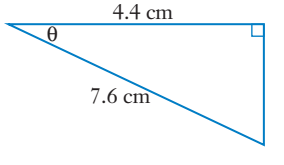
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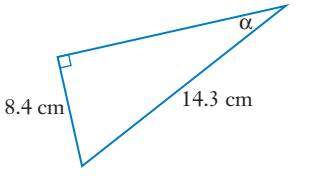
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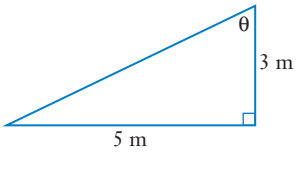
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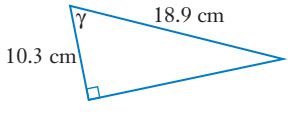
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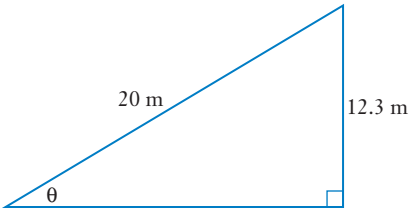


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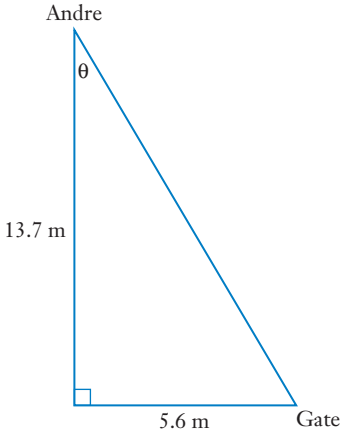


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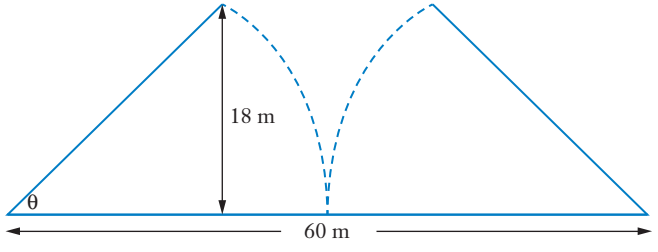
2 A kite is flying at an angle of θ above the ground as shown. If the kite is 12.3 m above the ground and has 20 m of string, find angle θ.



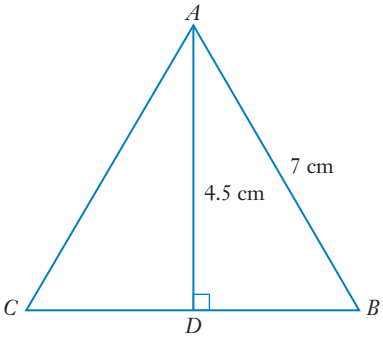
3 A field is 13.7 m wide and Andre is on one side. There is a gate on the opposite side and 5.6 m along from where Andre is. At what angle will he walk to get to the gate?



4 A 60 m long bridge has an opening in the middle and both sides open up to let boats pass underneath. The two parts of the bridge floor rise up to a height of 18 m. Through what angle do they move?



5 An equilateral triangle *ABC* with side 7 cm has an altitude *AD* 4.5 cm long. Evaluate the angle the altitude makes with vertex *A* (∠*DAB*).



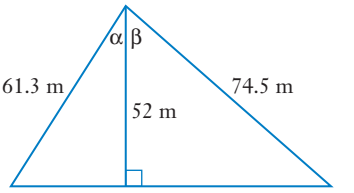
6 Rectangle *ABCD* has dimensions 18 cm × 7 cm. A line *AE* is drawn so that *E* bisects *DC*.

a How long is line *AE*? (Answer to 1 decimal place.)

b Evaluate ∠*DEA*.

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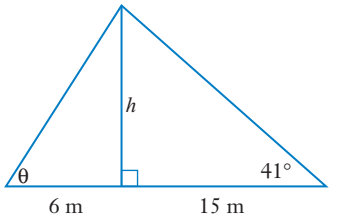
7 A 52 m tall tower has wire stays on either side to minimise wind movement. One stay is 61.3 m long and the other is 74.5 m long, as shown. Find the angles that the tower makes with each stay.



8 The angle up from the ground to the top of a pole is 41° from a position 15 m to one side.

a Find the height h of the pole, to the nearest metre.

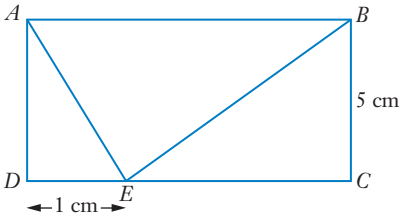
b If Sarah stands 6 m away on the other side, find the angle of elevation θ from Sarah to the top of the pole.



9 Rectangle $ABCD$ has a line BE drawn so that $\angle AEB = 90°$ and $DE = 1$ cm. The width of the rectangle is 5 cm.

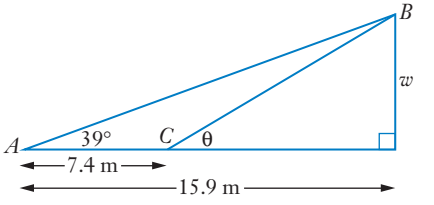
a Find $\angle BEC$.

b Find the length of the rectangle.



10 **a** Frankie is standing at the side of a road at point A, 15.9 m away from an intersection. She is at an angle of 39° from point B on the other side of the road. What is the width w of the road?

b Frankie walks 7.4 m to point C. At what angle is she from point B?



INVESTIGATION

LEANING TOWER OF PISA

The Tower of Pisa was built as a belltower for the cathedral nearby. Work started in 1174, but when it was half-completed the soil underneath one side of it sank. This made the tower lean to one side. Work stopped, and it wasn't until 100 years later that architects found a way of completing the tower. The third and fifth storeys were built close to the vertical to compensate for the lean. Later a vertical top storey was added. The tower is about 55 m tall and 16 m in diameter. It is tilted about 5 m from the vertical at the top, and tilts by an extra 6 mm each year.

Discuss some of the problems with the Leaning Tower of Pisa.

1 Find the angle at which it is tilted from the vertical.

2 Work out how far it will be tilted in 10 years.

3 Use research to find out if the tower will fall over, and if so, when.

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Right-angled trigonometry

Angles of elevation and depression

4.04 Applications of trigonometry

Angle of elevation

The **angle of elevation** can be used to measure the height of tall objects that cannot be measured directly; for example, a tree, cliff, tower or building. Stand outside a tall building and look up to the top of the building. Think about what angle your eyes pass through to look up to the top of the building.

Angle of elevation

The angle of elevation, θ, is the angle measured when looking from the ground up to the top of the object. We assume that the ground is horizontal.



EXAMPLE 7

The angle of elevation of a tree from a point 50 m out from its base is 38°14′.

Find the height of the tree to the nearest metre.

Solution

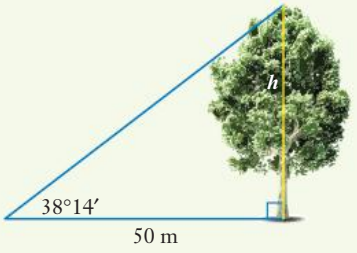
We assume that the tree is vertical.

$$\tan 38°14' = \frac{h}{50}$$

$$50 \tan 38°14' = h$$

$$39 \approx h$$

So the tree is 39 m tall.



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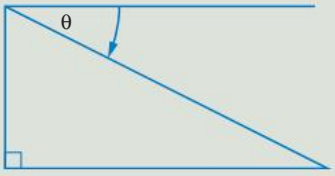
Angle of depression

The **angle of depression** is the angle formed when looking down from a high place to an object below. Find a tall building, hill or other high place, and look down to something below. Through what angle do your eyes pass as you look down?

Angle of depression

Angle of depression

The angle of depression, θ, is the angle measured when looking down from the horizontal to an object below.



Angle of depression

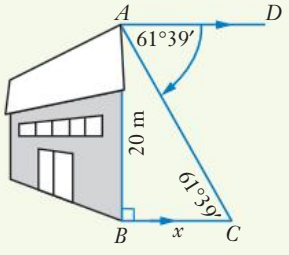
EXAMPLE 8

a The angle of depression from the top of a 20 m building to Gina below is 61°39′. How far is Gina from the building, to 1 decimal place?

b A bird sitting on top of an 8 m tall tree looks down at a possum 3.5 m out from the base of the tree. Find the angle of depression to the nearest minute.

Solution

a



$\angle DAC = \angle ACB = 61°39'$ (alternate angles, $AD \parallel BC$)

$$\tan 61°39' = \frac{20}{x}$$

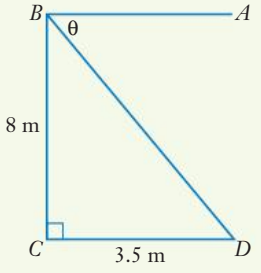
$$x \tan 61°39' = 20$$

$$x = \frac{20}{\tan 61°39'}$$

$$\approx 10.8$$

So Gina is 10.8 m from the building.

b



The angle of depression is θ.

$\angle ABD = \angle BDC = \theta$ (alternate angles, $AB \parallel DC$).

$$\tan \theta = \frac{8}{3.5}$$

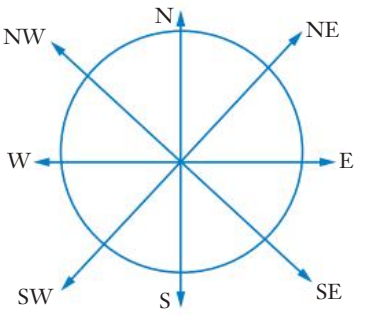
$$\therefore \theta \approx 66°22'$$

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Compass bearings

Compass bearings

A **bearing** is a direction according to a compass. The main points on a compass are north (N), south (S), east (E) and west (W). Halfway between these are NE, NW, SE, SW. We write **compass bearings** with north or south first, followed by an angle and then east or west.



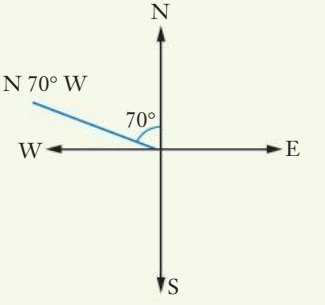
EXAMPLE 9

a Draw a compass bearing of N 70° W.

b Eli walks from his house on a bearing of S 25° E. If he walks 5.7 km, how far south is he from his house?

Solution

a Start at north and turn 70° towards west.



b Start at south and turn 25° towards east.

The hypotenuse is 5.7 and we want to measure the adjacent side (x).

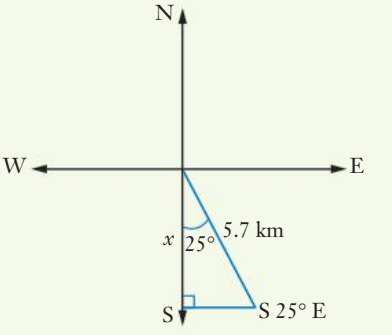
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 25° = \frac{x}{5.7}$$

$$5.7\cos 25° = x$$

$$x \approx 5.2$$

So Eli is 5.2 km south of his house.



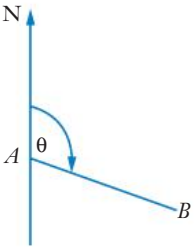
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True bearings

True bearings measure angles clockwise from north.

We say B is on a bearing of θ from A.

A true bearing uses 3 digits from 000° to 360°.




Bearings


True bearings



A page of bearings



Bearings match-up



Elevations and bearings

EXAMPLE 10

a X is on a bearing of 030° from Y. Sketch this diagram.

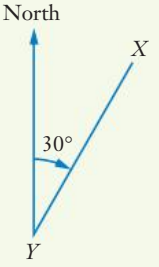
b A house is on a bearing of 305° from a school. What is the bearing of the school from the house?

c A plane leaves Sydney and flies 100 km due east, then 125 km due north. Find the bearing of the plane from Sydney, to the nearest degree.

d A ship sails on a bearing of 140° from Sydney for 250 km. How far east of Sydney is the ship now, to the nearest km?

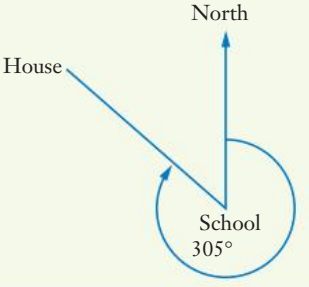
Solution

a



Note: Bearings use 3 digits so a bearing of 030° is a 30° angle.

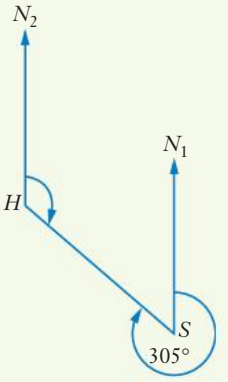
b



The diagram below shows the bearing of the house from the school.

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To find the bearing of the school from the house, draw in north from the house and use geometry to find the bearing as follows:



The bearing of the school from the house is $\angle N_2HS$.

$\angle N_1SH = 360° - 305°$ (angles in a revolution)

$= 55°$

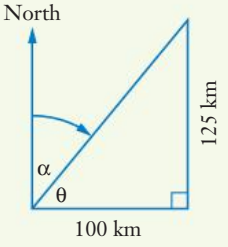
With parallel lines, the sum of cointerior angles is 180°.

$\angle N_2HS = 180° - 55°$

$= 125°$

So the bearing of the school from the house is 125°.

c



$$\tan\theta = \frac{125}{100}$$

$\theta \approx 51°$

$\alpha = 90° - 51° = 39°$

So the bearing of the plane from Sydney is 039°.

d

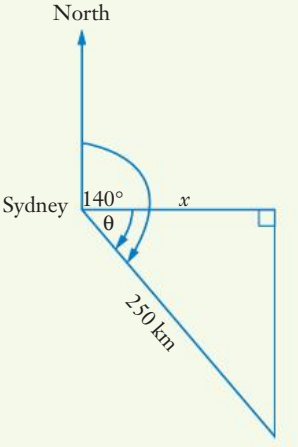
$\theta = 140° - 90° = 50°$

$$\cos 50° = \frac{x}{250}$$

$250\cos 50° = x$

$x \approx 161$

So the ship is 161 km east of Sydney.



Note: A navigator on a ship uses a sextant to measure angles. A clinometer measures angles of elevation and depression.

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Exercise 4.04 Applications of trigonometry

1 Draw a diagram to show the bearing in each question.

a N 50° E **b** S 60° W **c** S 80° E **d** N 40° W

e A boat is on a bearing of 100° from a beach house.

f Jamie is on a bearing of 320° from a campsite.

g A seagull is on a bearing of 200° from a jetty.

h Alistair is on a bearing of 050° from the bus stop.

i A plane is on a bearing of 285° from Broken Hill.

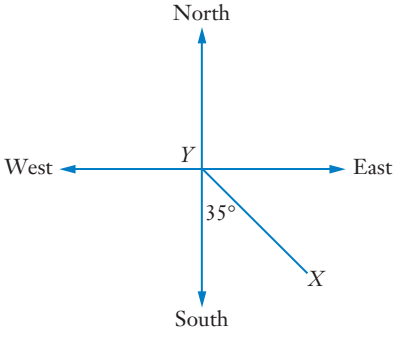
j A farmhouse is on a bearing of 012° from a dam.

k Mohammed is on a bearing of 160° from his house.

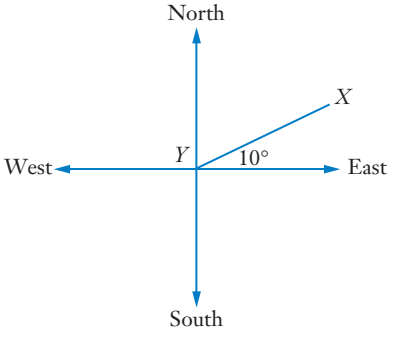
2 Find the bearing of X from Y in each question using:

i compass bearings **ii** true bearings.

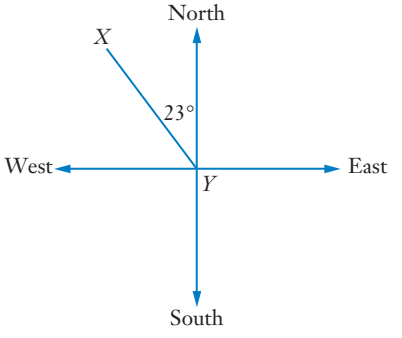
a



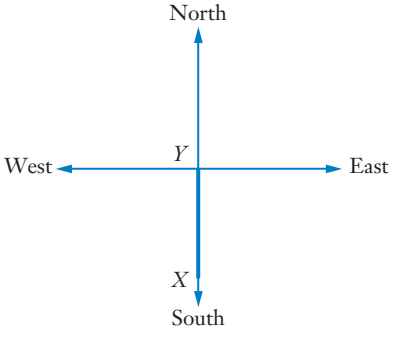
b



c



d



3 Jack is on a bearing of 260° from Jill. What is Jill's bearing from Jack?

4 A tower is on a bearing of 030° from a house. What is the bearing of the house from the tower?

5 Tamworth is on a bearing of 340° from Newcastle. What is the bearing of Newcastle from Tamworth?

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6 The angle of elevation from a point 11.5 m away from the base of a tree up to the top of the tree is 42°12′. Find the height of the tree to one decimal place.

7 Geoff stands 25.8 m away from the base of a tower and measures the angle of elevation as 39°20′. Find the height of the tower to the nearest metre.

8 A wire is suspended from the top of a 100 m tall bridge tower down to the bridge at an angle of elevation of 52°. How long is the wire, to 1 decimal place?

9 A cat crouches at the top of a 4.2 m high cliff and looks down at a mouse 1.3 m out from the foot (base) of the cliff. What is the angle of depression, to the nearest minute?

10 A plane leaves Melbourne and flies on a bearing of 065° for 2500 km.

- **a** How far north of Melbourne is the plane?
- **b** How far east of Melbourne is it?
- **c** What is the bearing of Melbourne from the plane?

11 The angle of elevation of a tower is 39°44′ when measured at a point 100 m from its base. Find the height of the tower, to 1 decimal place.

12 Kim leaves her house and walks for 2 km on a bearing of 155°. How far south is Kim from her house now, to 1 decimal place?

13 The angle of depression from the top of an 8 m tree down to a rabbit is 43°52′. If an eagle is perched in the top of the tree, how far does it need to fly to reach the rabbit, to the nearest metre?

14 Sanjay rides a motorbike through his property, starting at his house. If he rides south for 1.3 km, then rides west for 2.4 km, what is his bearing from the house, to the nearest degree?

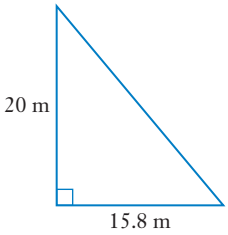
15 A plane flies north from Sydney for 560 km, then turns and flies east for 390 km. What is its bearing from Sydney, to the nearest degree?

16 Find the height of a pole, correct to 1 decimal place, if a 10 m rope tied to it at the top and stretched out straight to reach the ground makes an angle of elevation of 67°13′.

17 The angle of depression from the top of a cliff down to a boat 100 m out from the foot of the cliff is 59°42′. How high is the cliff, to the nearest metre?

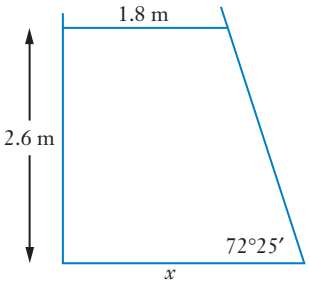
18 A group of students are bushwalking. They walk north from their camp for 7.5 km, then walk west until their bearing from camp is 320°. How far are they from camp, to 1 decimal place?

19 A 20 m tall tower casts a shadow 15.8 m long at a certain time of day. What is the angle of elevation from the edge of the shadow up to the top of the tower at this time?



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20 A flat verandah roof 1.8 m deep is 2.6 m up from the ground. At a certain time of day, the sun makes an angle of elevation of 72°25′. How much shade is provided on the ground by the verandah roof at that time, to 1 decimal place?



21 Find the angle of elevation of a 15.9 m cliff from a point 100 m out from its base.

22 A plane leaves Sydney and flies for 2000 km on a bearing of 195°. How far due south of Sydney is it?

23 The angle of depression from the top of a 15 m tree down to a pond is 25°41′. If a bird is perched in the top of the tree, how far does it need to fly to reach the pond, to the nearest metre?

24 Robin starts at her house, walks south for 2.7 km then walks east for 1.6 km. What is her bearing from the house, to the nearest degree?

25 The angle of depression from the top of a tower down to a car 250 m out from the foot of the tower is 38°19′. How high is the tower, to the nearest metre?

26 A blimp flies south for 3.6 km then turns and flies east until it is on a bearing of 127° from where it started. How far east does it fly?

27 A 24 m wire is attached to the top of a pole and runs down to the ground where the angle of elevation is 22°32′. Find the height of the pole.

28 A train depot has train tracks running north for 7.8 km where they meet another set of tracks going east for 5.8 km into a station. What is the bearing of the depot from the station, to the nearest degree?

29 Jessica leaves home and walks for 4.7 km on a bearing of 075°. She then turns and walks for 2.9 km on a bearing of 115° and she is then due east of her home.

a What is the furthest north that Jessica walks?

b How far is she from home?

30 Builder Jo stands 4.5 m out from the foot of a building and looks up to the top of the building where the angle of elevation is 71°. Builder Ben stands at the top of the building looking down at his wheelbarrow that is 10.8 m out from the foot of the building on the opposite side from where Jo is standing.

a Find the height of the building.

b Find the angle of depression from Ben down to his wheelbarrow.

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WS

Sine rule problems

4.05 The sine rule

The sin, cos and tan of angles greater than 90° give some interesting results. You will explore these in Chapter 9, *Trigonometric functions*. For now, we just need to know about **obtuse angles** (between 90° and 180°) so we can solve problems involving obtuse-angled triangles.

INVESTIGATION

LARGER ANGLES

1 Use your calculator to find the sin, cos and tan of some angles greater than 90°. What do you notice?

2 Can you see a pattern for angles between 90° and 180° for:

i sin? **ii** cos? **iii** tan?

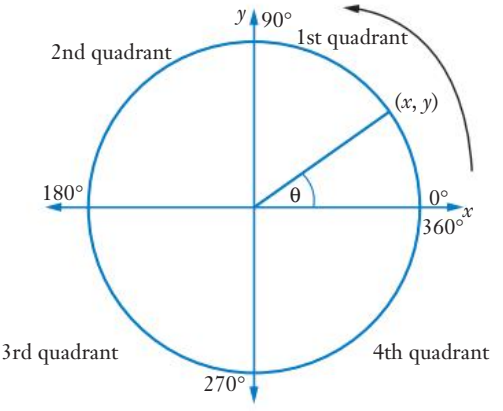
We can use a circle to show angles, starting with 0° at the x-axis and turning anticlockwise to show other angles. We divide the number plane into 4 **quadrants** as shown:

1st quadrant: 0° to 90°

2nd quadrant: 90° to 180°

3rd quadrant: 180° to 270°

4th quadrant: 270° to 360°



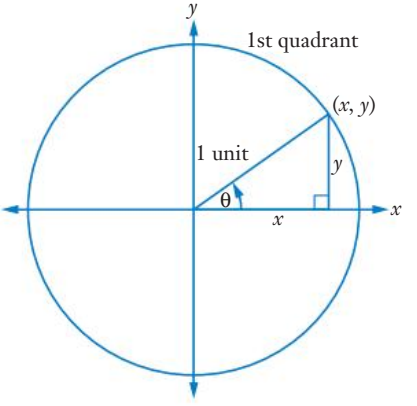
To make it easier to explore these results, we use a **unit circle** with radius 1.

We can find the trigonometric ratios for angle θ.

$$\sin\theta = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$



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In the 2nd quadrant, notice that x values are negative and y values are positive.

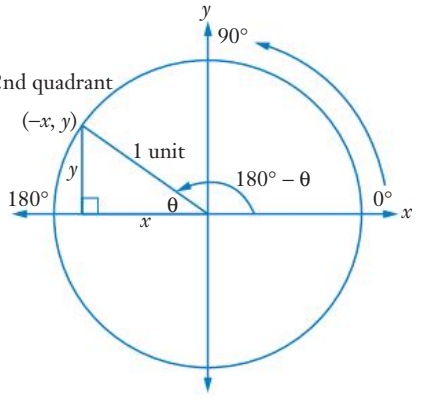
So the point in the 2nd quadrant will be $(-x, y)$.

Since $\sin\theta = y$, sin will be **positive** in the 2nd quadrant.

Since $\cos\theta = -x$, cos will **negative** in the 2nd quadrant.

Since $\tan\theta = \frac{y}{-x}$, tan will be negative in the 2nd quadrant (positive divided by negative).

To have an angle of θ in the triangle, the obtuse angle in the 2nd quadrant is $180° - \theta$.



Trigonometric ratios of obtuse angles

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\tan(180° - \theta) = -\tan\theta$$

EXAMPLE 11

a If $\cos 80° = 0.174$, evaluate $\cos 100°$.

b If $\sin 55° = 0.819$, find the value of $\sin 125°$.

Solution

a
$$\cos(180° - \theta) = -\cos\theta$$
So $\cos(180° - 80°) = -\cos 80°$
$$\cos 100° = -\cos 80°$$
$$= -0.174$$

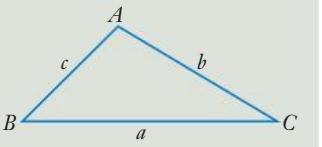
b
$$\sin(180° - \theta) = \sin\theta$$
So $\sin(180° - 55°) = \sin 55°$
$$\sin 125° = \sin 55°$$
$$= 0.819$$

You can check that this is true by finding both ratios on the calculator.

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Naming the sides and angles of a triangle

Side a is opposite angle A, side b is opposite angle B and side c is opposite angle C.



The shortest side is opposite the smallest angle.

The longest side is opposite the largest angle.

The sine rule

The **sine rule** is used to find unknown sides and angles in non-right-angled triangles.

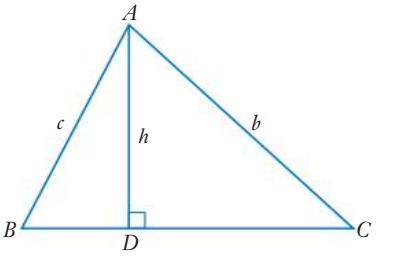
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof

In $\triangle ABC$ draw perpendicular AD and call it h.



From $\triangle ABD$,

$$\sin B = \frac{h}{c}$$

$$\therefore h = c \sin B \qquad [1]$$

From $\triangle ACD$,

$$\sin C = \frac{h}{b}$$

$$\therefore h = b \sin C \qquad [2]$$

From [1] and [2],

$$c \sin B = b \sin C$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

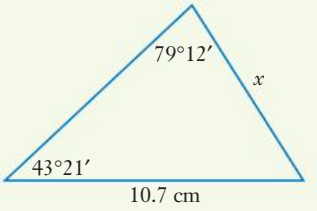
Similarly, by drawing a perpendicular from C it can be proved that

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

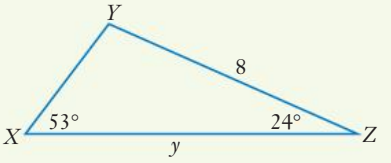
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EXAMPLE 12

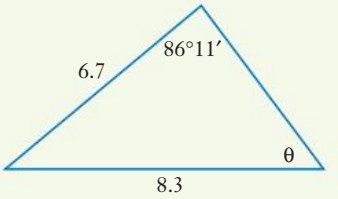
a Find the value of x, correct to 1 decimal place.



b Find the value of y, to the nearest whole number.



c Find the value of θ, in degrees and minutes, given θ is acute.



Solution

a Name the sides a and b, and opposite angles A and B.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 43°21'} = \frac{10.7}{\sin 79°12'}$$

$$x = \frac{10.7 \sin 43°21'}{\sin 79°12'}$$

$$\approx 7.5 \text{ cm}$$

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b First we need to find angle Y since it is opposite side y.

$\angle Y = 180° - (53° + 24°) = 103°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{y}{\sin 103°} = \frac{8}{\sin 53°}$$

$$y = \frac{8 \sin 103°}{\sin 53°}$$

$$\approx 10$$

c

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{6.7} = \frac{\sin 86°11'}{8.3}$$

$$\sin \theta = \frac{6.7 \sin 86°11'}{8.3}$$

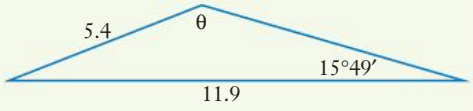
$$= 0.8054...$$

$\theta = \sin^{-1}(0.8054...)$ SHIFT sin ANS

$\approx 53°39'$

EXAMPLE 13

Find the value of θ, in degrees and minutes, given θ is obtuse.



Solution

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{11.9} = \frac{\sin 15°49'}{5.4}$$

$$\sin \theta = \frac{11.9 \sin 15°49'}{5.4}$$

$$= 0.6006...$$

$\theta = \sin^{-1}(0.6006...)$

$\approx 36°55'$

But θ is obtuse.

$\therefore \theta = 180° - 36°55'$

$= 143°05'$

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Ambiguous case

When using the sine rule to find an unknown angle, there are 2 possible solutions: one acute and one obtuse. This is called the **ambiguous case** of the sine rule.

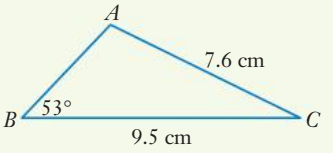
EXAMPLE 14

a Triangle ABC has $\angle B = 53°$, $AC = 7.6$ cm and $BC = 9.5$ cm. Find $\angle A$ to the nearest degree.

b In triangle XYZ, $\angle Y = 118°35'$, $YZ = 12.5$ mm and $XZ = 14.3$ mm. Find $\angle X$ in degrees and minutes.

Solution

a Draw a diagram.



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{9.5} = \frac{\sin 53°}{7.6}$$

$$\sin A = \frac{9.5 \sin 53°}{7.6}$$

$$= 0.998$$

$$A = \sin^{-1}(0.998)$$

$$\approx 87°$$

But $\angle A$ could be obtuse.

So $\angle A = 180° - 87°$

$$= 93°$$

Checking angle sum of a triangle:

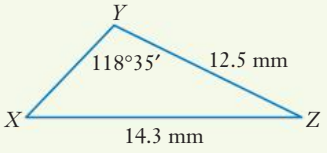
$53° + 87° = 140° < 180°$, so $87°$ is a possible answer.

$53° + 93° = 146° < 180°$, so $93°$ is a possible answer.

So $\angle A = 87°$ or $93°$.

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b



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin X}{12.5} = \frac{\sin 118°35'}{14.3}$$

$$\sin X = \frac{12.5 \sin 118°35'}{14.3}$$

$$= 0.768$$

$$X = \sin^{-1}(0.768)$$

$$\approx 50°8'$$

But $\angle X$ could be obtuse.

So $\angle X = 180° - 50°8'$

$$= 129°52'$$

Checking angle sum of a triangle:

$118°35' + 50°8' = 168°43' < 180°$, so a possible answer.

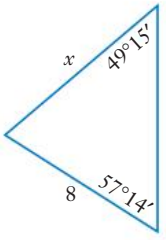
$118°35' + 129°52' = 248°27' > 180°$, so an impossible answer.

So $\angle X = 50°8'$.

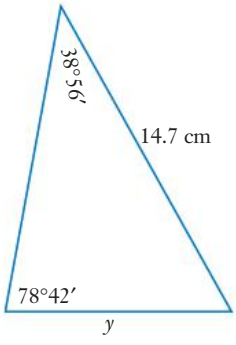
Exercise 4.05 The sine rule

1 Evaluate each pronumeral, correct to 1 decimal place:

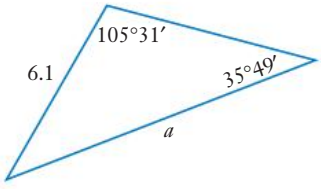
a



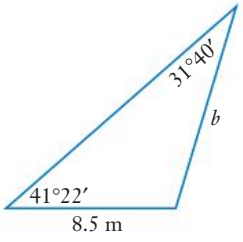
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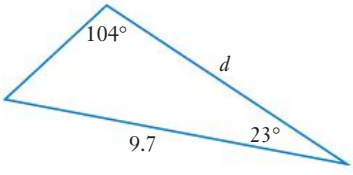
c



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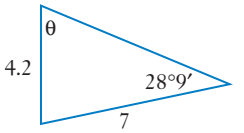
e



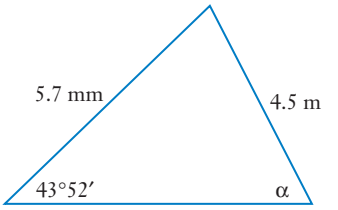
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2 Find the value of all pronumerals, in degrees and minutes (triangles not to scale):

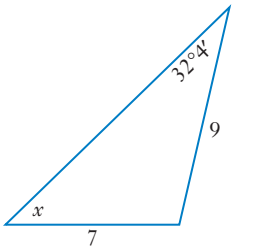
a



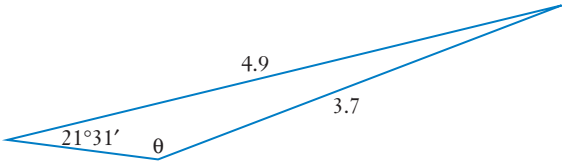
b



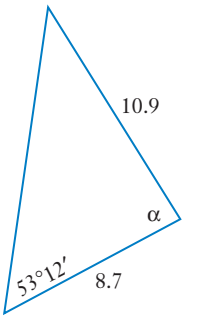
c



d



e



3 Triangle ABC has an obtuse angle at A. Evaluate this angle to the nearest minute if $AB = 3.2$ cm, $BC = 4.6$ cm and $\angle ACB = 33°47'$.

4 Triangle EFG has $\angle FEG = 48°$, $\angle EGF = 32°$ and $FG = 18.9$ mm. Find the length of:

a the shortest side **b** the longest side.

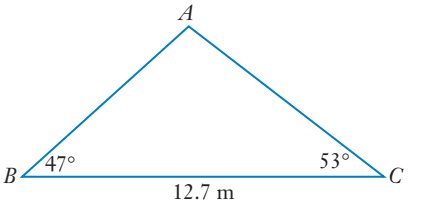
5 Triangle XYZ has $\angle XYZ = 51°$, $\angle YXZ = 86°$ and $XZ = 2.1$ m. Find the length of:

a the shortest side **b** the longest side.

6 Triangle XYZ has $XY = 5.4$ cm, $\angle ZXY = 48°$ and $\angle XZY = 63°$. Find the length of XZ.

7 Triangle ABC has $BC = 12.7$ m, $\angle ABC = 47°$and $\angle ACB = 53°$ as shown. Find the length of:

a AB **b** AC



8 Triangle PQR has sides $PQ = 15$ mm, $QR = 14.7$ mm and $\angle PRQ = 62°29'$. Find to the nearest minute:

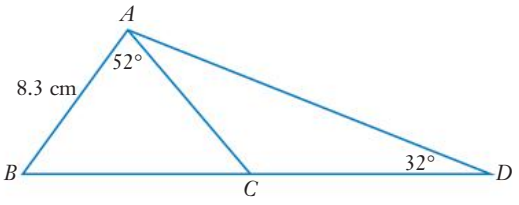
a $\angle QPR$ **b** $\angle PQR$

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9 Triangle ABC is isosceles with $AB = AC$. BC is produced to D as shown.

If $AB = 8.3$ cm, $\angle BAC = 52°$ and $\angle ADC = 32°$, find the length of:

a AD **b** BD



10 Triangle ABC is equilateral with side 63 mm. A line is drawn from A to BC where it meets BC at D and $\angle DAB = 26°15'$. Find the length of:

a AD **b** DC

11 In triangle ABC, find $\angle B$ to the nearest degree given:

a $\angle C = 67°, AB = 7.2, AC = 7.5$

b $\angle A = 92°, BC = 10.7, AC = 8.4$

c $\angle A = 29°, BC = 4.9, AC = 8.3$

WS Cosine rule problems

WS The sine and cosine rules

WS Finding an unknown side

WS Finding an unknown angle

4.06 The cosine rule

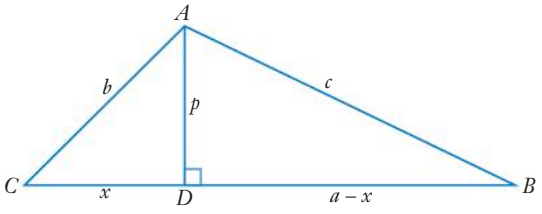
The cosine rule

The **cosine rule** is also used to find unknown sides and angles in non-right-angled triangles.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof

In triangle ABC, draw perpendicular AD with length p and let $CD = x$.



Since $BC = a$, $BD = a - x$

From triangle ACD:

$$b^2 = x^2 + p^2 \qquad [1]$$

$$\cos C = \frac{x}{b}$$

$$\therefore b\cos C = x \qquad [2]$$

From triangle DAB:

$$c^2 = p^2 + (a - x)^2$$

$$= p^2 + a^2 - 2ax + x^2$$

$$= p^2 + x^2 + a^2 - 2ax \qquad [3]$$

Substitute [1] into [3]:

$$c^2 = b^2 + a^2 - 2ax \qquad [4]$$

Substituting [2] into [4]:

$$c^2 = b^2 + a^2 - 2ab\cos C$$

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DID YOU KNOW?

The cosine rule for right-angled triangles

Pythagoras' theorem is a special case of the cosine rule when the triangle is right-angled.

$c^2 = a^2 + b^2 - 2ab \cos C$

When $C = 90°$

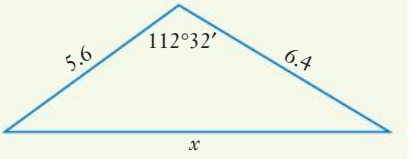
$c^2 = a^2 + b^2 - 2ab \cos 90°$

$= a^2 + b^2 - 2ab \times 0$

$= a^2 + b^2$

EXAMPLE 15

Find the value of x, correct to the nearest whole number.



Solution

$c^2 = a^2 + b^2 - 2ab \cos C$

$x^2 = 5.6^2 + 6.4^2 - 2(5.6)(6.4) \cos 112°32'$

$= 99.7892 \ldots$

$x = \sqrt{99.7892 \ldots}$

$= 9.9894 \ldots$

≈ 10

When using the cosine rule to find an unknown angle, it may be more convenient to change the subject of this formula to $\cos C$.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$2ab \cos C = a^2 + b^2 - c^2$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The cosine rule for angles

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

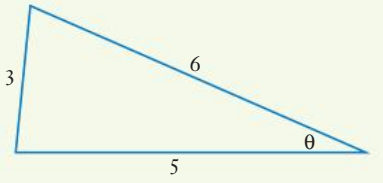


The cosine rule for angles

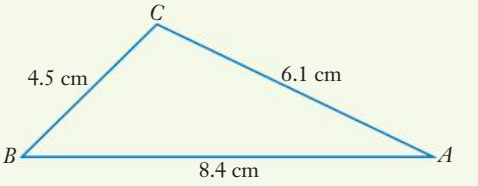
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EXAMPLE 16

a Find θ, in degrees and minutes.



b Evaluate $\angle BCA$ in degrees and minutes.



Solution

Naming sides and opposite angles, side c is opposite the unknown angle C.

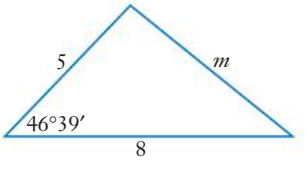
a
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta = \frac{5^2+6^2-3^2}{2(5)(6)}$$
$$= \frac{52}{60}$$
$$\theta = \cos^{-1}\left(\frac{52}{60}\right)$$
$$\approx 29°56'$$

b
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos\angle BCA = \frac{4.5^2+6.1^2-8.4^2}{2(4.5)(6.1)}$$
$$= -0.2386\ldots$$
$$\angle BCA = \cos^{-1}(-0.2386\ldots)$$
$$\approx 103°48'$$

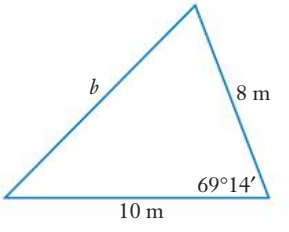
Exercise 4.06 The cosine rule

1 Find the value of each pronumeral, correct to 1 decimal place:

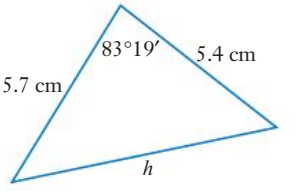
a



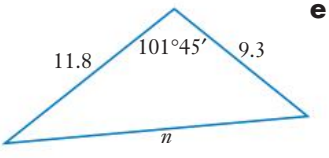
b



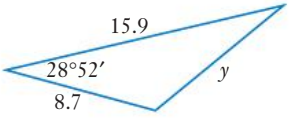
c



d



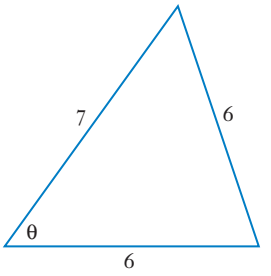
e



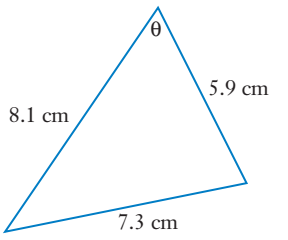
ISBN 9780170413152

2 Evaluate each pronumeral, correct to the nearest minute:

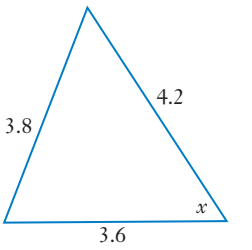
a



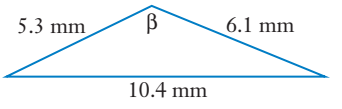
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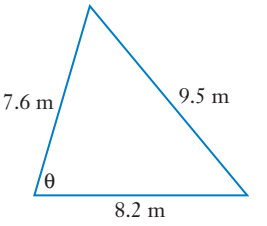
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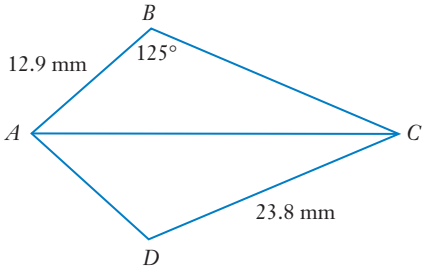
d



e



3 Kite $ABCD$ has $AB = 12.9$ mm, $CD = 23.8$ mm and $\angle ABC = 125°$ as shown. Find the length of diagonal AC.



4 Parallelogram $ABCD$ has sides 11 cm and 5 cm, and one interior angle 79°25′. Find the length of the diagonals.

5 Quadrilateral $ABCD$ has sides $AB = 12$ cm, $BC = 10.4$ cm, $CD = 8.4$ cm and $AD = 9.7$ cm with $\angle ABC = 63°57′$. Find:

a the length of diagonal AC **b** $\angle DAC$ **c** $\angle ADC$

6 Triangle XYZ is isosceles with $XY = XZ = 7.3$ cm and $YZ = 5.9$ cm. Find the value of all angles, to the nearest minute.

7 Quadrilateral $MNOP$ has $MP = 12$ mm, $NO = 12.7$ mm, $MN = 8.9$ mm, $OP = 15.6$ mm and $\angle NMP = 119°15′$. Find:

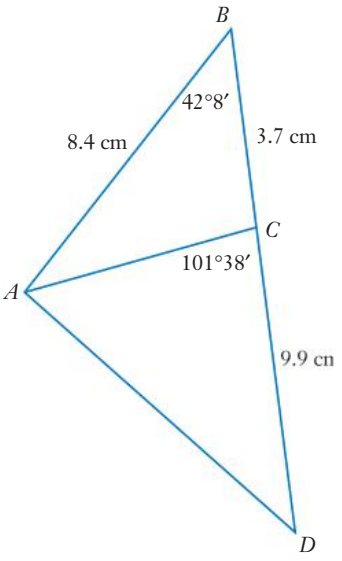
a the length of diagonal NP **b** $\angle NOP$

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8 Given the figure, find the length of:

a AC

b AD



9 In a regular pentagon $ABCDE$ with sides 8 cm, find the length of diagonal AD.

10 A regular hexagon $ABCDEF$ has sides 5.5 cm. Find:

a the length of AD **b** $\angle ADF$

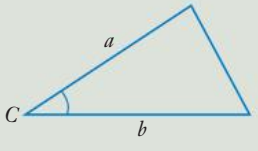
WS
Areas of triangles

4.07 Area of a triangle

Trigonometry allows us to find the area of a triangle if we know 2 sides and their included angle.

Sine formula for the area of a triangle

$$A = \frac{1}{2}ab\sin C$$



Proof

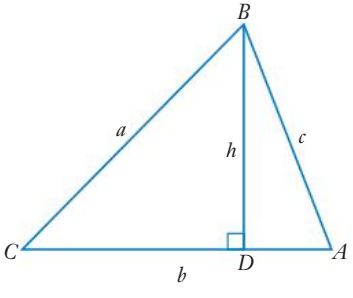
From $\triangle BCD$,

$\sin C = \dfrac{h}{a}$

$\therefore h = a\sin C$

$A = \dfrac{1}{2}bh$

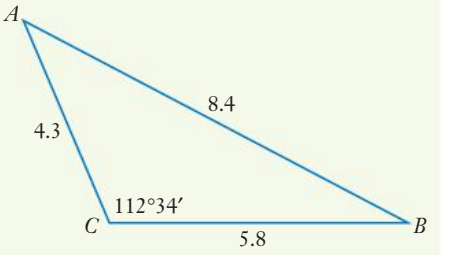
$= \dfrac{1}{2}ba\sin C$



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EXAMPLE 17

Find the area of $\triangle ABC$ correct to 2 decimal places.



Solution

$$A = \frac{1}{2}ab \sin C$$

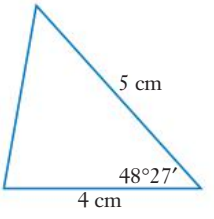
$$= \frac{1}{2}(4.3)(5.8) \sin 112°34'$$

$$\approx 11.52 \text{ units}^2$$

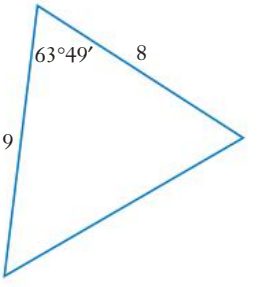
Exercise 4.07 Area of a triangle

1 Find the area of each triangle correct to 1 decimal place.

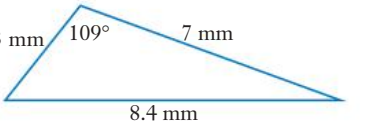
a



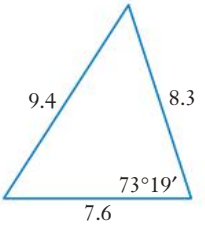
b



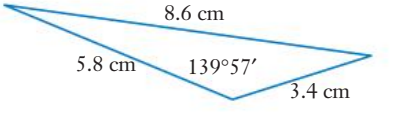
c



d

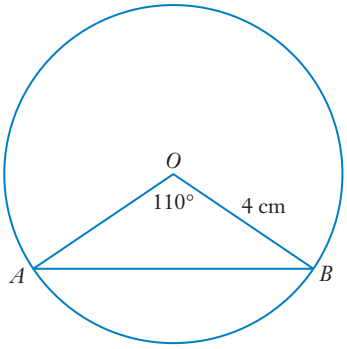


e



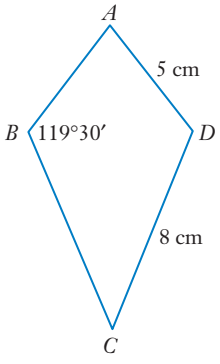
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2 Find the area of $\triangle OAB$ correct to 1 decimal place (O is the centre of the circle):

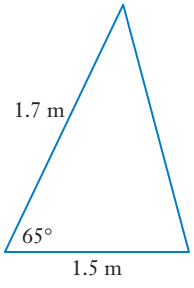


3 Find the area of a parallelogram with sides 3.5 cm and 4.8 cm and with one of its internal angles 67°13′, correct to 1 decimal place.

4 Find the area of kite $ABCD$, correct to 3 significant figures:



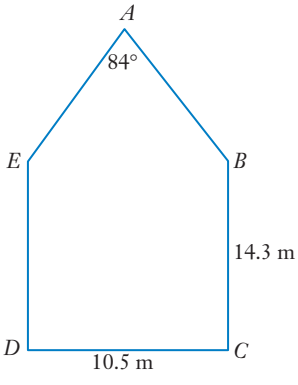
5 Find the area of this sail, correct to 1 decimal place:



6 This pentagon is made from a rectangle and isosceles triangle with $AE = AB$, as shown. Find:

a the length of AE

b the area of the figure.



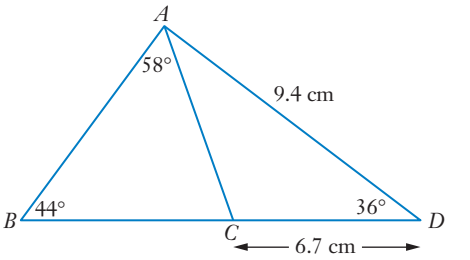
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7 For this figure, find:

a the length of AC

b the area of triangle ACD

c the area of triangle ABC.



8 Find the exact area of an equilateral triangle with sides 5 cm.

4.08 Mixed problems

The sine and cosine rules

Use the **sine rule** to find:

- a side, given one side and 2 angles
- an angle, given 2 sides and one angle

Use the **cosine rule** to find:

- a side, given 2 sides and one angle
- an angle, given 3 sides

EXAMPLE 18

a The angle of elevation of a tower from point A is 72°. From point B, 50 m further away from the tower than A, the angle of elevation is 47°.

i Find the exact length of AT, the distance from A to the top of the tower.

ii Hence, or otherwise, find the height h of the tower to 1 decimal place.

b A ship sails from Sydney for 200 km on a bearing of 040° then sails on a bearing of 157° for 345 km.

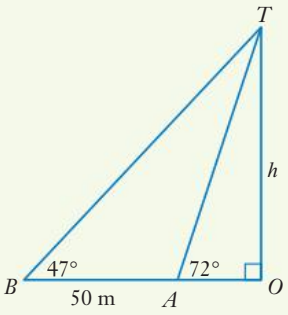
i How far from Sydney is the ship, to the nearest km?

ii What is the bearing of the ship from Sydney, to the nearest degree?

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Solution

a



i $\angle BAT = 180° - 72° = 108°$ (straight angle)

$\angle BTA = 180° - (47° + 108°)$ (angle sum of $\triangle BTA$)

$= 25°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AT}{\sin 47°} = \frac{50}{\sin 25°}$$

$$\therefore AT = \frac{50 \sin 47°}{\sin 25°}$$

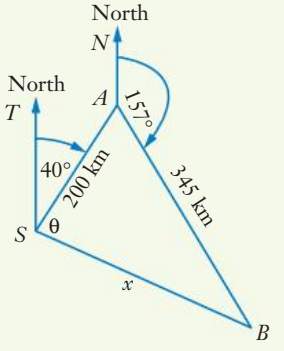
ii $$\sin 72° = \frac{h}{AT}$$

$$\therefore h = AT \sin 72°$$

$$= \frac{50 \sin 47°}{\sin 25°} \times \sin 72°$$

≈ 82.3 m

b



i $\angle SAN = 180° - 40° = 140°$ (cointerior angles)

$\therefore \angle SAB = 360° - (140° + 157°)$ (angle of revolution)

$= 63°$

$c^2 = a^2 + b^2 - 2ab \cos C$

$x^2 = 200^2 + 345^2 - 2(200)(345) \cos 63°$

$= 96\,374.3110\ldots$

$x = \sqrt{96374.3110\ldots}$

$= 310.4421...$

≈ 310

So the ship is 310 km from Sydney.

ii $$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{345} = \frac{\sin 63°}{310.4421...}$$

$$\therefore \sin \theta = \frac{345 \sin 63°}{310.4421...}$$

$= 0.9901\ldots$

$\theta \approx 82°$

The bearing from Sydney $= 40° + 82° = 122°$.

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We can also use trigonometry to solve 3-dimensional problems.

WS
3D trigonometry

EXAMPLE 19

a From point X, 25 m due south of the base of a tower, the angle of elevation is 47°. Point Y is 15 m due east of the tower. Find:

i the height, h, of the tower, correct to 1 decimal place

ii the angle of elevation, θ, of the tower from point Y.

b A cone has a base diameter of 18 cm and a slant height of 15 cm. Find the vertical angle at the top of the cone.

Solution

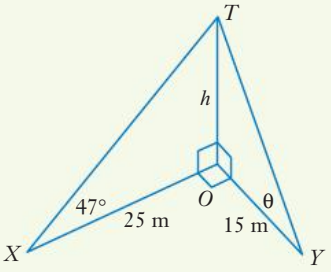
a i From $\triangle XTO$

$$\tan 47° = \frac{h}{25}$$

$$25 \tan 47° = h$$

$$26.8 = h$$

So the tower is 26.8 m high.



ii From $\triangle YTO$

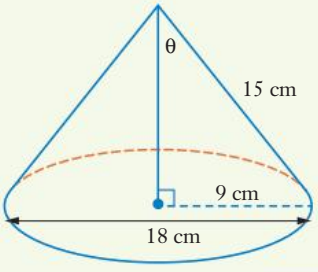
$$\tan \theta = \frac{26.8}{15}$$

$$\therefore \theta = \tan^{-1} \frac{26.8}{15}$$

$$= 60°46'$$

So the angle of elevation from Y is 60°46′.

b The radius of the base is 9 cm.



$$\sin \theta = \frac{9}{15}$$

$$\therefore \theta = \sin^{-1} \frac{9}{15}$$

$$= 36°52'$$

$$\text{Vertical angle} = 2\theta$$

$$= 73°44'$$

The vertical angle is the angle at the vertex of the cone.

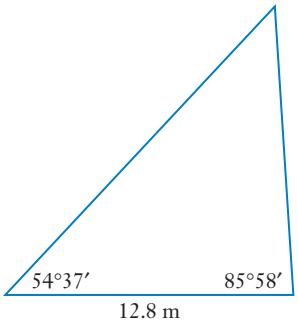
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Exercise 4.08 Mixed problems

1 A car is broken down to the north of 2 towns. The car is 39 km from town A and 52 km from town B. If A is due west of B and the 2 towns are 68 km apart, what is the bearing, to the nearest degree, of the car from:

a town A **b** town B?

2 The angle of elevation to the top of a tower is 54°37′ from a point 12.8 m out from its base. The tower is leaning at an angle of 85°58′ as shown. Find the height of the tower.



3 Rugby league goal posts are 5.5 m apart. If a footballer is standing 8 m from one post and 11 m from the other, find the angle within which the ball must be kicked to score a goal, to the nearest degree.

4 A boat is sinking 1.3 km out to sea from a marina. Its bearing is 041° from the marina and 324° from a rescue boat. The rescue boat is due east of the marina.

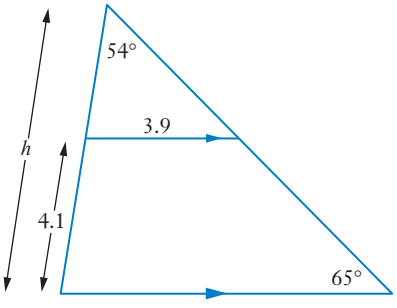
a How far, correct to 2 decimal places, is the rescue boat from the sinking boat?

b How long will it take the rescue boat, to the nearest minute, to reach the other boat if it travels at 80 km/h?

5 The angle of elevation of the top of a flagpole is 20° from where Thuy stands a certain distance away from its base. After walking 80 m towards the flagpole, Thuy finds the angle of elevation is 75°. Find the height of the flagpole, to the nearest metre.

6 A triangular field ABC has sides $AB = 85$ m and $AC = 50$ m. If B is on a bearing of 065° from A and C is on a bearing of 166° from A, find the length of BC, correct to the nearest metre.

7 Find the value of h, correct to 1 decimal place.



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8 A motorbike and a car leave a service station at the same time. The motorbike travels on a bearing of 080° and the car travels for 15.7 km on a bearing of 108° until the bearing of the motorbike from the car is 310°. How far, correct to 1 decimal place, has the motorbike travelled?

9 A submarine is being followed by two ships, A and B, 3.8 km apart, with A due east of B. If A is on a bearing of 165° from the submarine and B is on a bearing of 205° from the submarine, find the distance from the submarine to both ships.

10 A plane flies from Dubbo on a bearing of 139° for 852 km, then turns and flies on a bearing of 285° until it is due west of Dubbo. How far from Dubbo is the plane, to the nearest km?

11 Rhombus $ABCD$ with side 8 cm has diagonal BD 11.3 cm long. Find $\angle DAB$.

12 Zeke leaves school and runs for 8.7 km on a bearing of 338°, then turns and runs on a bearing of 061° until he is due north of school. How far north of school is he?

13 A car drives due east for 83.7 km then turns and travels for 105.6 km on a bearing of 029°. How far is the car from its starting point?

14 A plane leaves Sydney and flies for 1280 km on a bearing of 050°. It then turns and flies for 3215 km on a bearing of 149°. How far is the plane from Sydney, to the nearest km?

15 Trapezium $ABCD$ has $AD \parallel BC$, with $AB = 4.6$ cm, $BC = 11.3$ cm, $CD = 6.4$ cm, $\angle DAC = 23°30'$ and $\angle ABC = 78°$. Find:

a the length of AC **b** $\angle ADC$ to the nearest minute

16 A plane leaves Adelaide and flies for 875 km on a bearing of 056°. It then turns and flies on a bearing of θ for 630 km until it is due east of Adelaide. Evaluate θ to the nearest degree.

17 Quadrilateral $ABCD$ has $AB = AD = 7.2$ cm, $BC = 8.9$ cm and $CD = 10.4$ cm, with $\angle DAB = 107°$. Find:

a the length of diagonal BD **b** $\angle BCD$

18 A wall leans inwards and makes an angle of 88° with the floor.

a A 4 m long ladder leans against the wall with its base 2.3 m out from the wall. Find the angle that the top of the ladder makes with the wall.

b A longer ladder is placed the same distance out from the wall and its top makes an angle of 31° with the wall.

i How long is this ladder?

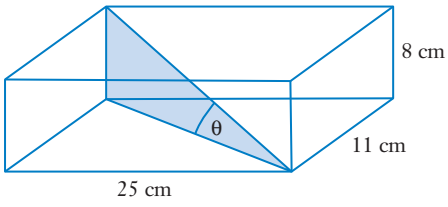
ii How much further does it reach up the wall than the first ladder?

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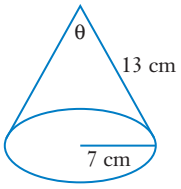
19 A 25 cm × 11 cm × 8 cm cardboard box contains an insert (the shaded area) made of foam.

a Find the area of foam in the insert, to the nearest cm^2.

b Find θ, the angle that the insert makes at the corner of the box.



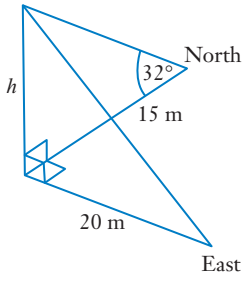
20 A cone has radius 7 cm and a slant height of 13 cm. Find the vertical angle, θ, at the top of the cone, in degrees and minutes.



21 From a point 15 m due north of a tower, the angle of elevation of the tower is 32°.

a Find the height of the tower, correct to 2 decimal places.

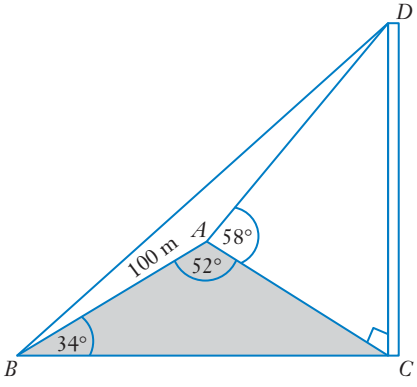
b Find correct to the nearest degree the angle of elevation of the tower at a point 20 m due east of the tower.



22 A pole DC is seen from two points A and B. The angle of elevation from A is 58°. If $\angle CAB = 52°$, $\angle ABC = 34°$, and A and B are 100 m apart, find:

a how far A is from the foot of the pole, to the nearest metre.

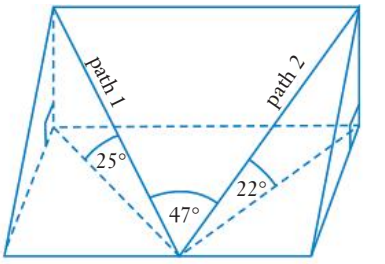
b the height of the pole, to 1 decimal place.



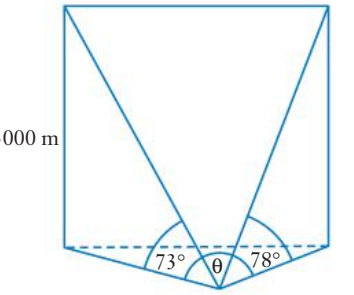
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23 Two straight paths to the top of a cliff are inclined at angles of 25° and 22° to the horizontal.

- **a** If path 1 is 114 m long, find the height of the cliff, to the nearest metre.
- **b** Find the length of path 2, to 1 decimal place.
- **c** If the paths meet at 47° at the base of the cliff, find their distance apart at the top of the cliff, correct to 1 decimal place.



24 A hot-air balloon floating at 950 m/h at a constant altitude of 3000 m is observed to have an angle of elevation of 78°. After 20 minutes, the angle of elevation is 73°. Calculate the angle through which the observer has turned during those 20 minutes.

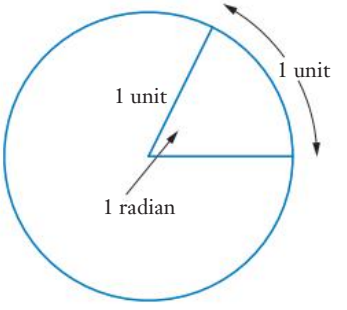


4.09 Radians

We use degrees to measure angles in geometry and trigonometry, but there are other units for measuring angles.

A **radian** is a unit for measuring angles based on the length of an arc in a circle.

One radian is the angle subtended by an arc with length 1 unit in a unit circle (of radius 1).



WS Radians

WS Converting degrees and radians

Conversions

We can change between radians and degrees using this equation:

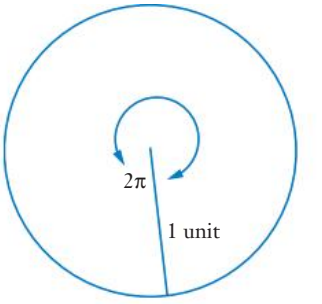
Radians and degrees

$$\pi \text{ radians} = 180°$$

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Proof

The circumference of a circle with radius 1 unit is:



$C = 2\pi r$

$= 2\pi(1)$

$= 2\pi$

The arc length of the whole circle is 2π.

$\therefore$ there are 2π radians in a whole circle.

But there are 360° in a whole circle (angle of revolution).

So $2\pi = 360°$

$\pi = 180°$

Converting between radians and degrees

To change from radians to degrees: multiply by $\frac{180}{\pi}$.

To change from degrees to radians: multiply by $\frac{\pi}{180}$.

Notice that $1° = \frac{\pi}{180} \approx 0.017$ radians

Also 1 radian $= \frac{180}{\pi} \approx 57°18'$

Degrees and radians

EXAMPLE 20

a Convert $\frac{3\pi}{2}$ into degrees.

b Change 60° to radians, leaving your answer in terms of π.

c Convert 50° into radians, correct to 2 decimal places.

d Change 1.145 radians into degrees, to the nearest minute.

e Convert 38°41′ into radians, correct to 3 decimal places.

f Evaluate cos 1.145 correct to 2 decimal places.

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Solution

a Since $\pi = 180°$,

$$\frac{3\pi}{2} = \frac{3(180°)}{2} = 270°$$

b $180° = \pi$ radians

So $1° = \frac{\pi}{180}$ radians

$$60° = \frac{\pi}{180} \times 60$$
$$= \frac{60\pi}{180}$$
$$= \frac{\pi}{3}$$

c $180° = \pi$ radians

So $1° = \frac{\pi}{180}$ radians

$$50° = \frac{\pi}{180} \times 50$$
$$= \frac{50\pi}{180}$$
$$\approx 0.87$$

d π radians $= 180°$

$$\therefore \text{ 1 radian} = \frac{180°}{\pi}$$

$$1.145 \text{ radians} = \frac{180°}{\pi} \times 1.145$$
$$\approx 65.6°$$
$$= 65°36'$$

e $180° = \pi$ radians

$$1° = \frac{\pi}{180°}$$

$$38°41' = \frac{\pi}{180°} \times 38°41'$$
$$= 0.675$$

f

Operation	Casio scientific	Sharp scientific
Make sure the calculator is in radians.	SHIFT SET UP Rad	Press DRG until rad is on the screen.
Enter data.	cos 1.145 =	cos 1.145 =

$\cos 1.145 = 0.4130 \ldots$

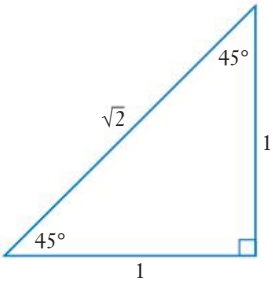
$$\approx 0.41$$

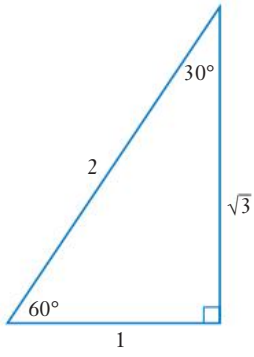
Special angles

$$30° = \frac{\pi}{6} \qquad 45° = \frac{\pi}{4} \qquad 60° = \frac{\pi}{3} \qquad 90° = \frac{\pi}{2}$$

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The angles 30°, 45° and 60° give exact results in trigonometry using 2 special triangles. You looked at these in Exercise 4.01, Questions 9 and 10, on page 175.





From these triangles we have the exact trigonometric ratios:

Exact values

Exact values 2

Exact trigonometric values

The exact ratios

$\sin 30° = \frac{1}{2}$ $\quad \sin 45° = \frac{1}{\sqrt{2}}$ $\quad \sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad \cos 45° = \frac{1}{\sqrt{2}}$ $\quad \cos 60° = \frac{1}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\quad \tan 45° = 1$ $\quad \tan 60° = \sqrt{3}$

We can write these same results in radians:

The exact ratios in radians

$\sin \frac{\pi}{6} = \frac{1}{2}$ $\quad \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \cos \frac{\pi}{3} = \frac{1}{2}$

$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad \tan \frac{\pi}{4} = 1$ $\quad \tan \frac{\pi}{3} = \sqrt{3}$

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EXAMPLE 21

a **i** Convert $\frac{\pi}{3}$ to degrees.

ii Find the exact value of $\tan\frac{\pi}{3}$.

b Find the exact value of $\cos\frac{\pi}{4}$.

Solution

a **i** $\frac{\pi}{3} = \frac{180°}{3}$

$= 60°$

ii $\tan\frac{\pi}{3} = \tan 60°$

$= \sqrt{3}$

b $\cos\frac{\pi}{4} = \cos 45°$

$= \frac{1}{\sqrt{2}}$

Exercise 4.09 Radians

1 Convert to degrees:

a $\frac{\pi}{5}$ **b** $\frac{2\pi}{3}$ **c** $\frac{5\pi}{4}$ **d** $\frac{7\pi}{6}$ **e** 3π

f $\frac{7\pi}{9}$ **g** $\frac{4\pi}{3}$ **h** $\frac{7\pi}{3}$ **i** $\frac{\pi}{9}$ **j** $\frac{5\pi}{18}$

2 Convert to radians in terms of π:

a $135°$ **b** $30°$ **c** $150°$ **d** $240°$ **e** $300°$

f $63°$ **g** $15°$ **h** $450°$ **i** $225°$ **j** $120°$

3 Change to radians, correct to 2 decimal places:

a $56°$ **b** $68°$ **c** $127°$ **d** $289°$ **e** $312°$

4 Change to radians, correct to 2 decimal places:

a $18°34'$ **b** $35°12'$ **c** $101°56'$ **d** $88°29'$ **e** $50°39'$

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5 Convert each radian measure into degrees and minutes, to the nearest minute:

a	1.09	**b**	0.768	**c**	1.16	**d**	0.99	**e**	0.32
f	3.2	**g**	2.7	**h**	4.31	**i**	5.6	**j**	0.11

6 Find correct to 2 decimal places:

a	sin 0.342	**b**	cos 1.5	**c**	tan 0.056	**d**	cos 0.589	**e**	tan 2.29
f	sin 2.8	**g**	tan 5.3	**h**	cos 4.77	**i**	cos 3.9	**j**	sin 2.98

7 Find the exact value of:

a	$\sin\frac{\pi}{4}$	**b**	$\cos\frac{\pi}{3}$	**c**	$\tan\frac{\pi}{6}$	**d**	$\sin\frac{\pi}{3}$	**e**	$\tan\frac{\pi}{4}$
f	$\sin\frac{\pi}{6}$	**g**	$\cos\frac{\pi}{4}$	**h**	$\cos\frac{\pi}{6}$	**i**	$\tan\frac{\pi}{3}$		

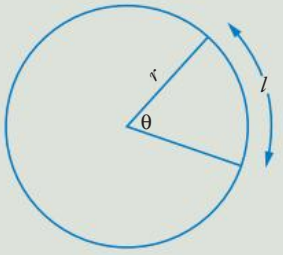
4.10 Length of an arc

Since radians are defined from the length of an arc of a circle, we can use radians to find the arc length of a circle.

You can find formulas for these using degrees, but they are not as simple. All the work on circles in this chapter uses radians.

Length of an arc

$l = r\theta$



Proof

$$\frac{\text{arc length } l}{\text{circumference}} = \frac{\text{angle } \theta}{\text{whole revolution}}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\therefore l = \frac{\theta 2\pi r}{2\pi}$$

$$= r\theta$$



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EXAMPLE 22

a Find the length of the arc formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

b Find the length of the arc formed given the angle subtended is 30° and the radius is 9 cm.

c The area of a circle is 450 cm^2. Find, in degrees and minutes, the angle subtended at the centre of the circle by a 2.7 cm arc.

Solution

a $l = r\theta$

$= 5\left(\frac{\pi}{4}\right)$

$= \frac{5\pi}{4}$ m

b First change 30° into radians.

$\theta = \frac{\pi}{6}$

$l = r\theta$

$= 9\left(\frac{\pi}{6}\right)$

$= \frac{3\pi}{2}$ cm

c $A = \pi r^2$

$450 = \pi r^2$

$\frac{450}{\pi} = r^2$

$\sqrt{\frac{450}{\pi}} = r$

$11.9682\ldots = r$

Now $l = r\theta$

$2.7 = 11.9682\ldots\theta$

$\frac{2.7}{11.9682\ldots} = \theta$

$0.2255\ldots = \theta$

π radians $= 180°$

1 radian $= \frac{180°}{\pi}$

0.2255... radians $= \frac{180°}{\pi} \times 0.2255\ldots$

$= 12.9257°$

$\approx 12°56'$

So $\theta = 12°56'$

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Exercise 4.10 Length of an arc

1 Find the exact arc length of a circle with:

- **a** radius 4 cm and angle subtended π
- **b** radius 3 m and angle subtended $\frac{\pi}{3}$
- **c** radius 10 cm and angle subtended $\frac{5\pi}{6}$
- **d** radius 3 cm and angle subtended 30°
- **e** radius 7 mm and angle subtended 45°.

2 Find the arc length, correct to 2 decimal places, given:

- **a** radius 1.5 m and angle subtended 0.43
- **b** radius 3.21 cm and angle subtended 1.22
- **c** radius 7.2 mm and angle subtended 55°
- **d** radius 5.9 cm and angle subtended 23°12′
- **e** radius 2.1 m and angle subtended 82°35′.

3 The angle subtended at the centre of a circle of radius 3.4 m is 29°51′. Find the length of the arc cut off by this angle, correct to 1 decimal place.

4 The arc length when a sector of a circle is subtended by an angle of $\frac{\pi}{5}$ at the centre is $\frac{3\pi}{2}$ m. Find the radius of the circle.

5 The radius of a circle is 3 cm and an arc is $\frac{2\pi}{7}$ cm long. Find the angle subtended at the centre of the circle by the arc.

6 The circumference of a circle is 300 mm. Find the length of the arc that is formed by an angle of $\frac{\pi}{6}$ subtended at the centre of the circle.

7 A circle with area 60 cm^2 has an arc 8 cm long. Find the angle that is subtended at the centre of the circle by the arc.

8 A circle with circumference 124 mm has a chord cut off it that subtends an angle of 40° at the centre. Find the length of the arc cut off by the chord.

9 A circle has a chord of 25 mm with an angle of $\frac{\pi}{6}$ subtended at the centre. Find, to 1 decimal place:

- **a** the radius
- **b** the length of the arc cut off by the chord.

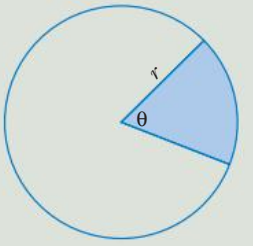
10 A sector of a circle with radius 5 cm and an angle of $\frac{\pi}{3}$ subtended at the centre is cut out of cardboard. It is then curved around to form an open cone. Find its exact volume.

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4.11 Area of a sector

Area of a sector

$$A = \frac{1}{2} r^2 \theta$$



Proof

$$\frac{\text{area of sector, } A}{\text{area of circle}} = \frac{\text{angle } \theta}{\text{whole revolution}}$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\therefore A = \frac{\theta \pi r^2}{2\pi} = \frac{1}{2} r^2 \theta$$

EXAMPLE 23

a Find the area of the sector formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

b The area of the sector of a circle with radius 4 cm is $\frac{6\pi}{5}$ cm^2. Find the angle, in degrees, that is subtended at the centre of the circle.

Solution

a
$$A = \frac{1}{2} r^2 \theta$$
$$= \frac{1}{2} (5)^2 \left(\frac{\pi}{4}\right)$$
$$= \frac{25\pi}{8} \text{ m}^2$$

b
$$A = \frac{1}{2} r^2 \theta$$
$$\frac{6\pi}{5} = \frac{1}{2} (4)^2 \theta$$
$$= 8\theta$$
$$\frac{6\pi}{40} = \theta$$
$$\theta = \frac{3\pi}{20} = \frac{3(180°)}{20} = 27°$$

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Exercise 4.11 Area of a sector

1 Find the exact area of the sector of a circle whose radius is:

a 4 cm and the subtended angle is π **b** 3 m and the subtended angle is $\frac{\pi}{3}$

c 10 cm and the subtended angle is $\frac{5\pi}{6}$ **d** 3 cm and the subtended angle is 30°

e 7 mm and the subtended angle is 45°.

2 Find the area of the sector, correct to 2 decimal places, given the radius is:

a 1.5 m and the subtended angle is 0.43 **b** 3.21 cm and the subtended angle is 1.22

c 7.2 mm and the subtended angle is 55° **d** 5.9 cm and the subtended angle is 23°12′

e 2.1 m and the subtended angle is 82°35′.

3 Find the area, correct to 3 significant figures, of the sector of a circle with radius 4.3 m and an angle of 1.8 subtended at the centre.

4 The area of a sector of a circle is 20 cm^2. If the radius of the circle is 3 cm, find the angle subtended at the centre of the circle by the sector.

5 The area of the sector of a circle that is subtended by an angle of $\frac{\pi}{3}$ at the centre is 6π m^2. Find the radius of the circle.

6 A circle with radius 7 cm has a sector cut off by an angle of 30° subtended at the centre of the circle. Find:

a the arc length **b** the area of the sector.

7 A circle has a circumference of 185 mm. Find the area of the sector cut off by an angle of $\frac{\pi}{5}$ subtended at the centre.

8 If the area of a circle is 200 cm^2 and a sector is cut off by an angle of $\frac{3\pi}{4}$ at the centre, find the area of the sector.

9 Find the area of the sector of a circle with radius 5.7 cm if the length of the arc formed by this sector is 4.2 cm.

10 The area of a sector is $\frac{3\pi}{10}$ cm^2 and the arc length cut off by the sector is $\frac{\pi}{5}$ cm. Find the angle subtended at the centre of the circle and the radius of the circle.

11 If an angle of $\frac{\pi}{7}$ is subtended at the centre of a circle with radius 3 cm, find:

a the exact arc length **b** the exact area of the sector.

12 An angle of $\frac{\pi}{6}$ is subtended at the centre of a circle with radius 5 cm. Find:

a the length of the arc **b** the area of the sector **c** the length of the chord.

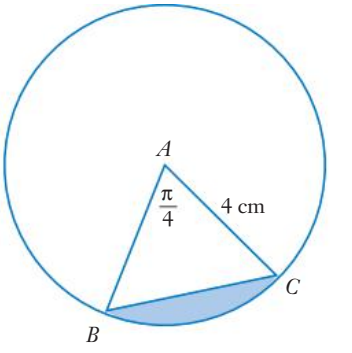
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13 A chord 8 mm long is formed by an angle of 45° subtended at the centre of a circle. Find, correct to 1 decimal place:

- **a** the radius of the circle
- **b** the area of the sector cut off by the angle.

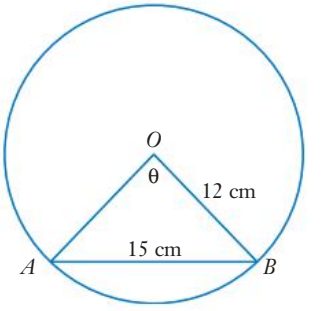
14 **a** Find the area of the sector of a circle with radius 4 cm if the angle subtended at the centre is $\frac{\pi}{4}$.

- **b** Find the length of BC to 1 decimal place.
- **c** Find the exact area of triangle ABC.
- **d** Hence find the exact area of the shaded minor segment of the circle.

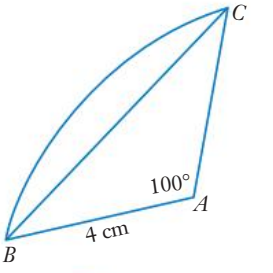


15 A triangle OAB is formed where O is the centre of a circle of radius 12 cm and A and B are endpoints of a 15 cm chord.

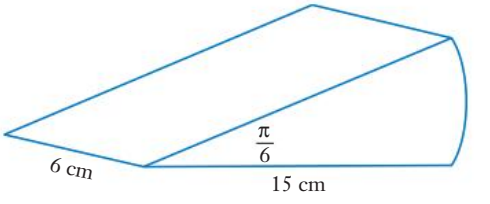
- **a** Find the angle subtended at the centre of the circle, in degrees and minutes.
- **b** Find the area of ΔOAB correct to 1 decimal place.
- **c** Find the area of the minor segment cut off by the chord, correct to 2 decimal places.
- **d** Find the area of the major segment cut off by the chord, correct to 2 decimal places.



16 Arc BC subtends an angle of 100° at the centre A of a circle with radius 4 cm. Find the perimeter of sector ABC.



17 A wedge is cut so that its cross-sectional area is a sector of a circle with radius 15 cm and subtending an angle of $\frac{\pi}{6}$ at the centre. Find the exact volume of the wedge.



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4. TEST YOURSELF

Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 Find the exact length of the radius of a circle if the arc length cut off by an angle of $\frac{5\pi}{4}$ is $\frac{25\pi}{8}$ cm.

A 5π cm **B** 5 cm **C** 2.5 cm **D** $\frac{5\pi}{2}$ cm

2 The cosine rule is (there is more than one answer):

A $c^2 = a^2 + b^2 - 2ab\cos C$

B $c^2 = a^2 + c^2 - 2ac\cos C$

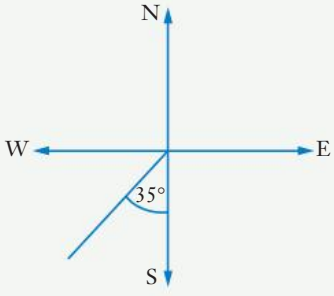
C $a^2 = b^2 + c^2 - 2bc\cos A$

D $a^2 = b^2 + c^2 - 2ab\cos A$

3 What bearing is shown on the diagram (there may be more than one answer)?

A 035° **B** W 35° S

C S 35° W **D** 215°



4 Find the exact value of $\cos\theta$ and $\sin\theta$ if $\tan\theta = \frac{3}{5}$.

5 Evaluate to 2 decimal places:

a $\sin 39°54'$ **b** $\tan 61°30'$ **c** $\cos 19°2'$ **d** $\sin 0.14$ **e** $\tan 3.5$

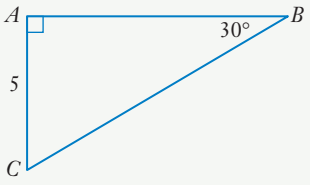
6 Find θ to the nearest minute if:

a $\sin\theta = 0.72$ **b** $\cos\theta = 0.286$ **c** $\tan\theta = \frac{5}{7}$

7 A ship sails on a bearing of 215° from port until it is 100 km due south of port. How far does it sail, to the nearest km?

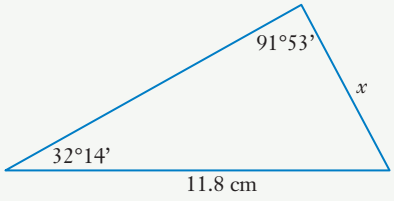
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8 Find the length of AB as a surd.

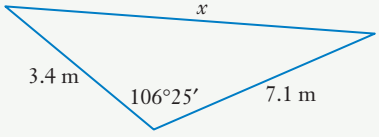


9 Evaluate x, correct to 2 significant figures.

a



b

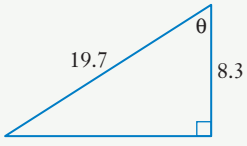


10 Convert each radian measure to degrees and minutes.

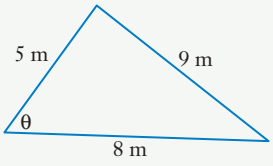
a 0.75 **b** 1.3 **c** 3.95 **d** 4.2 **e** 5.66

11 Evaluate θ to the nearest minute.

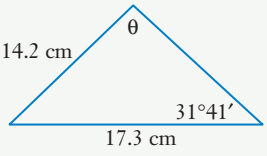
a



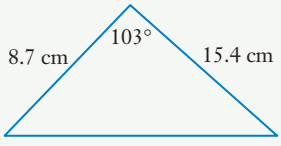
b



c



12 Find the area of this triangle.



13 Jacquie walks south from home for 3.2 km, then turns and walks west for 1.8 km. What is the bearing, to the nearest degree, of:

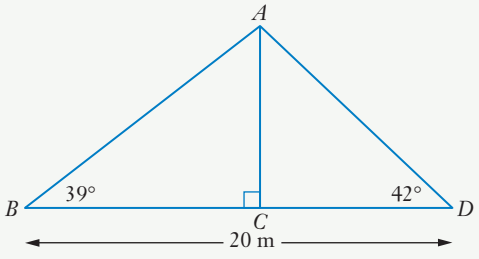
a Jacquie from her home?

b her home from where Jacquie is now?

14 The angle of elevation from point B to the top of a pole AC is 39° and the angle of elevation from D, on the other side of the pole, is 42°. B and D are 20 m apart.

a Find an expression for the length of AD.

b Find the height of the pole, to 1 decimal place.



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15 A plane flies from Orange for 1800 km on a bearing of 300°. It then turns and flies for 2500 km on a bearing of 205°. How far is the plane from Orange, to the nearest km?

16 Convert to radians, leaving in terms of π:

a 60° **b** 45° **c** 150° **d** 180° **e** 20°

17 A circle with radius 5 cm has an angle of $\frac{\pi}{6}$ subtended at the centre. Find:

a the exact arc length

b the exact area of the sector.

18 Find the exact value of:

a $\tan\frac{\pi}{3}$ **b** $\cos\frac{\pi}{6}$ **c** $\sin\frac{\pi}{4}$ **d** $\tan\frac{\pi}{6}$ **e** $\cos\frac{\pi}{4}$

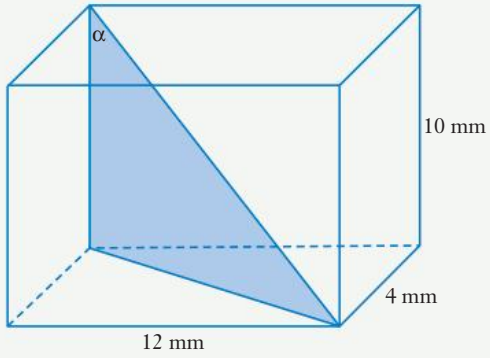
f $\sin\frac{\pi}{6}$ **g** $\tan\frac{\pi}{4}$ **h** $\cos\frac{\pi}{3}$ **i** $\sin\frac{\pi}{3}$

19 A circle has a circumference of 8π cm. If an angle of $\frac{\pi}{7}$ is subtended at the centre of the circle, find:

a the exact area of the sector

b the area of the minor segment, to 2 decimal places.

20 Evaluate α in this figure.



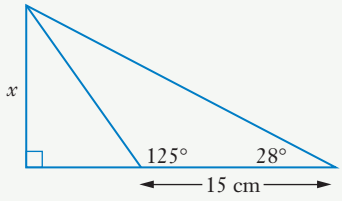
21 In triangle MNP, $NP = 14.9$ cm, $MP = 12.7$ cm and $\angle N = 43°49'$. Find $\angle M$ in degrees and minutes.

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4. CHALLENGE EXERCISE

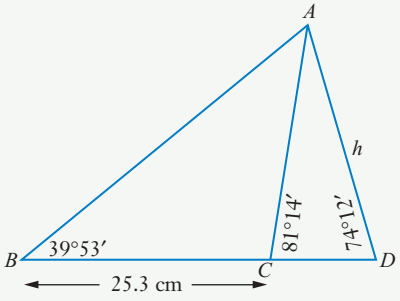
1 Two cars leave an intersection at the same time, one travelling at 70 km/h along one straight road and the other car travelling at 80 km/h along another straight road. After 2 hours they are 218 km apart. At what angle, to the nearest minute, do the roads meet at the intersection?

2 Evaluate x correct to 3 significant figures.



3 **a** Find an exact expression for the length of AC.

b Hence, or otherwise, find the value of h correct to 1 decimal place.



4 From the top of a vertical pole the angle of depression to Ian standing at the foot of the pole is 43°. Liam is on the other side of the pole, and the angle of depression from the top of the pole to Liam is 52°. The boys are standing 58 m apart. Find the height of the pole, to the nearest metre.

5 From point A, 93 m due south of the base of a tower, the angle of elevation is 35°. Point B is 124 m due east of the tower. Find:

a the height of the tower, to the nearest metre

b the angle of elevation of the tower from point B.

6 A cable car 100 m above the ground is seen to have an angle of elevation of 65° when it is on a bearing of 345°. After a minute, it has an angle of elevation of 69° and is on a bearing of 025°. Find:

a how far it travels in that minute

b its speed in m s^{-1}.

7 Find the area of a regular hexagon with sides 4 cm, to the nearest cm^2.

8 Calculate correct to one decimal place the area of a regular pentagon with sides 12 mm.

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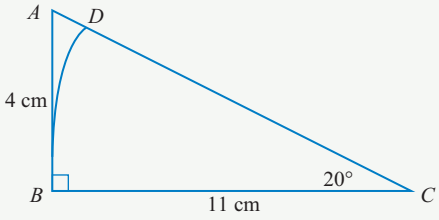
9 The length of an arc is 8.9 cm and the area of the sector is 24.3 cm^2 when an angle of θ is subtended at the centre of a circle. Find the area of the minor segment cut off by θ, correct to 1 decimal place.

10 *BD* is the arc of a circle with centre *C*. Find, correct to 2 decimal places:

a the length of arc *BD*

b the area of region *ABD*

c the perimeter of sector *BDC*.

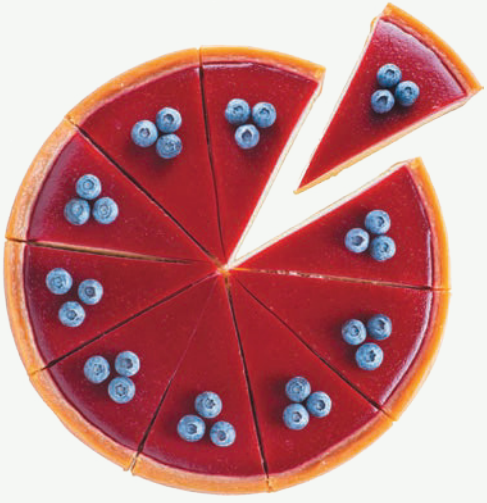


11 David walks along a straight road. At one point he notices a tower on a bearing of 053° with an angle of elevation of 21°. After David walks 230 m, the tower is on a bearing of 342° with an angle of elevation of 26°. Find the height of the tower correct to the nearest metre.

12 The hour hand of a clock is 12 cm long. Find:

a the length of the arc through which the hand would turn in 5 hours

b the area through which the hand would pass in 2 hours.



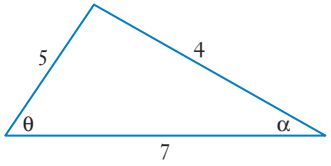
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Practice set 2

In Questions **1** to **6**, select the correct answer **A**, **B**, **C** or **D**.

1 Find an expression involving θ for this triangle (there may be more than one answer).



A $\cos\theta = \dfrac{5^2+4^2-7^2}{2\times5\times4}$

B $\dfrac{\sin\theta}{4} = \dfrac{\sin\alpha}{5}$

C $\dfrac{\sin\theta}{5} = \dfrac{\sin\alpha}{4}$

D $\cos\theta = \dfrac{5^2+7^2-4^2}{2\times5\times7}$

2 If $f(x) = \begin{cases} 8x^3 & \text{if } x > 3 \\ 3x^2 - 2 & \text{if } 0 \le x \le 3 \\ 9 & \text{if } x < 0 \end{cases}$ evaluate $f(3) + f(1) + f(-1)$.

A 35 **B** 226 **C** 233 **D** 53

3 The linear function with equation $4x - 2y + 3 = 0$ has:

A gradient -2, y-intercept $-1\frac{1}{2}$

B gradient $\frac{1}{2}$, y-intercept $\frac{3}{4}$

C gradient 2, y-intercept $1\frac{1}{2}$

D gradient 4, y-intercept 3.

4 For the quadratic function $y = ax^2 + bx + c > 0$ for all x:

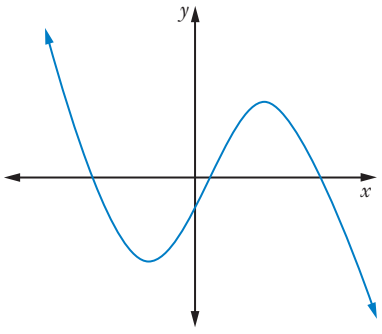
A $a > 0, b^2 - 4ac > 0$ **B** $a < 0, b^2 - 4ac > 0$

C $a > 0, b^2 - 4ac < 0$ **D** $a < 0, b^2 - 4ac < 0$

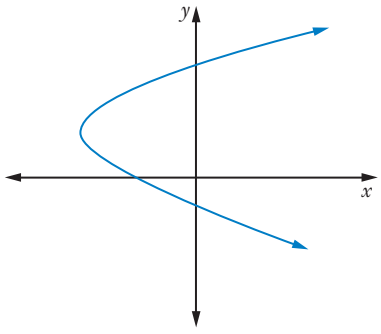
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5 Which of the following is not the graph of a function?

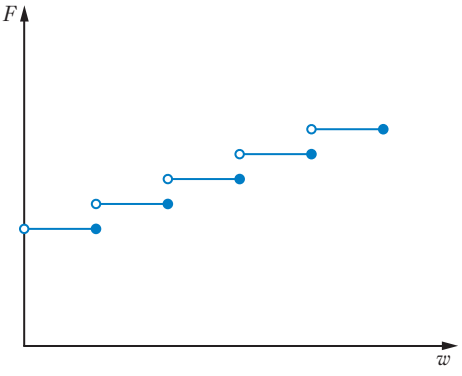
A



B



C



D (0, 3), (1, 3), (2, 5), (3, 1)

6 The polynomial $P(x) = x^3 - 5x^2 + 3x - 8$ (there is more than one answer):

A is monic

B has degree 3

C has leading coefficient –8

D has constant term –8.

7 A triangle has sides of length 5.1 m, 6.5 m and 8.2 m.

a Find the size of the angle opposite the 6.5 m side, correct to the nearest minute.

b Find the area of the triangle correct to one decimal place.

8 Find the equation of the straight line:

a with gradient –2 and y-intercept 3

b with x-intercept 5 and y-intercept –1

c passing through (2, 0) and (–3, –4)

d through (5, –4) and parallel to the line through (7, 4) and (3, –1)

e through (3, –1) perpendicular to the line $3x - 2y - 7 = 0$

f through (1, 2) parallel to the line through (–3, 4) and (5, 5)

g through (1, 3) and an angle of inclination of 135°.

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9 Simplify:

a $\dfrac{6x}{2x-8}$ **b** $\dfrac{5y+10}{xy^2} \div \dfrac{y^2-4}{x^2y}$ **c** $\dfrac{4a-3}{5} - \dfrac{a+1}{4}$

10 Convert these angles into radians in terms of π:

a 60° **b** 150° **c** 90°

d 10° **e** 315°

11 Sketch the graph of:

a $5x - 2y - 10 = 0$ **b** $x = 2$ **c** $f(x) = (x-3)^2$

d $y = x^2 - 5x + 4$ **e** $y = (x-1)^3 + 2$

12 Convert each value in radians into degrees and minutes:

a 1.7 **b** 0.36 **c** 2.54

13 The lines AB and AC have equations $3x - 4y + 9 = 0$ and $8x + 6y - 1 = 0$ respectively.

a Show that the lines are perpendicular.

b Find the coordinates of A.

14 Find the gradient of the line through the origin and (–3, 5).

15 If $g(x) = \begin{cases} 3-x & \text{if } x > 1 \\ 2x & \text{if } x \leq 1 \end{cases}$:

a find $g(2)$ and $g(-3)$

b sketch the graph of $y = g(x)$.

16 Find the value of x if $f(x) = 7$ where $f(x) = 2^x - 1$.

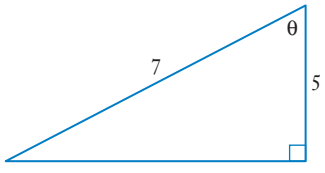
17 If $f(x) = 9 - 2x^2$, find the value of $f(-1)$.

18 Show that $3x - 4y + 10 = 0$ is a tangent to the circle $x^2 + y^2 = 4$.

19 Change each value in radians into degrees:

a $\dfrac{\pi}{4}$ **b** $\dfrac{3\pi}{2}$ **c** $\dfrac{\pi}{5}$ **d** $\dfrac{7\pi}{8}$ **e** 6π

20 Given the triangle ABC, find exact values of cos θ, sin θ and tan θ.



21 Show that:

a $-x^2 + x - 9 < 0$ for all x

b $x^2 - x + 3 > 0$ for all x.

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22 The distance travelled by a runner is directly proportional to the time she takes. If Vesna runs 12 km in 2 hours 30 minutes, find:

a an equation for distance d in terms of time t

b how far Vesna runs in:

i 2 hours **ii** 5 hours

c how long it takes Vesna to run:

i 30 km **ii** 19 km

d her average speed.

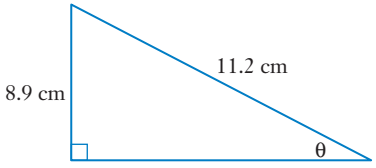
23 Find the equation of the parabola with x-intercepts 3 and -1 and y-intercept -3.

24 Show that the quadratic equation $6x^2 + x - 15 = 0$ has 2 real, rational roots.

25 The area of a circle is 5π and an arc 3 cm long cuts off a sector with an angle of θ subtended at the centre. Find θ in degrees and minutes.

26 A soccer goal is 8 m wide. Tim shoots for goal when he is 9 m from one post and 11 m from the other. Within what angle must a shot be made in order to score a goal?

27 Evaluate θ in degrees and minutes, to the nearest minute:



28 **a** Find the equation of the straight line l through $(-1, 2)$ that is perpendicular to the line $3x + 6y - 7 = 0$.

b Line l cuts the x-axis at P and the y-axis at Q. Find the coordinates of P and Q.

29 Show that $f(x) = x^6 - x^2 - 3$ is an even function.

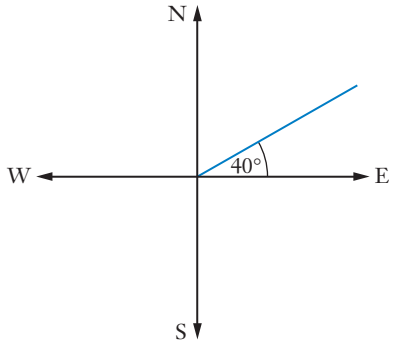
30 Find the angle of depression from the top of a 5.6 m tall cliff down to a boat that is 150 m out from the base of the cliff.

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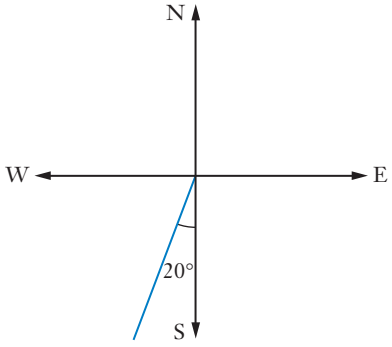
31 Write each direction shown as:

i a compass bearing **ii** a true bearing

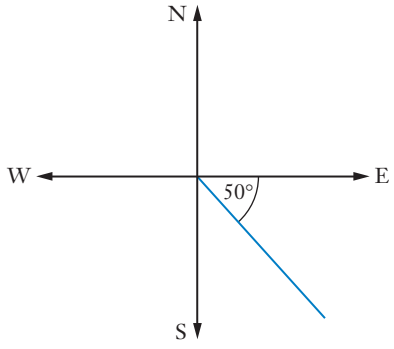
a



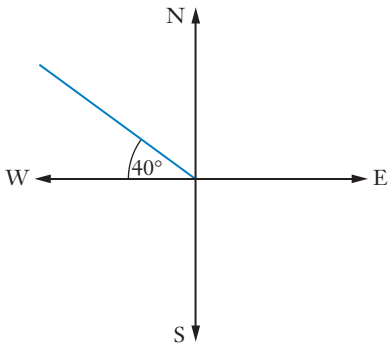
b



c



d



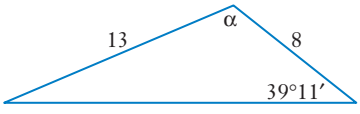
32 An angle of 30° is subtended at the centre of a circle with radius 5 cm. Find the exact:

a arc length **b** area of the sector.

33 Factorise:

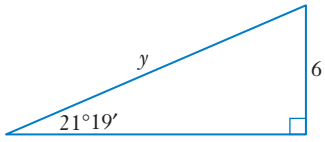
a $x^2 - 4x + 4$ **b** $9x^2 - 1$

34 Find α in degrees and minutes.



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35 Find the value of y correct to 3 significant figures:



36 Find the intersection of the graphs:

a $x + 3y - 1 = 0$ and $x - 2y - 6 = 0$

b $y = x^2$ and $x - 2y + 15 = 0$

37 For each quadratic function:

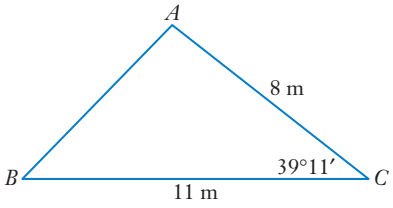
i find the equation of the axis of symmetry

ii state whether it has a maximum or minimum turning point and find its coordinates.

a $y = x^2 - 6x + 1$ **b** $y = -2x^2 - 4x - 3$

38 A hawk at the top of a 10 m tree sees a mouse on the ground. If the angle of depression is 34°51′, how far, to 1 decimal place, does the bird need to fly to reach the mouse?

39



a Find AB, to the nearest metre.

b Find the area of ΔABC, to 3 significant figures.

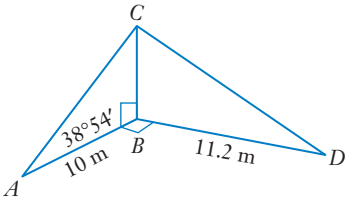
40 Two points A and B are 100 m apart on the same side of a tower. The angle of elevation of A to the top of the tower is 20° and the angle of elevation from B is 27°. Find the height of the tower, to the nearest metre.

41 The length of an arc in a circle of radius 6 cm is 7π cm. Find the area of the sector cut off by this arc.

42 Jordan walks for 3.1 km due west, then turns and walks for 2.7 km on a bearing of 205°. How far is he from his starting point?

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43 The angle of elevation from a point A to the top of a tower BC is 38°54′. A is 10 m due south of the tower.



a Find the height of the tower, to 1 decimal place.

b If point D is 11.2 m due east of the tower, find the angle of elevation from D to the tower.

44 Find the domain and range of:

a $f(x) = \dfrac{3}{x+4}$ **b** $y = |x| + 2$

c $y = 4$ **d** $y = x^2 - 3$

45 Nalini leaves home and cycles west for 12.5 km then turns and rides south for 11.3 km.

a How far is Nalini from home?

b Find the bearing of Nalini from home.

46 Show that $f(x) = x^3 - 5x$ is an odd function.

47 Sketch the graph of:

a $3x - 2y + 6 = 0$ **b** $y = x^2 - x - 2$

c $y = x^3 - 1$ **d** $y = x\,(x + 2)(x - 3)$

48 The length of an arc in a circle of radius 2 cm is 1.6 cm. Find the area of the sector.

49 Change each angle size from radians into degrees.

a 2π **b** $\dfrac{\pi}{6}$ **c** $\dfrac{9\pi}{4}$

50 A plane flies on a bearing of 034° from Sydney for 875 km. How far due east of Sydney is the plane?

51 Solve:

a $5b - 3 \geq 7$ **b** $x^2 - 3x = 0$ **c** $|2n + 5| = 9$

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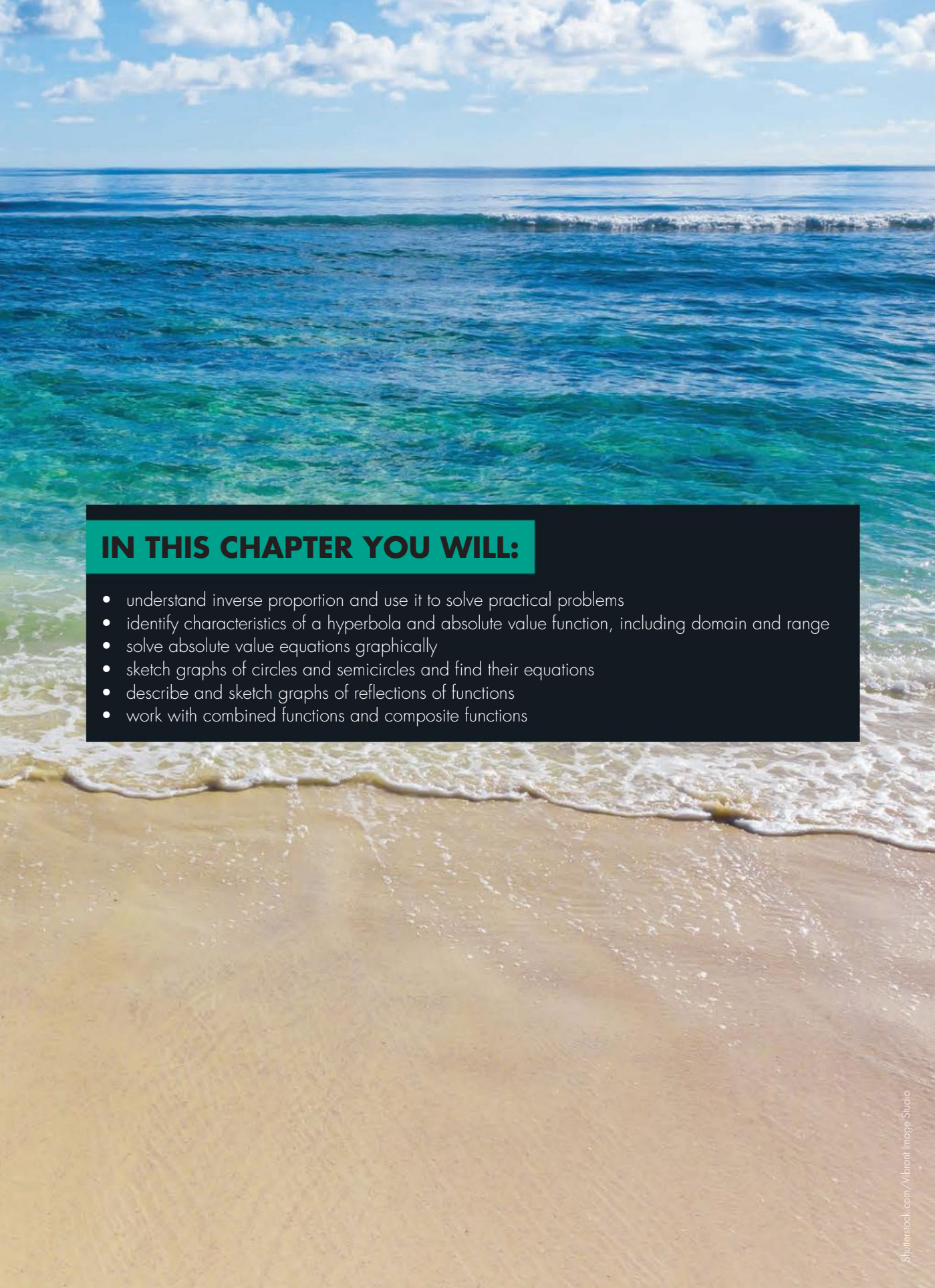
5.

FURTHER FUNCTIONS

In this chapter, we look at functions and relations that are not polynomials, including the hyperbola, absolute value, circles and semicircles. We will also study reflections and relationships between functions, including combined functions and composite functions.

CHAPTER OUTLINE

5.01 The hyperbola
5.02 Absolute value functions
5.03 Circles and semicircles
5.04 Reflections of functions
5.05 Combined and composite functions



IN THIS CHAPTER YOU WILL:

- understand inverse proportion and use it to solve practical problems
- identify characteristics of a hyperbola and absolute value function, including domain and range
- solve absolute value equations graphically
- sketch graphs of circles and semicircles and find their equations
- describe and sketch graphs of reflections of functions
- work with combined functions and composite functions

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TERMINOLOGY

asymptote: A line that a curve approaches but doesn't touch

composite function: A function of a function, where the output of one function becomes the input of a second function, written as $f(g(x))$. For example, if $f(x) = x^2$ and $g(x) = 3x + 1$ then $f(g(x)) = (3x + 1)^2$

continuous function: A function whose graph is smooth and does not have gaps or breaks

discontinuous function: A function whose graph has a gap or break in it; for example, $f(x) = \frac{1}{x}$, whose graph is a hyperbola

hyperbola: The graph of the function $y = \frac{k}{x}$, which is made up of 2 separate curves

inverse variation: A relationship between 2 variables such that as one variable increases the other variable decreases, or as one variable decreases the other variable increases. One variable is a multiple of the reciprocal of the other, with equation $y = \frac{k}{x}$. Also called **inverse proportion**

5.01 The hyperbola

Inverse variation

The hyperbola

Graphing hyperbolas

Graphing $y = \frac{10}{x}$

We looked at direct variation and the equation $y = kx$ in Chapter 3, *Functions*. When one variable is in **inverse variation** (or **inverse proportion**) with another variable, one is a constant multiple of the **reciprocal** of the other. This means that as one variable increases, the other decreases and when one decreases, the other increases.

For example:

- The more slices you cut a pizza into, the smaller the size of each slice
- The more workers there are on a project, the less time it takes to complete
- The fewer people sharing a house, the higher the rent each person pays.

Inverse variation

If variables x and y are in inverse variation, we can write the equation $y = \frac{k}{x}$ where k is called the **constant of variation**.

EXAMPLE 1

a Building a shed in 12 hours requires 3 builders. If the number of builders, N, is in inverse variation to the amount of time, t hours:

i find the equation for N in terms of t

ii find the number of builders it would take to build the shed in 9 hours

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iii find how long it would take 2 builders to build the shed

iv graph the equation for N after completing the table below.

t	1	2	3	4	5	6	7	8	9
N									

b The faster a car travels, the less time it takes to travel a certain distance. It takes the car 2 hours to travel this distance at a speed of 80 km/h. If the time taken, t hours, is in inverse proportion to the speed s km/h, then:

i find the equation for t in terms of s

ii find the time it would take if travelling at 100 km/h

iii find the speed at which the trip would take $2\frac{1}{2}$ hours

iv graph the equation.

Solution

a **i** For inverse variation, the equation is in the form $N = \frac{k}{t}$.

