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MATHSQUEST 11
SPECIALIST MATHEMATICS

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MATHSQUEST 11
SPECIALIST MATHEMATICS

VCE UNITS 1 AND 2

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Contents

Introduction

At Jacaranda, we are deeply committed to the ideal that learning brings life-changing benefits to all students. By continuing to provide resources for Mathematics of exceptional and proven quality, we ensure that all VCE students have the best opportunity to excel and to realise their full potential.

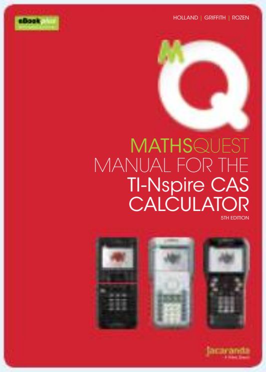
Maths Quest 11 Specialist Mathematics VCE Units 1 and 2 comprehensively covers the requirements of the revised Study Design 2016–2018.

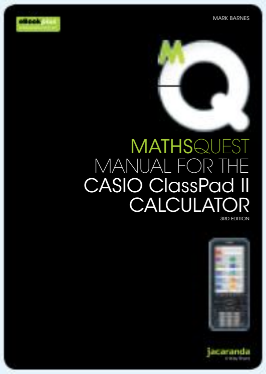
Features of the new *Maths Quest* series

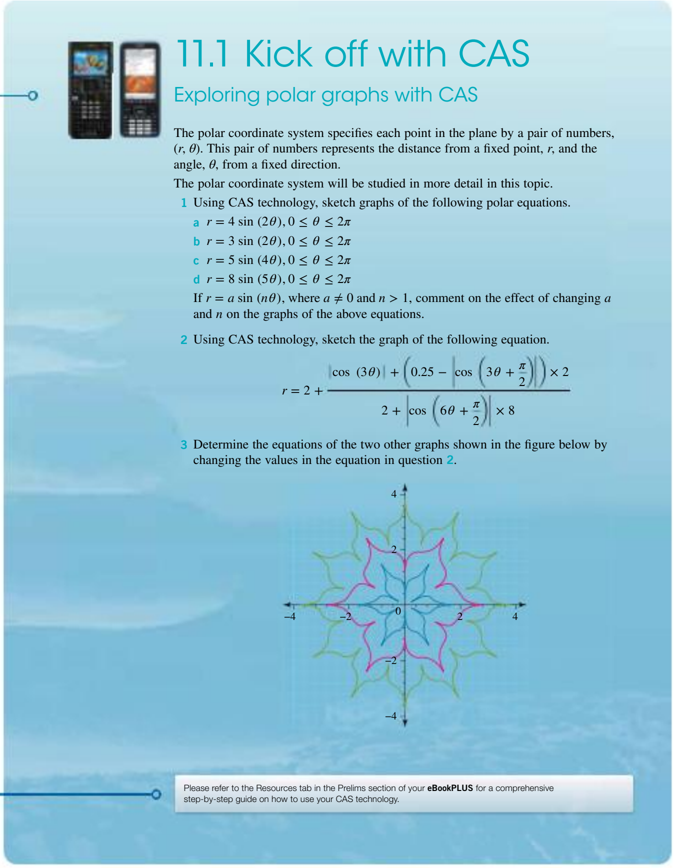
CAS technology

Each topic opens with an engaging **Kick off with CAS** activity designed to stimulate students' interest and curiosity and to highlight the important applications of CAS technology in developing deep understanding of the mathematical concepts presented.

For up-to-date, step-by-step instructions on how to use CAS technology, we have provided the *Manual for the TI-Nspire CAS calculator* and the *Manual for the Casio ClassPad II* in the Prelims section of the eBook.







11.1 Kick off with CAS

Exploring polar graphs with CAS

The polar coordinate system specifies each point in the plane by a pair of numbers, (r, θ). This pair of numbers represents the distance from a fixed point, r, and the angle, θ, from a fixed direction.

The polar coordinate system will be studied in more detail in this topic.

1 Using CAS technology, sketch graphs of the following polar equations.

a $r = 4\sin(2\theta), 0 \le \theta \le 2\pi$

b $r = 3\sin(2\theta), 0 \le \theta \le 2\pi$

c $r = 5\sin(4\theta), 0 \le \theta \le 2\pi$

d $r = 8\sin(5\theta), 0 \le \theta \le 2\pi$

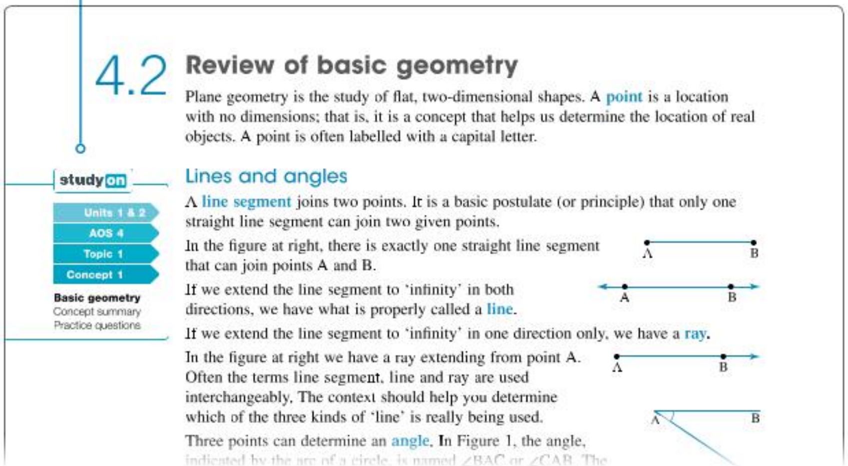
If $r = a\sin(n\theta)$, where $a \ne 0$ and $n > 1$, comment on the effect of changing a and n on the graphs of the above equations.

2 Using CAS technology, sketch the graph of the following equation.

$$r = 2 + \frac{|\cos(3\theta)| + \left(0.25 - \left|\cos\left(3\theta + \frac{\pi}{2}\right)\right|\right) \times 2}{2 + \left|\cos\left(6\theta + \frac{\pi}{2}\right)\right| \times 8}$$

3 Determine the equations of the two other graphs shown in the figure below by changing the values in the equation in question 2.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.



4.2 Review of basic geometry

Plane geometry is the study of flat, two-dimensional shapes. A point is a location with no dimensions; that is, it is a concept that helps us determine the location of real objects. A point is often labelled with a capital letter.

Lines and angles

A line segment joins two points. It is a basic postulate (or principle) that only one straight line segment can join two given points.

In the figure at right, there is exactly one straight line segment that can join points A and B.

If we extend the line segment to 'infinity' in both directions, we have what is properly called a line.

If we extend the line segment to 'infinity' in one direction only, we have a ray.

In the figure at right we have a ray extending from point A. Often the terms line segment, line and ray are used interchangeably. The context should help you determine which of the three kinds of 'line' is really being used.

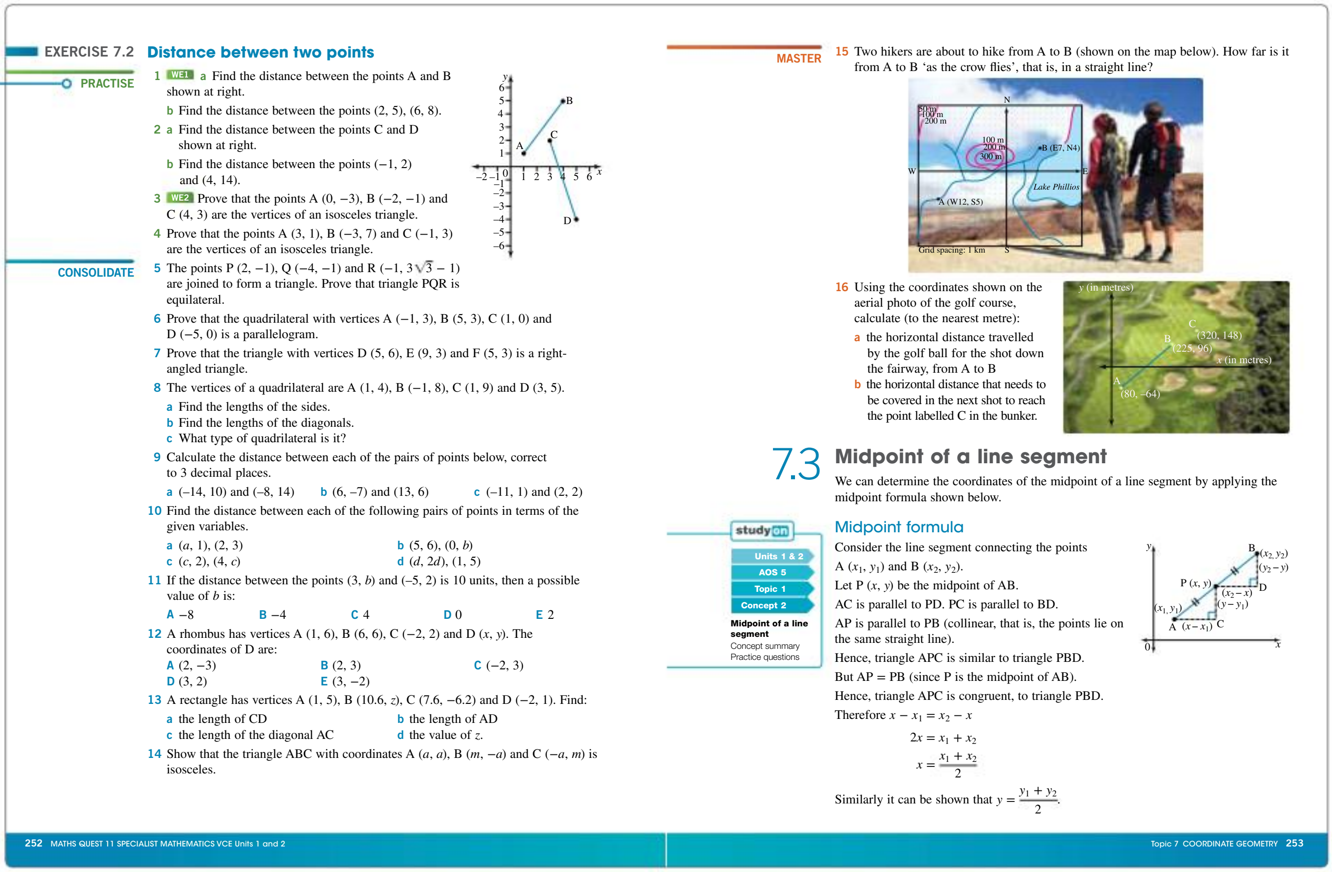
Three points can determine an angle. In Figure 1, the angle,

studyON links

Link to **studyON,** an interactive and highly visual study, revision and exam practice tool for instant feedback and on-demand progress reports.

Graded questions

A wide variety of questions at **Practise**, **Consolidate** and **Master** levels allows students to build, apply and extend their knowledge independently and progressively.



EXERCISE 7.2 Distance between two points

PRACTISE

1 WE1 a Find the distance between the points A and B shown at right.
b Find the distance between the points (2, 5), (6, 8).

2 a Find the distance between the points C and D shown at right.
b Find the distance between the points (−1, 2) and (4, 14).

3 WE2 Prove that the points A (0, −3), B (−2, −1) and C (4, 3) are the vertices of an isosceles triangle.

4 Prove that the points A (3, 1), B (−3, 7) and C (−1, 3) are the vertices of an isosceles triangle.

CONSOLIDATE

5 The points P (2, −1), Q (−4, −1) and R $(-1, 3\sqrt{3} - 1)$ are joined to form a triangle. Prove that triangle PQR is equilateral.

6 Prove that the quadrilateral with vertices A (−1, 3), B (5, 3), C (1, 0) and D (−5, 0) is a parallelogram.

7 Prove that the triangle with vertices D (5, 6), E (9, 3) and F (5, 3) is a right-angled triangle.

8 The vertices of a quadrilateral are A (1, 4), B (−1, 8), C (1, 9) and D (3, 5).
a Find the lengths of the sides.
b Find the lengths of the diagonals.
c What type of quadrilateral is it?

9 Calculate the distance between each of the pairs of points below, correct to 3 decimal places.
a (−14, 10) and (−8, 14) b (6, −7) and (13, 6) c (−11, 1) and (2, 2)

10 Find the distance between each of the following pairs of points in terms of the given variables.
a $(a, 1), (2, 3)$ b $(5, 6), (0, b)$
c $(c, 2), (4, c)$ d $(d, 2d), (1, 5)$

11 If the distance between the points $(3, b)$ and (−5, 2) is 10 units, then a possible value of b is:
A −8 B −4 C 4 D 0 E 2

12 A rhombus has vertices A (1, 6), B (6, 6), C (−2, 2) and D (x, y). The coordinates of D are:
A (2, −3) B (2, 3) C (−2, 3)
D (3, 2) E (3, −2)

13 A rectangle has vertices A (1, 5), B $(10.6, z)$, C (7.6, −6.2) and D (−2, 1). Find:
a the length of CD b the length of AD
c the length of the diagonal AC d the value of z.

14 Show that the triangle ABC with coordinates A (a, a), B $(m, -a)$ and C $(-a, m)$ is isosceles.

MASTER

15 Two hikers are about to hike from A to B (shown on the map below). How far is it from A to B 'as the crow flies', that is, in a straight line?

16 Using the coordinates shown on the aerial photo of the golf course, calculate (to the nearest metre):
a the horizontal distance travelled by the golf ball for the shot down the fairway, from A to B
b the horizontal distance that needs to be covered in the next shot to reach the point labelled C in the bunker.

7.3 Midpoint of a line segment

We can determine the coordinates of the midpoint of a line segment by applying the midpoint formula shown below.

studyon
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Topic 1
Concept 2
Midpoint of a line segment
Concept summary
Practice questions

Midpoint formula

Consider the line segment connecting the points A (x_1, y_1) and B (x_2, y_2).
Let P (x, y) be the midpoint of AB.
AC is parallel to PD. PC is parallel to BD.
AP is parallel to PB (collinear, that is, the points lie on the same straight line).
Hence, triangle APC is similar to triangle PBD.
But AP = PB (since P is the midpoint of AB).
Hence, triangle APC is congruent, to triangle PBD.
Therefore $x - x_1 = x_2 - x$

$$2x = x_1 + x_2$$
$$x = \frac{x_1 + x_2}{2}$$

Similarly it can be shown that $y = \frac{y_1 + y_2}{2}$.

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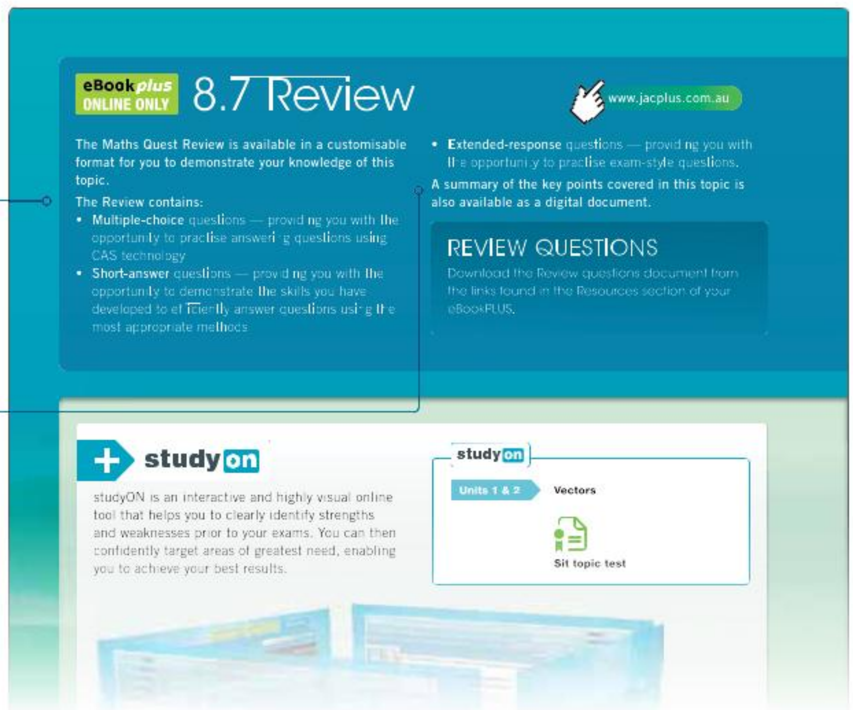
Review

Each topic concludes with a customisable **Review**, available in the Resources tab of the **eBookPLUS**, giving students the opportunity to revise key concepts covered throughout the topic. A variety of typical question types is available including short-answer, multiple-choice and extended response.

Summary

A comprehensive and fully customisable topic summary is available in the Resources tab of the **eBookPLUS**, enabling students to add study notes and key information relevant to their personal study needs.



eBookplus ONLINE ONLY **8.7 Review**

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS
Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Sit topic test

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The **eBookPLUS** is available for students and teachers and contains:

- the full text online in HTML format, including PDFs of all topics
- the *Manual for the TI-Nspire CAS calculator* for step-by-step instructions
- the *Manual for the Casio ClassPad II calculator* for step-by-step instructions
- topic reviews in a customisable format
- topic summaries in a customisable format
- links to **studyON**.

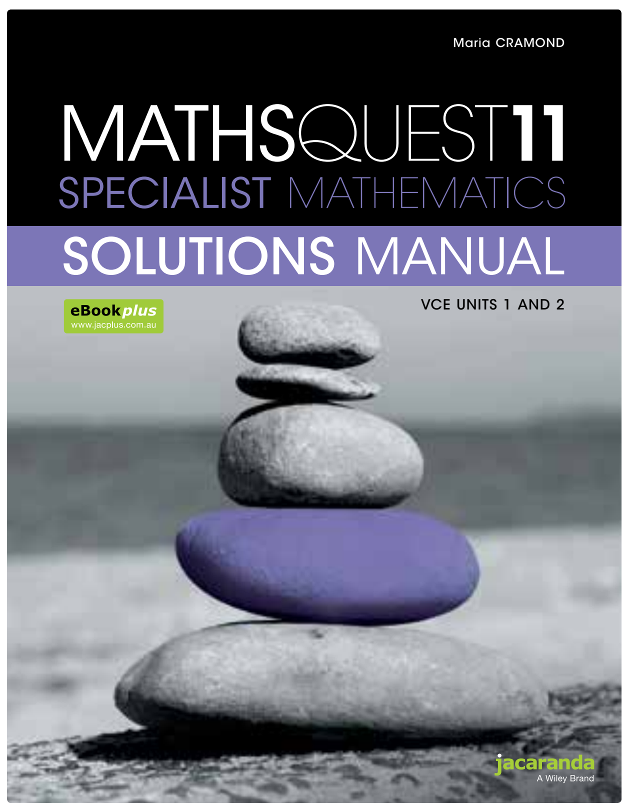
eGuide*plus*

The **eGuidePLUS** is available for teachers and contains:

- the full **eBookPLUS**
- a Work Program to assist with planning and preparation
- School-assessed Coursework — Application task and Modelling and Problem-solving tasks, including fully worked solutions
- two tests per topic with fully worked solutions.

Maths Quest 11 Specialist Mathematics Solutions Manual VCE Units 1 and 2

Available to students and teachers to purchase separately, the Solutions Manual provides fully worked solutions to every question in the corresponding student text. The Solutions Manual is designed to encourage student independence and to model best practice. Teachers will benefit by saving preparation and correction time.



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  Email: support@jacplus.com.au
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JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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1

Number systems: real and complex

1.1 Kick off with CAS

Exploring imaginary numbers

1 Enter $\sqrt{-1}$ into your CAS technology.

If the calculator output mode is set to real numbers, then the result for $\sqrt{-1}$ will be 'Error: Non-real result'.

If the calculator output mode is set to rectangular or polar, then the result for $\sqrt{-1}$ will be i.

When exploring numbers within the real number field only, $\sqrt{-1}$ is undefined. Within the field of complex numbers, $\sqrt{-1} = i$.

Therefore $i^2 = 1$.

2 With the calculator output mode set to rectangular or polar, simplify the following:

a i^3 b i^4 c i^5

d i^6 e i^7 f i^8.

3 Complete the following:

a $i^{2n} = \square$, for $n = 1, 2, 3, 4, \ldots$

b $i^{2n+1} = \square$, for $n = 1, 2, 3, 4, \ldots$

4 With the calculator output mode set to rectangular or polar, expand the following:

a $(2 - 3i)(3 - 4i)$

b $(5 - i)(2 + 2i)$

c $(3 + 4i)(5 - i)$

d $(2 - 3i)(2 + 3i)$

e $(i - 1)(1 - i)$

f $(x - yi)(x + yi)$.

5 With the calculator output mode set to rectangular or polar, factorise the following over the complex number field:

a $x^2 - 10z + 5$

b $x^2 + 4z + 6$

c $3x^2 - 7z + 7$

d $2x^2 + 5z + 4$

e $-3x^2 - 2z - 1$.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

1.2 Review of set notation

study on

Units 1 & 2
AOS 2
Topic 1
Concept 1

Set notation
Concept summary
Practice questions

Sets contain **elements**. In this topic the elements are numbers.

For example, the following are six elements: 1, 2, 3, 4, 5, 6.

ξ is the universal set — the set of all elements under consideration.

So, in this example, $\xi = \{1, 2, 3, 4, 5, 6\}$.

$\varnothing$ is the **empty** or **null set**. This set contains no elements. $\varnothing = \{\}$.

An upper case letter, such as A, represents a subset of ξ.

In our example, $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4\}$.

$\in$ is read as 'is an element of'. For example, $3 \in A$.

$\notin$ is read as 'is not an element of'. For example, $2 \notin A$.

$\subset$ is read as 'is a subset of'. For example, $\{1, 3\} \subset A$.

$\supset$ is read as 'is a superset of'. For example, $A \supset \{1, 3\}$.

Related symbols, such as $\supseteq$, $\subseteq$ and $\not\subset$, are also used.

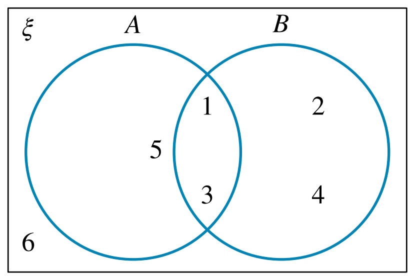
A' is the complement of A. This set contains all the elements not in A that are in ξ. For example, given $\xi = \{1, 2, 3, 4, 5, 6\}$, if $A = \{1, 3, 5\}$, then $A' = \{2, 4, 6\}$.

$A \cup B$ is the union of A and B. This set contains all the elements in sets A and B. For the example above, $A \cup B = \{1, 2, 3, 4, 5\}$.

$A \cap B$ is the intersection of A and B. This set contains all the elements in both A and B. For the example above, $A \cap B = \{1, 3\}$.

$C \backslash D$ is read as 'C slash D'. This set contains all the elements in C that are not in D. If $C = \{1, 2, 5, 6\}$ and $D = \{2, 5\}$, then $C \backslash D = \{1, 6\}$. This notation is particularly useful in modifying a given set to exclude a small number of elements.

A **Venn diagram** may be used to illustrate set notation.



WORKED EXAMPLE 1

$\xi = \{2, 4, 6, 8, 10, 12\}$, $C = \{4, 8, 12\}$ and $D = \{2, 6, 10, 12\}$.

a Illustrate these sets on a Venn diagram.

Then state:

b C'

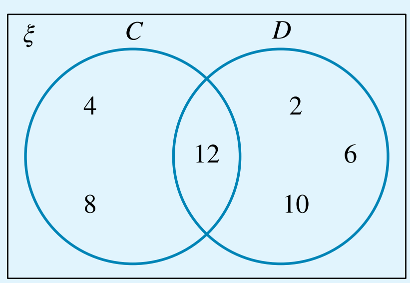
c $C \cup D$

d $C \cap D$

e $(C \cup D)'$

f $C' \cap D'$

g $C' \backslash \{2\}$.

THINK	WRITE/DRAW
a Draw a Venn diagram and enter the elements in the appropriate region.	**a** 

b The set C' is the complement of C and contains all elements not in the set C. — **b** $C' = \{2, 6, 10\}$

c The set $C \cup D$ is the union of C and D and contains all elements in sets C and D. — **c** $C \cup D = \{2, 4, 6, 8, 10, 12\}$

d The set $C \cap D$ is the intersection of C and D and contains elements common to sets C and D. — **d** $C \cap D = \{12\}$

e The set $(C \cup D)'$ is the complement of the union of sets C and D. It contains elements not in the union of sets C and D. In this case, there are no elements not in the union of sets C and D. — **e** $(C \cup D)' = \varnothing$

f The set $C' \cap D'$ is the intersection of C' and D'. It contains elements common to the sets C' and D'. There are no common elements to C' and D'. — **f** $C' \cap D' = \varnothing$

g The set $C' \backslash \{2\}$ is the set C' without the element 2. It contains all the elements of the set C' but not 2. — **g** $C' \backslash \{2\} = \{6, 10\}$

Natural numbers

study on
Units 1 & 2
AOS 2
Topic 1
Concept 2
Sets of numbers
Concept summary
Practice questions

Numbers were invented to quantify objects in the environment. Hunter–gatherers used counting or **natural numbers** to communicate how many of a particular animal were seen on a hunting trip. The set of natural numbers is given as $N = \{1, 2, 3, \ldots\}$. Natural numbers are positive whole numbers.

Proof of divisibility by mathematical induction

Worked example 2 demonstrates the proof of divisibility by mathematical induction.

WORKED EXAMPLE 2

Prove that $6^n + 4$ is divisible by 5 for all $n \in N$.

THINK	WRITE
1 Let $n = 1$.	$6^1 + 4 = 10$, which is divisible by 5
2 Assume that it is true for $n = k$.	Assume that when $n = k$, $6^k + 4$ is divisible by 5
3 Consider $n = k + 1$.	When $n = k + 1$, $6^{k+1} + 4 = (6^k \times 6^1) + 4$
4 Need to add and subtract a number that is a multiple of 6.	$(6^k \times 6^1) + 4 = (6^k \times 6^1) + 24 - 24 + 4$
5 Factorise the expression so that $6^k + 4$ is inside the bracket.	$= 6^1(6^k + 4) - 24 + 4$ $= 6(6^k + 4) - 20$

6 From step 2, $6^k + 4$ is divisible by 5.	Since both $6^k + 4$ and 20 are divisible by 5, $6^{k+1} + 4$ is divisible by 5.
7 State your conclusion.	Since $6^n + 4$ is divisible by 5 for $n = 1$ and it is assumed to be true for $n = k$, which implies that it is true for $n = k + 1$, the statement is true for all whole numbers, n.

Prime and composite numbers

A **prime number** is a positive natural number that has exactly two factors: itself and one.

Composite numbers are positive natural numbers that have more than two factors, including at least one prime factor.

Is there an infinite number of prime numbers?

Let us assume that there is a finite number of prime numbers, say $p_1, p_2, p_3, \ldots, p_n$.

Let $M = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$.

M could either be a prime number or a composite number. If M is a prime number, then p_n cannot be the last prime number. If M is a composite number, then none of the primes $p_1, p_2, p_3, \ldots, p_n$ can divide into it, since there will always be a remainder of 1. Hence a prime factor exists that is not in $p_1, p_2, p_3, \ldots, p_n$. Therefore, we can conclude that there are infinitely many prime numbers.

Integers and rational numbers

The systematic consideration of the concept of number in algebra, and the numbers required to solve equations of the form $x + 2 = 0$ and $3x + 1 = 0$, resulted in the invention of **integers** and **rational numbers**.

The set of integers is given by $Z = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$. They are positive and negative whole numbers, including zero.

Z^- is the set of negative integers: $Z^- = \{\ldots, -3, -2, -1\}$.

Z^+ is the set of positive integers: $Z^+ = \{1, 2, 3, \ldots\}$.

Therefore, $Z = Z^- \cup \{0\} \cup Z^+$.

The set of rational numbers is given by Q. These are numbers of the form $\frac{p}{q}$, where $p \in Z$ and $q \in Z \setminus \{0\}$. Whole numbers are also rational numbers.

Consistent with the definition of Q, $Z \subset Q$.

Q^- is the set of negative rational numbers.

Q^+ is the set of positive rational numbers.

Therefore, $Q = Q^- \cup \{0\} \cup Q^+$.

Rational numbers in their simplest form with denominators such as 2, 8, 16, 64 produce terminating decimals. Some examples include:

$$\frac{1}{2} = 0.5, \frac{3}{8} = 0.375, \frac{7}{16} = 0.4375, \frac{89}{125} = 0.712, \frac{123}{64} = 1.921\,875$$

Rational numbers in their simplest form with denominators such as 3, 6, 7, 9, 11, 13, 14, 15, 17 produce non-terminating recurring or repeating decimals. Some examples include:

$$\frac{1}{3} = 0.333\ldots = 0.\dot{3}, \frac{1}{6} = 0.1666\ldots = 0.1\dot{6}, \frac{5}{12} = 0.416\,66\ldots = 0.41\dot{6}$$

$$\frac{17}{99} = 0.171\,717\ldots = 0.\dot{1}\dot{7}, \frac{3}{7} = 0.428\,571\,428\,571\ldots = 0.\dot{4}28\,57\dot{1}, \frac{17}{13} = 1.\dot{3}07\,69\dot{2}$$

WORKED EXAMPLE 3

Using a calculator, express the following rational numbers in decimal form.

a $\frac{5}{16}$

b $\frac{4}{7}$

THINK	WRITE
a Since the denominator is 16, expect a terminating decimal.	**a** $\frac{5}{16} = 0.3125$
b **1** Since the denominator is 7, expect a non-terminating, repeating decimal.	**b** $\frac{4}{7} = 0.571\,428\,571\,4\ldots$
2 Indicate the repeating sequence using dot notation.	$\frac{4}{7} = 0.\dot{5}71\,42\dot{8}$

Irrational numbers

Irrational numbers are given by I. They are numbers that can be placed on a number line and may be expressed as non-terminating, non-recurring decimals. For example:

$$-\sqrt{2}, -\frac{\sqrt{3}}{2}, -\sqrt{5} + 1, 4^{\frac{1}{3}}, 5^{\frac{3}{4}}, \pi.$$

Irrational numbers cannot be written in the form $\frac{p}{q}$, where $p \in Z$ and $q \in Z \setminus \{0\}$.

Many irrational numbers in decimal form, such as $\sqrt{2}$ and π, have digits that have no pattern. For these numbers, it is impossible to predict the next digit from the preceding digits. However, other irrational numbers can be constructed with a pattern; for example:

$$0.101\,100\,111\,000\,111\,100\,00\ldots \text{ and } 0.010\,110\,111\,011\,11\ldots$$

There are two important subsets of the set of irrational numbers: the set of algebraic numbers and the set of transcendental numbers.

Algebraic numbers are those that are the solution of an algebraic polynomial equation of the form:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in Z$. For example, algebraic numbers include $3^{\frac{1}{3}}$ from one of the solutions of $x^3 - 3 = 0$ and $2^{\frac{3}{4}}$ from $x^4 - 8 = 0$.

Transcendental numbers occur in the evaluation of some functions, such as trigonometric functions. For example, sin (32.1°) and π are transcendental numbers. The functions that produce these numbers are often called transcendental functions.

Why is $\sqrt{7}$ an irrational number?

Assume that $\sqrt{7}$ is a rational number and can be written in the form $\frac{p}{q}$, a fraction in simplest form, where $p \in Z$ and $q \in Z \setminus \{0\}$.

$$\sqrt{7} = \frac{p}{q}$$

$$7 = \frac{p^2}{q^2}$$

$$p^2 = 7q^2$$

Therefore, p^2 is divisible by 7, which means that p is divisible by 7.

$$\text{Let } p = 7k,\ k \in Z$$

$$(7k)^2 = 7q^2$$

$$49k^2 = 7q^2$$

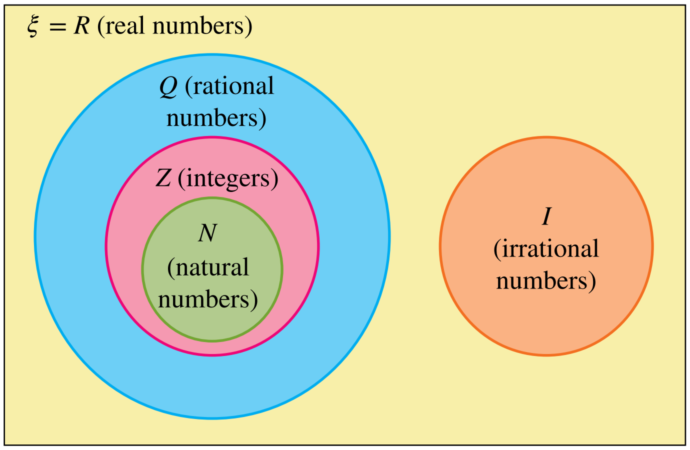
$$q^2 = 7k^2$$

Therefore, q^2 is divisible by 7, which means that q is divisible by 7.

As p and q have a common factor of 7, this contradicts the fact that $\frac{p}{q}$ is a fraction written in simplest form. The assumption that $\sqrt{7}$ is rational is incorrect, hence $\sqrt{7}$ is an irrational number.

Real numbers

Finally, the set of **real numbers** is given as R. R includes all numbers that can be put on a number line, where $R = Q \cup I$. The Venn diagram shows the relationships between R, Q, I, Z and N.



Absolute value of a real number

The absolute value $|a|$ of a real number a is the distance from a to zero.

$$|-8| = 8$$

$$|11| = 11$$

$$|-\sqrt{2}| = \sqrt{2}$$

WORKED EXAMPLE 4

For each of the numbers below, using R, Q, I, Z and N, state all the sets for which they are a member.

a -5

b $-\frac{17}{3}$

c $3\sqrt{2}$

d $27.\dot{1}7\dot{9}$

e 4.153

f $17.1354...$

g $1.011\,011\,101\,111...$

h $32^{\frac{1}{5}}$

i $17^{\frac{1}{4}}$

THINK	WRITE
a -5 is an integer.	a -5 is a negative integer (Z^-). It is also a rational number (Q) and a real number (R).
b $-\frac{17}{3}$ is a rational number, as it can be written as a fraction.	b $-\frac{17}{3}$ is a rational number (Q) and a real number (R).
c $3\sqrt{2}$ is an irrational number.	c $3\sqrt{2}$ is an irrational number (I) and a real number (R).
d $27.\dot{1}7\dot{9}$ is a rational number, as it is a recurring decimal.	d $27.\dot{1}7\dot{9}$ is a rational number (Q) and a real number (R).
e 4.153 is a rational number, as it is a terminating decimal.	e 4.153 is a rational number (Q) and a real number (R).
f 17.1354... is an irrational number as there is no indication that there is a recurring pattern.	f 17.1354... is an irrational number (I) and a real number (R).
g 1.011 011 101 111... is an irrational number.	g 1.011 011 101 111... is an irrational number (I) and a real number (R).
h $32^{\frac{1}{5}}$ can be simplified to 2 and is therefore a natural number.	h $32^{\frac{1}{5}}$ is a natural number (N). It is also an integer (Z), a rational number (Q) and a real number (R).
i $17^{\frac{1}{4}}$ is an irrational number.	i $17^{\frac{1}{4}}$ is an irrational number (I) and a real number (R).

WORKED EXAMPLE 5

Express each of the following in the form $\frac{a}{b}$, where $a \in Z$ and $b \in Z \setminus\{0\}$.

a $0.\dot{6}$

b $0.\dot{2}\dot{3}$

THINK	WRITE
a 1 Write $0.\dot{6}$ in expanded form.	a $0.\dot{6} = 0.666\,666...$ [1]
2 Multiply [1] by 10.	$10 \times 0.\dot{6} = 6.666\,66...$ [2]

3 Subtract [1] from [2].

$9 \times 0.\dot{6} = 6$

$0.\dot{6} = \frac{6}{9}$

4 State the simplest answer.

$= \frac{2}{3}$

b 1 Write $0.\dot{2}\dot{3}$ in the expanded form.

b $0.\dot{2}\dot{3} = 0.232\,323\ldots$ [1]

2 Multiply [1] by 100.

$100 \times 0.\dot{2}\dot{3} = 23.232\,323\ldots$ [2]

3 Subtract [1] from [2].

$99 \times 0.\dot{2}\dot{3} = 23$

4 State the simplest answer.

$0.\dot{2}\dot{3} = \frac{23}{99}$

The basic properties of number are assumed to be true if a counterexample cannot be found. For example, the statement 'the product of two integers is an integer' is accepted as true because a counterexample has not been found, but the statement 'the quotient of two integers is an integer' is false because a counterexample $\left(\frac{2}{3}\right)$ is not an integer.

WORKED EXAMPLE 6

Determine counterexamples for the following.

a The product of two irrational numbers is irrational.

b The sum of two irrational numbers is irrational.

THINK	WRITE
a Take a simple irrational number such as $\sqrt{2}$. Multiply by an irrational number, say $\sqrt{2}$. State your answer.	a Because $\sqrt{2} \times \sqrt{2} = 2$, which is a rational number, the statement 'the product of two irrational numbers is irrational' is shown to be false.
b 1 Take two irrational numbers such as $0.1011\,001\,110\,00\ldots$ and $0.010\,011\,000\,111\ldots$. Add these numbers.	b $0.101\,100\,111\,000\ldots + 0.010\,011\,000\,111\ldots$ $= 0.111\,111\,111\,111\ldots$ The digits form a pattern so the sum is a rational number.
2 State your answer.	Because $0.111\,111\,111\,111\ldots$ is a rational number, the statement 'the sum of two irrational numbers is irrational' has been shown to be false.

Standard form or scientific notation

Very large or very small numbers are conveniently expressed in standard form, $a \times 10^b$, where $a \in R$, $1 \leq a < 10$ and $b \in Z$. For example, $1\,234\,111 = 1.234\,111 \times 10^6$ and $0.000\,000\,000\,045 = 4.5 \times 10^{-11}$.

Decimal places and significant figures

The numerical answer to a calculation may be required to be given correct to a set number of **decimal places**, and this is done through a process of rounding. To determine the number of decimal places contained in a number, count the number of digits after the decimal point. For example, 0.35 has 2 decimal places. For numbers expressed to a given number of decimal places, remember to round up if the next digit is 5 or more. For example, rounded to 2 decimal places, 2.234 becomes 2.23 and 2.236 becomes 2.24.

To determine the number of **significant figures** contained in a number, count the number of digits from the first non-zero digit. For example, 0.035 contains 2 significant figures. Any zeros at the end of a number *after* a decimal point are considered to be significant. For example, 1.40 has 3 significant figures. The trailing zeros at the end of a number are not considered to be significant. For example, 24 000 has 2 significant figures.

For numbers expressed to a given number of significant figures, remember to round. For example, rounded to 2 significant figures, 2.234 becomes 2.2 and 2.236 also becomes 2.2.

Some examples are shown in the following table.

Number	2 significant figures	3 significant figures	2 decimal places	3 decimal places
471 860.237 8	470 000	472 000	471 860.24	471 860.238
1.238 9	1.2	1.24	1.24	1.239
1.006 8	1.0	1.01	1.01	1.007
0.016 78	0.017	0.016 8	0.02	0.017
0.001 556	0.0016	0.001 56	0.00	0.002
0.199 1	0.20	0.199	0.20	0.199

WORKED EXAMPLE 7

Calculate the following products and quotients without using a calculator, expressing your answer in scientific notation correct to 1 significant figure.

a $8 \times 10^{24} \times 3 \times 10^{-10}$ **b** $\dfrac{7 \times 10^{17}}{8 \times 10^{-10}}$

THINK	WRITE
a 1 Multiply the terms by using the properties of indices: $a^n \times a^m = a^{n+m}$.	**a** $8 \times 10^{24} \times 3 \times 10^{-10} = 24 \times 10^{14}$
2 Write the answer in standard form, correct to 1 significant figure.	$24 \times 10^{14} = 2.4 \times 10 \times 10^{14}$ $= 2.4 \times 10^{15}$ $= 2 \times 10^{15}$
b 1 Multiply the terms by using the properties of indices: $a^n \div a^m = a^{n-m}$.	**b** $\dfrac{7 \times 10^{17}}{8 \times 10^{-10}} = 0.875 \times 10^{27}$
2 Write the answer in standard form, correct to 1 significant figure.	$0.875 \times 10^{27} = 0.9 \times 10^{27}$ or $= 9 \times 10^{26}$

Subsets of the set of real numbers

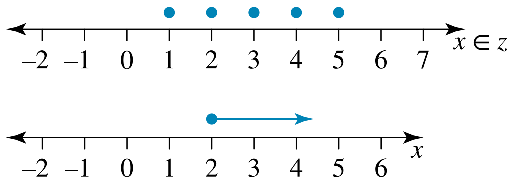
Notation

There are different forms of notation for representing subsets.

1. **Set notation**

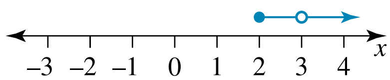
   For example $\{x: x \in Z, 1 \le x \le 5\}$, which is read as 'the set of numbers x such that x is an element of the set of integers and x is greater than or equal to 1 and less than or equal to 5'.

   If $x \in R$, it is not necessary to include the nature of x. For example, $\{x: x \ge 2\}$ represents the set of real numbers greater than or equal to 2. Each of the two sets above may be represented on a number line as follows.



   If $x \in Q$, the graph on the number line appears to look like the corresponding graph for $x \in R$ because the number line appears to be continuous (although all irrational numbers are missing). For example, $\{x: x \in Q, x \ge 2\}$ would appear to be identical to the graph of $\{x: x \ge 2\}$ shown above.

   If individual numbers are excluded from a given set, indicate this on a number line by an open circle. If individual numbers are included in a given set, indicate this on the number line by a closed circle. For example, $\{x: x \ge 2\}\backslash\{3\}$ is represented on a number line below.



   A given set can be stated in more than one way using set notation.

   For example, $\{1, 2, 3, 4, 5\}$ can be written as $\{x: x \in Z, 0 < x < 6\}$, $\{x: 1 \le x \le 5\}$ or $\{x: x \in Z^+, x \le 5\}$.

2. **Interval notation**

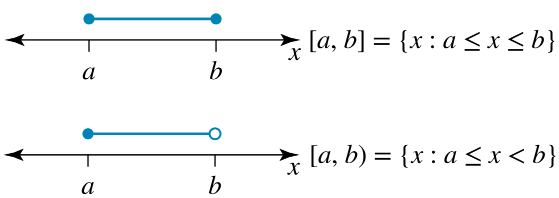
   Interval notation uses brackets, either square brackets $[-1, 6]$ or curved brackets $(1, 6)$ to describe a range of numbers between two numbers. Square brackets include the numbers at the end of the interval; curved brackets exclude them. For example:

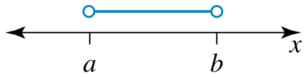
   $(3, 8)$ represents all real numbers between 3 and 8, excluding both 3 and 8.

   $[-2, 6]$ represents all real numbers between -2 and 6, including both -2 and 6.

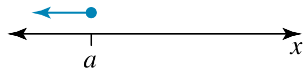
   $[-4, 10)$ represents all real numbers between -4 and 10, including -4 and excluding 10.

   Example sets are illustrated on the following number lines.

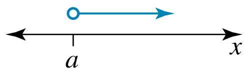




$(a, b) = \{x : a < x < b\}$



$(-\infty, a] = \{x : x \le a\}$



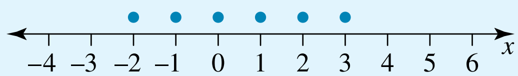
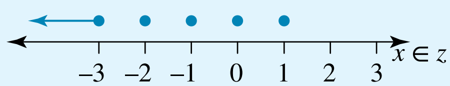
$(a, \infty) = \{x : x > a\}$

Note: Curved brackets are always used for $\pm\infty$.

WORKED EXAMPLE 8

List the following sets and then express each set using set notation. Illustrate each set on a number line.

a {Integers between −3 and 4} **b** {Integers less than 2}

THINK	WRITE/DRAW
a 1 This set involves integers. List the set of integers. Express the set using set notation.	**a** $\{-2, -1, 0, 1, 2, 3\} = \{x: x \in Z, -3 < x < 4\}$
2 Draw a number line showing arrowheads on each end. Ensure that the numbers from −3 to 4 are shown using an appropriate scale. Since the set of integers is to be represented, do not join the dots.	
b 1 This set involves integers. List the set of integers. Express the set using set notation.	**b** $\{\ldots, -2, -1, 0, 1\} = \{x: x \in Z, x < 2\}$
2 Draw a number line showing arrowheads on each end. Ensure that the numbers from 2 and below are shown, using an appropriate scale. Since the set of integers is to be represented, do not join the dots, but show an arrow on the left side of −3.	

WORKED EXAMPLE 9

Use set notation to represent the following sets.

a {Rational numbers greater than 27}

b {Integers between and including both 100 and 300, except for 200}

c {Positive integers less than 9 and greater than 50}

d {Real numbers that are less than 7 and greater than 2}

e {Positive real numbers that are less than 2 or greater than 7}

THINK | **WRITE**

a The numbers in this set belong to the set of rational numbers, Q.
a $\{x: x \in Q, x > 27\}$

b The numbers in this set belong to the set of integers, Z. Exclude 200.
b $\{x: x \in Z, 100 \le x \le 300\}\backslash\{200\}$

c The numbers in this set belong to the set of positive integers, Z^+. Express the set of positive integers less than 9 and greater than 50 as the union of two sets.
c $\{x: x \in Z^+, x < 9\} \cup \{x: x \in Z^+, x > 50\}$

d The numbers in this set belong to the set of real numbers, R, that are less than 7 and greater than 2.
d $\{x: x < 7\} \cap \{x: x > 2\}$ or, more simply, $\{x: 2 < x < 7\}$

e The numbers in this set belong to the set of positive real numbers, R^+, that are less than 2 or greater than 7.
e $\{x: 0 < x < 2\} \cup \{x: x > 7\}$ or $R^+\backslash\{x: 2 \le x \le 7\}$

WORKED EXAMPLE 10

Use interval notation to represent the following sets.

a $\{x: -2 < x \le 3\}$

b $\{x: x \le 4\}$

c $\{x: 3 < x \le 5\} \cup \{x: 4 \le x < 7\}$

d $\{x: 3 < x \le 5\} \cap \{x: 4 \le x < 7\}$

THINK | **WRITE**

a $x \in R$. Only the end point 3 is included; therefore, use a square bracket.
a $(-2, 3]$

b $x \in R$. Negative infinity is always preceded by a round bracket when using interval notation.
b $(-\infty, 4]$

c $x \in R$. Only the inner end points are included.
c $(3, 5] \cup [4, 7) = (3, 7)$

d $x \in R$. Only the inner end points are included.
d $(3, 5] \cap [4, 7) = [4, 5]$

EXERCISE 1.2 Review of set notation

PRACTISE

1 WE1 If $\xi = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2\}$ and $B = \{2, 3\}$, show these on a Venn diagram, and then state the following sets.

a A' b $A \cup B$ c $A \cap B$ d $A\backslash\{2\}$

2 If $\xi = \{4, 8, 12, 16, 20, 24, 28, 32, 36\}$, $A = \{4, 8, 20\}$ and $B = \{20, 24, 28, 32, 36\}$, show these on a Venn diagram, and then state the following sets.

a B' b $A \cup B'$ c $A' \cup B'$ d $(A \cap B)'$

3 WE2 Prove that $4^n - 1$ is divisible by 3 for all $n \in N$.

4 Prove that $n^3 + 2n$ is divisible by 3 for all $n \in N$.

5 WE3 Use a calculator to express the following rational numbers in decimal form.

a $\frac{213}{64}$ b $\frac{15}{44}$

6 Use a calculator to express the following rational numbers in decimal form.

a $\frac{17}{12}$ b $\frac{16}{13}$

7 WE4 For each number, using Z, N, Q, I and R, state all the sets of which it is a member.

$-2, \frac{16}{8}, \frac{21}{16}, -3\frac{2}{7}, 6\sqrt{3}, 16^{\frac{1}{4}}$

8 Specify to which sets (Z, N, Q, I and R) each of the following numbers belong.

$5^{\frac{1}{5}}, \pi, 21.72, 2.\dot{5}6\dot{7}, 4.135\,218\,974\ldots, \ 4.232\,332\,333.$

9 WE5 Express each of the following in the form $\frac{a}{b}$, where $a \in Z$ and $b \in Z\backslash\{0\}$.

a $0.4\dot{1}$ b $2.1\dot{2}3\dot{4}$

10 Express the following in their simplest rational form.

a $0.\dot{2}\dot{4}$ b $1.\dot{1}2\dot{3}$ c $0.1\dot{2}\dot{3}$ d $2.1\dot{1}2\dot{3}$

11 WE6 Find a counterexample, if possible, for the following statements. If a counterexample is found, the statement is false (F). If a counterexample is not found, accept the statement to be true (T).

a The product of two integers is an integer.
b The division of an integer by an integer is a rational number.

12 Find a counterexample, if possible, for the following statements. If a counterexample is found, the statement is false (F). If a counterexample is not found, accept the statement to be true (T).

a The difference of two irrational numbers is irrational.
b The sum of an irrational number and a rational number is irrational.

13 WE7 Calculate the following products and quotients without using a calculator, expressing your answer in scientific notation to 1 significant figure.

a $1.5 \times 10^{16} \times 4 \times 10^{12}$ b $1.2 \times 10^{24} \times 3 \times 10^{-10}$
c $3.2 \times 10^{25} \times 2 \times 10^{15}$

14 Calculate the following products and quotients without using a calculator, expressing your answer in scientific notation to 1 significant figure.

a $7 \times 10^{14} \times 9 \times 10^{-8}$ b $\frac{8 \times 10^{17}}{4 \times 10^{-10}}$ c $\frac{2.5 \times 10^{12} \times 8 \times 10^{-7}}{5 \times 10^{8}}$

15 WE8 List the following sets and then express each set using set notation. Then illustrate each set on a number line.

a {Integers between −6 and 1} b {Integers from −3 to 4}
c {Integers greater than −6 and less than or equal to 4}
d {Positive integers less than 5}

16 List the following sets and then express each set using set notation. Then illustrate each set on a number line.

a {Integers less than 5} b {Integers greater than 2}
c {Negative integers greater than −5}

17 WE9 Use set notation to represent the following sets.

a {Rational numbers greater than 5}
b {Rational numbers greater than 5 and less than or equal to 20}
c {Positive rational numbers less than 20}
d {Integers between 5 and 20, except for 8 and 9}
e {Positive integers less than 100, except for integers between 40 and 50}

18 Use set notation to represent the following sets.

- **a** {Real numbers from 2 to 5, including 2}
- **b** {Real numbers that are less than 5 and greater than 3}
- **c** {Real numbers that are less than 3 and greater than 7}
- **d** {Positive real numbers that are less than 3 and greater than 7}

19 **WE10** Use interval notation to represent the following sets, then illustrate the sets on a number line.

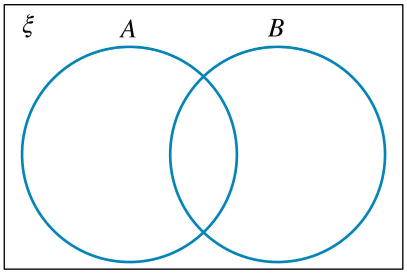
- **a** $\{x: -3 \le x \le 1\}$
- **b** $\{x: x < 2\}$
- **c** $\{x: -2 < x < 1\}$
- **d** $\{x: x \ge 2\}$

20 Use interval notation to represent the following sets, then illustrate the sets on a number line.

- **a** $\{x: 2 \le x < 5\} \cup \{x: 4 \le x < 6\}$
- **b** $\{x: x < 5\} \cup \{x: 4 \le x < 6\}$
- **c** $\{x: 2 \le x < 5\} \cup \{x: 4 < x \le 6\}$
- **d** $\{x: x > 5\} \cap \{x: 4 < x \le 6\}$

CONSOLIDATE

21 Copy the Venn diagram at right and then shade the region represented by each of the following sets.



- **a** A'
- **b** $A \cup B$
- **c** $A \cap B$
- **d** $(A \cup B) \setminus (A \cap B)$
- **e** $A' \cap B$
- **f** $A' \cap B'$
- **g** $(A \cup B)'$

22 Complete the following table.

Number	3 significant figures	4 significant figures	2 decimal places	3 decimal places
1267.1066				
7.6699				
8.000 56				
0.999 87				
0.076 768				
0.000 174 95				

23 Simplify the following.

- **a** $-|7| - |-4|$
- **b** $\left|\dfrac{|-3| \times -|5| \times |4|}{|-6| + |6|}\right| - |-8|$

24 The smallest subset of R in which $4 - 2\sqrt[3]{27}$ belongs is:

- **A** Z^+
- **B** Z^-
- **C** Q^+
- **D** Q^-
- **E** I

25 The smallest subset of R in which $\dfrac{\frac{9}{4}}{4.45\dot{6}\dot{7}}$ belongs is:

- **A** Z^+
- **B** Z^-
- **C** Q^+
- **D** Q^-
- **E** I

26 If $\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$ and $B = \{5\}$, then $A' \setminus B$ is:

- **A** {1, 2, 3, 4, 5}
- **B** {5, 6, 7, 8}
- **C** ∅
- **D** {6, 7, 8}
- **E** {1, 2, 3, 4, 5, 6, 7, 8}

27 3.0102 and 92 457 to 4 significant figures are:

- **A** 3.01 and 92 450
- **B** 3.010 and 92 450
- **C** 3.01 and 92 460
- **D** 3.010 and 92 460
- **E** 3.0102 and 92 457

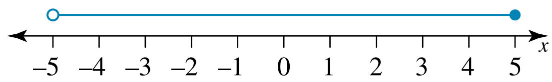
28 0.23, $0.\dot{2}\dot{3}$ and 0.232 233 222 333… respectively belong to which of the following sets?

A Z, Z, I B Q, Q, I C Q, I, I D Z, Q, I E Q, Q, Q

29 Which of the following sets is an incorrect representation of the set {all integers from 1 to 5}?

A {1, 2, 3, 4, 5} B $\{x: x \in Z, 1 \le x \le 5\}$ C $\{x: x \in Z, 1 \le x < 6\}$
D $Z^+ \backslash \{x: x \in Z, x \ge 6\}$ E [1, 5]

30 For the set illustrated on the given number line, which of the following cannot be true?



A (−5, 5] B $\{x: -5 < x \le 5\}$ C $\{x: x \in Q, -5 < x \le 5\}$
D {Real numbers from −5 to 5, not including −5} E [−5, 5]

MASTER

31 Calculate the following products and quotients using a calculator, expressing your answer in scientific notation to 3 significant figures.

a $1.4574 \times 10^{21} \times 3.6677 \times 10^{9}$ b $8.2583 \times 10^{25} \times 9.2527 \times 10^{-7}$

c $\dfrac{5.7789 \times 10^{17}}{4.6999 \times 10^{10}}$ d $\dfrac{2.578 \times 10^{12}(8.775 \times 10^{-7} + 7.342 \times 10^{-6})}{5.878 \times 10^{13}}$

32 a Using your calculator, investigate the percentage of prime numbers:

i between 1 and 9 ii between 10 and 99 iii between 100 and 999.

b What conclusion can you make about the percentage of prime numbers between 10^n and $10^{n+1} - 1$ as $n \longrightarrow \infty$?

1.3 Properties of surds

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 3

Surds
Concept summary
Practice questions

A surd is an irrational number of the form $\sqrt[n]{a}$, where $a > 0$ and $n \in Z^+$. In this section we will focus on the surds of the form $\sqrt{a}$, where $a \in Q$.

For example, $\sqrt{21}$ is a surd, but $\sqrt{36} = 6$ is a rational number and not a surd.

Simplifying surds

$\sqrt{2}$ cannot be simplified because it does not have a perfect square factor, but $\sqrt{8}$ can be simplified since $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2} = 2\sqrt{2}$. A surd is not simplified until all perfect square factors are removed, so the simplified version of $\sqrt{32}$ is not $2\sqrt{8}$ but $4\sqrt{2}$.

WORKED EXAMPLE 11

Simplify the following surds.

a $\sqrt{384}$ b $3\sqrt{405}$ c $-\frac{1}{8}\sqrt{175}$

THINK	WRITE
a 1 Express 384 as a product of two factors where one factor is the largest possible perfect square.	a $\sqrt{384} = \sqrt{64 \times 6}$
2 Express $\sqrt{64 \times 6}$ as the product of two surds.	$= \sqrt{64} \times \sqrt{6}$
3 Simplify the square root from the perfect square (that is, $\sqrt{64} = 8$).	$= 8\sqrt{6}$
b 1 Express 405 as a product of two factors, one of which is the largest possible perfect square.	b $3\sqrt{405} = 3\sqrt{81 \times 5}$

2 Express $\sqrt{81 \times 5}$ as a product of two surds.	$= 3\sqrt{81} \times \sqrt{5}$
3 Simplify $\sqrt{81}$.	$= 3 \times 9\sqrt{5}$
4 Multiply the whole numbers outside the root.	$= 27\sqrt{5}$
c 1 Express 175 as a product of two factors where one factor is the largest possible perfect square.	**c** $-\frac{1}{8}\sqrt{175} = -\frac{1}{8}\sqrt{25 \times 7}$
2 Express $\sqrt{25 \times 7}$ as a product of 2 surds.	$= -\frac{1}{8} \times \sqrt{25} \times \sqrt{7}$
3 Simplify $\sqrt{25}$.	$= -\frac{1}{8} \times 5\sqrt{7}$
4 Multiply the numbers outside the square root.	$= -\frac{5}{8}\sqrt{7}$

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 4

Addition and subtraction of surds
Concept summary
Practice questions

Addition and subtraction of surds

Only like surds may be added or subtracted. Like surds, in their simplest form, have the same number under the square root sign. For example, $5\sqrt{3} + 7\sqrt{3} = (5 + 7)\sqrt{3} = 12\sqrt{3}$ and $5\sqrt{3} - 7\sqrt{3} = (5 - 7)\sqrt{3} = -2\sqrt{3}$.

WORKED EXAMPLE 12

Simplify each of the following expressions involving surds. Assume that *a* and *b* are positive real numbers.

a $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6}$ **b** $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$

c $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$

THINK	WRITE
a All three terms contain the same surd ($\sqrt{6}$), so group like terms and simplify.	**a** $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6} = (3 + 17 - 2)\sqrt{6}$
	$= 18\sqrt{6}$
b 1 Simplify the surds where possible.	**b** $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$
	$= 5\sqrt{3} + 2\sqrt{4 \times 3} - 5\sqrt{2} + 3\sqrt{4 \times 2}$
	$= 5\sqrt{3} + 2 \times 2\sqrt{3} - 5\sqrt{2} + 3 \times 2\sqrt{2}$
	$= 5\sqrt{3} + 4\sqrt{3} - 5\sqrt{2} + 6\sqrt{2}$
2 Collect the like terms.	$= 9\sqrt{3} + \sqrt{2}$
c 1 Simplify the surds where possible.	**c** $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$
	$= \frac{1}{2} \times 10\sqrt{a^2 \times a \times b^2} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$
	$= \frac{1}{2} \times 10 \times a \times b\sqrt{a} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$
	$= 5ab\sqrt{a} + 6ab\sqrt{a} - 10a\sqrt{b}$
2 Add the like terms.	$= 11ab\sqrt{a} - 10a\sqrt{b}$

Multiplication of surds

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 5

Multiplication and division of surds
Concept summary
Practice questions

Using the property $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, where $a, b \in R^+$,
$\sqrt{2} \times \sqrt{6} = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$.

Using the distributive property $a(b + c) = ab + ac$,

$$\sqrt{2}(\sqrt{3} + \sqrt{6}) = \sqrt{2 \times 3} + \sqrt{2 \times 6} = \sqrt{6} + \sqrt{12} = \sqrt{6} + 2\sqrt{3}.$$

Using an extension of the distributive property,

$$(\sqrt{3} + 1)(\sqrt{3} - 2) = \sqrt{3 \times 3} - 2\sqrt{3} + \sqrt{3} - 2 = 3 - \sqrt{3} - 2 = 1 - \sqrt{3}.$$

When appropriate, the expansion of a perfect square may be used; that is,
$(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. For example,

$$(\sqrt{3} - \sqrt{2})^2 = 3 - 2\sqrt{3 \times 2} + 2 = 5 - 2\sqrt{6}.$$

Definition of the conjugate

The conjugate of $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$. The conjugate of $3 - 2\sqrt{5}$ is $3 + 2\sqrt{5}$. The product of a conjugate pair is rational if the numbers under the square root are rational. For example,

$$\begin{aligned}(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) &= \sqrt{3 \times 3} - \sqrt{3 \times 2} + \sqrt{2 \times 3} - \sqrt{2 \times 2}\\ &= 3 - \sqrt{6} + \sqrt{6} - 2 = 1.\end{aligned}$$

This is a special case of the difference of perfect squares expansion,
$(a + b)(a - b) = a^2 - b^2$.

WORKED EXAMPLE 13

Multiply the following surds, expressing answers in simplest form.

a $6\sqrt{12} \times 2\sqrt{6}$ **b** $\frac{3}{5}\sqrt{70} \times \frac{1}{4}\sqrt{10}$

THINK	WRITE
a 1 Write the expression and simplify $\sqrt{12}$.	**a** $6\sqrt{12} \times 2\sqrt{6} = 6\sqrt{4 \times 3} \times 2\sqrt{6}$
	$= 6 \times 2\sqrt{3} \times 2\sqrt{6}$
	$= 12\sqrt{3} \times 2\sqrt{6}$
2 Multiply the coefficients and multiply the surds.	$= 24\sqrt{18}$
3 Simplify the product surd.	$= 24\sqrt{9 \times 2}$
	$= 24 \times 3\sqrt{2}$
	$= 72\sqrt{2}$
b 1 Multiply the coefficients and multiply the surds.	**b** $\frac{3}{5}\sqrt{70} \times \frac{1}{4}\sqrt{10} = \frac{3}{5} \times \frac{1}{4} \times \sqrt{70} \times \sqrt{10}$
	$= \frac{3}{20}\sqrt{700}$
2 Simplify the product surd.	$= \frac{3}{20}\sqrt{100 \times 7}$
	$= \frac{3}{20} \times 10\sqrt{7}$
3 Simplify by dividing both 10 and 20 by 10 (cross-cancel).	$= \frac{3}{2}\sqrt{7}$ or $\frac{3\sqrt{7}}{2}$

WORKED EXAMPLE 14

Expand and simplify the following where possible.

a $\sqrt{7}(\sqrt{18}-3)$ **b** $-2\sqrt{3}(\sqrt{10}-5\sqrt{3})$

c $(\sqrt{5}+3\sqrt{6})(2\sqrt{3}-\sqrt{2})$

THINK	WRITE
a 1 Write the expression.	**a** $\sqrt{7}(\sqrt{18}-3)$
2 Simplify $\sqrt{18}$.	$=\sqrt{7}(3\sqrt{2}-3)$
3 Expand the bracket.	$=\sqrt{7}\times 3\sqrt{2}+\sqrt{7}\times -3$
4 Simplify.	$=3\sqrt{14}-3\sqrt{7}$
b 1 Write the expression.	**b** $-2\sqrt{3}(\sqrt{10}-5\sqrt{3})$
2 i Expand the brackets.	$=-2\sqrt{3}\times\sqrt{10}-2\sqrt{3}\times -5\sqrt{3}$
ii Be sure to multiply through with the negative.	$=-2\sqrt{30}+10\sqrt{9}$
3 Simplify.	$=-2\sqrt{30}+10\times 3$ $=-2\sqrt{30}+30$
c 1 Write the expression.	**c** $(\sqrt{5}+3\sqrt{6})(2\sqrt{3}-\sqrt{2})$
2 Expand the brackets.	$=\sqrt{5}\times 2\sqrt{3}+\sqrt{5}\times -\sqrt{2}+3\sqrt{6}\times 2\sqrt{3}$ $+3\sqrt{6}\times -\sqrt{2}$
3 Simplify.	$=2\sqrt{15}-\sqrt{10}+6\sqrt{18}-3\sqrt{12}$ $=2\sqrt{15}-\sqrt{10}+6\times 3\sqrt{2}-3\times 2\sqrt{3}$ $=2\sqrt{15}-\sqrt{10}+18\sqrt{2}-6\sqrt{3}$

Division of surds

$\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$, where $a, b \in R^+$. For example, $\dfrac{\sqrt{6}}{\sqrt{2}}=\sqrt{\dfrac{6}{2}}=\sqrt{3}$.

Using the property $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}\times\dfrac{\sqrt{b}}{\sqrt{b}}=\dfrac{\sqrt{ab}}{b}$, where a and b are rational, we can express answers with rational denominators. For example,

$$\frac{\sqrt{2}}{\sqrt{6}}=\frac{\sqrt{2}}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}=\frac{\sqrt{12}}{6}=\frac{2\sqrt{3}}{6}=\frac{\sqrt{3}}{3}$$

Using the property of conjugates, binomial surds in the denominator may be rationalised. For example,

$$\frac{\sqrt{7}-2\sqrt{2}}{\sqrt{7}+\sqrt{2}}=\frac{\sqrt{7}-2\sqrt{2}}{\sqrt{7}+\sqrt{2}}\times\frac{\sqrt{7}-\sqrt{2}}{\sqrt{7}-\sqrt{2}}=\frac{7-\sqrt{14}-2\sqrt{14}+2\times 2}{7-2}=\frac{11-3\sqrt{14}}{5}$$

By multiplying the original surd by $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{7}-\sqrt{2}}$, we are multiplying by 1, so the number is unchanged but is finally expressed in its rational denominator form.

WORKED EXAMPLE 15

Express the following in their simplest form with a rational denominator.

a $\dfrac{9\sqrt{88}}{6\sqrt{99}}$ b $\dfrac{\sqrt{6}}{\sqrt{13}}$ c $\dfrac{1}{2\sqrt{6}-\sqrt{3}}+\dfrac{1}{3\sqrt{6}+2\sqrt{3}}$

THINK	WRITE
a 1 Rewrite the surds, using $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$.	a $\dfrac{9\sqrt{88}}{6\sqrt{99}}=\dfrac{9}{6}\sqrt{\dfrac{88}{99}}$
2 Simplify the fraction under the root.	$=\dfrac{9}{6}\sqrt{\dfrac{8}{9}}$
3 Simplify the surds.	$=\dfrac{9\times 2\sqrt{2}}{6\times 3}$
4 Multiply the whole numbers in the numerator and those in the denominator and simplify.	$=\sqrt{2}$
b 1 Write the fraction.	b $\dfrac{\sqrt{6}}{\sqrt{13}}$
2 Multiply both the numerator and the denominator by the surd $\sqrt{13}$.	$=\dfrac{\sqrt{6}}{\sqrt{13}}\times\dfrac{\sqrt{13}}{\sqrt{13}}$ $=\dfrac{\sqrt{78}}{13}$
c 1 Write the first fraction.	c $=\dfrac{1}{2\sqrt{6}-\sqrt{3}}$
2 Multiply the numerator and the denominator by the conjugate of the denominator.	$=\dfrac{1}{2\sqrt{6}-\sqrt{3}}\times\dfrac{2\sqrt{6}+\sqrt{3}}{2\sqrt{6}+\sqrt{3}}$
3 Expand the denominator.	$=\dfrac{2\sqrt{6}+\sqrt{3}}{(2)^2\times 6-3}$
4 Simplify the denominator.	$=\dfrac{2\sqrt{6}+\sqrt{3}}{21}$
5 Write the second fraction.	$\dfrac{1}{3\sqrt{6}+2\sqrt{3}}$
6 Multiply the numerator and the denominator by the conjugate of the denominator.	$=\dfrac{1}{3\sqrt{6}+2\sqrt{3}}\times\dfrac{3\sqrt{6}-2\sqrt{3}}{3\sqrt{6}-2\sqrt{3}}$
7 Expand the denominator.	$=\dfrac{3\sqrt{6}-2\sqrt{3}}{3^2\times 6-2^2\times 3}$
8 Simplify the denominator.	$=\dfrac{3\sqrt{6}-2\sqrt{3}}{42}$

9 Add the two fractions together.
Find the lowest common denominator first.

$$\frac{2\sqrt{6}+\sqrt{3}}{21}+\frac{3\sqrt{6}-2\sqrt{3}}{42}$$
$$=\frac{2\sqrt{6}+\sqrt{3}}{21}\times\left(\frac{2}{2}\right)+\frac{3\sqrt{6}-2\sqrt{3}}{42}$$
$$=\frac{4\sqrt{6}+2\sqrt{3}}{42}+\frac{3\sqrt{6}-2\sqrt{3}}{42}$$

10 Add the numerators.

$$=\frac{7\sqrt{6}}{42}$$

11 Simplify where appropriate.

$$=\frac{\sqrt{6}}{6}$$

EXERCISE 1.3 Properties of surds

PRACTISE

1 WE11 Simplify the following surds.
a $\sqrt{24}$ b $\sqrt{56}$ c $\sqrt{125}$ d $\sqrt{98}$ e $\sqrt{48}$

2 Simplify the following surds.
a $\sqrt{300}$ b $7\sqrt{80}$ c $\frac{\sqrt{128}}{4}$ d $\frac{2\sqrt{18}}{5}$ e $\frac{-3\sqrt{50}}{10}$

3 WE12 Simplify the following expressions.
a $7\sqrt{2}+4\sqrt{3}-5\sqrt{2}-6\sqrt{3}$
b $2+5\sqrt{7}-6-4\sqrt{7}$
c $3\sqrt{5}-6\sqrt{3}+5\sqrt{5}-4\sqrt{2}-8\sqrt{5}$
d $\sqrt{18}-\sqrt{12}+\sqrt{75}+\sqrt{27}$

4 Simplify the following expressions.
a $\sqrt{50}-\sqrt{72}+\sqrt{80}+\sqrt{45}$
b $3\sqrt{12}-5\sqrt{18}+4\sqrt{27}+5\sqrt{98}$
c $\frac{2\sqrt{3}}{4}-\frac{3\sqrt{2}}{8}+\frac{5\sqrt{3}}{8}-\frac{5\sqrt{2}}{4}$
d $\frac{2\sqrt{27}}{5}-\frac{3\sqrt{32}}{5}+\frac{5\sqrt{48}}{3}-\frac{5\sqrt{2}}{2}$

5 WE13 Express the following surds in their simplest form.
a $\sqrt{6}\times\sqrt{15}$
b $2\sqrt{3}\times5\sqrt{7}$

6 Simplify the following surds.
a $4\sqrt{7}\times3\sqrt{14}$
b $\frac{\sqrt{20}}{3}\times\frac{\sqrt{15}}{4}$

7 WE14 Expand, giving your answers in their simplest form.
a $\sqrt{3}(\sqrt{5}-\sqrt{2})$
b $2\sqrt{3}(3\sqrt{3}+\sqrt{2})$
c $(\sqrt{5}-\sqrt{3})(\sqrt{5}-\sqrt{2})$
d $(\sqrt{18}-\sqrt{12})(\sqrt{3}-2\sqrt{2})$

8 Expand and simplify.
a $(\sqrt{5}+\sqrt{7})^2$
b $(2\sqrt{12}+3\sqrt{18})^2$
c $(2\sqrt{5}-\sqrt{3})(2\sqrt{5}+\sqrt{3})$
d $(5\sqrt{5}-10)(5\sqrt{5}+10)$

9 WE15 Express the following surds in their simplest form with a rational denominator.
a $\frac{\sqrt{18}}{\sqrt{3}}$
b $\frac{2\sqrt{24}}{3\sqrt{3}}$
c $\frac{\sqrt{5}}{\sqrt{3}}$
d $\frac{4\sqrt{3}}{7\sqrt{5}}$
e $\frac{2\sqrt{8}}{3\sqrt{12}}$
f $\frac{1}{\sqrt{5}-\sqrt{3}}$

10 Rationalise the denominators.

a $\dfrac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ b $\dfrac{2\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$ c $\dfrac{5+\sqrt{3}}{5-\sqrt{3}}$

d $\dfrac{\sqrt{12}-\sqrt{8}}{\sqrt{12}+\sqrt{8}}$ e $\dfrac{2\sqrt{5}-\sqrt{3}}{\sqrt{5}-2\sqrt{3}}$ f $\dfrac{2\sqrt{18}-\sqrt{24}}{3\sqrt{8}-\sqrt{54}}$

CONSOLIDATE

11 Express the following surds in their simplest form with a rational denominator.

a $\dfrac{1}{2\sqrt{2}-3}+\dfrac{1}{2\sqrt{2}+3}$ b $\dfrac{1}{3\sqrt{2}-2\sqrt{3}}-\dfrac{1}{2\sqrt{2}+3\sqrt{3}}$

c $\dfrac{3\sqrt{5}}{3\sqrt{2}-2\sqrt{3}}-\dfrac{2\sqrt{5}-1}{2\sqrt{2}+3\sqrt{3}}$ d $\dfrac{2\sqrt{5}+3\sqrt{3}}{3\sqrt{5}-2\sqrt{3}}-\dfrac{3\sqrt{5}-2\sqrt{3}}{2\sqrt{5}+3\sqrt{3}}$

e $\dfrac{4\sqrt{2}+3\sqrt{2}}{4\sqrt{2}-2\sqrt{3}}\times\dfrac{5\sqrt{2}-2\sqrt{3}}{6\sqrt{2}+3\sqrt{3}}$ f $\dfrac{2\sqrt{5}+3\sqrt{3}}{3\sqrt{5}-4\sqrt{3}}\div\dfrac{3\sqrt{5}+4\sqrt{3}}{2\sqrt{5}+3\sqrt{3}}$

12 Given that $x=2-3\sqrt{2}$, find each of the following, giving the answer in surd form with a rational denominator.

a $x+\dfrac{1}{x}$ b $x-\dfrac{1}{x}$ c $\dfrac{x^2-2x}{x+2}$

d $\dfrac{x^2+2x}{x+3}$ e $x^2-4x-14$ f $2x^2-2x-9$

g Using your answers to **e** and **f**, state if $2-3\sqrt{2}$ is a solution of $x^2-4x-14=0$ and $2x^2-2x-9=0$.

13 Show that $5-2\sqrt{3}$ is a solution of one of the following equations: $x^2-13x+10=0$ or $x^2-10x+13=0$.

14 Show that $\sqrt{2}+1$ is a solution of both of the following equations: $x^2-2\sqrt{2}x+1=0$ and $x^2-(2\sqrt{2}+3)x+4+3\sqrt{2}=0$.

15 Expressed in its simplest form, $\dfrac{3}{5}\sqrt{75}-\dfrac{2}{3}\sqrt{27}-\dfrac{1}{2}\sqrt{48}$ equals:

A $\sqrt{3}$ B $-\sqrt{3}$ C $7\sqrt{3}$ D 0 E $-3\sqrt{3}$

16 Expressed in its simplest form, $\dfrac{\sqrt{14a^3b^2}}{\sqrt{7ab^2}}$ equals:

A $\sqrt{2a}$ B $2a$ C $\sqrt{2ab}$
D $\sqrt{2a^2b^0}$ E $\sqrt{2a^2}$

17 Expressed in its simplest form, $(3\sqrt{3}-4\sqrt{8})(2\sqrt{3}-3\sqrt{8})$ equals:

A $114-34\sqrt{6}$ B $120-34\sqrt{6}$ C $-78-17\sqrt{24}$
D $18-24\sqrt{2}$ E $-18-34\sqrt{6}$

18 Expressed in its simplest form, $\dfrac{15\sqrt{21}}{6\sqrt{14}}$ equals:

A $\dfrac{5\sqrt{3}}{2\sqrt{2}}$ B $\dfrac{5\sqrt{6}}{2\sqrt{2}}$ C $\dfrac{5\sqrt{3}}{2}$

D $\dfrac{5\sqrt{6}}{4}$ E $\dfrac{5\sqrt{6}}{2}$

19 Expressed in its simplest form, $\dfrac{2\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ equals:

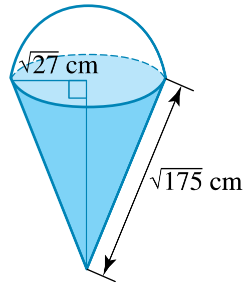
A $\dfrac{13+3\sqrt{15}}{2}$
B $\dfrac{12-\sqrt{15}}{2}$
C $\dfrac{18-3\sqrt{15}}{2}$
D $\dfrac{12+3\sqrt{15}}{2}$
E $\dfrac{13+\sqrt{15}}{2}$

20 Expressed in its simplest form, $\dfrac{3\sqrt{5}-5}{3\sqrt{5}+5}-\dfrac{3\sqrt{5}+5}{3\sqrt{5}-5}$ equals:

A $3\sqrt{5}$
B $-30\sqrt{5}$
C $30\sqrt{5}$
D $52-30\sqrt{5}$
E $-3\sqrt{5}$

MASTER

21 An ice-cream cone with measurements as shown is completely filled with ice-cream, and has a hemisphere of ice-cream on top.



a Determine the height of the ice-cream cone in simplest surd form.
b Determine the volume of the ice-cream in the cone.
c Determine the volume of the ice-cream in the hemisphere.
d Hence, find the total volume of ice-cream.

22 A gold bar with dimensions of $5\sqrt{20}$, $3\sqrt{12}$ and $2\sqrt{6}$ cm is to be melted down into a cylinder of height $4\sqrt{10}$ cm.
a Find the volume of the gold, expressing the answer in the simplest surd form and specifying the appropriate unit.
b Find the radius of the cylinder, expressing the answer in the simplest surd form and specifying the appropriate unit.
c If the height of the cylinder was $3\sqrt{40}$ cm, what would be the new radius? Express your answer in the simplest surd form.

1.4 The set of complex numbers

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 6

Complex numbers
Concept summary
Practice questions

The need to invent further numbers became clear when equations such as $x^2 = -1$ and $x^2 = -9$ were considered. Clearly there are no real solutions, so imaginary numbers were invented, using the symbol i, where $i^2 = -1$. The equation $x^2 = -1$ has two solutions, $x = -i$ and $x = i$. As $\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1} = 3 \times \sqrt{i^2} = 3i$, $x^2 = -9$ has the solutions $x = \pm 3i$.

Quadratic equations such as $x^2 - 4x + 5 = 0$ were investigated further. Using the general formula for the solution of a quadratic equation, that is, if $ax^2 + bx + c = 0$, then

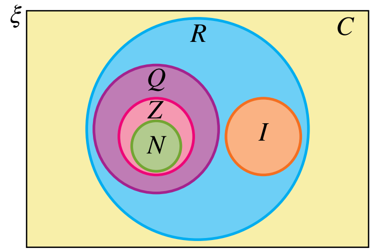
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $x = \dfrac{4 \pm \sqrt{16-20}}{2} = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i$. If the discriminant, $b^2 - 4ac$, is negative, the equation has no real solutions, but it does have two complex solutions.

A **complex number** is any number of the form $x + yi$, where $x, y \in R$.

C is the set of complex numbers where $C = \{x + yi : x, y \in R\}$.

Just as x is commonly used in algebra to represent a real number, z is commonly used to represent a complex number, where $z = x + yi$.

If $x = 0$, $z = yi$ is a pure imaginary number. If $y = 0$, $z = x$ is a real number, so that $R \subset C$. This is represented on the Venn diagram at right.



Notation

If $z = a + bi$, the real component of z is $\text{Re}(z) = a$, and the imaginary component of z is $\text{Im}(z) = b$. For example, if $z = -2 - 2\sqrt{3}i$, $\text{Re}(z) = -2$ and $\text{Im}(z) = -2\sqrt{3}$ (not $-2\sqrt{3}i$). Similarly, $\text{Re}(4 + 6i) = 4$ and $\text{Im}(4 + 6i) = 6$.

Equality of complex numbers

If $a + bi = c + di$, then $a = c$ and $b = d$.

For two complex numbers z_1 and z_2 to be equal, both their real and imaginary components must be equal.

WORKED EXAMPLE 16

If $(x + 2) + (y - 4)i = (2x + y) + xi$, find x and y.

THINK	WRITE	
1 Let the real parts be equal and the imaginary parts be equal to form two equations.	Re: $x + 2 = 2x + y$	[1]
	Im: $y - 4 = x$	[2]
2 Rearrange the linear simultaneous equations.	$-x - y = -2$	[3]
	$-x + y = 4$	[4]
3 Add equations [3] and [4] to solve for x.	$-2x = 2$	
	$x = -1$	
4 Substitute $x = -1$ into equation [2] to find the value of y.	$y - 4 = -1$	
	$y = 3$	
5 State the answer.	$x = -1, y = 3$	

studyon

Units 1 & 2
AOS 2
Topic 1
Concept 7

Operations with complex numbers
Concept summary
Practice questions

Multiplication of a complex number by a real constant

If $z = a + bi$, then $kz = k(a + bi) = ka + kbi$.

For example, if $z = -2 + 3i$, then $-3z = -3(-2 + 3i) = 6 - 9i$.

Adding and subtracting complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$ and $z_1 - z_2 = (a - c) + (b - d)i$.

Modulus of a complex number

The magnitude (or modulus or absolute value) of a complex number $z = x + yi$ is denoted by $|z|$ and $|z| = \sqrt{x^2 + y^2}$.

WORKED EXAMPLE 17

If $z_1 = 2 - 3i$ and $z_2 = -3 + 4i$, find:

a $z_1 + z_2$

b $3z_1 - 4z_2$

c $|z_1|$

d $|z_2|^2$.

THINK	WRITE
a Use the definition for addition of complex numbers: $z_1 + z_2 = (a + c) + (b + d)i$	a $z_1 + z_2 = (2 - 3) + (-3 + 4)i$ $= -1 + i$
b First multiply each complex number by the constant and then use the definition for subtraction of complex numbers to answer the question.	b $3z_1 - 4z_2 = 3(2 - 3i) - 4(-3 + 4i)$ $= 6 - 9i - (-12 + 16i)$ $= 18 - 25i$
c Use $|z| = \sqrt{x^2 + y^2}$.	c $|z_1| = \sqrt{2^2 + 3^2}$ $= \sqrt{13}$
d Use $|z| = \sqrt{x^2 + y^2}$.	d $|z_2| = \sqrt{(-3)^2 + 4^2}$ $= \sqrt{25}$ $|z_2|^2 = 25$

EXERCISE 1.4 The set of complex numbers

PRACTISE

1 WE16 Solve to find x and y in the following.

a $(x + 1) + (y - 1)i = 2 + 3i$

b $(x + 4) - (3 + yi) = 2 + 5i$

c $(2x + i) + (3 - 2yi) = x + 3i$

d $(x + 2i) + 2(y + xi) = 7 + 4i$

2 Solve the following.

a $(2x + 3yi) + 2(x + 2yi) = 3 + 2i$

b $(x + i) + (2 + yi) = 2x + 3yi$

c $(2x - 3i) + (-3 + 2y)i = y - xi$

3 WE17 If $z_1 = 2 - i$ and $z_2 = 3 + 4i$, find:

a $z_1 + z_2$

b $2z_1 + 3z_2$

c $|z_1|$

d $|z_2|^2$.

4 If $z_1 = 3 - 4i$ and $z_2 = 2 - 3i$, evaluate the following.

a $2z_1 - 3z_2$

b $\sqrt{2}z_1 + 2\sqrt{2}z_2$

c $2|z_1| + 3|z_2|$

CONSOLIDATE

5 Express the following in terms of i.

a $\sqrt{-16}$

b $\sqrt{-7}$

c $2 + \sqrt{-20}$

6 Simplify.

a $\sqrt{-10} + \sqrt{10}$

b $\dfrac{1 - \sqrt{-28}}{2}$

7 Simplify the following numbers.

a $-\sqrt{-25}$

b $\sqrt{-49} + \sqrt{4}$

c $11\sqrt{-81}$

8 State the values of Re(z) and Im(z) for the following.

a $3 + 4i$

b $-2 + \sqrt{2}i$

c $(\sqrt{2} - 1) + (\sqrt{2} + 1)i$

9 State the values of Re(z) and Im(z) for the following.

a $\sqrt{8} - \sqrt{-40}$

b -6

c $13i$

10 Find the following components.

a $\text{Re}(2 + 3i + 3(4 - 2i))$
b $\text{Re}(\sqrt{3} + 2\sqrt{2}i + \sqrt{2}(-3 - \sqrt{3}i))$
c $\text{Im}(2(2 - 3i) - 3(4 - 2i))$
d $\text{Im}(2\sqrt{3} - 2\sqrt{2}i + \sqrt{2}(-3 - \sqrt{6}i))$

11 If $z_1 = 2 - i$ and $z_2 = 3 - 2i$, then $\text{Re}(2z_1 - 3z_2)$ equals:

A 13 B -13 C 5 D -5 E 4

12 If $z_1 = 2 - i$ and $z_2 = 3 - 2i$, then $\text{Im}(2z_1 - 3z_2)$ equals:

A $4i$ B 4 C -4 D -8 E $-8i$

13 If $(2 + xi) + (4 - 3i) = x + 3yi$, then the respective values of x and y are:

A 6, 1 B 3, 6 C 6, -3 D 6, 3 E 1, 6

14 Simplify $(12 - 4i) + (3 + 6i) - (10 + 10i)$.

15 Find the error in the student's work shown below.

$$(7 - 3i) + (2 + 2i)$$
$$9 - i^2$$
$$9 - (-1)$$
$$9 + 1$$
$$10$$

MASTER

16 Simplify $(13 - 5i) + (7 - 5i)$, giving the answer in simplest form of $a + bi$.

17 Find the imaginary part of $21 - 2i + 2(10 - 5i)$.

1.5 Multiplication and division of complex numbers

Multiplication of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$ where $a, b, c, d \in R$, then

$$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Note that this is an application of the distributive property.

WORKED EXAMPLE 18

Simplify:

a $2i(2 - 3i)$ b $(2 - 3i)(-3 + 4i)$.

THINK	WRITE
a Expand the brackets.	a $2i(2 - 3i) = 4i - 6i^2$ $= 6 + 4i$
b Expand the brackets as for binomial expansion and simplify.	b $(2 - 3i)(-3 + 4i) = -6 + 8i + 9i - 12i^2$ $= 6 + 17i$

The conjugate of a complex number

If $z = a + bi$, then its conjugate, $\bar{z}$, is $\bar{z} = a - bi$.

The sum of a complex number and its conjugate $z + \bar{z} = a + bi + a - bi = 2a$, which is a real number.

The product of a complex number and its conjugate:

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 - (bi)^2 \\ &= a^2 + b^2, \text{ which is a real number.} \end{aligned}$$

WORKED EXAMPLE 19

If $z_1 = 2 + 3i$ and $z_2 = -4 - 5i$, find:

a $\bar{z}_1 + \bar{z}_2$

b $\overline{z_1 + z_2}$

c $\bar{z}_1\bar{z}_2$

d $\overline{z_1 z_2}$.

THINK	WRITE
a 1 Determine the conjugate of each complex number using the definition: if $z = a + bi$, then its conjugate, $\bar{z} = a - bi$.	**a** $z_1 = 2 + 3i$ $\Rightarrow \bar{z}_1 = 2 - 3i$ $z_2 = -4 - 5i$ $\Rightarrow \bar{z}_2 = -4 + 5i$
2 Evaluate $\bar{z}_1 + \bar{z}_2$.	$\bar{z}_1 + \bar{z}_2 = (2 - 3i) + (-4 + 5i)$ $= -2 + 2i$
b 1 To evaluate $\overline{z_1 + z_2}$, first evaluate $z_1 + z_2$.	**b** $z_1 + z_2 = (2 + 3i) + (-4 - 5i)$ $= -2 - 2i$
2 Evaluate $\overline{z_1 + z_2}$.	$\overline{z_1 + z_2} = -2 + 2i$
c Evaluate $\bar{z}_1\bar{z}_2$ using binomial expansion.	**c** $\bar{z}_1\bar{z}_2 = (2 - 3i)(-4 + 5i)$ $= -8 + 10i + 12i - 15i^2$ $= 7 + 22i$
d Evaluate $z_1 z_2$ first and then evaluate $\overline{z_1 z_2}$.	**d** $z_1 z_2 = (2 + 3i)(-4 - 5i)$ $= -8 - 10i - 12i - 15i^2$ $= 7 - 22i$ $\overline{z_1 z_2} = 7 + 22i$

Division of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$ where $a, b, c, d \in R$, then using the conjugate:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + bi}{c + di} \\ &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \end{aligned}$$

$$= \frac{ac - adi + bci - bdi^2}{c^2 - (di)^2}$$

$$= \frac{(ac + bd) - (ad - bc)i}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} - \frac{ad - bc}{c^2 + d^2}i$$

WORKED EXAMPLE 20

Express each of the following in the form $a + bi$.

a $\dfrac{4 - i}{2}$

b $\dfrac{3 - 4i}{3i}$

c $(3 - 2i)^{-1}$

d $\dfrac{2 - 3i}{2 + i}$

THINK	WRITE
a Divide each term of the numerator by 2.	a $\dfrac{4 - i}{2} = 2 - \dfrac{1}{2}i$
b Multiply the numerator and denominator by i and then divide each term of the numerator by 3. Write the answer in the required form $a + bi$.	b $\dfrac{3 - 4i}{3i} \times \dfrac{i}{i} = \dfrac{-3i - 4}{3}$ $= \dfrac{-4}{3} - i$
c 1 Express $(3 - 2i)^{-1}$ as a reciprocal.	c $(3 - 2i)^{-1} = \dfrac{1}{3 - 2i}$
2 Multiply the numerator and denominator by the conjugate of the denominator.	$= \dfrac{1}{3 - 2i} \times \dfrac{3 + 2i}{3 + 2i}$ $= \dfrac{3 + 2i}{9 - (2i)^2}$ $= \dfrac{3 + 2i}{9 - 4i^2}$ $= \dfrac{3 + 2i}{13}$
3 Write the answer in the required form $a + bi$.	$= \dfrac{3}{13} + \dfrac{2}{13}i$
d 1 Multiply the numerator and denominator by the conjugate of the denominator.	d $\dfrac{2 - 3i}{2 + i} = \dfrac{2 - 3i}{2 + i} \times \dfrac{2 - i}{2 - i}$ $= \dfrac{4 - 2i - 6i + 3i^2}{4 - i^2}$ $= \dfrac{1 - 8i}{5}$
2 Write the answer in the required form $a + bi$.	$= \dfrac{1}{5} - \dfrac{8}{5}i$

EXERCISE 1.5 Multiplication and division of complex numbers

PRACTISE

1 WE18 Simplify the following, giving each answer in its simplest $a + bi$ form.

a $2i(2 + 3i)$
b $(2 - 3i)(1 + i)$
c $(-2 - i)(1 - 3i)$
d $(2 - 3i)^2$
e $(6 + 7i)(6 - 7i)$

2 Expand and simplify:

a $-2i(3 - 4i)$
b $(3 + 5i)(3 - 3i)$
c $(a + bi)(a - bi)$.

3 WE19 Give the conjugate of the following complex numbers.

a $3 + 2i$
b $-4 + 3i$
c $\sqrt{2} - 2i$
d $-8i$

4 If $z_1 = 4 - 3i$ and $z_2 = 3 - 4i$, evaluate the following, giving each answer in its simplest $a + bi$ form.

a $\bar{z}_1$
b $\bar{z}_1 z_2$
c $z_1 \bar{z}_2$
d $(\bar{z}_1)^2$
e $2i\,\bar{z}_2$
f $(\overline{z_1 + z_2})^2$

5 WE20 Express each of the following in the form $a + bi$.

a $\dfrac{3 - 4i}{-5i}$
b $\dfrac{3 + 4i}{3 - 4i}$
c $\dfrac{1 + 2i}{2 + i}$
d $\dfrac{(2 + i)^2}{1 + 2i}$
e $(3 + 2i)^{-1}$
f $(3 + 2i)^{-2}$

6 Simplify each of the following if $z_1 = 4 - 3i$ and $z_2 = 3 - 4i$.

a $(\bar{z}_1)^{-1}$
b $\dfrac{z_1}{\bar{z}_2}$
c $\left(\dfrac{\bar{z}_1}{\bar{z}_2}\right)^{-1}$
d $\left(\dfrac{\bar{z}_1}{\bar{z}_2}\right)^2$
e $z_1 + \dfrac{1}{\bar{z}_1}$
f $\dfrac{z_1}{\bar{z}_1} - \dfrac{z_2}{\bar{z}_2}$

CONSOLIDATE

7 If $z_1 = a + bi$ and $z_2 = c + di$, show that:

a $\bar{z}_1 \bar{z}_2 = \overline{z_1 z_2}$
b $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$
c $\dfrac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\dfrac{z_1}{z_2}\right)}$.

8 Find x and y in each of the following.

a $(x + yi)(2 + i) = 3 + 6i$
b $\dfrac{x + yi}{1 + 2i} = 1 + i$

9 Solve for z.

a $(4 + 3i)z = 2 - i$
b $(2 - 3i)z = -3 - 2i$

10 For each of the following, state $\bar{z}$ and find z^{-1}, then state z^{-1} in terms of $\bar{z}$.

a $z = 4 + 5i$
b $z = a + bi$

11 Expressed in $a + bi$ form, $(2\sqrt{3} - 3i)\,(3\sqrt{3} - 2i)$ equals:

A $24 - 13\sqrt{3}i$
B $12 - 13\sqrt{3}i$
C $(6\sqrt{3} + 6) - 13\sqrt{3}i$
D $(6\sqrt{3} - 6) - 13\sqrt{3}i$
E $12 - 5\sqrt{3}i$

12 Expressed in $a + bi$ form, $\dfrac{2\sqrt{3} - 3i}{3\sqrt{3} - 2i}$ equals:

A $\dfrac{24}{31} - \dfrac{5\sqrt{3}}{31}i$
B $\dfrac{12}{31} - \dfrac{5\sqrt{3}}{31}i$
C $\dfrac{24}{31} - \dfrac{13\sqrt{3}}{31}i$
D $\dfrac{24}{23} - \dfrac{5\sqrt{3}}{31}i$
E $\dfrac{212}{23} - \dfrac{5\sqrt{3}}{23}i$

13 Expressed in $a + bi$ form, $(1 + i)^2 + (1 + i)^{-2}$ equals:

A $\frac{5}{2}i$ B $\frac{9}{4} - \frac{7}{4}i$ C 0

D $\frac{9}{4} - \frac{9}{4}i$ E $\frac{3}{2}i$

14 If $(2x + yi)(3 - 2i) = 4 + 5i$, then the respective values of x and y are:

A $\frac{2}{13}$ and $\frac{23}{13}$ B $\frac{1}{13}$ and $\frac{23}{13}$ C $\frac{7}{13}$ and $\frac{23}{13}$

D $\frac{1}{5}$ and $\frac{23}{5}$ E $\frac{2}{5}$ and $\frac{23}{5}$

15 Evaluate the following, giving each answer in the form $a + bi$.

a $\frac{1}{2 - 3i} + \frac{1}{2 + 3i}$ b $\frac{(3 - 2i)^2}{(2 - i)^2}$ c $2 + \sqrt{3}i + \frac{1}{2 + \sqrt{3}i}$

16 Evaluate the following, giving each answer in the form $a + bi$.

a $\frac{\sqrt{3} - i}{\sqrt{3} + i} + \frac{\sqrt{3} + i}{\sqrt{3} - i}$ b $\frac{2 - i}{3 + 2i} - \frac{3 + i}{2 - i}$

MASTER

17 If $z = 1 + i$, simplify $z^3 + z^2 + z + 1$, giving answer in the form $a + bi$.

18 Write $\frac{2 + 5i}{3 + i}$ in the form $a + bi$.

1.6 Representing complex numbers on an Argand diagram

studyon

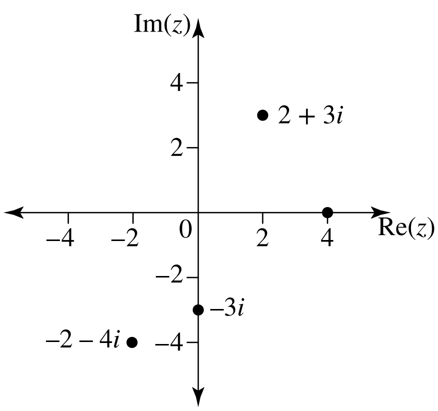
Units 1 & 2
AOS 2
Topic 1
Concept 8

Argand diagrams
Concept summary
Practice questions

Real numbers can be represented on a number line, but complex numbers, with their real and imaginary components, require a plane.

The Argand diagram or Argand plane has a horizontal axis, Re(z), and a vertical axis, Im(z). The complex number $z = a + bi$ is represented by the point (a, b). Because of the similarities with the Cartesian plane, $a + bi$ is referred to as the Cartesian or rectangular form.

The complex numbers $2 + 3i$, 4, $-3i$ and $-2 - 4i$ are shown on the Argand plane at right.



WORKED EXAMPLE 21

a Express the following in their simplest form: $i^0, i, i^2, i^3, i^4, i^5$.

b Use the pattern in these results to find the simplest form for i^8, i^{21} and i^{-63}.

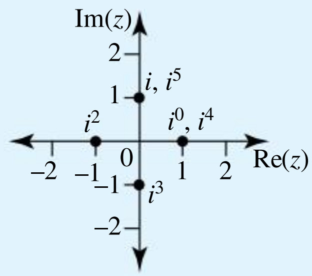
c Illustrate the points from part a on an Argand plane, and state their distance from the origin and the angle of rotation about the origin to rotate from one power of i to the next.

THINK	WRITE/DRAW
a Use $i^2 = -1$ and index law rules to simplify each term.	a $i^0 = 1$ $\quad i^1 = i$ $i^2 = -1$ $\quad i^3 = i^2 \times i = -i$ $i^4 = i^2 \times i^2 = 1$ $\quad i^5 = i^4 \times i = i$

b The pattern repeats as shown in part a.

b $i^8 = (i^4)^2$
$= 1$
$i^{21} = (i^5)^4 \times i$
$= i$
$i^{-63} = (i^{-15})^4 \times i^{-3}$
$= i^{-3}$
$= i$

c 1 Rule up a pair of labelled, scaled axes for the Argand plane.
Place each of the points from part a onto the plane and label them.

c 

2 Determine the distance of each point from the origin.

All points are 1 unit from the origin.

3 State the angle of rotation about the origin to rotate from one power of i to the next.

The angle of rotation about the origin to rotate from one power of i to the next is 90° in an anticlockwise direction.

EXERCISE 1.6 Representing complex numbers on an Argand diagram

PRACTISE

1 WE21 Give the following in their simplest form.

a i^7 b i^{37} c i^{-4} d i^{-15}

2 Give the following in their simplest form.

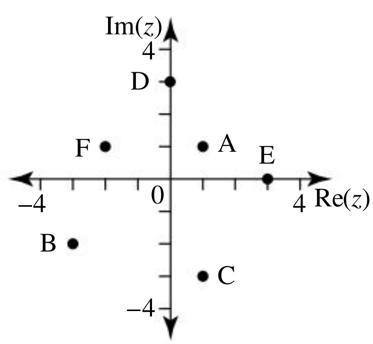
a $(2i)^6$ b $(-2i)^8$ c $-(2i)^9$ d $-(-2i)^{-9}$

CONSOLIDATE

3 Plot the following points on an Argand plane.

a $2 + 3i$ b $2 - 3i$ c $-2 + 3i$
d $-2 - 3i$ e $-3i$ f 2

4 Write down the complex number represented by the points A to F on the Argand diagram at right.



5 If $z_1 = 2 + 5i$ and $z_2 = 5 - 7i$, find the simplest algebraic expression for each of the following and represent them on an Argand diagram.

a $z_1 + z_2$
b $z_1 - z_2$

6 Assume $z_1 = 3 + 5i$ and $z_2 = -z_1$ to answer the following.

a Sketch z_1 and z_2 on an Argand diagram.
b State the angle of rotation about the origin to rotate from z_1 to z_2.

7 If $z = 4 - 3i$, show on an Argand diagram:

a z b $-z$ c $\bar{z}$.

8 Represent the following on an Argand diagram if $z = 2 - i$.

a zi b z^2 c z^{-1}

9 a If $z = 3 + 2i$, state $\bar{z}$ and calculate z^{-1}.

b Plot z, $\bar{z}$, and z^{-1} on an Argand plane.

c What transformation plots z onto z^{-1}?

d What is the relationship between the origin and the points representing $\bar{z}$ and z^{-1}?

10 a If $z = 2 + 3i$, calculate iz, i^2z, i^3z and i^4z.

b Plot z, iz, i^2z, i^3z and i^4z on an Argand plane.

c State a transformation that will plot the point i^nz onto $i^{n+1}z$ for $n \in Z^+$.

11 a If $z = 1 + i$, calculate z^{-2}, z^{-1}, z^0, z^2, z^3 and z^4.

b Plot z^{-2}, z^{-1}, z^0, z, z^2, z^3 and z^4 on an Argand plane.

c State the rotation and the change in distance from the origin required to plot the point z^n onto z^{n+1} for $n \in Z$.

d State the rotation and the distance from the origin required to plot the point z^0 onto z^n for $n \in Z^+$.

12 State, using the results from question 11, the following powers of z in their simplest Cartesian form.

a z^5 b z^{-3} c z^{10} d z^{17} e z^{-13}

MASTER

13 If $z = x + yi$ then mod $z = |z| = \sqrt{x^2 + y^2}$, which is the distance of z from the origin. What does $|z| = 1$ represent?

14 Describe the graph of the following if $z = x + yi$.

a $|z - 1| = 1$ b $|z + 1| = 2$ c $|z - 2 + 3i| = 3$.

1.7 Factorising quadratic expressions and solving quadratic equations over the complex number field

studyon

Units 1 & 2

AOS 2

Topic 1

Concept 9

Factorising quadratic expressions over the complex number field
Concept summary
Practice questions

In the introduction to complex numbers, a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \in R\backslash\{0\}$, $b, c \in R$, was solved using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The expression under the square root sign is called the **discriminant** and is represented by Δ, where $\Delta = b^2 - 4ac$. The discriminant can be used to determine the nature of the solutions. It can also be used to determine possible methods for factorising a quadratic expression.

The following table gives the method for factorising a quadratic expression and the nature of the solutions for given values of Δ, where $a, b, c \in Q\backslash\{0\}$.

Values of the discriminant	Factorising methods	Nature of solution(s)
$\Delta = 0$	A perfect square. State the answer.	Two equal rational solutions
Δ is a perfect square.	Factorise over Q or complete the square.	Two rational solutions
$\Delta > 0$ and is not a perfect square.	Complete the square.	Two irrational solutions
$\Delta < 0$	Complete the square.	Two complex solutions

Factorising quadratic expressions over the complex number field

Factorising over R implies that all the coefficients must be real numbers and factorising over C implies that all the coefficients must be complex numbers. As factors over C are required in this section, the variable label will be z. In Worked example 22 below, the factors for the expressions in parts a and b are factors over both R and C, but the factors for the expression in part c are factors over C only. It is still correct to say that $2z^2 + 3$ does not factorise over R.

If $c = 0$ and $a, b \in R\backslash\{0\}$, then factorise $az^2 + bz$ by taking out the common factor $z(az + b)$.

If $b = 0$ and $a, c \in R\backslash\{0\}$, then factorise $az^2 + c$ using completing the squares to factorise.

WORKED EXAMPLE 22

Factorise each of the following quadratic expressions over C.

a $2z^2 + 6z$ b $2z^2 - 6$ c $2z^2 + 3$

THINK	WRITE
a Factorise $2z^2 + 6z$ by taking out the highest common factor.	a $2z^2 + 6z = 2z(z + 3)$
b 1 Factorise $2z^2 - 6$ by taking out the highest common factor.	b $2z^2 - 6 = 2(z^2 - 3)$
2 Factorise further using the difference of two squares.	$= 2(z - \sqrt{3})(z + \sqrt{3})$
c 1 Factorise $2z^2 + 3$ by taking out the common factor of 2.	c $2z^2 + 3 = 2\left(z^2 + \frac{3}{2}\right)$
2 Factorise further using the difference of two squares. Let $\frac{3}{2} = -\frac{3}{2}i^2$	$= 2\left(z^2 - \frac{3}{2}i^2\right)$ $= 2\left(z - \frac{\sqrt{3}}{\sqrt{2}}i\right)\left(z + \frac{\sqrt{3}}{\sqrt{2}}i\right)$
3 Rationalise the denominators by multiplying the relevant terms by $\frac{\sqrt{2}}{\sqrt{2}}$.	$= 2\left(z - \frac{\sqrt{6}}{2}i\right)\left(z + \frac{\sqrt{6}}{2}i\right)$

WORKED EXAMPLE 23

Factorise each of the following quadratic expressions over C.

a $z^2 - 6z + 9$ b $z^2 - 4z - 60$

c $2z^2 - 6z - 6$ d $-2z^2 - 3z - 5$

THINK	WRITE
a 1 Calculate the value of the discriminant to determine the nature of the factors.	a $\Delta = b^2 - 4ac$ $= (-6)^2 - 4(1)(9)$ $= 0$

2 Since $\Delta = 0$, the expression is a perfect square.

$z^2 - 6z + 9 = (z - 3)^2$

b 1 Calculate the value of the discriminant to determine the nature of the factors.

b $\Delta = b^2 - 4ac$
$= (-4)^2 - 4(1)(-60)$
$= 256$

2 Since $\Delta = 256$, which is a perfect square, the factors will be rational.

$z^2 - 4z - 60 = (z - 10)(z + 6)$

c 1 Calculate the value of the discriminant to determine the nature of the factors.

c $\Delta = b^2 - 4ac$
$= (-6)^2 - 4(2)(-6)$
$= 84$

2 Since $\Delta = 84$, which is not a perfect square but is positive, use completing the square to find two factors over R.

$$2z^2 - 6z - 6 = 2(z^2 - 3z - 3)$$
$$= 2\left(\left(z^2 - 3z + \left(\frac{3}{2}\right)^2\right) - 3 - \left(\frac{3}{2}\right)^2\right)$$
$$= 2\left(\left(z - \frac{3}{2}\right)^2 - \frac{21}{4}\right)$$
$$= 2\left(z - \frac{3}{2} - \frac{\sqrt{21}}{2}\right)\left(z - \frac{3}{2} + \frac{\sqrt{21}}{2}\right)$$

d 1 Calculate the value of the discriminant to determine the nature of the factors.

d $\Delta = b^2 - 4ac$
$= (-3)^2 - 4(-2)(-5)$
$= -31$

2 Since $\Delta = -31$, which is not a perfect square but is negative, use completing the square to find two factors over C.

$$-2z^2 - 3z - 5 = -2\left(z^2 + \frac{3}{2}z + \frac{5}{2}\right)$$
$$= -2\left(\left(z^2 + \frac{3}{2}z + \left(\frac{3}{4}\right)^2\right) + \frac{5}{2} - \frac{9}{16}\right)$$
$$= -2\left(\left(z + \frac{3}{4}\right)^2 + \frac{31}{16}\right)$$
$$= -2\left(\left(z + \frac{3}{4}\right)^2 - \frac{31}{16}i^2\right)$$
$$= -2\left(z + \frac{3}{4} - \frac{\sqrt{31}}{4}i\right)\left(z + \frac{3}{4} + \frac{\sqrt{31}}{4}i\right)$$

Solving quadratic equations over the complex number field

Two methods can be used to solve quadratic equations over the complex number field:

1. Factorise first and use the null factor property to state solutions.
2. Use the formula for the solution of a quadratic equation.

The null factor property states that if $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$. From Worked example 23 d,

study on

Units 1 & 2
AOS 2
Topic 1
Concept 10

Solving quadratic equations over the complex number field
Concept summary
Practice questions

$-2z^2 - 3z - 5 = -2\left(z + \frac{3}{4} - \frac{\sqrt{31}}{4}i\right)\left(z + \frac{3}{4} + \frac{\sqrt{31}}{4}i\right)$, so the solutions of $-2z^2 - 3z - 5 = 0$ are from $z + \frac{3}{4} - \frac{\sqrt{31}}{4}i = 0$ and $z + \frac{3}{4} + \frac{\sqrt{31}}{4}i = 0$.

The solutions are $z = \frac{-3}{4} \pm \frac{\sqrt{31}}{4}i$.

If $az^2 + bz + c = 0$, where $a \in C \setminus \{0\}$, $b, c \in C$, the formula for the solution of the quadratic equation is $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

WORKED EXAMPLE 24

Solve the following using the formula for the solution of a quadratic equation.

a $2z^2 + 4z + 5 = 0$ **b** $2iz^2 + 4z - 5i = 0$

THINK	WRITE
a 1 Use the quadratic formula $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve over C, where $a = 2$, $b = 4$, $c = 5$.	**a** $z = \frac{-4 \pm \sqrt{16 - 40}}{4}$ $= \frac{-4 \pm \sqrt{-24}}{4}$ $= \frac{-4 \pm \sqrt{4 \times -6}}{4}$ $= \frac{-4 \pm 2\sqrt{6i^2}}{4}$
2 Express the answer in the form $a + bi$.	$= -1 \pm \frac{\sqrt{6}}{2}i$
b 1 Use the quadratic formula $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve over C, where $a = 2i$, $b = 4$, $c = -5i$.	**b** $z = \frac{-4 \pm \sqrt{16 - 4 \times -10i^2}}{4i}$ $= \frac{-4 \pm \sqrt{16 - 40}}{4i}$ $= \frac{-4 \pm \sqrt{-24}}{4i}$ $= \frac{-4 \pm \sqrt{4 \times -6}}{4i} \times \frac{i}{i}$ $= \frac{(-4 \pm 2\sqrt{6}i)i}{-4}$
2 Express the answer in the form $a + bi$.	$= i \pm \frac{\sqrt{6}}{2}$ $= \frac{\sqrt{6}}{2} + i$ or $\frac{-\sqrt{6}}{2} + i$

EXERCISE 1.7 Factorising quadratic expressions and solving quadratic equations over the complex number field

PRACTISE

1 **WE22** Factorise the following quadratic expressions over C.

a $2z^2 - 6$ b $2z^2 - 3$ c $3z^2 + 6$ d $2z^2 + \frac{1}{2}$

2 Factorise the following quadratic expressions over C.

a $z^2 - 4z$ b $6z^2 - 2z$ c $2\sqrt{2}z^2 - \sqrt{2}z$ d $-4z^2 - 3z$

3 **WE23** Factorise the following quadratic expressions over C using the completion of the square method.

a $z^2 + 4z + 14$ b $z^2 + 10z + 16$
c $2z^2 + 5z - 3$ d $z^2 + z - 3$
e $z^2 + 8z + 16$

4 Factorise the following quadratic expressions over C using the completion of the square method.

a $z^2 + 2z + 3$ b $2z^2 - 5z + 2$
c $2z^2 + 8z + 8$ d $-2z^2 + 5z + 4$
e $-4z^2 + 4z - 1$

5 **WE24** Factorise the following quadratic expressions over C, and then solve the given quadratic equations.

a $3z^2 - 2 = 0$ b $2z^2 + 5 = 0$
c $2z^2 - 7z = 0$ d $z^2 - 6z + 5 = 0$
e $z^2 - 5z + 6 = 0$

6 Solve the following quadratic equations over C using the formula for the solution of a quadratic equation.

a $z^2 - 10z + 25 = 0$ b $z^2 - 10z + 5 = 0$
c $z^2 + 4z + 7 = 0$ d $2z^2 - 7z + 6 = 0$
e $3z^2 - 7z + 7 = 0$ f $-2z^2 + 4z - 6 = 0$

CONSOLIDATE

7 Factorise the following quadratic expressions over C without using the completion of the square method.

a $z^2 + 8z + 16$ b $2z^2 - 8z + 8$ c $2z^2 + 3z - 2$
d $z^2 + 2z - 3$ e $2z^2 - 2z - 24$ f $-12z^2 + 10z + 12$

8 Factorise the following quadratic expressions over C, and then solve the given quadratic equations.

a $2z^2 - 5z + 3 = 0$ b $z^2 - 4z + 2 = 0$
c $2z^2 + 5z + 4 = 0$ d $z^2 - 6z + 5 = 0$
e $-3z^2 - 2z - 1 = 0$

9 Expand the following.

a $(z - (2 + 3i))\,(z - (2 - 3i))$ b $(z - (2 + 3i))^2$
c $(z - 2 + 3i)\,(z - 3 - 2i)$

10 Solve the following quadratic equations over C, using the formula for the solution of a quadratic equation.

a $iz^2 - 6z + 5i = 0$ b $(2 + i)z^2 - iz - (2 - i) = 0$

11 Solve $-3iz^2 - (1 + i)z + 5i = 0$.

12 Using the smallest set from Q, I and C, the solutions of $2z^2 - 5z + 6 = 0$ and $5z^2 - 11z + 5 = 0$, respectively, belong to the sets:

A C, C **B** C, Q **C** C, I **D** I, I **E** I, Q

13 The factors of $z^2 + 6z + 11$ and $2z^2 - 4z + 3$, respectively, are:

A $(z + 3 - \sqrt{2}i)(z + 3 + \sqrt{2}i), 2\left(z - 1 - \frac{\sqrt{2}}{2}i\right)\left(z - 1 + \frac{\sqrt{2}}{2}i\right)$

B $(z + 3 - \sqrt{2}i)(z + 3 + \sqrt{2}i), \left(z - 1 - \frac{\sqrt{2}}{2}i\right)\left(z - 1 + \frac{\sqrt{2}}{2}i\right)$

C $(z - 3 - \sqrt{2}i)(z - 3 + \sqrt{2}i), 2\left(z - 1 - \frac{\sqrt{2}}{2}i\right)\left(z - 1 + \frac{\sqrt{2}}{2}i\right)$

D $(z + 3 - \sqrt{2}i)(z + 3 + \sqrt{2}i), \left(z + 1 - \frac{\sqrt{2}}{2}i\right)\left(z + 1 + \frac{\sqrt{2}}{2}i\right)$

E $(z - 3 - \sqrt{2}i)(z - 3 + \sqrt{2}i), 2\left(z + 1 - \frac{\sqrt{2}}{2}i\right)\left(z + 1 + \frac{\sqrt{2}}{2}i\right)$

14 $2 - i$ is a solution of $x^2 - 4x + k = 0$. What is the value of k?

A 3 **B** 5 **C** $\sqrt{5}$ **D** -3 **E** $2 + i$

15 The solutions to the quadratic equation: $x^2 - 2x + 3 = 0$ are:

A $x = 3$ or $x = -1$

B $x = 2 + \sqrt{2}i$ or $x = 2 - \sqrt{2}i$

C $x = 1 + \sqrt{2}i$ or $x = 1 - \sqrt{2}i$

D $x = 1 + 2i$ or $x = 1 - 2i$

E $x = 3i$ or $x = -i$

16 Solve $x^2 + 4x + 7 = 0$.

MASTER

17 If $\frac{10x^2 - 2x + 4}{x^3 + x} = \frac{A(Bx^2 + Cx + D)}{x(x - i)(x + i)}$, find A, B, C and D.

18 Solve $x^4 + 13x^2 + 36 = 0$.

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1.8 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

studyon

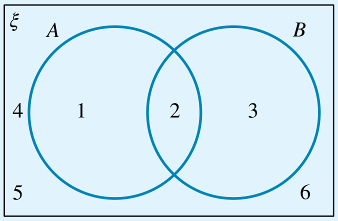
Units 1 & 2 — **Number systems: real and complex**

Sit topic test

1 Answers

EXERCISE 1.2

1



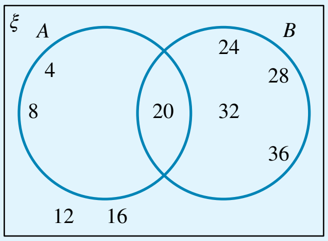
a {3, 4, 5, 6}

b {1, 2, 3}

c {2}

d {1}

2



a {4, 8, 12, 16}

b {4, 8, 12, 16, 20}

c {4, 8, 12, 16, 24, 28, 32, 36}

d {4, 8, 12, 16, 24, 28, 32, 36}

3 Answers will vary.

4 Answers will vary.

5 a 3.328 125 **b** $0.34\dot{0}\dot{9}$

6 a $1.41\dot{6}$ **b** $1.\dot{2}30\,76\dot{9}$

7 $-2 \Rightarrow$ Integer: Z, Q, R

$\frac{16}{8} \Rightarrow$ Natural number: N, Z, Q, R

$\frac{21}{16} \Rightarrow$ Rational: Q, R

$-3\frac{2}{7} \Rightarrow$ Rational: Q, R

$6\sqrt{3} \Rightarrow$ Irrational: I, R

$16^{\frac{1}{4}} \Rightarrow$ Natural number: N, Z, Q, R

8 $5^{\frac{1}{5}} \Rightarrow$ Irrational: I, R

$\pi \Rightarrow$ Irrational: I, R

$21.72 \Rightarrow$ Rational: Q, R

$2.\dot{5}6\dot{7} \Rightarrow$ Rational: Q, R

$4.135\,218\,976 \Rightarrow$ Irrational: I, R

$4.232\,332\,333 \Rightarrow$ Irrational: I, R

9 a $\frac{37}{90}$ **b** $\frac{2357}{1110}$

10 a $\frac{8}{33}$ **b** $\frac{374}{333}$ **c** $\frac{61}{495}$ **d** $\frac{3517}{1665}$

11 a True

b True

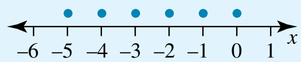
12 a False, e.g. $\sqrt{2} - \sqrt{2} = 0$.

b True

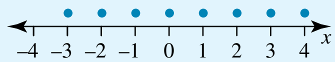
13 a 6×10^{28} **b** 4×10^{14} **c** 6×10^{40}

14 a 6×10^{7} **b** 2×10^{27} **c** 4×10^{-3}

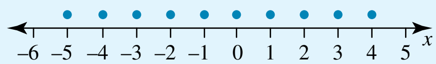
15 a $\{x : x \in Z, -6 < x < 1\}$



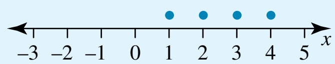
b $\{x : x \in Z, -3 \le x \le 4\}$



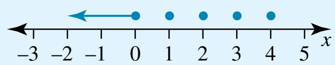
c $\{x : x \in Z, -6 < x \le 4\}$



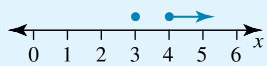
d $\{x : x \in Z, 0 < x < 5\}$



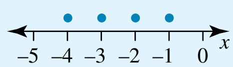
16 a $\{x : x \in Z, x < 5\}$



b $\{x : x \in Z, x > 2\}$



c $\{x : x \in Z, -5 < x < 0\}$



17 a $\{x : x \in Q, x > 5\}$

b $\{x : x \in Q, 5 < x \le 20\}$

c $\{x : x \in Q^+, x < 20\}$

d $\{x : x \in Z, 5 < x < 20\}\backslash\{8, 9\}$

e $\{x : x \in Z^+, x < 100\}\backslash\{40 < x < 50\}$

18 a $\{x : 2 \le x < 5\}$

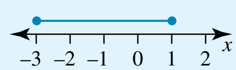
b $\{x : 3 < x < 5\}$

c $\{x : x < 3\} \cup \{x : x > 7\}$

d $\{x : x \in R^+, x < 3\} \cup \{x : x \in R, x > 7\}$

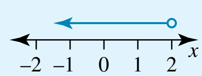
19 a $\{x : -3 \le x \le 1\}$

Interval notation $[-3, 1]$



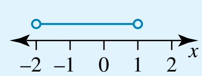
b $\{x : x < 2\}$

Interval notation $(-\infty, 2)$



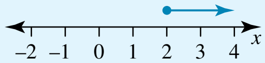
c $\{x : -2 < x < 1\}$

Interval notation $(-2, 1)$



d $\{x : x \geq 2\}$

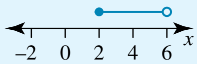
Interval notation $[2, \infty)$



20 a $\{x : 2 \leq x < 5\} \cup \{x : 4 \leq x < 6\}$

Interval notation

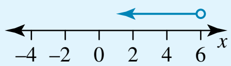
$[2, 5) \cup [4, 6) = [2, 6)$



b $\{x : x < 5\} \cup \{x : 4 \leq x < 6\}$

Interval notation

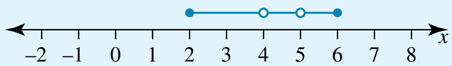
$(-\infty, 5) \cup [4, 6) = (-\infty, 6)$



c $\{x : 2 \leq x < 5\} \cup \{x : 4 < x \leq 6\}$

Interval notation

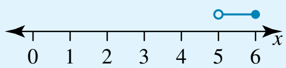
$[2, 5) \cup (4, 6] = [2, 6]\backslash(4, 5)$



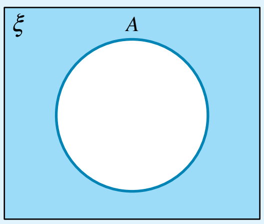
d $\{x : x > 5\} \cap \{x : 4 < x \leq 6\}$

Interval notation

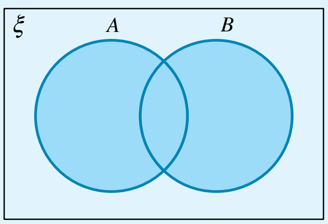
$(5, \infty) \cap (4, 6] = (5, 6]$



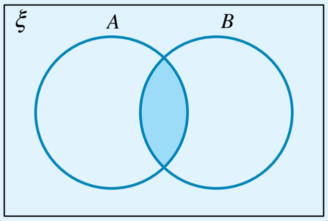
21 a A'



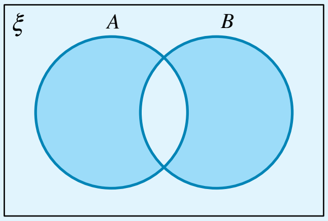
b $A \cup B$



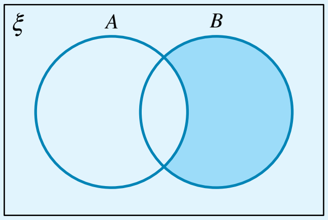
c $A \cap B$



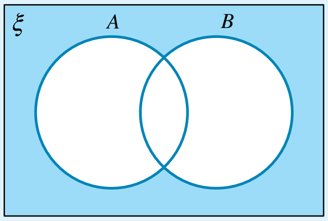
d $(A \cup B)/(A \cap B)$



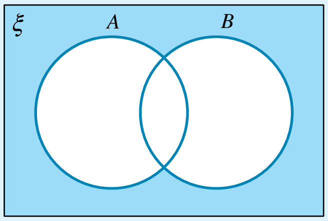
e $A' \cap B$



f $A' \cap B'$



g $(A \cup B)'$



22 See the table at the foot of page.*

23 a –11 **b** –3

24 B **25** E **26** D

*22

Number	3 sig. fig.	4 sig. fig.	2 d.p.	3 d.p.
1267.1066	1270	1267	1267.11	1267.107
7.6699	7.67	7.670	7.67	7.670
8.000 56	8.00	8.001	8.00	8.001
0.999 87	1.00	1.000	1.00	1.000
0.076 768	0.0768	0.076 77	0.08	0.077
0.000 174 95	0.000 175	0.000 175 0	0.00	0.000

27 D 28 B 29 E 30 C

31 a 5.35×10^{30} b 7.64×10^{19}
c 1.23×10^{7} d 3.60×10^{-7}

32 a i 40% ii 23% iii 16%
b The percentage of prime numbers between 10^n and 10^{n-1} decreases as $n \to \infty$.

EXERCISE 1.3

1 a $2\sqrt{6}$ b $2\sqrt{14}$ c $5\sqrt{5}$
d $7\sqrt{2}$ e $4\sqrt{3}$

2 a $10\sqrt{3}$ b $28\sqrt{5}$ c $2\sqrt{2}$
d $\frac{6\sqrt{2}}{5}$ e $\frac{-3\sqrt{2}}{2}$

3 a $2\sqrt{2} - 2\sqrt{3}$ b $\sqrt{7} - 4$
c $-2(3\sqrt{3} + 2\sqrt{2})$ d $3(\sqrt{2} + 2\sqrt{3})$

4 a $7\sqrt{5} - \sqrt{2}$ b $2(9\sqrt{3} + 10\sqrt{2})$
c $\frac{9\sqrt{3}}{8} - \frac{13\sqrt{2}}{8}$ d $\frac{118\sqrt{3}}{15} - \frac{49\sqrt{2}}{10}$

5 a $3\sqrt{10}$ b $10\sqrt{21}$

6 a $84\sqrt{2}$ b $\frac{5\sqrt{3}}{6}$

7 a $\sqrt{15} - \sqrt{6}$ b $18 + 2\sqrt{6}$
c $5 - \sqrt{10} - \sqrt{15} + \sqrt{6}$ d $7\sqrt{6} - 18$

8 a $2(6 + \sqrt{35})$ b $2(105 + 36\sqrt{6})$
c 17 d 25

9 a $\sqrt{6}$ b $\frac{4\sqrt{2}}{3}$ c $\frac{\sqrt{15}}{3}$
d $\frac{4\sqrt{15}}{35}$ e $\frac{2\sqrt{6}}{9}$ f $\frac{\sqrt{5} + \sqrt{3}}{2}$

10 a $3 + \sqrt{6}$ b $2\sqrt{10} - 6$ c $\frac{14 + 5\sqrt{3}}{11}$
d $5 - 2\sqrt{6}$ e $\frac{-4 - 3\sqrt{15}}{7}$ f $\frac{6 + 2\sqrt{3}}{3}$

11 a $-4\sqrt{2}$
b $\frac{69\sqrt{2} + 20\sqrt{3}}{114}$
c $\frac{195\sqrt{10} + 78\sqrt{15} - 12\sqrt{2} + 18\sqrt{3}}{114}$
d $\frac{1920 - 338\sqrt{15}}{231}$
e $\frac{35 - 7\sqrt{6}}{15}$
f $\frac{-47 - 12\sqrt{15}}{3}$

12 a $\frac{26 - 45\sqrt{2}}{14}$
b $\frac{30 - 39\sqrt{2}}{14}$
c $-3(6 + 5\sqrt{2})$
d $\frac{22 - 12\sqrt{2}}{7}$
e 0
f $31 - 18\sqrt{2}$
g $x = 2 - 3\sqrt{2}$ is a solution to $x^2 - 4x - 14$. However, since there was no remainder, it is not a solution to $2x^2 - 2x - 9$.

13 Check with your teacher.

14 Check with your teacher.

15 B 16 A 17 A
18 D 19 A 20 E

21 a $2\sqrt{37}$ cm b $18\pi\sqrt{37}$ cm^3
c $54\pi\sqrt{3}$ cm^3 d $18\pi(\sqrt{37} + 3\sqrt{3})$ cm^3

22 a $360\sqrt{10}$ cm^3 b $\frac{3\sqrt{10\pi}}{\pi}$ cm c $\frac{2\sqrt{15\pi}}{\pi}$ cm

EXERCISE 1.4

1 a $x = 1$ $y = 4$ b $x = 1$ $y = -5$
c $x = -3$ $y = -1$ d $x = 1$ $y = 3$

2 a $x = \frac{3}{4}$ $y = \frac{2}{7}$ b $x = 2$ $y = \frac{1}{2}$ c $x = \frac{6}{5}$ $y = \frac{12}{5}$

3 a $5 + 3i$ b $13 + 10i$
c $\sqrt{5}$ d 25

4 a i b $7\sqrt{2} - 10\sqrt{2}i$ c $10 + 3\sqrt{13}$

5 a $4i$ b $\sqrt{7}i$ c $2 + 2\sqrt{5}i$

6 a $\sqrt{10} + \sqrt{10}i$
b $\frac{1 - 2\sqrt{7}i}{2}$

7 a $-5i$ b $7i + 2$ c $99i$

8 a $\text{Re}(3 + 4i) = 3$
$\text{Im}(3 + 4i) = 4$
b $\text{Re}(-2 + \sqrt{2}i) = -2$
$\text{Im}(-2 + \sqrt{2}i) = \sqrt{2}$
c $\text{Re}(z) = \sqrt{2} - 1$
$\text{Im}(z) = \sqrt{2} + 1$

9 a $\text{Re}(z) = 2\sqrt{2}$
$\text{Im}(z) = -2\sqrt{10}$
b $\text{Re}(z) = -6$
$\text{Im}(z) = 0$
c $\text{Re}(z) = 0$
$\text{Im}(z) = 13$

10 a 14 b $\sqrt{3} - 3\sqrt{2}$
c 0 d $-2(\sqrt{2} + \sqrt{3})$

11 D 12 B 13 A

14 $5 - 8i$

15 Multiplied i by i instead of subtracting like terms

16 $10(2 - i)$ 17 -12

EXERCISE 1.5

1 a $-6 + 4i$ **b** $5 - i$ **c** $-5 + 5i$
d $-5 - 12i$ **e** 85

2 a $-8 - 6i$ **b** $24 + 6i$ **c** $a^2 + b^2$

3 a $3 - 2i$ **b** $-4 - 3i$ **c** $\sqrt{2} + 2i$ **d** $8i$

4 a $4 + 3i$ **b** $24 - 7i$ **c** $24 + 7i$
d $7 + 24i$ **e** $-8 + 6i$ **f** $28i$

5 a $\frac{4}{5} + \frac{3}{5}i$ **b** $\frac{-7}{25} + \frac{24}{25}i$ **c** $\frac{4}{5} + \frac{3}{5}i$
d $\frac{11}{5} - \frac{2}{5}i$ **e** $\frac{3}{13} - \frac{2}{13}i$ **f** $\frac{5}{169} - \frac{12}{169}i$

6 a $\frac{4}{25} + \frac{3}{25}i$ **b** $-i$ **c** $\frac{24}{25} + \frac{7i}{25}$
d $\frac{527}{625} - \frac{326}{625}i$ **e** $\frac{104}{25} + \frac{78}{25}i$ **f** $\frac{14}{25}$

7 a Check with your teacher.
b Check with your teacher.
c Check with your teacher.

8 a $x = \frac{12}{5}$
$y = \frac{9}{5}$
b $\frac{y - 2x}{5} = 1$
$y = 3$
$x = -1$

9 a $z = \frac{1}{5} - \frac{2}{5}i$ **b** $z = -i$

10 a $z^{-1} = \frac{\bar{z}}{41}$
b $z^{-1} = \frac{\bar{z}}{a^2 + b^2}$ or $z^{-1} = \frac{\bar{z}}{z\bar{z}}$

11 B **12** A **13** E **14** B

15 a $\frac{4}{13}$ **b** $\frac{63}{25} - \frac{16}{25}i$
c $\frac{16}{7} - \frac{8\sqrt{3}}{7}i$

16 a 1 **b** $-\frac{9}{13} - \frac{20}{13}i$

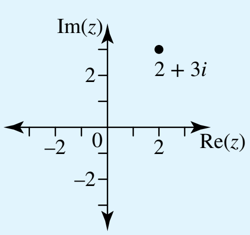
17 $0 + 5i$

18 $\frac{11}{10} + \frac{13}{10}i$

EXERCISE 1.6

1 a $-i$ **b** i **c** 1 **d** i

2 a -64 **b** 256 **c** $-512i$ **d** $\frac{-i}{512}$

3 a


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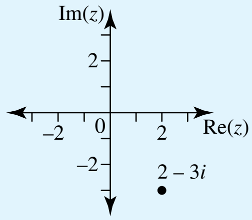
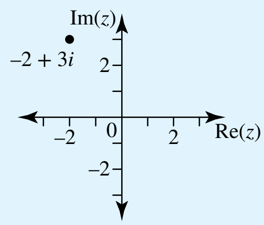


c


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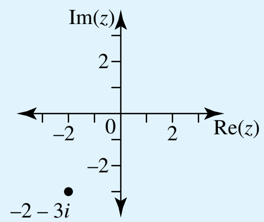

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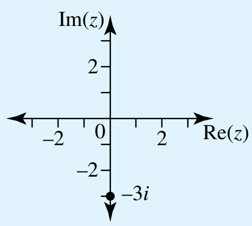
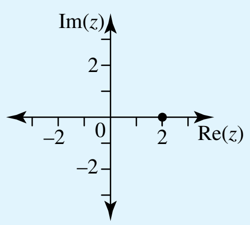
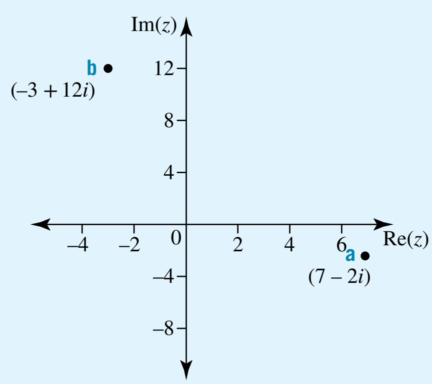


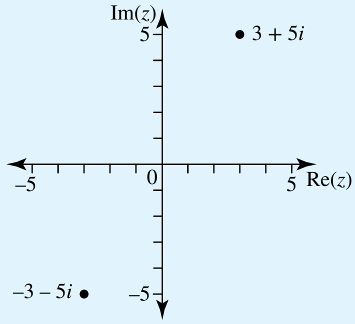
f


4 A $1 + i$ **B** $-3 - 2i$ **C** $1 - 3i$
D $3i$ **E** 3 **F** $-2 + i$

5 a $7 - 2i$ **b** $-3 + 12i$

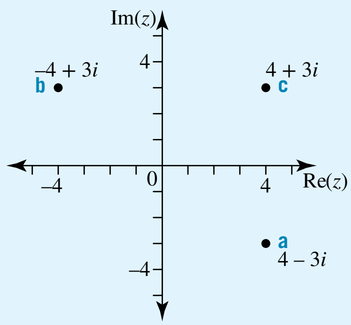


6 a



b A rotation of 180° about the origin.

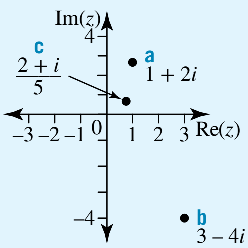
7 a $z = 4 - 3i$



b $-z = -4 + 3i$

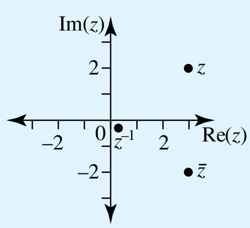
c $\bar{z} = 4 + 3i$

8



9 a $z^{-1} = \frac{3}{13} - \frac{2i}{13}, \bar{z} = 3 - 2i$

b



c
- Dilated by a factor of $\frac{1}{13}$ from the real and imaginary axes
- Reflected about the real axis

d z^{-1} is $\frac{1}{13}$ of the distance of $\bar{z}$ from the origin.

10 a $z = 2 + 3i$

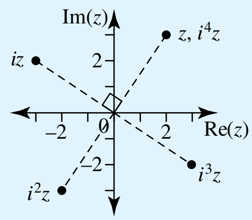
$iz = i(2 + 3i) = -3 + 2i$

$i^2 z = i(-3 + 2i) = -2 - 3i$

$i^3 z = i(-2 - 3i) = 3 - 2i$

$i^4 z = i(3 - 2i) = 2 + 3i$

b



c The distance from the origin remains the same with a 90° anticlockwise rotation.

11 a $z^{-2} = \frac{1}{(1+i)^2} = \frac{1}{1 + 2i - 1} = \frac{1}{2i} = \frac{2i}{-4} = \frac{-1}{2}i$

$z^{-1} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{1+1} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$

$z^0 = (1 + i)^0 = 1$

$z^1 = 1 + i$

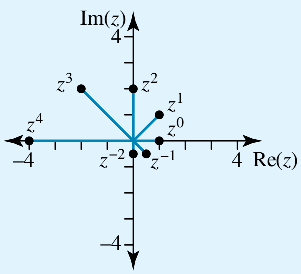
$z^2 = (1 + i)^2 = 1 + 2i - 1 = 2i$

$z^3 = (1 + i)^3 = 2i(1 + i) = 2i - 2 = -2 + 2i$

$z^4 = (1 + i)^4 = (-2 + 2i)(1 + i)$

$= -2 - 2i + 2i - 2 = -4$

b



c Plotting z^n compared to z^{n+1}, where $n \in Z$, undergoes a rotation 45° anticlockwise and its distance from the origin is increased by a factor of $\sqrt{2}$.

d
- Rotated $n \times 45°$ or $\frac{n\pi}{4}$ anticlockwise
- Distance from the origin multiplied by a factor of $(\sqrt{2})^n$

12 a $-4 - 4i$

b $\frac{-1}{4} - \frac{1}{4}i$

c $32i$

d $256 + 256i$

e $\frac{-1}{128} + \frac{1}{128}i$

13 A circle with centre $0 + 0i$ and a radius of 1

14 a A circle with centre $1 + 0i$ and a radius of 1

b A circle with centre $-1 + 0i$ and a radius of 2

c A circle with centre $2 - 3i$ and a radius of 3

EXERCISE 1.7

1 a $2(z - \sqrt{3})(z + \sqrt{3})$

b $(\sqrt{2}z - \sqrt{3})(\sqrt{2}z + \sqrt{3})$

c $3(z-\sqrt{2}i)(z+\sqrt{2}i)$

d $\frac{1}{2}(2z-i)(2z+i)$

2 a $z(z-4)$ b $2z(3z-1)$

c $\sqrt{2}z(2z-1)$ d $-z(4z+3)$

3 a $(z+2-\sqrt{10}i)\ (z+2+\sqrt{10}i)$

b $(z+2)(z+8)$

c $2\left(z-\frac{1}{2}\right)(z+3)$

d $\left(z+\frac{1}{2}-\frac{\sqrt{13}}{2}\right)\left(z+\frac{1}{2}+\frac{\sqrt{13}}{2}\right)$

e $(z+4)^2$

4 a $(z+1-\sqrt{2}i)\ (z+1+\sqrt{2}i)$

b $2(z-2)\left(z-\frac{1}{2}\right)$

c $2(z+2)^2$

d $-2\left(z-\frac{5}{4}-\frac{\sqrt{57}}{4}\right)\left(z-\frac{5}{4}+\frac{\sqrt{57}}{4}\right)$

e $-4\left(z-\frac{1}{2}\right)^2$

5 a $(\sqrt{3}z-\sqrt{2})-(\sqrt{3}z+\sqrt{2})=0$

$z=\pm\frac{\sqrt{6}}{2}$

b $(\sqrt{2}z-\sqrt{5}i)(\sqrt{2}z+\sqrt{5}i)=0$

$z=\pm\frac{\sqrt{10}i}{2}$

c $z(2z-7)=0$

$z=0, z=\frac{7}{2}$

d $(z-5)(z-1)=0$

$z=5, z=1$

e $(z-3)(z-2)=0$

$z=3, z=2$

6 a $z=5$

b $z=5\pm 2\sqrt{5}$

c $z=-2\ \pm\ i\sqrt{3}$

d $z=2, z=\frac{3}{2}$

e $z=\frac{7\ \pm\ i\sqrt{35}}{6}$

f $z=1\pm\ i\sqrt{2}$

7 a $(z+4)^2$ b $2(z-2)(z-2)$

c $(2z-1)(z+2)$ d $(z+3)(z-1)$

e $2(z+3)(z-4)$ f $-2(2z-3)(3z+2)$

8 a $(2z-3)(z-1)=0$

$z=\frac{3}{2}, z=1$

b $(z-2-\sqrt{2})\ (z-2+\sqrt{2})=0$

$z=2\pm\sqrt{2}$

c $2\left(z+\frac{5}{4}-\frac{\sqrt{7}i}{4}\right)\left(z+\frac{5}{4}+\frac{\sqrt{7}i}{4}\right)=0$

$z=\frac{-5}{4}\pm\frac{\sqrt{7}i}{4}$

d $(z-5)(z-1)=0$

$z=5, z=1$

e $-3\left(z+\frac{1}{3}-\frac{\sqrt{2}i}{3}\right)\left(z+\frac{1}{3}+\frac{\sqrt{2}i}{3}\right)=0$

$z=\frac{-1}{3}\pm\frac{\sqrt{2}i}{3}$

9 a $z^2-4z+13$

b z^2+6i-9

c $z^2-5z+iz-5i+12$

10 a $z=-3i\pm\ i\sqrt{14}$

b $z=\frac{(2-i)(i\pm\sqrt{19})}{10}$

11 $z=\frac{(i-1)\pm i\sqrt{2(i-30)}}{6}$

12 A 13 A 14 B 15 C

16 $x=-2\pm\sqrt{3i}$

17 $A=2, B=5$

$C=-1, D=2$

18 $x=\pm 3i$ or $x=\pm 2i$

2

Algebra and logic

2.1 Kick off with CAS

The Tower of Hanoi

The Tower of Hanoi puzzle is thought to have originated in China. It consists of three rods with a number of graduated discs stacked on one rod. The challenge is to determine the smallest number of moves to stack all the discs in order onto another rod. At no stage can a larger disc rest on a smaller disc.

1 If there are 3 discs on a rod, determine the smallest number of moves to arrange them in order on another rod.

2 If there are 4 discs on a rod, determine the smallest number of moves to arrange them in order on another rod.

3 Using CAS technology, construct and complete a spreadsheet similar to this example.

Number of discs	Minimum number of moves
1	1
2	3
3	
4	
5	

4 Using CAS technology, graph the scatterplot of *Minimum number of moves* against *Number of discs*.

5 Describe the shape of the graph.

6 Determine the general rule stating the *Minimum number of moves* in terms of the *Number of discs*.

7 Prove this rule by mathematical induction.

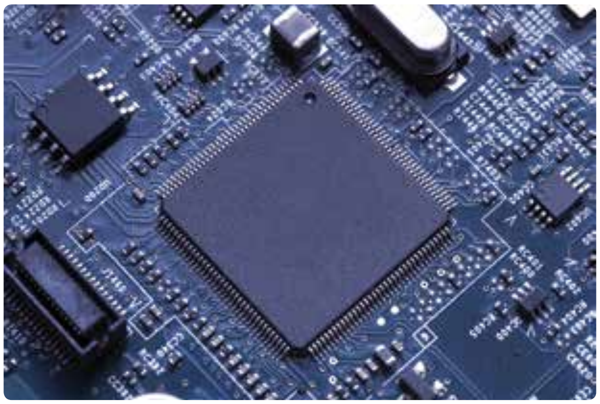
Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

2.2 Statements (propositions), connectives and truth tables

studyon

Units 1 & 2
AOS 1
Topic 1
Concept 1

Statement (propositions), connectives and truth tables
Concept summary
Practice questions



Older than calculus (17th century CE), algebra (9th century CE) and even geometry (300 years BCE) is the study of logic. Some of the material described in this section was developed by Aristotle, one of the most famous of the ancient Greek philosophers, yet it is still used today by people as diverse as mathematicians, lawyers, engineers and computer scientists. All of our modern digital technology owes its birth to the application of the principles of logic; every meaningful computer program ever written has relied on the principles you will learn in this topic.

Furthermore, logic can be seen as the study of **argument**. You will be able to analyse logically the arguments of teachers, politicians and advertisers to determine if they should convince you of their ideas, programs and products.

Statements

A **statement** is a sentence that is either true or false. For example, 'This book is about mathematics' is a true (T) statement, whereas 'The capital of Australia is Perth' is a false (F) statement.

Some sentences are not statements at all. 'Go to the store' is an instruction; 'How old are you?' is a question; 'See you later!' is an exclamation, and 'You should see the latest Spielberg movie' is a suggestion. 'The bus is the best way to get to work' is an opinion.

To determine whether a sentence is a statement, put the expression 'It is true that . . .' (or 'It is false that . . .') at the front of the sentence. If it still makes sense, then it is a statement.

Beware of some 'near-statements' such as 'I am tall' or 'She is rich' because these are relative sentences; they require more information to be complete. They can be turned into statements by saying 'I am tall compared to Mary' or 'She is rich compared to Peter'.

In some textbooks, statements are called **propositions**.

WORKED EXAMPLE 1

Classify the following sentences as either statements, instructions, suggestions, questions, opinions, exclamations or 'near-statements'. If they are statements, indicate whether they are true (T), false (F) or indeterminate without further information (T/F).

a **Germany won World War II.**

b **Would you like to read my new book?**

c **The most money that Mary can earn in one day is $400.**

d When it rains, I wear rubber boots.

e Hello!

f You will need to purchase a calculator in order to survive Year 11 Mathematics.

g Do not run in the hallways.

h You should read this book.

i I am short.

THINK	WRITE
a Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	a This is a (false) statement.
b Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	b This is a question.
c Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	c This is a statement. We cannot at this time determine if it is true or false without further information.
d Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	d This is a (presumably true) statement.
e Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	e This is an exclamation.
f Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	f This is a (true) statement.
g Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	g This is an instruction.
h Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	h This is a suggestion.
i Put the phrase 'It is true that . . .' in front of the sentence. If the new sentence makes sense, it is classed as a statement.	i This is a near-statement because it requires additional information to be complete. It can be turned into a statement by saying 'I am shorter than Karen'.

Connectives and truth tables

Two (or more) statements can be combined into compound statements using a connective. For example, the statement 'The book is new and about mathematics' is a compound of the single statements 'The book is new' and 'The book is about mathematics'.

Notice the **connective**, 'and', that is used to join the two statements. Two main connectives — 'and' and 'or' — are used in compound sentences. Other connectives are 'not', 'if . . . then . . .' and 'if . . . and only if . . .'.

The truth of a compound statement is determined by the truth of the separate single statements. These separate single statements are sometimes called atomic sentences.

Considering this example, there are four cases:

Case 1: 'The book is new' is true. 'The book is about mathematics' is true.
Case 2: 'The book is new' is true. 'The book is about mathematics' is false.
Case 3: 'The book is new' is false. 'The book is about mathematics' is true.
Case 4: 'The book is new' is false. 'The book is about mathematics' is false.

This list can be summarised using a truth table.

Let p = 'The book is new' and

q = 'The book is about mathematics'.

p	q	p and q
T	T	
T	F	
F	T	
F	F	

What about the third column? This represents the **truth value** of the compound statement 'p and q'. To determine this truth value we need to examine the logical definition of the connective 'and'. For the compound statement to be true, both single statements must be true. If either is false, then the whole statement is false. Therefore, we can complete the truth table for 'and' (using the common symbol $\wedge$ to represent 'and').

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Similarly, the truth table for 'or', using the symbol $\vee$, is shown. The implication here is that it takes only one (or both) of the statements to be true for a statement such as 'Mary went to the store or the library' to be true. If she went to the store, then certainly she went to the store or the library. Similarly, if she went to the library in this example, the statement would be true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

There are some compound statements where it is not possible for both statements to be true at the same time. For example: 'John is 15 or 16 years old'. Clearly, in this case John cannot be both 15 and 16. This is an example of 'exclusive or'.

Also be careful not to confuse the logical use of 'and' with the common English usage. For example, the sentence 'Boys and girls are allowed in the swimming pool after 6.00 pm' is made up of the compound sentences 'Boys are allowed . . .' and 'Girls are allowed . . .'. In reality, what is being said is that either boys *or* girls or both are allowed, so logically the sentence should be 'Both boys and girls are allowed in the swimming pool after 6.00 pm'.

In some textbooks 'and' is called the **conjunction** and 'or' is called the **disjunction**.

WORKED EXAMPLE 2 **Determine the truth table for the compound statement:**

'The suspect wore black shoes or was a female wearing a skirt'.

THINK	WRITE
1 Identify and label the individual statements.	p = 'The suspect wore black shoes.' q = 'The suspect was female.' r = 'The suspect wore a skirt.'
2 Form a compound statement. Clearly p is separate from q and r.	p or (q and r) $p \vee (q \wedge r)$ *Note:* Use brackets to indicate the separation.
3 Create a truth table. Because there are three statements and each can have two values (T or F), there are $2 \times 2 \times 2 = 8$ rows in the table. The $(q \wedge r)$ column is completed by looking at the q and r columns. 4 The last column is completed by looking at just the p column and at the $(q \wedge r)$ column.	

p	q	r	$(q \wedge r)$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

As can be observed in the last column above, p 'dominates' the table. Regardless of the truth of q and r, the entire statement is true if p is true (rows 1–4). Otherwise, if p is false then both q and r must be true (row 5).

Negation

Another connective is the negation, or 'not', which is denoted by the symbol ~. This is merely the opposite of the original statement. For example, if p = 'It is raining', then $\sim p$ = 'It is not raining'.

Be careful when negating English sentences. For example, the negation of 'I am over 21' isn't 'I am under 21', but 'I am not over 21'. Can you see the difference?

WORKED EXAMPLE 3 **Complete the truth table for the compound statement $p \vee \sim p$.**

THINK

Set up a truth table. Since there is only one statement here (p), we need only two rows, either p or not p.

WRITE

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Note: The compound statement in Worked example 3 is always true! An English sentence equivalent to this statement could be 'I will be there on Monday or I will not be there on Monday'.

Equivalent statements

Two statements are equivalent if their truth tables are identical. Each row of the truth tables must match. If there is even one difference, then the statements are not equivalent. The symbol ⇔ is used to indicate equivalence, as in $p \Leftrightarrow q$. This is read as 'p is true if and only if q is true'.

WORKED EXAMPLE 4

By completing truth tables, show that $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$.

THINK

1. Set up a truth table. Because there are two statements, we need $2 \times 2 = 4$ rows.
2. Complete the $\sim p$ and $\sim q$ columns by negating p and q separately.
3. Complete the $(p \wedge q)$ column.
4. Negate the $(p \wedge q)$ column.
5. Find $(\sim p \vee \sim q)$ using columns 3 and 4.
6. Observe that the final two columns are equal in every row.

WRITE

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Note that the equivalence operators, $p \Leftrightarrow q$, have a truth table of their own, as shown at right.

This clearly demonstrates that $p \Leftrightarrow q$ is true when the truth value of p equals the truth value of q.

That is, if both p and q are true or
if both p and q are false.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXERCISE 2.2 Statements (propositions), connectives and truth tables

PRACTISE

1 WE1 Classify the following sentences as statements (propositions), instructions, suggestions, exclamations, opinions or 'near-statements'. If they are statements, indicate whether they are true (T), false (F) or indeterminate without further information (T/F).

a That was the best Hollywood movie of 2015.
b That movie won the most Oscar nominations in 2015.
c When the power fails, candles are a good source of light and heat.
d Why did you use that candle?

2 Classify the following sentences as statements (propositions), instructions, suggestions, questions, opinions, exclamations or 'near-statements'. If they are statements, indicate whether they are true (T), false (F) or indeterminate without further information (T/F).

a Please go to the store before it closes.
b The store closes at 6.00 pm.
c A dingo is considered to be a native Australian animal.
d Mary is tall for her age.

3 WE2 Determine the truth table for the following compound statements.

a The Sydney flight was on time and the Perth flight was fully booked.
b John, Zia and David passed Specialist Mathematics.

4 Determine the truth table for the following compound statements.

a Either Alice and Renzo, or Carla will have to do the dishes. (Note use of the comma.)
b The committee requires two new members. One must be a female; the other must be either a student or a professor.

5 WE3 Complete the truth tables for the following compound statements.

a $p \wedge \sim q$ b $\sim p \wedge \sim q$ c $(p \wedge q) \wedge r$

6 Complete the truth tables for the following compound statements.

a $p \vee \sim q$ b $\sim p \vee \sim q$ c $(p \vee q) \vee r$

7 WE4 By completing truth tables, show that $\sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$.

8 By completing truth tables, show that $(p \vee q) \vee \sim p \Leftrightarrow (p \vee \sim p)$.

CONSOLIDATE

9 Break up the following compound statements into individual single statements.

a The car has four seats and air conditioning.
b The Departments of Finance and Defence were both over budget in 2015.
c Bob, Carol, Ted and Alice went to the hotel.
d To be a best-seller a novel must be interesting and relevant to the reader.
e Either Sam or Nancy will win the trophy.
f You can choose from ice-cream or fruit for dessert. We have vanilla or strawberry ice-cream.
g There are some statements that cannot be proved to be true or false.
h Most of my friends studied Mathematics, Physics, Engineering, Law or Arts.

10 Convert each of the following pairs of simple sentences into a compound sentence. Be sure to use 'and' and 'or' carefully.

a In a recent poll, 45% of people preferred jazz. In a recent poll, 35% preferred classical music.
b The book you want is in row 3. The book you want is in row 4.

c The weather is cold. The weather is cloudy.
d Many people read novels. Many people read history.
e John rode his bicycle to school. Mary rode her bicycle to school.
f Two is an even number. Two is a prime number.

11 For the compound statement $p \wedge q$ (p and q), the number of different ways for this statement to be true is:

A 0 B 1 C 2 D 3 E 4

12 For the compound statement $p \vee q$ (p or q), the number of different ways for this statement to be true is:

A 0 B 1 C 2 D 3 E 4

13 As you saw in Worked example 4, if there is a compound statement with two single statements, p and q, then there are $2 \times 2 = 4$ rows in the truth table. List all the different rows for compound statements made up of:

a three single statements
b four single statements
c five single statements.

You should be able to develop a pattern of completing the Ts and Fs in a logical sequence.

14 Find the truth table for the compound statement $p \wedge \sim p$.

15 Let p = 'It is raining' and q = 'I bring my umbrella'. Write a sentence for each of the following compound statements.

a $p \wedge q$ b $p \vee q$ c $\sim p \wedge q$

16 Let p = 'Peter likes football' and q = 'Quentin likes football'. Write a sentence for each of the following compound statements.

a $p \wedge q$ b $p \vee q$ c $p \vee \sim q$

17 Determine if the compound statement $\sim(p \vee q)$ is equivalent to $\sim p \vee \sim q$.

18 Determine if each of the following compound statement pairs are equivalent.

$(p \wedge q) \vee \sim p$
$(p \vee q) \wedge \sim p$

MASTER

19 Determine if the brackets in an expression alter the truth table by comparing $(p \wedge q) \vee r$ with $p \wedge (q \vee r)$.

20 Repeat question 19 with the following statement pairs.

a Compare $(p \wedge q) \wedge r$ with $p \wedge (q \wedge r)$.
b Compare $(p \vee q) \vee r$ with $p \vee (q \vee r)$.
c Based upon the results of questions 19 and 20, what might you conclude about the effect of brackets on a compound expression?

2.3 Valid and invalid arguments

The purpose of the logical connectives 'and', 'or' and 'not' is to form statements, true or false, in order to evaluate the truth, or otherwise, of something called an argument. An argument is a set of one or more propositions (statements). Before we can evaluate arguments, we need one more connective: the implication (or conditional) statement.

Implication

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Units 1 & 2

AOS 1

Topic 1

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Valid and invalid arguments
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Practice questions

Consider the following 'classical' statement: 'If it is raining, then I bring my umbrella.' This is the combination of the two statements 'It is raining' and 'I bring my umbrella', connected by two words: 'if' and 'then'. Each of the two statements has individual truth values; either could be true or false. The first statement is called the **antecedent**, the second is called the **consequent**, and in symbolic form this is written as $p \Rightarrow q$.

This is called **implication** because the first statement implies the second; it is also called **conditional**, because the outcome of the second statement is conditional on the first.

How can we determine the truth table of $p \Rightarrow q$? This is not as simple as employing a mere definition.

Referring to our example, consider the question 'Under what conditions would $p \Rightarrow q$ be a lie?'

1. If it *is* indeed raining and I bring my umbrella then, clearly $p \Rightarrow q$ is true.
2. If it is raining and I *don't* bring my umbrella, then I lied to you! Thus, $p \Rightarrow q$ is false.
3. What if it is *not* raining? I have told you *nothing* about what I would do in that case. I might either bring my umbrella, or I might not. In either case you *cannot* say I lied to you, so $p \Rightarrow q$ is true.

The truth table at right can be constructed to summarise this situation.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

This leads us immediately to ask the question: Is $p \Rightarrow q$ the same as $q \Rightarrow p$?

WORKED EXAMPLE 5

Determine by using truth tables if $(p \Rightarrow q) \Leftrightarrow (q \Rightarrow p)$.

THINK

1 Set up a truth table for p, q and $p \Rightarrow q$. $p \Rightarrow q$ is shown in the 3rd column.

2 Exchange the roles of p and q to determine the truth table for $q \Rightarrow p$. This is shown in the last column.

3 Clearly, they are *not* equivalent.

WRITE

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

This is a most important result; it is a result that people who think they are arguing logically often mistake for a valid statement. In this example, 'If I bring my umbrella, then it is raining' says a very different thing from the original statement and is called its **converse**. It seems to be making the argument that my bringing the umbrella can control the weather!

Converse, contrapositive and inverse

As we have just seen, there are alternative forms of $p \Rightarrow q$, such as the converse. These and their relationships to $p \Rightarrow q$ are shown in the table below.

Name	Symbol	Relationship to $p \Rightarrow q$
Implication	$p \Rightarrow q$	(assumed) True
Converse	$q \Rightarrow p$	False
Contrapositive	$\sim q \Rightarrow \sim p$	True
Inverse	$\sim p \Rightarrow \sim q$	False

Often the contrapositive is a more realistic way of stating an implication than the original statement is. Be careful, however, not to use the converse or inverse as they are (generally) false when $p \Rightarrow q$ is true.

Arguments

An **argument** is a series of statements divided into two parts — the premises and the conclusion. The **premises** are a series of statements intended to justify the **conclusion**. For example, consider the following argument:

A terrier is a breed of dog.	Premise
Rover is a terrier.	Premise
Therefore, Rover is a dog.	Conclusion

Generally, an argument will have only one conclusion and (usually) two premises.

Conclusion and premise indicators

To help identify the conclusion, look for words or phrases such as:

therefore, accordingly, hence, thus, consequently,
it must be so that, so, it follows that, implies that.

What follows one of these conclusion indicators is the conclusion; by default everything else is a premise. There are also premise indicators:

because, given that, since, seeing that,
may be inferred from, owing to, for, in that.

In a formal argument, the conclusion comes after the premises.

WORKED EXAMPLE 6

Identify the premises and conclusions for each of the following arguments.

a **A Commodore is a model of a Holden car.**
My car is a white Commodore.
Therefore, my car is a Holden.

b **Military defence depends upon adequate government funding.**
Adequate government funding depends on a healthy economy.
A healthy economy depends upon an intelligent fiscal policy.
Military defence depends upon an intelligent fiscal policy.

c **Pregnant mothers should not smoke.**
Cigarettes can harm the foetus.

THINK	WRITE	
a Examine each sentence looking for the conclusion indicators, or examine the sequence of the sentences.	**a** A Commodore is a model of a Holden car.	Premise
	My car is a white Commodore.	Premise
	Therefore, my car is a Holden.	Conclusion
b Note how the sequence of statements connects one with the next. The last is therefore the conclusion.	**b** Military defence depends upon adequate government funding.	Premise
	Adequate government funding depends on a healthy economy.	Premise
	A healthy economy depends upon an intelligent fiscal policy.	Premise
	Military defence depends upon an intelligent fiscal policy.	Conclusion
c In this case the sentences have been reversed. This is a common mistake.	**c** Pregnant mothers should not smoke.	Conclusion
	Cigarettes can harm the foetus.	Premise

In some textbooks, statements are called **propositions** and arguments are called **inferences**.

Categorical propositions and the deductive argument

The standard argument consists of two premises and a conclusion:

All dogs are mammals.	Premise
Rover is a dog.	Premise
Therefore, Rover is a mammal.	Conclusion

Note: Observe the use of the key word ‘All’. Beware of arguments that use the key word ‘some’, as in ‘Some journalists are hard-working’. This is a weaker form of argument, the study of which is beyond the scope of this course.

The first premise is called a **categorical statement** or **proposition**, and this form of argument can be called the classical **deductive argument**. However, as we shall see, there are many cases where we will not have a valid deductive argument, even if everything looks correct; these situations are called **fallacies**. As an example, consider the following argument:

All dogs are mammals.	Premise
Rover is a mammal.	Premise
Therefore, Rover is a dog.	Conclusion

Clearly, no one should be convinced by this argument. Both premises might be true, but the conclusion does not follow logically from them, and we would say that this is an **invalid** argument. This is an example of a formal, or structural, fallacy.

Some categorical propositions can be turned into implications. For instance, the statement ‘All dogs are mammals’ can be written as ‘If it is a dog, then it is a mammal’. This says exactly the same thing.

Beware of statements such as ‘If it is sunny tomorrow, I will go to the beach’. This is not the same as saying ‘On all sunny days I will go to the beach’. The key word here is 'tomorrow' — this restricts the statement so that the key word 'all' cannot be used. However, the implication can still be used in a valid argument:

If it is sunny tomorrow, I will go to the beach.

After checking the weather tomorrow:

It is sunny.

I will go to the beach.

This is certainly a valid argument. At this point, we can define a symbolic form for this kind of deductive argument:

$$\begin{array}{l} p \Rightarrow q \\ \underline{p \qquad} \\ q \end{array}$$

In other words, we start with an implication, which we assert to be true, then follow by stating that the antecedent is true, and argue that the conclusion is true. Can you see how the 'Rover' argument fits into this pattern? Note that this is only one form of (potentially) valid argument.

WORKED EXAMPLE 7

Determine if the following arguments are valid.

a All mathematics books are interesting.
This is a book about mathematics.
Therefore, this book is interesting.

b If I study hard, I will pass Physics.
I passed Physics.
I must have studied hard.

c Some history books are boring.
This book is about history.
Therefore, this book is boring.

d If I don't study, I will fail Physics.
I didn't study.
I will fail Physics.

THINK	WRITE
a 1 Change the first statement to: 'If ... then ...'.	**a** If it is a mathematics book, then it is interesting.
2 (a) Assign each statement a symbol.	p = It is a mathematics book. q = It is interesting.
(b) Put the argument into symbolic form.	$\begin{array}{l} p \Rightarrow q \\ \underline{p \qquad} \\ q \end{array}$
3 Determine if it is a valid form.	Yes, this is a valid form for an argument.
b 1 (a) Assign each statement a symbol.	**b** p = I study hard. q = I will pass Physics.
(b) Put the argument into symbolic form.	$\begin{array}{l} p \Rightarrow q \\ \underline{q \qquad} \\ p \end{array}$
2 Determine if it is a valid form.	No, this is not a valid form for an argument.
c Consider the first statement. Note the use of the word 'some'.	**c** The use of the word 'some' means that the statement cannot be put into this form. Thus, the entire argument is not valid.

d	**1** (a) Assign each statement a symbol.	**d** p = I don't study. q = I will fail Physics.
	(b) Put the argument into symbolic form.	$p \Rightarrow q$ $\underline{p}$ q
	2 Determine if it is a valid form.	Yes, this is a valid form for an argument.

Note: Even if the statements are expressed in negative form: 'I don't study' . . . 'I will fail Physics', it is still possible to have a valid argument. Can you devise a 'positive' argument that is the equivalent to the one in part d of Worked example 7?

It is important to note that an argument may be valid even if the truth of the component statements cannot be established. Consider the following (nonsense) argument:

All fribbles are granches.
A hommie is a fribble.
Therefore, a hommie is a granch.

We certainly cannot establish the truth of the two premises (let alone know what fribbles, granches or hommies are), but presuming they are true, the argument is valid. Furthermore, consider the argument:

If it is a dog, then it can do algebra.
Rover is a dog.
Therefore, Rover can do algebra.

This is a valid form of argument, but one (or more) of the premises is (are) false. In this case we do not have a sound argument and would certainly not use it to convince anyone of the mathematical ability of dogs. In other words, 'All sound arguments are valid, but all valid arguments are not necessarily sound'.

Valid forms of argument

There are many valid forms of argument. We shall limit our discussion to the most important ones, five of which are tabulated below.

Argument form and name	Example
$p \Rightarrow q$ $\underline{p}$ q *Modus ponens*	If Mary is elected, then she must be honest. <u>Mary was elected.</u> Mary must be honest. This is our standard form.
$p \vee q$ $\underline{\sim p}$ q Disjunctive syllogism	Either John or Jemma was born in Canada. <u>John was not born in Canada.</u> Jemma was born in Canada. Note that the roles of p and q can be interchanged here.

(*continued*)

Argument form and name	Example
$p \Rightarrow q$ $q \Rightarrow r$ ――― $p \Rightarrow r$ Hypothetical syllogism	If it is raining, I will bring my umbrella. If I bring my umbrella, then I will not get wet. ――― If it is raining, I will not get wet. Many statements (p, q, r, . . .) can be linked together this way to form a valid argument.
$p \Rightarrow q$ $\sim q$ ――― $\sim p$ *Modus tollens*	If I study hard, I will pass Physics. I did not pass Physics. ――― I did not study hard. This is a valid form of a negative argument.
$p \Rightarrow q \wedge r \Rightarrow s$ $p \wedge r$ ――― $q \wedge s$ Constructive dilemma	If we holiday in France, we will have to practise speaking French, and if we holiday in Germany, we will have to practise German. We will holiday in France and Germany. ――― We will have to practise speaking French and German.

There are several other forms more complex than these that are beyond the scope of this course.

Proving the validity of an argument form

It may not be satisfactory to merely declare that the five arguments in the previous table are automatically valid. There is a way to mathematically establish their validity using a truth table. The procedure is as follows.

Step 1. Set up a single truth table for all the premises and for the conclusion.

Step 2. Examine the row (or rows) in the table where *all* the premises are true.

Step 3. If the conclusion is true in each of the cases in step 2, then the argument is valid. Otherwise it is invalid.

WORKED EXAMPLE 8

Establish the validity of the *modus ponens* argument, namely:

$$\begin{array}{l} p \Rightarrow q \\ \underline{p \qquad} \\ q \end{array}$$

THINK

1 Set up a truth table for each of the premises, namely p and $p \Rightarrow q$, and the conclusion q. Note that p and q are set up first in the usual way, and $p \Rightarrow q$ is completed from them.

WRITE

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

2 Find the rows where all the premises are true.

The premises are all true in the 1st row only.

3 Compare with the conclusion column (q).

The conclusion is also true, so the argument is valid.

WORKED EXAMPLE 9

Show that the following argument is invalid.

$$\begin{array}{l} p \Rightarrow q \\ \underline{q \quad\quad} \\ p \end{array}$$

THINK

1 Set up a truth table for each of the premises, namely q and $p \Rightarrow q$, and the conclusion p. Note that p and q are set up first in the usual way, and $p \Rightarrow q$ is completed from them.

WRITE

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

2 Find the rows where all the premises are true.

The premises are all true in the 1st row and 3rd row.

3 Compare with the conclusion column (p).

The conclusion is true in the 1st row but false in the 3rd, so the argument is invalid.

The argument shown in Worked example 9 is a common error in logical argument and is called **affirming the consequent**.

In conclusion, if an argument fits exactly one of the five given forms, then it is immediately assumed to be valid; otherwise it must be established to be valid using truth tables.

EXERCISE 2.3 Valid and invalid arguments

PRACTISE

1 WE5 Determine using truth tables if $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$.

2 Determine using truth tables if $(p \Rightarrow q)$ is equivalent to $(\sim p \Rightarrow \sim q)$.

3 WE6 Identify the premises and conclusion in the following arguments.

a All cats are fluffy.
My pet is a cat.
My pet is fluffy.

b Two is the only even prime number.
Prime numbers are divisible by themselves and 1.
All even numbers are divisible by themselves and by 2.

4 Identify the premises and conclusion for each of these arguments.

a Growing apples depends on good water.
Growing apples depends on good irrigation.
Good water depends on good irrigation.

b The weather report on Channel 9 is accurate.
I will bring an umbrella tomorrow.
The weather report on Channel 9 predicts rain for tomorrow.

5 WE7 Determine if the following arguments are valid.

a If you are a mathematician, you can do algebra.
You are a mathematician.
You can do algebra.

b All footballers are fit.
David is not a footballer.
David is not fit.

c If it is a native Australian mammal, then it is a marsupial.
A wombat is a native Australian mammal.
A wombat is a marsupial.

6 Determine if the following arguments are valid.

a Some TV shows are boring.
Neighbours is a TV show.
Neighbours is boring.

b All musicians can read music.
Louise can read music.
Louise is a musician.

c If it is a sunny day, I play tennis.
It is a sunny day.
I must have played tennis.

7 WE8 Establish the validity of the disjunctive syllogism argument, namely:

$$\begin{array}{l} p \vee q \\ \underline{\sim p \quad} \\ q \end{array}$$

8 Establish the validity of the following three valid forms of argument:

a hypothetical syllogism

$$\begin{array}{l} p \Rightarrow q \\ \underline{q \Rightarrow r} \\ p \Rightarrow r \end{array}$$

b *modus tollens*

$$\begin{array}{l} p \Rightarrow q \\ \underline{\sim q \quad\;} \\ \sim p \end{array}$$

c constructive dilemma

$$\begin{array}{l} p \Rightarrow q \wedge r \Rightarrow s \\ \underline{p \wedge r \qquad\qquad} \\ q \wedge s \end{array}$$

9 WE9 a Show that the following argument is invalid.

$$\begin{array}{l} p \Rightarrow q \\ \underline{\sim p \quad\;} \\ \sim q \end{array}$$

b Show that the following is an example of this argument.
If elected with a majority, my government will introduce new tax laws.
My government was not elected with a majority.
Therefore, my government will not introduce new tax laws.

10 Show that the following argument is invalid.

$$\begin{array}{l} \sim p \Rightarrow \sim q \\ \underline{p \qquad\qquad} \\ q \end{array}$$

CONSOLIDATE

11 Let p = 'It is bread' and q = 'It is made with flour'. Write out the implication, converse, contrapositive and inverse in sentences.

12 The contrapositive of the statement 'If a child is playing quietly, then it is doing something bad' is:

A If a child is playing quietly, then it is not doing something bad.
B If a child is not playing quietly, then it is not doing something bad.
C If a child is not doing something bad, then it is not playing quietly.
D If a child is doing something bad, then it is not playing quietly.
E If a child is doing something bad, then it is playing quietly.

13 The inverse of the statement 'If you are not careful, then you will get hurt' is:

A If you are careful, then you will not get hurt.
B If you do not get hurt, then you are careful.
C If you get hurt, then you are not careful.
D If you do not get hurt, then you are not careful.
E If you are not careful, then you will not get hurt.

14 Look again at the arguments in questions **5** and **6** that were not valid. If possible, turn them into valid arguments. Assume that the first statement in each argument is always correct.

15 Which of the following statements about logic is false?

A An argument must have a conclusion.
B An argument consists of premises and a conclusion.
C An 'If . . . then . . .' compound statement is called a conditional statement.
D **A**, **B** and **C** are all false.
E **A**, **B** and **C** are all true.

16 The following are valid arguments. Determine which of the five forms of argument have been used.

a Either you clean up your room or you will not watch any television tonight.
You did not clean up your room.
Therefore, you will not watch any television tonight.

b If you help your mother with the dishes, I will take you to the football game tomorrow.
I didn't take you to the football game.
Therefore, you didn't help your mother with the dishes.

c If you study statistics, then you will understand what standard deviation means.
You studied statistics.
Therefore, you will understand what standard deviation means.

17 Consider the following valid argument.
If John plays for us on Saturday, then we will win.
If we win on Saturday, then we will come in first place on the ladder.
If we come in first place on the ladder, then we play our first final at home.
Therefore, if John plays for us on Saturday, then we play our first final at home.

This is an example of:

A *modus ponens*
B disjunctive syllogism
C hypothetical syllogism
D *modus tollens*
E constructive dilemma

18 The following argument is an example of which valid form?
If a lawyer is honest, then you should hire him.
You shouldn't hire John Smith.
John Smith is not honest.

A *Modus ponens*
B Disjunctive syllogism
C Hypothetical syllogism
D *Modus tollens*
E Constructive dilemma

19 Write the converse, contrapositive and inverse of the following statement.
If a politician is intelligent, she sends her children to good schools.

20 Determine the validity of the following arguments.

a All dogs have five legs.
All five-legged creatures are called chickens.
Therefore, all dogs are chickens.

b All dogs have five legs.
All chickens have five legs.
Therefore, all dogs are chickens.

c If you deposit money in the bank, then you will earn interest.
You didn't earn any interest.
Therefore, you didn't deposit any money in the bank.

d If I wanted an easy course to study, I would choose Human Development, and if I wanted an interesting course to study, I would choose Specialist Mathematics.
I can choose an easy course and an interesting one.
Therefore, I will study Human Development and Specialist Mathematics.

e If the team plays well, either the offence was good or the defence was good.
The defence wasn't good and the team did not play well.
Therefore, the offence wasn't good.

MASTER

21 A common argument is of the form:
If you work hard, then you will become rich.
You don't work hard.
Therefore, you will not become rich.

a Put this argument in symbolic form.
b Show that it is an invalid form of argument. (This is called 'denying the antecedent'.)

22 Determine the validity of the following arguments.

a
$$\begin{array}{l} p \Rightarrow q \\ \underline{r \Rightarrow \sim q} \\ p \Rightarrow \sim r \end{array}$$

b
$$\begin{array}{l} \sim p \wedge \sim q \\ \underline{r \Rightarrow p} \\ r \end{array}$$

2.4 Techniques of proof

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Techniques of proof
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As mentioned at the beginning of this topic, not only is logic used to establish the validity of arguments, but its techniques are also used to establish the truth (or otherwise) of mathematical statements. For example, it is not satisfactory to say that Pythagoras' theorem is true; it must be proved to be true as well. The tools of logic are also the tools of proof.

Tautologies

A **tautology** is a compound statement that is always true. Although this might seem like a useful thing, such statements are to be avoided in arguments, as they contribute nothing towards an argument. An example of a tautology is:

'The game will be won or lost in the last 30 minutes'.

A statement like this, while perhaps favoured by sports announcers, is of no value in establishing the truth of whether a particular team will win. A tautology is more useful as a technique of proof.

To establish if a compound statement is a tautology, construct a truth table from its component parts. If the compound statement is always true, then it is a tautology.

WORKED EXAMPLE 10

Consider the statement:
'If John and Jim are qualified lawyers, then John is a qualified lawyer'.
Establish whether or not this statement is a tautology.

THINK

1 Define symbols for each part of the statement.

2 Convert the statement to its symbolic form.

3 Set up a truth table.

4 Establish whether or not the statement is a tautology.

WRITE

1 Let p = John is a qualified lawyer.
Let q = Jim is a qualified lawyer.

2 $(p \wedge q) \Rightarrow p$

3

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

4 The last column shows all 'T', therefore the statement is a tautology.

As can be seen, the original statement would contribute very little to any argument as to the qualifications of either John or Jim as lawyers.

The 'opposite' of a tautology is a self-contradictory statement; one which is always false. It too has little use in arguments. However, the concept of a tautology can be used in establishing validity or, in mathematical language, proving arguments.

Proof using tautologies

An argument is valid under the following condition: 'If all the premises are true, then the conclusion is true'. Let p and q be the premises of an argument and r be the conclusion. If $(p \wedge q) \Rightarrow r$ is a tautology, then the argument is valid, or proved.

Note that p, q and r can be, themselves, compound statements. In fact, this method is exactly the same as that presented in the previous section but is a more mechanical technique.

WORKED EXAMPLE 11

Using tautology, establish the validity of the *modus tollens* argument, namely $p \Rightarrow q$

$$\begin{array}{l} \sim q \\ \hline \sim p \end{array}$$

THINK

1. Set up a standard truth table for p and q.
2. Determine the truth table for $p \Rightarrow q$.
3. Form the truth table of the 'and' of both statements, namely: $p \Rightarrow q \wedge \sim q$ (column 4).
4. Form the truth table of the implication using the first two statements along with the conclusion, namely: $(p \Rightarrow q \wedge \sim q) \Rightarrow \sim p$ (column 5).
5. Determine the validity of the argument.

WRITE

p	q	$p \Rightarrow q$	$p \Rightarrow q \wedge \sim q$	$(p \Rightarrow q \wedge \sim q) \Rightarrow \sim p$
T	T	T	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

Since the last column is always true, this is a tautology and the original argument is valid.

The other valid forms of argument can also be established using this technique. This is left as an exercise.

Mathematical proofs using valid argument forms

Any of the valid forms of argument can be used to prove theorems in mathematics. Although this can be a tedious way of proving things, it certainly establishes a theorem beyond doubt.

WORKED EXAMPLE 12

Prove the following: 'If two straight lines have equal gradients, then they do not intersect'.

THINK	WRITE
1 Find a property of lines with equal gradients.	If lines have equal gradients then they are parallel. This is a well-known result from linear graphs.
2 Find a property of parallel lines.	If lines are parallel then they do not intersect. This is a well-known result from geometry.
3 Define p, q and r from compound statements.	Let p = Two lines have equal gradients. Let q = Two lines are parallel. Let r = Two lines do not intersect.
Create a symbolic form of steps 1 and 2.	Step 1: If p then q, or $p \Rightarrow q$. Step 2: If q then r, or $q \Rightarrow r$.
4 Determine a conclusion from the valid argument form (hypothetical syllogism).	$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline p \Rightarrow r \end{array}$
5 Write the conclusion.	If two straight lines have equal gradients, then they do not intersect. QED

Whenever a theorem is proved, state *quod erat demonstrandum* (QED). This means 'It is demonstrated', but it could also mean 'quite easily done'!

Proof by contrapositive

As mentioned in the section on logical connectives, an alternative to $p \Rightarrow q$ is its contrapositive, $\sim q \Rightarrow \sim p$. If we can establish that the contrapositive statement is true, then the original implication is true.

WORKED EXAMPLE 13 **Prove, using the contrapositive, that 'If n^3 is odd, then n is odd (where n is any integer)'.**

THINK	WRITE
1 Write the statement in symbolic form.	Let p = 'n^3 is odd'; q = 'n is odd'.
2 Write the contrapositve as a statement.	$\sim q \Rightarrow \sim p$; 'If n is not odd then n^3 is not odd'.
3 Prove the truth of the contrapositive.	If n is not odd, it is even (or 0) and can be represented by $2x$, where x is any integer. Let $n = 2x$ then $n^3 = (2x)^3 = 8x^3$.
4 This relies on the fact that multiplying anything by 2 results in an even number.	But $8x^3 = 2 \times 2 \times 2 \times x^3$ and, hence, must be even. Since $8x^3 = n^3$ is even, it is not odd. Since the contrapositive statement is shown to be true, the original argument, namely that if n^3 is odd, then n is odd, is also proved. QED

Proof by contradiction

Another method of proof involves assuming the statement that we are trying to prove is false. If this leads to an apparent contradiction, we can assume that the statement is true.

WORKED EXAMPLE 14 **Prove, by contradiction, that the product of two negative numbers (non-zero) is positive.**

THINK	WRITE
1 Assume that the statement is false.	Let a and b be two negative numbers. Assume that $a \times b$ is negative.
2 Determine the magnitude of the product. Invoke the assumption that the product is negative.	The magnitude of $a \times b = \lvert ab \rvert$ $= \lvert a \rvert \times \lvert b \rvert$ $ab = -\lvert a \rvert \times \lvert b \rvert$
3 Consider the case $c = -a$, so that c is positive, and a positive number is being multiplied by a negative number.	$cb = -\lvert c \rvert \times \lvert b \rvert$ since c is > 0 and $b < 0$ $\lvert c \rvert = \lvert a \rvert$ since $c = -a$ $cb = -\lvert a \rvert \times \lvert b \rvert$ by substitution

4 Now from steps 2 and 3, the two expressions are equal.	$-\lvert a \rvert \times \lvert b \rvert = ab = cb$ from steps 2 and 3 $a = c$ divide both sides by b
5 This is a contradiction, since $c = -a$.	Contradiction implies that the original statement is true. QED

The contradiction must be as a **direct result** of the assumption of the original statement being false. In Worked example 14, this occurred in steps 2 and 3, leading to two different expressions for the same thing ($-|a| \times |b|$). This proof technique is based upon the logical argument form:

$$(p \wedge \sim q) \Rightarrow (r \wedge \sim r) \Leftrightarrow (p \Rightarrow q)$$

where $p \Rightarrow q$ (or more specifically q) is the statement you are trying to prove, and $(r \wedge \sim r)$ is the contradiction that arises by assuming q was false ($\sim q$). This method of proof is also called **indirect proof**, or **reductio ad absurdum** (reducing to an absurdity).

Proof by counterexample

Perhaps the simplest method of proof is that by assuming the statement to be true, an example arises that shows that the statement is false. Therefore the original statement cannot be true.

WORKED EXAMPLE 15 **Prove, by counterexample, that the statement 'the square root of x^2 is x' is false.**

THINK	WRITE
1 Consider a single case; let $x = 8$.	$x^2 = 64$
2 Invoke the rule for square roots.	The square root of 64 is ± 8.
3 Substitute back.	The square root of x^2 is $\pm x$, thus the original statement is false.

Remember, you need only a single example where the statement is false and hence, by extension, the entire statement is false. In the above example, the proof relied on the fact that there were two answers, not one as implied in the statement.

There are many other methods of proof, but the ones you have seen in this topic will provide you with a toolbox of techniques for proving a large number of mathematical statements.

Proof by mathematical induction

Mathematical induction is a method of proof, although it is considered less effective than the ones you have already learned. It is used to prove formulas, results and similar things where there is a sequence of results for different values.

For example, the sum of the series $1 + 2 + 3 + \ldots + n = \dfrac{(n)(n+1)}{2}$ has different results depending on the value of n.

We will use the method of induction on the above proof.

Step 1.

Show that the result is true for $n = 1$ (or in some proofs $n = 0$).

If $n = 1$, then the sum = 1 and $\frac{(n)(n+1)}{2} = \frac{1 \times 2}{2} = 1$

Step 2.

Assume it is true for any value n.

Assume: $1 + 2 + 3 + \ldots + n = \frac{(n)(n+1)}{2}$

Step 3.

Prove it true for $n + 1$ (add one more term to the series).

$(1 + 2 + 3 + \ldots + n) + (n + 1)$

$= \frac{(n)(n+1)}{2} + (n + 1)$

This step is as a result of adding with common denominators.

$= \frac{(n)(n+1) + 2(n+1)}{2}$

This step is as a result of the common factor of $(n + 1)$ in the numerator.

$= \frac{(n+1)(n+2)}{2}$

Let $m = n + 1$, so $m + 1 = n + 2$.

$= \frac{(m)(m+1)}{2}$

Step 4.

Clearly, this is the same formula as the one we assumed true.

$\frac{(m)(m+1)}{2}$

$= 1 + 2 + 3 + \ldots m$

$= 1 + 2 + 3 + \ldots (n + 1)$

The logic behind induction is that we can keep on increasing the value of n by one at a time until all (possible) values have been 'proved'. Thus the statement is proved for all values!

EXERCISE 2.4 Techniques of proof

PRACTISE

1 WE10 Determine if the following statement is a tautology.
'If she plays well she will win or she will lose.'

2 a Modify the sentence in question 1 to:
'If she plays well she will win, or if she plays poorly she will lose'.
Determine if this sentence is a tautology.

b Modify the sentence in question 1 to:
'Either she plays well and wins or she will lose'.
Determine if this sentence is a tautology.

3 WE11 Using tautology, establish the validity of the hypothetical syllogism argument, namely:

$$\begin{array}{c} p \Rightarrow q \\ \underline{q \Rightarrow r} \\ p \Rightarrow r \end{array}$$

4 Using tautology, establish the validity or otherwise of the following argument.

$$\begin{array}{c} \sim p \Rightarrow \sim q \\ \underline{\sim q \Rightarrow r} \\ \sim p \Rightarrow r \end{array}$$

5 WE12 Using a valid argument, prove the following mathematical result.
If a number x is even, then x^2 is even.

6 Using a valid argument, prove that 24^2 is even.

7 WE13 Prove, using the contrapositive, that if n^2 is odd then n is odd.

8 Prove the following results using the contrapositive.
a If $ax = bx$ and $x \neq 0$, then $a = b$.
b If $n^2 > 4$, then $n > 2$ (n is positive).
c If n^2 is divisible by 2, then n is divisible by 2.

9 WE14 Prove, by contradiction, that the product of a negative number and a positive number is negative.

10 Prove, by contradiction, that there is no smallest positive real number (a, such that $a > 0$).

11 WE15 Prove, by counterexample, the statement that '$x^2 = 4$ has only one solution' is false.

12 Prove, by counterexample, the fact that all prime numbers are odd.

CONSOLIDATE

13 Using tautology, establish the validity or otherwise of the following arguments.

a $\begin{array}{l} p \Rightarrow \sim q \\ \underline{q \Rightarrow \sim r} \\ p \Rightarrow \sim r \end{array}$

b $\begin{array}{l} \sim p \Rightarrow \sim q \\ \underline{q \qquad\quad} \\ p \end{array}$

c $\begin{array}{l} \sim p \Rightarrow q \\ \underline{p \qquad\quad} \\ \sim q \end{array}$

14 By proving $\sim q \Rightarrow \sim p$, you have proved $p \Rightarrow q$ is an example of proof by:
A counterexample
B contradiction
C contrapositive
D deduction
E induction

15 Express the following arguments in symbolic form.
a If you are a loser, then you didn't train hard enough.
If you didn't train hard enough, then you were distracted.
Therefore, if you are a loser, then you were distracted.

b If it is not raining, then I will wash my car.
It is raining.
Therefore, I will not wash my car.
c If it is cloudy, then I do not bring my umbrella.
I did bring my umbrella.
Therefore, it is not cloudy.
d If the hard drive isn't working, then the program will not work.
If the program is not working, then the printer will not work.
Therefore, if the hard drive isn't working, the printer will not work.

16 Prove, by contradiction, that $\sqrt{2}$ is irrational.

17 Consider the formula $p = n^2 + n + 11$. Let $n = 1, 2 \ldots$
$n = 1, p = 13; \quad n = 2, p = 17; \quad n = 3, p = 23; \quad n = 4, p = 31; \quad n = 5, p = 41$
It seems that: If n is a positive integer, then p is a prime number.
Prove, or disprove, this statement.

18 The missing truth values in each column in the following table are (from left to right):

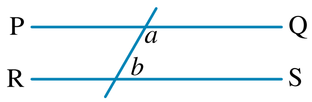
A F, T, T, F
B F, F, T, T
C T, F, T, F
D T, T, T, T
E T, T, F, T

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F			F
F	T		F	F	
F	F	T	T	T	T

19 Prove, by contradiction, that there is no largest positive integer (n, such that $n > 0$).

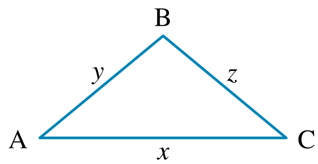
20 Consider the two lines PQ and RS shown below. A third line crossing both of them, called a transversal, forms two angles a and b. Investigate a proof for the statement that:

If $a + b = 180°$, then PQ is parallel to RS.



21 Consider *any* triangle ABC as shown in the figure below. Let the longest side of the triangle be labelled x and the other two sides be labelled y and z. Investigate a proof for the statement that:

$$x \leq y + z.$$



22 Examine some well-known results in mathematics. Can you prove them by induction? Here are some ideas to get you started. Some will work, some won't.

a The odd number series $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.
(*Hint:* How many odd numbers are there in this series?)
b $(1 + x)^n \geq (1 + xn)$ for $n \geq 1$ and $x > 0$
c Pythagoras' theorem
d Prove $(n)(n + 1)$ is an even number for any integer value of n.
e $2^n \geq n^2$ for all integers ≥ 4
f Let a, b and c be three consecutive integers. Prove $(a + b + c)^3$ is always divisible by 3.
g Prove the quadratic formula. Given $ax^2 + bx + c = 0$, prove that the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

h Prove that a polygon of n sides can be composed of $n - 2$ triangles.
(*Hint:* Start with $n = 3$.)
i $1 + 2 + 4 + 8 + \ldots + 2^n = 2^{n+1} - 1$
j $4^n \geq n^4$ for all integers ≥ 2
k Prove that the sum of the first n square numbers is $\frac{n}{6}(n + 1)(2n + 1)$.

MASTER

23 a Can you (reasonably) predict what type of proof is likely to be solvable by induction?

b What are the limitations of the proofs in terms of values of n that can be used?

c In some scientific research, particularly medicine, the concept of induction is used as follows:

If I treat Patient 1 with Drug X, she is cured.

If I treat Patient 2 with Drug X, she is cured.

If I treat Patient n with Drug X, she is cured.

What conclusion(s) is the researcher trying to draw? Comment on its (their) validity.

24 Express your thoughts on the usefulness or otherwise of proof by induction, paying attention not only to mathematical proofs but those used in areas such as science, commerce and politics.

2.5 Sets and Boolean algebra

studyon

Units 1 & 2
AOS 1
Topic 1
Concept 4

Sets and Boolean algebra
Concept summary
Practice questions

Many of the rules of logic that we have seen thus far can be collected into a single set of rules and procedures called **Boolean algebra**, named after the 19th century English mathematician, George Boole. Boole was also responsible for the introduction of sets into mathematics.

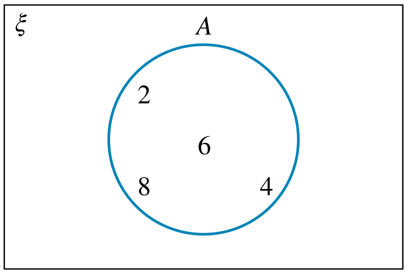
Sets and their properties

As covered in Topic 1, a set is a collection of objects (or members) that have something in common.

Sets can be **finite**, containing a fixed number of members, such as the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with 10 members, or **infinite**, such as the set of positive integers, $N = \{1, 2, 3, \ldots\}$.

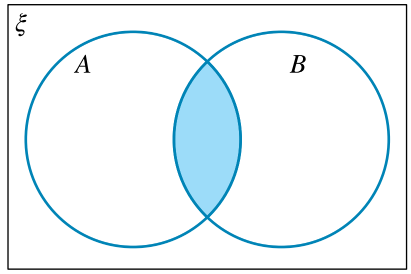
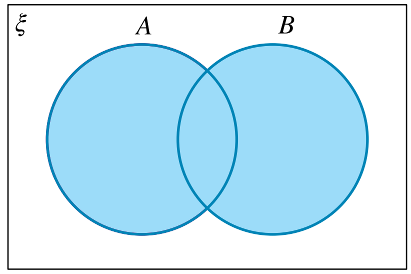
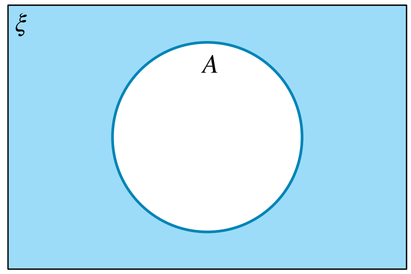
Implicit in sets is the concept that there are objects in the set and objects not in the set. If an object x is in set A, we write $x \in A$, and if object y is not in set A, we write $y \notin A$.

Remember that sets can be displayed visually using a Venn diagram, as shown at right. The area inside the circle represents the set with its members $A = \{2, 4, 6, 8\}$. The white area outside the circle represents all objects *not* in the set. In future we will not generally show the members in the set, but state its 'rule'. What could be the rule for the set in this figure?



The rectangle itself represents the **universal** set, the set of all possible members (some are in A, some are not), and is denoted by the symbol ξ. In this example the universal set could be *all* the integers.

As in arithmetic, there are a series of operations and properties that enable us to manipulate sets. Consider two sets, A and B, and the possible operations on them.

Intersection: Symbol:	The area in common between two sets is known as the intersection and is shown here in blue. $A \cap B$ 'A intersection B' or 'in both A and B'	
Union: Symbol:	The area in either A or B is the union and is shown here in blue. $A \cup B$ 'A union B' or 'in either A or B or both'	
Negation: Symbol:	The area not in A is the negation or complement and is shown here in blue. A' 'Complement of A' or 'A-prime' or 'not in A'	

Given these operations, we can now look at the rules of sets, comparing them to the rules of arithmetic. For some laws we will need three sets.

Name	Symbolic form	Description	Corresponding arithmetic
Commutative Law	1. $A \cup B = B \cup A$ 2. $A \cap B = B \cap A$	Order of a single operation is not important.	$a + b = b + a$ $a \times b = b \times a$
Identity sets	1. $A \cup \varnothing = A$ 2. $A \cap \xi = A$	The null set ($\varnothing$) has no effect on 'union'; the universal set has no effect on 'intersection'.	$a + 0 = a$ $a \times 1 = a$
Complements	1. $A \cup A' = \xi$ 2. $A \cap A' = \varnothing$	Inverse	$a + (-a) = 0$ $a \times \frac{1}{a} = 1$
Associative Law	1. $A \cup (B \cup C) = (A \cup B) \cup C$ 2. $A \cap (B \cap C) = (A \cap B) \cap C$	The placement of brackets has no effect on the final result when the operations are the same.	$a + (b + c) = (a + b) + c$ $a \times (b \times c) = (a \times b) \times c$

(*continued*)

Name	Symbolic form	Description	Corresponding arithmetic
Distributive Law	1. $A \cup (B \cap C)$ $= (A \cup B) \cap (A \cup C)$ 2. $A \cap (B \cup C)$ $= (A \cap B) \cup (A \cap C)$	Bracketed expressions can be expanded when different operations are involved.	$a \times (b + c) = a \times b + a \times c$ (Note that there is only one of these laws in arithmetic.)
Closure	Consider sets A, B and S. If $A, B \subset S$, then $A \cup B \subset S$ $A \cap B \subset S$	Performing operations on a set will create a result that still belongs to the same class of sets (S).	If a and b are real numbers, then: $a + b$ is a real number $a \times b$ is a real number.

It is important to note that union (∪) acts similarly to addition, and intersection (∩) is similar to multiplication, except in the complements, where their roles are reversed.

Although the commutative laws are self-evident, the remaining laws can be demonstrated using Venn diagrams. Closure is a concept that, for now, will have to be taken for granted. For example, closure applies for integers with the operations of addition and multiplication. It does not apply for division (for example $\frac{1}{2}$), as the result (0.5) is not an integer, even though 1 and 2 are.

Boolean algebra

By replacing the set symbols with Boolean ones, we get the laws of Boolean algebra, which are exactly the same as those for sets.

Set name	Set symbol	Boolean name	Boolean symbol
Intersection	$\cap$	and	$\cdot$
Union	$\cup$	or	$+$
Complement	$'$	not	$'$
Universal set	ξ	'everything'	I
Null set	$\varnothing$	'nothing'	O

Thus the set laws can be restated as Boolean laws:

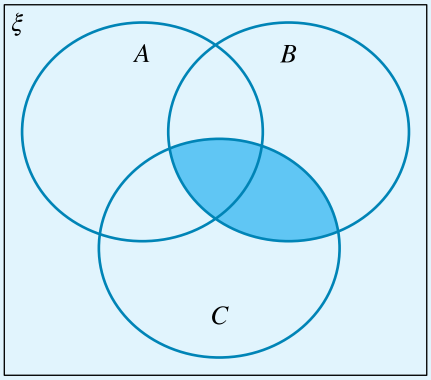
Name	Set law	Boolean law
Commutative Law	1. $A \cup B = B \cup A$ 2. $A \cap B = B \cap A$	$A + B = B + A$ $A \cdot B = B \cdot A$
Identity	1. $A \cup \varnothing = A$ 2. $A \cap \xi = A$	$A + \text{O} = A$ $A \cdot I = A$
Complements	1. $A \cup A' = \xi$ 2. $A \cap A' = \varnothing$	$A + A' = I$ $A \cdot A' = \text{O}$
Associative Law	1. $A \cup (B \cup C) = (A \cup B) \cup C$ 2. $A \cap (B \cap C) = (A \cap B) \cap C$	$A + (B + C) = (A + B) + C$ $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
Distributive Law	1. $A \cup (B \cap C)$ $= (A \cup B) \cap (A \cup C)$ 2. $A \cap (B \cup C)$ $= (A \cap B) \cup (A \cap C)$	$A + (B \cdot C) = (A + B) \cdot (A + C)$ $A \cdot (B + C) = A \cdot B + A \cdot C$
Closure	Whatever applies to sets also applies to Boolean algebra.	

Only the first distributive law may require some explanation. Do not confuse the Boolean '+' sign with addition!

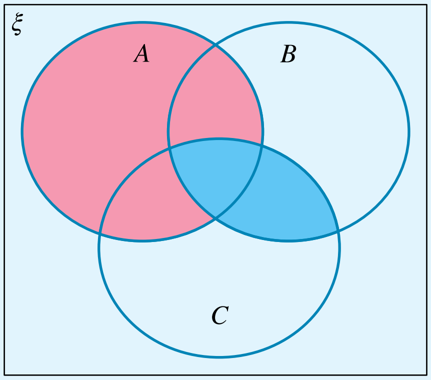
WORKED EXAMPLE 16 **Establish the distributive law, namely $A + (B \cdot C) = (A + B) \cdot (A + C)$, using a Venn diagram.**

THINK | **WRITE/DRAW**

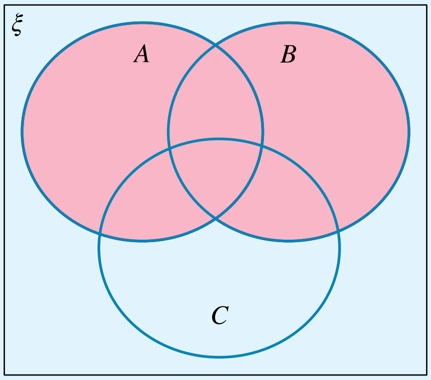
1 Consider the left-hand side term $(B \cdot C)$, which is the *intersection* of B and C.



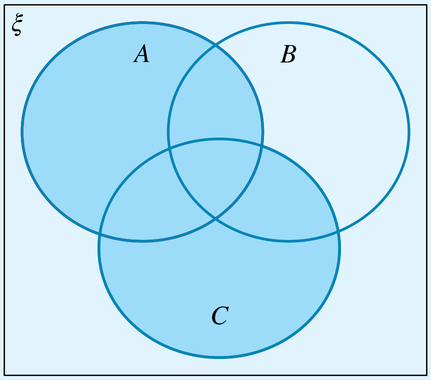
2 Now, create the *union* with A, namely $A + (B \cdot C)$. In this figure, the red shading shows the 'new' area added. The final result is the region that has either colour.



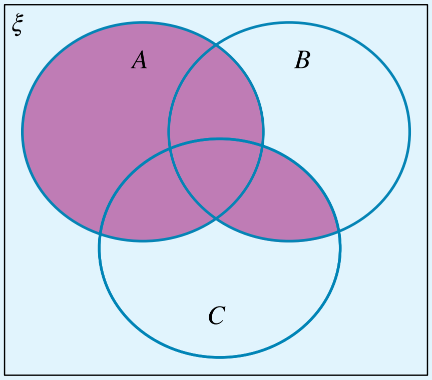
3 Now, consider the first 'term' of the right-hand side, namely $(A + B)$.



4 Now, consider the second term of the right-hand side, namely $(A + C)$.



5 Now, consider the intersection of the two regions in steps 3 and 4, which produces the region $(A + B) \cdot (A + C)$. The purple area is the resultant region.



6 Compare the two results.

Clearly the area in step 2 equals the area in step 5, thus $A + (B \cdot C) = (A + B) \cdot (A + C)$.

DeMorgan's laws and additional results

There are two further important results in Boolean algebra involving the negation of the union and intersection operations. These rules, called deMorgan's laws, can be proved using the results from Boolean algebra, or can be demonstrated using Venn diagrams.

DeMorgan's First Law states:

$$(A + B)' = A' \cdot B'$$

DeMorgan's Second Law states:

$$(A \cdot B)' = A' + B'$$

These laws can be interpreted as saying that 'the complement of union is intersection' and 'the complement of intersection is union'.

WORKED EXAMPLE 17

Prove the first of deMorgan's laws, namely that the complement of the union of two sets is the intersection of their complements using:

a the rules of Boolean algebra

b Venn diagrams.

THINK	WRITE/DRAW
a 1 State the requirements of proof in Boolean algebra terms. Since $(A + B)'$ is the complement of $(A + B)$, then $A' \cdot B'$ must satisfy both the complement laws.	a If $(A + B)' = A' \cdot B'$, then the two complement laws must be satisfied. Therefore, we must show that: $(A + B) + (A' \cdot B') = I$ 1st Complement Law [1] $(A + B) \cdot (A' \cdot B') = O$ 2nd Complement Law [2]
2 Simplify the left side of equation [1].	1st Complement Law
(a) This is as a result of the First Distributive Law.	$\text{LHS} = (A + B) + (A' \cdot B')$ $= (A + B + A') \cdot (A + B + B')$

(b) This is as a result of the First Commutative Law. $= (A + A' + B) \cdot (A + B + B')$

(c) This is as a result of the First Complement Law. $= (I + B) \cdot (A + I)$

Note: The term $(I + B)$ represents the union of B with I, which is 'everything'. Similarly, the term $(A + I)$ represents the union of A with I, which is 'everything'.

3 Complete the simplification.

$= I \cdot I$
$= I$
$=$ RHS QED

4 Simplify the left side of equation [2].

2nd Complement Law

$\text{LHS} = (A + B) + (A' \cdot B')$

(a) This is as a result of the Second Commutative Law. $= (A' \cdot B') \cdot (A + B)$

(b) This is as a result of the Second Distributive Law. $= A' \cdot B' \cdot A + A' \cdot B' \cdot B$

(c) This is as a result of the Second Commutative Law. $= A' \cdot A \cdot B' + A' \cdot B' \cdot B$

Note: The term $A' \cdot A$ is the intersection of A and its complement, which is 'nothing' or O. Similarly, $B' \cdot B = \text{O}$.

5 Complete the simplification. Note that the intersection and the union of O with any set must be O, since there is nothing in O.

$= \text{O} \cdot B' + A' \cdot \text{O}$
$= \text{O} + \text{O}$
$= \text{O}$
$=$ RHS QED

b 1 Draw a Venn diagram representing the left-hand side of the equation, that is, $(A + B)'$.

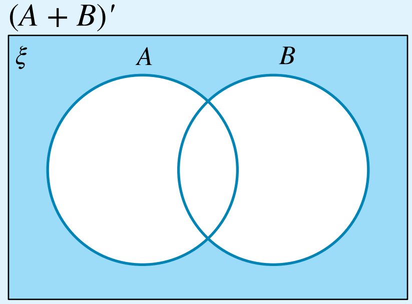
(a) Draw a rectangle with two large, partly intersecting circles. Label one of the circles as A and the other as B.

(b) Identify the portion required.

Note: $A + B$, that is, $A \cup B$, represents the portion inside the two circles. Therefore, its complement $(A + B)'$ is represented by the portion outside the two circles.

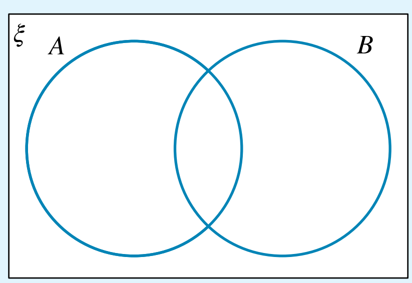
(c) Shade the required portion.

b $(A + B)'$

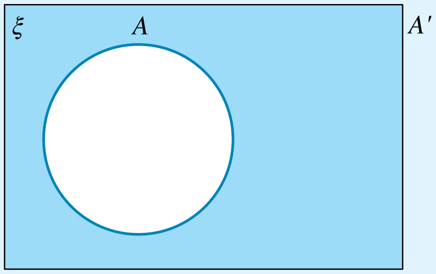


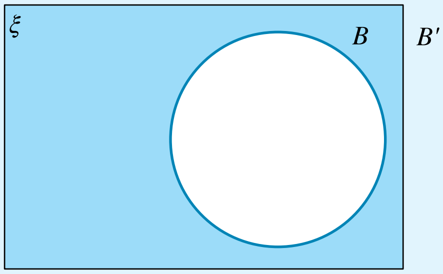
2 Draw a Venn diagram representing the right-hand side of the equation, that is, $A' \cdot B'$.

(a) Draw a rectangle with two large, partly intersecting circles. Label one of the circles as A and the other as B.

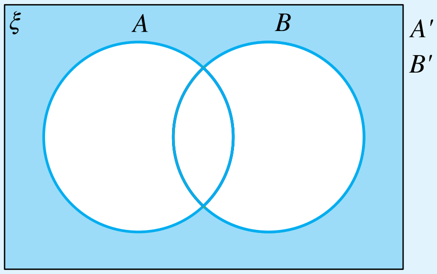


(b) Identify the portion required.
Note: A', the complement of A, represents the portion outside the two circles and the non-intersecting part of circle B, B', the complement of B, represents the portion outside the two circles and the non-intersecting part of circle A.
$A' \cdot B'$ (that is $A' \cap B'$) is represented by the common shaded portion. $A' \cdot B'$ is represented by the portion outside the two circles.





(c) Shade the required portion.



3 Comment on the Venn diagrams obtained.

The Venn diagrams obtained are identical, therefore, deMorgan's First Law, $(A + B)' = A' \cdot B'$, holds true.

The results in Worked example 17 establish the first of deMorgan's laws. The second law can be proved in a similar fashion.

Based upon the rules for Boolean algebra, some important additional results can be tabulated.

Rule	Explanation
$A + A = A$	The union of any set with itself must still be itself.
$A \cdot A = A$	The intersection of any set with itself must still be itself.
$A + I = I$	The union of any set with 'everything' must be 'everything' — I.
$A \cdot O = O$	The intersection of any set with 'nothing' must be 'nothing' — O.
$A \cdot (A + B) = A$	Consider that the only part of $(A + B)$ that intersects with A must be just A itself.
$A + (A \cdot B) = A$	Consider the fact that $A \cdot B$ is within A if $B \subset A$, or is A if $A' \subset B$, so that its union with A must be just A itself.

These results are easily established with Venn diagrams and are left as an exercise.

At this point it is worth noting that the key operations of sets and Boolean algebra are intimately related to those of deductive logic. These can be summarised by adding columns to an earlier table.

Set name	Set symbol	Logic name	Logic symbol	Boolean name	Boolean symbol
Intersection	$\cap$	and	$\wedge$	and	$\cdot$
Union	$\cup$	or	$\vee$	or	$+$
Complement	$'$	not	$\sim$	not	$'$
Universal set	ξ			'everything'	I
Null set	$\varnothing$			'nothing'	O

There are no logical equivalents to 'everything' or 'nothing'.

Let us use the rules of Boolean algebra to prove an earlier result.

WORKED EXAMPLE 18

The following pair of logical statements were established to be equivalent:

$$(p \vee q) \vee \sim p$$
$$p \vee \sim p$$

Establish this fact using Boolean algebra.

THINK	WRITE	
1 Write the first logic statement and equate it with its corresponding statement using Boolean algebra.	$(p \vee q) \vee \sim p = (P + Q) + P'$	
2 Simplify the right-hand side of the equation.	$= (P + P') + Q$	1st Commutative Law
	$= I + Q$	Identity Law
	$= I$	
3 Write the second logic statement and equate it with its corresponding statement using Boolean algebra.	$p \vee \sim p = P + P'$	Complements
4 Simplify the right-hand side of the equation.	$= I$	
5 Comment on the results obtained.	The two statements are both equal to I and therefore equivalent to each other. QED	

EXERCISE 2.5 Sets and Boolean algebra

PRACTISE

1 **WE16** Establish the Second Associative Law, namely:
$A \cap (B \cap C) = (A \cap B) \cap C$
using Venn diagrams.

2 Establish the Second Distributive Law, namely:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ or $A \cdot (B + C) = A \cdot B + A \cdot C$
using Venn diagrams.

3 **WE17** Using the rules for Boolean algebra, prove the second of deMorgan's laws:
$(A \cdot B)' = A' + B'$.

4 Show, using Venn diagrams, that:

a $(A + B) \cdot A = A$ b $(A + B) \cdot B' = A$ c $A + B \cdot A' = A + B$.

5 **WE18** Determine, using Boolean algebra, if the following two statements are equivalent.

$$(p \wedge q) \wedge \sim p$$
$$p \wedge \sim p$$

6 Use Boolean algebra to determine if the following pair of logical statements are equivalent.

$$(p \wedge q) \vee \sim p$$
$$(p \vee q) \wedge \sim p$$

CONSOLIDATE

7 Which of the following statements about sets is false?

A The symbol ∪ represents the union of two or more sets.
B All sets must have at least one member.
C Some sets can have an infinite number of members.
D A member cannot be in both set A and set A' at the same time.
E The set represented by the symbol ∅ has no members.

8 Write the following sets using the notation $A = \{\ldots\}$.

a A = the set of all even positive integers less than 20
b B = the set of all positive integers divisible by 4
c C = the set of all even prime numbers
d D = the set of court cards in a deck of playing cards
e E = the set of integers less than 0 that are square numbers
f F = the set of integers less than 10

9 Which of the sets in question 8 are finite?

10 Demonstrate, using a Venn diagram, the intersection of the following two sets:
A = the set of two-digit positive odd numbers
B = the set of two-digit square numbers.
List the members of the intersection on the diagram.

11 Demonstrate, using a Venn diagram, the intersection of the following two sets:
A = the set of two-digit positive even numbers
B = the set of two-digit palindromes (numbers that are the same backwards and forwards).
List the members of the intersection on the diagram.

12 Demonstrate on a Venn diagram the regions defined by:

a $A \cap B'$ b $A' \cap B'$ c $A' \cap (B \cap C)$.

13 The laws of sets can be demonstrated with specific sets.
Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{1, 4, 9\}$.
Consider the First Distributive Law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

a Find the set represented by the expression $(B \cap C)$.
b Find the set represented by $A \cup (B \cap C)$.
c Find the set represented by $(A \cup B)$.
d Find the set represented by $(A \cup C)$.
e Find the set represented by $(A \cup B) \cap (A \cup C)$ and show that this is the same set as that in the answer to part b.

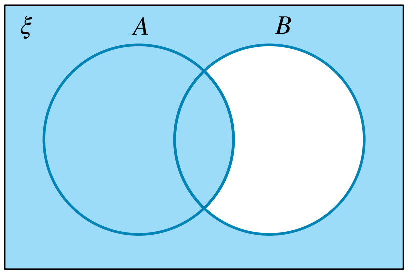
14 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{1, 4, 9\}$.

a Find the set represented by the expression $(B \cup C)$.
b Find the set represented by $A \cap (B \cup C)$.
c Find the set represented by $(A \cap C)$.
d Find the set represented by $(A \cap B)$.
e Find the set represented by $(A \cap B) \cup (A \cap C)$ and show that this is the same set as that in the answer to part b.

15 Let A = the set of all positive prime numbers less than 100.
Let B = the set of all positive two-digit numbers with the digit 1 in them.
Let C = the set of all positive two-digit numbers whose sum of digits = 7.
List the following sets.

a $A \cap B$ b $A \cup (B \cap C)$ c $A \cap B \cap C$

16 The shaded area in the figure at right represents:



A $A \cup B$
B $A' \cup B$
C $A \cup B'$
D $A \cap B'$
E $(A \cup B)'$

MASTER

17 Simplify the following logical expressions, using the rules of Boolean algebra.

a $A + A' \cdot B + A \cdot B$ b $(A + B + A') + B'$
c $A + A' \cdot B$ d $A \cdot B \cdot (A + C)$

18 Prove the following using Boolean algebra or Venn diagrams.

a $A + B + A' + B' = I$ b $(A + B) \cdot A' \cdot B' = O$
c $(A + B) \cdot (A + B') = A$ d $A \cdot B + C \cdot (A' + B') = A \cdot B + C$
(*Hint:* Use the results from question 4 to shorten your proofs.)

2.6 Digital logic

The contribution of logic and Boolean algebra to the design of digital computers is immense. All digital circuits rely on the application of the basic principles we have learned in this topic. Computer software is constructed using **logic gates** based on some of the rules of logic laid down by Aristotle.

Digital truth values

Digital circuits consist of electrical current flowing through wires that connect the various components. The computer recognises the presence of electricity as 'True' and the absence of electricity as 'False'.

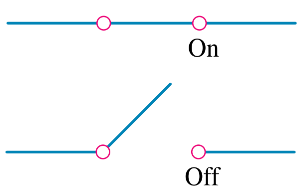
Furthermore, it is the accepted convention that we denote the presence of electricity by 1 and the absence by 0. (In some systems the value of 1 is given to positive electricity and 0 to negative electricity.) Thus we have the basic conversion rule that we will apply here as shown in the table below.

Logical value	Digital value	Spoken value
False	0	Off
True	1	On

The so-called 'on–off' values come from the notion of a switch: if the current flows through, the switch is on; otherwise it is off — as with a light switch.

Gates

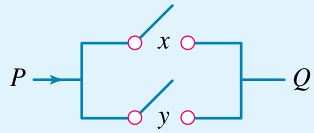
A gate is an electrical component that controls the flow of electricity in some way. It is similar to a gate on a farm, which sometimes lets the sheep through and sometimes doesn't. The simplest possible gate is the switch itself. It has two states, on and off, as shown in the figures at right.



When drawing a switch on a diagram it is conventional to show the ‘off’ position. By combining switches in certain configurations, we can create simple logic circuits.

WORKED EXAMPLE 19

Consider the pair of switches arranged (in parallel) as shown in the figure at right. Assume there is electricity at P. What positions of the two switches, x and y, will allow a current to flow through?



THINK

1 List the possible positions for each switch. Switch x can be either off or on (0 or 1) independently of y, so there are $2 \times 2 = 4$ possible positions.

2 Consider $x = 0$, $y = 0$. There will be no current at Q. Otherwise, if $x = 1$ there will be a current at Q. Similarly, if $y = 1$ there will be a current at Q. If both $x = 1$ and $y = 1$ there will be a current at Q.

Note: We can consider this as the ‘truth table’ for this circuit.

WRITE

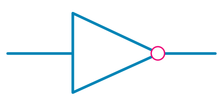
x	y	Q
0	0	0
0	1	1
1	0	1
1	1	1

Because of the similarity of this truth table to the Boolean operator ‘+’ (‘or’), we can symbolise this circuit as $Q = x + y$.

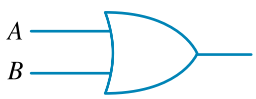
In theory, a computer could be constructed from nothing more than thousands (millions, billions . . .) of switches. However, the design of a logic circuit would be a long, time-consuming process. Furthermore, it is not clear ‘who’ turns the switches on or off. Hence, more complex logic gates were constructed as ‘black box’ components that could be combined quickly to perform relatively complex operations.

A logic gate consists of one or two inputs and one output. The inputs are ‘wires’ that are either off (0) or on (1). Similarly, the output is either 0 or 1. Inputs require a continuous source of electricity in order to remain at either 0 or 1.

The following table shows the gates we will use. Note that inputs are always on the left; output is always on the right.

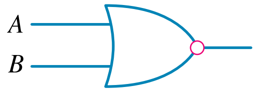
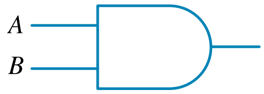
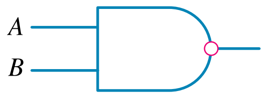
<table>
<tr><th>Name</th><th>Symbol</th><th>Truth table</th><th>Comments</th></tr>
<tr><td>NOT</td><td></td><td>

Input	Output
0	1
1	0

</td><td>Equivalent to Boolean ‘not’</td></tr>
<tr><td>OR</td><td>
</td><td>

Input A	Input B	Output
0	0	0
0	1	1
1	0	1
1	1	1

</td><td>Equivalent to Boolean ‘or’</td></tr>
</table>

<table>
<tr><th>Name</th><th>Symbol</th><th>Truth table</th><th>Comments</th></tr>
<tr><td>NOR</td><td></td><td><table><tr><th>Input A</th><th>Input B</th><th>Output</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td><td>Equivalent to Boolean ‘or’ followed by ‘not’</td></tr>
<tr><td>AND</td><td></td><td><table><tr><th>Input A</th><th>Input B</th><th>Output</th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table></td><td>Equivalent to Boolean ‘and’</td></tr>
<tr><td>NAND</td><td></td><td><table><tr><th>Input A</th><th>Input B</th><th>Output</th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table></td><td>Equivalent to Boolean ‘and’ followed by ‘not’</td></tr>
</table>

studyon

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Digital logic
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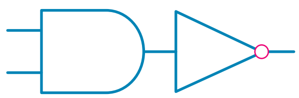
Practice questions

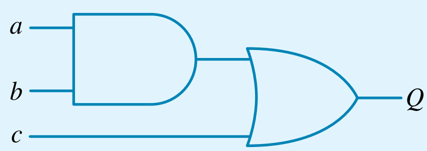
NAND and NOR gates, although they lack equivalent Boolean expressions, are convenient ways of combining AND or OR with NOT. For example, a NAND gate is equivalent to the combination shown at right.



Very sophisticated circuits can be constructed from combinations of these five gates, and the truth table of the output for all possible inputs can be determined.

WORKED EXAMPLE 20

Determine the truth table for the output Q in terms of the inputs a, b and c.

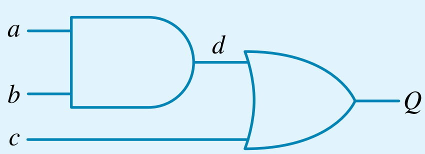


THINK

1 Working from left to right, determine the truth table for the output d in terms of inputs a and b.

2 Use the truth table for an AND gate.

WRITE/DRAW



a	*b*	*d*
0	0	0
0	1	0
1	0	0
1	1	1

3 Now consider the output d to be the input to the OR gate, combined with c to determine the truth table at Q.

Note that the first four rows correspond to step 2 for the case of $c = 0$, and the second four rows correspond to step 2 for the case of $c = 1$.

a	*b*	*d*	*c*	*Q*
0	0	0	0	0
0	1	0	0	0
1	0	0	0	0
1	1	1	0	1
0	0	0	1	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1

An alternative approach is to start with all inputs (a, b and c) and lay out a 'blank' truth table for these three inputs. Add columns for each gate as required.

A 'blank' truth table for three inputs.

a	*b*	*c*
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

The completed truth table.

a	*b*	*c*	*d*	*Q*
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

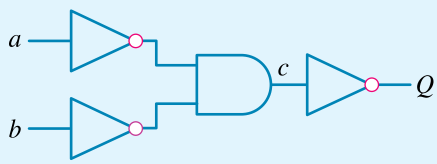
It should be clear that this truth table is equivalent to the one in step 3 of Worked example 20, with the rows in different order. Furthermore, this circuit of an AND and an OR gate is logically equivalent to the statement $(a \wedge b) \vee c$, or in Boolean algebra terms $(a \cdot b) + c$.

Simplifying logic circuits

In some cases an apparently complex circuit can be reduced to a simpler one.

WORKED EXAMPLE 21

Find a circuit equivalent to the one shown at right.



THINK

1 Determine the truth table of the circuit. Start by determining the output at c. Note that the inputs to the AND gate are 'inverted' by the two NOT gates.

WRITE/DRAW

a	*b*	*c*
0	0	1
0	1	0
1	0	0
1	1	0

Boolean expression $= (a' \cdot b')$

2 Complete the truth table by determining the output at Q. This is just the negation of c.

a	b	c	Q
0	0	1	0
0	1	0	1
1	0	0	1
1	1	0	1

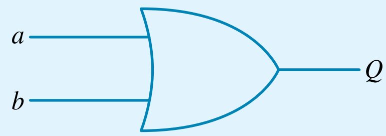
3 Write out the Boolean expression for Q by working backwards from Q.

$Q = c'$ (but $c = a' \cdot b'$)
$= (a' \cdot b')'$

4 Simplify, using the rules for Boolean algebra.

$Q = a'' + b''$ 2nd deMorgan's law
$= a + b$ 'double' negative

5 Create the equivalent circuit. In this case it is a single OR gate.



The original, more complicated circuit might have been used because of availability or cost of components. Otherwise, it would be advantageous to use the circuit in step 5.

Often, one has to design a logic circuit given a Boolean expression.

WORKED EXAMPLE 22 **Determine the logic circuit for the Boolean expression $Q = (a + b') \cdot (a + c')$.**

THINK / **WRITE/DRAW**

1 Determine the number of 'independent' inputs.

There are three inputs: a, b and c.

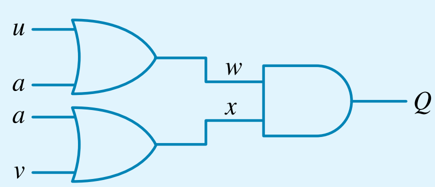
2 Reduce the original Boolean expression to simpler component parts.

Let $u = b'$ and $v = c'$
$Q = (a + u) \cdot (a + v)$
Let $w = a + u$
Let $x = a + v$
$Q = w \cdot x$

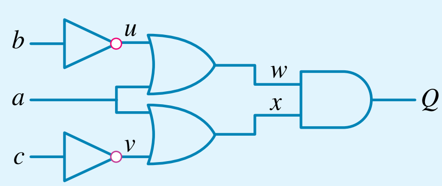
This last expression is as simple as possible.

3 Begin with the last, 'simplest' expression.
This is an AND gate with w and x as inputs, Q as output.

4 (a) Using $w = a + u$, add an OR gate with a and u as inputs, w as output.
(b) Using $x = a + v$, add an OR gate with a and v as inputs, x as output.
Note that input a has been 'duplicated' for each OR gate.



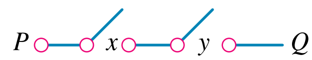
5 (a) Using $u = b'$ and $v = c'$, add two NOT gates to complete the circuit.
(b) The two a inputs must be connected.
What would be the truth table for Q?