Substitute $t = 12$, $N = 3$ to find the value of k:

$$3 = \frac{k}{12}$$
$$36 = k$$
$$\therefore N = \frac{36}{t}$$

ii Substitute $t = 9$.

$$N = \frac{36}{9}$$
$$= 4$$

So it takes 4 builders to build the shed in 9 hours.

iii Substitute $N = 2$.

$$2 = \frac{36}{t}$$
$$2t = 36$$
$$t = 18$$

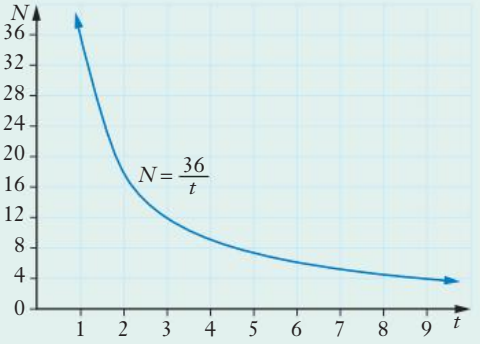
So it takes 18 hours for 2 builders to build the shed.

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iv $N = \dfrac{36}{t}$

Completing a table of values:

t	1	2	3	4	5	6	7	8	9
N	36	18	12	9	7.2	6	5.1	4.5	4



b **i** For inverse proportion, the equation is in the form $t = \dfrac{k}{s}$.

Substitute $s = 80$, $t = 2$ to find k.

$$2 = \frac{k}{80}$$

$$k = 160$$

$$\therefore \quad t = \frac{160}{s}$$

ii Substitute $s = 100$.

$$t = \frac{160}{100}$$

$$= 1.6 \text{ hours}$$

$$= 1 \text{ h } 36 \text{ min}$$

So the car takes 1 h 36 min to travel the distance if travelling at 100 km/h.

iii Substitute $t = 2.5$.

$$2.5 = \frac{160}{s}$$

$$2.5s = 160$$

$$s = \frac{160}{2.5}$$

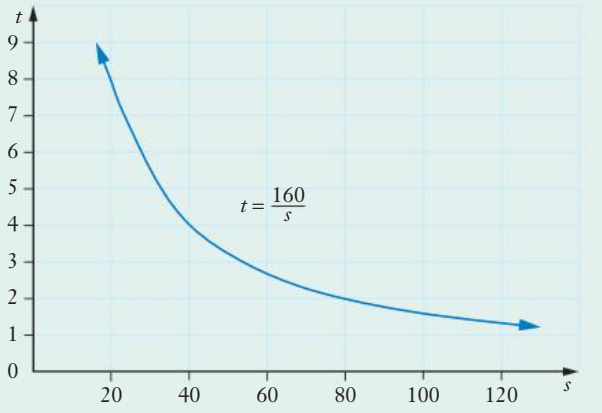
$$= 64$$

So the car travels at 64 km/h if the trip takes $2\frac{1}{2}$ hours.

iv To graph this function, complete a table of values for $t = \dfrac{160}{s}$.

s	20	40	60	80	100	120
t	8	4	2.667	2	1.6	1.333

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The graph of the function $y = \frac{k}{x}$ is a **hyperbola**.

Hyperbolas

A hyperbola is the graph of a function of the form $y = \frac{k}{x}$.

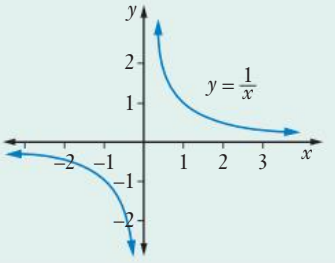
EXAMPLE 2

Sketch the graph of $y = \frac{1}{x}$. What is the domain and range?

Solution

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	–	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

When $x = 0$ the value of y is undefined.



Domain: x can be any real number except 0.
$(-\infty, 0) \cup (0, \infty)$

Range: y can be any real number except 0.
$(-\infty, 0) \cup (0, \infty)$

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CLASS DISCUSSION

LIMITS OF THE HYPERBOLA

What happens to the graph as x becomes closer to 0? What happens as x becomes very large in both positive and negative directions? The value of y is never 0. Why?

Continuity

Most functions have graphs that are smooth unbroken curves (or lines). They are called **continuous functions**. However, some functions have discontinuities, meaning that their graphs have gaps or breaks. These are called **discontinuous functions**.

The hyperbola is discontinuous because there is a gap in the graph and it has two separate parts. The graph of $y = \frac{1}{x}$ also does not touch the x- or y-axes, but it gets closer and closer to them. We call the x- and y-axes **asymptotes**: lines that the curve approaches but never touches.

To find the shape of the graph close to the asymptotes or as $x \to \pm\infty$, we can check points nearby.

EXAMPLE 3

Find the domain and range of $f(x) = \frac{3}{x-3}$ and sketch the graph of the function.

Solution

To find the domain, we notice that $x - 3 \neq 0$. So $x \neq 3$.

Domain $(-\infty, 3) \cup (3, \infty)$

Also y cannot be zero: $y \neq 0$.

Range $(-\infty, 0) \cup (0, \infty)$

The lines $x = 3$ and $y = 0$ (the x-axis) are the asymptotes of the hyperbola.

To find the limiting behaviour of the graph, look at what is happening as $x \to \pm\infty$.

As x increases and approaches ∞, $\frac{3}{x-3}$ becomes closer to 0 and is positive.
Substitute large values of x into the function, for example, $x = 1000$.

As $x \to \infty$, $y \to 0^+$ (as x approaches infinity, y approaches 0 from above, the positive side).

Similarly, as x decreases and approaches $-\infty$, y becomes closer to 0 and is negative.
Substitute $x = -1000$, for example.

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As $x \to -\infty, y \to 0^-$ (as x approaches negative infinity, y approaches 0 from below, the negative side).

To see the behaviour of the function near the asymptote $x = 3$ we can test values either side.

LHS: When $x = 2.999$,

$$\frac{3}{2.999-3} = -3000$$
$$< 0$$

As $x \to 3^-, y \to -\infty$

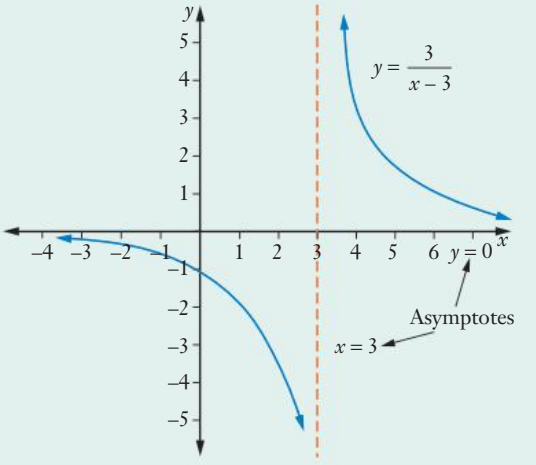
RHS: When $x = 3.001$,

$$\frac{3}{3.001-3} = 3000$$
$$> 0$$

As $x \to 3^+, y \to \infty$

For y-intercept, $x = 0$:

$$y = \frac{3}{0-3}$$
$$= -1$$



EXAMPLE 4

Sketch the graph of $y = -\frac{1}{2x+4}$.

Solution

To find the domain, notice that:

$2x + 4 \neq 0$

$2x \neq -4$

$x \neq -2$

Domain $(-\infty, -2) \cup (-2, \infty)$

For the range, $y \neq 0$.

Range $(-\infty, 0) \cup (0, \infty)$

So there are asymptotes at $x = -2$ and $y = 0$.

Limiting behaviour:

Substitute, say, $x = 5000$ and $x = -5000$.

As $x \to \infty, y \to 0^-$.

As $x \to -\infty, y \to 0^+$.

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To see the shape of the graph near the asymptote $x = -2$, we can test values either side.

LHS: When $x = -2.0001$,

$$y = -\frac{1}{2(-2.0001)+4} = 5000$$
$$> 0$$

As $x \to 2^-, y \to \infty$

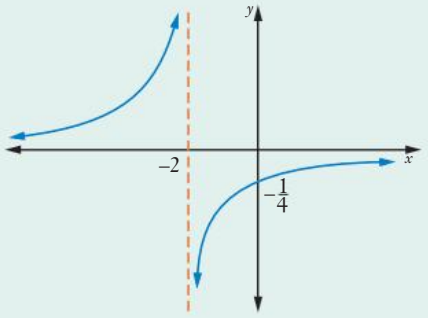
RHS: When $x = -1.9999$,

$$y = -\frac{1}{2(-1.9999)+4} = -5000$$
$$< 0$$

As $x \to 2^+, y \to -\infty$

For y-intercept, $x = 0$

$$y = -\frac{1}{2(0)+4}$$
$$= -\frac{1}{4}$$



The hyperbola

The hyperbola $y = \frac{k}{bx+c}$ is a discontinuous function with 2 parts, separated by vertical and horizontal asymptotes.

Exercise 5.01 The hyperbola

1 The diameter of a balloon varies inversely with the thickness of the rubber. The diameter of the balloon is 80 mm when the rubber is 2 mm thick.

- **a** Find an equation for the diameter D in terms of the thickness x.
- **b** Find the diameter when the thickness is 0.8 mm.
- **c** Find the thickness correct to one decimal place when the diameter is 115.3 mm.
- **d** Sketch the graph showing this information.

2 The more boxes a factory produces, the less it costs to produce each box. When 128 boxes are produced, it costs \$2 per box.

- **a** Write an equation for the cost c to produce each box when manufacturing n boxes.
- **b** Find the cost of each box when 100 boxes are produced.
- **c** Find how many boxes must be produced for the cost for each box to be 50 cents.
- **d** Sketch the graph of this information.

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3 For each function:

i state the domain and range

ii find the y-intercept if it exists

iii sketch the graph.

a $y=\frac{2}{x}$ **b** $y=-\frac{1}{x}$ **c** $f(x)=\frac{1}{x+1}$

d $f(x)=\frac{3}{x-2}$ **e** $y=\frac{1}{3x+6}$ **f** $f(x)=-\frac{2}{x-3}$

g $f(x)=\frac{4}{x-1}$ **h** $y=-\frac{2}{x+1}$ **i** $f(x)=\frac{2}{6x-3}$

4 Show that $f(x)=\frac{2}{x}$ is an odd function.

5 **a** Is the hyperbola $y=-\frac{2}{x+1}$:

i a function? **ii** even, odd or neither? **iii** continuous?

b What are the equations of the asymptotes?

c State its domain and range.

5.02 Absolute value functions

WS
Absolute value functions

An absolute value function is an example of a piecewise function with 2 sections.
We were introduced to absolute value in Chapter 3, *Functions*.

The absolute value function

$$|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

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EXAMPLE 5

Sketch the graph of $y = |x|$ and state its domain and range.

Solution

$y = |x|$ gives the piecewise function:

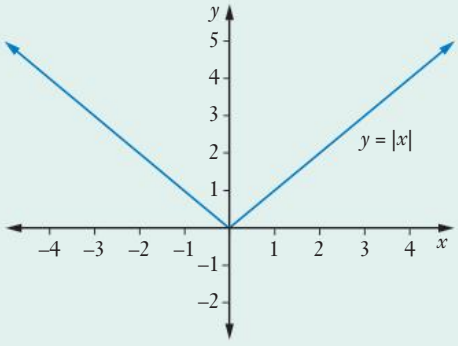
$$y = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

We can draw $y = x$ for $x \geq 0$ and $y = -x$ for $x < 0$ on the same set of axes.

From the graph, notice that x can be any real number while $y \geq 0$.

Domain $(-\infty, \infty)$

Range $[0, \infty)$



Absolute value graphs

EXAMPLE 6

a Sketch the graph of $f(x) = |x| - 1$ and state its domain and range.

b Sketch the graph of $y = |x + 2|$.

Solution

a Using the definition of absolute value:

$$y = \begin{cases} x - 1 & \text{for } x \geq 0 \\ -x - 1 & \text{for } x < 0 \end{cases}$$

Draw $y = x - 1$ for $x \geq 0$ and $y = -x - 1$ for $x < 0$.

For x-intercepts, $y = 0$:

$y = x - 1$ $\qquad$ $y = -x - 1$

$0 = x - 1$ $\qquad$ $0 = -x - 1$

$1 = x$ $\qquad$ $x = -1$

For y-intercept, $x = 0$:

$y = x - 1$ for $x = 0$

$= 0 - 1$

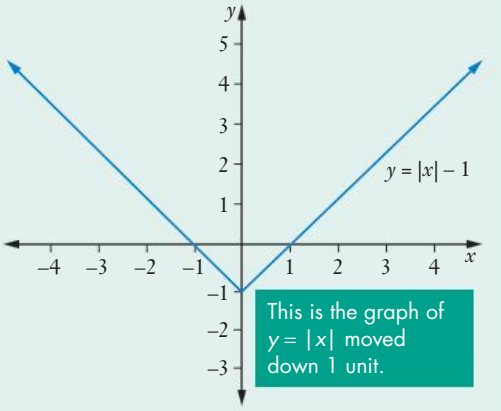
$= -1$

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From the graph, notice that x can be any real number while $y \geq -1$.

Domain: $(-\infty, \infty)$

Range: $[-1, \infty)$



b Using the definition of absolute value:

$$y = \begin{cases} x+2 & \text{for } x+2 \geq 0 \\ -(x+2) & \text{for } x+2 < 0 \end{cases}$$

Simplifying this gives:

$$y = \begin{cases} x+2 & \text{for } x \geq -2 \\ -x-2 & \text{for } x < -2 \end{cases}$$

For x-intercepts, $y = 0$:

$y = x + 2$ $\qquad$ $y = -x - 2$

$0 = x + 2$ $\qquad$ $0 = -x - 2$

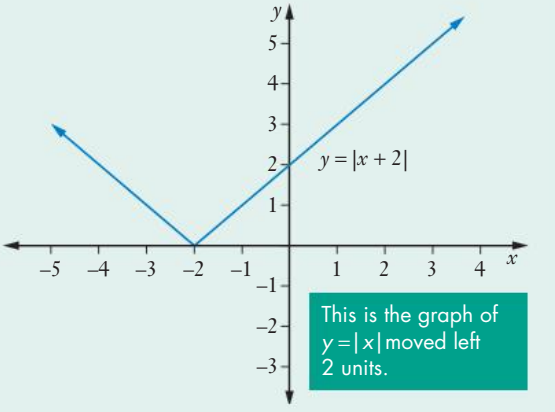
$-2 = x$ $\qquad$ $x = -2$

For y-intercept, $x = 0$:

$y = x + 2$ for $x = 0$

$= 0 + 2$

$= 2$



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INVESTIGATION

TRANSFORMATIONS OF THE ABSOLUTE VALUE FUNCTION

Use a graphics calculator or graphing software to explore each absolute value graph.

1	$y = \|x\|$	**2**	$y = 2\|x\|$	**3**	$y = 3\|x\|$
4	$y = -\|x\|$	**5**	$y = -2\|x\|$	**6**	$y = \|x\| + 1$
7	$y = \|x\| + 2$	**8**	$y = \|x\| - 1$	**9**	$y = \|x\| - 2$
10	$y = \|x + 1\|$	**11**	$y = \|x + 2\|$	**12**	$y = \|x + 3\|$
13	$y = \|x - 1\|$	**14**	$y = \|x - 2\|$	**15**	$y = \|x - 3\|$

Are graphs that involve absolute value always functions? Can you find an example of one that is not a function?

Are any of them odd or even? Are they continuous? Could you predict what the graph $y = 2|x - 7|$ would look like?

Equations involving absolute values

We learned how to solve equations involving absolute values using algebra in Chapter 2, *Equations and inequalities*. We can also solve these equations graphically.

EXAMPLE 7

Solve $|2x - 1| = 3$ graphically.

Solution

Sketch the graphs of $y = |2x - 1|$ and $y = 3$ on the same number plane.

$$y = \begin{cases} 2x - 1 & \text{for } 2x - 1 \geq 0 \\ -(2x - 1) & \text{for } 2x - 1 < 0 \end{cases}$$

Simplifying this gives:

$$y = \begin{cases} 2x - 1 & \text{for } x \geq \frac{1}{2} \\ -2x + 1 & \text{for } x < \frac{1}{2} \end{cases}$$

For x-intercepts, $y = 0$:

$y = 2x - 1$	$y = -2x + 1$
$0 = 2x - 1$	$0 = -2x + 1$
$1 = 2x$	$2x = 1$
$\frac{1}{2} = x$	$x = \frac{1}{2}$

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For y-intercept, $x = 0$:

$y = -2x + 1$ for $x = 0$

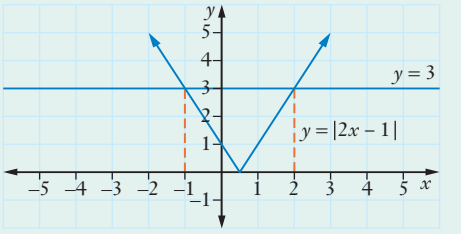
$= -2(0) + 1$

$= 1$

The graph of $y = 3$ is a horizontal line through 3 on the y-axis.

The solutions of $|2x - 1| = 3$ are the values of x at the point of intersection of the graphs.

$x = -1, 2$.



We can check that our solutions are correct by substituting them back into the equation.

Exercise 5.02 Absolute value functions

1 Find the x- and y-intercepts of the graph of each function.

a $f(x) = |x| + 7$ **b** $f(x) = |x| - 2$ **c** $y = 5|x|$

d $f(x) = -|x| + 3$ **e** $y = |x + 6|$ **f** $f(x) = |3x - 2|$

g $y = |5x + 4|$ **h** $y = |7x - 1|$ **i** $f(x) = |2x| + 9$

2 Sketch the graph of each function.

a $y = |x|$ **b** $f(x) = |x| + 1$ **c** $f(x) = |x| - 3$

d $y = 2|x|$ **e** $f(x) = -|x|$ **f** $y = |x + 1|$

g $f(x) = -|x - 1|$ **h** $y = |2x - 3|$ **i** $f(x) = |3x| + 1$

3 Find the domain and range of each function.

a $y = |x - 1|$ **b** $f(x) = |x| - 8$ **c** $f(x) = |2x + 5|$

d $y = 2|x| - 3$ **e** $f(x) = -|x - 3|$

4 Solve each equation graphically.

a $|x| = 3$ **b** $|x + 2| = 1$ **c** $|x - 3| = 0$

d $|2x - 3| = 1$ **e** $|2x + 3| = 11$ **f** $|5b - 2| = 8$

g $|3x + 1| = 2$ **h** $5 = |2x + 1|$ **i** $0 = |6t - 3|$

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WS

Equations of circles

5.03 Circles and semicircles

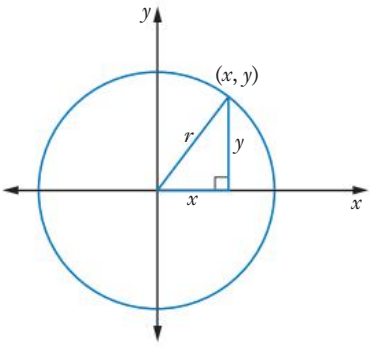
The circle is not a function. It does not pass the vertical line test.

Circle with centre (0, 0)

We can use Pythagoras' theorem to find the equation of a circle, using a general point (x, y) on a circle with centre (0, 0) and radius r.

$$c^2 = a^2 + b^2$$

$$\therefore \quad r^2 = x^2 + y^2$$



Equation of a circle with centre (0, 0)

The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

EXAMPLE 8

a Sketch the graph of $x^2 + y^2 = 4$.

b Why is it not a function?

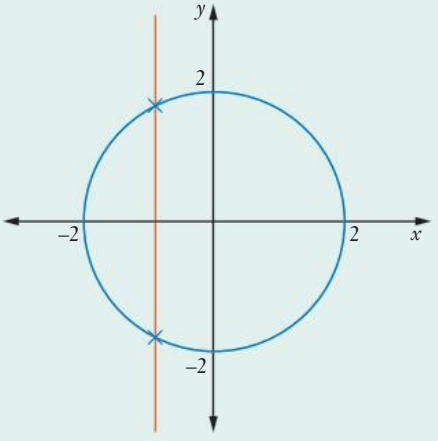
c State its domain and range.

Solution

a The equation is in the form $x^2 + y^2 = r^2$ where $r^2 = 4$.

Radius $r = \sqrt{4} = 2$

This is a circle with radius 2 and centre (0, 0).



b The circle is not a function because a vertical line will cut the graph in more than one place.

c The x values for this graph lie between –2 and 2 and the y values also lie between –2 and 2.

Domain: [–2, 2]

Range: [–2, 2]

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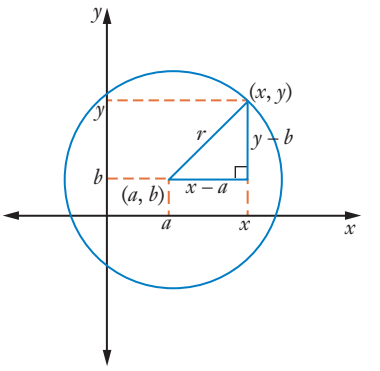
Circle with centre (a, b)

We can use Pythagoras' theorem to find the equation of a circle using a general point (x, y) on a circle with centre (a, b) and radius r.

The smaller sides of the triangle are $x - a$ and $y - b$ and the hypotenuse is r, the radius.

$c^2 = a^2 + b^2$

$r^2 = (x - a)^2 + (y - b)^2$



Equation of a circle with centre (a, b)

The equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

EXAMPLE 9

a **i** Sketch the graph of the circle $(x - 1)^2 + (y + 2)^2 = 4$.

ii State its domain and range.

b Find the equation of a circle with radius 3 and centre $(-2, 1)$ in expanded form.

c Find the centre and radius of the circle with equation $x^2 + 2x + y^2 - 6y - 6 = 0$.

Solution

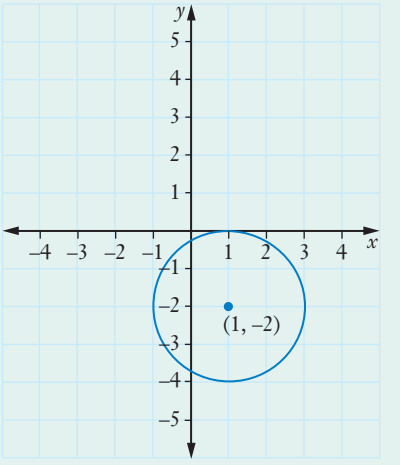
a **i** The equation is in the form $(x - a)^2 + (y - b)^2 = r^2$.

$(x - 1)^2 + (y + 2)^2 = 4$

$(x - 1)^2 + (y - (-2))^2 = 2^2$

So $a = 1$, $b = -2$ and $r = 2$.

This is a circle with centre $(1, -2)$ and radius 2.



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ii From the graph, we can see that all x values lie between -1 and 3 and all y values lie between -4 and 0.

Domain: $[-1, 3]$

Range: $[-4, 0]$

b Centre is $(-2, 1)$ so $a = -2$ and $b = 1$.

Radius is 3 so $r = 3$.

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x - (-2))^2 + (y - 1)^2 = 3^2$$

$$(x + 2)^2 + (y - 1)^2 = 9$$

Expanding:

$$x^2 + 4x + 4 + y^2 - 2y + 1 = 9$$

$$x^2 + 4x + y^2 - 2y - 4 = 0$$

c The equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$.

We need to complete the square to put the equation into this form.

To complete the square on $x^2 + 2x$, we add $\left(\frac{2}{2}\right)^2 = 1$.

To complete the square on $y^2 - 6y$, we add $\left(\frac{6}{2}\right)^2 = 9$.

$$x^2 + 2x + y^2 - 6y - 6 = 0$$

$$x^2 + 2x + y^2 - 6y = 6$$

$$x^2 + 2x + 1 + y^2 - 6y + 9 = 6 + 1 + 9$$

$$(x + 1)^2 + (y - 3)^2 = 16$$

$$(x - (-1))^2 + (y - 3)^2 = 4^2$$

This is in the form $(x - a)^2 + (y - b)^2 = r^2$ where $a = -1$, $b = 3$ and $r = 4$.

So it is a circle with centre $(-1, 3)$ and radius 4 units.

Semicircles

By rearranging the equation of a circle, we can find the equations of 2 semicircles.

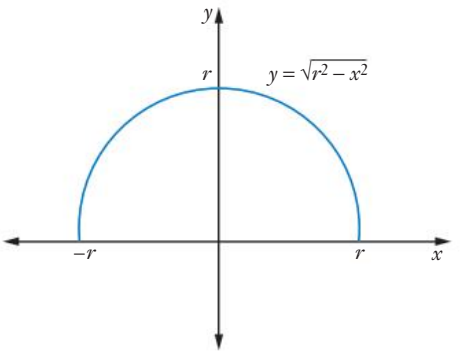
$$x^2 + y^2 = r^2$$

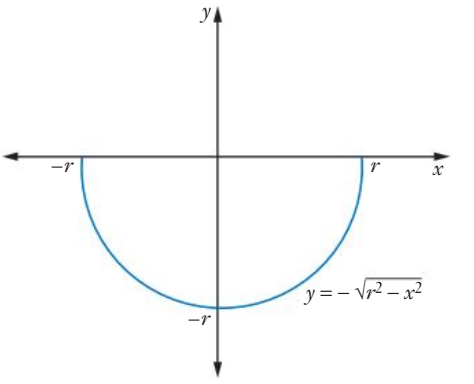
$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

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This gives 2 separate functions:





$y = \sqrt{r^2 - x^2}$ is the semicircle above the x-axis since $y \geq 0$.

$y = -\sqrt{r^2 - x^2}$ is the semicircle below the x-axis since $y \leq 0$.

Equations of a semicircle with centre (0, 0)

The equation of a semicircle above the x-axis with centre (0, 0) and radius r is

$$y = \sqrt{r^2 - x^2}.$$

The equation of a semicircle below the x-axis with centre (0, 0) and radius r is

$$y = -\sqrt{r^2 - x^2}.$$

EXAMPLE 10

Sketch the graph of each function and state the domain and range.

a $f(x) = \sqrt{9 - x^2}$ **b** $y = -\sqrt{4 - x^2}$

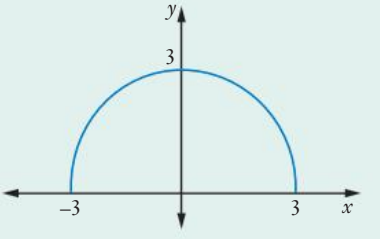
Solution

a This is in the form $f(x) = \sqrt{r^2 - x^2}$ where $r^2 = 9$, so $r = 3$.

It is a semicircle above the x-axis with centre (0, 0) and radius 3.

Domain: [−3, 3]

Range: [0, 3]



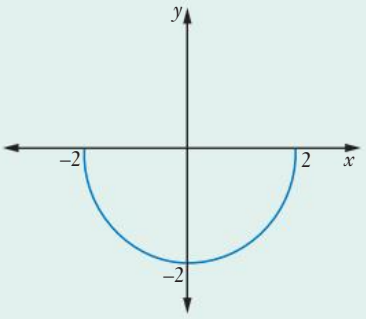
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b This is in the form $y = -\sqrt{r^2 - x^2}$ where $r^2 = 4$, so $r = 2$.

It is a semicircle below the x-axis with centre (0, 0) and radius 2.

Domain: [–2, 2]

Range: [–2, 0]



Exercise 5.03 Circles and semicircles

1 For each equation:

i sketch the graph **ii** state the domain and range.

a $x^2 + y^2 = 9$

b $x^2 + y^2 - 16 = 0$

c $(x - 2)^2 + (y - 1)^2 = 4$

d $(x + 1)^2 + y^2 = 9$

e $(x + 2)^2 + (y - 1)^2 = 1$

2 For each semicircle:

i state whether it is above or below the x-axis

ii sketch the graph

iii state the domain and range.

a $y = -\sqrt{25 - x^2}$

b $y = \sqrt{1 - x^2}$

c $y = \sqrt{36 - x^2}$

d $y = -\sqrt{64 - x^2}$

e $y = -\sqrt{7 - x^2}$

3 Find the radius and the centre of each circle.

a $x^2 + y^2 = 100$

b $x^2 + y^2 = 5$

c $(x - 4)^2 + (y - 5)^2 = 16$

d $(x - 5)^2 + (y + 6)^2 = 49$

e $x^2 + (y - 3)^2 = 81$

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4 Find the equation of each circle in expanded form.

a centre (0, 0) and radius 4
b centre (3, 2) and radius 5
c centre (–1, 5) and radius 3
d centre (2, 3) and radius 6
e centre (–4, 2) and radius 5
f centre (0, –2) and radius 1
g centre (4, 2) and radius 7
h centre (–3, –4) and radius 9
i centre (–2, 0) and radius $\sqrt{5}$
j centre (–4, –7) and radius $\sqrt{3}$

5 Find the radius and the centre of each circle.

a $x^2 - 4x + y^2 - 2y - 4 = 0$
b $x^2 + 8x + y^2 - 4y - 5 = 0$
c $x^2 + y^2 - 2y = 0$
d $x^2 - 10x + y^2 + 6y - 2 = 0$
e $x^2 + 2x + y^2 - 2y + 1 = 0$
f $x^2 - 12x + y^2 = 0$
g $x^2 + 6x + y^2 - 8y = 0$
h $x^2 + 20x + y^2 - 4y + 40 = 0$
i $x^2 - 14x + y^2 + 2y + 25 = 0$
j $x^2 + 2x + y^2 + 4y - 5 = 0$

6 Find the centre and radius of the circle with equation:

a $x^2 - 6x + y^2 + 2y - 6 = 0$
b $x^2 - 4x + y^2 - 10y + 4 = 0$
c $x^2 + 2x + y^2 + 12y - 12 = 0$
d $x^2 - 8x + y^2 - 14y + 1 = 0$

7 Sketch the circle whose equation is given by $x^2 + 4x + y^2 - 2y + 1 = 0$.

5.04 Reflections of functions

The graph of $y = -f(x)$



Advanced graphs



Matching graphs (Advanced)

EXAMPLE 11

For each function, sketch the graph of $y = f(x)$ and $y = -f(x)$ on the same number plane.

a $f(x) = x^2 - 2x$ **b** $f(x) = x^3$

Solution

a $f(x) = x^2 - 2x$ is a concave upwards parabola.

For x-intercepts: $f(x) = 0$

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0, 2$

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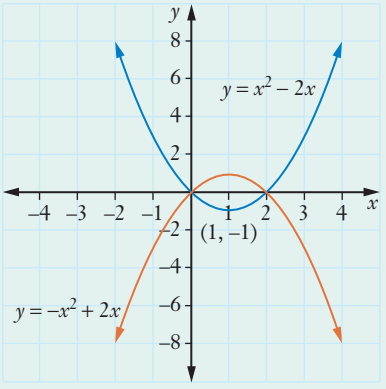
For y-intercept, $x = 0$:

$f(0) = 0^2 - 2(0) = 0$

Axis of symmetry at $x = 1$ (halfway between 0 and 2):

$f(1) = 1^2 - 2(1) = -1$

Minimum turning point at $(1, -1)$.



$y = -f(x)$

$= -(x^2 - 2x)$

$= -x^2 + 2x$

$y = -x^2 + 2x$ is a concave downwards parabola also with x-intercepts 0, 2 with y-intercept 0 and axis of symmetry at $x = 1$.

$f(1) = -1^2 + 2(1) = 1$

Maximum turning point at $(1, 1)$.

Draw both graphs on the same set of axes.

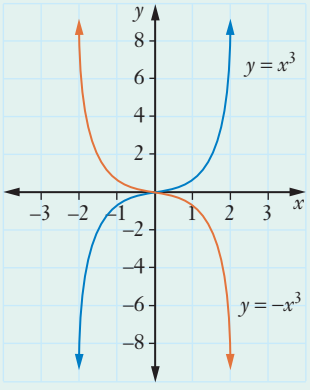
b $f(x) = x^3$ is a cubic curve with a point of inflection at $(0, 0)$.

x	−3	−2	−1	0	1	2	3
y	−27	−8	−1	0	1	8	27

$y = -f(x)$

$= -x^3$

x	−3	−2	−1	0	1	2	3
y	27	8	1	0	−1	−8	−27



$y = -f(x)$ changes the sign of the y values of the original function: from positive to negative, or negative to positive. On the number plane, this means reflecting the graph in the x-axis.

The graph of $y = -f(x)$

The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

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The graph of $y = f(-x)$

We have already seen that some functions are even or odd by finding $f(-x)$. We can see the relationship between $f(x)$ and $f(-x)$ by drawing their graphs.

EXAMPLE 12

For each function, sketch the graph of $y = f(x)$ and $y = f(-x)$ on a number plane.

a $f(x) = x^3 + 1$ **b** $f(x) = \dfrac{1}{x-2}$

Solution

a $f(x) = x^3 + 1$ is a cubic curve with point of inflection at (0, 1).

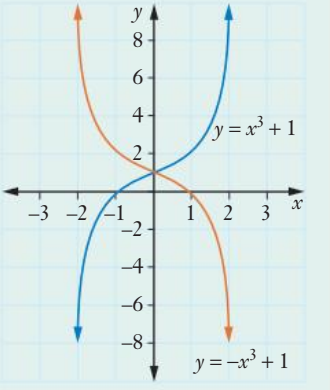
x	−3	−2	−1	0	1	2	3
y	−26	−7	0	1	2	9	28

$y = f(-x)$

$= (-x)^3 + 1$

$= -x^3 + 1$

x	−3	−2	−1	0	1	2	3
y	28	9	2	1	0	−7	−26



Draw $y = x^3 + 1$ and $y = -x^3 + 1$ on the same set of axes.

b $f(x) = \dfrac{1}{x-2}$ is a hyperbola with asymptotes at $x = 2$ and $y = 0$.

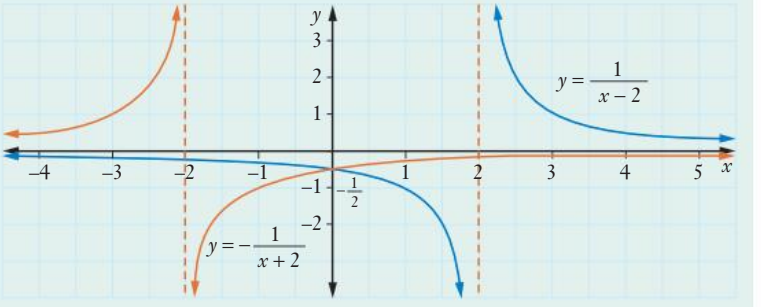
y-intercept, $x = 0$:

$f(0) = \dfrac{1}{0-2} = -\dfrac{1}{2}$

$y = f(-x)$

$= \dfrac{1}{-x-2}$

$= -\dfrac{1}{x+2}$



Asymptotes at $x = -2$ and $y = 0$.

y-intercept, $x = 0$:

$f(0) = -\dfrac{1}{0+2} = -\dfrac{1}{2}$

Draw $y = \dfrac{1}{x-2}$ and $y = -\dfrac{1}{x+2}$ on the same set of axes.

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$y = f(-x)$ changes the sign of the x value of the original function: from positive to negative, or negative to positive. On the number plane, this means reflecting the graph in the y-axis.

The graph of $y = f(-x)$

The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

The graph of $y = -f(-x)$

EXAMPLE 13

For each function, sketch the graph of $y = f(x)$ and $y = -f(-x)$ on the same number plane.

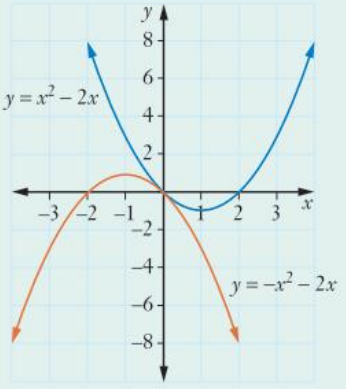
a $f(x) = x^2 - 2x$ **b** $f(x) = \dfrac{2}{x+1}$

Solution

a From Example **12a**, $f(x) = x^2 - 2x$ is a concave upwards parabola with x-intercepts 0, 2.

$$\begin{aligned} y &= -f(-x) \\ &= -\left([-x]^2 - 2[-x]\right) \\ &= -\left(x^2 + 2x\right) \\ &= -x^2 - 2x \end{aligned}$$

A concave downwards parabola with x-intercepts 0, –2.

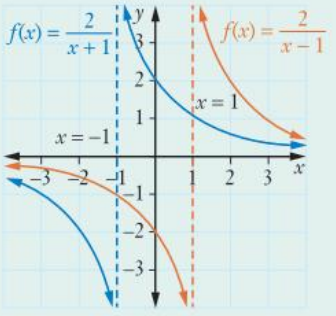


b $f(x) = \dfrac{2}{x+1}$

This is a hyperbola with asymptotes at $x = -1$, $y = 0$ and y-intercept $f(0) = 2$.

$$\begin{aligned} y &= -f(-x) \\ &= -\frac{2}{-x+1} \\ &= \frac{2}{x-1} \end{aligned}$$

This is a hyperbola with asymptotes at $x = 1$, $y = 0$ and y-intercept $f(0) = -2$.



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The graph of $y = -f(-x)$

$y = -f(-x)$ is a reflection of the graph of $y = f(x)$ in both the x- and y-axes.

CLASS DISCUSSION

REFLECTIONS OF FUNCTIONS

Use a graphics calculator or graphing software to draw the graphs of different functions $y = f(x)$ together with:

1 $y = -f(x)$

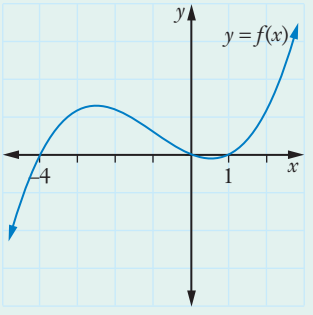
2 $y = f(-x)$

3 $y = -f(-x)$.

Are any of these functions the same as $y = f(x)$ if $f(x)$ is an even or odd function? Why?

EXAMPLE 14

The graph of $y = f(x)$ is shown below.

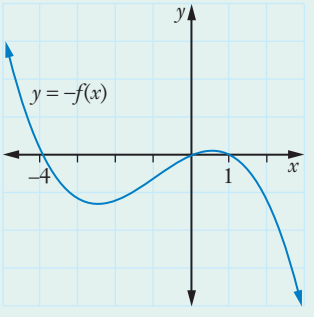


Sketch the graph of:

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

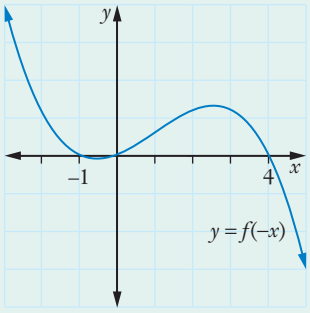
Solution

a $y = -f(x)$ is a reflection in the x-axis.

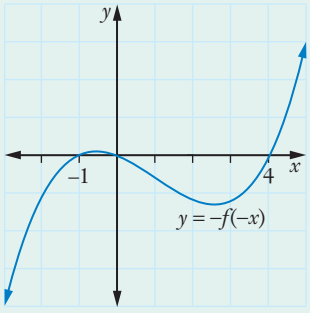


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b $y = f(-x)$ is a reflection in the y-axis.



c $y = -f(-x)$ is a reflection in both the x-and y-axes. Using the graph from **a** that has been reflected in the x-axis and reflecting it in the y-axis gives the graph below.



Exercise 5.04 Reflections of functions

1 For each function, find the equation of:

i $y = -f(x)$ **ii** $y = f(-x)$ **iii** $y = -f(-x)$

a $f(x) = x^2 - 2$ **b** $f(x) = (x + 1)^3$ **c** $y = 5x - 3$

d $y = |2x + 5|$ **e** $f(x) = \dfrac{1}{x - 1}$

2 Describe the type of reflection that each function has on $y = f(x)$.

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

3 Sketch the graphs of the function $f(x) = (x - 1)^2$ and $y = -f(-x)$ on the same number plane.

4 Sketch the graphs of the function $f(x) = 1 - x^3$ and $y = -f(x)$ on the same number plane.

5 For the function $f(x) = x^2 + 2x$, sketch the graph of:

a $y = f(x)$ **b** $y = -f(x)$ **c** $y = f(-x)$ **d** $y = -f(-x)$

6 **a** Show that $f(x) = 2x^2$ is an even function.

b Find the equation of:

i $y = f(-x)$ **ii** $y = -f(x)$

c Sketch the graph of $y = -f(-x)$.

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7 **a** Show that $f(x) = -x^3$ is an odd function.

b Find the equation of:

i $y = -f(x)$ **ii** $y = -f(-x)$

c Sketch the graph of $y = f(-x)$.

8 **a** Find the x- and y-intercepts of the graph of $f(x) = x^3 - 7x^2 + 12x$ and sketch the graph.

b Sketch the graph of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5.05 Combined and composite functions

Sometimes we use different operations to combine 2 different functions.

WS

Circles and composite functions

EXAMPLE 15

For $f(x) = 2x^2 - x + 1$ and $g(x) = x^3 - 2$, write each combined function below as a polynomial and find its degree and constant term.

a $y = f(x) + g(x)$ **b** $y = f(x) - g(x)$ **c** $y = f(x)g(x)$

Solution

a $y = f(x) + g(x)$

$= 2x^2 - x + 1 + x^3 - 2$

$= x^3 + 2x^2 - x - 1$

This polynomial has degree 3 and constant term -1.

b $y = f(x) - g(x)$

$= 2x^2 - x + 1 - (x^3 - 2)$

$= 2x^2 - x + 1 - x^3 + 2$

$= -x^3 + 2x^2 - x + 3$

This polynomial has degree 3 and constant term 3.

c $y = f(x)g(x)$

$= (2x^2 - x + 1)(x^3 - 2)$

$= 2x^5 - 4x^2 - x^4 + 2x + x^3 - 2$

$= 2x^5 - x^4 + x^3 - 4x^2 + 2x - 2$

We could also find the degree by multiplying just the 2 leading terms: $2x^2 \times x^3 = 2x^5$ and find the constant term by multiplying just the 2 constant terms: $1 \times (-2) = (-2)$.

This polynomial has degree 5 and constant term -2.

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EXAMPLE 16

a Find the domain and range of each function below given $f(x) = x^2 - x - 2$ and $g(x) = x - 2$.

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$ **iii** $y = f(x)g(x)$

b Find the domain of $y = \dfrac{f(x)}{g(x)}$ if $f(x) = x^3 + 1$ and $g(x) = x^2 - x - 6$.

Solution

a **i** $y = f(x) + g(x)$

$= x^2 - x - 2 + x - 2$

$= x^2 - 4$

This is a quadratic function with a minimum turning point at $(0, -4)$.

Domain $(-\infty, \infty)$, range $[-4, \infty)$

ii $y = f(x) - g(x)$

$= x^2 - x - 2 - (x - 2)$

$= x^2 - x - 2 - x + 2$

$= x^2 - 2x$

This is a quadratic function with x-intercepts 0, 2.

Axis of symmetry: $x = 1$

Minimum value:

$f(1) = 1^2 - 2(1)$

$= -1$

Domain $(-\infty, \infty)$, range $[-1, \infty)$

iii $y = f(x)g(x)$

$= (x^2 - x - 2)(x - 2)$

$= x^3 - 2x^2 - x^2 + 2x - 2x + 4$

$= x^3 - 3x^2 + 4$

This is a cubic function.

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

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b $y = \dfrac{f(x)}{g(x)}$

$= \dfrac{x^3 + 1}{x^2 - x - 6}$

For domain: $x^2 - x - 6 \neq 0$

$(x - 3)(x + 2) \neq 0$

$x \neq 3, -2$

So domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

Composite functions

A **composite function** $f(g(x))$ is a relationship between functions where the output of one function $g(x)$ becomes the input of a second function $f(x)$.

EXAMPLE 17

a Find the composite function $f(g(x))$ given:

i $f(x) = x^2$ and $g(x) = 2x - 5$

ii $f(x) = x^3$ and $g(x) = x^2 + 3$

iii $f(x) = 5x - 3$ and $g(x) = x^3 + 2$

b Given $f(x) = 5x + 2$ and $g(x) = \dfrac{1}{x}$, find:

i $f(g(x))$ **ii** $g(f(x))$

c Find the domain and range of $f(g(x))$ given $f(x) = \sqrt{x}$ and $g(x) = 9 - x^2$.

Solution

a **i** $f(g(x)) = (2x - 5)^2$

$= 4x^2 - 20x + 25$

ii $f(g(x)) = (x^2 + 3)^3$

$= (x^2 + 3)(x^2 + 3)^2$

$= (x^2 + 3)(x^4 + 6x^2 + 9)$

$= x^6 + 9x^4 + 27x^2 + 27$

iii $f(g(x)) = 5(x^3 + 2) - 3$

$= 5x^3 + 10 - 3$

$= 5x^3 + 7$

b **i** $f(g(x)) = 5\left(\dfrac{1}{x}\right) + 2$

$= \dfrac{5}{x} + 2$

ii $g(f(x)) = \dfrac{1}{5x + 2}$

c $f(g(x)) = \sqrt{9 - x^2}$

This is a semicircle above the x-axis with centre (0, 0) and radius 3.

Domain $[-3, 3]$, range $[0, 3]$

Exercise 5.05 Combined and composite functions

1 For each pair of functions, find the combined function:

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$

iii $y = f(x)g(x)$ **iv** $y = \dfrac{f(x)}{g(x)}$

a $f(x) = 4x + 1$ and $g(x) = 2x^2 + x$

b $f(x) = x^4 + 5x - 4$ and $g(x) = x^3 + 5$

c $f(x) = x^2 + 3$ and $g(x) = 5x^2 - 7x - 2$

d $f(x) = 3x^2 + 2x - 1$ and $g(x) = x^2 - x + 5$

e $f(x) = 4x^5 + 7$ and $g(x) = 3x - 4$

2 For each pair of functions, find the degree of:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 2x + 1$ and $g(x) = 5x - 7$

b $f(x) = x^2$ and $g(x) = 3x + 4$

c $f(x) = (x - 3)^2$ and $g(x) = x^2 - 6x + 1$

d $f(x) = 2x^3$ and $g(x) = x - 2$

3 For each pair of functions, find the constant term of:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 5x^2 + 4$ and $g(x) = x - 7$

b $f(x) = 3x^2 + 1$ and $g(x) = 2x - 5$

c $f(x) = (2x - 5)^2$ and $g(x) = 4x - 3$

d $f(x) = x^3 + 7$ and $g(x) = 2x^2$

4 Find the domain and range of $y = f(x) + g(x)$ given:

a $f(x) = x + 2$ and $g(x) = x - 4$

b $f(x) = 2x^2 + x - 1$ and $g(x) = -x - 1$

c $f(x) = x^3$ and $g(x) = x + 2$

d $f(x) = x^2 - 1$ and $g(x) = x - 1$

5 Find the domain and range of $y = f(x) - g(x)$ given:

a $f(x) = 3x + 2$ and $g(x) = x - 1$

b $f(x) = x^2 - 1$ and $g(x) = x - 1$

c $f(x) = x^3 + x$ and $g(x) = x + 2$

d $f(x) = 3x^2 - x - 1$ and $g(x) = x^2 + x + 3$

6 Find the domain and range of $y = f(x)g(x)$ given:

a $f(x) = x + 2$ and $g(x) = x - 4$

b $f(x) = x - 5$ and $g(x) = x + 5$

c $f(x) = x^2$ and $g(x) = x$

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7 Find the domain of $y = \dfrac{f(x)}{g(x)}$ given:

a $f(x) = 5$ and $g(x) = x - 4$ **b** $f(x) = x - 1$ and $g(x) = x + 1$

c $f(x) = 2x$ and $g(x) = x - 3$ **d** $f(x) = x + 3$ and $g(x) = x^3$

8 Find the composite function $f(g(x))$ given:

a $f(x) = x^2$ and $g(x) = x^2 + 1$ **b** $f(x) = x^3$ and $g(x) = 5x - 3$

c $f(x) = x^7$ and $g(x) = x^2 - 3x + 2$ **d** $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$

e $f(x) = \sqrt[3]{x}$ and $g(x) = x^4 + 7x^2 - 4$ **f** $f(x) = 3x$ and $g(x) = 2x + 1$

g $f(x) = 2x - 7$ and $g(x) = x^3$ **h** $f(x) = 6x - 5$ and $g(x) = x^2$

i $f(x) = 2x^2$ and $g(x) = 3x$ **j** $f(x) = 4x^2 + 1$ and $g(x) = x^2 + 3$

9 Find the domain and range of the composite function $f(g(x))$ given that:

a $f(x) = x^2$ and $g(x) = x - 1$ **b** $f(x) = x^3$ and $g(x) = x + 5$

c $f(x) = \sqrt{x}$ and $g(x) = x - 2$ **d** $f(x) = -\sqrt{x}$ and $g(x) = 3x + 9$

e $f(x) = \sqrt{x}$ and $g(x) = 4 - x^2$ **f** $f(x) = -\sqrt{x}$ and $g(x) = 1 - x^2$

10 If $f(x) = \sqrt{x}$ and $g(x) = x^3$, find:

a $f(g(x))$ **b** $g(f(x))$

11 If $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 + 3$, find:

a $y = f(x)g(x)$ **b** $y = f(g(x))$

c $y = \dfrac{f(x)}{g(x)}$ **d** $y = \dfrac{g(x)}{f(x)}$

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5. TEST YOURSELF

Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 The domain of $y = -\dfrac{3}{x-4}$ is:

A (-4) **B** $(-\infty, 4) \cup (4, \infty)$

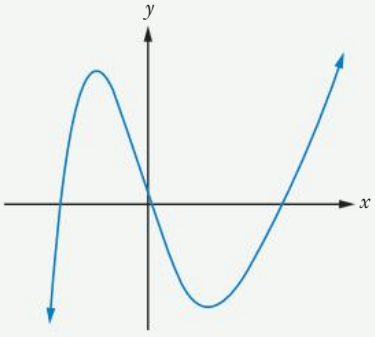
C $(-\infty, -4) \cup (-4, \infty)$ **D** $(-\infty, 4)$

2 The equation of a circle with radius 3 and centre $(1, -2)$ is:

A $(x-1)^2 + (y+2)^2 = 9$ **B** $(x+1)^2 + (y-2)^2 = 9$

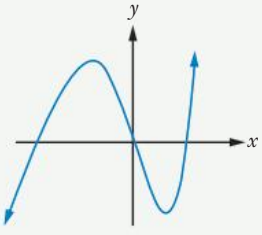
C $(x-1)^2 + (y+2)^2 = 3$ **D** $(x+1)^2 + (y-2)^2 = 3$

3 The graph of $y = f(x)$ is shown below.

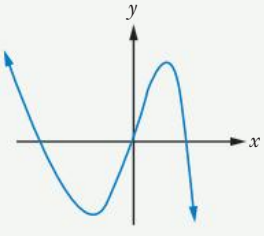


Which one of these is the graph of $y = -f(-x)$?

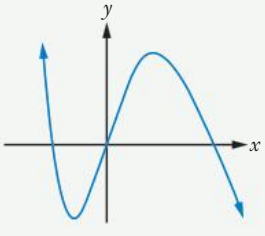
A



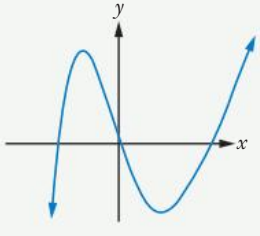
B



C



D



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4 The area of a pizza slice decreases as the number of people sharing it evenly increases. When 5 people share the pizza, the area of each slice is 30 cm^2.

a Find the equation of the area, A, of a pizza slice in terms of the number of people sharing, n.

b What is the area of one pizza slice when:

i 10 people share? **ii** 8 people share?

c How many people are sharing the pizza when each slice has an area of:

i 16. 67 cm^2? **ii** 25 cm^2?

5 Sketch the graph of each function or relation.

a $x^2 + y^2 = 1$ **b** $y = \frac{2}{x}$ **c** $y = |x + 2|$

d $y = -\sqrt{4 - x^2}$ **e** $y = f(-x)$ given $f(x) = \frac{2}{x-1}$

f $y = -f(x)$ if $f(x) = 3x - 6$ **g** $y = -f(-x)$ given $f(x) = x^2 + x$

6 Find the radius and centre of the circle $x^2 - 6x + y^2 - 2y - 6 = 0$.

7 If $f(x) = x^3$ and $g(x) = 3x - 1$, find the equation of:

a $y = f(x) + g(x)$ **b** $y = f(x)g(x)$

c $y = f(g(x))$ **d** $y = g(f(x))$

8 **a** Is the circle $x^2 + y^2 = 1$ a function?

b Change the subject of the equation to y in terms of x.

c Sketch the graphs of 2 separate functions that together make up the circle $x^2 + y^2 = 1$.

9 Find the domain and range of each relation.

a $x^2 + y^2 = 16$ **b** $y = \frac{1}{x+2}$

c $f(x) = |x| + 3$ **d** $y = \sqrt{9 - x^2}$

10 Find the domain and range of $y = f(x) + g(x)$ given $f(x) = x^2 - 4x$ and $g(x) = 2x - 3$.

11 **a** Write down the domain and range of the curve $y = \frac{2}{x-3}$.

b Sketch the graph of $y = \frac{2}{x-3}$.

12 **a** Sketch the graph of $y = |x + 1|$.

b From the graph, solve $|x + 1| = 3$

13 Solve graphically: $|x - 3| = 2$.

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14 Find the centre and radius of the circle with equation:

a $x^2 + y^2 = 100$

b $(x - 3)^2 + (y - 2)^2 = 121$

c $x^2 + 6x + y^2 + 2y + 1 = 0$

15 Find the x- and y-intercepts (where they exist) of:

a $P(x) = x^3 - 4x$ **b** $y = -\dfrac{2}{x+1}$ **c** $x^2 + y^2 = 9$

d $y = \sqrt{25 - x^2}$ **e** $f(x) = |x - 2| + 3$

16 If $f(x) = 2x^2 + x - 6$ and $g(x) = 5x^3 + 1$, find:

a the degree of $y = f(x) + g(x)$

b the leading term of $y = f(x)g(x)$

c the constant term of $y = f(x) - g(x)$

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5. CHALLENGE EXERCISE

1 Solve $|2x + 1| = 3x - 2$ graphically.

2 Given $f(x) = |x| + 3x - 4$, sketch the graph of:

a $y = f(x)$ **b** $y = -f(x)$

3 A variable a is inversely proportional to the square of b. When $b = 3$, $a = 2$.

a Find the equation of a in terms of b.

b Evaluate a when $b = 2$.

c Evaluate b when $a = 10$, correct to 2 decimal places, if $b > 0$.

4 Find the centre and radius of the circle with equation given by $x^2 + 3x + y^2 - 2y - 3 = 0$.

5 Find the equation of the straight line through the centres of the circles with equations $x^2 + 4x + y^2 - 8y - 5 = 0$ and $x^2 - 2x + y^2 + 10y + 10 = 0$.

6 Sketch the graph of $y = \dfrac{|x|}{x^2}$.

7 **a** Show that $\dfrac{2x+7}{x+3} = 2 + \dfrac{1}{x+3}$.

b Find the domain and range of $y = \dfrac{2x+7}{x+3}$.

c Hence sketch the graph of $y = \dfrac{2x+7}{x+3}$.

8 Show that $x^2 - 2x + y^2 + 4y + 1 = 0$ and $x^2 - 2x + y^2 + 4y - 4 = 0$ are concentric.

9 Sketch the graph of $f(x) = 1 - \dfrac{1}{x^2}$.

10 **a** Sketch the graph of $f(x) = \begin{cases} x & \text{for } x < -2 \\ x^2 & \text{for } -2 \le x \le 0 \\ 2 & \text{for } x > 0 \end{cases}$.

b Find any x values for which the function is discontinuous.

c Find the domain and range of the function.

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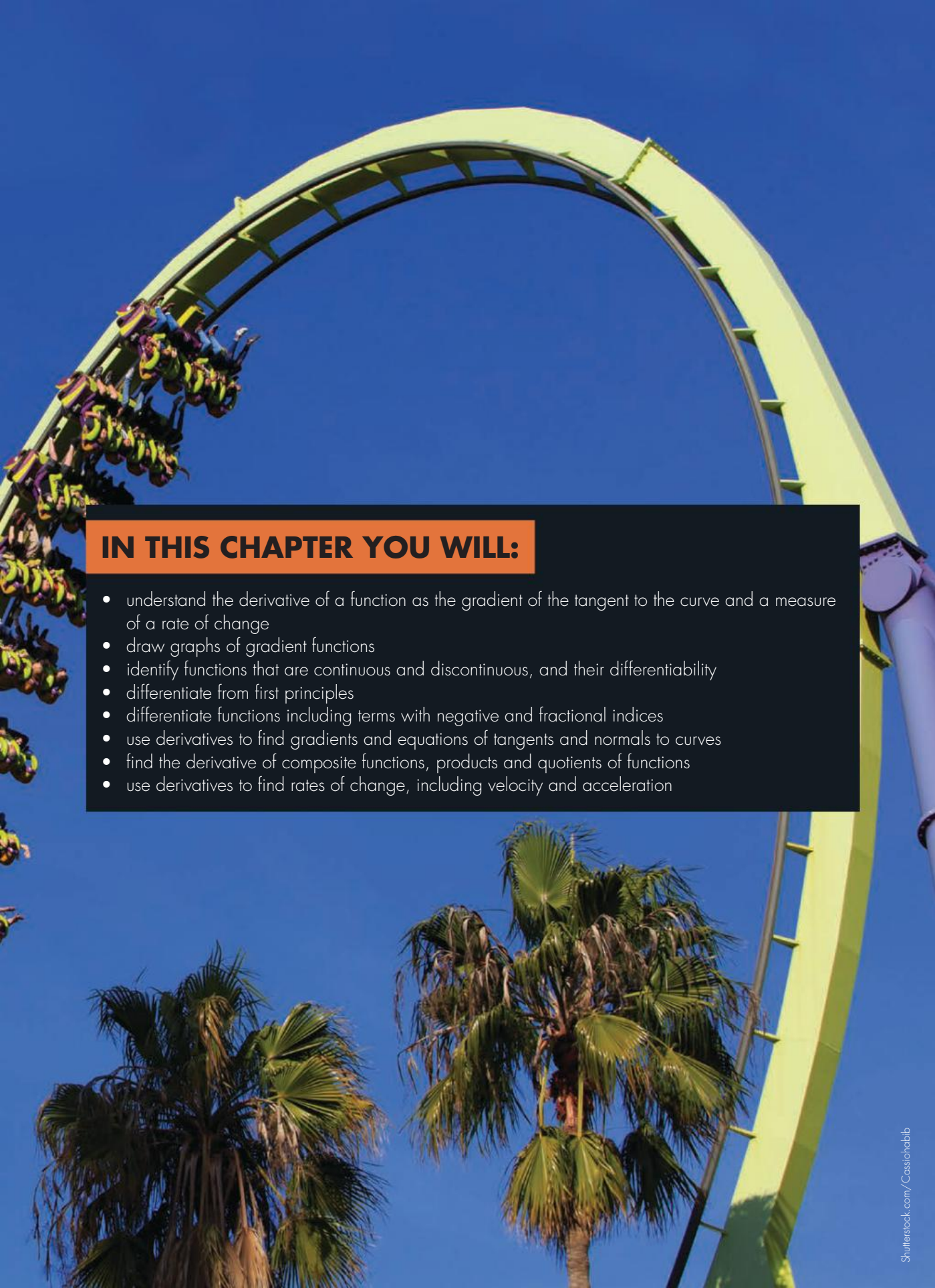
6.

INTRODUCTION TO CALCULUS

Calculus is a very important branch of mathematics that involves the measurement of change. It can be applied to many areas such as science, economics, engineering, astronomy, sociology and medicine. Differentiation, a part of calculus, has many applications involving rates of change: the spread of infectious diseases, population growth, inflation, unemployment, filling of our water reservoirs.

CHAPTER OUTLINE

6.01 Gradient of a curve
6.02 Differentiability
6.03 Differentiation from first principles
6.04 Short methods of differentiation
6.05 Derivatives and indices
6.06 Tangents and normals
6.07 Chain rule
6.08 Product rule
6.09 Quotient rule
6.10 Rates of change



IN THIS CHAPTER YOU WILL:

- understand the derivative of a function as the gradient of the tangent to the curve and a measure of a rate of change
- draw graphs of gradient functions
- identify functions that are continuous and discontinuous, and their differentiability
- differentiate from first principles
- differentiate functions including terms with negative and fractional indices
- use derivatives to find gradients and equations of tangents and normals to curves
- find the derivative of composite functions, products and quotients of functions
- use derivatives to find rates of change, including velocity and acceleration

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TERMINOLOGY

acceleration: The rate of change of velocity with respect to time

average rate of change: The rate of change between 2 points on a function; the gradient of the line (secant) passing through those points

chain rule: A method for differentiating composite functions

derivative function: The gradient function $y = f'(x)$ of a function $y = f(x)$ obtained through differentiation

differentiability: A function is differentiable wherever its gradient is defined

differentiation: The process of finding the gradient function

differentiation from first principles: The process of finding the gradient of a tangent to a curve by finding the gradient of the secant between 2 points and finding the limit as the secant becomes a tangent

displacement: The distance and direction of an object in relation to the origin

gradient of a secant: The gradient (slope) of the line between 2 points on a function; measures the average rate of change between the 2 points

gradient of a tangent: The gradient of a line that is a tangent to the curve at a point on a function; measures the instantaneous rate of change of the function at that point

instantaneous rate of change: The rate of change at a particular point on a function; the gradient of the tangent at this point

limit: The value that a function approaches as the independent variable approaches some value

normal: A line that is perpendicular to the tangent at a given point on a curve

product rule: A method for differentiating the product of 2 functions

quotient rule: A method for differentiating the quotient of 2 functions

secant: A straight line passing through 2 points on the graph of a function

stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection

tangent: A straight line that just touches a curve at one point. The curve has the same gradient or direction as the tangent at that point

turning point: A maximum or minimum point on a curve, where the curve turns around

velocity: The rate of change of displacement of an object with respect to time; involves speed and direction

DID YOU KNOW?

Newton and Leibniz

'Calculus' comes from the Latin meaning 'pebble' or 'small stone'. In many ancient civilisations stones were used for counting, but the mathematics they practised was quite sophisticated.



Isaac Newton

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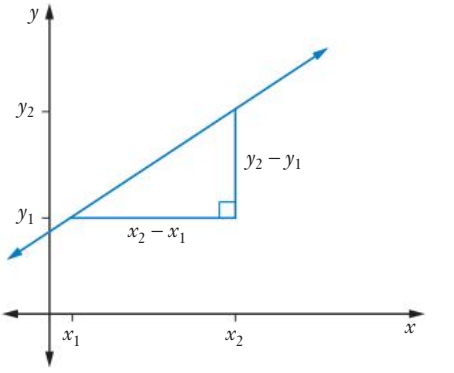
It was not until the 17th century that there was a breakthrough in calculus when scientists were searching for ways of measuring motion of objects such as planets, pendulums and projectiles.

Isaac Newton (1642–1727), an Englishman, discovered the main principles of calculus when he was 23 years old. At this time an epidemic of bubonic plague had closed Cambridge University where he was studying, so many of his discoveries were made at home. He first wrote about his calculus methods, which he called fluxions, in 1671, but his *Method of fluxions* was not published until 1704.

Gottfried Leibniz (1646–1716), in Germany, was studying the same methods and there was intense rivalry between the two countries over who was first to discover calculus!

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6.01 Gradient of a curve



The **gradient** of a straight line measures the **rate of change** of y (the dependent variable) with respect to the change in x (the independent variable).

Gradient

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Notice that when the gradient of a straight line is positive the line is increasing, and when the gradient is negative the line is decreasing. Straight lines increase or decrease at a constant rate and the gradient is the same everywhere along the line.

CLASS DISCUSSION

Remember that an **increasing** line has a positive gradient and a **decreasing** line has a negative gradient.



What is the gradient of a horizontal line?

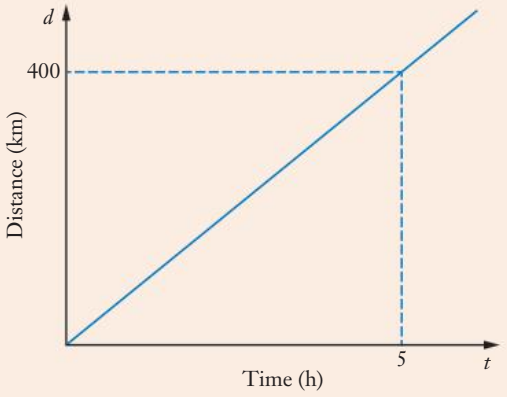


Can you find the gradient of a vertical line? Why?

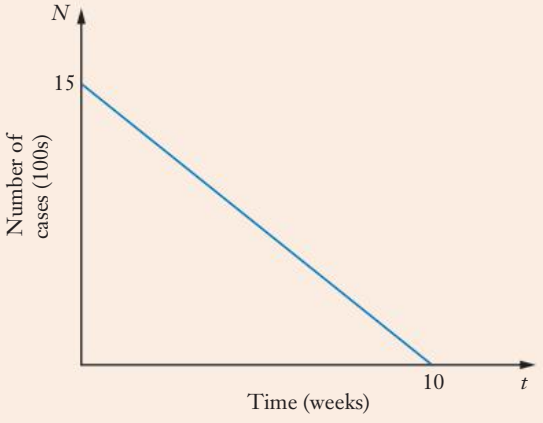
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EXAMPLE 1

a The graph shows the distance travelled by a car over time. Find the gradient and describe it as a rate.



b The graph shows the number of cases of flu reported in a town over several weeks. Find the gradient and describe it as a rate.



Solution

a $m = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{400}{5}$

$= 80$

The line is increasing, so it has a positive gradient.

This means that the car is travelling at a constant rate (speed) of 80 km/h.

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b $m = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{-1500}{10}$

$= -150$

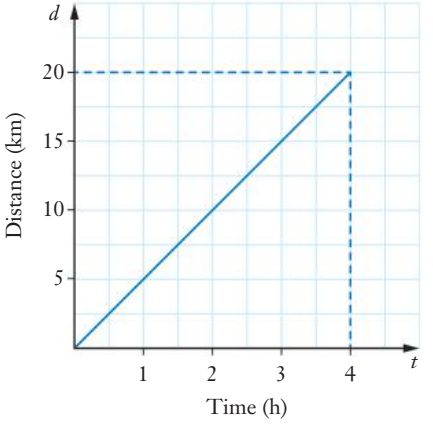
The line is decreasing, so it will have a negative gradient.

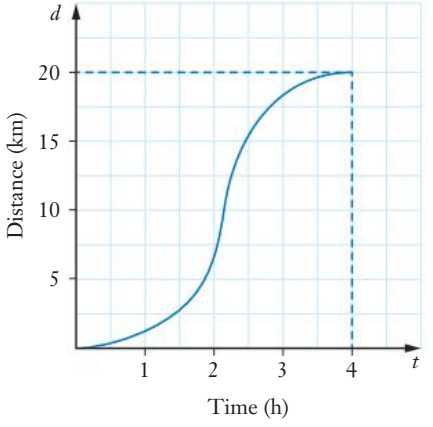
The 'rise' is a drop so it's negative.

This means that the rate is –150 cases/week, or the number of cases reported is decreasing by 150 cases/week.

CLASS DISCUSSION

The 2 graphs below show the distance that a bicycle travels over time. One is a straight line and the other is a curve.





Is the average speed of the bicycle the same in both cases? What is different about the speed in the 2 graphs?

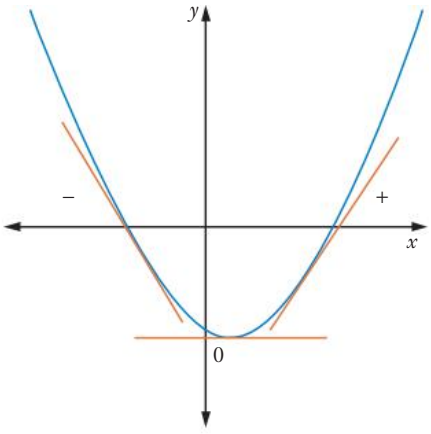
How could you measure the speed in the second graph at any one time? Does it change? If so, how does it change?

We can start finding rates of change along a curve by looking at its shape and how it behaves. We started looking at this in Chapter 3, *Functions*.

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The gradient of a curve shows the **rate of change of y** as x changes. A **tangent** to a curve is a straight line that just touches the curve at one point. We can see where the gradient of a curve is positive, negative or zero by drawing **tangents to the curve** at different places around the curve and finding the gradients of the tangents.

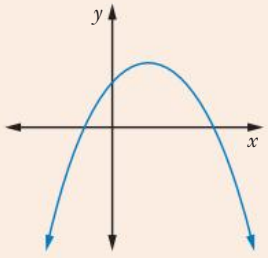
Notice that when the curve increases it has a positive gradient, when it decreases it has a negative gradient, and when it is a **turning point** the gradient is zero.



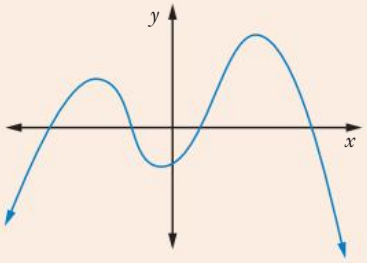
EXAMPLE 2

Copy each curve and write the sign of its gradient along the curve.

a



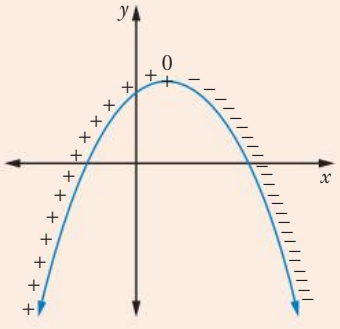
b



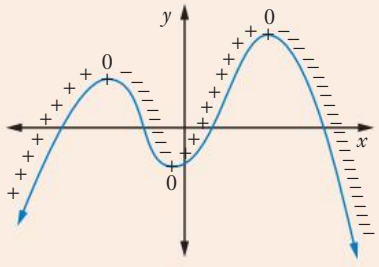
Solution

Where the curve increases, the gradient is positive. Where it decreases, it is negative. Where it is a turning point, it has a zero gradient.

a



b



We find the gradient of a curve by measuring the **gradient of a tangent** to the curve at different points around the curve.

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We can then sketch the graph of these gradient values, which we call $y = f'(x)$, the **gradient function** or the **derivative function**.

EXAMPLE 3

a Make an accurate sketch of $f(x) = x^2$ on graph paper, or use graphing software.

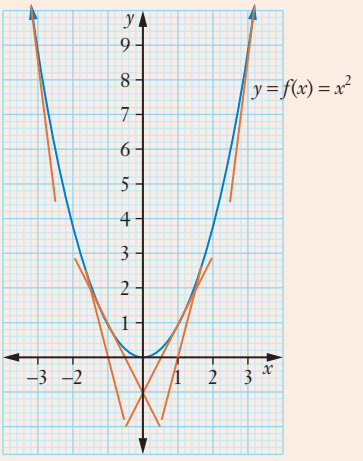
b Draw tangents to this curve at the points where $x = -3$, $x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

c Find the gradient of each of these tangents.

d Draw the graph of $y = f'(x)$ (the derivative or gradient function).

Solution

a and **b**



c At $x = -3$, $m = -6$

At $x = -2$, $m = -4$

At $x = -1$, $m = -2$

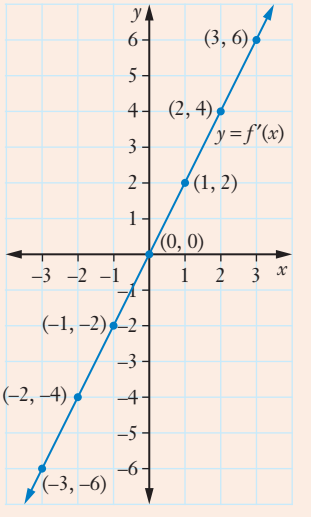
At $x = 0$, $m = 0$

At $x = 1$, $m = 2$

At $x = 2$, $m = 4$

At $x = 3$, $m = 6$

d Using the values from part **c**, $y = f'(x)$ is a linear function.



Notice in Example 3 that where $m > 0$, the gradient function is above the x-axis; where $m = 0$, the gradient function is on the x-axis; and where $m < 0$, the gradient function is below the x-axis. Since $m = f'(x)$, we can write the following:

Sketching gradient (derivative) functions

$f'(x) > 0$: gradient function is above the x-axis

$f'(x) < 0$: gradient function is below the x-axis

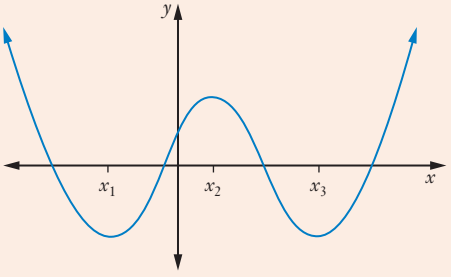
$f'(x) = 0$: gradient function is on the x-axis

Sketching gradient functions

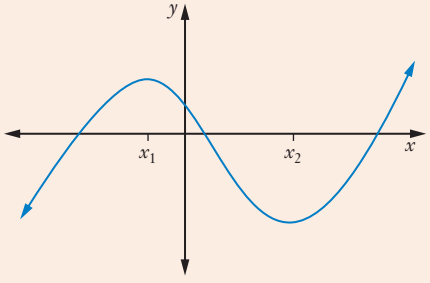
EXAMPLE 4

Sketch a gradient function for each curve.

a

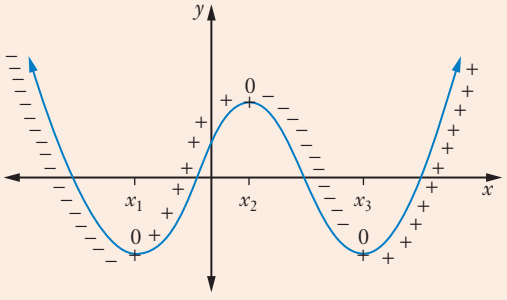


b

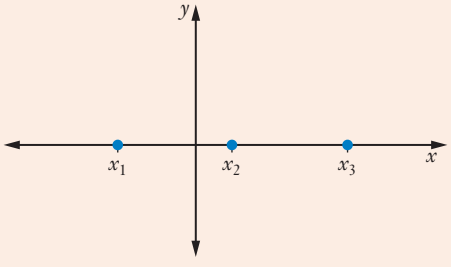


Solution

a First we mark in where the gradient is positive, negative and zero.



$f'(x) = 0$ at x_1, x_2 and x_3, so on the gradient graph these points will be on the x-axis (the x-intercepts of the gradient graph).

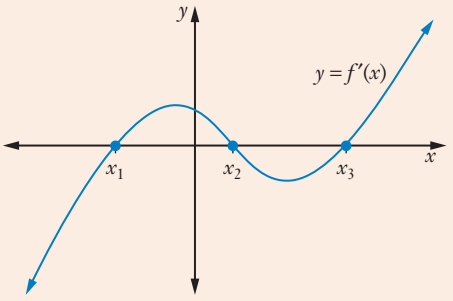


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$f'(x) < 0$ to the left of x_1, so this part of the gradient graph will be below the x-axis.

$f'(x) > 0$ between x_1 and x_2, so the graph will be above the x-axis here.

$f'(x) < 0$ between x_2 and x_3, so the graph will be below the x-axis here.



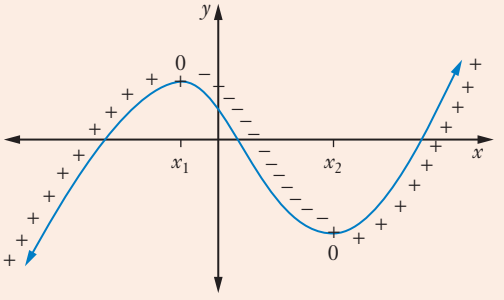
$f'(x) > 0$ to the right of x_3, so this part of the graph will be above the x-axis.

Sketching this information gives the graph of the gradient function $y = f'(x)$. Note that this is only a rough graph that shows the shape and sign rather than precise values.

b First mark in where the gradient is positive, negative and zero.

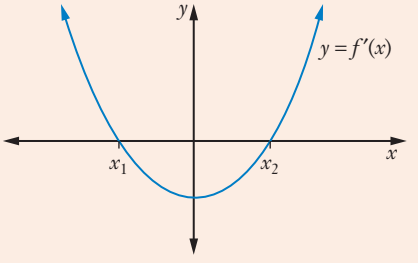
$f'(x) = 0$ at x_1 and x_2. These points will be the x-intercepts of the gradient function graph.

$f'(x) > 0$ to the left of x_1, so the graph will be above the x-axis here.



$f'(x) < 0$ between x_1 and x_2, so the graph will be below the x-axis here.

$f'(x) > 0$ to the right of x_2, so the graph will be above the x-axis here.



TECHNOLOGY

Tangents to a curve

There are some excellent graphing software, online apps and websites that will draw tangents to a curve and sketch the gradient function.

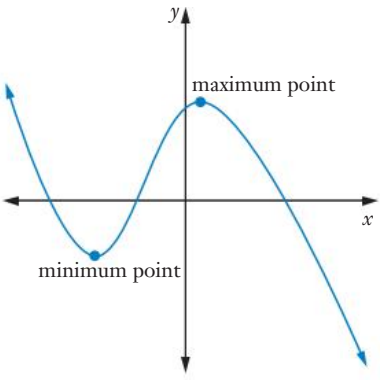
Explore how to sketch gradient functions from the previous examples.

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Stationary points

The points on a curve where the gradient $f'(x) = 0$ are called **stationary points** because the gradient there is neither increasing nor decreasing.

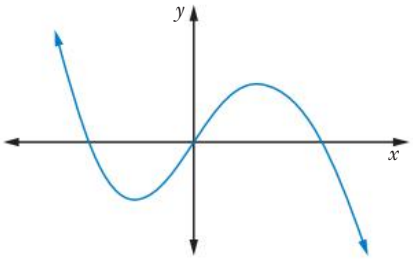
For example, the curve shown decreases to a **minimum turning point**, which is a type of **stationary point**. It then increases to a **maximum turning point** (also a stationary point) and then decreases again.



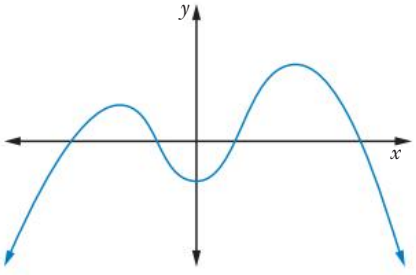
Exercise 6.01 Gradient of a curve

Sketch a gradient function for each curve.

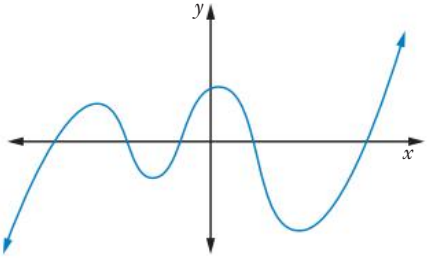
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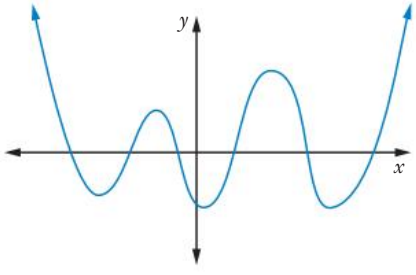
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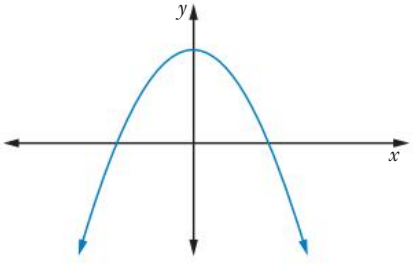
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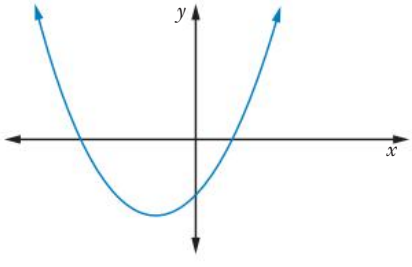
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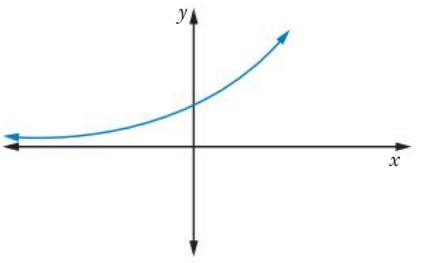


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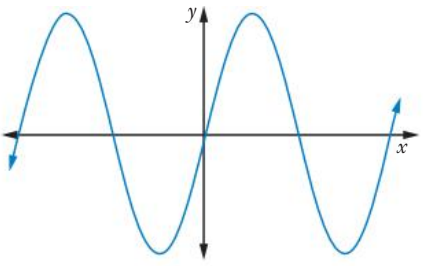


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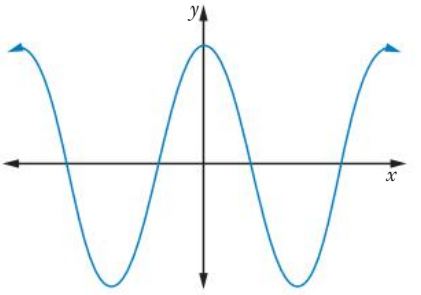
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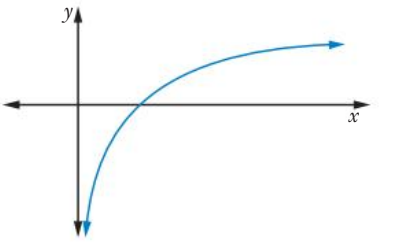
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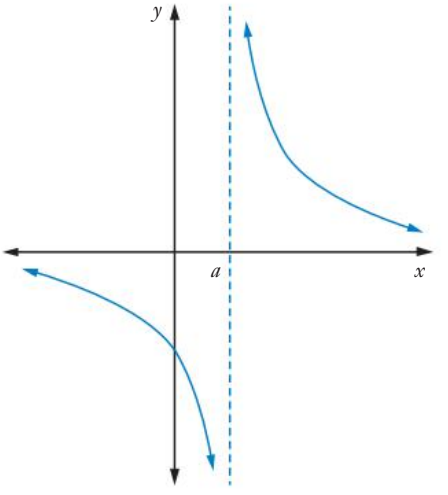
6.02 Differentiability

The process of finding the gradient function $y = f'(x)$ is called **differentiation**.

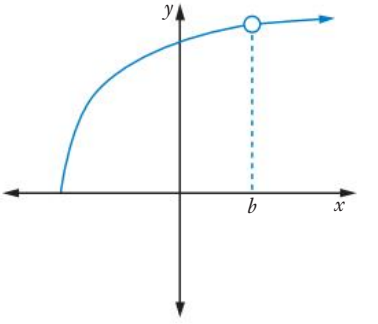
$y = f'(x)$ is called the **derivative function**, or just the **derivative**.

A function is **differentiable** at any point where it is continuous because we can find its gradient at that point. Linear, quadratic, cubic and other polynomial functions are differentiable at all points because their graphs are smooth and unbroken. A function is **not differentiable** at any point where it is **discontinuous**, where there is a gap or break in its graph.

This hyperbola is not differentiable at $x = a$ because the curve is discontinuous at this point.



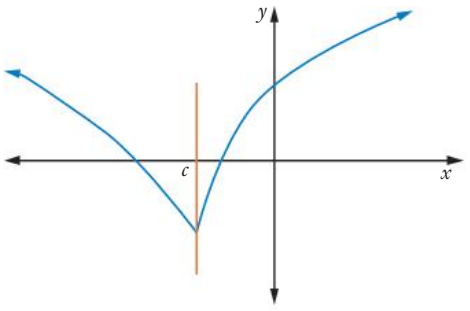
This function is not differentiable at $x = b$ because the curve is discontinuous at this point.



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A function is also **not differentiable** where it is not smooth.

This function is not **differentiable** at $x = c$ since it is not smooth at that point. We cannot draw a unique tangent there so we cannot find the gradient of the function at that point.

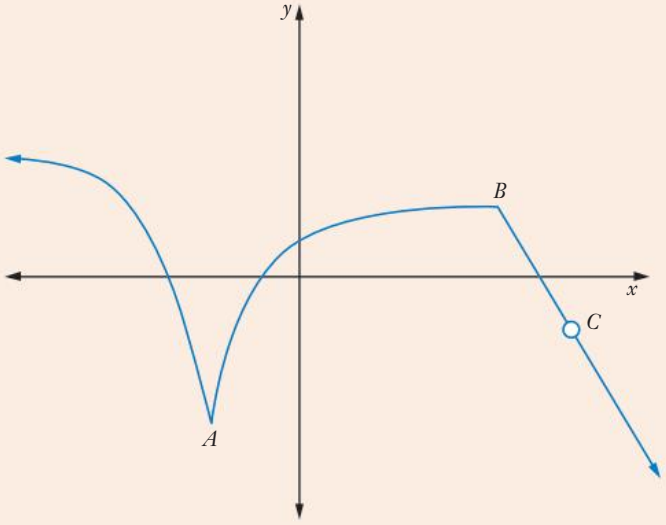


Differentiability at a point

A function $y = f(x)$ is **differentiable** at the point $x = a$ if its graph is **continuous** and **smooth** at $x = a$.

EXAMPLE 5

a Find all points where the function below is not differentiable.



b Is the function $f(x) = \begin{cases} x^2 & \text{for } x \geq 1 \\ 3x - 2 & \text{for } x < 1 \end{cases}$ differentiable at all points?

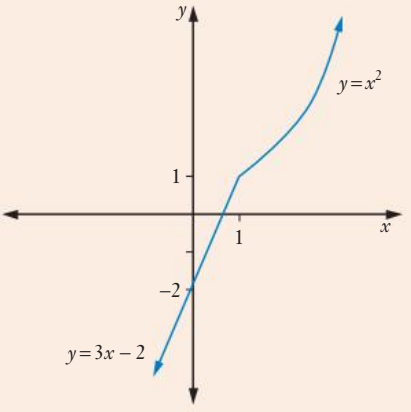
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Solution

a The function is not differentiable at points A and B because the curve is not smooth at these points.

It is not differentiable at point C because the function is discontinuous at this point.

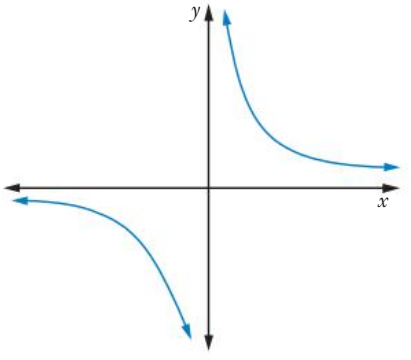
b Sketching this piecewise function shows that it is not smooth where the 2 parts meet, so it is not differentiable at $x = 1$.



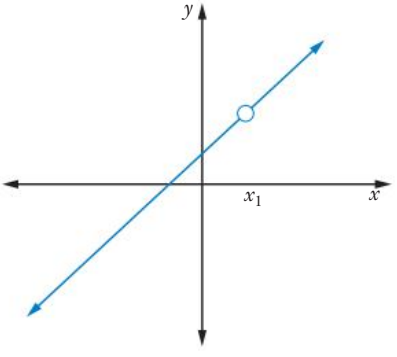
Exercise 6.02 Differentiability

For each graph of a function, state any x values where the function is not differentiable.

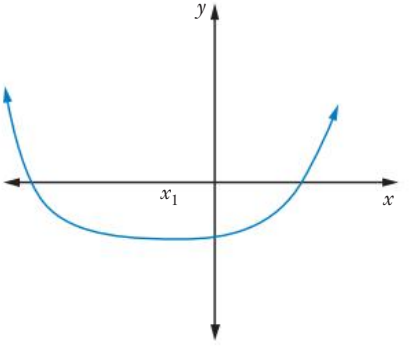
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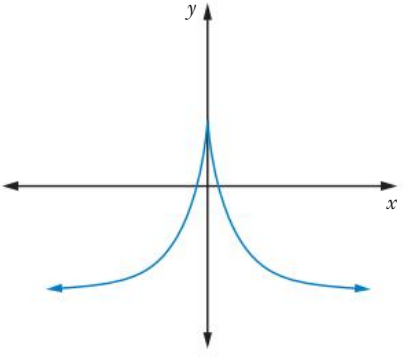
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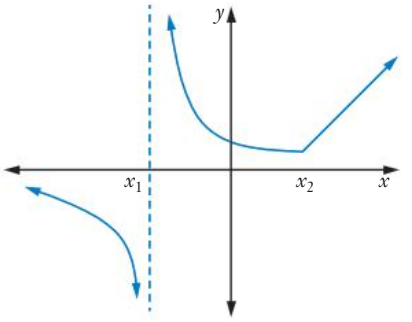


4



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5



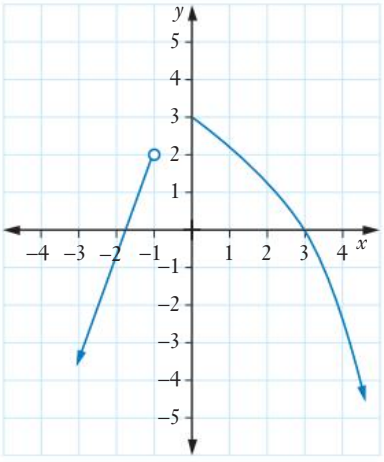
6 $f(x) = \dfrac{4}{x}$

7 $y = -\dfrac{1}{x+3}$

8 $f(x) = \begin{cases} x^3 & \text{for } x > 2 \\ x+1 & \text{for } x \leq 2 \end{cases}$

9 $f(x) = \begin{cases} 2x & \text{for } x > 3 \\ 3 & \text{for } -2 \leq x \leq 3 \\ 1-x^2 & \text{for } x < -2 \end{cases}$

10



11 $f(x) = \dfrac{|x|}{x}$

WS Differentiation from first principles

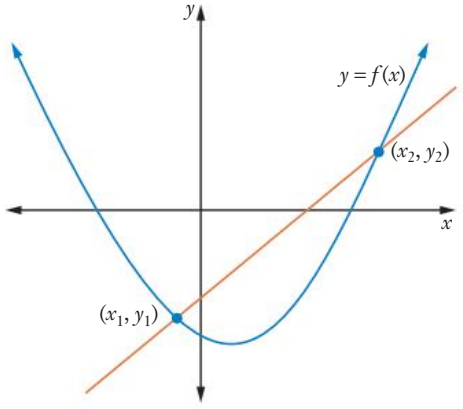
WS Limits

WS Finding derivatives from first principles

WS Rates of change - Gradients of secants

6.03 Differentiation from first principles

Gradient of a secant



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The line passing through the 2 points (x_1, y_1) and (x_2, y_2) on the graph of a function $y = f(x)$ is called a **secant**.



Differentiation from first principles



Limits

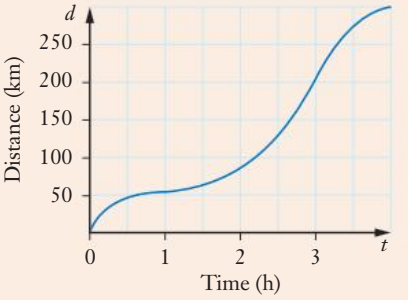
Gradient of the secant

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **gradient of a secant** gives the **average rate of change** between the 2 points.

EXAMPLE 6

a This graph shows the distance d in km that a car travels over time t in hours. After 1 hour the car has travelled 55 km and after 3 hours the car has travelled 205 km. Find the average speed of the car.



b Given the function $f(x) = x^2$, find the average rate of change between $x = 1$ and $x = 1.1$.

Solution

a Speed is the change in distance over time.

The gradient of the secant will give the average speed.

Average rate of change:

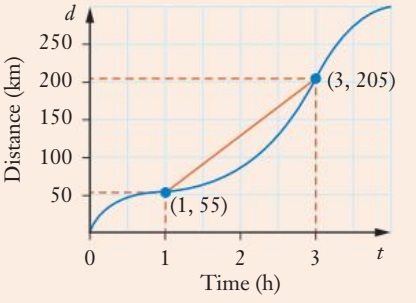
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{205 - 55}{3 - 1}$$

$$= \frac{150}{2}$$

$$= 75$$

So the average speed is 75 km/h.



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b When $x = 1$:

$f(1) = 1^2$

$= 1$

When $x = 1.1$:

$f(1.1) = 1.1^2$

$= 1.21$

So points are (1, 1) and (1.1, 1.21).

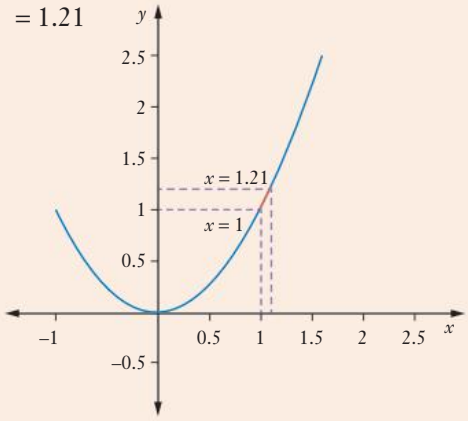
Average rate of change:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1.21 - 1}{1.1 - 1}$$

$$= 2.1$$

So the average rate of change is 2.1.



Notice that the secant (orange interval) is very close to the shape of the curve itself. This is because the 2 points chosen are close together.

Estimating the gradient of a tangent

By taking 2 points close together, the **average rate of change** is quite close to the gradient of the tangent to the curve at one of those points, which is called the **instantaneous rate of change** at that point.

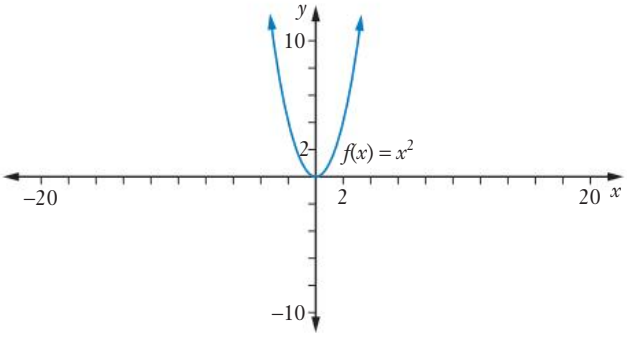
If you look at a close-up of a graph, you can get some idea of this concept. When the curve is magnified, any 2 points close together appear to be joined by a straight line. We say the curve is **locally straight**.

TECHNOLOGY

Locally straight curves

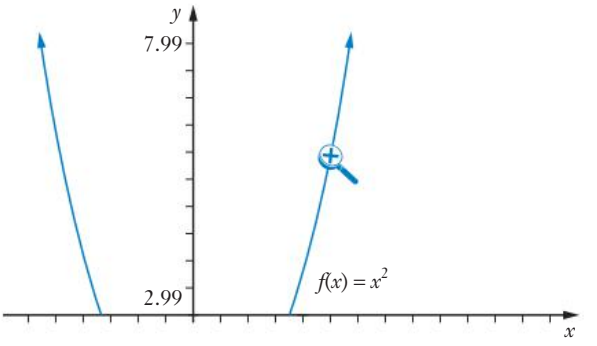
Use a graphics calculator or graphing software to sketch a curve and then zoom in on a section of the curve to see that it is locally straight.

For example, here is the parabola $y = x^2$.



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Notice how it looks straight when we zoom in on a point on the parabola.

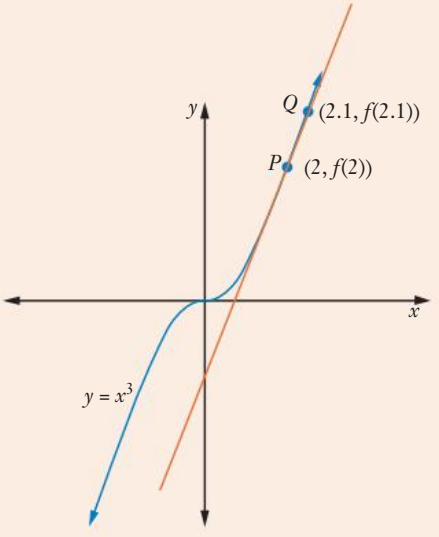


Use technology to sketch other curves and zoom in to show that they are locally straight.

We can calculate an approximate value for the gradient of the tangent at a point on a curve by taking another point close by, then calculating the gradient of the secant joining those 2 points.

EXAMPLE 7

a For $f(x) = x^3$, find the gradient of the secant PQ where P is the point on the curve where $x = 2$ and Q is another point on the curve where $x = 2.1$. Then choose different values for Q and use these results to estimate $f'(2)$, the gradient of the tangent to the curve at P.



b For the curve $y = x^2$, find the gradient of the secant AB where A is the point on the curve where $x = 5$ and point B is close to A. Find an estimate of the gradient of the tangent to the curve at A by using 3 different values for B.

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Solution

a P is $(2, f(2))$. Take different values of x for point Q, starting with $x = 2.1$, and find the gradient of the secant using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Point Q	Gradient of secant PQ	Point Q	Gradient of secant PQ
$(2.1, f(2.1))$	$m = \dfrac{f(2.1) - f(2)}{2.1 - 2}$ $= \dfrac{2.1^3 - 2^3}{0.1}$ $= 12.61$	$(1.9, f(1.9))$	$m = \dfrac{f(1.9) - f(2)}{1.9 - 2}$ $= \dfrac{1.9^3 - 2^3}{-0.1}$ $= 11.41$
$(2.01, f(2.01))$	$m = \dfrac{f(2.01) - f(2)}{2.01 - 2}$ $= \dfrac{2.01^3 - 2^3}{0.01}$ $= 12.0601$	$(1.99, f(1.99))$	$m = \dfrac{f(1.99) - f(2)}{1.99 - 2}$ $= \dfrac{1.99^3 - 2^3}{-0.01}$ $= 11.9401$
$(2.001, f(2.001))$	$m = \dfrac{f(2.001) - f(2)}{2.001 - 2}$ $= \dfrac{2.001^3 - 2^3}{0.001}$ $= 12.006\,001$	$(1.999, f(1.999))$	$m = \dfrac{f(1.999) - f(2)}{1.999 - 2}$ $= \dfrac{1.999^3 - 2^3}{-0.001}$ $= 11.994\,001$

From these results, we can see that a good estimate for $f'(2)$, the gradient at P, is 12.

As $x \to 2, f'(2) \to 12$.

We use a special notation for **limits** to show this.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$
$$= 12$$

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b $A = (5, f(5))$

Take 3 different values of x for point B; for example, $x = 4.9$, $x = 5.1$ and $x = 5.01$.

$B = (4.9, f(4.9))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(4.9) - f(5)}{4.9 - 5}$$
$$= \frac{4.9^2 - 5^2}{-0.1}$$
$$= 9.9$$

$B = (5.1, f(5.1))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(5.1) - f(5)}{5.1 - 5}$$
$$= \frac{5.1^2 - 5^2}{0.1}$$
$$= 10.1$$

$B = (5.01, f(5.01))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(5.01) - f(5)}{5.01 - 5}$$
$$= \frac{5.01^2 - 5^2}{0.01}$$
$$= 10.01$$

As $x \to 5, f'(5) \to 10$.

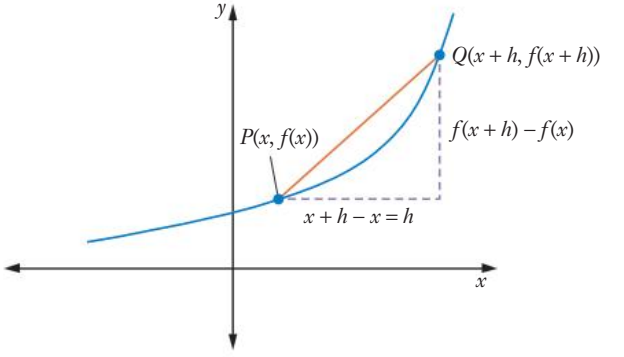
$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5}$$
$$= 10$$

The difference quotient

We measure the instantaneous rate of change of any point on the graph of a function by using limits to find the gradient of the tangent to the curve at that point. This is called **differentiation from first principles**. Using the method from the examples above, we can find a general formula for the derivative function $y = f'(x)$.

We want to find the instantaneous rate of change or gradient of the tangent to a curve $y = f(x)$ at point $P(x, f(x))$.

We choose a second point Q close to P with coordinates $(x + h, f(x + h))$ where h is small.



Now find the gradient of the secant PQ.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{x + h - x}$$

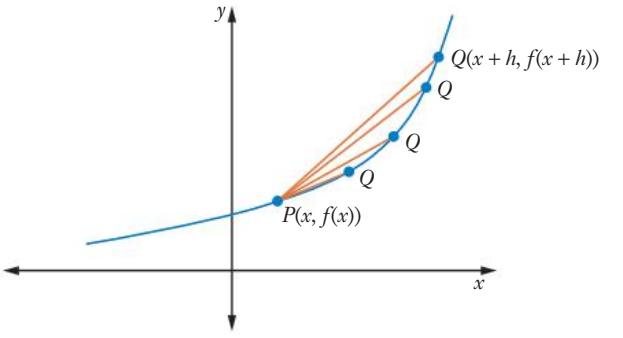
$$= \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h) - f(x)}{h}$ is called the **difference quotient** and it gives an **average rate of change**.

To find the gradient of the tangent at P, we make h smaller as shown, so that Q becomes closer and closer to P.

As h approaches 0, the gradient of the tangent becomes $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

We call this $f'(x)$ or $\frac{dy}{dx}$ or y'.



Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

INVESTIGATION

CALCULUS NOTATION

On p.274, we learned about the mathematicians Isaac Newton and Gottfried Leibniz. Newton used the notation $f'(x)$ for the derivative function while Leibniz used the notation $\frac{dy}{dx}$ where d stood for 'difference'. Can you see why he would have used this? Use the Internet to explore the different notations used in calculus and where they came from.

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EXAMPLE 8

a Differentiate from first principles to find the gradient of the tangent to the curve $y = x^2 + 3$ at the point where $x = 1$.

b Differentiate $f(x) = 2x^2 + 7x - 3$ from first principles.

Solution

a

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 + 3$$

$$f(x+h) = (x+h)^2 + 3$$

$$= x^2 + 2xh + h^2 + 3$$

Substitute $x = 1$:

$$f(1) = 1^2 + 3$$

$$= 4$$

$$f(1+h) = 1^2 + 2(1)h + h^2 + 3$$

$$= 4 + 2h + h^2$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 2h + h^2 - 4}{h}$$

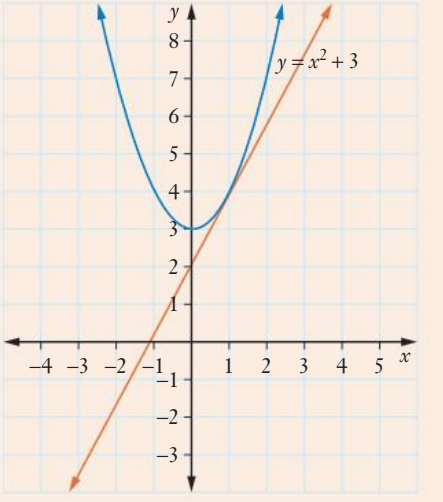
$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} (2+h)$$

$$= 2 + 0$$

$$= 2$$



So the gradient of the tangent to the curve $y = x^2 + 3$ at the point (1, 4) is 2.

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b

$$f(x) = 2x^2 + 7x - 3$$

$$f(x+h) = 2(x+h)^2 + 7(x+h) - 3$$

$$= 2(x^2 + 2xh + h^2) + 7x + 7h - 3$$

$$= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3$$

$$f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - (2x^2 + 7x - 3)$$

$$= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - 2x^2 - 7x + 3$$

$$= 4xh + 2h^2 + 7h$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 + 7h}{h}$$

$$= \lim_{h \to 0} \frac{h(4x + 2h + 7)}{h}$$

$$= \lim_{h \to 0} (4x + 2h + 7)$$

$$= 4x + 0 + 7$$

$$= 4x + 7$$

So the gradient function (derivative) of $f(x) = 2x^2 + 7x - 3$ is $f'(x) = 4x + 7$.

Exercise 6.03 Differentiation from first principles

1 **a** For the curve $y = x^4 + 1$, find the gradient of the secant between the point (1, 2) and the point where $x = 1.01$.

b Find the gradient of the secant between (1, 2) and the point where $x = 0.999$ on the curve.

c Use these results to find an approximation to the gradient of the tangent to the curve $y = x^4 + 1$ at the point (1, 2).

2 For the function $f(x) = x^3 + x$, find the average rate of change between the point (2, 10) and the point on the curve where:

a $x = 2.1$ **b** $x = 2.01$ **c** $x = 1.99$

Hence find an approximation to the gradient of the tangent at the point (2, 10).

3 For the function $f(x) = x^2 - 4$, find the gradient of the tangent at point P where $x = 3$ by selecting points near P and finding the gradient of the secant.

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4 A function is given by $f(x) = x^2 + x + 5$.

a Find $f(2)$.

b Find $f(2 + h)$.

c Find $f(2 + h) - f(2)$.

d Show that $\frac{f(2+h)-f(2)}{h} = 5 + h$.

e Find $f'(2)$.

5 Given the curve $f(x) = 4x^2 - 3$, find:

a $f(-1)$ **b** $f(-1 + h) - f(-1)$

c the gradient of the tangent to the curve at the point where $x = -1$.

6 For the parabola $y = x^2 - 1$, find:

a $f(3)$ **b** $f(3 + h) - f(3)$ **c** $f'(3)$.

7 For the function $f(x) = 4 - 3x - 5x^2$, find:

a $f'(1)$ **b** the gradient of the tangent at the point $(-2, -10)$.

8 If $f(x) = x^2$:

a find $f(x + h)$ **b** show that $f(x + h) - f(x) = 2xh + h^2$

c show that $\frac{f(x+h)-f(x)}{h} = 2x + h$ **d** show that $f'(x) = 2x$.

9 A function is given by $f(x) = 2x^2 - 7x + 3$.

a Show that $f(x + h) = 2x^2 + 4xh + 2h^2 - 7x - 7h + 3$.

b Show that $f(x + h) - f(x) = 4xh + 2h^2 - 7h$.

c Show that $\frac{f(x+h)-f(x)}{h} = 4x + 2h - 7$.

d Find $f'(x)$.

10 Differentiate from first principles to find the gradient of the tangent to the curve:

a $f(x) = x^2$ at the point where $x = 1$

b $y = x^2 + x$ at the point $(2, 6)$

c $f(x) = 2x^2 - 5$ at the point where $x = -3$

d $y = 3x^2 + 3x + 1$ at the point where $x = 2$

e $f(x) = x^2 - 7x - 4$ at the point $(-1, 4)$.

11 Find the derivative function for each function from first principles.

a $f(x) = x^2$ **b** $y = x^2 + 5x$

c $f(x) = 4x^2 - 4x - 3$ **d** $y = 5x^2 - x - 1$

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WS Derivatives of linear products

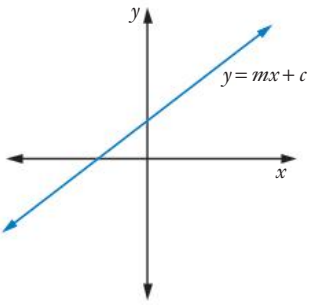
WS Derivatives of polynomials

6.04 Short methods of differentiation

Derivative of x^n

Remember that the gradient of a straight line $y = mx + c$ is m. The tangent to the line is the line itself, so the gradient of the tangent is m everywhere along the line.

So if $y = mx$, $\frac{dy}{dx} = m$.

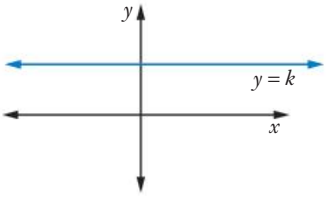


Derivative of kx

$$\frac{d}{dx}(kx) = k$$

A horizontal line $y = k$ has a gradient of zero.

So if $y = k$, $\frac{dy}{dx} = 0$.



Derivative of k

$$\frac{d}{dx}(k) = 0$$

TECHNOLOGY

Differentiation of powers of X

Find an approximation to the derivative of power functions such as $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$ by drawing the graph of $y = \frac{f(x+0.01) - f(x)}{0.01}$. You could use a graphics calculator or graphing software/website to sketch the derivative for these functions and find its equation. Can you find a pattern? Could you predict what the result would be for x^n?

When differentiating $y = x^n$ from first principles, a simple pattern appears:

- For $y = x$, $f'(x) = 1x^0 = 1$
- For $y = x^2$, $f'(x) = 2x^1 = 2x$
- For $y = x^3$, $f'(x) = 3x^2$
- For $y = x^4$, $f'(x) = 4x^3$
- For $y = x^5$, $f'(x) = 5x^4$

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Derivative of x^n

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.

There are some more properties of differentiation.