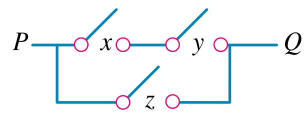
EXERCISE 2.6 Digital logic

PRACTISE

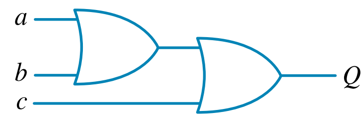
1 WE19 Consider the pair of switches arranged (in series) as shown at right. Assuming that there is electricity at P, when is there current at Q for various positions of the switches x and y?



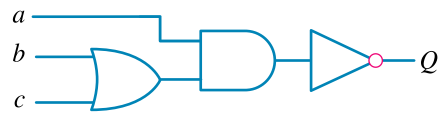
2 Consider the three switches arranged as shown at right. Assuming that there is electricity at P, when is there current at Q for various positions of the switches x, y and z?



3 WE20 Determine the truth table for the output Q in terms of the inputs a, b and c for the circuit at right.

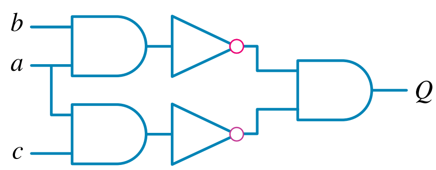


4 Determine the truth table for the output Q in terms of the inputs a, b and c for the circuit at right.

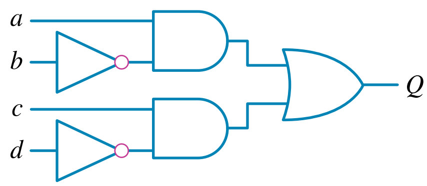


5 WE21 **a** Determine the truth table for the output Q in terms of the inputs a, b and c for the circuit at right.

b Hence, show that this circuit is equivalent to the one in question **4**.



6 Determine the truth table for the output Q in terms of the inputs a, b, c and d for the circuit at right.



7 WE22 Determine the logic circuit for the Boolean expression $Q = a \cdot (b + c)'$.

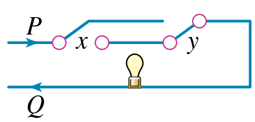
8 Design a logic circuit equivalent to the Boolean expression $Q = [A \cdot (B' \cdot C')] + [A \cdot (B \cdot C)]$.

CONSOLIDATE

9 The Boolean equivalent of the circuit in question **1** is:

A $Q = x + y$ **B** $Q = x - y$ **C** $Q = x \cdot y$
D $Q = x' + y$ **E** $Q = (x \cdot y)'$

10 Consider the circuit depicted at right, which represents a light fixture in a hallway connected to two switches, x and y. The light is 'on' whenever there is a direct connection between P and Q.



a Determine the truth table for this circuit.
b What would be an application for this?

11 A Boolean expression for Q in question **10** in terms of x and y is:

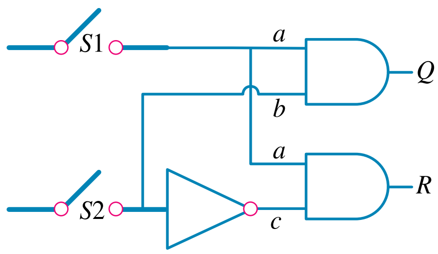
A $x + y$ **B** $x \cdot y$ **C** $(x + y) \cdot (x' + y')$
D $x \cdot y + x' \cdot y'$ **E** $(x + y) \cdot (x + y)'$

12 Modify the circuit in question **10** so that the light comes on only when either (or both) of the two switches is in the 'on' position.

13 The Boolean equivalent to the circuit in question 4 is:

A $[a \cdot (b + c)]'$
B $a' + (b' \cdot c')$
C $(a' + b') \cdot (a' + c')$
D both A and B are correct
E A, B and C are all correct

14 Consider the circuit shown at right, which can be used as an alarm system to protect a safe.



If there is a '1' at R, the alarm rings.

If there is a '1' at Q, the safe can be unlocked.

a Determine the truth table for this circuit.
b Hence, describe the operation of this circuit.

15 a Use deMorgan's laws to show that $a \cdot b = (a' + b')'$.
b Hence, construct a logic circuit equivalent to an AND gate.

16 Show how a single NAND gate can be the equivalent of a NOT gate.

17 By simplifying the expression for Q in question 8, design a circuit with fewer components.
(*Hint:* Consider using NAND or NOR gates.)

18 The designer of the circuit in question 7 does not have any NOT gates available. Redesign the circuit using NOR and/or NAND gates to replace the NOT gate.

MASTER

19 Design a logic circuit for the Boolean expression $Q = a \cdot b + a' \cdot c$ without using any NOT gates.

20 A conditional circuit: Up until now, we have not seen a digital equivalent, or even a Boolean equivalent, of the important logical expression $a \Rightarrow b$. The truth table for the conditional statement is shown at right.

a	b	$a \Rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

a From the following list of statements, find the one that has the same truth table as $a \Rightarrow b$.
i $a \cdot b'$
ii $a' \cdot b$
iii $a' + b$
iv $a + b' \cdot a \cdot b' \cdot b$

b Design a logic circuit equivalent to $a \Rightarrow b$.
c Design a logic circuit equivalent to $b \Rightarrow a$.
d Find a Boolean statement equivalent to $(a \Rightarrow b) \cdot a$.
e Find a Boolean statement equivalent to the *modus ponens* argument, namely:

$$\begin{array}{l} a \Rightarrow b \\ \underline{a \qquad} \\ b \end{array}$$

and simplify, as much as possible, using the result from part d.

f Design a circuit equivalent to the Boolean statement from part e and show that the output is always 1. Thus you have established the validity of the *modus ponens* argument.

eBook*plus* **ONLINE ONLY**

2.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyon

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

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Units 1 & 2 **Algebra and logic**

Sit topic test

2 Answers

EXERCISE 2.2

1 a Opinion

b T/F

c T

d Question

2 a Instruction

b T/F

c F

d Near-statement

3 a p = Sydney on time

q = Perth fully booked

$p \wedge q$

$2^2 = 4$ rows

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

b p = John passed.

q = Zia passed.

r = David passed.

$p \wedge q \wedge r$

$2^3 = 8$ rows

p	q	r	$p \wedge q \wedge r$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

4 a p = Alice does the dishes.

q = Renzo does the dishes.

r = Carla does the dishes.

$(p \wedge q) \vee r$

$2^3 = 8$ rows

p	q	r	$(p \wedge q) \vee r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

b p = female member

q = student

r = professor

$p \wedge (q \vee r)$

$2^3 = 8$ rows

p	q	r	$p \wedge (q \vee r)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

5 a

p	q	$p \wedge \sim q$
T	T	F
T	F	T
F	T	F
F	F	F

b

p	q	$\sim p \wedge \sim q$
T	T	F
T	F	F
F	T	F
F	F	T

c

p	q	r	$(p \wedge q) \wedge r$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

6 a

p	q	$p \vee \sim q$
T	T	T
T	F	T
F	T	F
F	F	T

b

p	q	$\sim p \vee \sim q$
T	T	F
T	F	T
F	T	T
F	F	T

c

p	q	r	$(p \vee q) \vee r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

7

p	q	$\sim(p \vee q)$	$(\sim p \wedge \sim q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Equivalent

8

p	q	$(p \vee q)$	$\sim p$	$(p \vee q) \vee \sim p$	$(p \vee \sim p)$
T	T	T	F	T	T
T	F	T	F	T	T
F	T	T	T	T	T
F	F	F	T	T	T

Equivalent

9 a The car has 4 seats.
The car has air conditioning.

b The Department of Finance was over budget in 2015.
The Department of Defence was over budget in 2015.

c Bob went to the hotel.
Carol went to the hotel.
Ted went to the hotel.
Alice went to the hotel.

d To be a best-seller a novel must be interesting to the reader.
To be a best-seller a novel must be relevant to the reader.

e Sam will win the trophy.
Nancy will win the trophy

f You can choose vanilla ice-cream for dessert.
You can choose strawberry ice-cream for dessert.
You can choose fruit for dessert.

g There are some statements that cannot be proved to be true.
There are some statements that cannot be proved to be false.

h Most of my friends studied Mathematics.
Most of my friends studied Physics.
Most of my friends studied Engineering.
Most of my friends studied Law.
Most of my friends studied Arts.

10 a John and Mary rode their bicycles to school.

b The book you want is in row 3 or 4.

c The weather is cold and cloudy.

d Many people read novels or history.

e In a recent poll 80% preferred jazz or classical music.

f Two is an even prime number.
Alternative answer: Two is the only even prime number.

11 B

12 D

13 a

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

8 ways

b

p	q	r	s
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F

p	q	r	s
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

16 ways

c See the table at the foot of the page.*

*13c

p	q	r	s	t
T	T	T	T	T
T	T	T	T	F
T	T	T	F	T
T	T	T	F	F
T	T	F	T	T
T	T	F	T	F
T	T	F	F	T
T	T	F	F	F

p	q	r	s	t
T	F	T	T	T
T	F	T	T	F
T	F	T	F	T
T	F	T	F	F
T	F	F	T	T
T	F	F	T	F
T	F	F	F	T
T	F	F	F	F

p	q	r	s	t
F	T	T	T	T
F	T	T	T	F
F	T	T	F	T
F	T	T	F	F
F	T	F	T	T
F	T	F	T	F
F	T	F	F	T
F	T	F	F	F

p	q	r	s	t
F	F	T	T	T
F	F	T	T	F
F	F	T	F	T
F	F	T	F	F
F	F	F	T	T
F	F	F	T	F
F	F	F	F	T
F	F	F	F	F

32 ways

14

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

15 a It is raining and I bring my umbrella.

b It is raining or I bring my umbrella.

c It is not raining and I bring my umbrella.

16 a Peter and Quentin like football.

b Peter or Quentin like football.

c Peter likes football or Quentin does not like football.

17

p	q	$\sim(p \vee q)$	$\sim p \vee \sim q$
T	T	F	F
T	F	F	T
F	T	F	T
F	F	T	T

Not equivalent

18

p	q	$(p \wedge q)$	$\sim p$	$(p \wedge q) \vee \sim p$	$(p \vee q)$	$(p \vee q) \wedge \sim p$
T	T	T	F	T	T	F
T	F	F	F	F	T	F
F	T	F	T	T	T	T
F	F	F	T	T	F	F

Not equivalent

19

p	q	r	$(p \wedge q) \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	T	F
F	F	F	F	F

Not equivalent

20 a

p	q	r	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Equivalent

b

p	q	r	$(p \vee q) \vee r$	$p \vee (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Equivalent

c Brackets have no effect on expressions with a single $\vee$ or $\wedge$ operator, but they do have an effect if they are mixed up together.

EXERCISE 2.3

1

p	q	$\sim p$	$\sim q$	$(p \Rightarrow q)$	$(\sim q \Rightarrow \sim p)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Equivalent

2

p	q	$\sim p$	$\sim q$	$(p \Rightarrow q)$	$(\sim p \Rightarrow \sim q)$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Not equivalent

3 a Conclusion: My pet is fluffy.

b Conclusion: Two is the only even prime number.

4 a Conclusion: Growing apples depends on good irrigation.

b Conclusion: I will bring an umbrella tomorrow.

5 a and c are valid.

6 c is valid.

7

p	q	$p \vee q$	$\sim p$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

The conclusion is true whenever all premises are true (3rd row), thus the argument is valid.

8 a

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

The conclusion is true whenever all premises are true (1st, 5th, 7th and 8th rows).

b

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

The conclusion is true whenever all premises are true (4th row), thus the argument is valid.

c

p	q	r	s	$p \Rightarrow q$	$r \Rightarrow s$	$p \Rightarrow q \wedge r \Rightarrow s$	$p \wedge r$	$q \wedge s$
T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T	F
T	T	F	T	T	T	T	F	T
T	T	F	F	T	T	T	F	F
T	F	T	T	F	T	F	T	F
T	F	T	F	F	F	F	T	F
T	F	F	T	F	T	F	F	F
T	F	F	F	F	T	F	F	F
F	T	T	T	T	T	T	F	T
F	T	T	F	T	F	F	F	F
F	T	F	T	T	T	T	F	T
F	T	F	F	T	T	T	F	F
F	F	T	T	T	T	T	F	F
F	F	T	F	T	F	F	F	F
F	F	F	T	T	T	T	F	F
F	F	F	F	T	T	T	F	F

The conclusion is true whenever all premises are true (1st row).

9 a

p	q	$p \Rightarrow q$	$\sim p$	$\sim q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

p = If elected with a majority

q = My government will introduce new tax laws

The conclusion is false when premises are true in the 3rd row, thus the argument is invalid.

b $p \Rightarrow q$

$\sim p$

$\overline{\sim q}$

10

p	q	$\sim p \Rightarrow \sim q$
T	T	T
T	F	T
F	T	F
F	F	T

Invalid argument

11 Implication: If it is bread then it is made with flour.

Converse: If it is made with flour then it is bread.

Contrapositive: If it is not made with flour then it is not bread.

Inverse: If it is not bread then it is not made with flour.

12 C

13 A

14 **5b** All footballers are fit.
David is not fit.
David is not a footballer.

6a Cannot be made into a valid argument.

6b All musicians can read music.
Louise is a musician.
Louise can read music.

15 E

16 a Disjunctive syllogism

b *Modus tollens*

c *Modus ponens*

17 C

18 D

19 Converse: If she sends her children to good schools the politician is intelligent.
Contrapositive: If she doesn't send her children to good schools the politician is not intelligent.
Inverse: If a politician isn't intelligent, she doesn't send her children to good schools.

20 a Valid — hypothetical syllogism

b $p \Rightarrow q$; $r \Rightarrow q$; $p \Rightarrow r$, invalid

c Valid — *modus tollens*

d Valid — constructive dilemma

e p = The team plays well.
q = The offence was good.
r = The defence was good.
$p \Rightarrow (q \vee r)$
$\sim p \wedge \sim p$
———
$\sim r$

p	q	r	$(q \vee r)$	$p \Rightarrow (q \vee r)$	$\sim q \wedge \sim p$	$\sim r$
T	T	T	T	T	F	F
T	T	F	T	T	F	T
T	F	T	T	T	F	F
T	F	F	F	F	F	T
F	T	T	T	T	F	F
F	T	F	T	T	F	T
F	F	T	T	T	T	F
F	F	F	F	T	T	T

Invalid argument

21 a $p \Rightarrow q$
$\sim p$
———
$\sim q$

b

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

The conclusion is false when premises are true (3rd row), therefore the argument is invalid.

22 a

p	q	r	$p \Rightarrow q$	$r \Rightarrow \sim q$	$p \Rightarrow \sim r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Valid argument

b

p	q	r	$\sim p \wedge \sim q$	$r \Rightarrow p$
T	T	T	F	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	T	F
F	F	F	T	T

Invalid argument

EXERCISE 2.4

1 p = She plays well.
q = She wins.

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \vee \sim q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

Therefore, this is a tautology.

2 a

p	q	$p \Rightarrow q$	$\sim p \Rightarrow \sim q$	$(p \Rightarrow q) \vee \sim p \Rightarrow \sim q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Therefore, this is a tautology.

b

p	q	$p \wedge q$	$(p \wedge q) \vee \sim q$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	T

Therefore, this is not a tautology.

3

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r) \rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The last column is always true. Thus, this is a tautology and the argument is valid.

4 See the table at the foot of the page.*

5 If x is even, then write it as $2n$.
$(2n)^2 = 4n^2$
If a number is multiplied by 4 then it is even.
Therefore $4n^2$ is even.

6 If a number x is even, then x^2 is even.
24 is even, therefore 24^2 is even.

7 Assume n is not odd and show that n^2 is even.

8 a Assume $a \neq b$, then multiply both sides by x.

b Assume $n < 2$, then write it as $2 - x$ and square it.

c Assume n is not divisible by 2, therefore it is odd, therefore write it as $(2x + 1)$, then square it.

9 Assume it is positive and compare it with the product of 2 positive numbers of the same magnitude.

10 Assume that a is the smallest positive real number.

Let $x = \frac{a}{2}$.

Since $a > 0$, then $x > 0$ and $x < a$ (property of division). This contradicts the assumption that a is the smallest positive number.

11 Let $x = 2$, then $x^2 = 4$; let $x = -2$, then $x^2 = 4$.

12 Demonstrate that 2 is both a prime number and is even.

13 a See the table at the foot of the page.*

b

p	q	$\sim p \Rightarrow \sim q$	$(\sim p \Rightarrow \sim q) \wedge q$	$(\sim p \Rightarrow \sim q) \wedge q \Rightarrow p$
T	T	T	T	T
T	F	T	F	T
F	T	F	F	T
F	F	T	F	T

Hence, this is a valid argument.

*4

p	q	r	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow r$	$(\sim p \Rightarrow \sim q) \wedge (\sim q \Rightarrow r)$	$\sim p \Rightarrow r$	$(\sim p \Rightarrow \sim q) \wedge (\sim q \Rightarrow r) \rightarrow (\sim p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	F	T	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	F	F	F	T

Hence, this is a valid argument.

*13a

p	q	r	$p \Rightarrow \sim q$	$q \Rightarrow \sim r$	$(\sim p \Rightarrow \sim q) \wedge (q \Rightarrow \sim r)$	$p \Rightarrow \sim r$	$(\sim p \Rightarrow \sim q) \wedge (q \Rightarrow \sim r) \rightarrow (p \Rightarrow \sim r)$
T	T	T	F	F	F	F	T
T	T	F	F	T	F	T	T
T	F	T	T	T	T	F	F
T	F	F	T	T	T	T	T
F	T	T	T	F	F	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence, this is an invalid argument (row 3).

c

p	q	$\sim p \Rightarrow q$	$(\sim p \Rightarrow q) \wedge p$	$(\sim p \Rightarrow q) \wedge p \Rightarrow \sim q$
T	T	T	T	F
T	F	T	T	T
F	T	T	F	T
F	F	F	F	T

Hence, this is an invalid argument (row 1).

14 C

15 a
$$\begin{array}{l} p \Rightarrow \sim q \\ \underline{\sim q \Rightarrow r} \\ p \Rightarrow r \end{array}$$

b
$$\begin{array}{l} \sim p \Rightarrow q \\ \underline{p \qquad\quad} \\ \sim q \end{array}$$

c
$$\begin{array}{l} p \Rightarrow \sim q \\ \underline{q \qquad\quad} \\ \sim p \end{array}$$

d
$$\begin{array}{l} \sim p \Rightarrow \sim q \\ \underline{\sim q \Rightarrow \sim r} \\ \sim p \Rightarrow \sim r \end{array}$$

16 Assume $\sqrt{2}$ is rational, so that $\sqrt{2} = \frac{a}{b}$, where a and b are integers that have no common factors.

Therefore $2 = \frac{a^2}{b^2}$, or $a^2 = 2b^2$.

Therefore a^2 is a multiple of 2 and therefore a is a multiple of 2 (from $a^2 = a \times a$)
Because it is a multiple of 2, write $a = 2x$.
Therefore $a^2 = 4x^2 = 2b^2$.
Therefore $b^2 = 2x^2$ and is thus a multiple of 2.
Therefore both a and b are multiples of 2 and have a common factor of 2.
This contradicts our initial statement, so it must be false.

17 Prove by counterexample.

18 D

19 Assume n is the largest possible integer.
Let $x = n + 1$.
Therefore $x > n$, which contradicts our initial statement.

20 Consider the contrapositive statement.

21 Consider what would happen if $x > y + z$. This would imply that the shortest distance from A to C is not a straight line!

22 Answers will vary.

23 a Mathematical induction is used particularly for sums of series.

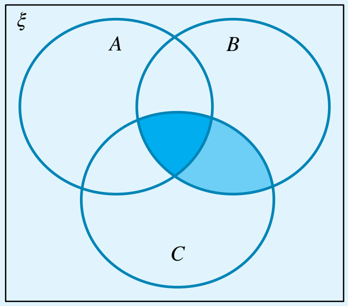
b The values of n are restricted to integer values.

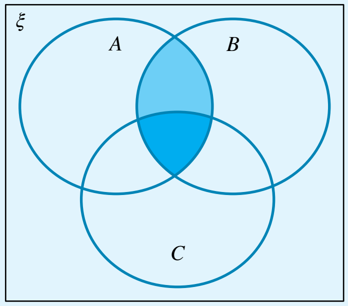
c Answers will vary.

24 Answers will vary.

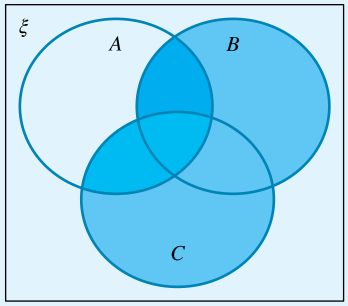
EXERCISE 2.5

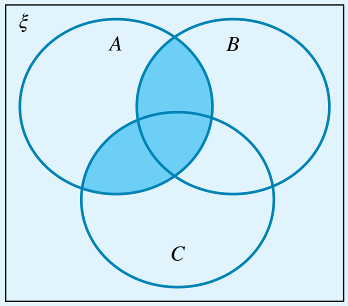
1





2





3 Part 1: Show that $(A \cdot B) + (A' + B') = I$.

$$\begin{aligned}(A \cdot B) + (A' + B') &= (A + A' + B') \cdot (B + A' + B') \\ &= (I + B') \cdot (A' + I) \\ &= (I) \cdot (I) \\ &= I \qquad \text{QED}\end{aligned}$$

Part 2: Show that $(A \cdot B) \cdot (A' + B') = \mathrm{O}$.

$$\begin{aligned}(A \cdot B) \cdot (A' + B') &= A \cdot B \cdot A' + A \cdot B \cdot B' \\ &= \mathrm{O} \cdot B + A \cdot \mathrm{O} \\ &= \mathrm{O} + \mathrm{O} \\ &= \mathrm{O} \qquad \text{QED}\end{aligned}$$

4 Answers will vary.

5 $(p \wedge q) \wedge \sim p = (p \cdot q) \cdot p' = (p \cdot p') \cdot q = \mathrm{O} \cdot q = \mathrm{O}$
$p \wedge \sim p = p \cdot p' = \mathrm{O}$ QED

6 $(p \wedge q) \vee \sim p = (p \cdot q) + p' = q$
$(p \vee q) \wedge \sim p = (p + q) \cdot p' \neq q$
Not equivalent

7 B

8 a $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

b $B = \{4, 8, 12, 16, \ldots\}$

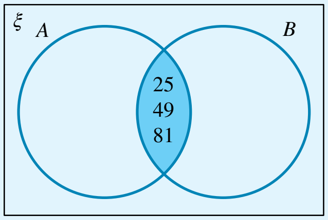
c $C = \{2\}$

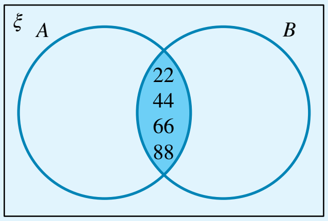
d $D = \{\text{jack, queen, king}\}$

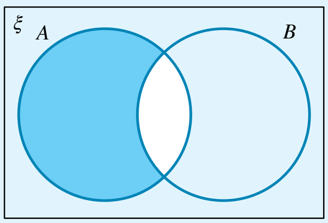
e $E = \varnothing$

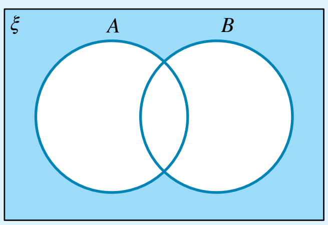
f $F = \{9, 8, 7, 6, \ldots\}$

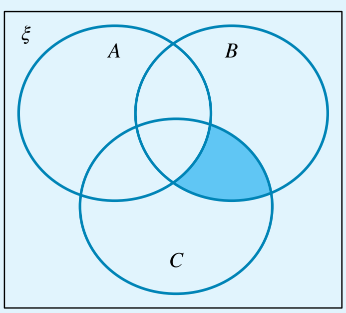
9 A, C, D, E

10


11


12 a


b


c


13 a $\{4\}$

b $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

d $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

e $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

14 a $(B \cup C) = \{1, 2, 4, 6, 8, 9, 10\}$

b $A \cap (B \cup C) = \{1, 2, 4, 6, 8, 9, 10\}$

c $(A \cap C) = \{1, 4, 9\}$

d $(A \cap B) = \{2, 4, 6, 8, 10\}$

e $(A \cap B) \cup (A \cap C) = \{1, 2, 4, 6, 8, 9, 10\}$

15 a $\{11, 13, 17, 19, 31, 41, 61, 71, 91\}$

b $\{2, 3, 5, 7, 11, 13, 16, 17, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 91, 97\}$

c $\{61\}$

16 C

17 a $A + B$

b I

c $A + B$

d $A \cdot B$

18 a $A + B + A' + B' = A + A' + B + B' = I + I = I$

b $(A + B) \cdot A' \cdot B' = A \cdot A' \cdot B' + B \cdot A' \cdot B'$
$= OB' + OA' = O + O = O$

c $(A + B) \cdot (A + B') = (A + B) \cdot A + (A + B) \cdot B'$
$= A + A = A$

d $A \cdot B + C \cdot (A' + B') = A \cdot B + C \cdot (A \cdot B)'$
$= A \cdot B + C$

EXERCISE 2.6

1

x	*y*	*Q*
0	0	0
0	1	0
1	0	0
1	1	1

2

x	*y*	*z*	*Q*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

3

a	*b*	*c*	Output
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

4

a	*b*	*c*	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

5 a

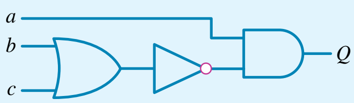
a	*b*	*c*	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

b It has the same truth table as question **4**.

6

a	*b*	*c*	*d*	Output
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	0

7



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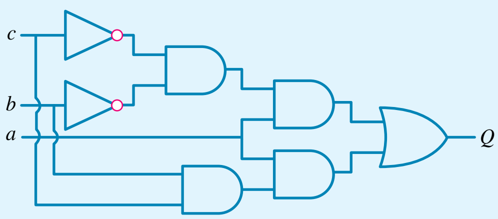


9 C

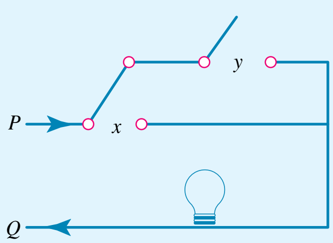
10 a

x	*y*	*Q*
0	0	1
0	1	0
1	0	0
1	1	1

b This circuit could be used where there are two people who can activate the light separately.

11 D

12



13 E

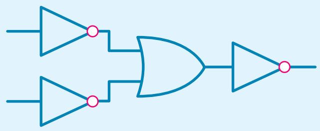
14 a

*S*1	*S*2	*a*	*b*	*c*	*Q*	*R*
0	0	0	0	1	0	0
0	1	0	1	0	0	0
1	0	1	0	1	0	1
1	1	1	1	0	1	0

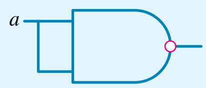
b When $S1 = 0$, the system is disabled; the safe can't be opened.
When $S1 = 1$ and $S2 = 0$, the alarm rings.
When $S1 = 1$ and $S2 = 1$, the safe can be opened.

15 a $a \cdot b = [(a \cdot b)']'$
$a \cdot b = (a' + b')'$

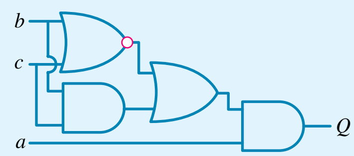
b



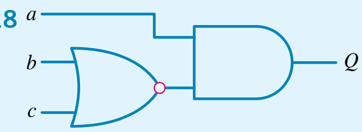
16 a



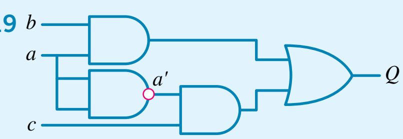
17 $Q = A \cdot [(B + C)' + B \cdot C]$



18

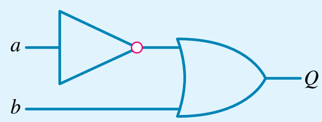


19

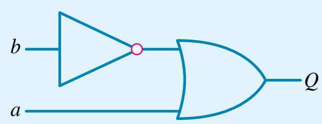


20 a iii $a' + b$

b



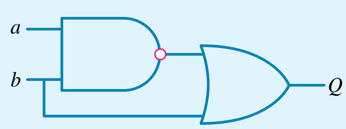
c



d $a \cdot b$

e $(a \cdot b)' + b$

f



a	*b*	$(a \cdot b)'$	$(a \cdot b)' + b$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	1

3

Sequences and series

3.1 Kick off with CAS

1 Add the following sets of numbers:

a $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$

b $9 + -6 + 4 + -\frac{8}{3} + \frac{16}{9} + -\frac{32}{27}$

c $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$

2 Using CAS technology, find the template or command to simplify the following:

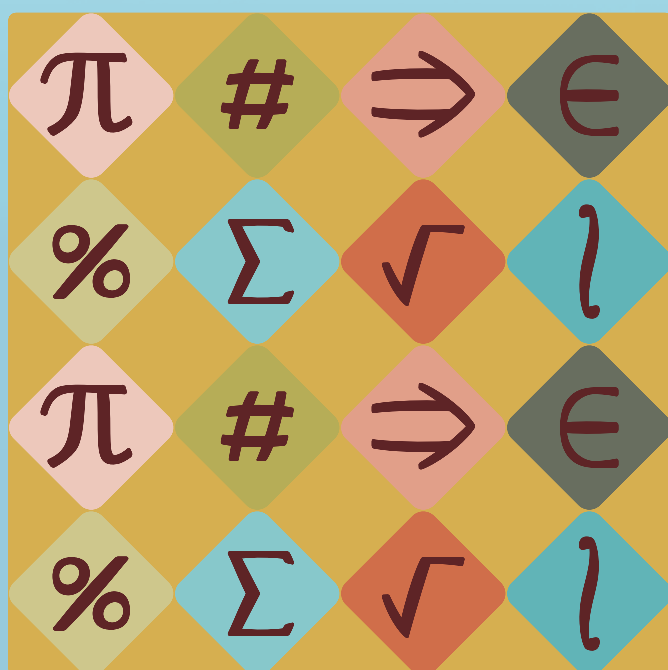
a $\sum_{n=1}^{10}(2n)$

b $\sum_{t=0}^{6} 9 \times \left(-\frac{2}{3}\right)^{t}$

c $\sum_{n=0}^{7}(2^{n})$

3 Compare the answers for questions 1 and 2, and explain how the sum (Σ) command was used to add these series of numbers.

4 Use the sum (Σ) command to add the following series:
$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100$



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

3.2 Describing sequences

studyon

Units 1 & 2

AOS 2

Topic 2

Concept 1

Describing sequences

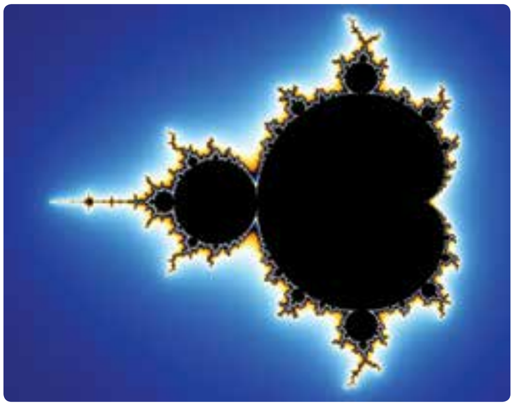
Concept summary

Practice questions

Sequences of numbers play an important part in our everyday life. For example, the following sequence:

$$2.25, 2.37, 2.58, 2.57, 2.63, \ldots$$

gives the end-of-day trading price (for 5 consecutive days) of a share in an electronics company. It looks like the price is on the rise, but is it possible to accurately predict the future price per share of the company?



The following sequence is more predictable:

$$10\,000, 9000, 8100, \ldots$$

This is the estimated number of radioactive decays of a medical compound each minute after administration to a patient. The compound is used to diagnose tumours. In the first minute, 10 000 radioactive decays are predicted; during the second minute, 9000, and so on. Can you predict the next number in the sequence? You're correct if you said 7290. Each successive term here is 90% of, or 0.90 times, the previous term.

Sequences are strings of numbers. They can be finite in number or infinite. Number sequences may follow an easily recognisable pattern or they may not. A great deal of recent mathematical work has gone into deciding whether certain strings follow a pattern (in which case subsequent terms could be predicted) or whether they are random (in which case subsequent terms cannot be predicted). This work forms the basis of chaos theory, speech recognition software for computers, weather prediction and stock market forecasting, to name but a few uses. The list is almost endless. The image above is a visual representation of a sequence of numbers called a Mandelbrot set.

Sequences that follow a pattern can be described in a number of different ways. They may be listed in sequential order; they may be described as a functional definition; or they may be described in an iterative definition.

Listing in sequential order

Consider the sequence of numbers t: $\{5, 7, 9, \ldots\}$. The numbers in sequential order are firstly 5 then 7 and 9, with the indication that there are more numbers to follow. The symbol t is the name of the sequence, and the first three terms in the sequence shown are $t_1 = 5$, $t_2 = 7$ and $t_3 = 9$. The fourth term, t_4, if the pattern were to continue, would be the number 11. In general, t_n is the nth term in the sequence. In this example, the next term is simply the previous term with 2 added to it, with the first term being the number 5.

Another possible sequence is t: $\{5, 10, 20, 40, \ldots\}$. In this case it appears that the next term is twice the previous term. The fifth term here, if the pattern continued, would be $t_5 = 80$. It can be difficult to determine whether or not a pattern exists in some sequences. Can you find the next term in the following sequence?

$$t: \{1, 1, 2, 3, 5, 8, \ldots\}$$

Here the next term is the sum of the previous *two terms*, hence the next term would be $5 + 8 = 13$, and so on. This sequence is called the Fibonacci sequence and is named after its discoverer Leonardo Fibonacci, a thirteenth century mathematician.

Here is another sequence; can you find the next term here?

$$t: \{7, 11, 16, 22, 29, \ldots\}$$

In this sequence the difference between successive terms increases by 1 for each pair. The first difference is 4, the next difference is 5 and so on. The sixth term is thus 37, which is 8 more than 29.

Functional definition

A functional definition of a sequence of numbers is expressed in the form: $t_n = f(n)$.

An example could be: $t_n = 2n - 7, n \in \{1, 2, 3, 4, \ldots\}$

Using this definition the nth term can be readily calculated. For this example $t_1 = 2 \times 1 - 7 = -5$, $t_2 = 2 \times 2 - 7 = -3$, $t_3 = 2 \times 3 - 7 = -1$ and so on. We can readily calculate the 100th term, $t_{100} = 2 \times 100 - 7 = 193$, simply by substituting the value $n = 100$ into the expression for t_n.

Look at the following example:

$$d_n = 4.9n^2, n \in \{1, 2, 3, \ldots\}$$

For this example, in which the sequence is given the name d, $d_1 = 4.9 \times 1^2 = 4.9$, and $d_2 = 4.9 \times 2^2 = 19.6$. Listing the sequence would yield d: $\{4.9, 19.6, 44.1, 78.4, \ldots\}$. The 10th term would be $4.9 \times 10^2 = 490$.

Here is another example:

$$c_n = \cos(n\pi) + 1, n \in \{1, 2, 3, \ldots\}$$

Here the sequence would be c: $\{0, 2, 0, 2, \ldots\}$.

Recursive definition

A sequence can be generated by the repeated use of an instruction. This is known as **recursion**. Term n is represented by t_n; the next term after this one is represented by t_{n+1}, while the term before t_n is t_{n-1}. For these sequences, the first term must be stated.

Look at the following example:

$$t_{n+1} = 3t_n - 2; t_1 = 6.$$

The first term, t_1, is 6 (this is given in the definition), so the next term, t_2, is $3 \times 6 - 2 = 16$, and the following term is $3 \times 16 - 2 = 46$. In each and all cases, the next term is found by multiplying the previous term by 3 and then subtracting 2. We could write the sequence out as a table:

n	t_n	Comment
1	$t_1 = 6$	Given in the definition
2	$t_2 = 3t_1 - 2$ $= 3 \times 6 - 2$ $= 16$	Using t_1 to find the next term, t_2
3	$t_3 = 3t_2 - 2$ $= 3 \times 16 - 2$ $= 46$	Using t_2 to find the next term, t_3
4	$t_4 = 3t_3 - 2$ $= 3 \times 46 - 2$ $= 136$	Using t_3 to find the next term, t_4

An example of this sequence using notation found in a spreadsheet would be:

A1 = 6 (the first term is equal to 6)

A2 = 3 × A1 − 2 (the next term is 3 times the previous term minus 2).

You could then apply the **Fill Down** option in the **Edit** menu of the spreadsheet from cell A2 downwards to generate as many terms in the sequence as required. This would result in the next cell down being three times the previous cell, less 2. The recursive definition finds a natural use in a spreadsheet environment and consequently is used often. A drawback is that you cannot find the nth term directly as in the functional definition, but an advantage is that more complicated systems can be successfully modelled using recursive descriptions.

WORKED EXAMPLE 1

a **Find the next three terms in the sequence: $\left\{14, 7, \frac{7}{2}, \ldots\right\}$.**

b **Find the 4th, 8th and 12th terms in the following sequence: $e_n = n^2 - 3n$, $n \in \{1, 2, 3, \ldots\}$.**

c **Find the second, third and fifth terms for the following sequence: $k_{n+1} = 2k_n + 1$, $k_1 = -0.50$.**

THINK	WRITE
a In this example the sequence is listed and a simple pattern is evident. From inspection, the next term is half the previous term and so the sequence would be $14, 7, \frac{7}{2}, \frac{7}{4}, \frac{7}{8}, \frac{7}{16}$.	**a** The next three terms are $\frac{7}{4}, \frac{7}{8}, \frac{7}{16}$.
b 1 This is an example of a functional definition. The nth term of the sequence is found simply by substitution into the expression $e_n = n^2 - 3n$.	**b** $e_n = n^2 - 3n$
2 Find the fourth term by substituting $n = 4$.	$e_4 = 4^2 - 3 \times 4$ $= 4$
3 Find the eighth term by substituting $n = 8$.	$e_8 = 8^2 - 3 \times 8$ $= 40$
4 Find the 12th term by substituting $n = 12$.	$e_{12} = 12^2 - 3 \times 12$ $= 108$
c 1 This is an example of a recursive definition. We can find the second, third and fifth terms for the sequence $k_{n+1} = 2k_n + 1$, $k_1 = -0.50$ by recursion.	**c** $k_{n+1} = 2k_n + 1$, $k_1 = -0.50$
2 Substitute $k_1 = -0.50$ into the formula to find k_2.	$k_2 = 2 \times -0.50 + 1$ $= 0$
3 Continue the process until the value of k_5 is found.	$k_3 = 2 \times 0 + 1$ $= 1$ $k_4 = 2 \times 1 + 1$ $= 3$ $k_5 = 2 \times 3 + 1$ $= 7$
4 Write the answer.	Thus $k_2 = 0$, $k_3 = 1$ and $k_5 = 7$.

Logistic equation

The **logistic equation** is a model of population growth. It gives the rule for determining the population in any year, based on the population in the previous year. Because we need the previous term in order to generate the next term of the sequence, the logistic equation is an example of a recursive definition. It is of the general form

$$t_{n+1} = at_n(1 - t_n)$$

where $0 < t_0 < 1$ and a is a constant.

Depending on the value of a, sequences generated by use of the logistic equation could be **convergent**, **divergent** or **oscillating**. A string of numbers that converges to (settles at) a certain fixed value is called a *convergent* sequence. A sequence t_n can converge to only one possible number, x, called the **limit** of the sequence. This can be written as $\lim_{n \to \infty} t_n = x$. (In this context the symbol → is read as 'tends to' or 'approaches'.) A sequence whose terms grow further and further apart is called *divergent*. That is, a sequence is divergent if $t_n \to \infty$ or $t_n \to -\infty$ as $n \to \infty$. Finally, a sequence whose terms tend to fluctuate between two (or more) values is called *oscillating*. An oscillating sequence is neither convergent nor divergent.

WORKED EXAMPLE 2

Given that $a = 2$ and $t_0 = 0.7$, use the logistic equation to generate a sequence of 6 terms and state whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, use $\lim_{n \to \infty} t_n$ to determine the limit.

THINK	WRITE
1 Write the logistic equation, replacing a with its given value (that is, 2).	$t_{n+1} = at_n(1 - t_n)$ $= 2t_n(1 - t_n)$
2 To find t_1, substitute the value of t_0 (that is, 0.7) in place of t_n and evaluate.	$t_1 = 2t_0(1 - t_0)$ $= 2 \times 0.7 \times (1 - 0.7)$ $= 0.42$
3 To find the next term, t_2, substitute the value of t_1 (that is, 0.42) in place of t_n and evaluate.	$t_2 = 2t_1(1 - t_1)$ $= 2 \times 0.42 \times (1 - 0.42)$ $= 0.4872$
4 Continue the iterative process four more times, each time substituting the value of the previous term into the logistic equation to find the next term.	$t_3 = 2t_2(1 - t_2)$ $= 2 \times 0.4872 \times (1 - 0.4872)$ $= 0.499\,672\,3$ $t_4 = 2t_3(1 - t_3)$ $= 2 \times 0.499\,672\,3 \times (1 - 0.499\,672\,3)$ $= 0.499\,999\,8$ $t_5 = 2t_4(1 - t_4)$ $= 2 \times 0.499\,999\,8 \times (1 - 0.499\,999\,8)$ $= 0.5$ $t_6 = 2t_5(1 - t_5)$ $= 2 \times 0.5 \times (1 - 0.5) = 0.5$
5 The terms of the sequence are growing closer and closer to each other, finally settling at 0.5.	The sequence is convergent; the limit of the sequence is 0.5.

Note that instead of saying 'the limit of the sequence is 0.5' in the previous example, we could simply write $t_n \to 0.5$.

EXERCISE 3.2 Describing sequences

PRACTISE

1 **WE1** **a** Find the next three terms in the sequence: $\left\{3, \frac{3}{2}, \frac{3}{4}, \ldots\right\}$.

b Find the second, fourth and sixth terms in the following sequence: $t_n = 4 \times 3^{n-2}$, $n \in \{1, 2, 3, \ldots\}$.

c Find the second, third and fifth terms for the following sequence: $k_{n+1} = k_n + 2$, $k_1 = -5$.

2 **a** Find the next three terms in the sequence: $\{2, -5, 8, -11, 14, \ldots\}$.

b Find the 4th, 8th and 12th terms in the following sequence: $t_n = n^2 - n + 41$, $n \in \{1, 2, 3, \ldots\}$.

c Find the second, third and fourth terms for the following sequence: $k_{n+1} = -\left(k_n{}^2\right) - 2$, $k_1 = 3$.

3 **WE2** Given that $a = 0.8$ and $t_0 = 0.5$, use the logistic equation to generate a sequence of 6 terms and state whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, use $\lim_{n\to\infty} t_n$ to determine the limit.

4 Given that $a = 1.1$ and $t_0 = 0.9$, use the logistic equation to generate a sequence of 6 terms and state whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, use $\lim_{n\to\infty} t_n$ to determine the limit.

CONSOLIDATE

5 For each of the following sequences, write a rule for obtaining the next term in the sequence and hence evaluate the next three terms.

a $\{1, 4, 7, \ldots\}$

b $\{1, 0, -1, -2, \ldots\}$

c $\{1, 4, 16, 64, \ldots\}$

d $\{2, 5, 9, 14, 20, \ldots\}$

e $\{3, 4, 7, 11, 18, \ldots\}$

f $\{2a - 5b, a - 2b, b, -a + 4b, \ldots\}$

g $\{1, 0, -1, 0, 1, \ldots\}$

h $\{1.0, 1.1, 1.11, \ldots\}$

i $\{1024, -512, 256, -128, \ldots\}$

6 Find the first, fifth and tenth terms in the following sequences.

a $t_n = 2n - 5, n \in \{1, 2, 3, \ldots\}$

b $t_n = \dfrac{n}{n+1}, n \in \{1, 2, 3, \ldots\}$

c $t_n = 17 - 3.7n, n \in \{1, 2, 3, \ldots\}$

d $t_n = 5 \times \left(\dfrac{1}{2}\right)^n, n \in \{1, 2, 3, \ldots\}$

e $t_n = 5 \times \left(\dfrac{1}{2}\right)^{(3-n)}, n \in \{1, 2, 3, \ldots\}$

f $t_n = (-1)^n + n, n \in \{1, 2, 3, \ldots\}$

g $t_n = 3^n 2^{-n}, n \in \{1, 2, 3, \ldots\}$

h $t_n = a + (n-1)d, n \in \{1, 2, 3, \ldots\}$

i $t_n = ar^{n-1}, n \in \{1, 2, 3, \ldots\}$

7 Using technology, find the third, eighth and tenth terms in the following sequences.

a $t_{n+1} = -2t_n, t_1 = -3$

b $t_{n+1} = t_n - 7, t_1 = 14$

c $t_{n+1} = -t_n + 2, t_1 = 3$

d $t_{n+1} = t_n + (-1)^n t_n, t_1 = 3$

8 For the sequences in question **7**, use technology to generate their graphs. Place the term number on the horizontal axis and the value of the term on the vertical axis.

9 Given the following values of a and t_0, use the logistic equation to generate a sequence of six terms. State whether the sequence is convergent, divergent or oscillating. If the sequence is convergent, state its limit.

a $a = 0.4, t_0 = 0.6$
b $a = 1.9, t_0 = 0.4$
c $a = 2.1, t_0 = 0.5$
d $a = 2.5, t_0 = 0.3$
e $a = 3, t_0 = 0.2$
f $a = 3.4, t_0 = 0.7$
g $a = 4.2, t_0 = 0.1$
h $a = 4.5, t_0 = 0.8$

10 Study the pattern in each of the following sequences and where possible write the next two terms in the sequence, describing the pattern that you use.

a 5, 6, 8, 11, . . .
b 4, 9, 12, 13, 12, 9, . . .
c 9, 8, 9, 0, . . .
d $6, 12, 12, 6, 1\frac{1}{2}, \ldots$
e 5, 8, 13, 21, . . .
f 1, 3, 7, 15, . . .
g 1, 3, 2, 4, 3, . . .

11 Which of the following functional definitions could be used to describe the sequence $\{3, 1, -1, \ldots\}$?

A $t_n = n - 2, n \in \{1, 2, 3, \ldots\}$
B $t_n = 2n - 5, n \in \{1, 2, 3, \ldots\}$
C $t_n = 5n - 2, n \in \{1, 2, 3, \ldots\}$
D $t_n = 5 - 2n, n \in \{1, 2, 3, \ldots\}$
E $t_n = 2(5 - n), n \in \{1, 2, 3, \ldots\}$

12 Which of the following recursive definitions could be used to describe the sequence $\{20, -10, 5, \ldots\}$?

A $t_{n+1} = t_n - 30, t_1 = 20$
B $t_{n+1} = \frac{t_n}{2}, t_1 = -20$
C $t_{n+1} = t_n - \frac{t_n}{2}, t_1 = 20$
D $t_{n+1} = t_n - 10, t_1 = 20$
E $t_{n+1} = \frac{-t_n}{2}, t_1 = 20$

13 Which of the following sequences is generated by the definition $t_n = \frac{6n^2 - 12}{2}$, $n \in \{1, 2, 3, \ldots\}$?

A $\{-3, 6, 15, \ldots\}$
B $\{-3, 6, -12, \ldots\}$
C $\{-3, 6, 21, \ldots\}$
D $\{-3, 6, 12, \ldots\}$
E $\{-3, 6, 18, \ldots\}$

14 Write the iterative definition for each of the following sequences.

a $\{7, 5, 3, 1, -1, \ldots\}$
b $\{12, 6, 3, 1.5, \ldots\}$
c $\{12, 12.6, 13.2, \ldots\}$
d $\{2, 11, 56, 281, \ldots\}$
e $\{4, -12, 36, \ldots\}$
f $\{2, 4, 16, 256, \ldots\}$

MASTER

15 In the township of Grizabella, the population of stray cats in any given year is given as p_{n+1}. This can be calculated using the formula $p_{n+1} = 1.3p_n(1 - p_n)$, where p_n is the number of cats (in hundreds) in the preceding year. If in 2005 there were 28 stray cats in Grizabella township, calculate:

a the expected number of stray cats for 2006 and 2007
b the limiting number of stray cats that Grizabella township can sustain.

16 In the township of Macavity, the population of stray cats follows the logistic equation $p_{n+1} = 0.3p_n(1 - p_n)$ where p_{n+1} and p_n refer to the population (in hundreds) in any given year and in the preceding year respectively. In 2005, there were 62 stray cats in the township. By generating the sequence of numbers using the above equation, decide what will happen in the long run to the population size of stray cats. (That is, will the population of cats keep increasing, keep decreasing or settle at a particular value?)

3.3 Arithmetic sequences

studyon

Units 1 & 2

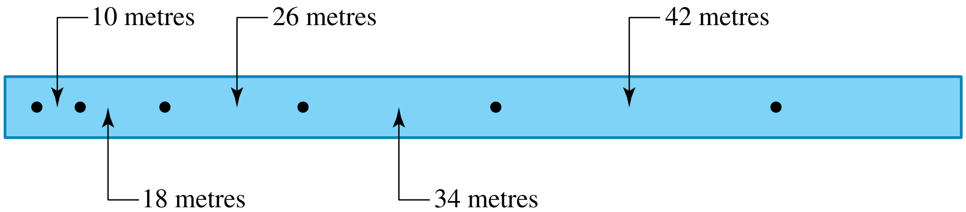
AOS 2

Topic 2

Concept 2

Arithmetic sequences
Concept summary
Practice questions

At a racetrack a new prototype racing car unfortunately develops an oil leak. Each second, a drop of oil hits the road. The driver of the car puts her foot on the accelerator and the car increases speed at a steady rate as it hurtles down the straight. The diagram below shows the pattern of oil drops on the road with the distances between the drops labelled.



The sequence of distances travelled in metres each second is $\{10, 18, 26, 34, 42, \ldots\}$. The first term in the sequence, t_1, is 10, and as you can see, each subsequent term is 8 more than the previous term. This type of sequence is given a special name — an **arithmetic sequence**.

An arithmetic sequence is a sequence where there is a common difference between any two successive terms.

We can list the sequence in a table as shown in table A below. From this table we can see that it is possible to write a **functional definition** for the sequence in terms of the first term, 10, and the common difference, 8, and thus:

$$t_n = 2 + 8n,\ n \in \{1, 2, 3, \ldots\}$$

This can be rewritten as: $t_n = 10 + (n - 1) \times 8$.

We can readily get a general formula for the nth term of an arithmetic sequence whose first term is a and whose common difference is d (see table B).

Table A

n	t_n	t_n
1	$10 + 0 \times 8$	10
2	$10 + 1 \times 8$	18
3	$10 + 2 \times 8$	26
4	$10 + 3 \times 8$	34
n	$10 + (n - 1) \times 8$ $= 10 + 8n - 8$ $= 2 + 8n$	$2 + 8n$

Table B

n	t_n
1	$a + 0 \times d$
2	$a + 1 \times d$
3	$a + 2 \times d$
4	$a + 3 \times d$
n	$a + (n - 1) \times d$ $= (a - d) + dn$

In general, then:

The nth term of an arithmetic sequence is given by

$$t_n = a + (n - 1) \times d$$
$$= (a - d) + nd, n \in \{1, 2, 3, \ldots\}$$

where a is the first term and d is the common difference.

If we consider three successive terms in an arithmetic sequence, namely x, y and z, then since $y - x =$ the common difference, d, and $z - y = d$, it follows that:

$$y - x = z - y \Rightarrow y = \frac{z + x}{2}$$

The middle term of any three consecutive terms in an arithmetic sequence is called an *arithmetic mean* and is the average of the outer two terms.

That is, $y = \frac{z + x}{2}$ for any 3 consecutive terms x, y, z of an arithmetic sequence.

WORKED EXAMPLE 3

State which of the following are arithmetic sequences by finding the difference between successive terms. For those that are arithmetic, find the next term in the sequence, t_4, and consequently find the functional definition for the nth term for the sequence, t_n.

a t: $\{4, 9, 15, \ldots\}$

b t: $\{-2, 1, 4, \ldots\}$

THINK	WRITE
a 1 To check that a sequence is arithmetic, see if a common difference exists.	a $9 - 4 = 5$ $15 - 9 = 6$
2 There is no common difference as $5 \neq 6$.	Since there is no common difference the sequence is not arithmetic.
b 1 To check that a sequence is arithmetic, see if a common difference exists.	b $1 - -2 = 3$ $4 - 1 = 3$
2 The common difference is 3.	The sequence is arithmetic with the common difference $d = 3$.
3 The next term in the sequence, t_4, can be found by adding 3 to the previous term, t_3.	$t_4 = t_3 + 3$ $= 4 + 3$ $= 7$
4 To find the functional definition, write the formula for the nth term of the arithmetic sequence.	$t_n = a + (n - 1) \times d$ $= (a - d) + nd$
5 Identify the values of a and d.	$a = -2$ and $d = 3$
6 Substitute $a = -2$ and $d = 3$ into the formula and simplify.	$t_n = (-2 - 3) + n \times 3$ $t_n = 3n - 5$

Graphing an arithmetic sequence

- Since an arithmetic sequence involves adding or subtracting the same value repeatedly, the relationship between the terms is a linear one.
- This means that the graph of terms of an arithmetic sequence forms a straight line.

WORKED EXAMPLE 4

For the arithmetic sequence 2, 4, 6, 8, 10, . . .

a **use CAS to draw up a table showing the term number with its value**

b **use CAS to graph the values in the table**

c **from your graph, determine the value of the tenth term in the sequence.**

THINK

a Draw up a table using CAS to show the term number and value.

b The value of the term depends on the term number, so 'value' is graphed on the y-axis. Draw up a suitable scale on both axes, plot the points and join with a straight line. Note that, even though the graph is a straight line, values on the line have meaning only for integer values of the term number.

c **1** Find the term number 10 on the x-axis, and draw a vertical line from the x-axis to meet the straight line. Read the y-value of this point.

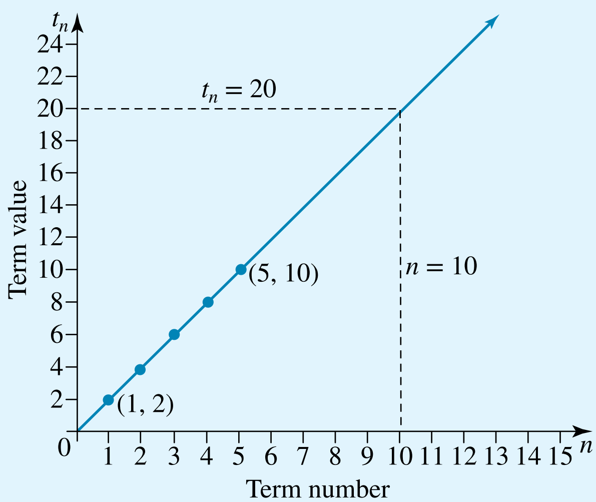
2 Write the answer.

WRITE/DRAW

a

Term number	1	2	3	4	5
Term value	2	4	6	8	10

b



c The tenth number in this sequence is 20.

WORKED EXAMPLE 5

Find the missing terms in this arithmetic sequence: {41, a, 55, b, . . .}.

THINK

1 The first three successive terms are 41, a, 55. Write the rule for the middle term of the three successive terms of an arithmetic sequence.

2 Identify the variables.

3 Substitute the values of x, y and z into the formula in step 1 and evaluate.

WRITE

For x, y, z: $y = \dfrac{x + z}{2}$

$x = 41;\ y = a;\ z = 55$

$$\begin{aligned} a &= \frac{41 + 55}{2} \\ &= 48 \end{aligned}$$

4 Find the common difference. (The second term is now known.)	$d = t_2 - t_1$ $= 48 - 41$ $= 7$
5 Find the value of b by adding the common difference to the preceding term.	$b = 55 + 7$ $= 62$
6 State your answer.	So $a = 48$, $b = 62$.

WORKED EXAMPLE 6 **Find the 16th and *n*th terms in an arithmetic sequence with the 4th term 15 and 8th term 37.**

THINK	WRITE
1 Write the formula for the nth term of the arithmetic sequence.	$t_n = a + (n - 1) \times d$
2 Substitute $n = 4$ and $t_4 = 15$ into the formula and label it equation [1].	t_4: $a + 3d = 15$ [1]
3 Substitute $n = 8$ and $t_8 = 37$ into the formula and label it equation [2].	t_8: $a + 7d = 37$ [2]
4 Solve the simultaneous equations: subtract equation [1] from equation [2] to eliminate a.	[2] − [1]: $a + 7d - a - 3d = 37 - 15$
5 Divide both sides by 4.	$4d = 22$ $d = \frac{22}{4}$ $= 5\frac{1}{2}$
6 Substitute $d = 5\frac{1}{2}$ into equation [1] and solve for a.	Substituting $d = 5\frac{1}{2}$ into [1]: $a + 3 \times 5\frac{1}{2} = 15$ $a = -1\frac{1}{2}$
7 To find the nth term of the arithmetic sequence, substitute the values of a and d into the general formula and simplify.	$t_n = -1\frac{1}{2} + (n - 1) \times 5\frac{1}{2}$ $= \frac{-3}{2} + (n - 1)\frac{11}{2}$ $= \frac{-3 + 11n - 11}{2}$ $= \frac{11n - 14}{2}$
8 To find the 16th term, substitute $n = 16$ into the formula established in the previous step and evaluate.	$t_n = \frac{11n - 14}{2}, n \in \{1, 2, 3, \ldots\}$ If $n = 16$, $t_{16} = \frac{11 \times 16 - 14}{2}$ $= 81$

EXERCISE 3.3 Arithmetic sequences

PRACTISE

1 WE3 State which of the following are arithmetic sequences by finding the difference between successive terms. For those that are arithmetic, find the next term in the sequence, t_4, and consequently find the functional definition for the nth term for the sequence, t_n.

a t: $\{3, 6, 12, \ldots\}$
b t: $\{-3, 0, 3, \ldots\}$

2 State which of the following are arithmetic sequences by finding the difference between successive terms. For those that are arithmetic, find the next term in the sequence, t_4, and consequently find the functional definition for the nth term for the sequence, t_n.

a t: $\{4, 7, 11, \ldots\}$
b t: $\{-2, -6, -10, \ldots\}$

3 WE4 For the arithmetic sequence 4, 8, 12, 16, 20, . . .

a use CAS technology to draw up a table showing the term number with its value
b use CAS technology to graph the values in the table
c from your graph, determine the value of the tenth term in the sequence.

4 For the arithmetic sequence 10, 13, 16, 19, 22, . . .

a use CAS technology to draw up a table showing the term number with its value
b use CAS technology to graph the values in the table
c from your graph, determine the value of the tenth term in the sequence.

5 WE5 Find the missing terms in this arithmetic sequence: $\{16, m, 27, n\}$.

6 Find the missing terms in this arithmetic sequence: $\{33, x, 61, y\}$.

7 WE6 Find the fourth term and nth term in the arithmetic sequence whose first term is 6 and whose seventh term is -10.

8 Find the eighth and nth terms in an arithmetic sequence with the third term 3.44 and the 12th term 5.42.

CONSOLIDATE

9 Find the term given in brackets for each of the following arithmetic sequences.

a $\{4, 9, 14, \ldots\}$, (t_{21})
b $\{-2, 10, 22, \ldots\}$, (t_{58})
c $\{-27, -12, 3, \ldots\}$ (t_{100})
d $\{2, -11, -24, \ldots\}$ (t_{2025})

10 Find the functional definition for the nth term of the arithmetic sequence:

a where the first term is 5 and the common difference is -3
b where the first term is 2.5 and the common difference is $\frac{1}{2}$
c where the first term is -3 and the common difference is 3
d where the first term is $2x$ and the common difference is $5x$.

11 Find the nth term in the arithmetic sequence where the first term is 6 and the third term is 10.

12 Find the nth term in the arithmetic sequence where the first term is 3 and the third term is 13.

13 Find the missing terms in the arithmetic sequence below.

$x - 3y$, ______________, $-3x + 5y$, ______________, . . .

14 If $t_{10} = 100$ and $t_{15} = 175$, find the first term, the common difference and hence the nth term for the arithmetic sequence.

15 If $t_{10} = \frac{-1}{2}$ and $t_{13} = \frac{3}{4}$, find the first term, the common difference and hence the nth term for the arithmetic sequence.

16 For the arithmetic sequence $\{22, m, n, 37, \ldots\}$, find the values for m and n.

17 The first three terms in an arithmetic sequence are 37, 32 and 27, and the kth term is -3. Find the value for k.

18 Find the value of x such that the following forms an arithmetic progression: $\ldots x, 3x + 4, 10x - 7 \ldots$

MASTER

19 For the following arithmetic sequences, find the recursive definition and use technology to generate the first 50 numbers in the sequence.

a t_n: $\{3, 7, 11, \ldots\}$

b t_n: $\{-3, 0, 3, \ldots\}$

c t_n: $\{-2, -6, -10, \ldots\}$

d t_n: $\left\{\frac{2}{7}, \frac{11}{14}, \frac{9}{7}, \ldots\right\}$

e t_n: $\left\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, \ldots\right\}$

f t_n: $\left\{\frac{1}{4}, \frac{-3}{2}, \frac{-13}{4}, \ldots\right\}$

g t_n: $\{2\pi + 3, 4\pi + 1, 6\pi - 1, \ldots\}$

20 The ratio between the first term and the second term in an arithmetic sequence is $\frac{3}{4}$. The ratio between the second term and the third term is $\frac{4}{5}$.

a Calculate the ratio of the third term to the fourth term.

b Find the ratio of the nth and the nth + 1 term in the sequence.

3.4 Arithmetic series

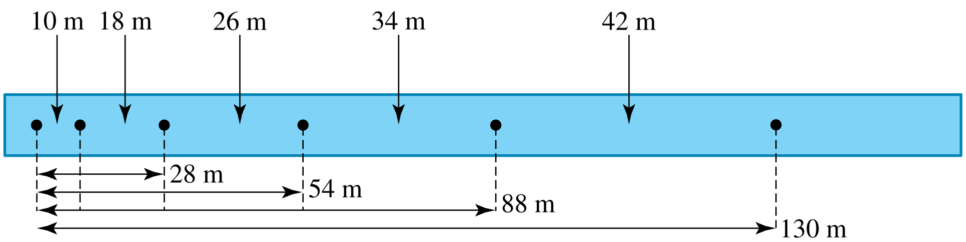
studyon

Units 1 & 2
AOS 2
Topic 2
Concept 3

Arithmetic series
Concept summary
Practice questions

Often we have a sequence of numbers and want to know their sum. For an example, we return to the oil drops on the racetrack from the start of the previous section on arithmetic sequences. The distance covered by the car each second illustrated the concept of an arithmetic sequence.

The total distance covered by the car is the sum of the individual distances covered each second. So after one second the car has travelled 10 m, after two seconds the car has travelled 10 + 18 m = 28 m, after three seconds the car has travelled a total distance of 10 + 18 + 26 m = 54 m, and so on.



A series, S_n, is the sum of a sequence of n terms $t_1 + t_2 + t_3 + \ldots + t_n$.

Thus:

$$S_1 = t_1$$
$$S_2 = t_1 + t_2$$
$$S_3 = t_1 + t_2 + t_3$$
$$S_n = t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n$$

For an arithmetic sequence, the sum of the first n terms, S_n, can be written in two ways.

1. The first term in the arithmetic sequence is a, the common difference is d, and the last term — that is, the nth term — in the sequence is l.

$$S_n = a + (a + d) + (a + 2d) + \ldots (a + (n - 3)d) + (a + (n - 2)d) + a + (n - 1)d$$
$$= a + (a + d) + (a + 2d) + \ldots (l - 3d) + (l - 2d) + (l - d) + l \qquad [1]$$

2. We can write the sum S_n in reverse order starting with the nth term and summing back to the first term a:

$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a \qquad [2]$$

If we add equation [1] and equation [2] together and recognise that there are n terms, each of which equals $(a + l)$, we get:

$$2S_n = (a + l) + (a + l) + \ldots n \text{ times}$$
$$= n(a + l)$$

and so: $$S_n = \frac{n}{2}(a + l)$$

or since l is the nth term, $l = a + (n - 1)d$, so $S_n = \frac{n}{2}[a + a + (n - 1)d]$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

The sum of the first n terms in an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a + l)$$

where a is the first term and l is the last term; or alternatively, since $l = a + (n - 1)d$, by

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

where a is the first term and d is the common difference.

If we know the first term, a, the common difference, d, and the number of terms, n, that we wish to add together, we can calculate the sum directly without having to add up all the individual terms.

It is worthwhile also to note that $S_{n+1} = S_n + t_{n+1}$. This tells us that the next term in the series S_{n+1} is the present sum, S_n, plus the next term in the sequence, t_{n+1}. This result is useful in spreadsheets where one column gives the sequence and an adjacent column is used to give the series.

WORKED EXAMPLE 7 Find the sum of the first 20 terms in the sequence t_n: {12, 25, 38, . . .}.

THINK	WRITE
1 Write the formula for the sum of the first n terms in the arithmetic sequence.	$S_n = \frac{n}{2}(2a + (n-1)d)$
2 Identify the variables.	$a = 12$, $d = 25 - 12 = 13$, $n = 20$
3 Substitute values of a, d and n into the formula and evaluate.	$S_{20} = \frac{20}{2}(2 \times 12 + 19 \times 13)$ $S_{20} = 2710$

EXERCISE 3.4 Arithmetic series

PRACTISE

1 WE7 Find the sum of the first 20 terms in the sequence t_n: {1, 3, 5, . . .}

2 Find the sum of the first 50 terms in the sequence $t_n = 3n + 7$, $n \in \{1, 2, 3, \ldots\}$

CONSOLIDATE

3 a Find the sum of the first 50 positive integers.
  b Find the sum of the first 100 positive integers.

4 a Find the sum of all the half-integers between 0 and 100.
  Note: The sequence of half-integers is $\left\{\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, \ldots\right\}$.
  b Compare your answer with that for question **3b**.

5 Find the sum of the first 12 terms of an arithmetic sequence in which the second term is 8 and 13th term is 41.

6 A sequence of numbers is defined by t_n: {15, 9, 3, −3, . . .}.
  a Find the sum of the first 13, 16 and 19 terms in the sequence.
  b Find the sum of all the terms between and including t_{10} and t_{15}.

7 A sequence of numbers is defined by $t_n = 2n - 7$, $n \in \{1, 2, 3, \ldots\}$. Find:
  a the sum of the first 20 terms
  b the sum of all the terms between and including t_{21} and t_{40}
  c the average of the first 40 terms. *Hint:* You need to find the sum first.

8 Find the equation that gives the sum of the first n positive integers.

9 a Show that the sum of the first n odd integers is equal to the perfect square n^2.
  b Show that the sum of the first n even integers is equal to $n^2 + n$.

10 A sequence is 5, 7, 9, 11, . . . How many consecutive terms need to be added to obtain 357?

11 Consider the sum of the first n integers. For what value of n will the sum first exceed 1000?

12 The first term in an arithmetic sequence is 5, and the sum of the first 20 terms is 1240. Find the common difference, d.

MASTER

13 The sum of the first four terms of an arithmetic sequence is 58, and the sum of the next four terms is twice that number. Find the sum of the following four terms.

14 The sum of a series is given by $S_n = 4n^2 + 3n$. Use the result that $t_{n+1} = S_{n+1} - S_n$ to prove that the sequence of numbers, t_n, whose series is $S_n = 4n^2 + 3n$ is arithmetic. Find both the functional and recursive equations for the sequence, t_n.

3.5 Geometric sequences

studyon

Units 1 & 2

AOS 2

Topic 2

Concept 4

Geometric sequences
Concept summary
Practice questions

A farmer is breeding worms that he hopes to sell to local shire councils to decompose waste at rubbish dumps. Worms reproduce readily and the farmer expects a 10% increase per week in the mass of worms that he is farming. A 10% increase per week would mean that the mass of worms would increase by a constant factor of $\left(1 + \frac{10}{100}\right)$ or 1.1.

He starts off with 10 kg of worms. By the beginning of the second week he will expect $10 \times 1.1 = 11$ kg of worms, by the start of the third week he would expect $11 \times 1.1 = 10 \times (1.1)^2 = 12.1$ kg of worms, and so on. This is an example of a **geometric sequence**.

> **A geometric sequence is a sequence where each term is obtained by multiplying the preceding term by a certain constant factor.**

The first term is 10, and the common factor is 1.10, which represents a 10% increase on the previous term. We can put the results of this example into a table.

n	t_n	t_n
1	$10\ (= 10 \times (1.1)^0)$	10
2	$10 \times (1.1)^1$	11
3	$10 \times (1.1)^2$	12.1
4	$10 \times (1.1)^3$	13.31
n	$10 \times (1.1)^{n-1}$	$10 \times (1.1)^{n-1}$

From this table we can see that

$$t_2 = 1.1 \times t_1,\ t_3 = 1.1 \times t_2$$

and so on. In general:

$$t_{n+1} = 1.1 \times t_n$$

The common factor or common ratio whose value is 1.1 for this example can be found by dividing any two successive terms: $\frac{t_{n+1}}{t_n}$.

A geometric sequence, t, can be written in terms of the first term, a, and the common ratio, r. Thus:

$$t\text{: } \{a, ar, ar^2, ar^3, \ldots, ar^{n-1}\}$$

The first term $t_1 = a$, the second term $t_2 = ar$, the third term $t_3 = ar^2$, and consequently the nth term, t_n is ar^{n-1}.

> **For a geometric sequence:**
>
> $$t_n = ar^{n-1}$$
>
> **where a is the first term and r the common ratio, given by**
>
> $$r = \frac{t_{n+1}}{t_n}.$$

If we consider three consecutive terms in a geometric sequence $\{x, y \text{ and } z\}$ then

$$\frac{y}{x} = r = \frac{z}{y}$$

where r is the common factor.

Thus the middle term, y, called the **geometric mean**, can be calculated in terms of the outer two terms, x and z.

> **For a geometric sequence $\{\ldots, x, y, z, \ldots\}$:**
>
> $$y^2 = xz$$

WORKED EXAMPLE 8

State whether the sequence t_n: $\{2, 6, 18, \ldots\}$ is geometric by finding the ratio of successive terms. If it is geometric, find the next term in the sequence, t_4, and the nth term for the sequence, t_n.

THINK	WRITE
1 Find the ratio $\frac{t_2}{t_1}$.	$\frac{t_2}{t_1} = \frac{6}{2}$ $= 3$
2 Find the ratio $\frac{t_3}{t_2}$.	$t_3 = \frac{18}{6}$ $= 3$
3 Compare the ratios and make your conclusion.	Since $\frac{t_2}{t_1} = \frac{t_3}{t_2} = 3$, the sequence is geometric with the common ratio $r = 3$.
4 Because the sequence is geometric, find the fourth term by multiplying the preceding (third) term by the common ratio.	$t_4 = t_3 \times r$ $= 18 \times 3$ $= 54$
5 Write the general formula for the nth term.	$t_n = ar^{n-1}$
6 Identify the values of a and r.	$a = 2$; $r = 3$
7 Substitute the values of a and r into the general formula.	$t_n = 2 \times 3^{n-1}$
8 Check the value for t_4.	$t_4 = 2 \times 3^{4-1} = 2 \times 27 = 54$

Graphs of geometric sequences

While the graph of an arithmetic sequence is a straight line, the graph of a geometric sequence is a curve for values of $r > 0$. (Different values of r produce graphs of different shapes.)

WORKED EXAMPLE 9

Consider the geometric sequence 2, 4, 8, 16, 32, . . .

a Using CAS, draw up a table showing the term number and its value.

b Using CAS, graph the entries in the table.

c Comment on the shape of the graph.

THINK

a Draw up a table showing the term number and its corresponding value.

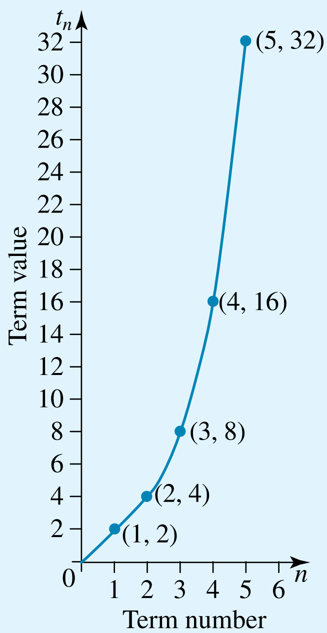
WRITE/DRAW

a

Term number	1	2	3	4	5
Term value	2	4	8	16	32

b The value of the term depends on the term number, so 'Term value' is graphed on the y-axis. Draw a set of axes with suitable scales. Plot the points and join with a smooth curve.

b

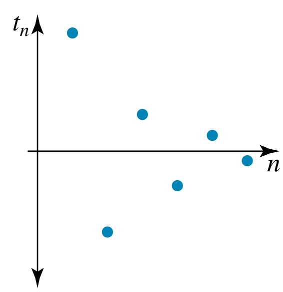


c Comment on the shape of the curve.

c The points lie on a smooth curve which increases rapidly. Because of this rapid increase in value, it would be difficult to use the graph to predict future values in the sequence with any accuracy.

Shape of graphs of geometric sequences

The shape of the graph of a geometric sequence depends on the value of r.



- When $r > 1$, the points lie on a curve, as shown in the graph in the previous worked example. Graphs of this kind are said to *diverge* (i.e. they move further and further away from the starting value).
- When $r < -1$, the points *oscillate* on either side of zero.
- When $-1 < r < 0$, the points oscillate and *converge* to a certain fixed number, as shown in the graph at right.
- When $0 < r < 1$, the points converge to a certain fixed number.

WORKED EXAMPLE 10 **Find the nth term and the tenth term in the geometric sequence where the first term is 3 and the third term is 12.**

THINK	WRITE
1 Write the general formula for the nth term in the geometric sequence.	$t_n = ar^{n-1}$
2 State the value of a (the first term in the sequence) and the value of the third term.	$a = 3$; $t_3 = 12$
3 Substitute all known values into the general formula.	$12 = 3 \times r^{3-1}$ $= 3 \times r^2$
4 Solve for r (note that there are two possible solutions).	$r^2 = \frac{12}{3}$ $= 4$ $r = \pm\sqrt{4}$ $= \pm 2$
5 Substitute the values of a and r into the general equation. Because there are two possible values for r, you must show both expressions for the nth term of the sequence.	So $t_n = 3 \times 2^{n-1}$, or $t_n = 3 \times (-2)^{n-1}$
6 Find the tenth term by substituting $n = 10$ into each of the two expressions for the nth term.	When $n = 10$, $t_{10} = 3 \times 2^{10-1}$(using $r = 2$) $= 3 \times 2^9$ $= 1536$ or $t_{10} = 3 \times (-2)^{10-1}$ (using $r = -2$) $= 3 \times (-2)^9 = -1536$

WORKED EXAMPLE 11 **The fifth term in a geometric sequence is 14 and the seventh term is 0.56. Find the common ratio, r, the first term, a, and the nth term for the sequence.**

THINK	WRITE
1 Write the general rule for the nth term of a geometric sequence.	$t_n = ar^{n-1}$
2 Use the information about the fifth term to form an equation. Label it [1].	When $n = 5$, $t_n = 14$ $14 = a \times r^{5-1}$ $14 = a \times r^4$ [1]
3 Similarly, use information about the seventh term to form an equation. Label it [2].	When $n = 7$, $t_n = 0.56$ $0.56 = a \times r^{7-1}$ $0.56 = a \times r^6$ [2]
4 Solve the equations simultaneously: divide equation [2] by equation [1] to eliminate a.	$\frac{[2]}{[1]}$ gives $\frac{ar^6}{ar^4} = \frac{0.56}{14}$

5 Solve for r.

$$r^2 = 0.04$$
$$r = \pm\sqrt{0.04}$$
$$= \pm 0.2$$

6 Because there are two solutions, we have to perform two sets of computations. Consider the positive value of r first. Substitute the value of r into either of the two equations, say equation [1], and solve for a.

If $r = 0.2$

Substitute r into [1]:

$$a \times (0.2)^4 = 14$$
$$0.0016a = 14$$
$$a = 14 \div 0.0016$$
$$= 8750$$

7 Substitute the values of r and a into the general equation to find the expression for the nth term.

The nth term is:

$$t_n = 8750 \times (0.2)^{n-1}$$

8 Now consider the negative value of r. Substitute the value of r into either of the two equations, say equation [1], and solve for a. (Note that the value of a is the same for both values of r.)

If $r = -0.2$

Substitute r into [1]

$$a = (-0.2)^4 = 14$$
$$0.0016a = 14$$
$$a = 14 \div 0.0016$$
$$= 8750$$

9 Substitute the values of r and a into the general formula to find the second expression for the nth term of the sequence.

The nth term is:

$$t_n = 8750 \times (-0.2)^{n-1}$$

EXERCISE 3.5 Geometric sequences

PRACTISE

1 WE8 State whether the sequence is geometric by finding the ratio of successive terms for t_n: $\{3, 6, 12, \ldots\}$. If the sequence is geometric, find the next term in the sequence, t_4, and the nth term for the sequence, t_n.

2 State whether the sequence is geometric by finding the ratio of successive terms for t_n: $\left\{-3, 1, \frac{-1}{3}, \ldots\right\}$. If the sequence is geometric, find the next term in the sequence, t_4, and the nth term for the sequence, t_n.

3 WE9 Consider the geometric sequence 1, 3, 9, 27, 81, . . .

a Use CAS technology to draw up a table showing the term number and its value.

b Use CAS technology to graph the entries in the table.

c Comment on the shape of the graph.

4 Consider the geometric sequence −20, 10, −5, 2.5, −1.25, . . .

a Use CAS technology to draw up a table showing the term number and its value.

b Use CAS technology to graph the entries in the table.

c Comment on the shape of the graph.

5 WE10 Find the nth term and the tenth term in the geometric sequence where the first term is 2 and the third term is 18.

6 Find the nth term and the tenth term in the geometric sequence where the first term is 1 and the third term is 4.

7 WE11 The third term in a geometric sequence is 100 and the fifth term is 400. Find the common ratio, r, the first term, a, and the nth term for the sequence.

8 The fourth term in a geometric sequence is 48 and the eighth term is 768. Find the common ratio, r, the first term, a, and the nth term for the sequence.

CONSOLIDATE

9 For each of the following:

i show that the sequence is geometric

ii find the nth term and consequently the sixth and tenth terms.

a $t: \{5, 10, 20, \ldots\}$

b $t: \{2, 5, 12.5, \ldots\}$

c $t: \{1, -3, 9, \ldots\}$

d $t: \{2, -4, 8, \ldots\}$

e $t: \{2.3, 3.45, 5.175, \ldots\}$

f $t: \left\{\frac{1}{2}, 1, 2, \ldots\right\}$

g $t: \left\{\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \ldots\right\}$

h $t: \left\{\frac{3}{5}, \frac{-1}{5}, \frac{1}{15}, \ldots\right\}$

i $t: \{x, 3x^4, 9x^7, \ldots\}$

j $t: \left\{\frac{1}{x}, \frac{2}{x^2}, \frac{4}{x^3}, \ldots\right\}$

10 Find the nth term and the tenth term in the geometric sequence where:

a the first term is 5 and the fourth term is 40

b the first term is -1 and the second term is 2

c the first term is 9 and the third term is $\frac{1}{81}$. (Why are there two possible answers?)

11 Find the fourth term in the geometric sequence where the first term is 6 and the seventh term is $\frac{3}{32}$.

12 Find the nth term in the geometric sequence where the first term is 3 and the fourth term is $6\sqrt{2}$.

13 For the geometric sequence $t: \{3, m, n, 192, \ldots\}$, find the values for m and n.

14 Consider the geometric sequence $t: \{16, m, 81, n, \ldots\}$. Find the values of m and n, if it is known that both are positive numbers.

15 For the geometric sequence $t: \{a, 15, b, 0.0375, \ldots\}$, find the values of a and b, given that they are positive numbers.

16 If $t_2 = \frac{1}{2}$ and $t_5 = \frac{27}{16}$, find the first term, a, the common factor, r, and hence the nth term for the geometric sequence.

17 Find the value of x such that the following sequence forms a geometric progression: $x - 1, 3x + 4, 6x + 8$.

18 Insert three terms in between 8, _, _, _, $\frac{1}{32}$ such that the sequence of numbers is geometric.

MASTER

19 The difference between the first term and the second term in a geometric sequence is 6. The difference between the second term and the third term is 3.

a Calculate the difference between the third term and the fourth term.

b Find the nth term in the sequence.

20 The first two terms in a geometric sequence are 120 and 24, and the kth term is 0.0384. Find the value for k.

3.6 Geometric series

When we add up or sum the terms in a sequence we get the series for that sequence. If we look at the geometric sequence {2, 6, 18, 54, . . .}, where the first term $t_1 = a = 2$ and the common ratio is 3, we can quickly calculate the first few terms in the series of this sequence.

study on

Units 1 & 2
AOS 2
Topic 2
Concept 5

Geometric series
Concept summary
Practice questions

$$S_1 = t_1 = 2$$
$$S_2 = t_1 + t_2 = 2 + 6 = 8$$
$$S_3 = t_1 + t_2 + t_3 = 2 + 6 + 18 = 26$$
$$S_4 = t_1 + t_2 + t_3 + t_4 = 2 + 6 + 18 + 54 = 80$$

In general the sum of the first n terms is:

$$S_n = t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n$$

For a geometric sequence the first term is a, the second term is ar, the third term is ar^2 and so on up to the nth term, which is ar^{n-1}. Thus:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \qquad [1]$$

If we multiply equation [1] by r we get:

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \qquad [2]$$

Note that on the right-hand side of equations [1] and [2] all but two terms are common, namely the first term in equation [1], a, and the last term in equation [2], ar^n. If we take the difference between equation [2] and equation [1] we get:

$$rS_n - S_n = ar^n - a \qquad [2] - [1]$$
$$\therefore \quad (r - 1)S_n = a(r^n - 1)$$
$$\therefore \quad S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1 \qquad (r \text{ cannot equal } 1)$$

We now have an equation that allows us to calculate the sum of the first n terms of a geometric sequence.

The sum of the first n terms of a geometric sequence is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1$$

where a is the first term of the sequence and r is the common ratio.

WORKED EXAMPLE 12

Find the sum of the first five terms (S_5) of these geometric sequences.

a t_n: {1, 4, 16, . . .}

b $t_n = 2(2)^{n-1}, n \in \{1, 2, 3, \ldots\}$

c $t_{n+1} = \frac{1}{4}t_n, t_1 = \frac{-1}{2}$

THINK	WRITE
a 1 Write the general formula for the sum of the first n terms of the geometric sequence.	a $S_n = \frac{a(r^n - 1)}{r - 1}$
2 Write the sequence.	t_n: {1, 4, 16, . . .}

3 Identify the variables: a is the first term; r can be established by finding the ratio; n is known from the question.	$a = 1;\ r = \frac{4}{1} = 4;\ n = 5$
4 Substitute the values of a, r and n into the formula and evaluate.	$S_5 = \frac{1(4^5 - 1)}{4 - 1}$ $= \frac{1024 - 1}{3}$ $= 341$
b 1 Write the sequence.	**b** $t_n = 2(2)^{n-1},\ n \in \{1, 2, 3, \ldots\}$
2 Compare the given rule with the general formula for the nth term of the geometric sequence $t_n = ar^{n-1}$ and identify values of a and r; the value of n is known from the question.	$a = 2;\ r = 2;\ n = 5$
3 Substitute values of a, r and n into the general formula for the sum and evaluate.	$S_5 = \frac{2(2^5 - 1)}{2 - 1}$ $= 62$
c 1 Write the sequence.	**c** $t_{n+1} = \frac{1}{4}t_n,\ t_1 = \frac{-1}{2}$
2 This is an iterative formula, so the coefficient of t_n is r; $a = t_1$; n is known from the question.	$r = \frac{1}{4};\ a = \frac{-1}{2};\ n = 5$
3 Substitute values of a, r and n into the general formula for the sum and evaluate.	$S_5 = \frac{\frac{-1}{2}\left[\left(\frac{1}{4}\right)^5 - 1\right]}{\frac{1}{4} - 1}$ $= \frac{\frac{-1}{2} \times \left(\frac{1}{1024} - 1\right)}{\frac{-3}{4}}$ $= \frac{-341}{512}$

The infinite sum of a geometric sequence where $r < 1$

studyon

Units 1 & 2
AOS 2
Topic 2
Concept 6

Infinite geometric series
Concept summary
Practice questions

When the constant ratio, r, is less than 1 and greater than -1, that is, $\{r: -1 < r < 1\}$, each successive term in the sequence gets closer to zero. This can readily be shown with the following two examples.

$$g: \{2, -1, \frac{1}{2}, \frac{-1}{4}, \ldots\} \text{ where } a = 2 \text{ and } r = \frac{-1}{2}$$

$$h: \{40, \frac{1}{2}, \frac{1}{160}, \ldots\} \text{ where } a = 40 \text{ and } r = \frac{1}{80}$$

In both the examples, successive terms approach zero as n increases. In the second case the approach is more rapid than in the first, and the first sequence alternates positive and negative. A simple investigation with a spreadsheet will quickly reveal that for geometric sequences with the size or magnitude of $r < 1$, the series eventually settles down to a near constant value. We say that the series converges to a value S_∞, which is the sum to infinity of all terms in the geometric sequence. We can find the value S_∞ by recognising that as $n \to \infty$ the term $r^n \to 0$, provided r is between -1 and 1. We write this technically as $-1 < r < 1$ or $|r| < 1$. The symbol $|r|$ means the magnitude or size of r. Using our equation for the sum of the first n terms:

$$S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1$$

Taking -1 as a common factor from the numerator and denominator:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

As $n \to \infty$, $r^n \to 0$ and hence $1 - r^n \to 1$. Thus the top line or numerator will equal a when $n \to \infty$:

$$S_\infty = \frac{a}{1 - r}; \quad |r| < 1$$

We now have an equation that allows us to calculate the sum to infinity, S_∞, of a geometric sequence.

The sum to infinity, S_∞, of a geometric sequence is given by:

$$S_\infty = \frac{a}{1 - r}; \quad |r| < 1$$

where a is the first term of the sequence and r is the common ratio whose magnitude is less than one.

WORKED EXAMPLE 13

a Find the sum to infinity for the sequence t_n: {10, 1, 0.1, . . .}.

b Find the fourth term in the geometric sequence whose first term is 6 and whose sum to infinity is 10.

THINK	WRITE
a 1 Write the formula for the nth term of the geometric sequence.	**a** $t_n = ar^{n-1}$
2 From the question we know that the first term, a, is 10 and $r = 0.1$.	$a = 10, r = 0.1$
3 Write the formula for the sum to infinity.	$S_\infty = \frac{a}{1 - r}; \ \lvert r \rvert < 1$
4 Substitute $a = 10$ and $r = 0.10$ into the formula and evaluate.	$S_\infty = \frac{10}{1 - 0.1}$ $S_\infty = \frac{10}{0.9} = \frac{100}{9}$ $= 11\frac{1}{9}$

b 1 Write the formula for the sum to infinity.

b $S_\infty = \dfrac{a}{1 - r}$; $|r| < 1$

2 From the question we know that the infinite sum is equal to 10 and that the first term, a, is 6.

$a = 6$; $S_\infty = 10$

3 Substitute known values into the formula.

$10 = \dfrac{6}{1 - r}$

4 Solve for r.

$10(1 - r) = 6$

$10 - 10r = 6$

$r = 0.4$

5 Write the general formula for the nth term.

$t_n = ar^{n-1}$

6 To find the fourth term substitute $a = 6$, $n = 4$ and $r = 0.4$ into the formula and evaluate.

$t_4 = 6 \times (0.4)^3$

$= 0.384$

EXERCISE 3.6 Geometric series

PRACTISE

1 WE12 Find the sum of the first five terms (S_5) of these geometric sequences.

a t_n: $\{1, 2, 4, \ldots\}$

b $t_n = 3(-2)^{n-1}, n \in \{1, 2, 3, \ldots\}$

c $t_{n+1} = 2t_n, t_1 = \dfrac{3}{2}$

2 Find the sum of the first five terms (S_5) of these geometric sequences.

a t_n: $\{1, 3, 9, \ldots\}$

b $t_n = -4(1.2)^{n-1},\ n \in \{1, 2, 3, \ldots\}$

c $t_{n+1} = \dfrac{1}{2}t_n, t_1 = \dfrac{-2}{3}$

3 WE13 a Find the sum to infinity for the sequence t_n: $\left\{1, \dfrac{1}{2}, \dfrac{1}{4}, \ldots\right\}$.

a Find the fourth term in the geometric sequence whose first term is 4 and whose sum to infinity is 6.

4 a Find the sum to infinity for the sequence t_n: $\left\{1, \dfrac{2}{3}, \dfrac{4}{9}, \ldots\right\}$.

a Find the fourth term in the geometric sequence whose first term is 1 and whose sum to infinity is $\dfrac{3}{5}$.

CONSOLIDATE

5 Consider the following geometric sequences and find the terms indicated.

a The first term is 440 and the 12th term is 880. Find S_6.

b The fifth term is 1 and the eighth term is 8. Find S_1, S_{10}, S_{20}.

6 What minimum number of terms of the series $2 + 3 + 4\frac{1}{2} + \ldots$ must be taken to give a sum in excess of 100?

7 For the infinite geometric sequence $\left\{1, \dfrac{1}{4}, \dfrac{1}{16}, \ldots\right\}$, find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

8 A sequence of numbers is defined by $t_n = 3\left(\dfrac{1}{2}\right)^{n-1}$, $n \in \{1, 2, 3, \ldots\}$.

a Find the sum of the first 20 terms.

b Find the sum of all the terms between and including t_{21} and t_{40}.

c Find the sum to infinity, S_∞.

9 A sequence of numbers is defined by t_n: {9, −3, 1, . . .}.

a Find the sum of the first nine terms.

b Find the sum of all the terms between and including t_{10} and t_{15}.

c Find the sum to infinity, S_∞.

10 The first term of a geometric sequence is 5 and the fourth term is 0.078 125. Find the sum to infinity.

11 The sum of the first four terms of a geometric sequence is 30 and the sum to infinity is 32. Find the first three terms of the sequence if the common ratio is positive.

12 For the geometric sequence $\sqrt{5} + \sqrt{3}$, $\sqrt{5} - \sqrt{3}$, . . ., find the common ratio, r, and the sum of the infinite series, S_∞.

13 If $1 + 3x + 9x^2 + \ldots = \frac{2}{3}$, find the value of x.

14 If the common ratio for a geometric sequence is 0.99 and the sum to infinity is 100, what is the value of the first and second terms in the sequence?

MASTER

15 Show that $x^n - 1$ always has a factor $(x - 1)$ for $n \in \{1, 2, 3, \ldots\}$.

16 A student stands at one side of a road 10 metres wide, and walks halfway across. The student then walks half of the remaining distance across the road, then half the remaining distance again and so on.

a Will the student ever make it *past* the other side of the road?

b Does the width of the road affect your answer?

3.7 Applications of sequences and series

This section consists of a mixture of problems where the work covered in the first five exercises is applied to a variety of situations.

study on

Units 1 & 2

AOS 2

Topic 2

Concept 7

Applications of sequences and series

Concept summary

Practice questions

The following general guidelines can assist you in solving the problems.

1. Read the question carefully.
2. Decide whether the information suggests an arithmetic or geometric sequence. Check to see if there is a constant difference between successive terms or a constant ratio. If there is neither, look for a simple number pattern such as the difference between successive terms changing in a regular way.
3. Write the information from the problem using appropriate notation. For example, if you are told that the fifth term is 12, write $t_5 = 12$. If the sequence is arithmetic, you then have an equation to work with, namely: $a + 4d = 12$. If you know the sequence is geometric, then $ar^4 = 12$.
4. Define what you have to calculate and write an appropriate formula or formulas. For example, if you have to find the tenth number in a sequence that you know is geometric, you have an equation: $t_{10} = ar^9$. This can be calculated if a and r are known or can be established.
5. Use algebra to find what is required in the problem.

WORKED EXAMPLE 14

In 1970 the cost of 1 megabyte of computer memory was $2025. In 1980 the cost for the same amount of memory had reduced to $45, and by 1990 the cost had dropped to $1.

a **Assuming the pattern continues through the years, what was the cost of 1 megabyte of memory in the year 2000?**

b **How much memory, in megabytes, could you buy for $10 in the year 2010 based on the trend?**

THINK

a 1 Present the given information in a table.

WRITE

a

Year	1970	1980	1990	2000	2010
Cost ($)	2025	45	1	?	?

2 Study the table. The information suggests a geometric sequence for the cost at each ten-year interval. Verify this by checking for a constant ratio between successive terms.

$45 \div 2025 = \frac{1}{45}$ and $1 \div 45 = \frac{1}{45}$, so the three terms form a geometric sequence with common ratio $r = \frac{1}{45}$.

3 To find the cost in the year 2000, find the fourth term in the sequence by multiplying the preceding (third) term by the common ratio.

$t_4 = t_3 \times r$

$t_4 = 1 \times \frac{1}{45}$

$= \frac{1}{45}$

$= 0.022\ldots$

4 Interpret the result and clearly answer the question.

In the year 2000 you would have paid about 2 cents for a megabyte of memory.

b 1 If the cost of 1 megabyte can be found in the year 2010, then the amount of memory purchased for $10 can be determined. To find the predicted cost in the year 2010, the fifth term in the sequence needs to be determined.

b $t_5 = t_4 \times r$

$= \frac{1}{45} \times \frac{1}{45}$

$= \frac{1}{2025}$ of a dollar per megabyte

2 Take the reciprocal of t_5 to get the amount of memory per dollar.

The amount of memory per dollar is 2025 megabytes.

3 Find the amount of memory that can be purchased for $10.

So $10 would buy 10 × 2025 = 20 250 megabytes.

WORKED EXAMPLE 15

Express the recurring decimal 0.131 313 13 . . . as a proper fraction.

THINK

1 Express the given number as a geometric series.

WRITE

$0.131\,313\ldots = 0.13 + 0.0013 + 0.000\,013\ldots$

2 State the values of a and r.

$a = 0.13$ and $r = \frac{0.0013}{0.13} = 0.01$

3 Find the sum to infinity, S_∞.
Write the formula for the sum to infinity.

$$S_\infty = \frac{a}{1-r}$$

4 Substitute values of a and r into the formula and simplify.

$$S_\infty = \frac{0.13}{1-0.01}$$

$$S_\infty = \frac{0.13}{0.99}$$

5 Multiply both numerator and denominator by 100 to get rid of the decimal point.

$$S_\infty = \frac{13}{99}$$

EXERCISE 3.7 Applications of sequences and series

PRACTISE

1 **WE14** In 1970 the Smith family purchased a small house for \$60 000. Over the following years, the value of their property rose steadily. In 1975 the value of the house was \$69 000 and in 1980 it reached \$79 350.

a Assuming that the pattern continues through the years, find (to the nearest dollar) the value of the Smiths' house in i 1985 and ii 1995.

b By what factor will the value of the house have increased by the year 2015, compared to the original value?

2 An accountant working with a company commenced on a salary of \$58 000 and has received a \$4200 increase each year.

a How much did she earn in her 15th year of employment?

b How much has she earned from the company altogether in those 15 years?

3 **WE15** Express the recurring decimal 0.1111 ... as a proper fraction.

4 Express the recurring decimal 0.575 757 ... as a proper fraction.

CONSOLIDATE

5 A chemist has been working with the same company for 15 years. He commenced on a salary of \$28 000 and has received a 4% increase each year.

a What type of sequence of numbers does his annual income follow?

b How much did he earn in his 15th year of employment?

c How much has he earned from the company altogether?

d What was his increase in salary at the end of i his 1st and ii his 14th year of employment?

6 A biologist is growing a tissue culture in a Petri dish. The initial mass of the culture was 20 milligrams. By the end of the first day the culture had a mass of 28 milligrams.

a Assuming that the daily growth is *arithmetic*, find the mass of the culture after the second, third, tenth and nth day.

b On what day will the culture mass first exceed 200 milligrams if its growth is arithmetic?

c Assuming that the daily growth is *geometric*, find the mass of the culture after the second, third, tenth and nth day.

d On what day will the culture mass first exceed 200 milligrams if its growth is geometric?

7 Logs of wood can be stacked so that there is one more log on each descending layer than on the previous layer. The top row has 6 logs and there are 20 rows.

a How many logs are in the stack altogether?

b The logs are to be separated into two equal piles. They are separated by removing logs from the top of the pile. How many rows down will workers take away before they remove half the stack?

8 Kind-hearted Kate has 200 movie tickets to give away to people at the shopping centre. She gives the first person one ticket, the next person two tickets, the third person four tickets and so on following a geometric progression until she can no longer give the nth person 2^{n-1} tickets. How many tickets did the last lucky person receive? How many tickets did Kate have left?

9 The King of Persia, so the story goes, offered Xanadu any reward to secure the safety of his kingdom. As his reward, Xanadu requested a chessboard with one grain of rice on the first square, two grains on the second, four on the third and so on until the 64th square had its share of rice deposited.

a Find the total number of grains of rice that the king needed to supply.

b If each grain of rice weighs 0.10 grams, how many kilograms of rice does this represent? (*Note:* There are 10^3 grams in 1 kilogram.)

10 As legend has it, the King of Constantinople offered Xanadu's cousin Yittrius any reward to secure the safety of his city. This Yittrius accepted: she requested a chessboard with one grain of rice on the first square, three grains of rice on the second square, five grains of rice on the third square and so on until the 64th square had its share of rice deposited.

a Find the total number of grains of rice that the king needed to supply.

b If each grain of rice weighs 0.10 grams, how many kilograms of rice does this represent? (*Note:* There are 10^3 grams in 1 kilogram.)

11 A hiker walks 36 km on the first day and $\frac{2}{3}$ that distance on the second. Every day thereafter she walks $\frac{2}{3}$ of the distance she walked on the day before. Will the hiker cover the distance of 100 km to complete the walk? If so, on what day will she complete the walk?

12 Find the fraction equivalent of the following recurring decimal numbers by writing the decimal number as a sum of infinite terms.

a 0.333 333 333 . . .

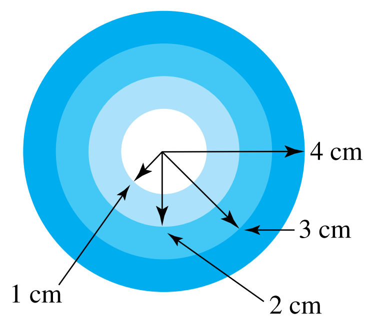
b 2.343 434 . . .

c 3.142 142 142 . . .

d 21.2121 . . .

e 16.666 . . .

13 A circular board is divided into a series of concentric circles of radius 1 cm, 2 cm, 3 cm and 4 cm as shown at right.



a Find the areas of each of the successive shaded regions and show that they form an arithmetic progression.

b A dart is fired at the board at random and hits the board. What is the probability of striking each of the four regions of the board?
(*Note:* The probability of striking a region = area of region ÷ total area.)

14 A bullet is fired vertically up into the air. In the first second it has an average speed of 180 m/s; that is, it travels 180 m up into the air during the first second. Each second its average speed diminishes by 12 m/s. Thus during the 2nd second the bullet has an average speed only 168 m/s and accordingly travels 168 m further up into the air.

a Find an equation for the average speed of the bullet for the *n*th second that it is in the air.

b Find the time when the average speed of the bullet is equal to zero.

c Find the maximum height of the bullet above where it was fired.

MASTER

15 Coffee cools according to Newton's Law of Cooling, in which the temperature of coffee *above* room temperature drops by a constant fraction each unit of time. The table below shows the temperature of a cup of coffee in a room at 20 °C each minute after it was made.

Time (min)	Temp. (°C)
1	80.0
2	74.0
3	68.6

Remember to subtract the room temperature from the temperature of the coffee before you do your calculations.

The person who made the coffee will drink it only if it has a temperature in excess of 50 °C. What is the minimum time after the cup of coffee has been made before it becomes undrinkable?

16 Two arithmetic sequences, t_n and u_n, are multiplied together. That is, each term is multiplied by the other to form a new term.

$$t_n = 2n - 3, n \in \{1, 2, 3, \ldots\} \text{ and}$$
$$u_n = 3n, n \in \{1, 2, 3, \ldots\}$$

Show that the new sequence of numbers $t_1 \times u_1, t_2 \times u_2, t_3 \times u_3, \ldots$ is an arithmetic series and hence find the arithmetic sequence for that new series.

(*Hint:* For a sequence a_n with a series A_n, $a_n = A_n - A_{n-1}$.)

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3.8 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

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Units 1 & 2 **Sequences and series**

Sit topic test

3 Answers

EXERCISE 3.2

1 a $\frac{3}{8}, \frac{3}{16}, \frac{3}{32}$

b 4, 36, 324

c −3, −1, 3

2 a −17, 20, −23

b 53, 97, 173

c −11, −123, −15 131

3 0.2, 0.128, 0.089, 0.065, 0.049, 0.037; the sequence converges to 0.

4 0.099, 0.0981, 0.0973, 0.0967, 0.096, 0.0955; the sequence converges to 0.0909.

5 a Add 3 (to the previous term); 10, 13, 16.

b Subtract 1 (from the previous term); −3, −4, −5.

c Multiply by 4; 256, 1024, 4096.

d The difference between the terms increases by 1 for each pair; 27, 35, 44.

e Add the preceding two terms; 29, 47, 76.

f Add $3b - a$; $-2a + 7b$, $-3a + 10b$, $-4a + 13b$.

g Many possible answers — assume the sequence repeats; 0, −1, 0.

h Append 1 to the decimal expansion of the preceding term; 1.111, 1.1111, 1.111 11.

i Divide by −2; 64, −32, 16.

6 a −3, 5, 15

b $\frac{1}{2}, \frac{5}{6}, \frac{10}{11}$

c 13.3, −1.5, −20

d $\frac{5}{2}, \frac{5}{32}, \frac{5}{1024}$

e $\frac{5}{4}$, 20, 640

f 0, 4, 11

g $\frac{3}{2}, \frac{243}{32}, \frac{59\,049}{1024}$

h a, $a + 4d$, $a + 9d$

i a, ar^4, ar^9

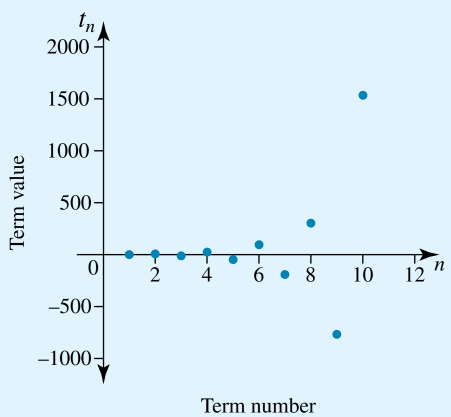
7 a −12, 384, 1536

b 0, −35, −49

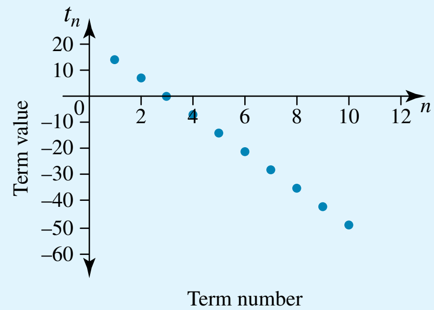
c 3, −1, −1

d 0, 0, 0

8 a



b



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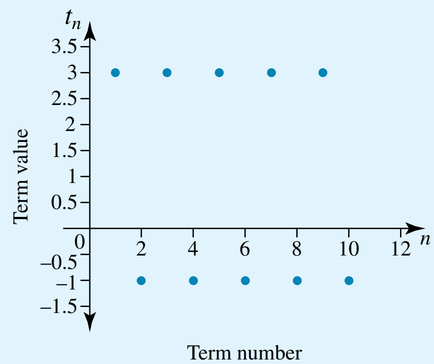

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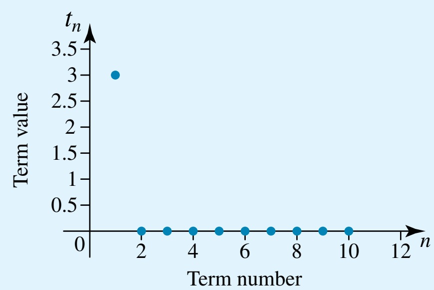


9 a 0.096, 0.0347, 0.0134, 0.0053, 0.0021, 0.0008; the sequence converges to 0.

b 0.456, 0.471 321, 0.473 437, 0.473 659, 0.473 682, 0.473 684; the sequence converges to $\frac{9}{19}$.

c 0.525, 0.523 687 5, 0.523 821 7, 0.523 808 3, 0.523 809 6, 0.523 809 5; the sequence converges to $\frac{11}{21}$.

d 0.525, 0.623, 0.587, 0.606, 0.597, 0.602; the sequence converges to $\frac{3}{5}$.

e 0.48, 0.749, 0.564, 0.738, 0.581, 0.73; oscillating

f 0.714, 0.694, 0.722, 0.683, 0.736, 0.66; oscillating

g 0.378, 0.987, 0.052, 0.207, 0.689, 0.901; divergent

h 0.72, 0.907, 0.379, 1.059, −0.281, −1.619; divergent

10 a 15, 20; the difference between subsequent terms increases by 1.

b There are many possible answers. A possible pattern is the addition of 5, then 3, then 1, then −1. The next two terms are 4, −3. Here the difference between successive terms follows an arithmetic sequence.

c Many possible answers as there is no obvious pattern. It could be the start of a telephone number.

d Each successive term is multiplied by an increasing factor of $\frac{1}{2}$, starting with $\left(\frac{1}{2}\right)^{-1} = 2$, then $\left(\frac{1}{2}\right)^{0} = 1$, and then $\left(\frac{1}{2}\right)^{1}$ followed by $\frac{1}{4}$; $\frac{3}{16}$, $\frac{3}{256}$.

e 34, 55; each subsequent term is the sum of the preceding two terms.

f 31, 63; terms are 1 less than powers of 2.

g 5, 4; add 2 to find the next term, then subtract 1 to find the subsequent term and repeat.

11 D **12** E **13** C

14 a $t_{n+1} = t_n - 2$, $t_1 = 7$

b $t_{n+1} = t_n \div 2$, $t_1 = 12$

c $t_{n+1} = t_n + 0.6$, $t_1 = 12$

d $t_{n+1} = t_n \times 5 + 1$, $t_1 = 2$

e $t_{n+1} = -3t_n$, $t_1 = 4$

f $t_{n+1} = (t_n)^2$, $t_1 = 2$

15 a 26 and 25 **b** 23 cats

16 The population size will rapidly decrease and by 2009, the stray cat population will be gone. (Happily, they were all taken in by good and loving households.)

EXERCISE 3.3

1 a Not arithmetic

b Arithmetic, difference = 3; $t_4 = 6$, $t_n = -6 + 3n$

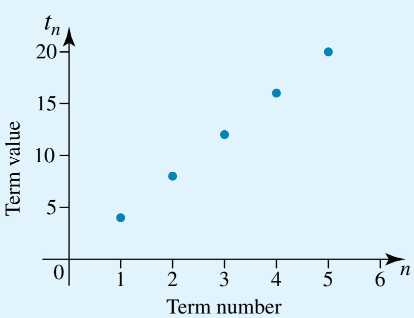
2 a Not arithmetic

b Arithmetic, difference = −4; $t_4 = -14$, $t_n = 2 - 4n$

3 a

Term number	1	2	3	4	5
Term value	4	8	12	16	20

b

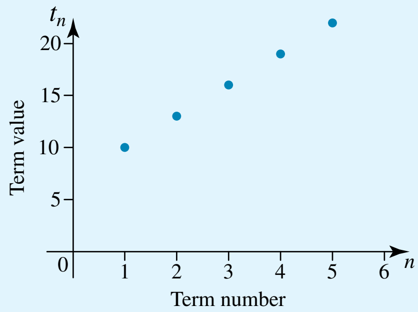


c $t_{10} = 40$

4 a

Term number	1	2	3	4	5
Term value	10	13	16	19	22

b



c $t_{10} = 37$

5 $m = 21.5$, $n = 32.5$

6 $x = 47$, $y = 75$

7 -2, $\frac{26}{3} - \frac{8}{3}n$

8 4.54, $t_n = 2.78 + 0.22n$

9 a 104 **b** 682

c 1458 **d** −26 310

10 a $t_n = 8 - 3n$, $n = 1, 2, 3, \ldots$

b $t_n = 2 + \frac{n}{2}$, $n = 1, 2, 3, \ldots$

c $t_n = -6 + 3n$, $n = 1, 2, 3, \ldots$

d $t_n = -3x + 5nx$, $n = 1, 2, 3, \ldots$

11 $t_n = 4 + 2n$, $n = 1, 2, 3, \ldots$

12 $5n - 2$

13 $-x + y$, $-5x + 9y$

14 −35; 15; $15n - 50$

15 $-4\frac{1}{4}$; $\frac{5}{12}$; $-4\frac{2}{3} + \frac{5}{12}n$

16 $m = 27$, $n = 32$

17 9 **18** 3

19 a $t_{n+1} = t_n + 4$; $t_1 = 3$

b $t_{n+1} = t_n + 3$; $t_1 = -3$

c $t_{n+1} = t_n - 4$; $t_1 = -2$

d $t_{n+1} = t_n + \frac{1}{2}$; $t_1 = \frac{2}{7}$

e $t_{n+1} = t_n + \frac{3}{4}$; $t_1 = \frac{3}{4}$

f $t_{n+1} = t_n - \frac{7}{4}$; $t_1 = \frac{1}{4}$

g $t_{n+1} = t_n + 2\pi - 2$; $t_1 = 2\pi + 3$

20 a $\frac{5}{6}$ **b** $\frac{n+2}{n+3}$

EXERCISE 3.4

1 400 **2** 4175

3 a 1275 **b** 5050

4 a 5000

b Each of the 100 terms is $\frac{1}{2}$ less than its corresponding term in question 3. There are 100 terms, so the answer to this question is 50 less than in question 3b.