Derivative of kx^n

$$\frac{d}{dx}(kx^n) = knx^{n-1}$$

More generally:

Derivative of a constant multiple of a function

$$\frac{d}{dx}\left(kf(x)\right) = kf'(x)$$

EXAMPLE 9

a Find the derivative of $3x^8$.

b Differentiate $f(x) = 7x^3$.

Solution

a
$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$
$$\frac{d}{dx}(3x^8) = 3 \times 8x^{8-1}$$
$$= 24x^7$$

b
$$f'(x) = knx^{n-1}$$
$$f'(x) = 7 \times 3x^{3-1}$$
$$= 21x^2$$

If there are several terms in an expression, we differentiate each one separately.

Derivative of a sum of functions

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

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EXAMPLE 10

a Differentiate $x^3 + x^4$.

b Find the derivative of $7x$.

c Differentiate $f(x) = x^4 - x^3 + 5$.

d Find the derivative of $y = 4x^7$.

e If $f(x) = 2x^5 - 7x^3 + 5x - 4$, evaluate $f'(-1)$.

f Find the derivative of $f(x) = 2x^2(3x - 7)$.

g Find the derivative of $\dfrac{3x^2 + 5x}{2x}$.

h Differentiate $S = 6r^2 - 12r$ with respect to r.

Solution

a $\dfrac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3$

b $\dfrac{d}{dx}(7x) = 7$

c $f'(x) = 4x^3 - 3x^2 + 0$

$= 4x^3 - 3x^2$

d $\dfrac{dy}{dx} = 4 \times 7x^6$

$= 28x^6$

e $f'(x) = 10x^4 - 21x^2 + 5$

$f'(-1) = 10(-1)^4 - 21(-1)^2 + 5$

$= -6$

f Expand first.

$f(x) = 2x^2(3x - 7)$

$= 6x^3 - 14x^2$

$f'(x) = 18x^2 - 28x$

g Simplify first.

$$\frac{3x^2 + 5x}{2x} = \frac{3x^2}{2x} + \frac{5x}{2x}$$

$$= \frac{3x}{2} + \frac{5}{2}$$

$$\frac{d}{dx}\left(\frac{3x^2 + 5x}{2x}\right) = \frac{3}{2}$$

$$= 1\frac{1}{2}$$

h Differentiating with respect to r rather than x:

$S = 6r^2 - 12r$

$\dfrac{dS}{dr} = 12r - 12$

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INVESTIGATION

FAMILIES OF CURVES

1 Differentiate:

a $x^2 + 1$ **b** $x^2 - 3$ **c** $x^2 + 7$
d x^2 **e** $x^2 + 20$ **f** $x^2 - 100$

What do you notice?

2 Differentiate:

a $x^3 + 5$ **b** $x^3 + 11$ **c** $x^3 - 1$
d $x^3 - 6$ **e** x^3 **f** $x^3 + 15$

What do you notice?

These groups of functions are families because they have the same derivatives. Can you find others?

Exercise 6.04 Short methods of differentiation

1 Differentiate:

a $x + 2$ **b** $5x - 9$ **c** $x^2 + 3x + 4$
d $5x^2 - x - 8$ **e** $x^3 + 2x^2 - 7x - 3$ **f** $2x^3 - 7x^2 + 7x - 1$
g $3x^4 - 2x^2 + 5x$ **h** $x^6 - 5x^5 - 2x^4$ **i** $2x^5 - 4x^3 + x^2 - 2x + 4$
j $4x^{10} - 7x^9$

2 Find the derivative of:

a $x(2x + 1)$ **b** $(2x - 3)^2$ **c** $(x + 4)(x - 4)$
d $(2x^2 - 3)^2$ **e** $(2x + 5)(x^2 - x + 1)$

3 Find the derivative of:

a $\frac{x^2}{6} - x$ **b** $\frac{x^4}{2} - \frac{x^3}{3} + 4$ **c** $\frac{1}{3}x^6(x^2 - 3)$

d $\frac{2x^3 + 5x}{x}$ **e** $\frac{x^2 + 2x}{4x}$ **f** $\frac{2x^5 - 3x^4 + 6x^3 - 2x^2}{3x^2}$

4 Find $f'(x)$ when $f(x) = 8x^2 - 7x + 4$.

5 If $y = x^4 - 2x^3 + 5$, find $\frac{dy}{dx}$ when $x = -2$.

6 Find $\frac{dy}{dx}$ if $y = 6x^{10} - 5x^8 + 7x^5 - 3x + 8$.

7 If $s = 5t^2 - 20t$, find $\frac{ds}{dt}$.

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8 Find $g'(x)$ given $g(x) = 5x^4$.

9 Find $\frac{dv}{dt}$ when $v = 15t^2 - 9$.

10 If $h = 40t - 2t^2$, find $\frac{dh}{dt}$.

11 Given $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$.

12 If $f(x) = 2x^3 - 3x + 4$, evaluate $f'(1)$.

13 Given $f(x) = x^2 - x + 5$, evaluate:

a $f'(3)$ **b** $f'(-2)$ **c** x when $f'(x) = 7$

14 If $y = x^3 - 7$, evaluate:

a the derivative when $x = 2$ **b** x when $\frac{dy}{dx} = 12$

15 Evaluate $g'(2)$ when $g(t) = 3t^3 - 4t^2 - 2t + 1$.

DID YOU KNOW?

Motion and calculus



Galileo

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Galileo (1564–1642) was very interested in the behaviour of bodies in motion. He dropped stones from the Leaning Tower of Pisa to try to prove that they would fall with equal speed. He rolled balls down slopes to prove that they move with uniform speed until friction slows them down. He showed that a body moving through the air follows a curved path at a fairly constant speed.

John Wallis (1616–1703) continued this study with his publication *Mechanica, sive Tractatus de Motu Geometricus*. He applied mathematical principles to the laws of motion and stimulated interest in the subject of mechanics.

Soon after Wallis' publication, **Christiaan Huygens** (1629–1695) wrote *Horologium Oscillatorium sive de Motu Pendulorum*, in which he described various mechanical principles. He invented the pendulum clock, improved the telescope and investigated circular motion and the descent of heavy bodies.

These three mathematicians provided the foundations of mechanics. **Sir Isaac Newton** (1642–1727) used calculus to increase the understanding of the laws of motion. He also used these concepts as a basis for his theories on gravity and inertia.

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6.05 Derivatives and indices

INVESTIGATION

DERIVATIVES AND INDICES

1 **a** Show that $\frac{1}{x+h}-\frac{1}{x}=\frac{-h}{x(x+h)}$.

b Hence differentiate $y=\frac{1}{x}$ from first principles.

c Differentiate $y=x^{-1}$ using the formula. Do you get the same answer as in part **b**?

2 **a** Show that $\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)=h$.

b Hence differentiate $y=\sqrt{x}$ from first principles.

c Differentiate $y=x^{\frac{1}{2}}$ and show that this gives the same answer as in part **b**.

EXAMPLE 11

a Differentiate $f(x)=7\sqrt[3]{x}$.

b Find the derivative of $y=\frac{4}{x^2}$ at the point where $x=2$.

Solution

a $f(x)=7\sqrt[3]{x}=7x^{\frac{1}{3}}$ ← Convert the function to a power of x first.

$f'(x)=7\times\frac{1}{3}x^{\frac{1}{3}-1}$

$=\frac{7}{3}x^{-\frac{2}{3}}$

$=\frac{7}{3}\times\frac{1}{x^{\frac{2}{3}}}$

$=\frac{7}{3}\times\frac{1}{\sqrt[3]{x^2}}$

$=\frac{7}{3\sqrt[3]{x^2}}$

b $y=\frac{4}{x^2}$

$=4x^{-2}$

$\frac{dy}{dx}=-8x^{-3}$

$=-\frac{8}{x^3}$

When $x=2$:

$\frac{dy}{dx}=-\frac{8}{2^3}$

$=-1$

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Exercise 6.05 Derivatives and indices

1 Differentiate:

a x^{-3} **b** $x^{1.4}$ **c** $6x^{0.2}$ **d** $x^{\frac{1}{2}}$

e $2x^{\frac{1}{2}} - 3x^{-1}$ **f** $3x^{\frac{1}{3}}$ **g** $8x^{\frac{3}{4}}$ **h** $-2x^{-\frac{1}{2}}$

2 Find the derivative function.

a $\frac{1}{x}$ **b** $5\sqrt{x}$ **c** $\sqrt[6]{x}$ **d** $\frac{2}{x^5}$

e $-\frac{5}{x^3}$ **f** $\frac{1}{\sqrt{x}}$ **g** $\frac{1}{2x^6}$ **h** $x\sqrt{x}$

i $\frac{2}{3x}$ **j** $\frac{1}{4x^2} + \frac{3}{x^4}$

3 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 27$.

4 If $x = \frac{12}{t}$, find $\frac{dx}{dt}$ when $t = 2$.

5 A function is given by $f(x) = \sqrt[4]{x}$. Evaluate $f'(16)$.

6 Find the derivative of $y = \frac{3}{2x^2}$ at the point $\left(1, 1\frac{1}{2}\right)$.

7 Find $\frac{dy}{dx}$ if $y = \left(x + \sqrt{x}\right)^2$.

8 A function $f(x) = \frac{\sqrt{x}}{2}$ has a tangent at (4, 1). Find its gradient.

9 **a** Differentiate $\frac{\sqrt{x}}{x}$.

b Hence find the derivative of $y = \frac{\sqrt{x}}{x}$ at the point where $x = 4$.

10 The function $f(x) = 3\sqrt{x}$ has $f'(x) = \frac{3}{4}$ at $x = a$. Find a.

11 The hyperbola $y = \frac{2}{x}$ has 2 tangents with gradient $-\frac{2}{25}$. Find the points where these tangents touch the hyperbola.

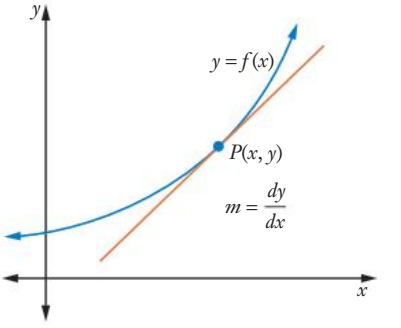
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6.06 Tangents and normals

Tangents to a curve

Remember that the derivative is a function that gives the instantaneous rate of change or gradient of the tangent to the curve.

A tangent is a line so we can use the formula $y = mx + c$ or $y - y_1 = m(x - x_1)$ to find its equation.



WS Tangents and normals

Equation of a tangent

WS Slopes of curves

WS Tangents to a curve

EXAMPLE 12

a Find the gradient of the tangent to the parabola $y = x^2 + 1$ at the point (1, 2).

b Find values of x for which the gradient of the tangent to the curve $y = 2x^3 - 6x^2 + 1$ is equal to 18.

c Find the equation of the tangent to the curve $y = x^4 - 3x^3 + 7x - 2$ at the point (2, 4).

Solution

a The gradient of a tangent to a curve is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x + 0$$
$$= 2x$$

Substitute $x = 1$ from the point (1, 2):

$$\frac{dy}{dx} = 2(1)$$
$$= 2$$

So the gradient of the tangent at (1, 2) is 2.

b $\frac{dy}{dx} = 6x^2 - 12x$

Gradient is 18 so $\frac{dy}{dx} = 18$.

$$18 = 6x^2 - 12x$$
$$0 = 6x^2 - 12x - 18$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$\therefore x = 3, -1$$

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c

$$\frac{dy}{dx} = 4x^3 - 9x^2 + 7$$

At (2, 4), $\frac{dy}{dx} = 4(2)^3 - 9(2)^2 + 7$

$= 3$

So the gradient of the tangent at (2, 4) is 3.

Equation of the tangent:

$y - y_1 = m(x - x_1)$

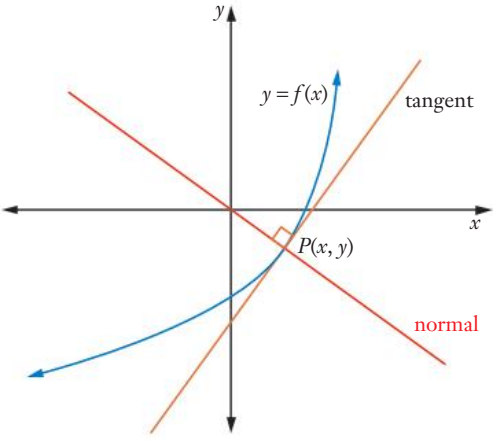
$y - 4 = 3(x - 2)$

$= 3x - 6$

$y = 3x - 2$ or $3x - y - 2 = 0$

Normals to a curve

The **normal** is a straight line **perpendicular** to the tangent at the same point of contact with the curve.



Remember the rule for perpendicular lines from Chapter 3, *Functions*:

Gradients of perpendicular lines

If 2 lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$ or $m_2 = -\frac{1}{m_1}$.

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EXAMPLE 13

a Find the gradient of the normal to the curve $y = 2x^2 - 3x + 5$ at the point where $x = 4$.

b Find the equation of the normal to the curve $y = x^3 + 3x^2 - 2x - 1$ at $(-1, 3)$.

Solution

a $\frac{dy}{dx} = 4x - 3$

When $x = 4$:

$\frac{dy}{dx} = 4 \times 4 - 3$

$= 13$

So $m_1 = 13$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$13m_2 = -1$

$m_2 = -\frac{1}{13}$

So the gradient of the normal is $-\frac{1}{13}$.

b $\frac{dy}{dx} = 3x^2 + 6x - 2$

When $x = -1$:

$\frac{dy}{dx} = 3(-1)^2 + 6(-1) - 2$

$= -5$

So $m_1 = -5$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$-5m_2 = -1$

$m_2 = \frac{1}{5}$

So the gradient of the normal is $\frac{1}{5}$.

Equation of the normal: $y - y_1 = m(x - x_1)$

$y - 3 = \frac{1}{5}(x - (-1))$

$5y - 15 = x + 1$

$x - 5y + 16 = 0$

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Exercise 6.06 Tangents and normals

1 Find the gradient of the tangent to the curve:

a $y = x^3 - 3x$ at the point where $x = 5$

b $f(x) = x^2 + x - 4$ at the point $(-7, 38)$

c $f(x) = 5x^3 - 4x - 1$ at the point where $x = -1$

d $y = 5x^2 + 2x + 3$ at $(-2, 19)$

e $y = 2x^9$ at the point where $x = 1$

f $f(x) = x^3 - 7$ at the point where $x = 3$

g $v = 2t^2 + 3t - 5$ at the point where $t = 2$

h $Q = 3r^3 - 2r^2 + 8r - 4$ at the point where $r = 4$

i $h = t^4 - 4t$ where $t = 0$

j $f(t) = 3t^5 - 8t^3 + 5t$ at the point where $t = 2$.

2 Find the gradient of the normal to the curve:

a $f(x) = 2x^3 + 2x - 1$ at the point where $x = -2$

b $y = 3x^2 + 5x - 2$ at $(-5, 48)$

c $f(x) = x^2 - 2x - 7$ at the point where $x = -9$

d $y = x^3 + x^2 + 3x - 2$ at $(-4, -62)$

e $f(x) = x^{10}$ at the point where $x = -1$

f $y = x^2 + 7x - 5$ at $(-7, -5)$

g $A = 2x^3 + 3x^2 - x + 1$ at the point where $x = 3$

h $f(a) = 3a^2 - 2a - 6$ at the point where $a = -3$.

i $V = h^3 - 4h + 9$ at $(2, 9)$

j $g(x) = x^4 - 2x^2 + 5x - 3$ at the point where $x = -1$.

3 Find the gradient of **i** the tangent and **ii** the normal to the curve:

a $y = x^2 + 1$ at $(3, 10)$

b $f(x) = 5 - x^2$ where $x = -4$

c $y = 2x^5 - 7x^2 + 4$ where $x = -1$

d $p(x) = x^6 - 3x^4 - 2x + 8$ where $x = 1$

e $f(x) = 4 - x - x^2$ at $(-6, 26)$

4 Find the equation of the tangent to the curve:

a $y = x^4 - 5x + 1$ at $(2, 7)$

b $f(x) = 5x^3 - 3x^2 - 2x + 6$ at $(1, 6)$

c $y = x^2 + 2x - 8$ at $(-3, -5)$

d $y = 3x^3 + 1$ where $x = 2$

e $v = 4t^4 - 7t^3 - 2$ where $t = 2$

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5 Find the equation of the normal to the curve:

a $f(x) = x^3 - 3x + 5$ at $(3, 23)$

b $y = x^2 - 4x - 5$ at $(-2, 7)$

c $f(x) = 7x - 2x^2$ where $x = 6$

d $y = 7x^2 - 3x - 3$ at $(-3, 69)$

e $y = x^4 - 2x^3 + 4x + 1$ where $x = 1$

6 Find the equation of **i** the tangent and **ii** the normal to the curve:

a $f(x) = 4x^2 - x + 8$ at $(1, 11)$

b $y = x^3 - 2x^2 - 5x$ at $(-3, -30)$

c $F(x) = x^5 - 5x^3$ where $x = 1$

d $y = x^2 - 8x + 7$ at $(3, -8)$.

7 For the curve $y = x^3 - 27x - 5$, find values of x for which $\frac{dy}{dx} = 0$.

8 Find the coordinates of the points at which the curve $y = x^3 + 1$ has a tangent with a gradient of 3.

9 A function $f(x) = x^2 + 4x - 12$ has a tangent with a gradient of -6 at point P on the curve. Find the coordinates of P.

10 The tangent at point P on the curve $y = 4x^2 + 1$ is parallel to the x-axis. Find the coordinates of P.

11 Find the coordinates of point Q where the tangent to the curve $y = 5x^2 - 3x$ is parallel to the line $7x - y + 3 = 0$.

12 Find the coordinates of point S where the tangent to the curve $y = x^2 + 4x - 1$ is perpendicular to the line $4x + 2y + 7 = 0$.

13 The curve $y = 3x^2 - 4$ has a gradient of 6 at point A.

a Find the coordinates of A.

b Find the equation of the tangent to the curve at A.

14 A function $h = 3t^2 - 2t + 5$ has a tangent at the point where $t = 2$. Find the equation of the tangent.

15 A function $f(x) = 2x^2 - 8x + 3$ has a tangent parallel to the line $4x - 2y + 1 = 0$ at point P. Find the equation of the tangent at P.

16 Find the equation of the tangent to the curve $y = \frac{1}{x^3}$ at $\left(2, \frac{1}{8}\right)$.

17 Find the equation of the tangent to $f(x) = 6\sqrt{x}$ at the point where $x = 9$.

18 Find the equation of the tangent to the curve $y = \frac{4}{x}$ at $\left(8, \frac{1}{2}\right)$.

19 If the gradient of the tangent to $y = \sqrt{x}$ is $\frac{1}{6}$ at point A, find the coordinates of A.

Chain rule

6.07 Chain rule

We looked at composite functions in Chapter 5, *Further functions*.

The **chain rule** is a method for differentiating composite functions. It is also called the **composite function** rule or the '**function of a function' rule**.

The chain rule

If a function y can be written as a composite function where $y = f(u(x))$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule

EXAMPLE 14

Differentiate:

a $y = (5x + 4)^7$ **b** $y = (3x^2 + 2x - 1)^9$ **c** $y = \sqrt{3 - x}$

Solution

a Let $u = 5x + 4$

Then $\frac{du}{dx} = 5$

$y = u^7$

$\therefore \frac{dy}{du} = 7u^6$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 7u^6 \times 5$$
$$= 35u^6$$
$$= 35(5x + 4)^6$$

b Let $u = 3x^2 + 2x - 1$

Then $\frac{du}{dx} = 6x + 2$

$y = u^9$

$\therefore \frac{dy}{du} = 9u^8$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 9u^8 \times (6x + 2)$$
$$= 9(3x^2 + 2x - 1)^8(6x + 2)$$
$$= 9(6x + 2)(3x^2 + 2x - 1)^8$$

c $y = \sqrt{3 - x} = (3 - x)^{\frac{1}{2}}$

Let $u = 3 - x$

Then $\frac{du}{dx} = -1$

$y = u^{\frac{1}{2}}$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{2}u^{-\frac{1}{2}} \times (-1)$$
$$= -\frac{1}{2}(3 - x)^{-\frac{1}{2}}$$
$$= -\frac{1}{2\sqrt{3 - x}}$$

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You might see a pattern when using the chain rule. The derivative of a composite function is the product of the derivatives of 2 functions.

The derivative of $[f(x)]^n$

$$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$$

EXAMPLE 15

Differentiate:

a $y = (8x^3 - 1)^5$ **b** $y = (3x + 8)^{11}$ **c** $y = \dfrac{1}{(6x+1)^2}$

Solution

a
$$\begin{aligned}\frac{dy}{dx} &= f'(x) \times n[f(x)]^{n-1}\\ &= 24x^2 \times 5(8x^3 - 1)^4\\ &= 120x^2(8x^3 - 1)^4\end{aligned}$$

b
$$\begin{aligned}\frac{dy}{dx} &= f'(x) \times n[f(x)]^{n-1}\\ &= 3 \times 11(3x + 8)^{10}\\ &= 33(3x + 8)^{10}\end{aligned}$$

c
$$y = \frac{1}{(6x+1)^2} = (6x+1)^{-2}$$

$$\begin{aligned}\frac{dy}{dx} &= f'(x) \times n[f(x)]^{n-1}\\ &= 6 \times (-2)(6x + 1)^{-3}\\ &= -12(6x + 1)^{-3}\\ &= -\frac{12}{(6x+1)^3}\end{aligned}$$

Exercise 6.07 Chain rule

1 Differentiate:

a $y = (x + 3)^4$ **b** $y = (2x - 1)^3$ **c** $y = (5x^2 - 4)^7$

d $y = (8x + 3)^6$ **e** $y = (1 - x)^5$ **f** $y = 3(5x + 9)^9$

g $y = 2(x - 4)^2$ **h** $y = (2x^3 + 3x)^4$ **i** $y = (x^2 + 5x - 1)^8$

j $y = (x^6 - 2x^2 + 3)^6$ **k** $y = (3x - 1)^{\frac{1}{2}}$ **l** $y = (4 - x)^{-2}$

m $y = (x^2 - 9)^{-3}$ **n** $y = (5x + 4)^{\frac{1}{3}}$ **o** $y = (x^3 - 7x^2 + x)^{\frac{3}{4}}$

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p $y = \sqrt{3x+4}$ **q** $y = \dfrac{1}{5x-2}$ **r** $y = \dfrac{1}{(x^2+1)^4}$

s $y = \sqrt[3]{(7-3x)^2}$ **t** $y = \dfrac{5}{\sqrt{4+x}}$ **u** $y = \dfrac{1}{2\sqrt{3x-1}}$

v $y = \dfrac{3}{4(2x+7)^9}$ **w** $y = \dfrac{1}{x^4-3x^3+3x}$ **x** $y = \sqrt[3]{(4x+1)^4}$

y $y = \dfrac{1}{\sqrt[4]{(7-x)^5}}$

2 Find the gradient of the tangent to the curve $y = (3x-2)^3$ at the point (1, 1).

3 If $f(x) = 2(x^2-3)^5$, evaluate $f'(2)$.

4 The curve $y = \sqrt{x-3}$ has a tangent with gradient $\dfrac{1}{2}$ at point N.
Find the coordinates of N.

5 For what values of x does the function $f(x) = \dfrac{1}{4x-1}$ have $f'(x) = -\dfrac{4}{49}$?

6 Find the equation of the tangent to $y = (2x+1)^4$ at the point where $x = -1$.

7 Find the equation of the tangent to the curve $y = (2x-1)^8$ at the point where $x = 1$.

8 Find the equation of the normal to the curve $y = (3x-4)^3$ at $(1, -1)$.

9 Find the equation of the normal to the curve $y = (x^2+1)^4$ at $(1, 16)$.

10 Find the equation of **a** the tangent and **b** the normal to the curve $f(x) = \dfrac{1}{2x+3}$ at the point where $x = -1$.

WS
Product rule

6.08 Product rule

The **product rule** is a method for differentiating the product of 2 functions.

The product rule

If $y = uv$ where u and v are functions, then:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

or $y' = u'v + v'u$.

We can also write the product rule the other way round (differentiating v first), but the above formulas will also help us to remember the quotient rule in the next section.

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EXAMPLE 16

Differentiate:

a $y = (3x + 1)(x - 5)$ **b** $y = 9x^3(2x - 7)$

Solution

a You could expand the brackets and then differentiate:

$$y = (3x + 1)(x - 5)$$
$$= 3x^2 - 15x + x - 5$$
$$= 3x^2 - 14x - 5$$
$$\frac{dy}{dx} = 6x - 14$$

Using the product rule:

$y = uv$ where $u = 3x + 1$ and $v = x - 5$

$u' = 3$ $v' = 1$

$$y' = u'v + v'u$$
$$= 3(x - 5) + 1(3x + 1)$$
$$= 3x - 15 + 3x + 1$$
$$= 6x - 14$$

b $y = uv$ where $u = 9x^3$ and $v = 2x - 7$

$u' = 27x^2$ $v' = 2$

$$y' = u'v + v'u$$
$$= 27x^2(2x - 7) + 2(9x^3)$$
$$= 54x^3 - 189x^2 + 18x^3$$
$$= 72x^3 - 189x^2$$

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We can use the product rule together with the chain rule.

EXAMPLE 17

Differentiate:

a $y = 2x^5(5x+3)^3$ **b** $y = (3x-4)\sqrt{5-2x}$

Solution

a $y = uv$ where $u = 2x^5$ and $v = (5x+3)^3$

$u' = 10x^4$ $v' = 5 \times 3(5x+3)^2$ using chain rule

$= 15(5x+3)^2$

$y' = u'v + v'u$

$= 10x^4(5x+3)^3 + 15(5x+3)^2\,2x^5$

$= 10x^4(5x+3)^3 + 30x^5(5x+3)^2$

$= 10x^4(5x+3)^2[(5x+3)+3x]$

$= 10x^4(5x+3)^2(8x+3)$

b $y = uv$ where $u = 3x-4$ and $v = \sqrt{5-2x} = (5-2x)^{\frac{1}{2}}$

$u' = 3$ $v' = -2 \times \frac{1}{2}(5-2x)^{-\frac{1}{2}}$ using chain rule

$= -(5-2x)^{-\frac{1}{2}}$

$= -\dfrac{1}{(5-2x)^{\frac{1}{2}}}$

$= -\dfrac{1}{\sqrt{5-2x}}$

$y' = u'v + v'u$

$= 3 \cdot \sqrt{5-2x} + -\dfrac{1}{\sqrt{5-2x}}(3x-4)$

$= 3\sqrt{5-2x} - \dfrac{3x-4}{\sqrt{5-2x}}$

$= \dfrac{3\sqrt{5-2x} \times \sqrt{5-2x}}{\sqrt{5-2x}} - \dfrac{3x-4}{\sqrt{5-2x}}$

$= \dfrac{3(5-2x)}{\sqrt{5-2x}} - \dfrac{3x-4}{\sqrt{5-2x}}$

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$$= \frac{3(5-2x)-(3x-4)}{\sqrt{5-2x}}$$

$$= \frac{15-6x-3x+4}{\sqrt{5-2x}}$$

$$= \frac{19-9x}{\sqrt{5-2x}}$$

Exercise 6.08 Product rule

1 Differentiate:

a $y = x^3(2x+3)$ **b** $y = (3x-2)(2x+1)$ **c** $y = 3x(5x+7)$

d $y = 4x^4(3x^2-1)$ **e** $y = 2x(3x^4-x)$ **f** $y = x^2(x+1)^3$

g $y = 4x(3x-2)^5$ **h** $y = 3x^4(4-x)^3$ **i** $y = (x+1)(2x+5)^4$

2 Find the gradient of the tangent to the curve $y = 2x(3x-2)^4$ at $(1, 2)$.

3 If $f(x) = (2x+3)(3x-1)^5$, evaluate $f'(1)$.

4 Find the exact gradient of the tangent to the curve $y = x\sqrt{2x+5}$ at the point where $x = 1$.

5 Find the gradient of the tangent where $t = 3$ given $x = (2t-5)(t+1)^3$.

6 Find the equation of the tangent to the curve $y = x^2(2x-1)^4$ at $(1, 1)$.

7 Find the equation of the tangent to $h = (t+1)^2(t-1)^7$ at $(2, 9)$.

8 Find exact values of x for which the gradient of the tangent to the curve $y = 2x(x+3)^2$ is 14.

9 Given $f(x) = (4x-1)(3x+2)^2$, find the equation of the tangent at the point where $x = -1$.

6.09 Quotient rule

The **quotient rule** is a method for differentiating the ratio of 2 functions.

The quotient rule

If $y = \frac{u}{v}$ where u and v are functions, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\text{or } y' = \frac{u'v - v'u}{v^2}.$$

WS Quotient rule

WS Rules of differentiation

WS Mixed differentiation problems

EXAMPLE 18

Differentiate:

a $y = \dfrac{3x-5}{5x+2}$ **b** $y = \dfrac{4x^3-5x+2}{x^3-1}$

Solution

a $y = \dfrac{u}{v}$ where $u = 3x-5$ and $v = 5x+2$

$u' = 3$ $v' = 5$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{3(5x+2) - 5(3x-5)}{(5x+2)^2}$$

$$= \frac{15x+6-15x+25}{(5x+2)^2}$$

$$= \frac{31}{(5x+2)^2}$$

b $y = \dfrac{u}{v}$ where $u = 4x^3 - 5x + 2$ and $v = x^3 - 1$

$u' = 12x^2 - 5$ $v' = 3x^2$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(12x^2-5)(x^3-1) - 3x^2(4x^3-5x+2)}{(x^3-1)^2}$$

$$= \frac{12x^5 - 12x^2 - 5x^3 + 5 - 12x^5 + 15x^3 - 6x^2}{(x^3-1)^2}$$

$$= \frac{10x^3 - 18x^2 + 5}{(x^3-1)^2}$$

Exercise 6.09 Quotient rule

1 Differentiate:

a $y = \dfrac{1}{2x-1}$ **b** $y = \dfrac{3x}{x+5}$ **c** $y = \dfrac{x^3}{x^2-4}$ **d** $y = \dfrac{x-3}{5x+1}$

e $y = \dfrac{x-7}{x^2}$ **f** $y = \dfrac{5x+4}{x+3}$ **g** $y = \dfrac{x}{2x^2-1}$ **h** $y = \dfrac{x+4}{x-2}$

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i $y = \dfrac{2x+7}{4x-3}$ **j** $y = \dfrac{x+5}{3x+1}$ **k** $y = \dfrac{x+1}{3x^2-7}$ **l** $y = \dfrac{2x^2}{2x-3}$

m $y = \dfrac{x^2+4}{x^2-5}$ **n** $y = \dfrac{x^3}{x+4}$ **o** $y = \dfrac{x^3+2x-1}{x+3}$ **p** $y = \dfrac{x^2-2x-1}{3x+4}$

q $y = \dfrac{2x}{(x+5)^{\frac{1}{2}}}$ **r** $y = \dfrac{x-1}{(7x+2)^4}$ **s** $y = \dfrac{3x+1}{\sqrt{x+1}}$ **t** $y = \dfrac{\sqrt{x-1}}{2x-3}$

2 Find the gradient of the tangent to the curve $y = \dfrac{2x}{3x+1}$ at $\left(1, \dfrac{1}{2}\right)$.

3 If $f(x) = \dfrac{4x+5}{2x-1}$, evaluate $f'(2)$.

4 Find values of x for which the gradient of the tangent to $y = \dfrac{4x-1}{2x-1}$ is -2.

5 Given $f(x) = \dfrac{2x}{x+3}$, find x if $f'(x) = \dfrac{1}{6}$.

6 Find the equation of the tangent to the curve $y = \dfrac{x}{x+2}$ at $\left(4, \dfrac{2}{3}\right)$.

7 Find the equation of the tangent to the curve $y = \dfrac{x^2-1}{x+3}$ at $x = 2$.

6.10 Rates of change



Rates of change



Instantaneous rates of change



Graphs of rates of change

We know that the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ of the secant passing through 2 points on the graph of a function gives the **average rate of change** between those 2 points.

Now consider a quantity Q that changes with time, giving the function $Q(t)$.

Average rate of change

The average rate of change of a quantity Q with respect to time t is $\dfrac{Q_2 - Q_1}{t_2 - t_1}$.

We know that the gradient $\dfrac{dy}{dx}$ of the tangent at a point on the graph of a function gives the **instantaneous rate of change** at that point.

Instantaneous rate of change

The instantaneous rate of change of a quantity Q with respect to time t is $\dfrac{dQ}{dt}$.

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EXAMPLE 19

a The number of bacteria in a culture increases according to the function $B = 2t^4 - t^2 + 2000$, where t is time in hours. Find:

- **i** the number of bacteria initially
- **ii** the average rate of change in number of bacteria between 2 and 3 hours
- **iii** the number of bacteria after 5 hours
- **iv** the rate at which the number of bacteria is increasing after 5 hours.

b An object travels a distance according to the function $D = t^2 + t + 5$, where D is in metres and t is in seconds. Find the speed at which it is travelling at:

i 4 s **ii** 10 s

Solution

a i $B = 2t^4 - t^2 + 2000$

Initially, $t = 0$:

$B = 2(0)^4 - (0)^2 + 2000$

$= 2000$

So there are 2000 bacteria initially.

ii When $t = 2$, $B = 2(2)^4 - (2)^2 + 2000$

$= 2028$

When $t = 3$, $B = 2(3)^4 - (3)^2 + 2000$

$= 2153$

Average rate of change $= \dfrac{B_2 - B_1}{t_2 - t_1}$

$= \dfrac{2153 - 2028}{3 - 2}$

$= 125$ bacteria/hour

So the average rate of change is 125 bacteria per hour.

iii When $t = 5$, $B = 2(5)^4 - (5)^2 + 2000$

$= 3225$

So there will be 3225 bacteria after 5 hours.

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iv The instantaneous rate of change is given by the derivative $\frac{dB}{dt} = 8t^3 - 2t$.

When $t = 5$, $\frac{dB}{dt} = 8(5)^3 - 2(5)$

$= 990$

So the rate of increase after 5 hours will be 990 bacteria per hour.

b Speed is the rate of change of distance over time: $\frac{dD}{dt} = 2t + 1$.

i When $t = 4$, $\frac{dD}{dt} = 2(4) + 1$

$= 9$

So speed after 4 s is 9 m/s.

ii When $t = 10$, $\frac{dD}{dt} = 2(10) + 1$

$= 21$

So speed after 10 s is 21 m/s.

Displacement, velocity and acceleration

Displacement (x) measures the distance of an object from a fixed point (origin). It can be positive or negative or 0, according to where the object is.

Velocity (v) is the rate of change of displacement with respect to time, and involves speed and direction.

Velocity

Velocity $v = \frac{dx}{dt}$ is the instantaneous rate of change of displacement x over time t.

Acceleration (a) is the rate of change of velocity with respect to time.

Acceleration

Acceleration $a = \frac{dv}{dt}$ is the instantaneous rate of change of velocity v over time t.

We usually write velocity units as km/h or m/s, but we can also use index notation and write km h^{-1} or m s^{-1}.

With acceleration units, we write km/h/h as km/h^2, or in index notation we write km h^{-2}.

EXAMPLE 20

A ball rolls down a ramp so that its displacement x cm in t seconds is $x = 16 - t^2$.

a Find its initial displacement.

b Find its displacement at 3 s.

c Find its velocity at 2 s.

d Show that the ball has a constant acceleration of -2 cm s^{-2}.

Solution

a $x = 16 - t^2$

Initially, $t = 0$:

$x = 16 - 0^2$

$= 16$

So the ball's initial displacement is 16 cm.

b When $t = 3$:

$x = 16 - 3^2$

$= 7$

So the ball's displacement at 3 s is 7 cm.

c $v = \dfrac{dx}{dt}$

$= -2t$

x is measured in cm, t is measured in s, so v is measured in cm/s or cm s^{-1}.

When $t = 2$:

$v = -2(2)$

$= -4$

So the ball's velocity at 2 s is -4 cm s^{-1}.

d $a = \dfrac{dv}{dt}$

$= -2$

So acceleration is constant at -2 cm s^{-2}.

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Exercise 6.10 Rates of change

1 Find the formula for the rate of change for each function.

a $h = 20t - 4t^2$ **b** $D = 5t^3 + 2t^2 + 1$ **c** $A = 16x - 2x^2$

d $x = 3t^5 - t^4 + 2t - 3$ **e** $V = \frac{4}{3}\pi r^3$ **f** $S = 2\pi r + \frac{50}{r^2}$

g $D = \sqrt{x^2 - 4}$ **h** $S = 800r + \frac{400}{r}$

2 If $h = t^3 - 7t + 5$, find:

a the average rate of change of h between $t = 3$ and $t = 4$

b the instantaneous rate of change of h when $t = 3$.

3 The volume of water V in litres flowing through a pipe after t seconds is given by $V = t^2 + 3t$. Find the rate at which the water is flowing when $t = 5$.

4 The mass in grams of a melting ice block is given by the formula $M = t - 2t^2 + 100$, where t is time in minutes.

a Find the average rate of change at which the ice block is melting between:

i 1 and 3 minutes **ii** 2 and 5 minutes.

b Find the rate at which it will be melting at 5 minutes.

5 The surface area in cm^2 of a balloon being inflated is given by $S = t^3 - 2t^2 + 5t + 2$, where t is time in seconds. Find the rate of increase in the balloon's surface area at 8 s.

6 A circular disc expands as it is heated. The area, in cm^2, of the disc increases according to the formula $A = 4t^2 + t$, where t is time in minutes. Find the rate of increase in the area after 5 minutes.

7 A car is d km from home after t hours according to the formula $d = 10t^2 + 5t + 11$.

a How far is the car from home:

i initially? **ii** after 3 hours? **iii** after 5 hours?

b At what speed is the car travelling after:

i 3 hours? **ii** 5 hours?

8 According to Boyle's Law, the pressure of a gas is given by the formula $P = \frac{k}{V}$ where k is a constant and V is the volume of the gas. If $k = 100$ for a certain gas, find the rate of change in the pressure when $V = 20$.

9 The displacement of a particle is $x = t^3 - 9t$ cm, where t is time in seconds.

a Find the velocity of the particle at 3 s.

b Find the acceleration at 2 s.

c Show that the particle is initially at the origin, and find any other times that the particle will be at the origin.

d At what time will the acceleration be 30 cm s^{-2}?

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10 A particle is moving with displacement $s = 2t^2 - 8t + 3$, where s is in metres and t is in seconds.

a Find its initial velocity.

b Show that its acceleration is constant and find its value.

c Find its displacement at 5 s.

d Find when the particle's velocity is zero.

e What will the particle's displacement be at that time?

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6. TEST YOURSELF

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

Derivatives find-a-word

1 Find the derivative of $\frac{2}{3x^4}$.

A $\frac{8}{3x^5}$ **B** $-\frac{8}{3x^3}$ **C** $-\frac{8}{3x^5}$ **D** $\frac{8}{3x^3}$

2 Differentiate $3x(x^3 - 5)$.

A $4x^3$ **B** $12x^3 - 15$ **C** $9x^2$ **D** $3x^4 - 15x$

3 The derivative of $y = f(x)$ is given by:

A $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{x-h}$ **B** $\lim\limits_{x\to 0} \frac{f(x+h)-f(x)}{x}$

C $\lim\limits_{h\to 0} \frac{f(x)-f(x+h)}{h}$ **D** $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

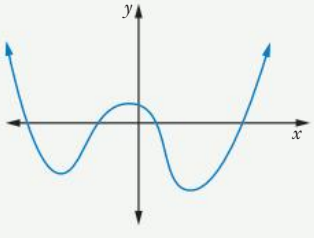
4 Which of the following is the chain rule (there is more than one answer)?

A $\frac{dy}{dx} = \frac{dy}{du} \times \frac{dx}{du}$ **B** $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

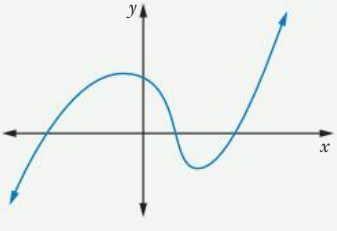
C $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$ **D** $\frac{dy}{dx} = nf(x)^{n-1}$

5 Sketch the derivative function of each graph.

a



b



6 Differentiate $y = 5x^2 - 3x + 2$ from first principles.

7 Differentiate:

a $y = 7x^6 - 3x^3 + x^2 - 8x - 4$ **b** $y = 3x^{-4}$ **c** $y = \frac{2}{(x+1)^4}$

d $y = x^2\sqrt{x}$ **e** $y = (x^2 + 4x - 2)^9$ **f** $y = \frac{3x-2}{2x+1}$

g $y = x^3(3x + 1)^6$

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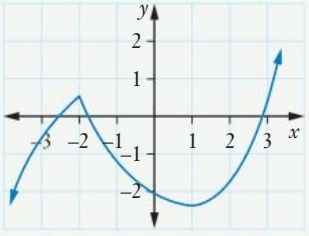
8 Find $\frac{dv}{dt}$ if $v = 2t^2 - 3t - 4$.

9 Find the gradient of the tangent to the curve $y = x^3 + 3x^2 + x - 5$ at $(1, 0)$.

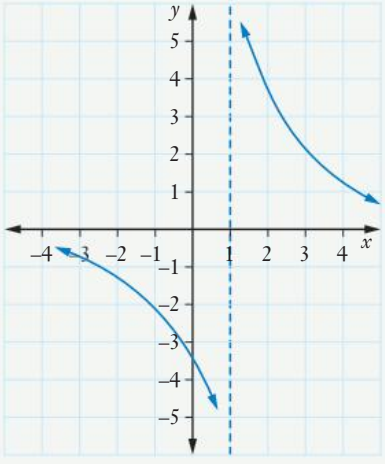
10 If $h = 60t - 3t^2$, find $\frac{dh}{dt}$ when $t = 3$.

11 For each graph of a function, find all values of x where it is not differentiable.

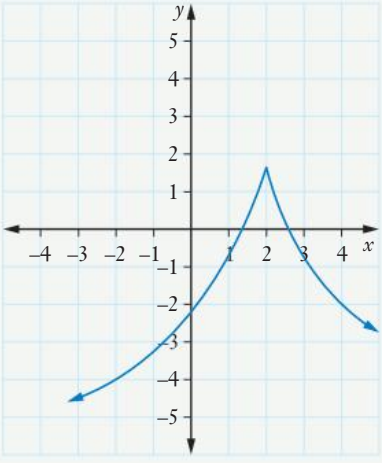
a



b



c

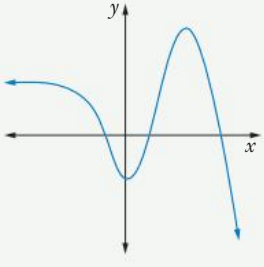


12 Differentiate:

a $y = \frac{4}{x}$ **b** $f(x) = \sqrt[5]{x}$ **c** $f(x) = 2(4x + 9)^4$

d $y = (3x + 2)(x - 1)^3$ **e** $f(x) = \frac{x^3 - 3}{2x + 5}$

13 Sketch the derivative function of this curve.



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14 Find the equation of the tangent to the curve $y = x^2 + 5x - 3$ at $(2, 11)$.

15 Find the point on the curve $y = x^2 - x + 1$ at which the tangent has a gradient of 3.

16 Find $\frac{dS}{dr}$ if $S = 4\pi r^2$.

17 Find the gradient of the secant on the curve $f(x) = x^2 - 3x + 1$ between the points where $x = 1$ and $x = 1.1$.

18 At which points on the curve $y = 2x^3 - 9x^2 - 60x + 3$ are the tangents horizontal?

19 Find the equation of the tangent to the curve $y = x^2 + 2x - 5$ that is parallel to the line $y = 4x - 1$.

20 **a** Differentiate $s = ut + \frac{1}{2}at^2$ with respect to t.

b Find the value of t for which $\frac{ds}{dt} = 5$, $u = 7$ and $a = -10$.

21 Find the equation of the tangent to the curve $y = \frac{1}{3x}$ at the point where $x = \frac{1}{6}$.

22 A ball is thrown into the air and its height h metres over t seconds is given by $h = 4t - t^2$.

a Find the height of the ball:

i initially **ii** at 2 s **iii** at 3 s **iv** at 3.5 s

b Find the average rate of change of the height between:

i 1 and 2 seconds **ii** 2 and 3 seconds

c Find the rate at which the ball is moving:

i initially **ii** at 2 s **iii** at 3 s

23 If $f(x) = x^2 - 3x + 5$, find:

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $f'(x)$

24 Given $f(x) = (4x - 3)^5$, find the value of:

a $f(1)$ **b** $f'(1)$

25 Find $f'(4)$ when $f(x) = (x - 3)^9$.

26 Differentiate:

a $y = 3(x^2 - 6x + 1)^4$ **b** $y = \frac{2}{\sqrt{3x - 1}}$

27 A particle moves so that its displacement after t seconds is $x = 4t^2 - 5t^3$ metres. Find:

a its initial displacement, velocity and acceleration

b when $x = 0$

c its velocity and acceleration at 2 s.

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6. CHALLENGE EXERCISE

1 Find the equations of the tangents to the curve $y = x(x - 1)(x + 2)$ at the points where the curve cuts the x-axis.

2 **a** Find the points on the curve $y = x^3 - 6$ where the tangents are parallel to the line $y = 12x - 1$.

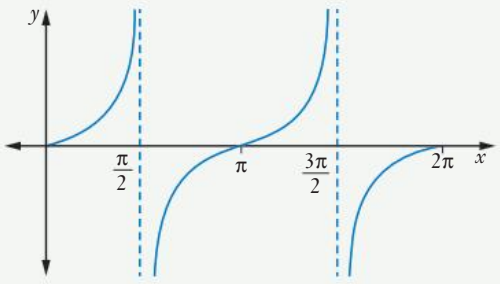
b Hence find the equations of the normals to the curve at those points.

3 The normal to the curve $y = x^2 + 1$ at the point where $x = 2$ cuts the curve again at point P. Find the coordinates of P.

4 The equation of the tangent to the curve $y = x^4 - nx^2 + 3x - 2$ at the point where $x = -2$ is given by $3x - y - 2 = 0$. Evaluate n.

5 **a** Find any points at which the graphed function is not differentiable.

b Sketch the derivative function for the graph.



6 Find the exact gradient of the tangent to the curve $y = \sqrt{x^2 - 3}$ at the point where $x = 5$.

7 Find the equation of the normal to the curve $y = 3\sqrt{x + 1}$ at the point where $x = 8$.

8 **a** Find the equations of the tangents to the parabola $y = 2x^2$ at the points where the line $6x - 8y + 1 = 0$ intersects with the parabola.

b Show that the tangents are perpendicular.

9 Find any x values of the function $f(x) = \dfrac{2}{x^3 - 8x^2 + 12x}$ where it is not differentiable.

10 Find the equation of the chord joining the points of contact of the tangents to the curve $y = x^2 - x - 4$ with gradients 3 and -1.

11 For the function $f(x) = ax^2 + bx + c$, $f(2) = 4$, $f'(1) = 0$ and $f'(-3) = 8$. Evaluate a, b and c.

12 For the function $f(x) = x^3$:

a Show that $f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3$.

b Show that $f'(x) = 3x^2$ by differentiating from first principles.

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13 Consider the function $f(x) = \frac{1}{x}$.

a Find the gradient of the secant between:

i $f(1)$ and $f(1.1)$ **ii** $f(1)$ and $f(1.01)$ **iii** $f(1)$ and $f(0.99)$

b Estimate the gradient of the tangent to the curve at the point where $x = 1$.

c Show that $\frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$.

d Hence show that $f'(x) = -\frac{1}{x^2}$ by differentiating from first principles.

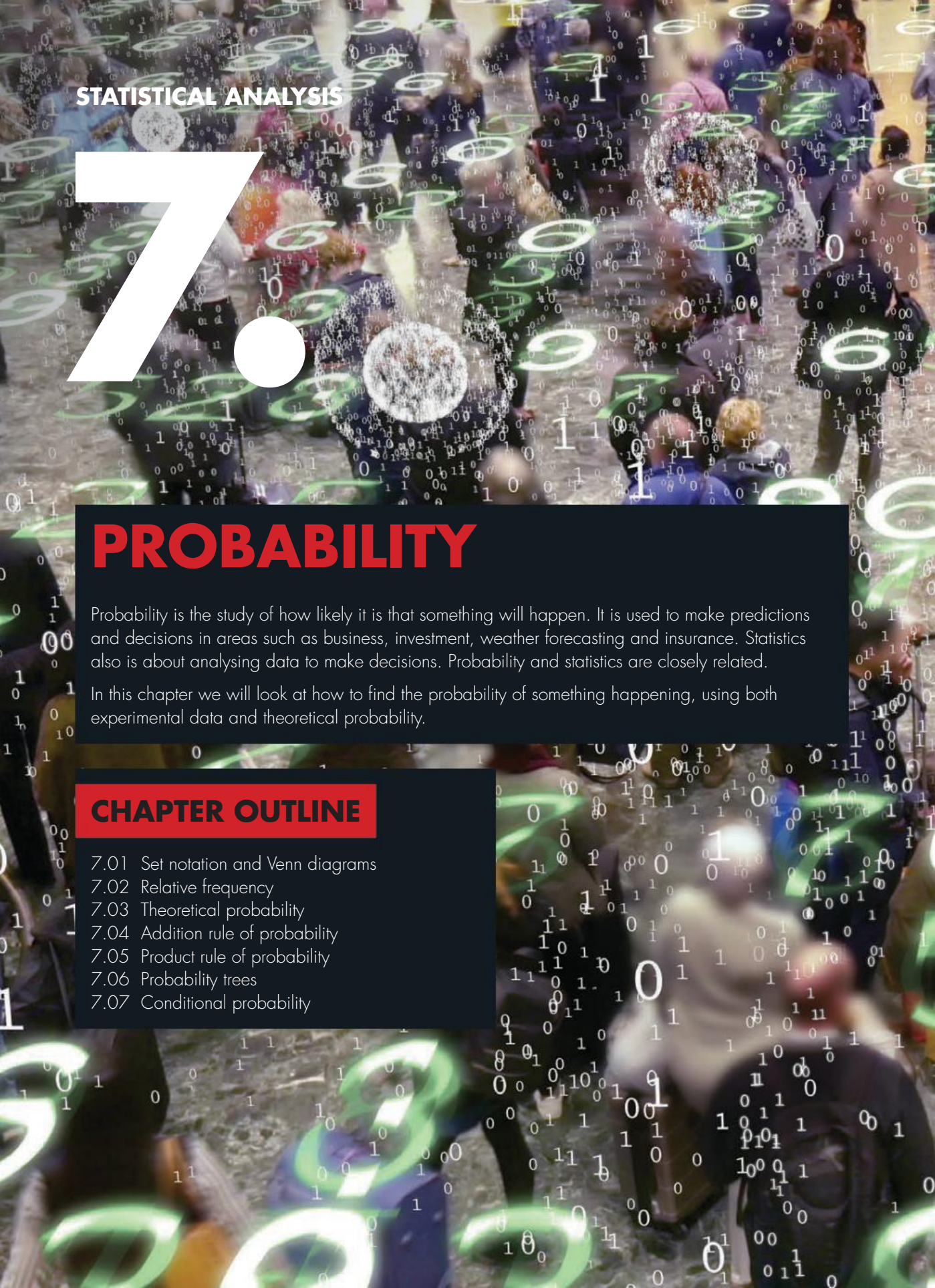
14 The displacement of a particle is given by $x = (t^3 + 1)^6$, where x is in metres and t is in seconds.

a Find the initial displacement and velocity of the particle.

b Find its acceleration after 2 s in scientific notation, correct to 3 significant figures.

c Show that the particle is never at the origin.

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STATISTICAL ANALYSIS

7.

PROBABILITY

Probability is the study of how likely it is that something will happen. It is used to make predictions and decisions in areas such as business, investment, weather forecasting and insurance. Statistics also is about analysing data to make decisions. Probability and statistics are closely related.

In this chapter we will look at how to find the probability of something happening, using both experimental data and theoretical probability.

CHAPTER OUTLINE

7.01 Set notation and Venn diagrams
7.02 Relative frequency
7.03 Theoretical probability
7.04 Addition rule of probability
7.05 Product rule of probability
7.06 Probability trees
7.07 Conditional probability



IN THIS CHAPTER YOU WILL:

- understand definitions of probability and associated terminology including set notation, events, outcomes and sample space
- identify differences between experimental and theoretical probability and their limitations
- recognise non-mutually exclusive events and use techniques to count outcomes in these cases
- identify conditional probability and calculate probabilities in these cases
- use probability trees, Venn diagrams and the addition and product rules to calculate probabilities

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TERMINOLOGY

complement: The complement of an event is when the event does not occur
conditional probability: The probability that an event A occurs when it is known that another event B has occurred
equally likely outcomes: Outcomes that have the same chance of occurring
independent events: Events where the occurrence of one event does not affect the probability of another event
mutually exclusive events: Events within the same sample space that cannot both occur at the same time; for example, rolling an even number on a die and rolling a 5 on the same die
non-mutually exclusive events: Events within the same sample space that can occur at the same time; for example, rolling a prime number on a die and rolling an odd number
probability tree: A diagram that uses branches to show multi-stage events and sets out the probability on each branch
relative frequency: The frequency of an event relative to the total frequency
sample space: The set of all possible outcomes in an event
set: A collection of distinct objects called elements or members.
For example, set $A = \{1, 2, 3, 4, 5, 6\}$
tree diagram: A diagram that uses branches to show multi-stage events
Venn diagram: A diagram that shows the relationship between 2 or more sets using circles (usually overlapping) drawn inside a rectangle

WS Venn diagrams

WS Venn diagrams matching activity

WS Set operations

Venn diagrams

7.01 Set notation and Venn diagrams

Probability and statistics do not provide exact answers in real life, but they can help in making decisions. Here are some examples of where statistics and probability are used.

- An actuary is a mathematician who looks at statistics and makes decisions for insurance companies. Life expectancy statistics help to decide the cost of life insurance for people of different ages. Statistics about car accidents will help set car insurance premiums.
- Stockbrokers use a chart or formula to predict when to buy and sell shares. This chart is usually based on statistics of past trends.
- A business does a feasibility study in a local area to decide whether to open up a new leisure centre. It uses this data to make a decision based on the likelihood that local people will want to join the centre.

Sample space

An **outcome** is a possible result of a random experiment.

The **sample space** is the **set** of all possible outcomes for an experiment.

An **event** is a set of one or more outcomes.

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EXAMPLE 1

In a survey, a TV show is given a rating from 1 to 10.

a Write down the sample space.

b Give an example of:

i an outcome **ii** an event.

Solution

a The sample space is the set of all possible ratings from 1 to 10.

Sample space = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

b **i** There are 10 different possible outcomes. One outcome is a rating of 7.

ii One example of an event is 'a rating higher than 6'.

To find the probability of an event happening, we compare the number of ways the event can occur with the total number of possible outcomes (the sample space).

$$\text{Probability of an event} = \frac{\text{Number of ways the event can occur}}{\text{Total number of possible outcomes}}$$

If we call the event E and the sample space S we can write this as:

Probability formula

$$P(E) = \frac{n(E)}{n(S)}$$

where $P(E)$ means the probability of E, $n(E)$ means the number of outcomes in E and $n(S)$ means the number of outcomes in the sample space S, where each **outcome** is **equally likely**.

We usually write a probability as a fraction, but we could also write it as a decimal or percentage.

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EXAMPLE 2

30 people were surveyed on their favourite sport: 11 liked football, 4 liked basketball, 7 liked tennis, 2 liked golf and 6 liked swimming.

Find the probability that any one of these people selected at random will like:

a swimming **b** football **c** golf.

Solution

The size of the sample space $n(S) = 30$ since 30 people were surveyed.

a $P(\text{swimming}) = \frac{6}{30}$
$= \frac{1}{5}$

b $P(\text{football}) = \frac{11}{30}$

c $P(\text{golf}) = \frac{2}{30}$
$= \frac{1}{15}$

Set notation

When working with probabilities, we often use **set notation**.

Union and intersection

$A \cup B$ means A **union** B and is the set of all elements in set A or set B.

$A \cap B$ means A **intersection** B and is the set of all elements that are in **both** sets A and B.

EXAMPLE 3

Set A contains the numbers 3, 7, 12 and 15.
Set B contains the numbers 2, 9, 12, 13 and 17.

a Write sets A and B in set notation.

b Find $A \cup B$ and $A \cap B$.

Solution

a Set $A = \{3, 7, 12, 15\}$.

Set $B = \{2, 9, 12, 13, 17\}$.

b $A \cup B = \{2, 3, 7, 9, 12, 13, 15, 17\}$. It includes all the numbers in either set A or set B.

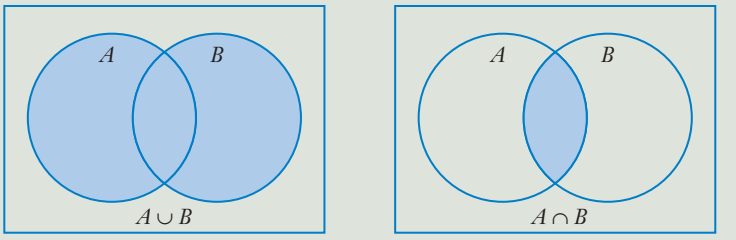
$A \cap B = \{12\}$. It includes any numbers that are in both set A and set B.

Venn diagrams

A **Venn diagram** is a special way to show the relationship between two or more sets.

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Venn diagram



EXAMPLE 4

Draw a Venn diagram for the integers from 1 to 10, given $A = \{1, 3, 4, 5, 8\}$ and $B = \{3, 6, 8, 9, 10\}$.

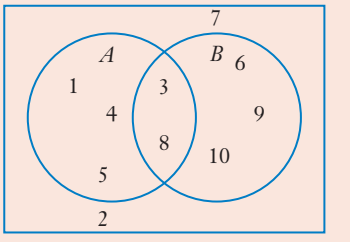
Solution

Draw two overlapping circles and name them A and B.

$A \cap B = \{3, 8\}$, so place these numbers in the overlapping part.

Place the remaining elements of A in the other part of circle A and the remaining elements of B in the other part of circle B.

The numbers are 2 and 7 are not in A or B, so place them outside the circles.



Exercise 7.01 Set notation and Venn diagrams

1 Write the sample space in set notation for each chance situation.

- **a** Tossing a coin
- **b** Rating a radio station between 1 and 5
- **c** Rolling a die
- **d** Selecting a jelly bean from a packet containing red, green, yellow and blue jelly beans
- **e** Rolling an 8-sided die with a different number from 1 to 8 on each face.

2 For each pair of sets, find:

i $X \cap Y$ **ii** $X \cup Y$.

- **a** $X = \{1, 2, 3, 4, 5\}$ and $Y = \{2, 4, 6\}$
- **b** $X = \{\text{red, yellow, white}\}$ and $Y = \{\text{red, white}\}$
- **c** $X = \{4, 5, 7, 11, 15\}$ and $Y = \{6, 8, 9, 10, 12\}$
- **d** $X = \{\text{blue, green, brown, hazel}\}$ and $Y = \{\text{brown, grey, blue}\}$
- **e** $X = \{1, 3, 5, 7, 9\}$ and $Y = \{2, 4, 6, 8, 10\}$

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3 Draw a Venn diagram for each pair of sets.

a $A = \{10, 12, 13, 14, 15\}$ and $B = \{12, 14, 15, 16\}$

b $P = \{\text{red, yellow, white}\}$ and $Q = \{\text{red, green, white}\}$

c $X = \{2, 3, 5, 7, 8\}$ and $Y = \{1, 2, 5, 7, 9, 10\}$

d $A = \{\text{Toyota, Mazda, BMW, Nissan, Porsche}\}$ and $B = \{\text{Mazda, Nissan, Holden, Ford, Porsche}\}$

e $X = \{\text{rectangle, square, trapezium}\}$ and $Y = \{\text{square, parallelogram, trapezium, kite}\}$

4 Discuss whether each probability statement is true.

a The probability of one particular horse winning the Melbourne Cup is $\frac{1}{20}$ if there are 20 horses in the race.

b The probability of a player winning a masters golf tournament is $\frac{1}{15}$ if there are 15 players in the tournament.

c A coin came up tails 8 times in a row. So the next toss must be a head.

d A family has 3 sons and is expecting a fourth child. There is more chance of the new baby being a daughter.

e The probability of a Ducati winning a MotoGP this year is $\frac{6}{47}$ because there are 6 Ducatis and 47 motobikes altogether.

5 To start playing a board game, Simone must roll an even number on a die.

a Write down the sample space for rolling a die.

b What is the set of even numbers on a die?

6 Draw a Venn diagram for each pair of sets.

a Event $K = \{\text{Monday, Thursday, Friday}\}$ and Event $L = \{\text{Tuesday, Thursday, Saturday}\}$ out of days of the week

b Event $A = \{3, 5, 6, 8\}$ and Event $B = \{4, 5, 7, 9\}$ with cards each with a number from 1 to 10 drawn out of a hat

DID YOU KNOW?

John Venn

Venn diagrams are named after **John Venn** (1834–1923), an English probabilist and logician.

WS Experimental probability

7.02 Relative frequency

We can use frequency distribution tables to find the probability of an event using **relative frequency**, the frequency of the event relative to the total frequency.

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EXAMPLE 5

This table shows the number of items bought by a group of people surveyed in a shopping centre.

Number of items	Frequency
0	6
1	4
2	5
3	3
4	7

a Find the relative frequency for each number of items as a fraction.

b Find the probability that a person surveyed at random would buy:

i no items

ii at least 3 items.

Solution

a The sum of the frequencies is 25. This means that 25 people were surveyed.

From the table, 0 has a frequency of 6. The relative frequency is 6 out of 25$= \frac{6}{25}$.
Similarly, other relative frequencies are:

1 item: $\frac{4}{25}$ 2 items: $\frac{5}{25} = \frac{1}{5}$ 3 items: $\frac{3}{25}$ 4 items: $\frac{7}{25}$

b **i** $P(0) = \frac{6}{25}$

ii At least 3 items means 3 or 4 items. Relative frequency of 3 or 4 is 3 + 7 = 10.

$P(\geq 3) = \frac{10}{25} = \frac{2}{5}$

Exercise 7.02 Relative frequency

1 The table shows the scores that a class earned on a maths test.

Score	Frequency
4	6
5	4
6	1
7	7
8	2
9	3

a Find the relative frequency for each score in the table, in fraction form.

b If a student is chosen at random from this class, find the probability that this student:

i scored 8

ii scored less than 7

iii passed, if the pass mark is 5.

c What score is:

i most likely?

ii least likely?

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2 The table shows the results of a survey into the number of days students study each week.

Number of days	Frequency
1	3
2	6
3	1
4	7
5	2
6	1

a Find the relative frequency as a percentage for each number of days.

b If a student was selected at random, find the most likely number of days this student studies.

c Find the probability that this student would study for:

i 1 day

ii 5 days

iii 3 or 4 days

iv at least 4 days

v fewer than 3 days.

3 The table shows the results of a trial HSC exam.

Class	Frequency
0–19	9
20–39	12
40–59	18
60–79	7
80–99	4

a Calculate the relative frequency as a decimal for each class.

b Find the probability that a student chosen at random from these students scored:

i between 20 and 39

ii between 60 and 99

iii less than 40.

4 This table shows the results of a science experiment to find the velocity of an object when it is rolled down a ramp.

Velocity (m/s)	Frequency
2–4	2
5–7	7
8–10	4
11–13	1
14–16	6

a Write the relative frequency of each velocity as a fraction.

b Find the probability that an object selected at random rolls down the ramp with a velocity between:

i 5 and 7 m/s **ii** 11 and 13 m/s **iii** 8 and 10 m/s

iv 11 and 16 m/s **v** 2 and 10 m/s.

c Find the probability that the object has a velocity:

i less than 8 m/s **ii** 5 m/s or more **iii** more than 7 m/s.

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5 A telemarketing company records the number of sales it makes per minute over a half-hour period. The results are in the table.

Sales/min	Frequency
0	4
1	12
2	6
3	3
4	0
5	5

a What percentage of the time were there 3 sales per minute?

b Write the relative frequencies as percentages.

c What is the most likely number of sales/minute?

d Find the probability of making:

i 2 sales/minute **ii** 5 sales/minute

iii more than 2 sales per minute.

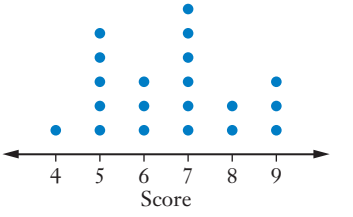
6 a Organise the scores below in a frequency distribution table.

9, 5, 4, 7, 7, 9, 4, 6, 5, 8, 9, 6, 7, 4, 4, 3, 8, 5, 6, 9

b Find the probability of an outcome chosen at random having a score of:

i 7 **ii** at least 8 **iii** less than 5 **iv** 7 or less

7 a From the dot plot, draw up a frequency distribution table.



b Find the probability (as a decimal) that an outcome chosen at random has a score of:

i 8 **ii** at least 6

iii less than 7 **iv** 5 or more

v 8 or less

8 The stem-and-leaf plot shows the ages of people attending a meeting.

Stem	Leaf
1	8 9 9
2	0 3 5 6
3	0 1 2 2 3 5 7 9
4	2 4 6 7 8
5	1 2 4 4 6

a Organise this data into a frequency distribution table, using groups of 10–19, 20–29 and so on.

b What percentage of people at the meeting were:

i in their 30s?

ii younger than 20?

iii in their 40s or 50s?

c Find the probability that a person selected at random from this meeting is:

i younger than 40

ii 50 or over

iii between 20 and 49

iv over 29

v between 10 and 49.

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9 The table shows the quantity of food that a pet shop uses each day for a month.

Food (kg)	Frequency
0–14	3
15–29	11
30–44	8
45–59	4
60–74	2

a In which month was this survey done?

b For what fraction of the month was 45–59 kg of food used?

c For what percentage of the month did the pet shop need more than 29 kg of pet food?

d Find the relative frequency for all groups as a fraction.

e If this survey is typical of the quantities of food that the pet shop uses, find the probability that on any day it will use between:

i 30 and 44 kg **ii** 45 and 74 kg **iii** 0 and 29 kg

iv 15 and 59 kg **v** 30 and 74 kg

WS Theoretical probability

Matching probabilities

7.03 Theoretical probability

While experiments and surveys can give a good prediction of the probability of future events, they are not very accurate. The larger the number of trials, the closer the results can become to the theoretical probability. However, this is not guaranteed.

For example, it is reasonable to assume that if you toss a coin many times you would get similar numbers of heads and tails. Yet in an experiment a coin may come up heads every time.

INVESTIGATION

TOSSING A COIN

Toss a coin 20 times and count the number of heads and tails. What would you expect to happen when tossing a coin this many times? Did your results surprise you?

Combine your results with others in your classroom into a table with relative frequencies.

1 Do the combined results differ from your own?

2 From the table, find the probability of tossing

a heads **b** tails.

If a coin came up tails every time it was tossed 20 times, do you think it would it be more likely to come up heads the next time? Why?

Even though we might think that theoretical probability should be more accurate than experiments, in real life these probabilities will not happen exactly as in theory!

Mutually exclusive events are events that cannot occur at the same time. For example, when throwing a die you cannot throw a number that is both a 5 and a 6.

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The addition rule for mutually exclusive events

When events A and B are mutually exclusive:

$$P(A \cup B) = P(A) + P(B)$$

EXAMPLE 6

A container holds 5 blue, 3 white and 7 yellow marbles. If one marble is selected at random, find the probability of selecting:

a a white marble

b a white or blue marble

c a yellow, white or blue marble

d a red marble.

Solution

Blue, white and yellow are mutually exclusive events.

$n(S) = 5 + 3 + 7 = 15$

a $P(\text{W}) = \frac{3}{15}$

$= \frac{1}{5}$

b $P(\text{W} \cup \text{B}) = P(\text{W}) + P(\text{B})$

$= \frac{3}{15} + \frac{5}{15}$

$= \frac{8}{15}$

c $P(\text{Y} \cup \text{W} \cup \text{B}) = P(\text{Y}) + P(\text{W}) + P(\text{B})$

$= \frac{7}{15} + \frac{3}{15} + \frac{5}{15}$

$= \frac{15}{15}$

$= 1$

d $P(\text{R}) = \frac{0}{15}$

$= 0$

The range of probabilities

If $P(E) = 0$ the event is impossible.

If $P(E) = 1$ the event is certain (it has to happen).

$0 \leq P(E) \leq 1$

The sum of all (mutually exclusive) probabilities is 1.

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Complementary events

The **complement** of set E is the set of all elements that are not in E. We write $\bar{E}$ or E^c.

The **complement** $\bar{E}$ of an event E happening is the event **not** happening.

$P(\bar{E}) = 1 - P(E)$

$P(E) + P(\bar{E}) = 1$

EXAMPLE 7

a The probability of winning a raffle is $\frac{1}{350}$. What is the probability of not winning?

b The probability of a tree surviving a fire is 72%. Find the probability of the tree failing to survive a fire.

Solution

a $P(\text{not win}) = 1 - P(\text{win})$

$= 1 - \frac{1}{350}$

$= \frac{349}{350}$

b $P(\text{failing to survive}) = 1 - P(\text{surviving})$

$= 100\% - 72\%$

$= 28\%$

We can use probability to make predictions or decisions.

EXAMPLE 8

The probability that a traffic light will turn green as a car approaches it is $\frac{5}{12}$. A taxi goes through 192 intersections where there are traffic lights. How many of these would be expected to turn green as the taxi approached?

Solution

It is expected that $\frac{5}{12}$ of the traffic lights would turn green.

$$\frac{5}{12} \times 192 = 80$$

So it would be expected that 80 traffic lights would turn green as the taxi approached.

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Exercise 7.03 Theoretical probability

1 Alannah is in a class of 30 students. If one student is chosen at random to make a speech, find the probability that the student chosen:

a will be Alannah **b** will not be Alannah.

2 A pack of cards contains 52 different cards, one of which is the ace of diamonds. If one card is chosen at random, find the probability that it:

a will be the ace of diamonds **b** will not be the ace of diamonds.

3 There are 6 different newspapers sold at the local newsagent each day. Wendy sends her little brother Rupert to buy her a newspaper one morning but forgets to tell him which one. What is the probability that Rupert will buy the correct newspaper?

4 A raffle is held in which 200 tickets are sold. If I buy 5 tickets, what is the probability of:

a my winning **b** my not winning the prize in the raffle?

5 In a lottery, 200 000 tickets are sold. If Lucia buys 10 tickets, what is the probability of her winning first prize?

6 A bag contains 6 red balls and 8 white balls. If Peter draws one ball out of the bag at random, find the probability that it will be:

a white **b** red.

7 A shoe shop orders in 20 pairs of black, 14 pairs of navy and 3 pairs of brown school shoes. If the boxes are all mixed up, find the probability that one box selected at random will contain brown shoes.

8 The probability of a bus arriving on time is estimated at $\frac{18}{33}$.

a What is the probability that the bus will not arrive on time?

b If there are 352 buses each day, how many would be expected to arrive on time?

9 A bag contains 5 black marbles, 4 yellow marbles and 11 green marbles. Find the probability of drawing 1 marble out at random and getting:

a a green marble **b** a yellow or a green marble.

10 The probability of a certain seed producing a plant with a pink flower is $\frac{7}{9}$.

a Find the probability of the seed producing a flower of a different colour.

b If 189 of these plants are grown, how many of them would be expected to have a pink flower?

11 If a baby has a 0.2% chance of being born with a disability, find the probability of the baby being born without a disability.

12 A die is thrown. Calculate the probability of throwing:

a a 6 **b** an even number **c** a number less than 3.

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13 A book has 124 pages. If any page is selected at random, find the probability of the page number being:

a either 80 or 90 **b** a multiple of 10

c an odd number **d** less than 100.

14 A machine has a 1.5% chance of breaking down at any given time.

a What is the probability of the machine not breaking down?

b If 2600 of these machines are manufactured, how many of them would be expected to:

i break down? **ii** not break down?

15 The probabilities when 3 coins are tossed are as follows:

$P(3 \text{ heads}) = \frac{1}{8}$ $P(2 \text{ heads}) = \frac{3}{8}$

$P(1 \text{ head}) = \frac{3}{8}$ $P(3 \text{ tails}) = \frac{1}{8}$

Find the probability of tossing at least one head.

16 In the game of pool, there are 15 balls, each with the number 1 to 15 on it. In Kelly pool, each person chooses a number at random to determine which ball to sink. If Tracey chooses a number, find the probability that her ball will be:

a an odd number **b** a number less than 8 **c** the 8 ball.

17 **a** Find the probability of a coin coming up heads when tossed.

b If the coin is double-headed, find the probability of tossing a head.

18 A student is chosen at random to write about his or her favourite sport. If 12 students like tennis best, 7 prefer soccer, 3 prefer squash, 5 prefer basketball and 4 prefer swimming, find the probability that the student chosen will write about:

a soccer **b** squash or swimming **c** tennis.

19 There are 29 red, 17 blue, 21 yellow and 19 green chocolate beans in a packet. If Kate chooses one at random, find the probability that it will be red or yellow.

20 The probability of breeding a white budgerigar is $\frac{2}{9}$. If Mr Seed breeds 153 budgerigars over the year, how many would be expected to be white?

21 A biased coin is weighted so that heads comes up twice as often as tails. Find the probability of tossing a tail.

22 A die has the centre dot painted white on the 5 so that it appears as a 4. Find the probability of throwing:

a a 2 **b** a 4 **c** a number less than 5.

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23 The probabilities of a certain number of seeds germinating when 4 seeds are planted are:

Number of seeds	0	1	2	3	4
Probability	$\frac{3}{49}$	$\frac{18}{49}$	$\frac{16}{49}$	$\frac{8}{49}$	$\frac{4}{49}$

Find the probability of at least one seed germinating.

24 The probabilities of 4 friends being chosen for a soccer team are:

$P(4 \text{ chosen}) = \frac{1}{15}$ $P(3 \text{ chosen}) = \frac{4}{15}$

$P(2 \text{ chosen}) = \frac{6}{15}$ $P(1 \text{ chosen}) = \frac{2}{15}$

Find the probability of:

a none of the friends being chosen

b at least 1 of the friends being chosen.

25 If 2 events are mutually exclusive, what could you say about $A \cap B$?

DID YOU KNOW?

The origins of probability

Girolamo Cardano (1501–76) was a doctor and mathematician who developed the first theory of probability. He was a great gambler, and he wrote *De Ludo Aleae* ('On Games of Chance'). This work was largely ignored, and it is said that the first book on probability was written by **Christiaan Huygens** (1629–95).

The main study of probability was done by **Blaise Pascal** (1623–62), and **Pierre de Fermat** (1601–65). Pascal developed the 'arithmetical triangle' called Pascal's triangle that has properties that are applicable to probability as well.

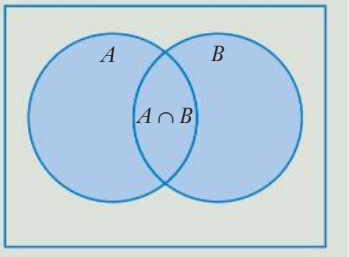
7.04 Addition rule of probability

Sometimes, there is an overlap where more than one event can occur at the same time. We call these **non-mutually exclusive events**. It is important to count the possible outcomes carefully when this happens. We need to be careful not to count the overlapping outcomes $A \cap B$ twice.

Addition rule of probability

For events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



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If A and B are mutually exclusive, then $P(A \cap B) = 0$ so $P(A \cup B) = P(A) + P(B)$.

EXAMPLE 9

One card is selected at random from a pack of 100 cards numbered from 1 to 100. Find the probability that the number on this card is even or less than 20.

Solution

Even: $A = \{2, 4, 6, \ldots, 100\}$

There are 50 even numbers between 1 and 100.

Less than 20: $B = \{1, 2, 3, \ldots, 19\}$

There are 19 numbers less than 20.

Even and less than 20: $A \cap B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

There are 9 numbers that are both even and less than 20.

P(Even or less than 20):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{50}{100} + \frac{19}{100} - \frac{9}{100}$$

This is to avoid counting the 9 'overlapping' numbers twice.

$$= \frac{60}{100}$$

$$= \frac{3}{5}$$

Sometimes for more complex problems, a Venn diagram is useful.

EXAMPLE 10

In Year 7 at Mt Random High School, every student must do art or music. In a group of 100 students surveyed, 47 do music and 59 do art. If one student is chosen at random from Year 7, find the probability that this student does:

a both art and music **b** only art **c** only music.

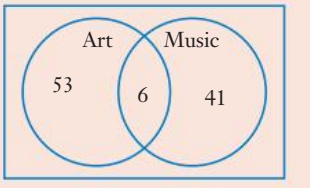
Solution

Number of students: $47 + 59 = 106$

But there are only 100 students!

This means 6 students have been counted twice.

That is, 6 students do both art and music.



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Students doing art only: $59 - 6 = 53$

Students doing music only: $47 - 6 = 41$

a $P(\text{both}) = \frac{6}{100} = \frac{3}{50}$ **b** $P(\text{art only}) = \frac{53}{100}$ **c** $P(\text{music only}) = \frac{41}{100}$

Exercise 7.04 Addition rule of probability

1 A number is chosen at random from the numbers 1 to 20. Find the probability that the number chosen will be:

a divisible by 3 **b** less than 10 or divisible by 3

c a composite number **d** a composite number or a number greater than 12.

2 A set of 50 cards is labelled from 1 to 50. One card is drawn out at random. Find the probability that the card will be:

a a multiple of 5

b an odd number

c a multiple of 5 or an odd number

d a number greater than 40 or an even number

e less than 20.

3 A set of 26 cards, each with a different letter of the alphabet on it, is placed in a box and one card is drawn out at random. Find the probability that the letter on the card is:

a a vowel **b** a vowel or one of the letters in the word 'random'

c a consonant or one of the letters in the word 'movies'.

4 A set of discs is numbered 1 to 100 and one is chosen at random. Find the probability that the number on the disc will be:

a less than 30 **b** an odd number or a number greater than 70

c divisible by 5 or less than 20.

5 In Lotto, a machine holds 45 balls, each with a different number between 1 and 45 on it. The machine draws out one ball at a time at random. Find the probability that the first ball drawn out will be:

a less than 10 or an even number

b between 1 and 15 inclusive, or divisible by 6

c greater than 30 or an odd number.

6 A class of 28 students puts on a concert with all class members performing. If 15 dance and 19 sing in the performance, find the probability that any one student chosen at random from the class will:

a both sing and dance **b** only sing **c** only dance.

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7 A survey of 80 people with dark hair or brown eyes showed that 63 had dark hair and 59 had brown eyes. Find the probability that one of the people surveyed chosen at random has:

- **a** dark hair but not brown eyes
- **b** brown eyes but not dark hair
- **c** both brown eyes and dark hair.

8 A list is made up of 30 people with experience in coding or graphical design. On the list, 13 have coding experience while 9 have graphical design experience. Find the probability that a person chosen at random from the list will have experience in:

- **a** both coding and graphical design
- **b** coding only
- **c** graphical design only.

9 Of a group of 75 students, all study either history or geography. Altogether 54 take history and 31 take geography. Find the probability that a student selected at random studies:

- **a** only geography
- **b** both history and geography
- **c** history but not geography.

10 In a group of 20 dogs at obedience school, 14 dogs will walk to heel and 12 will stay when told. All dogs will do one or the other, or both. If one dog is chosen at random, find the probability that it will:

- **a** both walk to heel and stay
- **b** walk to heel but not stay
- **c** stay but not walk to heel.

WS
Multi-stage problems

7.05 Product rule of probability

CLASS DISCUSSION

TWO-STAGE EVENTS

Work in pairs and try these experiments with one person doing the activity and one recording the results. Toss two coins as many times as you can in a 5-minute period and record the results in a table:

Result	2 heads	One head and one tail	2 tails
Tally			

Compare your results with others in the class. What do you notice? Is this surprising?

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Roll 2 dice as many times as you can in a 5-minute period, find the total of the 2 numbers rolled and record the results in a table:

Total	2	3	4	5	6	7	8	9	10	11	12
Tally											

Compare your results with others in the class. What do you notice? Is this surprising?

Tossing 2 coins and rolling 2 dice are examples of **multi-stage experiments**, where two outcomes happen together. The sample space becomes more complicated, so to list all possible outcomes we use tables and **tree diagrams**.

EXAMPLE 11

Find the sample space and the probability of each outcome for:

a tossing 2 coins

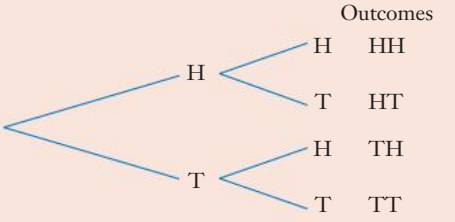
b rolling 2 dice and calculating their sum.

Solution

a Using a table gives:

		2nd coin	
		H	**T**
1st coin	**H**	HH	HT
	T	TH	TT

Using a tree diagram gives:



Since there are four possible outcomes (HH, HT, TH, TT), each outcome has a probability of $\frac{1}{4}$.

Remember that each outcome when tossing 1 coin is $\frac{1}{2}$.

Notice that $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

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b A tree diagram would be too big to draw for this question.

Using a table:

		2nd die					
		1	**2**	**3**	**4**	**5**	**6**
1st die	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Since there are 36 outcomes, each has a probability of $\frac{1}{36}$.

Remember that each outcome when rolling 1 die is $\frac{1}{6}$.

Notice that $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

If A and B are **independent events**, then A occurring does not affect the probability of B occurring. The probability of both occurring is the product of their probabilities.

The product rule for independent events

$$P(A \cap B) = P(A)P(B)$$

EXAMPLE 12

a Find the probability of rolling a double 6 on 2 dice.

b The probability that an archer will hit a target is $\frac{7}{8}$. Find the probability that the archer will:

i hit the target twice

ii miss the target twice.

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Solution

a $P(A \cap B) = P(A)P(B)$

$P(6 \cap 6) = P(6)P(6)$

$= \frac{1}{6} \times \frac{1}{6}$

$= \frac{1}{36}$

b **i** $P(A \cap B) = P(A)P(B)$

Let H = hit, M = miss

$P(\text{H} \cap \text{H}) = P(\text{H})P(\text{H})$

$= \frac{7}{8} \times \frac{7}{8}$

$= \frac{49}{64}$

ii $P(\text{M}) = P(\bar{\text{H}})$

$= 1 - \frac{7}{8}$

$= \frac{1}{8}$

$P(\text{M} \cap \text{M}) = P(\text{M})P(\text{M})$

$= \frac{1}{8} \times \frac{1}{8}$

$= \frac{1}{64}$

The sample space changes when events are not independent. The second event is **conditional** on the first event.

EXAMPLE 13

Maryam buys 5 tickets in a raffle in which 95 tickets are sold altogether. There are 2 prizes in the raffle. What is the probability of her winning:

a both first and second prizes?

b neither prize?

c at least one of the prizes?

Solution

a Probability of winning first prize: $P(\text{W}_1) = \frac{5}{95}$

After winning first prize she has 4 tickets left in the raffle out of a total of 94 tickets left.

Probability of winning second prize: $P(\text{W}_2) = \frac{4}{94}$

Probability of winning both prizes:

$$P(W_1 \cap W_2) = \frac{5}{95} \times \frac{4}{94}$$
$$= \frac{20}{8930}$$
$$= \frac{2}{893}$$

b Probability of not winning first prize:

$$P(\overline{W}_1) = 1 - \frac{5}{95}$$
$$= \frac{90}{95}$$
$$= \frac{18}{19}$$

After not winning first prize, Maryam's 5 tickets are all left in the draw, but the winning ticket is taken out, leaving 94 tickets in the raffle.

Probability of winning second prize: $P(W_2) = \frac{5}{94}$

Probability of not winning second prize:

$$P(\overline{W_2}) = 1 - \frac{5}{94}$$
$$= \frac{89}{94}$$

Probability of winning neither prize:

$$P(\overline{W}_1 \cap \overline{W}_2) = P(\overline{W}_1)P(\overline{W}_2)$$
$$= \frac{90}{95} \times \frac{89}{94}$$
$$= \frac{8010}{8930}$$
$$= \frac{801}{893}$$

c Probability of at least one win:

'at least one' is the same as 'not none'.

$$P(\geq 1 \text{ win}) = 1 - P(0 \text{ wins})$$
$$= 1 - \frac{801}{893} \quad \text{from } \mathbf{b}$$
$$= \frac{92}{893}$$

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Exercise 7.05 Product rule of probability

1 Find the probability of getting 2 heads if a coin is tossed twice.

2 A coin is tossed 3 times. Find the probability of tossing 3 tails.

3 A family has 2 children. What is the probability that they are both girls?

4 A box contains 2 black balls, 5 red balls and 4 green balls. If I draw out 2 balls at random, replacing the first before drawing out the second, find the probability that they will both be red.

5 The probability of a conveyor belt in a factory breaking down at any one time is 0.21. If the factory has 2 conveyor belts, find the probability that at any one time:

a both conveyor belts will break down

b neither conveyor belt will break down.

6 The probability of a certain plant flowering is 93%. If a nursery has 3 of these plants, find the probability that they will all flower.

7 An archery student has a 69% chance of hitting a target. If she fires 3 arrows at a target, find the probability that she will hit the target each time.

8 The probability of a pair of small parrots breeding an albino bird is $\frac{2}{33}$. If they lay 3 eggs, find the probability of the pair:

a not breeding any albinos **b** having all 3 albinos

c breeding at least one albino.

9 A photocopier has a paper jam on average around once every 2400 sheets of paper.

a What is the probability that a particular sheet of paper will jam?

b What is the probability that 2 particular sheets of paper will jam?

c What is the probability that 2 particular sheets of paper will both not jam?

10 In the game Yahtzee, 5 dice are rolled. Find the probability of rolling:

a five 6s **b** no 6s **c** at least one 6.

11 The probability of a faulty computer part being manufactured at Omikron Computer Factory is $\frac{3}{5000}$. If 2 computer parts are examined, find the probability that:

a both are faulty **b** neither is faulty

c at least one is faulty.

12 A set of 10 cards is numbered 1 to 10 and 2 cards are drawn out at random with replacement. Find the probability that the numbers on both cards are:

a odd numbers **b** divisible by 3

c less than 4.

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13 The probability of an arrow hitting a target is 85%. If 3 arrows are shot, find the probability as a percentage, correct to 2 decimal places, of:

a all arrows hitting the target

b no arrows hitting the target

c at least one arrow hitting the target.

14 A coin is tossed n times. Find the probability in terms of n of tossing:

a no tails **b** at least one tail.

15 A bag contains 8 yellow and 6 green lollies. If I choose 2 lollies at random, find the probability that they will both be green:

a if I replace the first lolly before selecting the second

b if I don't replace the first lolly.

16 Mala buys 10 tickets in a raffle in which 250 tickets are sold. Find the probability that she wins both first and second prizes.

17 Two cards are drawn from a deck of 20 red and 25 blue cards (without replacement). Find the probability that they will both be red.

18 A bag contains 100 cards numbered 1 to 100. Scott draws 2 cards out of the bag. Find the probability that:

a both cards are less than 10

b both cards are even

c neither card is a multiple of 5.

19 A box of pegs contains 23 green pegs and 19 red pegs. If 2 pegs are taken out of the box at random, find the probability that both will be:

a green **b** red

20 Find the probability of selecting 2 apples at random from a fruit bowl that contains 8 apples, 9 oranges and 3 peaches.



Tree diagrams

7.06 Probability trees


Tree diagrams

A **probability tree** is a tree diagram that shows the probabilities on the branches.


Tree diagrams

Probability trees

Use the product rule along the branches to find $P(A \cap B)$, the probability of 'A and B'.

Use the addition rule for different branches to find $P(A \cup B)$, the probability of 'A or B'.

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EXAMPLE 14

WS Probability trees

a Robert has a chance of 0.2 of winning a prize in a Taekwondo competition. If he enters 3 competitions, find the probability of his winning:

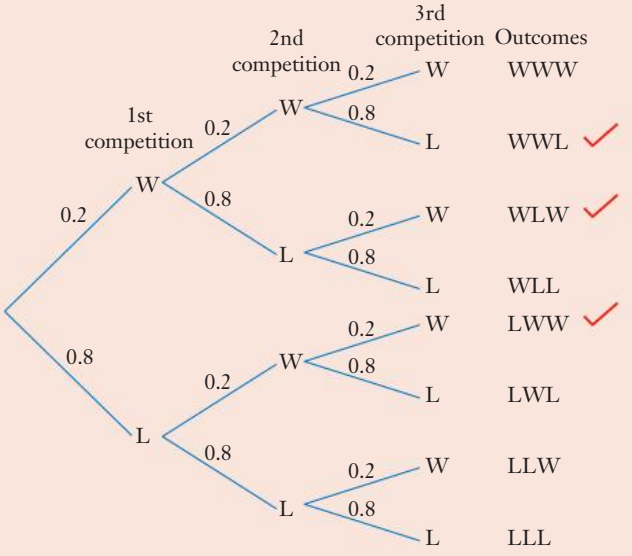
i 2 competitions **ii** at least 1 competition.

b A bag contains 3 red, 4 white and 7 blue marbles. Two marbles are drawn at random from the bag without replacement. Find the probability that the marbles are red and white.

Solution

a $P(\text{W}) = 0.2$, $P(\text{L}) = 1 - 0.2 = 0.8$. W = win, L = lose

Draw a probability tree with 3 levels of branches as shown.



i There are 3 different ways of winning 2 competitions (WWL, WLW and LWW, shown by the red ticks).

Using the product rule along the branches:

$P(\text{WWL}) = 0.2 \times 0.2 \times 0.8 = 0.032$ '*P*(win and win and lose)'

$P(\text{WLW}) = 0.2 \times 0.8 \times 0.2 = 0.032$

$P(\text{LWW}) = 0.8 \times 0.2 \times 0.2 = 0.032$

Using the addition rule for the different results:

$P(2 \text{ wins}) = P(\text{WWL}) + P(\text{WLW}) + P(\text{LWW})$ '*P*(WWL or WLW or LWW)'

$= 0.032 + 0.032 + 0.032$

$= 0.096$

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ii $P(\geq 1\text{W}) = 1 - P(\text{LLL})$

$= 1 - 0.8 \times 0.8 \times 0.8$

$= 0.488$

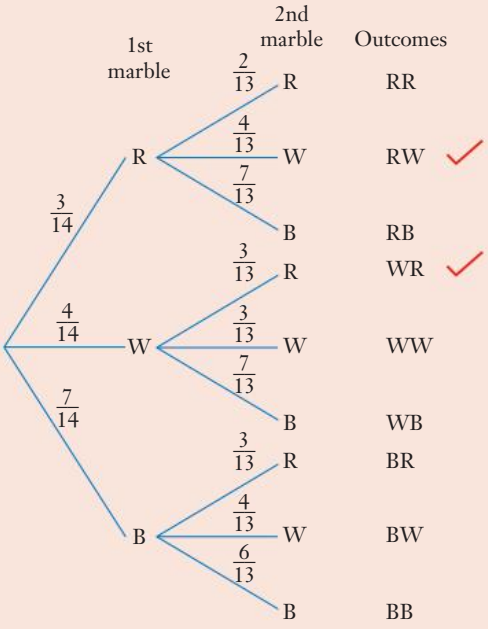
b First marble:

$P(\text{R}) = \frac{3}{14}$ $\quad P(\text{W}) = \frac{4}{14}$

$P(\text{B}) = \frac{7}{14}$



The probabilities for the second marble are dependent on the outcome of the first draw.



There are 2 different ways of drawing out a red and a white marble, as shown by the red ticks: RW, WR.

Using the product rule along the branches:

$P(\text{RW}) = \frac{3}{14} \times \frac{4}{13}$

$= \frac{12}{182}$

$= \frac{6}{91}$

$P(\text{WR}) = \frac{4}{14} \times \frac{3}{13}$

$= \frac{12}{182}$

$= \frac{6}{91}$

Using the addition rule for the different results:

$P(\text{RW or WR}) = \frac{6}{91} + \frac{6}{91}$

$= \frac{12}{91}$

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Exercise 7.06 Probability trees

1 Three coins are tossed. Find the probability of getting:

a 3 tails **b** 2 heads and 1 tail **c** at least 1 head.

2 In a set of 30 cards, each one has a number on it from 1 to 30. If 1 card is drawn out, then replaced and another drawn out, find the probability of getting:

a two 8s

b a 3 on the first card and an 18 on the second card

c a 3 on one card and an 18 on the other card.

3 A bag contains 5 red marbles and 8 blue marbles. If 2 marbles are chosen at random, with the first replaced before the second is drawn out, find the probability of getting:

a 2 red marbles **b** a red and a blue marble.

4 A certain breed of cat has a 35% probability of producing a white kitten. If a cat has 3 kittens, find the probability that she will produce:

a no white kittens **b** 2 white kittens **c** at least 1 white kitten.

5 The probability of rain on any day in May each year is given by $\frac{3}{10}$. A school holds a fete on a Sunday in May for 3 years running. Find the probability that it will rain:

a during 2 of the fetes **b** during 1 fete **c** during least 1 fete.

6 A certain type of plant has a probability of 0.85 of producing a variegated leaf. If I grow 3 of these plants, find the probability of getting a variegated leaf in:

a 2 of the plants **b** none of the plants **c** at least 1 plant.

7 A bag contains 3 yellow balls, 4 pink balls and 2 black balls. If 2 balls are chosen at random, find the probability of getting a yellow and a black ball:

a with replacement **b** without replacement.

8 Anh buys 4 tickets in a raffle in which 100 tickets are sold altogether. There are 2 prizes in the raffle. Find the probability that Anh will win:

a first prize **b** both prizes **c** 1 prize

d no prizes **e** at least 1 prize.

9 Two singers are selected at random to compete against each other in a TV singing contest. One person is chosen from Team A, which has 8 females and 7 males, and the other is chosen from Team B, which has 6 females and 9 males. Find the probability of choosing:

a 2 females **b** 1 female and 1 male.

10 Two tennis players are said to have a probability of $\frac{2}{5}$ and $\frac{3}{4}$ respectively of winning a tournament. Find the probability that:

a 1 of them will win **b** neither one will win.

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11 In a batch of 100 cars, past experience would suggest that 3 could be faulty. If 3 cars are selected at random, find the probability that:

a 1 is faulty **b** none is faulty **c** all 3 cars are faulty.

12 In a certain poll, 46% of people surveyed liked the current government, 42% liked the Opposition and 12% had no preference. If 2 people from the survey are selected at random, find the probability that:

a both will prefer the Opposition

b one will prefer the government and the other will have no preference

c both will prefer the government.

13 A manufacturer of X energy drink surveyed a group of people and found that 31 people liked X drinks best, 19 liked another brand better and 5 did not drink energy drinks. If any 2 people are selected at random from that group, find the probability that:

a one person likes the X brand of energy drink

b both people do not drink energy drinks.

14 In a group of people, 32 are Australian-born, 12 were born in Asia and 7 were born in Europe. If 2 of the people are selected at random, find the probability that:

a they were both born in Asia

b at least 1 of them will be Australian-born

c both were born in Europe.

15 There are 34 men and 32 women at a party. Of these, 13 men and 19 women are married. If 2 people are chosen at random, find the probability that:

a both will be men

b one will be a married woman and the other an unmarried man

c both will be married.

16 Frankie rolls 3 dice. Find the probability she rolls:

a 3 sixes **b** 2 sixes **c** at least 1 six.

17 A set of 5 cards, each labelled with one of the letters A, B, C, D and E, is placed in a hat and 2 cards are selected at random without replacement. Find the probability of getting:

a D and E

b neither D nor E on either card

c at least one D.

18 The ratio of girls to boys at a school is 4 : 5. Two students are surveyed at random from the school. Find the probability that the students are:

a both boys **b** a girl and a boy **c** at least one girl.

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7.07 Conditional probability

Conditional probability is the probability that an event A occurs when it is known that another event B has already occurred. You have already used conditional probability in multi-stage events when the outcome of the second event was dependent on the outcome of the first event. Examples include selections **without replacement**.

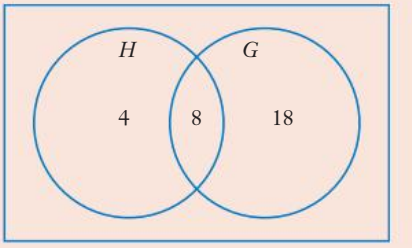
We write the probability of event A happening given that event B has happened as $P(A|B)$.

EXAMPLE 15

All 30 students in a class study either history or geography. If 18 only do geography and 8 do both subjects, find the probability that a student does geography, given that the student does history.

Solution

Draw a Venn diagram using H = history and G = geography.



8 students do both history and geography.

18 students only do geography.

So $n(G) = 18 + 8 = 26$.

So $n(H \text{ only}) = 30 - 26 = 4$

There are $8 + 4 = 12$ students doing history, of whom 8 also do geography.

So $P(G|H) = \frac{8}{12}$

$= \frac{2}{3}$

With conditional probability, knowing that an event has already occurred reduces the sample space. In the example above, the sample space changed from 30 to 12.

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EXAMPLE 16

The table shows the results of a survey into vaccinations against a new virus.

	Vaccinated	Not vaccinated	Totals
Infected	13	159	172
Not infected	227	38	265
Totals	240	197	437

Find the probability that a person selected at random is:

a not vaccinated

b infected given that the person is vaccinated

c not infected given that the person is not vaccinated

d vaccinated given that the person is infected.

Solution

a $n(S) = 437$, $n(\text{not vaccinated}) = 197$

$$P(\text{not vaccinated}) = \frac{197}{437}$$

b $n(\text{vaccinated}) = 240$

$n(\text{infected} \mid \text{vaccinated}) = 13$

$$P(\text{infected} \mid \text{vaccinated}) = \frac{13}{240}$$

c $n(\text{not vaccinated}) = 197$

$n(\text{not infected} \mid \text{not vaccinated}) = 38$

$$P(\text{not infected} \mid \text{not vaccinated}) = \frac{38}{197}$$

d $n(\text{infected}) = 172$

$n(\text{vaccinated} \mid \text{infected}) = 13$

$$P(\text{vaccinated} \mid \text{infected}) = \frac{13}{172}$$

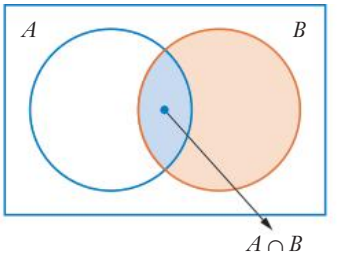
Notice that $P(\text{infected} \mid \text{vaccinated}) \neq P(\text{vaccinated} \mid \text{infected})$.

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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The $P(B)$ in the denominator is a result of the sample space being reduced to B (the orange circle in the Venn diagram).



Proof

$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$
$$= \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$
$$= \frac{P(A \cap B)}{P(B)}$$

For conditional probability, the product rule becomes $P(A \cap B) = P(A|B)P(B)$.

EXAMPLE 17

Lara is an athlete who enters a swimming and running race. She has a 44% chance of winning the swimming race and a 37% chance of winning both races. Find to the nearest whole percentage the probability that she wins the running race if she has won the swimming race.

Solution

If Lara has already won the swimming race (S), then the probability of her winning the running race (R) is conditional.

$$P(S) = 44\%$$
$$= 0.44$$
$$P(R \text{ and } S) = P(R \cap S)$$
$$= 37\%$$
$$= 0.37$$
$$P(R|S) = \frac{P(R \cap S)}{P(S)}$$
$$= \frac{0.37}{0.44}$$
$$\approx 0.84\,09...$$
$$\approx 84\%$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

So the probability of Lara winning the running race given that she has won the swimming race is 84%.

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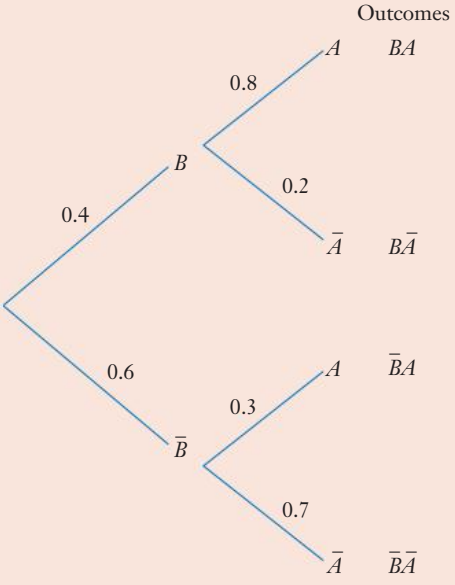
EXAMPLE 18

A zoo has a probability of 80% of having an article published in the newspaper when there is a birth of a baby animal. When there is no birth, the zoo has a probability of only 30% of having an article published. The probability of the zoo having an animal born at any one time is 40%.

Find the percentage probability that a baby animal was born given that an article was published.

Solution

We can draw up a probability tree showing the probabilities of having an article published (A) and a baby animal being born (B).



We want $P(B|A)$. According to the formula:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

From the probability tree:

$P(B \cap A) = P(BA)$ The event (numerator)

$= 0.4 \times 0.8$

$= 0.32$

$P(A) = 0.4 \times 0.8 + 0.6 \times 0.3$ The sample space (denominator)

$= 0.5$

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$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$
$$= \frac{0.32}{0.5}$$
$$= 0.64$$
$$= 64\%$$

So the probability that an animal was born given that an article was published is 64%.

Conditional probability and independent events

We saw earlier that:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Rearranging this gives $P(A \cap B) = P(A|B)\ P(B)$.

But if A and B are **independent events**, $P(A \cap B) = P(A)P(B)$ (the product rule), which means:

$P(A|B) = P(A)$.

Similarly, $P(B|A) = P(B)$.

Conditional probability and independent events

For independent events A and B:

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$
$$P(A \cap B) = P(A)P(B)$$

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EXAMPLE 19

a $P(X) = 0.2$ and $P(X \cap Y) = 0.06$. Determine whether X and Y are independent if:

i $P(Y) = 0.6$ **ii** $P(Y) = 0.3$

b Show that A and B are independent given that $P(A) = 0.6$, $P(B) = 0.45$, $P(A \cup B) = 0.78$.

Solution

a For independent events, the product rule is $P(X \cap Y) = P(X)P(Y)$.

i $P(X \cap Y) = 0.06$

$P(X)P(Y) = 0.2 \times 0.6$

$= 0.12$

$\neq P(X \cap Y)$

$\therefore X$ and Y are not independent.

ii $P(X \cap Y) = 0.06$

$P(X)P(Y) = 0.2 \times 0.3$

$= 0.06$

$= P(X \cap Y)$

$\therefore X$ and Y are independent

b Using the addition rule:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.78 = 0.6 + 0.45 - P(A \cap B)$

$0.78 = 1.05 - P(A \cap B)$

$P(A \cap B) = 1.05 - 0.78$

$= 0.27$

Using the product rule for independent events:

$P(A \cap B) = P(A)\,P(B)$

$= 0.6 \times 0.45$

$= 0.27$, as above

$\therefore A$ and B are independent.

Exercise 7.07 Conditional probability

1 A bag contains 9 black and 8 white balls. I draw out two at random. If the first ball is white, find the probability that the next ball is:

a black **b** white

2 A class has 13 boys and 15 girls. Two students are chosen at random to carry a box of equipment. Find the probability that the second person chosen is a boy given that the first student chosen was a girl.

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3 Two dice are rolled. Find the probability of rolling:

a a double six if the first die was a six.

b a total of 8 or more if the first die was a 3.

4 A team has a probability of 52% of winning its first season and a 39% chance of winning both seasons 1 and 2. What is the probability of the team winning the second season given that it wins the first season?

5 A missile has a probability of 0.75 of hitting a target. It has a probability of 0.65 of hitting two targets in a row. What is the probability that the missile will hit the second target given that it has hit the first target?

6 Danuta has an 80% probability of passing her first English assessment and she has a 45% probability of passing both the first and second assessments. Find the probability that Danuta will pass the second assessment given that she passes the first one.

7 A group of 10 friends all prepared to go out in the sun by putting on either sunscreen or a hat. If 5 put on only sunscreen and 3 put on both sunscreen and a hat, find the probability that a friend who:

a put on sunscreen also put on a hat

b put on a hat didn't put on sunscreen.

8 A container holds 20 cards numbered 1 to 20. Two cards are selected at random. Find the probability that the second card is:

a an odd number given that the first card was a 7

b a number less than 5 given that the first card was a 12

c a number divisible by 3 if the first number was 6.

9 A group of 12 people met at a café for lunch. If 9 people had a pie and 7 had chips, find the probability that one of the people:

a had chips, given that this person had a pie

b did not have a pie given that the person had chips.

10 All except for 3 people out of 25 on a European tour had studied either French or Spanish. Nine people studied only French and 5 studied both French and Spanish. Find the probability that one of these people:

a studied Spanish if that person studied French

b did not study French given that the person studied Spanish.

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11 The two-way table shows the numbers of students who own smartphones and tablets.

	Smartphone	No smartphone	Totals
Tablet	23	8	31
No tablet	65	3	68
Totals	88	11	99

Find the probability that a person selected at random:

a owns a smartphone given that the person:

i owns a tablet **ii** doesn't own a tablet.

b owns a tablet given that the person:

i owns a smartphone **ii** doesn't own a smartphone.

12 The table below shows the number of local people with casual and permanent jobs.

	Women	Men
Permanent	23	38
Casual	79	64

Find the probability that a person chosen at random:

a has a permanent job given that she is a woman

b has a casual job given that he is a man

c is a man given that the person has a casual job

d is a man if the person has a permanent job.

13 In a group of 35 friends, all either play sport or a musical instrument. If 14 play both and 8 only play sport, find the probability that a friend chosen at random will:

a play a musical instrument given that the friend plays sport

b not play sport given that the friend plays a musical instrument.

14 The two-way table shows the results of a survey into attendance at a local TAFE college.

	Under 25	Between 25 and 50	Over 50	Totals
At TAFE	53	68	34	155
Not at TAFE	85	105	88	278
Totals	138	173	122	433

Find the probability that a person:

a attends TAFE given that this person is over 50

b is between 25 and 50 if that person does not attend TAFE

c is not at TAFE given that this person is under 25

d is over 50 if the person is at TAFE

e is at TAFE given the person is aged 25 or over.

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15 A tennis team has a probability of 76% of winning a match when they are at home and 45% of winning a match when they are away. If the team plays 58% of their matches away, find the probability that the team:

- **a** wins their match given that they are away
- **b** are at home given that they win a match
- **c** are away given that they lose a match.

16 A factory produces solar batteries. The probability of a new battery being defective is 3%. However, if the manager is on duty, the probability of a new battery being defective changes to 2%. The manager is on duty 39% of the time. Find the probability that the manager is on duty if a new battery is defective.

17 The chance of a bushfire is 85% after a period of no rain and 21% after rain. The chance of rain is 46%. Find the probability that:

- **a** there is not a bushfire given that it has rained
- **b** it has rained given that there is a bushfire
- **c** it has not rained given there is a bushfire
- **d** it has rained given there is not a bushfire.

18 If $P(A|B) = 0.67$ and $P(B) = 0.31$, find the value of $P(A \cap B)$.

19 If $P(L) = 0.17$, $P(L \cap M) = 0.0204$ and $P(M) = 0.12$, show that L and M are independent.

20 Given $P(X) = 0.3$, $P(Y) = 0.42$ and $P(X \cup Y) = 0.594$, show that X and Y are independent.

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7. TEST YOURSELF

Practice quiz

Probability crossword

Probability find-a-word

For Questions **1**–**4**, select the correct answer **A**, **B**, **C** or **D**.

1 The probability of getting at least one 1 when rolling two dice is:

A $\frac{1}{3}$ **B** $\frac{1}{6}$ **C** $\frac{11}{36}$ **D** $\frac{5}{18}$

2 A bag contains 7 white and 5 blue balls. Two balls are selected at random without replacement. The probability of selecting a white and a blue ball is:

A $\frac{35}{132}$ **B** $\frac{35}{72}$ **C** $\frac{35}{144}$ **D** $\frac{35}{66}$

3 If $A = \{5, 7, 8\}$ and $B = \{3, 7, 9\}$ then the set $\{7\}$ represents:

A $A - B$ **B** $A \cap B$ **C** $A + B$ **D** $A \cup B$

4 For the table, the relative frequency of a score of 11 is (there may be more than one answer):

A 32% **B** $\frac{8}{11}$

C 0.032 **D** $\frac{8}{25}$

Score	Frequency
8	5
9	2
10	9
11	8
12	1

5 **a** Given event $A = \{3, 5, 6, 8, 10\}$ and event $B = \{5, 7, 8, 9, 11, 12\}$, find:
i $A \cup B$ **ii** $A \cap B$

b Draw a Venn diagram showing this information.

6 Find the sample space for each situation:

a Tossing two coins

b Choosing a colour from the Australian flag.

7 The table shows the results of an experiment when throwing a die.

a Add a column for relative frequencies (as fractions).

b From the table, find the probability of throwing:
i 3 **ii** more than 4
iii 6 **iv** 1 or 2
v less than 4

Face	Frequency
1	17
2	21
3	14
4	20
5	18
6	10

8 The probability that a certain type of seed will germinate is 93%. If 3 of this type of seeds are planted, find the probability that:

a all will germinate **b** just 1 will germinate

c at least 1 will germinate.

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9 A game is played where the differences of the numbers rolled on 2 dice are taken.

a Draw a table showing the sample space (all possibilities).

b Find the probability of rolling a difference of:

i 3 **ii** 0 **iii** 1 or 2

10 Mark buys 5 tickets in a raffle in which 200 are sold altogether.

a What is the probability that he will:

i win the raffle? **ii** not win the raffle?

b If the raffle has 2 prizes, find the probability that Mark will win just 1 prize.

11 In a class of 30 students, 17 study history, 11 study geography and 5 study neither. Find the probability that a student chosen at random studies:

a geography but not history

b both history and geography

c geography, given that the student studies history

d history, given that the student studies geography.

12 'In the casino, when tossing 2 coins, 2 tails came up 10 times in a row. So there is less chance that 2 tails will come up next time.' Is this statement true? Why?

13 A set of 100 cards numbered 1 to 100 is placed in a box and one is drawn at random. Find the probability that the card chosen is:

a odd **b** less than 30 **c** a multiple of 5

d less than 30 or a multiple of 5 **e** odd or less than 30.

14 Jenny has a probability of $\frac{3}{5}$ of winning a game of chess and a probability of $\frac{2}{3}$ of winning a card game. If she plays one of each game, find the probability that she wins:

a both games **b** one game **c** neither game.

15 A bag contains 5 black and 7 white marbles. Two are chosen at random from the bag without replacement. Find the probability of getting a black and a white marble.

16 There are 7 different colours and 8 different sizes of leather jackets in a shop. If Brady selects a jacket at random, find the probability that he will select one the same size and colour as his friend does.

17 Each machine in a factory has a probability of 4.5% of breaking down at any time. If the factory has 3 of these machines, find the probability that:

a all will be broken down

b at least one will be broken down.

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18 A bag contains 4 yellow, 3 red and 6 blue balls. Two are chosen at random.

a Find the probability of choosing:
i 2 yellow balls **ii** a red and a blue ball **iii** 2 blue balls.

b Find the probability that the second ball is:
i yellow, given that the first ball is blue
ii red, given the first ball is yellow.

19 In a group of 12 friends, 8 have seen the movie *Star Wars 20* and 9 have seen the movie *Mission Impossible 9*. Everyone in the group has seen at least one of these movies. If one of the friends is chosen at random, find the probability that this person has seen:

a both movies **b** only *Mission Impossible 9*.

20 A game of chance offers a $\frac{2}{5}$ probability of a win or a $\frac{3}{8}$ probability of a draw.

a If Billal plays one of these games, find the probability that he loses.

b If Sonya plays 2 of these games, find the probability of:
i a win and a draw **ii** a loss and a draw **iii** 2 wins.

21 A card is chosen at random from a set of 10 cards numbered 1 to 10. A second card is chosen from a set of 20 cards numbered 1 to 20. The 2 cards are placed together in order to make a number, for example 715. Find the probability that the combination number these cards make is:

a 911 **b** less than 100 **c** between 300 and 500.

22 A loaded die has a $\frac{2}{3}$ probability of coming up 6. The other numbers have an equal probability of coming up. If the die is rolled, find the probability that it comes up:

a 2 **b** even.

23 Amie buys 3 raffle tickets. If 150 tickets are sold altogether, find the probability that Amie wins:

a 1st prize **b** only 2nd prize
c 1st and 2nd prizes **d** neither prize.

24 A bag contains 6 white, 8 red and 5 blue balls. If 2 balls are selected at random, find the probability of choosing a red and a blue ball:

a with replacement **b** without replacement.

25 A group of 9 friends go to the movies. All buy popcorn or an ice-cream. If 5 buy popcorn and 7 buy ice-creams, find the probability that one friend chosen at random will have:

a popcorn but not ice-cream **b** both popcorn and ice-cream
c popcorn given that the friend has an ice-cream

26 Ed's probability of winning at tennis is $\frac{3}{5}$ and his probability of winning at squash is $\frac{7}{10}$. Find the probability of Ed winning:

a both games **b** neither game **c** one game.

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7. CHALLENGE EXERCISE

1 In a group of 35 students, 25 go to the movies and 15 go to the league game. If all the students like at least one of these activities, and two students are chosen from this group at random, find the probability that:

a both only go to the movies

b one only goes to the league game and the other goes to both the game and movies.

2 A certain soccer team has a probability of 0.5 of winning a match and a probability of 0.2 of a draw. If the team plays 2 matches, find the probability that it will:

a draw both matches **b** win at least 1 match **c** not win either match.

3 A game of poker uses a deck of 52 cards with 4 suits (hearts, diamonds, spades and clubs). Each suit has 13 cards, consisting of an ace, cards numbered from 2 to 10, a jack, queen and king. If a person is dealt 5 cards, find the probability of getting four aces.

4 If a card is drawn out at random from a set of playing cards find the probability that it will be:

a an ace or a heart

b a diamond or an odd number not including aces

c a jack or a spade.

5 Bill does not select the numbers 1, 2, 3, 4, 5 and 6 for Lotto because he says this combination would never win. Is he correct?

6 Out of a class of 30 students, 19 play a musical instrument and 7 play both a musical instrument and a sport. Two students play neither.

a One student is selected from the class at random. Find the probability that this person plays a sport but not a musical instrument.

b Two people are selected at random from the class. Find the probability that both these people only play a sport.

7 A game involves tossing 2 coins and rolling 2 dice. The scoring is shown in the table.

Result	Score (points)
2 heads and double 6	5
2 heads and double (not 6)	3
2 tails and double 6	4
2 tails and double (not 6)	2

a Find the probability of getting 2 heads and a double 6.

b Find the probability of getting 2 tails and a double that is not 6.

c What is the probability that Andre will score 13 in three moves?

8 Silvana has a 3.8% probability of passing on a defective gene to a daughter and a 0.6% probability of passing a defective gene on to a son. The probability of Silvana having a son is 52%. Find the probability that she has a son, given that Silvana passes on a defective gene.

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Practice set 3

For Questions **1** to **11**, select the correct answer **A**, **B**, **C** or **D**.

1 The quotient rule for differentiating $y = \frac{u}{v}$ is:

A $y' = \frac{uv' - vu'}{v^2}$ **B** $\frac{u'v - v'u}{v^2}$

C $y' = u'v + v'u$ **D** $y' = uv' + vu'$

2 If $f(x) = x^2$ and $g(x) = 2x + 1$, the composite function $g(f(x))$ is given by:

A $(2x + 1)^2$ **B** $(2x)^2 + 1$

C $2x + 1^2$ **D** $2x^2 + 1$

3 The number of employees, N, is inversely proportional to the time, t, it takes to do a stocktake. What is the equation showing this information?

A $N = kt$ **B** $N = t + k$ **C** $N = \frac{k}{t}$ **D** $N = \frac{t}{k}$

4 Find the derivative of $(3x - 2)^8$.

A $(3x - 2)^7$ **B** $8(3x - 2)^7$

C $8x^7(3x - 2)$ **D** $24(3x - 2)^7$

5 Find the probability of drawing out a blue and a white ball from a bag containing 7 blue and 5 white balls if the first ball is not replaced before taking out the second.

A $\frac{70}{121}$ **B** $\frac{70}{144}$ **C** $\frac{1225}{17\,424}$ **D** $\frac{35}{66}$

6 The equation of a circle with radius 3 and centre (–1, 4) is:

A $(x - 1)^2 + (y + 4)^2 = 3$ **B** $(x - 1)^2 + (y + 4)^2 = 9$

C $(x + 1)^2 + (y - 4)^2 = 9$ **D** $(x + 1)^2 + (y - 4)^2 = 3$

7 If $f(x) = 2x^2 - 3x + 1$ and $g(x) = (x + 3)^2$ find the degree of $y = f(x)g(x)$.

A 2 **B** 4 **C** 3 **D** 5

8 Find the domain of $f(x) = \frac{2}{x + 7}$.

A $(-\infty, 7) \cup (7, \infty)$ **B** $(-\infty, -7) \cup (-7, \infty)$

C $(-\infty, 7) \cap (7, \infty)$ **D** $(-\infty, -7) \cap (-7, \infty)$

9 If the displacement of a particle is given by $x = 2t^3 + 6t^2 - 4t + 10$, the initial velocity is:

A –4 **B** 10 **C** 12 **D** 14

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10 In a group of 25 students, 19 catch a train to school and 21 catch a bus. If one of these students is chosen at random, find the probability that the student only catches a bus to school.

A $\frac{6}{25}$ **B** $\frac{21}{25}$ **C** $\frac{3}{5}$ **D** $\frac{3}{20}$

11 Conditional probability $P(A|B)$ is given by:

A $\frac{P(A \cup B)}{P(B)}$ **B** $\frac{P(A \cap B)}{P(A)}$

C $\frac{P(A \cup B)}{P(A)}$ **D** $\frac{P(A \cap B)}{P(B)}$

12 Differentiate:

a $y = x^9 - 4x^2 + 7x + 3$ **b** $y = 2x(x^2 - 1)$ **c** $y = 3x^{-4}$

d $y = \frac{5}{2x^5}$ **e** $y = \sqrt{x^3}$ **f** $y = (2x + 3)^7$

g $y = \frac{1}{(x^2 - 7)^4}$ **h** $y = \sqrt[3]{5x + 1}$ **i** $y = \frac{5x^2 - 1}{2x + 3}$

13 Sketch the graph of:

a $y = \frac{4}{2x - 4}$ **b** $P(x) = x^3 + x^2 - 2x$ **c** $y = |x - 1|$

d $x^2 + y^2 = 25$ **e** $f(x) = -\sqrt{1 - x^2}$

14 In a class of 25 students, 11 play guitar, 9 play the piano, while 8 don't play either instrument. If one student is selected at random from the class, find the probability that this student will play:

a both guitar and piano

b neither guitar or piano

c only guitar.

15 The volume in litres of a rectangular container that is leaking over time t minutes is given by $V = -t^2 + 4t + 100$. Find:

a the initial volume

b the volume after 10 minutes

c the rate of change in volume after 10 minutes

d how long it will take, to 1 decimal place, until the container is empty.

16 **a** Find the equation of the tangent to the curve $y = x^3 - 3x$ at the point $P = (-2, -2)$.

b Find the equation of the normal to $y = x^3 - 3x$ at P.

c Find the point Q where this normal cuts the x-axis.

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17 Two dice are thrown. Find the probability of throwing:

a double 1 **b** any double **c** at least one 3

d a total of 6 **e** a total of at least 8.

18 The function $f(x) = ax^2 + bx + c$ has a tangent at $(1, -3)$ with a gradient of -1. It also passes through $(4, 3)$. Find the values of a, b and c.

19 Find the equation of the circle with centre $(-2, -3)$ and radius 5 units.

20 Find the centre and radius of the circle with equation:

a $x^2 + 6x + y^2 - 10y - 15 = 0$ **b** $x^2 + 10x + y^2 - 6y + 30 = 0$

21 $f(x) = 3x^2 - 4x + 9$.

a Find $f(x + h) - f(x)$.

b Show by differentiating from first principles that $f'(x) = 6x - 4$.

22 **a** Find the equation of the tangent to the curve $y = x^3 - 2$ at the point $P(1, -1)$.

b The curve $y = x^3 - 2$ meets the y-axis at Q. Find the equation of PQ.

c Find the equation of the normal to $y = x^3 - 2$ at the point $(-1, -3)$.

d Find the point R where this normal cuts the x-axis.

23 100 cards are numbered from 1 to 100. If one card is chosen at random, find the probability of selecting:

a an even number less than 30

b an odd number or a number divisible by 9.

24 A bag contains 5 white, 6 yellow and 3 blue balls. Two balls are chosen at random from the bag without replacement. Find the probability of choosing:

a 2 blue balls **b** a white ball and a yellow ball

25 If Scott buys 10 tickets, find the probability that he wins both first and second prizes in a raffle in which 100 tickets are sold.

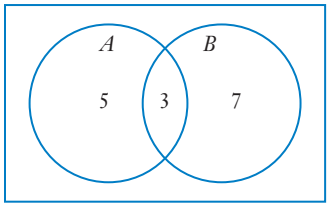
26 Two dice are rolled. Find the probability of rolling a total:

a of 8 **b** less than 7 **c** greater than 9

d of 4 or 5 **e** that is an odd number.

27 For the Venn diagram, find:

a $P(A|B)$ **b** $P(B|A)$



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28 A bag contains 5 red, 7 blue and 9 yellow balls. Cherylanne chooses 2 balls at random from the bag. Find the probability of that she chooses:

a blue given the first ball was yellow

b red given the first ball was blue.

29 If $f(x) = 2x^3 - 5x^2 + 4x - 1$, find $f(-2)$ and $f'(-2)$.

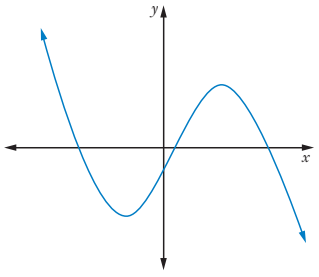
30 **a** Find the gradient of the secant to the curve $f(x) = 2x^3 - 7$ between the point $(2, 9)$ and the point where:

i $x = 2.01$ **ii** $x = 1.99$

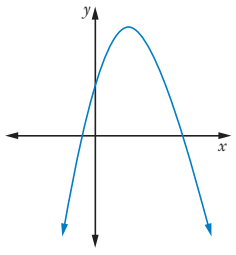
b Hence estimate the gradient of the tangent to the curve at $(2, 9)$.

31 Sketch the gradient function for each curve.

a



b



32 The area of a community garden in m^2 is given by $A = 7x - x^2$ where x is the length of the garden.

a Find the area when the length is:

i 3 m **ii** 4.5 m.

b Find the length when the area is 8 m^2, to 1 decimal place.

c Sketch the graph of the area function.

d Find the maximum possible area.

33 Solve graphically:

$$|x + 2| = 3.$$

34 If $f(x) = x^2 - 1$ and $g(x) = x^3 + 3$, find:

a the degree of $y = f(x)g(x)$

b the leading coefficient of $y = f(x)g(x)$

c the constant term of $y = f(x)g(x)$.

35 The displacement x cm of an object moving along a straight line over time t seconds is given by $x = 2t^3 - 13t^2 + 17t + 12$.

a Find the initial displacement, velocity and acceleration.

b Find the displacement, velocity and acceleration after 2 seconds.

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36 If $A = \{1, 3, 4, 5\}$ and $B = \{2, 3, 5, 6\}$:

a find $A \cup B$

b find $A \cap B$

c draw a Venn diagram showing this information.

37 Find the equation of the tangent to the curve $y = 3x^2 - 6x + 7$ at the point $(2, 7)$.

38 Find the derivative of:

a $y = x^{-3}$ **b** $y = \sqrt{x^3}$ **c** $y = \dfrac{1}{x^2}$ **d** $y = \dfrac{(7x+4)^2}{3x-1}$

e $y = (5x^2 + 1)(2x - 3)^4$ **f** $y = (3x + 1)^5$ **g** $y = \sqrt{2x-1}$

39 $f(x) = x^2 - 2$ and $g(x) = 2x - 1$.

a Find the equation of:

i $y = f(x) + g(x)$ **ii** $y = f(x)g(x)$

iii $y = g(x) - f(x)$ **iv** $y = \dfrac{g(x)}{f(x)}$

b Sketch the graph of:

i $y = -f(x)$ **ii** $y = g(-x)$

iii $y = -g(-x)$

40 **a** Find the centre and radius of the circle $x^2 + 2x + y^2 - 6y - 6 = 0$.

b Find its domain and range.

41 Find the equation of the normal to the curve $y = x^2 - 4x + 1$ at the point $(3, -2)$.

42 Differentiate:

a $y = 2x^4 - 5x^3 + 3x^2 - x - 4$ **b** $y = \dfrac{1}{2x^5}$ **c** $y = \sqrt{x}$

d $y = (2x - 3)^7$ **e** $y = 3x^4(2x - 5)^7$ **f** $y = \dfrac{5x+7}{3x-2}$

43 If $f(x) = x^2 + 1$ and $g(x) = x - 3$, find the degree of:

a $f(x) + g(x)$ **b** $f(x)g(x)$

44 A coin is tossed and a die thrown. Find the probability of getting:

a a head and a 6 **b** a tail and an odd number.

45 Find the domain and range of:

a $y = x^3 + 1$ **b** $y = 1 - x^2$

c $x^2 + 4x + y^2 - 2y - 20 = 0$ **d** $y = \dfrac{4}{x+2}$

46 If $f(x) = x^3$ and $g(x) = 2x + 5$, find:

a $f(g(x))$ **b** $g(f(x))$

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47 The table below shows the results of an experiment in tossing 2 coins.

Result	Frequency
HH	24
HT	15
TH	38
TT	23

a Add a column for relative frequencies as fractions.

b From the table, find the probability of tossing:

i 2 tails **ii** a head and a tail in any order

c What is the theoretical probability of tossing:

i 2 tails? **ii** a head and a tail in any order?

48 Find the equation of the tangent to the curve $y = x^3 - 7x + 3$ at the point where $x = 2$.

49 Find in exact form:

a the length of the arc

b the area of the sector

cut off by an angle of 40° at the centre of a circle with radius 4 cm.

50 If $f(x) = |x| - 2$ find:

a $f(-2)$ **b** $f(0)$ **c** $f(m + 1)$

51 The probability that Despina passes her first maths test is 64% and the probability that she will pass both the first and second tests is 48%. Find the probability that Despina passes the second test given that she passes the first test.

52 If $P(L) = 45\%$, $P(L \cap M) = 5.4\%$ and $P(M) = 12\%$, show that L and M are independent.

53 Given $P(X) = 0.26$, $P(Y) = 0.15$ and $P(X \cup Y) = 0.371$, show that X and Y are independent.

54 State whether events A and B are mutually exclusive if $P(A) = 0.18$, $P(A \cup B) = 0.5$ and $P(B) = 0.32$.

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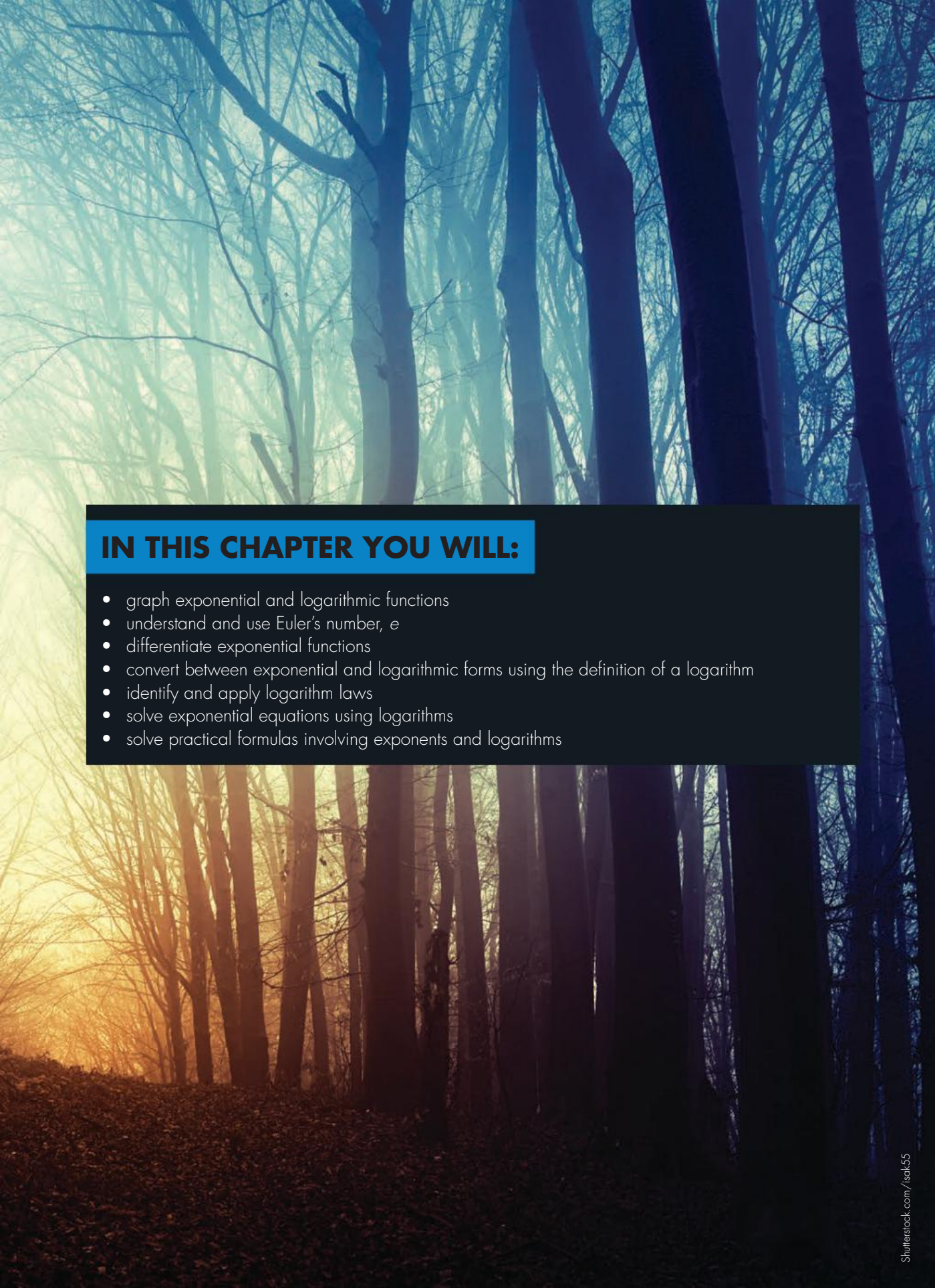
8.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this chapter you will study the definition and laws of logarithms and their relationship with the exponential and logarithmic functions. You will meet a new irrational number, e, that has special properties, solve exponential and logarithmic equations, and examine applications of exponential and logarithmic functions.

CHAPTER OUTLINE

8.01 Exponential functions
8.02 Euler's number, e
8.03 Differentiation of exponential functions
8.04 Logarithms
8.05 Logarithm laws
8.06 Logarithmic functions
8.07 Exponential equations



IN THIS CHAPTER YOU WILL:

- graph exponential and logarithmic functions
- understand and use Euler's number, e
- differentiate exponential functions
- convert between exponential and logarithmic forms using the definition of a logarithm
- identify and apply logarithm laws
- solve exponential equations using logarithms
- solve practical formulas involving exponents and logarithms

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TERMINOLOGY

Euler's number: This number, e, approximately 2.718 28, is an important constant that is the base of natural logarithms

exponential function: A function in the form $y = a^x$

logarithm: The logarithm of a positive number y is the power to which a given number a, called the base, must be raised in order to produce the number y, so $\log_a y = x$ means $y = a^x$

logarithmic function: A function in the form $y = \log_a x$

WS Graphing exponentials

WS Exponential functions

WS Translating exponential graphs

8.01 Exponential functions

An **exponential function** is in the form $y = a^x$, where $a > 0$.

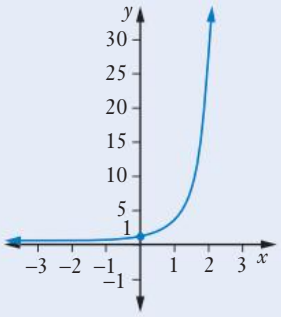
EXAMPLE 1

Sketch the graph of the function $y = 5^x$ and state its domain and range.

Solution

Complete a table of values for $y = 5^x$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125



Notice that a^x is always positive. So there is no x-intercept and $y > 0$.

For the y-intercept, when $x = 0$, $y = 5^0 = 1$.

The y-intercept is 1.

From the graph, the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

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INVESTIGATION

THE VALUE OF a IN $y = a^x$

Notice that the exponential function $y = a^x$ is only defined for $a > 0$.

1 Suppose $a = 0$. What would the function $y = 0^x$ look like? Try completing a table of values or use technology to sketch the graph. Is the function defined for positive values of x, negative values of x or when $x = 0$? What if x is a fraction?

2 Suppose $a < 0$. What would the function $y = (-2)^x$ look like?

3 For $y = 0^x$ and $y = (-2)^x$:

a is it possible to graph these functions at all?

b are there any discontinuities on the graphs?

c do they have a domain and range?

The exponential function $y = a^x$

- Domain $(-\infty, \infty)$, range $(0, \infty)$.
- The y-intercept ($x = 0$) is always 1 because $a^0 = 1$.
- The graph is always above the x-axis and there is no x-intercept ($y = 0$) because $a^x > 0$ for all values of x.
- The x-axis is an **asymptote**.

EXAMPLE 2

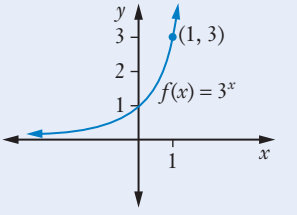
Sketch the graph of:

a $f(x) = 3^x$ **b** $y = 2^x + 1$

Solution

a The curve is above the x-axis with y-intercept 1. We must show another point on the curve.

$f(1) = 3^1 = 3$.



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b For y-intercept, $x = 0$:

$y = 2^0 + 1$

$= 1 + 1$

$= 2$

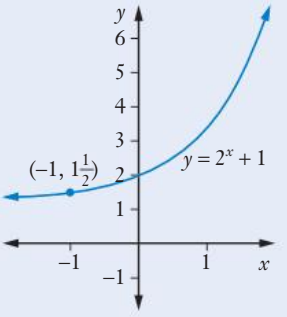
Find another point, when $x = -1$:

$y = 2^{-1} + 1$

$= \frac{1}{2} + 1$

$= 1\frac{1}{2}$

Notice that this is the graph of $y = 2^x$ moved up 1 unit.



EXAMPLE 3

Sketch the graph of:

a $f(x) = 3(4^x)$ **b** $y = 2^{x+1}$

Solution

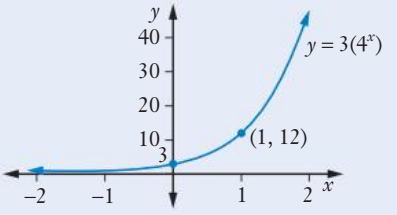
a The values of $f(x) = 3(4^x)$ will be 3 times greater than 4^x so its curve will be steeper.

$f(-1) = 3(4^{-1}) = 0.75$

$f(0) = 3(4^0) = 3$

$f(1) = 3(4^1) = 12$

$f(2) = 3(4^2) = 48$

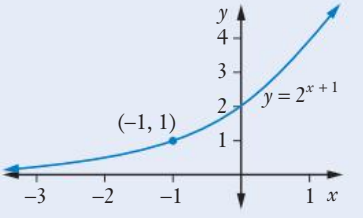


b $f(-1) = 2^{-1+1} = 1$

$f(0) = 2^{0+1} = 2$

$f(1) = 2^{1+1} = 4$

$f(2) = 2^{2+1} = 8$



This is the graph of $y = 2^x$ shifted 1 unit left.

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Reflections of exponential functions

We can reflect the graph of $y = a^x$ using what we learned in Chapter 5, *Further functions*.

EXAMPLE 4

Given $f(x) = 3^x$, sketch the graph of:

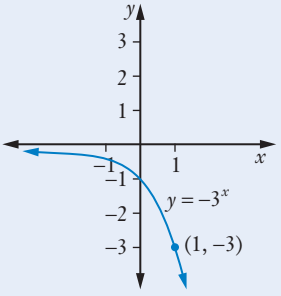
a $y = -3^x$ **b** $y = 3^{-x}$ **c** $y = -3^{-x}$

Solution

a Given $f(x) = 3^x$, then $y = -f(x) = -3^x$.

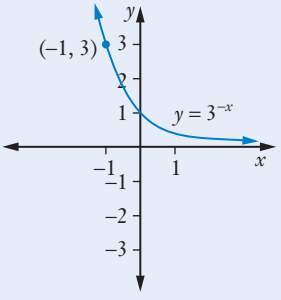
This is a reflection of $f(x)$ in the x-axis.

Note: -3^x means $-(3^x)$, *not* $(-3)^x$.



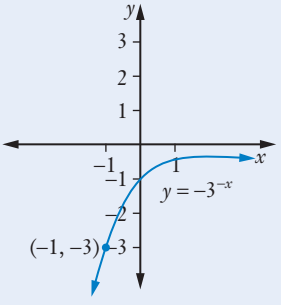
b Given $f(x) = 3^x$, then $y = f(-x) = 3^{-x}$.

This is a reflection of $f(x)$ in the y-axis.



c Given $f(x) = 3^x$, then $y = -f(-x) = -3^{-x}$.

This is a reflection of $f(x)$ in both the x- and y-axes.



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INVESTIGATION

GRAPHS OF EXPONENTIAL FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of the exponential functions below. Look for similarities and differences within each set.

a $y = 2^x, y = 2^x + 1, y = 2^x + 3, y = 2^x - 5$

b $y = 3(2^x), y = 4(2^x), y = -2^x, y = -3(2^x)$

c $y = 3(2^x) + 1, y = 4(2^x) + 3, y = -2^x + 1, y = -3(2^x) - 3$

d $y = 2^{x+1}, y = 2^{x+2}, y = 2^{x-1}, y = 2^{x-3}, y = 2^{-x}$

e $y = 2^{-x}, y = 2\left(2^{-x}\right), y = -2^{-x}, y = -3\left(2^{-x}\right), y = 2^{-x-1}$

Exercise 8.01 Exponential functions

1 Sketch each exponential function.

a $y = 2^x$ **b** $y = 4^x$ **c** $f(x) = 3^x + 2$ **d** $y = 2^x - 1$

e $f(x) = 3(2^x)$ **f** $y = 4^{x+1}$ **g** $y = 3(4^{2x}) - 1$ **h** $f(x) = -2^x$

i $y = 2(4^{-x})$ **j** $f(x) = -3(5^{-x}) + 4$

2 State the domain and range of each function.

a $f(x) = 2^x$ **b** $y = 3^x + 5$ **c** $f(x) = 10^{-x}$ **d** $f(x) = -5^x + 1$

3 Given $f(x) = 2^x$ and $g(x) = 3x - 4$, find:

a $f(g(x))$ **b** $g(f(x))$

4 **a** Sketch the graph of $f(x) = 4(3^x) + 1$.

b Sketch the graph of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5 Sales numbers, N, of a new solar battery are growing over t years according to the formula $N = 450(3^{0.9t})$.

a Draw a graph of this function.

b Find the initial number of sales when $t = 0$.

c Find the number of sales after:

i 3 years

ii 5 years

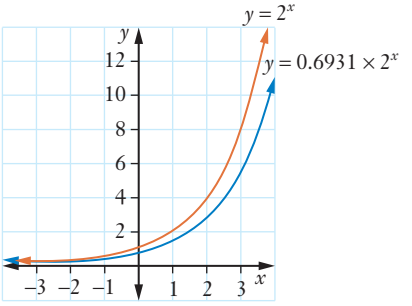
iii 10 years

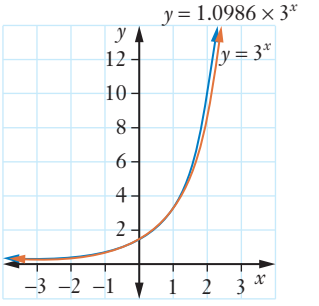
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8.02 Euler's number, e

The gradient function of exponential functions is interesting. Notice that the gradient of an exponential function is always increasing, and increases at an increasing rate.

If you sketch the derivative function of an exponential function, then it too is an exponential function! Here are the graphs of the derivative functions (in blue) of $y = 2^x$ and $y = 3^x$ (in red) together with their equations.





Notice that the graph of the derivative function of $y = 3^x$ is very close to the graph of the original function.

We can find a number close to 3 that gives exactly the same derivative function as the original graph. This number is approximately 2.718 28, and is called **Euler's number**, e. Like π, the number e is irrational.

Euler's number

$$e \approx 2.718\ 28$$

DID YOU KNOW?

Leonhard Euler

Like π, Euler's number, e, is a **transcendental** number, which is an irrational number that is not a surd. This was proven by a French mathematician, **Charles Hermite**, in 1873. The Swiss mathematician **Leonhard Euler** (1707–83) gave e its symbol, and he gave an approximation of e to 23 decimal places. Now e has been calculated to over a trillion decimal places.

Euler gave mathematics much of its important notation. He caused π to become standard notation for pi and used i for the square root of –1. He also introduced the symbol Σ for sums and $f(x)$ notation for functions.

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EXAMPLE 5

Sketch the graph of the exponential function $y = e^x$.

Solution

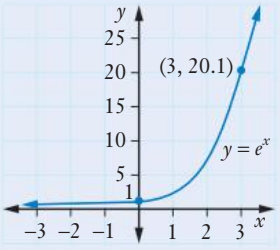
Use e^x on your calculator to draw up a table of values. For example, to calculate e^{-3}:

Casio scientific	Sharp scientific
SHIFT e^x (–) 3 =	2ndF e^x +/– 3 =

$e^{-3} = 0.049\ 78\ \ldots$

x	–3	–2	–1	0	1	2	3
y	0.05	0.1	0.4	1	2.7	7.4	20.1

(rounded figures)



EXAMPLE 6

The salmon population in a river over time can be described by the exponential function $P = 200e^{0.3t}$ where t is time in years.

a Find the population after 3 years.

b Draw the graph of the population.

Solution

a $P = 200e^{0.3t}$

When $t = 3$:

$P = 200e^{0.3 \times 3}$

$= 491.9206\ldots$

≈ 492

So after 3 years there are 492 salmon.

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b The graph is an exponential curve. Finding some points will help us graph it accurately.

When $t = 0$: $P = 200e^{0.3\times 0}$

$= 200$ This is also the P-intercept.

When $t = 1$: $P = 200e^{0.3\times 1}$

$= 269.9717\ldots$

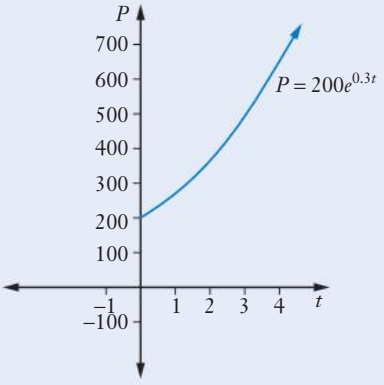
≈ 270

When $t = 2$: $P = 200e^{0.3\times 2}$

$= 364.4237\ldots$

≈ 364

We already know $P \approx 492$ when $t = 3$.



Time, $t \geq 0$, so don't sketch the curve for negative values of t.

Exercise 8.02 Euler's number, *e*

1 Sketch the curve $f(x) = 2e^{x-2}$.

2 Evaluate, correct to 2 decimal places:

a $e^{1.5}$ **b** e^{-2} **c** $2e^{0.3}$ **d** $\frac{1}{e^3}$ **e** $-3e^{-3.1}$

3 Sketch each exponential function.

a $y = 2e^x$ **b** $f(x) = e^x + 1$ **c** $y = -e^x$ **d** $y = e^{-x}$ **e** $y = -e^{-x}$

4 State the domain and range of $f(x) = e^x - 2$.

5 If $f(x) = e^x$ and $g(x) = x^3 + 3$, find:

a $f(g(x))$ **b** $g(f(x))$

6 The volume V of a metal in mm^3 expands as it is heated over time according to the formula $V = 25e^{0.7t}$, where t is in minutes.

a Sketch the graph of $V = 25e^{0.7t}$.

b Find the volume of the metal at:

i 3 minutes **ii** 8 minutes

c Is this formula a good model for the rise in volume? Why?

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7 The mass of a radioactive substance in g is given by $M = 150e^{-0.014t}$ where t is in years. Find the mass after:

a 10 years **b** 50 years **c** 250 years

8 The number of koalas in a forest is declining according to the formula $N = 873e^{-0.078t}$ where t is the time in years.

a Sketch a graph showing this decline in numbers of koalas for the first 6 years.

b Find the number of koalas:

i initially **ii** after 5 years **iii** after 10 years



Photo courtesy Margaret Grove

9 An object is cooling down according to the exponential function $T = 23 + 125e^{-0.06t}$ where T is the temperature in °C and t is time in minutes.

a Find the initial temperature.

b Find the temperature at:

i 2 minutes **ii** 5 minutes **iii** 10 minutes **iv** 2 hours

c What temperature is the object tending towards? Can you explain why?

10 A population is growing exponentially. If the initial population is 20 000 and after 5 years the population is 80 000, draw a graph showing this information.

11 The temperature of a piece of iron in a smelter is 1000°C and it is cooling down exponentially. After 10 minutes the temperature is 650°C. Draw a graph showing this information.

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8.03 Differentiation of exponential functions

WS
Differentiating exponential functions

Euler's number, e, is the special number such that the derivative function of $y = e^x$ is itself. The derivative of e^x is e^x.

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 7

a Differentiate $y = e^x - 5x^2$.

b Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

Solution

a $\frac{dy}{dx} = e^x - 10x$

b Gradient of the tangent:

$\frac{dy}{dx} = e^x$

At $(1, e)$:

$\frac{dy}{dx} = e^1$

$= e$

So $m = e$

Equation:

$y - y_1 = m(x - x_1)$

$y - e = e(x - 1)$

$= ex - e$

$y = ex$

The rule for differentiating $kf(x)$ works with the rule for e^x as well.

Derivative of ke^x

$$\frac{d}{dx}(ke^x) = ke^x$$

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EXAMPLE 8

a Differentiate $y = 5e^x$.

b Find the gradient of the normal to the curve $y = 3e^x$ at the point (0, 3).

Solution

a $\frac{dy}{dx} = 5e^x$

b Gradient of tangent:

$\frac{dy}{dx} = 3e^x$

At (0, 3):

$\frac{dy}{dx} = 3e^0$

$= 3$ since $e^0 = 1$

So $m_1 = 3$

For normal:

$m_1 m_2 = -1$

$3m_2 = -1$

$m_2 = -\frac{1}{3}$

So the gradient of the normal at (0, 3) is $-\frac{1}{3}$.

We can also use other differentiation rules such as the chain rule, product rule and quotient rule with the exponential function.

EXAMPLE 9

Differentiate:

a $y = e^{9x}$

b $y = e^{-5x}$

Solution

a Let $u = 9x$

Then $\frac{du}{dx} = 9$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times 9$

$= 9e^u$

$= 9e^{9x}$

b Let $u = -5x$

Then $\frac{du}{dx} = -5$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times (-5)$

$= -5e^u$

$= -5e^{-5x}$

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The derivative of e^{ax}

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

Proof

Let $u = ax$

Then $\frac{du}{dx} = a$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= e^u \times a$$

$$= ae^u$$

$$= ae^{ax}$$

EXAMPLE 10

Differentiate:

a $y = (1 + e^x)^3$

b $y = \frac{2x+3}{e^x}$

Solution

a $\frac{dy}{dx} = 3(1 + e^x)^2 \times e^x$

$= 3e^x(1 + e^x)^2$

b $\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$

$= \frac{2e^x - e^x(2x+3)}{(e^x)^2}$

$= \frac{2e^x - 2xe^x - 3e^x}{e^{2x}}$

$= \frac{-e^x - 2xe^x}{e^{2x}}$

$= \frac{-e^x(1+2x)}{e^{2x}}$

$= \frac{-(1+2x)}{e^x}$

Exercise 8.03 Differentiation of exponential functions

1 Differentiate:

a $y = 9e^x$ **b** $y = -e^x$ **c** $y = e^x + x^2$

d $y = 2x^3 - 3x^2 + 5x - e^x$ **e** $y = (e^x + 1)^3$ **f** $y = (e^x + 5)^7$

g $y = (2e^x - 3)^2$ **h** $y = xe^x$ **i** $y = \dfrac{e^x}{x}$

j $y = x^2e^x$ **k** $y = e^x(2x + 1)$ **l** $y = \dfrac{e^x}{7x - 3}$

m $y = \dfrac{5x}{e^x}$

2 Find the derivative of:

a $y = e^{2x}$ **b** $y = e^{-x}$ **c** $y = 2e^{3x}$

d $y = -e^{7x}$ **e** $y = -3e^{2x} + x^2$ **f** $y = e^{2x} - e^{-2x}$

g $y = 5e^{-x} - 3x + 2$ **h** $y = xe^{4x}$ **i** $y = \dfrac{2e^{3x} - 3}{x + 1}$

j $y = (9e^{3x} + 2)^5$

3 If $f(x) = x^3 + 3x - e^x$, find $f'(1)$ in terms of e.

4 Find the exact gradient of the tangent to the curve $y = e^x$ at the point $(1, e)$.

5 Find the exact gradient of the normal to the curve $y = e^{2x}$ at the point where $x = 5$.

6 Find the gradient of the tangent to the curve $y = 4e^x$ at the point where $x = 1.6$ correct to 2 decimal places.

7 Find the equation of the tangent to the curve $y = -e^x$ at the point $(1, -e)$.

8 Find the equation of the normal to the curve $y = e^{-x}$ at the point where $x = 3$ in exact form.

9 A population P of insects over time t weeks is given by $P = 3e^{1.4t} + 12\ 569$.

a What is the initial population?

b Find the rate of change in the number of insects after:

i 3 weeks **ii** 7 weeks

10 The displacement of a particle over time t seconds is given by $x = 2e^{4t}$ m.

a What is the initial displacement?

b What is the exact velocity after 10 s?

c Find the acceleration after 2 s correct to 1 decimal place.

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11 The displacement of an object in cm over time t seconds is given by $x = 6e^{-0.34t} - 5$. Find:

- **a** the initial displacement
- **b** the initial velocity
- **c** the displacement after 4 s
- **d** the velocity after 9 s
- **e** the acceleration after 2 s

12 The volume V of a balloon in mm^3 as it expands over time t seconds is given by $V = 3e^{0.8t}$.

- **a** Find the volume of the balloon at:
  - **i** 3 s **ii** 5 s
- **b** Find the rate at which the volume is increasing at:
  - **i** 3 s **ii** 5 s

13 The population of a city is changing over t years according to the formula $P = 34\,500e^{0.025t}$.

- **a** Find (to the nearest whole number) the population after:
  - **i** 5 years **ii** 10 years
- **b** Find the rate at which the population is changing after:
  - **i** 5 years **ii** 10 years

14 The depth of water (in metres) in a dam is decreasing over t months according to the formula $D = 3e^{-0.017t}$.

- **a** Find correct to 2 decimal places the depth after:
  - **i** 1 month **ii** 2 months **iii** 3 months
- **b** Find correct to 3 decimal places the rate at which the depth is changing after:
  - **i** 1 month **ii** 2 months **iii** 3 months

8.04 Logarithms

The **logarithm** of a positive number, y, is the **power** to which a **base**, a, must be raised in order to produce the number y. For example, $\log_2 8 = 3$ because $2^3 = 8$.



Logarithms


Logarithms

If $y = a^x$ then x is called the **logarithm of y to the base a**.

Just as the exponential function $y = a^x$ is defined for positive bases only ($a > 0$), logarithms are also defined for $a > 0$. Furthermore, $a \neq 1$ because $1^x = 1$ for all values of x.

Logarithms

$$\text{If } y = a^x \text{ then } x = \log_a y \qquad (a > 0, a \neq 1, y > 0)$$

Logarithms are related to exponential functions and allow us to solve equations like $2^x = 5$.

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EXAMPLE 11

a Write $\log_4 x = 3$ in index form and solve for x.

b Write $5^2 = 25$ in logarithm form.

c Solve $\log_x 36 = 2$.

d Evaluate $\log_3 81$.

e Find the value of $\log_2 \frac{1}{4}$.

Solution

a $\log_a y = x$ means $y = a^x$

$\log_4 x = 3$ means $x = 4^3$

So $x = 64$

b $y = a^x$ means $\log_a y = x$

So $25 = 5^2$ means $\log_5 25 = 2$

c $\log_x 36 = 2$ means $36 = x^2$

$x = \sqrt{36}$

$= 6$

Note: x is the base, so $x > 0$.

d $\log_3 81 = x$ means $81 = 3^x$

Solving $3^x = 81$:

$3^x = 3^4$

So $x = 4$

$\log_3 81 = 4$

e Let $\log_2 \frac{1}{4} = x$

Then $2^x = \frac{1}{4}$

$= \frac{1}{2^2}$

$= 2^{-2}$

$\therefore x = -2$

So $\log_2 \frac{1}{4} = -2$

EXAMPLE 12

Simplify:

a $\log_8 1$ **b** $\log_8 8$ **c** $\log_8 8^3$

d $\log_a a^x$ **e** $3^{\log_3 7}$ **f** $a^{\log_a x}$

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Solution

a $\log_8 1 = 0$ because $8^0 = 1$

b $\log_8 8 = 1$ because $8^1 = 8$

c $\log_8 8^3 = 3$ because $8^3 = 8^3$

d $\log_a a^x = x$ because $a^x = a^x$

e Let $\log_3 7 = y$

Then $3^y = 7$

So substituting for y:

$3^{\log_3 7} = 7$

f Let $\log_a x = y$

Then $a^y = x$

So substituting for y:

$a^{\log_a x} = x$

Notice that logarithms and exponentials are inverse operations.

Properties of logarithms

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

Common logarithms and natural logarithms

There are 2 types of logarithms that you can find on your calculator.

- **Common logarithms (base 10)**: $\log_{10} x$ or $\log x$
- **Natural (Naperian) logarithms (base *e*)**: $\log_e x$ or $\ln x$

EXAMPLE 13

a Find $\log_{10} 5.3$ correct to 1 decimal place.

b Evaluate $\log_e 80$ correct to 3 significant figures.

c Loudness in decibels is given by the formula $L = 10 \log_{10}\left(\frac{I}{I_0}\right)$ where I_0 is threshold sound, or sound that can barely be heard. Sound louder than 85 decibels can cause hearing damage.

i The loudness of a vacuum cleaner is 10 000 000 times the threshold level, or 10 000 000I_0. How many decibels is this?

ii If the loudness of the sound of rustling leaves is 20 dB, find its loudness in terms of I_0.

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Solution

a $\log_{10} 5.3 = 0.724\,2\ldots$ log 5.3 =

≈ 0.7

b $\log_e 80 = 4.382\,0\ldots$ ln 80 =

≈ 4.38

c **i** $L = 10 \log_{10}\left(\frac{10\,000\,000 I_0}{I_0}\right)$

$= 10 \log_{10} (10\,000\,000)$

$= 10 \times 7$

$= 70$

So the loudness of the vacuum cleaner is 70 dB.

ii $L = 10 \log_{10}\left(\frac{I}{I_0}\right)$

$20 = 10 \log_{10}\left(\frac{I}{I_0}\right)$

$2 = \log_{10}\left(\frac{I}{I_0}\right)$

Using the definition of a logarithm:

$10^2 = \frac{I}{I_0}$

$100 = \frac{I}{I_0}$

$100 I_0 = I$

So the loudness of rustling leaves is 100 times threshold sound.

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DID YOU KNOW?

The origins of logarithms

John Napier (1550–1617), a Scottish theologian and an amateur mathematician, was the first to invent logarithms. These 'natural', or 'Naperian', logarithms were based on e. Napier originally used the compound interest formula to find the value of e.

Napier was also one of the first mathematicians to use decimals rather than fractions. He invented decimal notation, using either a comma or a point. The point was used in England, but some European countries use a comma.

Henry Briggs (1561–1630), an Englishman who was a professor at Oxford, decided that logarithms would be more useful if they were based on 10 (our decimal system). Briggs painstakingly produced a table of common logarithms correct to 14 decimal places.

The work on logarithms was greatly appreciated by **Kepler**, **Galileo** and other astronomers at the time, since they allowed the computation of very large numbers.

Exercise 8.04 Logarithms

1 Evaluate:

a $\log_2 16$ **b** $\log_4 16$ **c** $\log_5 125$

d $\log_3 3$ **e** $\log_7 49$ **f** $\log_7 7$

g $\log_5 1$ **h** $\log_2 128$ **i** $\log_8 8$

2 Evaluate:

a $2^{\log_2 3}$ **b** $7^{\log_7 4}$ **c** $3^{\log_3 29}$

3 Evaluate:

a $3 \log_2 8$ **b** $\log_5 25 + 1$ **c** $3 - \log_3 81$

d $4 \log_3 27$ **e** $2 \log_{10} 10\,000$ **f** $1 + \log_4 64$

g $3 \log_4 64 + 5$ **h** $\dfrac{\log_3 9}{2}$ **i** $\dfrac{\log_8 64 + 4}{\log_2 8}$

4 Evaluate:

a $\log_2 \dfrac{1}{2}$ **b** $\log_3 \sqrt{3}$ **c** $\log_4 2$ **d** $\log_5 \dfrac{1}{25}$

e $\log_7 \sqrt[4]{7}$ **f** $\log_3 \dfrac{1}{\sqrt[3]{3}}$ **g** $\log_4 \dfrac{1}{2}$ **h** $\log_8 2$

i $\log_6 6\sqrt{6}$ **j** $\log_2 \dfrac{\sqrt{2}}{4}$

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5 Evaluate correct to 2 decimal places:

a $\log_{10} 1200$ **b** $\log_{10} 875$ **c** $\log_e 25$

d $\ln 140$ **e** $5 \ln 8$ **f** $\log_{10} 350 + 4.5$

g $\dfrac{\log_{10} 15}{2}$ **h** $\ln 9.8 + \log_{10} 17$ **i** $\dfrac{\log_{10} 30}{\log_e 30}$

6 Write in logarithmic form:

a $3^x = y$ **b** $5^x = z$ **c** $x^2 = y$ **d** $2^b = a$

e $b^3 = d$ **f** $y = 8^x$ **g** $y = 6^x$ **h** $y = e^x$

i $y = a^x$ **j** $Q = e^x$

7 Write in index form:

a $\log_3 5 = x$ **b** $\log_a 7 = x$ **c** $\log_3 a = b$

d $\log_x y = 9$ **e** $\log_a b = y$ **f** $y = \log_2 6$

g $y = \log_3 x$ **h** $y = \log_{10} 9$ **i** $y = \ln 4$

8 Solve for x, correct to 1 decimal place where necessary:

a $\log_{10} x = 6$ **b** $\log_3 x = 5$ **c** $\log_x 343 = 3$

d $\log_x 64 = 6$ **e** $\log_5 \dfrac{1}{5} = x$ **f** $\log_x \sqrt{3} = \dfrac{1}{2}$

g $\ln x = 3.8$ **h** $3 \log_{10} x - 2 = 10$ **i** $\log_4 x = \dfrac{3}{2}$

9 Evaluate y given that $\log_y 125 = 3$.

10 If $\log_{10} x = 1.65$, evaluate x correct to 1 decimal place.

11 Evaluate b to 3 significant figures if $\log_e b = 0.894$.

12 Find the value of $\log_2 1$. What is the value of $\log_a 1$?

13 Evaluate $\log_5 5$. What is the value of $\log_a a$?

14 **a** Evaluate $\ln e$ without a calculator.

b Using a calculator, evaluate:

i $\log_e e^3$ **ii** $\log_e e^2$ **iii** $\ln_e e^5$ **iv** $\log_e \sqrt{e}$

v $\ln_e \dfrac{1}{e}$ **vi** $e^{\ln 2}$ **vii** $e^{\ln 3}$ **viii** $e^{\ln 5}$

ix $e^{\ln 7}$ **x** $e^{\ln 1}$ **xi** $e^{\ln e}$

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15 A class was given musical facts to learn. The students were then tested on these facts and each week they were given similar tests to find out how much they were able to remember. The formula $A = 85 - 55 \log_{10}(t + 2)$ seemed to model the average score after t weeks.

a What was the initial average score?

b What was the average score after:

i 1 week? **ii** 3 weeks?

c After how many weeks was the average score 30?

16 The pH of a solution is defined as pH = $-\log [H^+]$ where $[H^+]$ is the hydrogen ion concentration. A solution is acidic if its pH is less than 7, alkaline if pH is greater than 7 and neutral if pH is 7. For each question find its pH and state whether it is acidic, alkaline or neutral.

a Fruit juice whose hydrogen ion concentration is 0.0035

b Water with $[H^+] = 10^{-7}$

c Baking soda with $[H^+] = 10^{-9}$

d Coca Cola whose hydrogen ion concentration is 0.01

e Bleach with $[H^+] = 1.2 \times 10^{-12}$

f Coffee with $[H^+] = 0.000\,01$

17 If $f(x) = \log x$ and $g(x) = 2x - 7$, find:

a $f(g(x))$ **b** $g(f(x))$

INVESTIGATION

HISTORY OF BASES AND NUMBER SYSTEMS

Common logarithms use base 10 like our decimal number system. We might have developed a different system if we had a different number of fingers! The Mayans, in ancient times, used base 20 for their number system since they counted with both their fingers and toes.

1 Research the history and types of other number systems, including those of Aboriginal and Torres Strait Islander peoples. Did any cultures use systems other than base 10? Why?

2 Explore computer-based systems. Computers have used both binary (base 2) and octal (base 8). Find out why these bases are used.

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Logarithm laws


Logarithm laws

8.05 Logarithm laws

Because logarithms are just another way of writing indices (powers), there are logarithm laws that correspond to the index laws.

$$\log_a (xy) = \log_a x + \log_a y$$

Proof

Let $x = a^m$ and $y = a^n$

Then $m = \log_a x$ and $n = \log_a y$

$$xy = a^m \times a^n$$
$$= a^{m+n}$$

$\therefore \log_a (xy) = m + n$ (by definition)

$$= \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Proof

Let $x = a^m$ and $y = a^n$

Then $m = \log_a x$ and $n = \log_a y$

$$\frac{x}{y} = a^m \div a^n$$
$$= a^{m-n}$$

$\therefore \log_a \left(\frac{x}{y}\right) = m - n$ (by definition)

$$= \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

Proof

Let $x = a^m$

Then $m = \log_a x$

$$x^n = (a^m)^n$$
$$= a^{mn}$$

$\therefore \log_a x^n = mn$ (by definition)

$$= n \log_a x$$

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$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$

Proof

$$\begin{aligned}\log_a\left(\frac{1}{x}\right) &= \log_a 1 - \log_a x \\ &= 0 - \log_a x \\ &= -\log_a x\end{aligned}$$



EXAMPLE 14

a Given $\log_5 3 = 0.68$ and $\log_5 4 = 0.86$, find:

i $\log_5 12$ **ii** $\log_5 0.75$ **iii** $\log_5 9$ **iv** $\log_5 20$

b Solve $\log_2 12 = \log_2 3 + \log_2 x$.

c Simplify $\log_a 21$ if $\log_a 3 = p$ and $\log_a 7 = q$.

d The formula for measuring R, the strength of an earthquake on the Richter scale, is $R = \log\left(\frac{I}{S}\right)$ where I is the maximum seismograph signal of the earthquake being measured and S is the signal of a standard earthquake.

Show that:

i $\log I = R + \log S$ **ii** $I = S(10^R)$

Solution

a i
$$\begin{aligned}\log_5 12 &= \log_5 (3 \times 4) \\ &= \log_5 3 + \log_5 4 \\ &= 0.68 + 0.86 \\ &= 1.54\end{aligned}$$

ii
$$\begin{aligned}\log_5 0.75 &= \log_5 \frac{3}{4} \\ &= \log_5 3 - \log_5 4 \\ &= 0.68 - 0.86 \\ &= -0.18\end{aligned}$$

iii
$$\begin{aligned}\log_5 9 &= \log_5 3^2 \\ &= 2\log_5 3 \\ &= 2 \times 0.68 \\ &= 1.36\end{aligned}$$

iv
$$\begin{aligned}\log_5 20 &= \log_5 (5 \times 4) \\ &= \log_5 5 + \log_5 4 \\ &= 1 + 0.86 \\ &= 1.86\end{aligned}$$

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b $\log_2 12 = \log_2 3 + \log_2 x$

$= \log_2 3x$

So $12 = 3x$

$4 = x$

c $\log_a 21 = \log_a (3 \times 7)$

$= \log_a 3 + \log_a 7$

$= p + q$

d **i** $R = \log\left(\frac{I}{S}\right)$

$= \log I - \log S$

$R + \log S = \log I$

ii $R = \log\left(\frac{I}{S}\right)$

$\frac{I}{S} = 10^R$

$I = S(10^R)$

Change of base

If we need to evaluate logarithms such as $\log_5 2$, we use the change of base formula.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof

Let $y = \log_a x$

Then $x = a^y$

Take logarithms to the base b of both sides of the equation:

$\log_b x = \log_b a^y$

$= y \log_b a$

$\therefore \quad \frac{\log_b x}{\log_b a} = y$

$= \log_a x$

To find the logarithm of any number, such as $\log_5 2$, you can change it to either $\log_{10} x$ or $\log_e x$.

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EXAMPLE 15

a Evaluate $\log_5 2$ correct to 2 decimal places.

b Find the value of $\log_2 3$ to 1 decimal place.

Solution

a $\log_5 2 = \dfrac{\log 2}{\log 5}$

≈ 0.43

b $\log_2 3 = \dfrac{\log 3}{\log 2}$

≈ 1.6

Exercise 8.05 Logarithm laws

1 Simplify:

a $\log_a 4 + \log_a y$ **b** $\log_a 4 + \log_a 5$

c $\log_a 12 - \log_a 3$ **d** $\log_a b - \log_a 5$

e $3 \log_x y + \log_x z$ **f** $2 \log_k 3 + 3 \log_k y$

g $5 \log_a x - 2 \log_a y$ **h** $\log_a x + \log_a y - \log_a z$

i $\log_{10} a + 4 \log_{10} b + 3 \log_{10} c$ **j** $3 \log_3 p + \log_3 q - 2 \log_3 r$

k $\log_4 \dfrac{1}{n}$ **l** $\log_x \dfrac{1}{6}$

2 Evaluate:

a $\log_5 5^2$ **b** $\log_7 7^6$

3 Given $\log_7 2 = 0.36$ and $\log_7 5 = 0.83$, find:

a $\log_7 10$ **b** $\log_7 0.4$ **c** $\log_7 20$

d $\log_7 25$ **e** $\log_7 8$ **f** $\log_7 14$

g $\log_7 50$ **h** $\log_7 35$ **i** $\log_7 98$

4 Use the logarithm laws to evaluate:

a $\log_5 50 - \log_5 2$ **b** $\log_2 16 + \log_2 4$

c $\log_4 2 + \log_4 8$ **d** $\log_5 500 - \log_5 4$

e $\log_9 117 - \log_9 13$ **f** $\log_8 32 + \log_8 16$

g $3 \log_2 2 + 2 \log_2 4$ **h** $2 \log_4 6 - (2 \log_4 3 + \log_4 2)$

i $\log_6 4 - 2 \log_6 12$ **j** $2 \log_3 6 + \log_3 18 - 3 \log_3 2$

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5 If $\log_a 3 = x$ and $\log_a 5 = y$, find an expression in terms of x and y for:

a $\log_a 15$ **b** $\log_a 0.6$ **c** $\log_a 27$

d $\log_a 25$ **e** $\log_a 9$ **f** $\log_a 75$

g $\log_a 3a$ **h** $\log_a \frac{a}{5}$ **i** $\log_a 9a$

6 If $\log_a x = p$ and $\log_a y = q$, find, in terms of p and q:

a $\log_a xy$ **b** $\log_a y^3$ **c** $\log_a \frac{y}{x}$ **d** $\log_a x^2$

e $\log_a xy^5$ **f** $\log_a \frac{x^2}{y}$ **g** $\log_a ax$ **h** $\log_a \frac{a}{y^2}$

i $\log_a a^3 y$ **j** $\log_a \frac{x}{ay}$

7 If $\log_a b = 3.4$ and $\log_a c = 4.7$, evaluate:

a $\log_a \frac{c}{b}$ **b** $\log_a bc^2$ **c** $\log_a (bc)^2$

d $\log_a abc$ **e** $\log_a a^2 c$ **f** $\log_a b^7$

g $\log_a \frac{a}{c}$ **h** $\log_a a^3$ **i** $\log_a bc^4$

8 Solve:

a $\log_4 12 = \log_4 x + \log_4 3$ **b** $\log_3 4 = \log_3 y - \log_3 7$

c $\log_a 6 = \log_a x - 3\log_a 2$ **d** $\log_2 81 = 4\log_2 x$

e $\log_x 54 = \log_x k + 2\log_x 3$

9 **a** Change the subject of $\text{dB} = 10\log\left(\frac{I}{I_0}\right)$ to I.

b Find the value of I in terms of I_0 when $\text{dB} = 45$.

10 **a** Show that the formula $A = 100 - 50\log(t+1)$ can be written as:

i $\log(t+1) = \frac{100-A}{50}$ **ii** $t = 10^{\frac{100-A}{50}} - 1$

b Hence find:

i A when $t = 3$ **ii** t when $A = 75$

11 Evaluate to 2 decimal places:

a $\log_4 9$ **b** $\log_6 25$ **c** $\log_9 200$ **d** $\log_2 12$

e $\log_3 23$ **f** $\log_8 250$ **g** $\log_5 9.5$ **h** $2\log_4 23.4$

i $7 - \log_7 108$ **j** $3\log_{11} 340$

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8.06 Logarithmic functions

A **logarithmic function** is a function of the form $y = \log_a x$.

Exponential and log graphs match-up

EXAMPLE 16

Sketch the graph of $y = \log_2 x$.

Solution

y-intercept ($x = 0$): No y-intercept because $x > 0$.

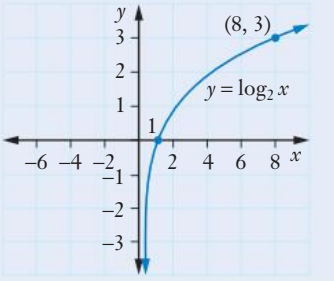
x-intercept ($y = 0$): $0 = \log_2 x$

$x = 2^0 = 1$, so x-intercept is 1 ($y = 0$).

Complete a table of values.

$y = \log_2 x$ means $x = 2^y$. For $x = 6$ in the table, use the change of base formula, $\log_2 x = \dfrac{\log x}{\log 2}$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	6	8
y	–2	–1	0	1	2	2.58	3



Logarithmic functions

- The logarithmic function $y = \log_a x$ is the inverse function of an exponential function $y = a^x$.
- Domain $(0, \infty)$, range $(-\infty, \infty)$.
- $x > 0$ so the curve is always to the right of the y-axis (no y-intercept).
- The y-axis is an **asymptote**.
- The x-intercept is always 1 because $\log_a 1 = 0$.

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EXAMPLE 17

Sketch the graph of:

a $y = \log_e x - 1$ **b** $y = 3 \log_{10} x + 4$

Solution

a No y-intercept ($x = 0$) because $\log_e 0$ is undefined. The y-axis is an asymptote.

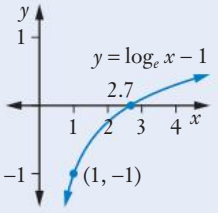
x-intercept ($y = 0$)

$0 = \log_e x - 1$

$1 = \log_e x$

$x = e^1$

≈ 2.7



Notice that this is the graph of $y = \log_e x$ moved down 1 unit.

Complete a table of values for this graph using the ln key on the calculator.

x	1	2	3	4
y	−1	−0.3	0.1	0.4

b Complete a table of values using the log key on the calculator.

$y = 3 \log_{10} x + 4$

x	1	2	3	4
y	4	4.9	5.4	5.8

No y-intercept.

For x-intercept, $y = 0$:

$0 = 3 \log_{10} x + 4$

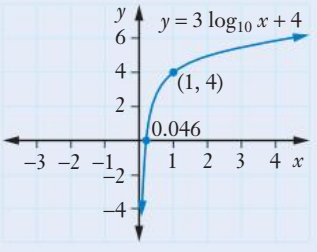
$-4 = 3 \log_{10} x$

$-\frac{4}{3} = \log_{10} x$

$10^{-\frac{4}{3}} = x$

$x = 0.04641\ldots$

≈ 0.046



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EXAMPLE 18

a Sketch the graphs of $y = e^x$, $y = \log_e x$ and $y = x$ on the same set of axes.

b What relationship do these graphs have?

c If $f(x) = \log_a x$, sketch the graph of $y = -f(x)$ and state its domain and range.

Solution

a Drawing $y = e^x$ gives an exponential curve with y-intercept 1.

Find another point, say $x = 2$:

$y = e^2$

$= 7.3890\ldots$

≈ 7.4

Drawing $y = \log_e x$ gives a logarithmic curve with x-intercept 1.

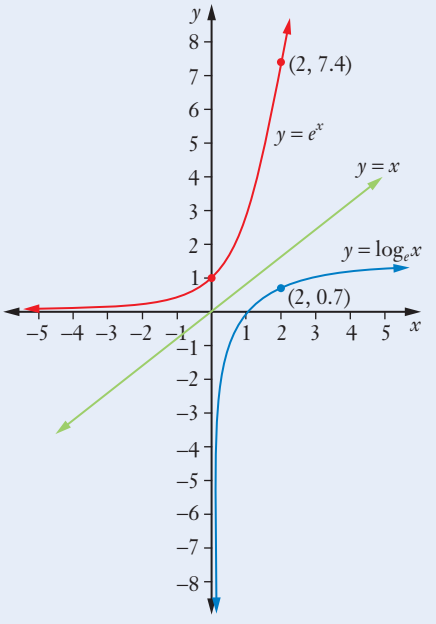
Find another point, say $x = 2$:

$y = \ln 2$

$= 0.6931\ldots$

≈ 0.7

$y = x$ is a linear function with gradient 1 and y-intercept 0.



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b The graphs of $y = e^x$ and $y = \log_e x$ are reflections of each other in the line $y = x$. They are **inverse functions**.

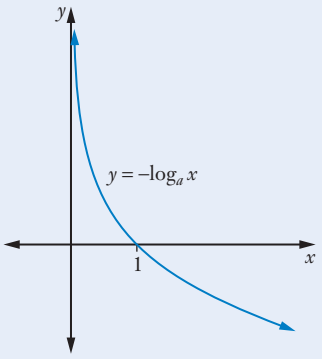
c Given $f(x) = \log_a x$,

$y = -f(x)$

$= -\log_a x$

This is a reflection of $f(x)$ in the x-axis.

Domain $(0, \infty)$, range $(-\infty, \infty)$



The exponential and logarithmic functions

$f(x) = a^x$ and $f(x) = \log_a x$ are inverse functions. Their graphs are reflections of each other in the line $y = x$.

INVESTIGATION

GRAPHS OF LOGARITHMIC FUNCTIONS

1 Substitute different values of x into the logarithmic function $y = \log x$: positive, negative and zero. What do you notice?

2 Use a graphics calculator or graphing software to sketch the graphs of different logarithmic functions such as

a $y = \log_2 x, y = \log_3 x, y = \log_4 x, y = \log_5 x, y = \log_6 x$

b $y = \log_2 x + 1, y = \log_2 x + 2, y = \log_2 x + 3, y = \log_2 x - 1, y = \log_2 x - 2$

c $y = 2 \log_2 x, y = 3 \log_2 x, y = -\log_2 x, y = -2 \log_2 x, y = -3 \log_2 x$

d $y = 2 \log_2 x + 1, y = 2 \log_2 x + 2, y = 2 \log_2 x + 3, y = 2 \log_2 x - 1, y = 2 \log_2 x - 2$

e $y = 3 \log_4 x + 1, y = 5 \log_3 x + 2, y = -\log_5 x + 3, y = -2 \log_2 x - 1, y = 4 \log_7 x - 2$

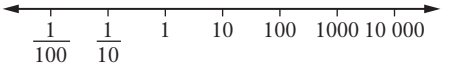
3 Try sketching the graph of $y = \log_{-2} x$. What does the table of values look like? Are there any discontinuities on the graph? Why? Could you find the domain and range? Use a graphics calculator or graphing software to sketch this graph. What do you find?

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Logarithmic scales

It is difficult to describe and graph exponential functions because their y values increase so quickly. We use logarithms and **logarithmic scales** to solve this problem.

On a base 10 logarithmic scale, an axis or number line has units that don't increase by 1, but by powers of 10.



Examples of base 10 logarithmic scales are:

- the Richter scale for measuring earthquake magnitude
- the pH scale for measuring acidity in chemistry
- the decibel scale for measuring loudness
- the octave (frequency) scale in music

EXAMPLE 19

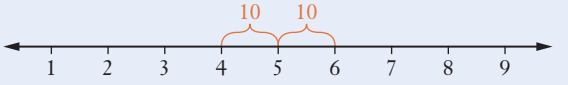
a Ged finds that the pH of soil is 4 in the eastern area of his garden and 6 in the western area. The pH formula is logarithmic and pH < 7 is acidic. What is the difference in acidity in these 2 areas of the garden?

b If Ged finds another area with a pH of 3.6, how much more acidic is this area than the eastern area?

Solution

a The difference in pH between 4 and 6 is 2. But this is a logarithmic scale.

Each interval on a logarithmic scale is a multiple of 10.



So the difference is $10 \times 10 = 10^2 = 100$.

The lower the pH, the more acidic. So the soil in the eastern area is 100 times more acidic than the soil in the western area.

b The difference in pH between 4 and 3.6 is 0.4.

So the difference is $10^{0.4} = 2.5118 \ldots \approx 2.5$.

The soil in this area is about 2.5 times more acidic than the soil in the eastern area.

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Exercise 8.06 Logarithmic functions

1 Sketch the graph of each logarithmic function and state its domain and range.

a $y = \log_3 x$ **b** $f(x) = 2 \log_4 x$ **c** $y = \log_2 x + 1$

d $y = \log_5 x - 1$ **e** $f(x) = \log_4 x - 2$ **f** $y = 5 \ln x + 3$

g $f(x) = -3 \log_{10} x + 2$

2 Sketch the graphs of $y = 10^x$, $y = \log_{10} x$ and $y = x$ on the same number plane. What do you notice about the relationship of the curves to the line?

3 Sketch the graph of $f(x) = \log_2 x$ and $y = \log_2 (-x)$ on the same set of axes and describe their relationship.

4 **a** Sketch the graphs of $y = \log_2 x$, $y = 2^x$ and $y = x$ on the same set of axes.

b Find the inverse function of $y = \log_2 x$.

5 This table lists some of the earthquakes experienced in Australia.

Year	Location	Strength on Richter scale
1989	Newcastle NSW	5.6
1997	Collier Bay WA	6.3
2001	Swan Hill Vic	4.8
2010	Kalgoorlie WA	5.2
2015	Coral Sea Qld	5.5
2017	Orange NSW	4.3
2018	Coffs Harbour NSW	4.2

The Richter scale for earthquakes is logarithmic. Use the table to find the difference in magnitude (correct to the nearest whole number) between the earthquakes in:

a Newcastle and Swan Hill

b Collier Bay and Orange

c Newcastle and Orange

d Coral Sea and Kalgoorlie

e Collier Bay and Coffs Harbour

6 The decibel (dB) scale for loudness is logarithmic. Find (correct to the nearest whole number) the difference in loudness between:

a 20 and 23 dB **b** 40 and 41 dB **c** 65.2 and 66.5 dB

d 85.4 and 88.9 dB **e** 52.3 and 58.6 dB

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8.07 Exponential equations

Exponential equations can be solved using logarithms or the change of base formula.

Exponential equations

Logarithmic and exponential equations

Solving exponential equations

Using exponential models

EXAMPLE 20

Solve $5^x = 7$ correct to 1 decimal place.

Solution

Method 1: Logarithms

Take logarithms of both sides:

$\log 5^x = \log 7$

$x \log 5 = \log 7$

$$x = \frac{\log 7}{\log 5}$$
$$= 1.2090\ldots$$
$$\approx 1.2$$

Method 2: Change of base formula

Convert to logarithm form:

$5^x = 7$ means $\log_5 7 = x$

Using the change of base to evaluate x:

$x = \log_5 7$

$$= \frac{\log 7}{\log 5}$$
$$= 1.2090\ldots$$
$$\approx 1.2$$

EXAMPLE 21

a Solve $e^{3.4x} = 100$ correct to 2 decimal places.

b The temperature T in °C of a metal as it cools down over t minutes is given by $T = 27 + 219e^{-0.032t}$. Find, correct to 1 decimal place, the time it takes to cool down to 100°C.

Exponential functions

Solution

a With an equation involving e we use $\ln x$, which is $\log_e x$.

Take natural logs of both sides:

$\ln e^{3.4x} = \ln 100$

$3.4x = \ln 100$ $\ln x$ and e^x are inverses

$$x = \frac{\ln 100}{3.4}$$
$$= 1.3544\ldots$$
$$\approx 1.35$$

b When $T = 100$:

$$100 = 27 + 219e^{-0.032t}$$

$$73 = 219e^{-0.032t}$$

$$\frac{73}{219} = e^{-0.032t}$$

$$\log_e\left(\frac{73}{219}\right) = \log_e(e^{-0.032t})$$

$$= -0.032t$$

$$t = \frac{\log_e\left(\frac{73}{219}\right)}{-0.032}$$

$$= 34.3316\ldots$$

$$= 34.3 \text{ to 1 d.p.}$$

So it takes 34.3 minutes to cool down to 100°C.

Exercise 8.07 Exponential equations

1 Solve each equation correct to 2 significant figures:

a $4^x = 9$ **b** $3^x = 5$ **c** $7^x = 14$ **d** $2^x = 15$

e $5^x = 34$ **f** $6^x = 60$ **g** $2^x = 76$ **h** $4^x = 50$

i $3^x = 23$ **j** $9^x = 210$

2 Solve, correct to 2 decimal places:

a $2^x = 6$ **b** $5^y = 15$ **c** $3^x = 20$ **d** $7^m = 32$

e $4^k = 50$ **f** $3^t = 4$ **g** $8^x = 11$ **h** $2^p = 57$

i $4^x = 81.3$ **j** $6^n = 102.6$

3 Solve, to 1 decimal place:

a $3^{x+1} = 8$ **b** $5^{3n} = 71$ **c** $2^{x-3} = 12$

d $4^{2n-1} = 7$ **e** $7^{5x+2} = 11$ **f** $8^{3-n} = 5.7$

g $2^{x+2} = 18.3$ **h** $3^{7k-3} = 32.9$ **i** $9^{\frac{x}{2}} = 50$

4 Solve each equation correct to 3 significant figures:

a $e^x = 200$ **b** $e^{3t} = 5$ **c** $2e^t = 75$

d $45 = e^x$ **e** $3000 = 100e^n$ **f** $100 = 20e^{3t}$

g $2000 = 50e^{0.15t}$ **h** $15\,000 = 2000e^{0.03k}$ **i** $3Q = Qe^{0.02t}$

5 The amount A of money in a bank account after n years grows with compound interest according to the formula $A = 850(1.025)^n$.

a Find:

i the initial amount in the bank **ii** the amount after 7 years.

b Find how many years it will take for the amount in the bank to be $1000.

6 The population of a city is given by $P = 35\,000e^{0.024t}$ where t is time in years.

a Find the population:

i initially **ii** after 10 years **iii** after 50 years.

b Find when the population will reach:

i 80 000 **ii** 200 000

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7 A species of wattle is gradually dying out in a Blue Mountains region. The number of wattle trees over time t years is given by $N = 8900e^{-0.048t}$.

a Find the number of wattle trees:

i initially

ii after 5 years

iii after 70 years.

b After how many years will there be:

i 5000 wattle trees?

ii 200 wattle trees?



Photo courtesy Margaret Grove

8 A formula for the mass M g of plutonium after t years is given by $M = 100e^{-0.000\,03t}$. Find:

a initial mass **b** mass after 50 years **c** mass after 500 years

d its half-life (the time taken to decay to half of its initial mass)

9 The temperature of an electronic sensor is given by the formula $T = 18 + 12e^{0.002t}$ where t is in hours.

a What is the temperature of the sensor after 5 hours?

b When the temperature reaches 50°C the sensor needs to be shut down to cool. After how many hours does this happen?

10 A particle is moving along a straight line with displacement x cm over time t s according to the formula $x = 5e^t + 23$.

a Find:

i the initial displacement

ii the exact velocity after 20 s

iii the displacement after 6 s

iv the time when displacement is 85 cm

v the time when the velocity is 1000 cm s^{-1}.

b Show that acceleration $a = x - 23$.

c Find the acceleration when displacement is 85 cm.

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8. TEST YOURSELF

Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 Simplify $\log_a 15 - \log_a 3$:

A $\log_a 45$ **B** $\dfrac{\log_a 15}{\log_a 3}$ **C** $\log_a 15 \times \log_a 3$ **D** $\log_a 5$

2 Write $a^x = y$ as a logarithm.

A $\log_y x = a$ **B** $\log_a y = x$ **C** $\log_a x = y$ **D** $\log_x a = y$

3 Solve $5^x = 4$ (there is more than one answer).

A $x = \dfrac{\log 4}{\log 5}$ **B** $x = \dfrac{\log 5}{\log 4}$ **C** $x = \dfrac{\ln 4}{\ln 5}$ **D** $x = \dfrac{\ln 5}{\ln 4}$

4 Evaluate:

a $\log_2 8$ **b** $\log_7 7$ **c** $\log_{10} 1000$ **d** $\log_9 81$

e $\log_e e$ **f** $\log_4 64$ **g** $\log_9 3$ **h** $\log_2 \dfrac{1}{2}$

i $\log_5 \dfrac{1}{25}$ **j** $\ln e^3$

5 Evaluate to 3 significant figures:

a $e^2 - 1$ **b** $\log_{10} 95$ **c** $\log_e 26$ **d** $\log_4 7$

e $\log_4 3$ **f** $\ln 50$ **g** $e + 3$ **h** $\dfrac{5e^3}{\ln 4}$

6 Evaluate:

a $e^{\ln 6}$ **b** $e^{\ln 2}$

7 Write in index form:

a $\log_3 a = x$ **b** $\ln b = y$ **c** $\log c = z$

8 If $\log_7 2 = 0.36$ and $\log_7 3 = 0.56$, find the value of:

a $\log_7 6$ **b** $\log_7 8$ **c** $\log_7 1.5$

d $\log_7 14$ **e** $\log_7 3.5$

9 Solve:

a $3^x = 8$ **b** $2^{3x-4} = 3$ **c** $\log_x 81 = 4$ **d** $\log_6 x = 2$

10 Solve $12 = 10e^{0.01t}$.

11 Evaluate $\log_9 8$ to 1 decimal place.

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12 Simplify:

a $5 \log_a x + 3 \log_a y$ **b** $2 \log_x k - \log_x 3 + \log_x p$

13 Evaluate to 2 significant figures:

a $\log_{10} 4.5$ **b** $\ln 3.7$

14 Sketch the graph of $y = 2^x + 1$ and state its domain and range.

15 Solve:

a $2^x = 9$ **b** $3^x = 7$ **c** $5^{x+1} = 6$ **d** $4^{2y} = 11$

e $8^{3n-2} = 5$ **f** $\log_x 16 = 4$ **g** $\log_3 y = 3$ **h** $\log_7 n = 2$

i $\log_x 64 = \frac{1}{2}$ **j** $\log_8 m = \frac{1}{3}$

16 Write as a logarithm:

a $2^x = y$ **b** $5^a = b$ **c** $10^x = y$

d $e^x = z$ **e** $3^{x+1} = y$

17 Sketch the graph of:

a $y = 5(3^{x+2})$ **b** $y = 2(3^x) - 5$ **c** $f(x) = -3^x$ **d** $y = 3(2^{-x})$

18 Sketch the graph of:

a $f(x) = \log_3 x$ **b** $y = 3 \ln x - 4$

19 If $\log_x 2 = a$ and $\log_x 3 = b$ find in terms of a and b:

a $\log_x 6$ **b** $\log_x 1.5$ **c** $\log_x 8$

d $\log_x 18$ **e** $\log_x 27$

20 The formula for loudness is $L = 10 \log\left(\frac{I}{I_0}\right)$ where I_0 is threshold sound and L is measured in decibels (dB). Find:

a the dB level of a $5500I_0$ sound

b the sound in terms of I_0 if its dB level is 32.

21 Simplify:

a $\log_a \frac{1}{x}$ **b** $\log_e \frac{1}{y}$

22 Evaluate:

a $\log_6 12 + \log_6 3$ **b** $\log 25 + \log 4$ **c** $2 \log_4 8$

d $\log_8 72 - \log_8 9$ **e** $\log 53\,000 - \log 53$

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23 Solve correct to 1 decimal place:

a $e^x = 15$ **b** $2.7^x = 21.8$ **c** $10^x = 128.7$

24 The amount of money in the bank after n years is given by $A = 5280(1.019)^n$.

a Find the amount in the bank:

i initially **ii** after 3 years **iii** after 4 years.

b Find how long it will take for the amount of money in the bank to reach:

i \$6000 **ii** \$10 000

25 Differentiate each function.

a $y = e^{3x}$ **b** $y = e^{-2x}$ **c** $y = 5e^{4x}$

d $y = -2e^{8x} + 5x^3 - 1$ **e** $y = x^2e^{2x}$ **f** $y = (4e^{3x} - 1)^9$

g $y = \dfrac{x}{e^{2x}}$

26 The formula for the number of wombats in a region of New South Wales after t years is $N = 1118 - 37e^{0.032t}$.

a Find the initial number of wombats in this region.

b How many wombats are there after 5 years?

c How long will it take until the number of wombats in the region is:

i 500? **ii** 100?

27 Differentiate:

a $y = e^x + x$ **b** $y = -4e^x$ **c** $y = 3e^{-x}$

d $y = (3 + e^x)^9$ **e** $y = 3x^5e^x$ **f** $y = \dfrac{e^x}{7x - 2}$

28 An earthquake has magnitude 6.7 and its aftershock has magnitude 4.7 on the base 10 logarithmic Richter scale. How much larger is the first earthquake?

29 Shampoo A has pH 7.2 and shampoo B has pH 8.5. The pH scale is base 10 logarithmic. How much more alkaline is shampoo B?

30 If $f(x) = \log_e x$, $g(x) = e^x$ and $h(x) = 6x^2 - 1$, find:

a $f(h(x))$ **b** $g(h(x))$ **c** $h(g(x))$

d $f(g(x))$ **e** $g(f(x))$

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8. CHALLENGE EXERCISE

1 If $\log_b 2 = 0.6$ and $\log_b 3 = 1.1$, find:

a $\log_b 6b$ **b** $\log_b 8b$ **c** $\log_b 1.5b^2$

2 Find the point of intersection of the curves $y = \log_e x$ and $y = \log_{10} x$.

3 Sketch the graph of $y = \log_2 (x - 1)$ and state its domain and range.

4 By substituting $u = 3^x$, solve $3^{2x} - 3^x - 2 = 0$ correct to 2 decimal places.

5 The pH of a solution is given by $\text{pH} = -\log [\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration.

a Show that pH could be given by $\text{pH} = \log \dfrac{1}{[\text{H}^+]}$.

b Show that $[\text{H}^+] = \dfrac{1}{10^{\text{pH}}}$.

c Find the hydrogen ion concentration, to 1 significant figure, of a substance with a pH of:

i 6.3

ii 7.7

6 If $y = 8 + \log_2 (x + 2)$:

a show that $x = 2(2^{y-9} - 1)$

b find, correct to 2 decimal places:

i y when $x = 5$

ii x when $y = 1$

7 Find the equation of **a** the tangent and **b** the normal to the curve $y = 3e^x - 5$ at the point $(2, 3e^2 - 5)$.

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9.

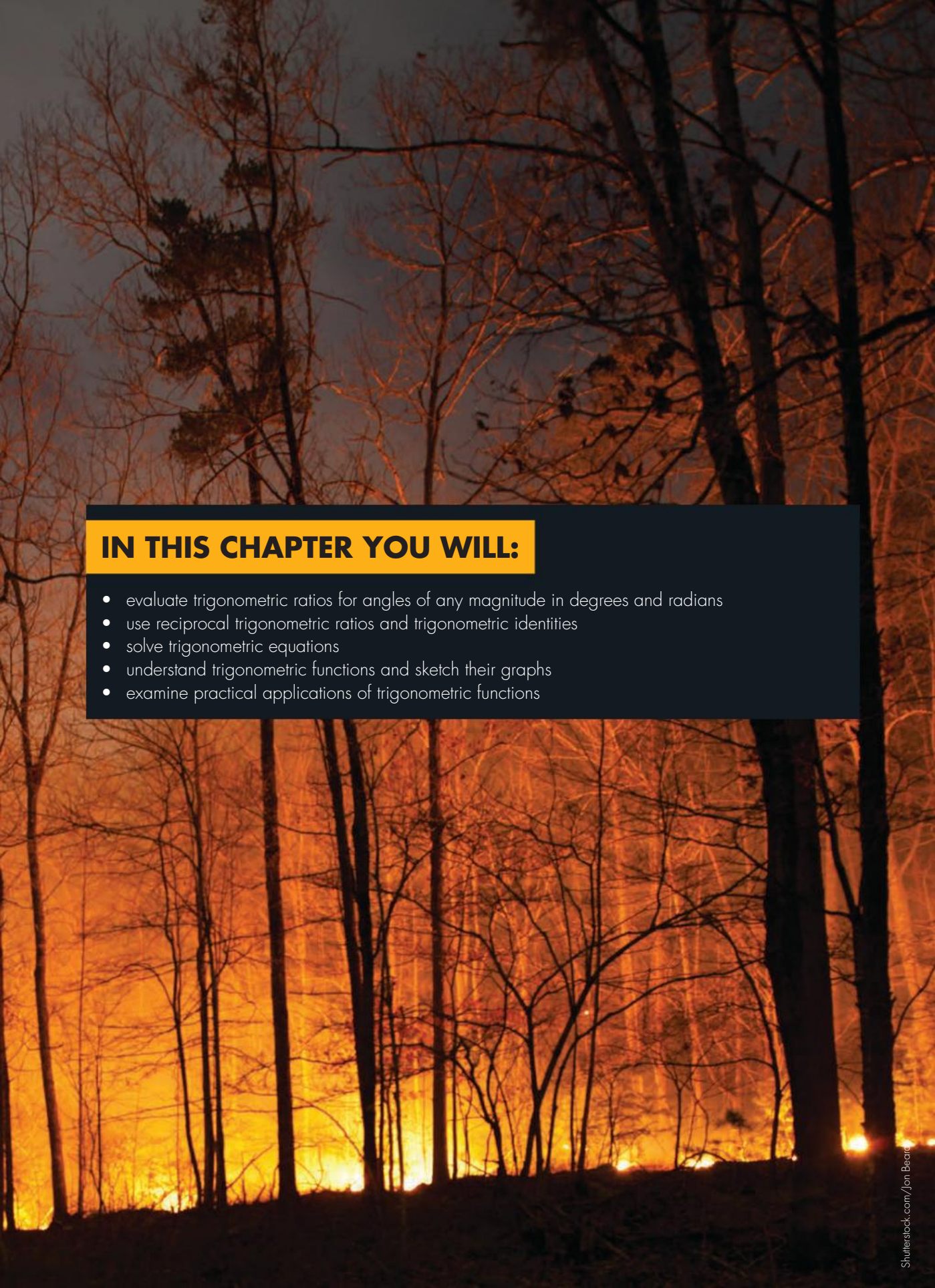
TRIGONOMETRIC FUNCTIONS

In this chapter, we will learn about trigonometric functions and their graphs, trigonometric identities and solving trigonometric equations.

Some physical changes such as tides, annual temperatures and phases of the Moon are described as cyclic or periodic because they repeat regularly. Trigonometric functions are also periodic and we can use them to model real-life situations.

CHAPTER OUTLINE

9.01 Angles of any magnitude
9.02 Trigonometric identities
9.03 Radians
9.04 Trigonometric functions
9.05 Trigonometric equations
9.06 Applications of trigonometric functions



IN THIS CHAPTER YOU WILL:

- evaluate trigonometric ratios for angles of any magnitude in degrees and radians
- use reciprocal trigonometric ratios and trigonometric identities
- solve trigonometric equations
- understand trigonometric functions and sketch their graphs
- examine practical applications of trigonometric functions

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TERMINOLOGY

amplitude: The height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively) For $y = k \sin ax$ the amplitude is k

centre: The mean value of a periodic function that is equidistant from the maximum and minimum values. For $y = k \sin ax + c$ the centre is c

identity: An equation that shows the equivalence of 2 algebraic expressions for all values of the variables

period: The length of one cycle of a periodic function on the x-axis, before the function repeats itself. For $y = k \sin ax$ the period is $\frac{2\pi}{a}$

periodic function: A function that repeats itself regularly

phase: A horizontal shift (translation). For $y = k \sin [a(x + b)]$, the phase is b, that is, the graph of $y = k \sin ax$ shifted b units to the left

reciprocal trigonometric ratios: The cosecant, secant and cotangent ratios, which are the reciprocals of sine, cosine and tangent respectively

WS
Angles of any magnitude

9.01 Angles of any magnitude

In Chapter 4, *Trigonometry*, we examined acute and obtuse obtuse angles by looking at angles turning around a unit circle. We can find angles of *any* size by continuing around the circle.

1st quadrant: acute angles (between 0° and 90°)

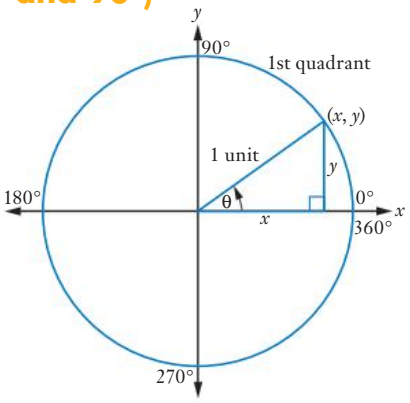
You can see from the triangle in the unit circle with angle θ that:

$$\sin \theta = y$$

$$\cos \theta = x$$

$$\tan \theta = \frac{y}{x}$$

In the 1st quadrant, x and y are both positive so all ratios are positive in the 1st quadrant.



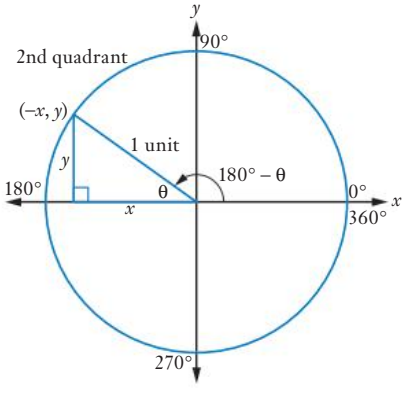
2nd quadrant: obtuse angles (between 90° and 180°)

$\sin \theta = y$ (positive)

$\cos \theta = -x$ (negative)

$\tan \theta = \frac{y}{-x}$ (negative)

The angle that gives θ in the triangle is $180° - \theta$.



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2nd quadrant

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\tan(180° - \theta) = -\tan\theta$$

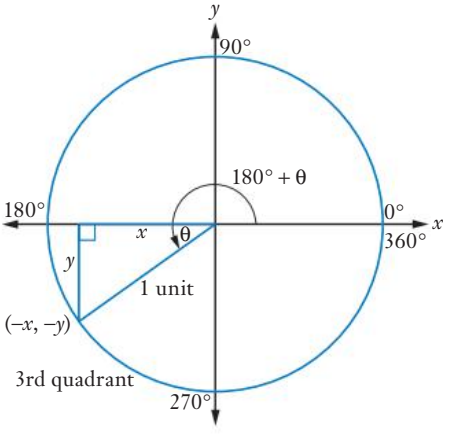
3rd quadrant: angles between 180° and 270°

$\sin\theta = -y$ (negative)

$\cos\theta = -x$ (negative)

$\tan\theta = \dfrac{-y}{-x} = \dfrac{y}{x}$ (positive)

The angle that gives θ in the triangle is $180° + \theta$.



3rd quadrant

$$\sin(180° + \theta) = -\sin\theta$$

$$\cos(180° + \theta) = -\cos\theta$$

$$\tan(180° + \theta) = \tan\theta$$

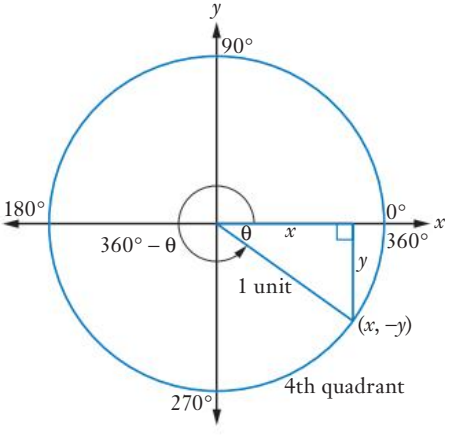
4th quadrant: angles between 270° and 360°

$\sin\theta = -y$ (negative)

$\cos\theta = x$ (positive)

$\tan\theta = \dfrac{-y}{x}$ (negative)

The angle that gives θ in the triangle is $360° - \theta$.



4th quadrant

$$\sin(360° - \theta) = -\sin\theta$$

$$\cos(360° - \theta) = \cos\theta$$

$$\tan(360° - \theta) = -\tan\theta$$

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Putting all of these results together gives a rule for all 4 quadrants that we usually call the **ASTC rule.**

ASTC rule

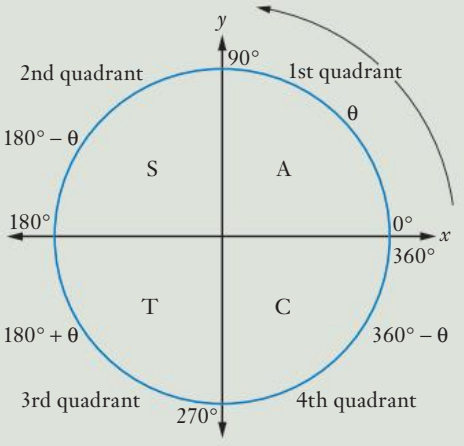
A: All ratios are positive in the 1st quadrant.

S: Sin is positive in the 2nd quadrant (cos and tan are negative).

T: Tan is positive in the 3rd quadrant (sin and cos are negative).

C: Cos is positive in the 4th quadrant (sin and tan are negative).

ASTC can be remembered using the phrase '**A**ll **S**tations **T**o **C**entral'.



EXAMPLE 1

a Find all quadrants where:

i $\sin\theta > 0$ **ii** $\cos\theta < 0$ **iii** $\tan\theta < 0$ and $\cos\theta > 0$

b Find the exact value of:

i $\tan 330°$ **ii** $\sin 225°$

c Simplify $\cos(180° + x)$.

d If $\sin x = -\frac{3}{5}$ and $\cos x > 0$, find the value of $\tan x$.

Solution

a **i** Using the ASTC rule, $\sin\theta > 0$ in the 1st and 2nd quadrants.

ii $\cos\theta > 0$ in the 1st and 4th quadrants, so $\cos\theta < 0$ in the 2nd and 3rd quadrants.

iii $\tan\theta > 0$ in the 1st and 3rd quadrants so $\tan\theta < 0$ in the 2nd and 4th quadrants. Also $\cos\theta > 0$ in the 1st and 4th quadrants.

So $\tan\theta < 0$ and $\cos\theta > 0$ in the 4th quadrant.

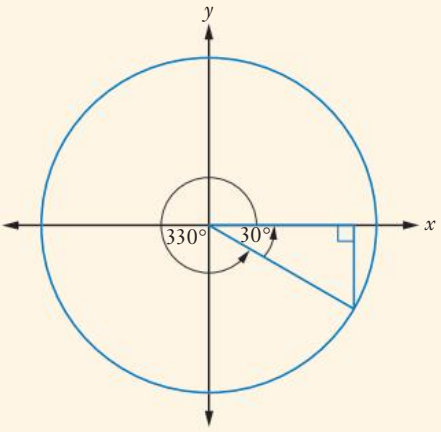
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b **i** 330° lies in the 4th quadrant.

The angle inside the triangle in the 4th quadrant is $360° - 330° = 30°$ and tan is negative in the 4th quadrant.

$\tan 330° = -\tan 30°$

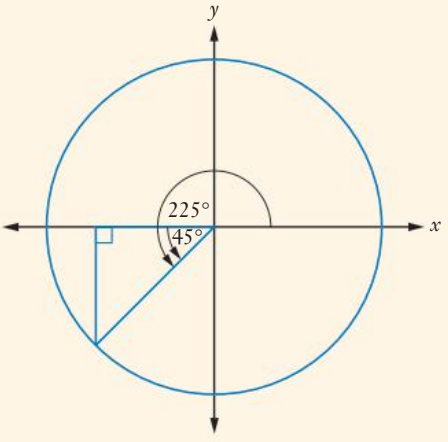
$= -\frac{1}{\sqrt{3}}$



ii 225° is in the 3rd quadrant.
The angle in the triangle in the 3rd quadrant is $225° - 180° = 45°$ and sin is negative in the 3rd quadrant.

$\sin 225° = -\sin 45°$

$= -\frac{1}{\sqrt{2}}$



c $180° + x$ is in the 3rd quadrant where $\cos x$ is negative.

So $\cos(180° + x) = -\cos x$.

d $\sin x < 0$ and $\cos x > 0$ so x is in the 4th quadrant.

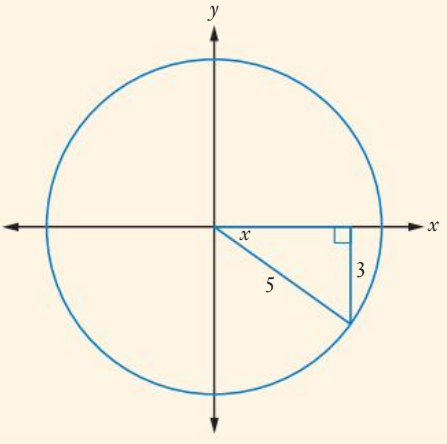
$\sin x = -\frac{3}{5}$

So the opposite side is 3 and the hypotenuse is 5.

By Pythagoras' theorem, the adjacent side is 4. (3, 4, 5 triangle).

$\tan x < 0$ in the 4th quadrant.

So $\tan x = -\frac{3}{4}$.

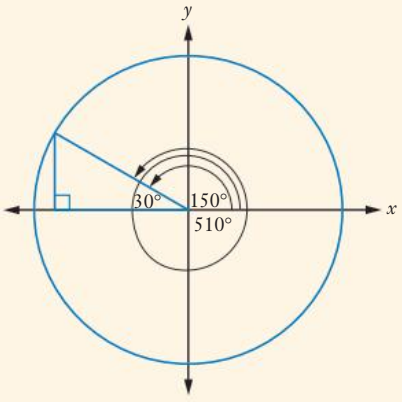


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We can find trigonometric ratios of angles greater than 360° by turning around the circle more than once.

EXAMPLE 2

Find the exact value of cos 510°.



Solution

To find cos 510°, we move around the circle more than once.

$\cos(510° - 360°) = \cos(150°)$

The angle is in the 2nd quadrant where cos is negative. The angle inside the triangle is $180° - 150° = 30°$.

So $\cos 510° = \cos 150°$

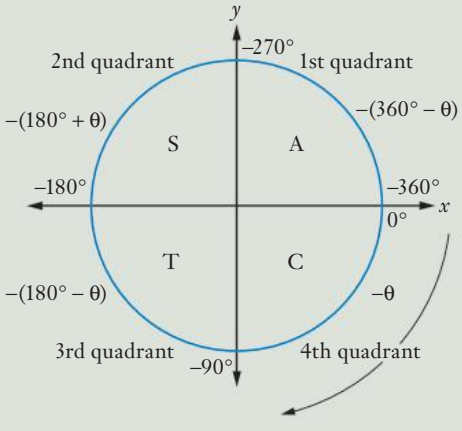
$= -\cos 30°$

$= -\frac{\sqrt{3}}{2}$.

Negative angles

The ASTC rule also works for negative angles. These are measured in the opposite direction (clockwise) from positive angles as shown.

Negative angles



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In the 4th quadrant:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

In the 3rd quadrant:

$\sin(-(180° - \theta)) = -\sin\theta$

$\cos(-(180° - \theta)) = -\cos\theta$

$\tan(-(180° - \theta)) = \tan\theta$

In the 2nd quadrant:

$\sin(-(180° + \theta)) = \sin\theta$

$\cos(-(180° + \theta)) = -\cos\theta$

$\tan(-(180° + \theta)) = -\tan\theta$

In the 1st quadrant:

$\sin(-(360° - \theta)) = \sin\theta$

$\cos(-(360° - \theta)) = \cos\theta$

$\tan(-(360° - \theta)) = \tan\theta$

EXAMPLE 3

Find the exact value of $\tan(-120°)$.

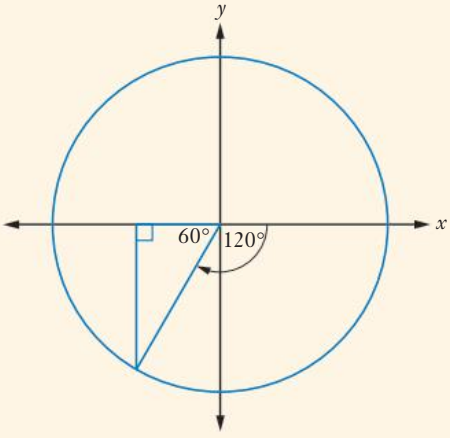
Solution

Moving clockwise around the circle, the angle is in the 3rd quadrant, with $180° - 120° = 60°$ in the triangle.

tan is positive in the 3rd quadrant.

$\tan(-120°) = \tan 60°$

$= \sqrt{3}$



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Exercise 9.01 Angles of any magnitude

1 Find all quadrants where:

a	$\cos\theta > 0$	**b**	$\tan\theta > 0$	**c**	$\sin\theta > 0$
d	$\tan\theta < 0$	**e**	$\sin\theta < 0$	**f**	$\cos\theta < 0$
g	$\sin\theta < 0$ and $\tan\theta > 0$	**h**	$\cos\theta < 0$ and $\tan\theta < 0$		
i	$\cos\theta > 0$ and $\tan\theta < 0$	**j**	$\sin\theta < 0$ and $\tan\theta < 0$		

2 **a** Which quadrant is the angle 240° in?

b Find the exact value of cos 240°.

3 **a** Which quadrant is the angle 315° in?

b Find the exact value of sin 315°.

4 **a** Which quadrant is the angle 120° in?

b Find the exact value of tan 120°.

5 **a** Which quadrant is the angle –225° in?

b Find the exact value of sin (–225°).

6 **a** Which quadrant is the angle –330° in?

b Find the exact value of cos (–330°).

7 Find the exact value of:

a	tan 225°	**b**	cos 315°	**c**	tan 300°	**d**	sin 150°
e	cos 120°	**f**	sin 210°	**g**	cos 330°	**h**	tan 150°
i	sin 300°	**j**	cos 135°				

8 Find the exact value of:

a	cos (–225°)	**b**	cos (–210°)	**c**	tan (–300°)	**d**	cos (–150°)
e	sin (–60°)	**f**	tan (–240°)	**g**	cos (–300°)	**h**	tan (–30°)
i	cos (–45°)	**j**	sin (–135°)				

9 Find the exact value of:

a	cos 570°	**b**	tan 420°	**c**	sin 480°	**d**	cos 660°
e	sin 690°	**f**	tan 600°	**g**	sin 495°	**h**	cos 405°
i	tan 675°	**j**	sin 390°				

10 If $\tan\theta = \frac{3}{4}$ and $\cos\theta < 0$, find $\sin\theta$ and $\cos\theta$ as fractions.

11 Given $\sin\theta = \frac{4}{7}$ and $\tan\theta < 0$, find the exact value of $\cos\theta$ and $\tan\theta$.

12 If $\sin x < 0$ and $\tan x = -\frac{5}{8}$, find the exact value of $\cos x$.

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13 Given $\cos x = \frac{2}{5}$ and $\tan x < 0$, find the exact value of $\sin x$ and $\tan x$.

14 If $\cos x < 0$ and $\sin x > 0$, find $\cos x$ and $\sin x$ in surd form if $\tan x = \frac{5}{7}$.

15 If $\sin \theta = -\frac{4}{9}$ and $270° < \theta < 360°$, find the exact value of $\tan \theta$ and $\cos \theta$.

16 If $\cos x = -\frac{3}{8}$ and $180° < x < 270°$, find the exact value of $\tan x$ and $\sin x$.

17 Given $\sin x = 0.3$ and $\tan x < 0$:

a express $\sin x$ as a fraction

b find the exact value of $\cos x$ and $\tan x$.

18 If $\tan \alpha = -1.2$ and $270° < \alpha < 360°$, find the exact values of $\cos \alpha$ and $\sin \alpha$.

19 Given that $\cos \theta = -0.7$ and $90° < \theta < 180°$, find the exact value of $\sin \theta$ and $\tan \theta$.

20 Simplify:

a $\sin(180° - \theta)$ **b** $\cos(360° - x)$ **c** $\tan(180° + \beta)$

d $\sin(180° + \alpha)$ **e** $\tan(360° - \theta)$ **f** $\sin(-\theta)$

g $\cos(-\alpha)$ **h** $\tan(-x)$

9.02 Trigonometric identities



Trigonometric identities


Simplifying trigonometric functions

The reciprocal trigonometric ratios

The **reciprocal trigonometric ratios** are the reciprocals of the sine, cosine and tangent ratios.

The reciprocal trigonometric ratios

Cosecant $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$

cosec θ can also be written as *csc* θ.

Secant $\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$

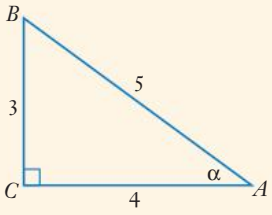
Cotangent $\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}}$

The reciprocal ratios have the same signs as their related ratios in the different quadrants. For example, in the 3rd and 4th quadrants, $\sin\theta < 0$, so $\operatorname{cosec}\theta < 0$.

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EXAMPLE 4

a Find cosec α, sec α and cot α for this triangle.



b If $\sin\theta = -\frac{2}{7}$ and $\tan > 0$, find the exact ratios of cot θ, sec θ and cosec θ.

c State the quadrants where cosec θ is negative.

Solution

a $\operatorname{cosec}\alpha = \dfrac{1}{\sin\alpha}$

$= \dfrac{\text{hypotenuse}}{\text{opposite}}$

$= \dfrac{5}{3}$

$\sec\alpha = \dfrac{1}{\cos\alpha}$

$= \dfrac{\text{hypotenuse}}{\text{adjacent}}$

$= \dfrac{5}{4}$

$\cot\alpha = \dfrac{1}{\tan\alpha}$

$= \dfrac{\text{adjacent}}{\text{opposite}}$

$= \dfrac{4}{3}$

b $\sin\theta < 0$ and $\tan\theta > 0$ in the 3rd quadrant. So $\cos\theta < 0$.

By Pythagoras' theorem:

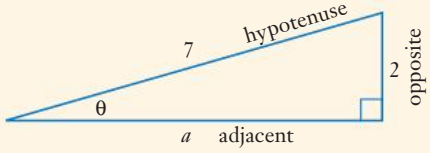
$7^2 = a^2 + 2^2$

$a^2 + 4 = 49$

$a^2 = 45$

$a = \sqrt{45}$

$= 3\sqrt{5}$



$\cot\theta = \dfrac{1}{\tan\theta}$

$= \dfrac{\text{adjacent}}{\text{opposite}}$

$= \dfrac{3\sqrt{5}}{2}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$= \dfrac{\text{hypotenuse}}{\text{adjacent}}$

$= -\dfrac{7}{3\sqrt{5}}$

$= -\dfrac{7\sqrt{5}}{15}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$= \dfrac{\text{hypotenuse}}{\text{opposite}}$

$= -\dfrac{7}{2}$

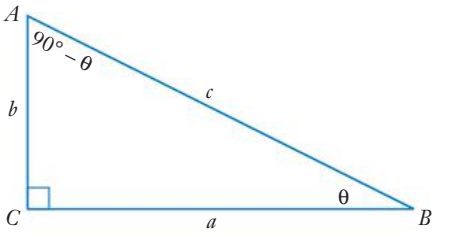
c $\sin\theta < 0$ in the 3rd and 4th quadrants.

So cosec $\theta < 0$ in the 3rd and 4th quadrants.

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Complementary angles

In $\triangle ABC$ if $\angle B = \theta$ then $\angle A = 90° - \theta$ (by the angle sum of a triangle). $\angle B$ and $\angle A$ are **complementary angles** because they add up to 90°.