5 258

6 a −273, −480, −741 b −324

7 a 280 b 1080

c 34

8 $\dfrac{n(n+1)}{2}$

9 a, b Various answers

10 17

11 45

12 6

13 174

14 The iterative equation is $t_{n+1} = t_n + 8$, $t_1 = 7$.
The functional equation is $t_n = 8n - 1$, $n = 1, 2, 3, \ldots$

EXERCISE 3.5

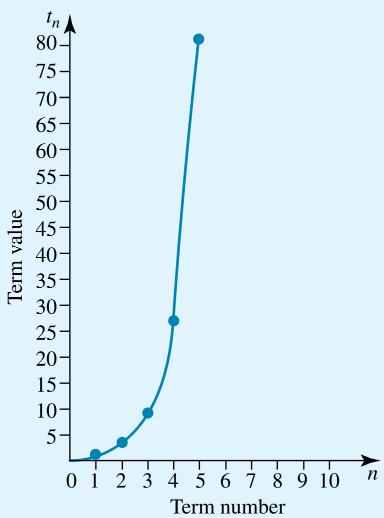
1 Geometric, ratio = 2; $t_4 = 24$; $t_n = 3 \times 2^{n-1}$

2 Geometric, ratio $= \frac{-1}{3}$; $t_4 = \frac{1}{9}$; $t_n = (-3)^{2-n}$

3 a

Term number	1	2	3	4	5
Term value	1	3	9	27	81

b

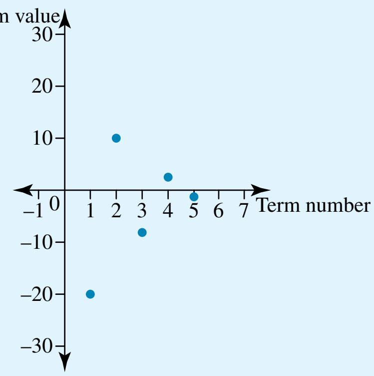


c The points lie on a smooth curve which increases rapidly. Because of this rapid increase in value, it would be difficult to use the graph to predict future values in the sequence with any accuracy.

4 a

Term number	1	2	3	4	5
Term value	−20	10	−5	2.5	−1.25

b



c The points are oscillating and converge on a point.

5 There are two possible answers because the ratio could be −3 or 3. The nth term is $t_n = 2 \times 3^{n-1}$ or $t_n = 2 \times (-3)^{n-1}$, $t_{10} = \pm 39\,366$.

6 There are two possible answers because the ratio could be −2 or 2. The nth term is $t_n = 2^{n-1}$ or $t_n = (-2)^{n-1}$, $t_{10} = \pm 512$.

7 $t_1 = 25$, $r = \pm 2$, $t_n = 25 \times 2^{n-1}$ or $t_n = 25 \times (-2)^{n-1}$

8 $a = \pm 6$, $r = \pm 2$, $t_n = \pm 6 \times (\pm 2)^{n-1}$

9 a $t_n = 5 \times 2^{n-1}$, $t_6 = 160$, $t_{10} = 2560$

b $t_n = 2 \times 2.5^{n-1}$, $t_6 = 195.31$, $t_{10} = 7629.39$

c $t_n = 1 \times (-3)^{n-1}$, $t_6 = -243$, $t_{10} = -19\,683$

d $t_n = 2 \times (-2)^{n-1}$, $t_6 = -64$, $t_{10} = -1024$

e $t_n = 2.3 \times (1.5)^{n-1}$, $t_6 = 17.47$, $t_{10} = 88.42$

f $t_n = \frac{1}{2} \times 2^{n-1}$, $t_6 = 16$, $t_{10} = 256$

g $t_n = \frac{1}{3} \times \left(\frac{1}{4}\right)^{n-1}$, $t_6 = \frac{1}{3072}$, $t_{10} = \frac{1}{786\,432}$

h $t_n = \frac{3}{5} \times \left(\frac{-1}{3}\right)^{n-1}$, $t_6 = \frac{-1}{405}$, $t_{10} = \frac{-1}{32\,805}$

i $t_n = x \times (3x^3)^{n-1}$, $t_6 = 243x^{16}$, $t_{10} = 19\,683x^{28}$

j $t_n = \dfrac{1}{x} \times \left(\dfrac{2}{x}\right)^{n-1}$, $t_6 = \dfrac{32}{x^6}$, $t_{10} = \dfrac{512}{x^{10}}$

10 a The nth term is $t_n = 5 \times 2^{n-1}$, $t_{10} = 2560$.

b The nth term is $t_n = -1 \times (-2)^{n-1}$, $t_{10} = 512$.

c There are two possible answers because the ratio could be $\frac{-1}{27}$ or $\frac{1}{27}$. The nth term is $t_n = 3^{5-3n}$ or $t_n = (-3)^{5-3n}$, $t_{10} = \pm 3^{-25}$.

11 $\frac{\pm 3}{4}$

12 $3 \times 2^{\frac{(n-1)}{2}}$

13 $m = 12$, $n = 48$

14 $m = 36$, $n = \frac{729}{4}$

15 $a = 300$, $b = 0.75$

16 $t_1 = \frac{1}{3}$, $r = \frac{3}{2}$, $t_n = 3^{n-2}2^{1-n}$

17 -6

18 $2, \frac{1}{2}, \frac{1}{8}$, or $-2, \frac{1}{2}, \frac{-1}{8}$

19 a $\frac{3}{2}$ b $\frac{24}{2^n}$

20 $k = 6$

EXERCISE 3.6

1 a 31

b 33

c 46.5

2 a 121

b -29.8

c $\frac{-31}{24}$

3 a 2 b $\frac{4}{27}$

4 a 3 b $\frac{-8}{27}$

5 a 3108 b $\frac{1}{16}, 63\frac{15}{16}, 66\,535\frac{15}{16}$

6 9

7 $\frac{4}{3}$; 75%, 18.75%, 4.6875%

8 a $6\left[1 - \left(\frac{1}{2}\right)^{20}\right] = 5.999\,994\,278$

b 5.722×10^{-6}

c 6

9 a $6\frac{3}{4}\left[1 - \left(\frac{-1}{3}\right)^{9}\right] = 6.750\,343$

b -3.425×10^{-4}

c $6\frac{3}{4}$

10 $6\frac{2}{3}$

11 16, 8, 4

12 $4 - \sqrt{15}, \dfrac{(\sqrt{3} + \sqrt{5})}{(\sqrt{15} - 3)} = \dfrac{(4\sqrt{3} + 3\sqrt{5})}{3}$

13 $\frac{-1}{6}$

14 1, 0.99

15 Check with your teacher.

16 a Mathematically, the student will never make it past the other side of the road. After each attempt, the distance remaining is halved, and this result is the extra distance walked at the next attempt. Thus the distance travelled across the road approaches but never reaches 10 metres.

b As shown in part **a**, the extra distance travelled at each attempt is equal to half the remaining distance from the previous attempt. Given that there will always be an amount remaining to travel, only half this amount can be achieved on the next attempt, regardless of the width of the road.

EXERCISE 3.7

1 a i \$91 253

ii \$120 681

b 3.518 times

2 a \$116 800 b \$1 311 000

3 $\frac{1}{9}$

4 $\frac{57}{99}$

5 a Geometric

b \$48 487

c \$560 660

d i \$1120

ii \$1865

6 a 36 mg, 44 mg, 100 mg, $20 + 8n$ mg

b 23rd day

c 39 mg, 55 mg, 579 mg, $28 \times (1.4)^{n-1}$ or 20×1.4^n

d Seventh day

7 a 310

b The workers must remove 12 full rows and 17 logs from the 13th row.

8 The last person received 64 tickets and Kate had 73 left.

9 a 1.8×10^{19} grains of rice

b 1.8×10^{15} kg

10 a 4096 grains of rice

b 0.41 kg

11 Yes, seventh day

12 a $\frac{1}{3}$ b $\frac{232}{99}$

c $\frac{3139}{999}$ d $\frac{700}{33}$

e $\frac{50}{3}$

13 a $\pi, 3\pi, 5\pi, 7\pi$ — arithmetic progression with $a = \pi$ and $d = 2\pi$

b $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}$

14 a $192 - 12n$ m/s

b During the 16th second

c 1440 m

15 After 7 minutes the coffee has cooled to below 50 °C.

16 The sequence for the arithmetic series $t_n u_n$ is $12n - 15$, $n \in \{1, 2, 3, \ldots\}$

4

Geometry in the plane

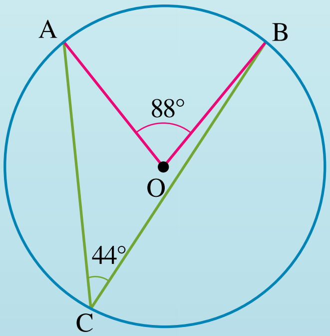
4.1 Kick off with CAS

Circle theorems

Circle theorem 1:

The angle subtended by the chord at the centre of a circle is twice the value of the angle subtended by the same chord at the circumference of the circle.

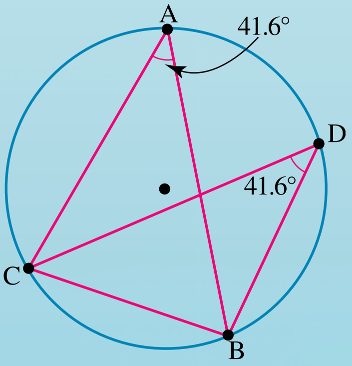
1 Use the dynamic geometry application of your CAS technology to verify circle theorem 1.



Circle theorem 2:

Angles subtended at the circumference in the same segment by the same chord are equal.

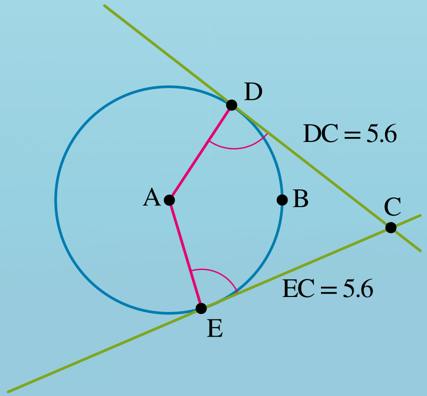
2 Use the dynamic geometry application of your CAS technology to verify circle theorem 2.



Circle theorem 3:

Two tangents that meet outside a circle are equidistant from the point of contact on the circle.

3 Use the dynamic geometry application of your CAS technology to verify circle theorem 3.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Review of basic geometry

Plane geometry is the study of flat, two-dimensional shapes. A **point** is a location with no dimensions; that is, it is a concept that helps us determine the location of real objects. A point is often labelled with a capital letter.

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Units 1 & 2
AOS 4
Topic 1
Concept 1

Basic geometry
Concept summary
Practice questions

Lines and angles

A **line segment** joins two points. It is a basic postulate (or principle) that only one straight line segment can join two given points.

In the figure at right, there is exactly one straight line segment that can join points A and B.

If we extend the line segment to 'infinity' in both directions, we have what is properly called a **line**.

If we extend the line segment to 'infinity' in one direction only, we have a **ray**.

In the figure at right we have a ray extending from point A. Often the terms line segment, line and ray are used interchangeably. The context should help you determine which of the three kinds of 'line' is really being used.

Three points can determine an **angle**. In Figure 1, the angle, indicated by the arc of a circle, is named ∠BAC or ∠CAB. The **vertex** of the angle is at point A and is placed in the middle of the name, so we do not write ∠ABC or ∠BCA in this example.

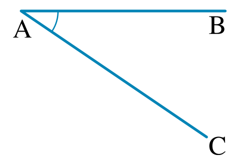


Figure 1

A **straight angle** is formed when the three points of the previous example are all in a line. In Figure 2, the straight angle is ACB, with the vertex at C. Note that another line segment can be drawn to point D. By convention, a straight angle equals 180 degrees and is written as ∠ACB = 180°. Therefore, it can be seen that ∠ACD + ∠DCB = ∠ACB = 180°.

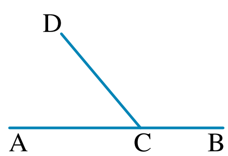


Figure 2

The two angles ∠ACD and ∠DCB are called **supplementary angles** as they sum to 180°.

If the point D is moved so that ∠ACD = ∠DCB, then we have created a **right angle**. In Figure 3, both ∠ACD and ∠DCB are right angles. From the previous equation, because ∠ACD = ∠DCB, we can rewrite

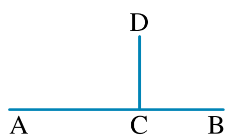


Figure 3

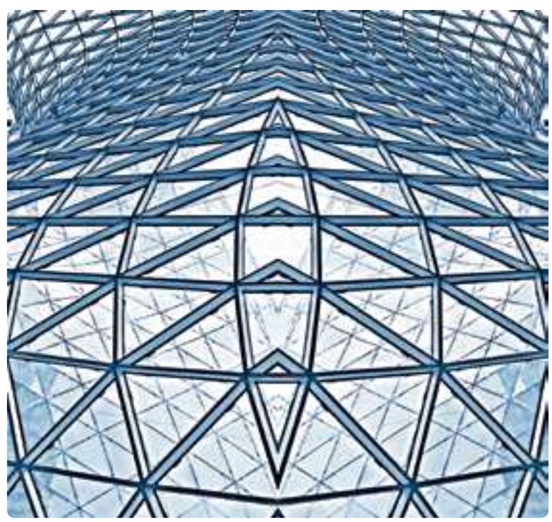
$$\angle ACD + \angle DCB = \angle ACB = 180°$$

to

$$\angle ACD + \angle ACD = 180°$$
$$2(\angle ACD) = 180°$$
$$\angle ACD = 90°$$

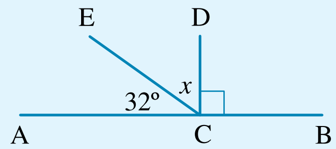
So a right angle equals 90°. Therefore, we can say that AB is perpendicular to CD.

$$\therefore AB \perp CD$$



WORKED EXAMPLE 1

Find the value of x in the given diagram.



THINK	WRITE
1 Write an equation involving the required angle. ∠ACB is a straight angle.	$\angle BCD + x + \angle ECA = 180°$
2 Replace angles with known values. ∠BCD = 90° (right angle)	$90° + x + 32° = 180°$
3 Solve for the missing angle.	$122° + x = 180°$ $122° - 122° + x = 180° - 122°$ $x = 58°$

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Units 1 & 2

AOS 4

Topic 1

Concept 2

Angles and parallel lines

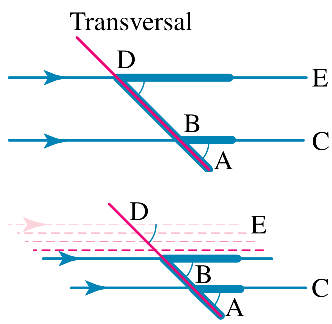
Concept summary

Practice questions

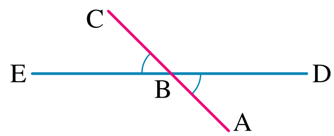
Parallel lines and angles

It is a basic postulate of geometry that parallel lines never meet. When parallel lines are intersected by a transversal, special angles are formed.

Consider the pair of parallel lines in the figure at right. Note the use of arrows to indicate that the lines are parallel, C‖E. Now consider the two angles ∠ABC and ∠BDE, marked with arcs. Because the parallel lines (E and C) never meet, we could easily move one on top of the other without affecting the angles. Thus it stands to reason that the angles are equal. These equal angles are called **corresponding angles**.



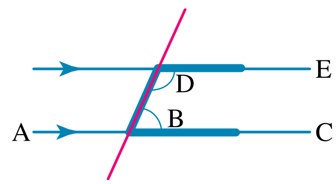
Now, consider the two crossing lines shown in the figure at right. Because ∠EBD and ∠ABC are both straight angles (180°), we can write:



$$\angle EBC + \angle CBD = \angle CBD + \angle ABD$$
$$\angle EBC = \angle ABD$$

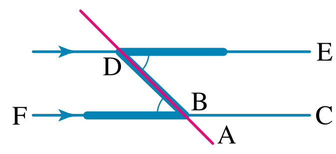
(the two angles indicated with arcs). These angles are called **vertically opposite angles**.

Next, consider the parallel lines at right, and the two angles ∠CBD and ∠EDB (indicated with arcs). They are positioned 'inside' the parallel lines, on the same side (to the right) of the transversal. These angles are called **co-interior angles** and sum to 180°.



Finally, consider the pair of parallel lines below right and the two angles ∠EDB and ∠FBD.

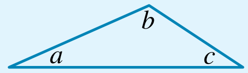
From the earlier result about parallel lines we know that ∠EDB = ∠CBA, and from the result about crossing lines, that ∠CBA = ∠FBD. Therefore, ∠EDB = ∠FBD (the two angles indicated with arcs). These are called **alternate angles**.

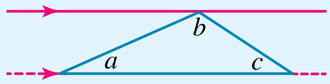
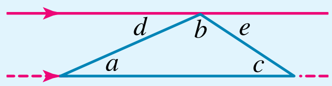


From the above results, we are in a position to perform our first 'proof'.

WORKED EXAMPLE 2

Prove that the sum of the interior angles of any triangle is 180 degrees. In other words, prove that $a + b + c = 180°$.

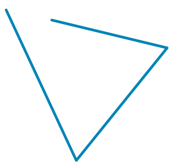


THINK	DRAW/WRITE
1 Construct a line parallel to one of the sides.	
2 Label some additional angles (d and e) that will be needed.	
3 The upper parallel line is also a straight angle.	$d + b + e = 180°$
4 But we have some alternate angles.	$d = a$ $e = c$
5 Substitute into the equation in step 3.	$a + b + c = 180°$ … QED

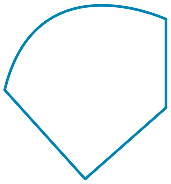
As described in Topic 2, QED is Latin for *quod erat demonstrandum* ('as has been demonstrated'), but could just as well be thought of as 'quite easily done'! Remember that, when proving something, you are allowed to use other facts that have already been established, without having to prove them each time.

Angles in polygons

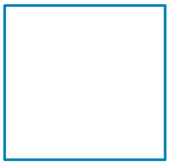
A **polygon** is a closed figure made up of three or more straight line segments.



This figure is not a polygon because it is not closed.

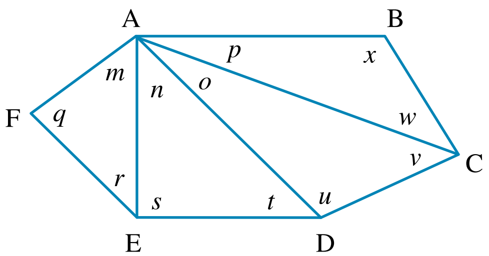


This figure is not a polygon because it has a curved line.



This figure is a polygon because it is closed and has four (more than two) line segments, all straight.

Now that we know that the sum of angles in a triangle is 180°, we can find the sum of angles in any polygon by dividing it into triangles as shown.



$m + q + r = 180°$

$n + s + t = 180°$

$o + u + v = 180°$

$p + w + x = 180°$

The sum of the angles in a hexagon

$= 4 \times 180°$ (since it contains 4 triangles)

$= 720°$

The general formula used to determine the sum of the interior angles in any polygon is:

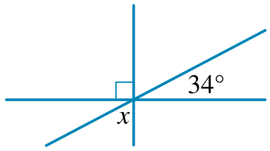
$$(n - 2) \times 180°, \text{ where } n = \text{number of sides.}$$

EXERCISE 4.2 Review of basic geometry

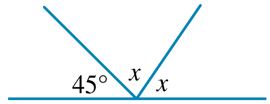
PRACTISE

1 **WE1** Find the values of the pronumerals in the following figures.

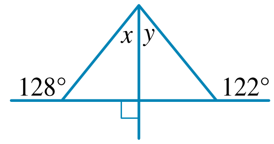
a



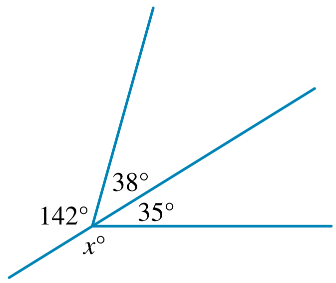
b



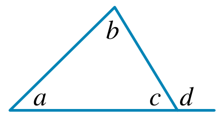
c



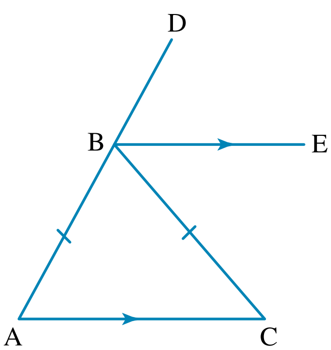
2 Find the value of x.



3 **WE2** Prove the following theorem: 'The value of the exterior angle of a triangle equals the sum of the two interior opposite angles'. In other words, in referring to the figure at right, $d = a + b$.



4 ΔABC shown below is isosceles. AB is produced to D, and BE is drawn through B, parallel to the base AC. Prove that BE bisects ∠CBD.



CONSOLIDATE

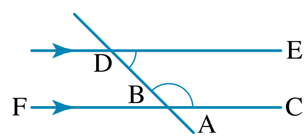
5 Angles that add up to 180 degrees are called:

A parallel
B isosceles
C complementary
D supplementary
E right-angled

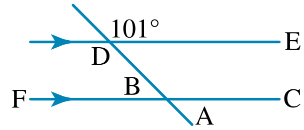
6 Which of the following statements is false?

A An angle is always formed when two lines meet.
B The sum of angles in a right-angled triangle is 180 degrees.
C A line segment extended to infinity in both directions is called a ray.
D Parallel lines never meet.
E Reflex angles are bigger than acute angles.

7 Prove that the co-interior (or allied) angles marked with arcs in the figure at right are supplementary. In other words, prove that ∠EDB + ∠DBC = 180°.

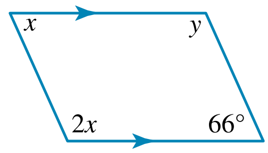


8 From the result in Worked example 2 and question **7**, show that there are only two different angle values in that figure: all angles must take on one or the other of these values. Furthermore, show that if you know only one of these angles, then all other angles can be determined. Demonstrate this using the value in the figure at right.

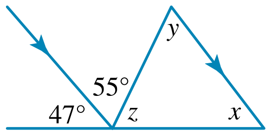


9 Find the values of the unknown angles in the following figures.

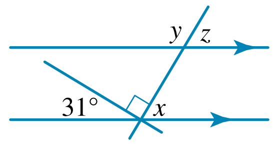
a



b

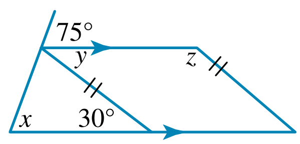


c

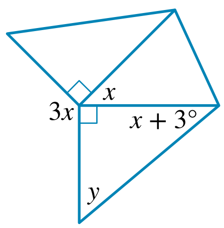


10 Find the values of the unknown angles in the following figures.

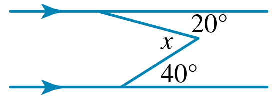
a



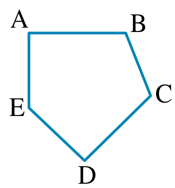
b



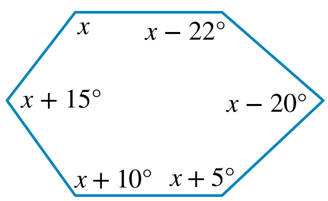
c



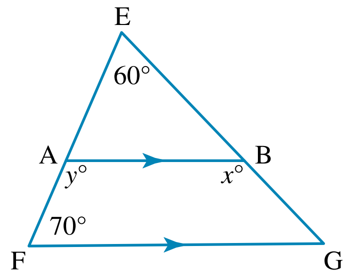
11 Show that the sum of the interior angles of the pentagon at right equals 540 degrees. In other words, show that: $\angle AED + \angle EDC + \angle DCB + \angle CBA + \angle BAE = 540°$.



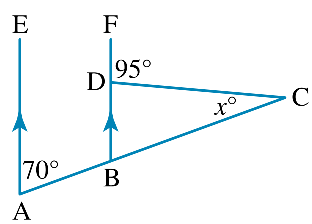
12 Find the value of x in the figure below.



13 Find the values of the pronumerals in the figure at right.

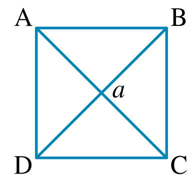


14 Using the figure below, find the value of x, giving reasons for your answer.



MASTER

15 Prove that the diagonals of a square form a right angle; that is, that angle a in the figure at right equals 90 degrees.



16 a Copy and complete the following table relating the sum of the interior angles to the number of sides of a polygon.

b Can you establish a general formula for the sum in terms of n, the number of sides?

Number of sides (n)	3	4	5	6	7	8	9	10	20
Sum of interior angles	180°		540°						

4.3 Congruence and similarity

studyon

Units 1 & 2
AOS 4
Topic 1
Concept 4

Congruent figures
Concept summary
Practice questions

Congruence

Congruent figures are shapes that have the exact same shape and size. If ΔABC and ΔDEF are congruent, we write ΔABC ≊ ΔDEF.

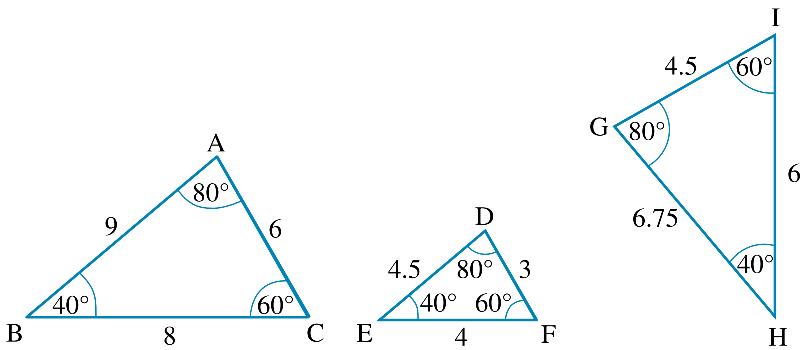
Congruent triangles have all angles and sides equal. There are four tests to prove that two triangles are congruent: three equal sides (SSS); two equal sides and the included angle (SAS); two equal angles and the included side (ASA); and a right-angle, equal hypotenuse and another equal side (RHS).

Similarity

Similar figures are shapes that have the same shape but different size. If triangles ΔABC and ΔDEF are similar, we write ΔABC ~ ΔDEF.

There are three tests to prove triangles are similar.

The following triangles will be used to prove the tests for similar triangles.



Test 1

Triangles are said to be similar if all the corresponding angles are equal (abbreviated to AAA). That is, the three angles in one of the triangles are equal to the three angles in the other triangle.

ΔABC ~ ΔDEF, because ∠A = ∠D
∠B = ∠E
∠C = ∠F

ΔDEF ~ ΔGHI, because ∠D = ∠G
∠E = ∠H
∠F = ∠I

Test 2

Triangles are said to be similar if the ratios between the corresponding side lengths are equal (abbreviated to SSS).

ΔABC ~ ΔDEF, because the ratio of corresponding side lengths is 2:

$$\frac{AB}{DE} = \frac{9}{4.5} = 2, \frac{AC}{DF} = \frac{6}{3} = 2, \text{ and } \frac{BC}{EF} = \frac{8}{4} = 2$$

$\Delta DEF \sim \Delta GHI$, because the ratio of corresponding side lengths is $\frac{2}{3}$:

$$\frac{DE}{GH} = \frac{4.5}{6.75} = \frac{2}{3}, \frac{DF}{GI} = \frac{3}{4.5} = \frac{2}{3}, \text{ and } \frac{EF}{HI} = \frac{4}{6} = \frac{2}{3}$$

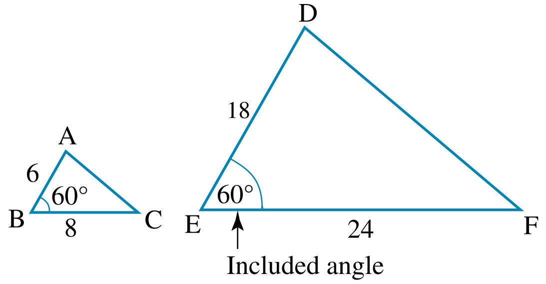
studyon

Units 1 & 2
AOS 4
Topic 1
Concept 3

Similar figures
Concept summary
Practice questions

Test 3

Triangles are said to be similar if two of their corresponding sides are in the same ratio (ratio value does not equate to 1) and the angle between these two sides (the included angle) is the same in both triangles (abbreviated to SAS).



$\Delta ABC \sim \Delta DEF$, because the corresponding sides have the same ratio value that does not equate to 1 and the included angle for both triangles is the same:

$$\frac{EF}{BC} = \frac{24}{8} = 3 \text{ and } \frac{ED}{AB} = \frac{18}{6} = 3; \angle B = \angle E$$

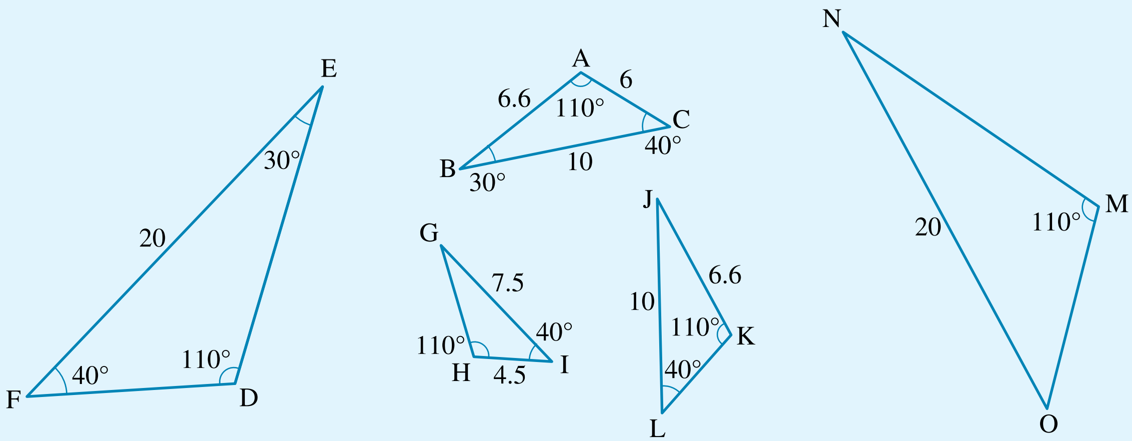
Sometimes it is hard to decide whether two triangles are similar or not because they are not oriented the same way. In such cases it is helpful to re-draw the triangles so that the sides and angles that might be corresponding are in the same order.

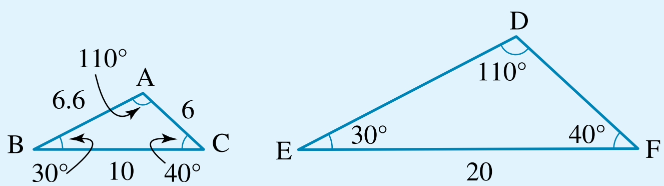
WORKED EXAMPLE 3

Compare each of the following triangles with ΔABC and state whether they are similar, congruent or there is not enough information given for a decision to be made. Justify your answers.

a ΔDEF **b** ΔGHI

c ΔJKL **d** ΔMNO



THINK	WRITE/DRAW
a 1 Re-draw ΔDEF so that its angles correspond to those of ΔABC.	**a** 

2 Compare the corresponding angles and write down what you observe.

$\angle A = \angle F$, $\angle B = \angle D$, $\angle C = \angle E$
All corresponding angles are equal.

3 Compare the side measurements given to test for congruency.

$\frac{FE}{CB} = \frac{20}{10} = 2$

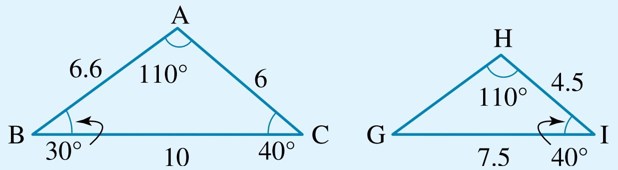
$2 \neq 1$, so the triangles are not congruent.

4 State whether the triangles are similar and specify the test upon which your conclusion is based.

$\Delta ABC \sim \Delta DEF$ because all corresponding angles are equal (AAA).

b **1** Re-draw ΔABC and ΔGHI so that angles/sides correspond.

b



2 Compare the side measurements.

$\frac{AC}{HI} = \frac{6}{4.5}$

$= 1.333$

$\frac{BC}{GI} = \frac{10}{7.5}$

$= 1.333$

3 Check whether the angles formed by the sides that have measurements are in the same ratio (that is, the included angles).

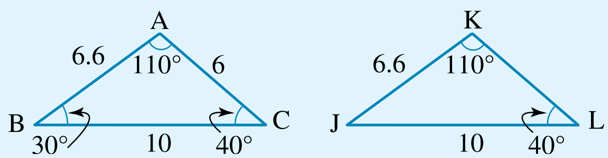
$\angle C = \angle I = 40°$

4 State and justify your conclusion.

ΔABC and ΔGHI are similar because two pairs of corresponding sides are in the same ratio that doesn't equate to 1 and the included angles are equal (SAS).

c **1** Re-draw ΔABC and ΔJKL so the angles/sides correspond.

c



2 Check if the corresponding angles are equal.

$\angle J = 180° - 150° = 30°$
$\angle J = \angle B$, $\angle A = \angle K$, $\angle C = \angle L$

3 As side measurements are given, check for congruency.

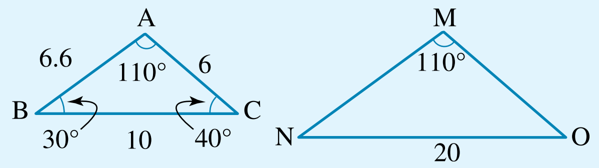
$\frac{BC}{JL} = \frac{10}{10} = 1$ and $\frac{AB}{KJ} = \frac{6.6}{6.6} = 1$

4 State your conclusion and justify.

Initially, the two triangles appeared to be similar because of AAA; however, as the side ratio equates to 1, these triangles are congruent.

d 1 Re-draw ΔABC and ΔMNO so the angles/sides correspond.

d



2 None of the three tests (AAA, SSS or SAS) can be performed since we know the measurements of only one side and one angle in the ΔMNO. State this in writing.

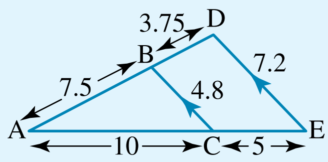
Unable to determine whether the triangles are similar, as ΔMNO does not provide enough information to test AAA, SSS or SAS.

WORKED EXAMPLE 4

Show that ΔABC ~ ΔADE, because:

a the corresponding angles are equal

b the corresponding side ratios are equal (to a value other than 1).

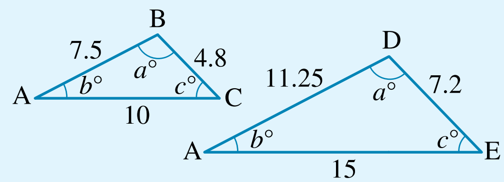


THINK

WRITE/DRAW

a 1 Draw the triangles separately with angles marked and side measurements shown.

a



2 State the corresponding pairs of angles which are equal in size. Specify the reason.

∠A = ∠A (shared)

∠B = ∠D (corresponding)

∠C = ∠E (corresponding)

3 Write your conclusion.

∴ ΔABC ~ ΔADE using AAA test.

b 1 Evaluate the ratio between the corresponding sides.

b $\frac{AD}{AB} = \frac{11.5}{7.5} = 1.5$

$\frac{AE}{AC} = \frac{15}{10} = 1.5$

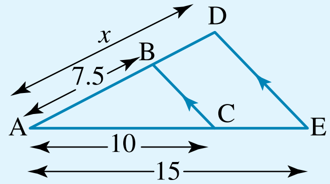
$\frac{DE}{BC} = \frac{7.2}{4.8} = 1.5$

2 Write your conclusion.

The corresponding side lengths of the triangles are in the same ratio. (The ratio is not equal to 1.)
∴ ΔABC ~ ΔADE (SSS test).

WORKED EXAMPLE 5

Find the value of the pronumeral x.



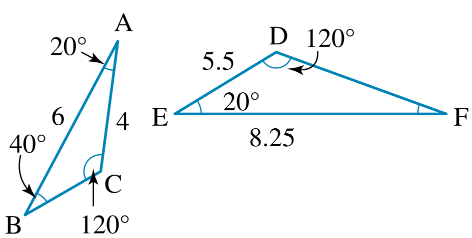
THINK	WRITE
1 Establish whether the triangles ABC and ADE are similar by applying the AAA test.	$\angle A = \angle A$ (shared) $\angle B = \angle D$ (corresponding) $\angle C = \angle E$ (corresponding) $\therefore \Delta ABC \sim \Delta ADE$ (AAA test)
2 Since the triangles are similar, their corresponding sides are in the same ratio. Find this ratio using the corresponding pair of sides whose lengths are known.	$\frac{AE}{AC} = \frac{15}{10}$ $= 1.5$
3 The missing length AD in ΔADE corresponds to the side AB in ΔABC. Use this information to form the ratio equation.	$\frac{AD}{AB} = 1.5$
4 Identify the values of AD and AB.	$AD = x$, $AB = 7.5$
5 Substitute the values of AD and AB into the equation and solve for x.	$\frac{x}{7.5} = 1.5$ $= 11.5$

EXERCISE 4.3 Congruence and similarity

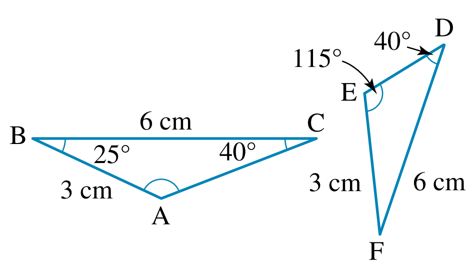
PRACTISE

1 WE3 For each of the following, compare ΔABC and ΔEFD and state whether they are similar, congruent, or there is not enough information given for a decision to be made. Justify your answers.

a

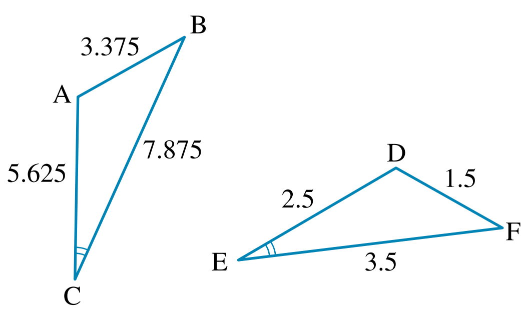


b

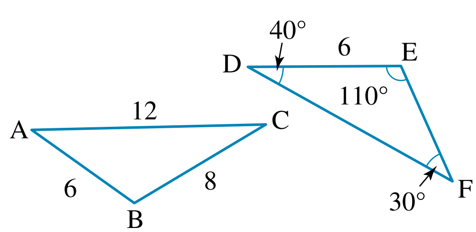


2 For each of the following, compare ΔABC and ΔEFD and state whether they are similar, congruent, or there is not enough information given for a decision to be made. Justify your answers.

a



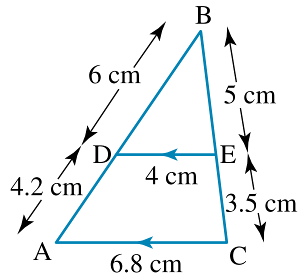
b



3 WE4 Show that ΔABC is similar to ΔDBE because:

a the corresponding angles are equal

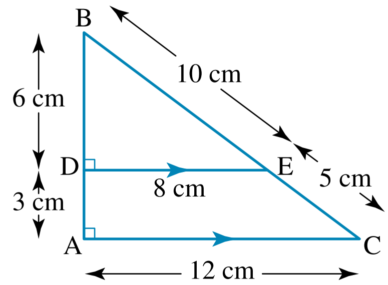
b the corresponding side ratios are equal (to a value other than 1).



4 Show that ΔABC is similar to ΔDBE because:

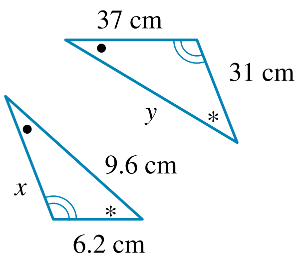
a the corresponding angles are equal

b the corresponding side ratios are equal (to a value other than 1).

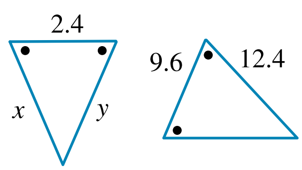


5 WE5 Find the values of the pronumerals (to 1 decimal place).

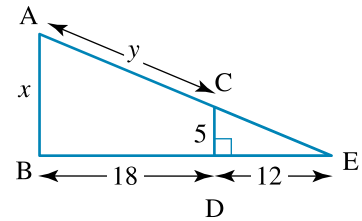
a



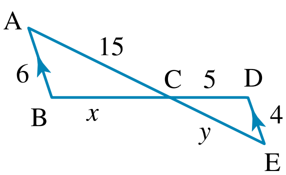
b



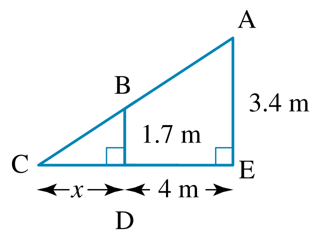
c



d

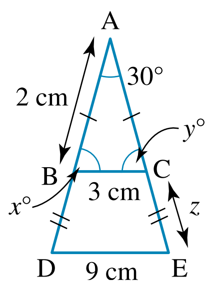


e



6 Find the values of the pronumerals (to 1 decimal place).

a



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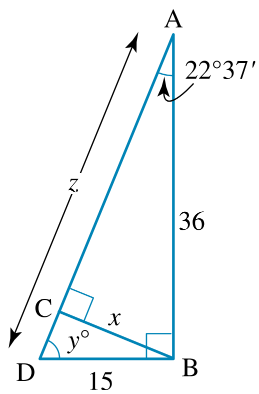

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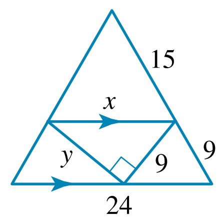

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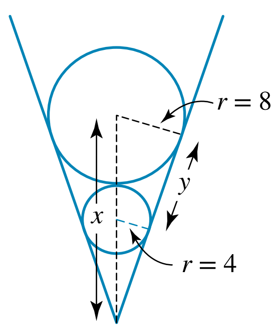
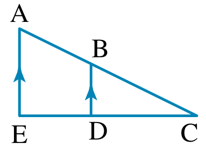


CONSOLIDATE

7 In each of the following diagrams, find and re-draw two triangles that are similar. Give reasons for your answer.

a



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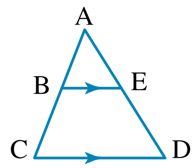

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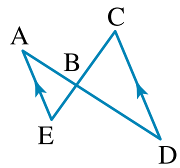
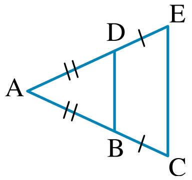
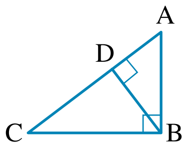


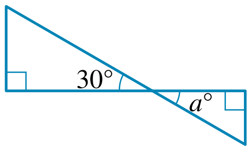
d



e

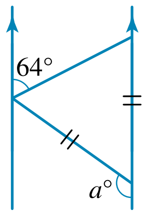


8 The magnitude of angle $a°$ is:



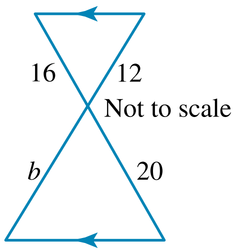
A 30° **B** 180° − 30°
C 60° **D** 180° − 60°
E 70°

9 The magnitude of angle $a°$ is:



A 64° **B** 128
C 52° **D** 154°
E 104°

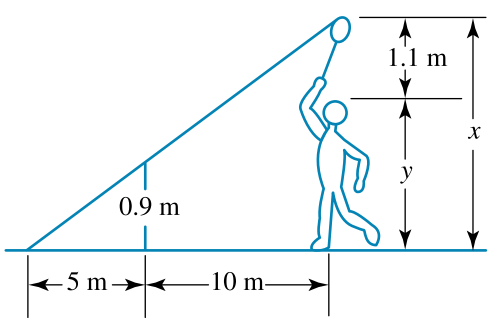
10 In the given diagram, the length of side b is closest to:



A 24 **B** 22
C 16 **D** 15
E 9.6

Questions **11** and **12** refer to the following information.

A young tennis player's serve is shown in the diagram at right. Assume the ball travels in a straight line.



11 The height of the ball just as it is hit, x, is closest to:

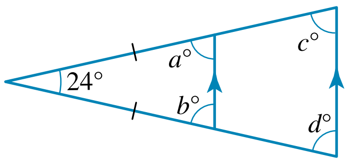
A 3.6 m **B** 2.7 m **C** 2.5 m
D 1.8 m **E** 1.6 m

12 The height of the player, y, is closest to:

A 190 cm **B** 180 cm **C** 170 cm
D 160 cm **E** 150 cm

13 Calculate the size of the missing angles. Justify your answer.

a



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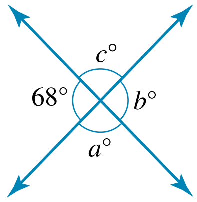

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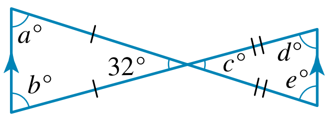

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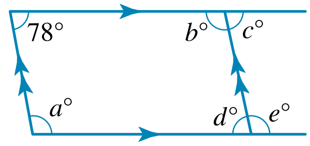
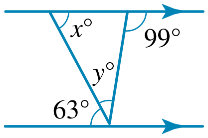
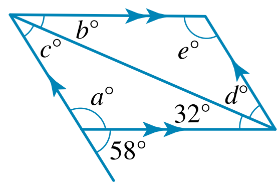


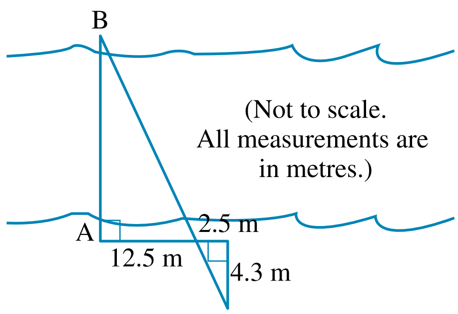
e



f



14 Find the length of the bridge, $\overline{AB}$, needed to span the river, using similar triangles as shown (to 1 decimal point).



15 A triangle with sides 12 cm, 24 cm and 32 cm is similar to a smaller triangle that has a longest side measurement of 8 cm. Draw a diagram to represent this situation and then calculate the perimeter of the smaller triangle.

16 It just happens that you always carry a wooden rod 1.5 m in length and a tape measure. For each of the following situations, draw a diagram showing the two similar triangles and calculate the height of the vertical object.

a You have always wanted to know how high the basketball players had to jump to touch the top of the ring. Your wooden rod held next to the basketball pole created a shadow half the length of the basketball pole's 1 m shadow. If the ring is 45 cm from the top of the board, how far up from the ground do the players have to reach to touch the ring?

b The measuring tape for the pole vault height is missing. How high did Australian National Champion Elizabeth Parnov pole vault if the pole vault creates a shadow of 2.4 m on the ground, while your wooden rod creates a shadow of 80 cm?

c Does the fire brigade need to bring an extension ladder longer than 15 m to rescue people from the top of a high-rise building if the shadow created by the building is 24 m and the shadow created by your pole is 2 m?

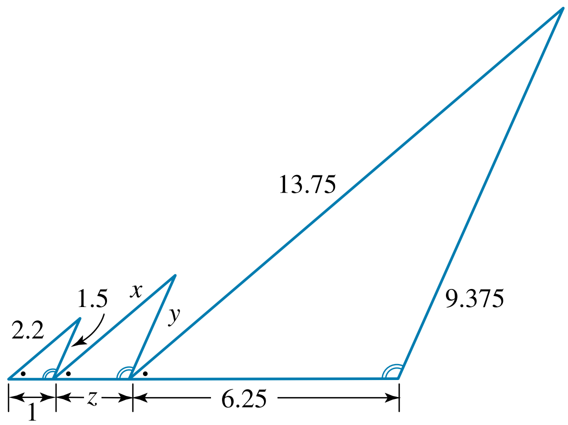
d The clearance sign is missing from a low bridge over a road. The sun has created an image of the bridge and its opening on the ground. The image of the opening is 1.44 m in length. The semi-trailer you are driving is 2.5 m high. You step out of the truck and place your trusty rod on the ground, producing a 90 cm shadow. Will you need to find an alternative route or can you proceed?

MASTER

17 A curved section of a fun-ride is to be supported by similar triangles.

a If the dimensions of the first and last triangle are as shown, calculate the dimensions of the triangle in between.

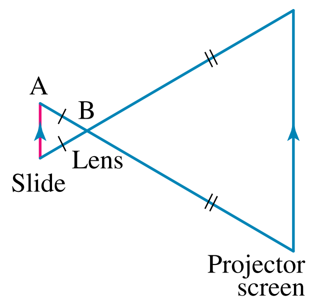
b The construction people decide to make the track even steeper by extending it with one more similar triangle. What are the dimensions of the next triangle?



18 At your 18th birthday party you want to show slides of you and your friends growing up. The diagram at right shows the set-up of the projector lens, slide and projector screen.

How far horizontally from the screen do you need to place the projector's lens so the image just covers the entire screen if:

- a slide is 5 cm × 5 cm
- the distance from A to B is 10 cm
- the projector screen is 1.5 m × 1.5 m?



4.4 Geometric constructions

A traditional part of any study of geometry includes the skill of producing constructions. In this section, you are to use only a straight edge (you can use a ruler, but no measuring!) and a pair of compasses.

Using our straight edge and compasses, we are able to construct a variety of geometric figures. We shall now look at constructions that show us how to bisect lines, bisect angles, and draw special angles (for example, 60°).

The only new definition required here is **bisection**. You can bisect a line by dividing it in half, or bisect an angle so that its measure is halved.

studyon

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Bisecting lines

In the following examples, constructions drawn with a ruler and a pair of compasses are shown in grey.

WORKED EXAMPLE 6

Use a ruler and a pair of compasses to bisect a line, AB.

THINK

1 a Draw a line AB.

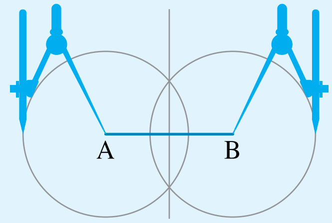
b Place the compass point at A, with any radius more than half the length of the line.

c Draw a circle.

2 With the same radius as in step 1, repeat for point B.

3 The two circles will intersect at two points. Join these points with a straight line. This line bisects AB and is at 90° to AB.

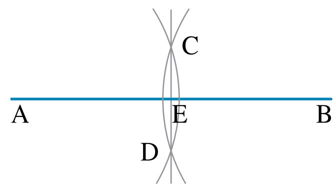
DRAW



With this construction, not only have you divided the original line (AB) in half, but the line that you drew in step 3 is perpendicular to AB. Therefore, this line is called the **perpendicular bisector** of AB.

Why does it work? It is useful to examine the geometry of the construction to help understand why it provides the correct result.

The figure at right shows the essential part of the construction. Points C and D are the intersections of the two circles.

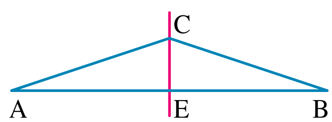


Join point A to C and B to C to create two triangles, as shown in the figure below.

Because the two circles had the same radius, AC = BC. The two triangles also share the common side CE. By symmetry, ∠ACE = ∠BCE and ∠CAE = ∠CBE.

Therefore, since two of the angles are equal, the third angles are also equal; that is, ∠AEC = ∠BEC.

From all this it is clear that the two triangles are identical. Therefore, AE = EB and we have effectively bisected AB (divided it into two equal parts).



WORKED EXAMPLE 7

Use a ruler and a pair of compasses to construct a line parallel to a given line.

THINK

1 Pick any two points, A and B, on the given line.

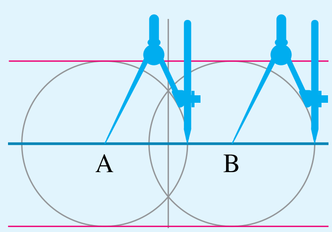
2 From point A, draw a circle of any radius more than half the distance from A to B.

3 With the same radius, repeat step 2 at point B.

4 Join the highest points (or lowest points) of the circles with a straight line. This line will be parallel to AB.

Note: The radius is the distance between the compass points, and this determines the distance between the parallel lines.

DRAW



Bisecting angles

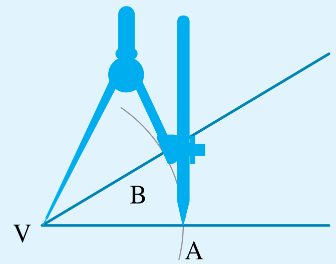
Another important construction is that employed to bisect angles.

WORKED EXAMPLE 8 Use a ruler and a pair of compasses to bisect any angle.

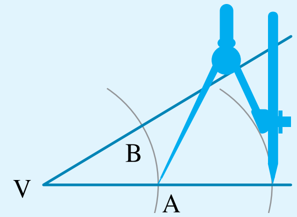
THINK

1 With any radius, and with the compass point at the vertex V, draw an arc of a circle that crosses both arms of the angle. The crossings are labelled A and B.

DRAW



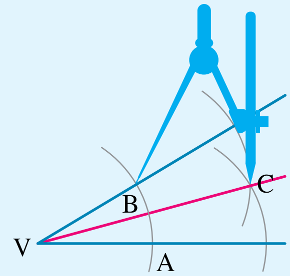
2 With any radius, and with the compass point at A, draw an arc inside the angle. This arc should be long enough so that the line representing half the angle would cross it.



3 With the same radius, repeat step 2, putting the point of the compass at B. The two arcs will cross at point C.

4 Join the vertex V to C. This line bisects the angle, namely:

$$\angle AVC = \frac{\angle AVB}{2}$$



When drawn carefully, this construction is an accurate way of halving an angle; it is even more accurate than with a protractor. For example, if the original angle was 68.3°, it would be difficult with a protractor to obtain an angle of 34.15°.

Constructing angles

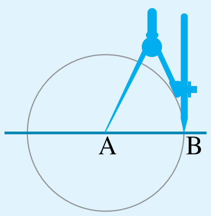
The last group of constructions involves the 'special angles' of 30°, 45°, 60° and 90°. You have already seen how to construct a 90° angle (see page 150); bisecting this will produce a 45° angle.

WORKED EXAMPLE 9 Use a ruler and compasses to construct a 60° angle.

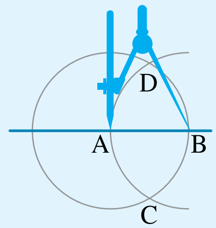
THINK

1 Draw a line as the base of the angle. Select a point for the vertex, A; put the compass point there and draw a circle of any radius, crossing the line at B.

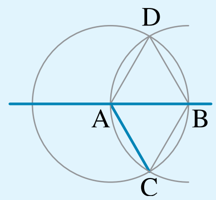
DRAW



2 With the same radius as in step 1, put the compass point at B and draw an arc, crossing the other circle at two places, C and D.



3 Join A to C. ∠CAB = 60°.



There are several other 60° angles in the construction above. Can you find them? (*Hint:* Use the symmetry of the construction.) A 30° angle can be constructed by bisecting a 60° angle.

EXERCISE 4.4 Geometric constructions

PRACTISE

1 WE6 Using a ruler and compasses, construct perpendicular lines AC and BD to the line AB.

2 Draw two line segments of different lengths and construct the perpendicular bisectors of each. Check your accuracy by measuring the two parts of each line segment.

3 WE7 Use a ruler and a pair of compasses to construct two parallel lines that are 3 cm apart.

4 Draw two parallel lines exactly 20 cm apart. (*Hint:* How can you do this if the compasses won't open wide enough?)

5 WE8 Draw an obtuse angle, and use a ruler and a pair of compasses to bisect it.

6 Draw a reflex angle, and use a ruler and a pair of compasses to bisect it.

7 WE9 Construct a 45° angle using a ruler and compasses only.

8 Construct a 30° angle using a ruler and compasses only.

CONSOLIDATE

9 Construct a right angle using a straight edge and a pair of compasses.

10 Draw a line AB. Use a straight edge and compasses to construct a line parallel to AB.

11 Use the following set of instructions to copy the angle ABC.

Step 1. With your compass point at B, trace an arc cutting AB and BC at D and E respectively.

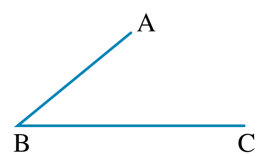
Step 2. Draw a line similar to line BC on another sheet of paper.

Step 3. With the same radius as in step 1, draw an arc on the new line.

Step 4. Use your compass to 'measure' the distance from D to E.

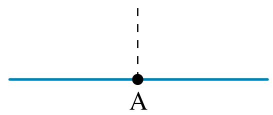
Step 5. Use this radius to draw an arc on the new line, putting the compass point where the first arc cut the line.

Step 6. Join the vertex of the new line with the point where the two arcs cross.

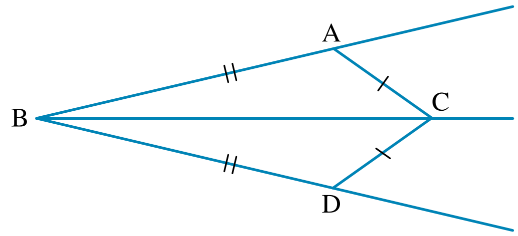


12 Draw an acute angle ABC and construct a copy of this angle.

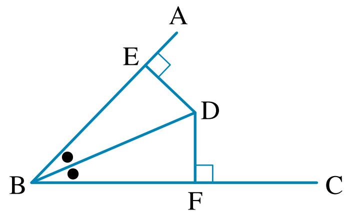
13 Devise a method of constructing a perpendicular through a point on the line.



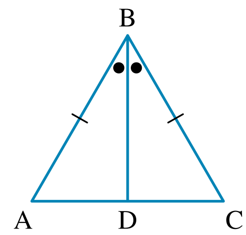
14 Prove that ∠ABC = ∠DBC.



15 Given an angle ABC, with BD the bisector of this angle, prove that the point D is the same distance from both arms of the angle.

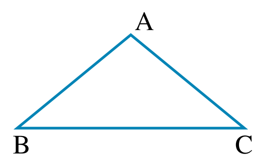


16 Given an isosceles triangle ΔABC, with BD the bisector of ∠ABC, prove that BD also bisects the base of the triangle, AC.



17 a Consider the isosceles triangle in the figure at right. Construct perpendicular bisectors of each of the three lines AB, BC, and CA.

b What do you notice about these bisectors?



18 a Repeat question 17 for any scalene triangle.

b What do you notice?

MASTER

19 A pair of circular pulleys of the same radius is connected by a closed band of rubber. The distance between the pulleys is equal to their diameter. Make a straight-edge-and-compass construction of this system.

20 Three circular pulleys of the same radius are arranged as follows: Pulley A is directly above Pulley B at a distance equal to twice the diameter. Pulley C is to the right of Pulley B at a distance equal to three times the diameter. The rubber band connecting them runs on the 'outside' of the system. Make a straight-edge-and-compass construction of this system.

4.5 Polygons

As we have seen, a polygon is a closed figure with straight sides. In this section, we look at the following aspects of polygons:

1. triangle constructions
2. quadrilaterals
3. regular polygons
4. star polygons.

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Triangle constructions

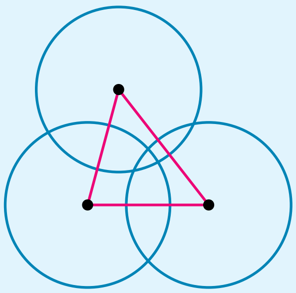
Let us investigate the properties of the perpendicular bisectors of each side of any triangle.

WORKED EXAMPLE 10

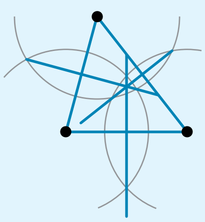
Construct the perpendicular bisectors and median bisectors of each side of any triangle and investigate their properties.

THINK | **DRAW**

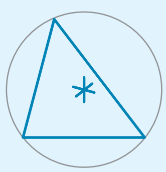
1 Draw any triangle and add circles centred at each vertex. The radius should be large enough so that perpendicular bisectors can be drawn.



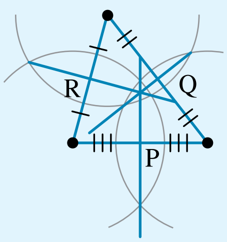
2 Use the construction circles to draw perpendicular bisectors. The added lines join pairs of intersecting arcs. It should be clear that the bisectors all meet at a point. This point is called the circumcentre.



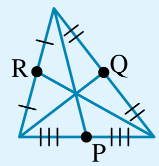
3 Use this point as a centre and draw a circle that just touches each vertex. To do this, put the compass point at the point where the bisectors meet, and put the pencil point at any vertex. You should observe that the resultant circumcircle (or outcircle) just touches each vertex.



4 Furthermore, from step 2, we can determine the midpoint of each side from the perpendicular bisectors. Note the use of short bars to indicate the bisection of the sides at points P, Q and R.



5 Join each midpoint to its opposite vertex. These lines also meet at a single point, called the centroid, which has applications in physics and engineering, as it is effectively the point of symmetry of the triangle.

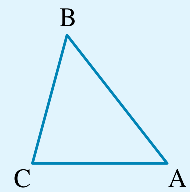


Imagine that the triangle in the previous worked example is made out of a thick piece of cardboard. The **centroid** is the point where you could place a finger and 'balance' the triangle. The reason for this is that the median bisectors in step 5 are the three **axes of symmetry** of the triangle, each axis dividing the triangle into two equal areas. Since they all meet, the centroid is the point of symmetry.

The **incentre** is the intersection of the angle bisectors of a triangle.

WORKED EXAMPLE 11

Construct the incentre of the triangle shown at right by hand.



THINK

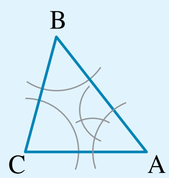
1 (a) Construct the angle bisectors by drawing arcs centred at each vertex (A, B and C).

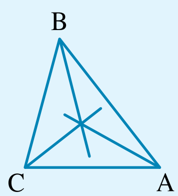
(b) From the intersection of these arcs and the sides of the triangles, draw intersecting arcs between pairs of sides. In the figure at right this has been done to vertex A only, to keep the drawing uncluttered.

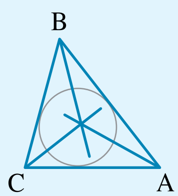
2 Complete the construction of the angle bisectors and observe that they meet at a point — the incentre.

3 By placing the compass point at the incentre and carefully drawing a circle, it is possible to construct the incircle, which just touches each side of the triangle.

DRAW







Quadrilaterals

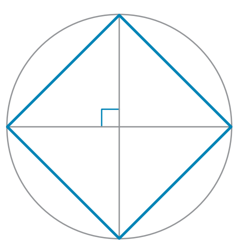
There are many names for the various kinds of quadrilaterals, and it is useful to look at their definitions in terms of their features.

Square

The square has:

1. four equal sides
2. four right angles.

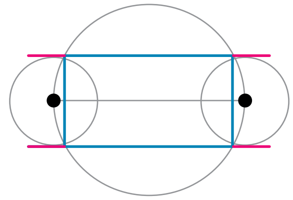
A square can be constructed easily from a circle by finding the perpendicular bisector of the diameter.



Rectangle

The rectangle has:

1. opposite sides equal
2. four right angles.

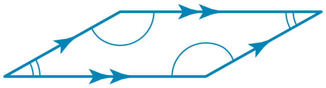


A rectangle can be constructed from a circle by using the construction shown to draw two lines parallel to a diameter. The rectangle is formed from the intersection of the parallel lines with the circle.

Parallelogram

A parallelogram has:

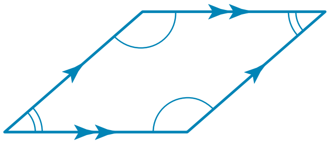
1. opposite sides equal in length
2. opposite sides parallel
3. opposite angles equal.



Rhombus

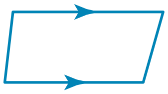
A rhombus has:

1. all four sides equal in length
2. opposite sides parallel
3. opposite angles equal.



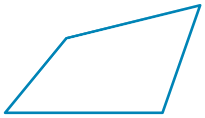
Trapezium

A **trapezium** (or trapezoid) has one pair of opposite sides parallel.



Other

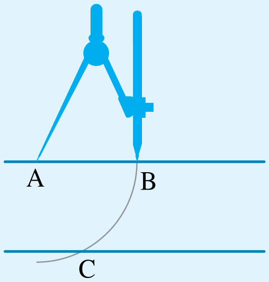
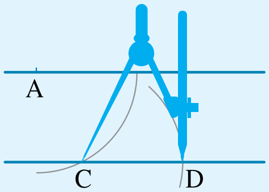
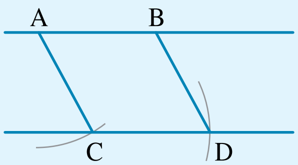
All other four-sided figures are generally just called **quadrilaterals**, even though the above figures are also quadrilaterals.



Obviously there is some overlap between these definitions; for example, a square is a kind of rectangle, just as a rhombus is a kind of parallelogram.

WORKED EXAMPLE 12

Construct a rhombus using compasses and a straight edge.

THINK	DRAW
1 (a) Construct a pair of parallel lines. (b) From any point on one line (A), draw an arc of radius equal to the length of the side of the rhombus. This arc cuts the first line at B and the second line at C.	
2 Using the same radius as in step 1, put the compass point at C and draw an arc cutting the same line at D.	
3 Join points A to C and B to D to form the rhombus ABDC.	

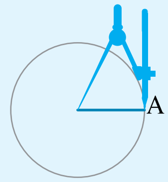
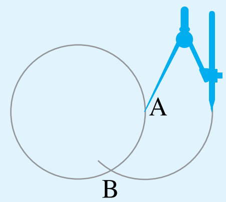
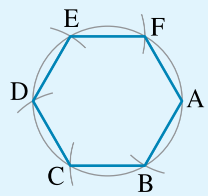
It is easy to extend this kind of construction to other types of quadrilaterals.

Regular polygons

A **regular polygon** is one with each side the same length and with each interior angle the same size. For triangles, the regular polygon is the equilateral triangle; for quadrilaterals, it is the square. Worked example 9 should give you an idea of how to construct an equilateral triangle; Worked example 12 should help you construct a square. The construction of a regular hexagon (6 sides) is particularly easy.

WORKED EXAMPLE 13

Construct a regular hexagon using a straight edge and compasses.

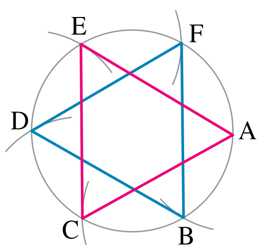
THINK	DRAW
1 Draw a circle whose radius is equal to the length of one side of the hexagon.	
2 Using the same radius as in step 1, and with the compass point anywhere on the circle (A), draw an arc that cuts the circle at point B.	
3 Repeat step 2 by: (a) putting the compass at B and cutting the circle at C (b) putting the compass at C and cutting the circle at D (c) putting the compass at D and cutting the circle at E (d) putting the compass at E and cutting the circle at F. 4 Join points A–B–C–D–E–F to form a hexagon.	

Several regular polygons, such as those with 7, 9, 11 and 13 sides, cannot be drawn using just a straight edge and ruler. By dividing up the circle's 360 degrees, it is possible to work out the angle 'between' each side. For example, the hexagon's 6 sides must be $\frac{360°}{6} = 60°$ apart. Because construction of a 60° angle is easy, the hexagon can be constructed. This does not occur with regular polygons such as the heptagon, where the angle would be $\frac{360°}{7} = 51.428\ldots°$. In cases like this, a protractor may be used.

Star polygons

The last step of Worked example 13 was to join the points, in order, to form the regular polygon. What would happen if you joined points by 'skipping' others?

Consider the figure at right. Point A has been joined to E and then to C and back to A again. A similar pattern has been used starting at point B. In other words, one point was 'skipped' for each line. In this case, the polygon formed (after removing the central lines) is the *only one* possible with 6 points: the 'Star of David'. These types of polygon are called star polygons because of the obvious resemblance.



Star polygons can be constructed with the aid of a protractor, as polygons that are 7-sided, 9-sided and so on have no exact construction method.

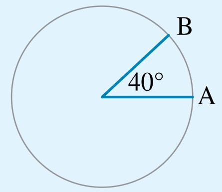
WORKED EXAMPLE 14

Construct all the star polygons from a regular nonagon (9 sides), using a straight edge, compasses and protractor.

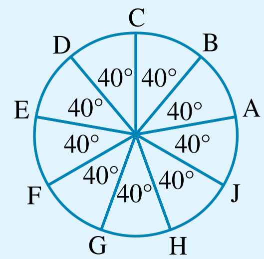
THINK | **DRAW**

1 (a) Draw a circle of any radius, and draw a line from the centre to the circumference (point A). Because there are to be 9 sides, the interior angle will be $\dfrac{360°}{9} = 40°$.

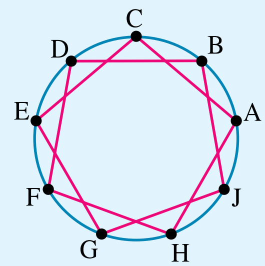
(b) Using a protractor, measure a 40° angle from A to cross the circle at B.



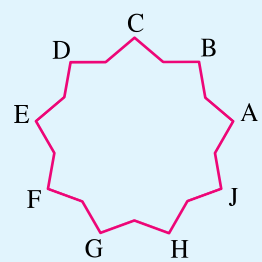
2 Measure a 40° angle from B to cross the circle at C. Repeat for the remaining 7 angles.



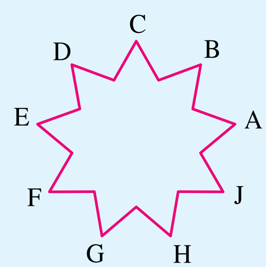
3 Instead of joining each point in order, try skipping one point, joining: A–C–E–G–J–B–D–F–H–A.



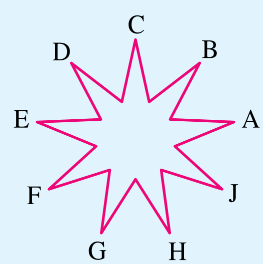
4 Remove the construction lines to form the star polygon.



5 Repeat steps 3 and 4, but skip two points, joining A–D–G–A and B–E–H–B and C–F–J–C.



6 Repeat steps 3 and 4, but skip three points, joining A–E–J–D–H–C–G–B–F–A.



7 Repeat steps 3 and 4, but skip four points, joining A–F–B–G–C–H–D–J–E–A. As this is the same figure as in step 6 with the order of the points reversed, there are no more different star polygons.

Note: The greater the number of points that are skipped, the sharper the points of the star polygon become. Furthermore, if we skip too many points (as in step 7) the same star polygon is produced; thus, there is a limited number of star polygons for a given regular polygon.

EXERCISE 4.5 Polygons

PRACTISE

1 WE10 Construct a scalene triangle with one angle greater than 90 degrees. Investigate the properties of perpendicular bisectors and side length bisectors.

2 Which of the basic constructions of the previous sections might be used to construct an isosceles triangle?

3 WE11 Construct the following triangle. Start with a base side of 6 cm. From the left-hand end point of the line draw a circle of radius 7 cm, and from the right-hand end point draw a circle of 5 cm. Join the two end points to the place where the two arcs meet, above the line, to form the triangle. Construct the incentre of this triangle.

4 Construct a scalene triangle and determine the incentre. What did you observe about the properties of the incentre?

5 WE12 Construct a parallelogram using a straight edge and compasses.

6 Construct a square of side length 5 cm using only a ruler and a pair of compasses.

7 WE13 Begin the construction of a regular dodecagon (12-sided polygon) using the following steps.

a Construct a regular hexagon.
b Join the vertices of the hexagon to the centre of the circle first used in the construction. What should the final three steps be?

8 Construct all the star polygons from the regular dodecagon drawn in question 7.

9 WE14 The stars on the Australian flag are 7-pointed (the points represent the 6 states and the territories). There is no compass-and-straight-edge construction, so use a protractor to help you construct the 7-pointed star polygon.

10 Construct all the different star polygons possible from an octagon. What is an easy compass-and-straight-edge construction of the octagon?

CONSOLIDATE

11 Construct a right-angled triangle (one with an angle of 90 degrees). Investigate the properties of the circumcentre and centroid.

12 Devise a method of constructing a parallelogram (with unequal sides) similar to that of Worked example 12.

13 State whether each of the following is true or false.

a An irregular quadrilateral cannot have a 90° angle.
b A rectangle is a quadrilateral because it has two pairs of parallel sides.
c A rhombus is a parallelogram because it has two pairs of parallel sides.
d Not every parallelogram has opposite sides equal in length.
e Not every square is a parallelogram.

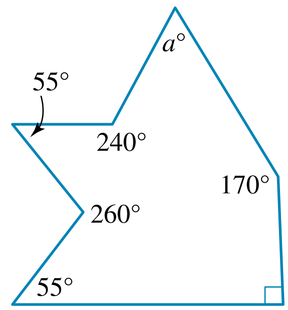
14 One angle in a rhombus is 40°. Find all other angles.

15 A quadrilateral has each angle 10° greater than the previous one. What size is the smallest angle?

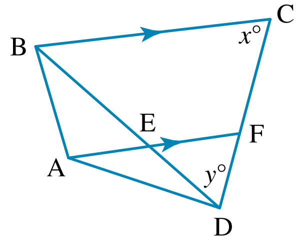
16 For the polygon shown, find:

a the sum of the interior angles

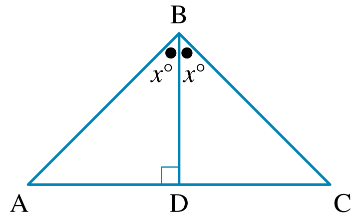
b the value of the pronumeral, a.



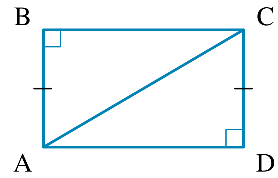
17 ABCD is a quadrilateral with AF parallel to BC. The diagonal BD intersects AF at E. Prove that $\angle AED = \angle BCD + \angle BDC$.



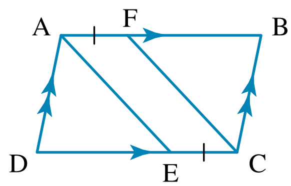
18 In ΔABC, the perpendicular line BD, from B to side AC, bisects ∠ABC. Prove that ΔABC is isosceles.



19 ABCD is a quadrilateral with right angles at B and D and diagonal AC, as shown. If AB = CD, prove that ABCD is a rectangle.



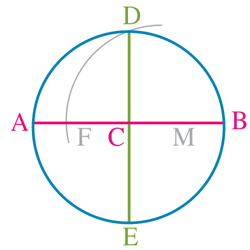
20 ABCD is a parallelogram and AF = EC. Prove that AE = CF.



MASTER

21 An ancient method of getting a right angle is to use the Pythagorean triple: a triangle with sides of 3, 4 and 5 units. Devise a method of constructing such a triangle with compasses and a straight edge only.

22 The construction of a regular pentagon is quite difficult. Use the following instructions to help in the compass-and-straight-edge construction. The various points defined in the steps are shown at right.



Step 1. Draw a circle of any radius, and mark the centre C and the diameter AB.

Step 2. Find the perpendicular bisector of AB meeting the circle at D and E.

Step 3. Find the midpoint (M) of CB, and draw an arc with radius MD cutting AB at F.

The side length of the pentagon is equal to the distance DF. Using this information, complete the construction of the pentagon. (*Hint:* This step is similar to the construction of a hexagon.)

4.6 Circle geometry

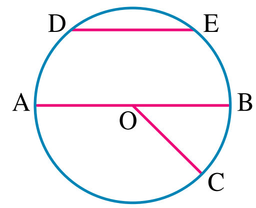
Until now, we have avoided the most ‘perfect’ form in geometry, the circle. There are several important properties and theorems of circles that form part of the classical study of geometry.

studyon

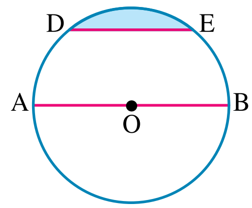
Units 1 & 2
AOS 4
Topic 1
Concept 7

Circle geometry
Concept summary
Practice questions

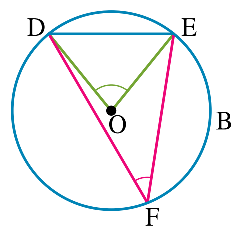
Review of circle definitions



A circle is formed by drawing the curved line that is at a given distance from a single point (O). This distance is called the **radius**. The line itself is called the **circumference**. A line from a point on the circumference (A), through the centre (O), to another point on the circumference (B) is called a **diameter**. A similar line (DE) that does not pass through the centre is called a **chord**. OC is a radius. Mathematically, it is easy to see that the length of AB is twice that of OC. In other words, diameter $= 2 \times$ radius.

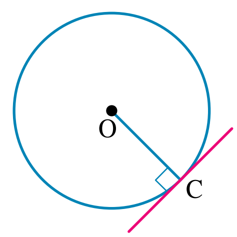


The length of any chord is less than that of the diameter (DE < AB). A chord divides the circle into two regions. In the figure at right, the shaded area is called the **minor segment**. The rest of the circle’s area is called the **major segment**. A minor segment’s area is always less than half the area of the circle. The part of the circumference joining points D and E is called an **arc**. The shorter arc (in the shaded region) is the **minor arc**; the longer one is the **major arc**.



A chord can define (or **subtend**) certain angles. The angle formed by joining the chord to the centre (∠DOE) is called the angle subtended by the chord at the centre. The angle formed by joining the chord to a third point on the circumference (∠DFE) is called the angle subtended by the chord at the circumference.

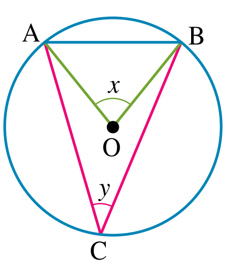
Obviously for a given chord (DE) there is only one angle subtended at the centre, but there are different angles subtended at the circumference as the point F is moved around the circle.



A **tangent** line is one that just touches the circumference at a single point (C) and is always **perpendicular** to the radius (OC). There is a close connection between this tangent and the one defined in trigonometry.

Circle theorems

Circle theorem 1



Consider the figure at right. A chord AB is drawn and a third point on the circumference (C) defines the angle ACB (marked angle y). From the same chord, lines are drawn to the centre (O), defining the angle x. It can be proven that $x = 2y$.

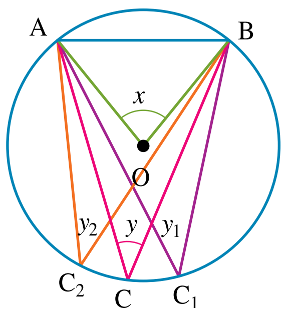
Circle theorem 1: The angle subtended by the chord at the centre is twice the value of the angle subtended by the same chord at the circumference.

Circle theorem 2

Now, imagine moving the point C in the figure anywhere along the circumference to, say, points C_1 or C_2. In each case 'new' angles (y_1, y_2) are subtended by the same chord.

But from circle theorem 1: $x = 2y_2$ and $x = 2y_1$, therefore $y_2 = y_1$.

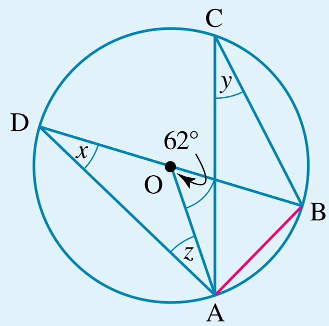
This is true as long as C_1 and C_2 stay on the same side of AB as C (that is, in the same segment as C).



Circle theorem 2: Angles subtended at the circumference, in the same segment, by the same chord are equal.

WORKED EXAMPLE 15

Find the values of the angles x, y and z.



THINK

1. The angles x and 62° are subtended by the same chord at the circumference and the centre respectively. The angle at the centre (62°) is twice the angle at the circumference (x).
2. The angles x and y are subtended by the same chord at the circumference, so they are equal.
3. Triangle ADO is isosceles, and angles on the base of an isosceles triangle are equal, so $x = z$.

WRITE

$\angle AOB = 2\angle ADB$

$62° = 2x$

$x = 31°$

$\angle ADB = 2\angle ACB$

$x = y$

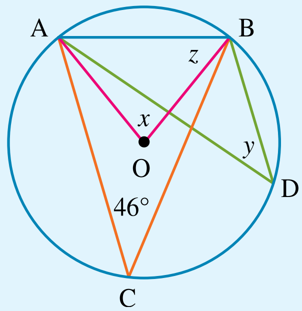
$y = 31°$

$x = z$

$z = 31°$

WORKED EXAMPLE 16

Find the values of the angles x, y and z.



THINK

1. Use circle theorem 1 to find x.

WRITE

$x = \angle AOB = 2\angle ACB$ (circle theorem 1)

$x = 2(46°)$

$= 92°$

2 Use circle theorem 2 to find y.

$y = \angle ADB = \angle ACB$ (circle theorem 2)
$y = 46°$

3 Use properties of isosceles triangle AOB to find z.

$x + z + \angle OAB = 180°$
$x + z + z = 180°$
$x + 2z = 180°$
$2z = 180° - 92°$
$z = 44°$

Circle theorem 3

Now, imagine moving the chord AB to the points A_1 and B_1 as shown in Figure 1. Notice that the values of both angles x and y increase.

What would happen if A_1 and B_1 were moved even further, so that the chord became a diameter? This is shown in Figure 2. Now, $\angle A_1OB_1$ is a straight angle (180°), and from circle theorem 1, $x = 2y$. Therefore, $\angle A_1CB_1 = 90°$. This special case of the theorem is circle theorem 3.

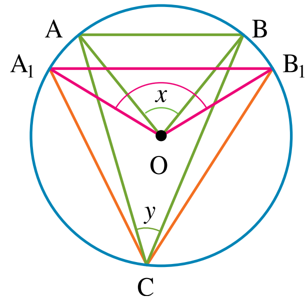


Figure 1

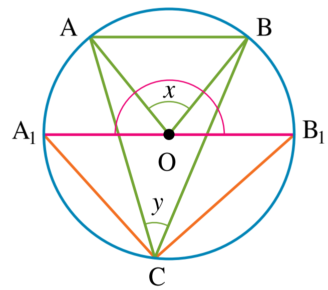
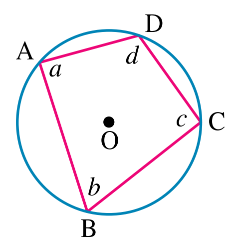


Figure 2

Circle theorem 3: The angle subtended by a diameter is a right angle (90°).

Circle theorem 4

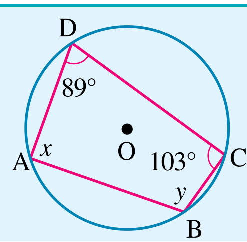
The figure at right is called a **cyclic quadrilateral**, because all four vertices touch the circumference and it encloses the centre. It can be proven that the sum of opposite angles is always 180°; that is, $a + c = 180°$ and $b + d = 180°$.



Circle theorem 4: Opposite angles in a cyclic quadrilateral add up to 180 degrees.

WORKED EXAMPLE 17

Find the values of x and y.



THINK | **WRITE**

1 ABCD is a cyclic quadrilateral, and opposite angles in such a shape add up to 180°.

2 Angles x and 103° are opposite angles.

$x + 103° = 180°$
$x = 77°$

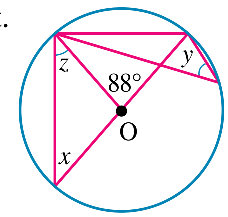
3 Angles y and 89° are opposite angles.

$y + 89° = 180°$
$y = 91°$

EXERCISE 4.6 Circle geometry

PRACTISE

1 **WE15** Find the values of the angles x, y and z in the figure at right.



2 The angle subtended by a chord at the centre is ____________ the angle subtended by the same chord at the ____________.

The above sentence can be completed using the words:

A half, circumference **B** twice, diameter
C twice, circumference **D** equal, circumference
E twice, edge

3 **WE16** Find the values of the angles x, y and z in Figure 1 below.

4 Using the result from circle theorem 2, prove the additional result that $x = z$ in Figure 2 below.

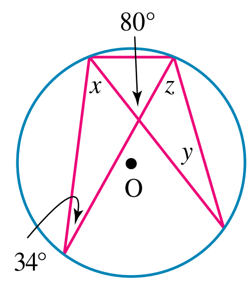


Figure 1

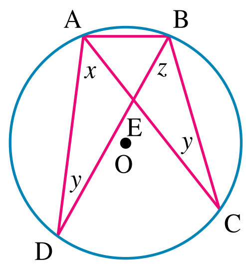


Figure 2

5 **WE17** Find the values of x and y in Figure 3 below.

6 In Figure 4 below, find the value of angle y in terms of angle x. (*Hint:* Use circle theorems 2 and 4 combined.)

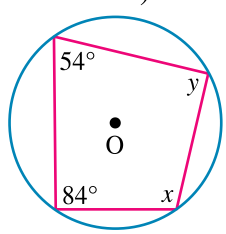


Figure 3

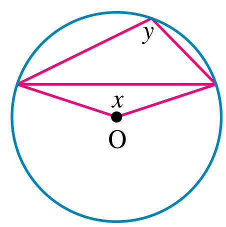


Figure 4

CONSOLIDATE

7 Find the value of the angle x in Figure 5 below.

8 In question 7, if angle $\angle OAC = 20°$, then $\angle OBC$ equals:

A 37° **B** 57° **C** 59° **D** 94° **E** 131°

9 In Figure 6 below, the angle (or angles) that is (are) half of $\angle AOC$ is (are):

A $\angle ABC$ only **B** $\angle ADC$ only **C** $\angle ABC$ and $\angle ADC$
D $\angle ADC$ and $\angle ACD$ **E** $\angle ACD$ only

10 The angle subtended at the circumference by a diameter is ________ degrees.

11 Find the values of x and y in Figure 7 below.

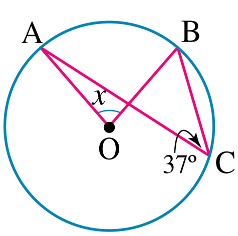


Figure 5

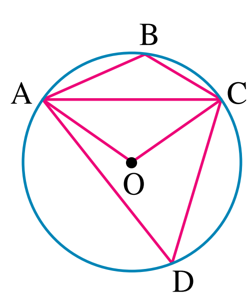


Figure 6

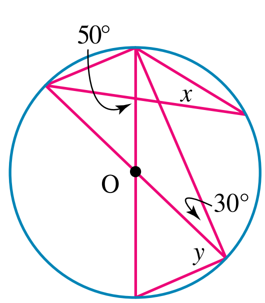
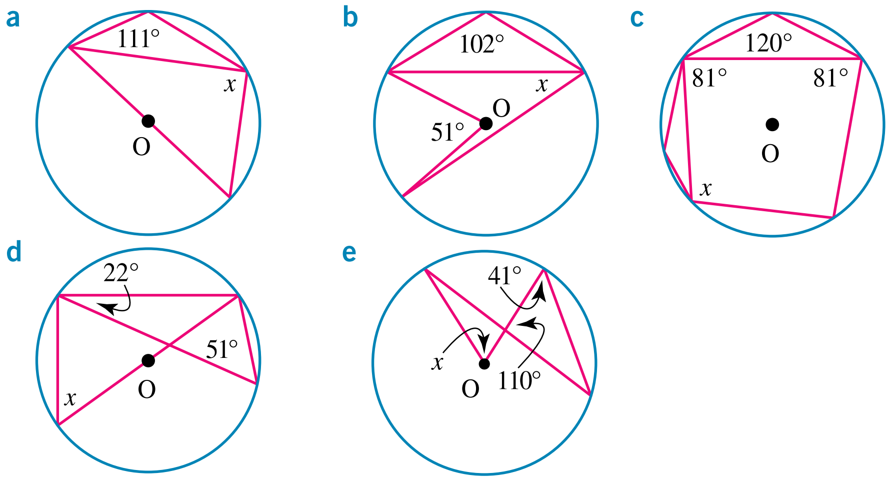


Figure 7

12 In the following five figures, identify the circle theorem that helps you find the value of the angle labelled x. Then find the actual value of x.



13 Find the values of the angles x, y and z in Figure 8 below.

14 Prove the general case of the result of question **13**, namely that: $x = y = z$.

15 In Figure 9 below, chords AB and CD are parallel. Find the values of the angles x, y and z.

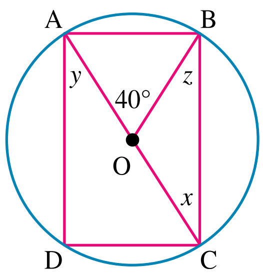


Figure 8

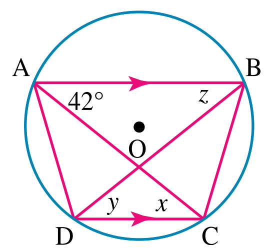
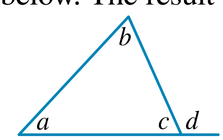


Figure 9

16 Prove the general case of the result of question **15**, namely that: $x = y = z$.

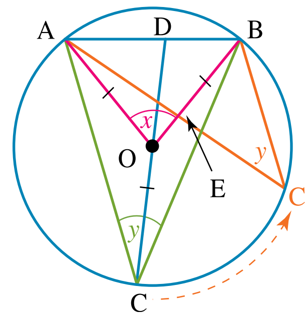
MASTER

17 The proof of circle theorem 1 relies on a result about the external angle of any triangle, as shown in the figure below. The result is that $d = a + b$. Prove this result.



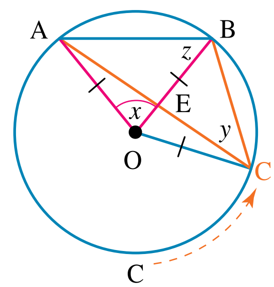
18 The proof of circle theorem 1 also relies on a particular construction whereby a line joining the vertices of the two angles in question are joined and extended towards the circumference.

This is the line COD in the figure at right. Imagine the point C being moved so that it was much closer to B as in the figure. Clearly, it is not possible to form the line COD, so how can we prove the theorem in this case?



Step 1. Draw a line from C to O (see the figure at right).

Step 2. Using the result about equal angles in isosceles triangles, find some equal angles. There are three isosceles triangles in the diagram.



To get you started, $\angle EBC = y + \angle ECO$. (Why?)

Complete the proof, namely that $x = 2y$.

4.7 Tangents, chords and circles

There are many geometric problems that use the four circle theorems to help in their solution. To begin, we need to recall the definition of a tangent to a circle as a line that just touches the circumference and forms a right angle to the radius at the point of contact.

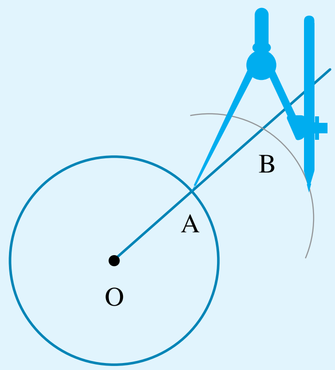
WORKED EXAMPLE 18

Construct, with a straight edge and compasses, a tangent to a circle at any point by hand. (This task is relatively easy, relying on the earlier construction of a perpendicular bisector.)

THINK | DRAW

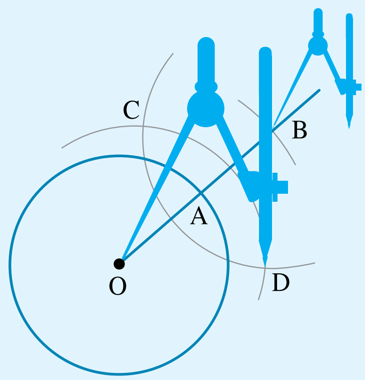
1 (a) Given the circle centred at O, and any point A on the circumference, extend the radius outward.

(b) Using the same radius as the circle (OA), draw an arc crossing the extended line at B. The result is that A is the midpoint of OB.

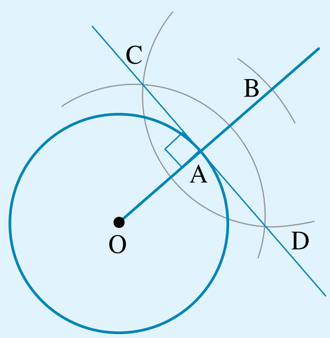


2 Construct the perpendicular bisector of the line OB.

You will need a larger radius for the compasses than in step 1. Draw arcs above and below the line OB by first placing the compass point at B, then at O. The arcs should cross at C and D.

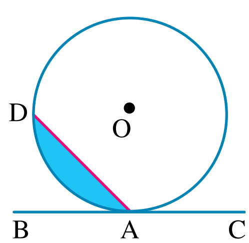


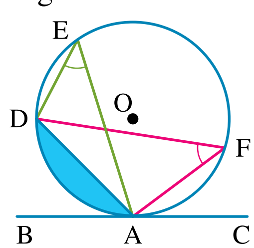
3 Join the points C and D. Because the line CD is perpendicular to OB, it is also perpendicular to OA. Because it just touches the circle at A, it must also be the tangent.



The alternate segment theorem

Consider the figures shown below. Line BC is a tangent to the circle at the point A.





studyon

Units 1 & 2
AOS 4
Topic 1
Concept 8

Tangents, chords and circles
Concept summary
Practice questions

A line is drawn from A to anywhere on the circle, point D. The angle ∠BAD defines a segment (the shaded area).

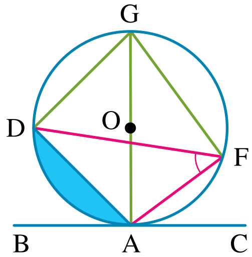
The unshaded part of the circle is called the alternate segment to ∠BAD.

Now consider angles subtended by the chord AD in the alternate segment, such as the angles marked in red and green. The alternate segment theorem states that these are equal to the angle that made the segment, namely:

$$\angle BAD = \angle AED \text{ and } \angle BAD = \angle AFD$$

Proof of the alternate segment theorem

We are required to prove that ∠BAD = ∠AFD.



1. Construct the diameter from A through O, meeting the circle at G. This can be done simply with a straight edge, as both points A and O are known. Join G to the points D and F.
2. Use circle theorem 3 to find some right angles. This refers to the property that angles subtended at the circumference by a diameter are right angles.

$$\angle BAG = \angle CAG = 90° \text{(property of tangents)}$$
$$\angle GFA = 90° \text{(circle theorem 3)}$$
$$\angle GDA = 90° \text{(circle theorem 3)}$$

3. Consider triangle GDA. We know that ∠GDA = 90°. Solve.

$$\angle GDA + \angle DAG + \angle AGD = 180°$$
$$90° + \angle DAG + \angle AGD = 180°$$
$$\angle DAG + \angle AGD = 90°$$

4. (a) ∠BAG is also a right angle.
   $\angle BAG = \angle BAD + \angle DAG = 90°$
   (b) Equate the two results.
   $\angle DAG + \angle AGD = \angle BAD + \angle DAG$
   (c) Cancel the equal angles on both sides.
   $\angle AGD = \angle BAD$
5. Now consider the fact that both triangles DAG and DAF are subtended from the same chord (DA).
   $\angle AGD = \angle AFD$ (circle theorem 2)
   Equate the two equations.
   $\angle AFD = \angle BAD$ QED

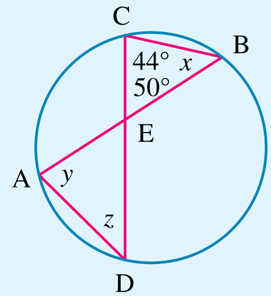
Chords and tangents in circles

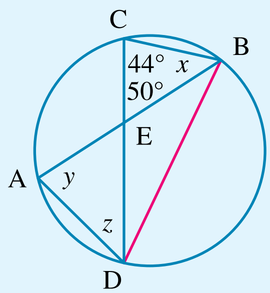
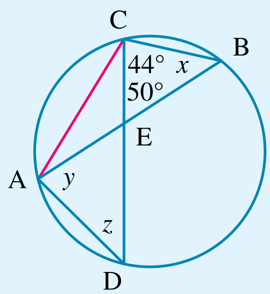
We shall now consider three situations: where chords intersect inside a circle, where they meet outside a circle and where one of the chords is a tangent. First, let us consider the case of two chords that meet inside a circle.

Chords meeting inside a circle

WORKED EXAMPLE 19

Two chords, AB and CD, meet at a 50° angle at point E. Find the values of the angles x, y and z.



THINK	WRITE/DRAW
1 Find the value of x using the sum of angles in a triangle.	$x + 50° + 44° = 180°$ $x = 86°$
2 a Construct a line joining B to D. b Now the angle y and the 44° angle (∠DCB) are subtended at the circumference by the same chord (BD).	
c Apply circle theorem 2.	$y = 44°$ (circle theorem 2)
3 a Now, construct a line joining C to A. b This time the angles x and z are subtended by the same chord (AC).	
c Apply circle theorem 2.	$z = x = 86°$ (circle theorem 2)

As a consequence of this example, the two triangles AED and CEB have all three angles equal and are therefore similar triangles. This can be stated as a theorem:

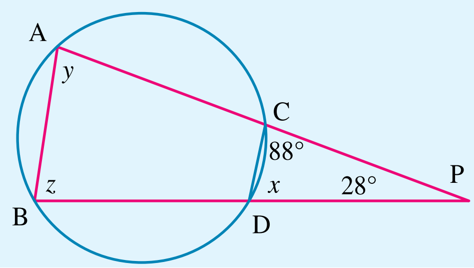
Two chords that meet inside a circle form two similar triangles.

Chords meeting outside a circle

Let us now consider the case where the two chords meet outside the circle.

WORKED EXAMPLE 20

Two chords, PA and PB, meeting at P, intersect the circumference of a circle at C and D. Find the values of the angles x, y and z.



THINK	WRITE
1 Use the sum of angles in a triangle to find x.	$x + 88° + 28° = 180°$ (sum of angles in a triangle) $x = 180° - 88° - 28°$ $= 64°$

2 (a) Find z. Consider the straight angle ∠ACP.

$\angle DCA + 88° = 180°$ (straight angle = 180°)

$\angle DCA = 92°$

(b) Consider the cyclic quadrilateral ABDC. Use circle theorem 4 to find z.

(*Note:* z will always equal ∠DCP.)

$\angle DCA + z = 180°$ (circle theorem 4)

$92° + z = 180°$

$z = 88°$

3 (a) Consider the straight angle ∠BDP.

$\angle BDC + x = 180°$ (straight angle = 180°)

(b) Use circle theorem 4 to find y.

(*Note:* y will always equal x (∠CDP).)

$\angle BDC + 64° = 180°$ (from step 1)

$\angle BDC = 116°$

$y + \angle BDC = 180°$ (circle theorem 4)

$y + 116° = 180°$

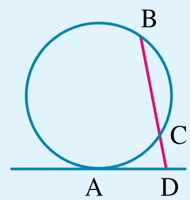
$y = 64°$

A chord meeting a tangent

Finally, consider what happens when one of the chords becomes a tangent.

WORKED EXAMPLE 21

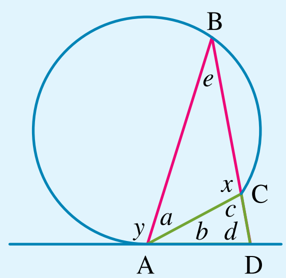
A chord (BD) meets a tangent (AD) at point D. The chord crosses the circumference at C. Construct a pair of similar triangles based on the diagram shown.



THINK | **DRAW/WRITE**

1 Join point A to B and A to C and label angles a, b, c, d, e, x and y.

To find similar triangles, attempt to find equal angles.



2 Because AD is a tangent, and AB and AC are chords, the alternate segment theorem can apply.

$y = x$ (alternate segment theorem on AB)

$b = e$ (alternate segment theorem on AC)

3 There are two straight angles along lines BD and AD.

$x + c = 180°$ (straight angle along AD)

$y + a + b = 180°$ (straight angle along BD)

$x + c = y + a + b$ (equate two straight angles)

$c = a + b$ (result from step 2, $x = y$)

4 Consider the angles in the triangle ACD and the triangle BAD.

ACD angles: b, c, d

BAD angles: $e, (a + b), d$

5 Use the results from steps 2 and 3 ($b = e$ and $c = a + b$).

$= b, c, d$

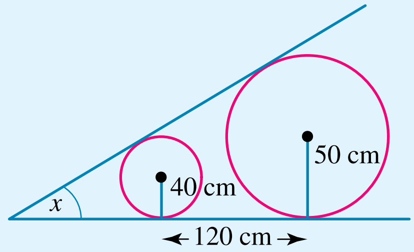
6 Because three angles are equal, the triangles are similar.

Hence, triangle ACD and triangle BAD are similar.

So, no matter what the geometry of the two chords (even where one is a tangent), a pair of similar triangles is always formed.

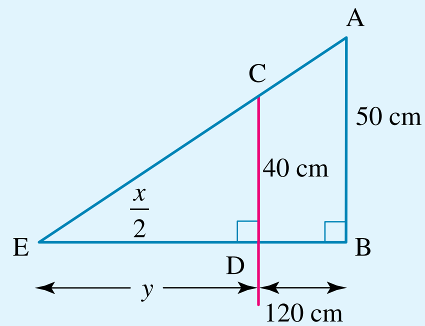
WORKED EXAMPLE 22

A plank rests on two cylindrical rollers, as shown in the diagram. The radii of the rollers are 40 cm and 50 cm and their distance apart on the ground is 120 cm. Find the value of the angle x.



THINK | **DRAW/WRITE**

1 Redraw the diagram as a similar triangle problem. Label the triangle and transfer all the relevant information onto the diagram.



2 Write a similarity statement.

Note: With any similar figures, the corresponding angles are equal and their corresponding sides are in equal ratio.

ΔCED is similar to ΔAEB.

$\Delta CED \sim \Delta AEB$

3 To find missing side lengths, establish the scale factor by calculating the ratio of two corresponding sides whose lengths are given.

$$\text{Scale factor} = \frac{AB}{CD} = \frac{50}{40} = 1.25$$

4 Use the scale factor to find the length of side ED.

(a) Write the relationship between EB and ED.

$$EB = 1.25\,ED$$

(b) Substitute the known values into the equation.

$$y + 120 = 1.25y$$
$$y - y + 120 = 1.25y - y$$
$$120 = 0.25y$$

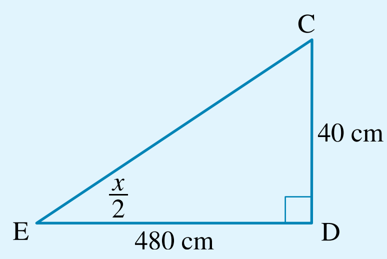
(c) Transpose the equation to make y the subject.

$$0.25y = 120$$

(d) Divide both sides by 0.25.

$$y = 480$$

5 Use triangle CED to find the magnitude of the angle required.



(a) Identify the appropriate ratio to use.
Note: The opposite and adjacent sides are known; therefore, choose the tangent ratio.

$$\tan(\theta) = \frac{O}{A}$$

(b) Substitute the known values into the ratio and evaluate.

$$\tan\left(\frac{x}{2}\right) = \frac{40}{480}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1}{12}$$

(c) Transpose the equation to make the angle $\frac{x}{2}$ the subject, using the inverse tangent function.

$$\frac{x}{2} = \tan^{-1}\left(\frac{1}{12}\right)$$

$$\frac{x}{2} = 4.7630°$$

(d) Evaluate for x and round the answer correct to 2 decimal places.

$$x = 2 \times 4.7636°$$
$$= 9.53°$$

6 Answer the question.

The required angle is 9.53°.

EXERCISE 4.7 Tangents, chords and circles

PRACTISE

1 WE18 Draw a circle of any radius. Construct a radius that points to the right (that is, due east). Construct a tangent to the circle at the point where this radius touches the circle.

2 Draw a circle of radius 3 cm and construct a tangent to the circle at any point, using a straight edge and compasses.

3 WE19 In Figure 1 below, find the values of the angles x, y and z.

4 Find the values of the angles x and y in Figure 2 below. (*Hint:* Use the alternate segment theorem.)

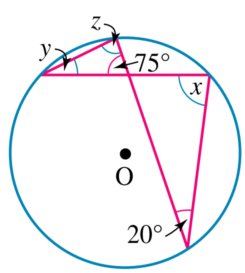


Figure 1

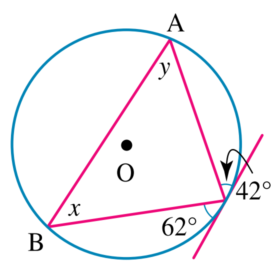


Figure 2

5 WE20 **a** Identify a pair of similar triangles in Figure 3 below.
b Find the values of the angles x and y.
c If AC = 10 cm, BC = 4 cm and EC = 11 cm, find ED.

6 Find the lengths of the sides labelled x and y, as shown in Figure 4 below.

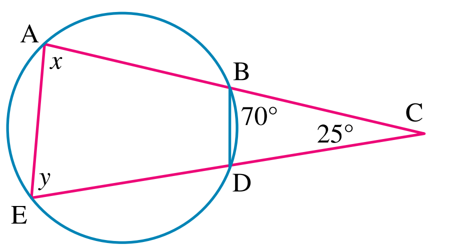


Figure 3

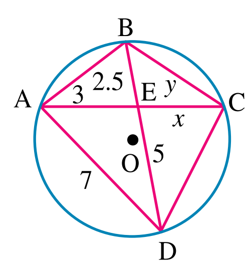


Figure 4

7 WE21 In Figure 5, chord SQ meets a tangent PQ at point Q. Construct a pair of similar triangles based on the diagram shown.

8 Find two pairs of similar triangles in Figure 6.

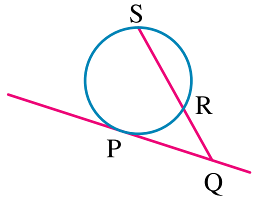


Figure 5

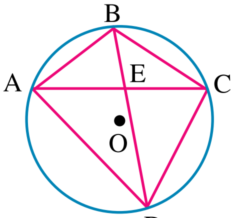
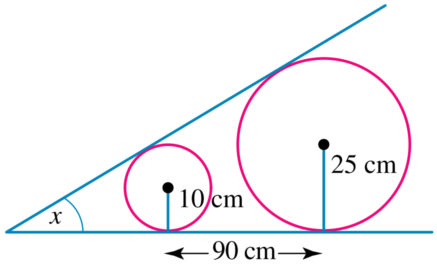
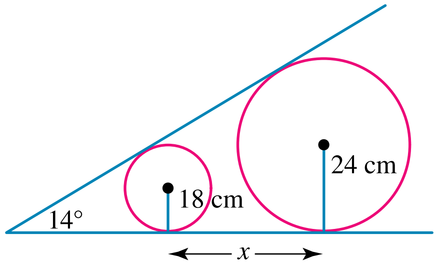


Figure 6

9 WE22 A plank rests on two cylindrical rollers, as shown. The radii of the rollers are 10 cm and 25 cm and their distance apart on the ground is 90 cm. Find the value of the angle x.



10 A plank rests on two cylindrical rollers, as shown. The radii of the rollers are 18 cm and 24 cm. The plank makes an angle of 14° with the horizontal ground. Find the distance, x cm, between the rollers.



CONSOLIDATE

11 Line AB is a tangent to the circle, as shown in Figure 7 below. Find the values of the angles labelled x and y.

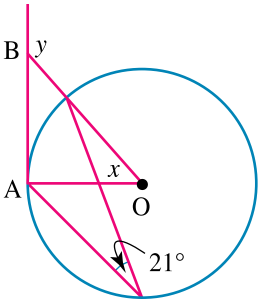


Figure 7

Questions 12, 13 and 14 refer to Figure 8 below. The line MN is a tangent to the circle and EA is a straight line. The circles have the same radii.

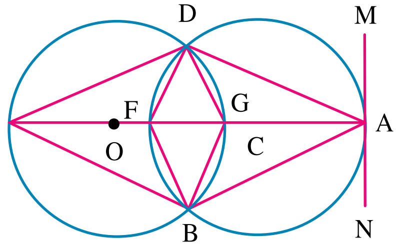


Figure 8

12 Find 6 different right angles.

13 If ∠DAC = 20°, then ∠CFD and ∠FDG are respectively:

A 70° and 50° **B** 70° and 40° **C** 40° and 70°
D 70° and 70° **E** 50° and 50°

14 A triangle similar to FDA is:

A FDG **B** FGB **C** EDA
D GDE **E** none of the above

15 In Figure 9 below, express x in terms of a and b.

16 Two tangent lines to a circle meet at an angle y, as shown in Figure 10 below. Find the values of the angles x, y and z.

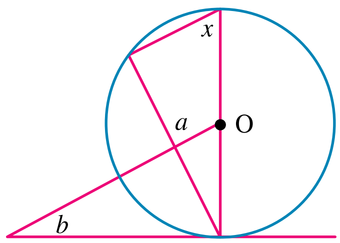


Figure 9

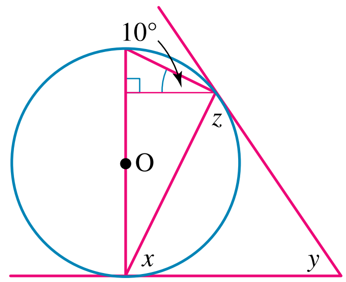


Figure 10

17 Solve question **16** in the general case (see Figure 11) and show that $y = 2a$. This result is important for space navigation (imagine the circle to be the Earth) in that an object at y can be seen by people at x and z at the same time.

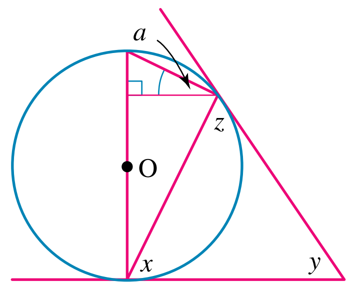


Figure 11

Questions **18**, **19** and **20** refer to Figure 12 at right. The line BA is a tangent to the circle at point B. Chord CD is extended to meet the tangent at A.

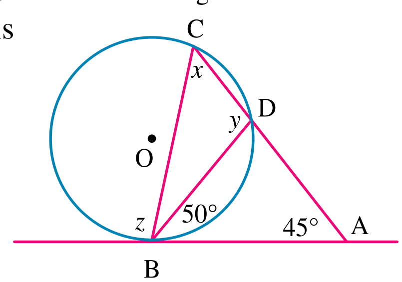


Figure 12

18 Find the values of the angles x and y.

19 The triangle that is similar to triangle BAD is:

A COD **B** CAB **C** BCD
D BDC **E** AOB

20 The value of the angle z is:

A 50° **B** 85° **C** 95°
D 100° **E** 130°

MASTER

21 Find the values of the angles x, y and z in Figure 13 below. The line AB is tangent to the circle at B. The line CD is a diameter.

22 Solve question **21** in the general case; that is, express angles x, y and z in terms of a (see Figure 14 below).

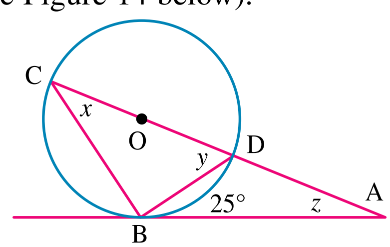


Figure 13

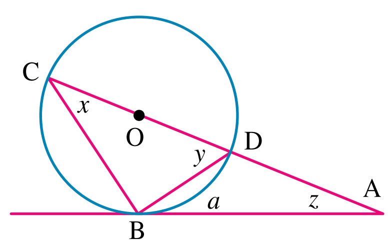


Figure 14

eBook*plus* ONLINE ONLY

4.8 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

studyon

Units 1 & 2 — **Geometry in the plane**

Sit topic test

4 Answers

EXERCISE 4.2

1 a 56°

b 67.5°

c $x = 38°$, $y = 32°$

2 $x = 145°$

3 Check with your teacher.

4 Check with your teacher.

5 D

6 C

7 Check with your teacher.

8 Check with your teacher.

9 a $x = 60°$, $y = 114°$

b $x = 47°$, $y = 55°$, $z = 78°$

c $x = 59°$, $y = 121°$, $z = 59°$

10 a $x = 75°$, $y = 30°$, $z = 150°$

b $x = 45°$, $y = 42°$

c 60°

11 Check with your teacher.

12 122°

13 $x = 130°$, $y = 110°$

14 $x = 25°$

15 AB = BC, therefore ABC is isosceles.

∠BAC = ∠BCA = 45° (property of isosceles triangles).

In triangle OAB, ∠OAB + ∠ABO + ∠BOA = 180° (sum of angles in a triangle).

∠OAB = ∠ABO = 45°, so ∠BOA = 90°.

16 a

Number of sides	Sum of interior angles
3	180°
4	360°
5	540°
6	720°
7	900°
8	1080°
9	1260°
10	1440°
20	3240°

b General formula: $180°(n - 2)$

EXERCISE 4.3

1 a Similar

b Congruent

2 a Similar

b Not enough information (unless cosine rule is known)

3 Check with your teacher.

4 Check with your teacher.

5 a $x = 7.4$ cm, $y = 48$ cm

b $x = 3.1$ cm, $y = 3.1$ cm

c $x = 12.5$ cm, $y = 19.5$ cm

d $x = 7.5$ cm, $y = 10$ cm

e $x = 4$

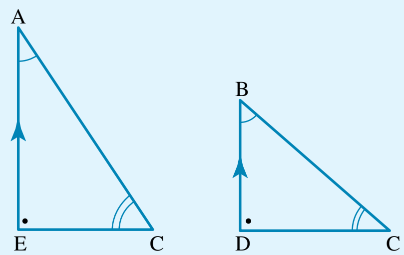
6 a $x = 75°$, $y = 75°$, $z = 4$ cm

b $x = 13.8$, $y = 67°23'$, $z = 39$

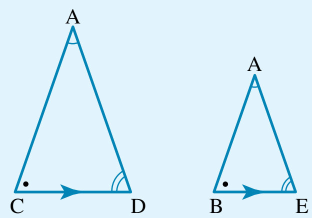
c $x = 15$, $y = 12$

d $x = 24$, $y = 11.3$

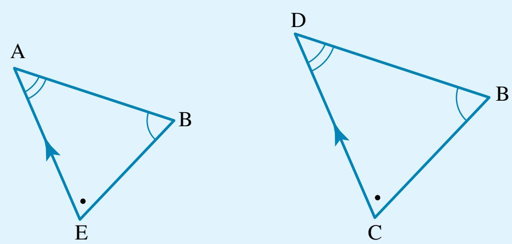
7 a



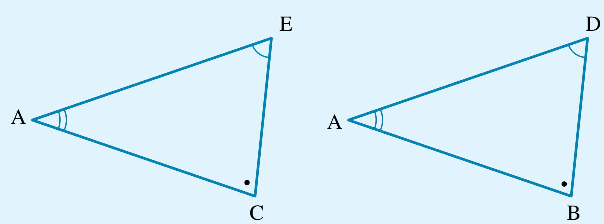
b



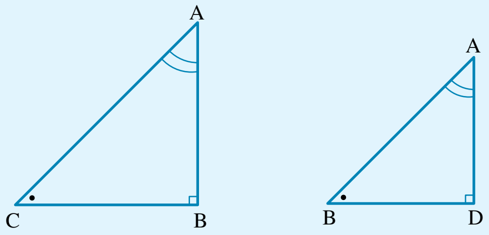
c



d



e



8 A

9 B

10 D

11 B

12 D

13 a $a = b = c = d = 78°$

b $a = c = 112°,\ b = 68°$

c $a = b = d = e = 74°,\ c = 32°$

d $a = b = e = 102°,\ c = d = 78°$

e $x = 63°,\ y = 36°$

f $a = e = 122°,\ b = 32°,\ c = d = 26°$

14 21.5 metres

15 17 cm

16 a 2.55 metres

b 4.5 metres

c 18 metres. Need a longer ladder.

d 2.4 metres. Need to find an alternative route.

17 a $x = 5.5,\ y = 3.75,\ z = 2.5$

b 34.38, 23.44, 15.63

18 2.9 metres

EXERCISE 4.4

1–12 Constructions. Check with your teacher.

13 Draw an arc with A as the centre, crossing the line at B and C.

With B as the centre, draw an arc above the line.

Repeat with the same radius at C. Join the points where the arcs meet to point A.

14–16 Constructions. Check with your teacher.

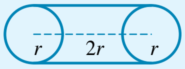
17 a Constructions. Check with your teacher.

b The bisectors meet at the same point.

18 a Constructions. Check with your teacher.

b The bisectors still meet at a point.

19



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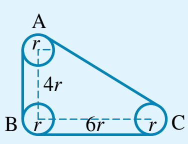
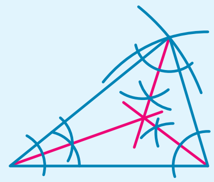


EXERCISE 4.5

1 Perpendicular bisectors meet outside the triangle.

2 Construct the perpendicular bisector of the base; vary the radius to get different triangles.

3



4 Check with your teacher.

5 Check with your teacher.

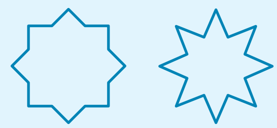
6 Check with your teacher.

7 Find the angle bisectors of the angles formed at the centre of the circle. Join them to the circumference to create 12 vertices. Join the 12 vertices to create the dodecagon.

8 Check with your teacher.

9 Constructions. Check with your teacher.

10 Create perpendicular bisectors, then angle bisectors to create 45° angles from the centre of circle, and a diameter.



11 Perpendicular bisectors meet on the hypotenuse.

12 Answers will vary.

13 a F

b F

c T

d F

e F

14 40°, 140°, 140°

15 75°

16 a 900° b 30°

17–20 Check with your teacher.

21 Answers will vary.

22 Constructions. Check with your teacher.

EXERCISE 4.6

1 $x = 44°,\ y = 44°,\ z = 44°$

2 C

3 $x = 46°,\ y = 34°,\ z = 46°$

4 $\angle AED = \angle BEC$ (opposite angles)
$\angle ADE = \angle BCE$ (circle theorem 2)

Thus, similar triangles, so $x = z$.

5 $x = 126°,\ y = 96°$

6 $y = 180° - \dfrac{x}{2}$

7 74°

8 B

9 B

10 90

11 $x = 30°$, $y = 60°$

12 a Theorem 3, 90°

b Theorem 1, 25.5°

c Theorem 4, 99°

d Theorem 2, 51°

e Theorem 1, 58°

13 $x = y = z = 20°$

14 Answers will vary.

15 $x = y = z = 42°$

16 Answers will vary.

17 $a + b + c = 180°$ — sum of angles in a triangle

$c + d = 180°$— straight angle

$a + b + c = c + d$ — equating the two equations

$a + b = d$ QED

18 y + ECO = EBC — isosceles triangle

BEC + EBC + y = 180° — sum of angles in triangle EBC

BEC + y + ECO + y = 180° — substitution

2y + ECO = 180°− BEC — rearranging

x + OAE + AEO = 180° — sum of angles in triangle OAE

AEO = BEC — opposite angles

x + OAE = 180° − BEC — rearranging

2y + ECO = x + OAE — equating two equations involving 180° − BEC

ECO = OAE — isosceles triangle

2y = x QED

EXERCISE 4.7

1 Construction

2 Check with your teacher.

3 $x = 85°$, $y = 20°$, $z = 85°$

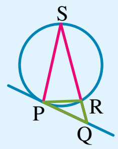
4 $x = 42°$, $y = 62°$

5 a CBD, CEA

b 85°, 70°

c 7.36 cm

6 4.17, 5.83

7 

ΔPQR and ΔPQS

8 ABE, DCE and ADE, BCE

9 18.92°

10 48.87 cm

11 $x = 42°$, $y = 132°$

12 MAC, NAC, FDA, FBA, EDG, EBG

13 B

14 D

15 $x = 180° - a - b$

16 $x = 80°$, $y = 20°$, $z = 80°$

17 $x = 90° - a$, $y = 2a$, $z = 90° - a$

18 $x = 50°$, $y = 95$

19 B

20 C

21 $x = 25°$, $y = 65°$, $z = 40°$

22 $x = a$, $y = 90° - a$, $z = 90° - 2a$

5

Trigonometric ratios and their applications

5.1 Kick off with CAS

Defining rules

Using CAS technology it is possible to define commonly used mathematical rules and save them so that they can be used at any stage to solve a problem.

1 Use CAS technology to define the rule for Pythagoras' theorem as $c = \sqrt{a^2 + b^2}$ or $b = \sqrt{c^2 - a^2}$ and hence answer the following questions.
   a If $a = 3$ and $b = 5$, determine the value of c.
   b If $a = 9.1$ and $c = 24.3$, determine the value of b.
   c If $b = 78.86$ and $c = 155.32$, determine the value of a.

In this topic, you will study the sine rule and the cosine rule, which can be used to calculate angles and side lengths in non–right-angled triangles.

Sine rule

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

2 After studying the sine rule, use CAS technology to rewrite and define it as: $a = \dfrac{b \times \sin(A)}{\sin(B)}$ or $A = \sin^{-1}\left(\dfrac{a \times \sin(B)}{b}\right)$ and answer the following questions regarding ΔABC.
   a Calculate a if $b = 15$ m, $A = 48°$ and $B = 74°$.
   b Calculate a if $b = 7$ m, $A = 32°$ and $B = 86°$.
   c Calculate A if $a = 12.7$ m, $b = 16.3$ m and $B = 45°$.

3 After studying the cosine rule, use CAS technology to rewrite and define it as: $a = \sqrt{b^2 + c^2 - 2bc\cos(A)}$ or $A = \cos^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)$ and answer the following questions regarding ΔABC.
   a Calculate a if $b = 10$ m, $c = 8$ m and $A = 30°$.
   b Calculate a if $b = 260$ cm, $c = 120$ cm and $A = 115°$.
   c Calculate A if $a = 20$ cm, $b = 12$ cm and $c = 13$ cm.
   d Calculate A if $a = 2$ cm, $b = 3.5$ cm and $c = 2.5$ cm.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

5.2 Trigonometry of right-angled triangles

Trigonometry, derived from the Greek words *trigon* (triangle) and *metron* (measurement), is the branch of mathematics that deals with the relationship between the sides and angles of a triangle. It involves finding unknown angles, side lengths and areas of triangles. The principles of trigonometry are used in many practical situations such as building, surveying, navigation and engineering. In previous years you will have studied the trigonometry of right-angled triangles. We will review this material before considering non–right-angled triangles.

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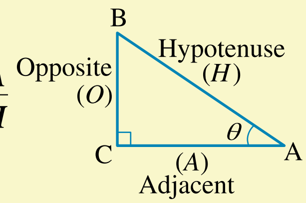
Units 1 & 2
AOS 4
Topic 3
Concept 1

Trigonometry of right-angled triangles
Concept summary
Practice questions

$\sin(\theta) = \dfrac{\text{opposite side}}{\text{hypotenuse}}$, **which is abbreviated to** $\sin(\theta) = \dfrac{O}{H}$

$\cos(\theta) = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$, **which is abbreviated to** $\cos(\theta) = \dfrac{A}{H}$

$\tan(\theta) = \dfrac{\text{opposite side}}{\text{adjacent side}}$, **which is abbreviated to** $\tan(\theta) = \dfrac{O}{A}$



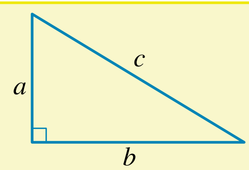
The symbol θ (theta) is one of the many letters of the Greek alphabet used to represent the angle. Other symbols include α (alpha), β (beta) and γ (gamma). Non-Greek letters may also be used.

Writing the mnemonic **SOH–CAH–TOA** each time we perform trigonometric calculations will help us to remember the ratios and solve the problem.

Pythagoras' theorem

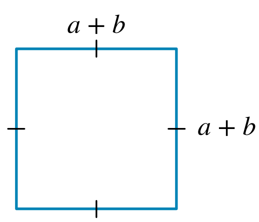
For specific problems it may be necessary to determine the side lengths of a right-angled triangle before calculating the trigonometric ratios. In these situations, Pythagoras' theorem is used. Pythagoras' theorem states:

In any right-angled triangle, $c^2 = a^2 + b^2$.



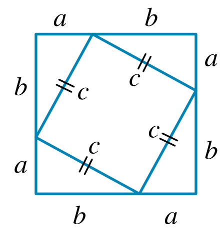
Proof of Pythagoras' theorem

Start with a square with side length $a + b$.



$$\text{Area } 1 = (a + b)^2$$

Divide this square into four congruent right-angled triangles, Δabc, and a square of length, c.



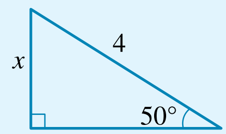
$$\begin{aligned}\text{Area } 2 &= \text{Area of four triangles} + \text{one square}\\ &= 4\left(\frac{1}{2}ab\right) + c^2\\ &= 2ab + c^2\end{aligned}$$

$$\therefore \quad \text{Area 1} = \text{Area 2}$$
$$(a+b)^2 = 2ab + c^2$$
$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$

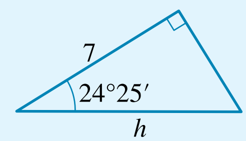
Pythagoras' theorem states that the square of the hypotenuse is equal to the sum of the squares of the other two sides.

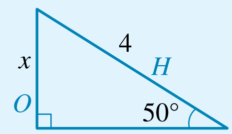
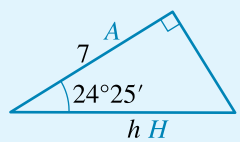
WORKED EXAMPLE 1 **Determine the value of the pronumerals, correct to 2 decimal places.**

a



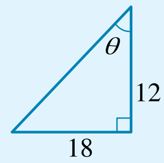
b

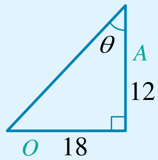


THINK	WRITE/DRAW
a 1 Label the sides, relative to the marked angles.	**a** 
2 Write what is given.	Have: angle and hypotenuse (H)
3 Write what is needed.	Need: opposite (O) side
4 Determine which of the trigonometric ratios is required, using SOH–CAH–TOA.	$\sin(\theta) = \frac{O}{H}$
5 Substitute the given values into the appropriate ratio.	$\sin(50°) = \frac{x}{4}$
6 Transpose the equation and solve for x.	$4 \times \sin(50°) = x$ $x = 4 \times \sin(50°)$
7 Round the answer to 2 decimal places.	$= 3.06$
b 1 Label the sides, relative to the marked angles.	**b** 
2 Write what is given.	Have: angle and adjacent (A) side
3 Write what is needed.	Need: hypotenuse (H)
4 Determine which of the trigonometric ratios is required, using SOH–CAH–TOA.	$\cos(\theta) = \frac{A}{H}$
5 Substitute the given values into the appropriate ratio.	$\cos(24°25') = \frac{7}{h}$ $h = \frac{7}{\cos(24°25')}$
6 Round the answer to 2 decimal places.	$= 7.69$

WORKED EXAMPLE 2

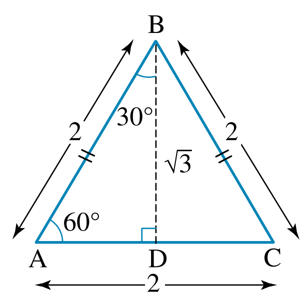
Find the angle θ, giving the answer in degrees and minutes.



THINK	WRITE/DRAW
1 Label the sides, relative to the marked angles.	
2 Write what is given.	Have: opposite (O) and adjacent (A) sides
3 Write what is needed.	Need: angle
4 Determine which of the trigonometric ratios is required, using SOH–CAH–TOA.	$\tan(\theta) = \dfrac{O}{A}$
5 Substitute the given values into the appropriate ratio.	$\tan(\theta) = \dfrac{18}{12}$
6 Write the answer to the nearest minute.	$\theta = \tan^{-1}\left(\dfrac{18}{12}\right)$ $= 56°19'$

Exact values

Most of the trigonometric values that we will deal with in this topic are only approximations. However, angles of 30°, 45° and 60° have exact values of sine, cosine and tangent. Consider an equilateral triangle, ABC, of side length 2 units.



If the triangle is perpendicularly bisected, then two congruent triangles, ABD and CBD, are obtained. From triangle ABD it can be seen that BD creates a right-angled triangle with angles of 60° and 30° and base length (AD) of 1 unit. The length of BD is obtained using Pythagoras' theorem.

Using triangle ABD and the three trigonometric ratios, the following exact values are obtained:

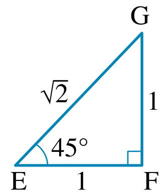
$$\sin(30°) = \frac{1}{2} \qquad \sin(60°) = \frac{\sqrt{3}}{2}$$

$$\cos(30°) = \frac{\sqrt{3}}{2} \qquad \cos(60°) = \frac{1}{2}$$

$$\tan(30°) = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3} \qquad \tan(60°) = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3}$$

Consider a right-angled isosceles triangle EFG whose equal sides are of 1 unit. The hypotenuse EG is obtained by using Pythagoras' theorem.

$$\begin{aligned}(EG)^2 &= (EF)^2 + (FG)^2 \\ &= 1^2 + 1^2 \\ &= 2 \\ EG &= \sqrt{2}\end{aligned}$$



Using triangle EFG and the three trigonometric ratios, the following exact values are obtained:

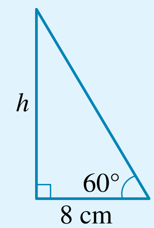
$$\sin(45°) = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

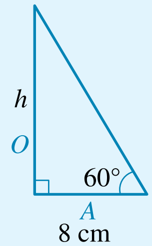
$$\cos(45°) = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

$$\tan(45°) = \frac{1}{1} \text{ or } 1$$

WORKED EXAMPLE 3

Determine the height of the triangle shown in surd form.

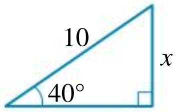


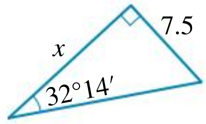
THINK	WRITE/DRAW
1 Label the sides relative to the marked angle.	
2 Write what is given.	Have: angle and adjacent (A) side
3 Write what is needed.	Need: opposite (O) side
4 Determine which of the trigonometric ratios is required, using SOH–CAH–TOA.	$\tan(\theta) = \frac{O}{A}$
5 Substitute the given values into the appropriate ratio.	$\tan(60°) = \frac{h}{8}$
6 Substitute exact values where appropriate.	$\sqrt{3} = \frac{h}{8}$
7 Transpose the equation to find the required value.	$h = 8\sqrt{3}$
8 State the answer.	The triangle's height is $8\sqrt{3}$ cm.

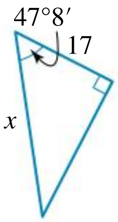
EXERCISE 5.2 Trigonometry of right-angled triangles

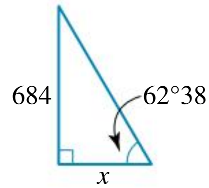
PRACTISE

1 **WE1** Find the value of the pronumerals, correct to 2 decimal places.

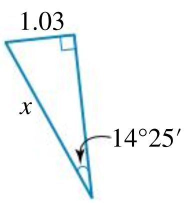
a


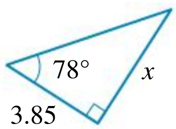
b


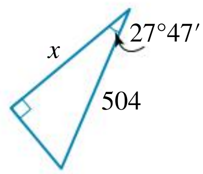
c


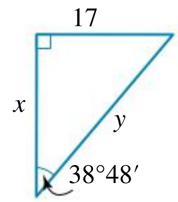
d


2 Find the value of x and y, correct to 2 decimal places.

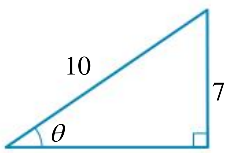
a


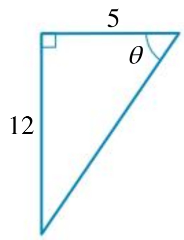
b


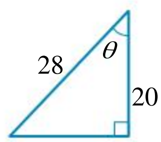
c


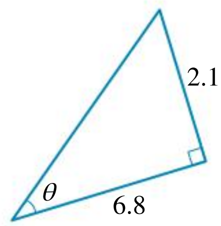
d


3 **WE2** Find the angle θ, giving the answer in degrees and minutes.

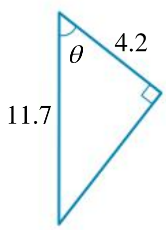
a


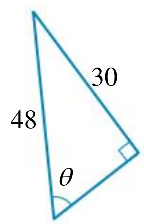
b


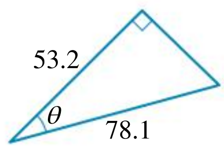
c


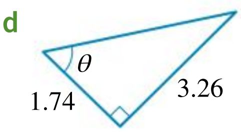
d


4 Find the angle θ, giving the answer in degrees and minutes.

a


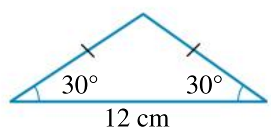
b


c


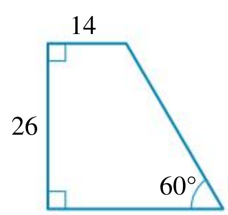
d


5 **WE3** An isosceles triangle has a base of 12 cm and equal angles of 30°. Find, in the simplest surd form:

a the height of the triangle
b the area of the triangle
c the perimeter of the triangle.



6 Find the perimeter of the composite shape below, in surd form. The length measurements are in metres.



CONSOLIDATE

7 A ladder 6.5 m long rests against a vertical wall and makes an angle of 50° to the horizontal ground.

a How high up the wall does the ladder reach?
b If the ladder needs to reach 1 m higher, what angle should it make to the ground, to the nearest minute?

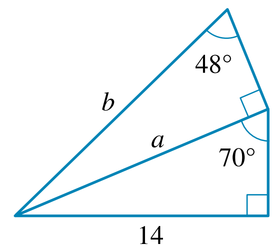
8 A road 400 m long goes straight up a slope. If the road rises 50 m vertically, what is the angle that the road makes with the horizontal?

9 An ice-cream cone has a diameter of 6 cm and a sloping edge of 15 cm. Find the angle at the bottom of the cone.

10 A vertical flagpole is supported by a wire attached from the top of the pole to the horizontal ground, 4 m from the base of the pole. Joanne measures the angle the wire makes with the ground as 65°. How tall is the flagpole?

11 A stepladder stands on a floor, with its feet 1.5 m apart. If the angle formed by the legs is 55°, how high above the floor is the top of the ladder?

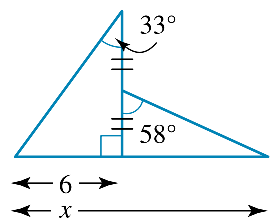
12 In the figure at right, find the value of the pronumerals, correct to 2 decimal places.



13 The angle formed by the diagonal of a rectangle and one of its shorter sides is 60°. If the diagonal is 8 cm long, find the dimensions of the rectangle, in surd form.

14 An advertising balloon is attached to a rope 120 m long. The rope makes an angle of 75° to level ground. How high above the ground is the balloon?

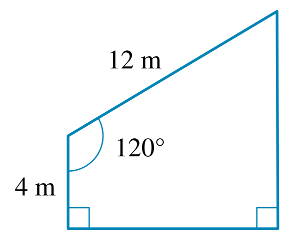
15 In the figure at right, find the value of the pronumeral x, correct to 2 decimal places.



16 An isosceles triangle has sides of 17 cm, 20 cm and 20 cm. Find the magnitude of the angles.

MASTER

17 A garden bed in the shape of a trapezium is shown at right. What volume of garden mulch is needed to cover it to a depth of 15 cm?



18 A ladder 10 m long rests against a vertical wall at an angle of 55° to the horizontal. It slides down the wall, so that it now makes an angle of 48° with the horizontal.

a Through what vertical distance did the top of the ladder slide?

b Does the foot of the ladder move through the same distance? Justify your answer.

5.3 Elevation, depression and bearings

Trigonometry is especially useful for measuring distances and heights that are difficult or impractical to access. For example, two important applications of right-angled triangles are:

1. angles of elevation and depression
2. bearings.

studyon

Units 1 & 2

AOS 4

Topic 3

Concept 2

Elevation and depression

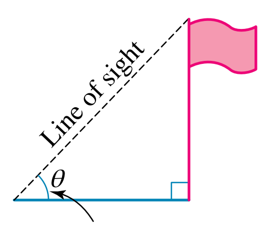
Concept summary

Practice questions

Angles of elevation and depression

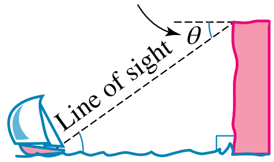
Angles of elevation and depression are employed when dealing with directions that require us to look up and down respectively.

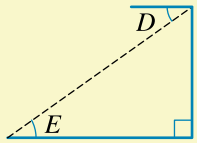
An **angle of elevation** is the angle between the horizontal and an object that is higher than the observer (for example, the top of a mountain or flagpole).



An **angle of depression** is the angle between the horizontal and an object that is lower than the observer (for example, a boat at sea when the observer is on a cliff).

Unless otherwise stated, the angle of elevation or depression is measured and drawn from the horizontal.



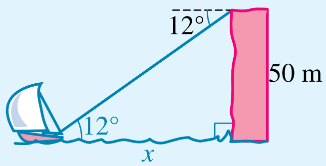
> **Angles of elevation and depression are each measured from the horizontal.**
>
> **When solving problems involving angles of elevation and depression, it is always best to draw a diagram.**
>
> **The angle of elevation is equal to the angle of depression because they are alternate 'Z' angles.**
>
> 
>
>
> D and E are alternate angles.
> $\therefore \angle D = \angle E$

WORKED EXAMPLE 4

From a cliff 50 metres high, the angle of depression to a boat at sea is 12°. How far is the boat from the base of the cliff?

THINK / **WRITE/DRAW**

1 Draw a diagram and label all the given information.
Include the unknown length, x, and the angle of elevation, 12°.



2 Write what is given.
Have: angle and opposite side

3 Write what is needed.
Need: adjacent side

4 Determine which of the trigonometric ratios is required (SOH–CAH–TOA).
$$\tan(\theta) = \frac{O}{A}$$

5 Substitute the given values into the appropriate ratio.
$$\tan(12^\circ) = \frac{50}{x}$$

6 Transpose the equation and solve for x.
$$x \times \tan(12^\circ) = 50$$
$$x = \frac{50}{\tan(12^\circ)}$$

7 Round the answer to 2 decimal places.
$$= 235.23$$

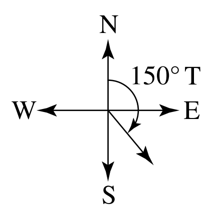
8 Answer the question.
The boat is 235.23 m from the base of the cliff.

Bearings

Bearings measure the direction of one object from another. There are two systems used for describing bearings.

True bearings are measured in a clockwise direction, starting from north (0° T).

Conventional or **compass bearings** are measured:
first, relative to north or south, and second, relative to east or west.



Compass bearing equivalent is S30°E.

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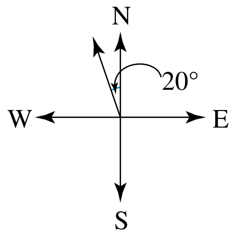
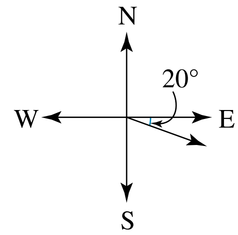
Units 1 & 2
AOS 4
Topic 3
Concept 5

Bearings
Concept summary
Practice questions



N20°W

True bearing equivalent is 340° T.

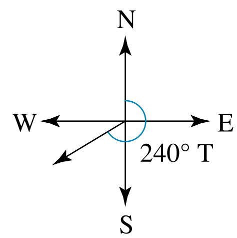


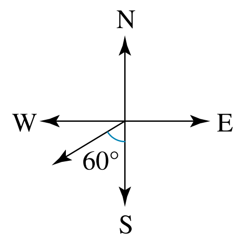
S70°E

True bearing equivalent is 110° T.

The two systems are interchangeable. For example, a bearing of 240° T is the same as S60°W.

When solving questions involving direction, always start with a diagram showing the basic compass points: north, south, east and west.

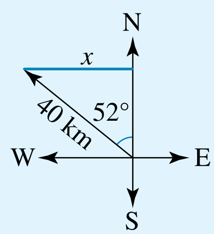




S60°W

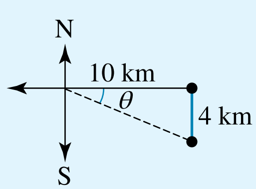
WORKED EXAMPLE 5

A ship sails 40 km in a direction of N52°W. How far west of the starting point is it?

THINK	WRITE/DRAW
1 Draw a diagram of the situation, labelling each of the compass points and the given information.	
2 Write what is given for the triangle.	Have: angle and hypotenuse
3 Write what is needed for the triangle.	Need: opposite side
4 Determine which of the trigonometric ratios is required (SOH–CAH–TOA).	$\sin(\theta) = \frac{O}{H}$
5 Substitute the given values into the appropriate ratio.	$\sin(52°) = \frac{x}{40}$
6 Transpose the equation and solve for x.	$40 \times \sin(52°) = x$
7 Round the answer to 2 decimal places.	$x = 31.52$
8 Answer the question.	The ship is 31.52 km west of the starting point.

WORKED EXAMPLE 6

A ship sails 10 km east, then 4 km south. What is its bearing from its starting point?

THINK	WRITE/DRAW
1 Draw a diagram of the situation, labelling each of the compass points and the given information.	

2 Write what is given for the triangle. | Have: adjacent and opposite sides

3 Write what is needed for the triangle. | Need: angle

4 Determine which of the trigonometric ratios is required (SOH–CAH–TOA).

$\tan(\theta) = \frac{O}{A}$

5 Substitute the given values into the appropriate ratio.

$\tan(\theta) = \frac{4}{10}$

6 Transpose the equation and solve for θ, using the inverse tan function.

$\theta = \tan^{-1}\left(\frac{4}{10}\right)$

7 Convert the angle to degrees and minutes.

$= 21.801\,409\,49°$
$= 21°48'$

8 Express the angle in bearings form. The bearing of the ship was initially 0° T; it has since rotated through an angle of 90° and an additional angle of 21°48′. To obtain the final bearing these values are added.

$\text{Bearing} = 90° + 21°48'$
$= 111°48'\text{ T}$

9 Answer the question.

The bearing of the ship from its starting point is 111°48′ T.

EXERCISE 5.3 Elevation, depression and bearings

PRACTISE

1 WE4 From a vertical fire tower 60 m high, the angle of depression to a fire is 6°. How far away, to the nearest metre, is the fire?

2 A person stands 20 m from the base of a building and measures the angle of elevation to the top of the building as 55°. If the person is 1.7 m tall, how high, to the nearest metre, is the building?

3 WE5 A pair of kayakers paddle 1800 m on a bearing of N20°E. How far north of their starting point are they, to the nearest metre?

4 A ship sails 230 km on a bearing of S20°W. How far west of its starting point has it travelled, correct to the nearest kilometre?

5 WE6 A ship sails 20 km south, then 8 km west. What is its bearing from the starting point?

6 A cross-country competitor runs 2 km west, then due north for 3 km. What is the true bearing of the runner from the starting point?

CONSOLIDATE

7 Express the following conventional bearings as true bearings, and the true bearings in conventional form.

a N35°W b S47°W c N58°E d S17°E
e 246° T f 107° T g 321° T h 074° T

8 a A bearing of S30°E is the same as:

A 030° T B 120° T C 150° T D 210° T E 240° T

b A bearing of 280° T is the same as:

A N10°W B S10°W C S80°W D N80°W E N10°E

9 An observer on a cliff top 57 m high observes a ship at sea. The angle of depression to the ship is 15°. The ship sails towards the cliff, and the angle of depression is then 25°. How far, to the nearest metre, did the ship sail between sightings?

10 A new skyscraper is proposed for the Melbourne Docklands region. It is to be 500 m tall. What would be the angle of depression, in degrees and minutes, from the top of the building to the island on Albert Park Lake, which is 4.2 km away?

11 From a rescue helicopter 2500 m above the ocean, the angles of depression to two shipwreck survivors are 48° (survivor 1) and 35° (survivor 2).

a Draw a labelled diagram that represents the situation.

b Calculate how far apart the two survivors are.

12 A lookout tower has been erected on top of a mountain. At a distance of 5.8 km, the angle of elevation from the ground to the base of the tower is 15.7°, and the angle of elevation to the observation deck (on the top of the tower) is 15.9°. How high, to the nearest metre, is the observation deck above the top of the mountain?

13 From point A on level ground, the angle of elevation to the top of a building 50 m high is 45°. From point B on the ground and in line with A and the foot of the building, the angle of elevation to the top of the building is 60°. Find, in simplest surd form, the distance from A to B.

14 A yacht race consists of four legs. The first three legs are 4 km due east, then 5 km south, followed by 2 km due west.

a How long is the final leg, if the race finishes at the starting point?

b On what bearing must the final leg be sailed?

15 Two hikers set out from the same campsite. One walks 7 km in the direction 043° T and the other walks 10 km in the direction 133° T.

a What is the distance between the two hikers?

b What is the bearing of the first hiker from the second?

16 A ship sails 30 km on a bearing of 220° T, then 20 km on a bearing of 250° T. Find:

a how far south of the original position it is

b how far west of the original position it is

c the true bearing of the ship from its original position, to the nearest degree.

MASTER

17 The town of Bracknaw is due west of Arley. Chris, in an ultralight plane, starts at a third town, Champton, which is due north of Bracknaw, and flies directly towards Arley at a speed of 40 km/h in a direction of 110° T. She reaches Arley in 3 hours. Find:

a the distance between Arley and Bracknaw

b the time to complete the journey from Champton to Bracknaw, via Arley, if she increases her speed to 45 km/h between Arley and Bracknaw.

18 A bird flying at 50 m above the ground was observed at noon from my front door at an angle of elevation of 5°. Two minutes later its angle of elevation was 4°.

a If the bird was flying straight and level, find the horizontal distance of the bird:
  i from my doorway at noon
  ii from my doorway at 12.02 pm.

b Hence, find:
  i the distance travelled by the bird in the two minutes
  ii its speed of flight in km/h.

5.4 The sine rule

studyon

Units 1 & 2

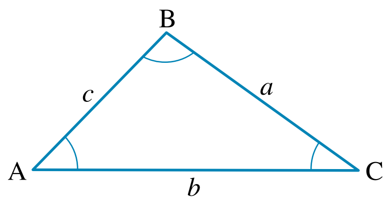
AOS 4

Topic 3

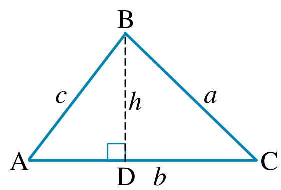
Concept 3

The sine rule
Concept summary
Practice questions

When working with non–right-angled triangles, it is usual to label the angles A, B and C, and the sides a, b and c, so that side a is the side opposite angle A, side b is opposite angle B and side c is opposite angle C.



In a non–right-angled triangle, a perpendicular line, h, can be drawn from the angle B to side b.



$\frac{h}{c} = \sin(A)$ and $\frac{h}{a} = \sin(C)$

Using triangle ABD, we obtain $\sin(A) = \frac{h}{c}$. Using triangle CBD, we obtain $\sin(C) = \frac{h}{a}$.

Transposing each equation to make h the subject, we obtain $h = c \times \sin(A)$ and $h = a \times \sin(C)$. Equate to get $c \times \sin(A) = a \times \sin(C)$.

Transpose to get

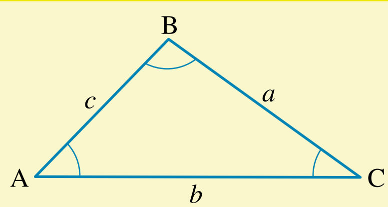
$$\frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$

In a similar way, if a perpendicular line is drawn from angle A to side a, we get

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

From this, the sine rule can be stated.

In any triangle ABC: $$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

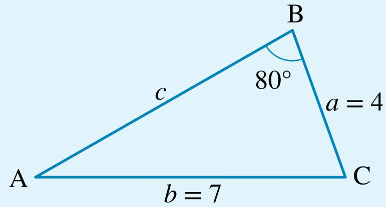


Notes

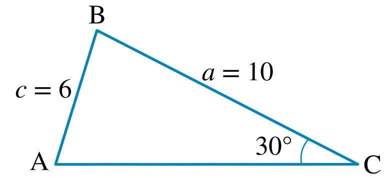
1. When using this rule, depending on the values given, any combination of the two equalities may be used to solve a particular triangle.
2. To solve a triangle means to find all unknown side lengths and angles.

The sine rule can be used to solve non–right-angled triangles if we are given:
1. two angles and one side length
2. two side lengths and an angle opposite one of these side lengths.

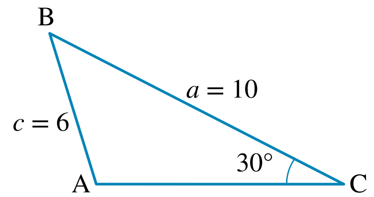
WORKED EXAMPLE 7 In the triangle ABC, $a = 4$ m, $b = 7$ m and $B = 80°$. Find A, C and c.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied.	The sine rule can be used since two side lengths and an angle opposite one of these side lengths have been given.
3 Write the sine rule to find A.	To find angle A: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$
4 Substitute the known values into the rule.	$\frac{4}{\sin(A)} = \frac{7}{\sin(80°)}$
5 Transpose the equation to make $\sin(A)$ the subject.	$4 \times \sin(80°) = 7 \times \sin(A)$ $\sin(A) = \frac{4 \times \sin(80°)}{7}$
6 Evaluate.	$A = \sin^{-1}\left(\frac{4 \times \sin(80°)}{7}\right)$ $= \sin^{-1}(0.562\,747\,287)$ $= 34.246\,004\,71°$
7 Round the answer to degrees and minutes.	$= 34°15'$
8 Determine the value of angle C using the fact that the angle sum of any triangle is 180°.	$C = 180° - (80° + 34°15')$ $= 65°45'$
9 Write the sine rule to find c.	To find side length c: $\frac{c}{\sin(C)} = \frac{b}{\sin(B)}$
10 Substitute the known values into the rule.	$\frac{c}{\sin(65°45')} = \frac{7}{\sin(80°)}$
11 Transpose the equation to make c the subject.	$c = \frac{7 \times \sin(65°45')}{\sin(80°)}$
12 Evaluate. Round the answer to 2 decimal places and include the appropriate unit.	$= \frac{7 \times 0.911\,762\,043}{0.984\,807\,753}$ $= \frac{6.382\,334\,305}{0.984\,807\,753}$ $= 6.480\,792\,099$ $= 6.48$ m

The ambiguous case



When using the sine rule there is one important issue to consider. If we are given two side lengths and an angle opposite one of these side lengths, then sometimes two different triangles can be drawn. For example, if $a = 10$, $c = 6$ and $C = 30°$, two possible triangles could be created.



In the first case, angle A is an acute angle, while in the second case, angle A is an obtuse angle. The two values for A will add to 180°.

The ambiguous case does not work for every example. It would be useful to know, before commencing a question, whether or not the ambiguous case exists and, if so, to then find both sets of solutions.

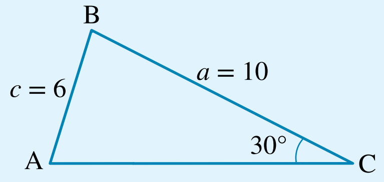
The ambiguous case exists if C is an acute angle and $a > c > a \times \sin(C)$, or any equivalent statement; for example, if B is an acute angle and $a > b > a \times \sin(B)$, and so on.

WORKED EXAMPLE 8

In the triangle ABC, $a = 10$ m, $c = 6$ m and $C = 30°$.

a Show that the ambiguous case exists.

b Find two possible values of A, and hence two possible values of B and b.

THINK	WRITE/DRAW
a 1 Check that the conditions for an ambiguous case exist, i.e. that C is an acute angle and that $a > c > a \times \sin(C)$.	a $C = 30°$ so C is an acute angle. $\sin(C) = \sin(30°) = 0.5$ $a > c > a \times \sin(C)$ $10 > 6 > 10 \sin(30°)$ $10 > 6 > 5$ This is correct.
2 State the answer.	This is an ambiguous case of the sine rule.
b **Case 1**	b
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Write the sine rule to find A.	To find angle A: $\dfrac{a}{\sin(A)} = \dfrac{c}{\sin(C)}$
3 Substitute the known values into the rule.	$\dfrac{10}{\sin(A)} = \dfrac{6}{\sin(30°)}$

4 Transpose the equation to make sin (A) the subject.

$$10 \times \sin(30°) = 6 \times \sin(A)$$

$$\sin(A) = \frac{10 \times \sin(30°)}{6}$$

5 Evaluate angle A, in degrees and minutes.

$$A = \sin^{-1}\left(\frac{10 \times \sin(30°)}{6}\right)$$

$$A = 56°27'$$

6 Determine the value of angle B, using the fact that the angle sum of any triangle is 180°.

$$B = 180° - (30° + 56°27')$$
$$= 93°33'$$

7 Write the sine rule to find b.

To find side length b:

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

8 Substitute the known values into the rule.

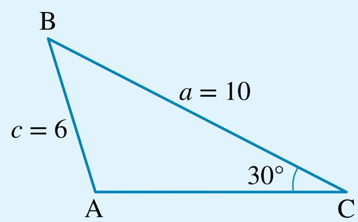
$$\frac{b}{\sin(93°33')} = \frac{6}{\sin(30°)}$$

9 Transpose the equation to make b the subject and evaluate.

$$b = \frac{6 \times \sin(93°33')}{\sin(30°)}$$
$$= 11.98 \text{ m}$$

Case 2

1 Draw a labelled diagram of the triangle ABC and fill in the given information.



2 Write the alternative value for angle A. Subtract the value obtained for A in Case 1 from 180°.

To find the alternative angle A:
If sin (A) = 0.8333, then A could also be:

$$A = 180° - 56°27'$$
$$= 123°33'$$

3 Determine the alternative value of angle B, using the fact that the angle sum of any triangle is 180°.

$$B = 180° - (30° + 123°33)'$$
$$= 26°27'$$

4 Write the sine rule to find the alternative b.

To find side length b:

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

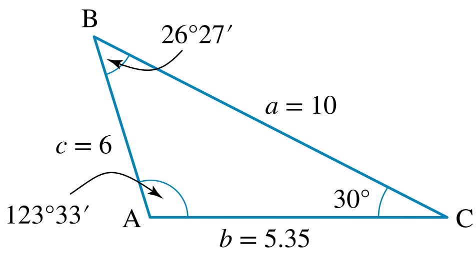
5 Substitute the known values into the rule.

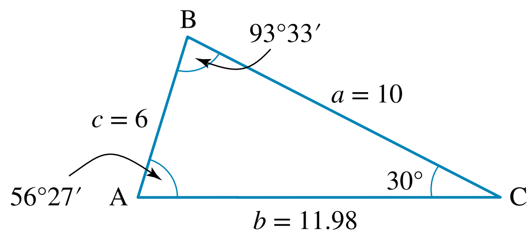
$$\frac{b}{\sin(26°27')} = \frac{6}{\sin(30°)}$$

6 Transpose the equation to make b the subject and evaluate.

$$b = \frac{6 \times \sin(26°27')}{\sin(30°)}$$
$$= 5.35 \text{ m}$$

Hence, for Worked example 8 there were two possible solutions as shown by the diagrams below.





EXERCISE 5.4 The sine rule

PRACTISE

1 WE7 In the triangle ABC, $a = 10$, $b = 12$ and $B = 58°$. Find A, C and c.

2 In the triangle ABC, $c = 17.35$, $a = 26.82$ and $A = 101°47'$. Find C, B and b.

3 WE8 In the triangle ABC, $a = 10$, $c = 8$ and $C = 50°$. Find two possible values of A and hence two possible values of b.

4 In the triangle ABC, $a = 20$, $b = 12$ and $B = 35°$. Find two possible values for the perimeter of the triangle.

CONSOLIDATE

5 In the triangle ABC, $c = 27$, $C = 42°$ and $A = 105°$. Find B, a and b.

6 In the triangle ABC, $a = 7$, $c = 5$ and $A = 68°$. Find the perimeter of the triangle.

7 Find all unknown sides and angles for the triangle ABC, given $a = 32$, $b = 51$ and $A = 28°$.

8 Find the perimeter of the triangle ABC if $a = 7.8$, $b = 6.2$ and $A = 50°$.

9 In a triangle ABC, $A = 40°$, $C = 80°$ and $c = 3$. The value of b is:

A 2.64 **B** 2.86 **C** 14 **D** 4.38 **E** 4.60

10 Find all unknown sides and angles for the triangle ABC, given $A = 27°$, $B = 43°$ and $c = 6.4$.

11 Find all unknown sides and angles for the triangle ABC, given $A = 25°$, $b = 17$ and $a = 13$.

12 To calculate the height of a building, Kevin measures the angle of elevation to the top as 48°. He then walks 18 m closer to the building and measures the angle of elevation as 64°. How high is the building?

13 A river has parallel banks that run directly east–west. From one bank Kylie takes a bearing to a tree on the oppsite bank. The bearing is 047° T. She then walks 10 m due east and takes a second bearing to the tree. This is 305° T. Find:

a her distance from the second measuring point to the tree

b the width of the river, to the nearest metre.

14 A ship sails on a bearing of S20°W for 14 km, then changes direction and sails for 20 km and drops anchor. Its bearing from the starting point is now N65°W.

a How far is it from the starting point?

b On what bearing did it sail the 20 km leg?

MASTER

15 **a** A cross-country runner runs at 8 km/h on a bearing of 150° T for 45 minutes, then changes direction to a bearing of 053° T and runs for 80 minutes until he is due east of the starting point.

i How far was the second part of the run?

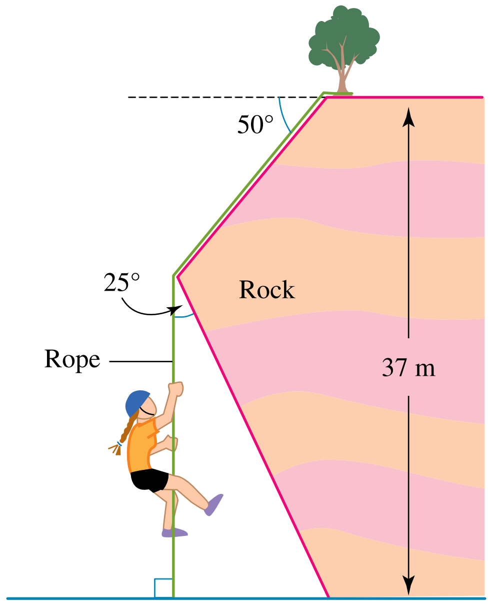
ii What was his speed for this section?

iii How far does he need to run to get back to the starting point?

b From a fire tower, A, a fire is spotted on a bearing of N42°E. From a second tower, B, the fire is on a bearing of N12°W. The two fire towers are 23 km apart, and A is N63°W of B. How far is the fire from each tower?

16 A cliff is 37 m high. The rock slopes outward at an angle of 50° to the horizontal, then cuts back at an angle of 25° to the vertical, meeting the ground directly below the top of the cliff.

Carol wishes to abseil from the top of the cliff to the ground as shown in the diagram below right. Her climbing rope is 45 m long, and she needs 2 m to secure it to a tree at the top of the cliff. Will the rope be long enough to allow her to reach the ground?



5.5 The cosine rule

studyon

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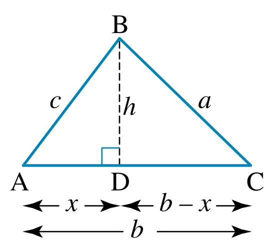
In any non–right-angled triangle ABC, a perpendicular line can be drawn from angle B to side b. Let D be the point where the perpendicular line meets side b, and the length of the perpendicular line be h. Let the length AD = x units. The perpendicular line creates two right-angled triangles, ADB and CDB.



Using triangle ADB and Pythagoras' theorem, we obtain:

$$c^2 = h^2 + x^2 \qquad [1]$$

Using triangle CDB and Pythagoras' theorem, we obtain:

$$a^2 = h^2 + (b - x)^2 \qquad [2]$$

Expanding the brackets in equation [2]:

$$a^2 = h^2 + b^2 - 2bx + x^2$$

Rearranging equation [2] and using $c^2 = h^2 + x^2$ from equation [1]:

$$\begin{aligned} a^2 &= h^2 + x^2 + b^2 - 2bx \\ &= c^2 + b^2 - 2bx \\ &= b^2 + c^2 - 2bx \end{aligned}$$

From triangle ABD, $x = c \times \cos(A)$, therefore $a^2 = b^2 + c^2 - 2bx$ becomes

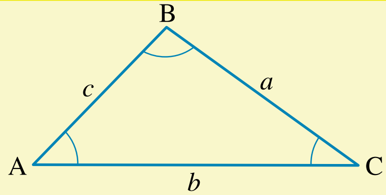
$$a^2 = b^2 + c^2 - 2bc \times \cos(A)$$

This is called the **cosine rule** and is a generalisation of Pythagoras' theorem.

In a similar way, if the perpendicular line was drawn from angle A to side a or from angle C to side c, the two right-angled triangles would give $c^2 = a^2 + b^2 - 2ab \times \cos(C)$ and $b^2 = a^2 + c^2 - 2ac \times \cos(B)$ respectively. From this, the cosine rule can be stated:

In any triangle ABC

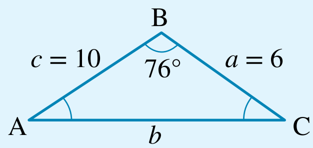
$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$



The cosine rule can be used to solve non–right-angled triangles if we are given:
1. **three sides of the triangle**
2. **two sides of the triangle and the included angle (the angle between the given sides).**

WORKED EXAMPLE 9

Find the third side of triangle ABC correct to 2 decimal places given $a = 6$, $c = 10$ and $B = 76°$.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the cosine rule has been satisfied.	Yes, the cosine rule can be used since two side lengths and the included angle have been given.
3 Write the appropriate cosine rule to find side b.	To find side b: $b^2 = a^2 + c^2 - 2ac\cos(B)$
4 Substitute the given values into the rule.	$= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos(76°)$
5 Evaluate.	$= 36 + 100 - 120 \times 0.241\,921\,895$ $= 106.969\,372\,5$ $b = \sqrt{106.969\,372\,5}$
6 Round the answer to 2 decimal places.	$= 10.34$ correct to 2 decimal places

Note: Once the third side has been found, the sine rule could be used to find other angles if necessary.

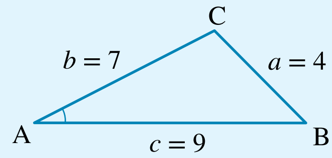
If three sides of a triangle are known, an angle could be found by transposing the cosine rule to make $\cos A$, $\cos B$ or $\cos C$ the subject.

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

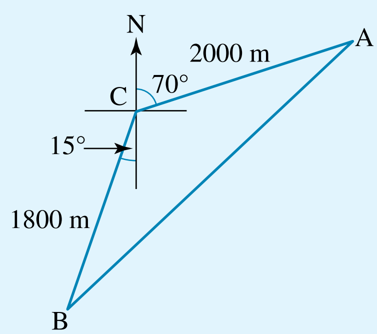
$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

WORKED EXAMPLE 10 **Find the smallest angle in the triangle with sides 4 cm, 7 cm and 9 cm.**

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information. *Note:* The smallest angle will be opposite the smallest side.	 Let $a = 4$ $b = 7$ $c = 9$
2 Check that one of the criteria for the cosine rule has been satisfied.	The cosine rule can be used since three side lengths have been given.
3 Write the appropriate cosine rule to find angle A.	$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$
4 Substitute the given values into the rearranged rule.	$= \frac{7^2 + 9^2 - 4^2}{2 \times 7 \times 9}$
5 Evaluate.	$= \frac{49 + 81 - 16}{126}$ $= \frac{114}{126}$
6 Transpose the equation to make A the subject by taking the inverse cos of both sides.	$A = \cos^{-1}\left(\frac{114}{126}\right)$ $= 25.208\,765\,3°$
7 Round the answer to degrees and minutes.	$= 25°13'$

WORKED EXAMPLE 11 **Two rowers set out from the same point. One rows N70°E for 2000 m and the other rows S15°W for 1800 m. How far apart are the two rowers?**

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	

2 Check that one of the criteria for the cosine rule has been satisfied.	The cosine rule can be used since two side lengths and the included angle have been given.
3 Write the appropriate cosine rule to find side c.	To find side c: $c^2 = a^2 + b^2 - 2ab\cos(C)$
4 Substitute the given values into the rule.	$= 2000^2 + 1800^2 - 2 \times 2000 \times 1800 \times \cos(125°)$
5 Evaluate.	$= 40\,000\,000 + 3\,240\,000 - 7\,200\,000 \times -0.573\,576\,436$ $= 11\,369\,750.342$ $c = \sqrt{11\,369\,750.342}$ $= 3371.906\,04$
6 Round the answer to 2 decimal places.	$= 3371.91$
7 Answer the question.	The rowers are 3371.91 m apart.

EXERCISE 5.5 The cosine rule

PRACTISE

1 WE9 Find the third side of triangle ABC given $a = 3.4$, $b = 7.8$ and $C = 80°$.

2 In triangle ABC, $b = 64.5$ cm, $c = 38.1$ cm and $A = 58°34'$. Find the third side, a.

3 WE10 Find the smallest angle in the triangle with sides 6 cm, 4 cm and 8 cm.

4 In triangle ABC, $a = 356$, $b = 207$ and $c = 296$. Find the smallest angle.

5 WE11 Two rowers set out from the same point. One rows N30°E for 1500 m and the other rows S40°E for 1200 m. How far apart are the two rowers?

6 Two rowers set out from the same point. One rows 16.2 km on a bearing of 053° T and the other rows 31.6 km on a bearing of 117° T. How far apart are the two rowers?

CONSOLIDATE

7 In triangle ABC, $a = 17$, $c = 10$ and $B = 115°$. Find b, and hence find A and C.

8 In triangle ABC, $a = 23.6$, $b = 17.3$ and $c = 26.4$. Find the size of all the angles.

9 In triangle DEF, $d = 3$ cm, $e = 7$ cm and $F = 60°$. Find f in exact form.

10 Maria cycles 12 km in a direction of N68°W, then 7 km in a direction of N34°E.

a How far is she from her starting point?

b What is the bearing of the starting point from her finishing point?

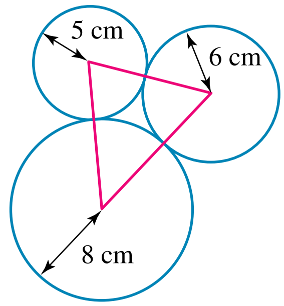
11 A garden bed is in the shape of a triangle with sides of length 3 m, 4.5 m and 5.2 m.

a Calculate the smallest angle.

b Hence, find the area of the garden. (*Hint:* Draw a diagram with the longest length as the base of the triangle.)

12 A hockey goal is 3 m wide. When Sophie is 7 m from one post and 5.2 m from the other, she shoots for goal. Within what angle, to the nearest degree, must the shot be made if it is to score a goal?

13 Three circles of radii 5 cm, 6 cm and 8 cm are positioned so that they just touch one another. Their centres form the vertices of a triangle. Find the largest angle in the triangle.



14 An advertising balloon is attached to two ropes 120 m and 100 m long. The ropes are anchored to level ground 35 m apart. How high can the balloon fly?

15 A plane flies N70°E for 80 km, then on a bearing of S10°W for 150 km.

a How far is the plane from its starting point?

b What direction is the plane from its starting point?

16 A plane takes off at 10:00 am from an airfield and flies at 120 km/h on a bearing of N35°W. A second plane takes off at 10:05 am from the same airfield and flies on a bearing of S80°E at a speed of 90 km/h. How far apart are the planes at 10:25 am?

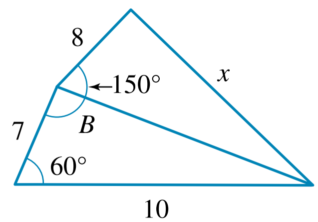
MASTER

17 For the given shape at right, determine:

a the length of the diagonal

b the magnitude (size) of angle B

c the length of x.



18 From the top of a vertical cliff 68 m high, an observer notices a yacht at sea. The angle of depression to the yacht is 47°. The yacht sails directly away from the cliff, and after 10 minutes the angle of depression is 15°. How fast does the yacht sail?

5.6 Arcs, sectors and segments

Radian measurement

studyon

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Arcs, sectors and segments
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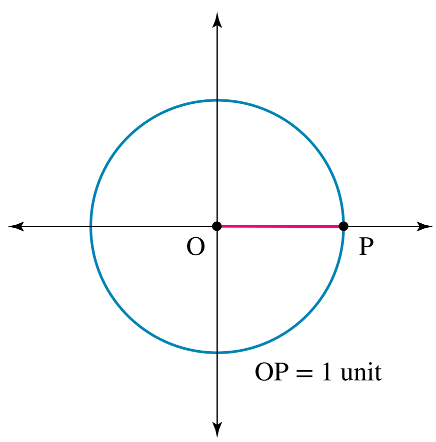
Practice questions

In all of the trigonometry tasks covered so far, the unit for measuring angles has been the degree. There is another commonly used measurement for angles: the **radian**. This is used in situations involving length and areas associated with circles.

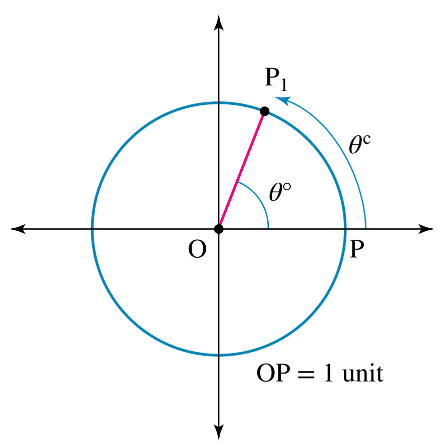


Consider the unit circle, a circle with a radius of 1 unit. OP is the radius.

If OP is rotated $\theta°$ anticlockwise, the point P traces a path along the circumference of the circle to a new point, P_1.

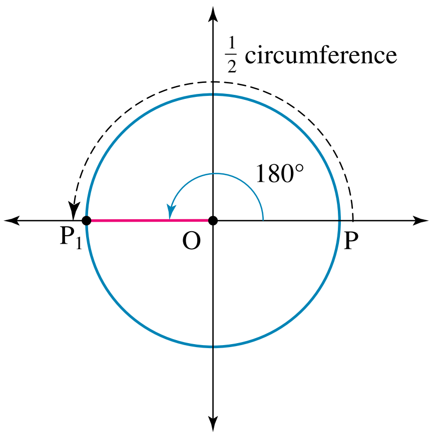
The arc length PP_1 is a radian measurement, symbolised by θ^c.

Note: 1^c is equivalent to the angle in degrees formed when the length of PP_1 is 1 unit; in other words, when the arc is the same length as the radius.



If the length OP is rotated 180°, the point P traces out half the circumference. Since the circle has a radius of 1 unit, and $C = 2\pi r$, the arc PP_1 has a length of π.

The relationship between degrees and radians is thus established.



$$180° = \pi^c$$

This relationship will be used to convert from one system to another. Rearranging the basic conversion factor gives:

$$180° = \pi^c$$

$$1° = \frac{\pi^c}{180}$$

To convert an angle in degrees to radian measure, multiply by $\frac{\pi}{180°}$.

Also, since $\pi^c = 180°$, it follows that $1^c = \frac{180°}{\pi}$.

To convert an angle in radian measure to degrees, multiply by $\frac{180°}{\pi}$.

Where possible, it is common to have radian values with π in them. It is usual to write radians without any symbol, but degrees must always have a symbol. For example, an angle of 25° must have the degree symbol written, but an angle of 1.5 is understood to be 1.5 radians.

WORKED EXAMPLE 12

a Convert 135° to radian measure, expressing the answer in terms of π.

b Convert the radian measurement $\frac{4\pi}{5}$ to degrees.

THINK	WRITE
a 1 To convert an angle in degrees to radian measure, multiply the angle by $\frac{\pi}{180}$.	**a** $135° = 135° \times \frac{\pi}{180°}$ $= \frac{135\pi}{180}$
2 Simplify, leaving the answer in terms of π.	$= \frac{3\pi}{4}$
b 1 To convert radian measure to an angle in degrees, multiply the angle by $\frac{180}{\pi}$.	**b** $\frac{4\pi}{5} = \frac{4\pi}{5} \times \frac{180°}{\pi}$ $= \frac{720°}{5}$
2 Simplify. *Note:* π cancels out.	$= 144°$

If the calculation does not simplify easily, write the answers in degrees and minutes, or radians to 4 decimal places. If angles are given in degrees and minutes, convert to degrees only before converting to radians.

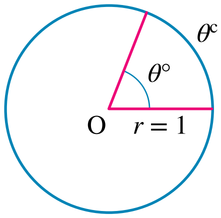
Arc length

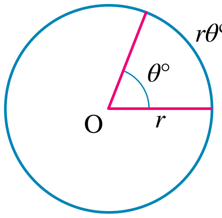
An arc is a section of the circumference of a circle. The length of an arc is proportional to the angle subtended at the centre. For example, an angle of 90° will create an arc that is $\frac{1}{4}$ the circumference.

We have already defined an arc length as equivalent to θ radians if the circle has a radius of 1 unit.

Therefore, a simple dilation of the unit circle will enable us to calculate the arc length for any sized circle, as long as the angle is expressed in radians.

If the radius is dilated by a factor of r, the arc length is also dilated by a factor of r.



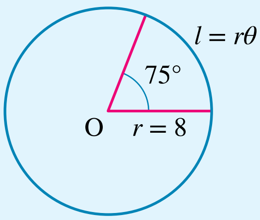


Dilation by factor of r

Therefore, $l = r\theta$, where l represents the arc length, r represents the radius and θ represents an angle measured in radians.

WORKED EXAMPLE 13

Find the length of the arc that subtends an angle of 75° at the centre of a circle with radius 8 cm.

THINK	WRITE/DRAW
1 Draw a diagram representing the situation and label it with the given values.	
2 Convert the angle from 75° to radian measure by multiplying the angle by $\frac{\pi}{180°}$.	$75° = 75° \times \frac{\pi}{180°}$ $= \frac{75\pi}{180}$
3 Evaluate to 4 decimal places.	$= 1.3090$
4 Write the rule for the length of the arc.	$l = r\theta$
5 Substitute the values into the formula.	$= 8 \times 1.3090$
6 Evaluate to 2 decimal places and include the appropriate unit.	$= 10.4720$ $= 10.47$ cm

Note: In order to use the formula for the length of the arc, the angle must be in radian measure.

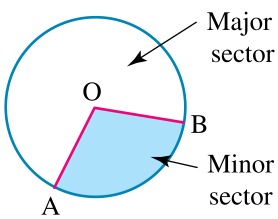
WORKED EXAMPLE 14 Find the angle subtended by a 17 cm arc in a circle of radius 14 cm:

a in radians **b** in degrees.

THINK	WRITE
a 1 Write the rule for the length of the arc.	**a** $l = r\theta$
2 Substitute the values into the formula.	$17 = 14\theta$
3 Transpose the equation to make θ the subject.	$\theta = \frac{17}{14}$
4 Evaluate to 4 decimal places and include the appropriate unit.	$= 1.214\,285\,714$ $= 1.2143^c$
b 1 To convert radian measure to an angle in degrees, multiply the angle by $\frac{180°}{\pi}$.	**b** $1.2143^c = 1.2143 \times \frac{180°}{\pi}$
2 Evaluate.	$= 69.573\,446\,55°$
3 Convert the angle to degrees and minutes.	$= 69°34'$

Area of a sector

In the diagram at right, the shaded area is the minor sector AOB, and the unshaded area is the major sector AOB.



The area of the sector is proportional to the arc length. For example, an area of $\frac{1}{4}$ of the circle contains an arc that is $\frac{1}{4}$ of the circumference.

Thus, in any circle: $\dfrac{\text{area of sector}}{\text{area of circle}} = \dfrac{\text{arc length}}{\text{circumference of circle}}$

$$\frac{A}{\pi r^2} = \frac{r\theta}{2\pi r} \text{ where } \theta \text{ is measured in radians.}$$

$$A = \frac{r\theta \times \pi r^2}{2\pi r}$$

$$= \frac{1}{2}r^2\theta$$

The area of a sector is: $A = \frac{1}{2}r^2\theta$

WORKED EXAMPLE 15 A sector has an area of 157 cm^2 and subtends an angle of 107°. What is the radius of the circle?

THINK	WRITE
1 Convert the angle from 107° to radian measure by multiplying the angle by $\frac{\pi}{180°}$.	$107° = 107° \times \frac{\pi}{180°}$ $= \frac{107\pi}{180}$

2 Evaluate to 4 decimal places. $= 1.8675$

3 Write the rule for the area of a sector. $A = \frac{1}{2}r^2\theta$

4 Substitute the values into the formula. $157 = \frac{1}{2} \times r^2 \times 1.8675$

5 Transpose the equation to make r^2 the subject. $r^2 = \frac{2 \times 157}{1.8675}$

$r^2 = 168.139\,016\,5$

6 Take the square root of both sides of the equation. $r = 12.966\,842\,97$

7 Evaluate to 2 decimal places and include the appropriate unit. $= 12.97$ cm

Area of a segment

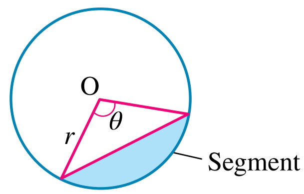
A segment is that part of a sector bounded by the arc and the chord.

As can be seen from the diagram at right:

Area of segment = area of sector – area of triangle

$$A = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin(\theta°)$$

$$= \frac{1}{2}r^2(\theta - \sin(\theta°))$$



Note: θ is in radians and $\theta°$ is in degrees.

The area of a segment: $A = \frac{1}{2}r^2(\theta - \sin(\theta°))$

WORKED EXAMPLE 16

Find the area of the segment in a circle of radius 5 cm, subtended by an angle of 40°.

THINK	WRITE
1 Convert the angle from 40° to radian measure by multiplying the angle by $\frac{\pi}{180°}$.	$40° = 40° \times \frac{\pi}{180°}$ $= \frac{40\pi}{180}$
2 Evaluate to 4 decimal places.	$= 0.6981$
3 Write the rule for the area of a segment.	$A = \frac{1}{2}r^2(\theta - \sin(\theta°))$
4 Identify each of the variables.	$r = 5,\ \theta = 0.6981,\ \theta° = 40°$
5 Substitute the values into the formula.	$A = \frac{1}{2} \times 5^2(0.6981 - \sin(40°))$
6 Evaluate.	$= \frac{1}{2} \times 25 \times 0.0553$ $= 0.691\,25$
7 Round to 2 decimal places and include the appropriate unit.	$= 0.69$ cm^2

EXERCISE 5.6 Arcs, sectors and segments

PRACTISE

1 **WE12** Convert the following angles to radian measure, expressing answers in terms of π.

a $30°$ b $60°$ c $120°$ d $150°$ e $225°$
f $270°$ g $315°$ h $480°$ i $72°$ j $200°$

2 Convert the following radian measurements into degrees.

a $\frac{\pi}{4}$ b $\frac{3\pi}{2}$ c $\frac{7\pi}{6}$ d $\frac{5\pi}{3}$ e $\frac{7\pi}{12}$
f $\frac{17\pi}{6}$ g $\frac{\pi}{12}$ h $\frac{13\pi}{10}$ i $\frac{11\pi}{8}$ j 8π

3 **WE13** Find the length of the arc that subtends an angle of $65°$ at the centre of a circle of radius 14 cm.

4 Find the length of the arc that subtends an angle of $153°$ at the centre of a circle of radius 75 mm.

5 **WE14** Find the angle subtended by a 20 cm arc in a circle of radius 75 cm:

a in radians b in degrees.

6 Find the angle subtended by an 8 cm arc in a circle of radius 5 cm:

a in radians b in degrees.

7 **WE15** A sector has an area of 825 cm^2 and subtends an angle of $70°$. What is the radius of the circle?

8 A sector has an area of 309 cm^2 and subtends an angle of $106°$. What is the radius of the circle?

9 **WE16** Find the area of the segment in a circle of radius 25 cm subtended by an angle of $100°$.

10 Find the area of the segment of a circle of radius 4.7 m that subtends an angle of $85°20'$ at the centre.

CONSOLIDATE

11 Convert the following angles in degrees to radians, giving answers to 4 decimal places.

a $27°$ b $109°$ c $243°$ d $351°$ e $7°$
f $63°42'$ g $138°21'$ h $274°8'$ i $326°53'$ j $47°2'$

12 Convert the following radian measurements into degrees and minutes.

a 2.345 b 0.6103 c 1 d 1.61 e 3.592
f 7.25 g 0.182 h 5.8402 i 4.073 j 6.167

13 Find the length of the arc that subtends an angle of $135°$ at the centre of a circle of radius 10 cm. Leave the answer in terms of π.

14 An arc of a circle is 27.8 cm long and subtends an angle of $205°$ at the centre of the circle. What is the radius of the circle?

15 An arc of length 8 cm is marked out on the circumference of a circle of radius 13 cm. What angle does the arc subtend at the centre of the circle?

16 The minute hand of a clock is 35 cm long. How far does the tip of the hand travel in 20 minutes?

17 A child's swing is suspended by a rope 3 m long. What is the length of the arc it travels if it swings through an angle of 42°?

18 Find the area of the sector of a circle of radius 6 cm with an angle of 100°. Write your answer in terms of π.

19 A garden bed is in the form of a sector of a circle of radius 4 m. The arc of the sector is 5 m long. Find:

a the area of the garden bed

b the volume of mulch needed to cover the bed to a depth of 10 cm.

20 A sector whose angle is 150° is cut from a circular piece of cardboard whose radius is 12 cm. The two straight edges of the sector are joined so as to form a cone.

a What is the surface area of the cone?

b What is the radius of the cone?

MASTER

21 Two irrigation sprinklers spread water in circular paths with radii of 7 m and 4 m. If the sprinklers are 10 m apart, find the area of crop that receives water from both sprinklers.

22 Two circles of radii 3 cm and 4 cm have their centres 5 cm apart. Find the area of the intersection of the two circles.

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5.7 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Units 1 & 2 | **Trigonometric ratios and their applications**

Sit topic test

5 Answers

EXERCISE 5.2

1 a 6.43 b 11.89
c 24.99 d 354.05

2 a 4.14 b 18.11
c 445.90 d $x = 21.14$, $y = 27.13$

3 a 44°26′ b 67°23′
c 44°25′ d 17°10′

4 a 68°58′ b 38°41′
c 47°4′ d 61°55′

5 a $2\sqrt{3}$ cm b $12\sqrt{3}\text{ cm}^2$ c $12 + 8\sqrt{3}$ cm

6 $26\sqrt{3} + 54$ m

7 a 4.98 m b 66°56′

8 7°11′

9 23°4′

10 8.58 m

11 1.44 m

12 $a = 14.90$, $b = 20.05$

13 4 and $4\sqrt{3}$

14 115.91 m

15 $x = 13.39$

16 64°51′, 64°51′, 50°18′

17 10.91 m^3

18 a 0.76 m
b No, the foot of the ladder moves through a distance of 0.95 m.

EXERCISE 5.3

1 571 m

2 30 m

3 1691 m

4 79 km

5 201°48′ T

6 326.3° T

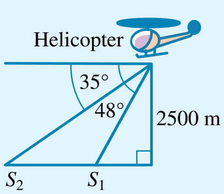
7 a 325° T b 227° T c 058° T
d 163° T e S66°W f S73°E
g N39°W h N74°E

8 a C b D

9 90 m

10 6°47′

11 a



b 1319.36 m

12 22 m

13 $50 - \frac{50\sqrt{3}}{3}$ m

14 a 5.39 km b N21°48′W

15 a 12.2 km b 348° T or N12°W

16 a 29.82 km b 38.08 km c 232° T

17 a 112.76 km b 5 hours 30 minutes

18 a i 571.5 m ii 715 m
b i 143.5 m ii 4.31 km/h

EXERCISE 5.4

1 44°58′, 77°2′, 13.79

2 39°18′, 38°55′, 17.21

3 $A = 73°15′$, $b = 8.83$; or $A = 106°45′$, $b = 4.12$

4 51.9 or 44.86

5 33°, 38.98, 21.98

6 19.12

7 $B = 48°26′$, $C = 103°34′$, $c = 66.26$; or $B = 131°34′$, $C = 20°26′$, $c = 23.8$

8 24.17

9 A

10 $C = 110°$, $a = 3.09$, $b = 4.64$

11 $B = 33°33′$, $C = 121°27′$, $c = 26.24$; or $B = 146°27′$, $C = 8°33′$, $c = 4.57$

12 43.62 m

13 a 6.97 m b 4 m

14 a 13.11 km b N20°47′W

15 a i 8.63 km ii 6.48 km/h
iii 9.90 km
b 22.09 km from A and 27.46 km from B

16 Yes, she needs 43 m altogether.

EXERCISE 5.5

1 7.95

2 55.22 cm

3 28°57′

4 35°32′

5 2218 m

6 28.5 km

7 23.08, 41°53′, 23°7′

8 $A = 61°15′$, $B = 40°$, $C = 78°45′$

9 $\sqrt{37}$ cm

10 a 12.57 km b S35°1′E

11 a 35°6′ b 6.73 m^2

12 23°

13 70°49′

14 89.12 m

15 a 130 km b S22°12′E

16 74.3 km

17 a 8.89 m b 76°59′ c $x = 10.07$ m

18 1.14 km/h

EXERCISE 5.6

1 a $\frac{\pi}{6}$ b $\frac{\pi}{3}$ c $\frac{2\pi}{3}$

d $\frac{5\pi}{6}$ e $\frac{5\pi}{4}$ f $\frac{3\pi}{2}$

g $\frac{7\pi}{4}$ h $\frac{8\pi}{3}$ i $\frac{2\pi}{5}$

j $\frac{10\pi}{9}$

2 a 45° b 270° c 210°

d 300° e 105° f 510°

g 15° h 234° i 247.5°

j 1440°

3 15.88 cm

4 200.28 mm

5 a 0.2667^c b 15°17′

6 a 1.6^c b 91°40′

7 36.75 cm

8 18.28 cm

9 237.66 cm^2

10 5.44 m^2

11 a 0.4712 b 1.9024 c 4.2412

d 6.1261 e 0.1222 f 1.1118

g 2.4147 h 4.7845 i 5.7052

j 0.8209

12 a 134°22′ b 34°58′ c 57°18′

d 92°15′ e 205°48′ f 415°24′

g 10°26′ h 334°37′ i 233°22′

j 353°21′

13 $\frac{15\pi}{2}$

14 7.77 cm

15 35°16′

16 73.3 cm

17 2.20 m

18 $A = 10\pi \text{ cm}^2$

19 a 10 m^2 b 1 m^3

20 a 188.5 cm^2 b 5 cm

21 2.95 m^2

22 6.64 cm^2

6

Simulation, sampling and sampling distributions

6.1 Kick off with CAS

Simulations and sampling with random numbers

1 At a particular high school, all Year 11 students have been given the opportunity to participate in an art and cultural tour of Italy. There are only 20 places available and 56 students apply. The teacher coordinating the tour decides that all students should have an equal chance of joining the group. Use the random number application on your CAS calculator to provide a fair means of selecting the 20 students.

2 a Using CAS, in a lists and spreadsheets application, simulate tossing a die 100 times.

b Graph your results on a dotplot.

c Repeat parts a and b, but this time simulate tossing a coin 500 times.

d Compare the two dotplots.

e Describe the distribution if you tossed a die 10 000 times.

3 Suppose this week's maths test consists of 20 multiple-choice questions. A student who did not study intends to guess all the answers.

a If there are five responses from which to choose, only one of which is correct, what is the likelihood that the student will answer any one question correctly?

b Using CAS and the random number application, simulate guessing the answers to all 20 questions. *Hint:* If there are ten digits (0, 1, 2, . . ., 9), then how many of these digits should represent a correct response?

According to your simulation, how many questions did the student answer correctly?

c Repeat the simulation 20 times. If the student needed to answer 12 questions correctly to pass the test, according to your simulation, what was the probability of this student passing?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

6.2 Random experiments, events and event spaces

In this topic we explore how to use modelling and sampling to inform us about the likelihood of events occurring.

Basic terminology and concepts

Below is a summary of terminology covered in previous years that will be used throughout this topic.

Key word	Definition
Experiment	An activity or situation that occurs involving probability; for example, a die is rolled.
Trial	The number of times an experiment is conducted; for example, if a coin is tossed 20 times, we say there are 20 trials.
Outcome	The results obtained when an experiment is conducted; for example, when a die is rolled, the outcome can be 1, 2, 3, 4, 5 or 6.
Equally likely	Outcomes that have the same chance of occurring; for example, when a fair coin is tossed, the two outcomes (Heads or Tails) are equally likely.
Frequency	The number of times an outcome occurs
Variable	The characteristic measured or observed when an experiment is conducted or an observation is made
Random variable	A variable that has a single numerical value, determined by chance, for each outcome of a trial

Random experiments, events and event spaces

Random experiments deal with situations where each outcome is unknown until the experiment is run. For example, before you roll a die, you don't know what the result will be, although you do know that it will be a number between 1 and 6.

The observable outcome of an experiment is defined as an **event**.

A **generating event** is a repeatable activity that has a number of different possible outcomes (or events), only one of which can occur at a time. For example, tossing a coin results in a Head or a Tail; rolling a single die results in a 1, 2, 3, 4, 5 or 6; drawing a card from a standard deck results in a specific card (e.g. ace of hearts, 8 of clubs). These can be described as the generating event of tossing a coin, the generating event of rolling a die, and the generating event of drawing a card. In each case there is no way of knowing, in advance, exactly what will happen (what the event, or observable outcome, will be), making it a random experiment.

An **event space** (or sample space) is a list of all possible and distinct outcomes for a generating event. This list is enclosed by braces { } and each element is separated by a comma. For example, if a coin is tossed once, the event space can be written as {Head, Tail} or abbreviated to {H, T}. The list of all possible outcomes for a generating event is given the symbol ε (epsilon). If a coin is tossed once, $\varepsilon = \{H, T\}$. If a single die is rolled, $\varepsilon = \{1, 2, 3, 4, 5, 6\}$. Capital letters are used to name other events. For example, $A = \{H\}$ means that A is the event of a Head landing uppermost when a coin is tossed once.

WORKED EXAMPLE 1

A die is rolled. List the elements of the event space and list the elements of *X*, the event of an odd number appearing uppermost.

THINK	WRITE
1 List the elements of the event space for rolling a die; that is, list all possible outcomes.	$\varepsilon = \{1, 2, 3, 4, 5, 6\}$
2 List the elements of X, the event of an odd number appearing uppermost on a roll of a die.	$X = \{1, 3, 5\}$

WORKED EXAMPLE 2

Two dice are rolled. List the elements of the event space and list the elements of *Y*, the event of the same number appearing on each die.

THINK	WRITE
1 List the elements of the event space for rolling two dice; that is, list all possible outcomes.	$\varepsilon = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$
2 List the elements of Y, the event of the same number appearing on each die when two dice are rolled.	$Y = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

WORKED EXAMPLE 3

A card is selected from a deck of 52 playing cards. List the elements of the event space and list the elements of *K*, the event of selecting a king *or* a spade.

***Note:* The joker is not included as one of the 52 playing cards.**

THINK	WRITE
1 List the event space for choosing a card from the deck; that is, list all the possible cards in the deck. Use the abbreviations A, 2, 3, … 10, J, Q, K for the numbers and S, C, H, D for the suits (spades, clubs, hearts, diamonds). *Note:* A, J, Q and K are abbreviations for the ace, jack, queen and king respectively.	ε = {AS, 2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AC, 2C, 3C, 4C, 5C, 6C, 7C, 8C, 9C, 10C, JC, QC, KC, AH, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH, AD, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, 10D, JD, QD, KD}
2 List all the cards that are a king *or* a spade. Note that KS is listed only once.	K = {AS, 2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, KC, KH, KD}

Worked example 3 introduced the idea of multiple events, such as selecting a king or a spade from a deck of 52 playing cards. In mathematics, the terms 'or' and 'and' have very specific definitions.

Let A and B be two events (for example, A is the event of drawing a king and B the event of drawing a spade). Then:

1. A OR B means that either A or B happens, or that *both* happen. You can get a king *or* a spade *or* the king of spades.
2. A AND B means *both must* happen. You must get a king *and* a spade; that is, the king of spades only.

A **random event** is one in which the outcome of any given trial is uncertain but the outcomes of a large number of trials follow a regular pattern. For example, when a die is rolled once, the outcome could be either a 1, 2, 3, 4, 5 or 6, but if it is rolled several thousand times it is evident that about 1 in 6 rolls will turn up a 6. If the die was weighted so that a 6 was more likely than a 2 to land uppermost, then the experiment would be **biased** and no longer random.

EXERCISE 6.2 Random experiments, events and event spaces

PRACTISE

1 WE1 A die is rolled. List the elements of the event space and list the elements of Y, the event of a number greater than 4 appearing uppermost.

2 A card is drawn from a deck of 52 playing cards and the suit is noted. List the elements of the event space and list the elements of Z, the event of a black card being selected.

3 WE2 Two dice are rolled. List the elements of the event space and list the elements of Z, the event of a total of greater than 4 appearing on the dice.

4 A die is rolled and a coin is tossed. List the elements of the event space and list the elements of X, the event of a Head appearing with an even number on the die.

5 WE3 A card is selected from a deck of 52 playing cards. List the elements of the event space and list the elements of P, the event of selecting a jack *or* a spade.

6 A card is selected from a deck of 52 playing cards. Use the list of elements of the event space from question 5 to list the elements of Q, the event of selecting a jack *and* a spade.

CONSOLIDATE

7 A card is drawn from a deck of 52 playing cards and the face value is noted. List the elements of the event space and list the elements of W, the event of a picture card (king, queen or jack) being selected.

8 A coin is tossed and a card selected from a standard pack, with the suit noted. List the elements of the event space, and list the elements of X, the event of a Tail appearing with a red card.

9 Two dice are rolled. List the sample space of D, the event of an odd number being rolled on each die.

10 Two dice are rolled. List the sample space of F, the event of both dice being greater than or equal to 4.

11 A die is rolled and a coin is tossed. List the elements of X, the event of a Head appearing or an even number on the die.

12 A coin is tossed and a card selected from a standard pack, with the suit noted. List the elements of Y, the event of a Tail appearing or a red card.

13 A coin is tossed three times. List the elements of the event space and the elements of D, the event of exactly two coins having the same result.

14 An urn contains four balls numbered 1–4. A ball is withdrawn, its number noted, and is put back in the urn. A second ball is then drawn out and its number noted. List the elements of the event space, and list the elements of Q, the event of the second ball having a value greater than the first ball.

15 An urn contains four balls numbered 1–4. A ball is withdrawn and its number noted. A second ball is then drawn out and its number noted (without replacement of the first ball). List the elements of the event space, and list the elements of R, the event of an odd number on both balls.

16 The three hearts picture cards are drawn at random. List the elements of the event space and list the elements of K, the event of a king being drawn first or a jack being drawn last.

MASTER

17 A card is drawn from a standard pack of 52 cards. List the elements of Q, the event of a red picture card or an even spade.

18 The event in question 17 could be described as (red card AND picture card) OR (even card AND spade). List the elements of R, the event of (red card OR picture card) AND (even card OR spade).

6.3 Simulation

Simulation is an activity employed in many areas, including business, engineering, medical and scientific research, and in specialist mathematical problem-solving activities, to name a few. It is a process by which experiments are conducted to model or imitate real-life situations. Simulations are often chosen because the real-life situation is dangerous, impractical, or too expensive or time consuming to carry out in full. An example is the training of airline pilots.

In this section we examine the basic tools of simulation and the steps that need to be followed for a simulation to be effective. We also look at various types of simulation.

Random numbers

Simulations often require the use of sequences of random numbers.

Consider the following result when rolling a die 12 times:

5, 3, 1, 4, 4, 3, 6, 1, 6, 4, 1, 3.

This sequence of numbers can be treated as a set of **random numbers**, whose possible values are 1, 2, 3, 4, 5, 6.

A deck of playing cards could be numbered from 1 to 52; by drawing a card, then replacing it in the deck, shuffling the deck and drawing another card, a sequence of random numbers between 1 and 52 would be generated. There are several other ways of generating random numbers; can you think of any?

There are three general rules that a set of random numbers must follow.

Rule 1 The set must be within a defined **range** (for example, whole numbers between 1 and 10, or decimals between 0 and 1). The range does not need to have a definite starting and finishing number, but in most cases it does.

Rule 2 All numbers within the defined range must be possible outcomes (for example, using a die but not counting 4s is not a proper sequence of random numbers between 1 and 6).

Rule 3 Each random number in a sequence is independent of any of the previous (or future) numbers in the sequence. (This is difficult to prove but is assumed with dice, spinners and so on.)

These rules can make it extremely difficult to obtain a proper set of random numbers in practice. Furthermore, to generate many random numbers, say 1000, by rolling dice could take a long time! Fortunately, computers can be used to generate random numbers for us. (Technically, they are known as pseudo-random numbers, because rules 2 and 3 above cannot be rigorously met.)

WORKED EXAMPLE 4 **Use a sequence of 20 random numbers to simulate rolling a die 20 times. Record your results in a frequency table.**

THINK

1. Using CAS, go to the random integer function.
2. Select 20 random integers between 1 and 6 inclusive and store them in a list.
3. Use CAS to count the number of times each element appears in the list.
   Note: Naturally, the values shown may differ each time this sequence is repeated.
4. Enter this information in a table as shown.

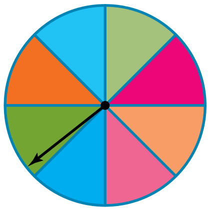
WRITE

Die value	1	2	3	4	5	6
Frequency	2	4	6	2	5	1

Basic simulation tools

The simulation option that you select may depend on the type and number of events or on the complexity of the problem to be simulated.

1. Coin tosses can be used when there are only two possible, equally likely, events — for example, to simulate the results of a set of tennis where the two players are equally matched. In this case each toss could represent a single game, the simulation ending when either 'H' or 'T' has enough games to win the set.
2. Spinners can be used when there are three, four, five or more possible, equally likely, events. Provided that your spinner is a fair one, where each sector has an equal area, this is an effective simulation device.



3. Dice — One die can be used to simulate an experiment with six equally likely outcomes. One die can also be used to simulate three equally likely outcomes by assigning a 1 or a 2 to the first outcome, a 3 or a 4 to the second outcome and a 5 or a 6 to the third outcome. Two or more dice can be used for more complex situations. For example, two dice can be used to simulate the number of customers who enter a bank during a 5-minute period.
4. Playing cards can be used to simulate extremely complicated experiments. There are 52 cards, arranged in 13 values (A, 2, 3, …, J, Q, K) and in 4 suits (spades, clubs, hearts, diamonds); they can be defined to represent all kinds of situations.
5. Random number generators are the most powerful simulation tool of all. Computers can tirelessly generate as many numbers in a given range as you wish. They are available on most scientific or graphics calculators (randInt), spreadsheets (randbetween) and computer programming software.

Steps to an effective simulation

Step 1 Understand the situation being simulated and choose the most effective simulation tool.

Step 2 Determine the basic, underlying, assumed probabilities, if there are any.

Step 3 Decide how many times the simulation needs to be repeated. The more repetitions, the more accurate the results.

Step 4 Perform the simulation, displaying and recording your results.

Step 5 Interpret the results, stating the resultant simulated probabilities. Sometimes you can repeat the entire simulation (steps 3 and 4) several times and compute 'averages'.

WORKED EXAMPLE 5 **Simulate the result of a best-of-3-sets match of tennis using simple coin tosses.**

THINK / **WRITE**

1 Understand the situation. The winners of games are recorded, not the winners of points.

We will need, at most, 13 coin tosses per set for, at most, 3 sets, assuming that if a set reaches 6–6 it will be resolved by a tie breaker.

2 Determine the probabilities.

Assume evenly matched players:
Player A = 'H', Player B = 'T'
(H represents Player A winning; T represents Player B winning.)

3 Decide how many times to simulate.

We will simulate one best-of-3-sets match.

4 Perform the simulation.

Toss	Result	Score	
H	A	1–0	
T	B	1–1	
H	A	2–1	
H	A	3–1	
H	A	4–1	
T	B	4–2	
H	A	5–2	
T	B	5–3	
H	A	6–3	Player A wins Set 1.
T	B	0–1	
T	B	0–2	
T	B	0–3	
T	B	0–4	
T	B	0–5	
T	B	0–6	Player B wins Set 2.

Toss	Result	Score	
T	B	0–1	
H	A	1–1	
H	A	2–1	
T	B	2–2	
H	A	3–2	
T	B	3–3	
H	A	4–3	
T	B	4–4	
T	B	4–5	
H	A	5–5	
T	B	5–6	
H	A	6–6	Tie breaker required.
T	B	6–7	Player B wins match.

5 Interpret your results.

28 coin tosses were required to simulate 1 match. Player A won 12 games, Player B won 16; thus we could say that Player A has a probability of winning of $\frac{12}{28} = 0.4286$ based upon this small simulation. However, since coins were used, 'true' probability = 0.5.

EXERCISE 6.3 Simulation

PRACTISE

1 WE4 Use a sequence of 6 random numbers to simulate rolling a die 9 times. Record your results in a frequency table.

2 Generate 100 numbers between 10 and 20. Record your results in a frequency table.

3 WE5 Simulate the result of a best-of-5-sets tennis match using a simple coin toss.

4 Use a die to simulate the two-player game described by the table at right. Perform enough simulations to convince yourself about which player has the better chance of making a profit.

Die value	Outcome
1	Player A wins \$1
2	Player A wins \$2
3	Player A wins \$4
4	Player A wins \$8
5	Player A wins \$16
6	Player B wins \$32

CONSOLIDATE

5 **a** Suggest how a graphics calculator could be used to simulate the tossing of a coin.
b Generate 10 'coin tosses' using this method.
c Sketch a frequency histogram for your results.

6 Repeat question **5** for 100 coin tosses. What do you notice about the histogram?

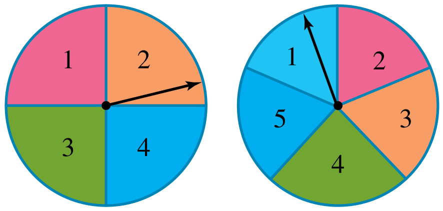
7 A mini-lottery game may be simulated as follows. Each game consists of choosing two numbers from the whole numbers 1 to 6. The cost to play one game is $1. A particular player always chooses the numbers 1 and 2. A prize of $10 is paid for both numbers correct. No other prizes are awarded.

a Simulate a game by generating two random numbers between 1 and 6. Do this 20 times (that is, 'play' 20 games of lotto). How many times does the player win?

b What is the player's profit/loss based on the simulation?

8 Simulate 20 tosses of two dice (die 1 and die 2). How many times did die 1 produce a lower number than die 2? (*Hint:* Generate two lists of 20 values between 1 and 6.)

9 A board game contains two spinners pictured at right.



a Simulate 10 spins of each spinner.

b How many times does the total (that is, both spinners' numbers added) equal 5?

c How often is there an even number on both spinners?

d How often does the highest possible total occur?

10 Use dice, a spreadsheet or other means to generate data that simulate the arrival and departure of customers from a bank according to the rules in the table at right (assume there are 10 customers to begin with).

Each total represents what happens over a 5-minute interval. Perform enough simulations to cover at least 2 hours' worth of data. Discuss your results.

Dice total	Outcome
2	1 customer leaves
3	1 customer arrives
4	2 customers leave
5	2 customers arrive
6	3 customers leave
7	3 customers arrive
8	4 customers leave
9	4 customers arrive
10	5 customers leave
11	5 customers arrive
12	6 customers leave

11 Design a spreadsheet to simulate the tennis match from Worked example 5. Create a version for a best-of-3-sets match and a best-of-5-sets match.

12 In the Smith Fish & Chip shop, the owner, Mary Jones, believes that her customers order various items according to the probabilities shown in the table.

Set up a spreadsheet to simulate the behaviour of 100 customers. How many orders of fish only and how many orders of chips only will Mary prepare?

Order	Probability
Chips only	0.20
1 fish and 1 chips	0.15
2 fish and 1 chips	0.26
1 fish only	0.14
2 fish only	0.11
3 fish and chips	0.14

13 Use a spreadsheet to simulate the tossing of 4 coins at once. Perform the simulation enough times to convince yourself that you can predict the probabilities of getting 0 Heads,1 Head, …, 4 Heads. Compare your results with the 'theoretical' probabilities.

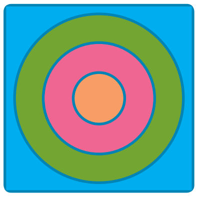
14 The data below show the number of bullseyes scored by 40 dart players after 5 throws each.

1	4	0	3	4	2	1	5	4	2
3	0	4	5	2	1	2	3	2	1
0	2	1	4	3	5	3	2	4	4
0	2	1	0	3	5	4	2	3	1

a Explain how a die or a calculator can be used to obtain the range of numbers given in the table.
b What proportion of players scored at least 3 bullseyes?
c Using a die (or by some other means) conduct 20 trials and obtain another possible value for the proportion of players who scored at least 3 bullseyes.
d Comment on your results.

MASTER

15 Use a spreadsheet to simulate the following situation. A target consists of 3 concentric circles. Let the smallest circle have a radius of r_1 cm, the next r_2 cm and the largest r_3 cm. They sit on a square board $y \times y$ cm ($y \geq 2r_3$).



Start with some numeric examples, say $r_1 = 5$, $r_2 = 10$, $r_3 = 15$ and $y = 32$.

If you 'hit' a circle, you score points: the inner circle, 5 points; the middle circle, 3 points; the outer circle, 1 point; and you score no points for hitting the board outside the target.

a Perform enough simulations so that you can predict the 'expected' number of points per throw.
b Experiment with different values of the radii and board size, and tabulate your results.
c Experiment with different scoring systems and tabulate your results.
d Can you calculate the 'theoretical' probabilities and expected scores?

Hints on setting up simulation
1. Generate two random numbers which represent x- and y-coordinates with the target at (0, 0).
2. For each random pair, calculate the distance from the origin and compare it to r_1, r_2 and r_3.
3. Allocate points appropriately.

16 A football league has 8 teams. Each team plays all the other teams once. Thus there are 28 games played in all.

a Simulate a full season's play, assuming that each team has a 50 : 50 chance of winning each game.

b Modify the probabilities so that they are *unequal* (*hint:* sum of probabilities = 4) and simulate a full season's play. Did the better teams reach the top of the ladder? Discuss your results with other students.

Hint: If Team 1 has a probability of winning of 0.7 and Team 2 has a probability of winning of 0.6, then when they play against each other, the probability of Team 1 winning is $\frac{0.7}{0.7 + 0.6}$.

6.4 Populations and samples

In statistics, objects or people are measured. For example, we could measure the heights of a group of 30 Year 11 students from a particular school. We could then find the average height of the students in the group.

However, we could measure the heights of the same group of students and for each student ask 'Is this student over 1.6 m tall?'. The answer to this question, although height was measured, is actually 'Yes' (that is, taller than 1.6 m) or 'No' (that is, shorter than 1.6 m). This sort of measurement is called an **attribute**, and subjects either have an attribute or do not have it. By counting those who have the attribute, we can find the **proportion** of the group with the attribute. In our example, we could find the proportion of Year 11 students who are over 1.6 m tall.

study on

Units 1 & 2

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Topic 1

Concept 1

Populations and samples
Concept summary
Practice questions

Populations and samples

It is important to distinguish between the **population**, which covers, in some way, everything or everyone in a grouping, and the **sample**, which comprises the members of the group that we actually measure for the attribute. In the above example, all the Year 11 students in the school make up the Year 11 population. The 30 students whose heights were measured represent a sample from that population.

When analysing data, we must be careful in drawing conclusions. It is crucial that the sample comes from the true population that it presumes to represent. For example, it is no good taking a sample of students from one school only and then making statements about all Australian students. Unless the sample is the entire population (which is very rare), knowledge about the population remains unknown. Suppose we measure the heights of 100 students in Tasmania and find that 45 are over 1.6 m tall. We cannot then make statements about the true proportion of all Australian students with this attribute, or even all Tasmanian students; we can say only that our sample proportion of 0.45 may be close to the true proportion. This true proportion is given the name **population proportion**, p. We can make positive statements only about the sample that we took, and hope that it is representative of the population. In reality, this is usually what happens if our sample is properly selected.

WORKED EXAMPLE 6

A scientist wishes to study pollution in a river. She takes a sample of 42 fish to see what proportion have mercury poisoning and finds that 12 are poisoned. What proportion of fish have mercury poisoning?

THINK	WRITE
1 Write down the rule for proportion.	$\text{Proportion} = \frac{\text{number who have an attribute}}{\text{total number sampled}}$

2 Write down the given values for the total number of fish sampled and the number of fish poisoned.	Total number of fish sampled = 42 Number of fish poisoned = 12
3 Substitute the given values into the rule.	$\text{Proportion} = \dfrac{12}{42}$
4 Evaluate and round off the answer to 3 decimal places.	$= \dfrac{2}{7} \; (\approx 0.286)$
5 Answer the question.	Approximately 0.286 (or 28.6%) of the fish were poisoned.

Simulating proportions

The Education Department wishes to know the proportion of students studying Specialist Mathematics. The table below illustrates a random sample of 50 students, where SM represents a student studying Specialist Mathematics and N represents a student not studying Specialist Mathematics.

study on

Units 1 & 2
AOS 6
Topic 1
Concept 2

Population and sample proportions
Concept summary
Practice questions

N	N	SM	N	N
N	N	SM	N	N
N	N	SM	SM	SM
SM	N	SM	SM	N
SM	N	N	SM	N
N	SM	N	N	N
N	N	SM	SM	N
N	N	N	N	N
N	SM	N	SM	N
N	N	SM	SM	SM

In order to determine the proportion of the sample studying Specialist Mathematics, we simply count the number of SMs (18), and compute the proportion as $\dfrac{18}{50} = 0.36$.

In this example we don't know the population proportion but can find the sample proportion and use it to make some statement about the population proportion. The strongest statement that could be made here is:

> Of a sample of 50 students, it was found that 36% (or 0.36) studied Specialist Mathematics.

This is known as the **sample proportion**, and is symbolised as $\hat{p}$ (read as 'p hat').

A more interesting situation arises when we do know the population proportion and are interested in the outcomes of one or more samples from such a population.

WORKED EXAMPLE 7

Toss two dice 80 times, record the total sum of each outcome and find the sample proportion of the number of times the total is a prime number.

THINK

1 Toss two dice 80 times and record the total sum of each outcome.

2 Highlight the results that are prime numbers, that is, 2, 3, 5, 7, 11.

Note: As this is an experiment, the results will differ each time a sample of 80 tosses is collected.

WRITE

6	9	8	8	8	10	5	11
4	12	9	6	7	12	6	4
8	6	2	7	9	9	9	10
6	6	8	8	5	9	3	7
7	11	8	6	5	7	6	9
3	5	12	11	9	2	6	6
8	9	7	4	8	8	8	6
5	10	8	6	11	12	8	5
11	6	11	5	6	5	3	6
7	11	4	7	11	6	6	7

3 Write down the rule for proportion.

$$\text{Proportion} = \frac{\text{number who have an attribute}}{\text{total number sampled}}$$

4 Write down the given values for the total number of tosses and the number of prime numbers.

Total number of tosses = 80
Number of prime numbers = 30

5 Substitute the given values into the rule.

$$\text{Proportion} = \frac{30}{80}$$

6 Evaluate and give answer to 3 decimal places.

$$= \frac{3}{8}$$
$$= 0.375$$

7 Answer the question.

In the above sample of 80 tosses 0.375 (or 37.5%) of the numbers were prime numbers.

The experiment in Worked example 7 was repeated another two times and the following results were obtained:

Experiment 2: $\frac{33}{80} \approx 0.413$

Experiment 3: $\frac{34}{80} = \frac{17}{40}$

≈ 0.425

Our experimental values compare fairly well with the theoretical value.

From theoretical probability we 'know' that the probability (or proportion) of a prime number is $\frac{15}{36}$ (or 0.417) as illustrated below.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

As demonstrated by our two additional experimental results, in comparing the result from Worked example 7 with your own and classmates' results, each will give a different answer. This is the risk one takes with sampling — you can make definite statements only about the sample and not about the entire population. In this case the entire population represents *all the totals of pairs of dice everywhere and for all time* — an impossible thing to calculate.

If Worked example 7 was repeated an additional number of times the following may be observed:

> **The sample proportion rarely equals the population proportion but is generally close to it.**
>
> **The sample proportion is a random variable whose value can not be predicted but lies between 0 and 1.**

Simulation by spreadsheet

In many of the questions and examples of the previous section, a spreadsheet could have been used to create a simulation of a situation with a known proportion. This is very easy to do, and relies on the built-in spreadsheet function **=RAND()**. The example below simulates a sample of babies in a hospital, given that the proportion of boys is 0.495.

A	B	C	D
3			
4		p	0.495
5		Yes	B
6		No	G
7			
8		G	G
9		B	G
10		G	B
11		B	G
12		B	G
13		G	G
14		G	B
15		G	G
16		B	G
17		G	G

A	B	C	D
3			
4		p	0.495
5		Yes	B
6		No	G
7			
8		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
9		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
10		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
11		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
12		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
13		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
14		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
15		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
16		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)
17		=IF(RAND()<=$D$4,$D$5,$D$6)	=IF(RAND()<=$D$4,$D$5,$D$6)

This spreadsheet works as follows.

Cell D4 contains the known proportion of an attribute (in this case, sex).

Cell D5 contains the symbol used to represent a person (or thing) with the attribute.

Cell D6 contains the symbol used to represent a person (or thing) without the attribute.

The formula in Cell C8 **=IF(RAND()<=$D$4,$D$5,$D$6)** is **FilledDown** and **FilledRight** as many times as you wish to create a 'table' of samples. The formula compares the value of the random number generator (**RAND()**) with the proportion in Cell D4. If it is less than or equal to the proportion value, it uses the symbol from D5, otherwise it uses the symbol from D6.

Sample means

When numerical data is recorded, we use a capital letter such as X to name the variable, for example the time take to run 100 m. Lower-case letters are used for the individual results. In Worked example 8 below, we could refer to the data as $x_1 = 11.48$, $x_2 = 12.09$ and so on. We say that the sample size is n. For Worked example 8, $n = 30$.

The mean of a sample, $\bar{x}$, can be found using $\bar{x} = \frac{\Sigma x_i}{n}$. This is the sample statistic. The corresponding population parameter is the population mean, μ.

WORKED EXAMPLE 8

How fast can the average under-16 boy run 100 m? A quick internet search (for 'U16 100 m times; Australian sites only') resulted in the following 30 times recorded from two different athletics carnivals.

11.48	12.09	12.37	12.41	12.53	12.7
12.73	12.87	13.16	13.37	11.32	11.39
11.50	11.61	11.67	11.78	11.71	11.72
11.82	11.83	11.85	12.10	12.15	12.19
12.31	12.43	13.07	10.67	10.88	11.02

a Find the sample mean correct to 2 decimal places.

b Can you make any conclusions about how fast the average under-16 boy can run 100 m from this sample?

THINK

a 1 Write down the formula for the sample mean.

2 Find the total of the times.

3 Find the average time.

b These times have been published. How representative of under-16 boys' running skills could they be?

WRITE

a $\bar{x} = \frac{\Sigma x_i}{n}$

$\Sigma x = 11.48 + 12.09 + 12.37 + \ldots + 10.67 + 10.88 + 11.02$

$= 360.73$

$\bar{x} = \frac{360.73}{30}$

$= 12.02$

The average time for the sample is 12.02 seconds.

b The results have been published and are therefore from various carnivals. Most under-16 boys do not have the opportunity to compete at these carnivals. Therefore, the sample does not reflect the general population.

EXERCISE 6.4 Populations and samples

PRACTISE

1 WE6 In a random sample of male students at Heartbreak High, it was found that 12 out of 47 were blond. What proportion of males is this?

2 In the sample of male students at Heartbreak High in question 1, it was found that 16 of them had blue eyes. What proportion of males is this?

3 WE7 Toss two dice 80 times, record the total sum of each outcome and find the sample proportion of the number of times the total is a perfect square.

4 Toss a die and a coin 50 times, record each outcome and find the sample proportion of the number of times a Head and 6 occur together.

5 **WE8** The following times were recorded at the 2014 Australian All Schools Championship for the under-16 girls 200 m. Find the sample mean.

24.42	24.89	25.29	24.95	25.31
25.80	25.17	25.45	25.80	25.86
25.99	26.09	26.54	26.61	26.79
27.02	27.03	27.42	28.30	28.36
24.57	24.70	24.87	25.12	25.41

6 Over the last decade, 30 men have represented Australia in Test Cricket matches against England. The number of 50s they have scored are recorded below. Find the sample mean.

7	2	8	6	3	2
1	3	1	1	3	0
2	1	0	1	1	0
1	0	0	0	0	0
0	0	0	0	0	0

CONSOLIDATE

7 Which of the following are populations, and which are samples?
- a The number of trout in a lake
- b The weights of 50 Year 7 students
- c The number of letters mailed yesterday
- d The letters delivered by Mr J. Postman
- e The grades received by 15 students
- f The voters of Australia

8 From a sample of 124 oranges, it was found that 31 were spoiled. Calculate the sample proportions of spoiled and unspoiled oranges.

9 In a group of 35 Year 11 students, 10 studied Mathematical Methods, 11 studied Specialist Mathematics and 14 studied no mathematics subjects. Find the sample proportion of students who studied mathematics.

10 At Wombat University there are 240 students studying Engineering, 31 of whom are girls.
- a Find the percentage of girls studying Engineering correct to 1 decimal place.
- b Is this a population or a sample of all students at Wombat University?
- c What can you conclude about the proportion of girls at Wombat University?

11 A manufacturer wishes to know what proportion of his transistor radios are defective. He takes a sample of 60 and finds 5 are defective.
- a What proportion of the sample was defective?
- b What can you say about this manufacturer's defective rate?

12 Two statisticians wish to know the proportion of cars that have CD players. Statistician A takes a sample of 40 cars and finds that 12 have CD players, while statistician B takes a sample of 214 cars and finds that 71 have CD players.
- a Find the sample proportions.
- b Comment on who is more likely to be close to the population proportion.

13 It is known that 24% of all cars in Victoria were made in Japan. A sample of 60 cars from the suburb of Frankston showed the following result (J = car made in Japan).

N	N	N	N	J	N	J	N	N	N
J	J	J	N	N	J	N	N	J	N
N	N	N	N	J	N	N	N	N	N
N	N	J	J	J	J	N	J	N	J
N	N	N	N	N	N	N	N	N	J
N	J	N	N	N	N	N	N	J	J

a Find $\hat{p}$, the proportion of cars made in Japan.
b Is this a representative (that is, a good) sample?

Another statistician samples 60 cars, this time using 2 cars each from 30 different suburbs (or towns) in Victoria. She obtains the following table.

N	J	J	J	N	N	N	N	N	J
N	N	N	J	N	N	J	N	N	J
N	N	N	N	N	J	N	J	N	N
N	N	N	N	J	J	J	J	N	N
N	J	J	N	J	J	N	N	N	N
N	N	N	N	J	N	N	N	N	J

c Find $\hat{p}$, the proportion of cars made in Japan.
d Is this a representative (that is, a good) sample?

14 A pair of dice is used to simulate the following situation: let a value of 7, 8 or 10 represent a person who suffers from asthma. A sample of 40 people yields the following table.

5	11	8	8	6	3	7	9
6	9	4	6	2	10	4	6
10	8	5	8	4	6	8	6
10	6	3	11	7	5	6	7
7	9	5	8	5	8	2	7

a Find the sample proportion of asthma sufferers.
b Compare this with the theoretical result (assuming that rolling a 7, 8 or 10 reflects the proportion of asthma sufferers).

15 Over the last decade, the following highest innings scores by Australian cricketers in Test matches against England have been recorded. Find the sample mean, correct to 2 decimal places.

195	148	136	103	196	124
115	119	153	100	64	156
102	71	43	53	55	31
50	39	43	18	37	34
26	29	16	8	7	2

16 Ailsa wants to borrow $400 000 to pay back over 25 years. She is interested in the interest rate that she might be charged. She decides to survey 10 banks. She recorded the following rates. Find the sample mean.

6.50%	5.63%	5.7%	4.70%	5.65%
5.91%	5.74%	4.43%	5.39%	4.39%

MASTER

17 It is said that the average song length is 3 minutes. Find a suitable representative sample of songs and decide if you agree with the statement.

18 Ashleigh is a newly appointed sports coordinator at a school. Her first task is to organise the school swimming carnival. Buses need to be ordered to transport the students to and from the school, so timing of events is important. Detail how she could find information about race times.

6.5 Distribution of sample proportions and means

studyon

Units 1 & 2
AOS 6
Topic 1
Concept 3

Population and sample means
Concept summary
Practice questions

As we observed in the previous section, a sample proportion or sample mean may or may not be close to the population proportion, p, or population mean, μ. Theory states that if we get many sample statistics then the average of these statistics will be much closer to the population parameters. Why? The answer is quite simple: by repeating the sampling we are effectively taking one large sample. That is, 10 samples of 12 objects each is roughly equivalent to a single sample of 120 (assuming that the population is significantly larger than 120 and that all sampling is random).

Simulation examples

WORKED EXAMPLE 9

The Department of Health wishes to know how many patients admitted to the hospital have private health insurance and decides on the following sampling technique. For each of 10 hospitals, H, in a city, a sample of 12 patients, P, will be taken. The number of these patients with private insurance, I, will be counted and recorded (see the table below).