$\sin\theta = \frac{b}{c}$ $\quad$ $\sin(90° - \theta) = \frac{a}{c}$

$\cos\theta = \frac{a}{c}$ $\quad$ $\cos(90° - \theta) = \frac{b}{c}$

$\tan\theta = \frac{b}{a}$ $\quad$ $\tan(90° - \theta) = \frac{a}{b}$

$\sec\theta = \frac{c}{a}$ $\quad$ $\sec(90° - \theta) = \frac{c}{b}$

$\operatorname{cosec}\theta = \frac{c}{b}$ $\quad$ $\operatorname{cosec}(90° - \theta) = \frac{c}{a}$

$\cot\theta = \frac{a}{b}$ $\quad$ $\cot(90° - \theta) = \frac{b}{a}$

Notice the pairs of trigonometric ratios that are equal:

Complementary angle results

$\sin\theta = \cos(90° - \theta)$ $\quad$ $\tan\theta = \cot(90° - \theta)$ $\quad$ $\sec\theta = \operatorname{cosec}(90° - \theta)$

$\cos\theta = \sin(90° - \theta)$ $\quad$ $\cot\theta = \tan(90° - \theta)$ $\quad$ $\operatorname{cosec}\theta = \sec(90° - \theta)$

EXAMPLE 5

a Simplify $\tan 50° - \cot 40°$.

b Find the value of m if $\sec 55° = \operatorname{cosec}(2m - 15)°$.

Solution

a $\tan 50° - \cot 40° = \tan 50° - \cot(90° - 50°)$

$= \tan 50° - \tan 50°$

$= 0$

b $\sec 55° = \operatorname{cosec}(90° - 55°)$

$= \operatorname{cosec} 35°$

So $2m - 15 = 35$

$2m = 50$

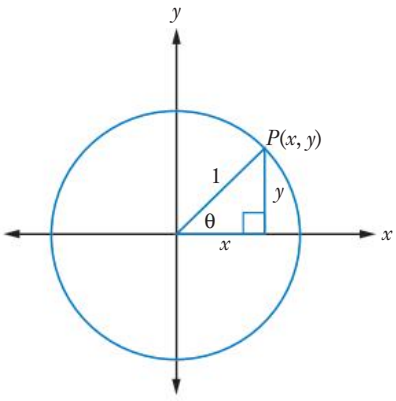
$m = 25$

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The tangent identity

In the work on angles of any magnitude, we saw that $\sin\theta = y$, $\cos\theta = x$ and $\tan\theta = \frac{y}{x}$.

From this we get the following trigonometric identities:



The tangent identity

For any value of θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

An **identity** is an equation that shows the equivalence of 2 algebraic expressions for all values of the variables, for example, $a^2 - b^2 = (a + b)(a - b)$ is an identity.

EXAMPLE 6

Simplify $\sin\theta \cot\theta$.

Solution

$$\sin\theta \cot\theta = \sin\theta \times \frac{\cos\theta}{\sin\theta}$$
$$= \cos\theta$$

The Pythagorean identities

The unit circle above has equation $x^2 + y^2 = 1$, because of Pythagoras' theorem.

But $\sin\theta = y$ and $\cos\theta = x$, so

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

A shorter way of writing this is:

$\cos^2\theta + \sin^2\theta = 1$

This formula is called a Pythagorean identity because it is based on Pythagoras' theorem in the unit circle.

There are 2 other identities that can be derived from this identity.

Dividing each term by $\cos^2\theta$:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$1 + \tan^2\theta = \sec^2\theta$$

Dividing each term by $\sin^2\theta$:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

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Pythagorean identities

For any value of θ:

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$\cos^2\theta + \sin^2\theta = 1$ can also be rearranged to give:

$$\cos^2\theta = 1 - \sin^2\theta \text{ or}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

EXAMPLE 7

Prove that:

a $\cot x + \tan x = \operatorname{cosec} x \sec x$

b $\dfrac{1-\cos x}{\sin^2 x} = \dfrac{1}{1+\cos x}$

Solution

a LHS $= \cot x + \tan x$

$$= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$$

$$= \frac{1}{\sin x \cos x}$$

$$= \frac{1}{\sin x} \times \frac{1}{\cos x}$$

$$= \operatorname{cosec} x \sec x$$

$$= \text{RHS}$$

$\therefore \cot x + \tan x = \operatorname{cosec} x \sec x$

b LHS $= \dfrac{1-\cos x}{\sin^2 x}$

$$= \frac{1-\cos x}{1-\cos^2 x}$$

$$= \frac{1-\cos x}{(1+\cos x)(1-\cos x)}$$

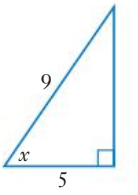
$$= \frac{1}{1+\cos x}$$

$$= \text{RHS}$$

$\therefore \dfrac{1-\cos x}{\sin^2 x} = \dfrac{1}{1+\cos x}$

Exercise 9.02 Trigonometric identities

1 For this triangle, find the exact ratios of sec x, cot x and cosec x.



2 If $\sin\theta = \dfrac{5}{13}$, find cosec θ, sec θ and cot θ.

3 If $\cos\theta = \dfrac{4}{7}$, find exact values of cosec θ, sec θ and cot θ.

4 If $\sec\theta = -\dfrac{6}{5}$ and $\sin\theta > 0$, find exact values of tan θ, cosec θ and cot θ.

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5 If $\cot\theta = 0.6$ and $\operatorname{cosec}\theta < 0$, find the exact values of $\sin\theta$, $\operatorname{cosec}\theta$, $\tan\theta$ and $\sec\theta$.

6 Show $\sin 67° = \cos 23°$.

7 Show $\sec 82° = \operatorname{cosec} 8°$.

8 Show $\tan 48° = \cot 42°$.

9 Simplify:

a $\cos 61° + \sin 29°$

b $\sec\theta - \operatorname{cosec}(90° - \theta)$

c $\tan 70° + \cot 20° - 2\tan 70°$

d $\dfrac{\sin 55°}{\cos 35°}$

e $\dfrac{\cot 25° + \tan 65°}{\cot 25°}$

10 Find the value of x if $\sin 80° = \cos(90 - x)°$

11 Find the value of y if $\tan 22° = \cot(90 - y)°$

12 Find the value of p if $\cos 49° = \sin(p + 10)°$

13 Find the value of b if $\sin 35° = \cos(b + 30)°$

14 Find the value of t if $\cot(2t + 5)° = \tan(3t - 15)°$

15 Find the value of k if $\tan(15 - k)° = \cot(2k + 60)°$

16 Simplify:

a $\tan\theta\cos\theta$

b $\tan\theta\operatorname{cosec}\theta$

c $\sec x\cot x$

d $1 - \sin^2 x$

e $\sqrt{1-\cos^2\alpha}$

f $\cot^2 x + 1$

g $1 + \tan^2 x$

h $\sec^2\theta - 1$

i $5\cot^2\theta + 5$

j $\dfrac{1}{\operatorname{cosec}^2 x}$

k $\sin^2\alpha\operatorname{cosec}^2\alpha$

l $\cot\theta - \cot\theta\cos^2\theta$

17 Prove that:

a $\cos^2 x - 1 = -\sin^2 x$

b $\sec\theta + \tan\theta = \dfrac{1+\sin\theta}{\cos\theta}$

c $3 + 3\tan^2\alpha = \dfrac{3}{1-\sin^2\alpha}$

d $\sec^2 x - \tan^2 x = \operatorname{cosec}^2 x - \cot^2 x$

e $(\sin x - \cos x)^3 = \sin x - \cos x - 2\sin^2 x\cos x + 2\sin x\cos^2 x$

f $\cot\theta + 2\sec\theta = \dfrac{1-\sin^2\theta+2\sin\theta}{\sin\theta\cos\theta}$

g $\cos^2(90° - \theta)\cot\theta = \sin\theta\cos\theta$

h $(\operatorname{cosec} x + \cot x)(\operatorname{cosec} x - \cot x) = 1$

i $\dfrac{1-\sin^2\theta\cos^2\theta}{\cos^2\theta} = \tan^2\theta + \cos^2\theta$

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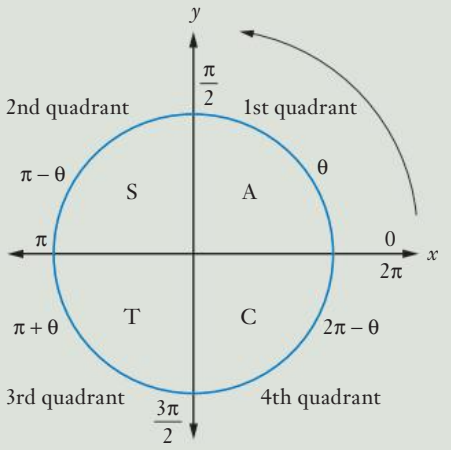
9.03 Radians

The rules and formulas learned in this chapter can also be expressed in radians, which we learned about in Chapter 4, *Trigonometry*.

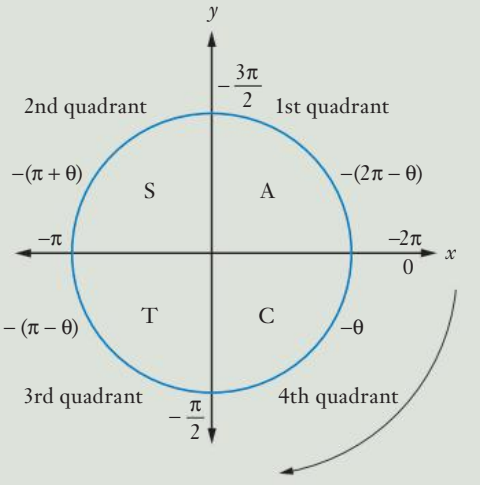
WS
Radians of any magnitude

ASTC rule

Positive angles



Negative angles



In the 2nd quadrant:

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

In the 3rd quadrant:

$\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta$

In the 4th quadrant:

$\sin(2\pi - \theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos\theta$

$\tan(2\pi - \theta) = -\tan\theta$

In the 4th quadrant:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

In the 3rd quadrant:

$\sin(-(\pi - \theta)) = -\sin\theta$

$\cos(-(\pi - \theta)) = -\cos\theta$

$\tan(-(\pi - \theta)) = \tan\theta$

In the 2nd quadrant:

$\sin(-(\pi + \theta)) = \sin\theta$

$\cos(-(\pi + \theta)) = -\cos\theta$

$\tan(-(\pi + \theta)) = -\tan\theta$

In the 1st quadrant:

$\sin(-(2\pi - \theta)) = \sin\theta$

$\cos(-(2\pi - \theta)) = \cos\theta$

$\tan(-(2\pi - \theta)) = \tan\theta$

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EXAMPLE 8

Find the exact value of:

a $\sin\frac{5\pi}{4}$ **b** $\cos\frac{11\pi}{6}$

Solution

a $\frac{5\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}$

$= \pi + \frac{\pi}{4}$

in the 3rd quadrant, so $\sin\theta < 0$.

$\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right)$

$= -\sin\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}}$

b $\frac{11\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6}$

$= 2\pi - \frac{\pi}{6}$

in the 4th quadrant, so $\cos\theta > 0$.

$\cos\left(\frac{11\pi}{6}\right) = \cos\left(2\pi - \frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$

Exercise 9.03 Radians

1 Find the exact value of each expression.

a $\operatorname{cosec}\frac{\pi}{4}$ **b** $\sec\frac{\pi}{6}$ **c** $\cot\frac{\pi}{3}$

d $\dfrac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}$ **e** $1 - \cos^2\frac{\pi}{4}$ **f** $\tan\frac{\pi}{3}\cos\frac{\pi}{3}$

g $\sqrt{1+\tan^2\frac{\pi}{4}}$ **h** $\operatorname{cosec}^2\frac{\pi}{6} - 1$ **i** $\dfrac{\cot\frac{\pi}{5} + \tan\frac{3\pi}{10}}{\cot\frac{\pi}{5}}$

2 **a** Show that $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$.

b In which quadrant is the angle $\frac{3\pi}{4}$?

c Find the exact value of $\cos\frac{3\pi}{4}$.

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3 **a** Show that $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$.

b In which quadrant is the angle $\frac{5\pi}{6}$?

c Find the exact value of $\sin\frac{5\pi}{6}$.

4 **a** Show that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$.

b In which quadrant is the angle $\frac{7\pi}{4}$?

c Find the exact value of $\tan\frac{7\pi}{4}$.

5 **a** Show that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$.

b In which quadrant is the angle $\frac{4\pi}{3}$?

c Find the exact value of $\cos\frac{4\pi}{3}$.

6 **a** Show that $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$.

b In which quadrant is the angle $\frac{5\pi}{3}$?

c Find the exact value of $\sin\frac{5\pi}{3}$.

7 **a** **i** Show that $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$.

ii In which quadrant is the angle $\frac{13\pi}{6}$?

iii Find the exact value of $\cos\frac{13\pi}{6}$.

b Find the exact value of:

i $\sin\frac{9\pi}{4}$ **ii** $\tan\frac{7\pi}{3}$ **iii** $\cos\frac{11\pi}{4}$

iv $\tan\frac{19\pi}{6}$ **v** $\sin\frac{10\pi}{3}$

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8 Copy and complete each table with exact values.

a

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	$\frac{10\pi}{3}$	$\frac{11\pi}{3}$
sin								
cos								
tan								

b

	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$	$\frac{13\pi}{4}$	$\frac{15\pi}{4}$
sin								
cos								
tan								

c

	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	$\frac{17\pi}{6}$	$\frac{19\pi}{6}$	$\frac{23\pi}{6}$
sin								
cos								
tan								

9 Copy and complete the table where possible.

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin									
cos									
tan									

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9.04 Trigonometric functions



Sine and cosine curves



Trigonometric graphs


Trigonometric graphs match-up



Sketching periodic functions: amplitude and period



Sketching periodic functions: phase and vertical shift


Amplitude and period

INVESTIGATION

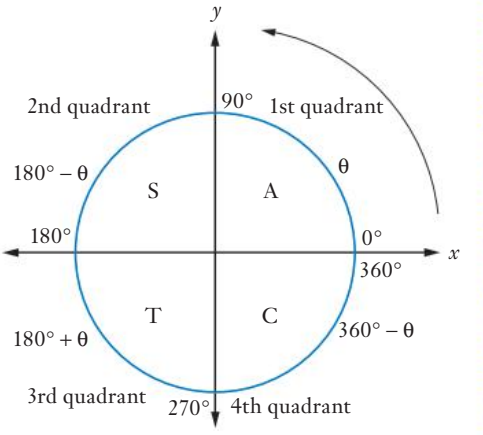
TRIGONOMETRIC RATIOS OF 0°, 90°, 180°, 270° AND 360°

Remember the results from the unit circle:

$\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \frac{y}{x}$



1 Angle 0° is at the point (1, 0) on the unit circle. Use the circle results to find sin 0°, cos 0° and tan 0°.

2 Angle 90° is at the point (0, 1). Use the circle results to find sin 90°, cos 90° and tan 90°. Discuss the result for tan 90° and why this happens.

3 Angle 180° is at the point (–1, 0). Find sin 180°, cos 180° and tan 180°.

4 Angle 270° is at the point (0, –1). Find sin 270°, cos 270° and tan 270°. Discuss the result for tan 270° and why this happens.

5 What are the results for sin 360°, cos 360° and tan 360°? Why?

6 Check these results on your calculator.

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The sine function

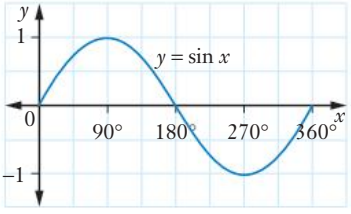
Using all the results from the investigation, we can draw up a table of values for $y = \sin x$.

x	0°	90°	180°	270°	360°
y	0	1	0	−1	0

We could add in all the exact value results we know for a more accurate graph. Remember that sin x is positive in the 1st and 2nd quadrants and negative in the 3rd and 4th quadrants.

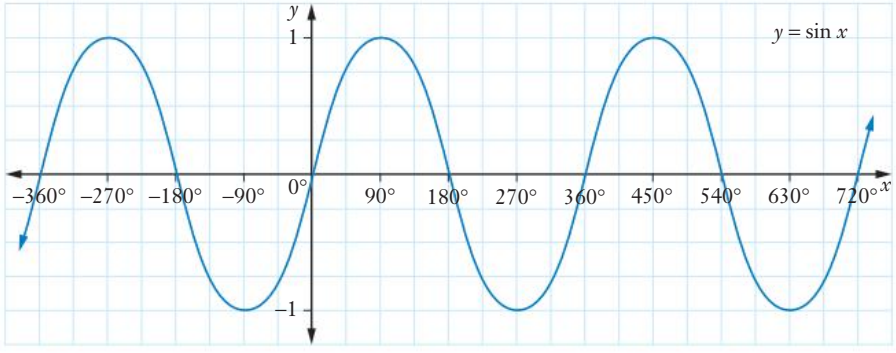
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0

Drawing the graph gives a smooth 'wave' curve.



As we go around the unit circle and graph the y values of the points on the circle, the graph should repeat itself every 360°.

$y = \sin x$ has domain $(-\infty, \infty)$ and range $[-1, 1]$. It is an odd function.

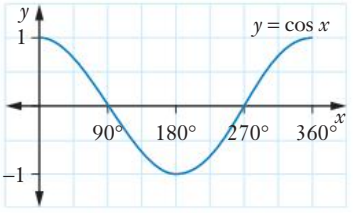


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The cosine function

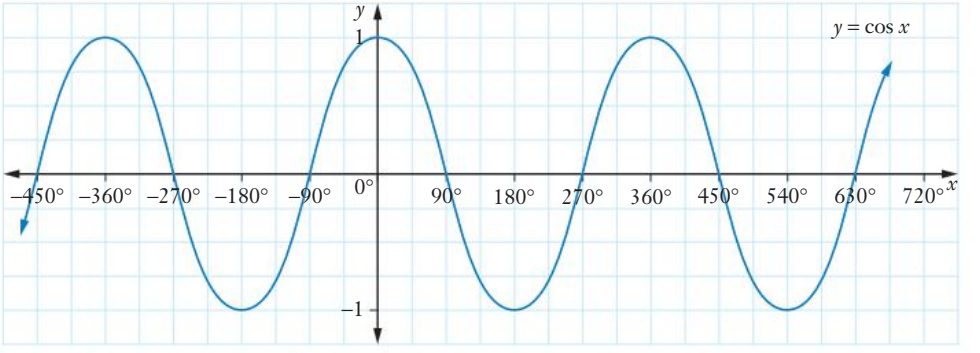
Similarly for $y = \cos x$, which is positive in the 1st and 4th quadrants and negative in the 2nd and 3rd quadrants. Its graph has the same shape as the graph of the sine function.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0



As we go around the unit circle and graph the x values of the points on the circle, the graph should repeat itself every 360°.

$y = \cos x$ has domain $(-\infty, \infty)$ and range $[-1, 1]$. It is an even function.



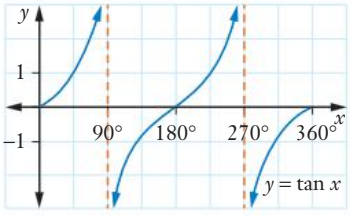
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The tangent function

$y = \tan x$ is positive in the 1st and 3rd quadrants and negative in the 2nd and 4th quadrants.

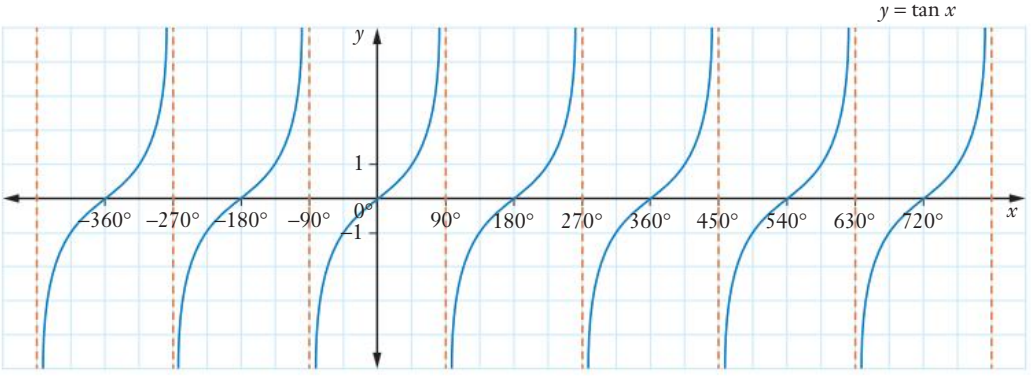
It is also undefined for 90° and 270° so there are vertical asymptotes at those x values, where the function is discontinuous.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0



As we go around the unit circle and graph the values of $\frac{y}{x}$ of the points on the circle, the graph repeats itself every 180°.

$y = \tan x$ has domain $(-\infty, \infty)$ except for 90°, 270°, 540°, … (odd multiples of 90°) and range $(-\infty, \infty)$. It is an odd function.



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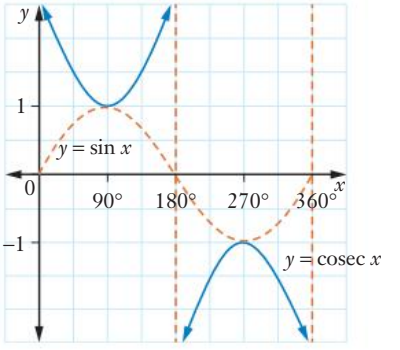
The cosecant function

$$\text{cosec } x = \frac{1}{\sin x}$$

Each y value of $y = \text{cosec } x$ will be the reciprocal of $y = \sin x$. Because $\sin x = 0$ at $x = 0°$, 180°, 360°, …, $y = \text{cosec } x$ will have vertical asymptotes at those values.

We can use a table of values and explore the limits as x approaches any asymptotes.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–	–2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	–1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	–2	–

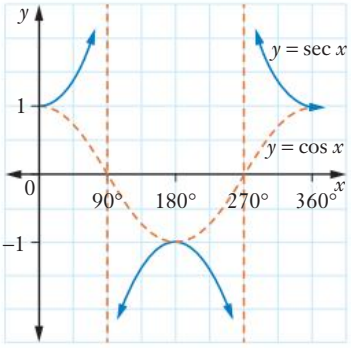


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The secant function

$\sec x = \dfrac{1}{\cos x}$, so each y value of $y = \sec x$ will be the reciprocal of $y = \cos x$. Because $\cos x = 0$ at $x = 90°, 270°, 450°, \ldots$, $y = \sec x$ will have vertical asymptotes at those values.

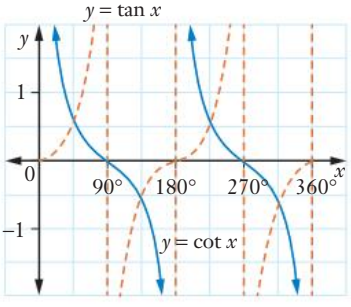
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–	–2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	–1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	–2	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1



The cotangent function

$\cot x = \dfrac{1}{\tan x}$, so each y value of $y = \cot x$ will be the reciprocal of $y = \tan x$. Because $\tan x = 0$ at $x = 0°, 180°, 360°, \ldots$, $y = \cot x$ will have vertical asymptotes at those values. Also, because $\tan x$ has asymptotes at $x = 90°, 270°, 450°, \ldots$, $y = \cot x = 0$ and there are x-intercepts at those values.

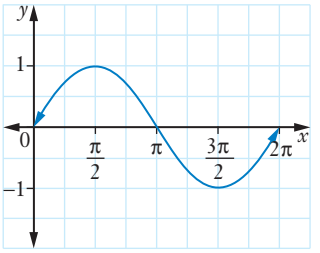
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	–



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It is more practical to express the trigonometric functions in terms of radians (not degrees), so here are the graphs in radians:

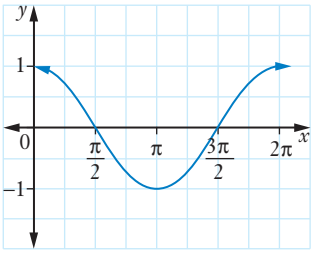
$y = \sin x$



Domain $(-\infty, \infty)$, range $[-1, 1]$

Odd function

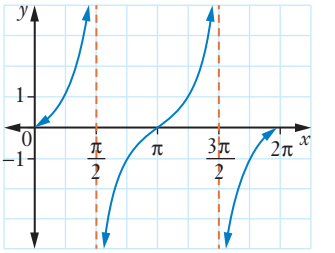
$y = \cos x$



Domain $(-\infty, \infty)$, range $[-1, 1]$

Even function

$y = \tan x$

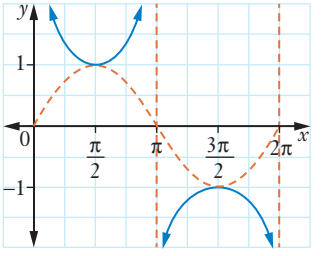


Domain $(-\infty, \infty)$

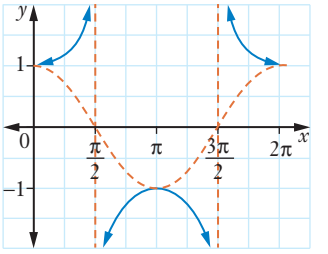
except for $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

(odd multiples of $\frac{\pi}{2}$),
range $(-\infty, \infty)$
Odd function

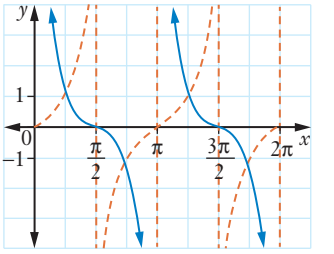
$y = \text{cosec } x$



$y = \sec x$



$y = \cot x$



Properties of the trigonometric functions

All the trigonometric functions have graphs that repeat at regular intervals, so they are called **periodic functions**. The **period** is the length of one cycle of a periodic function on the x-axis, before the function repeats itself.

The **centre** of a periodic function is its mean value and is equidistant from the maximum and minimum values. The mean value of $y = \sin x$, $y = \cos x$ and $y = \tan x$ is 0, represented by the x-axis.

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The **amplitude** is the height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively). The range of $y = \sin x$ and $y = \cos x$ is $[-1, 1]$.

$y = \sin x$ has period 2π and amplitude 1.

$y = \cos x$ has period 2π and amplitude 1.

$y = \tan x$ has period π and no amplitude.

INVESTIGATION

TRANSFORMING TRIGONOMETRIC GRAPHS

Use a graphics calculator or graphing software to draw the graphs of trigonometric functions with different values.

1 Graphs in the form $y = k \sin x$, $y = k \cos x$ and $y = k \tan x$ where $k = \ldots, -3, -2, -1, 2, 3, \ldots$

2 Graphs in the form $y = \sin ax$, $y = \cos ax$ and $y = \tan ax$ where $a = \ldots, -3, -2, -1, 2, 3, \ldots$

3 Graphs in the form $y = \sin x + c$, $y = \cos x + c$ and $y = \tan x + c$ where $c = \ldots, -3, -2, -1, 2, 3, \ldots$

4 Graphs in the form $y = \sin (x + b)$, $y = \cos (x + b)$ and $y = \tan (x + b)$ where $b = \ldots, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{\pi}{4}, \ldots$

Can you see patterns? Could you predict what different graphs look like?

Now we shall examine more general trigonometric functions of the form $y = k \sin ax$, $y = k \cos ax$ and $y = k \tan ax$, where k and a are constants.

Period and amplitude of trigonometric functions

$y = k \sin ax$ has amplitude k and period $\frac{2\pi}{a}$.

$y = k \cos ax$ has amplitude k and period $\frac{2\pi}{a}$.

$y = k \tan ax$ has no amplitude and has period $\frac{\pi}{a}$.

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EXAMPLE 9

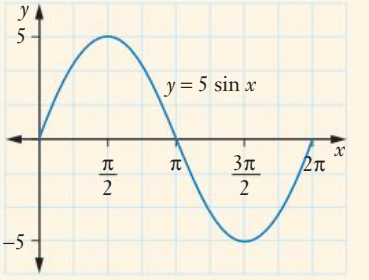
a Sketch each function in the domain $[0, 2\pi]$.

i $y = 5 \sin x$ **ii** $y = \sin 4x$ **iii** $y = 5 \sin 4x$

b Sketch the graph of $y = 2 \tan \frac{x}{2}$ for $[0, 2\pi]$.

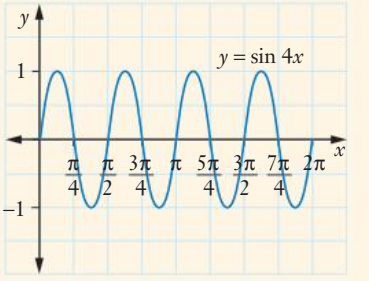
Solution

a **i** The graph of $y = 5 \sin x$ has y values that are 5 times as much as $y = \sin x$, so this function has amplitude 5 and period 2π. We draw one period of the sine 'shape' between ± 5.

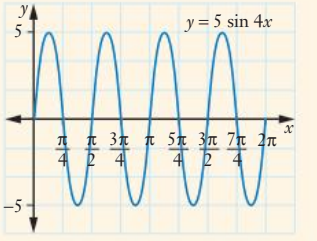


ii The graph $y = \sin 4x$ has amplitude 1 and period $\frac{2\pi}{4} = \frac{\pi}{2}$.

The curve repeats every $\frac{\pi}{2}$, so in the domain $[0, 2\pi]$ there will be 4 repetitions. The '4' in $\sin 4x$ compresses the graph of $y = \sin x$ horizontally.



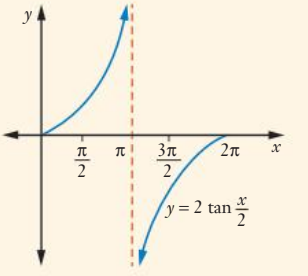
iii The graph $y = 5 \sin 4x$ has amplitude 5 and period $\frac{\pi}{2}$. It is a combination of graphs **i** and **ii**.



b $y = 2 \tan \frac{x}{2}$ has no amplitude.

Period $= \frac{\pi}{\frac{1}{2}} = 2\pi$

So there will be one period in the domain $[0, 2\pi]$.



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The graphs of trigonometric functions can change their **phase**, a shift to the left or right.

Phase shift of trigonometric functions

$y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ have phase b, which is a shift b units from $y = \sin x$, $y = \cos x$ and $y = \tan x$ respectively, to the left if $b > 0$ and to the right if $b < 0$.

EXAMPLE 10

Sketch the graph of $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ for $[0, 2\pi]$.

Solution

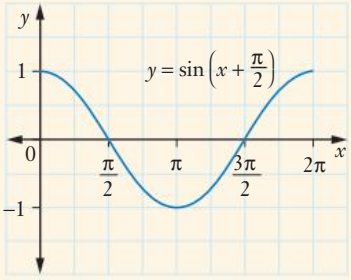
Amplitude = 1

Period $= \frac{2\pi}{1} = 2\pi$

Phase: $b = \frac{\pi}{2}$

This is the graph of $y = \sin x$ moved $\frac{\pi}{2}$ units to the left. If you're unsure how the phase affects the graph, draw a table of values.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	−1	0	1



The graphs of trigonometric functions can change their centre, a shift up or down.

Centre of trigonometric functions

$y = \sin x + c$, $y = \cos x + c$ and $y = \tan x + c$ have centre c, which is a shift up from $y = \sin x$, $y = \cos x$ and $y = \tan x$ respectively if $c > 0$ and a shift down if $c < 0$.

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EXAMPLE 11

Graphing the cosine function

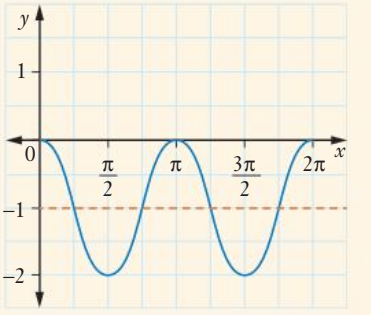
Sketch the graph of $y = \cos 2x - 1$ in the domain $[0, 2\pi]$.

Solution

Amplitude = 1, period $\frac{2\pi}{2} = \pi$.

$c = -1$ so the centre of the graph moves down 1 unit to -1.

Instead of moving between -1 and 1, the graph moves between -2 and 0.



General trigonometric functions

	Amplitude	Period	Phase	Centre
$y = k \sin [a(x + b)] + c$	k	$\frac{2\pi}{a}$	b Shift left if $b > 0$ Shift right if $b < 0$	$y = c$ Shift up if $c > 0$ Shift down if $c < 0$
$y = k \cos [a(x + b)] + c$	k	$\frac{2\pi}{a}$		
$y = k \tan [a(x + b)] + c$	No amplitude	$\frac{\pi}{a}$		

EXAMPLE 12

For the function $y = 3 \cos (2x - \pi)$, find:

a the amplitude **b** the period **c** the phase

Solution

$y = 3 \cos (2x - \pi)$

$= 3 \cos \left[2\left(x - \frac{\pi}{2}\right)\right]$

a Amplitude = 3

b Period $= \frac{2\pi}{2}$
$= \pi$

c Phase $= \frac{\pi}{2}$ units

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EXAMPLE 13

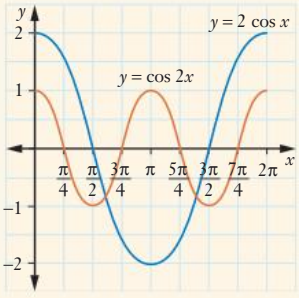
a Sketch the graph of $y = 2\cos x$ and $y = \cos 2x$ on the same set of axes for $[0, 2\pi]$.

b Hence, sketch the graph of $y = \cos 2x + 2\cos x$ for $[0, 2\pi]$.

Solution

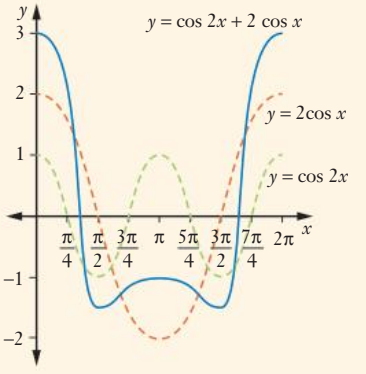
a $y = 2\cos x$ has amplitude 2 and period 2π.

$y = \cos 2x$ has amplitude 1 and period $\dfrac{2\pi}{2}$ or π.



b Add y values on the graph, using a table of values if more accuracy is needed.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos 2x$	1	0	−1	0	1	0	−1	0	1
$2\cos x$	2	$\sqrt{2}$	0	$-\sqrt{2}$	−2	$-\sqrt{2}$	0	$\sqrt{2}$	2
$\cos 2x + 2\cos x$	3	$\sqrt{2}$	−1	$-\sqrt{2}$	−1	$-\sqrt{2}$	−1	$\sqrt{2}$	3



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Exercise 9.04 Trigonometric functions

1 **a** Sketch the graph of $f(x) = \cos x$ in the domain $[0, 2\pi]$.

b Sketch the graph of $y = -f(x)$ in the same domain.

2 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $f(x) = 2 \sin x$ **b** $y = 1 + \sin x$ **c** $y = 2 - \sin x$

d $f(x) = -3 \cos x$ **e** $y = 4 \sin x$ **f** $f(x) = \cos x + 3$

g $y = 5 \tan x$ **h** $f(x) = \tan x + 3$ **i** $y = 1 - 2 \tan x$

3 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos 2x$ **b** $y = \tan 2x$ **c** $y = \sin 3x$

d $f(x) = 3 \cos 4x$ **e** $y = 6 \cos 3x$ **f** $y = \tan \frac{x}{2}$

g $f(x) = 2 \tan 3x$ **h** $y = 3 \cos \frac{x}{2}$ **i** $y = 2 \sin \frac{x}{2}$

4 Sketch the graph of each function in the domain $[-\pi, \pi]$:

a $y = -\sin 2x$ **b** $y = 7 \cos 4x$ **c** $f(x) = -\tan 4x$

d $y = 5 \sin 4x$ **e** $f(x) = 2 \cos 2x$ **f** $f(x) = 3 \tan x - 1$

5 Sketch the graph of $y = 8 \sin \frac{x}{2}$ in the domain $[0, 4\pi]$.

6 Sketch over the interval $[0, 2\pi]$ the graph of:

a $y = \sin (x + \pi)$ **b** $y = \tan\left(x + \frac{\pi}{2}\right)$ **c** $f(x) = \cos (x - \pi)$

d $y = 3 \sin\left(x - \frac{\pi}{2}\right)$ **e** $f(x) = 2 \cos\left(x + \frac{\pi}{2}\right)$ **f** $y = 4 \sin\left(2x + \frac{\pi}{2}\right)$

g $y = \cos\left(x - \frac{\pi}{4}\right)$ **h** $y = \tan\left(x + \frac{\pi}{4}\right)$

7 Sketch over the interval $[-2, 2]$ the graph of:

a $y = \sin \pi x$ **b** $y = 3 \cos 2\pi x$

8 For each function, find:

i the amplitude **ii** the period **iii** the centre **iv** the phase

a $y = 5 \sin 2x$ **b** $f(x) = -\cos (x - \pi)$ **c** $y = 2 \tan (4x) - 2$

d $y = 3 \sin\left(x + \frac{\pi}{4}\right) + 1$ **e** $y = 8 \cos (\pi x - 2) - 3$ **f** $f(x) = 3 \tan\left(5x + \frac{\pi}{2}\right) + 2$

9 Find the domain and range of each function.

a $y = 4 \sin x - 1$ **b** $f(x) = -3 \cos 5x + 7$

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10 Sketch in the domain $[0, 2\pi]$ the graphs of:

a $y = \sin x$ and $y = \sin 2x$ on the same set of axes

b $y = \sin x + \sin 2x$

11 Sketch for the interval $[0, 2\pi]$ the graphs of:

a $y = 2\cos x$ and $y = 3\sin x$ on the same set of axes

b $y = 2\cos x + 3\sin x$

12 By sketching the graphs of $y = \cos x$ and $y = \cos 2x$ on the same set of axes for $[0, 2\pi]$, sketch the graph of $y = \cos 2x - \cos x$.

13 Sketch the graph of $y = \cos x + \sin x$.

9.05 Trigonometric equations

WS Trigonometric equations

EXAMPLE 14

Solve each equation for $[0°, 360°]$.

a $\sin x = 0.34$ **b** $\cos x = \frac{\sqrt{3}}{2}$ **c** $\tan \theta = -1$

Solution

Make sure that your calculator is in **degrees mode**. Check your solution by substituting back into the equation.

a 0.34 is positive and $\sin x > 0$ in 1st and 2nd quadrants.

$\sin x = 0.34$

$x \approx 19°53', 180° - 19°53'$

$= 19°53', 160°7'$

19°53′ is the **principal solution** but there is another solution in the 2nd quadrant.

b $\cos x > 0$ in the 1st and 4th quadrants.

$\cos x = \frac{\sqrt{3}}{2}$

$x = 30°, 360° - 30°$

$= 30°, 330°$

c $\tan \theta < 0$ in the 2nd and 4th quadrants.

For $\tan \theta = -1$

$\theta = 180° - 45°, 360° - 45°$

$= 135°, 315°$

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EXAMPLE 15

Solve $\tan x = \sqrt{3}$ for $[-180°, 180°]$.

Solution

In the domain $[-180°, 180°]$, we use positive angles for $0° \leq x \leq 180°$ and negative angles for $-180° \leq x \leq 0°$.

tan > 0 in the 1st and 3rd quadrants.

$$\tan x = \sqrt{3}$$
$$x = 60°, -(180° - 60°)$$
$$= 60°, -120°$$

EXAMPLE 16

Solve $2\sin^2 x - 1 = 0$ for $0° \leq x \leq 360°$.

Solution

$$2\sin^2 x - 1 = 0$$
$$2\sin^2 x = 1$$
$$\sin^2 x = \frac{1}{2}$$
$$\sin x = \pm\frac{\sqrt{1}}{\sqrt{2}}$$
$$= \pm\frac{1}{\sqrt{2}}$$

Since the ratio could be positive or negative, there are solutions in all 4 quadrants.

$x = 45°, 180° - 45°, 180° + 45°, 360° - 45°$

$= 45°, 135°, 225°, 315°$

If we are solving an equation involving $2x$ or $3x$, for example, we need to change the domain to find all possible solutions.

EXAMPLE 17

Solve $2\sin 2x - 1 = 0$ for $[0°, 360°]$.

Solution

Notice that the angle is $2x$ but the domain is for x.

If $0° \leq x \leq 360°$

then $0° \leq 2x \leq 720°$

This means that we can find the solutions by going around the circle twice.

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$2 \sin 2x - 1 = 0$

$2 \sin 2x = 1$

$\sin 2x = \frac{1}{2}$

Sin is positive in the 1st and 2nd quadrants.

First time around the circle: 1st quadrant is θ and the 2nd quadrant is $180° - \theta$.

Second time around the circle: add $360°$ to θ and $180° - \theta$.

$2x = 30°, 180° - 30°, 360° + 30°, 360° + 180° - 30°$

$= 30°, 150°, 390°, 510°$

$\therefore x = 15°, 75°, 195°, 255°$

You can solve trigonometric equations involving **radians**. You can recognise these because the domain is in radians.



Trigonometric equations

EXAMPLE 18

Solve each equation for $[0, 2\pi]$.

a $\cos x = 0.34$

b $\sin \alpha = -\frac{1}{\sqrt{2}}$

c $\sin^2 x - \sin x = 2$

The domain $[0, 2\pi]$ tells us that the solutions will be in **radians**. Make sure that your calculator is in radians mode here.

Solution

a $\cos x > 0$ in the 1st and 4th quadrants.

$\cos x = 0.34$

$x \approx 1.224, 2\pi - 1.224$

$= 1.224, 5.059$

b $\sin \alpha$ is negative in the 3rd and 4th quadrants.

$\sin \alpha = -\frac{1}{\sqrt{2}}$

$\alpha = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$= \frac{5\pi}{4}, \frac{7\pi}{4}$

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c $\sin^2 x - \sin x = 2$

$\sin^2 x - \sin x - 2 = 0$ This is a quadratic equation.

$(\sin x - 2)(\sin x + 1) = 0$

$\sin x = 2$ $\sin x = -1$

$\sin x = 2$ has no solutions since $-1 \leq \sin x \leq 1$

$\sin x = -1$ has solution $x = \frac{3\pi}{2}$

$x = \frac{3\pi}{2}$

Exercise 9.05 Trigonometric equations

1 Solve each equation for $[0°, 360°]$:

a $\sin\theta = 0.35$ **b** $\cos\theta = -\frac{1}{2}$ **c** $\tan\theta = -1$

d $\sin\theta = \frac{\sqrt{3}}{2}$ **e** $\tan\theta = -\frac{1}{\sqrt{3}}$ **f** $2\cos\theta = \sqrt{3}$

g $\tan 2\theta = \sqrt{3}$ **h** $2\cos 2\theta - 1 = 0$ **i** $2\sin 3\theta = -1$

j $\tan^2 3\theta = 1$ **k** $\sin^2 x = 1$ **l** $2\cos^2 x - \cos x = 0$

2 Solve for $0° \leq x \leq 360°$:

a $\cos x = 1$ **b** $\sin x + 1 = 0$ **c** $\cos^2 x = 1$

d $\sin x = 1$ **e** $\tan x = 0$ **f** $\sin^2 x + \sin x = 0$

g $\cos^2 x - \cos x = 0$ **h** $\tan^2 x = \tan x$ **i** $\tan^2 x = 3$

3 Solve for $[0, 2\pi]$:

a $\sin x = 0$ **b** $\tan 2x = 0$ **c** $\sin x = -1$

d $\cos x - 1 = 0$ **e** $\cos x = -1$

4 Solve for $[-180°, 180°]$:

a $\cos\theta = 0.187$ **b** $\sin\theta = \frac{1}{2}$ **c** $\tan\theta = 1$

d $\sin\theta = -\frac{\sqrt{3}}{2}$ **e** $\tan\theta = -\frac{1}{\sqrt{3}}$ **f** $3\tan^2\theta = 1$

g $\tan\theta + 1 = 0$ **h** $\tan 2\theta = 1$

5 Solve for $0 \leq x \leq 2\pi$:

a $\cos x = \frac{1}{2}$ **b** $\sin x = -\frac{1}{\sqrt{2}}$ **c** $\tan x = 1$

d $\tan x = \sqrt{3}$ **e** $\cos x = -\frac{\sqrt{3}}{2}$

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6 Solve for $-\pi \leq x \leq \pi$:

a $2 \sin x = \sqrt{3}$ **b** $2 \cos x = 0$ **c** $3 \tan^2 x = 1$

7 Solve $2 \cos x = -1$ in the domain $[-2\pi, 2\pi]$.

8 Solve for $[0, 2\pi]$:

a $\tan^2 x + \tan x = 0$ **b** $\sin^2 x - \sin x = 0$ **c** $2 \cos^2 x - \cos x - 1 = 0$

d $4 \sin^2 x = 1$ **e** $\tan x \cos x + \tan x = 0$ **f** $\sin^2 x + 2 \cos x - 2 = 0$

Applications of trigonometric functions

WS

Applying trigonometric functions

9.06 Applications of trigonometric functions

Trigonometric graphs can model real-life situations.

EXAMPLE 19

This table shows the average maximum monthly temperatures in Sydney.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
°C	26.1	26.1	25.1	22.8	19.8	17.4	16.8	18.0	20.1	22.2	23.9	25.6

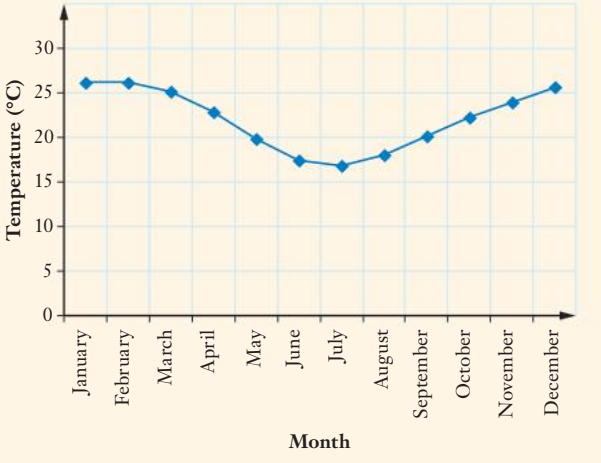
a Draw a graph of this data.

b Is it periodic? Would you expect it to be periodic?

c What is the period and amplitude?

Solution

a



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b The graph looks like it is periodic, and we would expect it to be, since the temperature varies with the seasons and these repeat every 12 months. It goes up and down, and reaches a highest value in summer and a lowest value in winter.

c This curve is approximately a cosine curve with a period of 12 months.

The highest maximum temperature is around 26° and the lowest maximum temperature is around 18°, so the centre of the graph is $\frac{26° + 18°}{2} = 22°$. So the amplitude is 26 – 22 (or 22 – 18) = 4.

DID YOU KNOW?

Waves

The sine and cosine curves are used in many applications including the study of waves. There are many different types of waves, including water, light and sound waves. Oscilloscopes display patterns of electrical waves on the screen of a cathode-ray tube.



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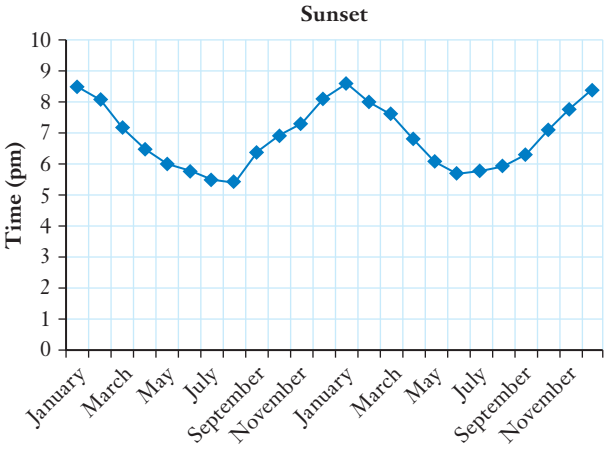
Simple harmonic motion (such as the movement of a pendulum) is a wave-like or oscillatory motion when graphed against time. In 1581, when he was 17 years old, the Italian scientist Galileo noticed a lamp swinging backwards and forwards in Pisa cathedral. He found that the lamp took the same time to swing to and fro, no matter how much weight it had on it. This led him to discover the pendulum.

Galileo also invented the telescope. Find out more about his life and work.

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Exercise 9.06 Applications of trigonometric functions

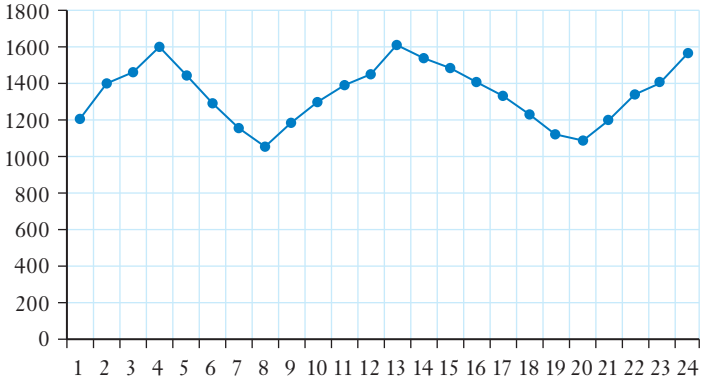
1 This graph shows the time of sunset in a city over a period of 2 years.



a Find the approximate period and amplitude of the graph.

b At approximately what time would you expect the Sun to set in July?

2 The graph shows the incidence of crimes committed over 24 years in Gotham City.



a Approximately how many crimes were committed in the 10th year?

b What was:

i the highest number of crimes? **ii** the lowest number of crimes?

c Find the approximate amplitude and the period of the graph.

3 This table shows the tides (in metres) at a jetty measured 4 times each day for 3 days.

Day	Friday				Saturday				Sunday			
Time	6:20 a.m.	11:55 a.m.	6:15 p.m.	11:48 p.m.	6:20 a.m.	11:55 a.m.	6:15 p.m.	11:48 p.m.	6:20 a.m.	11:55 a.m.	6:15 p.m.	11:48 p.m.
Tide (m)	3.2	1.1	3.4	1.3	3.2	1.2	3.5	1.1	3.4	1.2	3.5	1.3

a Draw a graph showing the tides.

b Find the period and amplitude.

c Estimate the height of the tide at around 8 a.m. on Friday.

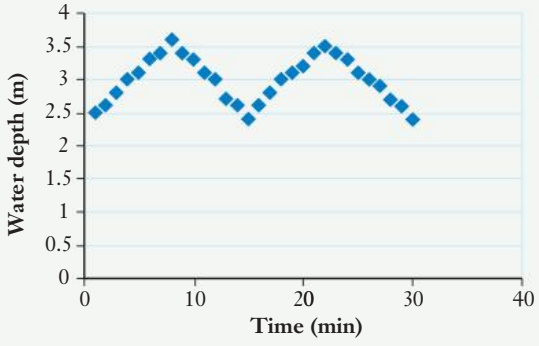
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9. TEST YOURSELF

For Questions **1** to **4** select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 This graph shows the water depth in metres as a lock opens and closes over time.



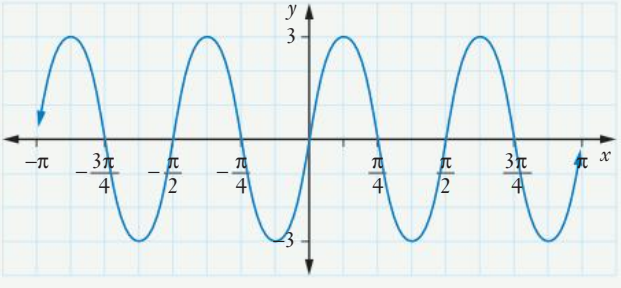
The approximate period and amplitude of the graph are:

A Amplitude 1, period 15 min
B Amplitude 0.5, period 7.5 min
C Amplitude 1, period 7.5 min
D Amplitude 0.5, period 15 min

2 The exact value of $\cos\frac{2\pi}{3}$ is:

A $\frac{1}{2}$ **B** $-\frac{\sqrt{3}}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** $-\frac{1}{2}$

3 The equation of the graph below is:



A $y = 3\cos 4x$ **B** $y = 3\sin 4x$ **C** $y = 4\sin 3x$ **D** $y = 4\cos 3x$

4 $\text{cosec}^2 x$ is equal to:

A $\cot^2 x - 1$ **B** $1 - \cos^2 x$ **C** $1 + \cot^2 x$ **D** $\tan^2 x + 1$

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5 Find the exact value of:

a $\cos 315°$ **b** $\sin(-60°)$ **c** $\tan 120°$

6 Solve for $0° \leq x \leq 360°$:

a $\sin x = \frac{\sqrt{3}}{2}$ **b** $\tan x = 1$ **c** $2\cos x + 1 = 0$

d $\sin^2 x = \frac{3}{4}$ **e** $\tan 2x = \frac{1}{\sqrt{3}}$

7 Solve for $0 \leq x \leq 2\pi$:

a $\tan x = -1$ **b** $2\sin x = 1$ **c** $\tan^2 x = 3$

d $\cos x = 1$ **e** $\sin x = -1$

8 For $0 \leq x \leq 2\pi$, sketch the graph of:

a $y = 3\cos 2x$ **b** $y = 7\sin\frac{x}{2}$

9 If $\sin x = -\frac{12}{13}$ and $\cos x > 0$, evaluate $\cos x$ and $\tan x$.

10 Simplify:

a $\cos(180° + \theta)$ **b** $\tan(-\theta)$

c $\sin(\pi - \theta)$ **d** $\tan x \cos x$

e $\sqrt{4 - 4\sin^2 A}$ **f** $\cos(90 - x)°$

g $\cot\beta \tan\beta$

11 Find the exact value of:

a $\sin\frac{5\pi}{4}$ **b** $\cos\frac{5\pi}{6}$ **c** $\tan\frac{4\pi}{3}$

12 Prove that $\frac{2\cos^2\theta}{1-\sin\theta} = 2 + 2\sin\theta$.

13 Find the value of b if $\sin b = \cos(2b - 30)°$.

14 Find the period, amplitude, centre and phase of $y = -2\cos\left(3x + \frac{\pi}{12}\right) + 5$.

15 Find the exact value of:

a $\sec\frac{\pi}{4}$ **b** $\cot\frac{\pi}{6}$

c $\operatorname{cosec}\frac{\pi}{3}$ **d** $\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}$

16 Find the domain and range of each function.

a $y = -6\sin(2x) + 5$ **b** $f(x) = 4\cos x - 3$

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9. CHALLENGE EXERCISE

1 Find the exact value of:

a $\sin 600°$ **b** $\tan(-405°)$

2 Solve $2\cos(\theta + 10°) = -1$ for $0° \leq \theta \leq 360°$.

3 If $f(x) = 3\cos \pi x$:

a find the period and amplitude of the function

b sketch the graph of $f(x)$ for $0 \leq x \leq 4$.

4 For $0 \leq x \leq 2\pi$, sketch the graph of:

a $f(x) = 2\cos\left(x + \frac{\pi}{2}\right) + 1$ **b** $y = 2 - 3\sin\left(x - \frac{\pi}{2}\right)$

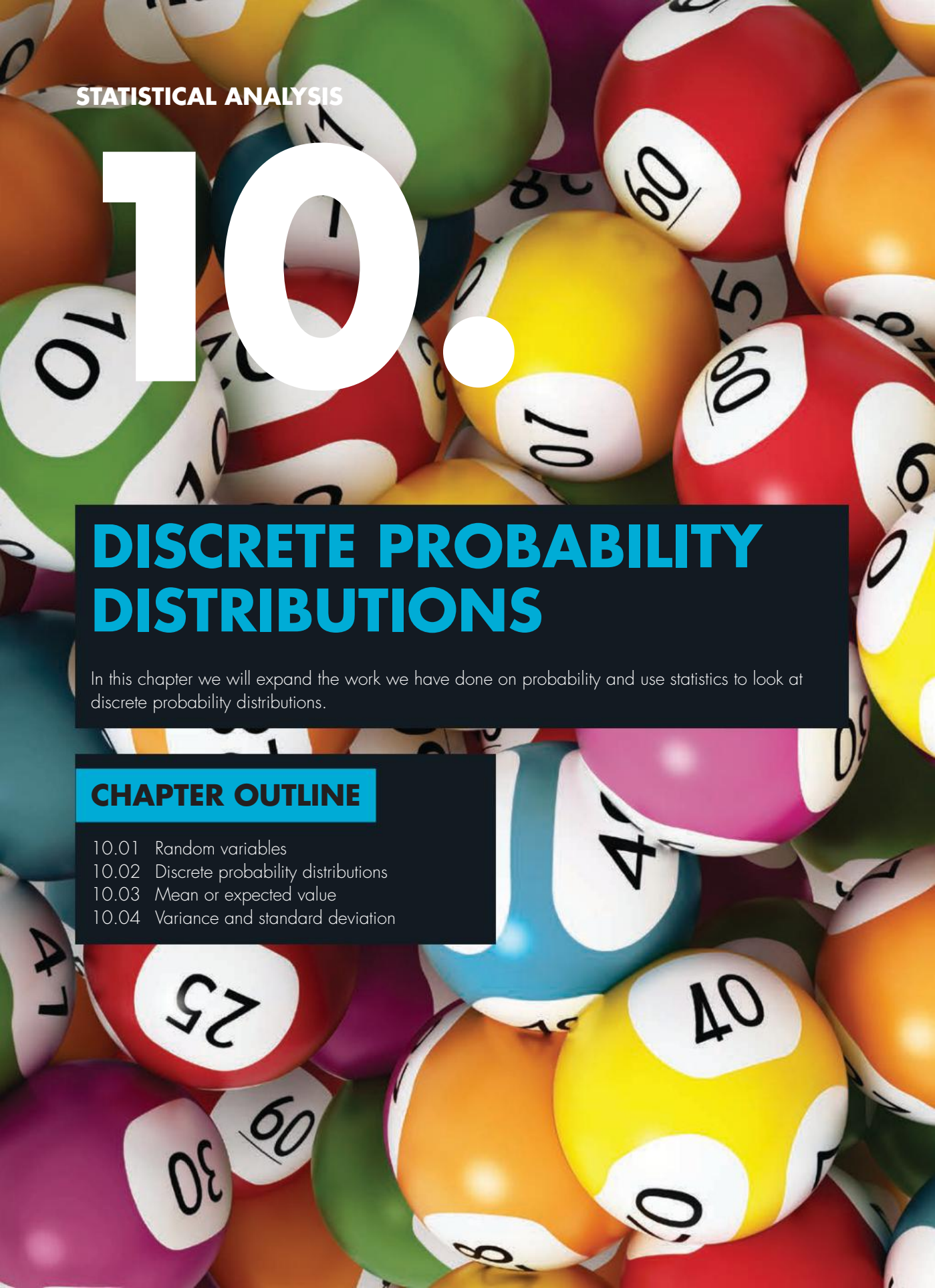
c $y = \sin 2x - \sin x$ **d** $y = \sin x + 2\cos 2x$

e $y = 3\cos x - \cos 2x$ **f** $y = \sin x - \sin\frac{x}{2}$

5 Solve $\cos^2 x - \cos x = 0$ for $0 \leq x \leq 2\pi$.

6 Find the exact value of $\sin 120° + \cos 135°$ as a surd with rational denominator.

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STATISTICAL ANALYSIS

10.

DISCRETE PROBABILITY DISTRIBUTIONS

In this chapter we will expand the work we have done on probability and use statistics to look at discrete probability distributions.

CHAPTER OUTLINE

10.01 Random variables
10.02 Discrete probability distributions
10.03 Mean or expected value
10.04 Variance and standard deviation



IN THIS CHAPTER YOU WILL:

- understand random variables and definitions of discrete, continuous, uniform, finite and infinite variables
- recognise discrete probability distributions and their properties
- use probability distributions to solve practical problems
- find expected values, variance and standard deviations of probability distributions

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TERMINOLOGY

discrete random variable: A random variable that can take on a number of discrete values; for example, the number of children in a family
expected value: Average or mean value of a probability distribution
population: The whole data set from which a sample can be taken
probability distribution: A function that sets out all possible values of a random variable together with their probabilities
random variable: A variable whose values are based on a chance experiment; for example, the number of road accidents in an hour
standard deviation: A measure of the spread of values from the mean of a distribution; the square root of the variance
uniform probability distribution: A probability distribution in which every outcome has the same probability
variance: A measure of the spread of values from the mean of a distribution; the square of the standard deviation

10.01 Random variables

We studied probability in Chapter 7. Now we will look at **probability distributions**, which use random variables to predict and model random situations in areas such as science, economics and medicine.

A **random variable** is a variable that can take on different values depending on the outcome of a random process, such as an experiment. Random variables can be **discrete** or **continuous**. Discrete variables such as goals scored or number of children take on specific finite values, while continuous variables such as length or temperature are measured along a continuous scale.

Discrete random variables

A **discrete random variable** is a variable whose values are specific and can be listed.

In this chapter we will look at **discrete random variables**. We will look at continuous random variables in Year 12.

EXAMPLE 1

Is each random variable discrete or continuous?

a The number of goals scored by a netball team

b The height of a student

c The shoe size of a Year 11 student.

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Solution

a The number of goals scored is a specific whole number so it is a discrete random variable.

b Height is measured on a continuous scale so the height of a student is a continuous random variable.

c Shoe sizes are specific values that can be listed so it is a discrete random variable.

We use a capital letter such as X for a random variable and a lower-case letter such as x for the values of X.

EXAMPLE 2

Find the set of possible values for each discrete random variable:

a The number rolled on a die

b The number of girls in a family of 3 children

c The number of heads when tossing a coin 8 times.

Solution

a Any number from 1 to 6 can be rolled on a die.

So $X = \{1, 2, 3, 4, 5, 6\}$

b It is possible to have no girls, 1 girl, 2 girls or 3 girls.

So $X = \{0, 1, 2, 3\}$

c The coin could come up heads 0, 1, 2, … 8 times.

So $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

Exercise 10.01 Random variables

1 For each random variable, state whether it is discrete or continuous:

a A film critic's rating of a film, from 0 to 4 stars

b The speed of a car

c The sum rolled on a pair of dice

d The winning ticket number drawn from a raffle

e The weight of parcels at a post office

f The size of jeans in a shop

g The temperature of a metal as it cools

h The amount of water in different types of fruit drink
i The number of cars passing the school over a 10-minute period
j The number of cities in each country in Europe
k The number of heads when tossing a coin 50 times
l The number of correct answers in a 10-question test.

2 Write the set of possible values for each discrete random variable:
a Number of daughters in a one-child family
b Number of 6s on 10 rolls of a die
c Number of people aged over 50 in a group of 20 people
d The number of days it rains in March
e The sum of the 2 numbers rolled on a pair of dice.

WS Discrete probability distributions 1

WS Discrete probability distributions 2

Probability density functions

10.02 Discrete probability distributions

Discrete probability distribution

A **discrete probability distribution** lists the probability for each value of a discrete random variable.

A discrete probability distribution can be displayed in a table or graph, or represented by an equation or set of ordered pairs. It is also called a **discrete probability function**.

We can write a probability function that uses X as the random variable as $P(X = x)$ or $p(x)$.

EXAMPLE 3

a In a random experiment, a die was rolled and the results recorded in the table below.

Number	Frequency
1	18
2	23
3	17
4	28
5	12
6	22

i How many times was the die rolled?
ii Draw up a probability distribution for this experiment.

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b Write the probability function of rolling a die as a set of ordered pairs $(x, P(X = x))$.

c A probability function for discrete random variable X is given by:

$$P(X=x)=\begin{cases}\frac{1}{16}(4-x) & \text{for } x=0,2\\ \frac{1}{8}(x-1) & \text{for } x=3,4\\ 0 & \text{for any other } x \text{ value}\end{cases}$$

i Complete a discrete probability distribution table.

ii Find the sum of all probabilities.

iii Evaluate $P(X = \text{odd})$.

iv Evaluate $P(X \leq 3)$.

Solution

a **i** Adding the frequencies, the die was rolled 120 times.

ii For the probability of each outcome, we use **relative frequencies** as we did in Chapter 7.

x	1	2	3	4	5	6
$P(X=x)$	$\frac{18}{120}=\frac{3}{20}$	$\frac{23}{120}$	$\frac{17}{120}$	$\frac{28}{120}=\frac{7}{30}$	$\frac{12}{120}=\frac{1}{10}$	$\frac{22}{120}=\frac{11}{60}$

b The probability of each number being rolled on a die is $\frac{1}{6}$. So the probability function $P(X = x)$ can be written as $\left(1,\frac{1}{6}\right),\left(2,\frac{1}{6}\right),\left(3,\frac{1}{6}\right),\left(4,\frac{1}{6}\right),\left(5,\frac{1}{6}\right),\left(6,\frac{1}{6}\right)$.

c **i** $p(0) = \frac{1}{16}(4-0)$

$= \frac{1}{4}$

$p(2) = \frac{1}{16}(4-2)$

$= \frac{1}{8}$

$p(3) = \frac{1}{8}(3-1)$

$= \frac{1}{4}$

$p(4) = \frac{1}{8}(4-1)$

$= \frac{3}{8}$

All other values of x give $p(x) = 0$. We cannot put all of these in a table. They will not make any difference to calculations anyway.

x	0	1	2	3	4
$p(x)$	$\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$

ii $p(0) + p(1) + p(2) + p(3) + p(4) = \frac{1}{4} + 0 + \frac{1}{8} + \frac{1}{4} + \frac{3}{8}$

$= 1$

iii $P(X = \text{odd}) = p(1) + p(3)$

$= 0 + \frac{1}{4}$

$= \frac{1}{4}$

iv $P(X \leq 3) = p(0) + p(1) + p(2) + p(3)$

$= \frac{1}{4} + 0 + \frac{1}{8} + \frac{1}{4}$

$= \frac{5}{8}$

Remember that all probabilities lie between 0 and 1 and their total is 1. These same rules apply to a probability distribution.

Properties of discrete probability distributions

For a discrete probability distribution:

- all possible values of X must be mutually exclusive
- the sum of all probabilities must be 1
- for each value of x, $0 \leq P(X = x) \leq 1$.

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EXAMPLE 4

Consider this discrete probability distribution.

x	1	2	3	4	5	6
$P(X = x)$	0.2	0.35	0.1	0.15	0.05	0.15

a Find:

i $P(X = 2)$ **ii** $P(X < 3)$

iii $P(X \geq 4)$ **iv** $P(2 \leq X < 5)$

b Show that the sum of probabilities is 1.

c Draw a histogram of the function.

Solution

a **i** From the table, $p(2) = 0.35$.

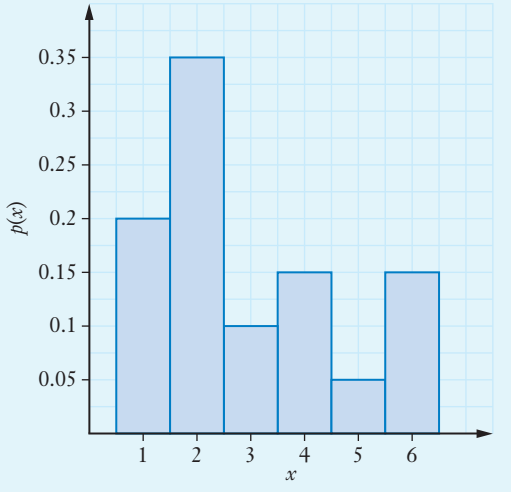
ii $P(X < 3) = p(1) + p(2)$
$= 0.2 + 0.35$
$= 0.55$

iii $P(X \geq 4) = p(4) + p(5) + p(6)$
$= 0.15 + 0.05 + 0.15$
$= 0.35$

iv $P(2 \leq X < 5) = p(2) + p(3) + p(4)$
$= 0.35 + 0.1 + 0.15$
$= 0.6$

b $p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 0.2 + 0.35 + 0.1 + 0.15 + 0.05 + 0.15$
$= 1$

c A histogram is the best type of graph to draw a discrete probability distribution.



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EXAMPLE 5

a A function is given by $p(x) = \dfrac{x-1}{3}$ where $x = 1, 2, 3$. Is the function a probability distribution?

b Find the value of n for which the table below is a discrete probability distribution.

x	0	1	2	3	4
$p(x)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{1}{12}$	n

Solution

a
$$p(1) = \frac{1-1}{3} = 0$$
$$p(2) = \frac{2-1}{3} = \frac{1}{3}$$
$$p(3) = \frac{3-1}{3} = \frac{2}{3}$$
$$p(1) + p(2) + p(3) = 0 + \frac{1}{3} + \frac{2}{3} = 1$$

So the function is a probability distribution.

b The sum of the probabilities must be 1.

$$\frac{1}{12} + \frac{1}{6} + \frac{5}{12} + \frac{1}{12} + n = 1$$
$$\frac{9}{12} + n = 1$$
$$n = 1 - \frac{9}{12}$$
$$n = \frac{1}{4}$$

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EXAMPLE 6

A game involves scoring points for selecting a card at random from a deck of 52 playing cards. The table shows the scores awarded for different selections.

Type of card	Score
Number 2–10	2
Picture card (jack, queen or king)	5
Ace	8

a Draw a probability distribution table for random variable Y for the different scores.

b Find:

i $P(Y < 8)$ **ii** $P(Y > 2)$

Solution

a There are 9 cards numbered 2–10 in each of the 4 suits (hearts, diamonds, spades and clubs). So there are $9 \times 4 = 36$ cards that will score 2 points.

There are 3 picture cards (jack, queen and king) in each suit, so there are $3 \times 4 = 12$ cards that will score 5 points.

There are 4 aces that will score 8 points.

y	2	5	8
$P(Y = y)$	$\frac{36}{52} = \frac{9}{13}$	$\frac{12}{52} = \frac{3}{13}$	$\frac{4}{52} = \frac{1}{13}$

b **i** $P(X < 8) = p(2) + p(5)$

$$= \frac{9}{13} + \frac{3}{13}$$

$$= \frac{12}{13}$$

ii $P(X > 2) = p(5) + p(8)$

$$= \frac{3}{13} + \frac{1}{13}$$

$$= \frac{4}{13}$$

Uniform distribution

In a **uniform probability distribution**, the probability for each value is the same.

Uniform probability distribution

A uniform probability distribution occurs if random variable X has n values where

$P(X = x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots, n$

EXAMPLE 7

Which probability distribution is uniform?

a The number of heads when tossing a coin

b The number of heads when tossing 2 coins.

Solution

a When tossing a coin, we could get either 0 or 1 head.

$X = \{0, 1\}$

$p(0) = \frac{1}{2}$

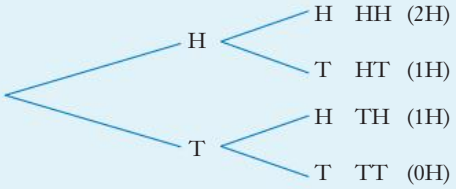
$p(1) = \frac{1}{2}$

Since both values have the same probability, it is a uniform distribution.

b When tossing 2 coins, the number of heads could be 0, 1 or 2.

$X = \{0, 1, 2\}$

Using a probability tree or table, we can find the probability for each outcome.



$p(0) = P(\text{TT})$

$= \frac{1}{4}$

$p(1) = P(\text{HT}) + P(\text{TH})$

$= \frac{2}{4}$

$= \frac{1}{2}$

$p(2) = P(\text{HH})$

$= \frac{1}{4}$

The probabilities are not all the same so it is not a uniform distribution.

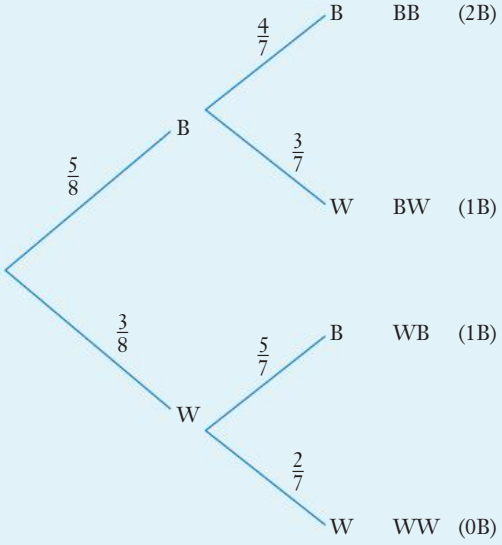
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EXAMPLE 8

Draw a probability distribution table for the number of black balls that could be drawn out of a bag containing 5 black and 3 white balls when 2 balls are selected randomly without replacement.

Solution

First draw a probability tree:



$$P(0B) = P(WW)$$
$$= \frac{3}{8} \times \frac{2}{7}$$
$$= \frac{6}{56}$$
$$= \frac{3}{28}$$

$$P(1B) = P(BW) + P(WB)$$
$$= \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$$
$$= \frac{30}{56}$$
$$= \frac{15}{28}$$

$$P(2B) = P(BB)$$
$$= \frac{5}{8} \times \frac{4}{7}$$
$$= \frac{20}{56}$$
$$= \frac{5}{14}$$

x	0	1	2
$P(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

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Exercise 10.02 Discrete probability distributions

1 Draw a probability distribution table for the sum of the numbers rolled on 2 dice.

2 Write the probability distribution as a set of ordered pairs $(x, P(X = x))$ for the number of heads when tossing:

a 1 coin **b** 2 coins **c** 3 coins.

3 A survey of a sample of bags of 50 jelly beans found that they didn't all hold exactly 50. The table shows the results of the study.

Number of jelly beans	Frequency
48	8
49	9
50	21
51	9
52	6

a Draw a probability distribution table for the results.

b If a bag of jelly beans is chosen at random, find the probability that the bag contains:

i at least 50 jelly beans

ii fewer than 51 jelly beans

4 A function is given by $p(x) = \frac{x-2}{6}$ for $x = 3, 4, 5$.

a Show that the function is a probability distribution.

b Draw up a probability distribution table.

c Find:

i $P(X > 3)$ **ii** $P(X = \text{odd})$ **iii** $P(3 \leq X < 5)$

5 Draw a histogram to show this discrete probability distribution.

x	0	1	2	3	4
$P(X = x)$	0.05	0.4	0.25	0.1	0.2

6 **a** Draw a probability distribution table for rolling a die.

b Is this a uniform distribution?

c Find:

i $P(X \geq 4)$ **ii** $P(X < 3)$ **iii** $P(1 < X \leq 4)$

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7 For each function, state whether it is a probability distribution:

a $\left(0,\frac{1}{5}\right)$, $\left(1,\frac{2}{5}\right)$, $(2, 0)$, $\left(3,\frac{2}{5}\right)$, $\left(4,\frac{1}{5}\right)$

b

x	1	2	3	4
$P(X = x)$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$

c $p(x) = \frac{x+2}{4}$ for $x = 0, 1, 2$

8 Find k if each function is a probability distribution:

a $p(x) = k(x + 1)$ for $x = 1, 2, 3, 4$

b

x	0	1	2	3	4
$P(X = x)$	0.2	k	0.15	0.34	0.12

c $(1, k), \left(2,\frac{1}{10}\right), (3, 0), \left(4,\frac{1}{5}\right), \left(5,\frac{3}{10}\right), \left(6,\frac{2}{5}\right)$

9 The probability function for the random variable X is given by $p(x) = \frac{kx^2}{x+5}$ for $x = 1, 2, 3, 4$.

a Construct a probability distribution table for the function.

b Find the value of k.

10 In a game, each player rolls 2 dice. The game pays \$1 if one of the numbers is a 6, \$3 for double 6 and \$2 for any other double. There is no payout for other results.

a Draw a probability distribution table for the game payout Y.

b Find the probability of winning:

i \$3 **ii** at least \$2 **iii** less than \$3.

11 Given the probability function below, evaluate p.

x	3	4	5	6
$P(X = x)$	$2p$	$3p$	$5p$	p

12 Simon plays a game where he selects a card at random from 100 cards numbered 1 to 100. He wins \$1 for selecting a number less than 20, \$2 for a number greater than 90, \$3 for any number from 61 and 69 (inclusive) and \$5 for any number from 41 to 50 (inclusive).

a Create a probability distribution table for the random variable X for the prize values.

b Find the probability of winning more than \$2.

c Find the probability of winning less than \$5.

13 The table below shows the probability function for random variable X.

x	5	6	7	8	9	10
$P(X = x)$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{7}$

Find:

a $P(X = 6)$ **b** $P(X = \text{even})$ **c** $P(X > 8)$ **d** $P(X \leq 7)$

e $P(6 < X < 9)$ **f** $P(7 \leq X < 10)$ **g** $P(6 \leq X \leq 9)$

14 The table below shows the probability function for random variable Q.

q	0	2	4	6	8	10
$P(Q = q)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$

Find:

a $p(8)$ **b** $P(Q \geq 4)$ **c** $P(2 < Q \leq 6)$

d $P(4 \leq Q \leq 10)$ **e** $P(0 \leq Q < 4)$ **f** $P(2 \leq Q \leq 8)$

15 A company makes washing machines. On average, there are 3 faulty machines made for every 1000 machines. Two washing machines are selected at random for a quality control inspection.

a Draw a probability distribution table for the number of these machines that could be faulty.

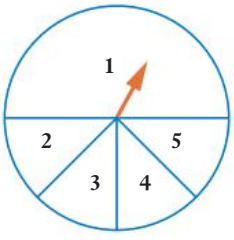
b Find the probability that:

i one will be faulty **ii** at least one will be faulty.

16 A bag contains 7 red, 6 white and 8 blue balls. Create a probability function table for the number of white balls selected when drawing 2 balls at random from the bag:

a with replacement **b** without replacement.

17 The spinner below has the numbers 1–5 distributed as shown:



a What is the probability that the arrow points to the number 3 when it is spun?

b Is the probability distribution of the spun numbers uniform?

c Draw a table showing the probability distribution for spinning the numbers.

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18 The probability of a traffic light showing green as a car approaches it is 12%. Draw a probability distribution table for the number of green traffic lights on approach when a car passes through 3 traffic lights.



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19 There is a 51% chance of giving birth to a boy. If a family has 4 children, construct a probability function to show the number of boys in the family.

20 A raffle has 2 prizes with 100 tickets sold altogether. Iris buys 5 tickets.

a Draw a probability distribution table to show the number of prizes Iris could win in the raffle.

b Find the probability that Iris wins at least one prize.

10.03 Mean or expected value

WS
Expected value

The **expected value** $E(X)$ of a probability distribution measures the centre of the distribution. It is the same as finding the **mean**, or average, which has symbol μ. It is the expected value of the random variable.

We use $\bar{x}$ for the mean of a **sample** and μ for the mean of a **population**. For probability distributions, we use the population mean, μ. The sample mean, $\bar{x}$, is an estimate of μ, and as the sample size increases, the sample represents the population better and the value of $\bar{x}$ approaches μ.

EXAMPLE 9

This table shows Harrison's diving scores (out of 10) over one year.

Score	Frequency
5	7
6	9
7	8
8	3
9	2
10	1

a Copy the table and add 2 columns to calculate the relative frequency for each score and the product of each score and its relative frequency.

b Calculate, correct to 2 decimal places, the sum of the last column (products of scores and their relative frequencies) to find Harrison's expected value (average score) for the year.

Solution

a Adding frequencies gives a total of 30.

Score	Frequency	Relative frequency	Score × relative frequency
5	7	$\frac{7}{30}$	$5 \times \frac{7}{30} = \frac{35}{30}$
6	9	$\frac{9}{30}$	$6 \times \frac{9}{30} = \frac{54}{30}$
7	8	$\frac{8}{30}$	$7 \times \frac{8}{30} = \frac{56}{30}$
8	3	$\frac{3}{30}$	$8 \times \frac{3}{30} = \frac{24}{30}$
9	2	$\frac{2}{30}$	$9 \times \frac{2}{30} = \frac{18}{30}$
10	1	$\frac{1}{30}$	$10 \times \frac{1}{30} = \frac{10}{30}$

b Expected value $= \frac{35}{30} + \frac{54}{30} + \frac{56}{30} + \frac{24}{30} + \frac{18}{30} + \frac{10}{30}$

$= \frac{197}{30}$

$= 6.5666\ldots$

$= 6.57$

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INVESTIGATION

MEAN

Find the mean in Example 9 using the formula $\bar{x} = \frac{\Sigma fx}{\Sigma f}$. Can you see why the sum of scores multiplied by relative frequencies also gives this mean?

Expected value

$$E(X) = \mu = \Sigma xp(x)$$

The symbol Σ means 'the sum of'. It is the Greek capital letter 'sigma'.

$\Sigma xp(x)$ is the sum of the products of x times $p(x)$.

Proof

$$\mu = \frac{\Sigma fx}{\Sigma f}$$

$$= \frac{\Sigma fx}{n} \text{ where } n \text{ is the sum of frequencies}$$

$$= \sum x \frac{f}{n}$$

$$= \Sigma xp(x)$$

EXAMPLE 10

a Find the expected value of this discrete probability distribution.

x	1	2	3	4	5
$P(X = x)$	0.1	0.3	0.2	0.1	0.3

b In a game of chance, Bethany tosses 2 coins. She wins \$10 for 2 heads, \$5 for 2 tails and nothing for a head and a tail.

i Find the expected value of this game.

ii If the game costs \$5 to play, would Bethany expect to win or lose money in the long term?

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Solution

a $E(X) = \sum xp(x)$
$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.1 + 5 \times 0.3$
$= 3.2$

b **i** We can make $X = \{0, 1, 2\}$ where 0, 1 and 2 is the number of heads.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The game pays different amounts of money for different outcomes.

We can use Y as the random variable for the payout of money in dollars.

$Y = \{0, 5, 10\}$

1 head earns \$0, 0 heads earn \$5, 2 heads earn \$10.

y	\$0	\$5	\$10
$P(Y = y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

$E(Y) = \sum yp(y)$
$= \$0 \times \frac{1}{2} + \$5 \times \frac{1}{4} + \$10 \times \frac{1}{4}$
$= \$3.75$

ii Since the game costs \$5 to play and the expected outcome is \$3.75, Bethany would expect to lose money in the long term.

You can find the expected value on your calculator using the statistics mode, in the same way you would find the mean of data presented in a frequency table.

EXAMPLE 11

Find the expected value of this discrete probability distribution.

x	0	2	4	6	8
$P(X = x)$	0.14	0.13	0.25	0.36	0.12

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Solution

Operation	Casio scientific	Sharp scientific
Place your calculator in statistical mode:	MODE STAT 1-VAR SHIFT MODE scroll down to STAT Frequency? ON	MODE STAT =
Clear the statistical memory:	SHIFT 1 Edit, Del-A	2ndF DEL
Enter data:	SHIFT 1 Data to get table. 0 = 2 = etc. to enter in x column. 0.14 = 0.13 = etc. to enter in FREQ column. AC to leave table.	0 2ndF STO 0.14 M+ 2 2ndF STO 0.13 M+ etc.
Calculate mean: ($\bar{x} = 4.38$)	SHIFT VAR $\bar{x}$ =	RCL $\bar{x}$
Check the number of scores: ($n = 1$)	SHIFT VAR n =	RCL n
Change back to normal mode:	MODE COMP	MODE 0

Expected value $E(X) = 4.38$.

You can solve problems using expected values.

EXAMPLE 12

a The probability of selling a red car, based on previous experience, is 35%. Find the expected number of red cars sold in one week if a dealer sells 2 cars.

b For the probability function below, evaluate a and b given that $E(X) = 2$.

x	1	2	3	4
$P(X = x)$	a	b	0.2	0.1

Solution

a $P(\text{R}) = 35\%$

$= 0.35$

So $P(\text{not R}) = P(\bar{\text{R}})$

$= 1 - 0.35$

$= 0.65$

$P(X = 0) = P(\bar{\text{R}}\,\bar{\text{R}})$

$= 0.65 \times 0.65$

$= 0.4225$

$P(X = 1) = P(\bar{\text{R}}\,\text{R}) + P(\text{R}\bar{\text{R}})$

$= 0.65 \times 0.35 + 0.35 \times 0.65$

$= 0.455$

$P(X = 2) = P(\text{RR})$

$= 0.35 \times 0.35$

$= 0.1225$

x	0	1	2
$P(X = x)$	0.4225	0.455	0.1225

$E(X) = \sum xp(x)$

$= 0 \times 0.4225 + 1 \times 0.455 + 2 \times 0.1225$

$= 0.7$

So it is expected that when 2 cars are sold, 0.7 of them will be red.

Rounded to the nearest whole number, we could expect around 1 car to be red.

Note: This answer would be more meaningful for a much larger number of sales!

b $E(X) = \sum xp(x)$

$2 = 1 \times a + 2 \times b + 3 \times 0.2 + 4 \times 0.1$

$2 = a + 2b + 1$

$a + 2b = 1$ [1]

Since the function is a probability distribution:

$a + b + 0.2 + 0.1 = 1$

$a + b + 0.3 = 1$

$a + b = 0.7$ [2]

[1] − [2]: $b = 0.3$

Substitute into [2]:

$a + 0.3 = 0.7$

$a = 0.4$

So $a = 0.4$, $b = 0.3$.

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Exercise 10.03 Mean or expected value

1 Find the expected value of each probability distribution.

a $\left(0, \frac{1}{4}\right), \left(1, \frac{1}{2}\right), \left(2, \frac{1}{4}\right)$

b

x	1	2	3	4	5
$P(X = x)$	0.31	0.16	0.15	0.2	0.18

c

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{16}$

d $p(x) = \frac{x+1}{8}$ for $x = 0, 1, 4$

e
$$p(x) = \begin{cases} \frac{x}{2} & \text{for } x = 1 \\ \frac{x}{8} & \text{for } x = 2 \\ \frac{x-3}{4} & \text{for } x = 4 \\ 0 & \text{for all other values of } x \end{cases}$$

2 For each question:

i evaluate k **ii** find the mean of the probability distribution.

a $(1, k), \left(2, \frac{1}{5}\right), \left(3, \frac{3}{10}\right), \left(4, \frac{2}{5}\right)$

b $p(x) = k(x + 3)$ for $x = 0, 1, 2$

c

x	1	2	3	4	5	6
$P(X = x)$	0.1	0.02	0.17	0.24	k	0.32

3 Find the expected value of each probability distribution:

a The number of heads when tossing 2 coins

b The sum of the 2 numbers rolled on a pair of dice

c The number of girls in a 3-child family

d The number of faulty cars when testing 3 cars if 1 in every 1000 cars is faulty

e The number of red counters when 2 counters are selected at random from a bag containing 7 red and 12 white counters:

i with replacement

ii without replacement.

4 The expected value $E(X) = 6.35$ for this probability function. Find p and q.

x	3	7	8	9
$p(x)$	p	0.25	0.35	q

5 The mean of the probability distribution below is $3\frac{3}{4}$. Evaluate a and b.

$(1, a), \left(2, \frac{1}{8}\right), (3, b), \left(4, \frac{1}{4}\right), \left(5, \frac{3}{8}\right)$

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4$.

a Draw up a probability distribution table for X.

b Find $E(X)$.

7 Find the expected number of heads when tossing 3 coins.

8 A bag contains 8 white and 3 yellow marbles. If 3 marbles are selected at random, find the mean number of white marbles:

a with replacement

b without replacement.

9 In a game, 2 dice are rolled and the difference between the 2 numbers is calculated. A player wins \$1 if the difference is 3, \$2 if it is 4 and \$3 if it is 5.

a Draw a probability distribution table for the winning values.

b Find the expected value.

c It costs \$1 for a player to roll the dice. How much would the player be expected to win or lose?

10 Staff at a call centre must make at least 1 phone sale every hour. The probability that Yasmin will make a sale on a phone call is $\frac{2}{5}$. She makes 4 phone calls in an hour.

a Draw a probability distribution for the number of sales Yasmin makes.

b Find the expected value.

c Will Yasmin make at least one phone sale in an hour?

11 A game uses a spinner with the numbers 1 to 12 equally spread around it. A player wins \$3 for spinning a number greater than 10, \$2 for a number less than 4 and loses \$1 for any other number. How much money would a player be expected to win or lose?

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10.04 Variance and standard deviation



Variance and standard deviation


Variance and standard deviation of a discrete random variable

Variance and **standard deviation** measure the spread of data in a distribution by finding the difference of each value from the mean. Variance is the square of the standard deviation.

The variance, σ^2, involves the average of the squared differences of each value from the mean.

σ is the Greek lower-case letter 'sigma'.

Variance, σ^2

$$Var(X) = \Sigma(x-\mu)^2 p(x)$$
$$= E[(X-\mu)^2]$$

Proof

$$\sigma^2 = \frac{\Sigma f(x-\mu)^2}{\Sigma f}$$
$$= \frac{\Sigma f(x-\mu)^2}{n} \quad \text{where } n = \Sigma f \text{ is the sum of frequencies}$$
$$= \Sigma(x-\mu)^2 \times \frac{f}{n}$$
$$= \Sigma(x-\mu)^2 p(x)$$
$$= E[(X-\mu)^2]$$

Standard deviation is the square root of variance.

Standard deviation, σ

$$\sigma = \sqrt{\Sigma(x-\mu)^2 p(x)}$$
$$= \sqrt{E[(X-\mu)^2]}$$

We use s for the standard deviation of a **sample** and σ (the lower-case Greek letter sigma) for the standard deviation of a **population**. For probability distributions, we use the population standard deviation, σ. The sample standard deviation, s, is an estimate of σ, and as the sample size increases, the sample represents the population better and the value of s approaches σ.

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EXAMPLE 13

Find the variance and standard deviation of this probability distribution.

x	1	2	3	4
$P(X = x)$	0.16	0.32	0.42	0.1

Solution

$E(X) = \sum xp(x)$

$= 1 \times 0.16 + 2 \times 0.32 + 3 \times 0.42 + 4 \times 0.1$

$= 2.46$

$Var(X) = \sum (x - \mu)^2 p(x)$

$= (1 - 2.46)^2 0.16 + (2 - 2.46)^2 0.32 + (3 - 2.46)^2 0.42 + (4 - 2.46)^2 0.1$

$= 0.7684$

Standard deviation:

$\sigma = \sqrt{0.7684}$

$= 0.8765\ldots$

≈ 0.8766

The formula for variance is a little tedious since we subtract the mean from every value. There is a simpler formula for variance.

Calculation formulas for variance and standard deviation

$$Var(X) = \sum [x^2 p(x)] - \mu^2$$
$$= E(X^2) - \mu^2$$
$$\sigma = \sqrt{Var(X)}$$

Proof

$\sigma^2 = \sum (x - \mu)^2 p(x)$

$= \sum [x^2 p(x) - 2\mu x p(x) + \mu^2 p(x)]$ expanding $(x - \mu)^2$

$= \sum x^2 p(x) - \sum 2\mu x p(x) + \sum \mu^2 p(x)$ taking separate sums of each part

$= \sum x^2 p(x) - 2\mu \sum x p(x) + \mu^2 \sum p(x)$ since μ is a constant

$= \sum x^2 p(x) - 2\mu \times \mu + \mu^2 \times 1$ since $\sum xp(x) = \mu$ and $\sum p(x) = 1$ is a constant

$= \sum [x^2 p(x)] - \mu^2$

$= E(X^2) - \mu^2$

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If we use the same probability distribution as Example 13, we can see that this formula gives us the same result.

EXAMPLE 14

Use the simpler calculation formulas to find the variance and standard deviation of this probability distribution.

x	1	2	3	4
$P(X = x)$	0.16	0.32	0.42	0.1

Solution

$$\mu = E(X) = \Sigma xp(x)$$
$$= 1 \times 0.16 + 2 \times 0.32 + 3 \times 0.42 + 4 \times 0.1$$
$$= 2.46$$
$$Var(X) = \Sigma[x^2 p(x)] - \mu^2$$
$$= (1)^2\ 0.16 + (2)^2\ 0.32 + (3)^2\ 0.42 + (4)^2\ 0.1 - 2.46^2$$
$$= 0.7684$$

Standard deviation:

$$\sigma = \sqrt{0.7684}$$
$$= 0.8765\ldots$$
$$\approx 0.8766$$

You can use a calculator to work out the variance and standard deviation.

EXAMPLE 15

Find the expected value, standard deviation and variance of this probability distribution.

x	1	2	3	4	5
$P(X = x)$	0.1	0.25	0.2	0.35	0.1

Solution

Operation	Casio scientific	Sharp scientific
Place your calculator in statistical mode:	MODE STAT 1-VAR SHIFT MODE scroll down to STAT Frequency? ON	MODE STAT =
Clear the statistical memory:	SHIFT 1 Edit, Del-A	2ndF DEL
Enter data:	SHIFT 1 Data to get table. 1 = 2 = etc. to enter in x column. 0.1 = 0.25 = etc to enter in FREQ column. AC to leave table.	1 2ndF STO 0.1 M+ 2 2ndF STO 0.25 M+ etc.
Calculate mean: ($\bar{x} = 3.1$)	SHIFT Var $\bar{x}$ =	RCL $\bar{x}$
Calculate the standard deviation: ($\sigma_x = 1.1789\ldots$)	SHIFT Var σ_x =	RCL σ_x
Change back to normal mode:	MODE COMP	MODE 0

Mean $\mu = 3.1$

Standard deviation $\sigma \approx 1.18$

Variance $\sigma^2 = 1.1789\ldots^2$

≈ 1.39

Exercise 10.04 Variance and standard deviation

In this exercise, round answers to 2 decimal places where necessary.

1 For each probability distribution, find:

i the standard deviation **ii** the variance.

a

x	1	2	3	4	5	6
$P(X = x)$	0.17	0.24	0.12	0.13	0.23	0.11

b

x	0	1	2	3
$P(X = x)$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{1}{14}$	$\frac{5}{14}$

c $\left(1, \frac{3}{8}\right), \left(2, \frac{1}{4}\right), \left(3, \frac{1}{8}\right), \left(4, \frac{1}{16}\right), \left(5, \frac{3}{16}\right)$

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2 Find the mean, variance and standard deviation of each probability function.

a

x	1	4	7	9	10
$p(x)$	0.09	0.18	0.26	0.32	0.15

b $P(x) = \frac{x+1}{9}$ for $x = 0, 2, 4$

3 Evaluate n and find the expected value and variance for this probability distribution.

x	1	2	3	4	5
$p(x)$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	n

4 For the probability distribution below with $E(X) = 3.32$, find:

x	1	2	3	4	5
$p(x)$	a	b	0.17	0.2	0.3

a the values of a and b

b the variance

c the standard deviation.

5 For each probability distribution find:

i the expected value

ii the standard deviation

iii the variance.

a The number of tails when tossing 3 coins.

b The number of blue marbles when 2 marbles are selected randomly from a bag containing 10 blue and 12 white marbles.

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4, 5$. Find:

a the mean

b the standard deviation

c the variance.

7 **a** Create a probability distribution table for the number of 6s rolled on a pair of dice.

b Find the mean, variance and standard deviation of this function.

8 The probability of selecting a black jelly bean at random from a packet is 4%. If 2 jelly beans are selected at random, find:

a the expected number of black jelly beans

b the standard deviation

c the variance.

9 A set of cards contains 5 blue and 7 white cards. If 3 are drawn out at random, the discrete random variable X is the number of blue cards drawn out. Find the mean and variance of X if the cards are drawn out:

a with replacement

b without replacement

10 In a game, 2 cards are drawn from a deck of 52 standard playing cards. A player wins 5 points if one of the cards is an ace and 10 points for double aces.

a If random variable X is the number of aces drawn:

i create a probability distribution for X

ii find the mean, variance and standard deviation for this distribution.

b If random variable Y is the number of points won:

i create a probability distribution for Y

ii find the mean, variance and standard deviation for this distribution.

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10. TEST YOURSELF

Practice quiz

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

1 The table shows a discrete probability distribution:

x	1	2	3	4	5	6
$P(X = x)$	0.24	0.16	0.08	0.14	0.21	0.17

Find $P(X \geq 2)$.

A 0.4 **B** 0.52 **C** 0.76 **D** 0.48

2 The expected value of the probability distribution below is:

x	1	2	3	4
$p(x)$	0.6	0.1	0.2	0.1

A $0.6 + 0.1 + 0.2 + 0.1$ **B** $\frac{1}{4}(1 \times 0.6 + 2 \times 0.1 + 3 \times 0.2 + 4 \times 0.1)$

C $\frac{1+2+3+4}{4}$ **D** $1 \times 0.6 + 2 \times 0.1 + 3 \times 0.2 + 4 \times 0.1$

3 The value of t in the probability distribution is:

x	1	2	3	4	5	6
$p(x)$	0.28	0.16	0.04	0.1	t	0.25

A 0.13 **B** 0.17 **C** 0.27 **D** 0.07

4 Which table represents a probability function?

A

x	$f(x)$
1	0.6
2	0.2
3	0.1
4	0.2

B

x	$f(x)$
1	0.3
2	0.25
3	0.1
4	0.25

C

x	$f(x)$
1	0.15
2	0.25
3	0.3
4	0.4

D

x	$f(x)$
1	0.25
2	0.4
3	0.15
4	0.2

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5 For each random variable, write the set of possible values.

a The number of 6s when rolling a die 5 times

b The number of heads when tossing a coin 10 times

c The first day the temperature rises above 28° in November

d The number of doubles when rolling 2 dice twice

e The number of red cards selected in 9 trials when pulling a card from a hat that contains 20 red and 20 blue cards.

6 This table shows a discrete probability distribution. Evaluate k.

x	0	1	2	3	4
$P(X = x)$	$5k$	$3k$	$4k - 1$	$2k - 3$	$6k$

7 A probability function is given by $p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$. Find its mean, variance and standard deviation.

8 The table represents a probability distribution.

x	4	7	8	9
$P(X = x)$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{12}$

Find:

a $P(X = 9)$ **b** $P(X < 8)$ **c** $P(X \geq 7)$

d $P(4 \leq X \leq 8)$ **e** $P(7 < X \leq 9)$

9 Draw a discrete probability distribution table for the number of tails when tossing 2 coins.

10 State whether each probability distribution is uniform.

a Number of tails when tossing a coin

b Number of heads when tossing 2 coins

c The number rolled on a die

d Number of 6s when rolling a die

11 State whether each random variable is discrete or continuous.

a The number of heads when tossing 5 coins

b The distances between cars parked in the street

c The number of correct answers in an exam

d The masses of babies

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12 Find the expected value, variance and standard deviation for this probability distribution.

x	0	1	2	3	4
$P(X = x)$	31%	22%	18%	24%	5%

13 A spinner has the numbers 1 to 7 evenly spaced around it.

a Draw a probability distribution table for the spinner.

b Is it a uniform distribution?

c Find the probability of spinning a number:

i greater than 5 **ii** 3 or less **iii** at least 4

d Find the expected value of the spinner.

14 A function is given by:

$$f(x) = \begin{cases} \dfrac{x-1}{10} & \text{for } x = 3 \\ \dfrac{x-4}{5} & \text{for } x = 5 \\ \dfrac{x}{15} & \text{for } x = 9 \end{cases}$$

a Find:

i $f(3)$ **ii** $f(5)$ **iii** $f(9)$

b Show that $f(x)$ is a probability function.

15 **a** Construct a probability distribution table for the number of tails when tossing 2 coins.

b Is it a uniform distribution?

c Find the probability of tossing:

i one tail **ii** at least one tail.

16 State whether each function is a probability function.

a

x	1	2	3	4	5	6
$f(x)$	0.2	0.07	0.15	0.2	0.3	0.08

b $f(x) = \dfrac{x+1}{6}$ for $x = 0, 1, 2, 3$

c $\left(0, \frac{1}{8}\right), \left(1, \frac{1}{4}\right), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{16}\right), \left(4, \frac{3}{16}\right)$

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17 In a game, Jonas pays \$1 to toss 3 coins together. He wins \$1.50 for 3 heads and \$2 for 3 tails.

a Find the expected value for this game.

b How much would you expect Jonas to win or lose in the long term?

18 **a** Show that the points (3, 21%), (5, 14%), (6, 47%) and (9, 18%) represent a discrete probability function.

b Find $E(X)$ and $Var(X)$.

19 Each table represents a probability distribution. Evaluate n.

a

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{8}$	n	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{1}{16}$

b

x	0	1	2	3
$P(X = x)$	0.27	0.51	0.14	n

c $P(x) = n(2x - 1)$ for $x = 1, 2, 3$

20 The table represents a probability distribution.

x	2	3	4	5	6
$P(X = x)$	0.2	0.2	0.3	a	b

If $E(X) = 3.8$, evaluate a and b.

21 A game involves tossing 2 coins. Tannika wins \$2 for 2 heads or 2 tails and loses \$1 for a head and a tail.

a Draw up a probability distribution table for random variable Y showing the winning amounts.

b If it costs Tannika \$1 to play, would you expect her to win or lose the game?

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10. CHALLENGE EXERCISE

1 For the probability distribution below, $E(X) = 2.94$ and $Var(X) = 2.2564$. Evaluate a, b and c.

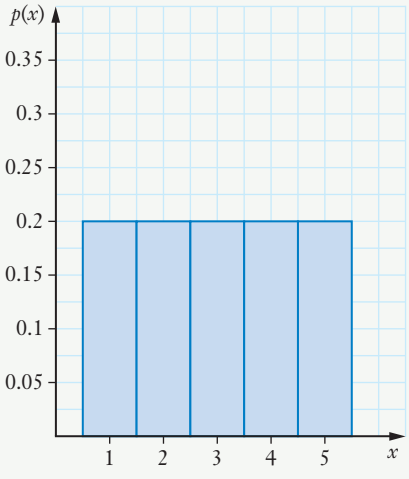
x	1	2	3	4	5
$p(x)$	a	b	c	0.16	0.24

2 The variance of the probability distribution $(1, a)$, $(2, 0.3)$, $(k, 0.4)$, $(5, 0.1)$ is 1.89 and the mean is 2.9. Evaluate a and k.

3 The probability distribution below has $E(X) = 3.34$ and $Var(X) = 4.3044$. Find the value of k and l.

x	1	3	k	6
$P(X = x)$	0.4	0.12	0.3	l

4



a Show that the graph above represents a discrete probability distribution.

b Is it a uniform distribution?

c Find:

i $P(X \leq 3)$ **ii** $P(X > 2)$ **iii** $P(1 \leq X < 5)$

d Find $E(X)$ and $Var(X)$.

e If this distribution changes so that $P(X = 1) = 0.35$, find $P(X = 2)$ if all the other probabilities remain the same.

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5 A sample of people were surveyed to rate a TV show on a scale of 1 to 5.

a How many people were surveyed?

b Draw a probability distribution table for the survey results.

c Is the sample mean from the survey a good estimate of the population mean of 2.5?

d Find the standard deviation. Is this a good estimate of the population standard deviation of 1?

e Can you explain these results from **c** and **d**?

Rating	Frequency
1	4
2	15
3	23
4	59
5	19

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Practice set 4

For Questions **1** to **5**, select the correct answer **A**, **B**, **C** or **D**.

1 Find the amplitude and period of $y = 5 \sin 3x$.

A Amplitude 3, period 5 **B** Amplitude 5, period 3

C Amplitude 5, period $\frac{2\pi}{3}$ **D** Amplitude 3, period $\frac{2\pi}{5}$

2 The table is a discrete probability distribution.

x	1	2	3	4	5	6
$P(X = x)$	0.14	0.16	0.08	0.14	0.31	0.17

Find $P(X \leq 4)$.

A 0.38 **B** 0.52 **C** 0.14 **D** 0.62

3 Find the exact value of $\sin 135° + \cos 120°$.

A $\frac{\sqrt{2} - \sqrt{3}}{2}$ **B** $\frac{\sqrt{2} + 1}{2}$

C $\frac{\sqrt{2} + \sqrt{3}}{2}$ **D** $\frac{\sqrt{2} - 1}{2}$

4 Which statement is the same as $3^x = 7$? There is more than one answer.

A $x = \log \frac{7}{3}$ **B** $\log_3 x = 7$

C $\log_3 7 = x$ **D** $x = \frac{\log 7}{\log 3}$

5 The derivative of $x^2(2x + 9)^2$ is:

A $4x(2x + 9)$ **B** $2x(2x + 9)^2 + 2x^2(2x + 9)$

C $2x(2x + 9)$ **D** $2x(2x + 9)^2 + 4x^2(2x + 9)$

6 Differentiate:

a $y = e^x - x$ **b** $y = 3e^x + 1$ **c** $y = (e^x - 2)^4$

d $y = e^x(4x + 1)^3$ **e** $y = \frac{e^x}{5x - 2}$ **f** $y = 5e^{7x}$

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7 A function is given by

$$f(x) = \begin{cases} \dfrac{x+1}{8} & \text{for } x = 0, 1, 2 \\ \dfrac{x-2}{4} & \text{for } x = 3 \end{cases}$$

a Find:

i $f(0)$ **ii** $f(3)$

b Show that $f(x)$ is a probability function.

8 Find $\log_5 \dfrac{1}{25}$.

9 The table represents a probability distribution.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$

Find:

a $P(X = 2)$ **b** $P(X < 4)$ **c** $P(X \geq 2)$

d $P(4 \leq X \leq 6)$ **e** $P(1 \leq X < 5)$

10 Simplify:

a $\tan(180° - \theta)$ **b** $\sin(-\theta)$ **c** $\cos(2\pi - \theta)$

11 For $0 \leq x \leq 2\pi$ sketch the graph of:

a $y = 2 \sin 4x$ **b** $y = \tan \dfrac{x}{2}$ **c** $y = -\cos x$

12 For each random variable X, write the set of possible values.

a The number of rolls of a die until a 6 turns up

b The number of red cards selected when choosing 12 cards from a bag containing 15 red and 15 black cards

c The first rainy day in January.

13 Solve $\log_x \dfrac{1}{16} = 4$.

14 The population of a city over t years is given by the formula $P = 100\,000e^{0.71t}$. After how many years, to 1 decimal place, will the population become 1 million?

15 A bag contains 7 white and 6 blue cards. Create a probability distribution table for the number of blue cards selected when randomly selecting 3 cards:

a with replacement **b** without replacement.

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16 If $\tan x = -\frac{4}{3}$ and $\cos x > 0$, evaluate $\sin x$ and $\cos x$.

17 Solve for $0 \leq x \leq 2\pi$:

a $2\cos x + 1 = 0$ **b** $\tan^2 x = 1$

c $\cos x = 0$ **d** $\sin 2x = \frac{1}{2}$

18 This table represents a probability distribution.

x	1	2	3	4	5
$P(X = x)$	0.16	0.23	0.22	a	b

If $E(X) = 3.04$, evaluate a and b.

19 Find the expected value, variance and standard deviation for the probability distribution below.

x	0	1	2	3	4
$P(X = x)$	0.2	0.1	0.3	0.1	0.3

20 Find the exact value of:

a $\cos \frac{7\pi}{4}$ **b** $\sin \frac{4\pi}{3}$ **c** $\tan \frac{5\pi}{6}$

21 Draw a discrete probability distribution table for the number of tails when tossing 3 coins.

22 Sketch the graph of:

a $y = \log_3 x$ **b** $y = 3\log_2 x - 1$

23 **a** Write $\log_e x$ as an equation with x in terms of y.

b Hence find the value of x, to 3 significant figures, when $y = 1.23$.

24 Solve $7^{2x} = 3$.

25 This table shows a discrete probability distribution. Evaluate k.

x	0	1	2	3	4
$P(X = x)$	$2k$	$3k$	$4k - 2$	$5k - 1$	$6k$

26 State whether each probability distribution is uniform.

a Number of heads when tossing 2 coins

b Number of heads when tossing a coin

c Number of even numbers when rolling one die

d Number of 1s when rolling one die.

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27 State whether each function is a probability function.

a $f(x) = \dfrac{x+1}{10}$ for $x = 0, 1, 2, 3$

b $f(x) = \begin{cases} \dfrac{x}{11} & \text{for } x = 1, 2 \\ \dfrac{x-1}{22} & \text{for } x = 3, 4, 5 \end{cases}$

28 Solve for $0° \leq x \leq 360°$:

a $\tan x = -1$ **b** $2 \sin x = 1$

c $2 \cos^2 x = 1$ **d** $\tan 2x = \sqrt{3}$

29 Evaluate, to 2 decimal places where appropriate.

a $\log_2 16$ **b** $\log_3 3$ **c** $\log_4 2$

d $\log_{10} 109.7$ **e** $\ln 43.1$ **f** $\log_3 11$

30 Sketch the graph of:

a $y = e^{-x}$

b $y = 2e^{3x} + 1$

31 The probability of winning a game is 65% and the probability of losing the game is 12%.

a Draw a probability distribution table showing 0 for a loss, 1 for a draw and 2 for a win.

b Find the expected value and variance.

32 Find the equation of the tangent to the curve $y = 5e^x$ at the point $(2, 5e^2)$.

33 In a game, Faizal pays \$1 to toss 2 coins. He wins \$2 for 2 heads or 2 tails and loses \$1 for a head and a tail.

a Find the expected value for this game.

b How much would you expect Faizal to win or lose in the long term?

34 A spinner has the numbers 1 to 8 equally placed around it.

a Draw a probability distribution table for the spinner.

b Is it a uniform distribution?

c Find the probability of spinning a number:

i greater than 4 **ii** 3 or less **iii** at least 4

d Find the expected value of the spinner.

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35 **a** Show that the points (1, 27%), (2, 31%), (3, 28%) and (4, 14%) represent a discrete probability function.

b Find $E(X)$ and $Var(X)$.

36 For the following probability distribution, evaluate k.

x	1	2	3	4	5	6
$p(x)$	$\frac{5}{16}$	k	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{1}{16}$	$\frac{1}{8}$

37 Simplify:

a $5 + 5\tan^2 x$ **b** $\dfrac{(1+\sin x)(1-\sin x)}{\sin x \cos x}$

38 Find the exact value of:

a $\tan 150°$ **b** $\cos(-45°)$ **c** $\sin 240°$

39 Find the value of x:

a $x^2 - 2x - 3 = 0$ **b** $1 < 2x - 3 \leq 7$ **c** $|3x + 1| = 4$

40 Find the centre and radius of the circle $x^2 - 4x + y^2 + 6y - 3 = 0$.

41 Amanda leaves home and cycles south for 3.6 km. She then turns and cycles for 5.4 km on a bearing of 243°.

a How far is Amanda from her house, to 1 decimal place?

b What is Amanda's bearing from her house, to the nearest degree?

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ANSWERS

Answers are based on full calculator values and only rounded at the end, even when different parts of a question require rounding. This gives more accurate answers. Answers based on reading graphs may not be accurate.

Chapter 1

Exercise 1.01

1 **a** 500 **b** 145 **c** $\frac{1}{64}$
d 3 **e** 2

2 **a** 13.7 **b** 1.1 **c** 0.8
d 2.7 **e** -2.6 **f** 0.5

3 **a** a^{17} **b** $y^0 = 1$ **c** a^{-4}
d w **e** x^5 **f** p^{10}
g y^6 **h** x^{21} **i** $4x^{10}$
j $81y^{-8}$ **k** a **l** $\frac{x^{10}}{y^{45}}$
m w^{10} **n** p^5 **o** x^{-3}
p $a^{-2}b^3$ or $\frac{b^3}{a^2}$ **q** $x^{-5}y^2$ or $\frac{y^2}{x^5}$

4 **a** x^{14} **b** a^{-7} **c** m^4
d k^{10} **e** a^{-8} **f** x
g mn^2 **h** p^{-1} **i** $9x^{22}$
j x^{21}

5 **a** p^5q^{15} **b** $\frac{a^8}{b^8}$ **c** $\frac{64a^3}{b^{12}}$
d $49a^{10}b^2$ **e** $8m^{17}$ **f** x^4y^{10}
g $\frac{2k^{23}}{27}$ **h** $16y^{47}$ **i** a^3
j $125x^{-21}y^{18}$

6 $4\frac{1}{2}$ **7** 324 **8** $2\frac{10}{27}$

9 **a** a^3b **b** $\frac{1}{25}$

10 **a** pq^2r^2 **b** $\frac{7}{32}$

11 $\frac{4}{9}$ **12** $\frac{1}{18}$ **13** $\frac{4}{27}$

14 $\frac{1}{81}$ **15** $\frac{1}{108}$ **16** $\frac{1}{12}$

Exercise 1.02

1 **a** $\frac{1}{27}$ **b** $\frac{1}{4}$ **c** $\frac{1}{343}$
d $\frac{1}{10\,000}$ **e** $\frac{1}{256}$ **f** 1
g $\frac{1}{32}$ **h** $\frac{1}{81}$ **i** $\frac{1}{7}$
j $\frac{1}{81}$ **k** $\frac{1}{64}$ **l** $\frac{1}{9}$
m 1 **n** $\frac{1}{36}$ **o** $\frac{1}{125}$
p $\frac{1}{100\,000}$ **q** $\frac{1}{128}$ **r** 1
s $\frac{1}{64}$ **t** $\frac{1}{64}$

2 **a** 1 **b** 16 **c** $1\frac{1}{2}$ **d** $1\frac{11}{25}$
e 1 **f** 125 **g** $1\frac{1}{3}$ **h** 49
i $3\frac{3}{8}$ **j** 32 **k** $2\frac{1}{3}$ **l** 1
m $1\frac{13}{36}$ **n** $1\frac{19}{81}$ **o** 1 **p** 16
q $-15\frac{5}{8}$ **r** $-\frac{7}{23}$ **s** 1 **t** $\frac{16}{25}$

3 **a** m^{-3} **b** x^{-1}
c p^{-7} **d** d^{-9}
e k^{-5} **f** x^{-2}
g $2x^{-4}$ **h** $3y^{-2}$
i $\frac{1}{2}z^{-6}$ or $\frac{z^{-6}}{2}$ **j** $\frac{3t^{-8}}{5}$

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k $\frac{2x^{-1}}{7}$ **l** $\frac{5m^{-6}}{2}$

m $\frac{2y^{-7}}{3}$ **n** $(3x+4)^{-2}$

o $(a+b)^{-8}$ **p** $(x-2)^{-1}$

q $(5p+1)^{-3}$ **r** $2(4t-9)^{-5}$

s $\frac{(x+1)^{-11}}{4}$ **t** $\frac{5(a+3b)^{-7}}{9}$

4 a $\frac{1}{t^5}$ **b** $\frac{1}{x^6}$ **c** $\frac{1}{y^3}$

d $\frac{1}{n^8}$ **e** $\frac{1}{w^{10}}$ **f** $\frac{2}{x}$

g $\frac{3}{m^4}$ **h** $\frac{5}{x^7}$ **i** $\frac{1}{8x^3}$

j $\frac{1}{4n}$ **k** $\frac{1}{(x+1)^6}$ **l** $\frac{1}{8y+z}$

m $\frac{1}{(k-3)^2}$ **n** $\frac{1}{(3x+2y)^9}$ **o** x^5

p y^{10} **q** $\frac{p}{2}$ **r** $(a+b)^2$

s $\frac{x-y}{x+y}$ **t** $\left(\frac{3x+y}{2w-z}\right)^7$

Exercise 1.03

1 a 9 **b** 3 **c** 4 **d** 2

e 7 **f** 10 **g** 2 **h** 8

i 4 **j** 1 **k** 3 **l** 2

m 0 **n** 5 **o** 7 **p** 2

q 4 **r** 25 **s** 32 **t** 4

u 27 **v** $\frac{1}{2}$ **w** $\frac{1}{3}$ **x** $\frac{1}{2}$

y $\frac{1}{16}$

2 a 2.19 **b** 2.60 **c** 1.53 **d** 0.60

e 0.90 **f** 0.29

3 a $\sqrt[3]{y}$ **b** $\sqrt[6]{x}$ **c** $\sqrt{a}$

d $\sqrt[9]{t}$ **e** $\sqrt[3]{y^2}$ or $\left(\sqrt[3]{y}\right)^2$ **f** $\sqrt[4]{x^3}$

g $\sqrt[5]{b^2}$ **h** $\sqrt[7]{a^4}$ **i** $\frac{1}{\sqrt{x}}$

j $\frac{1}{\sqrt[3]{d}}$ **k** $\frac{1}{\sqrt[8]{x}}$ **l** $\frac{1}{\sqrt[3]{y}}$

m $\frac{1}{\sqrt[4]{a}}$ **n** $\frac{1}{\sqrt[4]{z^3}}$ **o** $\frac{1}{\sqrt[5]{y^3}}$

p $\sqrt{2x+5}$ **q** $\sqrt[3]{6q+r}$ **r** $\sqrt[9]{a+b}$

s $\frac{1}{\sqrt{3x-1}}$ **t** $\frac{1}{\sqrt[5]{(x+7)^2}}$ or $\frac{1}{\left(\sqrt[5]{x+7}\right)^2}$

4 a $t^{\frac{1}{2}}$ **b** $y^{\frac{1}{5}}$ **c** $x^{\frac{3}{2}}$

d $(9-x)^{\frac{1}{3}}$ **e** $(4s+1)^{\frac{1}{2}}$ **f** $(3x+1)^{\frac{5}{2}}$

g $(2t+3)^{-\frac{1}{2}}$ **h** $(5x-y)^{-\frac{3}{2}}$ **i** $(x-2)^{-\frac{2}{3}}$

j $\frac{1}{2}(y+7)^{-\frac{1}{2}}$ **k** $5(x+4)^{-\frac{1}{3}}$ **l** $\frac{2}{3}(y^2-1)^{-\frac{1}{2}}$

m $\frac{3}{5}(x^2+2)^{-\frac{3}{4}}$

5 a $x^{\frac{3}{2}}$ **b** $x^{-\frac{1}{2}}$ **c** $x^{\frac{2}{3}}$

d $x^{\frac{5}{3}}$ **e** $x^{\frac{5}{4}}$

6 a $\frac{1}{\sqrt[3]{a-2b}}$ **b** $\frac{1}{\sqrt[3]{(y-3)^2}}$ **c** $\frac{4}{\sqrt[7]{(6a+1)^4}}$

d $\frac{1}{3\sqrt[4]{(x+y)^5}}$ **e** $\frac{6}{7\sqrt[9]{(3x+8)^2}}$

Exercise 1.04

1 a $3a$ **b** z **c** $3b$

d $-3r$ **e** $-y$ **f** $-5x$

g 0 **h** $3k$ **i** $9t$

j $10w$ **k** $-m$ **l** $-x$

m 0 **n** $11b$ **o** $-10x$

p $6x-6y$ **q** $a-3b$ **r** $4xy+2y$

s $-6ab^2$ **t** $m^2-6m+12$

u p^2-2p-6 **v** $-2ab+10b$

w $2bc-ac$ **x** $2a^5-9x^3+1$

y $x^3-2xy^2+3x^2y+2y^3$

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2 **a** $10b$ **b** $8xy$ **c** $10p^2$
d $-6wz$ **e** $15ab$ **f** $14xyz$
g $48abc$ **h** $12d^2$ **i** $12a^3$
j $-27y^3$ **k** $32x^{10}$ **l** $6a^2b^3$
m $-10a^3b^2$ **n** $21p^3q^4$ **o** $5a^3b^3$
p $-8h^{10}$ **q** k^3p^3 **r** $81t^{12}$
s $-14m^{11}$ **t** $24x^6y^3$

3 **a** $6x$ **b** 2 **c** $4a^2$
d $8a$ **e** $4a$ **f** $\frac{y}{2}$
g $3p$ **h** $\frac{ab}{2}$ **i** $\frac{4}{3y}$
j $-3x^3$ **k** $3a$ **l** $\frac{1}{3ab^2}$
m $\frac{-2}{qs}$ **n** $\frac{2}{3c^2d}$ **o** $\frac{z^2}{2x^2}$
p $6p^4q$ **q** $\frac{a^4b^7}{4c}$ **r** $\frac{b^6}{2a}$
s $-\frac{x^3z^3}{3y}$ **t** $\frac{a^{13}}{2b^6}$

Exercise 1.05

1 $2x - 8$
2 $6h + 9$
3 $-5a + 10$
4 $2xy + 3x$
5 $x^2 - 2x$
6 $6a^2 - 16ab$
7 $2a^2b + ab^2$
8 $5n^2 - 20n$
9 $3x^3y^2 + 6x^2y^3$
10 $4k + 7$
11 $2t - 17$
12 $4y^2 + 11y$
13 $-5b - 6$
14 $8 - 2x$
15 $-3m + 1$
16 $8h - 19$
17 $d - 6$
18 $a^2 - 2a + 4$
19 $3x^2 - 9x - 5$
20 $2ab - 2a^2b + b$
21 $4x - 1$
22 $-7y + 4$
23 $2b$
24 $5t - 6$

Exercise 1.06

1 $a^2 + 7a + 10$
2 $x^2 + 2x - 3$
3 $2y^2 + 7y - 15$
4 $m^2 - 6m + 8$
5 $x^2 + 7x + 12$
6 $y^2 - 3y - 10$
7 $2x^2 + x - 6$
8 $h^2 - 10h + 21$
9 $x^2 - 25$
10 $15a^2 - 17a + 4$
11 $8y^2 + 6y - 9$
12 $xy + 7x - 4y - 28$
13 $x^3 - 2x^2 + 3x - 6$
14 $n^2 - 4$
15 $4x^2 - 9$
16 $16 - 49y^2$
17 $a^2 - 4b^2$
18 $9x^2 - 16y^2$
19 $x^2 - 9$
20 $y^2 - 36$
21 $9a^2 - 1$
22 $4z^2 - 49$
23 $x^2 - 2xy + 11x - 18y + 18$
24 $2ab + 2b^2 - 7b - 6a + 3$
25 $x^3 + 8$
26 $a^3 - 27$
27 $a^2 + 18a + 81$
28 $k^2 - 8k + 16$
29 $x^2 + 4x + 4$
30 $y^2 - 14y + 49$
31 $4x^2 + 12x + 9$
32 $4t^2 - 4t + 1$
33 $9a^2 + 24ab + 16b^2$
34 $x^2 - 10xy + 25y^2$
35 $4a^2 + 4ab + b^2$
36 $a^2 - b^2$
37 $a^2 + 2ab + b^2$
38 $a^2 - 2ab + b^2$
39 $a^3 + b^3$
40 $a^3 - b^3$

Exercise 1.07

1 $t^2 + 8t + 16$
2 $z^2 - 12z + 36$
3 $x^2 - 2x + 1$
4 $y^2 + 16y + 64$
5 $q^2 + 6q + 9$
6 $k^2 - 14k + 49$
7 $n^2 + 2n + 1$
8 $4b^2 + 20b + 25$
9 $9 - 6x + x^2$
10 $9y^2 - 6y + 1$
11 $x^2 + 2xy + y^2$
12 $9a^2 - 6ab + b^2$
13 $16d^2 + 40de + 25e^2$
14 $t^2 - 16$
15 $x^2 - 9$
16 $p^2 - 1$
17 $r^2 - 36$
18 $x^2 - 100$
19 $4a^2 - 9$
20 $x^2 - 25y^2$
21 $16a^2 - 1$
22 $49 - 9x^2$
23 $x^4 - 4$
24 $x^4 + 10x^2 + 25$
25 $9a^2b^2 - 16c^2$
26 $x^2 + 4 + \frac{4}{x^2}$
27 $a^2 - \frac{1}{a^2}$
28 $x^2 - y^2 + 4y - 4$
29 $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$
30 $x^2 + 2x + 1 - 2xy - 2y + y^2$
31 $12a$
32 $32 - z^2$
33 $9x^2 + 8x - 3$
34 $x^2 + 3xy + y^2 - 2x$
35 $14n^2 - 4$
36 $x^3 - 12x^2 + 48x - 64$
37 x^2
38 $x^4 - 2x^2y^2 + y^4$
39 $8a^3 + 60a^2 + 150a + 125$

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Exercise 1.08

1 $2(y+3)$
2 $5(x-2)$
3 $3(m-3)$
4 $2(4x+1)$
5 $6(4-3y)$
6 $x(x+2)$
7 $m(m-3)$
8 $2y(y+2)$
9 $3a(5-a)$
10 $ab(b+1)$
11 $2xy(2x-1)$
12 $3mn(n^2+3)$
13 $2xz(4x-z)$
14 $a(6b+3-2a)$
15 $x(5x-2+y)$
16 $q^2(3q^3-2)$
17 $5b^2(b+3)$
18 $3a^2b^2(2b-a)$
19 $(m+5)(x+7)$
20 $(y-1)(2-y)$
21 $(7+y)(4-3x)$
22 $(a-2)(6x+5)$
23 $(2t+1)(x-y)$
24 $(3x-2)(a+2b-3c)$
25 $3x^2(2x+3)$
26 $3q^3(pq^2-2)$
27 $3ab(5a^3b^2+1)$
28 $4x^2(x-6)$
29 $5m^2n(7mn^3-5)$
30 $8ab^2(3ab^3+2)$
31 $2\pi r(r+h)$
32 $(x-3)(x+2)$
33 $(x+4)(y^2+2)$
34 $-(a+1)$

Exercise 1.09

1 $(x+4)(2+b)$
2 $(y-3)(a+b)$
3 $(x+5)(x+2)$
4 $(m-2)(m+3)$
5 $(d-c)(a+b)$
6 $(x+1)(x^2+3)$
7 $(5a-3)(b+2)$
8 $(2y-x)(x+y)$
9 $(y+1)(a+1)$
10 $(x+5)(x-1)$
11 $(y+3)(1+a)$
12 $(m-2)(1-2y)$
13 $(x+5y)(2x-3y)$
14 $(a+b^2)(ab-4)$
15 $(5-x)(x+3)$
16 $(x+7)(x^3-4)$
17 $(x-3)(7-y)$
18 $(d+3)(4-e)$
19 $(x-4)(3+y)$
20 $(a+3)(2-b)$
21 $(x-3)(x^2+6)$
22 $(q-3)(p+q)$
23 $(x-2)(3x^2-5)$
24 $(a-3b)(4+c)$
25 $(y+7)(x-4)$
26 $(x-4)(x^3-5)$
27 $(2x-3)(2x^2+4)=2(2x-3)(x^2+2)$
28 $3(a+2b)(a+3)$
29 $5(y-3)(1+2x)$
30 $(r+2)(\pi r-3)$

Exercise 1.10

1 $(x+3)(x+1)$
2 $(y+4)(y+3)$
3 $(m+1)^2$
4 $(t+4)^2$
5 $(z+3)(z-2)$
6 $(x+1)(x-6)$
7 $(v-3)(v-5)$
8 $(t-3)^2$
9 $(x+10)(x-1)$
10 $(y-7)(y-3)$
11 $(m-6)(m-3)$
12 $(y+12)(y-3)$
13 $(x-8)(x+3)$
14 $(a-2)^2$
15 $(x-2)(x+16)$
16 $(y+4)(y-9)$
17 $(n-6)(n-4)$
18 $(x-5)^2$
19 $(p+9)(p-1)$
20 $(k-2)(k-5)$
21 $(x+4)(x-3)$
22 $(m-7)(m+1)$
23 $(q+10)(q+2)$
24 $(d-5)(d+1)$

Exercise 1.11

1 $(2a+1)(a+5)$
2 $(5y+2)(y+1)$
3 $(3x+7)(x+1)$
4 $(3x+2)(x+2)$
5 $(2b-3)(b-1)$
6 $(7x-2)(x-1)$
7 $(3y-1)(y+2)$
8 $(2x+3)(x+4)$
9 $(5p-2)(p+3)$
10 $(3x+5)(2x+1)$
11 $(2y+1)(y-6)$
12 $(5x-1)(2x+1)$
13 $(4t-1)(2t-3)$
14 $(3x+4)(2x-3)$
15 $(6y-1)(y+8)$
16 $(4n-3)(n-2)$
17 $(4t-1)(2t+5)$
18 $(3q+2)(4q+5)$
19 $(4r-1)(r+3)$
20 $(2x-5)(2x+3)$
21 $(6y-1)(y-2)$
22 $(2p-3)(3p+2)$
23 $(8x+7)(x+3)$
24 $(3b-4)(4b-9)$
25 $(6x+1)(x-9)$
26 $(3x+5)^2$
27 $(4y+3)^2$
28 $(5k-2)^2$
29 $(6a-1)^2$
30 $(7m+6)^2$

Exercise 1.12

1 $(y-1)^2$
2 $(x+3)^2$
3 $(m+5)^2$
4 $(t-2)^2$
5 $(x-6)^2$
6 $(2x+3)^2$
7 $(4b-1)^2$
8 $(3a+2)^2$
9 $(5x-4)^2$
10 $(7y+1)^2$
11 $(3y-5)^2$
12 $(4k-3)^2$
13 $(5x+1)^2$
14 $(9a-2)^2$
15 $(7m+6)^2$
16 $\left(t+\frac{1}{2}\right)^2$
17 $\left(x-\frac{2}{3}\right)^2$
18 $\left(3y+\frac{1}{5}\right)^2$
19 $\left(x+\frac{1}{x}\right)^2$
20 $\left(5k-\frac{2}{k}\right)^2$

Exercise 1.13

1 $(a+2)(a-2)$
2 $(x+3)(x-3)$
3 $(y+1)(y-1)$
4 $(x+5)(x-5)$
5 $(2x+7)(2x-7)$
6 $(4y+3)(4y-3)$
7 $(1+2z)(1-2z)$
8 $(5t+1)(5t-1)$

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9 $(3t+2)(3t-2)$ **10** $(3+4x)(3-4x)$
11 $(x+2y)(x-2y)$ **12** $(6x+y)(6x-y)$
13 $(2a+3b)(2-3b)$ **14** $(x+10y)(x-10y)$
15 $(2a+9b)(2a-9b)$ **16** $(x+2+y)(x+2-y)$
17 $(a+b-3)(a-b+1)$ **18** $(z+w+1)(z-w-1)$
19 $\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$ **20** $\left(\frac{y}{3}+1\right)\left(\frac{y}{3}-1\right)$
21 $(x+2y+3)(x-2y+1)$
22 $(x^2+1)(x-1)(x+1)$
23 $(3x^3+2y)(3x^3-2y)$
24 $(x^2+4y^2)(x+2y)(x-2y)$

Exercise 1.14

1 $4a(a+3)(a-3)$ **2** $2(x+3)(x-3)$
3 $3(p+3)(p-4)$ **4** $5(y+1)(y-1)$
5 $5(a-1)^2$ **6** $3z(z+5)(z+4)$
7 $ab(3+2ab)(3-2ab)$ **8** $x(x+1)(x-1)$
9 $2(3x-2)(x+2)$ **10** $(y+5)(y+4)(y-4)$
11 $x(x+1)(x-1)(x+8)$ **12** $(y^3+2)(y^3-2)$
13 $x(x+2)(x-5)$ **14** $(x+3)(x-3)^2$
15 $y(2xy+1)(2xy-1)$ **16** $6(2+b)(2-b)$
17 $3(3x-2)(2x+5)$ **18** $3(x-1)^2$
19 $(x+2)(x+5)(x-5)$ **20** $z(z+3)^2$
21 $3(y+5)^2$ **22** $a(b+3)(b-3)$
23 $4k(k+5)^2$ **24** $3(x+1)(x-1)(x+3)$
25 $2ab(a+2b)(2a-1)$

Exercise 1.15

1 $a+2$ **2** $2t-1$ **3** $\frac{4y+1}{3}$ **4** $\frac{4}{2d-1}$
5 $\frac{x}{5x-2}$ **6** $\frac{1}{y-4}$ **7** $\frac{2(b-2a)}{a-3}$ **8** $\frac{s-1}{s+3}$
9 b^2+1 **10** $\frac{p+5}{3}$ **11** $\frac{a+1}{a+3}$ **12** $\frac{3+y}{x+2}$
13 $x-3$ **14** $\frac{p-2}{p}$ **15** $\frac{a+b}{2a-b}$

Exercise 1.16

1 **a** $\frac{5x}{4}$ **b** $\frac{13y+3}{15}$ **c** $\frac{a+8}{12}$
d $\frac{4p+3}{6}$ **e** $\frac{x-13}{6}$

2 **a** 6 **b** $\frac{5b(a-2)}{3}$ **c** $\frac{2(t+5)}{5y}$
d $\frac{5(a-3)}{4}$ **e** $\frac{5-y}{35}$ **f** $\frac{b}{2a-1}$
g $\frac{b^2(x+2y)}{10(2b-1)}$ **h** $\frac{xy}{ab}$
i $\frac{(x-3)(x-1)}{(x-5)(x-2)}$ **j** $\frac{5(p-2)}{3(q+1)}$

3 **a** $\frac{5}{x}$ **b** $\frac{-x+2}{x(x-1)}$
c $\frac{a+b+3}{a+b}$ **d** $\frac{2x}{x+2}$
e $\frac{(p+q)(p-q)+1}{p+q}$ **f** $\frac{2(x-1)}{(x+1)(x-3)}$
g $\frac{-3x+8}{(x+2)(x-2)}$ **h** $\frac{a+2}{(a+1)^2}$

4 **a** $\frac{(y+2)(y+1)}{15y}$ **b** $\frac{x^2+10x-24}{2(x-3)(x-4)}$
c $\frac{3b^2-5b-10}{2b(b+1)}$ **d** x

5 **a** $\frac{3-5x}{(x+2)(x-2)}$ **b** $\frac{3p^2+5pq-2q^2}{pq(p+q)(p-q)}$
c $\frac{a^2-2ab-b^2+1}{(a+b)(a-b)}$

Exercise 1.17

1 **a** -7.1 **b** -6.9 **c** 48.1 **d** -37.7
e 0.6 **f** 2.3 **g** -5.3
2 $T=47$ **3** $y=-7$ **4** $h=375$
5 $v=-196$ **6** $y=5.5$ **7** $S=377$
8 $A=284$ **9** $u=-40$ **10** $V=51.935$
11 $m=-1\frac{3}{4}$ **12** $A=22.4$ **13** $V=1838.8$
14 $v=\frac{3}{4}$ **15** $S=15$ **16** $c=10$
17 $y=\sqrt{12}=2\sqrt{3}$ **18** $E=23.987$ **19** $A=352.47$
20 $S=93$

Exercise 1.18

1 **a** $2\sqrt{3}$ **b** $3\sqrt{7}$ **c** $2\sqrt{6}$ **d** $5\sqrt{2}$
e $6\sqrt{2}$ **f** $10\sqrt{2}$ **g** $4\sqrt{3}$ **h** $5\sqrt{3}$
i $4\sqrt{2}$ **j** $3\sqrt{6}$ **k** $4\sqrt{7}$ **l** $10\sqrt{3}$

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m $8\sqrt{2}$ **n** $9\sqrt{3}$ **o** $7\sqrt{5}$ **p** $6\sqrt{3}$
q $3\sqrt{11}$ **r** $5\sqrt{5}$

2 **a** $6\sqrt{3}$ **b** $20\sqrt{5}$ **c** $28\sqrt{2}$ **d** $4\sqrt{7}$
e $16\sqrt{5}$ **f** $8\sqrt{14}$ **g** $72\sqrt{5}$ **h** $30\sqrt{2}$
i $14\sqrt{10}$ **j** $24\sqrt{5}$

3 **a** $\sqrt{18}$ **b** $\sqrt{20}$ **c** $\sqrt{176}$ **d** $\sqrt{128}$
e $\sqrt{75}$ **f** $\sqrt{160}$ **g** $\sqrt{117}$ **h** $\sqrt{98}$
i $\sqrt{363}$ **j** $\sqrt{1008}$

4 **a** 45 **b** 12 **c** 63 **d** 50
e 44 **f** 147 **g** 304 **h** 828
i 775 **j** 960

Exercise 1.19

1 **a** $3\sqrt{5}$ **b** $\sqrt{2}$ **c** $6\sqrt{3}$
d $3\sqrt{3}$ **e** $-3\sqrt{5}$ **f** $3\sqrt{6}$
g $-7\sqrt{2}$ **h** $8\sqrt{5}$ **i** $-4\sqrt{2}$
j $4\sqrt{5}$ **k** $\sqrt{2}$ **l** $5\sqrt{3}$
m $-\sqrt{3}$ **n** $\sqrt{2}$ **o** $5\sqrt{7}$
p $\sqrt{2}$ **q** $13\sqrt{6}$ **r** $-9\sqrt{10}$
s $47\sqrt{3}$ **t** $5\sqrt{2}-2\sqrt{3}$ **u** $\sqrt{7}-5\sqrt{2}$
v $-2\sqrt{3}-4\sqrt{5}$

2 **a** $\sqrt{21}$ **b** $\sqrt{15}$ **c** $3\sqrt{6}$
d $10\sqrt{14}$ **e** $-6\sqrt{6}$ **f** 30
g $-12\sqrt{55}$ **h** 14 **i** 60
j $\sqrt{12}=2\sqrt{3}$ **k** 2 **l** 28
m $\sqrt{30}$ **n** $-2\sqrt{105}$ **o** 18

3 **a** $2\sqrt{6}$ **b** $4\sqrt{3}$ **c** 1
d $\frac{8}{\sqrt{6}}$ **e** $2\sqrt{3}$ **f** $\frac{1}{3\sqrt{10}}$
g $\frac{1}{2\sqrt{5}}$ **h** $\frac{1}{3\sqrt{5}}$ **i** $\frac{1}{2}$
j $\frac{\sqrt{3}}{2\sqrt{2}}$ **k** $\frac{\sqrt{3}}{\sqrt{2}}$ **l** $\frac{9}{2\sqrt{5}}$
m $\frac{\sqrt{5}}{2\sqrt{2}}$ **n** $\frac{2}{3}$ **o** $\frac{5}{7}$

4 **a** $\sqrt{10}+\sqrt{6}$ **b** $2\sqrt{6}-\sqrt{15}$
c $12+8\sqrt{15}$ **d** $5\sqrt{14}-2\sqrt{21}$
e $-\sqrt{6}+12\sqrt{2}$ **f** $5\sqrt{33}+3\sqrt{21}$
g $-6-12\sqrt{6}$ **h** $5-5\sqrt{15}$
i $6+\sqrt{30}$ **j** $6\sqrt{6}+6$
k $-8+24\sqrt{3}$ **l** $210-14\sqrt{15}$
m $10\sqrt{6}-120$ **n** $-\sqrt{10}-2\sqrt{2}$
o $4\sqrt{3}-12$

5 **a** $\sqrt{10}+3\sqrt{6}+3\sqrt{5}+9\sqrt{3}$
b $\sqrt{10}-\sqrt{35}-2+\sqrt{14}$
c $2\sqrt{10}-6+10\sqrt{15}-15\sqrt{6}$
d $24\sqrt{5}+36\sqrt{15}-8\sqrt{10}-12\sqrt{30}$
e $52-13\sqrt{10}$
f $15-\sqrt{15}+18\sqrt{10}-6\sqrt{6}$
g 4 **h** -1
i -12 **j** 43
k 3 **l** -241
m -6 **n** $7+2\sqrt{10}$
o $11-4\sqrt{6}$ **p** $25+6\sqrt{14}$
q $57+12\sqrt{15}$ **r** $27-4\sqrt{35}$
s $77-12\sqrt{40}=77-24\sqrt{10}$
t $53+12\sqrt{10}$

6 **a** 18 **b** $108\sqrt{2}$ **c** $432\sqrt{2}$
d $19+6\sqrt{2}$ **e** 9

7 **a** $a=21, b=80$ **b** $a=19, b=-7$

8 **a** $a-1$ **b** $2p-1-2\sqrt{p(p-1)}$

9 25 **10** $2x-3y-5\sqrt{xy}$

11 $a=17, b=240$ **12** $a=107, b=-42$

Exercise 1.20

1 **a** $\frac{\sqrt{7}}{7}$ **b** $\frac{\sqrt{6}}{4}$
c $\frac{2\sqrt{15}}{5}$ **d** $\frac{6\sqrt{14}}{10}=\frac{3\sqrt{14}}{5}$
e $\frac{\sqrt{3}+\sqrt{6}}{3}$ **f** $\frac{2\sqrt{3}-5\sqrt{2}}{2}$
g $\frac{5+2\sqrt{10}}{5}$ **h** $\frac{3\sqrt{14}-4\sqrt{7}}{14}$
i $\frac{8\sqrt{5}+3\sqrt{10}}{20}$ **j** $\frac{4\sqrt{15}-2\sqrt{10}}{35}$

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2 **a** $4\sqrt{3}-4\sqrt{2}=4\left(\sqrt{3}-\sqrt{2}\right)$

b $\dfrac{-\left(\sqrt{6}+7\sqrt{3}\right)}{47}$

c $\dfrac{-\left(2\sqrt{15}-4\sqrt{18}\right)}{19}=\dfrac{-2\left(\sqrt{15}-6\sqrt{2}\right)}{19}$

d $\dfrac{-\left(19-8\sqrt{3}\right)}{13}=\dfrac{8\sqrt{3}-19}{13}$

e $\sqrt{6}+2+5\sqrt{3}+5\sqrt{2}$

f $\dfrac{6\sqrt{15}-9\sqrt{6}+2\sqrt{10}-6}{2}$

3 **a** $2\sqrt{2}$

b $-2-\sqrt{6}+3\sqrt{2}-3\sqrt{3}$

c -4

d $4\sqrt{2}$

e $\dfrac{6+9\sqrt{2}+2\sqrt{3}}{6}$

f $\dfrac{4\sqrt{6}+9\sqrt{3}}{21}$

g $\dfrac{15\sqrt{30}-30\sqrt{5}-4\sqrt{3}}{30}$

h $\dfrac{28-2\sqrt{6}-7\sqrt{3}}{13}$

i $\dfrac{2\sqrt{15}+2\sqrt{10}-2\sqrt{6}-\sqrt{3}-5}{2}$

4 **a** $a=45, b=10$ **b** $a=1, b=8$

c $a=-\dfrac{1}{2}, b=\dfrac{1}{2}$ **d** $a=-1\dfrac{5}{9}, b=-\dfrac{8}{9}$

e $a=5, b=32$

5 3, so rational.

6 **a** 4 **b** 14 **c** 16

Test yourself 1

1 B, C **2** D **3** A, D

4 C **5** C **6** B

7 A **8** D

9 **a** $\dfrac{1}{49}$ **b** $\dfrac{1}{5}$ **c** $\dfrac{1}{3}$

10 **a** x^9 **b** $25y^6$ **c** $a^{11}b^6$

d $\dfrac{8x^{18}}{27}$ **e** 1

11 **a** 6 **b** $\dfrac{1}{64}$ **c** 4

d $\dfrac{1}{7}$ **e** 2 **f** 1

12 **a** a^5 **b** $x^{30}y^{18}$ **c** p^9

d $16b^{36}$ **e** $8x^{11}y$

13 **a** $n^{\frac{1}{2}}$ **b** x^{-5} **c** $(x+y)^{-1}$

d $(x+1)^{\frac{1}{4}}$ **e** $(a+b)^{\frac{1}{7}}$ **f** $2x^{-1}$

g $\dfrac{1}{2}x^{-3}$ **h** $x^{\frac{4}{3}}$ **i** $(5x+3)^{\frac{9}{7}}$

j $m^{-\frac{3}{4}}$

14 **a** $\dfrac{1}{a^5}$ **b** $\sqrt[4]{n}$ **c** $\sqrt{x+1}$

d $\dfrac{1}{x-y}$ **e** $\dfrac{1}{(4t-7)^4}$ **f** $\sqrt[5]{a+b}$

g $\dfrac{1}{\sqrt[3]{x}}$ **h** $\sqrt[4]{b^3}$ **i** $\sqrt[3]{(2x+3)^4}$

j $\dfrac{1}{\sqrt{x^3}}$

15 1 **16** $\dfrac{1}{192}$

17 **a** $x^{\frac{1}{2}}$ **b** y^{-1} **c** $(x+3)^{\frac{1}{6}}$

d $(2x-3)^{-11}$ **e** $y^{\frac{7}{3}}$

18 **a** $\dfrac{1}{x^3}$ **b** $\dfrac{1}{2a+5}$ **c** $\left(\dfrac{b}{a}\right)^5$

19 **a** $-2y$ **b** $a+4$ **c** $-6k^5$

d $\dfrac{5x+3y}{15}$ **e** $3a-8b$ **f** $6\sqrt{2}$

g $4\sqrt{5}$

20 **a** $(x+6)(x-6)$ **b** $(a+3)(a-1)$

c $4ab(b-2)$ **d** $(y-3)(5+x)$

e $2(2n-p+3)$

21 **a** $4b-6$ **b** $2x^2+5x-3$

c $4m+17$ **d** $16x^2-24x+9$

e p^2-25 **f** $-1-7a$

g $2\sqrt{6}-5\sqrt{3}$ **h** $3\sqrt{3}-6+\sqrt{21}-2\sqrt{7}$

22 **a** $\dfrac{8}{b^2(a+3)}$ **b** $\dfrac{15}{(m-2)^2}$

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23 $V = 157.464$

24 **a** 17 **b** $\frac{6\sqrt{15}-9}{17}$

25 $\frac{4x+5}{(x+3)(x-2)}$

26 **a** 36 **b** -2 **c** 2
d 216 **e** 2

27 **a** $\frac{1}{\sqrt{5}}$ **b** 8

28 $d = 11.25$

29 **a** $\frac{2\sqrt{3}}{15}$ **b** $\frac{\sqrt{2}+\sqrt{6}}{2}$

30 **a** $3\sqrt{6}-6-4\sqrt{3}+4\sqrt{2}$ **b** $11+4\sqrt{7}$

31 **a** $3(x-3)(x+3)$ **b** $6(x-3)(x+1)$
c $5(y-3)^2$

32 **a** $\frac{x^3}{3y^4}$ **b** $\frac{1}{3x-1}$

33 **a** 99 **b** $24\sqrt{3}$

34 **a** a^2-b^2 **b** $a^2+2ab+b^2$

35 **a** $(a-b)^2$ **b** $(a+b)(a-b)$

36 $\frac{3\sqrt{3}+1}{2}$

37 **a** $\frac{4b+3a}{ab}$ **b** $\frac{3x-11}{10}$

38 $\frac{21\sqrt{5}-46-\sqrt{2}}{7}$

39 **a** $6\sqrt{2}$ **b** $-8\sqrt{6}$ **c** $2\sqrt{3}$
d $\frac{4}{\sqrt{3}}$ **e** $30a^2b$ **f** $\frac{m}{3n^4}$
g $2x-3y$

40 **a** $2\sqrt{6}+4$
b $10\sqrt{14}-5\sqrt{21}-6\sqrt{10}+3\sqrt{15}$
c 7 **d** 43 **e** $65-6\sqrt{14}$

41 **a** $\frac{3\sqrt{7}}{7}$ **b** $\frac{\sqrt{6}}{15}$
c $\frac{\sqrt{5}+1}{2}$ **d** $\frac{12-2\sqrt{6}}{15}$
e $\frac{20+3\sqrt{15}+4\sqrt{10}+3\sqrt{6}}{53}$

42 **a** $\frac{x+10}{10}$ **b** $\frac{17a-15}{21}$
c $\frac{3-2x}{(x+1)(x-1)}$ **d** $\frac{1}{k-1}$
e $\frac{\sqrt{15}-\sqrt{6}-15\sqrt{3}-15\sqrt{2}}{3}$

43 **a** $n = 48$ **b** $n = 175$ **c** $n = 392$
d $n = 5547$ **e** $n = 1445$

Challenge exercise 1

1 $\frac{1}{16}$

2 Proof involves $2 \times 2^{k+1} - 2$ (see worked solutions)

3 $-2^4 \times 3^5$

4 **a** $2a^2b - 8ab^2 + 6a^3$ **b** $y^4 - 4$
c $8x^3 - 60x^2 + 150x - 125$

5 $\frac{11\sqrt{3}+2\sqrt{5}+14}{11}$ **6** $\frac{1}{2\sqrt{2}}$ or $\frac{\sqrt{2}}{4}$

7 **a** $(x+4)(x+9)$ **b** $(x^2-3y)(x^2+2y)$
c $(b-2)(a+2)(a-2)$

8 $\frac{y+1}{2(x-1)}$ **9** $\frac{(a+1)^2}{a-1}$

10 $\left(\frac{2}{x}+\frac{a}{b}\right)\left(\frac{2}{x}-\frac{a}{b}\right)$

11 **a** $8x^3 - 12x^2 + 6x - 1$ **b** $\frac{3x+4}{(2x-1)^2}$

12 $r = \frac{3\sqrt{\pi}}{4\pi}$ **13** $s = 2 + 6\sqrt{3}$

14 **a** $x + x^2 + 2x^{\frac{3}{2}}$ **b** $a^{\frac{2}{3}} - b^{\frac{2}{3}}$
c $p^2 + p^{-1} + 2p^{\frac{1}{2}}$ **d** $x + x^{-1} + 2$

15 4

Chapter 2

Exercise 2.01

1 $t = -5$ **2** $z = -5.6$ **3** $y = 1$
4 $w = 6.7$ **5** $x = 12$ **6** $x = 4$
7 $y = \frac{1}{15}$ **8** $b = 35$ **9** $n = -16$

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10 $r = 4$ **11** $y = 9$ **12** $k = 6$
13 $d = 2$ **14** $x = 5$ **15** $y = 15$
16 $x = 20$ **17** $m = 20$ **18** $x = 4$
19 $a = -7$ **20** $y = 3$ **21** $x = 3$
22 $a = -1\frac{2}{3}$ **23** $t = -4$ **24** $x = 1.2$
25 $a = 1.6$ **26** $b = \frac{1}{8}$ **27** $t = 39$
28 $p = 5$ **29** $x \approx 4.41$ **30** $b = 3\frac{1}{3}$
31 $x = 1\frac{9}{35}$ **32** $x = 36$ **33** $x = -3$
34 $y = -1.2$ **35** $x = 69$ **36** $w = 13$
37 $t = 30$ **38** $x = 14$ **39** $x = -1$
40 $x = -0.4$ **41** $p = 3$ **42** $t = 8.2$
43 $x = -9.5$ **44** $q = 22$ **45** $x = -3$

Exercise 2.02

1 a $x > 3$



b $y \leq 4$

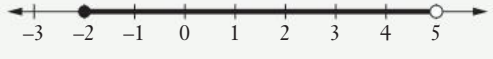


2 a $t > 7$ **b** $x \geq 3$ **c** $p > -1$
d $x \geq -2$ **e** $y > -9$ **f** $a \geq -1$
g $y \geq -2\frac{1}{2}$ **h** $x < -2$ **i** $a \leq -6$
j $y < 12$ **k** $b < -18$ **l** $x > 30$

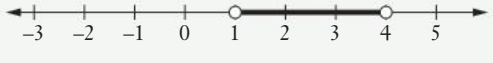
3 a $1 < x < 7$



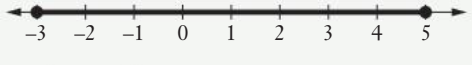
b $-2 \leq p < 5$



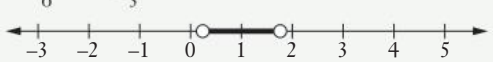
c $1 < x < 4$



d $-3 \leq y \leq 5$



e $\frac{1}{6} < y < 1\frac{2}{3}$



Exercise 2.03

1 a 7 **b** 5 **c** 6 **d** 0
e 2 **f** 11 **g** 6 **h** 24
i 25 **j** 125
2 a 5 **b** −1 **c** 2 **d** 14
e 4 **f** −67 **g** 7 **h** 12
i −6 **j** 10
3 a 3 **b** 3 **c** 1
d 3 **e** 1
4 a a **b** $-a$ **c** 0
d $3a$ **e** $-3a$ **f** 0
g $a + 1$ **h** $-a - 1$ **i** $x - 2$
5 a $6 \leq 6$ **b** $3 \leq 3$ **c** $1 \leq 5$
d $1 \leq 9$ **e** $10 \leq 10$
6 See worked solutions.
7 a $x + 5$ for $x \geq -5$ and $-x - 5$ for $x < -5$
b $b - 3$ for $b \geq 3$ and $3 - b$ for $b < 3$
c $a + 4$ for $a \geq -4$ and $-a - 4$ for $a < -4$
d $2y - 6$ for $y \geq 3$ and $6 - 2y$ for $y < 3$
e $3x + 9$ for $x \geq -3$ and $-3x - 9$ for $x < -3$
f $4 - x$ for $x \leq 4$ and $x - 4$ for $x > 4$
g $2k + 1$ for $k \geq -\frac{1}{2}$ and $-2k - 1$ for $k < -\frac{1}{2}$
h $5x - 2$ for $x \geq \frac{2}{5}$ and $-5x + 2$ for $x < \frac{2}{5}$
i $a + b$ for $a \geq -b$ and $-a - b$ for $a < -b$
8 ± 3 **9** ± 1 **10** $\pm 1, x \neq 2$

Exercise 2.04

1 a $x = \pm 5$ **b** $y = \pm 8$ **c** $x = 0$
2 a $x = 5, -9$ **b** $n = 4, -2$ **c** $x = 3, -6$
d $x = 5, -4\frac{5}{7}$ **e** $x = \pm 12$
3 a $x = 2, -0.75$ **b** $n = 1\frac{1}{3}, 2$ **c** $t = 2.4, -4$
d $y = -6, 15$ **e** $x = -3, 1\frac{2}{3}$

Exercise 2.05

1 a $n = 4$ **b** $y = 5$ **c** $m = 9$ **d** $x = 5$
e $m = 0$ **f** $x = 3$ **g** $x = 2$ **h** $x = 2$
i $x = 1$ **j** $k = 2$

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2 **a** $x = 2$ **b** $x = 1$ **c** $x = -2$ **d** $n = 2$
e $x = 0$ **f** $x = 6$ **g** $y = \frac{1}{3}$ **h** $x = 2$
i $x = 2$ **j** $a = 0$

3 **a** $m = \frac{1}{2}$ **b** $x = \frac{1}{3}$ **c** $x = \frac{1}{3}$
d $k = -\frac{1}{2}$ **e** $k = -\frac{2}{3}$ **f** $n = \frac{3}{4}$
g $x = 1\frac{1}{2}$ **h** $n = \frac{2}{3}$ **i** $k = -\frac{1}{6}$
j $x = 1\frac{2}{3}$

4 **a** $x = -1$ **b** $x = -1\frac{1}{3}$ **c** $k = -4$
d $n = 3$ **e** $x = -2\frac{1}{2}$ **f** $x = -\frac{2}{3}$
g $x = -4\frac{1}{2}$ **h** $x = -1\frac{7}{11}$ **i** $x = 1\frac{4}{5}$

5 **a** $m = \frac{1}{4}$ **b** $k = -2\frac{3}{4}$ **c** $x = 2\frac{3}{8}$
d $k = 1\frac{1}{2}$ **e** $n = \frac{1}{18}$ **f** $n = -\frac{1}{2}$
g $x = \frac{4}{5}$ **h** $b = -3\frac{1}{6}$ **i** $x = -1\frac{1}{7}$

6 **a** $x = 3$ **b** $y = \pm 8$ **c** $n = \pm 2$
d $x = \pm 2\sqrt{5}$ **e** $p = 10$ **f** $x = \pm 5$
g $y = \pm 3$ **h** $w = 2$ **i** $n = \pm 4$

7 **a** $p = \pm 6.71$ **b** $x = 4.64$ **c** $n = 2.99$
d $x = \pm 5.92$ **e** $y = 1.89$ **f** $d = \pm 2.55$
g $k = \pm 4.47$ **h** $x = 2.22$ **i** $y = \pm 3.81$

8 **a** $x = \frac{1}{5}$ **b** $a = \frac{1}{2}$ **c** $y = \frac{1}{2}$
d $x = \pm\frac{1}{7}$ **e** $n = \frac{2}{3}$ **f** $a = 2$
g $x = \pm 2$ **h** $b = 9$ **i** $x = \pm\frac{2}{3}$
j $b = \pm 1\frac{1}{2}$

Puzzle

1 All months have 28 days. Some months have more days as well.
2 10
3 Bottle $1.05; cork 5 cents
4 16 each time **5** Friday

Exercise 2.06

1 $y = 0, -1$ **2** $b = 2, -1$
3 $p = 3, -5$ **4** $t = 0, 5$
5 $x = -2, -7$ **6** $q = \pm 3$
7 $x = \pm 1$ **8** $a = 0, -3$
9 $x = 0, -4$ **10** $x = \pm\frac{1}{2}$
11 $x = -1, -1\frac{1}{3}$ **12** $y = 1, -1\frac{1}{2}$
13 $b = \frac{3}{4}, \frac{1}{2}$ **14** $x = 5, -2$
15 $x = 0, \frac{2}{3}$ **16** $x = 1, 2\frac{1}{2}$
17 $x = 0, 5$ **18** $y = -1, 2$
19 $n = 3, 5$ **20** $x = 3, 4$
21 $m = -6, 1$ **22** $x = 0, -1, -2$
23 $y = 1, -5, -2$ **24** $x = 5, -7$
25 $m = 8, -1$

Exercise 2.07

1 **a** $x = \pm\sqrt{7} - 1$ **b** $y = \pm\sqrt{5} - 5$
c $a = \pm\sqrt{6} + 3$ **d** $x = \pm\sqrt{13} + 2$
e $y = \frac{\pm\sqrt{2} - 3}{2}$

2 **a** $h = 1.9, -5.9$ **b** $a = 3.8, -1.8$
c $x = 8.1, -0.1$ **d** $y = -2.4, -11.6$
e $x = 1.5, -0.8$

3 **a** $x = \pm\sqrt{5} - 2$ **b** $a = \pm\sqrt{7} + 3$
c $y = \pm\sqrt{23} + 4$ **d** $x = \pm\sqrt{13} - 1$
e $p = \pm\sqrt{44} - 7 = \pm 2\sqrt{11} - 7$
f $x = \pm\sqrt{28} + 5 = \pm 2\sqrt{7} + 5$
g $x = \pm\sqrt{88} - 10 = \pm 2\sqrt{22} - 10 = 2\left(\pm\sqrt{22} - 5\right)$
h $x = \pm\sqrt{2} + 1$
i $n = \pm\sqrt{137} - 12$

4 **a** $x = 3.45, -1.45$ **b** $x = -4.59, -7.41$
c $q = 0.0554, -18.1$ **d** $x = 4.45, -0.449$
e $b = -4.26, -11.7$ **f** $x = 17.7, 6.34$
g $r = 22.3, -0.314$ **h** $x = -0.683, -7.32$
i $a = 0.162, -6.16$

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Exercise 2.08

1 **a** $y = -0.354, -5.65$ **b** $x = 1, 1.5$
c $b = 3.54, -2.54$ **d** $x = 1, -0.5$
e $x = -0.553, 0.678$ **f** $n = 0.243, -8.24$
g $m = -2, -5$ **h** $x = 0, 7$
i $x = 1, -6$

2 **a** $x = \frac{-1 \pm \sqrt{17}}{2}$ **b** $x = \frac{5 \pm \sqrt{13}}{6}$
c $q = 2 \pm \sqrt{7}$ **d** $h = \frac{-3 \pm 2\sqrt{2}}{2}$
e $s = \frac{4 \pm \sqrt{10}}{3}$ **f** $x = \frac{-11 \pm \sqrt{133}}{2}$
g $d = \frac{-5 \pm \sqrt{73}}{12}$ **h** $x = 1 \pm 2\sqrt{2}$
i $t = \frac{1 \pm \sqrt{5}}{2}$

Exercise 2.09

1 $t = 8.5$ **2** $l = 122$
3 $b = 8$ **4** $a = 41$
5 $y = 4$ **6** $r = 6.68$
7 $x = 6.44$ **8** $n = 15$
9 $y_1 = 3\frac{2}{3}$ **10** $h = 3.7$
11 **a** BMI = 25.39 **b** $w = 69.66$
c $h = 1.94$
12 $r = 0.072$ **13** $x_1 = -9$
14 $t = 2.14$ **15** $x = \pm 2$
16 $r = 2.12$ **17** $r = 10.46$
18 $x = 1.19$ **19** $x = 5.5$
20 $r = 3.3$

Exercise 2.10

1 $a = 1, b = 3$ **2** $x = 2, y = 1$
3 $p = 2, q = -1$ **4** $x = 6, y = 17$
5 $x = -10, y = 2$ **6** $t = 3, v = 1$
7 $x = -3, y = 2$ **8** $x = -64, y = -39$
9 $x = 3, y = -4$ **10** $m = 2, n = 3$
11 $w_1 = -1, w_2 = 5$ **12** $a = 0, b = 4$
13 $p = -4, q = 1$ **14** $x_1 = 1, x_2 = -1$
15 $x = -1, y = -4$ **16** $s = 2, t = -1$
17 $a = -2, b = 0$ **18** $k = -4, h = 1$

Problem

23 adults and 16 children.

Exercise 2.11

1 $x = 0, y = 0$ and $x = 1, y = 1$
2 $x = 0, y = 0$ and $x = -2, y = 4$
3 $x = 0, y = 3$ and $x = 3, y = 0$
4 $x = 4, y = -3$ and $x = 3, y = -4$
5 $x = -1, y = -3$ **6** $x = 3, y = 9$
7 $t = -2, x = 4$ and $t = 1, x = 1$
8 $m = -4, n = 0$ and $m = 0, n = -4$
9 $x = 1, y = 2$ and $x = -1, y = -2$
10 $x = 0, y = 0$ and $x = 1, y = 1$
11 $x = 2, y = 1$ and $x = -1, y = -2$
12 $x = 0, y = 1$
13 $x = 1, y = 5$ and $x = 4, y = 11$
14 $x = \frac{1}{4}, y = 4$ and $x = -1, y = -1$
15 $t = -\frac{1}{2}, h = \frac{1}{4}$ **16** $x = 2, y = 0$
17 $x = 0, y = 0$ and $x = -2, y = -8$ and $x = 3, y = 27$
18 $x = 0, y = 0$ and $x = 1, y = 1$ and $x = -1, y = 1$
19 $x = \frac{1}{2}, y = 2\frac{3}{4}$ **20** $x = -\frac{5}{13}, y = -\frac{12}{13}$

Exercise 2.12

1 $x = -2, y = -8, z = -1$ **2** $a = -2, b = -1, c = 2$
3 $a = -4, b = 2, c = 7$ **4** $a = 1, b = 2, c = -3$
5 $x = 5, y = 0, z = -2$ **6** $p = -3, q = 7, r = 4$
7 $x = 1, y = -1, z = 2$ **8** $x = 0, y = -5, z = 4$
9 $h = -3, j = 2, k = -4$ **10** $a = 3, b = -1, c = -2$

Test yourself 2

1 C **2** A, D
3 B
4 **a** $b = 10$ **b** $a = -116$
c $x = -7$ **d** $p \leq 4$
5 **a** $A = 1262.48$ **b** $P = 8558.59$
6 **a** $x = -2, y = 5$
b $x = 4, y = 1$ and $x = -\frac{1}{2}, y = -8$
7 **a** $x = 2$ **b** $y = \frac{1}{4}$

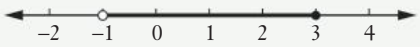
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8 $b = 2, -1\frac{1}{3}$

9 **a** $A = 36$ **b** $b = 12$

10 $x = \frac{1}{2}, 1$

11 $-1 < y \leq 3$



12 **a** $x = -0.298, -6.70$ **b** $y = 4.16, -2.16$

c $n = 0.869, -1.54$

13 **a** $A = 764.5$ **b** $r = 2.9$

14 $x = 2.4, y = 3.2$

15 **a** $V = 2100$ **b** $r = 3.9$

16 **a** B **b** A **c** A

d C **e** B

17 $a = 3, b = 2, c = -4$

18 $x = -2$

19 **a** $y > 3$ **b** $x = 2$

c $x = 2$ **d** $x = 3, -1\frac{2}{5}$

e $x = -3$ **f** $x = \frac{5}{6}$

Challenge exercise 2

1 $y = 1$ **2** $a = 3, b = \pm 2$

3 **a** $(x + 3)(x - 3)(x^3 - 8)$

b $x = \pm 3, 2$

4 $x = 1, y = 2$ and $x = -1, y = 0$

5 $b = 16, x = 4 \pm \sqrt{17}$

6 $x = 1$ **7** $x = \frac{1}{4}$

8 $x = \pm\sqrt{b + a^2} + a$

9 $x = \frac{2\left(4 \pm \sqrt{10}\right)}{3}$ **10** $x = 2, -4$

Practice set 1

1 B **2** A **3** B **4** B

5 D **6** D

7 **a** $x = 10$ **b** $b = 6$ **c** $x = 3$

d $y = 16$ **e** $z = 6$ **f** $x = 5$

g $y = 1.5$ **h** $x = 0, 3$ **i** $x = 3, -7$

j $a = 2, -1.2$

8 $p = 9$ **9** $4\sqrt{3}$

10 $2(5 + y)(x - y)$

11 **a** x^{-1} **b** $x^{\frac{4}{3}}$

12 $6y - 10$ **13** $\frac{25 + 5\sqrt{2}}{23}$

14 $x = 1.78, -0.281$

15 $\frac{2}{x - 3}$ **16** 231.3

17 $-\sqrt{3}$

18 $x^3 + 2x^2 - 16x + 3$

19 $3\sqrt{10} - 4$

20 $x + 3$ **21** $a > -3$

22 $r = 1.55$

23 **a** $a = 3, b = 2$ **b** $a = 3, b = 5, c = 0$

24 $x > 1$



25 $x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$ **26** $\frac{1}{49}$

27 $x = 4, y = 11$ or $x = -1, y = -4$

28 $x = 2, y = -1$

29 7 **30** $8(x + 2)(x - 2)$

31 $\frac{6\sqrt{15} + 2\sqrt{6}}{43}$ **32** 7

33 $-2\sqrt{10} + 3\sqrt{5} - 2\sqrt{2} + 3$ **34** $a^{-21}b^{10} = \frac{b^{10}}{a^{21}}$

35 $\frac{1}{8}$

36 $-x - 7$ **37** $x = \frac{1}{4}$

38 $(x + 3)^{-1}$ **39** $\frac{5^4}{2^{17}}$

40 $\frac{1}{\sqrt{3x + 2}}$

41 **a** $12x - 8y$ **b** $2\sqrt{31}$

c $\frac{x - 3}{2x - 1}$ **d** $3\sqrt{2} + 1$

e $\frac{-(x + 5)}{(x + 1)(x - 1)}$ **f** $\frac{11\sqrt{3}}{6}$

g $x^{-14}y^7z^{-11}$ or $\frac{y^7}{x^{14}z^{11}}$ **h** $\frac{3}{5a(a + b)(1 + 2b)}$

i $8\sqrt{5}$ **j** $13\frac{1}{2}$

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42 $r = \frac{2}{\sqrt[3]{\pi}}$ cm

43 $k = 20$

44 $9xy\sqrt{y}$

45 **a** $5(a-2)(a+6)$ **b** $(3a+4b)(a-6b+2c)$

46 $-1\frac{1}{8} \leq x < 5\frac{3}{4}$

47 $\frac{-2x-7}{15}$

48 $x = 0, 5$

49 $x = 5.19, -0.19$

50 $9\sqrt{2}$

51 $\frac{-2x-17}{x(x+5)}$

52 $x = -\frac{1}{4}$

53 **a** $(x-4)(x+2)$ **b** $(a+3)(a-3)$
c $(y+3)^2$ **d** $(t+4)^2$
e $(3x-2)(x-3)$

54 **a** $x = 5$ **b** $y = 4.74, -2.74$
c $x = 0.75$ **d** $b = 2, -5$

Chapter 3

Exercise 3.01

1 **a** (Wade, black), (Scott, blond), (Geoff, grey), (Deng, black), (Mila, brown), (Stevie, blond); many-to-one
b (1, 1), (1, 4), (2, 3), (3, 1), (4, 4); many-to-many
c (1, A), (2, D), (3, A), (4, B), (5, C); many-to-one
d (3, 5), (5, −2), (5, 2), (8, −7), (9, 3), (5, 6), (8, 0); one-to-many
e (1, 9), (2, 15), (3, 27), (4, 33), (5, 45); one-to-one

2 **a** Yes (one-to-one) **b** No
c No **d** Yes
e Yes **f** Yes
g No **h** Yes
i Yes **j** No
k Yes **l** No
m Yes (one-to-one) **n** No
o Yes

3 **a** {−3, −1, 0, 1, 6} **b** {−2, 4, 5, 8}
c many-to-one **d** Yes

Exercise 3.02

1 $f(1) = 4, f(-3) = 0$

2 $h(0) = -2, h(2) = 2, h(-4) = 14$

3 $f(5) = -25, f(-1) = -1, f(3) = -9, f(-2) = -4$

4 14 **5** −35

6 $x = 9$ **7** $x = \pm 5$

8 $x = -3$ **9** $z = 1, -4$

10 $f(p) = 2p - 9, f(x+h) = 2x + 2h - 9$

11 $g(x-1) = x^2 + 2$ **12** $f(k) = (k-1)(k+1)$

13 **a** $t = 1$ **b** $t = 4, -2$

14 0

15 $f(5) = 125, f(1) = 1, f(-1) = -1$

16 −2 **17** 10

18 7 **19** −28

20 $f(x+h) - f(x) = 2xh + h^2 - 5h$

21 $4x + 2h + 1$

22 $5(x-c)$ **23** $3k^2 + 5$

24 **a** 2 **b** 0 **c** $n^4 + n^2 + 2$

25 **a** 3
b Denominator cannot be 0 so $x - 3 \neq 0$, $\therefore x \neq 3$.
c 4

Exercise 3.03

1 **a** x-intercept $\frac{2}{3}$, y-intercept −2
b x-intercept −10, y-intercept 4
c x-intercept 12, y-intercept 4
d x-intercepts 0, −3, y-intercept 0
e x-intercepts ± 2, y-intercept −4
f x-intercepts −2, −3, y-intercept 6
g x-intercepts 3, 5, y-intercept 15
h x-intercept $-\sqrt[3]{5}$, y-intercept 5
i x-intercept −3, no y-intercept
j x-intercepts ± 3, y-intercept 9

2 **a** $x = 2$
b x-intercept 2, y-intercept −6

3 $f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$ so even.

4 **a** $f(x^2) = x^6 + 1$
b $[f(x)]^2 = x^6 + 2x^3 + 1$
c $f(-x) = -x^3 + 1$
d Neither odd nor even
e $x = -1$
f x-intercept −1, y-intercept 1

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5 $g(-x) = (-x)^8 + 3(-x)^4 - 2(-x)^2 = x^8 + 3x^4 - 2x^2$
$= g(x)$ so even.

6 $f(-x) = -x = -f(x)$ so odd.

7 $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$ so even.

8 $f(-x) = 4(-x) - (-x)^3 = -4x + x^3 = -(4x - x^3)$
$= -f(x)$ so odd.

9 **a** $f(-x) = (-x)^4 + (-x)^2 = x^4 + x^2 = f(x)$ so even
b 0

10 **a** Odd **b** Neither
c Even **d** Neither
e Neither

11 **a** Even values, i.e. $n = 2, 4, 6, \ldots$
b Odd values, i.e. $n = 1, 3, 5, \ldots$

12 **a** No value of n
b Yes, when n is odd $(1, 3, 5, \ldots)$

13 **a** **i** $(0, \infty)$ **ii** $(-\infty, 0)$ **iii** Even
b **i** $(-\infty, 2)$ **ii** $(2, \infty)$ **iii** Neither
c **i** $(-2, 2)$ **ii** $(-\infty, -2) \cup (2, \infty)$
iii Neither
d **i** $(-\infty, 0) \cup (0, \infty)$
ii None **iii** Odd
e **i** None **ii** $(-\infty, \infty)$ **iii** Neither

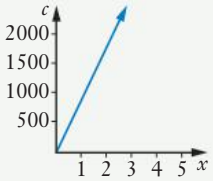
14 **a** Domain $(-\infty, \infty)$, range $[1, \infty)$
b Domain $(-\infty, \infty)$, range $(-\infty, \infty)$
c Domain $[0, \infty)$, range $[0, \infty)$
d Domain $[-5, \infty)$, range $[0, \infty)$
e Domain $[3, \infty)$, range $(-\infty, 0]$

15 **a** 1 **b** 49 **c** $x = 2$
d x-intercept 2, y-intercept 4
e Domain $(-\infty, \infty)$, range $[0, \infty)$
f $(-x - 2)^2$ **g** Neither

Exercise 3.04

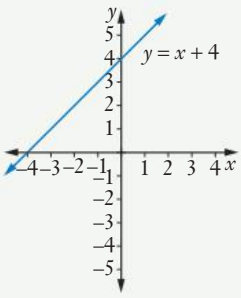
1 **a** $N = 12x$ **b** $A = 2n$ **c** $c = 1.5x$
d $y = 4x$ **e** $w = 400x$

2 $c = 850x$

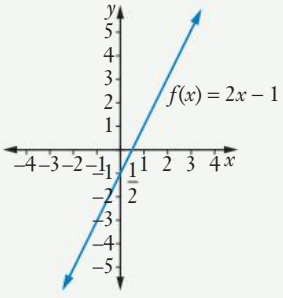


3 **a** x-intercept 2, y-intercept -2
b x-intercept -3, y-intercept 9
c x-intercept 2, y-intercept 4
d x-intercept $-1\frac{1}{2}$, y-intercept 3
e x-intercept $\frac{4}{5}$, y-intercept -4
f x-intercept $-\frac{1}{2}$, y-intercept 5
g x-intercept 2, y-intercept 2
h x-intercept -2, y-intercept 4
i x-intercept -3, y-intercept 3
j x-intercept $\frac{2}{3}$, y-intercept $-\frac{1}{3}$

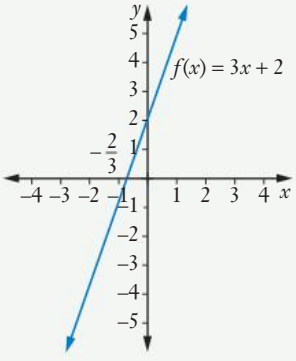
4 **a**



b

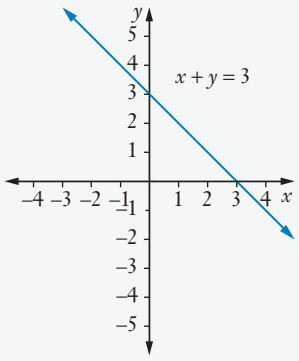


c

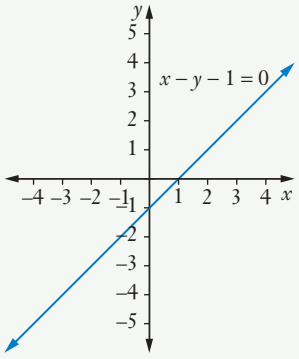


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d



e



5 **a** Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

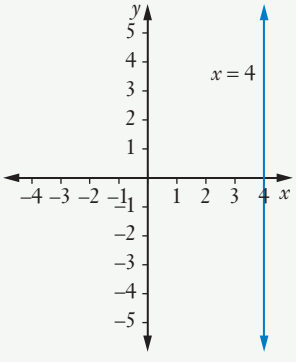
b Domain $(-\infty, \infty)$, range [2]

c Domain $[-4]$, range $(-\infty, \infty)$

d Domain [2], range $(-\infty, \infty)$

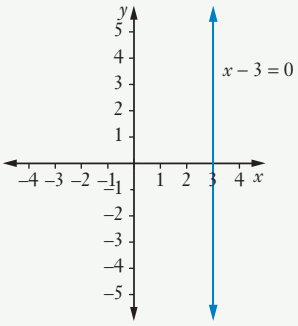
e Domain $(-\infty, \infty)$, range [3]

6 **a**



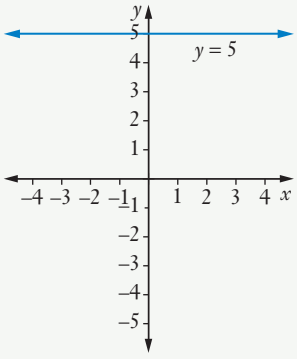
Domain [4], range $(-\infty, \infty)$

b



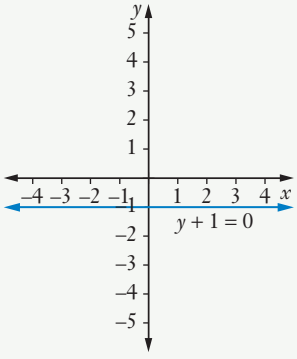
Domain [3], range $(-\infty, \infty)$

c



Domain $(-\infty, \infty)$, range [5]

d

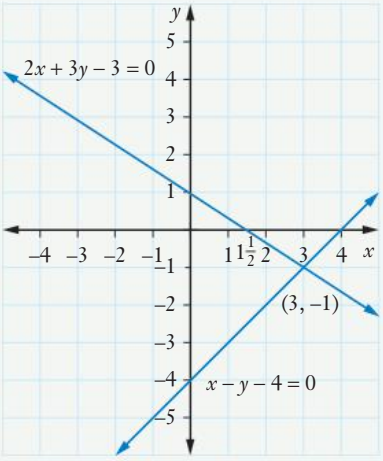


Domain $(-\infty, \infty)$, range $[-1]$

7 **a** $N = 36x$ **b** 1008 **c** 17

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8



Exercise 3.05

1 **a** 2 **b** $1\frac{1}{3}$ **c** $-1\frac{1}{3}$
d $-2\frac{2}{5}$ **e** $\frac{2}{3}$ **f** $-\frac{1}{8}$
g $-4\frac{1}{2}$ **h** $-\frac{2}{3}$ **i** $2\frac{1}{4}$

2 **a** 0.5 **b** 7.1 **c** 2.5
d −5.7 **e** −1.2 **f** −0.3

3 **a** **i** 3 **ii** 5
b **i** 2 **ii** 1
c **i** 6 **ii** −7
d **i** −1 **ii** 0
e **i** −4 **ii** 3
f **i** 1 **ii** −2
g **i** −2 **ii** 6
h **i** −1 **ii** 1
i **i** 9 **ii** 0

4 **a** $\frac{1}{3}$ **b** $-1\frac{1}{3}$ **c** −2
d $\frac{3}{5}$ **e** $\frac{1}{3}$

5 **a** 63°26′ **b** 59°32′ **c** 80°32′
d 101°19′ **e** 139°38′ **f** 129°48′

6 **a** **i** −2 **ii** 3
b **i** −5 **ii** −6
c **i** 6 **ii** −1
d **i** 1 **ii** 4
e **i** −2 **ii** $\frac{1}{2}$
f **i** 3 **ii** $1\frac{1}{2}$
g **i** $-\frac{1}{3}$ **ii** −2
h **i** $-\frac{4}{5}$ **ii** 2
i **i** $3\frac{1}{2}$ **ii** $-\frac{1}{2}$

7 **a** −2 **b** 0 **c** −1
d −3 **e** 2 **f** $-\frac{1}{4}$
g $1\frac{1}{2}$ **h** $1\frac{1}{4}$ **i** $\frac{2}{3}$
j $\frac{1}{5}$ **k** $\frac{2}{7}$ **l** $-\frac{3}{5}$
m $-\frac{1}{14}$ **n** 15 **o** $-1\frac{1}{2}$

8 $y_1 = 21$ **9** $x = 1.8$ **10** $x = 9$

11 **a** $P = \frac{d}{3}$ **b** 280 **c** $171

12 $m_{AB} = m_{CD} = 1.5$, $m_{AD} = m_{BC} = 0$. Opposite sides are parallel so $ABCD$ is a parallelogram.

Exercise 3.06

1 **a** $y = 4x - 1$ **b** $y = -3x + 4$ **c** $y = 5x$
d $y = 4x + 20$ **e** $3x + y - 3 = 0$
f $4x - 3y - 12 = 0$

2 **a** $4x - 3y + 7 = 0$ **b** $3x - 4y + 4 = 0$
c $4x - 5y + 13 = 0$ **d** $3x + 4y - 25 = 0$
e $x - 2y + 2 = 0$

3 **a** −4 **b** $y = -4x + 8$

4 **a** $y = 3$ **b** $x = -1$

5 **a** 0 **b** $y = -2x$

6 **a** $P = 15x + 20$
b **i** 380 **ii** 845 **iii** 3725
c **i** 145 **ii** 512 **iii** 843

7 **a** $c = 250n + 7000$
b **i** $32 000 **ii** $74 500 **iii** $307 000
c **i** 180 **ii** 285 **iii** 1440
d 10

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8 **a** $A = 450 - 8h$

b **i** 426 L **ii** 258 L

c 56.25 h or 2 days 8 hours 15 minutes

9 **a** $C = 20 - 1.69x$ **b** 11 songs

10 **a** $A = 20\,000 - 320x$

b **i** \$18 400 **ii** \$16 160 **iii** \$800

c 63 months or 5 years 3 months

11 **a** $P = 5x - 100$ **b** \$1400

c 231 **d** 20 balloons

Exercise 3.07

1 **a** -3 **b** $\frac{1}{3}$ **c** $\frac{3}{4}$

d $1\frac{1}{2}$ **e** 1 **f** $-\frac{5}{6}$

g $\frac{1}{3}$ **h** $\frac{1}{5}$

2 **a** $x - y + 1 = 0$ **b** $x - 3y + 16 = 0$

c $x + y - 5 = 0$ **d** $x + 2y + 5 = 0$

e $x - 2y + 4 = 0$ **f** $x + 3y - 1 = 0$

g $3x + 4y + 13 = 0$ **h** $x + y = 0$

i $5x - y - 8 = 0$ **j** $2x + y + 2 = 0$

k $2x - 3y + 16 = 0$

3 $m_1 = m_2 = 3$ so parallel

4 $m_1 m_2 = -\frac{1}{5} \times 5 = -1$ so perpendicular

5 $m_1 = m_2 = 1\frac{1}{5}$ so parallel

6 $m_1 m_2 = -\frac{7}{3} \times \frac{3}{7} = -1$ so perpendicular

7 $k = -\frac{2}{3}$

8 $m_1 = m_2 = 4$ so parallel

9 $AB \parallel CD$ ($m_1 = m_2 = 3$) and $BC \parallel AD$ $\left(m_1 = m_2 = -\frac{5}{8}\right)$. So $ABCD$ is a parallelogram.

10 Gradient $AC = m_1 = \frac{1}{2}$, gradient $BD = m_2 = -2$.

$m_1 m_2 = \frac{1}{2} \times -2 = -1$ so diagonals are perpendicular.

11 $7x + 6y - 24 = 0$

Exercise 3.08

1 **a** x-intercepts 0, –2, y-intercept 0

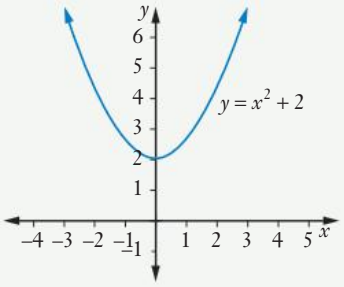
b x-intercepts 0, 3, y-intercept 0

c x-intercepts ± 1, y-intercept –1

d x-intercepts –1, 2, y-intercept –2

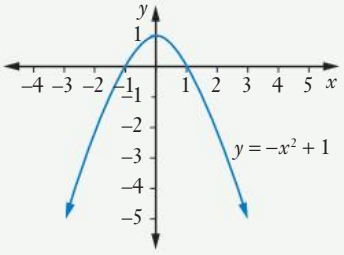
e x-intercepts 1, 8, y-intercept 8

2 **a**



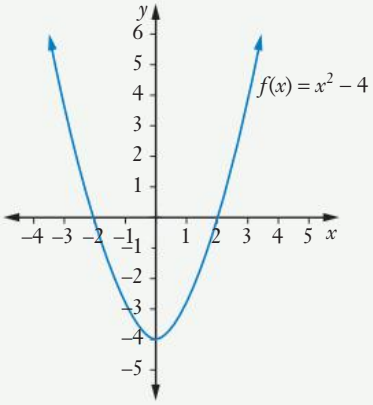
Minimum 2

b



Maximum 1

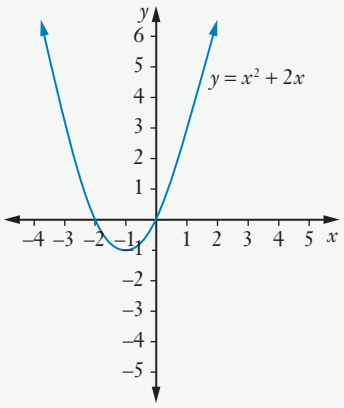
c



Minimum –4

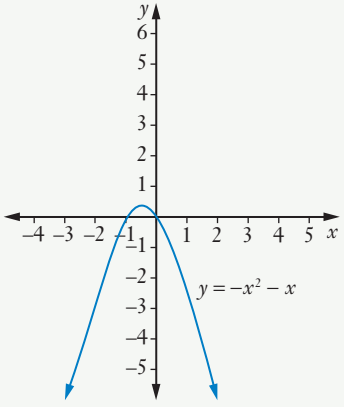
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d



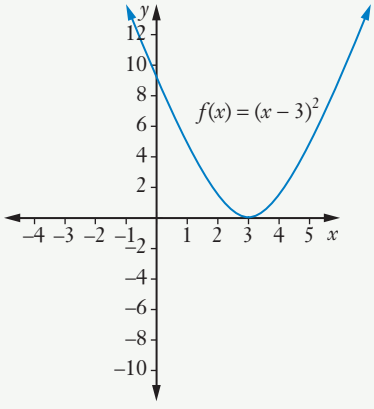
Minimum –1

e



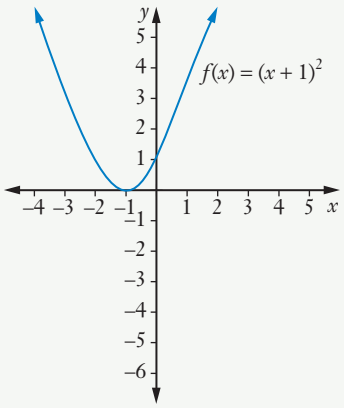
Maximum 0.25

f



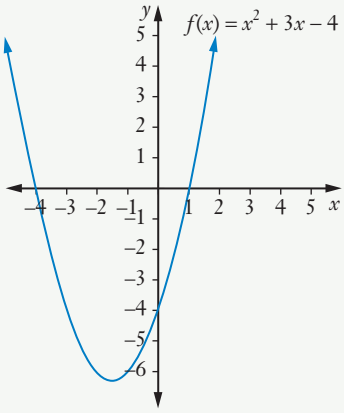
Minimum 0

g



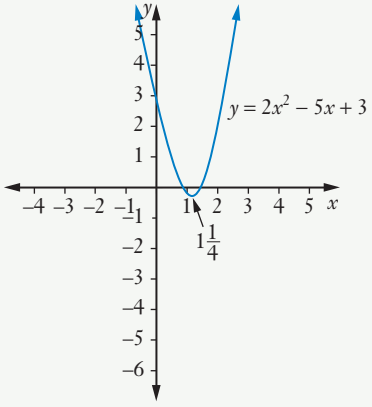
Minimum 0

h



Minimum –6.25

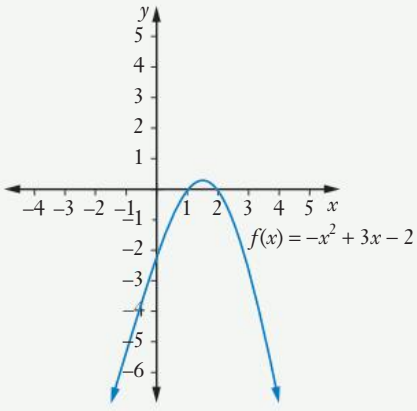
i



Minimum –0.125

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j



Maximum 0.25

3 **a** **i** x-intercepts 3, 4, y-intercept 12

ii Domain $(-\infty, \infty)$, range $[-\frac{1}{4}, \infty)$

b **i** x-intercepts 0, –4, y-intercept 0

ii Domain $(-\infty, \infty)$, range $[-4, \infty)$

c **i** x-intercepts –2, 4, y-intercept –8

ii Domain $(-\infty, \infty)$, range $[-9, \infty)$

d **i** x-intercept 3, y-intercept 9

ii Domain $(-\infty, \infty)$, range $[0, \infty)$

e **i** x-intercepts ± 2, y-intercept 4

ii Domain $(-\infty, \infty)$, range $(-\infty, 4]$

4 **a** Domain $(-\infty, \infty)$, range $[-5, \infty)$

b Domain $(-\infty, \infty)$, range $[-9, \infty)$

c Domain $(-\infty, \infty)$, range $\left[-2\frac{1}{4}, \infty\right)$

d Domain $(-\infty, \infty)$, range $(-\infty, 0]$

e Domain $(-\infty, \infty)$, range $[0, \infty)$

5 **a** $d = 6$ m **b** $w = 2.8$ m

Exercise 3.09

1 Axis of symmetry $x = -1$, minimum value –1

2 Axis of symmetry $x = 0$, minimum value –4

3 Axis of symmetry $x = \frac{3}{8}$, minimum turning point $\left(\frac{3}{8}, \frac{7}{16}\right)$

4 Axis of symmetry $x = 1$, maximum value –6

5 Axis of symmetry $x = -1$, vertex (–1, 7)

6 Axis of symmetry $x = -1.5$, minimum value –0.25

7 **a** $x = -3$, $(-3, -12)$ **b** $x = -4$, $(-4, 17)$

c $x = -3$, $(-3, -23)$ **d** $x = 1\frac{1}{4}, \left(1\frac{1}{4}, 3\frac{1}{8}\right)$

e $x = -1\frac{1}{4}, \left(-1\frac{1}{4}, -13\frac{1}{4}\right)$

8 **a** **i** $x = -1$ **ii** –3 **iii** (–1, –3)

b **i** $x = 1$ **ii** 1 **iii** (1, 1)

9 **a** (–1, 0), min **b** (4, –23), min

c (–2, –7), min **d** (1, –1), min

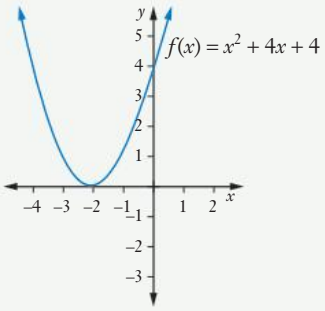
e (2, –11), min **f** $\left(-\frac{1}{4}, -3\frac{1}{8}\right)$, min

g (–1, 6), max **h** (2, 11), max

i $\left(\frac{1}{2}, 7\frac{3}{4}\right)$, max

10 **a** **i** –2 **ii** Minimum 0

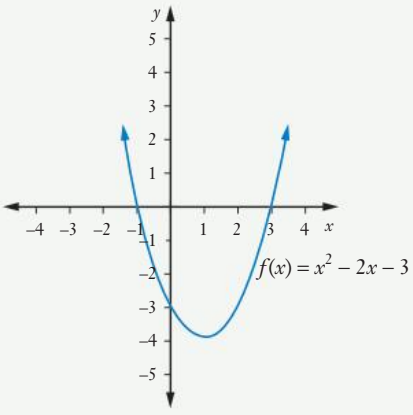
iii



iv $x = -2$

b **i** –1, 3 **ii** Minimum –4

iii

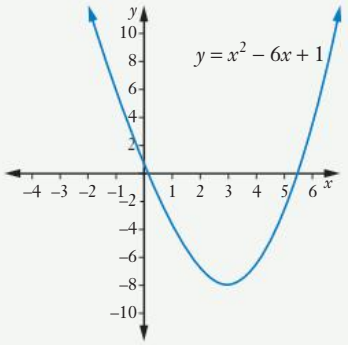


iv $x = 3, -1$

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c **i** 5.83, 0.17 **ii** Minimum −8

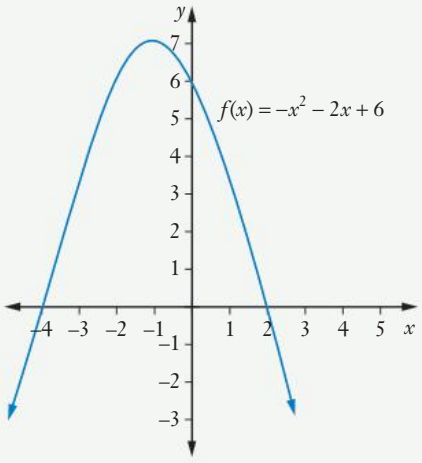
iii



iv $x = 5.8, 0.2$

d **i** 1.65, −3.65 **ii** Maximum 7

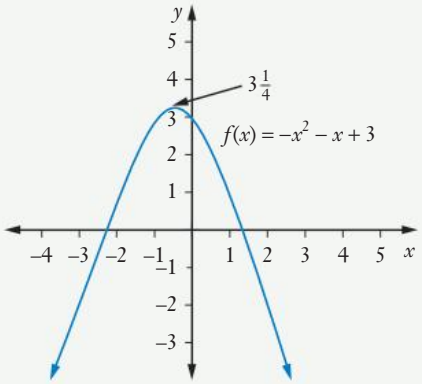
iii



iv $x = -3.65, 1.65$

e **i** 1.3, −2.3 **ii** Maximum $3\frac{1}{4}$

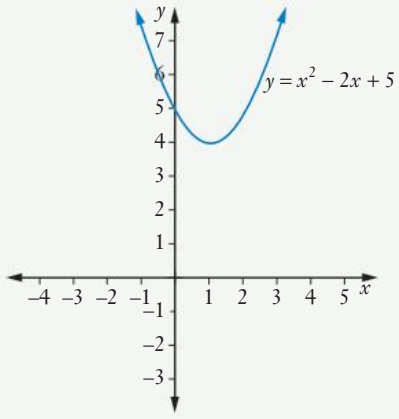
iii



iv $x = -2.3, 1.3$

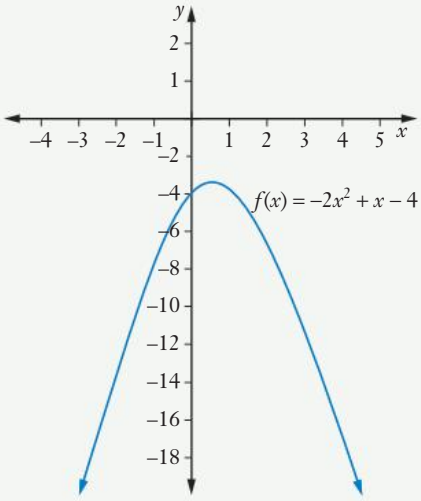
11 **a** 4 **b** None

c



12 **a** $-3\frac{7}{8}$ **b** None

c



13 $f(-x) = -(-x)^2 = -x^2 = f(x)$ $\therefore$ even

14 a, b, c, f

15 **a** $f(-w) = -\dfrac{(-w)^2}{800} + 200 = -\dfrac{w^2}{800} + 200 = f(w)$

b 200 m **c** 800 m

d 9.875 m **e** 565.7 m

Exercise 3.10

1 **a** 20 **b** −47 **c** −12

d 49 **e** 9 **f** −16

g 0 **h** 64 **i** 17

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2 **a** 17, unequal real irrational roots
b −39, no real roots
c 1, unequal real rational roots
d 0, equal real rational roots
e 33, unequal real irrational roots
f −16, no real roots
g 49, unequal real rational roots
h −116, no real roots
i 1, unequal real rational roots

3 $p = 1$ **4** $k = \pm 2$

5 $b \leq -\frac{7}{8}$ **6** $p > 2$

7 $k > -2\frac{1}{12}$

8 $a = 3 > 0$
$\Delta = -83 < 0$

9 Solving simultaneously:
$x^2 - 2x - 3 = 0$
$\Delta = 16 > 0$
So there are 2 points of intersection.

10 $\Delta = 68 > 0$
So there are 2 points of intersection.

11 $\Delta = -15 < 0$
So there are no points of intersection.

12 $\Delta = 0$
So there is 1 point of intersection
$\therefore$ the line is a tangent to the parabola.

Exercise 3.11

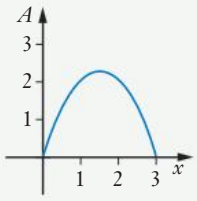
1 **a** $k = 0.004$ **b** $d = 25.6$ m
c Yes ($d = 10$ m) **d** No ($d = 48.4$ m)

2 **a** $A = 5x^2$ **b** 88.2 cm^2 **c** 7.1 cm

3 **a** $V = 8\pi r^2$ **b** 628.3 cm^3 **c** 2 cm

4 **a** $A = -x^2 + 3x$

b



c $x = 1.5$ **d** $A = 2.25$ units^2

5 **a** $y = x^2 - x - 5$ **b** $y = x^2 - 3x$
c $y = 2x^2 - 3x + 7$ **d** $y = x^2 + 4x - 9$
e $y = -x^2 - 2x + 1$

6 **a** $h = -5t^2 + 17.5t + 10$
b $h = 25$ m **c** At $t = 0, 3.5$ s

7 **a** $225y = -32x^2 + 1800$ **b** 6.72 cm
c 4.6 cm

8 **a** $y = x^2 - 4x$
b **i** $y = 5$ **ii** $y = 32$
c $x = 2$ **d** $x = 2 \pm \sqrt{6}$

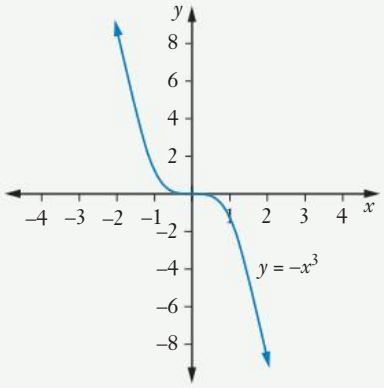
9 **a** $f(x) = x^2 + 2x + 7$ **b** $y = 22$
c $a = 1 > 0, \Delta = -24 < 0$

10 $y = 2x^2 - 4x$ **11** $y = -x^2 + 6x + 4$

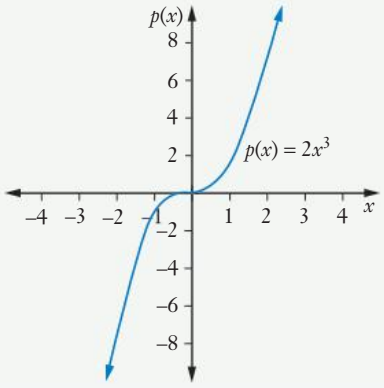
Exercise 3.12

1 **a** x-intercept 1, y-intercept −1
b x-intercept 2, y-intercept 8
c x-intercept −5, y-intercept 125
d x-intercept 4, y-intercept 64
e x-intercept −6, y-intercept 1026
f x-intercepts −5, 1, 2, y-intercept 10

2 **a**

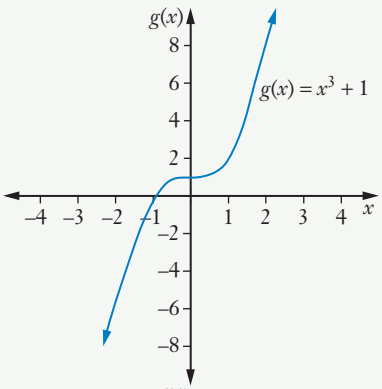


b

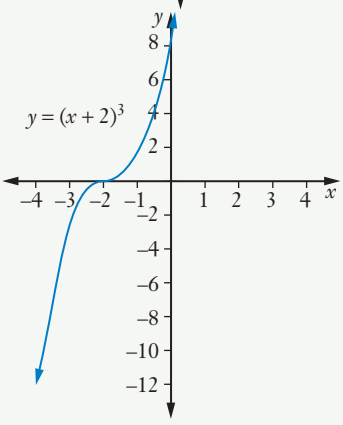


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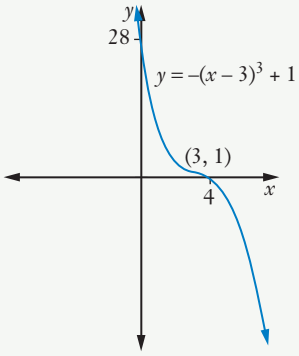
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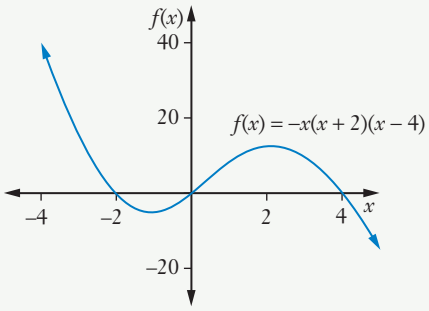
d



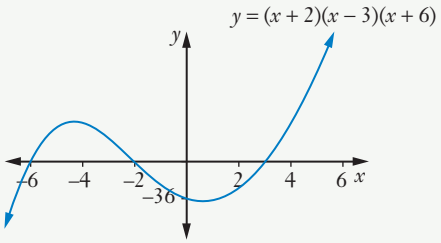
e



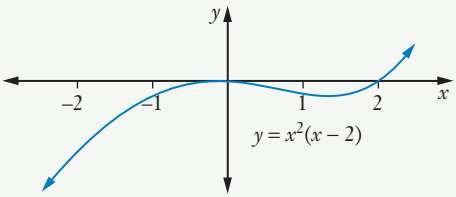
f



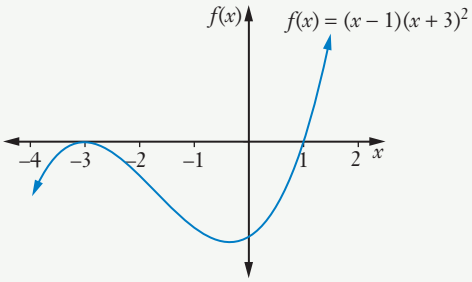
g



h



i



3 **a** (0, 1) **b** (0, 27) **c** (−2, 0)
d (1, −16) **e** (−1, 1)

4 **a** 1.4 **b** −0.3 **c** 0.7
d −1.9 **e** 0.9

5 **a** Increasing curve with x-intercept 4, point of inflection (0, −64) and y-intercept −64

b Decreasing curve with x-intercept 3, point of inflection (3, 0) and y-intercept 27

c LHS increasing to a maximum turning point between $x = -4$ and $x = -2$, then decreasing to a minimum turning point between $x = -2$ and $x = 0$, then increasing again on the RHS, x-intercepts −4, −2, 0, y-intercept 0

d LHS decreasing to a minimum turning point between $x = -3$ and $x = -1$, then increasing to a maximum turning point between $x = -1$ and $x = 4$, then decreasing again on the RHS, x-intercepts −3, −1, 4, y-intercept 24

e LHS increasing to a maximum turning point at $x = -5$, then decreasing to a minimum turning point between $x = -5$ and $x = 0$, then increasing again on the RHS, x-intercepts −5, 0, y-intercept 0

6 **a** $x = 1.7$ **b** $x = -1.3$ **c** $x = 1.65$
d $x = -1.1$ **e** $x = -0.8$ **f** $x = -2, 0, 1$

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7 **a** $V = 2.8x^3$ **b** 604.5 cm^3 **c** 4.5 cm

8 **a** $V = 4.2r^3$ **b** 65.4 mm^3 **c** 11.9 mm

9 $f(-x) = -(-x)^3 = x^3 = -f(-x)$ So an odd function.

10 a, d

11 **a** $y = 2x^3$ **b** $y = -x^3 + 5$ **c** $y = -3x^3 - 1$

d $y = 5x^3 - 7$ **e** $y = -4x^3 + 3$

12 **a** $y = -2(x - 2)(x - 3)(x + 5)$

b $y = 8(x + 1)(x - 4)(x - 6)$

c $y = -3(x - 1)(x - 3)(x + 3)$

Exercise 3.13

1 **a** 7 **b** 4 **c** 1 **d** 11

e 3 **f** 0

2 **a** −19 **b** −10 **c** −1

3 **a** −6 **b** 5 **c** 2 **d** 1 **e** 2

4 **a** 5 **b** 4 **c** −3 **d** 0

5 **a** ±3 **b** −5 **c** −2, 1

d 4 **e** 0

6 a, b, f

7 **a** $a = 0$ **b** $b = 10$ **c** $c = -6$

d $a = -1$ **e** $a = 4$

8 **a** $x = -2\frac{1}{2}$ **b** $x = 2, -1$ **c** 3

d 3 **e** x^5

9 **a** $\Delta = -8 < 0$

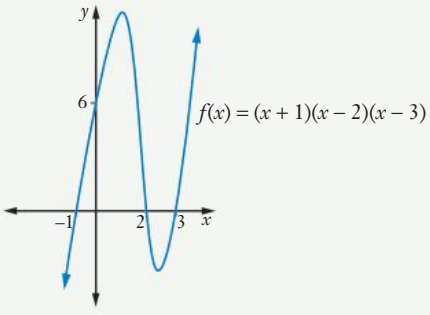
b $9x^3$ **c** −2 **d** 9

e $x = \frac{2}{3}, -1$

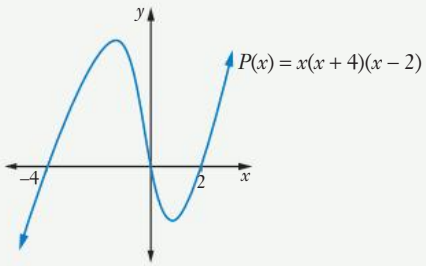
10 **a** 2 **b** 0 **c** 2 **d** 0

e 2 **f** 4

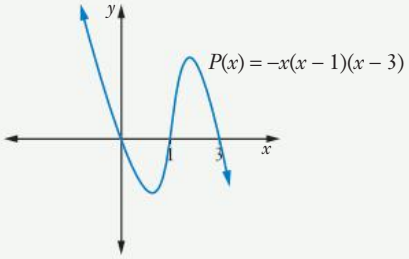
11 **a**



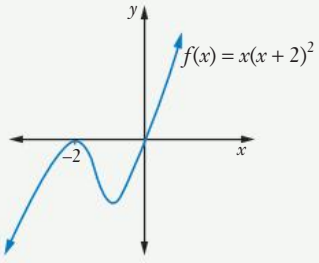
b



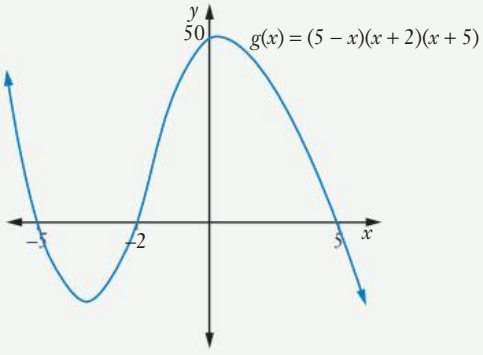
c



d



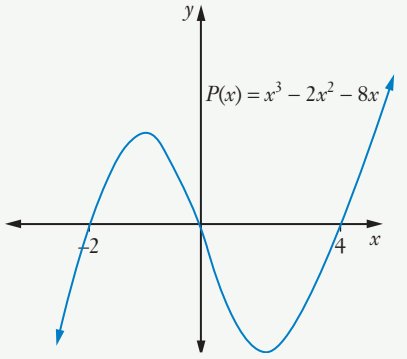
e



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12 a i $P(x) = x(x-4)(x+2)$

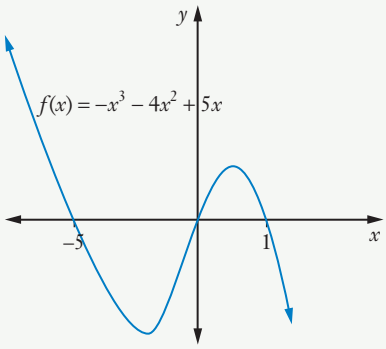
ii



Increases to a maximum turning point, then decreases to a minimum turning point, then increases.

b i $f(x) = -x(x-1)(x+5)$

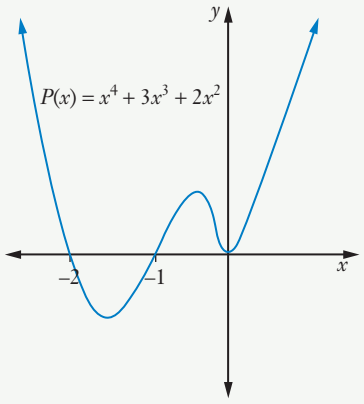
ii



Decreases to a minimum turning point, then increases to a maximum turning point, then decreases.

c i $P(x) = x^2(x+1)(x+2)$

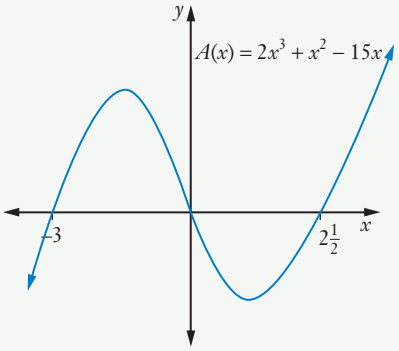
ii



Decreases to a minimum turning point, then increases to a maximum turning point, then decreases to a minimum turning point, then increases.

d i $A(x) = x(2x-5)(x+3)$

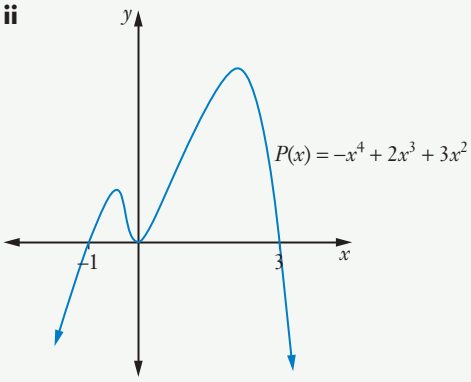
ii



Increases to a maximum turning point, then decreases to a minimum turning point, then increases.

e i $P(x) = -x^2(x-3)(x+1)$

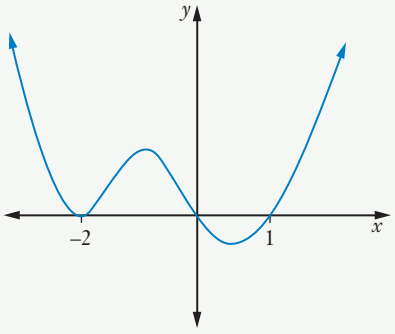
ii



Increases to a maximum turning point, then decreases to a minimum turning point, then increases to a maximum point then decreases.

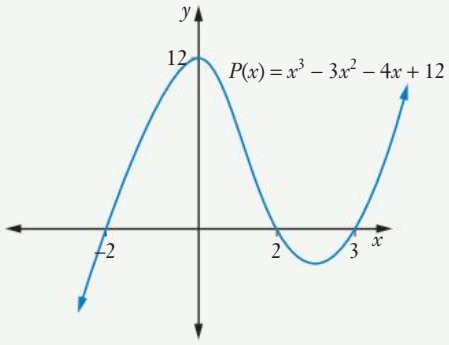
13 a $x = 0, 1, -2$

b



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14 b



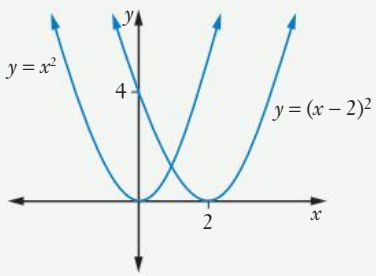
Exercise 3.14

1 a i $x = 2$ **ii** $x = 1$ **iii** $x = 4$
b i $x = 0, 2$ **ii** $x = 3, -1$
c i $x = 1$ **ii** $x = 2$ **iii** $x = -1$

2 a (1, 4) **b** (–1, –4)
c (0, 2) **d** (1, –1)
e (2, 1)

3 a (0, 0), (1, 1) **b** (–2, 4), (2, 4)
c (2, 4), (–1, 1) **d** (–3, 9), (1, 1)
e (5, 20), (–1, –4)

4 a



b 1 **c** (1, 1)

5 (–1, 1)

6 (–3, 4), (–2, –1)

7 (–1, 3)

8 a (140, 1400) i.e. 140 roses at $1400
b $343 **c** $665 loss

9 71 calculators

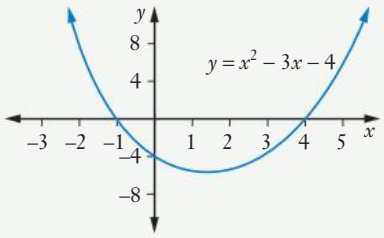
10 a Income: $y = 5x$, Costs: $y = x + 264$
b 66 cupcakes **c** $736 **d** $64 loss

11 a $y = 18x$ **b** $y = 12x^2$ **c** $x = 1.5$

Test yourself 3

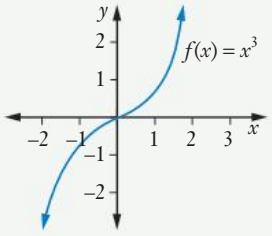
1 C **2** A **3** B
4 B **5** D
6 a $f(-2) = 6$ **b** $f(a) = a^2 - 3a - 4$
c $x = 4, -1$

7 a



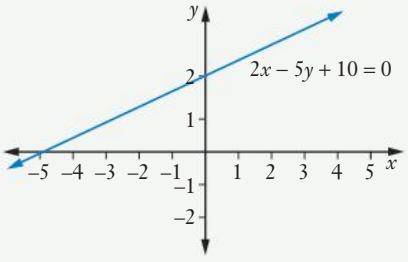
Domain $(-\infty, \infty)$, range $[-6\frac{1}{4}, \infty)$

b



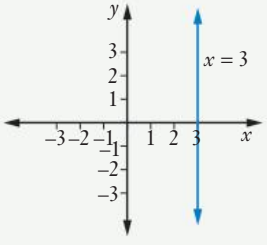
Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

c



Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

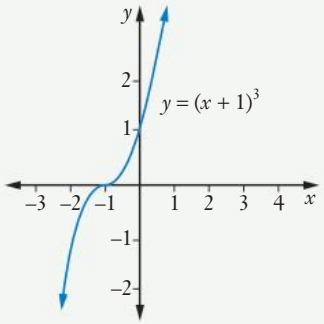
d



Domain [3], range $(-\infty, \infty)$

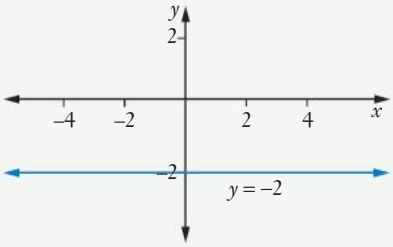
ISBN 9780170413152

e



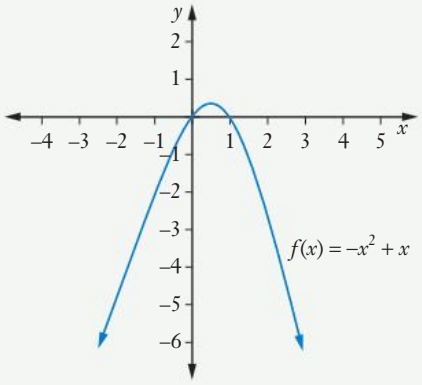
Domain $(-\infty, \infty)$, range: $(-\infty, \infty)$

f



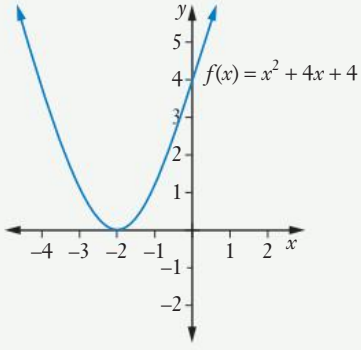
Domain $(-\infty, \infty)$, range $[-2]$

g



Domain $(-\infty, \infty)$, range $(-\infty, 0.25]$

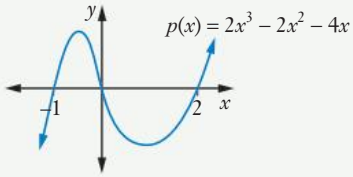
h



Domain $(-\infty, \infty)$, range $[0, \infty)$

8 **a** 2 **b** $x = 3\frac{2}{3}$ **c** $x = 1\frac{1}{3}$

9

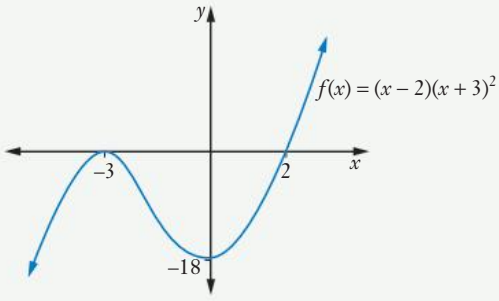


10 **a** $-1\frac{1}{5}$ **b** 2

c $\frac{3}{5}$ **d** 1

11 **a** $x = 2$ **b** -3

12



13 **a** 3 **b** -3

c $x = 0, -3, 1$ **d** x^3

14 **a** x-intercept -10, y-intercept 4

b x-intercepts -2, 7, y-intercept -14

c x-intercept -2, y-intercept 8

d x-intercept 5, y-intercept -2

15 $(-1, 1)$

16 **a** $x = -\frac{1}{4}$ **b** $6\frac{1}{8}$

17 Domain $(-\infty, \infty)$, range $(-\infty, 6\frac{1}{8}]$

18 **a** D **b** B **c** C

d B **e** A

19 **a** $7x - y - 11 = 0$ **b** $5x + y - 6 = 0$

c $3x + 2y = 0$ **d** $3x + 5y - 14 = 0$

e $x - 3y - 3 = 0$

20 $a = 2, b = -18, c = 40$

21 **a** Even **b** Neither **c** Odd

d Neither **e** Odd

22 $f(-x) = -(x^3 - x) = -f(x)$.

23 $m_1 = -\frac{1}{4}$, $m_2 = 4$ so $m_1 m_2 = -1$

24 $a = -1 < 0$

$\Delta = -7 < 0$

$\therefore -4 + 3x - x^2 < 0$ for all x.

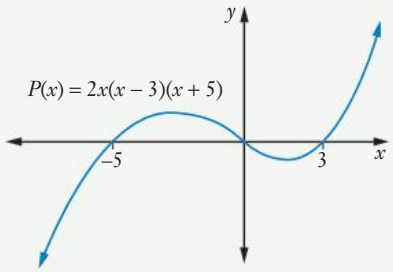
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25 **a** 2 **b** 2 **c** 0
d 1

26 $m_1 = m_2 = 5$ so lines are parallel.