Estimate the population proportion of those with private insurance, using the information in the table.

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12
H1	I	N	I	N	I	N	N	N	N	N	I	I
H2	I	N	I	N	N	N	N	N	N	N	I	I
H3	N	N	I	N	I	N	N	I	N	I	N	N
H4	I	N	N	I	N	N	N	N	N	N	I	N
H5	N	I	N	N	I	I	I	I	I	N	N	N
H6	N	N	N	N	I	I	I	N	N	I	N	N
H7	N	I	N	N	N	I	I	N	N	I	N	N
H8	N	N	N	N	I	I	I	I	N	N	N	N
H9	N	N	N	N	N	N	N	N	I	I	I	N
H10	N	N	N	N	I	N	N	I	N	N	N	I

THINK	WRITE
1 Count the number of Is from each of the 10 hospitals.	Hospital 1 count = 5; Hospital 2 count = 4 Hospital 3 count = 4; Hospital 4 count = 3 Hospital 5 count = 6; Hospital 6 count = 4 Hospital 7 count = 4; Hospital 8 count = 4 Hospital 9 count = 3; Hospital 10 count = 3
2 Write down the rule for proportion.	$\text{Proportion} = \dfrac{\text{number who have an attribute}}{\text{total number sampled}}$
3 Calculate the sample proportion for each hospital. Evaluate and round off each answer to 3 decimal places.	Hospital 1 $\hat{p} = \dfrac{5}{12} = 0.417$ Hospital 2 $\hat{p} = \dfrac{4}{12} = 0.333$ Hospital 3 $\hat{p} = \dfrac{4}{12} = 0.333$ Hospital 4 $\hat{p} = \dfrac{3}{12} = 0.250$ Hospital 5 $\hat{p} = \dfrac{6}{12} = 0.500$ Hospital 6 $\hat{p} = \dfrac{4}{12} = 0.333$ Hospital 7 $\hat{p} = \dfrac{4}{12} = 0.333$ Hospital 8 $\hat{p} = \dfrac{4}{12} = 0.333$ Hospital 9 $\hat{p} = \dfrac{3}{12} = 0.250$ Hospital 10 $\hat{p} = \dfrac{3}{12} = 0.250$
4 Calculate the average $\hat{p}$.	Average $\hat{p}$ = $\dfrac{0.417 + 0.333 + 0.333 + 0.25 + 0.5 + 0.333 + 0.333 + 0.333 + 0.25 + 0.25}{10}$
Method 1 Add each sample proportion and divide by 10.	$= \dfrac{3.332}{10}$ $= 0.3332$
Method 2 (a) Obtain the sum of the total counts and the sum of the total samples.	Total counts $= 5 + 4 + 4 + 3 + 6 + 4 + 4 + 4 + 3 + 3$ $= 40$ Total samples $= 12 \times 10$ $= 120$

(b) Divide the sum of the total counts by the sum of the total samples.

Note: It is better to calculate $\hat{p}$ using method 2 and avoid any inaccuracies which may be incurred by the rounding-off process of method 1.

$$\text{Average } \hat{p} = \frac{40}{120}$$
$$= 0.3333$$

5 Answer the question. An estimate of the population proportion using the 10 samples is 0.3333 (or 33.33%).

To average sample proportions, each individual sample *must* be of the *same size*. In Worked example 9 above, each sample had 12 patients.

WORKED EXAMPLE 10

To try to find the mean length of a television show on Foxtel, 12 channels (C) were selected. The lengths of the first 10 shows (S) starting after 10 am were recorded (in minutes). Estimate the population mean program length.

C	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
101	55	60	60	60	60	55	5	30	30	30
103	35	30	30	60	60	130	35	60	30	35
105	85	30	50	55	60	60	55	25	55	30
108	30	30	30	30	30	30	30	30	30	30
111	30	30	30	30	30	25	25	25	25	25
112	60	60	60	60	60	60	60	30	30	60
113	30	60	60	60	90	60	35	35	35	30
114	55	55	60	35	35	55	55	55	50	55
115	85	60	60	30	30	60	60	60	60	60
116	30	30	30	45	45	45	45	45	45	45
117	70	70	130	35	35	70	70	35	35	70
118	60	105	60	120	60	60	60	60	60	105

THINK | WRITE

1 Write down the formula for mean.

$$\bar{x} = \frac{\Sigma x_i}{n}$$

2 Find the mean for each channel, each of which is a sample.	Channel 101: $\bar{x} = \frac{445}{10} = 44.5$ Channel 103: $\bar{x} = \frac{505}{10} = 50.5$ Channel 105: $\bar{x} = \frac{505}{10} = 50.5$ Channel 108: $\bar{x} = \frac{300}{10} = 30$ Channel 111: $\bar{x} = \frac{275}{10} = 27.5$ Channel 112: $\bar{x} = \frac{540}{10} = 54$ Channel 113: $\bar{x} = \frac{495}{10} = 49.5$ Channel 114: $\bar{x} = \frac{510}{10} = 51$ Channel 115: $\bar{x} = \frac{565}{10} = 56.5$ Channel 116: $\bar{x} = \frac{405}{10} = 40.5$ Channel 117: $\bar{x} = \frac{620}{10} = 62$ Channel 118: $\bar{x} = \frac{750}{10} = 75$
3 Calculate the average $\bar{x}$.	Average $\bar{x} = \frac{44.5 + 50.5 + 50.5 + 30 + 27.5 + 54 + 49.5 + 51 + 56.5 + 40.5 + 62 + 75}{12}$ $= \frac{591.5}{12}$ $= 49.3$
4 Answer the question.	An estimate of the population mean is 49.3 minutes.

Finding the average of sample means is only valid if the samples are of the same size.

Frequency dotplots

Consider the data from Worked examples 9 and 10. Each sample proportion we obtained could have been graphed using a dotplot. This would give us a pictorial view of the distribution of the sample proportions. The pictorial view is called a **frequency dotplot**. In a similar fashion, the frequency dotplots for the distribution of values of $\bar{x}$ can be constructed.

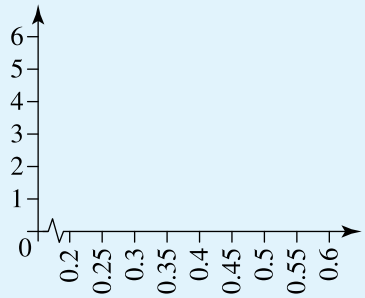
WORKED EXAMPLE 11 Construct a frequency dotplot for the distribution of values of $\hat{p}$ from Worked example 9.

THINK

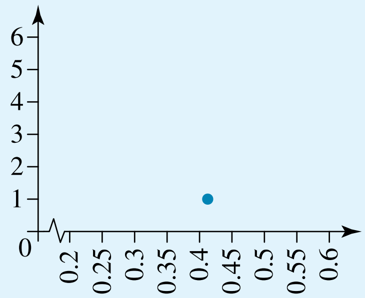
1 Draw a horizontal line which allows for the range of proportions to be represented (that is, 0.2 to 0.6).

2 Draw a vertical line which allows for the frequency of proportions to be represented (that is, 0 to 6).

WRITE/DRAW

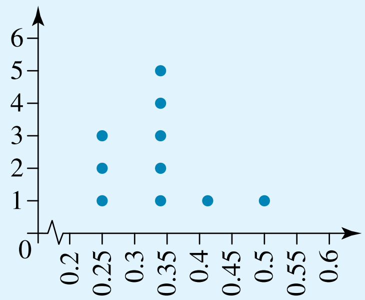


3 Plot the first sample proportion (0.417).



4 Repeat for the remaining 9 sample proportions.

Note: Each dot represents 1 sample's proportion. Thus there were 3 samples with a proportion of 0.25, 5 samples with a proportion of 0.33 and so on.



At first glance at the results for Worked example 11, it appears that the population proportion lies somewhere between about 0.25 and 0.35, since 8 of the 10 sample proportions were in this range. In fact, the samples were taken from a population where the population proportion was 0.37.

EXERCISE 6.5 Distribution of sample proportions and means

PRACTISE

1 WE9 The following data come from five samples of 10 students each. If a student studied Specialist Mathematics, an M appears in the box; otherwise an X appears. Using the information in the table, estimate the population proportion.

Sample 1	M	M	X	M	X	X	M	X	M	X
Sample 2	X	M	M	M	X	M	X	M	X	M
Sample 3	X	X	X	X	X	M	X	X	M	M
Sample 4	M	X	M	X	M	X	X	M	X	M
Sample 5	X	M	X	M	X	X	M	X	M	X

2 Twenty shoe stores are surveyed to find out if either ladies' shoes (L) or men's shoes (M) have been purchased. Each store provides a random sample of 15 sales. The data are shown in the table below. Estimate the population of ladies' shoes from the 20 samples.

Sample 1	M	L	L	M	L	L	L	M	L	L	L	M	L	L	M
Sample 2	L	L	L	L	L	L	L	L	M	L	M	M	L	M	M
Sample 3	M	M	L	M	M	L	L	L	L	L	L	L	M	M	L
Sample 4	M	L	M	M	M	L	L	M	L	M	M	L	M	L	M
Sample 5	M	L	M	M	M	L	M	L	M	M	L	L	L	L	L
Sample 6	L	L	L	L	M	M	L	L	L	L	M	M	L	L	L
Sample 7	L	L	M	L	L	L	L	L	M	L	M	L	L	L	L
Sample 8	L	L	M	M	L	M	L	L	M	L	M	L	L	M	L
Sample 9	L	L	L	L	M	L	M	M	M	M	L	M	M	L	M
Sample 10	M	L	L	M	L	M	L	M	M	M	L	M	L	L	L
Sample 11	L	L	L	L	L	L	L	M	M	M	L	L	M	L	M
Sample 12	L	L	L	L	L	L	L	L	L	L	L	M	L	L	M
Sample 13	L	M	M	L	L	L	L	M	L	M	L	M	M	M	L
Sample 14	M	M	M	M	L	M	M	M	M	L	M	M	L	M	L
Sample 15	M	L	M	M	L	L	L	L	M	M	L	M	M	M	L
Sample 16	L	M	M	L	L	L	L	M	L	M	M	M	M	M	L
Sample 17	M	M	L	L	M	L	L	L	M	M	M	L	L	L	M
Sample 18	M	L	M	L	M	L	L	L	L	M	M	L	L	L	L
Sample 19	L	L	L	L	M	M	M	M	M	L	L	M	L	M	L
Sample 20	L	M	M	L	L	L	M	M	M	L	L	M	L	L	L

3 WE10 Diego decides to measure his mean drive time to work. He measures the time taken every day for 7 weeks and records the time in minutes. Estimate the population mean drive time.

Week	Monday	Tuesday	Wednesday	Thursday	Friday
1	92	43	41	39	35
2	118	81	46	51	38
3	62	48	46	41	49
4	82	48	42	43	41
5	78	51	42	41	38
6	63	62	41	43	44
7	55	41	46	41	32

4 Millicent wants to know the mean movie length for children's movies. She records the lengths for 8 movies every day for a week. Her results are recorded in minutes. Estimate the population mean movie length, correct to 1 decimal place.

Day	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5	Movie 6	Movie 7	Movie 8
Monday	115	95	105	95	115	100	90	95
Tuesday	95	85	90	90	105	95	75	95
Wednesday	110	95	80	110	95	90	105	80
Thursday	95	100	90	95	105	100	90	85
Friday	105	95	90	100	105	100	90	105
Saturday	90	85	90	110	80	100	90	90
Sunday	105	100	100	95	90	90	90	110

5 WE11 Construct a frequency dotplot for the distribution of values of $\hat{p}$ from question 1.

6 Construct a frequency dotplot for the distribution of values of $\hat{p}$ from question 2.

CONSOLIDATE

Questions 7 to 10 use the following table, a simulation of the total obtained when a pair of dice were tossed. There were 9 people who each tossed the dice 8 times.

	Toss 1	Toss 2	Toss 3	Toss 4	Toss 5	Toss 6	Toss 7	Toss 8
Player 1	7	6	10	7	11	2	10	8
Player 2	8	9	8	6	6	11	4	10
Player 3	3	4	2	10	6	8	8	6
Player 4	8	5	5	6	11	7	6	2
Player 5	10	12	6	8	10	8	3	4
Player 6	4	10	9	5	3	6	5	5
Player 7	7	6	5	6	9	10	4	2
Player 8	11	8	9	8	9	9	6	10
Player 9	4	7	10	10	7	4	12	8

7 Find an estimate of the population mean and compare it to the theoretical mean.

8 Find an estimate of the population proportion of prime numbers (that is, 2, 3, 5, 7, 11) and compare it with the theoretical proportion.

9 Construct a dotplot for the distribution of values of $\hat{p}$.

10 Repeat the experiment with your own simulation of dice tosses. Estimate the population mean and population parameter. Compare your results with those of other students as well as those included here.

11 Use the random number generator on your calculator to simulate the following situation: if the number is between 0.25 and 0.47 inclusive, it represents a 4-wheel drive vehicle; otherwise it represents a 2-wheel drive vehicle.

a Perform the simulation 50 times, calculate $\hat{p}$, the proportion of 4-wheel drive vehicles, and comment on your result.

b Compare your results with those of your classmates. How close were they to the theoretical result of 0.22?

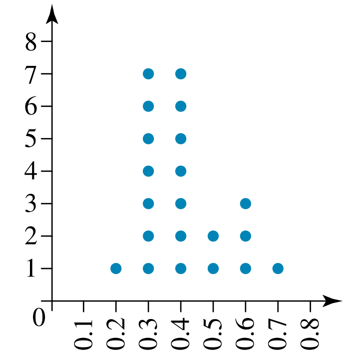
c Find the average of $\hat{p}$ for your class. How close is it to 0.22?

12 The following sample proportions were obtained when 12 service stations were sampled for the proportion of cars buying unleaded petrol. Each sample was of 20 cars. Find an estimate of the population proportion, correct to 3 decimal places.

Service station	1	2	3	4	5	6	7	8	9	10	11	12
$\hat{p}$	0.75	0.8	0.65	0.7	0.75	0.7	0.85	0.75	0.85	0.7	0.65	0.8

13 The number of samples in the frequency dotplot at right is:

A 17 **B** 20
C 19 **D** 21
E 18



14 Eighteen bakeries were sampled to determine the proportion of raspberry tarts (R) compared to the proportion of strawberry tarts (S) that were sold that day.

Sample 1	R	R	R	R	S	R	S	R	R	S
Sample 2	S	R	R	S	S	R	R	S	S	R
Sample 3	R	R	R	R	S	R	R	R	R	S
Sample 4	S	R	R	S	S	R	R	R	R	S
Sample 5	R	R	S	S	R	S	R	R	R	R
Sample 6	R	S	S	R	S	S	R	S	R	R
Sample 7	R	R	S	S	R	R	R	R	R	R
Sample 8	R	R	S	S	R	R	R	R	R	R
Sample 9	S	R	S	R	R	S	R	R	R	R
Sample 10	S	S	R	R	R	R	R	R	R	R
Sample 11	R	R	R	R	S	R	S	R	S	S
Sample 12	S	R	R	R	R	R	R	S	S	R
Sample 13	R	R	R	S	R	S	S	R	S	R
Sample 14	R	R	R	S	R	S	S	S	R	S
Sample 15	R	R	R	S	R	S	R	S	R	R
Sample 16	S	R	R	R	S	S	R	S	R	S
Sample 17	R	S	R	R	R	R	R	R	S	S
Sample 18	R	R	S	S	R	R	S	R	R	S

a Find an estimate of the population proportion of raspberry tarts.
b Construct a dotplot of the distribution of $\hat{p}$.

15 Use two coins to simulate the following situation. If both coins are tossed and come down Heads, a student with blond hair is represented. Perform the simulation 40 times and calculate $\hat{p}$, the proportion of blond students. Compare your result with that of other students in your class. What was the average value of $\hat{p}$?

16 Construct a frequency dotplot for the distribution of values from question 3.

MASTER

17 It was stated that to accurately determine the estimate of the population proportion from a set of sample proportions, the size of each sample must be the same. This is not strictly true. Consider the following example of the proportion of medical cases requiring hospitalisation at a group of medical clinics.

a Devise a method of 'normalising' the data so that they are effectively the same sample size.
(*Hint:* Consider the number of patients in each clinic requiring hospitalisation.)

b Find an estimate of the population proportion.

c Would a dotplot be an appropriate way to display the spread of the sample proportions?

Clinic	$\hat{p}$	Sample size
Abbotsford	0.429	7
Brunswick	0.385	13
Carlton	0.300	10
Dandenong	0.333	15
Eltham	0.400	10
Frankston	0.250	8
Geelong	0.381	21
Hawthorn	0.273	11
Inner Melbourne	0.133	15
N. Melbourne	0.231	13
S. Melbourne	0.294	17
E. Melbourne	0.333	9
W. Melbourne	0.357	14
St Kilda	0.375	8

18 a Use a spreadsheet to simulate drawing cards from a deck of 52 playing cards.

i Generate random numbers between 1 and 52.

ii Divide the numbers into 4 groups (1–13 = spades, 14–26 = clubs, 27–39 = diamonds, 40–52 = hearts).

iii Subdivide further (1 = ace, 2 = 2, …, 10 = 10, 11 = jack, 12 = queen, 13 = king).

iv Show the card drawn (that is, 'ace of hearts', not '40').

(*Hint:* You will need to learn about the MOD function of a spreadsheet for part iii and the VLOOKUP function for part iv.)

b Simulate the drawing of up to 10 cards, so that no 2 cards are the same. To do this you will have to 'shuffle' the 52 cards in some way.

c Simulate 12 drawings of 10 cards so that no 2 cards are the same in each draw (that is, the cards are put back and the deck shuffled after each set of 10 cards).

d Estimate the population proportion of red aces. Compare this to the theoretical population proportion.

6.6 Measuring central tendency and spread of sample distributions

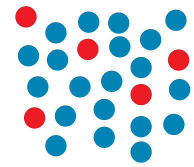
study on

Units 1 & 2
AOS 6
Topic 1
Concept 4

Measures of central tendency and spread
Concept summary
Practice questions

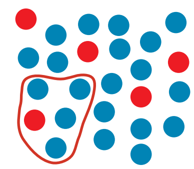
As seen in the previous section, the dotplot for the sample proportions, $\hat{p}$, and sample means, $\bar{x}$, can give a pictorial indication of where the population proportion or mean lies — somewhere in the middle of the plot. In other words, the sample proportions or means group around the population proportion or mean. The range of values can give an indication of the possible values of the population proportion or mean.

The distribution of $\hat{p}$

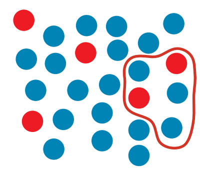


Let's say we are interested in the collection of balls shown in the figure at right. As you can see, there are 25 balls and $\frac{1}{5}$ of them are red. This means that the population proportion, p, is $\frac{1}{5}$. The population size, N, is 25. *Note:* This is a very small population for demonstration purposes only.

Normally we wouldn't know the population parameter, but would choose a sample from the population and use the sample proportion to estimate the population proportion. In this case, we are going to use a sample size of 5, that is $n = 5$.



If the circled balls in the figure at right are our sample, then, as there is 1 red ball, the sample proportion would be $\hat{p} = \frac{1}{5}$, where $\hat{p} = \frac{X}{n}$ and X is the number of items with a certain characteristic in a sample of size n.



A different sample could have a different sample proportion. In the case at right, $\hat{p} = \frac{2}{5}$. And in the case below that $\hat{p} = 0$.

It would also be possible to have samples where $\hat{p} = \frac{3}{5}$, $\hat{p} = \frac{4}{5}$ or $\hat{p} = 1$, although these samples are less likely to occur. We are going to model what would happen if we took a large number of samples of the same size (but were able to return each sample back to the population before selecting again).

We have just claimed that samples with 0, 1 or 2 red balls are more likely to occur than samples with 3, 4 or 5 red balls. How is it possible to make claims about the likelihood of different samples occurring?

WORKED EXAMPLE 12

Consider a container with 25 balls and $\frac{1}{5}$ of them are red. Samples of 5 balls are drawn. Determine the likelihood of each different possible sample being drawn.

THINK	WRITE
1 Determine the number of different samples that could be selected. There are 25 balls and we are choosing 5 balls.	Total number of samples $= {}^{25}C_5$ $= 53\,130$

2 Determine the different sample proportions that could be drawn.

There are 20 blue balls and 5 red balls.
Each draw could have between 0 and 5 red balls.
Possible. $\hat{p}$: $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$.

3 Determine the probability of choosing a sample with 0 red balls. This means that 5 blue balls were chosen from 20.

$$\text{P(0 red balls)} = \frac{{}^{20}C_5 \, {}^5C_0}{53\,130} = \frac{15\,504}{53\,130}$$

4 Determine the probability of choosing a sample with 1 red ball. This means that 4 blue balls were chosen from 20 and 1 red ball was chosen from 5.

$$\text{P(1 red ball)} = \frac{{}^{20}C_4 \, {}^5C_1}{53\,130} = \frac{24\,225}{53\,130}$$

5 Determine the probability of choosing a sample with 2 red balls. This means that 3 blue balls were chosen from 20 and 2 red balls were chosen from 5.

$$\text{P(2 red balls)} = \frac{{}^{20}C_3 \, {}^5C_2}{53\,130} = \frac{11\,400}{53\,130}$$

6 Determine the probability of choosing a sample with 3 red balls. This means that 2 blue balls were chosen from 20 and 3 red balls were chosen from 5.

$$\text{P(3 red balls)} = \frac{{}^{20}C_2 \, {}^5C_3}{53\,130} = \frac{1900}{53\,130}$$

7 Determine the probability of choosing a sample with 4 red balls. This means that 1 blue ball was chosen from 20 and 4 red balls were chosen from 5.

$$\text{P(4 red balls)} = \frac{{}^{20}C_1 \, {}^5C_4}{53\,130} = \frac{100}{53\,130}$$

8 Determine the probability of choosing a sample with 5 red balls. This means that 5 red balls were chosen from 5.

$$\text{P(5 red balls)} = \frac{{}^{20}C_0 \, {}^5C_5}{53\,130} = \frac{1}{53\,130}$$

9 Present the information in a table.

X	$\hat{p}$	Probability
0	0	$\frac{15\,504}{53\,130}$
1	$\frac{1}{5}$	$\frac{24\,225}{53\,130}$
2	$\frac{2}{5}$	$\frac{11\,400}{53\,130}$
3	$\frac{3}{5}$	$\frac{1900}{53\,130}$
4	$\frac{4}{5}$	$\frac{100}{53\,130}$
5	1	$\frac{1}{53\,130}$

Worked example 12 found the probability distribution of $\hat{p}$, that is, the likelihood of different values of $\hat{p}$ occurring in samples. This means that the mean of $\hat{p}$ can be calculated. In this instance:

$$\mu_{\hat{p}} = \frac{15\,504}{53\,130} \times 0 + \frac{24\,225}{53\,130} \times \frac{1}{5} + \frac{11\,400}{53\,130} \times \frac{2}{5} + \frac{1900}{53\,130} \times \frac{3}{5} + \frac{100}{53\,130} \times \frac{4}{5} + \frac{1}{53\,130} \times 1$$

$$= 0.2$$

Notice that this is the population proportion. This is always true. The mean of the distribution of sample proportions will be the population proportion. That is, $\mu_{\hat{p}} = p$.

The standard deviation of the distribution of sample proportions can be found using $\sigma_p = \sqrt{\dfrac{p(1-p)}{n}}$. Why is this true? The standard deviation of the number of successes in a sample of size n is found using $\sigma = \sqrt{np(1-p)}$. In this instance, we are not looking for the number of successes, but the proportion of successes. Therefore, the standard deviation of the sample proportions will be $\dfrac{1}{n}$ th of this value.

$$\sigma_p = \frac{\sqrt{np(1-p)}}{n}$$

$$= \sqrt{\frac{np(1-p)}{n^2}}$$

$$= \sqrt{\frac{p(1-p)}{n}}$$

WORKED EXAMPLE 13

Samples of size 50 are taken from a population of 100 000. The population proportion $p = 0.3$.

a Find the mean and standard deviation of the distribution of sample proportions.

b If the sample size were increased to 100, how would the mean and standard deviation of the distribution of sample proportions be affected?

THINK	WRITE
a 1 The mean of the distribution of sample proportions is the same as the population proportion.	**a** $\mu_{\hat{p}} = p$ $= 0.3$
2 Write down the formula for standard deviation of the distribution of sample proportions.	$\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$
3 Calculate the standard deviation.	$\sigma_{\hat{p}} = \sqrt{\dfrac{0.3 \times 0.7}{50}}$ $= 0.065$
b 1 The mean is not dependent on sample size.	**b** $\mu_{\hat{p}} = 0.3$

2 Calculate the standard deviation using $n = 100$.

$$\mu_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$
$$= \sqrt{\frac{0.3 \times 0.7}{100}}$$
$$= 0.046$$

3 Write your conclusions.

Increasing the sample size does not change the mean of the distribution, but the standard deviation is reduced.

The mean of the distribution of sample proportions is the same as the population proportion. This means that on average the sample will have a sample proportion that will be the population proportion.

The standard deviation of the distribution of sample proportions becomes smaller as the sample size increases. This means that the distribution has less variability as the sample size increases and larger samples will have results closer to the population proportion.

The distribution of $\bar{x}$

It is also possible to find a distribution of sample means. In this case, the mean of the sample distribution is the same as the mean of the population. Why is this true? Some samples will have means higher than the population, but other samples will have means lower than the population. These samples tend to cancel each other out, meaning that the mean of the distribution, $\mu_{\bar{x}}$, is the same as the population mean, μ.

$$\mu_{\bar{x}} = \mu$$

The standard deviation of the distribution of sample means can be found using $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$. Why is this true? The population standard deviation is σ. This means that the variance of the random variables in the population is σ^2. Each sample is a selection of n of these variables. The total of the variables in the sample will have a variance found by adding the variance of the variables together. This means that the variance of the sample total is $n\sigma^2$.

$$\sigma_{\Sigma x}{}^2 = n\sigma^2$$
$$\sigma_{\Sigma x} = \sqrt{n\sigma^2}$$
$$= \sigma\sqrt{n}$$

The sample mean is found by dividing the total by n. This means that the standard deviation of the sample means is the standard deviation of the sample totals divided by n.

$$\sigma_{\bar{x}} = \frac{\sigma_{\Sigma x}}{n}$$
$$= \frac{\sigma\sqrt{n}}{n}$$
$$= \frac{\sigma}{\sqrt{n}}$$

Note: Larger sample sizes will not change the mean of the distribution of sample means, but will reduce the standard deviation, meaning that larger samples will have results closer to the population parameters.

WORKED EXAMPLE 14

A total of 500 students are studying a statistics course at Parker University. On a recent exam, the mean score was 73.4 with a standard deviation of 20.

a Samples of 5 students are selected and the means found. Find the mean and standard deviation of the distribution of $\bar{x}$.

b What effect would increasing the sample size to 30 have on the mean and standard deviation of the distribution of sample means?

THINK	WRITE
a 1 The mean of the distribution of sample means is the same as the population mean.	**a** $\mu_{\bar{x}} = \mu$ $= 73.4$
2 Write down the formula for standard deviation of the distribution of sample means.	$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$
3 Calculate the standard deviation.	$\sigma_{\bar{x}} = \dfrac{20}{\sqrt{5}}$ $= 8.94$
b 1 The mean is not dependent on sample size.	**b** $\mu_{\bar{x}} = \mu$ $= 73.4$
2 Calculate the standard deviation using $n = 30$.	$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ $= \dfrac{20}{\sqrt{30}}$ $= 3.65$
3 Write your conclusions.	Increasing the sample size does not change the mean of the distribution, but the standard deviation is reduced.

EXERCISE 6.6 Measuring central tendency and spread of sample distributions

PRACTISE

1 **WE12** Consider a container with 20 balls and $\frac{1}{5}$ of them are red. Samples of 5 balls are drawn. Determine the likelihood of each different possible sample being drawn.

2 Consider a container with 20 balls and $\frac{1}{4}$ of them are red. Samples of 4 balls are drawn. Determine the likelihood of each different possible sample being drawn.

3 **WE13** Samples of size 25 are taken from a population of 100 000. The population proportion, p, is 0.5.
   a Find the mean and standard deviation of the distribution of sample proportions.
   b What effect would increasing the sample size to 40 have on the mean and standard deviation of the distribution of sample proportions?

4 Samples of size 50 are taken from a population of 200 000. The population proportion, p, is 0.9.

a Find the mean and standard deviation of the distribution of sample proportions.

b What effect would increasing the sample size to 100 have on the mean and standard deviation of the distribution of sample proportions?

5 WE14 Every year 500 students apply for a place at Maccas University. The average enrolment test score is 600 with a standard deviation of $\sqrt{300}$.

a Samples of 10 students are selected and the means found. Find the mean and standard deviation of the distribution of $\bar{x}$.

b What effect would increasing the sample size to 20 have on the mean and standard deviation of the distribution of sample means?

6 A total of 1000 students are studying a statistics course at Minitrees University. On a recent exam, the mean score was 90.5 with a standard deviation of 10.

a Samples of 15 students are selected and the mean scores found. Find the mean and standard deviation of the distribution of $\bar{x}$.

b What effect would increasing the sample size to 30 have on the mean and standard deviation of the distribution of sample means?

CONSOLIDATE

Questions 7 and 8 refer to a population of 100 where $p = 0.3$.

7 What is the likelihood of selecting a sample of size 10 that has the same distribution as the population?

8 Calculate the mean and standard deviation of the distribution of sample proportions, that is, size 10.

Questions 9, 10 and 11 refer to a population of 200 where $p = 0.5$.

9 What is the likelihood of selecting a sample of size 10 where $0.4 \leq \hat{p} \leq 0.6$?

10 Calculate the mean and standard deviation of the distribution of sample proportions, that is, size 10.

11 If the sample size was increased to 20, how do the answers to questions 9 and 10 change?

Questions 12, 13 and 14 refer to a population of 200 where $p = 0.1$.

12 What is the likelihood of selecting a sample of size 10 where $0 \leq \hat{p} \leq 0.2$?

13 Calculate the mean and standard deviation of the distribution of sample proportions.

14 If the sample size was increased to 20, how do the answers to questions 12 and 13 change?

Questions 15 and 16 refer to a population with a mean of 67 and a standard deviation of 15.

15 If samples of size 20 are selected, find the mean and standard deviation of the distribution of sample means.

16 What sample size would be needed to reduce the standard deviation of the distribution of sample means to less than 2?

MASTER

17 Answer the following questions for a population and samples with $N = 600$, $n = 60$ and $p = 0.3$.

a Use Excel or another similar program to graph the distribution for $\hat{p}$.

b Find the mean of the distribution.

c Find the standard deviation of the distribution.

18 Assuming $N = 600$ and $p = 0.3$, use Excel or another similar program to graph the distribution of $\hat{p}$, demonstrating the effect of sample size on the distribution. Use samples of size 20, 30 and 60.

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6.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

studyON

Units 1 & 2 **Simulation, sampling and sampling distributions**

Sit topic test

6 Answers

EXERCISE 6.2

1 $\varepsilon = \{1, 2, 3, 4, 5, 6\}$, $Y = \{5, 6\}$

2 $\varepsilon = \{$spade, club, heart, diamond$\}$, $Z = \{$spade, club$\}$

3 $\varepsilon = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$z = \{5, 6, 7, 8, 9, 10, 11, 12\}$

4 $\varepsilon = \{$H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6$\}$, $X = \{$H2, H4, H6$\}$

5 $\varepsilon = \{$AS, 2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AC, 2C, 3C, 4C, 5C, 6C, 7C, 8C, 9C, 10C, JC, QC, KC, AH, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH, AD, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, 10D, JD, QD, KD$\}$,

$P = \{$AS, 2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, QS, KS, JC, JD, JH$\}$

6 $Q = \{$JS$\}$

7 $\varepsilon = \{$A, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$\}$, $W = \{$J, Q, K$\}$

8 $\varepsilon = \{$H Heart, H Diamond, H Spade, H Club, T Heart, T Diamond, T Spade, T Club$\}$

$X = \{$T Heart, T Diamond$\}$

9 $D = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

10 $F = \{(4, 4), (4, 5), (4, 6), (5.4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$

11 $X = \{$H1, H2, H3, H4, H5, H6, T2, T4, T6$\}$

12 $Y = \{$H Heart, H Diamond, T Heart, T Diamond, T Spade, T Club$\}$

13 $\varepsilon = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$
$D = \{$HHT, HTH, HTT, THH, THT, TTH$\}$

14 $\varepsilon = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

$Q = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

15 $\varepsilon = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$

$R = \{(1, 3), (3, 1)\}$

16 $\varepsilon = \{$KQJ, KJQ, QKJ, QJK, JKQ, JQK$\}$

$K = \{$KQJ, KJQ, QKJ$\}$

17 $Q = \{$JH, QH, KH, JD, QD, KD, 2S, 4S, 6S, 8S, 10S$\}$

18 $R = \{$2H, 4H, 6H, 8H, 10H, 2D, 4D, 6D, 8D, 10D, JS, QS, KS$\}$

EXERCISE 6.3

1–3 Results will vary.

4 Player B

5 a One way is to use **randInt**(0,1,10) to generate 10 values that are either equal to 0 or 1, and let 0s represent Heads, and 1s represent Tails.

b Answers will vary.

c Answers will vary.

6 Generally, the histogram for 100 tosses will be more even than that for 10 tosses, approximately symmetrical.

7 a, b The player can expect to win about once every 15 games, spending \$15 to win \$10 (a loss of \$5), but answers will vary.

8–16 Answers will vary.

EXERCISE 6.4

1 $\frac{12}{47}$

2 $\frac{16}{47}$

3–4 Answers will vary.

5 25.9104 seconds

6 1.43

7 a Population

b Sample

c Population

d Sample

e Sample

f Population

8 0.25 spoiled; 0.75 unspoiled

9 0.6

10 a 12.9%

b Sample

c Probably not a representative sample — proportion of girls studying Engineering probably not the same as proportion of girls at Wombat University.

11 a $\frac{1}{12}$

b Very little — it is a very small sample; perhaps 5–10% defective rate in the population

12 a 0.3 and 0.33

b Statistician B used a much larger sample and is therefore more likely to be close.

13 a 0.3

b No, because only Frankston was sampled, not the entire state.

c $\frac{19}{60}$

d Better than the previous sample — a more representative sample

14 a 0.375

b Theoretical result = 0.388

15 75.77

16 5.404%

17–18 Answers will vary.

EXERCISE 6.5

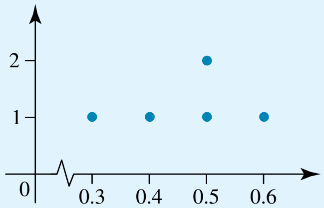
1 0.46

2 0.58

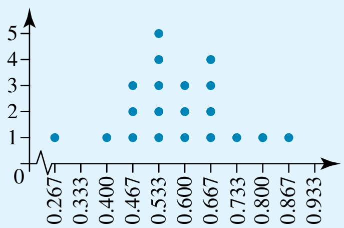
3 51.5 minutes

4 95.9 minutes

5



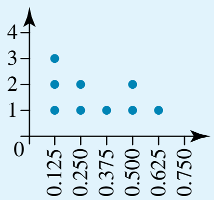
6



7 6.93; true mean 7

8 0.319; true proportion 0.417

9



10 As this is a random experiment, answers will differ from sample to sample.

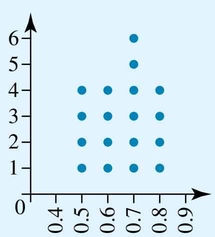
11 Answers will vary.

12 0.746

13 D

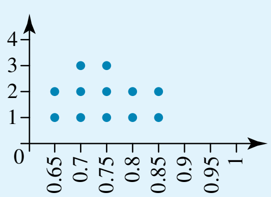
14 a 0.65

b



15 Answers will vary.

16



17 a Multiply the sample size by $\hat{p}$ for each clinic — this is the number of patients. Then add, and divide by the total number in the entire survey.

b $\frac{54}{171} = 0.316$ (or approximately 32%)

c No, because sample sizes are not the same, each $\hat{p}$ is not really significant; only the total count matters.

Clinic	$\hat{p}$	Sample size	Patients
Abbotsford	0.429	7	3
Brunswick	0.385	13	5
Carlton	0.300	10	3
Dandenong	0.333	15	5
Eltham	0.400	10	4
Frankston	0.250	8	2
Geelong	0.381	21	8
Hawthorn	0.273	11	3
Inner Melbourne	0.133	15	2
N. Melbourne	0.231	13	3
S. Melbourne	0.294	17	5
E. Melbourne	0.333	9	3
W. Melbourne	0.357	14	5
St Kilda	0.375	8	3

18 a–c Answers will vary.

d Theoretical proportion $\frac{1}{26}$

EXERCISE 6.6

1

X	$\hat{p}$	Probability
0	0	$\frac{4368}{15\,504}$
1	$\frac{1}{5}$	$\frac{7280}{15\,504}$
2	$\frac{2}{5}$	$\frac{3360}{15\,504}$
3	$\frac{3}{5}$	$\frac{480}{15\,504}$
4	$\frac{4}{5}$	$\frac{16}{15\,504}$
5	1	0

2

X	$\hat{p}$	Probability
0	0	$\frac{1365}{4845}$
1	$\frac{1}{4}$	$\frac{2275}{4845}$
2	$\frac{1}{2}$	$\frac{1050}{4845}$
3	$\frac{3}{4}$	$\frac{150}{4845}$
4	1	$\frac{5}{4845}$

3 a $\mu_{\hat{p}} = 0.5$, $\sigma_{\hat{p}} = 0.1$

b $\mu_{\hat{p}} = 0.5$, $\sigma_{\hat{p}} = \frac{\sqrt{10}}{40}$. No effect on the mean; the standard deviation is reduced.

4 a $\mu_{\hat{p}} = 0.9, \sigma_{\hat{p}} = \frac{3\sqrt{2}}{100}$

b $\mu_{\hat{p}} = 0.9, \sigma_{\hat{p}} = 0.03$. No effect on mean; SD reduced.

5 a $\mu_{\bar{x}} = 600, \sigma_{\bar{x}} = \sqrt{30}$

b $\mu_{\bar{x}} = 600, \sigma_{\bar{x}} = \sqrt{15}$. No effect on mean; SD reduced.

6 a $\mu_{\bar{x}} = 90.5, \sigma_{\bar{x}} = \frac{2\sqrt{15}}{3}$

b $\mu_{\bar{x}} = 90.5, \sigma_{\bar{x}} = \frac{\sqrt{30}}{3}$. No effect on mean; SD reduced.

7 0.28

8 $\mu_{\hat{p}} = 0.3, \sigma_{\hat{p}} = \frac{\sqrt{210}}{100}$

9 0.67

10 $\mu_{\hat{p}} = 0.5, \sigma_{\hat{p}} = \frac{\sqrt{10}}{20}$

11 0.76, $\mu_{\hat{p}} = 0.5, \sigma_{\hat{p}} = \frac{\sqrt{5}}{20}$. No effect on mean; SD reduced.

12 0.93

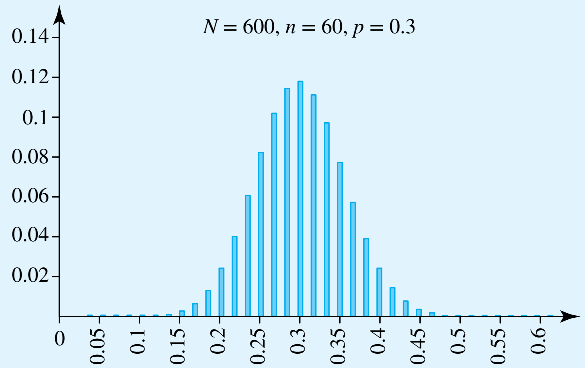
13 $\mu_{\hat{p}} = 0.1, \sigma_{\hat{p}} = 0.09$

14 0.97, $\mu_{\hat{p}} = 0.1, \sigma_{\hat{p}} = 0.07$

15 $\mu_{\bar{x}} = 67, \sigma_{\bar{x}} = \frac{3\sqrt{5}}{2}$

16 57

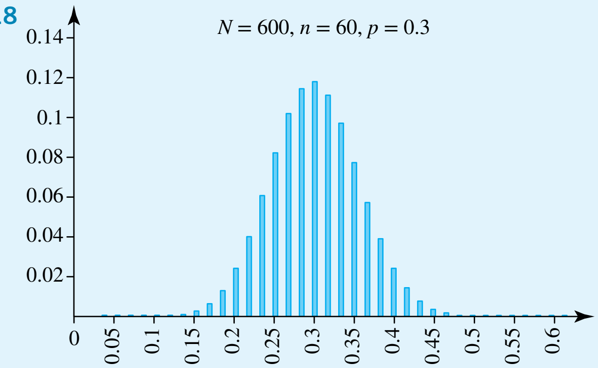
17 a

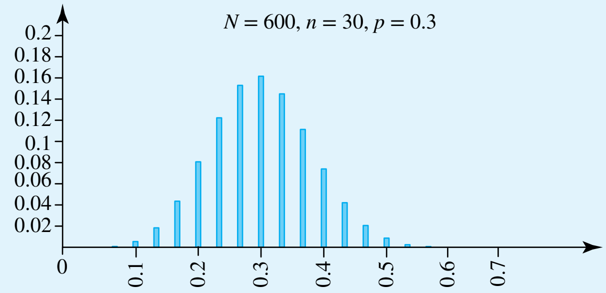


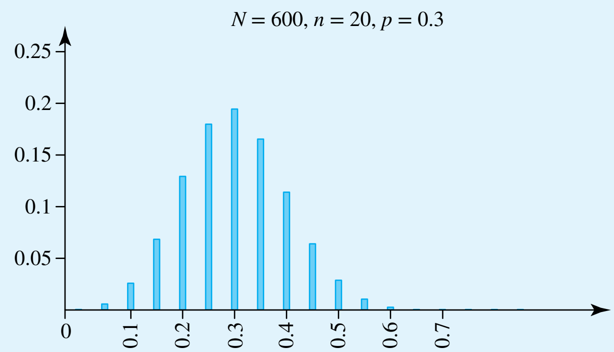
b $\mu_{\hat{p}} = 0.3$

c $\sigma_{\hat{p}} = 0.06$

18







7

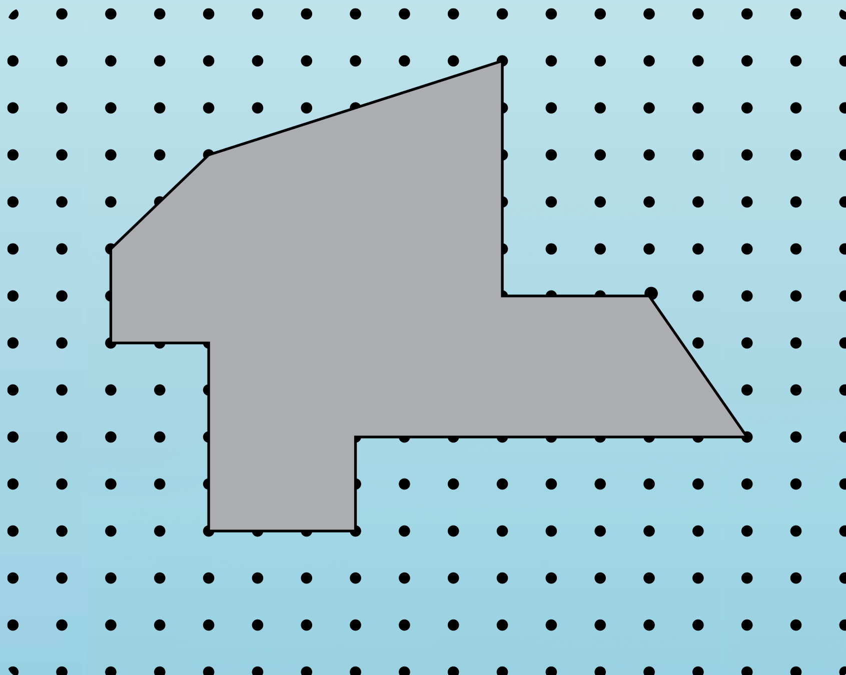
Coordinate geometry

7.1 Kick off with CAS

Geometry with CAS

1 Using CAS, sketch the shape as shown in the diagram below.

2 Using the geometry tools within the CAS technology, determine the perimeter and area of the shape.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

7.2 Distance between two points

studyon

Units 1 & 2

AOS 5

Topic 1

Concept 1

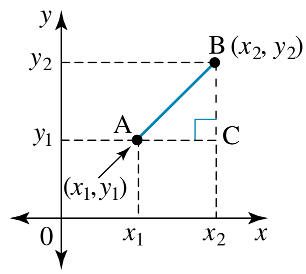
Distance between two points

Concept summary

Practice questions

Coordinate geometry is a branch of mathematics with many practical applications. The distance between two points can be calculated easily using Pythagoras' theorem. It is particularly useful when trying to find a distance that is difficult to measure directly; for example, finding the distance from a point on one side of a lake to a point on the other side.

Let A (x_1, y_1) and B (x_2, y_2) be two points on the Cartesian plane as shown below right.



Triangle ABC is a right-angled triangle.

$$AC = x_2 - x_1$$
$$BC = y_2 - y_1$$

By Pythagoras' theorem:

$$AB^2 = AC^2 + BC^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Hence, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

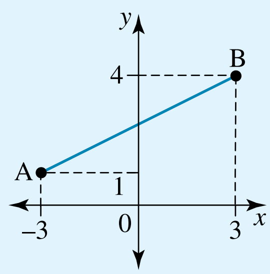
The distance between two points A (x_1, y_1) and B (x_2, y_2) is:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

WORKED EXAMPLE 1

a Find the distance between the points A and B in the figure at right, correct to 2 decimal places.

b Find the distance between the points P (−1, 5) and Q (3, −2), correct to 2 decimal places.



THINK	WRITE
a 1 From the graph find points A and B.	**a** A (−3, 1) and B (3, 4)
2 Let A have coordinates (x_1, y_1).	Let $(x_1, y_1) = (-3, 1)$.
3 Let B have coordinates (x_2, y_2).	Let $(x_2, y_2) = (3, 4)$.
4 Find the length AB by applying the formula for the distance between two points.	$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(3 - (-3))^2 + (4 - 1)^2}$ $= \sqrt{(6)^2 + (3)^2}$ $= \sqrt{36 + 9}$ $= \sqrt{45}$ $= 3\sqrt{5}$ units = 6.71 units (correct to 2 decimal places)

b **1** Let P have coordinates (x_1, y_1).

2 Let Q have coordinates (x_2, y_2).

3 Find the length PQ by applying the formula for the distance between two points.

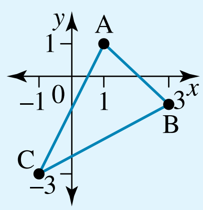
b Let $(x_1, y_1) = (-1, 5)$.

Let $(x_2, y_2) = (3, -2)$.

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-1))^2 + (-2 - 5)^2} \\ &= \sqrt{(4)^2 + (-7)^2} \\ &= \sqrt{16 + 49} \\ &= \sqrt{65} \text{ units} \\ &= 8.06 \text{ units (correct to 2 decimal places)} \end{aligned}$$

WORKED EXAMPLE 2

Prove that the points A (1, 1), B (3, −1) and C (−1, −3) are the vertices of an isosceles triangle.

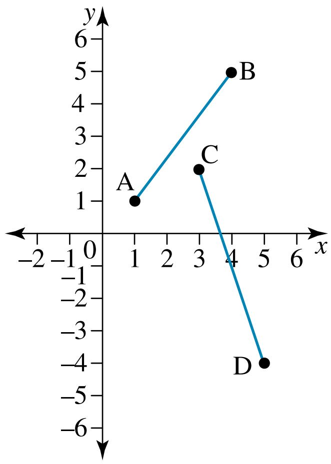
THINK	WRITE/DRAW
1 Plot the points. *Note:* For triangle ABC to be isosceles, two sides must have the same magnitude.	 From the diagram, AC appears to have the same length as BC.
2 Find the length AC.	$AC = \sqrt{[1 - (-1)]^2 + [1 - (-3)]^2} = \sqrt{(2)^2 + (4)^2} = \sqrt{20} = 2\sqrt{5}$
3 Find the length BC.	$BC = \sqrt{[3 - (-1)]^2 + [-1-(-3)]^2} = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$
4 Find the length AB.	$AB = \sqrt{[3 - (1)]^2 + [-1 - (1)]^2} = \sqrt{(2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$
5 State your proof.	Since $AC = BC \neq AB$, triangle ABC is an isosceles triangle.

EXERCISE 7.2 Distance between two points

PRACTISE

1 WE1 a Find the distance between the points A and B shown at right.

b Find the distance between the points (2, 5), (6, 8).



2 a Find the distance between the points C and D shown at right.

b Find the distance between the points (−1, 2) and (4, 14).

3 WE2 Prove that the points A (0, −3), B (−2, −1) and C (4, 3) are the vertices of an isosceles triangle.

4 Prove that the points A (3, 1), B (−3, 7) and C (−1, 3) are the vertices of an isosceles triangle.

CONSOLIDATE

5 The points P (2, −1), Q (−4, −1) and R $(-1, 3\sqrt{3} - 1)$ are joined to form a triangle. Prove that triangle PQR is equilateral.

6 Prove that the quadrilateral with vertices A (−1, 3), B (5, 3), C (1, 0) and D (−5, 0) is a parallelogram.

7 Prove that the triangle with vertices D (5, 6), E (9, 3) and F (5, 3) is a right-angled triangle.

8 The vertices of a quadrilateral are A (1, 4), B (−1, 8), C (1, 9) and D (3, 5).

a Find the lengths of the sides.

b Find the lengths of the diagonals.

c What type of quadrilateral is it?

9 Calculate the distance between each of the pairs of points below, correct to 3 decimal places.

a (–14, 10) and (–8, 14) b (6, –7) and (13, 6) c (–11, 1) and (2, 2)

10 Find the distance between each of the following pairs of points in terms of the given variables.

a $(a, 1)$, $(2, 3)$ b $(5, 6)$, $(0, b)$

c $(c, 2)$, $(4, c)$ d $(d, 2d)$, $(1, 5)$

11 If the distance between the points $(3, b)$ and (–5, 2) is 10 units, then a possible value of b is:

A −8 B −4 C 4 D 0 E 2

12 A rhombus has vertices A (1, 6), B (6, 6), C (−2, 2) and D (x, y). The coordinates of D are:

A (2, −3) B (2, 3) C (−2, 3)

D (3, 2) E (3, −2)

13 A rectangle has vertices A (1, 5), B (10.6, z), C (7.6, −6.2) and D (−2, 1). Find:

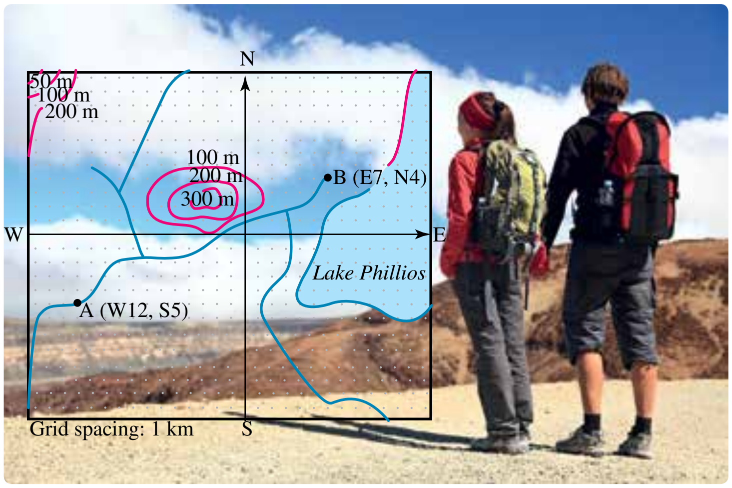
a the length of CD b the length of AD

c the length of the diagonal AC d the value of z.

14 Show that the triangle ABC with coordinates A (a, a), B $(m, -a)$ and C $(-a, m)$ is isosceles.

MASTER

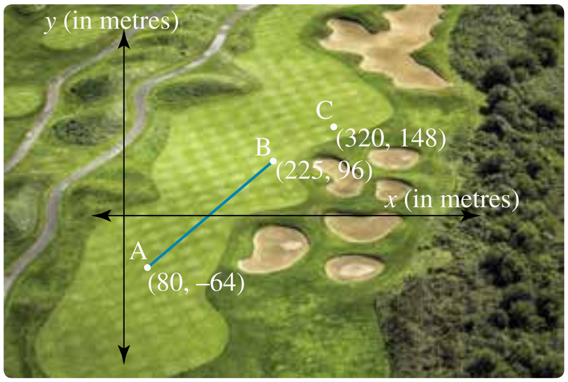
15 Two hikers are about to hike from A to B (shown on the map below). How far is it from A to B 'as the crow flies', that is, in a straight line?



16 Using the coordinates shown on the aerial photo of the golf course, calculate (to the nearest metre):

a the horizontal distance travelled by the golf ball for the shot down the fairway, from A to B

b the horizontal distance that needs to be covered in the next shot to reach the point labelled C in the bunker.



7.3 Midpoint of a line segment

We can determine the coordinates of the midpoint of a line segment by applying the midpoint formula shown below.

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Midpoint formula

Consider the line segment connecting the points A (x_1, y_1) and B (x_2, y_2).

Let P (x, y) be the midpoint of AB.

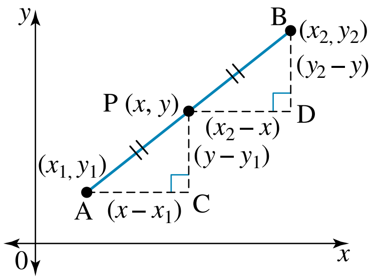
AC is parallel to PD. PC is parallel to BD.

AP is parallel to PB (collinear, that is, the points lie on the same straight line).

Hence, triangle APC is similar to triangle PBD.

But AP = PB (since P is the midpoint of AB).

Hence, triangle APC is congruent, to triangle PBD.



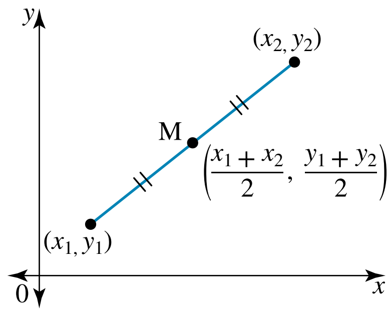
Therefore $x - x_1 = x_2 - x$

$$2x = x_1 + x_2$$

$$x = \frac{x_1 + x_2}{2}$$

Similarly it can be shown that $y = \dfrac{y_1 + y_2}{2}$.

In general, the coordinates of the midpoint of a line segment joining the points (x_1, y_1) and (x_2, y_2) can be found by averaging the x- and y-coordinates of the end points, respectively.



The coordinates of the midpoint of the line segment joining (x_1, y_1) and (x_2, y_2) are: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

WORKED EXAMPLE 3

Find the coordinates of the midpoint of the line segment joining (−2, 5) and (7, 1).

THINK	WRITE
1 Label the given points (x_1, y_1) and (x_2, y_2).	Let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (7, 1)$.
2 Find the x-coordinate of the midpoint.	$x = \frac{x_1 + x_2}{2}$ $= \frac{-2 + 7}{2}$ $= 2\frac{1}{2}$
3 Find the y-coordinate of the midpoint.	$y = \frac{y_1 + y_2}{2}$ $= \frac{5 + 1}{2}$ $= 3$
4 Give the coordinates of the midpoint.	Hence, the coordinates of the midpoint are $\left(2\frac{1}{2}, 3\right)$.

WORKED EXAMPLE 4

The coordinates of the midpoint, M, of the line segment AB are (7, 2). If the coordinates of A are (1, −4), find the coordinates of B.

THINK	WRITE/DRAW
1 Label the start of the line segment (x_1, y_1) and the midpoint (x, y).	Let $(x_1, y_1) = (1, -4)$ and $(x, y) = (7, 2)$.
2 Find the x-coordinate of the end point.	$x = \frac{x_1 + x_2}{2}$ $7 = \frac{1 + x_2}{2}$ $14 = 1 + x_2$ $x_2 = 13$

3 Find the y-coordinate of the end point.

$$y = \frac{y_1 + y_2}{2}$$

$$2 = \frac{-4 + y_2}{2}$$

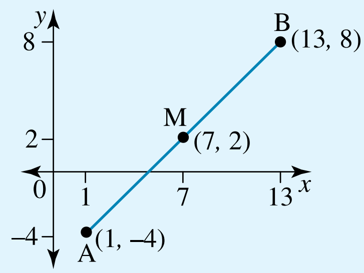
$$4 = -4 + y_2$$

$$y_2 = 8$$

4 Give the coordinates of the end point.

Hence, the coordinates of point B are (13, 8).

5 Check that the coordinates are feasible.



EXERCISE 7.3 Midpoint of a line segment

PRACTISE

1 WE3 Find the coordinates of the midpoint of the line segment joining (−5, 1) and (−1, −8).

2 Find the coordinates of the midpoint of the line segment joining (4, 2) and (11, −2).

3 WE4 The coordinates of the midpoint, M, of the line segment AB are (2, −3). If the coordinates of A are (7, 4), find the coordinates of B.

4 The coordinates of the midpoint, M, of the line segment AB are (−2, 4). If the coordinates of A are (1, 8), find the coordinates of B.

CONSOLIDATE

5 The vertices of a square are A (0, 0), B (2, 4), C (6, 2) and D (4, −2). Find:

a the coordinates of the centre of the square

b the side length

c the length of the diagonals.

6 The midpoint of the line segment joining the points (−2, 1) and (8, −3) is:

A (6, −2) B (5, 2) C (6, 2) D (3, −1) E (5, −2)

7 If the midpoint of AB is (−1, 5) and the coordinates of B are (3, 8), then A has coordinates:

A (1, 6.5) B (2, 13) C (−5, 2) D (4, 3) E (7, 11)

8 Find the coordinates of the midpoint of each of the following pairs of points, in terms of a variable or variables where appropriate.

a $(2a, a), (6a, 5a)$ b $(5, 3c), (11, 3c)$ c $(3f, 5), (g, -1)$

9 Find the value of a so M is the midpoint of the segment joining points A and B.

a A $(-2, a)$, B $(-6, 5)$, M $(-4, 5)$ b A $(a, 0)$, B $(7, 3)$, M $\left(8, \frac{3}{2}\right)$

10 a The vertices of a triangle are A (2, 5), B (1, −3) and C (−4, 3). Find:

i the coordinates of P, the midpoint of AC

ii the coordinates of Q, the midpoint of AB

iii the length of PQ

iv the length of BC.

b Hence show that BC = 2PQ.

11 a A quadrilateral has vertices A (6, 2), B (4, −3), C (−4, −3) and D (−2, 2). Find:
i the midpoint of the diagonal AC
ii the midpoint of the diagonal BD.
b Comment on your finding.

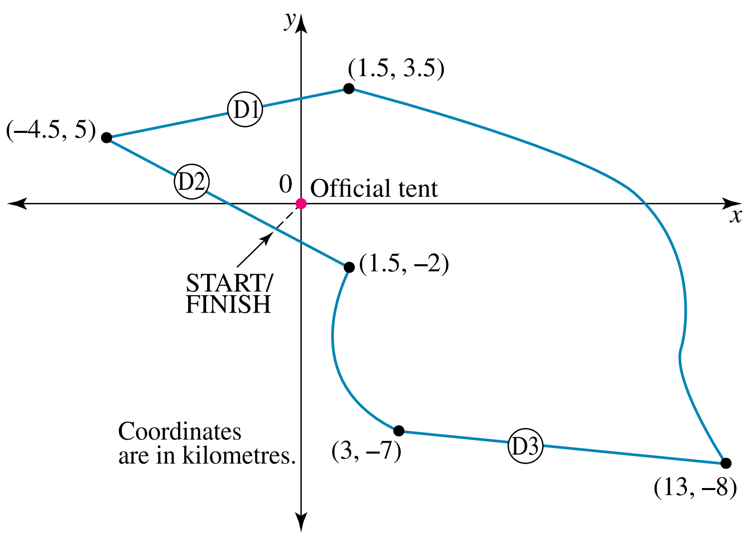
12 a The points A (−5, 3.5), B (1, 0.5) and C (−6, −6) are the vertices of a triangle. Find:
i the midpoint, P, of AB
ii the length of PC
iii the length of AC
iv the length of BC.
b Describe the triangle ABC. What could PC represent?

13 Find the equation of the straight line that passes through the midpoint of A (−2, 5) and B (−2, 3) and has a gradient of −3.

14 Find the equation of the straight line that passes through the midpoint of A (−1, −3) and B (3, −5) and has a gradient of $\frac{2}{3}$.

MASTER

15 A fun-run course is drawn (not to scale) at right. If drink stations D1, D2 and D3 are to be placed at the middle of each straight section, give the map coordinates of each drink station.



16 Find the equation of a line that has a gradient of 5 and passes through the midpoint of the segment joining (−1, −7) and (3, 3).

7.4 Parallel lines and perpendicular lines

Parallel lines

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The equation of a straight line may be expressed in the form:

$$y = mx + c$$

where m is the gradient of the line and c is the y-intercept.

The gradient can be calculated if two points, (x_1, y_1) and (x_2, y_2), are given.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

An alternative form for the equation of a straight line is:

$$ax + by + c = 0$$

where a, b and c are constants.

Another alternative form is:

$$y - y_1 = m(x - x_1)$$

where m is the gradient and (x_1, y_1) is a point on the line.

WORKED EXAMPLE 5 Show that AB is parallel to CD given that A has coordinates (−1, −5), B has coordinates (5, 7), C has coordinates (−3, 1), and D has coordinates (4, 15).

THINK	WRITE
1 Find the gradient of AB.	Let A (−1, −5) = (x_1, y_1) and B (5, 7) = (x_2, y_2). $m = \frac{y_2 - y_1}{x_2 - x_1}$ $m_{AB} = \frac{7 - (-5)}{5 - (-1)} = \frac{12}{6} = 2$
2 Find the gradient of CD.	Let C (−3, 1) = (x_1, y_1) and D (4, 15) = (x_2, y_2). $m_{CD} = \frac{15 - 1}{4 - (-3)} = \frac{14}{7} = 2$
3 Compare the gradients to determine if they are parallel. (*Note:* ‖ means 'is parallel to'.)	Parallel lines have the same gradient. $m_{AB} = m_{CD} = 2$, hence AB‖CD.

Collinear points lie on the same straight line.

WORKED EXAMPLE 6 Show that the points A (2, 0), B (4, 1) and C (10, 4) are collinear.

THINK	WRITE
1 Find the gradient of AB.	Let A (2, 0) = (x_1, y_1) and B (4, 1) = (x_2, y_2). Since $m = \frac{y_2 - y_1}{x_2 - x_1}$ $m_{AB} = \frac{1 - 0}{4 - 2} = \frac{1}{2}$
2 Find the gradient of BC.	Let B (4, 1) = (x_1, y_1) and C (10, 4) = (x_2, y_2). $m_{BC} = \frac{4 - 1}{10 - 4} = \frac{3}{6} = \frac{1}{2}$

3 Show that A, B and C are collinear.

Since $m_{AB} = m_{BC} = \frac{1}{2}$
then AB||BC.
Since B is common to both line segments, A, B and C must lie on the same straight line. That is, A, B and C are collinear.

WORKED EXAMPLE 7 **Find the equation of the straight line that passes through the point (2, 5) and is parallel to the line $y = 3x + 1$.**

THINK	WRITE
1 In order to find the equation of a straight line, we need to know the gradient and a point on the line. One point is given, and because the line is parallel to $y = 3x + 1$, the gradients will be the same.	Point on the line: (2, 5) Gradient: $m = 3$.
2 Use the formula $y - y_1 = m(x - x_1)$ and substitute the coordinates of the point and the gradient to find the equation of the line.	$y - y_1 = m(x + x_1)$ $y - 5 = 3(x - 2)$ $y - 5 = 3x - 6$ $y = 3x - 1$

Perpendicular lines

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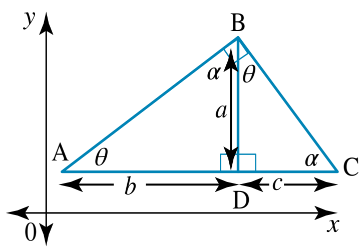
In this section, we examine some of the properties of perpendicular lines. Observing the graphs can be very useful in investigating these properties.

Consider the diagram below right, where the line segment AB is perpendicular to the line segment BC. Line AC is parallel to the x-axis. Line BD is the height of the resulting triangle ABC.



Let $m_{AB} = m_1$

$= \frac{a}{b}$

$= \tan(\theta)$

Let $m_{BC} = m_2$

$= -\frac{a}{c}$

$= -\tan(\theta)$

$= -\frac{b}{a}$

$= -\frac{1}{m_1}$

Hence $m_2 = -\frac{1}{m_1}$

or $m_1 m_2 = -1$

Hence, if two lines are perpendicular to each other, then the product of their gradients is –1.

Two lines are perpendicular if and only if:

$$m_1 m_2 = -1$$

or

$$m_2 = -\frac{1}{m_1}$$

WORKED EXAMPLE 8

Show that the lines $y = -5x + 2$ and $5y - x + 15 = 0$ are perpendicular to one another.

THINK	WRITE
1 Find the gradient of equation 1.	$y = -5x + 2$ Hence $m_1 = -5$
2 Find the gradient of equation 2.	$5y - x + 15 = 0$ Rewrite in the form $y = mx + c$: $5y = x - 15$ $y = \frac{x}{5} - 3$ Hence $m_2 = \frac{1}{5}$
3 Test for perpendicularity. (The two lines are perpendicular if the product of their gradients is –1.)	$m_1 m_2 = -5 \times \frac{1}{5}$ $= -1$ Hence, the two lines are perpendicular to each other.

EXERCISE 7.4 Parallel lines and perpendicular lines

PRACTISE

1 WE5 Show that AB is parallel to CD given that A has coordinates (2, 4), B has coordinates (8, 1), C has coordinates (−6, −2) and D has coordinates (2, −6).

2 Show that AB is parallel to CD given that A has coordinates (1, 0), B has coordinates (2, 5), C has coordinates (3, 15) and D has coordinates (7, 35).

3 WE6 Show that the points A (0, −2), B (5, 1) and C (−5, −5) are collinear.

4 Show that the points A (3, 1), B (5, 2) and C (11, 5) are collinear.

5 WE7 Find the equation of a straight line given the following conditions. The line passes through the point (−1, 3) and is parallel to $y = -2x + 5$.

6 Find the equation of a straight line given that the line passes through the point (4, −3) and is parallel to $3y + 2x = -3$.

7 WE8 Show that the lines $y = 6x - 3$ and $x + 6y - 6 = 0$ are perpendicular to one another.

8 Show that the lines $y = 2x - 4$ and $x + 2y - 10 = 0$ are perpendicular to one another.

CONSOLIDATE

9 Which pairs of the following straight lines are parallel?

a $2x + y + 1 = 0$
b $y = 3x - 1$
c $2y - x = 3$
d $y = 4x + 3$
e $y = \frac{x}{2} - 1$
f $6x - 2y = 0$
g $3y = x + 4$
h $2y = 5 - x$

10 Show that the line that passes through the points (−4, 9) and (0, 3) also passes through the point (6, −6).

11 In each of the following, show that ABCD is a parallelogram.

a A (2, 0), B (4, −3), C (2, −4), D (0, −1)
b A (2, 2), B (0, −2), C (−2, −3), D (0, 1)
c A (2.5, 3.5), B (10, −4), C (2.5, −2.5), D (−5, 5)

12 In each of the following, show that ABCD is a trapezium.

a A (0, 6), B (2, 2), C (0, −4), D (−5, −9)
b A (26, 32), B (18, 16), C (1, −1), D (−3, 3)
c A (2, 7), B (1, −1), C (−0.6, −2.6), D (−2, 3)

13 The line that passes through the points (0, −6) and (7, 8) also passes through:

A (4, 3) B (5, 4) C (−2, 10) D (1, −8) E (1, 4)

14 The point (−1, 5) lies on a line parallel to $4x + y + 5 = 0$. Another point on the same line as (−1, 5) is:

A (2, 9) B (4, 2) C (4, 0) D (−2, 3) E (3, −11)

15 Determine which pairs of the following straight lines are perpendicular.

a $x + 3y - 5 = 0$
b $y = 4x - 7$
c $y = x$
d $2y = x + 1$
e $y = 3x + 2$
f $x + 4y - 9 = 0$
g $2x + y = 6$
h $x + y = 0$

16 Show that the following sets of points form the vertices of a right-angled triangle.

a A (1, −4), B (2, −3), C (4, −7)
b A (3, 13), B (1, 3), C (−4, 4)
c A (0, 5), B (9, 12), C (3, 14)

17 Prove that the quadrilateral ABCD is a rectangle when A is (2, 5), B (6, 1), C (3, −2) and D (−1, 2).

18 Find the equation of the straight line that cuts the x-axis at 3 and is perpendicular to the line with equation $3y - 6x = 12$.

MASTER

19 Calculate the value of m for which the following pairs of equations are i parallel ii perpendicular to each other.

a $2y - 5x = 7$ and $4y + 12 = mx$
b $5x - 6y = -27$ and $15 + mx = -3y$

20 Prove that the quadrilateral ABCD is a rhombus, given A (2, 3), B (3, 5), C (5, 6) and D (4, 4).

Hint: The diagonals of a rhombus intersect at right angles.

7.5 Applications

In this section we look at two important applications: the equation of a straight line, and equations of horizontal and vertical lines.

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The equation of a straight line

The equation of a straight line can be determined by two methods.

The $y = mx + c$ method requires the gradient, m, and a given point to be known, in order to establish the value of c.

Note: Because the value of c represents the y-intercept, it can be substituted directly if known.

WORKED EXAMPLE 9 **Find the equation of the straight line that passes through the point (3, −1) and is parallel to the straight line with equation $y = 2x + 1$.**

THINK	WRITE
1 Write the general equation.	$y = mx + c$
2 Find the gradient of the given line.	$y = 2x + 1$ has a gradient of 2. Hence $m = 2$.
3 Substitute for m in the general equation.	so $y = 2x + c$
4 Substitute the given point to find c.	$(x, y) = (3, -1)$ $\therefore -1 = 2(3) + c$ $= 6 + c$ $c = -7$
5 Substitute for c in the general equation.	$y = 2x - 7$ or $2x - y - 7 = 0$

The alternative method comes from the gradient definition:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence $$m(x_2 - x_1) = y_2 - y_1$$

Using the general point (x, y) instead of the specific point (x_2, y_2) gives the general equation:

$$y - y_1 = m(x - x_1)$$

This requires the gradient, m, and a given point (x_1, y_1) to be known.

WORKED EXAMPLE 10 **Find the equation of the line that passes through the point (0, 3) and is perpendicular to a straight line with a gradient of 5.**

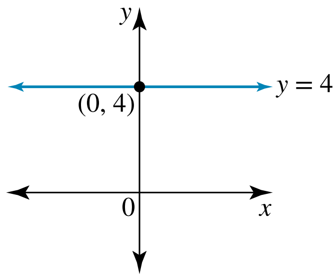
THINK	WRITE
1 Find the gradient of the perpendicular line.	Given $m = 5$ $m_1 = -\frac{1}{5}$

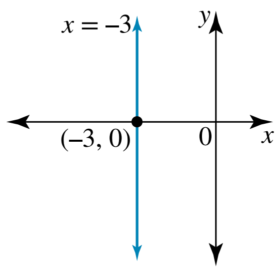
2 Substitute for m and (x_1, y_1) in the general equation.

Since $y - y_1 = m(x - x_1)$
and $(x_1, y_1) = (0, 3)$

$$\text{then } y - 3 = -\frac{1}{5}(x - 0)$$
$$= -\frac{x}{5}$$
$$5(y - 3) = -x$$
$$5y - 15 = -x$$
$$x + 5y - 15 = 0$$

Horizontal and vertical lines





For horizontal lines the gradient is equal to zero, so the equation $y = mx + c$ becomes $y = c$. Notice that x does not appear in the equation because there is no x-intercept. Horizontal lines are parallel to the x-axis. In the case of vertical lines, the gradient is infinite or undefined. The general equation for a vertical line is given by $x = a$. In this case, just as the equation suggests, a represents the x-intercept. Notice that y does not appear in the equation because there is no y-intercept. Vertical lines are parallel to the y-axis. The graphs of $y = 4$ and $x = -3$ are shown at right to highlight this information.

WORKED EXAMPLE 11

Find the equation of:

a the vertical line that passes through the point (2, −3)

b the horizontal line that passes through the point (−2, 6).

THINK

a A vertical line has no y-intercept, so y does not appear in the equation. The x-coordinate of the point is 2.

b A horizontal line has no x-intercept, so x does not appear in the equation. The y-coordinate of the point is 6.

WRITE

a $x = 2$

b $y = 6$

WORKED EXAMPLE 12

Find the equation of the perpendicular bisector of the line joining the points (0, −4) and (6, 5).

THINK

1 Find the gradient of the line joining the given points using the general equation.

WRITE

Let $(0, -4) = (x_1, y_1)$.
Let $(6, 5) = (x_2, y_2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{5 - (-4)}{6 - 0}$$
$$= \frac{9}{6}$$
$$= \frac{3}{2}$$

2 Find the gradient of the perpendicular line.

For lines to be perpendicular, $m_2 = -\frac{1}{m_1}$.

$m_1 = -\frac{2}{3}$

3 Find the midpoint of the line joining the given points.

$$x = \frac{x_1 + x_2}{2}$$
$$= \frac{0 + 6}{2}$$
$$= 3$$
$$y = \frac{y_1 + y_2}{2}$$
$$= \frac{-4 + 5}{2}$$
$$= \frac{1}{2}$$

Hence the coordinates of the midpoint are $\left(3, \frac{1}{2}\right)$.

4 Substitute for m and (x_1, y_1) in the general equation.

Since $y - y_1 = m(x - x_1)$ and $(x_1, y_1) = \left(3, \frac{1}{2}\right)$ and $m_1 = -\frac{2}{3}$

5 Simplify by removing the fractions.

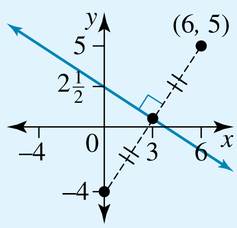
(a) Multiply both sides by 3.

(b) Multiply both sides by 2.

then $y - \frac{1}{2} = -\frac{2}{3}(x - 3)$

$$3\left(y - \frac{1}{2}\right) = -2(x - 3)$$
$$3y - \frac{3}{2} = -2x + 6$$
$$6y - 3 = -4x + 12$$
$$4x + 6y - 15 = 0$$

Note: The diagram at right shows the geometric situation.



WORKED EXAMPLE 13

ABCD is a parallelogram. The coordinates of A, B and C are (1, 5), (4, 2) and (2, −2) respectively. Find:

a **the equation of AD**

b **the equation of DC**

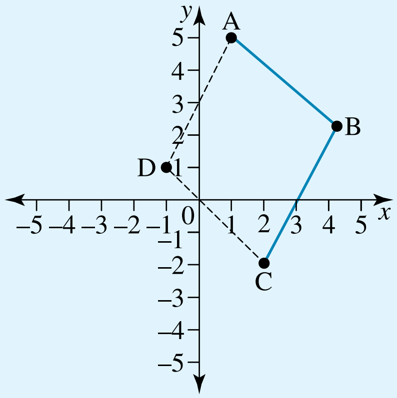
c **the coordinates of D.**

THINK | **WRITE/DRAW**

a **1** Draw the parallelogram ABCD.

Note: The order of the lettering of the geometric shape determines the links in the diagram. For example: ABCD means join A to B to C to D to A. This avoids any ambiguity.

a



2 Find the gradient of BC.

$$m_{BC} = \frac{-2 - 2}{2 - 4} = \frac{-4}{-2} = 2$$

3 State the gradient of AD.

Since $m_{BC} = 2$
and AD||BC
then $m_{AD} = 2$.

4 Using the given coordinates of A and the gradient of AD, find the equation of AD.

$y = 2x + c$
Let $(x, y) = (1, 5)$:
$5 = 2(1) + c$
$c = 3$
Hence, the equation of AD is $y = 2x + 3$.

b **1** Find the gradient of AB.

b $$m_{AB} = \frac{2 - 5}{4 - 1} = \frac{-3}{3} = -1$$

2 State the gradient of DC.

Since $m_{AB} = -1$
and DC||AB
then $m_{DC} = -1$.

3 Using the given coordinates of C and the gradient of DC, find the equation of DC.

$y = -x + c$
Let $(x, y) = (2, -2)$:
$-2 = -(2) + c$
$c = 0$
Hence, the equation of DC is $y = -x$.

c To find D, solve simultaneously the point of intersection of the equations AD and DC.

Note: Alternatively, a calculator could be used to determine the point of intersection of AD.

c Equation of AD: $y = 2x + 3$ [1]
Equation of DC: $y = -x$ [2]
[1] – [2]: $0 = 3x + 3$
$3x = -3$
$x = -1$
Substituting $x = -1$ in [2]:
$y = -(-1)$
$= 1$

Hence, the coordinates of D are $(-1, 1)$.

EXERCISE 7.5 Applications

PRACTISE

1 **WE9** Find the equation of the straight line that passes through the point $(4, -1)$ and is parallel to the straight line with equation $y = 2x - 5$.

2 Find the equation of the line that passes through the point $(3, -4)$ and is parallel to the straight line with equation $y = -x - 5$.

3 **WE10** Find the equation of the line that passes through the point $(-2, 7)$ and is perpendicular to a straight line with a gradient of $\frac{2}{3}$.

4 Find the equation of the line that passes through the point $(2, 0)$ and is perpendicular to a straight line with a gradient of -2.

5 **WE11** Find the equation of:
a the vertical line that passes through the point $(1, -8)$
b the horizontal line that passes through the point $(-5, -7)$.

6 Find the equation of:
a the vertical line that passes through the point $(-1, 4)$
b the horizontal line that passes through the point $(5, -2)$.

7 **WE12** Find the equation of the perpendicular bisector of the line joining the points $(1, 2)$ and $(-5, -4)$.

8 Find the equation of the perpendicular bisector of the line joining the points $(-4, 0)$ and $(2, -6)$.

9 **WE13** ABCD is a parallelogram. The coordinates of A, B and C are $(4, 1)$, $(1, -2)$ and $(-2, 1)$ respectively. Find:
a the equation of AD
b the equation of DC
c the coordinates of D.

10 ABCD is a parallelogram. The coordinates of A, B and C are $\left(\frac{-1}{3}, \frac{-5}{3}\right)$, $(1, 1)$ and $\left(\frac{3}{2}, -1\right)$ respectively. Find:
a the equation of AD
b the equation of DC
c the coordinates of D.

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11 Find the equations of the following straight lines.
a Gradient 3 and passing through the point $(1, 5)$
b Gradient -4 and passing through the point $(2, 1)$
c Passing through the points $(2, -1)$ and $(4, 2)$
d Passing through the points $(1, -3)$ and $(6, -5)$
e Passing through the point $(5, -2)$ and parallel to $x + 5y + 5 = 0$
f Passing through the point $(1, 6)$ and parallel to $x - 3y - 2 = 0$
g Passing through the point $(-1, -5)$ and perpendicular to $3x + y + 2 = 0$

12 Find the equation of the line that passes through the point $(-2, 1)$ and is:
a parallel to the straight line with equation $2x - y - 3 = 0$
b perpendicular to the straight line with equation $2x - y - 3 = 0$.

13 Find the equation of the line that contains the point $(1, 1)$ and is:
a parallel to the straight line with equation $3x - 5y = 0$
b perpendicular to the straight line with equation $3x - 5y = 0$.

14 The vertical line passing through the point $(3, -4)$ is given by:
A $y = -4$
B $x = 3$
C $y = 3x - 4$
D $y = -4x + 3$
E $x = -4$

15 Which of the following points does the horizontal line given by the equation $y = -5$ pass through?
A $(-5, 4)$
B $(4, 5)$
C $(3, -5)$
D $(5, -4)$
E $(5, 5)$

16 Which of the following statements is true?
A Vertical lines have a gradient of zero.
B The y-coordinates of all points on a vertical line are the same.
C Horizontal lines have an undefined gradient.
D The x-coordinates of all points on a vertical line are the same.
E A horizontal line has the general equation $x = a$.

17 Which of the following statements is false?
A Horizontal lines have a gradient of zero.
B The straight line joining the points $(1, -1)$ and $(-7, -1)$ is vertical.
C Vertical lines have an undefined gradient.
D The straight line joining the points $(1, 1)$ and $(-7, 1)$ is horizontal.
E A horizontal line has the general equation $y = c$.

18 The triangle ABC has vertices A $(9, -2)$, B $(3, 6)$ and C $(1, 4)$.
a Find the midpoint, M, of BC.
b Find the gradient of BC.
c Show that AM is the perpendicular bisector of BC.
d Describe triangle ABC.

19 Find the equation of the perpendicular bisector of the line joining the points $(-2, 9)$ and $(4, 0)$.

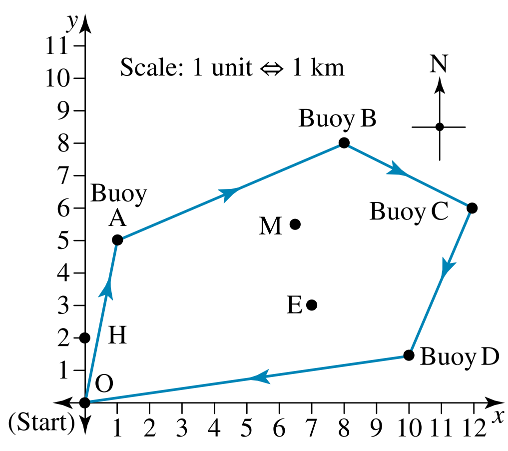
20 a The equation of the line passing through the point $(4, 3)$ and parallel to the line $2x - 4y + 1 = 0$ is:
A $x - 2y + 2 = 0$ B $2x - y - 5 = 0$ C $2x - y - 10 = 0$
D $2x - y - 11 = 0$ E $2y + x + 2 = 0$

b The equation of the perpendicular bisector of the line segment AB where A is $(-3, 5)$ and B is $(1, 7)$ is:
A $2y = x + 13$ B $y = 2x - 8$ C $2y = x + 11$
D $y = -2x + 4$ E $y = 2x - 4$

c The coordinates of the centroid of triangle ABC with vertices A $(1, 8)$, B $(9, 6)$ and C $(-1, 4)$ are:
A $(4, 5)$ B $(0, 6)$ C $(3, 6)$ D $(5, 7)$ E $(2, 7)$

MASTER

21 The map below right shows the proposed course for a yacht race. Buoys have been positioned at A $(1, 5)$, B $(8, 8)$ and C $(12, 6)$, but the last buoy's placement, D $(10, w)$, is yet to be finalised.
a How far is the first stage of the race, that is, from the start, O, to buoy A?
b The race marshall boat, M, is situated halfway between buoys A and C. What are the coordinates of the boat?
c Stage 4 of the race (from C to D) is perpendicular to stage 3 (from B to C). What is the gradient of CD?
d Find the linear equation that describes stage 4.
e Hence, determine the exact position of buoy D.
f An emergency boat is to be placed at point E, $\frac{2}{3}$ of the way from buoy A to buoy D. Into what internal ratio does point E divide the distance from A to D?



g Determine the coordinates of the emergency boat.
h How far is the emergency boat from the hospital, located at H, 2 km north of the start?

22 To supply cities with water when the source is a long distance away, artificial channels called aqueducts may be built. More than 2000 years after it was built, a Roman aqueduct still stands in southern France. It brought water from a source in Uzès to the city of Nîmes. The aqueduct does not follow a direct route between these two locations as there is a mountain range between them. The table shows the approximate distance from Uzès along the aqueduct to each town (or in the case of Pont du Gard, a bridge) and the aqueduct's height above sea level at each location.

Location	Distance from Uzès (km)	Height of aqueduct above sea level (m)
Uzès	0	76
Pont du Gard (bridge)	16	65
Saint-Bonnet	25	64
Saint-Gervasy	40	61.5
Nîmes	50	59

a Show the information in the table as a graph with the distance from Uzès along the horizontal axis. Join the plotted points with straight lines.
b Calculate the gradient of the steepest part of the aqueduct (in m/km).
c Suppose the aqueduct started at Uzès and ended at Nîmes but had a constant gradient. Write a linear equation to describe its course.
d Using the equation found in part **c**, calculate the height of the aqueduct at the Pont du Gard. This calculated height is higher than the actual height. How much higher?
e Why do you think the Romans made the first part of the aqueduct steeper than the rest?

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7.6 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

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Units 1 & 2 **Coordinate geometry**

Sit topic test

7 Answers

EXERCISE 7.2

1 a $AB = 5$
 b 5
2 a $CD = 2\sqrt{10}$
 b 13
3–7 Answers will vary.
8 a $AB = 2\sqrt{5}$, $BC = \sqrt{5}$, $CD = 2\sqrt{5}$, $DA = \sqrt{5}$
 b $AC = 5$, $BD = 5$
 c Rectangle
9 a 7.211
 b 14.765
 c 13.038
10 a $\sqrt{a^2 - 4a + 8}$
 b $\sqrt{b^2 - 12b + 61}$
 c $\sqrt{2(c^2 - 6c + 10)}$
 d $\sqrt{5d^2 - 22d + 26}$
11 B
12 D
13 a 12
 b 5
 c 13
 d −2.2
14 Answers will vary.
15 $\sqrt{442}$ km
16 a $5\sqrt{1865}$ m
 b $\sqrt{11\,729}$ m

EXERCISE 7.3

1 $(-3, -3\frac{1}{2})$
2 $(7\frac{1}{2}, 0)$
3 $(-3, -10)$
4 $(-5, 0)$
5 a $(3, 1)$
 b $2\sqrt{5}$ units
 c $3\sqrt{10}$ units
6 D
7 C
8 a $(4a, 3a)$
 b $(8, 3c)$
 c $\left(\frac{3f + g}{2}, 2\right)$
9 a 5
 b 9
10 a i $(-1, 4)$
 ii $(1\frac{1}{2}, 1)$
 iii $\frac{\sqrt{61}}{2}$ units
 iv $\sqrt{61}$
 b Answers will vary.
11 a i $(1, -0.5)$
 ii $(1, -0.5)$
 b Answers will vary.
12 a i $(-2, 2)$
 ii $4\sqrt{5}$ units
 iii $\frac{\sqrt{365}}{2}$ units
 iv $\frac{\sqrt{365}}{2}$ units
 b Isosceles triangle, perpendicular height
13 $y = -3x - 2$
14 $3y - 2x + 14 = 0$
15 D1 $(-1.5, 4.25)$, D2 $(-1.5, 1.5)$, D3 $(8, -7.5)$
16 $y = 5x - 7$

EXERCISE 7.4

1–4 Answers will vary.
5 $y = -2x + 1$
6 $3y + 2x + 1 = 0$
7 and 8 Answers will vary.
9 b, f; c, e
10–12 Answers will vary.
13 B
14 E
15 a, e; b, f; c, h; d, g
16, 17 Answers will vary.
18 $y = -\frac{1}{2}x + \frac{3}{2}$
19 a i $m = 10$ ii $m = -\frac{8}{5}$
 b i $m = \frac{-5}{2}$ ii $m = \frac{18}{5}$
20 Answers will vary.

EXERCISE 7.5

1 $y = 2x - 9$
2 $y = -x - 1$
3 $3x + 2y - 8 = 0$
4 $x - 2y - 2 = 0$
5 a $x = 1$
 b $y = -7$

6 a $x = -1$

b $y = -2$

7 $y = -x - 3$

8 $y = x - 2$

9 a $y = -x + 5$

b $y = x + 3$

c (1, 4)

10 a $y = -4x - 3$

b $y = 2x - 4$

c $\left(\frac{1}{6}, -\frac{11}{3}\right)$

11 a $y = 3x + 2$

b $y = -4x + 9$

c $3x - 2y - 8 = 0$

d $2x + 5y + 13 = 0$

e $x + 5y + 5 = 0$

f $x - 3y + 17 = 0$

g $x - 3y - 14 = 0$

12 a $2x - y + 5 = 0$

b $x + 2y = 0$

13 a $3x - 5y + 2 = 0$

b $5x + 3y - 8 = 0$

14 B

15 C

16 D

17 B

18 a (2, 5)

b 1

c Answers will vary.

d Isosceles triangle

19 $4x - 6y + 23 = 0$

20 a A

b D

c C

21 a $\sqrt{26}$ km

b (6.5, 5.5)

c 2

d $y = 2x - 18$

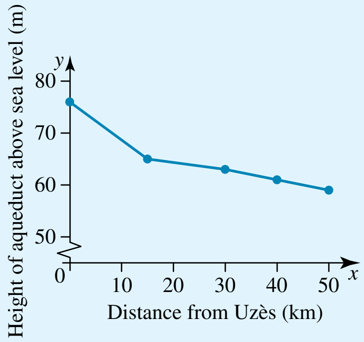
e (10, 2)

f 2:1

g (7, 3)

h $5\sqrt{2}$ km

22 a



b -0.69 m/km

c $y = -0.34x + 76$, where 76 is the height in metres above sea level and x is the distance in km from Uzès

d 5.56 m

e Check with your teacher.

8

Vectors

8.1 Kick off with CAS

Exploring the lengths of vectors with CAS

A vector in two dimensions from the origin to the terminal point P (x, y) is represented as $\overrightarrow{\text{OP}} = \underset{\sim}{r} = x\underset{\sim}{i} - y\underset{\sim}{j}$. We often need to calculate the length of the vector. Using CAS this can be done using the norm command and representing a vector as a 1×2 matrix: norm $([x \;\; y])$

1 Use CAS to find the length of each of the following vectors.

a $\underset{\sim}{i} + 2\underset{\sim}{j}$
b $3\underset{\sim}{i} - \underset{\sim}{j}$
c $4\underset{\sim}{i} - 4\underset{\sim}{j}$
d $3\underset{\sim}{i} + 4\underset{\sim}{j}$
e $-8\underset{\sim}{i} - 6\underset{\sim}{j}$
f $-5\underset{\sim}{i} + 12\underset{\sim}{j}$

2 Using CAS, define the following vectors:

$a = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, $b = -4\underset{\sim}{i} - \underset{\sim}{j}$, $c = -10\underset{\sim}{j}$

Then calculate:

a $2\underset{\sim}{a} - 2\underset{\sim}{b}$
b $\underset{\sim}{a} - 3\underset{\sim}{c}$
c $2\underset{\sim}{c} - 5\underset{\sim}{a}$
d $\underset{\sim}{a} + 2\underset{\sim}{b}$
e $\underset{\sim}{c} - \underset{\sim}{b} - \underset{\sim}{a}$
f $-3\underset{\sim}{a} + 5\underset{\sim}{b} - 2\underset{\sim}{c}$.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

8.2 Introduction to vectors

A scalar quantity is one that is specified by size, or magnitude, only.

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Units 1 & 2
AOS 4
Topic 2
Concept 1

Introduction to vectors
Concept summary
Practice questions

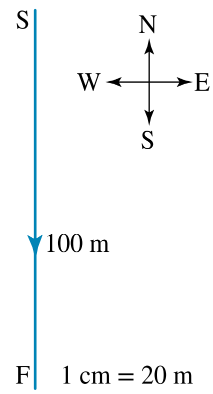
Distance is an example of a **scalar** quantity; it needs only a number to specify its size or magnitude. Time, length, volume, temperature and mass are scalars.

A vector quantity is specified by both magnitude and direction.

Displacement measures the final position compared to the starting position and requires both a magnitude (e.g. distance 800 m) and a direction (e.g. 230°T). Displacement is an example of a **vector** quantity. Force, velocity and acceleration are also vectors. They all require a size and a direction to be specified completely.

Representation of vectors

Vectors can be represented by directed line segments.



For example, if north is straight up the page and a scale of 1 cm = 20 m is used, then a displacement of 100 m south is represented by a 5 cm line straight down the page. We place an arrow on the line to indicate the direction of the vector, as shown at right.

The start and end points of a vector can be labelled with capital letters.

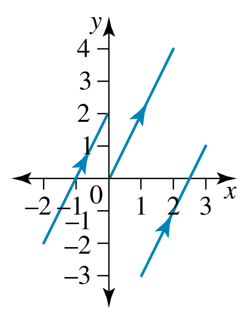
For example, the vector shown at right can have the starting point, or tail, labelled S and the end point, or head, labelled F.

This vector can then be referred to as $\overrightarrow{SF}$.

The vector can also be represented by a lower-case letter over a tilde, for example, $\underset{\sim}{s}$.

Representing a vector as an ordered pair (*a*, *b*)

A vector in the *x*–*y* plane can be described by an ordered pair (*a*, *b*).



The values *a* and *b* are called **components**; *a* gives the change of position relative to the positive *x*-axis and *b* gives the change of position relative to the positive *y*-axis of the end of the vector compared to the start.

For example (2, 4) represents a change of position of 2 units in the positive *x*-direction and 4 units in the positive *y*-direction.

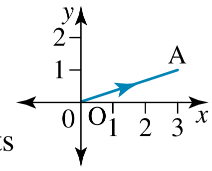
Note that the vector represented by (2, 4) doesn't necessarily start at the origin. It can be in any position on the Cartesian plane.

Representing a vector as a column matrix $\begin{bmatrix} a \\ b \end{bmatrix}$

Any vector can be written as a column matrix, which is a matrix consisting of a single column with two elements. For example, the vector represented by the directed line segments described in the previous section can be written as the column matrix $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. The top number gives the displacement relative to the positive direction on the *x*-axis and the bottom number gives the displacement relative to the positive direction on the *y*-axis.

Position vectors

A position vector describes a point in the Cartesian plane. Position vectors start at the origin O (0, 0). For example, for A (3, 1) the position vector $\overrightarrow{OA}$ is shown at right.



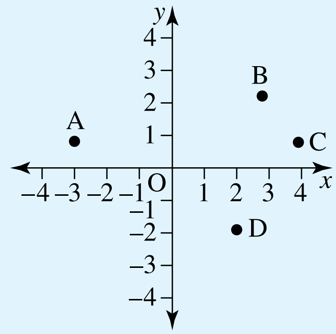
Note we can also use (3, 1) to describe any vector that travels three units across and one up, but it is only a position vector if it starts at (0, 0).

WORKED EXAMPLE 1

Write the following vectors in the form (a, b) and $\begin{bmatrix} a \\ b \end{bmatrix}$.

a $\overrightarrow{OC}$

b $\overrightarrow{DA}$



THINK	WRITE
a From O to C, we travel +4 units in the positive x-direction and +1 unit in the positive y-direction.	**a** $\overrightarrow{OC} = (4, 1)$ and $\overrightarrow{OC} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
b From D to A, we travel −5 units in the positive x-direction and +3 units in the positive y-direction.	**b** $\overrightarrow{DA} = (-5, 3)$ and $\overrightarrow{DA} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

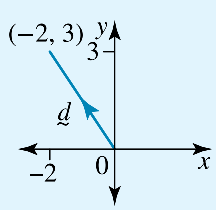
WORKED EXAMPLE 2

If we started at (5, −2), where would we end up after a displacement of (3, 2)?

THINK	WRITE
1 Write (5, −2) + (3, 2).	(5, −2) + (3, 2)
2 We start at (5, −2) and move +3 units in the positive x-direction and +2 units in the positive y-direction.	= (8, 0)
3 Write the answer.	We would end up at the point (8, 0).

WORKED EXAMPLE 3

Draw $\underset{\sim}{d}$, the position vector of (−2, 3), on a set of axes.

THINK	DRAW
1 A position vector must start at (0, 0) and end at the point specified. Make sure the arrow is pointing away from the origin.	
2 Label the vector.	

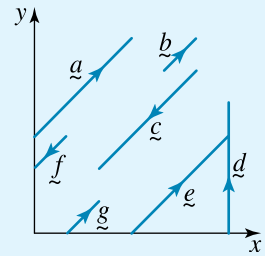
Equality of vectors

Two vectors are equal if they are:

1. equal in magnitude
2. parallel, and
3. point in the same direction.

WORKED EXAMPLE 4

Which of the vectors shown at right are equal?



THINK

1 Vectors $\underset{\sim}{a}$ and $\underset{\sim}{e}$ are of equal length, parallel and point in the same direction.

2 Vectors $\underset{\sim}{b}$ and $\underset{\sim}{g}$ are of equal length, parallel and point in the same direction.

WRITE

$\underset{\sim}{a} = \underset{\sim}{e}$

$\underset{\sim}{b} = \underset{\sim}{g}$

WORKED EXAMPLE 5

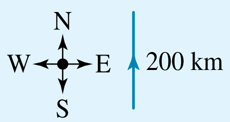
An aircraft flies 200 km north, then 400 km east.

Draw a vector diagram to represent the path taken by the aircraft and also the displacement of the aircraft from its starting point to its finishing point.

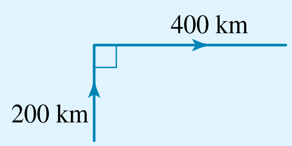
THINK

1 Take north as vertically up the page and east to the right.

2 Draw a short vertical directed line segment to represent a displacement of 200 km north.

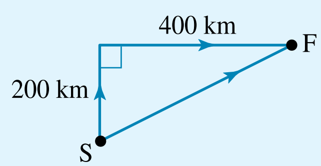


3 Draw a horizontal directed line segment with its tail joined to the head of the first. This represents a displacement of 400 km east.



4 Draw a directed line segment from the tail of the 'north' vector (point S) to the head of the 'east' vector (point F). This represents the displacement of the aircraft from its starting point to its finishing point.

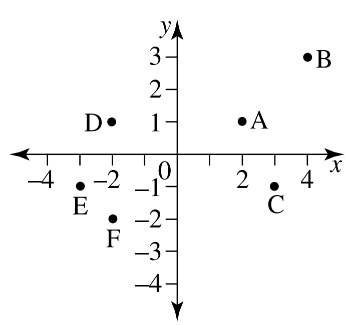
DRAW



EXERCISE 8.2 Introduction to vectors

PRACTISE

1 **WE1** Examine the diagram at right. Represent each of the following vectors as an ordered pair (a, b).



a $\overrightarrow{AB}$

b $\overrightarrow{AC}$

c $\overrightarrow{AF}$

d $\overrightarrow{BC}$

e $\overrightarrow{BD}$

2 Examine the diagram in question 1. Represent each of the following vectors as a column matrix $\begin{bmatrix} a \\ b \end{bmatrix}$.

a $\overrightarrow{CD}$ b $\overrightarrow{CA}$ c $\overrightarrow{ED}$

d $\overrightarrow{EF}$ e $\overrightarrow{FE}$

3 WE2 If we started at the point (1, 1) where would we end up after a displacement of (−3, 6)?

4 If we started at the point (2, −5), where would we end up after each of these displacements?

a (3, −2) b (−3, 5)

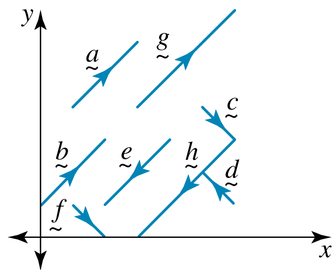
c (0, 4) d (2, −5)

5 WE3 Draw $\underset{\sim}{d}$, the position vector (−1, 4), on a set of axes.

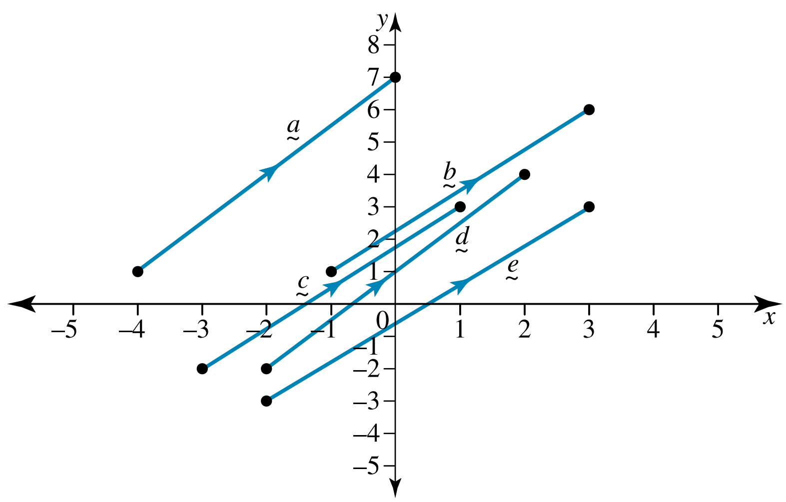
6 Draw the position vector for each of the following points on the same set of axes.

A (4, 1), B (−3, 2), C (0, −3), D (−2, −2)

7 WE4 Which of the vectors shown in the diagram below are equal?



8 Which of the vectors shown in the diagram below are equal?

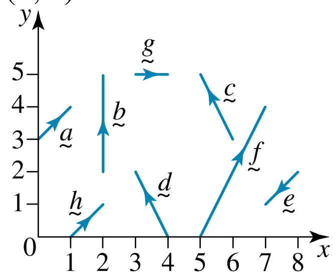


9 WE5 An aeroplane flies 1000 km north from airport A to airport B. It then travels to airport C, which is 1200 km north-east of B. Draw a vector diagram to represent the path taken by the aeroplane and the displacement of the finishing point from the starting point.

10 A boat travels 30 km north and then 40 km west. Draw a vector diagram showing the path of the boat and the displacement of the finishing point from the starting point.

CONSOLIDATE

11 Examine the diagram below. Represent the change of position of each of the vectors shown in the form (a, b).



12 Represent the change of position of each of the vectors shown in question **11** in the form $\begin{bmatrix} a \\ b \end{bmatrix}$.

13 Represent each of the following vectors on separate diagrams.

a The position vector of (2, 3)

b The position vector of (0, 5)

c The position vector of (−3, 2)

14 Represent each of the following vectors on separate diagrams.

a A displacement of (2, −8) starting from the point (4, 4)

b A displacement of (−2, 5) starting from the point (3, −6)

c A displacement of (0, 3) starting from the point (2, 5)

d The position vector of (4, −2) followed by (3, 5)

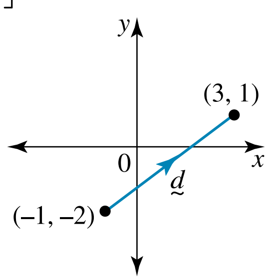
15 A vector that starts at the point (−2, 1) and finishes at the point (3, −3) is represented by a displacement of:

A (4, −5) **B** (5, −4) **C** (1, −2) **D** (−5, 4) **E** (3, 2)

16 Draw two directed line segments represented by the vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

17 The directed line segment shown in the diagram represents the vector $\underset{\sim}{d}$.

Find a and b if $\underset{\sim}{d} = \begin{bmatrix} a \\ b \end{bmatrix}$.



18 Sketch the following vectors on separate axes, if A = (2, –1), B = (0, 2), C = (4, 1) and O is the origin.

a $\overrightarrow{OA}$

b $\overrightarrow{AB}$

c $\overrightarrow{AC}$

d $\overrightarrow{BC}$

19 Express each of the vectors from question **18** in the form $\begin{bmatrix} a \\ b \end{bmatrix}$.

20 Marcus cycles 20 km in an easterly direction and then travels 30 km due south. Draw a vector diagram to represent Marcus's path and the displacement of the finishing point from the starting point.

MASTER

In questions **21** and **22**, draw vector diagrams to represent the paths described and the displacement of the finishing point from the starting point.

21 Bianca rows straight across a river in which a current is flowing at 3.5 km/h. Bianca can row at 11.5 km/h.

22 An aeroplane takes off and flies at an angle of elevation of 25° for 25 km. It then flies horizontally for 300 km.

8.3 Operations on vectors

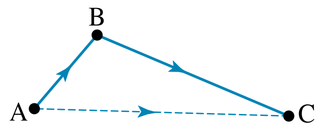
Addition of vectors

studyon

Units 1 & 2
AOS 4
Topic 2
Concept 2

Operations on vectors
Concept summary
Practice questions

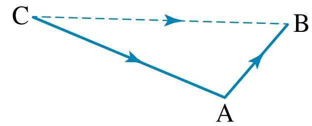
If we travel from A to B and then from B to C, the combined effect is to start from A and finish at C. We write



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Notice that the tail of the second vector $\overrightarrow{BC}$ is joined to the head of the first vector $\overrightarrow{AB}$.

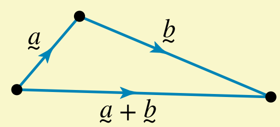
If the addition is reversed, so that the tail of the first vector is joined to the head of the second vector, the combined effect is also a vector equal to $\overrightarrow{AC}$. So $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}$.



This shows that changing the order in which vectors are added does not alter the combined effect of the vectors.

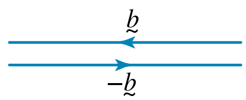
This method for adding two vectors is called the **triangle rule** for vectors.

The addition of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ can be shown by forming a vector from the tail of $\underset{\sim}{a}$ to the head of $\underset{\sim}{b}$.



Negative vectors

Just as moving −2 units on the *x*-axis is opposite in direction to moving 2 units along the *x*-axis, the negative of a given vector is opposite in direction to the original vector.



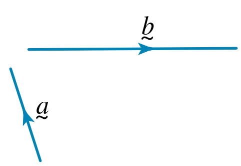
The vector $-\underset{\sim}{b}$ has the same magnitude as $\underset{\sim}{b}$ but is in the opposite direction.

Subtraction of vectors

Subtraction of vectors can be performed by combining vector addition and negative vectors.

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$$

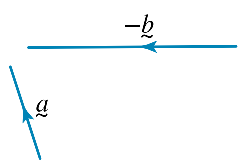
For example, if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are vectors as shown at right, then we can find $\underset{\sim}{a} - \underset{\sim}{b}$ by:



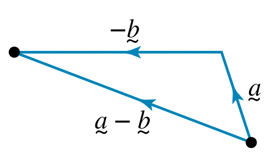
1. expressing it as an addition:

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$$

2. reversing the arrow on vector $\underset{\sim}{b}$ so that it becomes $-\underset{\sim}{b}$



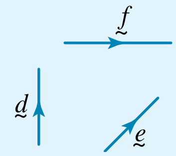
3. adding $-\underset{\sim}{b}$ to $\underset{\sim}{a}$ as shown to form $\underset{\sim}{a} - \underset{\sim}{b}$.

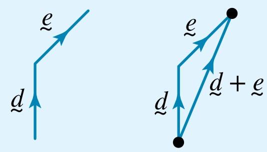
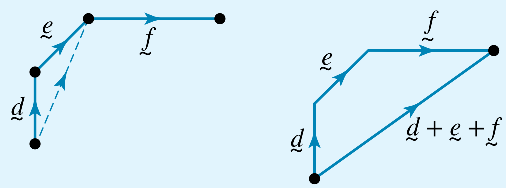
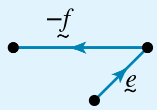
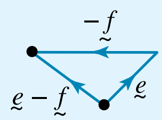


WORKED EXAMPLE 6

Using $\underset{\sim}{d}$, $\underset{\sim}{e}$ and $\underset{\sim}{f}$ as shown in the diagram, draw vector diagrams to show:

a $\underset{\sim}{d} + \underset{\sim}{e}$ **b** $\underset{\sim}{d} + \underset{\sim}{e} + \underset{\sim}{f}$ **c** $\underset{\sim}{e} - \underset{\sim}{f}$.



THINK	DRAW
a **1** Draw the vector $\underset{\sim}{d}$ and join the tail of $\underset{\sim}{e}$ to the head of $\underset{\sim}{d}$. **2** $\underset{\sim}{d} + \underset{\sim}{e}$ is shown by the vector drawn from the tail of $\underset{\sim}{d}$ to the head of $\underset{\sim}{e}$.	**a** 
b **1** $\underset{\sim}{d} + \underset{\sim}{e} + \underset{\sim}{f}$ is obtained by joining the head of $\underset{\sim}{d} + \underset{\sim}{e}$ (from part **a**) with the tail of $\underset{\sim}{f}$. **2** $\underset{\sim}{d} + \underset{\sim}{e} + \underset{\sim}{f}$ is shown by the vector drawn from the tail of $\underset{\sim}{d}$ (or $\underset{\sim}{d} + \underset{\sim}{e}$) to the head of $\underset{\sim}{f}$.	**b** 
c **1** Reverse the arrow on $\underset{\sim}{f}$ to obtain $-\underset{\sim}{f}$ and join the head of $\underset{\sim}{e}$ to the tail of $-\underset{\sim}{f}$.	**c** 
2 $\underset{\sim}{e} - \underset{\sim}{f}$ is shown by the vector drawn from the tail of $\underset{\sim}{e}$ to the head of $-\underset{\sim}{f}$.	

WORKED EXAMPLE 7

If $\underset{\sim}{a} = (1, 4)$, $\underset{\sim}{b} = (-5, 2)$ and $\underset{\sim}{c} = (-2, 3)$, find each of the following:

a $\underset{\sim}{a} + \underset{\sim}{b}$ **b** $\underset{\sim}{a} - \underset{\sim}{c}$ **c** $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$.

THINK	WRITE
a Add the corresponding components of each vector to give the answer for $\underset{\sim}{a} + \underset{\sim}{b}$.	**a** $\underset{\sim}{a} + \underset{\sim}{b} = (1, 4) + (-5, 2)$ $= (-4, 6)$
b Subtract the corresponding components of each vector to give the answer for $\underset{\sim}{a} - \underset{\sim}{c}$.	**b** $\underset{\sim}{a} - \underset{\sim}{c} = (1, 4) - (-2, 3)$ $= (3, 1)$
c $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ may be calculated by adding the corresponding components of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and $\underset{\sim}{c}$.	**c** $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = (1, 4) + (-5, 2) + (-2, 3)$ $= (-6, 9)$

Scalar multiplication

A displacement of (2, 3) followed by another displacement of (2, 3) equals a displacement of (4, 6).

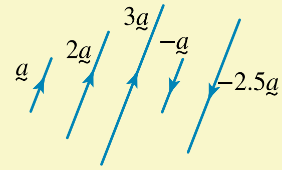
We could write this as 2(2, 3) = (4, 6).

The vector represented by (2, 3) has been multiplied by the number 2 to give the vector represented by (4, 6).

This process is called multiplication by a scalar or **scalar multiplication**. Scalar multiplication means that the vector is made larger or smaller by a scale factor. In the case above, the scalar is 2.