27 $x = 4, 5$

28



29 $x = 0, 2$

30 $x = 1\frac{4}{5}$

31 15

32 **a** 4 **b** 5 **c** 9
d 3 **e** 2

33 **a** $y = x^2 - 5x + 4$
b $y = -2x^2 + 2x + 12$

34 **a** $A = 7x^2$ **b** 700 cm^2
c 12.5 cm

35 **a** function, not one-to-one
b not a function
c not a function
d function, one-to-one
e not a function

36 $f(x) = -3x^3 + 5$

37 (3, 36)

38 **a** $2x + y - 1 = 0$ **b** $\frac{1}{2}$

39 $m < -\frac{9}{16}$

40 **a** (1, −1) **b** (1, 0), (2, 2)
c (−3, 9), (3, 9)

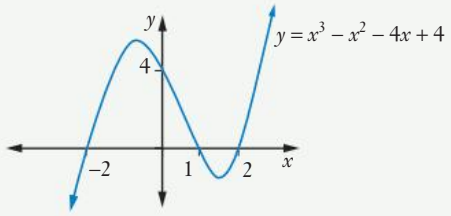
41 $3x - 4y = 0$

42 $3x - 7y - 14 = 0$

43 **a** $A = 0.12d$ **b** 31 L **c** 296 km

44 **a** $x = 1, \pm 2$
b x-intercepts $1, \pm 2$; y-intercept 4

c

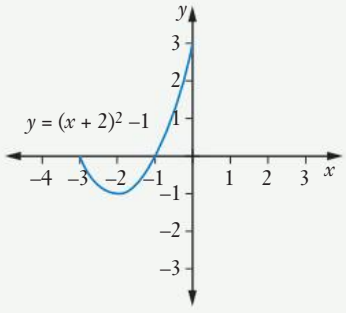


d **i** 3 **ii** 1

Challenge exercise 3

1 $b = -\frac{2}{3}, 3$

2



3 $k = -2$

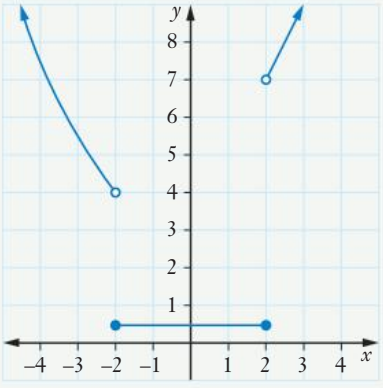
4 $2x + 3y + 13 = 0$

5 $a = 2, b = 3$

6 **a** $x = 1\frac{2}{3}, \frac{1}{3}$ **b** $x = 0, 2$ **c** $x = 1, 3$
d $x = \pm 1$

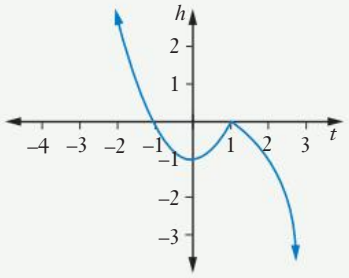
7 $y = 3$

8 $f(3) = 9, f(-4) = 16, f(0) = 1$



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9 $h(2) + h(-1) - h(0) = -3 + 0 - (-1) = -2$



10 $x = 0, 3, -2$

11 $\Delta = (k-4)^2 \geq 0$ and a perfect square
$\therefore$ real, rational roots

12 $p > 0.75$

13 $f((-a)^2) = 2(-a^2) - 1 = 2a^2 - 1 = f(a^2)$

14 $2x + 5y + 14 = 0$

15 (0, 0), (1, 1)

16 $y = x^3 + 2x^2 - x + 1$

17 $b^2 - 4ac = 0$. So equal roots.

18 $P(x) = (x+2)(x-1)(x-6)$

Chapter 4

Exercise 4.01

1 $\cos\theta = \frac{5}{13}, \sin\theta = \frac{12}{13}, \tan\theta = \frac{12}{5}$

2 $\sin\beta = \frac{4}{5}, \tan\beta = \frac{4}{3}, \cos\beta = \frac{3}{5}$

3 $\sin\beta = \frac{7}{\sqrt{74}}, \tan\beta = \frac{7}{5}, \cos\beta = \frac{5}{\sqrt{74}}$

4 $\cos x = \frac{5}{9}, \tan x = \frac{\sqrt{56}}{5}, \sin x = \frac{\sqrt{56}}{9}$

5 $\cos\theta = \frac{3}{5}, \sin\theta = \frac{4}{5}$

6 $\tan\theta = \frac{\sqrt{5}}{2}, \sin\theta = \frac{\sqrt{5}}{3}$

7 $\cos\theta = \frac{\sqrt{35}}{6}, \tan\theta = \frac{1}{\sqrt{35}}$

8 $\tan\theta = \frac{\sqrt{51}}{7}, \sin\theta = \frac{\sqrt{51}}{10}$

9 a $\sqrt{2}$ **b** 45°

c $\sin 45° = \frac{1}{\sqrt{2}}, \cos 45° = \frac{1}{\sqrt{2}}, \tan 45° = 1$

10 a $\sqrt{3}$

b $\sin 30° = \frac{1}{2}, \cos 30° = \frac{\sqrt{3}}{2}, \tan 30° = \frac{1}{\sqrt{3}}$

c $\sin 60° = \frac{\sqrt{3}}{2}, \cos 60° = \frac{1}{2}, \tan 60° = \sqrt{3}$

11 **a** 47° **b** 82° **c** 19°
d 77° **e** 52°

12 **a** 47°13′ **b** 81°46′ **c** 19°26′
d 76°37′ **e** 52°30′

13 **a** 77.75° **b** 65.5° **c** 24.85°
d 68.35° **e** 82.517°

14 **a** 59°32′ **b** 72°14′ **c** 85°53′
d 46°54′ **e** 73°13′

15 **a** 0.635 **b** 0.697 **c** 0.339
d 0.928 **e** 1.393

16 **a** 17°20′ **b** 34°20′ **c** 34°12′
d 46°34′ **e** 79°10′

Exercise 4.02

1 **a** $x = 6.3$ **b** $y = 5.6$ **c** $b = 3.9$ cm
d $x = 5.6$ m **e** $m = 2.9$ **f** $x = 13.5$
g $y = 10.0$ **h** $p = 3.3$ **i** $x = 5.1$ cm
j $t = 28.3$ **k** $x = 3.3$ cm **l** $x = 2.9$ cm
m $x = 20.7$ cm **n** $x = 20.5$ mm **o** $y = 4.4$ m
p $k = 20.6$ cm **q** $h = 17.3$ m **r** $d = 1.2$ m
s $x = 17.4$ cm **t** $b = 163.2$ m

2 1.6 m **3** 20.3 cm **4** 13.9 m

5 **a** 18.4 cm **b** 13.8 cm

6 10 cm and 10.5 cm

7 47.4 mm **8** 20.3 m

9 **a** 7.4 cm **b** 6.6 cm **c** 9.0 cm

10 **a** 12.6 cm **b** 22.2 cm **11** 38 cm

Exercise 4.03

1 **a** $x = 39°48′$ **b** $\alpha = 35°06′$ **c** $\theta = 37°59′$
d $\alpha = 50°37′$ **e** $\alpha = 38°54′$ **f** $\beta = 50°42′$
g $x = 44°50′$ **h** $\theta = 30°51′$ **i** $\alpha = 29°43′$
j $\theta = 45°37′$ **k** $\alpha = 57°43′$ **l** $\theta = 43°22′$
m $\theta = 37°38′$ **n** $\theta = 64°37′$ **o** $\beta = 66°16′$
p $\alpha = 29°56′$ **q** $\theta = 54°37′$ **r** $\alpha = 35°58′$
s $\theta = 59°2′$ **t** $\gamma = 56°59′$

2 37°57′ **3** 22°14′ **4** 36°52′ **5** 50°

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6 **a** 11.4 cm **b** 37°52′

7 $\alpha = 31°58'$, $\beta = 45°44'$

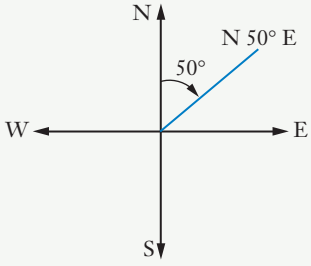
8 **a** 13 m **b** 65°13′

9 **a** 11°19′ **b** 26 cm

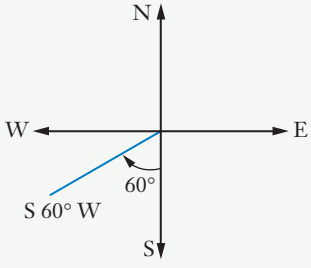
10 **a** 12.9 m **b** 56°37′

Exercise 4.04

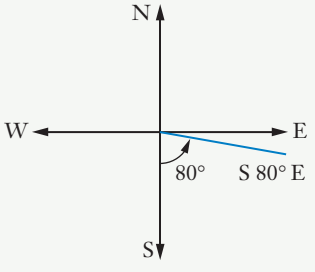
1 **a**



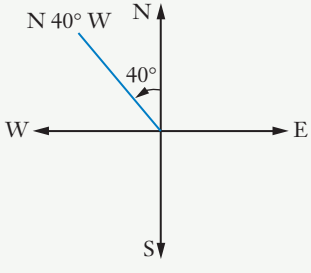
b



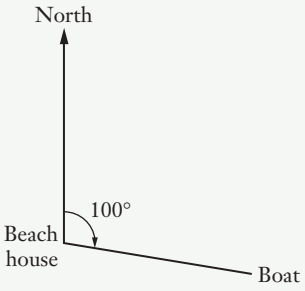
c



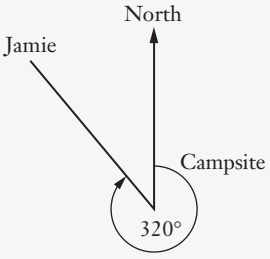
d



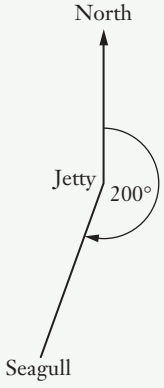
e



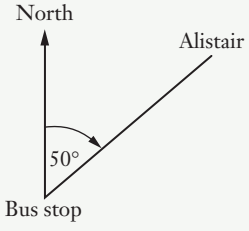
f



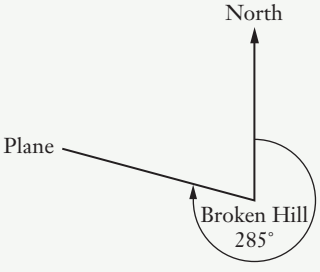
g



h

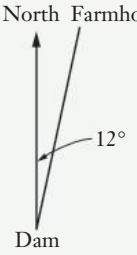


i

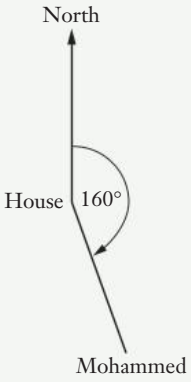


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j



k



2 **a** **i** S 35° E **ii** 145°
b **i** N 80° E **ii** 080°
c **i** N 23° W **ii** 337°
d **i** S **ii** 180°

3 080° **4** 210° **5** 160° **6** 10.4 m
7 21 m **8** 126.9 m **9** 72°48′
10 **a** 1056.5 km **b** 2265.8 km **c** 245°
11 83.1 m **12** 1.8 km **13** 12 m **14** 242°
15 035° **16** 9.2 m **17** 171 m
18 9.8 km **19** 51°41′ **20** 2.6 m
21 9°2′ **22** 1931.9 km **23** 35 m
24 149° **25** 198 m **26** 4.8 km
27 9.2 m **28** 217°
29 **a** 1.2 km **b** 7.2 km
30 **a** 13.1 m **b** 50°26′

Exercise 4.05

1 **a** $x = 8.9$ **b** $y = 9.4$ cm **c** $a = 10.0$
d $b = 10.7$ m **e** $d = 8.0$
2 **a** $\theta = 51°50′$ or $128°10′$
b $\alpha = 61°23′$ or $118°37′$
c $x = 43°03′$ or $136°57′$
d $\theta = 29°4′$ or $150°56′$
e $\alpha = 87°4′$
3 $126°56′$
4 **a** 13.5 mm **b** 25 mm
5 **a** 1.8 m **b** 2.7 m **6** 5.7 cm
7 **a** 10.3 m **b** 9.4 m
8 **a** 60°22′ **b** 57°9′
9 **a** 14.1 cm **b** 15.6 cm
10 **a** 54.7 mm **b** 35.1 mm
11 **a** 74° or 106° **b** 52° **c** 55° or 125°

Exercise 4.06

1 **a** $m = 5.8$ **b** $b = 10.4$ m **c** $h = 7.4$ cm
d $n = 16.4$ **e** $y = 9.3$
2 **a** $\theta = 54°19′$ **b** $\theta = 60°27′$ **c** $x = 57°42′$
d $\beta = 131°31′$ **e** $\theta = 73°49′$
3 32.94 mm **4** 11.2 cm and 12.9 cm
5 **a** 11.9 cm **b** 44°11′ **c** 82°12′
6 $\angle XYZ = \angle XZY = 66°10′$, $\angle YXZ = 47°40′$
7 **a** 18.1 mm **b** 78°47′
8 **a** 6.2 cm **b** 12.7 cm
9 12.9 cm
10 **a** 11 cm **b** 30°

Exercise 4.07

1 **a** 7.5 cm^2 **b** 32.3 $units^2$ **c** 9.9 mm^2
d 30.2 $units^2$ **e** 6.3 cm^2
2 7.5 cm^2 **3** 15.5 cm^2 **4** 34.8 cm^2
5 1.2 m^2
6 **a** 7.8 m **b** 180.7 m^2
7 **a** 5.6 cm **b** 18.5 cm^2 **c** 18.9 cm^2
8 $\frac{25\sqrt{3}}{4}$ cm^2

Exercise 4.08

1 **a** 040° **b** 305°
2 16.4 m **3** 28°
4 **a** 1.21 km **b** 1 minute
5 32 m **6** 107 m
7 $h = 8.5$ **8** 7.7 km
9 5.4 km from A and 5.7 km from B
10 1841 km **11** 89°52′
12 9.9 km **13** 163.5 km
14 3269 km
15 **a** 11.3 cm **b** 44°45′ or 135°15′

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16 141°

17 **a** 11.6 cm **b** 73°14′

18 **a** 35°5′ **b** **i** 4.5 m **ii** 0.55 m

19 **a** 109 cm^2 **b** 16°20′

20 65°9′

21 **a** 9.37 m **b** 25°

22 **a** 56 m **b** 89.7 m

23 **a** 48 m **b** 128.6 m **c** 97.7 m

24 11°10′

Exercise 4.09

1 **a** 36° **b** 120° **c** 225°
d 210° **e** 540° **f** 140°
g 240° **h** 420° **i** 20°
j 50°

2 **a** $\frac{3\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $\frac{5\pi}{6}$
d $\frac{4\pi}{3}$ **e** $\frac{5\pi}{3}$ **f** $\frac{7\pi}{20}$
g $\frac{\pi}{12}$ **h** $\frac{5\pi}{2}$ **i** $\frac{5\pi}{4}$
j $\frac{2\pi}{3}$

3 **a** 0.98 **b** 1.19 **c** 2.22
d 5.04 **e** 5.45

4 **a** 0.32 **b** 0.61 **c** 1.78
d 1.54 **e** 0.88

5 **a** 62°27′ **b** 44°0′ **c** 66°28′
d 56°43′ **e** 18°20′ **f** 183°21′
g 154°42′ **h** 246°57′ **i** 320°51′
j 6°18′

6 **a** 0.34 **b** 0.07 **c** 0.06
d 0.83 **e** −1.14 **f** 0.33
g −1.50 **h** 0.06 **i** −0.73
j 0.16

7 **a** $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $\frac{1}{\sqrt{3}}$
d $\frac{\sqrt{3}}{2}$ **e** 1 **f** $\frac{1}{2}$
g $\frac{1}{\sqrt{2}}$ **h** $\frac{\sqrt{3}}{2}$ **i** $\sqrt{3}$

Exercise 4.10

1 **a** 4π cm **b** π m **c** $\frac{25\pi}{3}$ cm
d $\frac{\pi}{2}$ cm **e** $\frac{7\pi}{4}$ mm

2 **a** 0.65 m **b** 3.92 cm **c** 6.91 mm
d 2.39 cm **e** 3.03 m

3 1.8 m **4** 7.5 m **5** $\frac{2\pi}{21}$

6 25 mm **7** 1.83 **8** $13\frac{7}{9}$ mm

9 **a** 48.3 mm **b** 25.3 mm

10 $\frac{125\sqrt{35}\,\pi}{648}$ cm^3

Exercise 4.11

1 **a** 8π cm^2 **b** $\frac{3\pi}{2}$ m^2 **c** $\frac{125\pi}{3}$ cm^2
d $\frac{3\pi}{4}$ cm^2 **e** $\frac{49\pi}{8}$ mm^2

2 **a** 0.48 m^2 **b** 6.29 cm^2 **c** 24.88 mm^2
d 7.05 cm^2 **e** 3.18 m^2

3 16.6 m^2 **4** 4.4 **5** 6 m

6 **a** $\frac{7\pi}{6}$ cm **b** $\frac{49\pi}{12}$ cm^2

7 $\frac{6845}{8\pi}$ mm^2 **8** 75 cm^2 **9** 11.97 cm^2

10 $\frac{\pi}{15}$, 3 cm

11 **a** $\frac{3\pi}{7}$ cm **b** $\frac{9\pi}{14}$ cm^2

12 **a** $\frac{5\pi}{6}$ cm **b** $\frac{25\pi}{12}$ cm^2 **c** 2.6 cm

13 **a** 10.5 mm **b** 42.9 mm^2

14 **a** 2π cm^2 **b** 3.1 cm **c** $4\sqrt{2}$ cm^2
d $2\pi - 4\sqrt{2}$ cm^2

15 **a** 77°22′ **b** 70.3 cm^2 **c** 26.96 cm^2
d 425.43 cm^2

16 14.98 cm **17** $\frac{225\pi}{2}$ cm^3

Test yourself 4

1 C **2** A, C **3** C, D

4 $\cos\theta = \frac{5}{\sqrt{34}}$, $\sin\theta = \frac{3}{\sqrt{34}}$

5 **a** 0.64 **b** 1.84 **c** 0.95
d 0.14 **e** 0.37

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6 **a** $\theta = 46°3'$ **b** $\theta = 73°23'$ **c** $\theta = 35°32'$

7 122 km **8** $5\sqrt{3}$

9 **a** 6.3 cm **b** 8.7 m

10 **a** 42°58′ **b** 74°29′ **c** 226°19′

d 240°39′ **e** 324°18′

11 **a** $\theta = 65°5'$ **b** $\theta = 84°16'$

c $\theta = 39°47', 140°13'$

12 65.3 cm^2

13 **a** 209° **b** 029°

14 **a** $AD = \dfrac{20\sin 39°}{\sin 99°}$ **b** 8.5 m

15 2951 km

16 **a** $\dfrac{\pi}{3}$ **b** $\dfrac{\pi}{4}$ **c** $\dfrac{5\pi}{6}$

d π **e** $\dfrac{\pi}{9}$

17 **a** $\dfrac{5\pi}{6}$ cm **b** $\dfrac{25\pi}{12}$ cm^2

18 **a** $\sqrt{3}$ **b** $\dfrac{\sqrt{3}}{2}$ **c** $\dfrac{1}{\sqrt{2}}$

d $\dfrac{1}{\sqrt{3}}$ **e** $\dfrac{1}{\sqrt{2}}$ **f** $\dfrac{1}{2}$

g 1 **h** $\dfrac{1}{2}$ **i** $\dfrac{\sqrt{3}}{2}$

19 **a** $\dfrac{8\pi}{7}$ cm^2 **b** 0.12 cm^2

20 $\alpha = 51°40'$ **21** 54°19′ or 125°41′

Challenge exercise 4

1 92°58′ **2** $x = 12.7$ cm

3 **a** $AC = \dfrac{25.3\sin 39°53'}{\sin 41°21'}$ **b** $h = 25.2$ cm

4 31 m

5 **a** 65 m **b** 27°42′

6 **a** 30.1 m **b** 0.5 m s^{-1}

7 42 cm^2 **8** 247.7 mm^2 **9** 9.4 cm^2

10 **a** 3.84 cm **b** 0.88 cm^2 **c** 25.84 cm

11 84 m

12 **a** 31.4 cm **b** 75.4 cm^2

Practice set 2

1 B, D **2** A **3** C

4 C **5** B **6** A, B, D

7 **a** 52°26′ **b** 16.6 m^2

8 **a** $y = -2x + 3$ **b** $x - 5y - 5 = 0$

c $4x - 5y - 8 = 0$ **d** $5x - 4y - 41 = 0$

e $2x + 3y - 3 = 0$ **f** $x - 8y + 15 = 0$

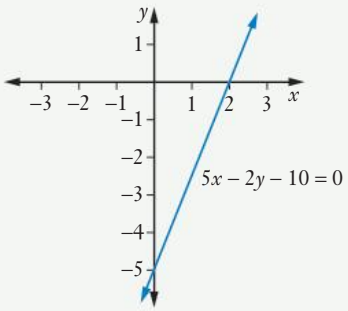
g $x + y - 4 = 0$

9 **a** $\dfrac{3x}{x-4}$ **b** $\dfrac{5x}{y(y-2)}$ **c** $\dfrac{11a-17}{20}$

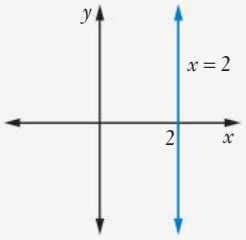
10 **a** $\dfrac{\pi}{3}$ **b** $\dfrac{5\pi}{6}$ **c** $\dfrac{\pi}{2}$

d $\dfrac{\pi}{18}$ **e** $\dfrac{7\pi}{4}$

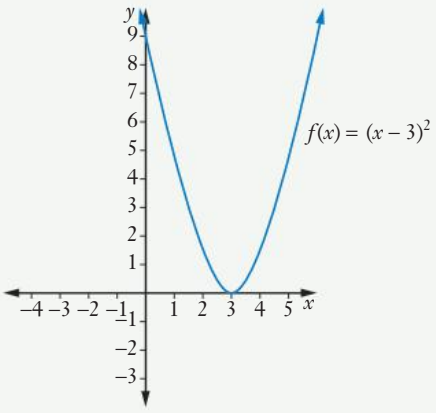
11 **a**



b

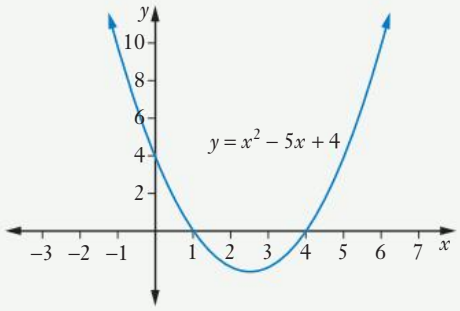


c

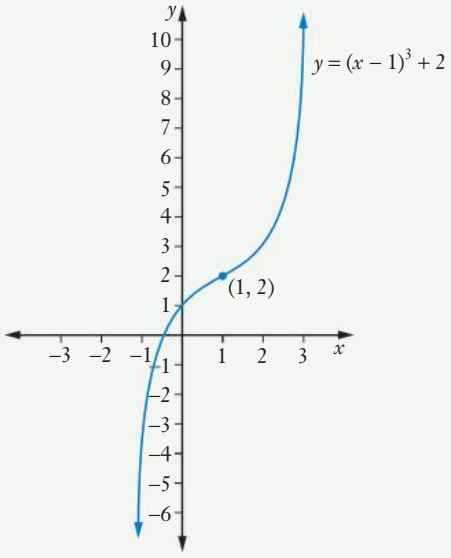


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d



e



12 **a** 97°24′ **b** 20°38′ **c** 145°32′

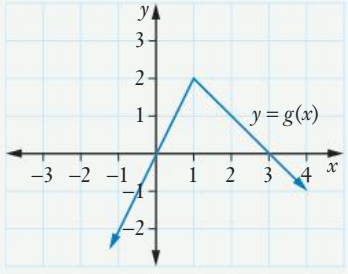
13 **a** $m_1m_2 = \frac{3}{4} \times \left(-\frac{8}{6}\right) = -1$. So perpendicular.

b $A = \left(-1, 1\frac{1}{2}\right)$

14 $-1\frac{2}{3}$

15 **a** $g(2) = 1, g(-3) = -6$

b



16 $x = 3$ **17** $f(-1) = 7$

18 Solve simultaneous equations:
$25x^2 + 60x + 36 = 0$
$\Delta = 0$
One solution so the line is a tangent to the circle.

19 **a** 45° **b** 270° **c** 36°
d 157°30′ **e** 1080°

20 $\cos\theta = \frac{5}{7}$, $\sin\theta = \frac{\sqrt{24}}{7}$, $\tan\theta = \frac{\sqrt{24}}{5}$

21 **a** $a < 0, \Delta < 0$
b $a > 0, \Delta < 0$

22 **a** $d = 4.8t$
b **i** 9.6 km **ii** 24 km
c **i** 6.25 h **ii** 3.96 h **d** 4.8 km/h

23 $y = x^2 - 2x - 3$

24 $\Delta = 361$ (> 0 and a perfect square)

25 76°52′ **26** 45°49′

27 52°37′

28 **a** $2x - y + 4 = 0$ **b** $P(-2, 0), Q(0, 4)$

29 $f(-x) = x^6 - x^2 - 3$

30 2°8′

31 **a** **i** N 50° E **ii** 050°
b **i** S 20° W **ii** 200°
c **i** S 40° E **ii** 140°
d **i** N 50° W **ii** 310°

32 **a** $\frac{5\pi}{6}$ cm **b** $\frac{25\pi}{12}$ cm^2

33 **a** $(x - 2)^2$ **b** $(3x + 1)(3x - 1)$

34 117°56′

35 $y = 16.5$

36 **a** (4, −1) **b** (3, 9), (−2.5, 6.25)

37 **a** **i** $x = 3$ **ii** Minimum (3, −8)
b **i** $x = -1$ **ii** Maximum (−1, −1)

38 17.5 m

39 **a** 7 m **b** 27.8 m^2

40 127 m

41 21π cm^2

42 4.9 km

43 **a** 8.1 m **b** 35°46′

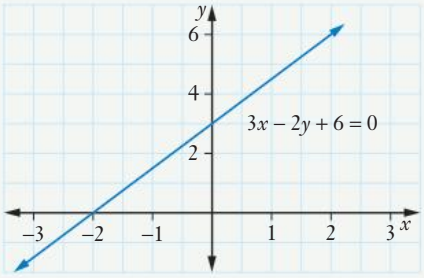
44 **a** Domain $(-\infty, -4) \cup (-4, \infty)$, range $(-\infty, 0) \cup (0, \infty)$
b Domain $(-\infty, \infty)$, range $[2, \infty)$
c Domain $(-\infty, \infty)$, range [4]
d Domain $(-\infty, \infty)$, range $[-3, \infty)$

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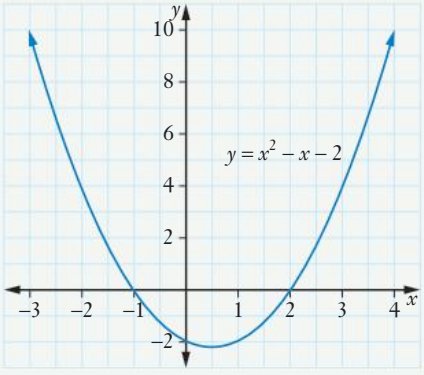
45 **a** 16.85 km **b** 228°

46 $f(-x) = -x^3 + 5x = -(x^3 - 5x) = -f(x)$

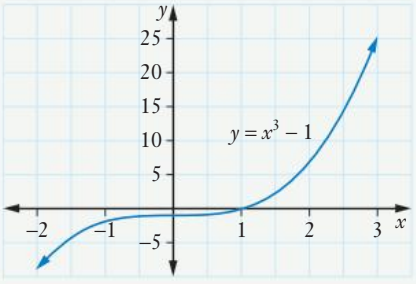
47 **a**



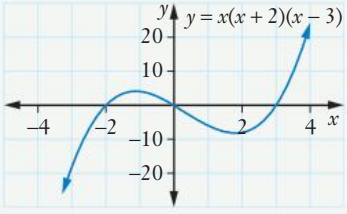
b



c



d



48 1.6 cm^2

49 **a** 360° **b** 30° **c** 405°

50 489.3 km

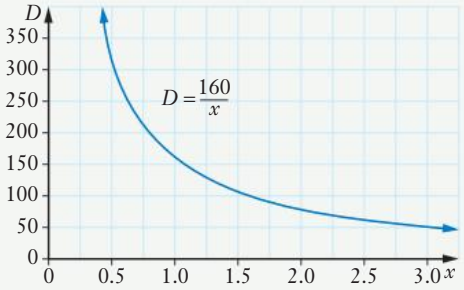
51 **a** $b \geq 2$ **b** $x = 0, 3$ **c** $n = 2, -7$

Chapter 5

Exercise 5.01

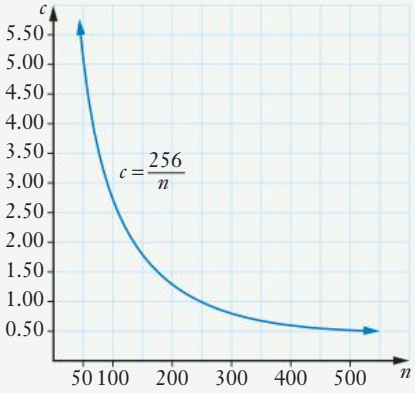
1 **a** $D = \frac{160}{x}$ **b** 200 mm **c** 1.4 mm

d



2 **a** $c = \frac{256}{n}$ **b** \$2.56 **c** 512 boxes

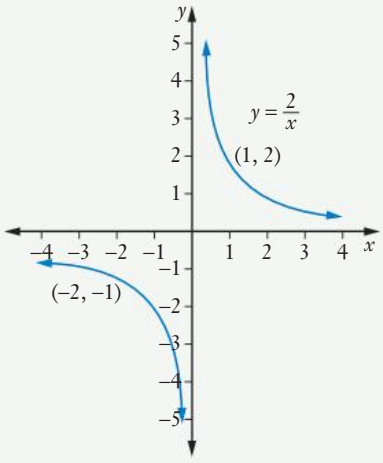
d



3 **a** **i** Domain $(-\infty, 0) \cup (0, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii no y-intercept

iii

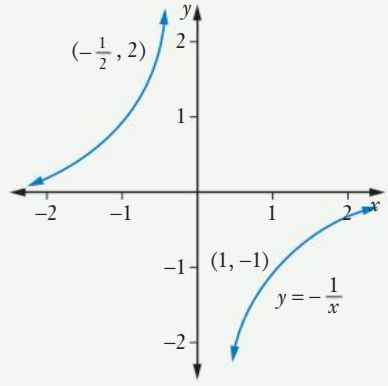


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b **i** Domain $(-\infty, 0) \cup (0, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii no y-intercept

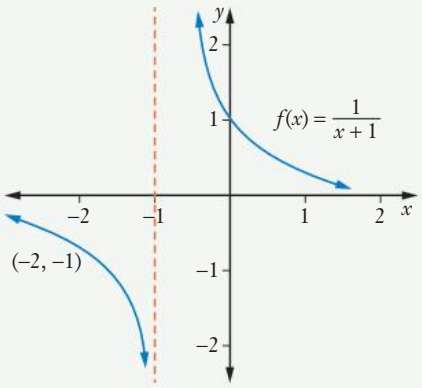
iii



c **i** Domain $(-\infty, -1) \cup (-1, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii 1

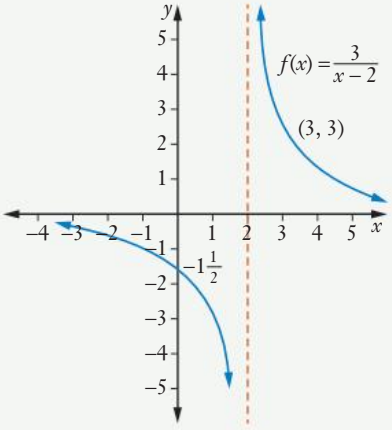
iii



d **i** Domain $(-\infty, 2) \cup (2, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii $-1\frac{1}{2}$

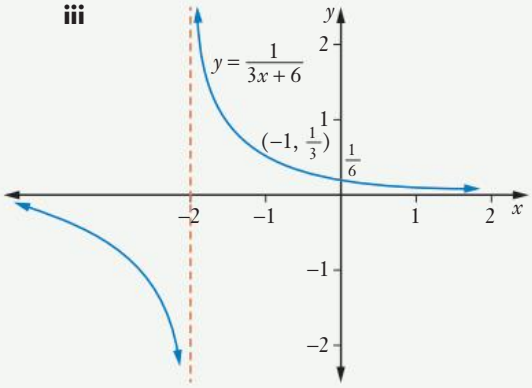
iii



e **i** Domain $(-\infty, -2) \cup (-2, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii $\frac{1}{6}$

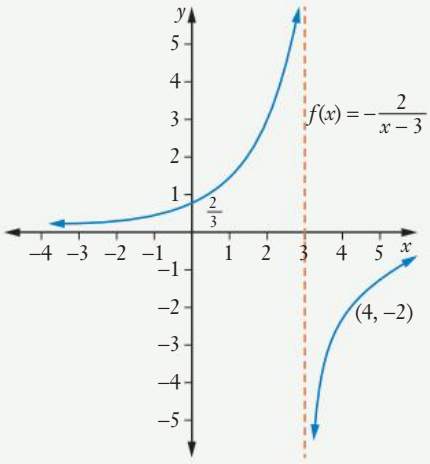
iii



f **i** Domain $(-\infty, 3) \cup (3, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii $\frac{2}{3}$

iii

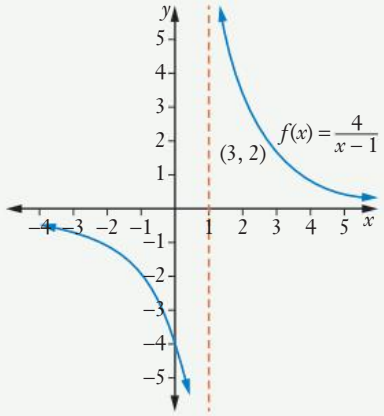


g **i** Domain $(-\infty, 1) \cup (1, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii −4

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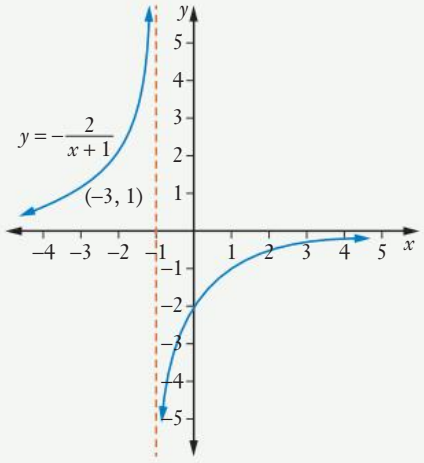
iii



h **i** Domain $(-\infty, -1) \cup (-1, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

ii -2

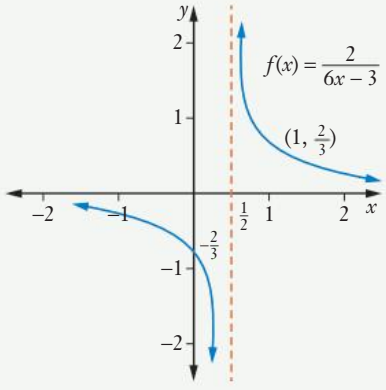
iii



i **i** Domain $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$, range $(-\infty, 0) \cup (0, \infty)$

ii $-\frac{2}{3}$

iii



4 $f(-x) = \frac{2}{-x} = -\frac{2}{x} = -f(x)$

$\therefore$ odd function

5 **a** **i** Yes **ii** Neither **iii** No

b $x = -1, y = 0$

c Domain $(-\infty, -1) \cup (-1, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

Exercise 5.02

1 **a** No x-intercepts, y-intercept 7

b x-intercepts ± 2, y-intercept -2

c x-intercept 0, y-intercept 0

d x-intercepts ± 3, y-intercept 3

e x-intercept -6, y-intercept 6

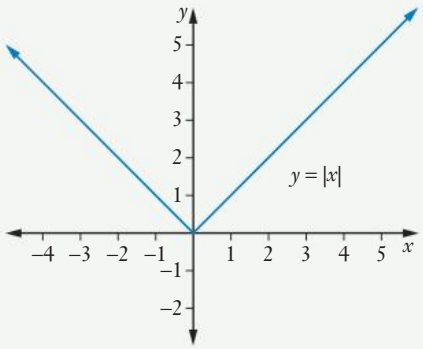
f x-intercept $\frac{2}{3}$, y-intercept 2

g x-intercept $-\frac{4}{5}$, y-intercept 4

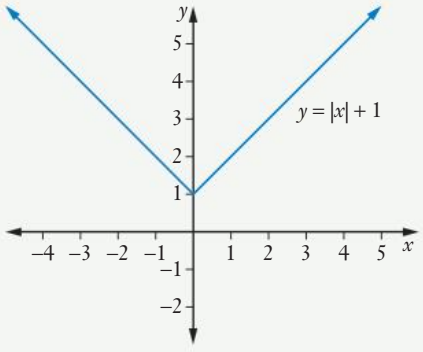
h x-intercept $\frac{1}{7}$, y-intercept 1

i No x-intercepts, y-intercept 9

2 **a**

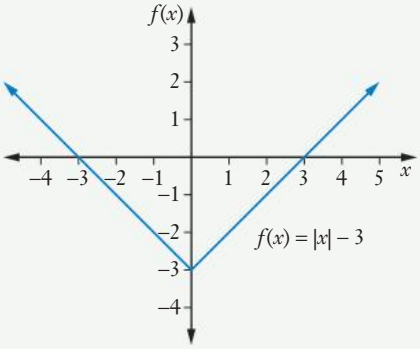


b

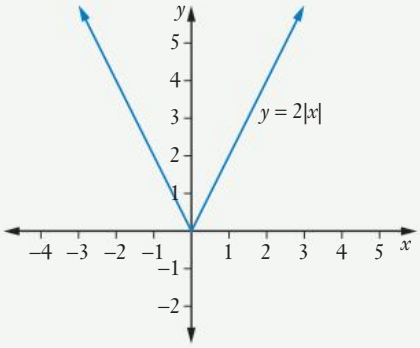


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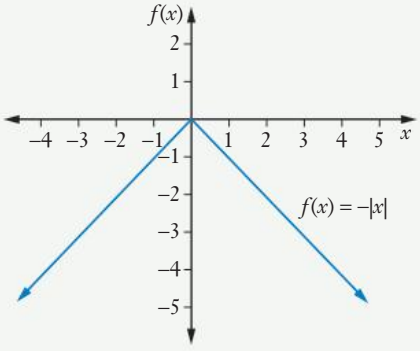
c



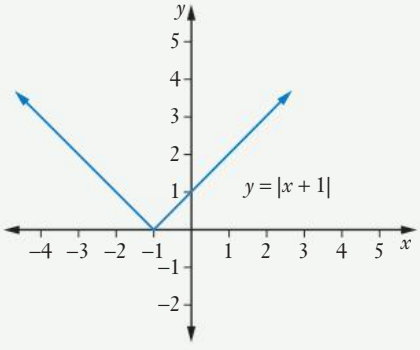
d



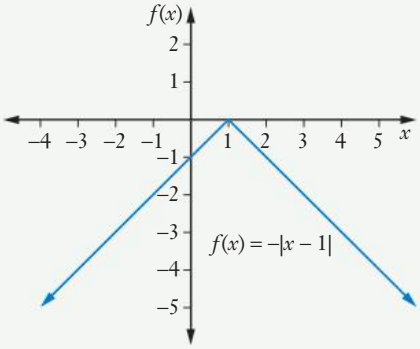
e



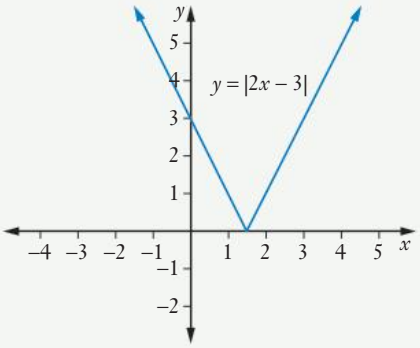
f



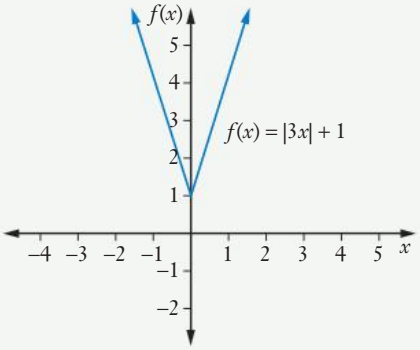
g



h



i



3 **a** Domain $(-\infty, \infty)$, range $[0, \infty)$

b Domain $(-\infty, \infty)$, range $[-8, \infty)$

c Domain $(-\infty, \infty)$, range $[0, \infty)$

d Domain $(-\infty, \infty)$, range $[-3, \infty)$

e Domain $(-\infty, \infty)$, range $(-\infty, 0]$

4 **a** $x = \pm 3$ **b** $x = -1, -3$ **c** $x = 3$

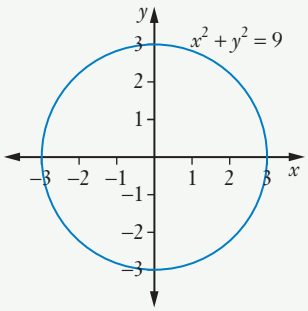
d $x = 1, 2$ **e** $x = 4, -7$ **f** $b = 2, -1.2$

g $x = \frac{1}{3}, -1$ **h** $x = 2, -3$ **i** $t = \frac{1}{2}$

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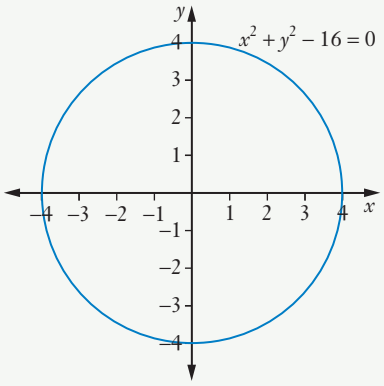
Exercise 5.03

1 a i



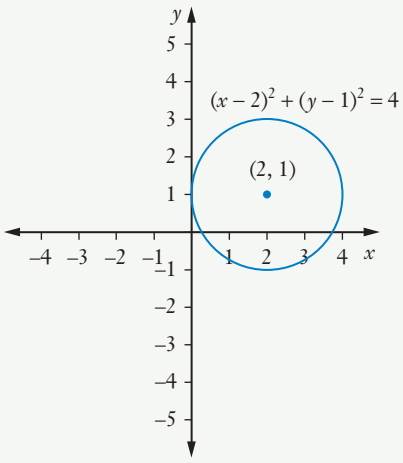
ii Domain $[-3, 3]$, range $[-3, 3]$

b i



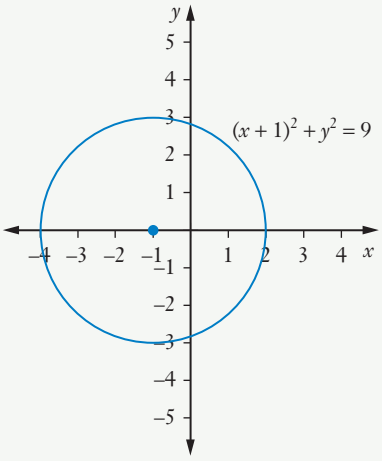
ii Domain $[-4, 4]$, range $[-4, 4]$

c i



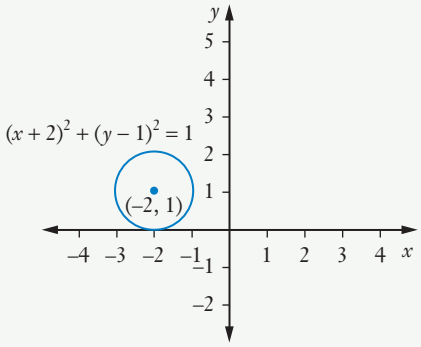
ii Domain $[0, 4]$, range $[-1, 3]$

d i



ii Domain $[-4, 2]$, range $[-3, 3]$

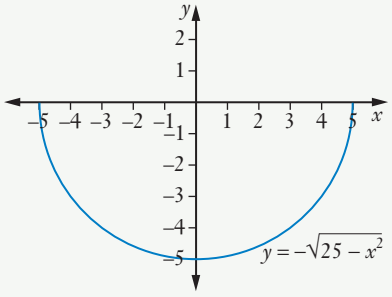
e i



ii Domain $[-3, -1]$, range $[0, 2]$

2 a i Below x-axis

ii

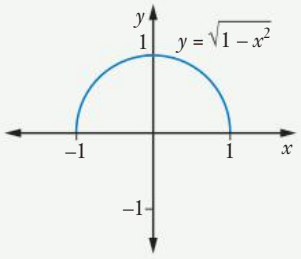


iii Domain $[-5, 5]$, range $[-5, 0]$

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b **i** Above x-axis

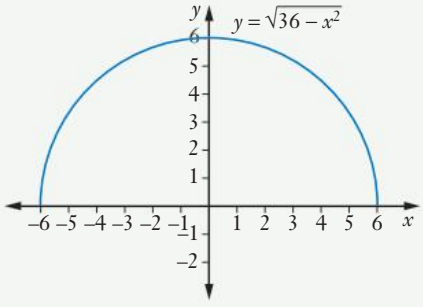
ii



iii Domain $[-1, 1]$, range $[0, 1]$

c **i** Above x-axis

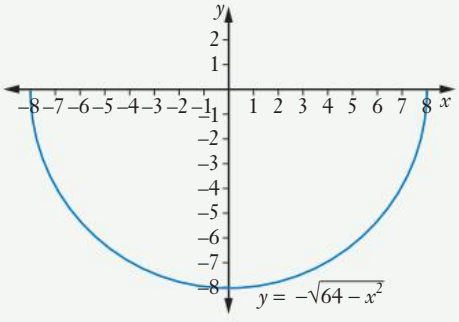
ii



iii Domain $[-6, 6]$, range $[0, 6]$

d **i** Below x-axis

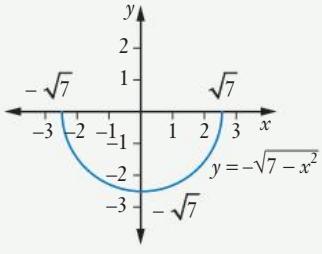
ii



iii Domain $[-8, 8]$, range $[-8, 0]$

e **i** Below x-axis

ii



iii Domain $[-\sqrt{7}, \sqrt{7}]$, range $[-\sqrt{7}, 0]$

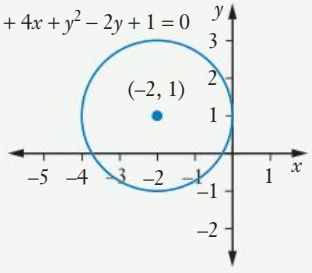
3 **a** 10, (0, 0) **b** $\sqrt{5}$, (0, 0)
c 4, (4, 5) **d** 7, (5, –6)
e 9, (0, 3)

4 **a** $x^2 + y^2 = 16$
b $x^2 - 6x + y^2 - 4y - 12 = 0$
c $x^2 + 2x + y^2 - 10y + 17 = 0$
d $x^2 - 4x + y^2 - 6y - 23 = 0$
e $x^2 + 8x + y^2 - 4y - 5 = 0$
f $x^2 + y^2 + 4y + 3 = 0$
g $x^2 - 8x + y^2 - 4y - 29 = 0$
h $x^2 + 6x + y^2 + 8y - 56 = 0$
i $x^2 + 4x + y^2 - 1 = 0$
j $x^2 + 8x + y^2 + 14y + 62 = 0$

5 **a** 3, (2, 1) **b** 5, (–4, 2)
c 1, (0, 1) **d** 6, (5, –3)
e 1, (–1, 1) **f** 6, (6, 0)
g 5, (–3, 4) **h** 8, (–10, 2)
i 5, (7, –1) **j** $\sqrt{10}$, (–1, –2)

6 **a** (3, –1), 4 **b** (2, 5), 5
c (–1, –6), 7 **d** (4, 7), 8

7



Exercise 5.04

1 **a** **i** $y = -x^2 + 2$ **ii** $y = x^2 - 2$
iii $y = -x^2 + 2$
b **i** $y = -(x + 1)^3$ **ii** $y = (-x + 1)^3$
iii $y = -(-x + 1)^3$
c **i** $y = -5x + 3$ **ii** $y = -5x - 3$
iii $y = 5x + 3$
d **i** $y = -|2x + 5|$ **ii** $y = |-2x + 5|$
iii $y = -|-2x + 5|$
e **i** $y = -\dfrac{1}{x - 1}$ **ii** $y = -\dfrac{1}{x + 1}$
iii $y = \dfrac{1}{x + 1}$

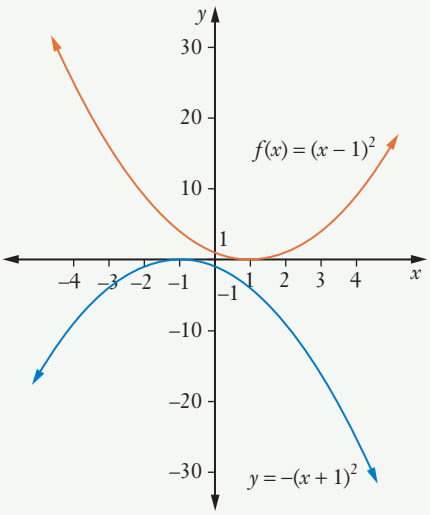
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2 **a** Reflection in x-axis

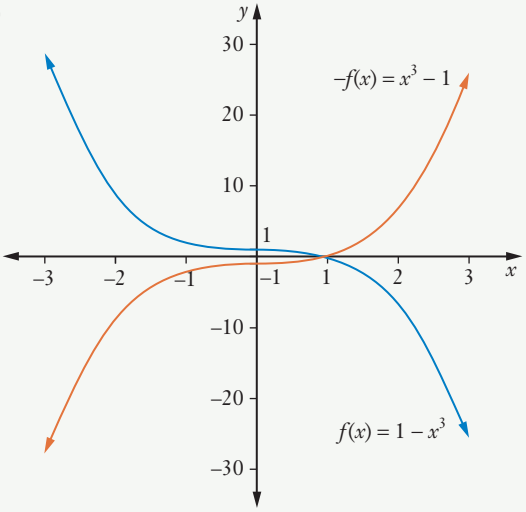
b Reflection in y-axis

c Reflection in both x-axis and y-axis

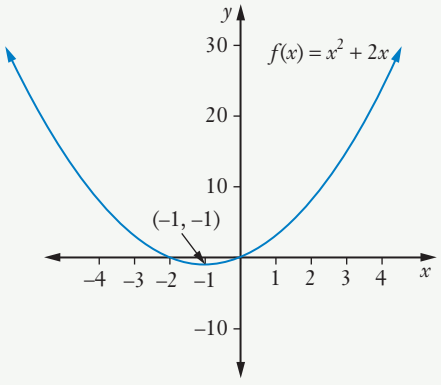
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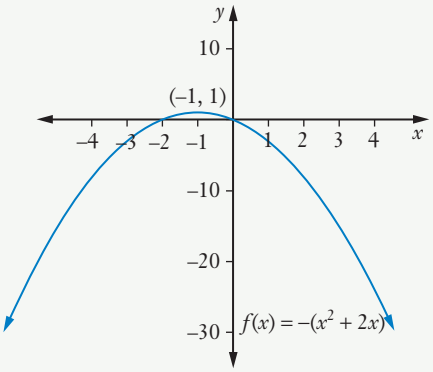
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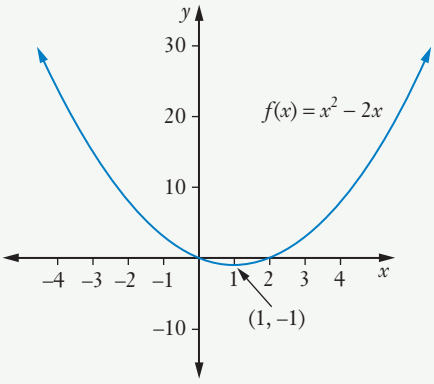
5 **a**



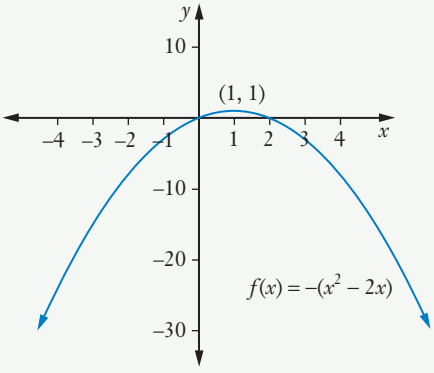
b



c



d

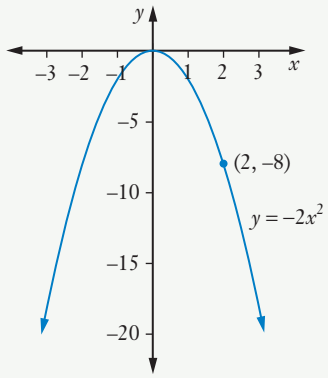


6 **a** $f(-x) = 2(-x)^2 = 2x^2 = f(x)$ so even.

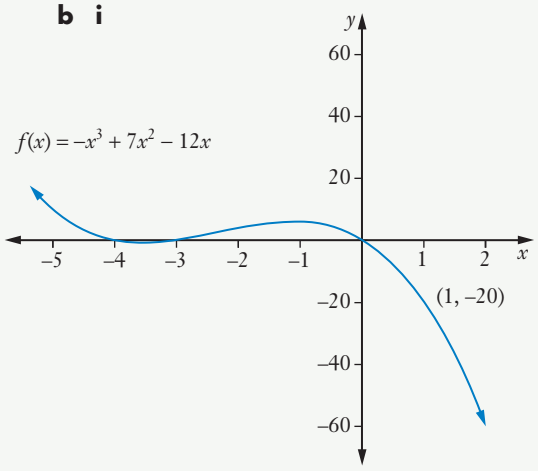
b **i** $y = 2x^2$ **ii** $y = -2x^2$

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c



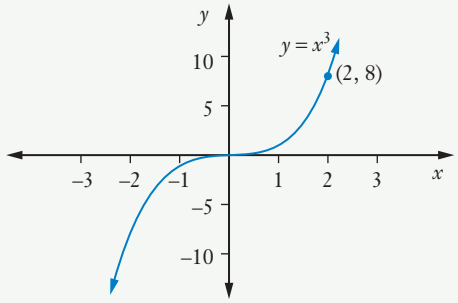
b i



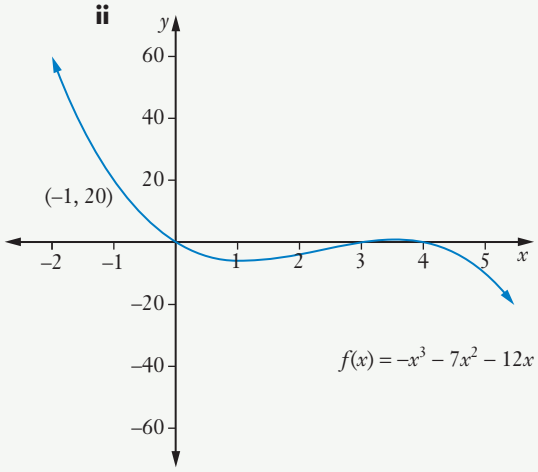
7 a $f(-x) = -(-x)^3 = x^3 = -f(x)$ so odd.

b i $y = x^3$ **ii** $y = -x^3$

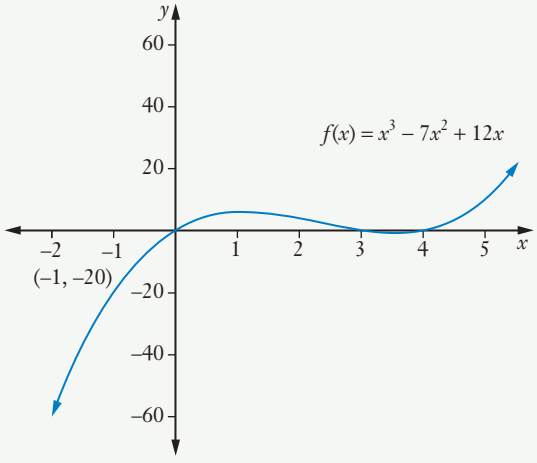
c



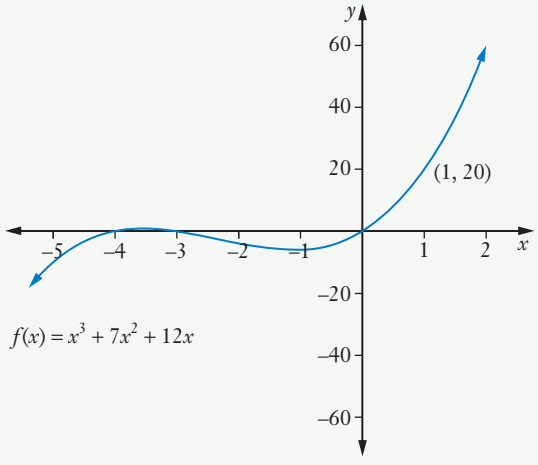
ii



8 a x-intercepts 0, 3, 4; y-intercept 0.



iii



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Exercise 5.05

1 **a** **i** $y = 2x^2 + 5x + 1$ **ii** $y = -2x^2 + 3x + 1$

iii $y = 8x^3 + 6x^2 + x$ **iv** $y = \dfrac{4x+1}{2x^2+x}$

b **i** $y = x^4 + x^3 + 5x + 1$ **ii** $y = x^4 - x^3 + 5x - 9$

iii $y = x^7 + 10x^4 - 4x^3 + 25x - 20$

iv $y = \dfrac{x^4+5x-4}{x^3+5}$

c **i** $y = 6x^2 - 7x + 1$ **ii** $y = -4x^2 + 7x + 5$

iii $y = 5x^4 - 7x^3 + 13x^2 - 21x - 6$

iv $y = \dfrac{x^2+3}{5x^2-7x-2}$

d **i** $y = 4x^2 + x + 4$ **ii** $y = 2x^2 + 3x - 6$

iii $y = 3x^4 - x^3 + 12x^2 + 11x - 5$

iv $y = \dfrac{3x^2+2x-1}{x^2-x+5}$

e **i** $y = 4x^5 + 3x + 3$ **ii** $y = 4x^5 - 3x + 11$

iii $y = 12x^6 - 16x^5 + 21x - 28$

iv $y = \dfrac{4x^5+7}{3x-4}$

2 **a** **i** 1 **ii** 1 **iii** 2

b **i** 2 **ii** 2 **iii** 3

c **i** 2 **ii** 1 **iii** 4

d **i** 3 **ii** 3 **iii** 4

3 **a** **i** –3 **ii** 11 **iii** –28

b **i** –4 **ii** 6 **iii** –5

c **i** 22 **ii** 28 **iii** –75

d **i** 7 **ii** 7

iii no constant term

4 **a** Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

b Domain $(-\infty,\infty)$, range $[-2, \infty)$

c Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

d Domain $(-\infty, \infty)$, range $[-2.25, \infty)$

5 **a** Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

b Domain $(-\infty, \infty)$, range $\left[-\frac{1}{4}, \infty\right)$

c Domain $(-\infty,\infty)$, range $(-\infty, \infty)$

d Domain $(-\infty, \infty)$, range $[-4.5, \infty)$

6 **a** Domain $(-\infty, \infty)$, range $[-9, \infty)$

b Domain $(-\infty, \infty)$, range $[-25, \infty)$

c Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

7 **a** Domain $(-\infty, 4) \cup (4, \infty)$

b Domain $(-\infty, -1) \cup (-1, \infty)$

c Domain $(-\infty, 3) \cup (3, \infty)$

d Domain $(-\infty, 0) \cup (0, \infty)$

8 **a** $y = (x^2 + 1)^2 = x^4 + 2x^2 + 1$

b $y = (5x - 3)^3 = 125x^3 - 225x^2 + 135x - 27$

c $y = (x^2 - 3x + 2)^7$ **d** $y = \sqrt{2x-1}$

e $y = \sqrt[3]{x^4 + 7x^2 - 4}$ **f** $y = 6x + 3$

g $y = 2x^3 - 7$ **h** $y = 6x^2 - 5$

i $y = 18x^2$ **j** $y = 4x^4 + 24x^2 + 37$

9 **a** Domain $(-\infty, \infty)$, range $[0, \infty)$

b Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

c Domain $[2, \infty)$, range $[0, \infty)$

d Domain $[-3, \infty)$, range $(-\infty, 0]$

e Domain $[-2, 2]$, range $[0, 2]$

f Domain $[-1, 1]$, range $[-1, 0]$

10 **a** $y = \sqrt{x^3}$ **b** $y = \left(\sqrt{x}\right)^3$

11 **a** $y = \dfrac{x^2+3}{x}$ **b** $y = \dfrac{1}{x^2+3}$

c $y = \dfrac{1}{x^3+3x}$ **d** $y = x^3 + 3x$

Test yourself 5

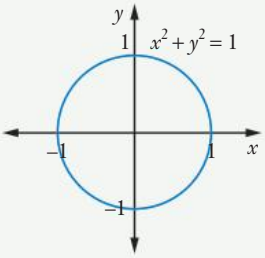
1 B **2** A **3** A

4 **a** $A = \dfrac{150}{n}$

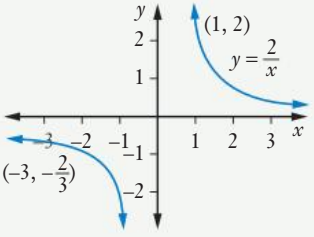
b **i** 15 cm^2 **ii** 18.75 cm^2

c **i** 9 **ii** 6

5 **a**

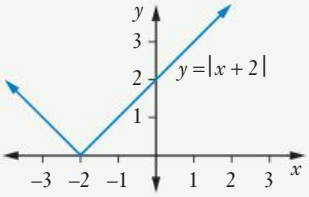


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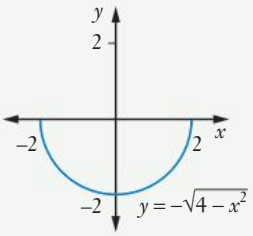


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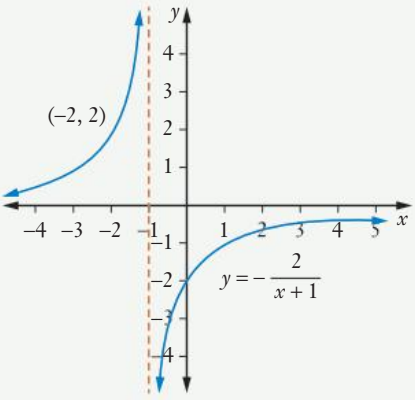
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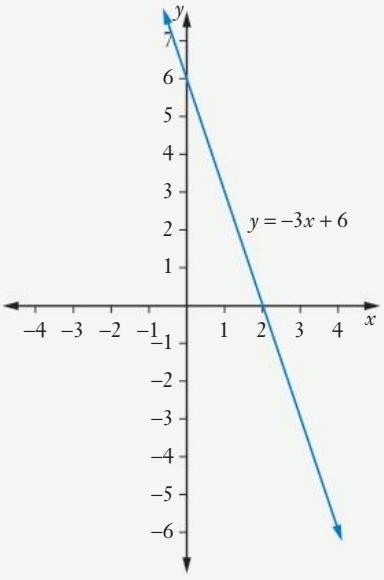
d



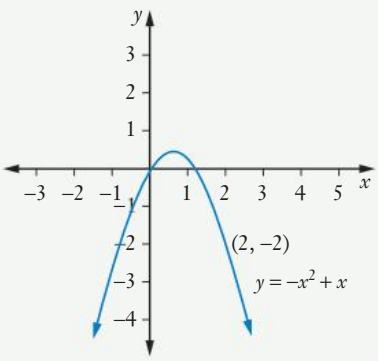
e



f



g



6 Radius 4, centre (3, 1)

7 **a** $y = x^3 + 3x - 1$

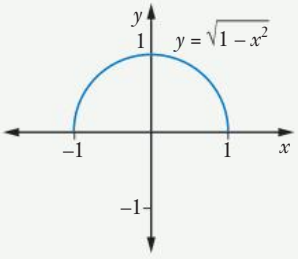
b $y = 3x^4 - x^3$

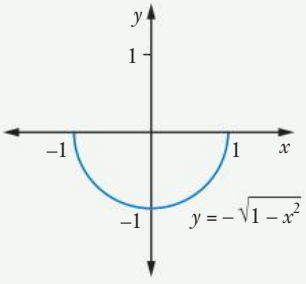
c $y = (3x - 1)^3 = 27x^3 - 27x^2 + 9x - 1$

d $y = 3x^3 - 1$

8 **a** No **b** $y = \pm\sqrt{1 - x^2}$

c





9 **a** Domain $[-4, 4]$, range $[-4, 4]$

b Domain $(-\infty, -2) \cup (-2, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

c Domain $(-\infty, \infty)$, range $[3, \infty)$

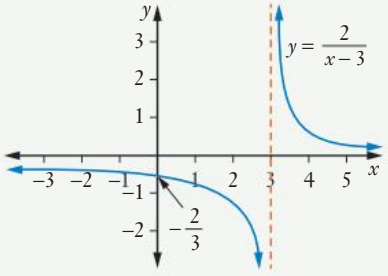
d Domain $[-3, 3]$, range $[0, 3]$

10 Domain $(-\infty, \infty)$, range $[-4, \infty)$

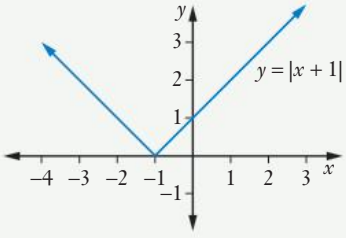
11 **a** Domain $(-\infty, 3) \cup (3, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

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b



12 a



b $x = 2, -4$

13 $x = 1, 5$

14 a Radius 10, centre (0, 0)

b Radius 11, centre (3, 2)

c Radius 3, centre (−3, −1)

15 a x-intercepts 0, ±2, y-intercept 0

b No x-intercepts, y-intercept −2

c x-intercepts ±3, y-intercepts ±3

d x-intercepts ±5, y-intercept 5

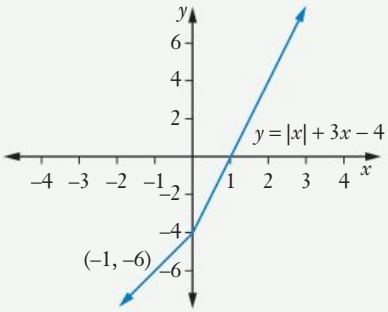
e No x-intercepts, y-intercept 5

16 a 3 **b** $10x^5$ **c** −7

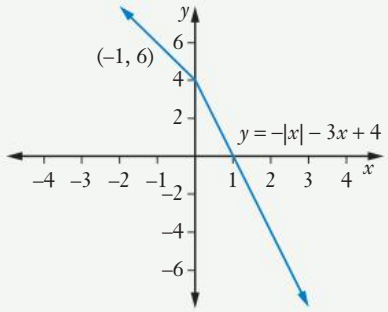
Challenge exercise 5

1 $x = 3$

2 a



b

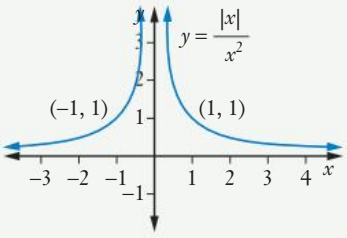


3 a $a = \frac{18}{b^2}$ **b** $a = 4.5$ **c** $b = 1.34$

4 Centre $\left(-1\frac{1}{2}, 1\right)$, radius $2\frac{1}{2}$

5 $3x + y + 2 = 0$

6

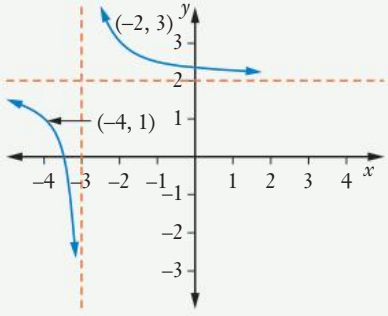


7 a $\frac{2(x+3)}{x+3} + \frac{1}{x+3} = \frac{2x+6+1}{x+3}$

$\therefore \frac{2x+7}{x+3} = 2 + \frac{1}{x+3}$

b Domain $(-\infty, -3) \cup (-3, \infty)$, range $(-\infty, 2) \cup (2, \infty)$

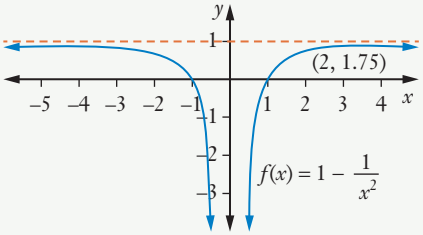
c



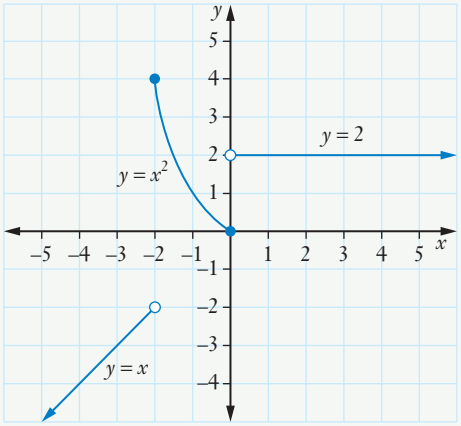
ISBN 9780170413152

8 Both circles have same centre (1, –2), so concentric.

9



10 a



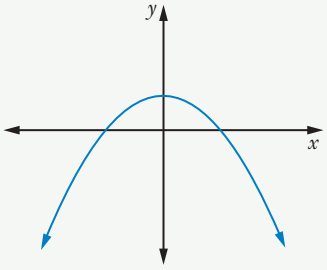
b $x = -2, 0$

c Domain $(-\infty, \infty)$, range $(-\infty, -2) \cup [0, 4]$

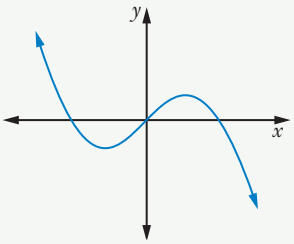
Chapter 6

Exercise 6.01

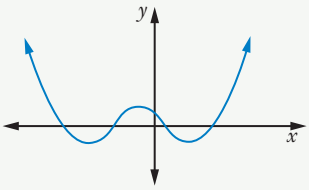
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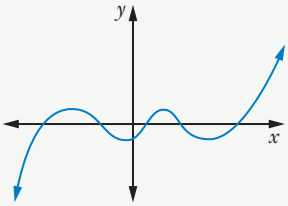
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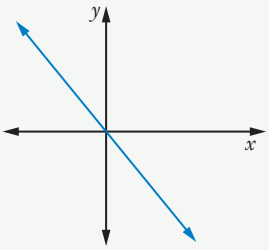
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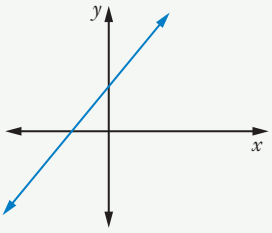
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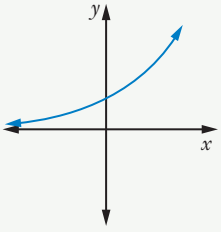
5



6

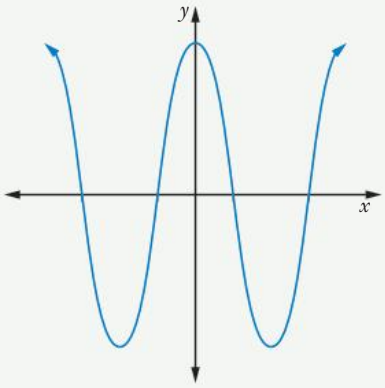


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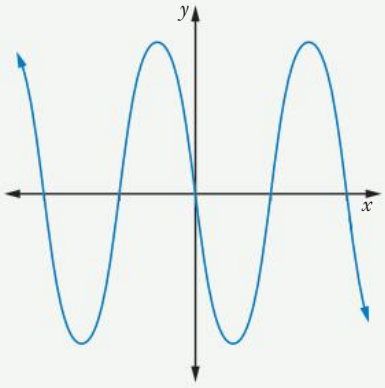


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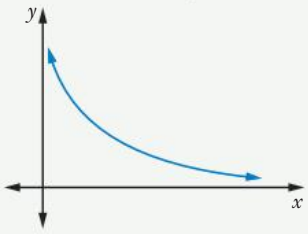
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9



10



Exercise 6.02

1 $x = 0$ **2** $x = x_1$ **3** None
4 $x = 0$ **5** $x = x_1, x_2$
6 $x = 0$ **7** $x = -3$ **8** $x = 2$
9 $x = -2, 3$ **10** $-1 \le x \le 0$ **11** $x = 0$

Exercise 6.03

1 a 4.06 **b** 3.994 **c** 4
2 a 13.61 **b** 13.0601 **c** 12.9401, 13
3 6
4 a 11 **b** $h^2 + 5h + 11$ **c** $h^2 + 5h$
d $\frac{h^2+5h}{h} = \frac{h(h+5)}{h} = h+5$ **e** 5
5 a 1 **b** $4h^2 - 8h$ **c** -8
6 a 8 **b** $6h + h^2$ **c** 6
7 a -13 **b** 17
8 a $x^2 + 2xh + h^2$
b $x^2 + 2xh + h^2 - x^2 = 2xh + h^2$
c $\frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x + h$
d $\lim_{h\to 0}(2x+h) = 2x$
9 a $2(x^2 + 2xh + h^2) - 7x - 7h + 3$
b $2x^2 + 4xh + 2h^2 - 7x - 7h + 3 - (2x^2 - 7x + 3)$
c $\frac{h(4x+2h-7)}{h}$ **d** $4x - 7$
10 a 2 **b** 5 **c** -12 **d** 15
e -9
11 a $2x$ **b** $2x + 5$ **c** $8x - 4$ **d** $10x - 1$

Exercise 6.04

1 a 1 **b** 5 **c** $2x + 3$ **d** $10x - 1$
e $3x^2 + 4x - 7$ **f** $6x^2 - 14x + 7$
g $12x^3 - 4x + 5$ **h** $6x^5 - 25x^4 - 8x^3$
i $10x^4 - 12x^2 + 2x - 2$ **j** $40x^9 - 63x^8$
2 a $4x + 1$ **b** $8x - 12$ **c** $2x$
d $16x^3 - 24x$ **e** $6x^2 + 6x - 3$
3 a $\frac{x}{3} - 1$ **b** $2x^3 - x^2$ **c** $\frac{8x^7}{3} - 6x^5$
d $4x$ **e** $\frac{1}{4}$ **f** $2x^2 - 2x + 2$
4 $16x - 7$ **5** -56
6 $60x^9 - 40x^7 + 35x^4 - 3$ **7** $10t - 20$ **8** $20x^3$
9 $30t$ **10** $40 - 4t$ **11** $4\pi r^2$ **12** 3
13 a 5 **b** -5 **c** 4
14 a 12 **b** ± 2 **15** 18

Exercise 6.05

1 a $-3x^{-4}$ **b** $1.4x^{0.4}$ **c** $1.2x^{-0.8}$ **d** $\frac{1}{2}x^{-\frac{1}{2}}$
e $x^{-\frac{1}{2}} + 3x^{-2}$ **f** $x^{-\frac{2}{3}}$ **g** $6x^{-\frac{1}{4}}$
h $x^{-\frac{3}{2}}$
2 a $-\frac{1}{x^2}$ **b** $\frac{5}{2\sqrt{x}}$ **c** $\frac{1}{6\sqrt[6]{x^5}}$

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d $-\frac{10}{x^6}$ **e** $\frac{15}{x^4}$ **f** $-\frac{1}{2\sqrt{x^3}}$

g $-\frac{3}{x^7}$ **h** $\frac{3\sqrt{x}}{2}$ **i** $-\frac{2}{3x^2}$

j $-\frac{1}{2x^3}-\frac{12}{x^5}$

3 $\frac{1}{27}$ **4** -3 **5** $\frac{1}{32}$ **6** -3

7 $2x+3\sqrt{x}+1$ **8** $\frac{1}{8}$

9 a $-\frac{1}{2\sqrt{x^3}}$ **b** $-\frac{1}{16}$

10 $a=4$ **11** $\left(5,\frac{2}{5}\right),\left(-5,-\frac{2}{5}\right)$

Exercise 6.06

1 a 72 **b** -13 **c** 11 **d** -18

e 18 **f** 27 **g** 11 **h** 136

i -4 **j** 149

2 a $-\frac{1}{26}$ **b** $\frac{1}{25}$ **c** $\frac{1}{20}$ **d** $-\frac{1}{43}$

e $\frac{1}{10}$ **f** $\frac{1}{7}$ **g** $-\frac{1}{71}$ **h** $\frac{1}{20}$

i $-\frac{1}{8}$ **j** $-\frac{1}{5}$

3 a i 6 **ii** $-\frac{1}{6}$

b i 8 **ii** $-\frac{1}{8}$

c i 24 **ii** $-\frac{1}{24}$

d i -8 **ii** $\frac{1}{8}$

e i 11 **ii** $-\frac{1}{11}$

4 a $27x-y-47=0$ **b** $7x-y-1=0$

c $4x+y+17=0$ **d** $36x-y-47=0$

e $44t-v-82=0$

5 a $x+24y-555=0$ **b** $x-8y+58=0$

c $x-17y-516=0$ **d** $x-45y+3108=0$

e $x+2y-9=0$

6 a i $7x-y+4=0$ **ii** $x+7y-78=0$

b i $34x-y+72=0$ **ii** $x+34y+1023=0$

c i $10x+y-6=0$ **ii** $x-10y-41=0$

d i $2x+y+2=0$ **ii** $x-2y-19=0$

7 $x=\pm3$ **8** (1, 2) and (−1, 0)

9 (−5, −7) **10** (0, 1)

11 (1, 2) **12** $\left(-1\frac{3}{4},-4\frac{15}{16}\right)$

13 a (1, −1) **b** $6x-y-7=0$

14 $10t-h-7=0$ **15** $4x-2y-19=0$

16 $3x+16y-8=0$ **17** $x-y+9=0$

18 $x+16y-16=0$ **19** (9, 3)

Exercise 6.07

1 a $4(x+3)^3$ **b** $6(2x-1)^2$

c $70x(5x^2-4)^6$ **d** $48(8x+3)^5$

e $-5(1-x)^4$ **f** $135(5x+9)^8$

g $4(x-4)=4x-16$ **h** $4(6x^2+3)(2x^3+3x)^3$

i $8(2x+5)(x^2+5x-1)^7$

j $6(6x^5-4x)(x^6-2x^2+3)^5$
$=12x(3x^4-2)(x^6-2x^2+3)^5$

k $\frac{3}{2}(3x-1)^{-\frac{1}{2}}$ **l** $2(4-x)^{-3}$

m $-6x(x^2-9)^{-4}$ **n** $\frac{5}{3}(5x+4)^{-\frac{2}{3}}$

o $\frac{3}{4}(3x^2-14x+1)(x^3-7x^2+x)^{-\frac{1}{4}}$

p $\frac{3}{2\sqrt{3x+4}}$ **q** $-\frac{5}{(5x-2)^2}$

r $-\frac{8x}{(x^2+1)^5}$ **s** $-\frac{2}{\sqrt[3]{7-3x}}$

t $-\frac{5}{2\sqrt{(4+x)^3}}$ **u** $-\frac{3}{4\sqrt{(3x-1)^3}}$

v $-\frac{27}{2(2x+7)^{10}}$ **w** $-\frac{4x^3-9x^2+3}{(x^4-3x^3+3x)^2}$

x $\frac{16\sqrt[3]{4x+1}}{3}$ **y** $\frac{5}{4\sqrt[4]{(7-x)^9}}$

2 9 **3** 40 **4** (4, 1)

5 $x=2,-1\frac{1}{2}$ **6** $8x+y+7=0$

7 $16x-y-15=0$ **8** $x+9y+8=0$

9 $x+64y-1025=0$

10 a $2x+y+1=0$ **b** $x-2y+3=0$

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Exercise 6.08

1 **a** $8x^3 + 9x^2$ **b** $12x - 1$
c $30x + 21$ **d** $72x^5 - 16x^3$
e $30x^4 - 4x$ **f** $x(5x + 2)(x + 1)^2$
g $8(9x - 1)(3x - 2)^4$ **h** $3x^3(16 - 7x)(4 - x)^2$
i $(10x + 13)(2x + 5)^3$

2 26 **3** 1264

4 $\frac{8\sqrt{7}}{7}$ **5** 176

6 $10x - y - 9 = 0$ **7** $69t - h - 129 = 0$

8 $\frac{-6 \pm \sqrt{30}}{3}$ **9** $34x - y + 29 = 0$

Exercise 6.09

1 **a** $\frac{-2}{(2x-1)^2}$ **b** $\frac{15}{(x+5)^2}$

c $\frac{x^4 - 12x^2}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}$

d $\frac{16}{(5x+1)^2}$ **e** $\frac{-x^2 + 14x}{x^4} = \frac{14 - x}{x^3}$

f $\frac{11}{(x+3)^2}$ **g** $\frac{-2x^2 - 1}{(2x^2-1)^2}$

h $\frac{-6}{(x-2)^2}$ **i** $\frac{-34}{(4x-3)^2}$

j $\frac{-14}{(3x+1)^2}$ **k** $\frac{-3x^2 - 6x - 7}{(3x^2-7)^2}$

l $\frac{4x^2 - 12x}{(2x-3)^2} = \frac{4x(x-3)}{(2x-3)^2}$

m $\frac{-18x}{(x^2-5)^2}$

n $\frac{2x^3 + 12x^2}{(x+4)^2} = \frac{2x^2(x+6)}{(x+4)^2}$

o $\frac{2x^3 + 9x^2 + 7}{(x+3)^2}$ **p** $\frac{3x^2 + 8x - 5}{(3x+4)^2}$

q $\frac{2(x+5)^{\frac{1}{2}} - x(x+5)^{-\frac{1}{2}}}{x+5}$

r $\frac{(7x+2)^4 - 28(x-1)(7x+2)^3}{(7x+2)^8} = \frac{30 - 21x}{(7x+2)^5}$

s $\frac{3\sqrt{x+1} - \frac{3x+1}{2\sqrt{x+1}}}{x+1} = \frac{3x+5}{2\sqrt{(x+1)^3}}$

t $\frac{\frac{2x-3}{2\sqrt{x-1}} - 2\sqrt{x-1}}{(2x-3)^2} = \frac{-2x+1}{2\sqrt{x-1}(2x-3)^2}$

2 $\frac{1}{8}$ **3** $-1\frac{5}{9}$ **4** $x = 0, 1$

5 $x = -9, 3$ **6** $x - 18y + 8 = 0$

7 $17x - 25y - 19 = 0$

Exercise 6.10

1 **a** $\frac{dh}{dt} = 20 - 8t$ **b** $\frac{dD}{dt} = 15t^2 + 4t$

c $\frac{dA}{dx} = 16 - 4x$ **d** $\frac{dx}{dt} = 15t^4 - 4t^3 + 2$

e $\frac{dV}{dr} = 4\pi r^2$ **f** $\frac{dS}{dr} = 2\pi - \frac{100}{r^3}$

g $\frac{dD}{dx} = \frac{x}{\sqrt{x^2-4}}$ **h** $\frac{dS}{dr} = 800 - \frac{400}{r^2}$

2 **a** 30 **b** 20

3 13 L/s

4 **a** **i** −7 g/min **ii** −13 g/min
b 19 g/min

5 181 cm^2/s **6** 41 cm^2/min

7 **a** **i** 11 km **ii** 116 km **iii** 286 km
b **i** 65 km/h **ii** 105 km/h

8 −0.25

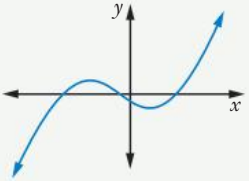
9 **a** 18 cm s^{-1} **b** 12 cm s^{-2}
c When $t = 0$, $x = 0$; at 3 s
d 5 s

10 **a** −8 m s^{-1}
b $a = 4$; constant acceleration of 4 m s^{-2}
c 13 m **d** 2 s **e** −5 m

Test yourself 6

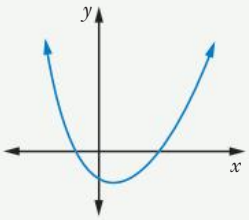
1 C **2** B **3** D **4** B, C

5 **a**



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b



6 $\frac{dy}{dx} = 10x - 3$

7 **a** $42x^5 - 9x^2 + 2x - 8$ **b** $-12x^{-5}$

c $-\frac{8}{(x+1)^5}$ **d** $\frac{dy}{dx} = \frac{5\sqrt{x^3}}{2}$

e $18(x+2)(x^2+4x-2)^8$

f $\frac{7}{(2x+1)^2}$

g $3x^2(3x+1)^5(9x+1)$

8 $4t - 3$ **9** 10 **10** 42

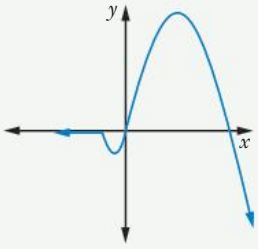
11 **a** $x = -2$ **b** $x = 1$ **c** $x = 2$

12 **a** $-\frac{4}{x^2}$ **b** $\frac{1}{5\sqrt[5]{x^4}}$ **c** $32(4x+9)^3$

d $3(x-1)^2(4x+1)$

e $\frac{4x^3+15x^2+6}{(2x+5)^2}$

13



14 $9x - y - 7 = 0$ **15** (2, 3)

16 $8\pi r$ **17** –0.9

18 (–2, 71), (5, –272) **19** $4x - y - 6 = 0$

20 **a** $u + at$ **b** $\frac{1}{5}$ **21** $12x + y - 4 = 0$

22 **a** **i** 0 m **ii** 4 m **iii** 3 m **iv** 1.75 m

b **i** 1 ms^{-1} **ii** –1 ms^{-1}

c **i** 4 ms^{-1} **ii** 0 ms^{-1} **iii** –2 ms^{-1}

23 **a** $x^2 + 2xh + h^2 - 3x - 3h + 5$

b $2xh + h^2 - 3h$ **c** $2x - 3$

24 **a** 1 **b** 20 **25** 9

26 **a** $24(x-3)(x^2-6x+1)^3$

b $-\frac{3}{(\sqrt{3x-1})^3}$

27 **a** 0 m, 0 ms^{-1}, 8 ms^{-2}

b 0, 0.8 s **c** –44 ms^{-1}, –52 ms^{-2}

Challenge exercise 6

1 $2x + y = 0, 3x - y - 3 = 0, 6x - y + 12 = 0$

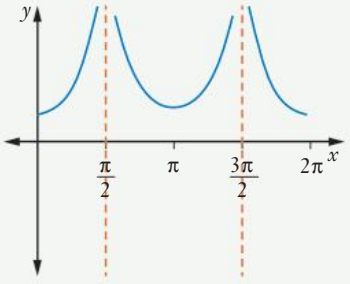
2 **a** (2, 2), (–2, –14)

b $x + 12y - 26 = 0, x + 12y + 170 = 0$

3 $\left(-2\frac{1}{4}, 6\frac{1}{16}\right)$ **4** $n = 8$

5 **a** $\frac{\pi}{2}, \frac{3\pi}{2}$

b



6 $\frac{5\sqrt{22}}{22}$ **7** $2x + y - 25 = 0$

8 **a** $16x + 32y + 1 = 0, 4x - 2y - 1 = 0$

b $m_1 m_2 = -\frac{1}{2} \times 2 = -1$, so perpendicular

9 $x = 0, 2, 6$

10 $x - y - 4 = 0$

11 $a = -1, b = 2, c = 4$

12 **a** Proof, includes $x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3$

b Proof, includes $\lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$

13 **a** **i** –0.909 **ii** –0.99 **iii** –1.01

b –1

c Proof, includes $\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}$

d Proof, includes $\lim_{h \to 0} \frac{-h}{x(x+h)} \times \frac{1}{h}$

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14 **a** 1 m, 0 m s^{-1} **b** 3.26×10^7 m s^{-2}
c $(t^3 + 1)^6 = 0$ has solution $t = -1$, but time $t \geq 0$.

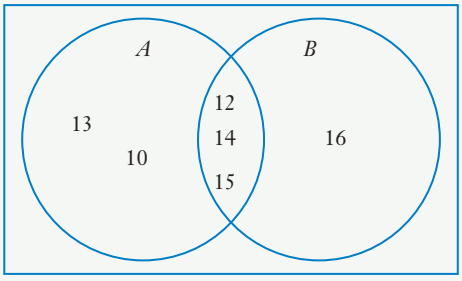
Chapter 7

Exercise 7.01

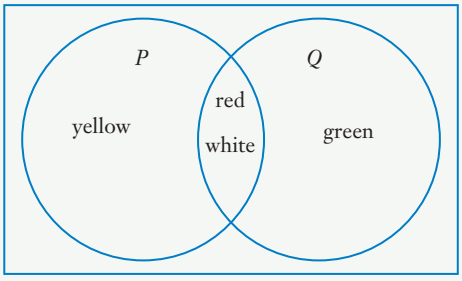
1 **a** {H, T} **b** {1, 2, 3, 4, 5}
c {1, 2, 3, 4, 5, 6}
d {red, green, yellow, blue}
e {1, 2, 3, 4, 5, 6, 7, 8}

2 **a** **i** {2, 4} **ii** {1, 2, 3, 4, 5, 6}
b **i** {red, white} **ii** {red, yellow, white}
c **i** { } **ii** {4, 5, 6, 7, 8, 9, 10, 11, 12, 15}
d **i** {brown, blue}
ii {blue, green, brown, hazel, grey}
e **i** { } **ii** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

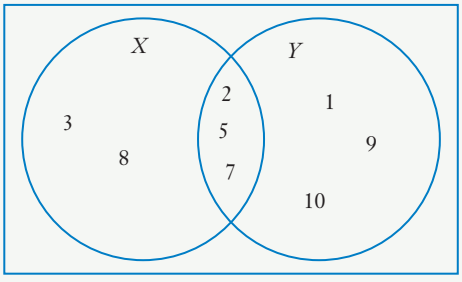
3 **a**



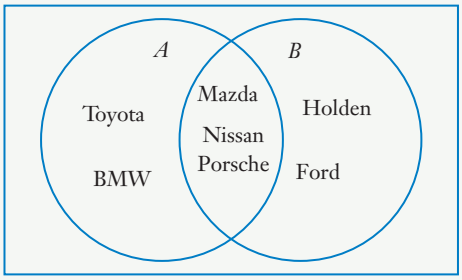
b



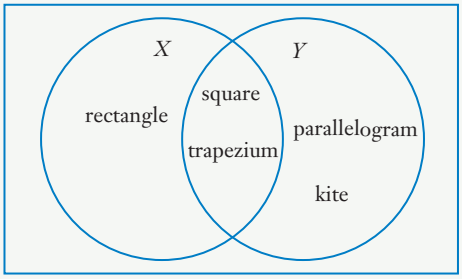
c



d



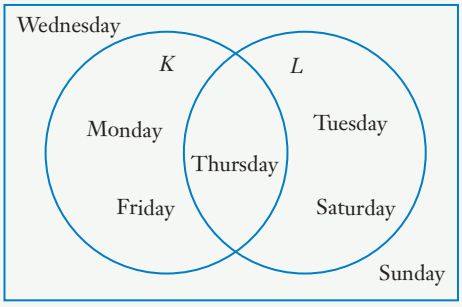
e



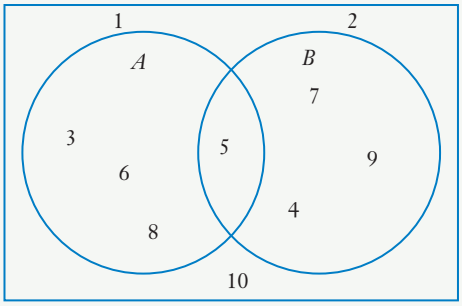
4 **a** False. Each horse has different abilities and chances.
b False. Each player has different abilities and chances.
c False. Previous outcomes have no influence.
d False. Previous outcomes have no influence.
e False. Each bike rider has different abilities and chances.

5 **a** {1, 2, 3, 4, 5, 6} **b** {2, 4, 6} **c** $\frac{1}{2}$

6 **a**

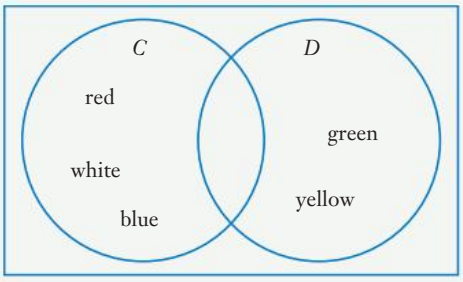


b

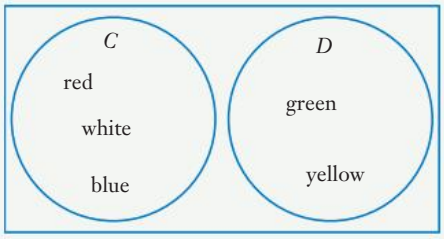


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c



or



Exercise 7.02

1 a

Score	Frequency	Relative frequency
4	6	$\frac{6}{23}$
5	4	$\frac{4}{23}$
6	1	$\frac{1}{23}$
7	7	$\frac{7}{23}$
8	2	$\frac{2}{23}$
9	3	$\frac{3}{23}$

b **i** $\frac{2}{23}$ **ii** $\frac{11}{23}$ **iii** $\frac{17}{23}$

c **i** 7 **ii** 6

2 a

Number of days	Frequency	Relative frequency
1	3	15%
2	6	30%
3	1	5%
4	7	35%
5	2	10%
6	1	5%

b 4

c **i** 15% **ii** 10% **iii** 40% **iv** 50%
v 45%

3 a

Class	Frequency	Relative frequency
0–19	9	0.18
20–39	12	0.24
40–59	18	0.36
60–79	7	0.14
80–99	4	0.08

b **i** 0.24 **ii** 0.22 **iii** 0.42

4 a

Velocity (m/s)	Frequency	Relative frequency
2–4	2	$\frac{1}{10}$
5–7	7	$\frac{7}{20}$
8–10	4	$\frac{1}{5}$
11–13	1	$\frac{1}{20}$
14–16	6	$\frac{3}{10}$

b **i** $\frac{7}{20}$ **ii** $\frac{1}{20}$ **iii** $\frac{1}{5}$ **iv** $\frac{7}{20}$
v $\frac{13}{20}$

c **i** $\frac{9}{20}$ **ii** $\frac{9}{10}$ **iii** $\frac{11}{20}$

5 a 10%

b

Sales/min	Frequency	Relative frequency
0	4	13.3%
1	12	40%
2	6	20%
3	3	10%
4	0	0%
5	5	16.7%

c 1 **d** **i** 20% **ii** 16.7% **iii** 26.7%

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6 a

Score	Frequency
3	1
4	4
5	3
6	3
7	3
8	2
9	4

b **i** $\frac{3}{20}$ **ii** $\frac{3}{10}$ **iii** $\frac{1}{4}$ **iv** $\frac{7}{10}$

7 a

Score	Frequency
4	1
5	5
6	3
7	6
8	2
9	3

b **i** 0.1 **ii** 0.7 **iii** 0.45 **iv** 0.95 **v** 0.85

8 a

Ages	Frequency
10–19	3
20–29	4
30–39	8
40–49	5
50–59	5

b **i** 32% **ii** 12% **iii** 40%
c **i** 60% **ii** 20% **iii** 68% **iv** 72% **v** 80%

9 a February (28 days) **b** $\frac{1}{7}$ **c** 50%

d

Food (kg)	Frequency	Relative frequency
0–14	3	$\frac{3}{28}$
15–29	11	$\frac{11}{28}$
30–44	8	$\frac{2}{7}$
45–59	4	$\frac{1}{7}$
60–74	2	$\frac{1}{14}$

e **i** $\frac{2}{7}$ **ii** $\frac{3}{14}$ **iii** $\frac{1}{2}$ **iv** $\frac{23}{28}$ **v** $\frac{1}{2}$

Exercise 7.03

1 a $\frac{1}{30}$ **b** $\frac{29}{30}$

2 a $\frac{1}{52}$ **b** $\frac{51}{52}$ **3** $\frac{1}{6}$

4 a $\frac{1}{40}$ **b** $\frac{39}{40}$ **5** $\frac{1}{20000}$

6 a $\frac{4}{7}$ **b** $\frac{3}{7}$ **7** $\frac{3}{37}$

8 a $\frac{5}{11}$ **b** 192 **9 a** $\frac{11}{20}$ **b** $\frac{3}{4}$

10 a $\frac{2}{9}$ **b** 147 **11** 99.8%

12 a $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$

13 a $\frac{1}{62}$ **b** $\frac{3}{31}$ **c** $\frac{1}{2}$ **d** $\frac{99}{124}$

14 a 98.5% **b** **i** 39 **ii** 2561

15 $\frac{7}{8}$ **16 a** $\frac{8}{15}$ **b** $\frac{7}{15}$ **c** $\frac{1}{15}$

17 a $\frac{1}{2}$ **b** 1

18 a $\frac{7}{31}$ **b** $\frac{7}{31}$ **c** $\frac{12}{31}$

19 $\frac{25}{43}$ **20** 34 **21** $\frac{1}{3}$

22 a $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{5}{6}$

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23 $\frac{46}{49}$ **24** **a** $\frac{2}{15}$ **b** $\frac{13}{15}$

25 $A \cap B = 0$

Exercise 7.04

1 **a** $\frac{3}{10}$ **b** $\frac{3}{5}$ **c** $\frac{11}{20}$ **d** $\frac{7}{10}$

2 **a** $\frac{1}{5}$ **b** $\frac{1}{2}$ **c** $\frac{3}{5}$ **d** $\frac{3}{5}$ **e** $\frac{19}{50}$

3 **a** $\frac{5}{26}$ **b** $\frac{9}{26}$ **c** $\frac{12}{13}$

4 **a** $\frac{29}{100}$ **b** $\frac{13}{20}$ **c** $\frac{9}{25}$

5 **a** $\frac{3}{5}$ **b** $\frac{4}{9}$ **c** $\frac{2}{3}$

6 **a** $\frac{3}{14}$ **b** $\frac{13}{28}$ **c** $\frac{9}{28}$

7 **a** $\frac{21}{80}$ **b** $\frac{17}{80}$ **c** $\frac{21}{40}$

8 **a** $\frac{1}{10}$ **b** $\frac{11}{20}$ **c** $\frac{7}{20}$

9 **a** $\frac{7}{25}$ **b** $\frac{2}{15}$ **c** $\frac{44}{75}$

10 **a** $\frac{3}{10}$ **b** $\frac{2}{5}$ **c** $\frac{3}{10}$

Exercise 7.05

1 $\frac{1}{4}$ **2** $\frac{1}{8}$

3 $\frac{1}{4}$ **4** $\frac{25}{121}$

5 **a** 0.0441 **b** 0.6241

6 80.4% **7** 32.9%

8 **a** $\frac{29\ 791}{35\ 937}$ **b** $\frac{8}{35\ 937}$ **c** $\frac{6146}{35\ 937}$

9 **a** $\frac{1}{2400}$ **b** $\frac{1}{5\ 760\ 000}$ **c** $\frac{5\ 755\ 201}{5\ 760\ 000}$

10 **a** $\frac{1}{7776}$ **b** $\frac{3125}{7776}$ **c** $\frac{4651}{7776}$

11 **a** $\frac{9}{25\,000\,000}$ **b** $\frac{24\,970\,009}{25\,000\,000}$ **c** $\frac{29\,991}{25\,000\,000}$

12 **a** $\frac{1}{4}$ **b** $\frac{9}{100}$ **c** $\frac{9}{100}$

13 **a** 61.41% **b** 0.34% **c** 99.66%

14 **a** $\frac{1}{2^n}$ **b** $1-\frac{1}{2^n}=\frac{2^n-1}{2^n}$

15 **a** $\frac{9}{49}$ **b** $\frac{15}{91}$ **16** $\frac{3}{2075}$

17 $\frac{19}{99}$

18 **a** $\frac{2}{275}$ **b** $\frac{49}{198}$ **c** $\frac{316}{495}$

19 **a** $\frac{253}{861}$ **b** $\frac{57}{287}$ **20** $\frac{14}{95}$

Exercise 7.06

1 **a** $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{7}{8}$

2 **a** $\frac{1}{900}$ **b** $\frac{1}{900}$ **c** $\frac{1}{450}$

3 **a** $\frac{25}{169}$ **b** $\frac{80}{169}$

4 **a** 27.5% **b** 23.9% **c** 72.5%

5 **a** $\frac{189}{1000}$ **b** $\frac{441}{1000}$ **c** $\frac{657}{1000}$

6 **a** 0.325 **b** 0.0034 **c** 0.997

7 **a** $\frac{4}{27}$ **b** $\frac{1}{6}$

8 **a** $\frac{1}{25}$ **b** $\frac{1}{825}$ **c** $\frac{64}{825}$ **d** $\frac{152}{165}$ **e** $\frac{13}{165}$

9 **a** $\frac{16}{75}$ **b** $\frac{38}{75}$

10 **a** $\frac{11}{20}$ **b** $\frac{3}{20}$

11 **a** $\frac{84\,681}{1\,000\,000}$ **b** $\frac{912\,673}{1\,000\,000}$ **c** $\frac{27}{1\,000\,000}$

12 **a** 17.6% **b** 11% **c** 21.2%

13 **a** $\frac{1488}{3025}$ **b** $\frac{1}{121}$

14 **a** $\frac{22}{425}$ **b** $\frac{368}{425}$ **c** $\frac{7}{425}$

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15 **a** $\frac{17}{65}$ **b** $\frac{133}{715}$ **c** $\frac{496}{2145}$

16 **a** $\frac{1}{216}$ **b** $\frac{5}{72}$ **c** $\frac{91}{216}$

17 **a** $\frac{1}{10}$ **b** $\frac{3}{10}$ **c** $\frac{2}{5}$

18 **a** $\frac{25}{81}$ **b** $\frac{40}{81}$ **c** $\frac{56}{81}$

Exercise 7.07

1 **a** $\frac{9}{16}$ **b** $\frac{7}{16}$

2 $\frac{13}{27}$ **3** **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **4** 75%

5 86.7% **6** 56.25% **7** **a** $\frac{3}{8}$ **b** $\frac{2}{5}$

8 **a** $\frac{9}{19}$ **b** $\frac{4}{19}$ **c** $\frac{5}{19}$

9 **a** $\frac{4}{9}$ **b** $\frac{3}{7}$ **10** **a** $\frac{5}{14}$ **b** $\frac{8}{13}$

11 **a** **i** $\frac{23}{31}$ **ii** $\frac{65}{68}$ **b** **i** $\frac{23}{88}$ **ii** $\frac{8}{11}$

12 **a** $\frac{23}{102}$ **b** $\frac{32}{51}$ **c** $\frac{64}{143}$ **d** $\frac{38}{61}$

13 **a** $\frac{7}{11}$ **b** $\frac{13}{27}$

14 **a** $\frac{17}{61}$ **b** $\frac{105}{278}$ **c** $\frac{85}{138}$ **d** $\frac{34}{155}$ **e** $\frac{102}{295}$

15 **a** 45% **b** 55% **c** 76% **16** 29.9%

17 **a** 79% **b** 17.4% **c** 82.6% **d** 81.8%

18 0.2077

19 $P(L \cap M) = 0.0204$

$P(L)P(M) = 0.17 \times 0.12 = 0.0204$

Since $P(L \cap M) = P(L)P(M)$, L and M are independent.

20 $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$0.594 = 0.3 + 0.42 - P(X \cap Y)$

$P(X \cap Y) = 0.3 + 0.42 - 0.594$

$= 0.126$

$P(X \cap Y) = P(X)\,P(Y|X)$

$0.126 = 0.3 \times P(Y|X)$

$0.42 = P(Y|X)$

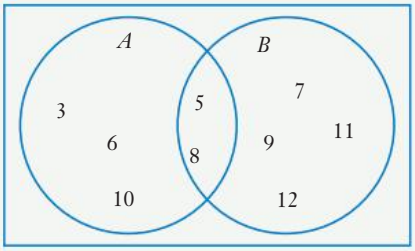
Since $P(Y|X) = P(Y) = 0.42$, X and Y are independent.