In general, we can say that if $k \in R$:

1. **$k\underset{\sim}{a}$ is a vector k times as big as $\underset{\sim}{a}$ and in the same direction as $\underset{\sim}{a}$ for $k > 0$.**
2. **$k\underset{\sim}{a}$ is in the opposite direction to $\underset{\sim}{a}$ for $k < 0$.**



WORKED EXAMPLE 8

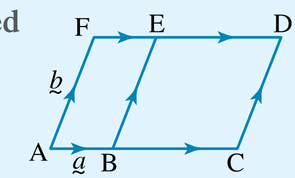
If $\underset{\sim}{a} = (5, -4)$ and $\underset{\sim}{b} = (-3, 2)$, calculate:

a $2\underset{\sim}{a} + \underset{\sim}{b}$ **b** $3(\underset{\sim}{b} - \underset{\sim}{a})$.

THINK	WRITE
a 1 Multiply each component of $\underset{\sim}{a}$ by 2 to obtain $2\underset{\sim}{a}$.	**a** $2\underset{\sim}{a} = 2(5, -4)$ $= (10, -8)$
2 Add the components of $2\underset{\sim}{a}$ and $\underset{\sim}{b}$ to obtain $2\underset{\sim}{a} + \underset{\sim}{b}$.	$2\underset{\sim}{a} + \underset{\sim}{b} = (10, -8) + (-3, 2)$ $= (7, -6)$
b 1 Subtract the components of $\underset{\sim}{a}$ from $\underset{\sim}{b}$ to obtain $\underset{\sim}{b} - \underset{\sim}{a}$.	**b** $\underset{\sim}{b} - \underset{\sim}{a} = (-3, 2) - (5, -4)$ $= (-8, 6)$
2 Multiply the components of $\underset{\sim}{b} - \underset{\sim}{a}$ by 3 to obtain $3(\underset{\sim}{b} - \underset{\sim}{a})$.	$3(\underset{\sim}{b} - \underset{\sim}{a}) = 3(-8, 6)$ $= (-24, 18)$

WORKED EXAMPLE 9

ABEF and BCDE are parallelograms with $\overrightarrow{AB}$ represented by $\underset{\sim}{a}$ and $\overrightarrow{AF}$ represented by $\underset{\sim}{b}$. The length of BC is twice the length of AB. Express the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



a $\overrightarrow{BC}$ **b** $\overrightarrow{AC}$ **c** $\overrightarrow{BD}$

THINK	WRITE
a 1 $\overrightarrow{BC}$ and $\overrightarrow{AB}$ are in the same direction and $\overrightarrow{BC}$ is twice as big as $\overrightarrow{AB}$.	**a** $\overrightarrow{BC} = 2\overrightarrow{AB}$
2 Replace $\overrightarrow{AB}$ by $\underset{\sim}{a}$.	$= 2\underset{\sim}{a}$
b 1 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ using vector addition.	**b** $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
2 Replace $\overrightarrow{AB}$ and $\overrightarrow{BC}$ by $\underset{\sim}{a}$ and $2\underset{\sim}{a}$ respectively.	$= \underset{\sim}{a} + 2\underset{\sim}{a}$
3 Simplify.	$= 3\underset{\sim}{a}$
c 1 $\overrightarrow{CD} = \overrightarrow{AF}$ because opposite sides of a parallelogram are parallel and the same size.	**c** $\overrightarrow{CD} = \underset{\sim}{b}$
2 $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$, using the triangle rule to add vectors.	$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
3 Replace $\overrightarrow{BC}$ and $\overrightarrow{CD}$ by $2\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively.	$= 2\underset{\sim}{a} + \underset{\sim}{b}$

WORKED EXAMPLE 10

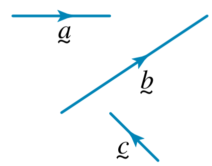
Simplify the expression $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{EC}$.

THINK	WRITE
1 $\overrightarrow{AB} + \overrightarrow{BC}$ represents a vector from A to B with the vector from B to C added on. This is the same as the vector from A to C.	$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{EC}$ $= \overrightarrow{AC} - \overrightarrow{EC}$
2 $-\overrightarrow{EC}$ is the same as vector $\overrightarrow{CE}$ because the negative of a vector reverses the direction.	$= \overrightarrow{AC} + \overrightarrow{CE}$
3 $\overrightarrow{AC} + \overrightarrow{CE}$ represents a vector starting at A going to C and then from C to E. This is the same as $\overrightarrow{AE}$.	$= \overrightarrow{AE}$

EXERCISE 8.3 Operations on vectors

PRACTISE

1 **WE6** Using vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ as shown, sketch:



a $3\underset{\sim}{a}$ b $2\underset{\sim}{b}$ c $-\underset{\sim}{c}$
d $\underset{\sim}{a} + \underset{\sim}{b}$ e $\underset{\sim}{a} + \underset{\sim}{c}$ f $\underset{\sim}{b} + \underset{\sim}{c}$.

2 Using $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ from question 1, sketch:
a $\underset{\sim}{a} + 2\underset{\sim}{b}$ b $2\underset{\sim}{a} + 3\underset{\sim}{c}$ c $\underset{\sim}{a} - \underset{\sim}{c}$
d $\underset{\sim}{b} - \underset{\sim}{c}$ e $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ f $\underset{\sim}{a} - \underset{\sim}{b} - \underset{\sim}{c}$.

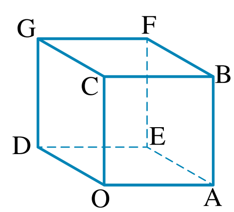
3 **WE7** If $\underset{\sim}{m} = (-2, 3)$, $\underset{\sim}{n} = (4, 0)$ and $\underset{\sim}{p} = (-1, 5)$, find each of the following.
a $\underset{\sim}{m} + \underset{\sim}{n}$ b $\underset{\sim}{m} + \underset{\sim}{n} + \underset{\sim}{p}$

4 Using $\underset{\sim}{m}$, $\underset{\sim}{n}$ and $\underset{\sim}{p}$ from question 3, find each of the following.
a $\underset{\sim}{n} - \underset{\sim}{p}$ b $\underset{\sim}{m} - \underset{\sim}{n} - \underset{\sim}{p}$

5 **WE8** Using $\underset{\sim}{m}$, $\underset{\sim}{n}$ and $\underset{\sim}{p}$ from question 3, calculate the following.
a $3\underset{\sim}{n} - \underset{\sim}{p}$ b $2\underset{\sim}{m} + \underset{\sim}{n} - \underset{\sim}{p}$

6 Using $\underset{\sim}{m}$, $\underset{\sim}{n}$ and $\underset{\sim}{p}$ from question 3, calculate the following.
a $2(\underset{\sim}{m} + \underset{\sim}{n})$ b $4\underset{\sim}{p} - 3\underset{\sim}{n}$

7 **WE9** Refer to the cube shown on the right. Let $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{c} = \overrightarrow{OC}$ and $\underset{\sim}{d} = \overrightarrow{OD}$. Write, in terms of $\underset{\sim}{a}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$, the vectors representing:



a $\overrightarrow{DE}$ b $\overrightarrow{OB}$
c $\overrightarrow{AC}$ d $\overrightarrow{AE}$
e $\overrightarrow{EA}$.

8 Using the cube from question 7, write the following vectors in terms of $\underset{\sim}{a}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$.
a $\overrightarrow{EG}$ b $\overrightarrow{DF}$ c $\overrightarrow{OF}$ d $\overrightarrow{AG}$ e $\overrightarrow{DB}$

9 **WE10** Show that $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OC}$.

10 In simplest form, $\overrightarrow{MN} - \overrightarrow{QP} + \overrightarrow{NP} + \overrightarrow{QR}$ equals:
A 0 B $\overrightarrow{MR}$ C $\overrightarrow{MQ}$ D $\overrightarrow{QN}$ E $\overrightarrow{NR}$

CONSOLIDATE

11 Draw two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ such that $\underset{\sim}{u} + \underset{\sim}{v} = (0, 0)$.

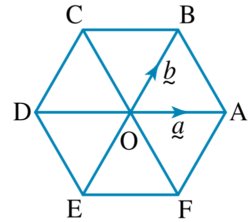
12 Draw two possible representations of $\underset{\sim}{u} + \underset{\sim}{v} = (3, 5)$.

13 Draw two possible representations of $\underset{\sim}{u} + \underset{\sim}{v} = (-3, 2)$.

14 Using the cube shown in question 7, write all the vectors that are equal to the following vectors.

a $\overrightarrow{OA}$ b $\overrightarrow{OC}$ c $\overrightarrow{OD}$ d $\overrightarrow{GF}$ e $\overrightarrow{OB}$ f $\overrightarrow{AD}$

15 ABCDEF is a regular hexagon with vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represented by $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Write, in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, the following vectors:



a $\overrightarrow{DO}$ b $\overrightarrow{DA}$ c $\overrightarrow{AD}$ d $\overrightarrow{AB}$

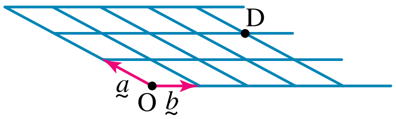
e $\overrightarrow{BC}$ f $\overrightarrow{AC}$ g $\overrightarrow{CD}$ h $\overrightarrow{ED}$

i $\overrightarrow{EA}$ j $\overrightarrow{DF}$.

16 Show that $\overrightarrow{EF} + \overrightarrow{GH} - \overrightarrow{GF} - \overrightarrow{EH} = 0$.

17 Express in simplest form $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{DE} - \overrightarrow{DC}$.

18 In terms of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the figure at right, the vector joining O to D is given by:

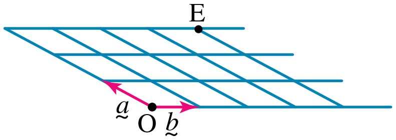


A $3\underset{\sim}{a} + 3\underset{\sim}{b}$ B $2\underset{\sim}{a} + 4\underset{\sim}{b}$

C $3\underset{\sim}{b} - 2\underset{\sim}{a}$ D $2\underset{\sim}{a} - 3\underset{\sim}{b}$

E none of these

19 In terms of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, the vector joining E to O at right is:

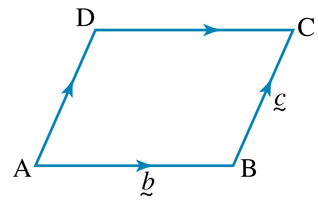


A $3\underset{\sim}{a} + 4\underset{\sim}{b}$ B $4\underset{\sim}{b} - 3\underset{\sim}{a}$

C $3\underset{\sim}{a} - 4\underset{\sim}{b}$ D $-3\underset{\sim}{a} - 4\underset{\sim}{b}$

E none of these

20 The parallelogram ABCD can be defined by the two vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$. In terms of these vectors, find:



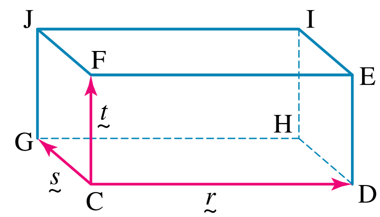
a the vector from A to D

b the vector from C to D

c the vector from D to B.

MASTER

21 A rectangular prism CDEFGHIJ can be defined by the vectors $\underset{\sim}{r}$, $\underset{\sim}{s}$ and $\underset{\sim}{t}$, as shown on the right. Express in terms of $\underset{\sim}{r}$, $\underset{\sim}{s}$ and $\underset{\sim}{t}$:



a the vector joining C to H

b the vector joining C to J

c the vector joining G to D

d the vector joining F to I

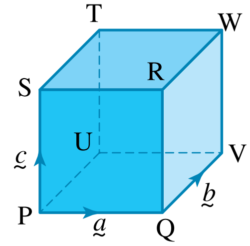
e the vector joining H to E

f the vector joining D to J

g the vector joining C to I

h the vector joining J to C.

22 A cube PQRSTUVW can be defined by the three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ as shown at right. Express in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$:



a the vector joining P to V

b the vector joining P to W

c the vector joining U to Q

d the vector joining S to W

e the vector joining Q to T.

8.4 Magnitude, direction and components of vectors

Magnitude

studyon

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AOS 4
Topic 2
Concept 3

Magnitude of a vector
Concept summary
Practice questions

The magnitude of a vector can be calculated from the length of the line segment representing the vector.

> **The magnitude of a vector $\underset{\sim}{a}$ is denoted by $|\underset{\sim}{a}|$ or a.**

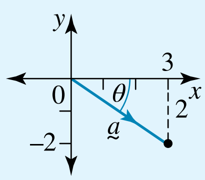
Direction

The direction of a vector can be found by applying appropriate trigonometric ratios to find a relevant angle.

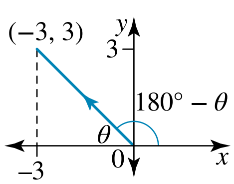
This angle is usually the angle that the vector makes with a given direction such as north, the positive x-axis, or the horizontal or vertical.

WORKED EXAMPLE 11

Find the magnitude and direction, relative to the positive x-axis, of the vector $(3, -2)$.

THINK	DRAW/WRITE
1 Draw a diagram of the vector and denote it as $\underset{\sim}{a}$ with the angle between $\underset{\sim}{a}$ and the positive x-axis as θ.	
2 The magnitude of $\underset{\sim}{a}$ is the length of the line segment representing the vector.	
3 Use Pythagoras' theorem to calculate this length.	$|\underset{\sim}{a}| = \sqrt{3^2 + 2^2}$ $= \sqrt{13}$
4 Calculate the angle θ using trigonometry.	$\tan(\theta) = \frac{2}{3}$ $\theta = 33.7°$ (correct to 1 decimal place)
5 State the solution with the angle down from the positive x-axis given as a negative.	The vector $(3, -2)$ has a magnitude of $\sqrt{13}$ units and makes an angle of $-33.7°$ with the positive x-axis.

The angle that a vector makes with the positive x-axis can be found using trigonometry. If the vector points in the negative x-direction, then you will need to add your found angle θ to 90° or subtract it from 180° to find the required angle. See the diagram at right.



Upward vectors are expressed as positive angles anticlockwise from the positive x-axis. Downward vectors are expressed as negative angles clockwise from the positive x-axis.

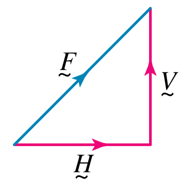
> **In general, if $\underset{\sim}{r} = (a, b)$, then the direction of $\underset{\sim}{r}$ compared to the positive x-axis is found by appropriately adjusting θ where $\tan(\theta) = \left|\frac{b}{a}\right|$.**

Vector components

We have seen that two vectors may be added to give one resultant vector. The reverse process may be used to express one vector as the sum of two other vectors. This process is called 'breaking the vector into two components'.

> **A vector can be broken into two perpendicular components such as x and y or north and east.**

It may be convenient to find the effect of a vector in a particular direction. We do this by breaking the vector into two components.



A force $\underset{\sim}{F}$ acting as shown will move an object to the right and upwards.

The force $\underset{\sim}{F}$ can be separated into two component parts; one in the horizontal direction, $\underset{\sim}{H}$, and the other in the vertical direction, $\underset{\sim}{V}$.

$$\underset{\sim}{F} = \underset{\sim}{H} + \underset{\sim}{V}$$

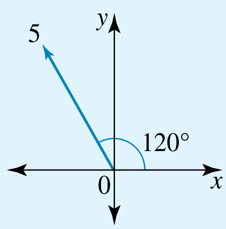
The effect of the force in the horizontal direction is given entirely by $\underset{\sim}{H}$ and the effect in the vertical direction is given by $\underset{\sim}{V}$.

By breaking $\underset{\sim}{F}$ into component parts in two perpendicular directions, we can analyse the effect of the vector in one or both of these directions.

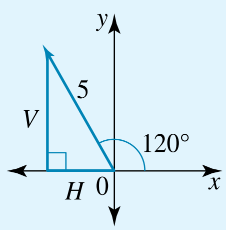
WORKED EXAMPLE 12 **Write the horizontal and vertical components of a vector of magnitude 5 and angle of 120° with the positive x-axis.**

THINK | **DRAW/WRITE**

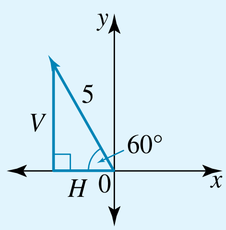
1 Represent the vector on the Cartesian plane.



2 Construct a right-angled triangle with the vector as the hypotenuse and the other sides H for horizontal and V for vertical.



3 Calculate the angle between the vector and the x-axis and indicate it on the graph.



$$\text{Angle} = 180° - 120° = 60°$$

4 Calculate V using the sine ratio.

$$\sin(60°) = \frac{V}{5}$$

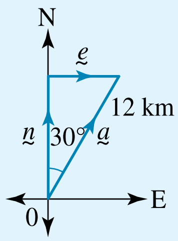
$$V = 5\sin(60°) = \frac{5\sqrt{3}}{2} \text{ (or 4.33)}$$

5 Calculate H using the cosine ratio.	$\cos(60°) = \dfrac{H}{5}$ $H = 5\cos(60°)$ $= \dfrac{5}{2}$ (or 2.5)
6 State the solution, adding negative signs where necessary.	The vector has a horizontal component of $-\dfrac{5}{2}$ and a vertical component of $\dfrac{5\sqrt{3}}{2}$.

WORKED EXAMPLE 13

A car travels 12 km in a direction N30°E. From its starting point, how far has it travelled:

a north **b east?**

THINK	DRAW/WRITE
a 1 Draw a vector diagram representing the motion of the car. Call the vector $\underset{\sim}{a}$ and its eastern and northern components $\underset{\sim}{e}$ and $\underset{\sim}{n}$ respectively.	**a** 
2 Calculate n (the magnitude of $\underset{\sim}{n}$) using the cosine ratio.	$\cos(30°) = \dfrac{n}{12}$ $n = 12\cos(30°) = 12\dfrac{\sqrt{3}}{2}$ $= 6\sqrt{3}$
3 State the distance travelled by the car to the north.	The car has travelled to approximately $6\sqrt{3}$ km north of its starting point.
b 1 Calculate e (the magnitude of $\underset{\sim}{e}$) using the sine ratio.	**b** $\sin(30°) = \dfrac{e}{12}$ $e = 12\sin(30°)$ $= 6$
2 State the distance travelled by the car to the east.	The car has travelled to 6 km east of its starting point.

EXERCISE 8.4 Magnitude, direction and components of vectors

PRACTISE

1 WE11 Calculate the exact magnitude and direction, relative to the positive x-axis, of the following displacements.

a (6, 2) **b** (4, −1) **c** (2, 4) **d** (1, 1)

2 Calculate the exact magnitude and direction, relative to the positive x-axis, of the following displacements.

a $(-2, 1)$ b $(-1, 4)$ c $(1, 0)$ d $(-2, -2)$

3 WE12 Write the horizontal and vertical components of these vectors. Write your answers in exact form where possible.

a Magnitude 2, angle of 60° with the x-axis
b Magnitude 3, angle of 150° with the x-axis
c Magnitude 10, angle of −60° with the x-axis
d Magnitude 2, angle of −120° with the x-axis
e Magnitude 20, angle of 45° with the x-axis
f Magnitude 4, parallel to the y-axis
g Magnitude 12, parallel to the x-axis

4 Write the horizontal and vertical components of these vectors. Write your answers in exact form where possible.

a A speed of 30 m/s for one second vertically downwards
b A move of magnitude 10 m at an angle of 30° anticlockwise from the negative direction of the x-axis
c A move of magnitude 20 m at an angle of 30° anticlockwise from the positive direction of the x-axis
d A speed of 50 m/s horizontally to the right
e A force of 40 N at an angle of 20° to the horizontal
f A force of 98 N vertically downwards
g A force of 1250 N at an angle of 15° to the horizontal

5 WE13 A yacht sails 32 km in a direction S25°E. From its starting point how far has it travelled:

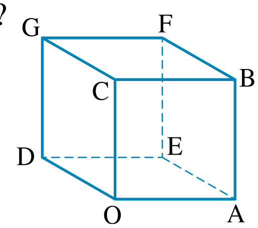
a south b east?

6 A car travels 20 km in a direction N45°W. From its starting point, how far has it travelled:

a north (to the nearest km) b west (to 1 decimal place)?

CONSOLIDATE

7 Refer to the diagram of the cube shown. If the sides of the cube are 1 unit in length, write the magnitudes of these vectors in exact form.

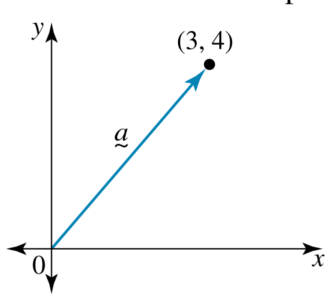


a $\overrightarrow{OA}$ b $\overrightarrow{AB}$ c $\overrightarrow{OB}$
d $\overrightarrow{OD}$ e $\overrightarrow{AD}$ f $\overrightarrow{DF}$
g $\overrightarrow{OE}$ h $\overrightarrow{EF}$ i $\overrightarrow{OF}$ j $\overrightarrow{AG}$

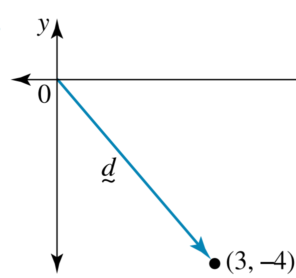
8 A vector has a horizontal component of $-x$ $(x > 0)$ and a vertical component of y $(y > 0)$. Write the magnitude and direction from the positive x-axis of the vector.

9 Find the magnitude and direction of each of the following vectors. Express the direction relative to the positive x-axis.

a



b

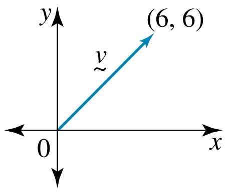


10 Write the horizontal and vertical components of a vector of magnitude 30 on an angle of 310° with the positive x-axis. Give answers correct to 1 decimal place.

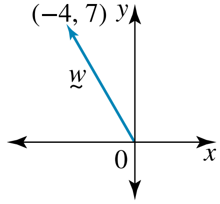
11 For each of the following, find:

- **i** the magnitude of the vector
- **ii** the direction of each vector. (Express the direction with respect to the positive x-axis)

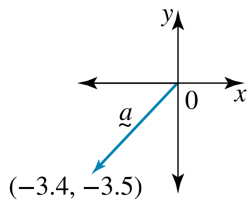
a



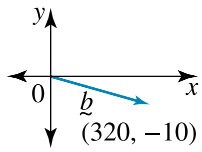
b



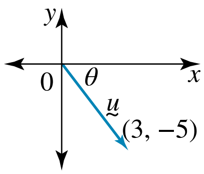
c



d



12 Using the vector shown at right, find:

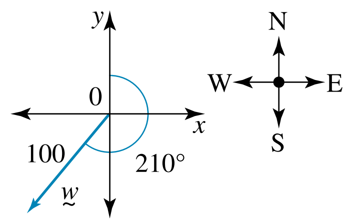


- **a** the magnitude of $\underset{\sim}{u}$
- **b** the direction of $\underset{\sim}{u}$ (express the angle with respect to the positive x-axis)
- **c** the true bearing of $\underset{\sim}{u}$.

13 A vector with a true bearing of 60 degrees and a magnitude of 10 has:

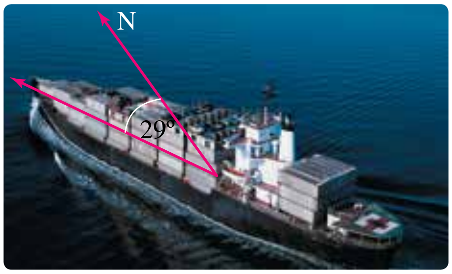
- **A** x-component $= \dfrac{\sqrt{3}}{2}$, y-component $= \dfrac{1}{2}$
- **B** x-component $= \dfrac{1}{2}$, y-component $= \dfrac{\sqrt{3}}{2}$
- **C** x-component $= 5\sqrt{3}$, y-component $= 5$
- **D** x-component $= 5$, y-component $= 5\sqrt{3}$
- **E** none of the above

14 Consider the vector $\underset{\sim}{w}$, shown on the right, with a magnitude of 100 and on a bearing of 210° T. Find the x and y components of $\underset{\sim}{w}$. Express answers as exact values.



15 Justine cycles 8 km in a northerly direction. She then travels 6 km in an easterly direction. Calculate the magnitude and direction of her displacement.

16 Express the horizontal and vertical components of a vector to the nearest whole number represented by a ship that sails on a bearing of 331° for 125 km.



MASTER

17 For the following pairs of vectors, calculate the magnitude and direction of $\underset{\sim}{a} + \underset{\sim}{b}$.

- **a** $\underset{\sim}{a}$ = 10 km north and $\underset{\sim}{b}$ = 6 km north-east
- **b** $\underset{\sim}{a}$ = 25 units east and $\underset{\sim}{b}$ = 20 units S30°W
- **c** $\underset{\sim}{a}$ = 10 units and $\underset{\sim}{b}$ = 8 units in the opposite direction

18 For the following pairs of vectors, calculate the magnitude and direction of $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$.

a $\underset{\sim}{a}$ = 12 km west and $\underset{\sim}{b}$ = 12 km south

b $\underset{\sim}{a}$ = 20 km and $\underset{\sim}{b}$ = 15 km in the same direction

c $\underset{\sim}{a}$ = 50 units in a direction 300° T and $\underset{\sim}{b}$ = 40 units in a direction 30° T

8.5 $\underset{\sim}{i}$, $\underset{\sim}{j}$ notation

Unit vectors

studyon

Units 1 & 2

AOS 4

Topic 2

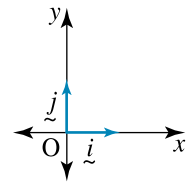
Concept 4

$\underset{\sim}{i}$, $\underset{\sim}{j}$ notation and resolving a vctor

Concept summary

Practice questions

1. A **unit vector** is any vector with a magnitude or length of 1 unit.
2. The vector $\underset{\sim}{i}$ is defined as the unit vector in the positive x-direction.
3. The vector $\underset{\sim}{j}$ is defined as the unit vector in the positive y-direction.



For example, a displacement of $\underset{\sim}{d} = (2, 5)$ represents a move of 2 units in the positive x-direction and 5 units in the positive y-direction.

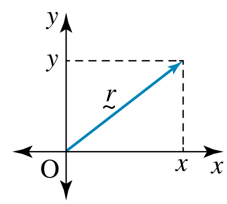
An alternative way of representing this is:

$$\underset{\sim}{d} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$$

Any vector in two dimensions can be represented as a combination of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ vectors, the coefficient of $\underset{\sim}{i}$ representing the magnitude of the horizontal component and the coefficient of $\underset{\sim}{j}$ representing the magnitude of the vertical component.

In general we may represent any two-dimensional vector $\underset{\sim}{r}$ as:

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} \text{ where } x, y \in R$$



This vector, $\underset{\sim}{r}$, may also be written as a column matrix:

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} = \begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ where } \underset{\sim}{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \underset{\sim}{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For example,

$$\underset{\sim}{d} = 2\underset{\sim}{i} + 5\underset{\sim}{j} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

WORKED EXAMPLE 14

a **Draw a vector to represent $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$.**

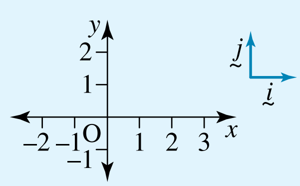
b **Find the magnitude and direction of the vector $\underset{\sim}{a}$.**

THINK

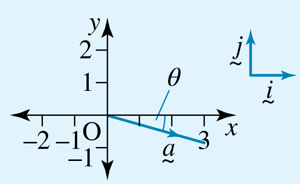
DRAW/WRITE

a 1 Draw axes with $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors in the x- and y-directions respectively.

a



2 Represent $3\underset{\sim}{i} - \underset{\sim}{j}$ as a vector from 0 that is 3 units in the positive x-direction and 1 unit in the negative y-direction, and mark the angle between $\underset{\sim}{a}$ and the x-axis as θ.



b 1 The magnitude of $\underset{\sim}{a}$ (that is, $|\underset{\sim}{a}|$) may be found using Pythagoras' theorem.

b $|\underset{\sim}{a}| = \sqrt{3^2 + (-1)^2}$
$= \sqrt{10}$

2 Find the value of angle θ using the tangent ratio.

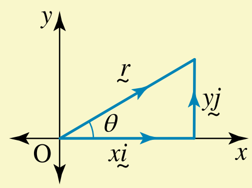
$\tan(\theta) = \dfrac{1}{3}$
$\theta \approx 18.4°$

3 Give the direction of vector $\underset{\sim}{a}$ relative to the positive x-axis.

Vector $\underset{\sim}{a}$ makes an angle of $-18.4°$ from the positive x-axis.

As we have seen, angles are usually given with respect to the positive x-direction. We may generalise this procedure:

For any vector, $\underset{\sim}{r}$:
1. $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$
2. **magnitude** $\underset{\sim}{r}$, $|\underset{\sim}{r}| = \sqrt{x^2 + y^2}$
3. **the direction from the positive x-axis is given by appropriately adjusting θ where** $\tan(\theta) = \left|\dfrac{y}{x}\right|$.



Addition, subtraction and multiplication by a scalar for a vector in $\underset{\sim}{i}$, $\underset{\sim}{j}$ form follow the rules of normal arithmetic, with each component treated separately.

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ **and** $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{a} + \underset{\sim}{b} = (x_1 + x_2)\underset{\sim}{i} + (y_1 + y_2)\underset{\sim}{j}$$
$$\underset{\sim}{a} - \underset{\sim}{b} = (x_1 - x_2)\underset{\sim}{i} + (y_1 - y_2)\underset{\sim}{j}$$
$$k\underset{\sim}{a} = kx_1\underset{\sim}{i} + ky_1\underset{\sim}{j}$$

WORKED EXAMPLE 15

If $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 5\underset{\sim}{j}$, express in $\underset{\sim}{i}$, $\underset{\sim}{j}$ form:

a $\underset{\sim}{a} + \underset{\sim}{b}$

b $2\underset{\sim}{a} - \underset{\sim}{b}$.

THINK

a Add the $\underset{\sim}{i}$ components and $\underset{\sim}{j}$ components separately.

b 1 $2\underset{\sim}{a}$ is calculated by multiplying the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of $\underset{\sim}{a}$ by 2.

2 $2\underset{\sim}{a} - \underset{\sim}{b}$ is calculated by subtracting the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of $\underset{\sim}{b}$ respectively from $2\underset{\sim}{a}$.

WRITE

a $\underset{\sim}{a} + \underset{\sim}{b} = (3\underset{\sim}{i} + \underset{\sim}{j}) + (-2\underset{\sim}{i} + 5\underset{\sim}{j})$
$= 3\underset{\sim}{i} - 2\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{j}$
$= \underset{\sim}{i} + 6\underset{\sim}{j}$

b $2\underset{\sim}{a} = 2(3\underset{\sim}{i} + \underset{\sim}{j})$
$= 6\underset{\sim}{i} + 2\underset{\sim}{j}$

$2\underset{\sim}{a} - \underset{\sim}{b} = 6\underset{\sim}{i} + 2\underset{\sim}{j} - (-2\underset{\sim}{i} + 5\underset{\sim}{j})$
$= 6\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{i} - 5\underset{\sim}{j}$
$= 8\underset{\sim}{i} - 3\underset{\sim}{j}$

WORKED EXAMPLE 16 $\overrightarrow{OA} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{OB} = -\underset{\sim}{i} + 4\underset{\sim}{j}$.

a Represent $\overrightarrow{OA}$ and $\overrightarrow{OB}$ on a diagram.

b Find, in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$, the vector $\overrightarrow{AB}$.

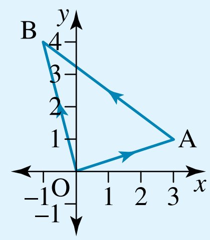
c If M is the midpoint of AB, find the vector $\overrightarrow{OM}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

THINK

a 1 Draw axes with $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors in the x- and y-directions respectively.

2 Represent $\overrightarrow{OA}$ as $3\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{OB}$ as $-\underset{\sim}{i} + 4\underset{\sim}{j}$ on the axes.

DRAW/WRITE

a 

$\overrightarrow{OA} = 3\underset{\sim}{i} + \underset{\sim}{j}$

$\overrightarrow{OB} = -\underset{\sim}{i} + 4\underset{\sim}{j}$

THINK

b 1 $\overrightarrow{AB}$ may be expressed as $\overrightarrow{AO} + \overrightarrow{OB}$ using the triangle rule for adding vectors.

2 Change $\overrightarrow{AO}$ to negative $\overrightarrow{OA}$.

3 Express this in $\underset{\sim}{i}$, $\underset{\sim}{j}$ form.

4 Simplify.

DRAW/WRITE

b $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$\overrightarrow{AB} = -(3\underset{\sim}{i} + \underset{\sim}{j}) + (-\underset{\sim}{i} + 4\underset{\sim}{j})$

$= -4\underset{\sim}{i} + 3\underset{\sim}{j}$

THINK

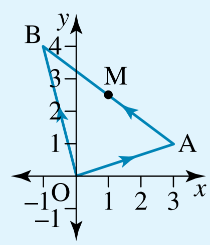
c 1 Mark the point M in the middle of AB.

2 Express $\overrightarrow{OM}$ as the sum of $\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$.

3 Express this in $\underset{\sim}{i}$, $\underset{\sim}{j}$ form.

4 Simplify.

DRAW/WRITE

c 

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

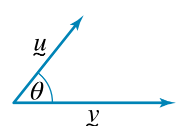
$= (3\underset{\sim}{i} + \underset{\sim}{j}) + \frac{1}{2}(-4\underset{\sim}{i} + 3\underset{\sim}{j})$

$= \underset{\sim}{i} + 2.5\underset{\sim}{j}$

Multiplying two vectors

The **dot product** is one method of multiplying one vector by another vector. It is also called a scalar product as the result of this multiplication is a scalar (magnitude only). The product of two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is denoted by $\underset{\sim}{u} \cdot \underset{\sim}{v}$.

Consider the two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$, as shown.



By definition, the dot product $\underset{\sim}{u} \cdot \underset{\sim}{v}$ is given by:

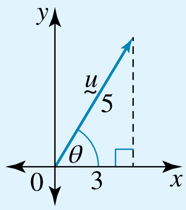
$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos(\theta)$$

where θ is the angle between (the positive directions of) $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Note: The vectors are not aligned as for addition or subtraction; instead their two tails are joined.

WORKED EXAMPLE 17

Let $\underset{\sim}{u} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{v} = 6i$. Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$.

THINK	WRITE/DRAW
1 Find the magnitudes of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.	$\lvert\underset{\sim}{u}\rvert = \sqrt{3^2 + 4^2}$ $= 5$ $\lvert\underset{\sim}{v}\rvert = \sqrt{6^2}$ $= 6$
2 Draw a right-angled triangle showing the angle that $\underset{\sim}{u}$ makes with the positive x-axis since $\underset{\sim}{v}$ is along the x-axis.	
3 Find cos (θ), knowing that $u = 5$ and the x-component of $\underset{\sim}{u}$ is 3.	$\cos(\theta) = \frac{3}{5}$
4 Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$ using the dot product equation.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = \lvert\underset{\sim}{u}\rvert \times \lvert\underset{\sim}{v}\rvert \times \cos(\theta)$
5 Simplify.	$= 5 \times 6 \times \frac{3}{5}$ $= 18$

Note: An easier method for finding the dot product is to multiply the corresponding x and y components of the two vectors.

WORKED EXAMPLE 18

Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$ if $\underset{\sim}{u} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} + \underset{\sim}{j}$.

THINK	WRITE
1 Write down $\underset{\sim}{u} \cdot \underset{\sim}{v}$.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = (2\underset{\sim}{i} + 5\underset{\sim}{j}) \cdot (3\underset{\sim}{i} + \underset{\sim}{j})$
2 Multiply the corresponding components.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = 2 \times 3 + 5 \times 1$
3 Simplify.	$= 11$

Perpendicular vectors

If two vectors are perpendicular then the angle between them is 90°.

$$\begin{aligned}\underset{\sim}{u} \cdot \underset{\sim}{v} &= \lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert\cos(90^\circ)\\ &= \lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert \times 0 \quad \text{since } \cos(90^\circ) = 0\\ &= 0\end{aligned}$$

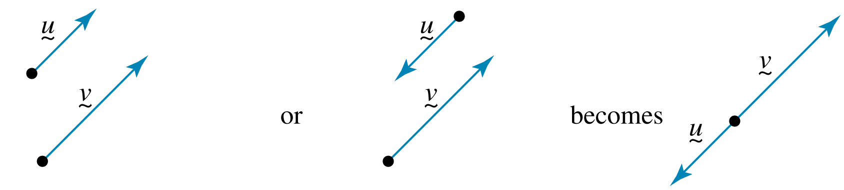
If $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$, then $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.

WORKED EXAMPLE 19 Find the constant a if the vectors $\underset{\sim}{u} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -3\underset{\sim}{i} - a\underset{\sim}{j}$ are perpendicular.

THINK	WRITE
1 Find the dot product.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = (4\underset{\sim}{i} + 3\underset{\sim}{j}) \cdot (-3\underset{\sim}{i} - a\underset{\sim}{j})$
2 Simplify.	$= -12 - 3a$
3 Set $\underset{\sim}{u} \cdot \underset{\sim}{v}$ equal to zero since $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = -12 - 3a = 0$
4 Solve the equation for a.	$a = -4$

Parallel vectors

If two vectors are parallel (||) then the angle between them is either 0° (if acting in the same direction) or 180° (if acting in opposite directions).



Same direction, $\theta = 0°$ Opposite directions, $\theta = 180°$

For $\theta = 0°$ (same direction)

$$\begin{aligned}\underset{\sim}{u} \cdot \underset{\sim}{v} &= |\underset{\sim}{u}||\underset{\sim}{v}|\cos(\theta) \\ &= |\underset{\sim}{u}||\underset{\sim}{v}|\cos(0°) \\ &= |\underset{\sim}{u}||\underset{\sim}{v}| \qquad \text{as } \cos(0°) = 1\end{aligned}$$

For $\theta = 180°$ (opposite directions)

$$\begin{aligned}\underset{\sim}{u} \cdot \underset{\sim}{v} &= |\underset{\sim}{u}||\underset{\sim}{v}|\cos(\theta) \\ &= |\underset{\sim}{u}||\underset{\sim}{v}|\cos(180°) \\ &= -|\underset{\sim}{u}||\underset{\sim}{v}| \qquad \text{as } \cos(180°) = -1\end{aligned}$$

WORKED EXAMPLE 20 Let $\underset{\sim}{u} = 5\underset{\sim}{i} + 2\underset{\sim}{j}$. Find a vector parallel to $\underset{\sim}{u}$ such that the dot product is 87.

THINK	WRITE
1 The dot product is positive so the vectors are in the same direction, with $\theta = 0°$.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = \lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert$ $\therefore \lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert = 87$
2 Find the magnitude of $\underset{\sim}{u}$.	$\lvert\underset{\sim}{u}\rvert = \sqrt{5^2 + 2^2}$ $= \sqrt{25 + 4}$ $= \sqrt{29}$
3 Substitute $\lvert\underset{\sim}{u}\rvert$ into the formula found in step 1.	$\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert = 87$ $\therefore \lvert\underset{\sim}{v}\rvert = \dfrac{87}{\sqrt{29}}$ $= 3\sqrt{29}$

4 The magnitude of $\underset{\sim}{v}$ is 3 times the magnitude of $\underset{\sim}{u}$, hence $\underset{\sim}{v}$ is 3 times larger than $\underset{\sim}{u}$. | $|\underset{\sim}{v}| = 3|\underset{\sim}{u}|$

5 Multiply $\underset{\sim}{u}$ by 3 to find $\underset{\sim}{v}$. | $3(5\underset{\sim}{i} + 2\underset{\sim}{j}) = 15\underset{\sim}{i} + 6\underset{\sim}{j}$

6 Answer the question. | $\underset{\sim}{v} = 15\underset{\sim}{i} + 6\underset{\sim}{j}$

Finding the angle between two vectors

The dot product formula can be used to find the angle between two vectors.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos(\theta)$$

By re-arranging this formula, we get:

$$\cos(\theta) = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

$$\therefore \theta = \cos^{-1}\left(\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}\right)$$

WORKED EXAMPLE 21

Let $\underset{\sim}{u} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$. Find the angle between them to the nearest degree.

THINK	WRITE
1 Find the dot product $\underset{\sim}{u} \cdot \underset{\sim}{v}$.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = (4\underset{\sim}{i} + 3\underset{\sim}{j}) \cdot (2\underset{\sim}{i} - 3\underset{\sim}{j})$ $= 4 \times 2 + 3 \times -3$ $= -1$
2 Find the magnitude of $\underset{\sim}{u}$.	$|\underset{\sim}{u}| = \sqrt{4^2 + 3^2}$ $= \sqrt{25}$ $= 5$
3 Find the magnitude of $\underset{\sim}{v}$.	$|\underset{\sim}{v}| = \sqrt{2^2 + (-3)^2}$ $= \sqrt{13}$
4 Substitute the results into the formula for the angle between two vectors.	$\theta = \cos^{-1}\left(\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}\right)$ $= \cos^{-1}\left(\frac{-1}{5\sqrt{13}}\right)$ $= \cos^{-1}(-0.055\,47)$ $= 93.1798°$
5 Answer to the nearest degree.	$= 93°$

EXERCISE 8.5 $\underset{\sim}{i}$, $\underset{\sim}{j}$ notation

PRACTISE

1 **WE14** Draw a vector to represent each of the following.

a $4\underset{\sim}{i} + 3\underset{\sim}{j}$
b $4\underset{\sim}{i} - 3\underset{\sim}{j}$
c $2\underset{\sim}{i} + 2\underset{\sim}{j}$
d $\underset{\sim}{i} - \underset{\sim}{j}$
e $4\underset{\sim}{i} + \underset{\sim}{j}$
f $5\underset{\sim}{i}$
g $-6\underset{\sim}{j}$
h $-2\underset{\sim}{i}$
i $-8\underset{\sim}{i} - 6\underset{\sim}{j}$
j $-5\underset{\sim}{i} + 12\underset{\sim}{j}$

2 Calculate the magnitude and direction of each of the vectors in question 1.

3 WE15 If $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{c} = -2\underset{\sim}{j}$, find the following in $\underset{\sim}{i}$, $\underset{\sim}{j}$ form.

a $3\underset{\sim}{a}$
b $\underset{\sim}{a} + \underset{\sim}{b}$
c $\underset{\sim}{a} - \underset{\sim}{c}$
d $2\underset{\sim}{b}$
e $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$
f $2\underset{\sim}{b} - \underset{\sim}{c}$
g $3\underset{\sim}{a} + 2\underset{\sim}{b} + \underset{\sim}{c}$
h $4\underset{\sim}{c}$
i $4\underset{\sim}{c} - \underset{\sim}{a}$
j $3\underset{\sim}{c} - \underset{\sim}{a} - \underset{\sim}{b}$

4 If $\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} + \underset{\sim}{j}$, find the following in exact form.

a $|\underset{\sim}{u}|$
b $|\underset{\sim}{u} + \underset{\sim}{v}|$
c $|3\underset{\sim}{v}|$
d $|\underset{\sim}{u} - \underset{\sim}{v}|$

5 WE16 $\overrightarrow{OA} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\overrightarrow{OB} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$.

a Represent $\overrightarrow{OA}$ and $\overrightarrow{OB}$ on a diagram.
b Find, in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$, the vector $\overrightarrow{AB}$.
c If M is the midpoint of AB, find the vector $\overrightarrow{OM}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

6 OACB is a rectangle in which the vector $\overrightarrow{OA} = 4\underset{\sim}{i}$ and $\overrightarrow{OB} = 6\underset{\sim}{j}$. Express the following in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

a $\overrightarrow{OC}$
b $\overrightarrow{OM}$ where M is the midpoint of $\overrightarrow{OA}$
c $\overrightarrow{AC}$
d $\overrightarrow{ON}$ where N is the midpoint of $\overrightarrow{OB}$
e $\overrightarrow{AB}$
f $\overrightarrow{MN}$

7 WE17 Find the dot product of $\underset{\sim}{u} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 5\underset{\sim}{j}$.

8 Let $\underset{\sim}{u} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} - 3\underset{\sim}{j}$. Find the dot product of $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

9 WE18 Let $\underset{\sim}{u} = \underset{\sim}{i} + 6\underset{\sim}{j}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} - \underset{\sim}{j}$. Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$.

10 Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$, when $\underset{\sim}{u} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$.

11 WE19 Find the constant a if the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} + 7\underset{\sim}{j}$ and $\underset{\sim}{v} = -7\underset{\sim}{i} - a\underset{\sim}{j}$ are perpendicular.

12 Find the constant a, if the vectors $\underset{\sim}{v} = a\underset{\sim}{i} + 3\underset{\sim}{j}$ and $u = 6\underset{\sim}{i} - 2\underset{\sim}{j}$ are perpendicular.

13 WE20 Let $\underset{\sim}{u} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$. Find a vector parallel to $\underset{\sim}{u}$ such that its dot product with $\underset{\sim}{u}$ is 40.

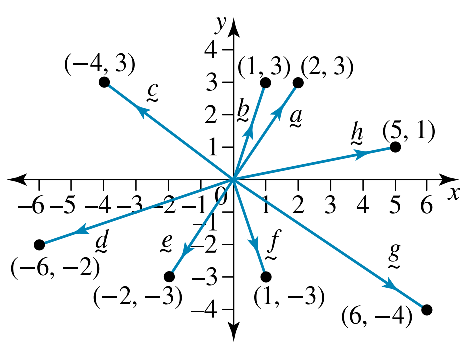
14 Let $\underset{\sim}{u} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$. Find a vector parallel to $\underset{\sim}{u}$ such that their dot product is 80.

15 WE21 Let $\underset{\sim}{u} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$. Find the angle between them to the nearest degree.

16 Find the angle, to the nearest degree, between the vectors $\underset{\sim}{i} - 6\underset{\sim}{j}$ and $4\underset{\sim}{i} + 3\underset{\sim}{j}$.

CONSOLIDATE

17 Represent the following position vectors in the form $x\underset{\sim}{i} + y\underset{\sim}{j}$.



18 The position of the points A, B and C is defined by: $\overrightarrow{OA} = 4\underset{\sim}{i}$, $\overrightarrow{OB} = 10\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\overrightarrow{OC} = 4\underset{\sim}{i} + 4\underset{\sim}{j}$.

a Find the vectors representing the three sides of the triangle ABC (that is, find in $\underset{\sim}{i}$, $\underset{\sim}{j}$ form the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$).

b Calculate the magnitude of these three sides. Leave answers in exact form.

c What type of triangle is ABC?

19 M, N and P are three points defined by: $\overrightarrow{OM} = -\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{ON} = \underset{\sim}{i} + 4\underset{\sim}{j}$ and $\overrightarrow{OP} = 5\underset{\sim}{i} + 10\underset{\sim}{j}$.

a Find $\overrightarrow{MN}$ and $\overrightarrow{NP}$.

b Show that $\overrightarrow{MN}$ and $\overrightarrow{NP}$ are parallel vectors.

20 $\underset{\sim}{a} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + \underset{\sim}{j}$.

a Find $3\underset{\sim}{a} - 2\underset{\sim}{b}$ and $3\underset{\sim}{a} + 4\underset{\sim}{b}$.

b Explain why $3\underset{\sim}{a} + 4\underset{\sim}{b}$ is parallel to the y-axis.

21 The magnitude of the vector $\sqrt{2}\underset{\sim}{i} + 2\underset{\sim}{j}$ is:

A $\sqrt{2} + 2$ B $2\sqrt{2}$ C $\sqrt{6}$ D 2 E $\sqrt{\sqrt{2} + 2}$

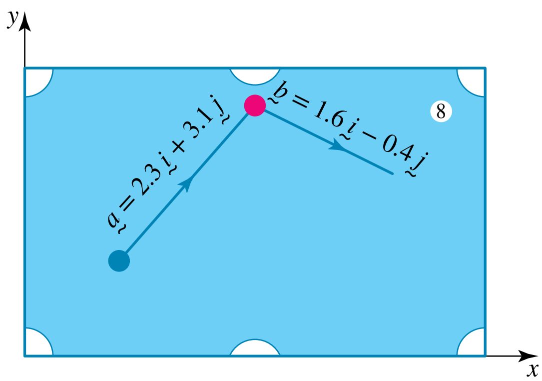
22 If $\underset{\sim}{a} = 3\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} - 2\underset{\sim}{j}$, then $\underset{\sim}{a} - 2\underset{\sim}{b}$ equals:

A $9\underset{\sim}{i} - \underset{\sim}{j}$ B $9\underset{\sim}{i} + \underset{\sim}{j}$ C $-3\underset{\sim}{i} - \underset{\sim}{j}$ D $-3\underset{\sim}{i} + \underset{\sim}{j}$ E $-4\underset{\sim}{i} - 9\underset{\sim}{j}$

23 The angle the vector $3\underset{\sim}{i} - 4\underset{\sim}{j}$ makes with the positive x-axis is nearest to:

A 37° B 53° C −53° D −37° E −127°

24 Find the vector $\underset{\sim}{a} + \underset{\sim}{b}$, which represents the planned shot of a pool player.



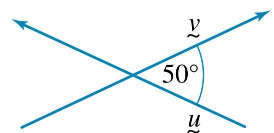
25 Vector $\underset{\sim}{m} = 12\underset{\sim}{i} + x\underset{\sim}{j}$. The magnitude of $\underset{\sim}{m}$ is 13. Find the value of x.

26 Find a vector perpendicular to $3\underset{\sim}{i} - 6\underset{\sim}{j}$.

MASTER

27 Let $\underset{\sim}{u} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$. Find cos ($\theta$), where θ is the angle between the two vectors. Give answer in exact form.

28 Consider the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ at right. Their magnitudes are 7 and 8 respectively. Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$ (to the nearest whole number).



8.6 Applications of vectors

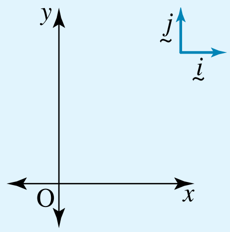
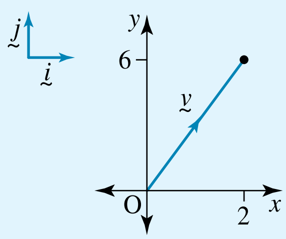
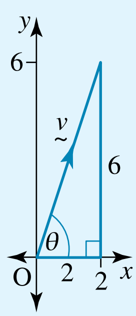
Vectors have a wide range of applications, such as in orienteering, navigation, mechanics and engineering. Vectors are applied whenever quantities specified by both magnitude and direction are involved.

When solving problems involving vectors:

1. Draw a vector diagram depicting the situation described.
2. Use the appropriate skills to answer the question being asked.

WORKED EXAMPLE 22

A boat is being rowed straight across a river at a speed of 6 km/h. The river is flowing at 2 km/h. If $\underset{\sim}{i}$ is the unit vector in the direction that the river is flowing and $\underset{\sim}{j}$ is the unit vector in the direction straight across the river, represent the velocity of the boat in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$. Hence, find the magnitude and direction of the velocity of the boat, correct to 1 decimal place.

THINK	DRAW/WRITE
1 Draw a set of axes with $\underset{\sim}{i}$ in the direction of the positive x-axis and $\underset{\sim}{j}$ in the direction of the positive y-axis.	
2 Indicate the velocity vector of the boat, $\underset{\sim}{v}$, starting at O and finishing at the point (2, 6).	
3 Represent the velocity of the boat in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.	$\underset{\sim}{v} = 2\underset{\sim}{i} + 6\underset{\sim}{j}$
4 The magnitude of $\underset{\sim}{v}$ is $\sqrt{2^2 + 6^2}$.	$\lvert\underset{\sim}{v}\rvert = \sqrt{2^2 + 6^2}$
5 Evaluate the magnitude correct to 1 decimal place.	$= \sqrt{40}$ ≈ 6.3 km/h
6 Draw a right-angled triangle with $\underset{\sim}{v}$ as the hypotenuse and θ as the angle between $\underset{\sim}{v}$ and the $\underset{\sim}{i}$ direction.	
7 Express θ using the tangent ratio.	$\tan(\theta) = \frac{6}{2}$ $= 3$
8 Evaluate θ correct to 1 decimal place.	$\theta = 71.6°$
9 State the magnitude and direction of the velocity of the boat.	The velocity of the boat has a magnitude of approximately 6.3 km/h and is directed at approximately 71.6° from the riverbank.

Note: The magnitude of **velocity** is referred to as **speed**.

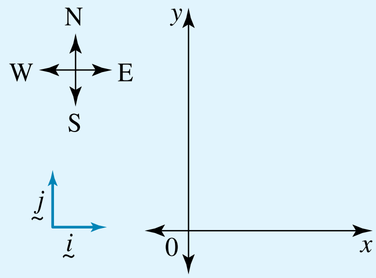
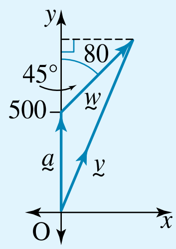
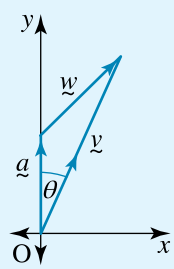
WORKED EXAMPLE 23

An aircraft is heading north with an airspeed of 500 km/h. A wind of 80 km/h is blowing from the south-west. Using $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors in the directions east and north respectively:

a **represent the aircraft's air velocity in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$**

b **represent the aircraft's exact ground velocity, $\underset{\sim}{v}$, in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$**

c **find the direction in which the aircraft is heading and its ground speed, correct to 1 decimal place.**

THINK	WRITE/DRAW
a Express $\underset{\sim}{a}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.	**a** $\underset{\sim}{a} = 500\underset{\sim}{j}$
b **1** Draw a set of axes with $\underset{\sim}{i}$ in the direction of the positive x-axis and $\underset{\sim}{j}$ in the direction of the positive y-axis. **2** Indicate the vector representing the aircraft's airspeed, $\underset{\sim}{a}$, starting at O and finishing at the point (0, 500).	**b** 
3 Indicate the vector representing the wind speed, $\underset{\sim}{w}$, by placing its tail at the head of the first vector, directed in a direction 45° from the north with a magnitude of 80, as the wind speed is 80 km/h from the south-west.	
4 Represent the combined effect of the two speeds with a vector, $\underset{\sim}{v}$, using the triangle rule.	
5 Express $\underset{\sim}{w}$, exactly, in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ using basic trigonometry.	$\underset{\sim}{w} = 80\sin(45°)\underset{\sim}{i} + 80\cos(45°)\underset{\sim}{j}$ $= 40\sqrt{2}\underset{\sim}{i} + 40\sqrt{2}\underset{\sim}{j}$
6 Express the aircraft's ground velocity, $\underset{\sim}{v}$, as the sum of $\underset{\sim}{a}$ and $\underset{\sim}{w}$.	$\underset{\sim}{v} = \underset{\sim}{a} + \underset{\sim}{w}$
7 Express $\underset{\sim}{v}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.	$= 40\sqrt{2}\underset{\sim}{i} + (500 + 40\sqrt{2})\underset{\sim}{j}$
c **1** Indicate the angle between $\underset{\sim}{v}$ and the y-axis as θ. **2** Use the tangent ratio to evaluate θ to 1 decimal place. The length of the horizontal component of $\underset{\sim}{v}$ is $40\sqrt{2}$. The length of the vertical component of $\underset{\sim}{v}$ is $500 + 40\sqrt{2}$.	**c**  $\tan(\theta) = \dfrac{40\sqrt{2}}{500 + 40\sqrt{2}}$ ≈ 0.1016 $\theta = 5.8°$
3 Calculate the magnitude of $\underset{\sim}{v}$ correct to 1 decimal place.	$\lvert\underset{\sim}{v}\rvert = \sqrt{(40\sqrt{2})^2 + (500 + 40\sqrt{2})^2}$ ≈ 559.4
4 State the direction and magnitude of the ground speed of the aircraft.	The aircraft is flying with a ground speed of 559.4 km/h in a N5.8°E direction.

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Units 1 & 2
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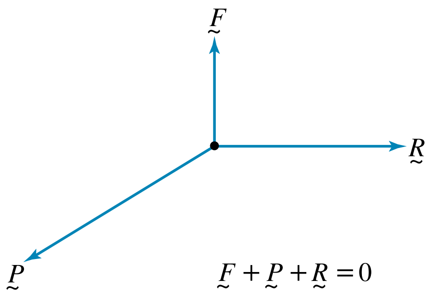
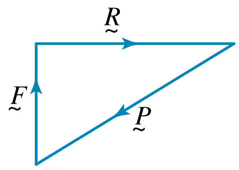
Statics

When the vector sum of the forces acting on a stationary particle is zero, the situation is said to be **static** and the particle will remain stationary. The particle is also said to be in **equilibrium**. In the case of two forces, we have the situation shown below.

In the case of three forces, we have the situation shown in the diagram below left. Where the three forces are acting so that the particle is in equilibrium, the lines representing the forces can be rearranged into a **triangle of forces** (as in the diagram below right) because their vector sum is zero. Hence, problems can be solved using trigonometry (including the **sine rule** and **cosine rule**) and sometimes Pythagoras' theorem.



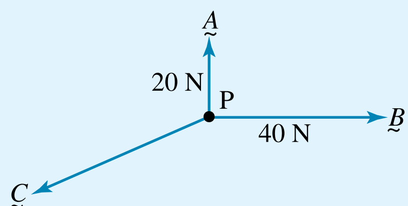
$\underset{\sim}{F} + \underset{\sim}{P} + \underset{\sim}{R} = 0$



Note: The three forces are still acting in the same direction and have the same magnitudes (or lengths) as they did in the 'real' situation.

WORKED EXAMPLE 24

Three forces are acting on the particle P as shown in the diagram below. Force $\underset{\sim}{A}$ is vertically up and has a magnitude of 20 N (20 newtons); force $\underset{\sim}{B}$ is horizontally to the right and has a magnitude of 40 N. If the particle is in equilibrium, find the magnitude of force $\underset{\sim}{C}$ to the nearest tenth of a newton and give its direction to the nearest tenth of a degree.

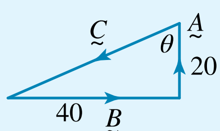


THINK

1 Draw the three forces as a triangle of forces.

2 Label the angle between forces $\underset{\sim}{A}$ and $\underset{\sim}{C}$ as θ.

DRAW/WRITE



3 Calculate $|\underset{\sim}{C}|$ using Pythagoras' theorem.

$$|\underset{\sim}{C}|^2 = |\underset{\sim}{A}|^2 + |\underset{\sim}{B}|^2$$
$$= 20^2 + 40^2$$
$$= 400 + 1600$$
$$= 2000$$
$$|\underset{\sim}{C}| = \sqrt{2000}$$

4 Evaluate $|\underset{\sim}{C}|$ correct to 1 decimal place.

$$|\underset{\sim}{C}| = 44.7 \text{ newtons}$$

5 Evaluate θ using the tangent ratio.

$$\tan(\theta) = \frac{40}{20}$$
$$\theta = \tan^{-1}(2)$$
$$\theta = 63.4°$$

6 State the answer to the question.

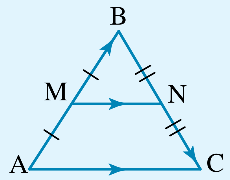
The force has a magnitude of 44.7 N and is acting downwards at an angle of 63.4° from the vertical.

Geometric proofs

Vectors can also be used to prove a range of geometric theorems. From earlier in the topic, you will remember that two vectors are equal if they are equal in magnitude, are parallel and point in the same direction. One important vector property that is useful in geometric proofs is that if $\underset{\sim}{a} = k\underset{\sim}{b}$, where $k \in R$ $(k \neq 0)$, then the two vectors, a and b are parallel.

WORKED EXAMPLE 25

Show that the line joining the midpoint of two sides of a triangle is parallel to the third side and equal to half of its length.



THINK	WRITE
1 Let side AB represent vector $\overrightarrow{AB}$ and side BC represent vector $\overrightarrow{BC}$. Use the symbol $\underset{\sim}{a}$ for vector $\overrightarrow{AB}$, and $\underset{\sim}{b}$ for vector $\overrightarrow{BC}$.	Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$ $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
2 Let side AC represent vector $\overrightarrow{AC}$. Express AC in terms of a and b.	$\overrightarrow{AC} = \underset{\sim}{a} + \underset{\sim}{b}$
3 Express $\overrightarrow{MN}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN}$ $= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$ $= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$
4 Simplify the expression by taking out $\frac{1}{2}$ as a common factor.	$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$
5 Express $\overrightarrow{MN}$ in terms of $\overrightarrow{AC}$.	$= \frac{1}{2}\overrightarrow{AC}$
6 $\overrightarrow{MN}$ is parallel to $\overrightarrow{AC}$ since $\overrightarrow{AC}$ is a multiple of $\overrightarrow{MN}$.	Therefore, $\overrightarrow{MN}$ is parallel to $\overrightarrow{AC}$ and its length is half the length of $\overrightarrow{AC}$.

EXERCISE 8.6 Applications of vectors

PRACTISE

1 WE22 A boat is being rowed straight across a river at a speed of 7 km/h. The river is flowing at 2.5 km/h. If $\underset{\sim}{i}$ is the unit vector in the direction that the river is flowing and $\underset{\sim}{j}$ is the unit vector in the direction straight across the river, represent the velocity of the boat in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$. Hence, find the magnitude and direction of the velocity of the boat correct to 1 decimal place.

2 A boat is being rowed straight across a river at a speed of 10 km/h. The river is flowing at 3.4 km/h. Find the magnitude and direction of the velocity of the boat.

3 **WE23** An aircraft is heading north with an airspeed of 650 km/h. A wind of 60 km/h is blowing from the south-west. Using $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors in the directions east and north respectively:

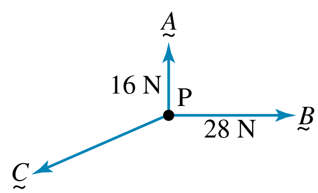
a represent the aircraft's airspeed

b represent the aircraft's ground speed in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$

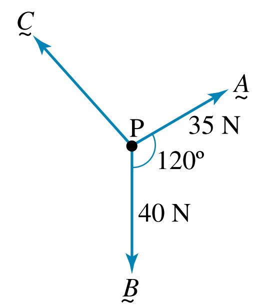
c find the direction in which the aircraft is heading and its ground speed.

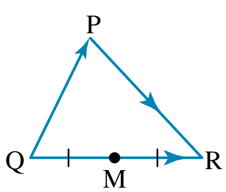
4 An aircraft is heading south with an airspeed of 600 km/h. A wind of 50 km/h is blowing in a S30°W direction. Find the direction in which the aircraft is heading and the ground speed.

5 **WE24** Three coplanar forces are acting on the particle P as shown. Force $\underset{\sim}{A}$ is vertically up and has magnitude of 16 N; force $\underset{\sim}{B}$ is horizontally to the right and has a magnitude of 28 N. If the particle is in equilibrium, find the magnitude of force $\underset{\sim}{C}$ to the nearest tenth of a newton and give its direction to the nearest tenth of a degree.



6 Three coplanar forces are acting on the particle P as shown at right. Force $\underset{\sim}{A}$ has a magnitude of 35 N, and force $\underset{\sim}{B}$ has a magnitude of 40 N. If the particle is in equilibrium, find the magnitude of force $\underset{\sim}{C}$ to the nearest tenth of a newton and give its direction to the nearest tenth of a degree.





7 **WE25** PQR is a triangle in which M is the midpoint of QR as shown on the left. Prove that $\overrightarrow{PM} = \frac{1}{2}(\overrightarrow{PR} - \overrightarrow{QP})$.

8 Prove that if the midpoints E, F, G and H of a rhombus ABCD are joined, then a parallelogram EFGH is formed.
(*Extension:* Show that the parallelogram is, in fact, a rectangle.)

CONSOLIDATE

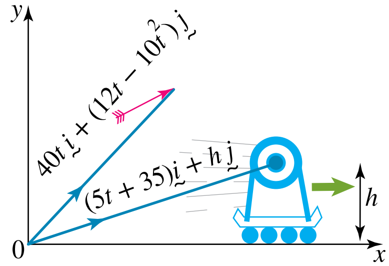
9 Forces of $3\underset{\sim}{i} + 4\underset{\sim}{j}$ and $2\underset{\sim}{i} + 2\underset{\sim}{j}$ act simultaneously on an object. Find the magnitude and direction of the resultant of the two forces.

10 Forces of $5\underset{\sim}{i} - 4\underset{\sim}{j}$, $3\underset{\sim}{i} - \underset{\sim}{j}$ and $-2\underset{\sim}{i} + 3\underset{\sim}{j}$ act simultaneously on an object. Find the magnitude and direction of the resultant of the three forces.

11 A hiker is located at a position given by (8, 6), where the coordinates represent the distances in kilometres east and north of O respectively. If a campsite is at a position given by (3, 2), find the distance and direction of the hiker from the campsite.

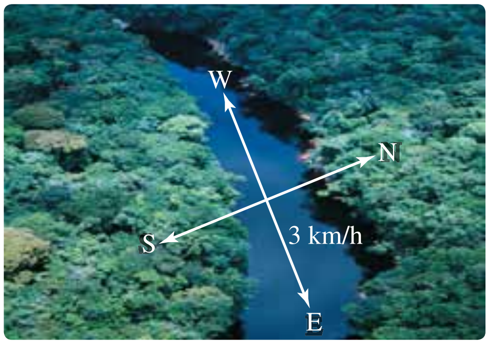
12 A hiker is located at a position given by (−5, 3), where the coordinates represent the distances in kilometres east and north of O respectively. If a campsite is at a position given by (3, −2), find the distance and direction of the hiker from the campsite.

13 The position vectors for an arrow and a moving target are shown at right, where t is the time in seconds since the target began to move, and h is the height of the target in metres. If the arrow is to hit the target, when must this happen and what must the value of h be for this to occur?



14 Forces of $-2\underset{\sim}{i} + 3\underset{\sim}{j}$, $4\underset{\sim}{i} - 5\underset{\sim}{j}$, $x\underset{\sim}{i} + \underset{\sim}{j}$ and $3\underset{\sim}{i} - y\underset{\sim}{j}$ act on a particle that is in equilibrium. Find the values of x and y.

15 A river flows through the jungle from west to east at a speed of 3 km/h. An explorer wishes to cross the river by boat, and attempts this by travelling at 5 km/h due north. Using $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors in the directions east and north respectively:



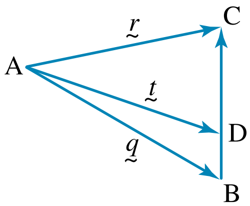
a express the velocity of the river and the velocity of the boat in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$
b draw vectors represented by the velocity of the river and the velocity of the boat
c calculate the magnitude of the resultant vector
d find the bearing of the boat's journey, correct to the nearest degree.

16 Boat A travels east at 20 km/h, while Boat B travels south from the same point at 15 km/h. Find the velocity of Boat A with respect to Boat B.

17 A river flows west–east at 5 m/s. A swimmer, in still water, can swim 3 m/s and tries to swim directly across the river from south to north.

a Draw a vector diagram to illustrate this situation.
b Find the resultant speed of the swimmer correct to 1 decimal point.
c Find the bearing of the swimmer correct to the nearest degree.
d If it took the swimmer 2 minutes to reach the opposite bank, how wide is the river?
e How far downstream would the swimmer be carried?

18 In the drawing at right, ABC is a triangle. Point D is along the line BC such that $BD = \frac{1}{3}BC$. The vectors $\underset{\sim}{q}$, $\underset{\sim}{r}$ and $\underset{\sim}{t}$ are as shown in the diagram. Prove that: $\underset{\sim}{t} = \frac{1}{3}(2\underset{\sim}{q} + \underset{\sim}{r})$.



MASTER

19 A bushwalker starts walking at 8.00 am from a campsite at $(-4, 8)$, where the coordinates represent the distances in kilometres east and north of O respectively. After 1 hour she is at $(-2, 6.5)$. Take $\underset{\sim}{i}$ and $\underset{\sim}{j}$ as unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$.

a Write, in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$, her position at the start and after 1 hour.
b Calculate the distance travelled in 1 hour.

c She then continues at the same rate and in the same direction. What is her position vector after:

i 2 hours

ii 3 hours?

d Show that her position t hours after 8.00 am is given by:

$$\underset{\sim}{r}_1 = (-4 + 2t)\underset{\sim}{i} + (8 - 1.5t)\underset{\sim}{j}$$

Another bushwalker commences walking from his campsite, also at 8.00 am. His position is given by:

$$\underset{\sim}{r}_2 = (7.4 - 1.8t)\underset{\sim}{i} + (2 + 0.5t)\underset{\sim}{j}$$

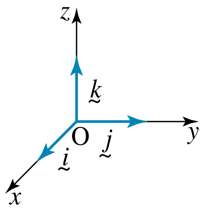
e What are the coordinates of this bushwalker's campsite?

f What is his position after 2 hours of walking?

g By equating $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components, show that the two bushwalkers meet.

h Find the distance from each campsite that each bushwalker has travelled when they meet.

20 The $\underset{\sim}{i}$, $\underset{\sim}{j}$ system may be extended to three dimensions with a unit vector $\underset{\sim}{k}$ in the z-direction.
Take $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ as unit vectors in the directions east, north and vertically up respectively.



Frank travels 2 km in a direction N30°E from O to a point A. He then climbs a 100 m high cliff.

a Write the vector $\overrightarrow{OA}$ in $\underset{\sim}{i}$, $\underset{\sim}{j}$ form.

b Calculate how far Frank has travelled to the north of his starting point.

c If T represents the top of the cliff, write down the vectors $\overrightarrow{AT}$ and $\overrightarrow{OT}$ using $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ components.

d Calculate the magnitude of $\overrightarrow{OT}$.

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8.7 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

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Units 1 & 2 — **Vectors**

Sit topic test

8 Answers

EXERCISE 8.2

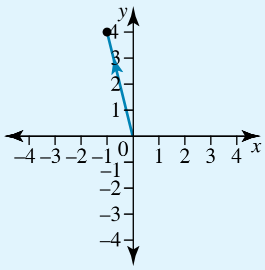
1 a (2, 2) **b** (1, −2) **c** (−4, −3)
d (−1, −4) **e** (−6, −2)

2 a $\begin{bmatrix}-5\\2\end{bmatrix}$ **b** $\begin{bmatrix}-1\\2\end{bmatrix}$ **c** $\begin{bmatrix}1\\2\end{bmatrix}$
d $\begin{bmatrix}1\\-1\end{bmatrix}$ **e** $\begin{bmatrix}-1\\1\end{bmatrix}$

3 (−2, 7)

4 a (5, −7) **b** (−1, 0)
c (2, −1) **d** (4, −10)

5



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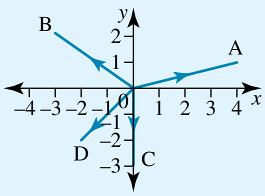
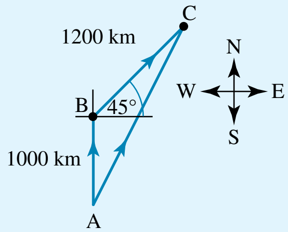


7 $\underset{\sim}{a} = \underset{\sim}{b}$; $\underset{\sim}{c} = \underset{\sim}{f}$

8 $\underset{\sim}{a} = \underset{\sim}{d}$; $\underset{\sim}{b} = \underset{\sim}{c}$

9



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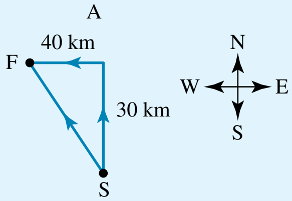
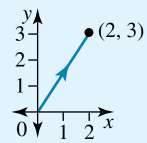


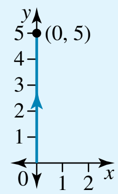
11 $\underset{\sim}{a} = (1, 1)$, $\underset{\sim}{b} = (0, 3)$, $\underset{\sim}{c} = (-1, 2)$, $\underset{\sim}{d} = (-1, 2)$, $\underset{\sim}{e} = (-1, -1)$, $\underset{\sim}{f} = (2, 4)$, $\underset{\sim}{g} = (1, 0)$, $\underset{\sim}{h} = (1, 1)$

12 $\underset{\sim}{a} = \begin{bmatrix}1\\1\end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix}0\\3\end{bmatrix}$ $\underset{\sim}{c} = \begin{bmatrix}-1\\2\end{bmatrix}$
$\underset{\sim}{d} = \begin{bmatrix}-1\\2\end{bmatrix}$ $\underset{\sim}{e} = \begin{bmatrix}-1\\-1\end{bmatrix}$ $\underset{\sim}{f} = \begin{bmatrix}2\\4\end{bmatrix}$
$\underset{\sim}{g} = \begin{bmatrix}1\\0\end{bmatrix}$ $\underset{\sim}{h} = \begin{bmatrix}1\\1\end{bmatrix}$

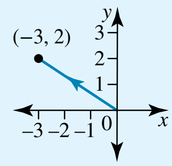
13 a



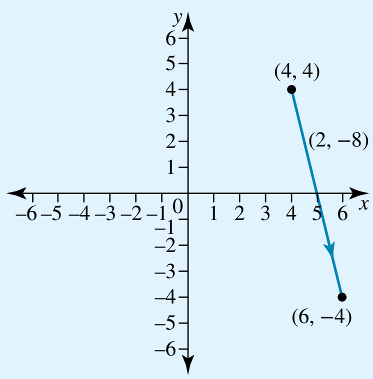
b



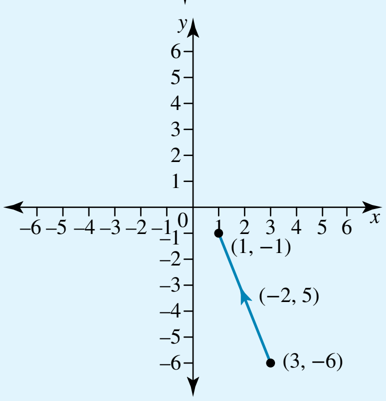
c



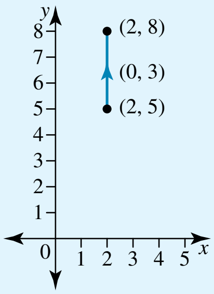
14 a



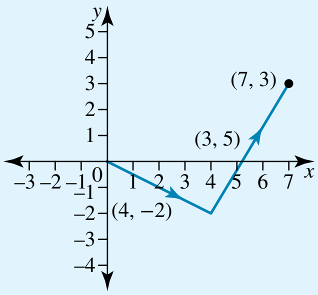
b



c

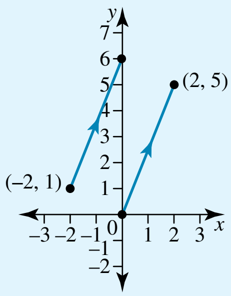


d



15 B

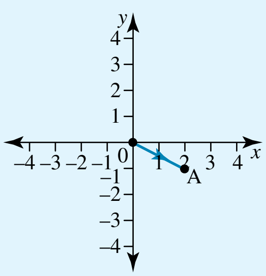
16



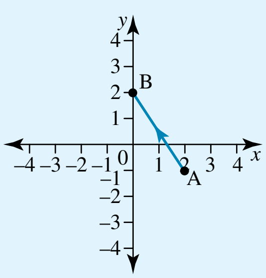
Note: Vector can start at any point. Started at (0, 0) and (–2, 1) to draw 2 different line segments for the vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

17 $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \underset{\sim}{d} \therefore a = 4,\ b = 3$

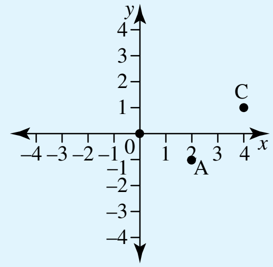
18 a



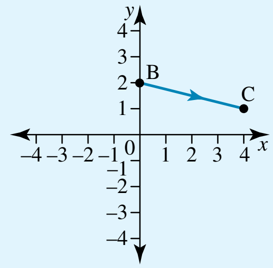
b



c



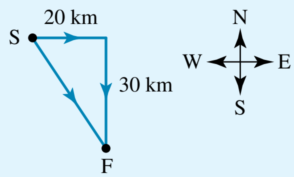
d



19 a $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

c $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **d** $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$

20



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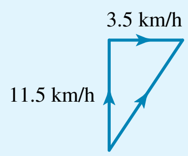

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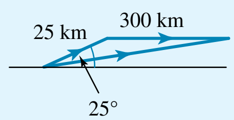
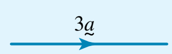
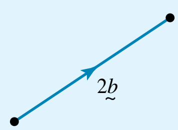


EXERCISE 8.3

1 a



b



c

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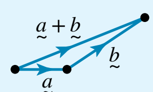
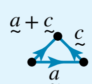


e



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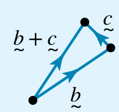
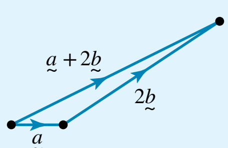


2 a



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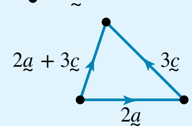

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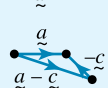

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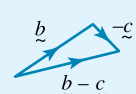
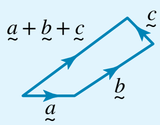
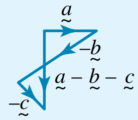


e 

f 

3 a (2, 3) **b** (1, 8)

4 a (5, –5) **b** (–5, –2)

5 a (13, –5) **b** (1, 1)

6 a (4, 6) **b** (–16, 20)

7 a $\underset{\sim}{a}$ **b** $\underset{\sim}{a} + \underset{\sim}{c}$ **c** $-\underset{\sim}{a} + \underset{\sim}{c}$

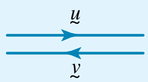
d $\underset{\sim}{d}$ **e** $-\underset{\sim}{d}$

8 a $-\underset{\sim}{a} + \underset{\sim}{c}$ **b** $\underset{\sim}{a} + \underset{\sim}{c}$ **c** $\underset{\sim}{a} + \underset{\sim}{c} + \underset{\sim}{d}$

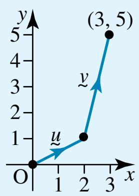
d $-\underset{\sim}{a} + \underset{\sim}{c} + \underset{\sim}{d}$ **e** $\underset{\sim}{a} + \underset{\sim}{c} - \underset{\sim}{d}$

9 Teacher to check student proofs.

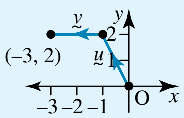
10 B

11 

or any two vectors equal in length but opposite in direction

12 

or any similar result

13 

or any similar result

14 a $\overrightarrow{CB}$, $\overrightarrow{DE}$, $\overrightarrow{GF}$ **b** $\overrightarrow{AB}$, $\overrightarrow{DG}$, $\overrightarrow{EF}$

c $\overrightarrow{AE}$, $\overrightarrow{CG}$, $\overrightarrow{BF}$ **d** $\overrightarrow{DE}$, $\overrightarrow{CB}$, $\overrightarrow{OA}$

e $\overrightarrow{DF}$ **f** $\overrightarrow{BG}$

15 a $\underset{\sim}{a}$ **b** $2\underset{\sim}{a}$ **c** $-2\underset{\sim}{a}$

d $-\underset{\sim}{a} + \underset{\sim}{b}$ **e** $-\underset{\sim}{a}$ **f** $-2\underset{\sim}{a} + \underset{\sim}{b}$

g $-\underset{\sim}{b}$ **h** $-\underset{\sim}{a} + \underset{\sim}{b}$ **i** $\underset{\sim}{a} + \underset{\sim}{b}$

j $2\underset{\sim}{a} - \underset{\sim}{b}$

16 Teacher to check student proofs.

17 $\overrightarrow{AE}$

18 B

19 D

20 a $\underset{\sim}{c}$ **b** $-\underset{\sim}{b}$ **c** $\underset{\sim}{b} - \underset{\sim}{c}$

21 a $\underset{\sim}{r} + \underset{\sim}{s}$ **b** $\underset{\sim}{s} + \underset{\sim}{t}$ **c** $\underset{\sim}{r} - \underset{\sim}{s}$

d $\underset{\sim}{r} + \underset{\sim}{s}$ **e** $\underset{\sim}{t} - \underset{\sim}{s}$ **f** $\underset{\sim}{s} + \underset{\sim}{t} - \underset{\sim}{r}$

g $\underset{\sim}{r} + \underset{\sim}{s} + \underset{\sim}{t}$ **h** $-\underset{\sim}{s} - \underset{\sim}{t}$

22 a $\underset{\sim}{a} + \underset{\sim}{b}$ **b** $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$

c $\underset{\sim}{a} - \underset{\sim}{b}$ **d** $\underset{\sim}{a} + \underset{\sim}{b}$

e $\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{a}$

EXERCISE 8.4

1

	Magnitude	Direction
a	$2\sqrt{10}$	18.4°
b	$\sqrt{17}$	–14.0°
c	$2\sqrt{5}$	63.4°
d	$\sqrt{2}$	45°

2

	Magnitude	Direction
a	$\sqrt{5}$	153.4°
b	$\sqrt{17}$	104.0°
c	1	Parallel to x-axis
d	$2\sqrt{2}$	–135°

3 a $1, \sqrt{3}$ **b** $\dfrac{-3\sqrt{3}}{2}, 1.5$ **c** $5, -5\sqrt{3}$

d $-1, -\sqrt{3}$ **e** $10\sqrt{2}, 10\sqrt{2}$ **f** 0, 4

g 12, 0

4 a 0, –30 **b** $-5\sqrt{3}, -5$ **c** $10\sqrt{3}, 10$

d 50, 0 **e** 37.6, 13.7 **f** 0, –98

g 1207.4, 323.5

5 a 29 **b** 13.5 km

6 a 14 km **b** 14 km

7 a 1 **b** 1 **c** $\sqrt{2}$

d 1 **e** $\sqrt{2}$ **f** $\sqrt{2}$

g $\sqrt{2}$ **h** 1 **i** $\sqrt{3}$

j $\sqrt{3}$

8 $\sqrt{x^2 + y^2}$, angle θ from the positive direction of the x-axis where $\tan(180° - \theta) = \dfrac{-y}{x}$

9 a 5, 53° **b** 5, –53°

10 19.3, –23

11 a i $6\sqrt{2}$ **ii** 45°

b i $\sqrt{65}$ **ii** 119.7°

c i 4.9 **ii** –134.2°

d i $50\sqrt{41}$ **ii** –1.8°

12 a $\sqrt{34}$ **b** –59° **c** 149° T

13 C

14 $-50, -50\sqrt{3}$

15 10 km, N36.9°E

16 –61, 109

17 a 14.86 km, N16.6°E

b 22.91 units, S40.9°E

c 2 units in direction of $\underset{\sim}{a}$

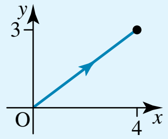
18 a 16.97 km, SW and 16.97 km, NW

b 35 km in direction of $\underset{\sim}{a}$ and 5 km in direction of $\underset{\sim}{a}$

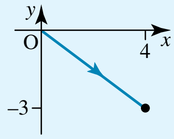
c 64.03 units, 338.7° T and 64.03 units, 261.3° T

EXERCISE 8.5

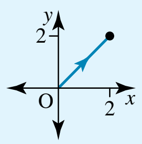
1 a



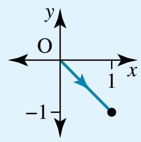
b



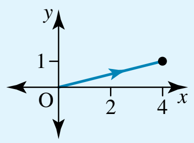
c



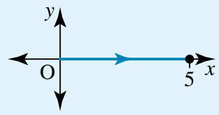
d



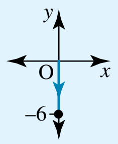
e



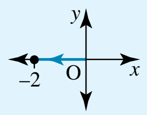
f



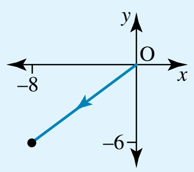
g



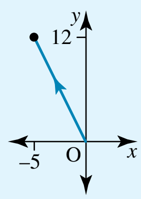
h



i



j

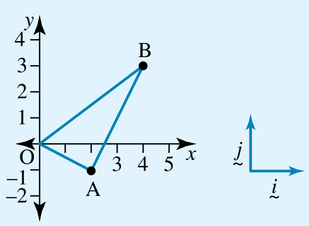


2 a 5, 36.9° **b** 5, −36.9° **c** $2\sqrt{2}$, 45°
d $\sqrt{2}$, −45° **e** $\sqrt{17}$, 14° **f** 5, 0°
g 6, −90° **h** 2, 180° **i** 10, −143.1°
j 13, 112.6°

3 a $9\underset{\sim}{i} + 6\underset{\sim}{j}$ **b** $4\underset{\sim}{i} + \underset{\sim}{j}$ **c** $3\underset{\sim}{i} + 4\underset{\sim}{j}$
d $2\underset{\sim}{i} - 2\underset{\sim}{j}$ **e** $4\underset{\sim}{i} - \underset{\sim}{j}$ **f** $2\underset{\sim}{i}$
g $11\underset{\sim}{i} + 2\underset{\sim}{j}$ **h** $-8\underset{\sim}{j}$ **i** $-3\underset{\sim}{i} - 10\underset{\sim}{j}$
j $-4\underset{\sim}{i} - 7\underset{\sim}{j}$

4 a $\sqrt{13}$ **b** $\sqrt{29}$
c $3\sqrt{10}$ **d** $\sqrt{17}$

5 a



b $2\underset{\sim}{i} + 4\underset{\sim}{j}$ **c** $3\underset{\sim}{i} + \underset{\sim}{j}$

6 a $4\underset{\sim}{i} + 6\underset{\sim}{j}$ **b** $2\underset{\sim}{i}$ **c** $6\underset{\sim}{j}$
d $3\underset{\sim}{j}$ **e** $-4\underset{\sim}{i} + 6\underset{\sim}{j}$ **f** $-2\underset{\sim}{i} + 3\underset{\sim}{j}$

7 22
8 3
9 −8
10 −6
11 −2
12 1
13 $4\underset{\sim}{i} - 8\underset{\sim}{j}$
14 $\frac{64}{5}\underset{\sim}{i} - \frac{48}{5}\underset{\sim}{j}$
15 87°
16 117°
17 $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{c} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{d} = -6\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{e} = -2\underset{\sim}{i} - 3\underset{\sim}{j}$, $\underset{\sim}{f} = \underset{\sim}{i} - 3\underset{\sim}{j}$, $\underset{\sim}{g} = 6\underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{h} = 5\underset{\sim}{i} + \underset{\sim}{j}$

18 a $6\underset{\sim}{i} + 2\underset{\sim}{j}$, $4\underset{\sim}{j}$, $-6\underset{\sim}{i} + 2\underset{\sim}{j}$ **b** $2\sqrt{10}$, 4, $2\sqrt{10}$
c Isosceles

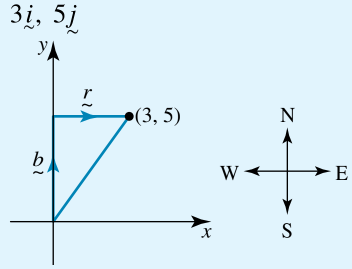
19 a $\overrightarrow{MN} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\overrightarrow{NP} = 4\underset{\sim}{i} + 6\underset{\sim}{j}$
b Parallel since $\overrightarrow{NP} = 2\overrightarrow{MN}$

20 a $18\underset{\sim}{i} - 8\underset{\sim}{j}$, $-2\underset{\sim}{j}$ **b** The $\underset{\sim}{i}$ component is zero.

21 C
22 A
23 C
24 $3.9\underset{\sim}{i} + 2.7\underset{\sim}{j}$
25 $x = \pm 5$
26 $6\underset{\sim}{i} + 3\underset{\sim}{j}$ or $-6\underset{\sim}{i} - 3\underset{\sim}{j}$
27 $\frac{2\sqrt{5}}{25}$
28 −36

EXERCISE 8.6

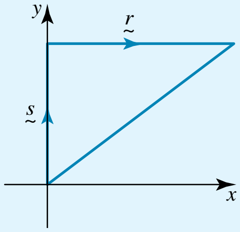
1 $2.5\underset{\sim}{i} + 7\underset{\sim}{j}$; 7.43 km/h; 70.3° from the river bank
2 10.6 km/h, 71.2° from the river bank
3 a $650\underset{\sim}{j}$
b $30\sqrt{2}\underset{\sim}{i} + (650 + 30\sqrt{2})\underset{\sim}{j}$
c N3.5°E, 693.7 km/h
4 S2.2°W, 643.8 km/h
5 32.3 N, 60.3° from the vertical
6 37.8 N, 53.4° from the vertical
7 Teacher to check student proofs.
8 Teacher to check student proofs.
9 7.8 units, 50.2° from the positive x-axis
10 6.3 units, −18.4° from the positive x-axis
11 6.4 km, N51.3°E or 51.3° T
12 9.4 km, N58°W
13 1 s, 2 m
14 $x = -5$, $y = -1$
15 a $3\underset{\sim}{i}$, $5\underset{\sim}{j}$
b



c $\sqrt{34}$ **d** N31°E

16 25 km/h on a bearing of N53°E

17 a



b 5.8 m/s

c N59°E

d 360 metres

e 600 metres

18 Teacher to check student proofs.

19 a $-4\underset{\sim}{i} + 8\underset{\sim}{j}, -2\underset{\sim}{i} + 6.5\underset{\sim}{j}$

b 2.5 km

c i $5\underset{\sim}{j}$

ii $2\underset{\sim}{i} + 3.5\underset{\sim}{j}$

d Check with your teacher.

e (7.4, 2)

f (3.8, 3)

h First 7.5 km, second 5.6 km

20 a $\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$

b $\sqrt{3}$ km

c $0.1\underset{\sim}{k}, \ \underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 0.1\underset{\sim}{k}$

d 2.002

9

Kinematics

9.1 Kick off with CAS

Calculus and the skateboarder

The motion of stellar and Earthly objects — the sweeping movement of a planet, the path of a car on a race track, the acceleration of a stone dropped into a river and the graceful turnings of a ballet dancer — have intrigued people for centuries. The study of the motion of bodies and particles is called kinematics.

1 A skateboarder starts on a 1 metre long flat section before coming down on an inclined ramp. When he reaches the top of the ramp, a stopwatch is started. The distance he travels is known to follow an exponential curve, $d = a^t$ for the first 2 seconds of motion, where d is in metres and t is in seconds. If his initial speed (at $t = 0$) is 1 m/s, find his equation of motion; that is, find a. Use the following steps to help.

a Use a CAS calculator to find the gradient of $d = 2^t$ at $t = 0$.

b Use a CAS calculator to find the gradient of $d = 3^t$ at $t = 0$.

Clearly the value of a must be between 2 and 3.

c Using trial and error, find the value of a correct to 2 decimal places.

d Show that the accurate value of a is 2.718 281 8 (correct to 7 decimal places).

e Find the value of $2.718\,281\,8^2$ (correct to 5 decimal places).

f Find the gradient of $d = 2.718\,281\,8^t$ at $t = 2$ (correct to 5 decimal places).

g What do you notice?

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

9.2 Introduction to kinematics

Our lives are perpetually involved in movement. Walking around the house, being transported to school, throwing a ball, riding a bicycle, picking up a pen, climbing stairs and going on a holiday are just a few examples. Most of our movements are routine, and we don't give them a second thought. However, sometimes we do need to think about what we are doing; for example, understanding motion can be a matter of life and death in situations such as crossing a road safely, deciding when it is prudent to overtake when driving or calculating where a cyclone is heading. Even in less dramatic situations like keeping an appointment on time, or judging how and when to throw a ball while playing sport, we give more thought to motion. Then we start to employ questions of judgement: How far is it? How long will it take? How will I get there?

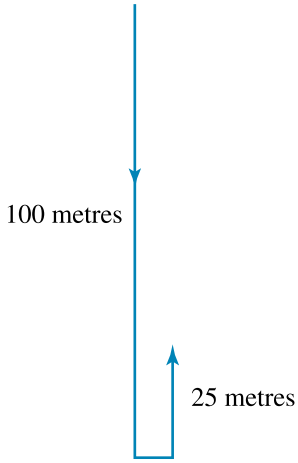
Our interest in analysing motion extends far beyond these examples taken from our daily lives. People have long been fascinated by movement in the world about them: by the motion of the planets and stars, by the flight of birds, by the oscillations of pendulums and by the growth of plants, to name a few examples. The study of motion is fundamental in all branches of science.

The name **kinematics** is given to the study of the motion of bodies, objects or particles.

In this topic, we consider motion that is only one-dimensional; that is, **straight-line motion**. This is called **rectilinear motion** (to distinguish it from **curvilinear motion**, which deals with curves). Examples of rectilinear motion include a ball travelling along a pool table in a single direction and an ice-hockey puck that has been hit along the ice.

For mathematical convenience, all moving objects that we consider in this topic will be treated as points; that is, the objects do not rotate or change shape.

To look at how we might analyse motion, let's consider Bill's latest bungee jump. Bill jumps from a bridge 120 metres above the ground and is attached to an 80-metre elastic rubber rope. He falls vertically towards the ground. In the first 2 seconds he falls 20 metres, and in the next 2 seconds he falls a further 60 metres. After 80 metres the bungee rope starts to stretch, and therefore slows the fall so that Bill falls a further 20 metres, in 2 seconds. The stretched bungee rope then pulls him up a distance of 15 metres in 2 seconds, passing what is called the 'equilibrium position'. (This is the position that Bill would eventually remain in, once he stopped bouncing on the rope.) He continues travelling up a further 10 metres in 2 seconds. This motion is shown in the figure at right. Bill continues bouncing until he is lowered safely to ground level.



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Units 1 & 2

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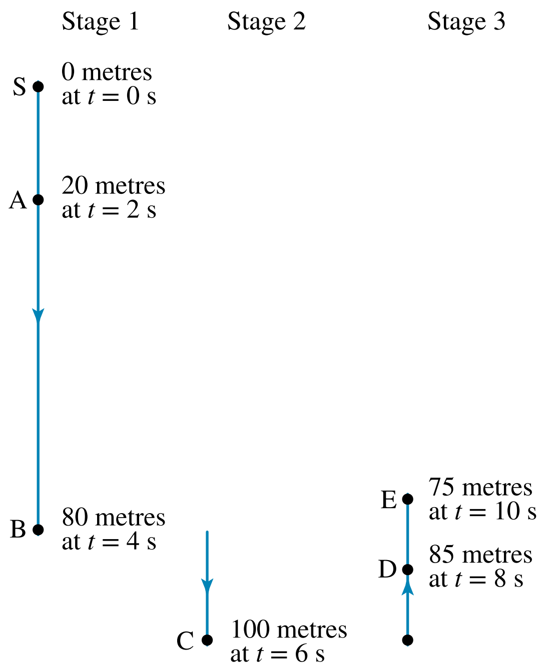
Topic 2

Concept 1

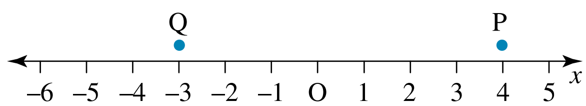
Introduction to kinematics
Concept summary
Practice questions

If we take the starting point, S, to be 0 metres, then the first 10 seconds of Bill's jump can be displayed as follows.



Position

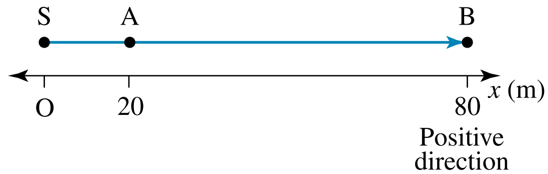
The **position** of a particle moving in a straight line is established by its distance from a fixed reference point on the line. This is usually the origin, O, with positions to the right of O normally being taken as positive.



Consider the particles P and Q above, which both start from the origin, O. The position of particle P is 4 units to the right, therefore $x = 4$. Particle Q is 3 units to the left of the origin and therefore has a position of $x = -3$.

We could describe Bill's motion by noting his position at various times.

We show this on a straight line (vertical or horizontal) by indicating his location relative to a reference point, usually the origin, O.

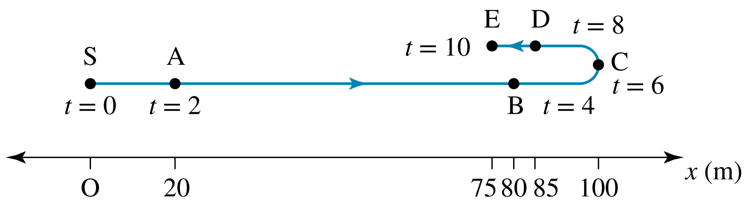


Point S, at the origin (actually 120 metres above the ground), shows Bill's starting position. Taking downwards as positive, point A is at 20 and point B is at 80.

Displacement

The **displacement** of a moving particle is its change in position relative to a fixed point. Displacement gives both the **distance** and **direction** that a particle is from a point.

This can be represented on a position–time line (or displacement–time line), as shown at right, for the first 10 seconds of Bill the bungee jumper's path.



Note: The direction of the motion is indicated by the arrows.

Bill travels from C (100 metres) to E (75 metres). The displacement from C to E is the change in position from C to E.

$$\begin{aligned}\text{Displacement} &= \text{final position} - \text{initial position}\\ &= 75 - 100\\ &= -25 \text{ metres}\end{aligned}$$

The distance from C to E is 25 metres but the displacement is −25 metres. Displacement is a vector quantity and has both magnitude and direction. (In this case the magnitude is 25 metres and the direction is negative.) Distance is a scalar quantity and has magnitude only.

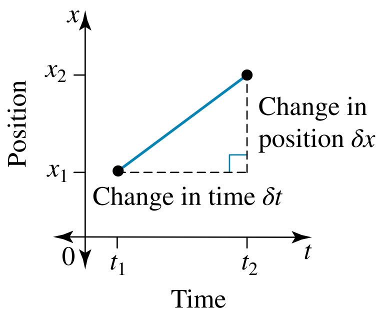
For the first 10 seconds of Bill's jump, his displacement is 75 metres (75 − 0). However, the distance Bill has moved is 125 metres.

Note: At point C, Bill is momentarily at a stop (his velocity is 0) and his motion changes direction from down to up.

Velocity

Velocity is also a vector quantity.

The average velocity of a particle is the rate of change of its position with respect to time. This can be shown on a position–time graph. The blue line shows the position of the particle, x, at time, t.



$$\begin{aligned}\text{Average velocity} &= \frac{\text{change in position}}{\text{change in time}}\\ &= \frac{\text{final position} - \text{initial position}}{\text{change in time}}\\ &= \frac{x_2 - x_1}{t_2 - t_1}\\ &= \frac{\delta x}{\delta t}\end{aligned}$$

Bill's average velocity over the first 10 seconds of his jump can be calculated as follows:

$$\begin{aligned}\text{Average velocity} &= \frac{x_2 - x_1}{t_2 - t_1}\\ &= \frac{75 - 0}{10 - 0}\\ &= \frac{75}{10}\\ &= 7.5 \text{ m/s}\end{aligned}$$

The commonly used units of velocity are cm/s, m/s and km/h.

Note: 1 m/s = 3.6 km/h.

The **instantaneous velocity** is the velocity at a given point of time. That is, it is the gradient of the displacement–time graph at a given point.

Speed

Speed is the magnitude of velocity, and therefore is a scalar quantity.

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Instantaneous speed is the magnitude of instantaneous velocity and is always positive.

Bill's average speed over the first 10 seconds of his jump can be calculated as follows:

$$\begin{aligned}\text{Average speed} &= \frac{125}{10}\\ &= 12.5 \text{ m/s (compared to the average velocity of 7.5 m/s)}\end{aligned}$$

WORKED EXAMPLE 1

The following position–time line shows a particle that moves from S to A in 2 seconds, then from A to F in 3 seconds. Find:

a **the starting position, S**

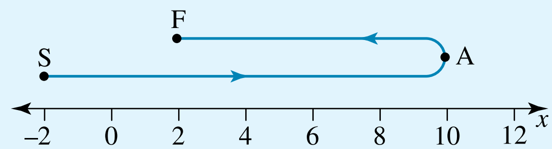
b **the final position, F**

c **the displacement of F from S**

d **the distance travelled from S to F**

e **the average velocity from S to F**

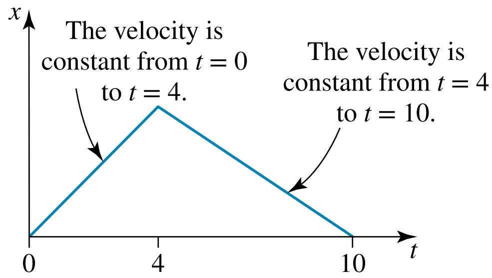
f **the average speed from S to F.**



THINK	WRITE
a Read the position of point S.	**a** The position of point S is −2.
b Read the position of point F.	**b** The position of point F is 2.
c Displacement = final position − initial position	**c** Displacement = 2 − −2 = 4 units to the right of S
d Add the distance from S to A to the distance from A to F.	**d** Distance = 12 + 8 = 20 units
e Average velocity $= \frac{\text{change in position}}{\text{change in time}}$	**e** Average velocity $= \frac{x_2 - x_1}{t_2 - t_1} = \frac{2 - -2}{5 - 0} = \frac{4}{5}$ = 0.8 units/second in the positive direction
f Average speed $= \frac{\text{distance travelled}}{\text{time taken}}$	**f** Average speed $= \frac{20}{5}$ = 4 units/second

Constant velocity

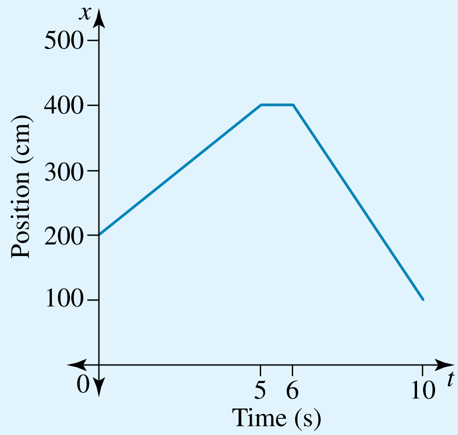
Velocity can be determined by the gradient of a position–time graph. If the position–time graph is a series of connected straight-line sections, then the velocity is constant over the duration of each straight-line section.



WORKED EXAMPLE 2

At Luna Park there is a new game called 'Hit the duck'. To win, you must knock down a mobile duck that moves back and forth in a straight line on a 5-metre track. You have three shots with small sandbags.

The position–time graph shows the position of the duck, x cm to the right of its starting point, along the track at various times, t seconds.



a What is the initial position of the duck?

b How long did the game last?

c What is the final displacement of the duck from its starting position?

d Write the times for which the velocity is:

i positive ii negative iii zero.

e Hence, find the velocity for each of the three time intervals in part **d**.

f What was the average speed of the duck during this game?

THINK	WRITE
a The initial position of the duck is when $t = 0$.	**a** When $t = 0$, the initial position of the duck is 200 cm to the right of its starting point.
b The graph finishes when $t = 10$.	**b** The game lasted for 10 seconds.
c Displacement = final position − initial position	**c** Displacement $= 100 - 200$ $= -100$ cm
d i Look for where the gradient slopes upwards to the right.	**d** i The gradient is positive from $t = 0$ to $t = 5$.
ii Look for where the gradient slopes downwards to the right.	ii The gradient is negative from $t = 6$ to $t = 10$.
iii Look for where the gradient is horizontal.	iii The gradient is zero from $t = 5$ to $t = 6$.
e Velocity $= \dfrac{\text{change in position}}{\text{change in time}}$	**e** i Velocity $= \dfrac{x_2 - x_1}{t_2 - t_1}$ $= \dfrac{400 - 200}{5 - 0}$ $= \dfrac{200}{5}$ $= 40$ cm/s

ii Velocity $= \dfrac{x_2 - x_1}{t_2 - t_1}$

$= \dfrac{100 - 400}{10 - 6}$

$= \dfrac{-300}{4}$

$= -75$ cm/s

iii Velocity $= \dfrac{x_2 - x_1}{t_2 - t_1}$

$= \dfrac{400 - 400}{6 - 5}$

$= \dfrac{0}{1}$

$= 0$ cm/s

f Average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

f Average speed $= \dfrac{500}{10}$

$= 50$ cm/s

Position expressed as a function of time

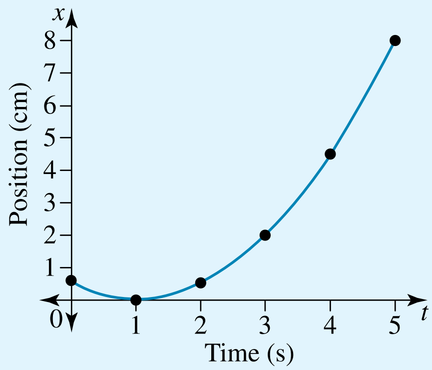
When the position is expressed as a function of time, the position–time graph can be sketched and the motion then analysed. If the position–time graph is curved, then the velocity (or gradient) is always changing and never constant.

WORKED EXAMPLE 3

A particle moves in a straight line so that its position, *x* cm, from a fixed point, O, on the line at time *t* seconds is given by the rule:

$$x = \frac{1}{2}(t - 1)^2, t \in [0, 5]$$

The position–time graph is shown at right.



a **Copy and complete the table below.**

t	0	1	2	3	4	5
x						

b **What is the initial position of the particle?**

c **What is the significance of the position at $t = 1$?**

d **Show the movement of the particle on a position–time line.**

e i **What is the displacement of the particle?**

ii **Hence, determine the average velocity of the particle.**

f i **What is the distance travelled by the particle?**

ii **Hence, determine the particle's average speed.**

THINK | **WRITE**

a **1** Substitute each value of t into the rule $x = \frac{1}{2}(t - 1)^2$ and evaluate for x.

a When $t = 0$, $x = \frac{1}{2}(0 - 1)^2$
$= \frac{1}{2}(-1)^2$
$= 0.5$

When $t = 1$, $x = \frac{1}{2}(1 - 1)^2$
$= \frac{1}{2}(0)^2$
$= 0$

When $t = 2$, $x = \frac{1}{2}(2 - 1)^2$
$= \frac{1}{2}(1)^2$
$= 0.5$

When $t = 3$, $x = \frac{1}{2}(3 - 1)^2$
$= \frac{1}{2}(2)^2$
$= 2$

When $t = 4$, $x = \frac{1}{2}(4 - 1)^2$
$= \frac{1}{2}(3)^2$
$= 4.5$

When $t = 5$, $x = \frac{1}{2}(5 - 1)^2$
$= \frac{1}{2}(4)^2$
$= 8$

2 Complete the table.

t	0	1	2	3	4	5
x	0.5	0	0.5	2	4.5	8

b State the position of the particle when $t = 0$.

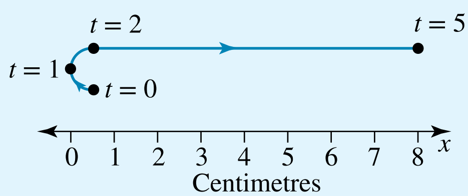
b The initial position is 0.5 cm from O.

c At $t = 1$ the particle is at $x = 0$, and the position–time graph shows it is changing direction.

c At $t = 1$ the particle is changing direction.

d The particle starts at $x = 0.5$, moves to $x = 0$, then turns and finishes at $x = 8$.

d



e **i** Displacement = final position − initial position

ii Average velocity $= \dfrac{\text{change in position}}{\text{change in time}}$

e **i** Displacement $= 8 - 0.5$
$= 7.5$ cm

ii Average velocity $= \dfrac{x_2 - x_1}{t_2 - t_1}$
$= \dfrac{8 - 0.5}{5 - 0}$
$= 1.5$ cm/s

f **i** Add the distance travelled from $t = 0$ to $t = 1$ to the distance travelled from $t = 1$ to $t = 5$.

f **i** The distance from $t = 0$ to $t = 1$ is 0.5 cm and the distance from $t = 1$ to $t = 5$ is 8 cm. The total distance is 8.5 cm.

ii $\text{Average speed} = \dfrac{\text{distance travelled}}{\text{time taken}}$

ii $\text{Average speed} = \dfrac{8.5}{5}$

$= 1.7 \text{ cm/s}$

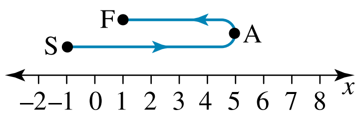
EXERCISE 9.2 Introduction to kinematics

PRACTISE

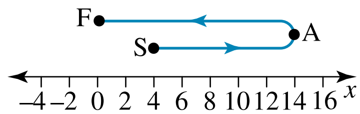
1 WE1 Each of the following position–time lines shows a particle that moves from S to A in 2 seconds, then from A to F in 3 seconds. In each case, find:

- **i** the starting position, S
- **ii** the final position, F
- **iii** the displacement of F from S
- **iv** the distance travelled from S to F
- **v** the average velocity from S to F
- **vi** the average speed from S to F.

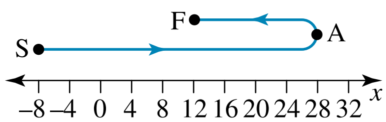
a



b



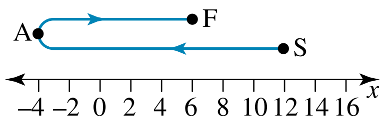
c



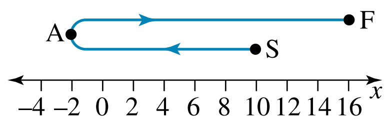
2 Each of the following position–time lines shows a particle that moves from S to A in 2 seconds, then from A to F in 3 seconds. In each case, find:

- **i** the starting position, S
- **ii** the final position, F
- **iii** the displacement of F from S
- **iv** the distance travelled from S to F
- **v** the average velocity from S to F
- **vi** the average speed from S to F.

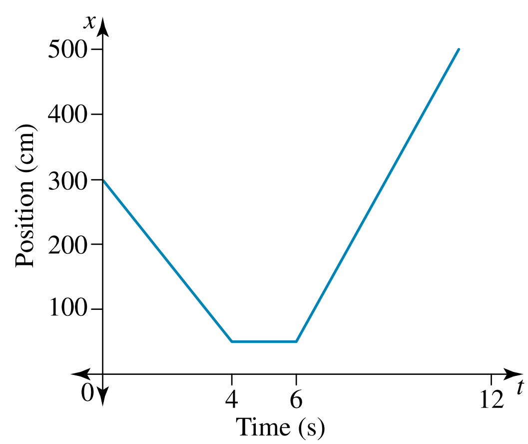
a



b



3 WE2 The position–time graph at right shows the position of a moving particle, x centimetres to the right of the origin, O, at various times, t seconds.



a What is the initial position of the particle?

b What is the final displacement of the particle from its starting position?