Test yourself 7

1 C **2** D **3** B **4** A, D

5 **a** **i** {3, 5, 6, 7, 8, 9, 10, 11, 12} **ii** {5, 8}

b



6 **a** {HH, HT, TH, TT} **b** {red, white, blue}

7 **a**

Face	Frequency	Relative frequency
1	17	$\frac{17}{100}$
2	21	$\frac{21}{100}$
3	14	$\frac{7}{50}$
4	20	$\frac{1}{5}$
5	18	$\frac{9}{50}$
6	10	$\frac{1}{10}$

b **i** $\frac{7}{50}$ **ii** $\frac{7}{25}$ **iii** $\frac{1}{10}$ **iv** $\frac{19}{50}$ **v** $\frac{13}{25}$

8 **a** 80.4% **b** 1.4% **c** 99.97%

9 **a**

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

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b **i** $\frac{1}{6}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{2}$

10 **a** **i** $\frac{1}{40}$ **ii** $\frac{39}{40}$ **b** $\frac{39}{796}$

11 **a** $\frac{4}{15}$ **b** $\frac{1}{10}$ **c** $\frac{3}{17}$ **d** $\frac{3}{11}$

12 False: the events are independent.

13 **a** $\frac{1}{2}$ **b** $\frac{29}{100}$ **c** $\frac{1}{5}$ **d** $\frac{11}{25}$

e $\frac{16}{25}$

14 **a** $\frac{2}{5}$ **b** $\frac{7}{15}$ **c** $\frac{2}{15}$

15 $\frac{35}{66}$ **16** $\frac{1}{56}$

17 **a** 0.009% **b** 12.9%

18 **a** **i** $\frac{1}{13}$ **ii** $\frac{3}{13}$ **iii** $\frac{5}{26}$

b **i** $\frac{1}{3}$ **ii** $\frac{1}{4}$

19 **a** $\frac{5}{12}$ **b** $\frac{1}{3}$

20 **a** $\frac{9}{40}$ **b** **i** $\frac{3}{10}$ **ii** $\frac{27}{160}$ **iii** $\frac{4}{25}$

21 **a** $\frac{1}{200}$ **b** $\frac{81}{200}$ **c** $\frac{11}{100}$

22 **a** $\frac{1}{15}$ **b** $\frac{4}{5}$

23 **a** $\frac{1}{50}$ **b** $\frac{147}{7450}$ **c** $\frac{1}{3725}$ **d** $\frac{3577}{3725}$

24 **a** $\frac{80}{361}$ **b** $\frac{40}{171}$

25 **a** $\frac{2}{9}$ **b** $\frac{1}{3}$ **c** $\frac{3}{7}$

26 **a** $\frac{21}{50}$ **b** $\frac{3}{25}$ **c** $\frac{23}{50}$

Challenge exercise 7

1 **a** $\frac{38}{119}$ **b** $\frac{10}{119}$

2 **a** 0.04 **b** 0.75 **c** 0.25 **3** $\frac{1}{54145}$

4 **a** $\frac{4}{13}$ **b** $\frac{25}{52}$ **c** $\frac{4}{13}$

5 No – all combinations are equally likely to win.

6 **a** $\frac{3}{10}$ **b** $\frac{12}{145}$

7 **a** $\frac{1}{144}$ **b** $\frac{5}{144}$ **c** $\frac{1}{165\,888}$

8 14.6%

Practice set 3

1 B **2** D **3** C **4** D

5 D **6** C **7** B **8** B

9 A **10** A **11** D

12 **a** $9x^8 - 8x + 7$ **b** $6x^2 - 2$ **c** $-12x^{-5}$

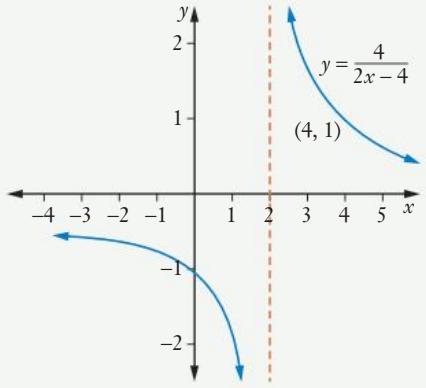
d $-\frac{25}{2x^6}$ **e** $\frac{3\sqrt{x}}{2}$

f $14(2x+3)^6$ **g** $-\frac{8x}{(x^2-7)^5}$

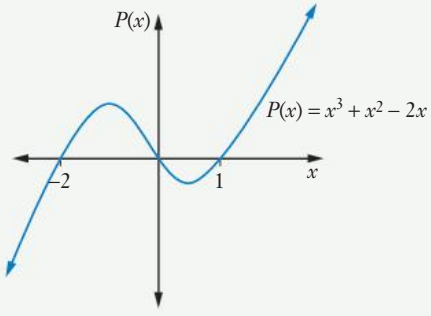
h $\frac{5}{3\sqrt[3]{(5x+1)^2}}$

i $\frac{2(5x^2+15x+1)}{(2x+3)^2}$

13 **a**

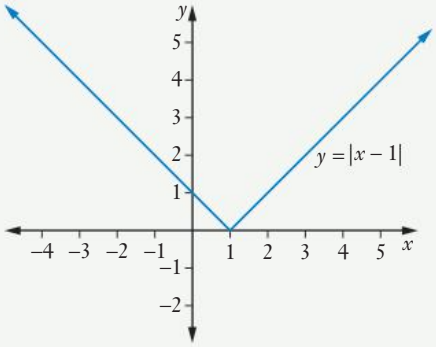


b

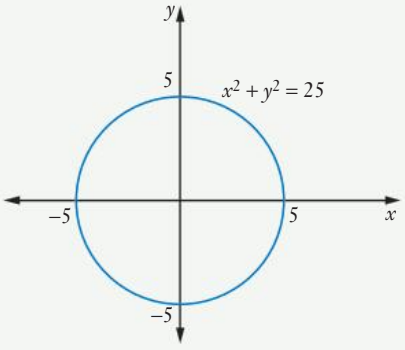


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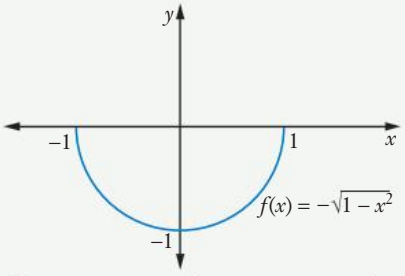
c



d



e



14 **a** $\frac{3}{25}$ **b** $\frac{8}{25}$ **c** $\frac{8}{25}$

15 **a** 100 L **b** 40 L
c –16 (leaking at the rate of 16 L/min)
d 12.2 min

16 **a** $9x - y + 16 = 0$ **b** $x + 9y + 20 = 0$
c $Q = (-20, 0)$

17 **a** $\frac{1}{36}$ **b** $\frac{1}{6}$ **c** $\frac{11}{36}$
d $\frac{5}{36}$ **e** $\frac{5}{12}$

18 $a = 1, b = -3, c = -1$

19 $x^2 + 4x + y^2 + 6y - 12 = 0$

20 **a** Centre (–3, 5), radius 7
b Centre (–5, 3), radius 2

21 **a** $3h^2 + 6xh - 4h$
b Proof involves $\lim_{h \to 0} \frac{3h^2 + 6xh - 4h}{h}$

22 **a** $3x - y - 4 = 0$ **b** $x - y - 2 = 0$
c $x + 3y + 10 = 0$ **d** $R = (-10, 0)$

23 **a** $\frac{7}{50}$ **b** $\frac{11}{20}$

24 **a** $\frac{3}{91}$ **b** $\frac{30}{91}$

25 $\frac{1}{110}$

26 **a** $\frac{5}{36}$ **b** $\frac{5}{12}$ **c** $\frac{1}{6}$ **d** $\frac{7}{36}$
e $\frac{1}{2}$

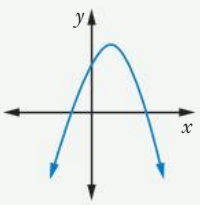
27 **a** $\frac{3}{10}$ **b** $\frac{3}{8}$

28 **a** $\frac{7}{20}$ **b** $\frac{1}{4}$

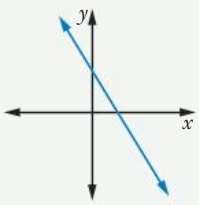
29 $f(-2) = -45, f'(-2) = 48$

30 **a** **i** 24.1202 **ii** 23.8802 **b** 24

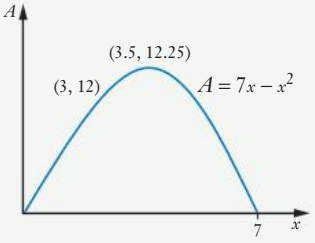
31 **a**



b



32 **a** **i** 12 m^2 **ii** 11.25 m^2
b 5.6 m or 1.4 m
c



d 12.25 m^2

33 $x = 1, -5$

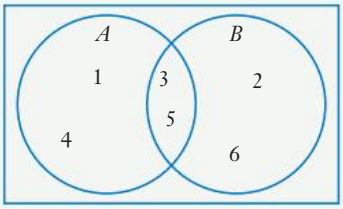
34 **a** 5 **b** 1 **c** –3

35 **a** 12 cm, 17 cm s^{-1}, –26 cm s^{-2}
b 10 cm, –11 cm s^{-1}, –2 cm s^{-2}

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36 **a** $\{1, 2, 3, 4, 5, 6\}$ **b** $\{3, 5\}$

c



37 $y = 6x - 5$

38 **a** $-3x^{-4}$ **b** $\dfrac{3\sqrt{x}}{2}$ **c** $-\dfrac{2}{x^3}$

d $\dfrac{(7x+4)(21x-26)}{(3x-1)^2}$

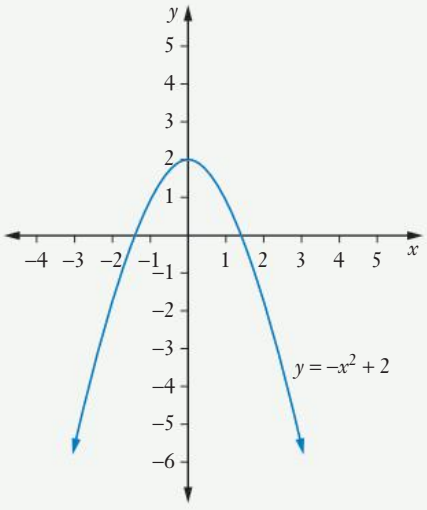
e $2(2x-3)^3(30x^2-15x+4)$

f $15(3x+1)^4$ **g** $\dfrac{1}{\sqrt{2x-1}}$

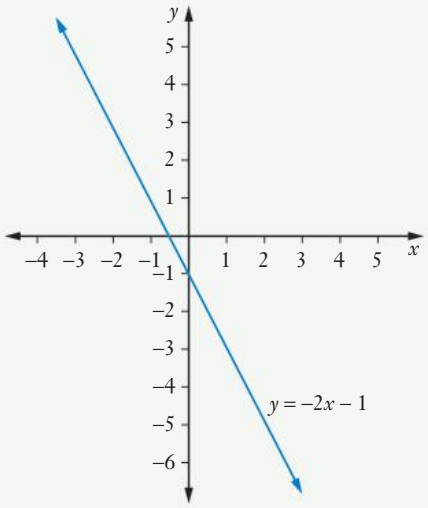
39 **a** **i** $y = x^2 + 2x - 3$ **ii** $y = 2x^3 - x^2 - 4x + 2$

iii $y = -x^2 + 2x + 1$ **iv** $y = \dfrac{2x-1}{x^2-2}$

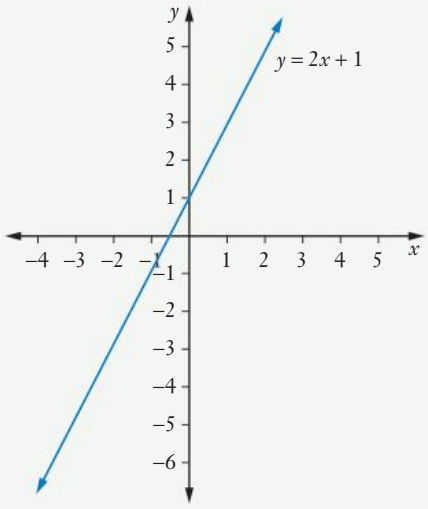
b **i**



ii



iii



40 **a** Centre $(-1, 3)$, radius 4

b Domain $[-5, 3]$, range $[-1, 7]$

41 $x + 2y + 1 = 0$

42 **a** $8x^3 - 15x^2 + 6x - 1$ **b** $-\dfrac{5}{2x^6}$

c $\dfrac{1}{2\sqrt{x}}$ **d** $14(2x-3)^6$

e $6x^3(2x-5)^6(11x-10)$ **f** $-\dfrac{31}{(3x-2)^2}$

43 **a** 2 **b** 3

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44 **a** $\frac{1}{12}$ **b** $\frac{1}{4}$

45 **a** Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

b Domain $(-\infty, \infty)$, range $(-\infty, 1]$

c Domain $[-7, 3]$, range $[-4, 6]$

d Domain $(-\infty, -2) \cup (-2, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

46 **a** $y = (2x + 5)^3$ **b** $y = 2x^3 + 5$

47 **a**

Result	Frequency	Relative frequency
HH	24	$\frac{6}{25}$
HT	15	$\frac{3}{20}$
TH	38	$\frac{19}{50}$
TT	23	$\frac{23}{100}$

b **i** $\frac{23}{100}$ **ii** $\frac{53}{100}$

c **i** $\frac{1}{4}$ **ii** $\frac{1}{2}$

48 $5x - y - 13 = 0$

49 **a** $\frac{8\pi}{9}$ cm **b** $\frac{16\pi}{9}$ cm^2

50 **a** 0 **b** -2 **c** $|m + 1| - 2$

51 75%

52 $P(L)P(M) = 0.45 \times 0.12 = 0.054 = P(L \cap M)$

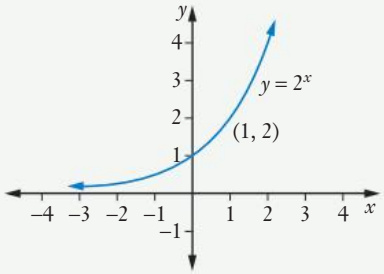
53 Show that $P(Y|X) = P(Y) = 0.15$.

54 Mutually exclusive.

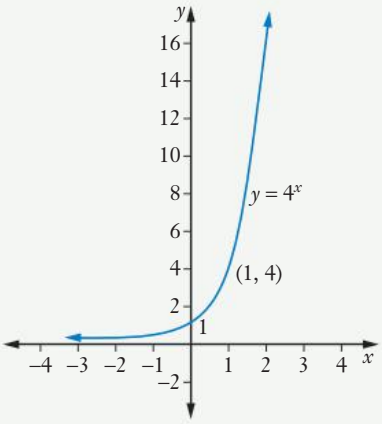
Chapter 8

Exercise 8.01

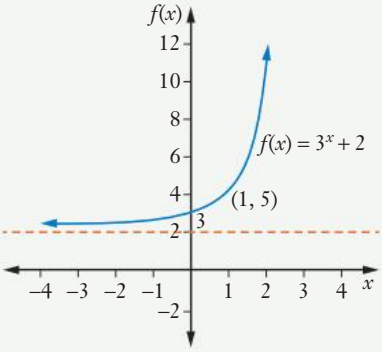
1 **a**



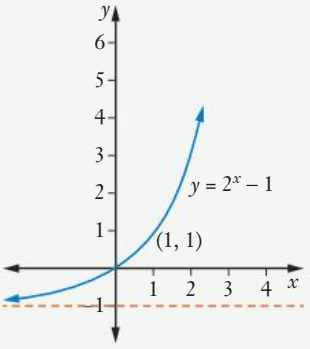
b



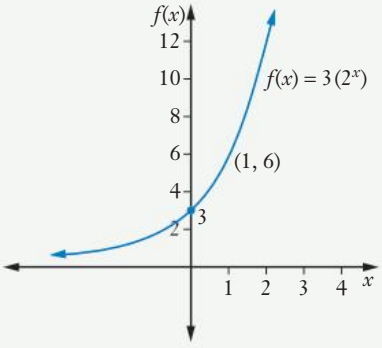
c



d

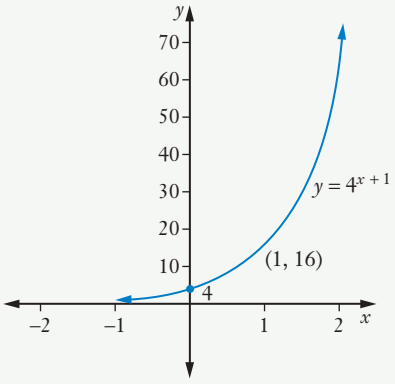


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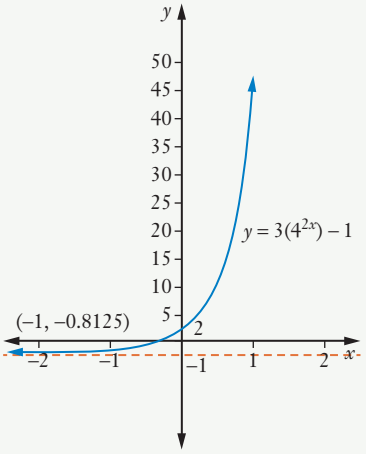


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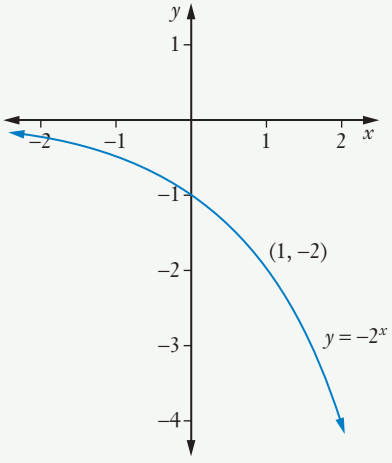
f



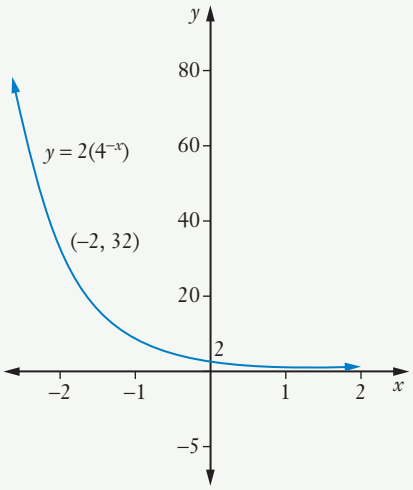
g



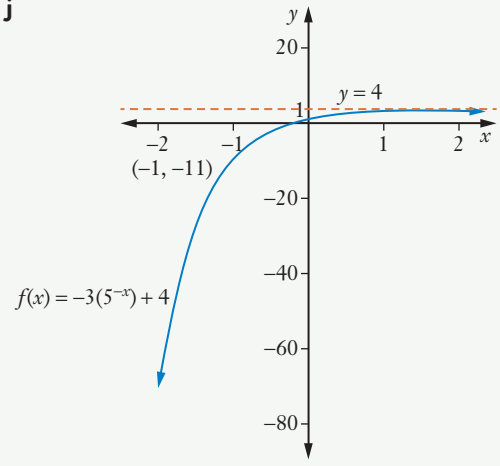
h



i



j



2 **a** Domain $(-\infty, \infty)$, range $(0, \infty)$

b Domain $(-\infty, \infty)$, range $(5, \infty)$

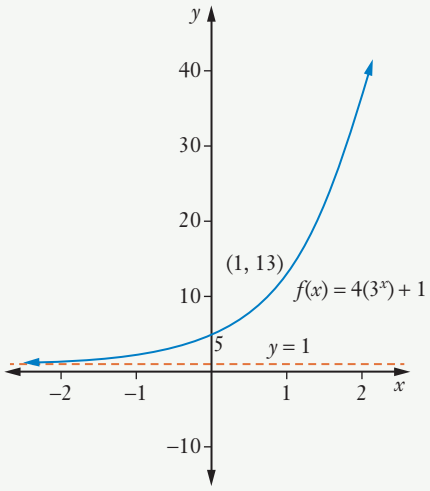
c Domain $(-\infty, \infty)$, range $(0, \infty)$

d Domain $(-\infty, \infty)$, range $(-\infty, 1)$

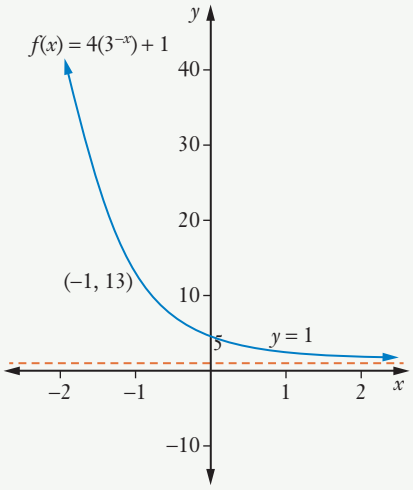
3 **a** $y = 2^{3x-4}$ **b** $y = 3(2^x) - 4$

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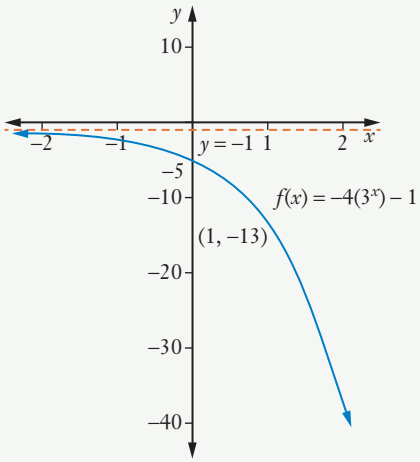
4 a



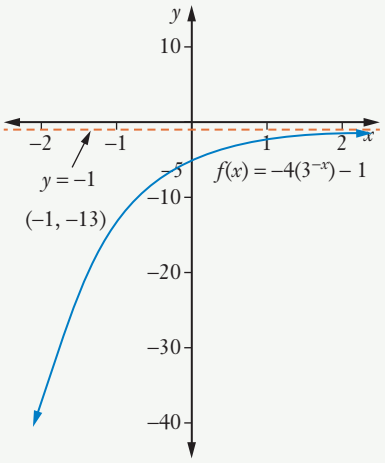
b i



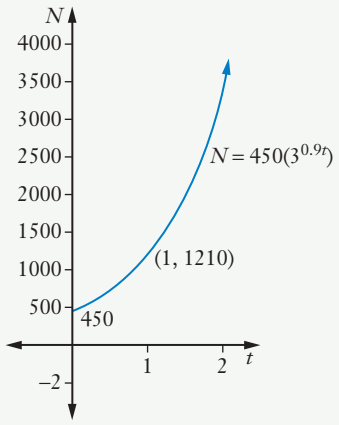
ii



iii



5 a

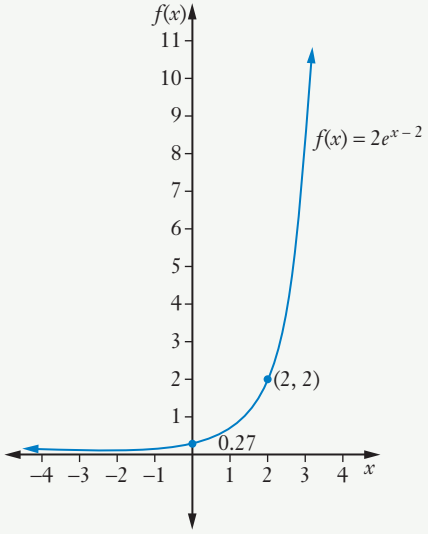


b 450 sales

c **i** 8739 **ii** 63 133 **iii** 8 857 350

Exercise 8.02

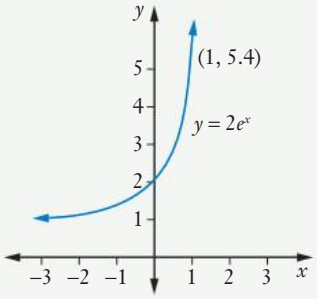
1



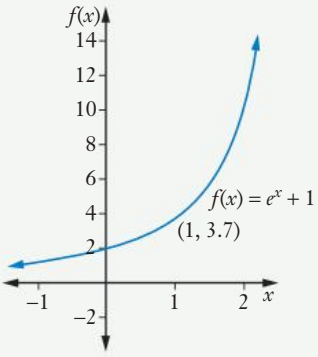
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2 **a** 4.48 **b** 0.14 **c** 2.70 **d** 0.05
e –0.14

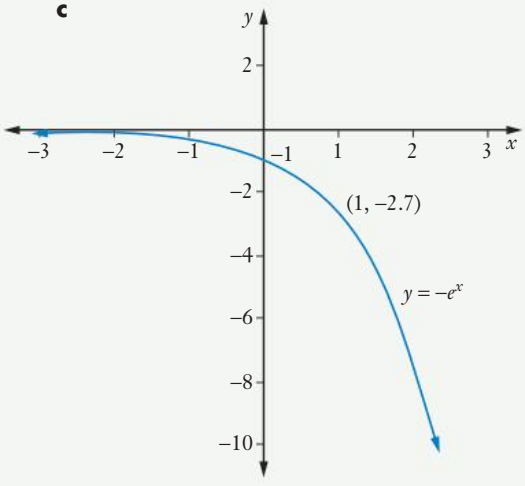
3 **a**



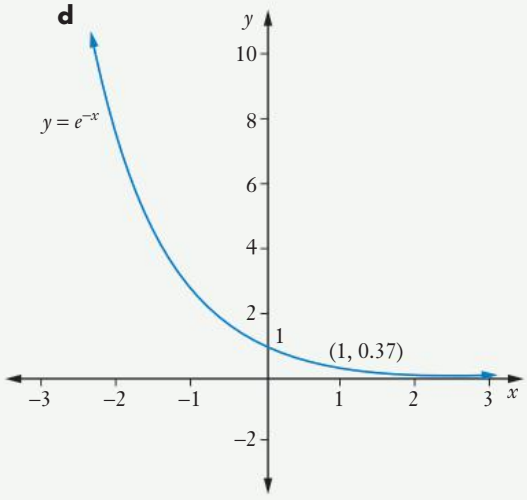
b



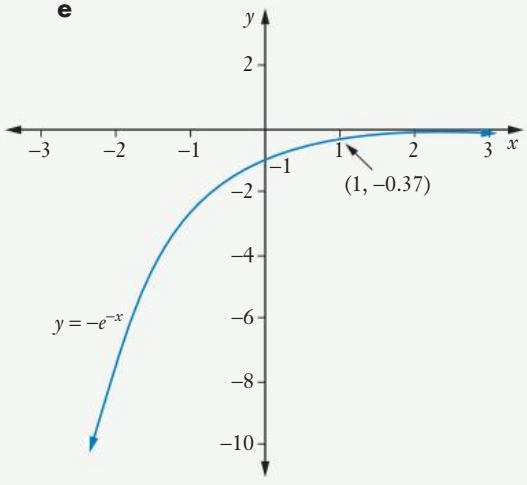
c



d



e

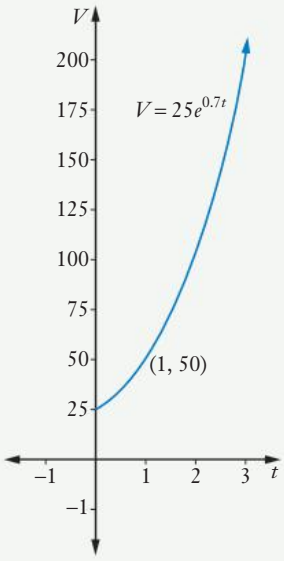


4 Domain $(-\infty, \infty)$, range $(-2, \infty)$

5 **a** $y = e^{x^3 + 3}$ **b** $y = e^{3x} + 3$

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6 a

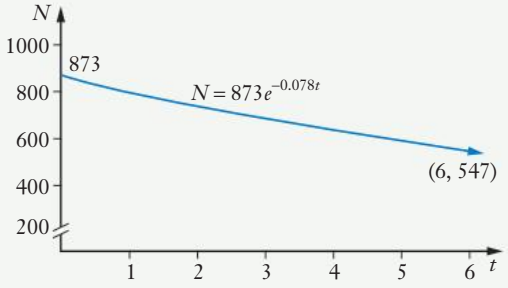


b **i** 204.2 mm^3 **ii** 6760.7 mm^3

c No, because it predicts the volume never stops increasing.

7 a 130.4 g **b** 74.5 g **c** 4.5 g

8 a



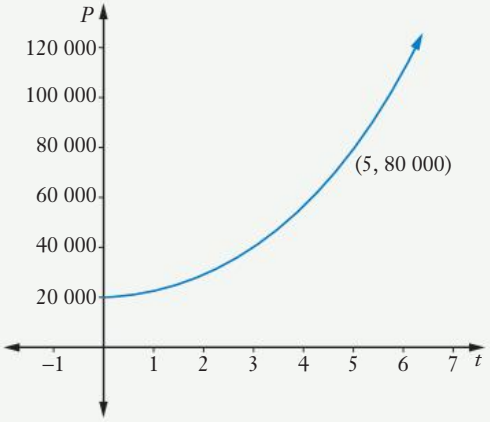
b **i** 873 **ii** 591 **iii** 400

9 a 148°C

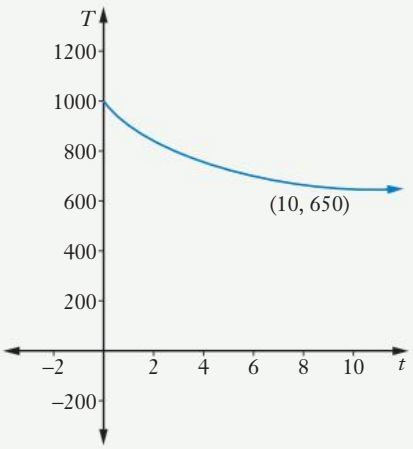
b **i** 133.9°C **ii** 115.6°C **iii** 91.6°C **iv** 23.1°C

c 23°C (room temperature)

10



11



Exercise 8.03

1 a $9e^x$ **b** $-e^x$ **c** $e^x + 2x$

d $6x^2 - 6x + 5 - e^x$ **e** $3e^x(e^x + 1)^2$

f $7e^x(e^x + 5)^6$ **g** $4e^x(2e^x - 3)$ **h** $e^x(x + 1)$

i $\dfrac{e^x(x-1)}{x^2}$ **j** $xe^x(x + 2)$

k $e^x(2x + 3)$

l $\dfrac{e^x(7x-10)}{(7x-3)^2}$ **m** $\dfrac{5e^x - 5xe^x}{e^{2x}} = \dfrac{5(1-x)}{e^x}$

2 a $2e^{2x}$ **b** $-e^{-x}$ **c** $6e^{3x}$

d $-7e^{7x}$ **e** $-6e^{2x} + 2x$

f $2e^{2x} + 2e^{-2x} = 2(e^{2x} + e^{-2x})$ **g** $-5e^{-x} - 3$

h $e^{4x}(4x + 1)$ **i** $\dfrac{6xe^{3x} + 4e^{3x} + 3}{(x+1)^2}$

j $135e^{3x}(9e^{3x} + 2)^4$

3 $6 - e$ **4** e **5** $-\dfrac{1}{2e^{10}}$

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6 19.81 **7** $y = -ex$

8 $x + e^3y - 3 - e^6 = 0$

9 **a** 12 572

b **i** 280 insects/week **ii** 75 742 insects/week

10 **a** 2 m **b** $8e^{40}$ m s^{-1}

c 95 390.7 m s^{-2}

11 **a** 1 cm **b** –2.04 cm s^{-1}

c –3.46 cm **d** –0.096 cm s^{-1}

e 0.35 cm s^{-2}

12 **a** **i** 33.1 mm^3 **ii** 163.8 mm^3

b **i** 26.5 mm^3 s^{-1} **ii** 131.0 mm^3 s^{-1}

13 **a** **i** 39 094 **ii** 44 299

b **i** 977 people/year **ii** 1107 people/year

14 **a** **i** 2.95 m **ii** 2.90 m **iii** 2.85 m

b **i** –0.050 m/month

ii –0.049 m/month **iii** –0.048 m/month

Exercise 8.04

1 **a** 4 **b** 2 **c** 3 **d** 1

e 2 **f** 1 **g** 0 **h** 7

i 1

2 **a** 3 **b** 4 **c** 29

3 **a** 9 **b** 3 **c** –1 **d** 12

e 8 **f** 4 **g** 14 **h** 1

i 2

4 **a** –1 **b** $\frac{1}{2}$ **c** $\frac{1}{2}$ **d** –2

e $\frac{1}{4}$ **f** $-\frac{1}{3}$ **g** $-\frac{1}{2}$ **h** $\frac{1}{3}$

i $1\frac{1}{2}$ **j** $-1\frac{1}{2}$

5 **a** 3.08 **b** 2.94 **c** 3.22 **d** 4.94

e 10.40 **f** 7.04 **g** 0.59 **h** 3.51

i 0.43

6 **a** $\log_3 y = x$ **b** $\log_5 z = x$ **c** $\log_x y = 2$

d $\log_2 a = b$ **e** $\log_b d = 3$ **f** $\log_8 y = x$

g $\log_6 y = x$ **h** $\log_e y = x$ **i** $\log_a y = x$

j $\log_e Q = x$

7 **a** $3^x = 5$ **b** $a^x = 7$ **c** $3^b = a$ **d** $x^9 = y$

e $a^y = b$ **f** $2^y = 6$ **g** $3^y = x$ **h** $10^y = 9$

i $e^y = 4$

8 **a** $x = 1\,000\,000$ **b** $x = 243$ **c** $x = 7$

d $x = 2$ **e** $x = -1$ **f** $x = 3$

g $x = 44.7$ **h** $x = 10\,000$ **i** $x = 8$

9 $y = 5$ **10** 44.7 **11** 2.44 **12** 0, 0

13 1, 1

14 **a** 1

b **i** 3 **ii** 2 **iii** 5 **iv** $\frac{1}{2}$

v –1 **vi** 2 **vii** 3 **viii** 5

ix 7 **x** 1 **xi** e

15 **a** 68.4 **b** **i** 58.8 **ii** 46.6

c 8 weeks

16 **a** 2.5, acidic **b** 7, neutral

c 9, alkaline **d** 2, acidic

e 11.9, alkaline **f** 5, acidic

17 **a** $y = \log(2x - 7)$ **b** $y = 2\log x - 7$

Exercise 8.05

1 **a** $\log_a 4y$ **b** $\log_a 20$ **c** $\log_a 4$ **d** $\log_a \frac{b}{5}$

e $\log_x y^3z$ **f** $\log_k 9y^3$ **g** $\log_a \frac{x^5}{y^2}$

h $\log_a \frac{xy}{z}$ **i** $\log_{10} ab^4c^3$

j $\log_3 \frac{p^3q}{r^2}$ **k** $-\log_4 n$ **l** $-\log_x 6$

2 **a** 2 **b** 6

3 **a** 1.19 **b** –0.47 **c** 1.55 **d** 1.66

e 1.08 **f** 1.36 **g** 2.02 **h** 1.83

i 2.36

4 **a** 2 **b** 6 **c** 2 **d** 3

e 1 **f** 3 **g** 7 **h** $\frac{1}{2}$

i –2 **j** 4

5 **a** $x + y$ **b** $x - y$ **c** $3x$ **d** $2y$

e $2x$ **f** $x + 2y$ **g** $x + 1$ **h** $1 - y$

i $2x + 1$

6 **a** $p + q$ **b** $3q$ **c** $q - p$ **d** $2p$

e $p + 5q$ **f** $2p - q$ **g** $p + 1$ **h** $1 - 2q$

i $3 + q$ **j** $p - 1 - q$

7 **a** 1.3 **b** 12.8 **c** 16.2 **d** 9.1

e 6.7 **f** 23.8 **g** –3.7 **h** 3

i 22.2

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8 **a** $x = 4$ **b** $y = 28$ **c** $x = 48$ **d** $x = 3$
e $k = 6$

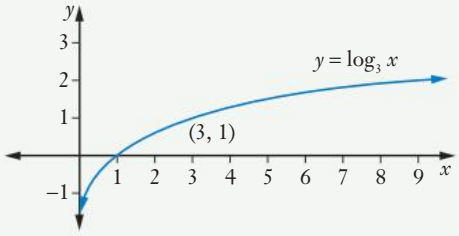
9 **a** $I = 10^{\frac{L}{10}} I_0$ **b** $31\,622.8 I_0$

10 **a** Proof: see worked solutions
b **i** 69.9 **ii** 2.2

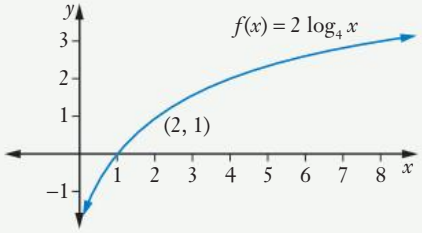
11 **a** 1.58 **b** 1.80 **c** 2.41
d 3.58 **e** 2.85 **f** 2.66
g 1.40 **h** 4.55 **i** 4.59
j 7.29

Exercise 8.06

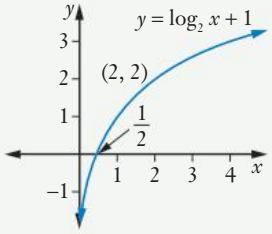
1 **a**



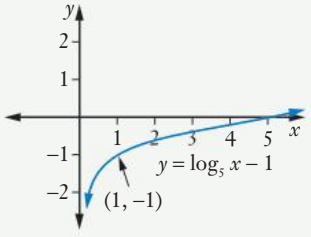
b



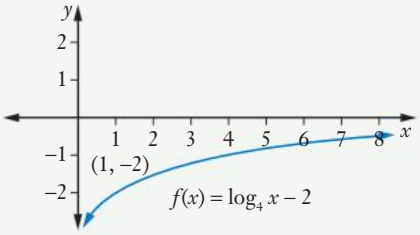
c



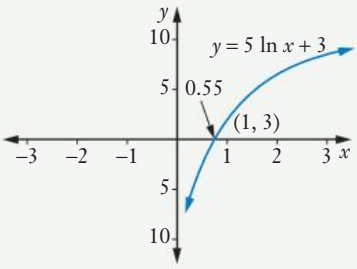
d



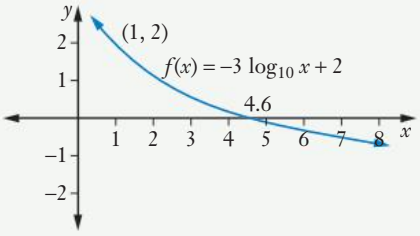
e



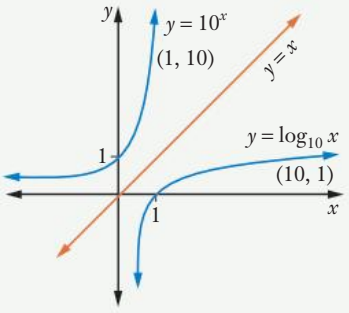
f



g

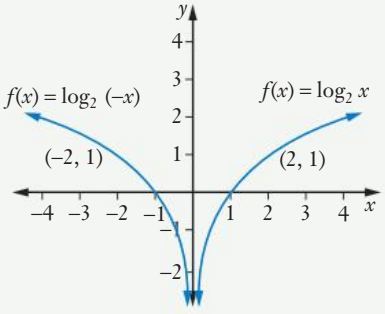


2



Curves are reflections of each other in the line $y = x$.

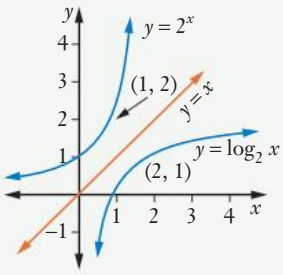
3



$y = \log_2(-x)$ is a reflection of $f(x) = \log_2 x$ in the y-axis.

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4 a



b $y = 2^x$

5 a 6 **b** 100 **c** 20 **d** 2
e 126

6 a 1000 **b** 10 **c** 20 **d** 3162
e 1 995 262

Exercise 8.07

1 a $x = 1.6$ **b** $x = 1.5$ **c** $x = 1.4$ **d** $x = 3.9$
e $x = 2.2$ **f** $x = 2.3$ **g** $x = 6.2$ **h** $x = 2.8$
i $x = 2.9$ **j** $x = 2.4$

2 a $x = 2.58$ **b** $y = 1.68$ **c** $x = 2.73$
d $m = 1.78$ **e** $k = 2.82$ **f** $t = 1.26$
g $x = 1.15$ **h** $p = 5.83$ **i** $x = 3.17$
j $n = 2.58$

3 a $x = 0.9$ **b** $n = 0.9$ **c** $x = 6.6$
d $n = 1.2$ **e** $x = -0.2$ **f** $n = 2.2$
g $x = 2.2$ **h** $k = 0.9$ **i** $x = 3.6$

4 a $x = 5.30$ **b** $t = 0.536$ **c** $t = 3.62$
d $x = 3.81$ **e** $n = 3.40$ **f** $t = 0.536$
g $t = 24.6$ **h** $k = 67.2$ **i** $t = 54.9$

5 a i \$850 **ii** \$1010.38 **b** 6.6 years

6 a i 35 000 **ii** 44 494 **iii** 116 204
b i after 34.4 years **ii** after 72.6 years

7 a i 8900 **ii** 7001 **iii** 309
b i 12 years **ii** 79 years

8 a 100 g **b** 99.85 g
c 98.5 g **d** 23 105 years

9 a 30.1°C **b** 490.4 hours

10 a i 28 cm **ii** $5e^{20}$ cm s^{-1} **iii** 2040.1 cm
iv 2.52 s **v** 5.30 s
b $a = 5e^t$
$= 5e^t + 23 - 23$
$= x - 23$
c 62 cm s^{-2}

Test yourself 8

1 D **2** B **3** A, C

4 a 3 **b** 1 **c** 3 **d** 2
e 1 **f** 3 **g** $\frac{1}{2}$ **h** –1
i –2 **j** 3

5 a 6.39 **b** 1.98 **c** 3.26
d 1.40 **e** 0.792 **f** 3.91
g 5.72 **h** 72.4

6 a 6 **b** 2

7 a $3^x = a$ **b** $e^y = b$ **c** $10^z = c$

8 a 0.92 **b** 1.08 **c** 0.2
d 1.36 **e** 0.64

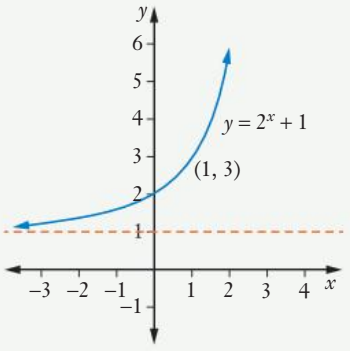
9 a $x = 1.9$ **b** $x = 1.9$ **c** $x = 3$ **d** $x = 36$

10 $t = 18.2$ **11** 0.9

12 a $\log_a x^5y^3$ **b** $\log_x \frac{k^2p}{3}$

13 a 0.65 **b** 1.3

14



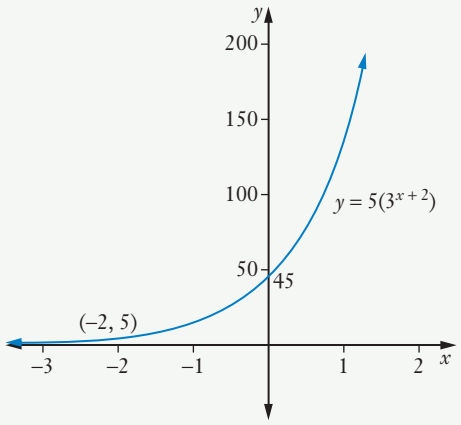
Domain $(-\infty, \infty)$, range $(1, \infty)$

15 a $x = 3.17$ **b** $x = 1.77$ **c** $x = 0.11$
d $y = 0.86$ **e** $n = 0.92$ **f** $x = 2$
g $y = 27$ **h** $n = 49$ **i** $x = 4096$
j $m = 2$

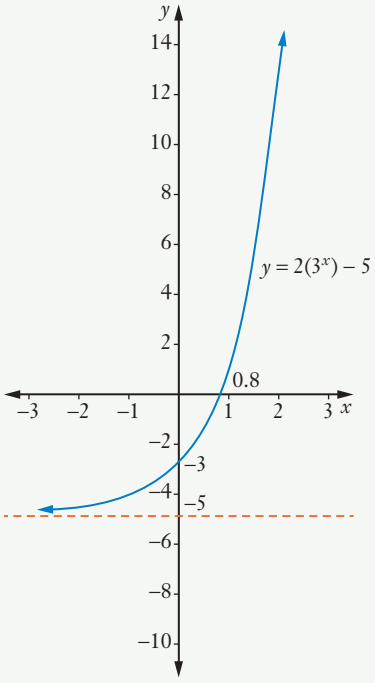
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16 **a** $\log_2 y = x$ **b** $\log_5 b = a$ **c** $\log y = x$
d $\ln z = x$ **e** $\log_3 y = x + 1$

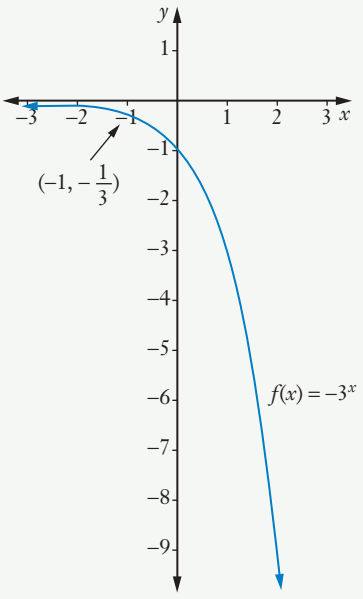
17 **a**



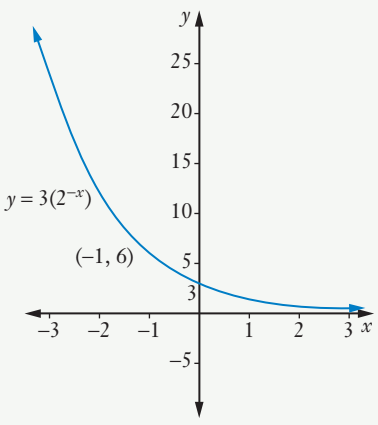
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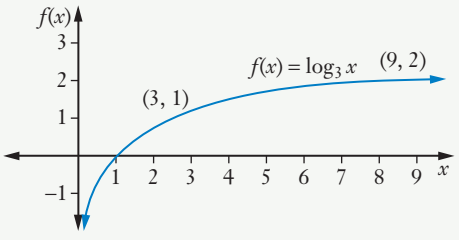
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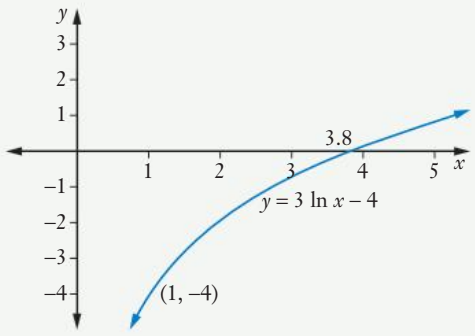


18 **a**



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b



19 **a** $a + b$ **b** $b - a$ **c** $3a$ **d** $a + 2b$ **e** $3b$

20 **a** 37.4 dB **b** $1585I_0$

21 **a** $-\log_a x$ **b** $-\log_e y$

22 **a** 2 **b** 2 **c** 3 **d** 1 **e** 3

23 **a** $x = 2.7$ **b** $x = 3.1$ **c** $x = 2.1$

24 **a** **i** \$5280 **ii** \$5586.71 **iii** \$5692.86
b **i** 6.8 years **ii** 33.9 years

25 **a** $3e^{3x}$ **b** $-2e^{-2x}$ **c** $20e^{4x}$
d $-16e^{8x} + 15x^2$ **e** $2xe^{2x}(x + 1)$
f $108e^{3x}(4e^{3x} - 1)^8$ **g** $\dfrac{1-2x}{e^{2x}}$

26 **a** 1081 **b** 1075
c **i** 88 years **ii** 103.6 years

27 **a** $e^x + 1$ **b** $-4e^x$ **c** $-3e^{-x}$
d $9e^x(3 + e^x)^8$ **e** $3x^4e^x(x + 5)$ **f** $\dfrac{e^x(7x-9)}{(7x-2)^2}$

28 100 times

29 20 times

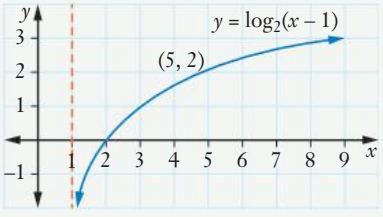
30 **a** $y = \log_e(6x^2 - 1)$ **b** $y = e^{6x^2-1}$
c $y = 6e^{2x} - 1$ **d** $y = x$
e $y = x$

Challenge exercise 8

1 **a** 2.7 **b** 2.8 **c** 2.5

2 (1, 0)

3



Domain $(1, \infty)$, range $(-\infty, \infty)$

4 $x = 0.63$

5 **a** $\text{pH} = -\log [H^+]$
$= \log 1 - \log [H^+]$
$= \log \dfrac{1}{[H^+]}$

b $\text{pH} = -\log [H^+]$
$-\text{pH} = \log [H^+]$
$10^{-\text{pH}} = [H^+]$
$\dfrac{1}{10^{\text{pH}}} = [H^+]$

c **i** 0.000 000 5 **ii** 0.000 000 02

6 **a** Proof includes $x + 2 = 2^{y-8}$ (see worked solutions)

b **i** $y = 10.81$ **ii** $x = -1.99$

7 **a** $3e^2x - y - 3e^2 - 5 = 0$

b $x + 3e^2y - 9e^4 + 15e^2 - 2 = 0$

Chapter 9

Exercise 9.01

1 **a** 1st, 4th **b** 1st, 3rd **c** 1st, 2nd
d 2nd, 4th **e** 3rd, 4th **f** 2nd, 3rd
g 3rd **h** 2nd **i** 4th
j 4th

2 **a** 3rd **b** $-\dfrac{1}{2}$

3 **a** 4th **b** $-\dfrac{1}{\sqrt{2}}$

4 **a** 2nd **b** $-\sqrt{3}$

5 **a** 2nd **b** $\dfrac{1}{\sqrt{2}}$

6 **a** 1st **b** $\dfrac{\sqrt{3}}{2}$

7 **a** 1 **b** $\dfrac{1}{\sqrt{2}}$ **c** $-\sqrt{3}$ **d** $\dfrac{1}{2}$
e $-\dfrac{1}{2}$ **f** $-\dfrac{1}{2}$ **g** $\dfrac{\sqrt{3}}{2}$ **h** $-\dfrac{1}{\sqrt{3}}$
i $-\dfrac{\sqrt{3}}{2}$ **j** $-\dfrac{1}{\sqrt{2}}$

8 **a** $-\dfrac{1}{\sqrt{2}}$ **b** $-\dfrac{\sqrt{3}}{2}$ **c** $\sqrt{3}$ **d** $-\dfrac{\sqrt{3}}{2}$
e $-\dfrac{\sqrt{3}}{2}$ **f** $-\sqrt{3}$ **g** $\dfrac{1}{2}$ **h** $-\dfrac{1}{\sqrt{3}}$
i $\dfrac{1}{\sqrt{2}}$ **j** $-\dfrac{1}{\sqrt{2}}$

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9 **a** $-\frac{\sqrt{3}}{2}$ **b** $\sqrt{3}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{1}{2}$
e $-\frac{1}{2}$ **f** $\sqrt{3}$ **g** $\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{2}}$
i -1 **j** $\frac{1}{2}$

10 $\sin\theta = -\frac{3}{5}, \cos\theta = -\frac{4}{5}$

11 $\cos\theta = -\frac{\sqrt{33}}{7}, \tan\theta = -\frac{4}{\sqrt{33}}$

12 $\cos x = \frac{8}{\sqrt{89}}$

13 $\sin x = -\frac{\sqrt{21}}{5}, \tan x = -\frac{\sqrt{21}}{2}$

14 $\sin x = \frac{5}{\sqrt{74}}, \cos x = -\frac{7}{\sqrt{74}}$

15 $\tan\theta = -\frac{4}{\sqrt{65}}, \cos\theta = \frac{\sqrt{65}}{9}$

16 $\tan x = \frac{\sqrt{55}}{3}, \sin x = -\frac{\sqrt{55}}{8}$

17 **a** $\sin x = \frac{3}{10}$ **b** $\cos x = -\frac{\sqrt{91}}{10}, \tan x = -\frac{3}{\sqrt{91}}$

18 $\cos\alpha = \frac{5}{\sqrt{61}}, \sin\alpha = -\frac{6}{\sqrt{61}}$

19 $\sin\theta = \frac{\sqrt{51}}{10}, \tan\theta = -\frac{\sqrt{51}}{7}$

20 **a** $\sin\theta$ **b** $\cos x$ **c** $\tan\beta$
d $-\sin\alpha$ **e** $-\tan\theta$ **f** $-\sin\theta$
g $\cos\alpha$ **h** $-\tan x$

Exercise 9.02

1 $\sec x = \frac{9}{5}, \cot x = \frac{5}{\sqrt{56}}, \operatorname{cosec} x = \frac{9}{\sqrt{56}}$

2 $\operatorname{cosec}\theta = \frac{13}{5}, \sec\theta = \frac{13}{12}, \cot\theta = \frac{12}{5}$

3 $\operatorname{cosec}\theta = \frac{7}{\sqrt{33}}, \sec\theta = \frac{7}{4}, \cot\theta = \frac{4}{\sqrt{33}}$

4 $\tan\theta = -\frac{\sqrt{11}}{5}, \operatorname{cosec}\theta = \frac{6}{\sqrt{11}}, \cot\theta = -\frac{5}{\sqrt{11}}$

5 $\sin\theta = -\frac{5}{\sqrt{34}}, \operatorname{cosec}\theta = -\frac{\sqrt{34}}{5}, \tan\theta = \frac{5}{3}, \sec\theta = -\frac{\sqrt{34}}{3}$

6 $\sin 67° = \cos(90° - 67°) = \cos 23°$

7 $\sec 82° = \operatorname{cosec}(90° - 82°) = \operatorname{cosec} 8°$

8 $\tan 48° = \cot(90° - 48°) = \cot 42°$

9 **a** $2\cos 61°$ or $2\sin 29°$
b 0 **c** 0 **d** 1 **e** 2

10 $x = 80$ **11** $y = 22$ **12** $p = 31$

13 $b = 25$ **14** $t = 20$ **15** $k = 15$

16 **a** $\sin\theta$ **b** $\sec\theta$ **c** $\operatorname{cosec} x$
d $\cos^2 x$ **e** $\sin\alpha$ **f** $\operatorname{cosec}^2 x$
g $\sec^2 x$ **h** $\tan^2\theta$ **i** $5\operatorname{cosec}^2\theta$
j $\sin^2 x$ **k** 1 **l** $\sin\theta\cos\theta$

17 See worked solutions.

Exercise 9.03

1 **a** $\sqrt{2}$ **b** $\frac{2}{\sqrt{3}}$ **c** $\frac{1}{\sqrt{3}}$
d $\sqrt{3}$ **e** $\frac{1}{2}$ **f** $\frac{\sqrt{3}}{2}$
g $\sqrt{2}$ **h** 3 **i** 2

2 **b** 2nd **c** $-\frac{1}{\sqrt{2}}$

3 **b** 2nd **c** $\frac{1}{2}$

4 **b** 4th **c** -1

5 **b** 3rd **c** $-\frac{1}{2}$

6 **b** 4th **c** $-\frac{\sqrt{3}}{2}$

7 **a** **ii** 1st **iii** $\frac{\sqrt{3}}{2}$
b **i** $\frac{1}{\sqrt{2}}$ **ii** $\sqrt{3}$ **iii** $-\frac{1}{\sqrt{2}}$ **iv** $\frac{1}{\sqrt{3}}$
v $-\frac{\sqrt{3}}{2}$

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8 a

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	$\frac{10\pi}{3}$	$\frac{11\pi}{3}$
sin	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
cos	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
tan	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$

b

	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$	$\frac{13\pi}{4}$	$\frac{15\pi}{4}$
sin	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
cos	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
tan	1	−1	1	−1	1	−1	1	−1

c

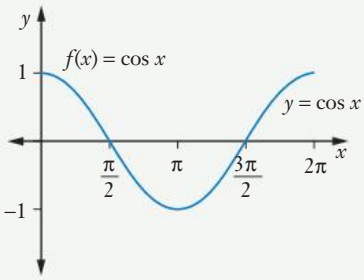
	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	$\frac{17\pi}{6}$	$\frac{19\pi}{6}$	$\frac{23\pi}{6}$
sin	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
cos	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
tan	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

9

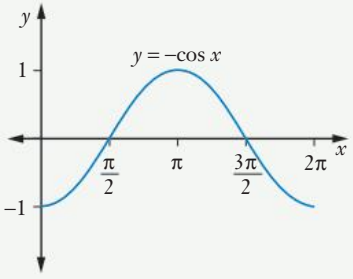
	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin	0	1	0	−1	0	1	0	−1	0
cos	1	0	−1	0	1	0	−1	0	1
tan	0	Not defined	0	Not defined	0	Not defined	0	Not defined	0

Exercise 9.04

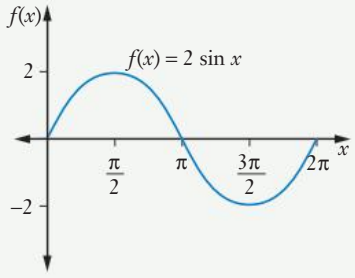
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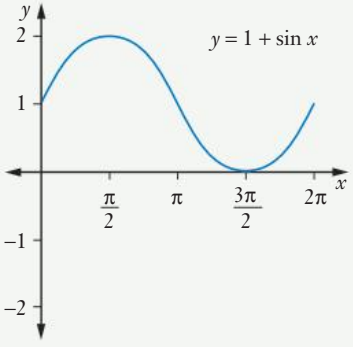
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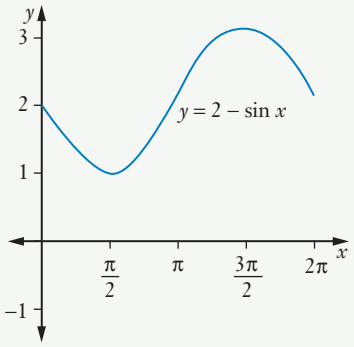


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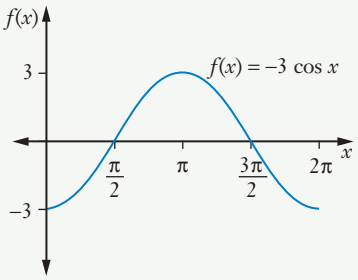


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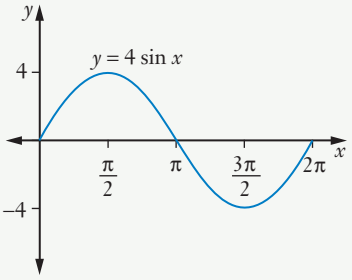
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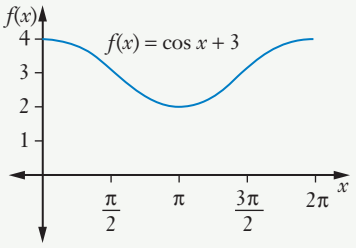
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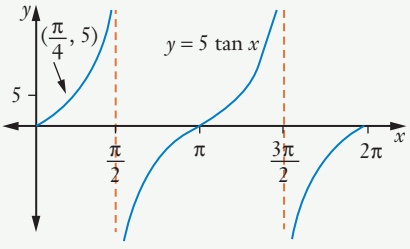
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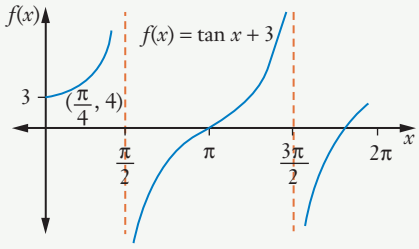
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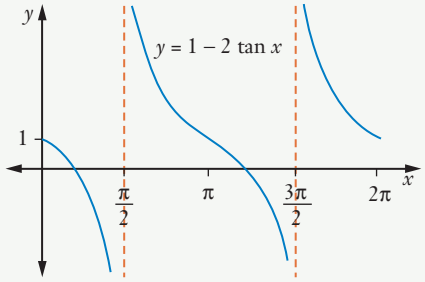
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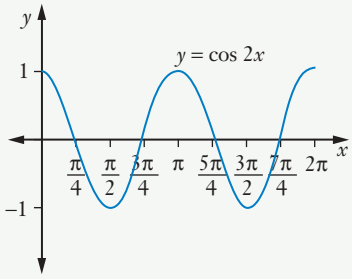
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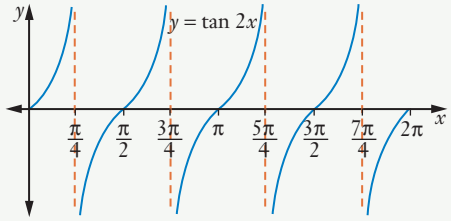
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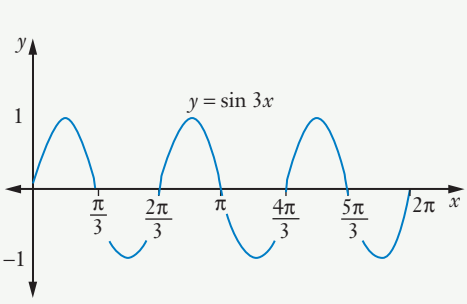
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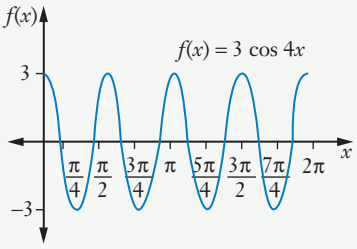


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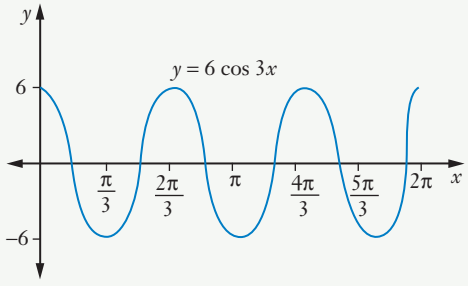


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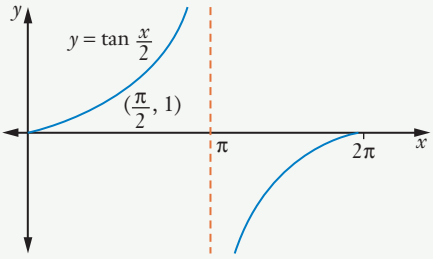
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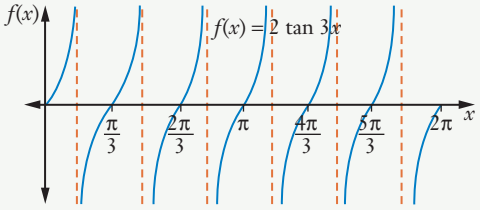
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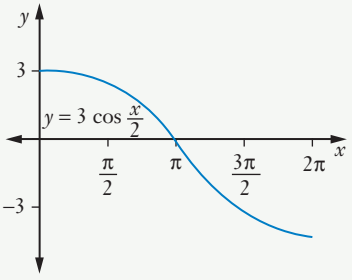
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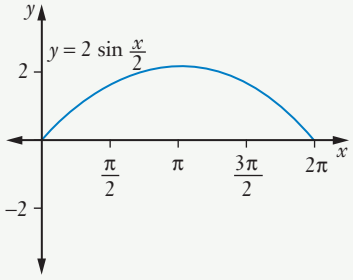
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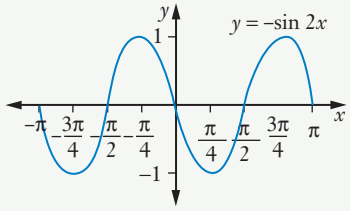
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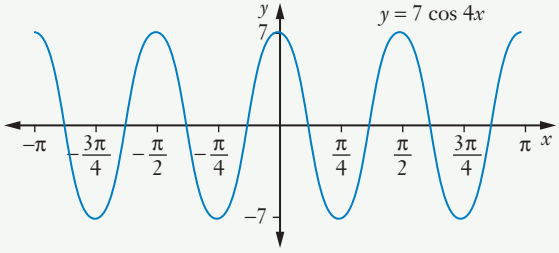
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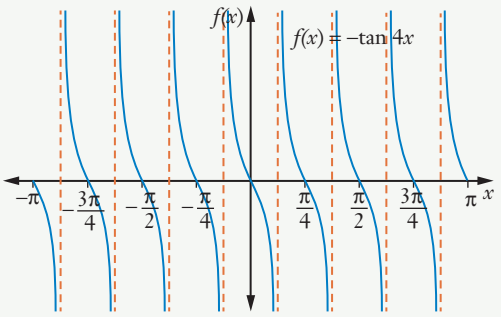
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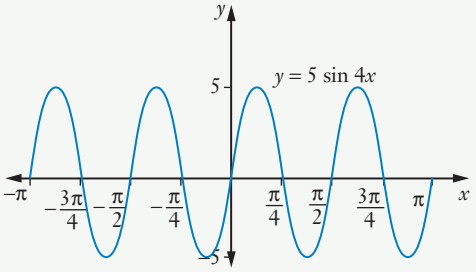
b



c

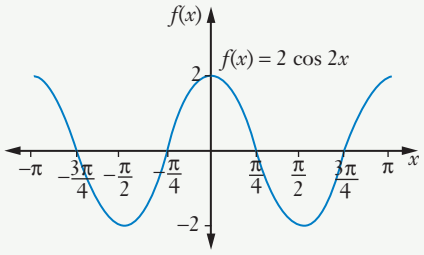


d

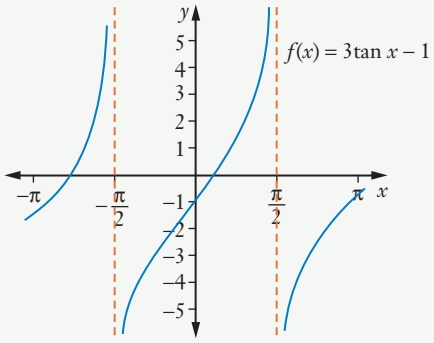


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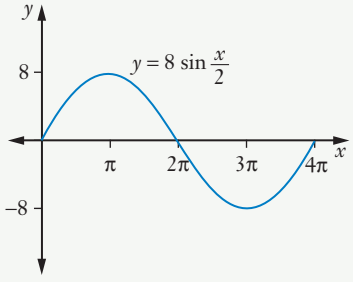
e



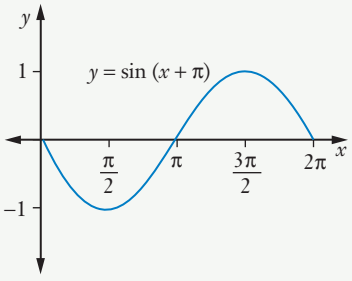
f



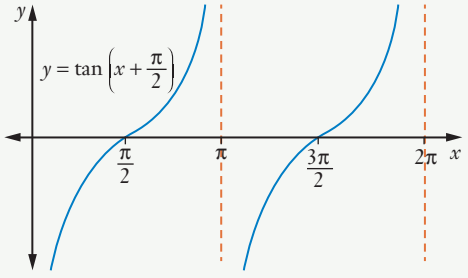
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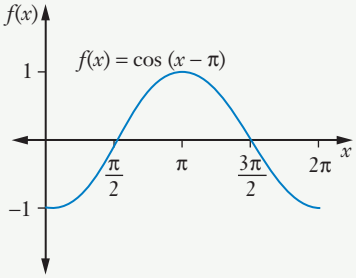
6 a



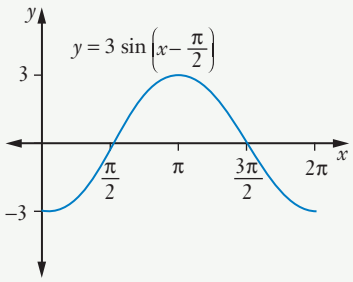
b



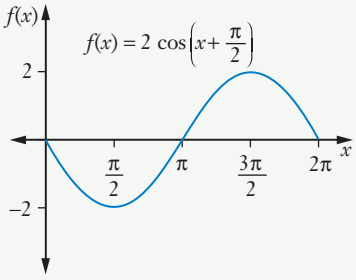
c



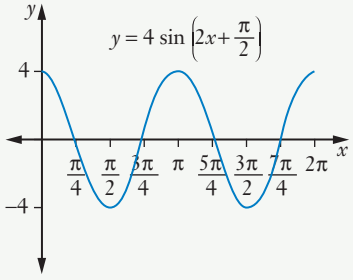
d



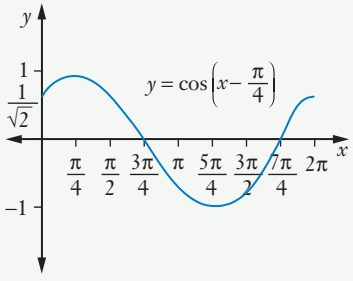
e



f

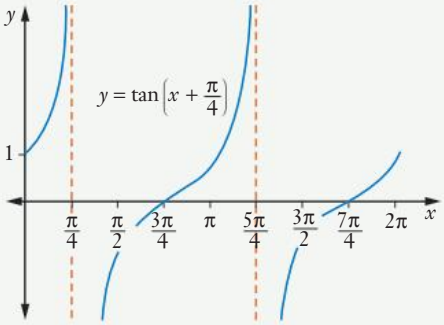


g

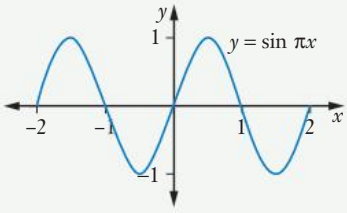


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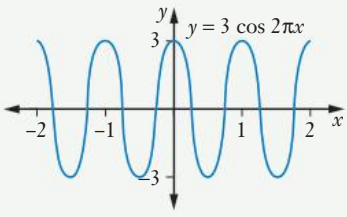
h



7 a



b



8 a i 5 **ii** π **iii** 0 **iv** 0

b i 1 **ii** 2π **iii** 0 **iv** −π

c i None **ii** $\frac{\pi}{4}$ **iii** −2 **iv** 0

d i 3 **ii** 2π **iii** 1 **iv** $\frac{\pi}{4}$

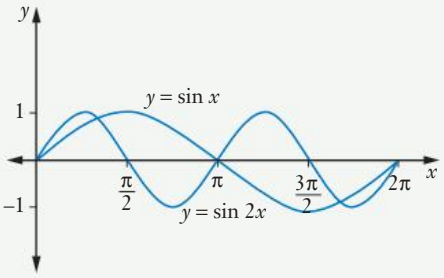
e i 8 **ii** 2 **iii** −3 **iv** $-\frac{2}{\pi}$

f i None **ii** $\frac{\pi}{5}$ **iii** 2 **iv** $\frac{\pi}{10}$

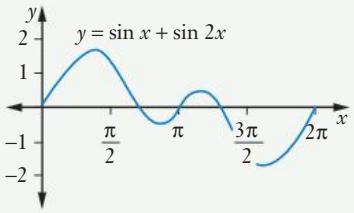
9 a Domain $(-\infty, \infty)$, range $[-5, 3]$

b Domain $(-\infty, \infty)$, range $[4, 10]$

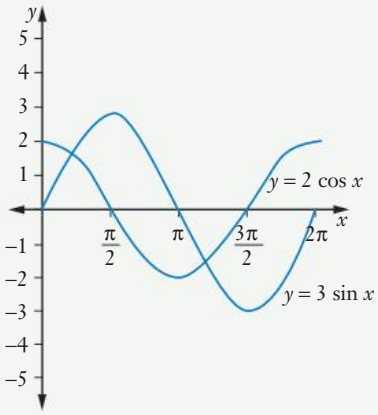
10 a



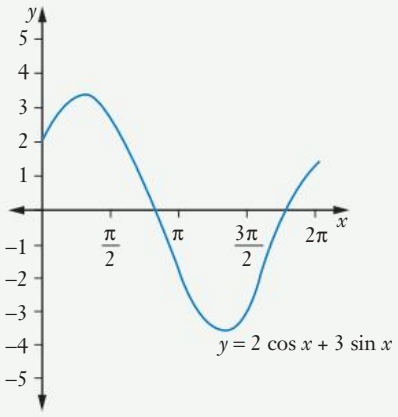
b



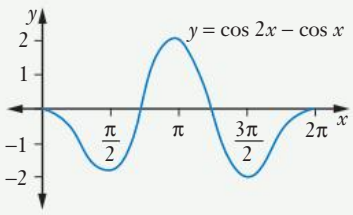
11 a



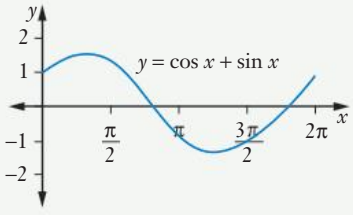
b



12



13



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Exercise 9.05

1 **a** $\theta = 20°29', 159°31'$

b $\theta = 120°, 240°$

c $\theta = 135°, 315°$

d $\theta = 60°, 120°$

e $\theta = 150°, 330°$

f $\theta = 30°, 330°$

g $\theta = 30°, 120°, 210°, 300°$

h $\theta = 30°, 150°, 210°, 330°$

i $\theta = 70°, 110°, 190°, 230°, 310°, 350°$

j $\theta = 15°, 45°, 75°, 105°, 135°, 165°, 195°, 225°, 255°, 285°, 315°, 345°$

k $x = 90°, 270°$

l $x = 60°, 90°, 270°, 300°$

2 **a** $x = 0°, 360°$ **b** $x = 270°$

c $x = 0°, 180°, 360°$ **d** $x = 90°$

e $x = 0°, 180°, 360°$

f $x = 0°, 180°, 270°, 360°$

g $x = 0°, 90°, 270°, 360°$

h $x = 0°, 45°, 180°, 225°, 360°$

i $x = 60°, 120°, 240°, 300°$

3 **a** $x = 0, \pi, 2\pi$ **b** $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

c $x = \frac{3\pi}{2}$ **d** $x = 0, 2\pi$ **e** $x = \pi$

4 **a** $\theta = \pm 79°13'$ **b** $\theta = 30°, 150°$

c $\theta = 45°, -135°$ **d** $\theta = -60°, -120°$

e $\theta = 150°, -30°$ **f** $\theta = \pm 30°, \pm 150°$

g $\theta = 135°, -45°$

h $\theta = 22.5°, 112.5°, -67.5°, -157.5°$

5 **a** $x = \frac{\pi}{3}, \frac{5\pi}{3}$ **b** $x = \frac{5\pi}{4}, \frac{7\pi}{4}$

c $x = \frac{\pi}{4}, \frac{5\pi}{4}$ **d** $x = \frac{\pi}{3}, \frac{4\pi}{3}$

e $x = \frac{5\pi}{6}, \frac{7\pi}{6}$

6 **a** $x = \frac{\pi}{3}, \frac{2\pi}{3}$ **b** $x = \pm\frac{\pi}{2}$

c $x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

7 $x = \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}$

8 **a** $x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$ **b** $x = 0, \frac{\pi}{2}, \pi, 2\pi$

c $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ **d** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

e $x = 0, \pi, 2\pi$ **f** $x = 0, 2\pi$

Exercise 9.06

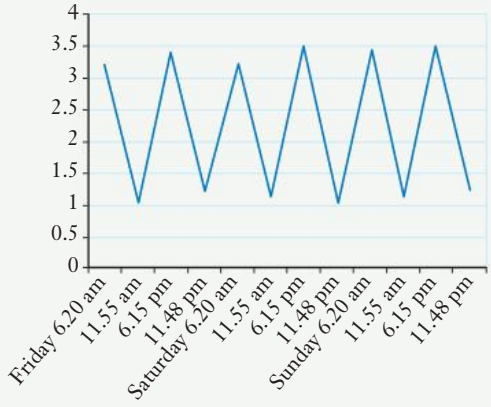
1 **a** Period 12 months, amplitude 1.5

b 5:30 p.m.

2 **a** 1300 **b** **i** 1600 **ii** 1100

c Amplitude 250, period 8 or 9 years

3 **a**



b Period 12 hours, amplitude 1.25 m

c 2.5 m

Test yourself 9

1 D **2** D **3** B **4** C

5 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $-\sqrt{3}$

6 **a** $x = 60°, 120°$ **b** $x = 45°, 225°$

c $x = 120°, 240°$

d $x = 60°, 120°, 240°, 300°$

e $x = 15°, 105°, 195°, 285°$

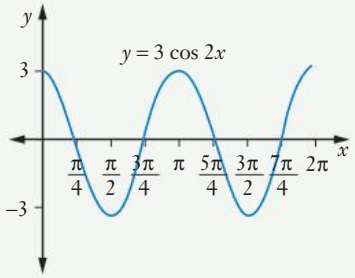
7 **a** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ **b** $x = \frac{\pi}{6}, \frac{5\pi}{6}$

c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **d** $x = 0, 2\pi$

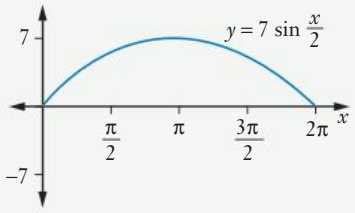
e $x = \frac{3\pi}{2}$

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8 a



b



9 $\cos x = \frac{5}{13}$, $\tan x = -\frac{12}{5}$

10 **a** $-\cos\theta$ **b** $-\tan\theta$ **c** $\sin\theta$
d $\sin x$ **e** $2\cos A$ **f** $\sin x$
g 1

11 **a** $-\frac{1}{\sqrt{2}}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $\sqrt{3}$

12 See worked solutions.

13 $b = 40$

14 Period $\frac{2\pi}{3}$, amplitude 2, centre 5, phase $\frac{\pi}{36}$

15 **a** $\sqrt{2}$ **b** $\sqrt{3}$ **c** $\frac{2}{\sqrt{3}}$
d $\sqrt{3}$

16 **a** Domain $(-\infty, \infty)$, range $[-1, 11]$
b Domain $(-\infty, \infty)$, range $[-7, 1]$

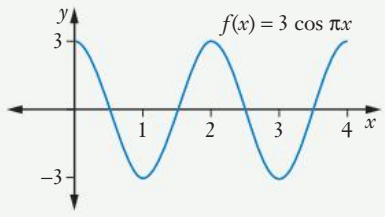
Challenge exercise 9

1 **a** $-\frac{\sqrt{3}}{2}$ **b** -1

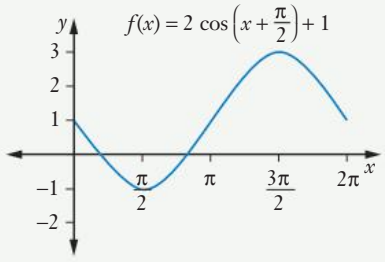
2 $\theta = 110°, 230°$

3 **a** Period = 2, amplitude = 3

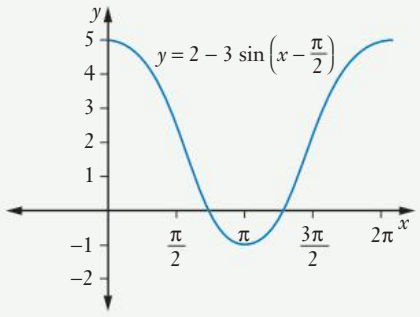
b



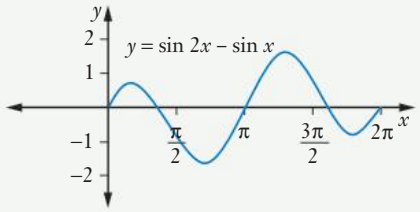
4 **a**



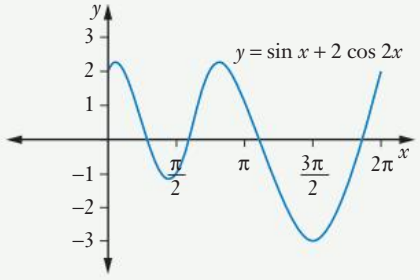
b



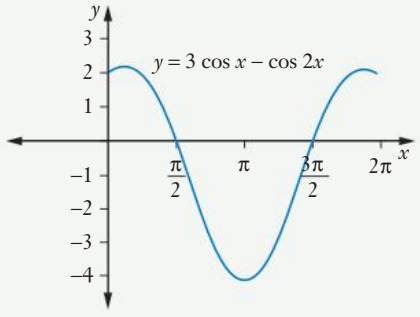
c



d

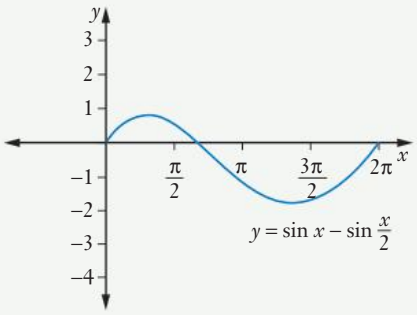


e



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f



5 $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$

6 $\frac{\sqrt{3}-\sqrt{2}}{2}$

Chapter 10

Exercise 10.01

1 **a** Discrete **b** Continuous **c** Discrete
d Discrete **e** Continuous **f** Discrete
g Continuous **h** Continuous **i** Discrete
j Discrete **k** Discrete **l** Discrete

2 **a** $X = \{0, 1\}$
b $X = \{0, 1, 2, \ldots, 10\}$
c $X = \{0, 1, 2, 3, \ldots, 20\}$
d $X = \{0, 1, 2, \ldots 31\}$
e $X = \{2, 3, 4, \ldots, 12\}$

Exercise 10.02

1

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

2 **a** $\left(0, \frac{1}{2}\right), \left(1, \frac{1}{2}\right)$ **b** $\left(0, \frac{1}{4}\right), \left(1, \frac{1}{2}\right), \left(2, \frac{1}{4}\right)$
c $\left(0, \frac{1}{8}\right), \left(1, \frac{3}{8}\right), \left(2, \frac{3}{8}\right), \left(3, \frac{1}{8}\right)$

3 **a**

x	48	49	50	51	52
$P(X=x)$	$\frac{8}{53}$	$\frac{9}{53}$	$\frac{21}{53}$	$\frac{9}{53}$	$\frac{6}{53}$

b **i** $\frac{36}{53}$ **ii** $\frac{38}{53}$

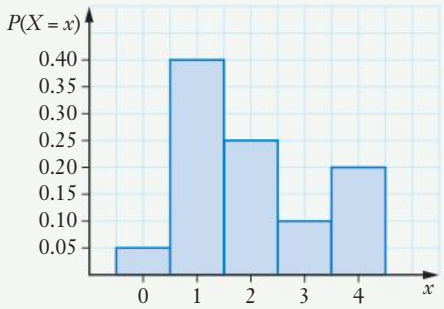
4 **a** $p(3) + p(4) + p(5) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6}$
$= 1$

b

x	3	4	5
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

c **i** $\frac{5}{6}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{2}$

5



6

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b Yes

c **i** $\frac{1}{2}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{2}$

7 **a** No **b** Yes **c** No

8 **a** $k = \frac{1}{14}$ **b** $k = 0.19$ **c** $k = 0$

9 **a**

x	1	2	3	4
$P(X=x)$	$\frac{k}{6}$	$\frac{4k}{7}$	$\frac{9k}{8}$	$\frac{16k}{9}$

b $k = \frac{504}{1835}$

10 **a**

y	0	1	2	3
$P(Y=y)$	$\frac{5}{9}$	$\frac{5}{18}$	$\frac{5}{36}$	$\frac{1}{36}$

b **i** $\frac{1}{36}$ **ii** $\frac{1}{6}$ **iii** $\frac{35}{36}$

11 $p = \frac{1}{11}$

12 **a**

x	0	1	2	3	5
$P(X=x)$	$\frac{13}{25}$	$\frac{19}{100}$	$\frac{1}{10}$	$\frac{9}{100}$	$\frac{1}{10}$

b $\frac{19}{100}$ **c** $\frac{9}{10}$

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13 **a** $\frac{2}{7}$ **b** $\frac{1}{2}$ **c** $\frac{5}{14}$ **d** $\frac{4}{7}$

e $\frac{1}{7}$ **f** $\frac{5}{14}$ **g** $\frac{9}{14}$

14 **a** $\frac{3}{16}$ **b** $\frac{3}{4}$ **c** $\frac{3}{8}$ **d** $\frac{3}{4}$

e $\frac{1}{4}$ **f** $\frac{3}{4}$

15 **a**

x	0	1	2
$P(X=x)$	$\frac{994\,009}{1\,000\,000}$	$\frac{5982}{1\,000\,000}$	$\frac{9}{1\,000\,000}$

b **i** $\frac{5982}{1\,000\,000}$ **ii** $\frac{5991}{1\,000\,000}$

16 **a**

x	0	1	2
$P(X=x)$	$\frac{25}{49}$	$\frac{20}{49}$	$\frac{4}{49}$

b

x	0	1	2
$P(X=x)$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{1}{14}$

17 **a** $\frac{1}{8}$ **b** No

c

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

18

x	0	1	2	3
$P(X=x)$	68.1%	27.9%	3.8%	0.2%

19

x	0	1	2	3	4
$P(X=x)$	5.76%	24%	37.47%	26%	6.77%

20 **a**

x	0	1	2
$P(X=x)$	$\frac{893}{990}$	$\frac{19}{198}$	$\frac{1}{495}$

b $\frac{97}{990}$

Exercise 10.03

1 **a** 1 **b** 2.78 **c** $3\frac{7}{16}$

d $2\frac{3}{4}$ **e** 2

2 **a** **i** $k=\frac{1}{10}$ **ii** 3

b **i** $k=\frac{1}{12}$ **ii** $1\frac{1}{6}$

c **i** $k=0.15$ **ii** 4.28

3 **a** 1 **b** 7 **c** $1\frac{1}{2}$ **d** $\frac{3}{1000}$

e **i** $\frac{14}{19}$ **ii** $\frac{14}{19}$

4 $p=0.3, q=0.1$

5 $a=\frac{1}{16}, b=\frac{3}{16}$

6 **a**

x	1	2	3	4
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b $2\frac{1}{2}$

7 $1\frac{1}{2}$

8 **a** $2\frac{2}{11}$ **b** $2\frac{2}{11}$

9 **a**

x	0	1	2	3
$P(X=x)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

b 56 cents **c** Lose 44 cents

10 **a**

x	0	1	2	3	4
$P(X=x)$	$\frac{81}{625}$	$\frac{216}{625}$	$\frac{216}{625}$	$\frac{96}{625}$	$\frac{16}{625}$

b 1.6

c Yes, Yasmin will make at least one phone sale in an hour.

11 Win 42 cents

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Exercise 10.04

1 **a** **i** 1.69 **ii** 2.85
b **i** 1.11 **ii** 1.23
c **i** 1.50 **ii** 2.25

2 **a** Mean 7.01, variance 7.49, σ 2.74
b Mean 2.89, variance 1.87, σ 1.37

3 $n = \frac{3}{10}$, $E(X) = 2.75$, variance = 2.89

4 **a** $a = 0.15, b = 0.18$ **b** 2.08 **c** 1.44

5 **a** **i** 1.5 **ii** 0.87 **iii** 0.75
b **i** 0.91 **ii** 0.69 **iii** 0.47

6 **a** 3 **b** 1.41 **c** 2

7 **a**

x	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

b Mean 0.33, variance 0.28, standard deviation 0.53

8 **a** 0.08 **b** 0.28 **c** 0.08

9 **a** Mean 1.25, variance 0.73
b Mean 1.25, variance 0.60

10 **a** **i**

x	0	1	2
$P(X = x)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

ii Mean 0.15, variance 0.14, standard deviation 0.37

b **i**

y	0	5	10
$P(Y = y)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

ii Mean 0.77, variance 3.48, standard deviation 1.87

Test yourself 10

1 C **2** D **3** B **4** D

5 **a** $X = \{0, 1, 2, 3, 4, 5\}$
b $X = \{0, 1, 2, 3, \ldots, 10\}$
c $X = \{0, 1, 2, 3, \ldots, 30\}$
d $X = \{0, 1, 2\}$
e $X = \{0, 1, 2, 3, \ldots, 9\}$

6 $k = \frac{1}{4}$

7 Mean 3.67, variance 1.56, standard deviation 1.25

8 **a** $\frac{1}{12}$ **b** $\frac{7}{12}$ **c** $\frac{5}{6}$
d $\frac{11}{12}$ **e** $\frac{5}{12}$

9

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

10 **a** uniform **b** not uniform
c uniform **d** not uniform

11 **a** Discrete **b** Continuous
c Discrete **d** Continuous

12 $E(X) = 1.5$, $Var(X) = 1.65$, standard deviation = 1.28

13 **a**

x	1	2	3	4	5	6	7
$P(X = x)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

b Yes **c** **i** $\frac{2}{7}$ **ii** $\frac{3}{7}$ **iii** $\frac{4}{7}$
d $E(X) = 4$

14 **a** **i** $\frac{1}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{3}{5}$

b $f(3) + f(5) + f(9) = \frac{1}{5} + \frac{1}{5} + \frac{3}{5} = 1.$
So it is a probability function.

15 **a**

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b No **c** **i** $\frac{1}{2}$ **ii** $\frac{3}{4}$

16 **a** Yes **b** No **c** No

17 **a** $E(X) = 0.44$ **b** Jonas loses 56 cents.

18 **a** 21% + 14% + 47% + 18% = 100% = 1
b $E(X) = 5.77$, $Var(X) = 3.5971$

19 **a** $n = \frac{1}{16}$ **b** $n = 0.08$ **c** $n = \frac{1}{9}$

20 $a = 0.2, b = 0.1$

21 **a**

y	\$2	–\$1	\$2
$P(Y = y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

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b $E(X) = 50$ cents so the player is expected to lose 50 cents.

Challenge exercise 10

1 $a = 0.22, b = 0.26, c = 0.12$

2 $a = 0.2, k = 4$ **3** $k = 5, l = 0.18$

4 **a** $5 \times 0.2 = 1$ **b** Yes

c **i** 0.6 **ii** 0.6 **iii** 0.8

d $E(X) = 3$, $Var(X) = 2$ **e** $P(X = 2) = 0.05$

5 **a** 120

b

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{30}$	$\frac{1}{8}$	$\frac{23}{120}$	$\frac{59}{120}$	$\frac{19}{120}$

c No: mean 3.62

d Yes, standard deviation 1

e Class discussion

Practice set 4

1 C **2** B **3** D

4 C, D **5** D

6 **a** $e^x - 1$ **b** $3e^x$ **c** $4e^x(e^x - 2)^3$

d $e^x(4x + 1)^2(4x + 13)$ **e** $\frac{e^x(5x - 7)}{(5x - 2)^2}$

f $35e^{7x}$

7 **a** **i** $\frac{1}{8}$ **ii** $\frac{1}{4}$

b $\frac{1}{8} + \frac{1}{4} + \frac{3}{8} + \frac{1}{4} = 1$

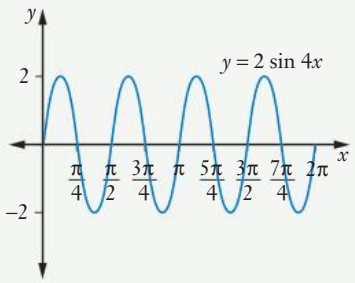
8 -2

9 **a** $\frac{3}{10}$ **b** $\frac{3}{5}$ **c** $\frac{9}{10}$

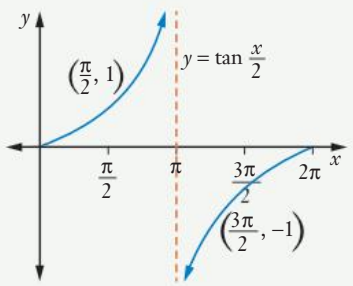
d $\frac{2}{5}$ **e** $\frac{7}{10}$

10 **a** $-\tan\theta$ **b** $-\sin\theta$ **c** $\cos\theta$

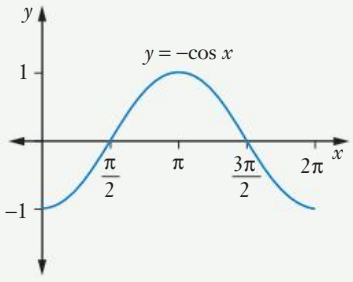
11 **a**



b



c



12 **a** $X = \{0, 1, 2, 3, \ldots\}$

b $X = \{0, 1, 2, \ldots, 12\}$

c $X = \{0, 1, 2, \ldots, 31\}$

13 $x = \frac{1}{2}$ **14** 3.2 years

15 **a**

x	0	1	2	3
$P(X = x)$	$\frac{343}{2197}$	$\frac{882}{2197}$	$\frac{756}{2197}$	$\frac{216}{2197}$

b

x	0	1	2	3
$P(X = x)$	$\frac{35}{286}$	$\frac{63}{143}$	$\frac{105}{286}$	$\frac{10}{143}$

16 $\sin x = -\frac{4}{5}, \cos x = \frac{3}{5}$

17 **a** $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ **b** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

c $x = \frac{\pi}{2}, \frac{3\pi}{2}$ **d** $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

18 $a = 0.19, b = 0.2$

19 $E(X) = 2.2$, $Var(X) = 2.16$, $\sigma = 1.47$

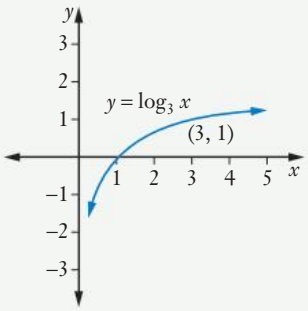
20 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $-\frac{1}{\sqrt{3}}$

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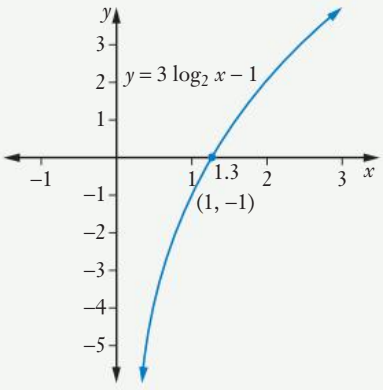
21

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

22 a



b



23 a $x = e^y$ **b** $x \approx 3.42$

24 $x \approx 0.28$

25 $k = \frac{1}{5}$

26 a No **b** Yes **c** Yes **d** No

27 a Yes **b** No

28 a $x = 135°, 315°$ **b** $x = 30°, 150°$

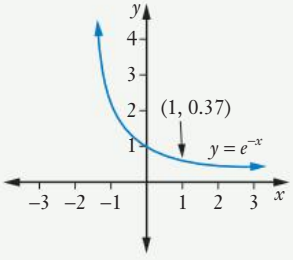
c $x = 45°, 135°, 225°, 315°$

d $x = 30°, 120°, 210°, 300°$

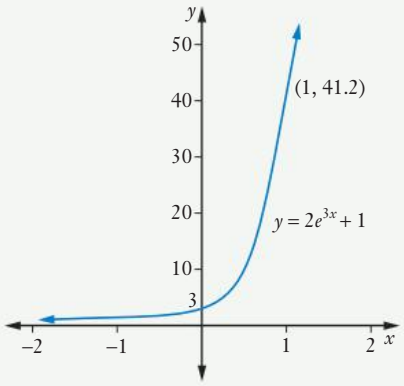
29 a 4 **b** 1 **c** $\frac{1}{2}$

d 2.04 **e** 3.76 **f** 2.18

30 a



b



31 a

x	0	1	2
$P(X = x)$	12%	23%	65%

b $E(X) = 1.53$, $Var(X) = 0.4891$

32 $y = 5e^2x - 5e^2$

33 a 50 cents **b** Lose 50 cents

34 a

x	1	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

b Yes **c** **i** $\frac{1}{2}$ **ii** $\frac{3}{8}$ **iii** $\frac{5}{8}$

d $4\frac{1}{2}$

35 a $0.27 + 0.31 + 0.28 + 0.14 = 1$

b $E(X) = 2.29$, $Var(X) = 1.0259$

36 $k = \frac{1}{16}$

37 a $5 \sec^2 x$ **b** $\cot x$

38 a $-\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\frac{\sqrt{3}}{2}$

39 a $x = -1, 3$ **b** $2 < x \leq 5$ **c** $x = -1\frac{2}{3}, 1$

40 Centre $(2, -3)$, radius 4

41 a 7.7 km **b** 218°

ISBN 9780170413152

INDEX

absolute value functions 243–51
absolute values 48, 52–3
  equations involving 56–7, 246–9
  properties 53
acceleration 270, 313, 314
acute angles 412
addition rule
  for mutually exclusive events 333
  of probability 337–9
adjacent side (triangle) 168
algebraic expressions
  expanding 16
  factorising 19, 20
  simplifying 14, 27
algebraic fractions, operations with 28
alternate angles 166
ambiguous case of the sine rule 166, 195–6
amplitude
  periodic functions 412, 435
  trigonometric functions 438–9
angle of depression 166, 183
angle of elevation 166, 182
angle of inclination of a line 84, 109–11
angles
  cosine rule for 199–201
  finding in a right-angled triangle 178
  measured in degrees, minutes and seconds 170
  and sides of a triangle 192
angles of any magnitude 412–13
  1st quadrant: acute angles (between 0° and 90°) 412
  2nd quadrant: obtuse angles (between 90° and 180°) 412–13
  3rd quadrant: angles between 180° and 270° 413
  4th quadrant: angles between 270° and 360° 413
  ASTC rule 414
  negative angle 416–17
arc, length of an 216–18
area of a sector 215–16
area of a triangle, sine formula for 201–3
ASTC rule 414–16
  negative angles 416–17, 425
  radians 425–6
asymptotes 236, 240–2, 399
average rate of change 270, 283, 284, 288, 311, 312
axis of symmetry 84
  of a parabola 126–28

bases 4, 386
  and number systems 391
bearings 166, 184
  compass 184
  true 185–6
binomial expressions 4, 17
binomial products 4, 17, 21
  special products 18
Body Mass Index 68
break-even points 84, 115, 154–5
Briggs, Henry 389

calculus 270
  and motion 296
  notation 288
Cardano, Girolamo 337
Cartesian plane 90
centre
  periodic functions 412, 436
  trigonometric functions 438
chain rule 270, 304–5
circles 248
  with centre (0, 0) 248
  with centre (a, b) 248–9
coefficients
  polynomials 84, 147
co-interior angles are supplementary 166
combined functions 259–61
common logarithms 387–8
compass bearings 164, 184
complement 324, 334
complementary angle results 421
complementary angles 421
complementary events 334
completing the square, to solve quadratic equations 62–3
composite function rule 304
composite functions 236, 261–2

ISBN 9780170413152

concavity
cubic functions, and point of inflection 140–1
parabola, and turning point 123
conditional probability 324, 351–4
and independent events 355–6
product rule 355–6
constant of variation 236
constant term (polynomial) 84, 147
continuous functions 236, 240
continuous random variables 454–5
corresponding angles 166
cosecant function 432
cosecant ratio 419
cosine function 431
cosine ratio 169
exact ratios 214–5
quadrants 190–1
cosine rule 199
applications 205–7
for angles 199–200
for right-angled triangles 199
cotangent function 434
cotangent ratio 419
cubic equations 144–5
cubic functions 84, 139–145
concavity 140–1
point of inflection 140–2
curve, gradient of a 274–8

decimal degrees, and degrees-minutes-seconds 170–2
decreasing graphs 98, 140
decreasing line 271
degree (polynomial) 84, 147
degree-minute-second (DMS) key (calculators) 170–2
degrees (angles) 170
converting to radians 212–3
decimal 170–2
exact trigonometric ratios 214, 215
and radians 211–12
dependent variable 84, 85
derivative
of a constant multiple of a function 293
of e^{ax} 383
of e^{x} 381
of $[f(x)]^{n}$ 305
of k 292
of ke^{x} 381–2
of kx 292
of kx^{n} 293
of a sum of functions 293
of x^{n} 292–4
derivative functions 270, 275–6, 279, 287
exponential functions 377
sketching 277–8
derivatives, and indices 297
Descartes, Rene 90
difference quotient 287–90
difference of two squares 18
factorising 25
differentiability 270, 279–81
at a point 280
differentiation 270, 279
chain rule 270, 304–5
exponential functions 381–2
product rule 270, 306–7
quotient rule 270, 309–10
short methods of 292–4
differentiation from first principles 270, 282, 287, 288–90
estimating the gradient of a tangent 284–5, 287
gradient of a secant 282–4, 284–6
Diophantus of Alexandria 13, 49
direct proportion 84, 103
direct variation 84, 103
discontinuous functions 236, 240–2, 279
discrete probability distributions 456–8
properties 458–60
discrete probability function 456
discrete random variables 454–5
discriminant 84, 129, 130–1
and the parabola 132–4
and roots of a quadratic equation 132–4
displacement 270, 313, 314
distributive law 16, 19
domain 84, 86, 97
hyperbolas 240–2
and increasing and decreasing graphs 98
linear functions 105–6

elimination method, simultaneous equations 69–70
equally likely outcomes 323, 324
equation of a circle
with centre (0, 0) 248
with centre (a, b) 249–50

ISBN 9780170413152

equations 48–9
cubic 144–5
exponential 48, 58–9, 403–4
involving absolute values 56–7, 246–7
linear 113–4
linear simultaneous 69–70
polynomial 148–9
quadratic 48, 61, 62–3, 64–5
substitution into formulas to solve 66
trigonometric 442–5
equations of a semicircle with centre (0, 0) 251–2
equilateral triangle 167
Euler, Leonhard 94, 377
Euler's number, e 372, 377–80
even functions 84, 99
event 324
exact trigonometric ratios 214
in radians 214
expanding expressions 16
expected value 454, 467–72
exponential equations 48, 58–9, 403–4
exponential functions 372–6
derivative functions 377
differentiation 381–3
and Euler's number, e 377
reflections 375
sketching 372–5
$y = a^x$ 373–4
$y = e^x$ 376
expressions 4
exterior angle in any triangle 167

factorisation 4, 19
by grouping in pairs 20
mixed 26
to solve quadratic equations 61
factorising
difference of two squares 25
perfect squares 24
trinomials 21–2, 23–4
factors 4
Fermat, Pierre de 337
1st quadrant: acute angles (between 0°and 90°) 192, 412
formulas
substitution into 30
substitution into to solve equations 66
4th quadrant – angles between 270° and 360° 413
fractional indices 9–12
'function of a function' rule 304
function notation 92–4
functions 84, 86–7
absolute value 243–7
combined 259–61
composite 261–2
cubic 84, 139–46
domain and range 97, 98
even and odd 84, 99–100
exponential 372–6
gradient (derivative) 275–7
horizontal line test 84, 88–90
hyperbolas 236, 239–40
increasing and decreasing graphs 98
intercepts 84, 96
linear 103–6
logarithmic 372, 397–400
many-to-one 85
one-to-one 84, 85, 86, 88–90
piecewise 84, 93, 243–4
polynomial 147–48
properties of 96–100
quadratic 84, 121–4, 125–7
reflection of 253–8
substituting pronumerals into 94
trigonometric 429–40
vertical line test 84, 87

Galileo 296, 389, 447
Gauss, Carl Friedrich 49
general cubic function 142–44
general trigonometric functions 439–40
geometry results 166–8
gradient 84
of a curve 271–8
of parallel lines 117–8
of perpendicular lines 119, 300
of a secant 270, 282–4, 285–6
of a straight line 107–10, 271–3
of a tangent 270, 274, 284–5, 287
gradient formula 108
gradient functions 275–6
sketching 276–7
gradient–intercept equation of a straight line 111
graphing
cubic functions 139–44
exponential functions 376
linear functions 104–5
logarithmic functions 400
polynomials functions 149–50
quadratic functions 121–4
see also sketching

ISBN 9780170413152

graphs
intersection of 152–6
$y = -f(-x)$ 256–7
$y = -f(x)$ 244–5, 253–4
$y = f(-x)$ 254–5
$y = f(x)$ 253–4
$y = kx^3$ 140

Hermite, Charles 377
horizontal line test 84, 88–90
horizontal lines 105–6
Huygens, Christiaan 296, 337
Hypatia 49
hyperbolas 236, 237, 239–40
asymptotes 240–2
as discontinuous functions 240–2
hypotenuse 168

identity 412, 422
increasing graphs 98, 140
increasing line 271
independent events 324, 342
and conditional probability 355–6
product rule 342–3
independent variable 84, 85
index laws 4–5
indices 4
and derivatives 296
fractional 9–12
zero and negative 7–8
inequalities 48
on a number line 51–2
quadratic 48
solving 51–2
instantaneous rate of change 270, 284, 287, 311, 313
intercepts of the graph of a function 84, 96
intersection of graphs
break-even points 154–5
intersecting lines 153
intersecting lines and parabolas 155–6
solving equations graphically 152–3
intersection (set notation) 326
interval notation 84, 97
inverse proportion 236
inverse variation 236–9
isosceles triangle 167

Kepler, Johannes 389

leading coefficient (polynomial) 84, 147
leading term (polynomial) 84, 147
Leaning Tower of Pisa 183
Leibniz, Gottfried 270, 288
length of an arc 216–8
limits 270, 286–7
linear equations, finding 113–4
linear functions 84, 103–6
applications 115
domain and range 105
gradient and y-intercept 111
graphing 104–5
horizontal and vertical lines 105–6
point of intersection 153
linear simultaneous equations 69–70, 153
locally straight curves 284–5
logarithm laws 392–4
logarithmic functions 372, 397–402
logarithmic scales 401
logarithms 372, 385–7
change of base 394–5
common 387–8
natural 387–8
origins 389
properties 387
longest side (triangle) 192
loss 155

many-to-many relation 85, 86
many-to-one function 86
many-to-one relation 85
mathematical verbs xiii
maximum turning point 123, 278
mean 467–72
minimum turning point 123, 278
minutes (angles) 170
monic polynomials 84, 147
motion
and calculus 296
Muhammad ibn Musa al-Khwarizmi 13
multi-stage experiments 340–4
conditional probability 343–4
independent events 342–3
mutually exclusive events 324, 332
addition rule for 333

Napier, John 389
natural (Naperian) logarithms 387–8
negative angles (ASTC rule) 113, 416
negative gradient 107, 271, 273
negative indices 7–8
Newton, Sir Isaac 270, 288, 296
non-linear simultaneous equations 70–1
non-mutually exclusive events 324, 337–9
normal 270, 300
normals to a curve 300–1
not differentiable functions 279–81
number plane, quadrants of 190–1

ISBN 9780170413152

obtuse angles 412
  trigonometric ratios 190, 191
odd functions 84, 99–100, 142
one-to-many relation 85
one-to-one function 84, 86, 88–90
one-to-one relation 85, 86
opposite side (triangle) 168, 192
ordered pairs 85
Oresme, Nicole 12
outcome 324

parabolas 84, 121, 122
  axis of symmetry 126–8
  concavity 123
  discriminant 132–3
  and intersecting lines 155–6
  turning points 123–4, 126–7
parallel lines, gradient 117–8
parallelogram 167
Pascal, Blaise 337
perfect squares 18
  factorising 24
period
  periodic functions 412, 435
  trigonometric functions 438–9
periodic functions 412, 435
perpendicular lines, gradients 119, 300
phase shifts, trigonometric functions 412, 438–9
piecewise functions 84, 93, 243–4
point–gradient equation of a straight line 113–4
point of inflection (cubic functions) 84, 140–2
polynomial expressions 147–8
polynomial functions 147–8
  graphing 149–50
polynomials 84
population 454
population mean 467
population standard deviation 475
positive angles (ASTC rule) 414, 425
positive gradient 107, 271, 274
power 4, 385
power functions 100
principal solution 442
probabilities, range of 333
probability
  addition rule 337–9
  conditional 323, 343–4, 351–5
  origins of 337
  product rule 340–2
  theoretical 332–4
probability distributions 454
  discrete 456–8
  uniform 461–3
probability formula 325–6
probability of an event 325
probability trees 324, 346–8, 354
product rule
  conditional probability 354–5
  differentiation 270, 306–9
  independent events 342–3
  probability 340–4
profit 155
pronumerals 4
proportionality constant 103
Ptolemy 162
Pythagorean identities 422

quadrants 190–1
  1st – acute angles (between 0°and 90°) 190, 412
  2nd – obtuse angles (between 90°and 180°) 191, 412–13
  3rd – angles between 180°and 270° 413
  4th – angles between 270° and 360° 413
  see also ASTC rule
quadratic equations 48, 61
  discriminant 129, 130–33
  finding 134–35
  solving by completing the square 62–3
  solving by factorisation 61
  solving by quadratic formula 64–5
quadratic formula 64–5
  discriminant 129–33
quadratic functions 84, 121–4
  and axis of symmetry 126–8
  concavity 123–4
  turning points 123–4, 126–7
quotient rule 270, 313–14

radians 168, 211
  ASTC rule 425–6
  converting to degrees 212–13
  and degrees 211–12
  exact trigonometric ratios 214–15
  special angles 213
  to find area of a sector 219–20
  to find length of an arc 216–18
  trigonometric equations 444–5
  trigonometric functions 435–6
random variables 455–6
range 84, 86, 97
  absolute value functions 245–6
  hyperbolas 240–2
  linear functions 103–4

ISBN 9780170413152

rate of change of y 271, 274
rates of change 311–13
displacement, velocity and acceleration 313–14
rationalising the binomial denominator (surds) 37–9
rationalising the denominator (surds) 36–7
reciprocal 235
reciprocal trigonometric ratios 414, 419–20
rectangle 167
reflections of exponential functions 375
reflections of functions 263–8
graph of $y = -f(-x)$ 256–7
graph of $y = -f(x)$ 263–4
graph of $y = f(-x)$ 254–6
relations 85
types of 85
relative frequency 324, 329, 457
rhombus 169
right-angled triangles
cosine rule 199
finding an angle 178
finding a side 174
sides of 164
root of a number 4
roots of the equation 84, 129

sample mean 466
sample space 314–5
sample standard deviation 475
secant 270, 283
secant function 434
secant ratio 419
2nd quadrant: obtuse angles (between 90°and 180°) 190, 412–13
seconds (angles) 170
sector, area of 219–20
semicircles 250–1
set 323
set notation 325
shortest side (triangle) 192
sides of a right-angled triangle 168
find a side 174
sides of a triangle 192
simple harmonic motion 447
simplifying
algebraic expressions 14, 27
surds 32
simultaneous equations 48
linear 69–70, 153
non-linear 70–1
solving using elimination method 69–70
solving using substitution method 69, 70–1
with three unknown variables 72–3
sine formula for the area of a triangle 202–3
sine function 430–1
sine ratio 169
exact ratios 214–5
quadrants 190–1
sine rule 166, 192–4
ambiguous case 166, 195–6
applications 205–7
sketching
exponential functions 371–4
gradient (derivative) functions 275–7
logarithmic functions 397–400
see also graphing
square (geometry) 167
standard deviation 455, 475–6
formula 476–8
stationary points
curves 270, 278
parabola 123
straight line
angle of inclination 84, 109
gradient 107–12, 271
gradient–intercept equation 111
point–gradient equation 113
substitution
into formulas 30
into formulas to solve equations 66
substitution method
non-linear simultaneous equations 70–1
simultaneous equations 69
sum of the interior angles in any triangle 167
surds 4
operations with 33–4
properties 32
rationalising the binomial denominator 37–9
rationalising the denominator 36–7
simplifying 32

tangent 270, 274
tangent function 431–2
tangent identity 422
tangent ratio 169
exact ratios 214–5
quadrants 190–1
tangents to a curve 274, 299–300
terms 4
theoretical probability 332–4
3rd quadrant: angles between 180 °and 270° 413
trapezium 168
tree diagrams 324, 341

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triangle
area of 202–3
cosine rule 198–200
naming the sides and angles 192
and sine rule 192–6
see also right-angled triangles
triangle measurement 169
trigonometric equations 442–4
involving radians 444–5
trigonometric functions 429–44
applications 446–7
centre of 438
general 438–9
period and amplitude 434–5
phase shifts 412, 437–8
properties 435–40
in terms of radians 435–6
trigonometric identities 419–23
trigonometric ratios 168, 169
exact 214–5
obtuse angles 190, 191
unit circle 190–1, 429
trigonometry
applications 182–6
origins 169
see also right-angled triangles
trinomials 4, 21
factorising 20–1, 23–4, 25
true bearings 166, 185–6
turning point 84, 270
parabola 123–4, 126–7

uniform probability distribution 454, 461–3
union (set notation) 322
unit circle
angles at any magnitude 412–16
negative angles 416–17
trigonometric ratios 190–2, 429

variables 85
variance 455, 475
formula 476–8
velocity 270, 313, 314
Venn diagrams 322, 325–6
vertex (parabola) 84, 123, 126
vertical line test 84, 87–8
vertical lines 105–6
vertically opposite angles 166

Wallis, John 12, 296
waves 447

x-intercept 96

$y = a^x$ 373–4
$y = e^x$ 378
$y = -f(-x)$, graph of 256–7
$y = -f(x)$, graph of 244–5
$y = f(-x)$, graph of 254–6
$y = f(x)$, graph of 253–7
$y = kx^3$, graph of 140
y-intercept 96

zero indices 7–8
zeros of the polynomial 84, 148

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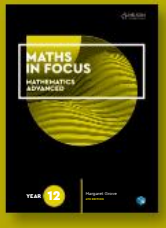
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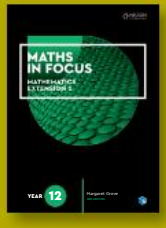
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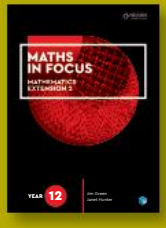
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