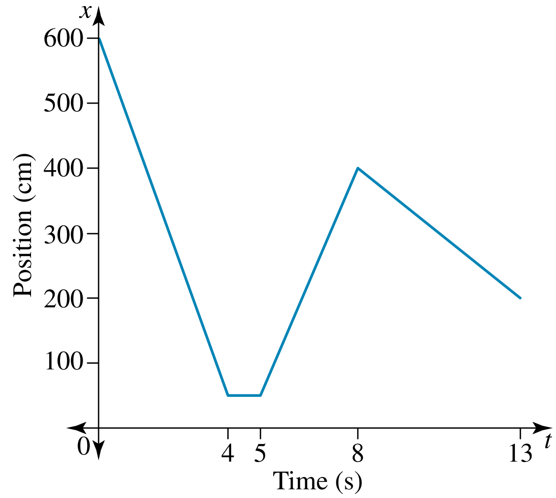
c Write the times for which the velocity is:

- **i** positive
- **ii** negative
- **iii** zero.

d Hence, find the velocity for each of the three time intervals in part **c**.

e What was the average speed of the particle?

4 The position–time graph at right shows the position of a moving particle, x centimetres to the right of the origin, O, at various times, t seconds.

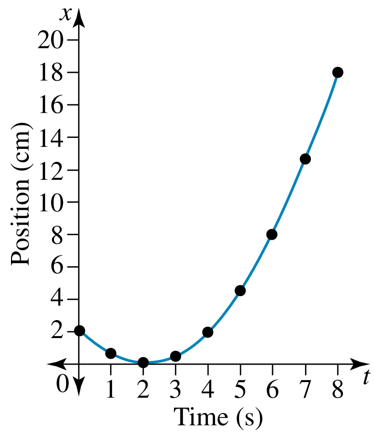


a What is the initial position of the particle?
b What is the final displacement of the particle from its starting position?
c Write the times for which the velocity is:
 i positive
 ii negative
 iii zero.
d Hence, find the velocity for each of the three time intervals in part c.
e What was the average speed of the particle?

5 WE3 A particle moves in a straight line so that its position, x cm, from a fixed point, O, on the line at time t seconds is given by the rule:

$$x = \frac{1}{2}(t - 2)^2, t \in [0, 8]$$

The position–time graph is shown at right.



a Copy and complete the table below.

t	0	2	4	6	8
x					

b What is the significance of the position at $t = 2$?
c Show the movement of the particle on a position–time line.
d Determine the average velocity of the particle.
e What is the particle's average speed?

6 A particle moves in a straight line so that its position, x cm, from a fixed point, O, on the line at time t seconds is given by the rule:

$$x = t^2 - 8t + 12, \quad t \in [0, 8]$$

a Copy and complete the table below.

t	0	2	4	6	8
x					

b Sketch the position–time graph for the particle. Check your answer using a CAS calculator.
c What is the significance of the position at $t = 4$?
d Show the movement of the particle on a position–time line.
e Determine the average velocity of the particle.
f What is the particle's average speed?

CONSOLIDATE

7 Represent each of the following situations on a position–time line.
a A particle starts at S, 2 units to the left of the origin. It is then displaced 10 units to A and undergoes a final displacement of −5 units to F.
b A particle starts at S, 3 units to the left of the origin. It is then displaced −10 units to A and undergoes a final displacement of 8 units to F.

c A particle starts at S, 6 units to the right of the origin. It is then displaced −8 units to A and undergoes a final displacement of 7 units to F.

d A particle starts at S, 4 units to the left of the origin. It is then displaced −11 units to A and undergoes a final displacement of 6 units to F.

e A particle starts at S, 3 units to the left of the origin. It is then displaced 8 units to A, followed by a displacement of −7 units to B and undergoes a final displacement of −5 units to F.

f A particle starts at S, 8 units to the right of the origin. It is then displaced 3 units to A, followed by a displacement of −4 units to B and undergoes a final displacement of 2 units to F.

8 Each movement from S to F described in question 7 takes 6 seconds and the measurements are in centimetres. In each case determine:

i the displacement of F from S
ii the total distance travelled by the particle
iii the average velocity
iv the average speed.

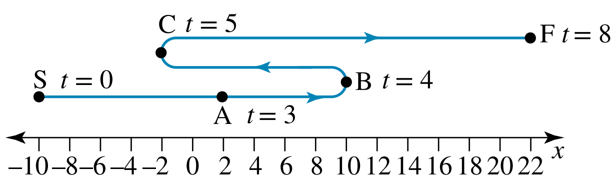
9 Draw a position–time graph for each of the following.

a An object moving with a constant positive velocity
b An object moving in a positive direction with a constant slow speed at first and then a faster constant speed

Use the position–time line below right to answer questions 10 to 13.

10 The displacement of F from S, in cm, is:

A −24
B 24
C 32
D 14
E 56



11 The distance travelled in moving from S to F, in cm, is:

A 24 B 34 C 44 D −34 E 56

12 The average speed in moving from S to F, in cm/s, is:

A 4.25 B 7 C 5.5 D −6.8 E 3

13 The average velocity in moving from A to C, in cm/s, is:

A 2 B 10 C −10 D −2 E −0.5

14 A train moves in a straight line so that its distance, x metres, from a fixed point, O, at t seconds is given by the rule:

$$x = t^2 - 6t + 9,\ t \in [0, 8]$$

a Draw a position–time graph for the movement of the train.
b Determine the average velocity of the train in the first 4 seconds.
c What is the train's average speed in the first 4 seconds?

15 A particle moves in a straight line so that its position, x cm, from a fixed point, O, on the line at time t seconds is given by the rule:

$$x = t^2 - 4t - 5,\ t \in [0, 6]$$

a Sketch the position–time graph for the particle. Check your answer using a CAS calculator.

b Show the movement of the particle on a position–time line.
c Determine the average velocity of the particle.
d What is the particle's average speed?

16 A particle moves in a straight line so that its position, x cm, from a fixed point, O, on the line at time t seconds is given by the rule:

$$x = -t^2 + 2t + 8,\ t \in [0, 6]$$

a Sketch the position–time graph for the particle. Check your answer using a CAS calculator.
b Show the movement of the particle on a position–time line.
c Determine the average velocity of the particle.
d What is the particle's average speed?

MASTER

17 A particle moves in a straight line so that its position, x cm, from a fixed point, O, on the line at time t seconds is given by the rule:

$$x = t^2 - 7t + 10,\ t \in [0, 8]$$

a Sketch the position–time graph for the particle.
b Show the movement of the particle on a position–time line.
c What is the displacement of the particle?
d Determine the average velocity of the particle.
e What is the distance travelled by the particle?
f Determine the particle's average speed.

18 A particle moves in a straight line so that its position, x cm, from a fixed point, O, on the line at time t seconds is given by the rule:

$$x = t^2 - 5t + 6,\ t \in [0, 6]$$

a Sketch the position–time graph for the particle.
b Show the movement of the particle on a position–time line.
c What is the displacement of the particle?
d Determine the average velocity of the particle.
e What is the distance travelled by the particle?
f Determine the particle's average speed, correct to 2 decimal places.

9.3 Velocity–time graphs and acceleration–time graphs

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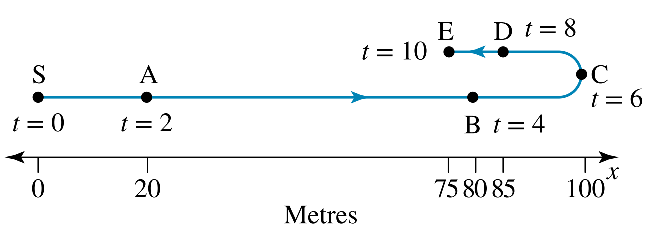
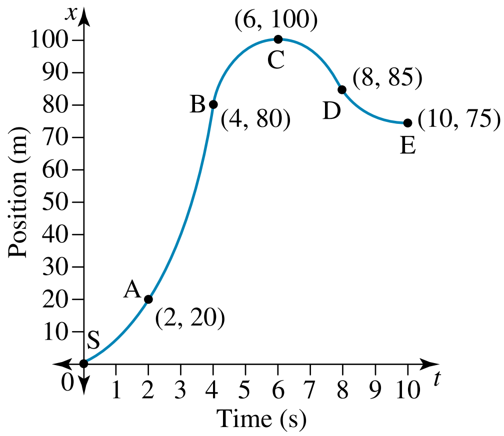
Velocity–time graphs

Let us take another look at the position–time line for the bungee jump performed by Bill that was described at the start of the topic.



This situation can be represented on a position–time graph as shown on the next page. The curve reflects the fact that the change of position over time (velocity) is not constant.



We can calculate the average velocity in each of the stages as follows:

From S to A: Avg. velocity $= \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{20 - 0}{2 - 0} = \dfrac{20}{2} = 10$ m/s

From A to B: Avg. velocity $= \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{80 - 20}{4 - 2} = \dfrac{60}{2} = 30$ m/s

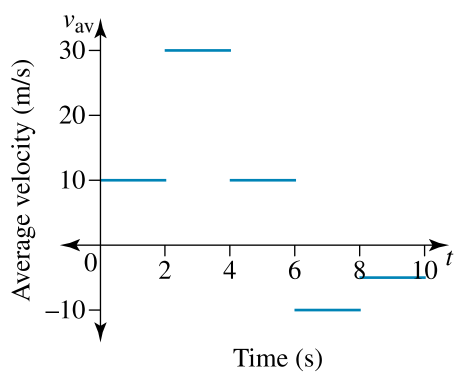
From B to C: Avg. velocity $= \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{100 - 80}{6 - 4} = \dfrac{20}{2} = 10$ m/s

From C to D: Avg. velocity $= \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{85 - 100}{8 - 6} = -\dfrac{15}{2} = -7.5$ m/s

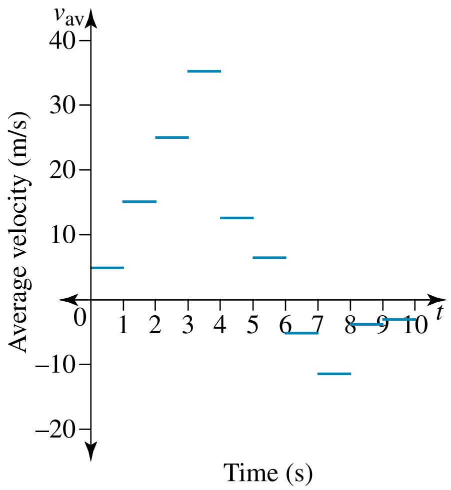
From D to E: Avg. velocity $= \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{75 - 85}{10 - 8} = -\dfrac{10}{2} = -5$ m/s

Note: The negative velocities occur when the motion is upwards, because we decided to define downwards as positive.

We can now represent the motion of Bill's bungee jump during each stage on a velocity–time graph (or more precisely, an average velocity–time graph).

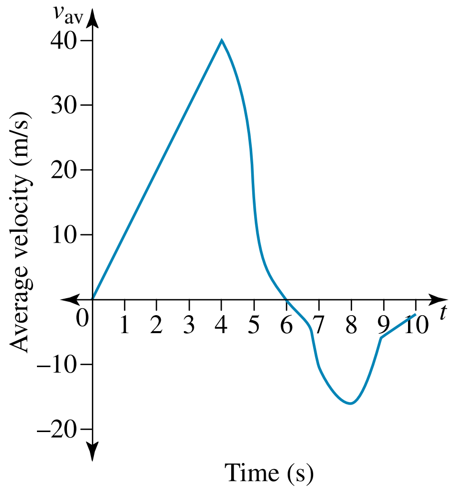


Notice that the graph shows that the velocity is constant during each of the stages (shown as the 'step formation' of the graph). This is because we have calculated the average velocity of each stage. If we were to analyse the average velocity over smaller time intervals, we would get *more* steps with *smaller* widths, as is displayed in the graph at right.



If we allowed these time intervals (step widths) to get closer and closer to zero, then the associated average velocities would effectively become a series of connected points that would collectively produce a velocity–time graph something like the one displayed below right.

This is a velocity–time graph as it shows Bill's velocity at every instance of the first 10 seconds of motion during his bungee jump. There are no horizontal lines (steps) because the velocity is changing every instant over the course of the motion. This change in velocity over time is called **acceleration**. Acceleration is also a vector quantity.



For the first 4 seconds of motion, the graph is a straight line because Bill is subjected only to acceleration due to gravity, which is constant at 9.8 m/s^2. This means that every second, Bill's velocity increases by 9.8 m/s while he is moving downwards.

For the period of time where the bungee rope is stretched (longer than 80 m), from $t = 4$ seconds to about $t = 9$ seconds, the elasticity of the rope causes the acceleration to continually change according to the tension in the bungee rope. That is why the velocity–time graph is curved during this time.

From about $t = 9$ seconds to $t = 10$ seconds (where the bungee rope is shorter than 80 m), the rope is again slack and Bill is subject to acceleration due only to gravity again. At this stage the motion is upwards, but because acceleration due to gravity acts downwards, Bill is slowing down or **decelerating**.

$$\text{Average acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

$$= \frac{v_2 - v_1}{t_2 - t_1}$$

$$= \frac{\delta v}{\delta t}$$

The most common units of acceleration are cm/s^2 and m/s^2.

For the moment we will consider only examples that involve constant acceleration.

WORKED EXAMPLE 4

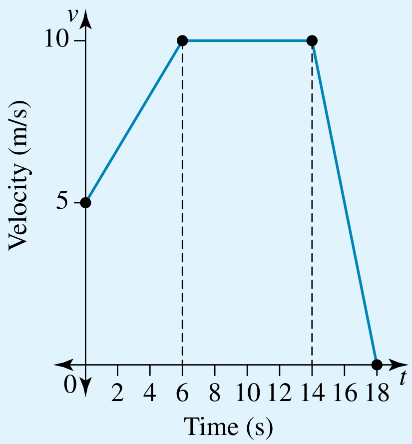
Draw a velocity–time graph to match the following description.

An object that is moving in a straight line has an initial velocity of 5 m/s. It accelerates at a constant rate until it reaches a velocity of 10 m/s after 6 seconds. It maintains this velocity for 8 seconds and then decelerates at a constant rate for a further 4 seconds until it comes to rest.

THINK

1. The velocity ranges from 0 m/s to 10 m/s.
2. The total time is 6 + 8 + 4 = 18 seconds.
3. Draw a set of axes with velocity on the vertical axis and time on the horizontal axis. Label each axis appropriately.
4. Sketch a straight line from (0, 5) to (6, 10) to show the acceleration in the first stage.
5. Draw a horizontal line from (6, 10) to (14, 10) to show the constant velocity during the second stage.
6. Draw a straight line from (14, 10) to (18, 0) to show the final stage of deceleration.

WRITE/DRAW



Notice that the gradient of each straight-line section of the velocity–time graph gives the acceleration of the object.

Analysing velocity–time graphs

The gradient of a velocity–time graph allows us to calculate the acceleration of an object moving in a straight line. In addition to this, the area between the velocity–time graph and the time axis provides useful information relating to displacement and distance.

Earlier, it was shown that:

$$\text{Average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

or

$$v_{av} = \frac{\delta x}{\delta t}$$

where v_{av} represents average velocity.

Rearranging this results in:

$$\delta x = v_{av} \times \delta t$$

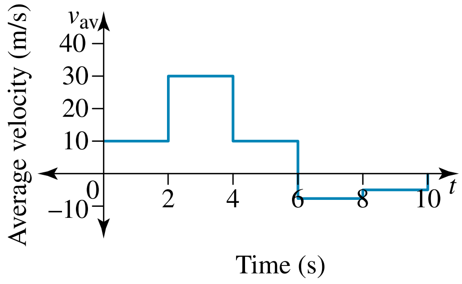
In other words, the signed area between a velocity–time graph and the time axis is equal to the change in position or displacement.

When we calculate the signed area, we take the area above the time axis as positive displacement and the area below the time axis as negative displacement.

If the distance (rather than the displacement that the particle has travelled) is required, then there is no need to sign the areas. That is, the distance travelled is the total area between the velocity–time graph and the time axis.

Using the average velocity–time graph describing Bill's bungee jump from earlier, the information described above can be highlighted as follows.

The displacement is equal to the sum of the signed areas of the rectangles.

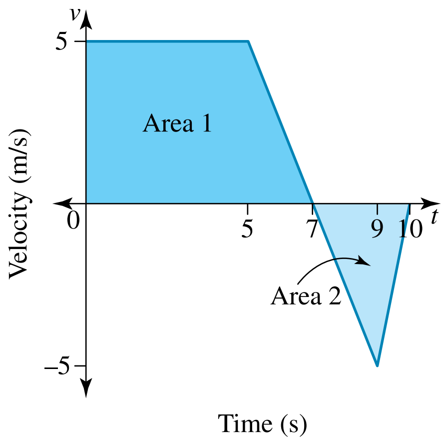


$$\begin{aligned}\text{Displacement} &= 10 \times 2 + 30 \times 2 + 10 \times 2 - 7.5 \times 2 - 5 \times 2\\ &= 20 + 60 + 20 - 15 - 10\\ &= 75 \text{ metres}\end{aligned}$$

The distance is equal to the sum of all the unsigned areas of the rectangles.

$$\begin{aligned}\text{Distance} &= 10 \times 2 + 30 \times 2 + 10 \times 2 + 7.5 \times 2 + 5 \times 2\\ &= 20 + 60 + 20 + 15 + 10\\ &= 125 \text{ metres}\end{aligned}$$

The following can be obtained from the figure shown below.



1. The object is travelling at a constant velocity of 5 m/s until $t = 5$ s. It slows down until it stops at $t = 7$ s, then it changes direction and increases its speed to 5 m/s at $t = 9$ s. The object then slows down and stops when $t = 10$ s.
2. The gradient of the line between $t = 0$ s and $t = 5$ s is zero, so the acceleration is 0 m/s^2.
   Between $t = 5$ s and $t = 9$ s the gradient is $-\dfrac{10}{4}$, so the acceleration is $-\dfrac{10}{4}$ m/s^2.
   Between $t = 9$ s and $t = 10$ s the gradient is 5, so the acceleration is 5 m/s^2.
3. Total displacement = Area 1 − Area 2.
4. Total distance = Area 1 + Area 2.

Note: When appropriate, break the area between the velocity–time graph and the time axis into simple shapes, for example rectangles, triangles or trapeziums.

$$\text{Area of a rectangle} = L \times W$$

$$\text{Area of a triangle} = \frac{1}{2}bh$$

$$\text{Area of a trapezium} = \frac{1}{2}(a + b)h$$

WORKED EXAMPLE 5

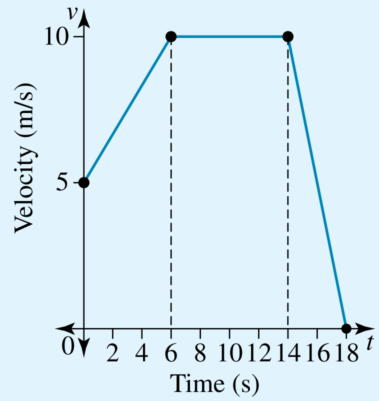
Consider the velocity–time graph obtained in Worked example 4 to find:

a **the acceleration in the first 6 seconds**

b **the acceleration in the last 4 seconds**

c **the total displacement**

d **the total distance travelled.**



THINK

a Average acceleration $= \dfrac{\text{change in velocity}}{\text{change in time}}$

b Average acceleration $= \dfrac{\text{change in velocity}}{\text{change in time}}$

c **1** The displacement is equal to the total signed area under the velocity–time graph.

2 Divide the given graph into two trapeziums, one from $t = 0$ to $t = 6$ and the other from $t = 6$ to $t = 18$.

3 Calculate the area of each trapezium.

WRITE/DRAW

a Average acceleration $= \dfrac{v_2 - v_1}{t_2 - t_1}$

$= \dfrac{10 - 5}{6 - 0}$

$= \dfrac{5}{6}$ m/s

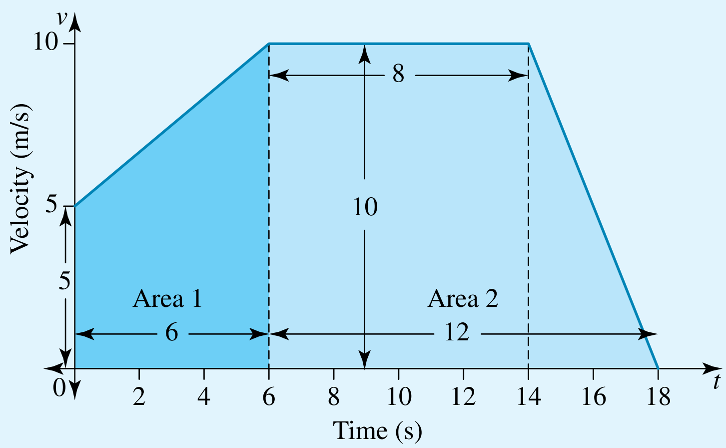
b Average acceleration $= \dfrac{v_2 - v_1}{t_2 - t_1}$

$= \dfrac{0 - 10}{18 - 14}$

$= -\dfrac{10}{4}$

$= -2.5\ \text{m/s}^2$

c



Area 1 $= \dfrac{1}{2}(5 + 10) \times 6$

$= \dfrac{1}{2} \times 15 \times 6$

$= 45\ \text{units}^2$

Area 2 $= \dfrac{1}{2}(8 + 12) \times 10$

$= \dfrac{1}{2} \times 20 \times 10$

$= 100\ \text{units}^2$

4 Find the displacement.

Displacement = Area 1 + Area 2
= 45 + 100
= 145 m

d The distance is equal to the total unsigned area under the velocity–time graph.

d The distance is equal to 145 m.
Note: Because the velocity is always positive in this example, the distance is equal to the displacement.

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Acceleration–time graphs

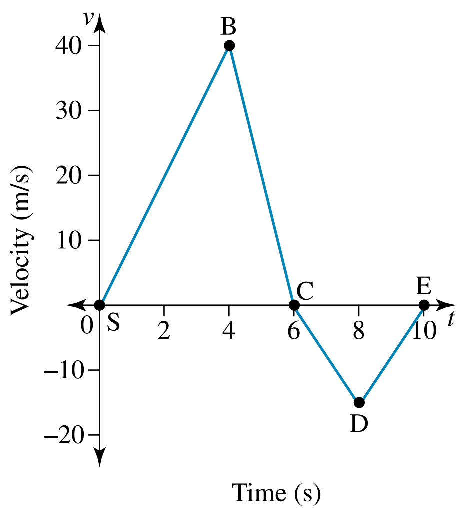
Just as the gradient of a position–time graph gives the rate of change of position, or velocity, the gradient of a velocity–time graph gives the rate of change of velocity, or acceleration.

Where the velocity is *increasing*, the acceleration is *positive*.

Where the velocity is *decreasing*, the acceleration is *negative*.

Where the velocity is not changing, the acceleration is *zero*.

Consider a modified velocity–time graph of the first 10 seconds of motion of Bill's bungee jump. We will assume the acceleration is constant, but different through each stage of the jump.



As average velocity $= \dfrac{\text{change in velocity}}{\text{change in time}}$, the acceleration for each stage is:

From S to B: Average acceleration $= \dfrac{v_2 - v_1}{t_2 - t_1}$

$$= \frac{40-0}{4-0}$$

$$= \frac{40}{4}$$

$$= 10\text{ m/s}^2$$

From B to C: Average acceleration $= \dfrac{v_2 - v_1}{t_2 - t_1}$

$$= \frac{0-40}{6-4}$$

$$= -\frac{40}{2}$$

$$= -20\text{ m/s}^2$$

From C to D: Average acceleration $= \dfrac{v_2 - v_1}{t_2 - t_1}$

$$= -\frac{15-0}{8-6}$$

$$= -\frac{15}{2}$$

$$= -7.5\text{ m/s}^2$$

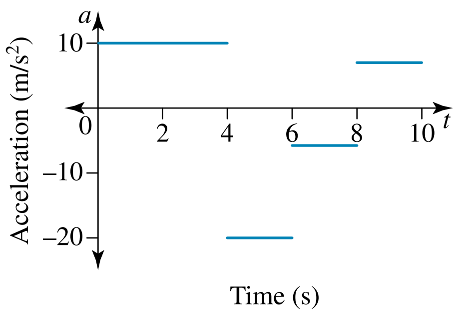
From D to E: Average acceleration $= \dfrac{v_2 - v_1}{t_2 - t_1}$

$$= \frac{0 - -15}{10 - 8}$$

$$= \frac{15}{2}$$

$$= 7.5 \text{ m/s}^2$$

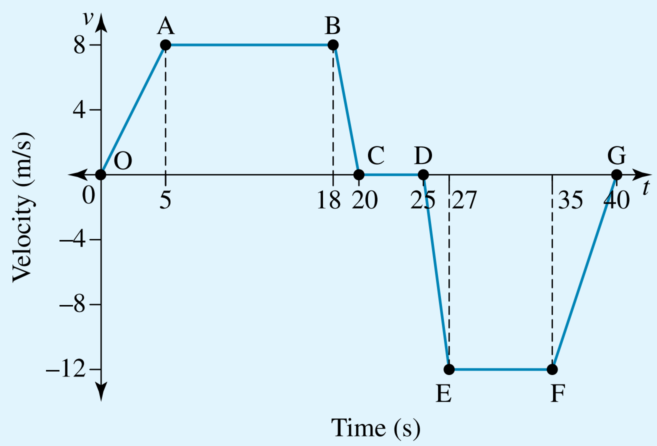
Therefore, the acceleration–time graph would look like the graph below.



Note: The signed area under the acceleration–time graph gives the change in velocity. In the graph above, the area between the graph and the time axis from $t = 0$ s to $t = 4$ s is 40, which is verified on the previous acceleration–time graph.

WORKED EXAMPLE 6

Consider the motion of an elevator that has the velocity–time graph shown. Take positive values to represent upward motion.



a **In what sections — OA, AB, BC etc. — is the lift:**
  i **accelerating positively**
  ii **accelerating negatively**
  iii **travelling at a constant velocity?**

b **Determine the acceleration for each section of the lift's journey.**

c **Sketch the acceleration–time graph.**

d **If the lift started at ground level, 0 metres, determine its position at:**
  i **C** **ii** **G.**

e **Determine the average velocity of the lift.**

f **How far did the lift travel?**

g **What was the lift's average speed?**

THINK	WRITE/DRAW
a **i** Acceleration is positive where the velocity is increasing.	**a** **i** The acceleration is positive from O to A and from F to G.

ii Acceleration is negative where the velocity is decreasing.

iii Acceleration is zero where the velocity is not changing.

b Average acceleration $= \dfrac{\text{change in velocity}}{\text{change in time}}$

$= \dfrac{v_2 - v_1}{t_2 - t_1}$

ii The acceleration is negative from B to C and from D to E.

iii The acceleration is zero from A to B, from C to D and from E to F.

b From O to A, average acceleration $= \dfrac{v_2 - v_1}{t_2 - t_1}$

$= \dfrac{8 - 0}{5 - 0}$

$= \dfrac{8}{5}$

$= 1.6 \text{ m/s}^2$

From A to B, average acceleration $= \dfrac{8 - 8}{18 - 5}$

$= \dfrac{0}{13}$

$= 0 \text{ m/s}^2$

From B to C, average acceleration $= \dfrac{0 - 8}{20 - 18}$

$= -\dfrac{8}{2}$

$= -4 \text{ m/s}^2$

From C to D, average acceleration $= \dfrac{0 - 0}{25 - 25}$

$= \dfrac{0}{5}$

$= 0 \text{ m/s}^2$

From D to E, average acceleration $= -\dfrac{12 - 0}{27 - 25}$

$= -\dfrac{12}{2}$

$= -6 \text{ m/s}^2$

From E to F, average acceleration $= \dfrac{-12 - -12}{35 - 27}$

$= \dfrac{0}{8}$

$= 0 \text{ m/s}^2$

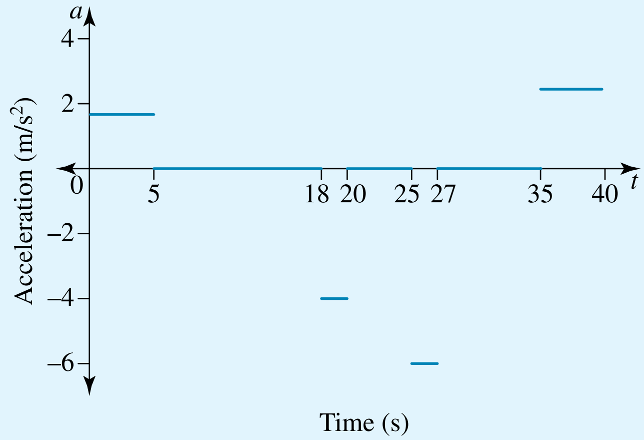
From F to G, average acceleration $= \dfrac{0 - -12}{40 - 35}$

$= \dfrac{12}{5}$

$= 2.4 \text{ m/s}^2$

c The acceleration is constant in each section, so the acceleration–time graph is a series of horizontal lines (steps).

c



d **i** Because the lift started at position 0 metres, the position at point C is the signed area under the trapezium OABC.

d **i** The position at C is the area of trapezium OABC

$= \frac{1}{2}(13 + 20) \times 8$

$= \frac{1}{2} \times 33 \times 8$

$= 132$ metres

ii The position at point G is the signed area under the trapezium DEFG plus position at point C.

ii The position at G is the signed area under the trapezium DEFG plus position at point C

$= -\frac{1}{2}(8 + 15) \times 12 + 132$

$= -\frac{1}{2} \times 23 \times 12 + 132$

$= -138 + 132$

$= -6$ metres (that is, the lift ends up 6 metres below ground level).

e Average velocity $= \dfrac{\text{change in position}}{\text{change in time}}$

e Average velocity $= \dfrac{x_2 - x_1}{t_2 - t_1}$

$= -\dfrac{6 - 0}{40 - 0}$

$= -\dfrac{6}{40}$

$= -0.15$ m/s

f The total distance travelled by the lift is the total area between the velocity–time graph and the time axis.

f The total distance travelled by the lift is $132 + 138 = 270$ metres.

g Average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

g Average speed $= \dfrac{270}{40}$

$= 6.75$ m/s

EXERCISE 9.3 Velocity–time graphs and acceleration–time graphs

PRACTISE

1 WE4 Draw a velocity–time graph to match each of the following descriptions.

a An object, which is moving in a straight line, has an initial velocity of 4 m/s. It accelerates at a constant rate until, after 5 seconds, it reaches a velocity of 9 m/s. It maintains this velocity for 10 seconds and then decelerates at a constant rate for a further 5 seconds, when it comes to rest.

b An object, which is moving in a straight line, has an initial velocity of 6 m/s. It accelerates at a constant rate until, after 8 seconds, it reaches a velocity of 12 m/s. It maintains this velocity for 15 seconds and then decelerates at a constant rate for a further 5 seconds until it reaches a velocity of 8 m/s.

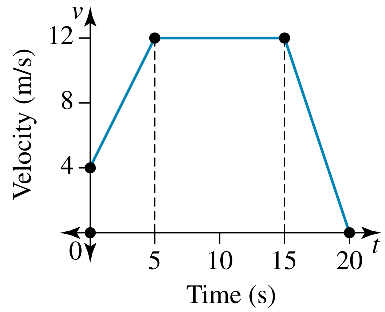
c An object, which is moving in a straight line, has an initial velocity of −5 m/s. It accelerates at a constant rate until, after 10 seconds, it reaches a velocity of 4 m/s. It maintains this velocity for 12 seconds and then decelerates at a constant rate for a further 9 seconds, when it comes to rest.

2 Draw a velocity–time graph to match each of the following descriptions.

a An object, which is moving in a straight line, has an initial velocity of 5 m/s. It decelerates at a constant rate until, after 6 seconds, it reaches a velocity of −5 m/s. It maintains this velocity for 4 seconds and then accelerates at a constant rate for a further 6 seconds, when it comes to rest.

b An object, which is moving in a straight line, has an initial velocity of −8 m/s. It maintains this velocity for 10 seconds and then accelerates at a constant rate until, after 8 seconds, it reaches a velocity of 4 m/s. It maintains this velocity for 12 seconds and then decelerates at a constant rate for a further 4 seconds, when it reaches a velocity of 2 m/s, which it maintains.

3 **WE5** Consider the velocity–time graph shown at right to find:



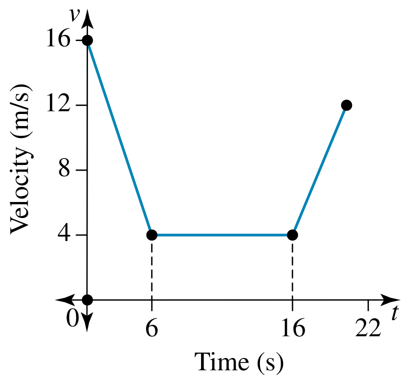
a the acceleration in the first 5 seconds

b the acceleration in the last 5 seconds

c the total displacement

d the total distance travelled.

4 Consider the velocity–time graph shown at right to find:



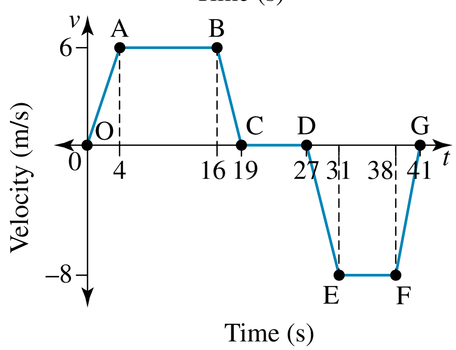
a the acceleration in the first 6 seconds

b the acceleration in the last 6 seconds

c the total displacement

d the total distance travelled.

5 **WE6** Consider the motion of an elevator whose velocity–time graph is as shown. Take positive values to represent upward motion.



a In what sections, OA, AB, BC etc., is the lift:

i accelerating positively

ii accelerating negatively

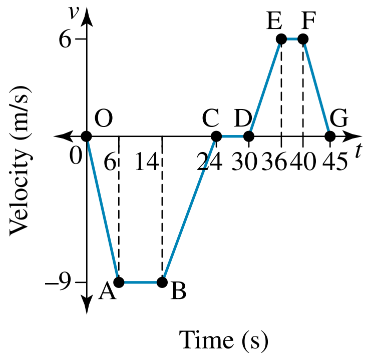
iii travelling at a constant velocity?

b Determine the acceleration for each section of the lift's journey.

c Sketch the acceleration–time graph.

d If the lift started at ground level, 0 metres, determine its position at:
  i C **ii** G.
e Determine the average velocity of the lift.
f How far did the lift travel?
g What was the lift's average speed?

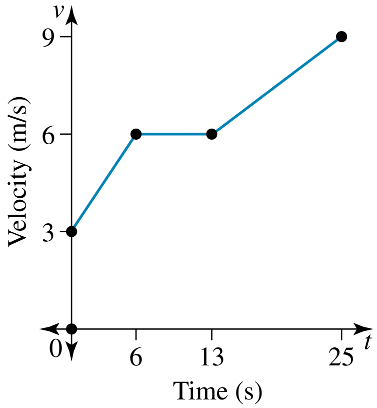
6 Consider the motion of a lift in a high-rise building. The lift's velocity–time graph is as shown. The lift starts from the 25th floor, which is 100 metres above ground level. Take positive values to represent upward motion.



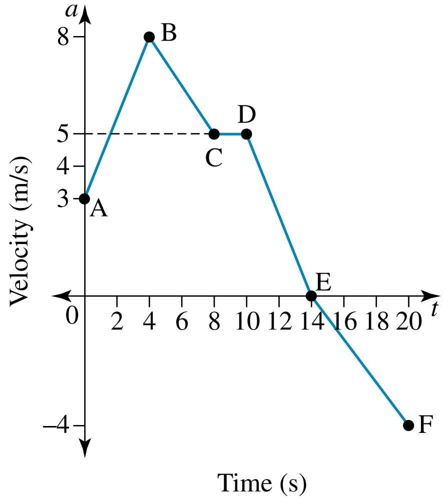
a In what sections, OA, AB, BC etc., is the lift:
  i accelerating positively
  ii accelerating negatively
  iii travelling at a constant velocity?
b Determine the acceleration for each section of the lift's journey.
c Sketch the acceleration–time graph.
d Determine the lift's position at:
  i C **ii** G.
e Determine the average velocity of the lift.
f How far did the lift travel?
g What was the lift's average speed?

CONSOLIDATE

7 Consider the velocity–time graph shown below to find:
a the acceleration in the first 6 seconds
b the acceleration in the last 12 seconds
c the total displacement
d the total distance travelled.



Use the velocity–time graph below to answer questions **8** to **10**.



8 The magnitude of the acceleration is greatest between the points:

A A and B **B** B and C **C** A and B and D and E
D D and E **E** E and F

9 The average velocity from A to F is equal to:

A 3.3 m/s **B** 2.3 m/s **C** 4 m/s **D** 2.8 m/s **E** −4 m/s

10 The average speed from A to F is equal to:

A 3.3 m/s **B** 2.3 m/s **C** 4 m/s **D** 2.8 m/s **E** −4 m/s

11 A learner driver starts her car from rest and drives for 8 seconds until she reaches a speed of 16 m/s. What is the average acceleration?

12 A racing car accelerates uniformly from 15 m/s to 45 m/s in 2.6 seconds. Determine the average acceleration of the car, correct to 1 decimal place.

13 A cyclist increases his speed from 4.2 m/s to 6.3 m/s over 5.3 seconds. What is the average acceleration, correct to 1 decimal place?

14 Sarah took a new car for a test drive. From rest, the car travelled a distance of 400 metres in 16 seconds.

a What is the average velocity?

b Assuming a constant acceleration:

i what is Sarah's final velocity?

ii what is the average acceleration of the car, correct to 1 decimal place?

15 A driver brakes suddenly to avoid hitting a kangaroo on the road. The car is travelling at 30 m/s and stops in 2.58 seconds.

a What is the average velocity from the time the driver brakes?

b What is the stopping distance of the car?

16 A car is travelling at a constant speed of 108 km/h when it passes a stationary police motorcycle. Four seconds later the motorcycle sets off in pursuit with a constant acceleration of 5 m/s^2 until it reaches a speed of 126 km/h, which it then maintains. (1 m/s = 3.6 km/h)

a For how long does the motorcycle accelerate?
b Sketch a velocity–time graph that represents the motion of both the car and the motorcycle.
c How long after the car first passes the motorcycle does the motorcycle catch up to the car?
d How far have they travelled?

MASTER

17 Polly is leading a 50-kilometre bicycle race when her bicycle gets a puncture 360 metres from the finish line. She changes her tyre, and the instant she takes off again, Molly passes her, travelling at a constant speed of 14 m/s. Polly accelerates at a constant rate for 5 seconds until she reaches a speed of 16 m/s, which she maintains until the finish.

a Sketch a velocity–time graph that represents the motion of both Polly and Molly.
b Verify that Polly still wins the race.
c How far from the finish line are they when Polly catches up to Molly?
d If Molly started to accelerate at a constant rate from the moment that Polly caught up to her, what would her acceleration be if they were to dead heat?

18 Max the monkey is climbing a coconut tree in a straight line to find a coconut for lunch. His motion is described as follows.

Max starts from rest at ground level with constant acceleration until he reaches a speed of 1.5 m/s after 4 seconds. He maintains this speed for 8 seconds, then he decelerates to a stop after another 2 seconds. After a further 9 seconds, Max heads back down the tree with constant acceleration, reaching a speed of 2.5 m/s in 2 seconds. He maintains this speed for 5 seconds, then he jumps from the tree. (Take positive as up.)

a Draw a velocity–time graph representing the motion of the monkey until he leaves the tree.
b At what height did Max leap off the tree?
c What was the total distance travelled by Max on the tree?
d What was Max the monkey's average speed:
i while on the tree ii while in motion on the tree?

Challenge: When Max begins his descent, a palm leaf falls from the tree at a height of 25 metres. It falls with a constant acceleration of 2 m/s^2.

e Verify that Max the monkey is still on the tree when the palm leaf hits the ground and determine where Max is at this time.

9.4 Constant acceleration formulas

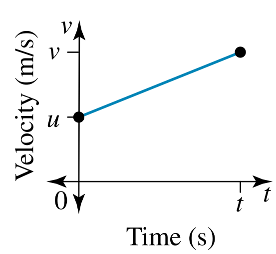
studyon

Units 1 & 2
AOS 5
Topic 2
Concept 4

Constant acceleration formulas
Concept summary
Practice questions

Acceleration due to gravity is usually 9.8 m/s^2. It varies slightly depending on the distance from the centre of the Earth.

This means that a falling object or an object thrown into the air is subject to a constant (or uniform) downward acceleration of 9.8 m/s^2. Since acceleration is a vector quantity, when the object is moving upwards, it is subject to an acceleration of −9.8 m/s^2; that is, a deceleration or retardation.



Consider an object moving in a straight line that has an initial velocity of u. It accelerates constantly until it reaches a velocity of v after t seconds. Its velocity–time graph is shown at right.

We can use this graph to derive various formulas that can be applied to problems involving constant acceleration.

Since acceleration, a, is the change in velocity over time:

$$a = \frac{\delta v}{\delta t}$$

$$= \frac{v - u}{t}$$

Multiply both sides by t:

$$at = v - u$$

Make v the subject, so:

$$v = u + at \qquad [1]$$

Furthermore, since average velocity is the change in position, s, over time:

$$\text{average velocity} = \frac{\delta s}{\delta t} \text{ or } \frac{u + v}{2}$$

So,
$$\frac{s}{t} = \frac{u + v}{2}$$

Therefore:
$$s = \frac{1}{2}(u + v)t \qquad [2]$$

Substituting $v = u + at$ (equation [1]) into equation [2]:

$$s = \frac{1}{2}(u + u + at)t$$

$$= \frac{1}{2}(2u + at)t$$

$$= \frac{1}{2}(2ut + at^2)$$

Therefore:
$$s = ut + \frac{1}{2}at^2 \qquad [3]$$

From [1],
$$t = \frac{v - u}{a}$$

Substituting $t = \dfrac{v - u}{a}$ into equation [2]:

$$s = \frac{1}{2}(u + v)\left(\frac{v - u}{a}\right)$$

$$= \frac{1}{2}\left(\frac{v^2 - u^2}{a}\right)$$

$$2as = v^2 - u^2$$

Therefore:
$$v^2 = u^2 + 2\text{as} \qquad [4]$$

In summary, if u is the initial velocity, v is the final velocity, s is the displacement, a is the constant acceleration and t is the time interval, then the following formulas apply for straight-line motion:

$$\boldsymbol{v = u + at} \qquad \textbf{[1]}$$

$$\boldsymbol{s = \frac{1}{2}(u + v)t} \qquad \textbf{[2]}$$

$$\boldsymbol{s = ut + \frac{1}{2}at^2} \qquad \textbf{[3]}$$

$$\boldsymbol{v^2 = u^2 + 2as} \qquad \textbf{[4]}$$

Notes

1. 'At rest' means the velocity is zero.
2. 1 m/s = 3.6 km/h. (Verify this.)
3. When an object is travelling in one direction, u can be treated as the initial speed, v as the final speed and s as the distance travelled.

WORKED EXAMPLE 7

A stone is dropped from a bridge that is 150 metres above a river. Find:

a the time taken for the stone to reach the river

b the stone's speed on impact.

Give answers correct to 1 decimal point.

THINK	WRITE
a 1 List the given information and what has to be found.	**a** Given: $s = 150$, $a = 9.8$ and $u = 0$ Require: $t = ?$
2 Find t using $s = ut + \frac{1}{2}at^2$ by substituting $s = 150$, $a = 9.8$ and $u = 0$.	$s = ut + \frac{1}{2}at^2$ $150 = 0 \times t + \frac{1}{2} \times 9.8 \times t^2$
3 Solve the equation for t.	$150 = 4.9t^2$ $30.6122 = t^2$ $t = \sqrt{30.6122}$ $= 5.533$
4 State the solution.	The stone reaches the river after approximately 5.5 seconds.
b 1 List the given information and what has to be found.	**b** Given: $s = 150$, $a = 9.8$ and $u = 0$ Require: $v = ?$
2 Find v using $v^2 = u^2 + 2as$ by substituting $u = 0$, $a = 9.8$ and $s = 150$.	$v^2 = u^2 + 2as$ $= 0^2 \times 2 \times 9.8 \times 150$
3 Solve the equation for v.	$v^2 = 2940$ $v = \sqrt{2940}$ $= 54.22$
4 State the solution.	The stone reaches the river at a speed of 54.2 m/s.

WORKED EXAMPLE 8

A driver is forced to suddenly apply the brakes when a dog appears in front of his car. The car skids in a straight line, stopping 2 centimetres short of the startled dog. The car skidded a distance of 12 metres for 2 seconds.

a At what speed was the car travelling as it began to skid?

b What was the acceleration of the car during the skid?

THINK | **WRITE**

a 1 List the given information and what has to be found.

a Given: $s = 12$, $t = 2$ and $v = 0$
Require: $u = ?$

2 Find u using $s = \frac{1}{2}(u + v)t$ by substituting $s = 12$, $t = 2$ and $v = 0$.

$$s = \frac{1}{2}(u + v)t$$
$$12 = \frac{1}{2}(u + 0) \times 2$$

3 Solve the equation for u.

$$12 = \frac{1}{2}u \times 2$$
$$u = 12$$

4 State the solution.

The initial speed of the car was 12 m/s.

b 1 List the given information and what has to be found.

b Given: $v = 0$, $u = 12$ and $t = 2$
Require: $a = ?$

2 Find a using $v = u + at$ by substituting $v = 0$, $u = 12$ and $t = 2$.

$$v = u + at$$
$$0 = 12 + a \times 2$$

3 Solve the equation for a.

$$-12 = 2a$$
$$a = -6$$

4 State the solution.

The acceleration of the car was -6 m/s^2.

WORKED EXAMPLE 9

A ball is thrown upwards at 14.7 m/s from a tower that is 50 metres above the ground.

a **Determine the total time that the ball is in the air before it reaches the ground.**

b **Find the maximum height reached by the ball.**

c **Find the ball's speed when it first strikes the ground.**

THINK | **WRITE**

a 1 List the given information and what has to be found.

a Given: $u = 14.7$, $a = -9.8$, $s = -50$
Require: $t = ?$

$$s = ut + \frac{1}{2}at^2$$

2 Find t using $s = ut + \frac{1}{2}at^2$ by substituting $u = 14.7$, $a = -9.8$ and $s = -50$.

$$-50 = 14.7t + \frac{1}{2}(-9.8)t^2$$
$$-50 = 14.7t - 4.9t^2$$
$$4.9t^2 - 14.7t - 50 = 0$$

3 Solve the quadratic equation by using the quadratic formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{14.7 \pm \sqrt{(14.7)^2 - 4(4.9)(-50)}}{2(4.9)}$$
$$t = -2.0 \text{ and } 5.0$$

$t = 5.0$ seconds, as time cannot be negative.

b 1 List the given information and what has to be found.

b Given: $u = 14.7$, $a = -9.8$ and $v = 0$
Require: $t = ?$ $s = ?$

2 Find t using $v = u + at$ by substituting $u = 14.7$, $a = -9.8$ and $v = 0$.

$$v = u + at$$
$$0 = 14.7 - 9.8t$$
$$t = \frac{14.7}{9.8}$$
$$= 1.5$$

3 Find s using $s = ut + \frac{1}{2}at^2$ by substituting $u = 14.7$, $a = -9.8$ and $t = 1.5$.

$$s = ut + \frac{1}{2}at^2$$
$$s = 14.7 \times 1.5 + \frac{1}{2} \times -9.8 \times 1.5^2$$
$$= 11.025$$

4 Add s to the height of the tower.

Maximum height $= 50 + 11.025$
$= 61.025$

5 State the solution.

The maximum height reached by the ball is 61.0 m.

c 1 List the given information and what has to be found.

c Given: $u = 0$, $a = 9.8$ and $s = 61.025$
Require: $v = ?$

2 Find v using $v^2 = u^2 + 2as$ by substituting $u = 0$, $a = 9.8$ and $s = 61.025$.

$$v^2 = u^2 + 2as$$
$$= 0^2 + 2 \times 9.8 \times 61.025$$
$$= 1196.09$$

3 Solve the equation for v.

$$v = \sqrt{1196.09}$$
$$= 34.5845$$

4 State the solution.

The ball first strikes the ground at a speed of 34.6 m/s.

EXERCISE 9.4 Constant acceleration formulas

PRACTISE

1 WE7 A stone is dropped from a bridge that is 98 metres above a river. Giving answers to the nearest tenth, find:
- a the time taken for the stone to reach the river
- b the stone's speed on impact.

2 An object travelling at 8 m/s accelerates uniformly over a distance of 20 metres until it reaches a speed of 18 m/s. Find:
- a the acceleration
- b the time taken, correct to 2 decimal places.

3 WE8 A driver is forced to suddenly apply the brakes when a cat appears in front of her car. The car skids in a straight line, stopping 8 cm short of the startled cat. The car skidded a distance of 15 metres for 3 seconds.
- a At what speed was the car travelling as it began to skid?
- b What was the acceleration of the car during the skid?

4 A falcon is hovering in the air when it suddenly dives vertically down to swoop on its prey, which is 150 metres directly below it. If the acceleration is uniform and it takes the falcon 5 seconds to reach its prey, find:

a the final speed of the falcon in m/s and km/h
b the acceleration of the falcon.

5 WE9 A ball is thrown upwards at 9.8 m/s from a tower that is 30 metres above the ground.

a Determine the total time that the ball is in the air before it reaches the ground.
b Find the ball's speed when it first strikes the ground. Give answers to the nearest tenth.

6 A ball is thrown upwards at 20 m/s from a tower that is 80 metres above the ground.

a Determine the total time that the ball is in the air before it reaches the ground.
b Find the ball's speed when it first strikes the ground. Give answers to the nearest tenth.

CONSOLIDATE

7 A parachutist free-falls from an aircraft for 6 seconds. Find:

a the speed of the parachutist after 6 seconds
b the distance travelled.

8 A particle moving from rest with constant acceleration reaches a speed of 16 m/s in 4 seconds. Find:

a the acceleration
b the distance travelled.

9 A ball is dropped from a tower and reaches the ground in 4 seconds. Find:

a the height of the tower
b the speed of the ball when it hits the ground.

10 How long does it take for:

a a car to accelerate on a straight road at a constant 6 m/s^2 from an initial speed of 17 m/s to a final speed of 28 m/s
b a downhill skier to accelerate from rest at a constant 2 m/s^2 to a speed of 10 m/s?

11 A skateboarder is travelling down a gently sloping path at a speed of 10 m/s when he stops skating. He rolls a further 60 metres before coming to a stop. Assuming the acceleration is uniform, find:

a the acceleration
b the time it takes to come to a stop.

12 A tram is travelling at 16 m/s when the brakes are applied, reducing the speed to 6 m/s in 2 seconds. Assuming the retardation is constant, find:

a the acceleration
b the distance travelled during the 2 seconds after the brakes are applied
c the braking distance of the tram.

13 A train travels a distance of 1800 metres in 90 seconds while accelerating uniformly from rest.

a The speed of the train after 90 seconds can be determined using the formula:
   A $v = u + at$
   B $s = \frac{1}{2}(u + v)t$
   C $C = 2\pi r$
   D $s = ut + \frac{1}{2}at^2$
   E $v^2 = u^2 + 2as$

b The speed in km/h after 90 seconds is:

A 2	B 36	C 144
D 216	E 40	

c The speed in km/h after 45 seconds is:

A 72	B 36	C 144
D 216	E 20	

d The distance travelled after 45 seconds is:

A 225 m	B 900 m	C 675 m
D 1350 m	E 450 m	

14 An object is projected vertically upwards from the top of a building that is 50 metres above the ground. Its initial speed is 28 m/s. If the object then falls to the ground, find:
   a its maximum height above the ground
   b the total time taken to reach the ground, correct to 1 decimal place
   c the speed of the object when it reaches the ground.

15 A car moving from rest with uniform acceleration takes 12 seconds to travel 144 metres. What is its speed after 6 seconds?

16 A bird's egg falls from a nest in a tree. If it is initially 39.2 metres above the ground, calculate:
   a its speed when it is halfway to the ground
   b its speed on striking the ground, correct to 1 decimal place
   c the time taken to reach the ground, correct to 1 decimal place.

MASTER

17 A cage is descending into a well at a constant speed of 2 m/s when a stone falls through the wire in the cage. If the stone reaches the water at the bottom of the well 10 seconds before the cage, find the height above the water at which the stone fell out of the cage.

18 A hot-air balloon is rising with a speed of 19.6 m/s when a gas cylinder falls off the balloon. If the balloon is 80 metres above the ground when the cylinder falls off, how long will it take the cylinder to reach the ground and what will its speed be then?

9.5 Instantaneous rates of change

Instantaneous velocity

studyon

Units 1 & 2
AOS 5
Topic 2
Concept 5

Instantaneous rates of change — differentiation
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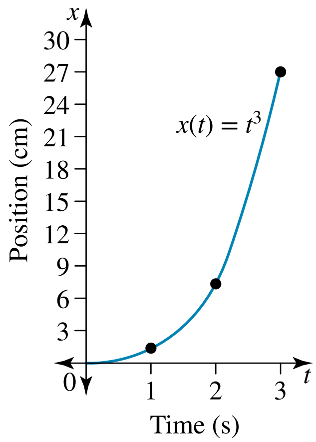
Practice questions

As we have discussed previously, the **instantaneous velocity** at a given time is in fact the **gradient** of the position–time graph at that time. We have also seen that when the velocity is variable the position–time graph will be curved.



Consider a particle moving in a straight line such that its position, x cm, at any time, t seconds, is described by the rule:

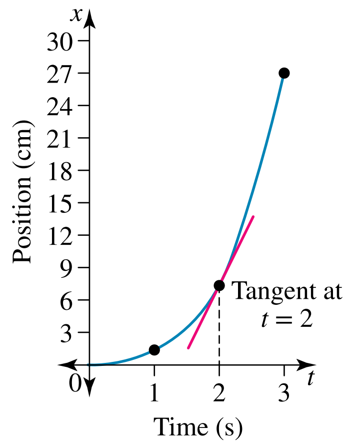
$$x(t) = t^3,\ t \in [0, 3]$$

Completing a table of values will give:

t	0	1	2	3
x	0	1	8	27

The position–time graph is shown above right.

The velocity at any given time (say at $t = 2$ seconds) is equal to the gradient of the curve at that given time ($t = 2$). The gradient of a curve at any given point is the gradient of the **tangent** to the curve at that point. So, the velocity at $t = 2$ is equal to the gradient of the tangent to the curve at $t = 2$.

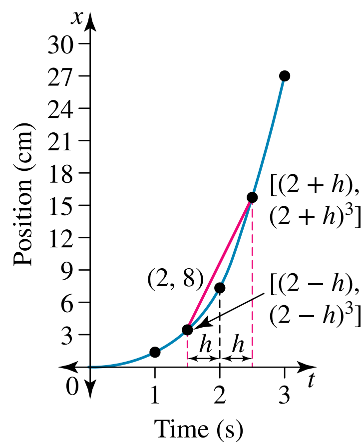


Physically determining the gradient of the tangent often leads to inaccurate results. Care needs to be taken, firstly to draw an accurate and smooth curve, then to place the tangent at exactly the right position. There is too much room for error with this process.

Instead, we can estimate the gradient (velocity) by applying the rule:

$$\text{Average velocity} = \frac{\delta x}{\delta t}$$

This involves taking two points on the curve on either side of $t = 2$. To ensure that the point at $t = 2$ is in the middle of the two points chosen, each point must be the same distance, h, either side of $t = 2$.



The gradient of the line that joins the two points on the curve at $t = 2 - h$ and $t = 2 + h$ estimates the gradient at $t = 2$.

Finding the rise and the run between the two points allows us to calculate the gradient as:

$$v(2) = \frac{(2 + h)^3 - (2 - h)^3}{2h}.$$

The smaller the value of h, the closer this gradient will be to the true gradient of the tangent.

For example, using a calculator to find $v(2)$ when $h = 1$, 0.1 and 0.01 produces the results shown in the table below.

h	1	0.1	0.01
$v(2)$	13	12.01	12.0001

It is quite clear from this table that as h gets smaller and smaller, the value of $v(2)$ approaches 12. If it is not already obvious, it becomes even more so if $h = 0.001$ or 0.0001 and so on.

In summary, the instantaneous velocity at $t = t_0$, $v(t_0)$, of a particle moving in a straight line with its position described as $x(t)$ is found by evaluating:

$$v(t_0) = \frac{x(t_0 + h) - x(t_0 - h)}{2h}$$

for very small values of h ($h > 0$). This method of approximating instantaneous velocity uses the central difference formula.

This technique uses the same process as that of differentiating from first principles, which was covered in Mathematical Methods (CAS) Units 1 and 2, and thus we can say:

$$v(t) = \frac{dx}{dt} \qquad \text{the derivative of } x \text{ with respect to } t$$

or $\quad v(t) = x'(t)$

WORKED EXAMPLE 10

A particle is travelling in a straight line with its position, x cm, at any time, t seconds, given as $x(t) = t^3 - t$, $t \in [0, 3]$.

Find the velocity of the particle after 1.5 seconds.

THINK	WRITE
1 Given the expression $x(t) = t^3 - t$, we want $v(1.5)$.	$x(t) = t^3 - t$
2 Find the velocity equation by differentiating position, x, with respect to time, t ($v(t) = x'(t)$).	$v(t) = x'(t)$ $v(t) = 3t^2 - 1$
3 Substitute $t = 1.5$ seconds.	$v(1.5) = 3(1.5)^2 - 1$ $v(1.5) = 5.75$
4 State the solution.	The velocity of the particle at $t = 1.5$ seconds is 5.75 cm/s.

Instantaneous acceleration

When the acceleration is variable, the velocity–time graph is curved. The instantaneous acceleration at a given time is the gradient of the velocity–time graph at that time. So, like the instantaneous velocity:

The **instantaneous acceleration** at $t = t_0$, $a(t_0)$, of a particle moving in a straight line with its velocity described as $v(t)$ is found by evaluating the central difference formula:

$$a(t_0) = \frac{v(t_0 + h) - v(t_0 - h)}{2h}$$

for very small values of h ($h > 0$).

Again the technique uses the same process to that of differentiating from first principles, and we can say:

$$a(t) = \frac{dv}{dt} \qquad \text{the derivative of } v \text{ with respect to } t$$

or $$a(t) = v'(t)$$

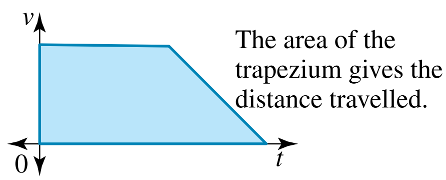
WORKED EXAMPLE 11

A particle is travelling in a straight line with its velocity, v cm/s, at any time, t seconds, given as $v(t) = \dfrac{8}{t+1}$, $t \geq 0$.

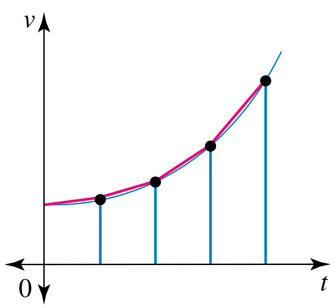
Find the acceleration of the particle after 1 second.

THINK	WRITE
1 Given the expression $v(t) = \dfrac{8}{t+1}$, we want $a(1)$.	$v(t) = \dfrac{8}{t+1}$
2 Find the acceleration equation by differentiating velocity with respect to time ($a(t) = v'(t)$) using a calculator.	$v'(t) = \dfrac{-8}{(t+1)^2}$
3 Substitute $t = 1$ second into the formula for $a(t)$.	$a(1) = \dfrac{-8}{(1+1)^2}$ $= -\dfrac{8}{4}$ $= -2$
4 State the solution.	The acceleration of the particle at $t = 1$ seconds is -2 cm/s^2.

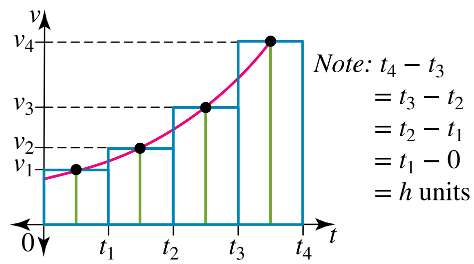
Approximating velocity–time graphs



We have already seen that the distance travelled by a particle travelling in a straight line is the unsigned area between the velocity–time graph and the time axis. When the acceleration is constant, we calculate the areas of rectangles, triangles or trapeziums.



If the acceleration is variable, the velocity–time graph is curved and so it needs to be approximated by straight-line functions. This will result in the area under the graph comprising either rectangles, triangles or trapeziums. Then the distance travelled can be estimated.



One way to approximate the velocity–time curve is to use a series of 'horizontal steps' over the required domain or time values. This can be achieved by first dividing the domain interval into n equally sized time intervals, each h units long. Next, evaluate the velocity at the midpoint of each of these intervals. Each of these velocities can be treated as the average velocity over its corresponding interval. The result will be a 'step function' graph something like the figure on the bottom of the previous page.

The unsigned area under this velocity–time graph can be found by determining the sum of each rectangular area ($h \times v_n$). This gives an estimate for the distance travelled over a given period of time.

As the rectangle width (or interval width), h, gets smaller and smaller, the number of rectangles, n, increases and therefore the estimate gets closer to the exact distance.

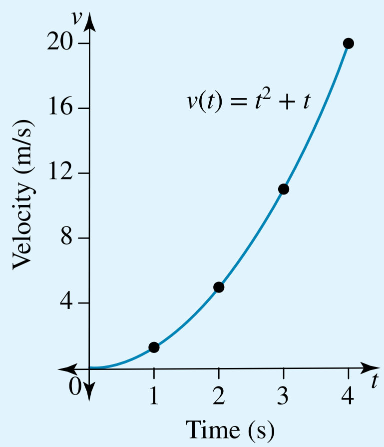
The following worked example outlines the steps involved, with the aid of graphs.

WORKED EXAMPLE 12

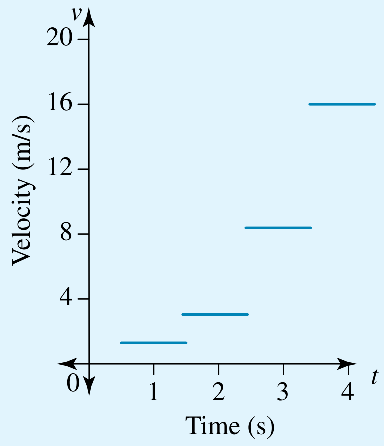
A particle is travelling in a straight line with its velocity, v m/s, at any time, t seconds, given as:

$$v(t) = t^2 + t, \quad t \geq 0$$

Estimate the distance travelled during the first 4 seconds of its motion by approximating the velocity with step functions each 1 unit wide.

THINK	DRAW/WRITE
1 Sketch the graph of $v(t) = t^2 + t$ over the domain [0, 4].	
2 As h is 1 and the domain is [0, 4], the intervals are from 0 to 1, 1 to 2, 2 to 3 and 3 to 4.	
3 The midpoints of each interval are 0.5, 1.5, 2.5 and 3.5.	
4 Evaluate $v(0.5)$, $v(1.5)$, $v(2.5)$ and $v(3.5)$. These represent the height of each rectangle.	$v(0.5) = 0.5^2 + 0.5$ $= 0.75$ $v(1.5) = 1.5^2 + 1.5$ $= 3.75$ $v(2.5) = 2.5^2 + 2.5$ $= 8.75$ $v(3.5) = 3.5^2 + 3.5$ $= 15.75$

5 Sketch the step function graph over the domain [0, 4] as an approximation of the velocity–time relationship.



6 Determine the sum of each rectangular area under the step function.

Area of each rectangle = length × width

$$\begin{aligned}\text{Total area} &= (0.75 \times 1) + (3.75 \times 1) + (8.75 \times 1) + (15.75 \times 1)\\ &= 1(0.75 + 3.75 + 8.75 + 15.75)\\ &= 29\end{aligned}$$

7 State the unsigned area as the approximate distance travelled.

(*Note:* The exact distance is $29\frac{1}{3}$ metres.)

The particle travels approximately 29 metres during the first 4 seconds.

In summary, if the acceleration is variable, then the distance, d, travelled by a particle can be estimated from a velocity–time function by evaluating:

$$d = h[v(t_1) + v(t_2) + v(t_3) + \ldots + v(t_n)]$$

where h = step function width (time interval width)

n = the number of intervals

t_n = midpoint of time interval n

$v(t_n)$ = velocity at time t_n.

The method shown above is an approximation of the displacement (area under the curve) that can be improved by reducing the step function width. However, to calculate the exact displacement (area under the curve), calculus is used. Using your knowledge from Mathematical Methods (CAS) Units 1 and 2, this can be achieved as shown in Worked example 13.

That is, a formula for the distance travelled by an object, $d(t)$, can be found by finding the antiderivative of the formula for its velocity, $v(t)$, with respect to time. A formula for its velocity, $v(t)$, can be found by finding the antiderivative of its acceleration, $a(t)$, with respect to time.

$$d(t) = \int v(t)dt$$

$$v(t) = \int a(t)dt$$

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Instantaneous rates of change — integration

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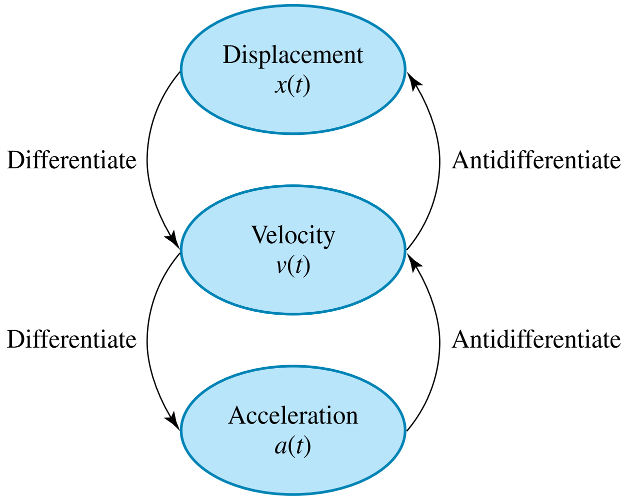

Notes

1. The signed area between a velocity–time curve and the t-axis gives the displacement.
2. If the velocity is positive over the given time interval, then the displacement is equal to the distance.

WORKED EXAMPLE 13

A particle is travelling in a straight line with its velocity, v m/s, at any time, t seconds, given as:

$$v(t) = t^2 + t, t \geq 0$$

Calculate the exact distance travelled during the first 4 seconds of its motion.

THINK	WRITE
1 In order to calculate the distance, $d(t)$, travelled during the first 4 seconds of motion, the antiderivative of the expression, $v(t) = t^2 + t$, from $t = 0$ to $t = 4$ needs to be found.	$d(t) = \int v(t)dt$ $d(t) = \int_0^4 (t^2 + t)dt$
2 Antidifferentiate the expression.	$d(t) = \left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^4$
3 Substitute the limits and solve.	$d(t) = \left[\left(\frac{4^3}{3} + \frac{4^2}{2}\right) - \left(\frac{0^3}{3} + \frac{0^2}{2}\right)\right]$ $d(t) = \left[\left(\frac{64}{3} + \frac{16}{2}\right) - 0\right]$ $d(t) = 29\frac{1}{3}$
4 State the exact distance travelled.	The exact distance travelled by the particle during the first 4 seconds of its motion is $29\frac{1}{3}$ metres.

WORKED EXAMPLE 14

A car accelerates from rest at 2 m/s^2 for 5 seconds.

a Write the equation for the acceleration.

b Write the equation for the velocity.

c Calculate the distance covered in the first 5 seconds.

THINK	WRITE
a The acceleration is 2 m/s^2.	**a** $a(t) = 2$
b 1 The velocity equation is produced by antidifferentiating the formula for acceleration.	**b** $a(t) = 2$ $v(t) = \int 2\,dt$ $v(t) = 2t + c$
2 It is given that $v = 0$ when $t = 0$. Calculate the constant, c, using this information.	$0 = 2(0) + c$ $c = 0$ $\therefore v(t) = 2t,\ 0 \le t \le 5$
c 1 To calculate the distance, $d(t)$, covered in the first 5 seconds, antidifferentiate the velocity equation with limits 0 and 5.	**c** $d(t) = \int_0^5 (2t)dt$ $d(t) = [t^2]_0^5$ $d(t) = [5^2 - 0^2]$ $d(t) = 25$
2 State the distance covered in the first 5 seconds.	The distance travelled in the first 5 seconds is 25 metres.

EXERCISE 9.5 Instantaneous rates of change

PRACTISE

1 **WE10** A particle is travelling in a straight line with its position, x cm, at any time, t seconds, given as $x(t) = t^3 + t,\ t \in [0, 5]$. Find the velocity of the particle after 2 seconds.

2 A particle is accelerating in a straight line with its position, x cm, at any time, t seconds, given as $x(t) = t^4,\ t \in [0, 4]$. Find the velocity of the particle after 3.5 seconds.

3 **WE11** A particle is travelling in a straight line with its velocity, v m/s, at any time, t seconds, given as $v(t) = \dfrac{16}{t + 2},\ t \ge 0$. Find the acceleration of the particle after 2 seconds.

4 An ant is travelling in a straight line with its velocity, v cm/s, at any time, t seconds, given as $v(t) = \dfrac{8}{(t + 1)^2},\ t \ge 0$. Find the ant's acceleration after 3.5 seconds.

5 **WE12** A particle is travelling in a straight line with its velocity, v m/s, at any time, t seconds, given as $v(t) = 2t^2 + 3t,\ t \ge 0$. Estimate the distance travelled during the first 3 seconds of its motion by approximating the velocity with step functions each 1 unit wide.

6 A particle is travelling in a straight line with its velocity, v m/s, at any time, t seconds, given as $v(t) = t^2 + 3t,\ t \ge 0$. Estimate the distance travelled during the first 6 seconds of its motion by approximating the velocity with step functions each 1 unit wide.

7 WE13 An object is travelling in a straight line with its velocity, v m/s, at any time, t seconds, given as $v(t) = t^3 + t, t \geq 0$. Calculate the exact distance travelled during the first 3 seconds of its motion.

8 A particle is travelling in a straight line with its velocity, v m/s, at any time, t seconds, given as $v(t) = t^2 + 3t, t \geq 0$. Calculate the exact distance travelled during the first 6 seconds of its motion.

9 WE14 An object initially starts from rest at the origin and accelerates in a straight line with acceleration, a m/s^2, at any time, t seconds, given as $a(t) = 2t + 1, t \geq 0$.

a Find the equation for the velocity of the object with respect to time.
b Find the equation for the position of the object with respect to time.
c Calculate the distance travelled in the first 4 seconds.

10 An object initially starts at the origin travelling in a straight line at 15 m/s and speeds up with acceleration, a m/s^2, at any time, t seconds, given as $a(t) = 12t^2 - 4t + 4, t \geq 0$.

a Find the equation for the velocity of the object with respect to time.
b Find the equation for the position of the object with respect to time.
c Calculate the distance travelled in the first 2 seconds.

CONSOLIDATE

11 A missile travelling in a straight line has its position, x m, at any time, t seconds, given by $x(t) = 2t^3 - 4t, t \in [0, 6]$. Find the velocity of the missile after 4 seconds.

12 A particle is travelling in a straight line with its position, x cm, at any time, t seconds, given as $x(t) = \dfrac{8}{t+1}, t \geq 0$. Find the velocity of the particle after 3 seconds.

13 The position of a lift, x m, at any time, t seconds, is given as $x(t) = \dfrac{1}{t+2} - t^2$, $t \in [0, 8]$. Find the velocity of the lift after 1.5 seconds.

14 A lift moves with its velocity, v m/s, at any time, t seconds, given as $v(t) = t^2 - \dfrac{4}{t+1}$, $t \in [0, 4]$.

Find the acceleration of the lift after 1 second.

15 An object is accelerating in a straight line such that its velocity, v cm/s, at any time, t seconds, is given as $v(t) = t^3 + 2t^2 - 3t, t \in [0, 6]$. Find the acceleration of the object after 2 seconds.

Questions 16 and 17 refer to the following information:

The position of an object travelling in a straight line is given by x m. At any time, t seconds, its position is $x(t) = 2\log_e(t + 1), t \geq 0$.

16 The velocity at $t = 3$ is nearest to:

A 0.92 m/s
B 2.77 m/s
C 0.51 m/s
D 1.37 m/s
E 0.50 m/s

17 The velocity at $t = 6$ is nearest to:

A 1 m/s
B 0.29 m/s
C 3.74 m/s
D 1.84 m/s
E 5.0 m/s

Questions **18** and **19** refer to the following information:

An object is travelling in a straight line such that its velocity, v m/s, at any time, t seconds, is given as $v(t) = 3e^{2-t}, t \geq 0$.

18 The acceleration, in m/s^2, after 2 seconds is equal to:

A -1.5 **B** -3 **C** 1.5
D 3 **E** -6

19 Using a step function 0.2 seconds wide to approximate the velocity, the distance travelled after 2 seconds is nearest to:

A 10.57 m **B** 9 m **C** 18 m
D 19.14 m **E** 16.8 m

20 A particle starts at rest at the origin and travels in a straight line with its velocity, v m/s, at any time, t seconds, given as $v(t) = t^3 + t, t \geq 0$.

a Find the equation for the position of the particle with respect to time.
b Calculate the distance covered in the first 3 seconds.

MASTER

21 A particle is travelling in a straight line with its position, x cm, at any time, t seconds, given as $x(t) = t^3 + 2t^2, t \geq 0$.

a Find an approximation for the instantaneous velocity, where $t = 1$ and $h = 0.001$, using the central difference formula. Give your answer correct to 2 decimal places.
b Find the instantaneous velocity at $t = 1$.
c Comment on the results found in parts **a** and **b**.

22 A rocket starts from rest and travels in a straight line with its velocity, v m/s, at any time, t seconds, given as $v(t) = 3t^3 + 2t, t \geq 0$.

a Find the equation for the position of the rocket with respect to time.
b Calculate the distance covered by the rocket in the first 5 seconds.

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9.6 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Units 1 & 2 — **Kinematics**

Sit topic test

9 Answers

EXERCISE 9.2

1 a i -1
ii 1
iii 2 units
iv 10 units
v $\frac{2}{5}$ units/second
vi 2 units/second

b i 4
ii 0
iii -4 units
iv 24 units
v $-\frac{4}{5}$ units/second
vi $4\frac{4}{5}$ units/second

c i -8
ii 12
iii 20 units
iv 52 units
v 4 units/second
vi $10\frac{2}{5}$ units/second

2 a i 12
ii 6
iii -6 units
iv 26 units
v $-1\frac{1}{5}$ units/second
vi $5\frac{1}{5}$ units/second

b i 10
ii 16
iii 6 units
iv 30 units
v $1\frac{1}{5}$ units/second
vi 6 units/second

3 a 300 cm

b 200 cm

c i $t = 6$ to $t = 11$
ii $t = 0$ to $t = 4$
iii $t = 4$ to $t = 6$

d i 90 cm/s
ii -62.5 cm/s
iii 0 cm/s

e 58.3 cm/s

4 a 600 cm

b -400 cm

c i $t = 5$ to $t = 8$
ii $t = 0$ to $t = 4$ and $t = 8$ to $t = 13$
iii $t = 4$ to $t = 5$

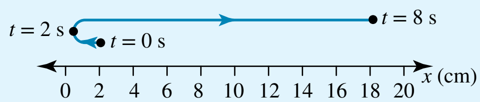
d i $116\frac{2}{3}$ cm/s
ii -137.5 cm/s and -40 m/s
iii 0 cm/s

e 84.6 cm/s

5 a

t	0	2	4	6	8
x	2	0	2	8	18

b The particle changes direction or turns.

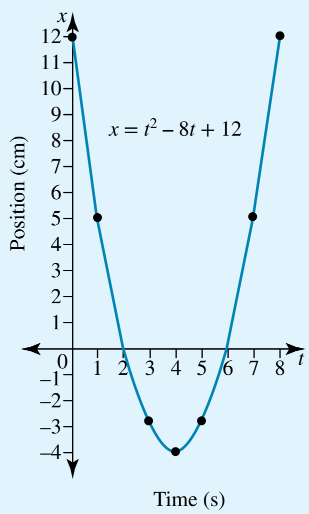
c 

d 2 cm/s

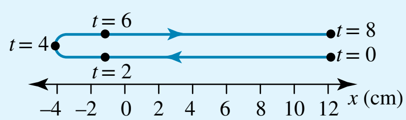
e 2.5 cm/s

6 a

t	0	2	4	6	8
x	12	0	-4	0	12

b 

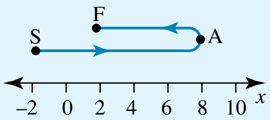
c The particle changes direction or turns.

d 

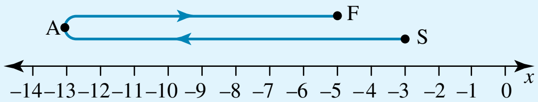
e 0 cm/s

f 4 cm/s

7 a



b



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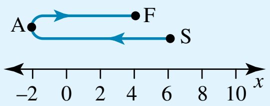
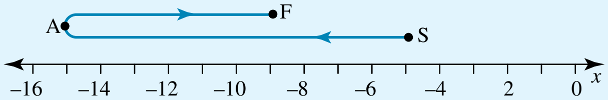


d



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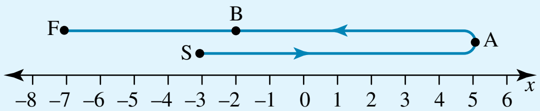
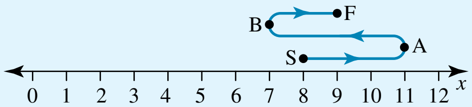


f



8 a i 5 cm
ii 15 cm
iii $\frac{5}{6}$ cm/s
iv $2\frac{1}{2}$ cm/s

b i −2 cm
ii 18 cm
iii $-\frac{1}{3}$ cm/s
iv 3 cm/s

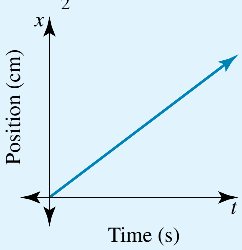
c i −1 cm
ii 15 cm
iii $-\frac{1}{6}$ cm/s
iv $2\frac{1}{2}$ cm/s

d i −5 cm
ii 17 cm
iii $-\frac{5}{6}$ cm/s
iv $2\frac{5}{6}$ cm/s

e i −4 cm
ii 20 cm
iii $-\frac{2}{3}$ cm/s
iv $3\frac{1}{3}$ cm/s

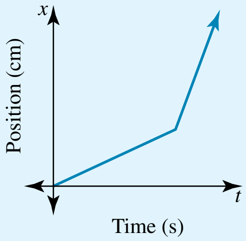
f i 1 cm
ii 9 cm
iii $\frac{1}{6}$ cm/s
iv $1\frac{1}{2}$ cm/s

9 a



Check with your teacher.

b



Check with your teacher.

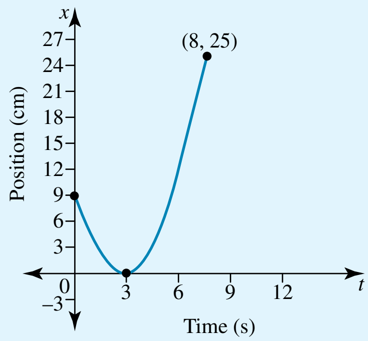
10 C

11 E

12 B

13 D

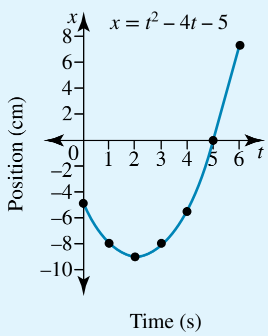
14 a



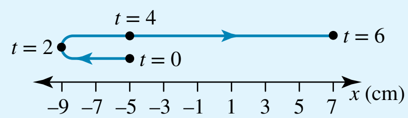
b −2 cm/s

c $4\frac{1}{4}$ cm/s

15 a



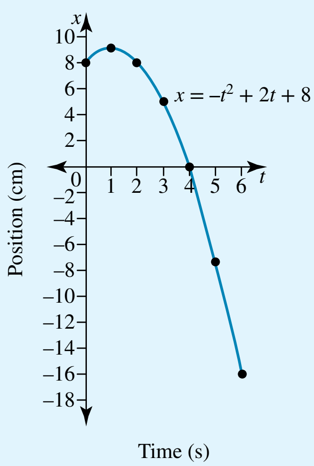
b



c 2 cm/s

d $3\frac{1}{3}$ cm/s

16 a



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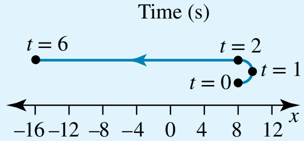
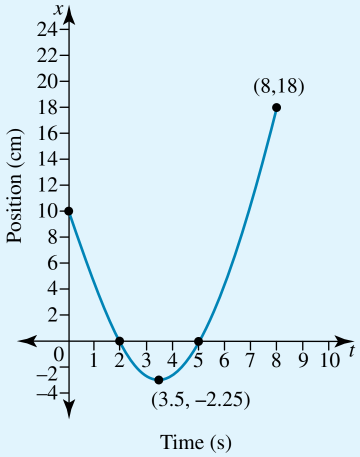


c −4 cm/s

d $4\frac{1}{3}$ cm/s

17 a



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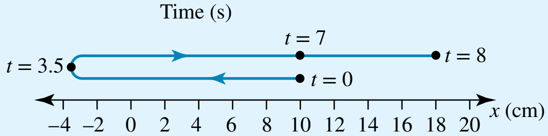


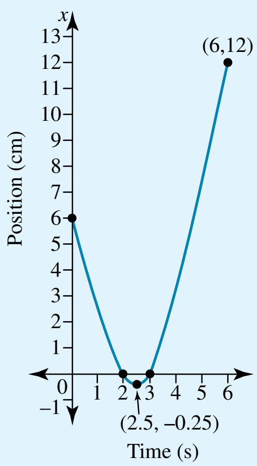
c 8 cm

d 1 cm/s

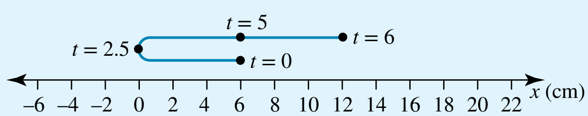
e 32.5 cm

f 4.1 cm/s

18 a



b



c 6 cm

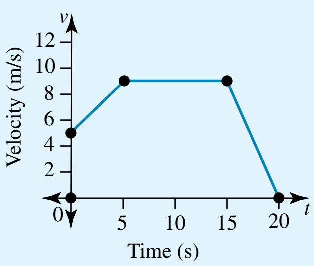
d 1 cm/s

e 18.5 cm

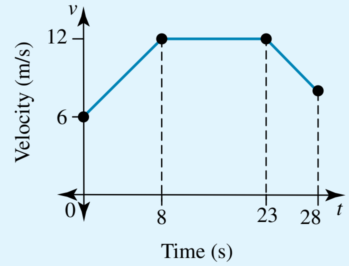
f 3.08 cm/s

EXERCISE 9.3

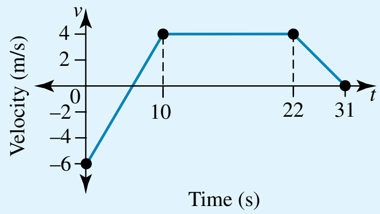
1 a



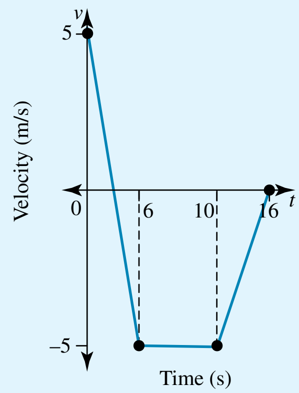
b



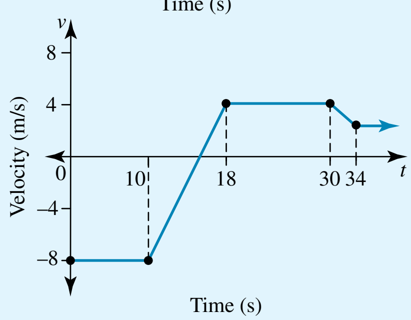
c



2 a



b



3 a 1.6 m/s^2

b −2.4 m/s^2

c 190 m

d 190 m

4 a −2 m/s^2

b $1\frac{1}{3}$ m/s^2

c 148 m

d 148 m

5 a **i** From O to A and F to G

ii From B to C and D to E

iii From A to B, C to D and E to F

b OA: 1.5 m/s^2

AB: 0 m/s^2

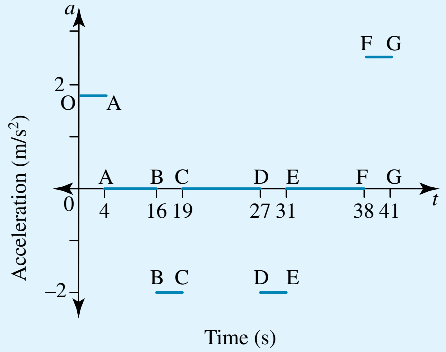
BC: −2 m/s^2

CD: 0 m/s^2

DE: −2 m/s^2

EF: 0 m/s^2

FG: $2\frac{2}{3}$ m/s^2

c 

d i 93 m above ground level

ii 9 m above ground level

e $\frac{9}{41}$ m/s

f 177 m

g $4\frac{13}{41}$ m/s

6 a i From B to C and D to E

ii From O to A and F to G

iii From A to B, C to D and E to F

b OA: −1.5 m/s^2

AB: 0 m/s^2

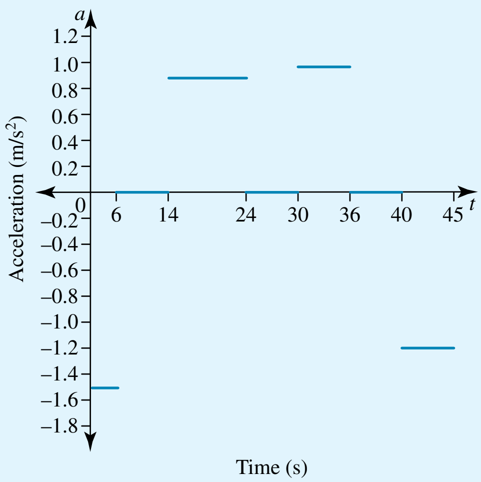
BC: 0.9 m/s^2

CD: 0 m/s^2

DE: 1 m/s^2

EF: 0 m/s^2

FG: −1.2 m/s^2

c 

d i 144 m below ground level

ii 13 m above ground level

e $\frac{-87}{45}$ m/s

f 201 m

g 4.47 m/s

7 a 0.5 m/s^2

b 0.25 m/s^2

c 159 m

d 159 m

8 C

9 D

10 C

11 2 m/s^2

12 11.5 m/s^2

13 0.4 m/s^2

14 a 25 m/s

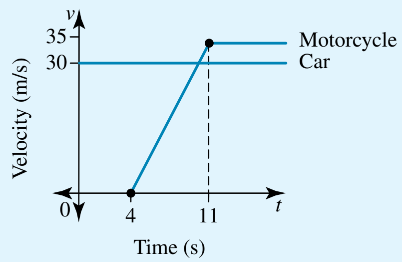
b i 50 m/s

ii 3.1 m/s^2

15 a 11.6 m/s

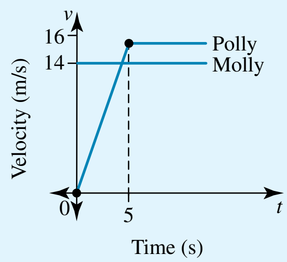
b 38.7 m

16 a 7 seconds

b 

c 52.5 seconds

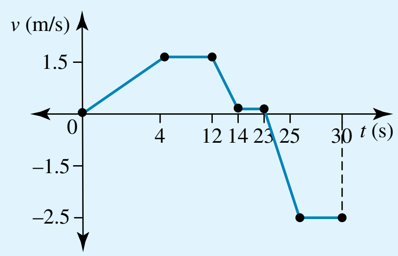
d 1575 m

17 a 

b Polly takes 25 seconds and Molly takes 25.7 seconds.

c 80 m

d 0.8 m/s^2

18 a 

b 1.5 m

c 31.5 m

d i 1.05 m/s

ii 1.5 m/s

e Yes, the monkey is 6.5 m above the ground.

EXERCISE 9.4

1 a 4.5 seconds

b 43.8 m/s

2 a 6.5 m/s^2

b 1.54 seconds

3 a 10 m/s

b $-3\frac{1}{3}$ m/s^2

4 a 60 m/s and 216 km/h

b 12 m/s^2

5 a 3.7 seconds

b 26.2 m/s

6 a 6.6 seconds

b 44.4 m/s

7 a 58.8 m/s

b 176.4 m

8 a 4 m/s^2

b 32 m

9 a 78.4 m

b 39.2 m/s

10 a $1\frac{5}{6}$ seconds

b 5 seconds

11 a $-\frac{5}{6}$ seconds

b 12 seconds

12 a -5 m/s^2

b 22 m

c 25.6 m

13 a B

b C

c A

d E

14 a 90 m

b 7.1 seconds

c 42 m/s

15 12 m/s

16 a 19.6 m/s

b 27.7 m/s

c 2.8 seconds

17 24.04 m

18 6.5 seconds; 44.2 m/s

EXERCISE 9.5

1 13 cm/s

2 171.5 cm/s

3 -1 m/s^2

4 -0.176 m/s^2

5 31 m

6 125.5 m

7 $24\frac{3}{4}$ m

8 126 m

9 a $v(t) = t^2 + t$

b $x(t) = \dfrac{t^3}{3} + \dfrac{t^2}{2}$

c $d = 29\frac{1}{3}$ m

10 a $v(t) = 4t^3 - 2t^2 + 4t + 15$

b $x(t) = t^4 - \dfrac{2t^3}{3} + 2t^2 + 15t$

c $d = 48\frac{2}{3}$ m

11 92 m/s

12 -0.5 cm/s

13 -3.082 m/s

14 3 m/s^2

15 17 cm/s^2

16 E

17 B

18 B

19 D

20 a $x(t) = \dfrac{t^4}{4} + \dfrac{t^2}{2}$

b $d = 24\frac{3}{4}$ m

21 a 5.65 m/s

b 7 m/s

c There is an error of 19%. This could be reduced by taking smaller values of h.

22 a $x(t) = \dfrac{3t^4}{4} + t^2$

b $493\frac{3}{4}$ m

10

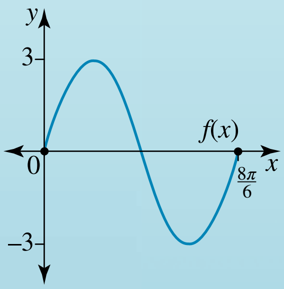
Circular functions

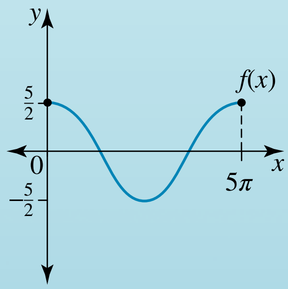
10.1 Kick off with CAS

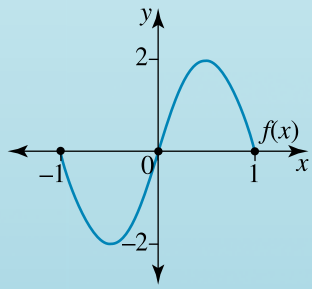
Modelling with trigonometric functions

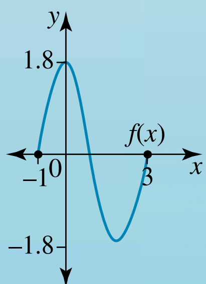
Where relationships between two variables exhibit periodic behavior, trigonometric functions can be used to model this behavior. Examples include tide heights, sound waves, minimum and maximum temperatures, and daily UV levels.

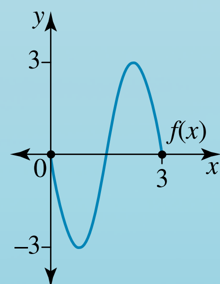
1 Using CAS to assist, state the rule for each of the functions below.

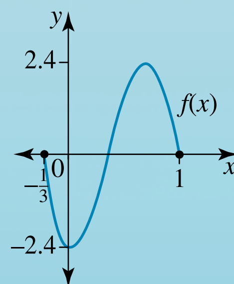
a


b


c


d


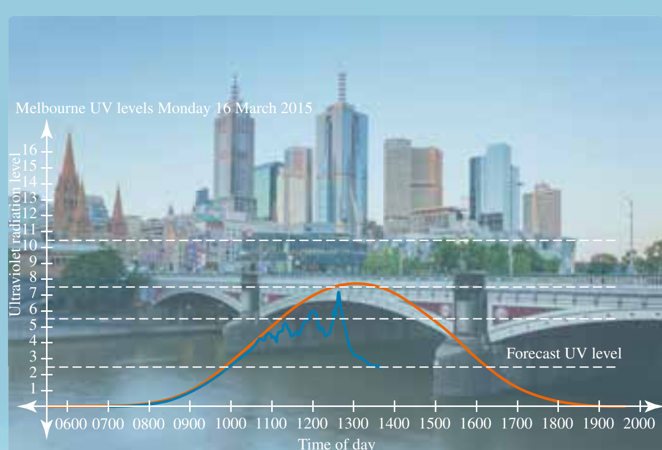
e


f


2 The Australian Radiation Protection and Nuclear Safety Agency provides daily updated UV levels for capital cities and other centres across Australia.

The graph below is for one particular day. It shows the predicted UV level as a sinusoidal curve and the actual UV level as the day progresses.

Use your knowledge of trigonometric functions and CAS to determine a rule that models the predicted UV level for Monday 16 March. From your rule, what is the expected UV level at 2 pm?



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

10.2 Modelling with trigonometric functions

studyon

Units 1 & 2
AOS 1
Topic 2
Concept 1

Trigonometry
Concept summary
Practice questions

Three trigonometric functions — $\sin(x)$, $\cos(x)$ and $\tan(x)$ — and their graphs have been studied in detail in Mathematical Methods (CAS) Units 1 and 2.

Trigonometric functions can be used to model the relationships between two variables that exhibit periodic behaviour. Tide heights, daily UV levels, sound waves, water storage levels and ovulation cycles are some examples.

The independent variable (x) is often a measurement such as time.

When modelling with trigonometric functions you should work in radians unless otherwise instructed.

WORKED EXAMPLE 1

E. coli **is a type of bacterium. Its concentration, P parts per million (ppm), at a particular beach over a 12-hour period t hours after 6 am is described by the function**

$$P = 0.05\sin\left(\frac{\pi t}{12}\right) + 0.1.$$

a **Find i the maximum and ii the minimum *E. coli* levels at this beach.**

b **What is the level at 3 pm?**

c **By sketching the graph determine for how long the level is above 0.125 ppm during the first 12 hours after 6 pm.**

THINK	WRITE/DRAW
a Write the function.	**a** $P = 0.05\sin\left(\frac{\pi t}{12}\right) + 0.1$
i **1** The maximum value of the sine function is 1.	**i** The maximum P occurs when $\sin\left(\frac{\pi t}{12}\right) = 1$.
2 Substitute $\sin\left(\frac{\pi t}{12}\right) = 1$ into the equation for P and evaluate.	Max. $P = 0.05(1) + 0.1$ $= 0.15$
3 State the solution.	The maximum *E. coli* level is 0.15 ppm.
ii **1** The minimum value of the sine function is −1.	**ii** The minimum P occurs when $\sin\left(\frac{\pi t}{12}\right) = -1$.
2 Substitute $\sin\left(\frac{\pi t}{12}\right) = -1$ into the equation for P and evaluate.	Min. $P = 0.05(-1) + 0.1$ $= 0.05$
3 State the solution.	The minimum *E. coli* level is 0.05 ppm.

b **1** At 3 pm it is 9 hours since 6 am.

2 Substitute $t = 9$ into the equation for P and evaluate.

b At 3 pm, $t = 9$.

When $t = 9$, $P = 0.05\sin\left(\frac{9\pi}{12}\right) + 0.1$

$= 0.05\sin\left(\frac{3\pi}{4}\right) + 0.1$

$= 0.05\frac{1}{\sqrt{2}} + 0.1$

$= 0.035 + 0.1$

$= 0.135$

3 State the solution.

The *E. coli* level at 3 pm is approximately 0.135 ppm.

c **1** Sketching a graph will give a better understanding of this question.

c

2 State the amplitude.

Amplitude = 0.05

3 Calculate the period.

$$\text{Period} = \frac{2\pi}{\frac{\pi}{12}}$$

Period = 24

4 Identify the basic graph.

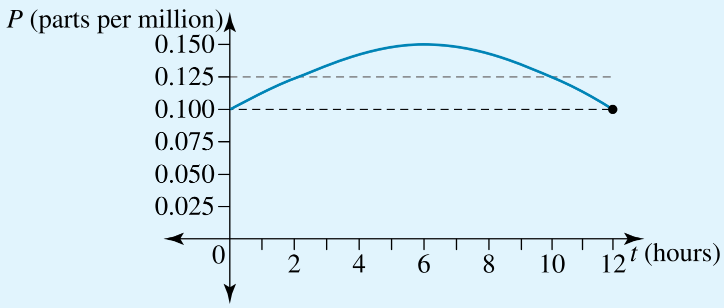
The basic graph is $P = 0.05\ \sin\left(\frac{\pi t}{12}\right)$.

5 State the translations needed.

No horizontal translation is needed; the vertical translation is 0.1 units up.

6 Sketch the graph of P.

7 Draw a horizontal line through $P = 0.125$.



8 Identify where $P > 0.125$ from the graph.

The graph shows that $P > 0.125$ between the first two points where $P = 0.125$.

9 Solve the equation $P = 0.125$ to find the first two values of t.

When $P = 0.125$,

$$0.05\sin\left(\frac{\pi t}{12}\right) + 0.1 = 0.125$$

$$0.05\sin\left(\frac{\pi t}{12}\right) = 0.025$$

$$\sin\left(\frac{\pi t}{12}\right) = 0.5$$

$$\frac{\pi t}{12} = \frac{\pi}{6} \text{ or } \pi - \frac{\pi}{6}$$

$$= \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\frac{t}{12} = \frac{1}{6} \text{ or } \frac{5}{6}$$

$$t = 2 \text{ or } 10$$

10 Find the difference between the solutions $t = 2$ and $t = 10$.

$P > 0.125$, for $10 - 2 = 8$ hours

11 State the solution.

The *E. coli* level is above 0.125 parts per million for 8 hours.

EXERCISE 10.2 Modelling with trigonometric functions

PRACTISE

1 WE1 The height above the ground, h metres, of a child on a swing at any time, t seconds, after being released is:

$$h = 1 + 0.6\cos\left(\frac{\pi t}{2}\right)$$

Find:

a the maximum height of the swing

b the height after

i 3 seconds and

ii $\frac{4}{3}$ seconds

c the length of time that the swing is below 1.5 metres, travelling from one side to the other, correct to 3 decimal places.

2 The height of a bungee jumper, h metres, above a pool of water at any time, t seconds, after jumping is described by the function:

$$h(t) = 20\cos(0.8t) + 20$$

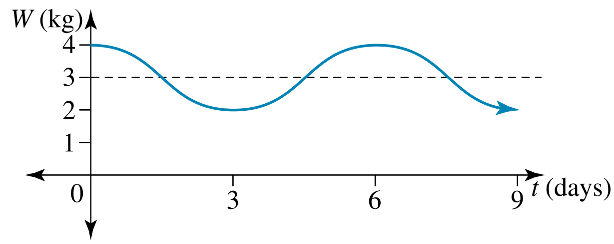
a What is the initial height of the bungee jumper?

b When, if at all, does the bungee jumper first touch the water?

c Assuming the cord is perfectly elastic, how long is it until the bungee jumper returns to the lowest position?

CONSOLIDATE

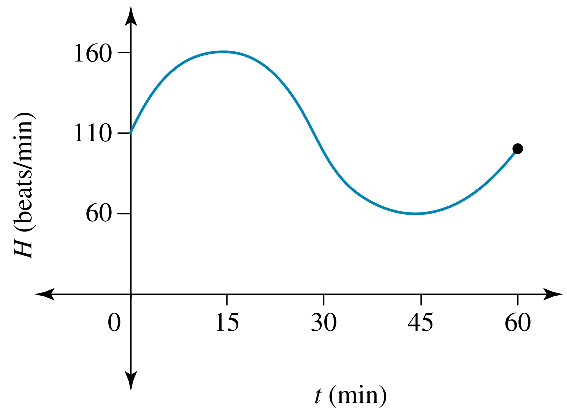
3 The weight of a rabbit over a period of time is modelled by the graph shown at right.



a State **i** the amplitude and **ii** the period.

b Express W as a function of t.

4 The graph at right shows the heart rate of an athlete during a particular hour of a workout.



a Find the initial heart rate.

b State **i** the amplitude and **ii** the period.

c Express H as a function of t.

5 The temperature, T (°C), inside a building on a given day is given by the function:

$$T = 8\ \sin\left(\frac{\pi t}{12}\right) + 18$$

where t is the number of hours after 8 am.

a What is the maximum temperature in the building and what time does it first occur?

b Find the temperature at **i** 8 pm, **ii** 6 pm and **iii** 12 am (midnight).

6 The displacement, x (mm), of a harp string t seconds after it is initially plucked is modelled by the function:

$$x(t) = 12\ \sin(20\pi t)$$

a What is **i** the amplitude and **ii** the period of this function?

b How many vibrations (i.e., cycles) will the harp string complete in 1 second?

c Find the displacement after 0.08 seconds.

d At what time will its displacement first be 6 mm?

7 A cyclist rides one lap of a circular track at a constant speed so that her distance, d metres, from her starting point at any time, t seconds, after starting is:

$$d = 50 - 50\cos\left(\frac{\pi t}{30}\right)$$

Find:

a the time taken to complete one lap

b the radius of the track

c the maximum distance from the start

d the length of the track

e the distance from her starting point after **i** 15 seconds and **ii** 40 seconds

f the times at which she is 93.3 metres from her starting point, to the nearest second.

8 The depth of water, d metres, at a port entrance is given by the function:

$$d\ (t) = 4.5 + 1.5\sin\left(\frac{\pi t}{12}\right)$$

where t is in hours.

a Find **i** the maximum and **ii** the minimum depth at the port entrance.
b A certain ship needs the depth at the port entrance to be more than 5 metres. The ship can be loaded and unloaded, and in and out of the port, in 9 hours. Assuming that the ship enters the port just as the depth at the entrance passes 5 metres, will the ship be able to exit 9 hours later? How long will it have to spare, or by how many minutes will it miss out?

9 The Australian dollar's value (a) in US dollars was observed to follow the equation $a = 0.9 + 0.01t + 0.02\sin(0.5\pi t)$ over a period of 8 days (t represents the number of days).

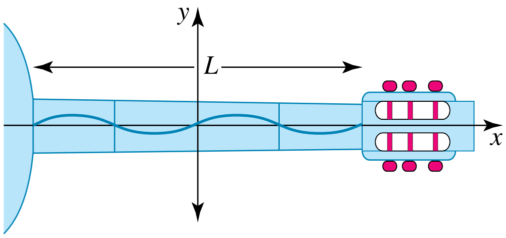
a Using a CAS calculator, sketch a graph of a for $0 \le t \le 8$.
b On which day will the Australian dollar first reach US$0.95?
c At what other times will it be worth US$0.95, correct to 3 decimal places?
d Find all of the maximum turning points of the graph (to 3 decimal places).
e What is the highest value reached?

10 The temperature in an office is controlled by a thermostat. The preferred temperature, P, can be set to values between 18 °C and 25 °C.

The temperature, T (°C), in the office at time t hours after 9 am is given by the rule $T = P + 2.4\sin(\pi t)$. If the preferred temperature on the thermostat has been set to 23 °C:

a find the maximum and minimum temperatures
b find the temperature at:
i noon **ii** 3.30 pm
c sketch the graph of the function between 9 am and 5 pm.
Freddy feels thirsty if the temperature is above 24.2 °C.
d Find the amount of time between 9 am and 5 pm that Freddy feels thirsty.

11 A 'standing wave' on a guitar string may be approximated by the function $y = 0.3\sin\left(\frac{\pi}{20}x\right)$, where x cm and y cm are defined on the diagram at right.



a Find the period of the standing wave.
b If the frets coincide with the mean positions of the wave, find the value of L.
c If the frets were to be spaced at 16 cm, what would be the equation of a similar standing wave of amplitude 0.3 cm such that a fret is at each mean position?

12 The number of rabbits in a national park is observed for one year. At any time t months after observation begins, the number is modelled by the function:

$$P = 2 - 0.8\sin\left(\frac{\pi t}{6}\right)$$

where P is in thousands.

a Find:

i the maximum number of rabbits

ii the minimum number of rabbits

iii the median number of rabbits.

b Find **i** the period and **ii** the amplitude of the function.

c Sketch the graph of the function for $t \in [0, 12]$.

d Find the population after 5 months.

e How long is the population below 1600?

f How long is the population above 2100?

MASTER

13 The height (in cm) that a clock's pendulum swings above its base can be approximated by the function $H = 14 + 5.9\cos\left(\frac{7\pi}{4}t\right)$ at any time t seconds after being released. Give all answers correct to 3 decimal places.

a Find **i** the maximum and **ii** the minimum heights that the pendulum reaches.

b Find the height after **i** 1.5 seconds and **ii** 1 minute.

c Sketch the graph of the function for the first 2 seconds.

d On the same set of axes, sketch the median position.

e Find the length of time that the pendulum is below 14 cm travelling from one side to the other.

f Find the number of times the pendulum swings in 1 minute.

The pendulum is found to be losing time and needs its swing adjusted to 75 swings per minute.

g Find the new function $H(t)$ that approximates the height of the pendulum.

14 The depth, $h(t)$, of water in metres at a point on the coast at a time t hours after noon on a certain day is given by $h(t) = 2.5 + 0.5\cos\left(\frac{2\pi(t+2)}{11}\right)$.

Use a CAS calculator to answer the following.

a What is the depth of the water at noon (correct to 2 decimal places)?

b What is the period of $h(t)$?

c What is the depth of the water (and what time does each occur) at:

i high tide

ii low tide?

d Sketch the graph of $h(t)$ for $0 \le t \le 12$.

e The local people wish to build a bonfire for New Year's celebrations on a rock shelf near the point. They estimate that they can pass the point safely and not get splashed by waves if the depth of water is less than 2.25 m. Between what times can they work?

f How long do they have?

10.3

Reciprocal trigonometric functions

studyon

Units 1 & 2
AOS 1
Topic 2
Concept 2

Reciprocal trigonometric functions
Concept summary
Practice questions

The reciprocals of the $\sin(x)$, $\cos(x)$ and $\tan(x)$ functions are often used to simplify trigonometric expressions or equations.

Definitions

The reciprocal of the sine function is called the **cosecant function**. It is abbreviated to 'cosec' and is defined as:

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}, \sin(x) \neq 0.$$

The reciprocal of the cosine function is called the **secant function**. It is abbreviated to 'sec' and is defined as:

$$\sec(x) = \frac{1}{\cos(x)}, \cos(x) \neq 0.$$

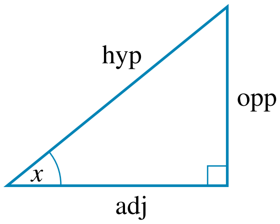
The reciprocal of the tangent function is called the **cotangent function**. It is abbreviated to 'cot' and is defined as:

$$\cot(x) = \frac{1}{\tan(x)}, \tan(x) \neq 0$$

or

$$\cot(x) = \frac{\cos(x)}{\sin(x)}, \sin(x) \neq 0.$$

For the right-angled triangle shown,



$$\sin(x) = \frac{\text{opp}}{\text{hyp}} \qquad \cos(x) = \frac{\text{adj}}{\text{hyp}} \qquad \tan(x) = \frac{\text{opp}}{\text{adj}}$$

or, using the reciprocal trigonometric functions:

$$\operatorname{cosec}(x) = \frac{\text{hyp}}{\text{opp}} \qquad \sec(x) = \frac{\text{hyp}}{\text{adj}} \qquad \cot(x) = \frac{\text{adj}}{\text{opp}}.$$

Note: In the triangle shown at right:

opp = opposite   adj = adjacent   hyp = hypotenuse

Symmetry and complementary properties

You have learned previously about symmetry and complementary properties of trigonometric functions. In summary, these were:

1. First quadrant:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \Leftrightarrow \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta) \Leftrightarrow \sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}(\theta)$$

2. Second quadrant:

$\sin(\pi - \theta) = \sin(\theta) \Leftrightarrow \operatorname{cosec}(\pi - \theta) = \operatorname{cosec}(\theta)$
$\cos(\pi - \theta) = -\cos(\theta) \Leftrightarrow \sec(\pi - \theta) = -\sec(\theta)$
$\tan(\pi - \theta) = -\tan(\theta) \Leftrightarrow \cot(\pi - \theta) = -\cot(\theta)$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta) \Leftrightarrow \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec(\theta)$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta) \Leftrightarrow \sec\left(\frac{\pi}{2} + \theta\right) = -\operatorname{cosec}(\theta)$$

3. Third quadrant:

$\sin(\pi + \theta) = -\sin(\theta) \Leftrightarrow \operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}(\theta)$
$\cos(\pi + \theta) = -\cos(\theta) \Leftrightarrow \sec(\pi + \theta) = -\sec(\theta)$
$\tan(\pi + \theta) = \tan(\theta) \Leftrightarrow \cot(\pi + \theta) = \cot(\theta)$

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos(\theta) \Leftrightarrow \operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec(\theta)$$

$$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin(\theta) \Leftrightarrow \sec\left(\frac{3\pi}{2} - \theta\right) = -\operatorname{cosec}(\theta)$$

4. Fourth quadrant:

$\sin(-\theta) = -\sin(\theta) \Leftrightarrow \operatorname{cosec}(-\theta) = -\operatorname{cosec}(\theta)$
$\cos(-\theta) = \cos(\theta) \Leftrightarrow \sec(-\theta) = \sec(\theta)$
$\tan(-\theta) = -\tan(\theta) \Leftrightarrow \cot(-\theta) = -\cot(\theta)$

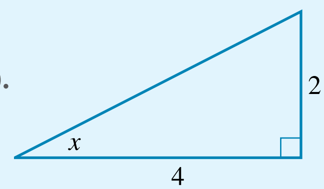
$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos(\theta) \Leftrightarrow \operatorname{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec(\theta)$$

$$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin(\theta) \Leftrightarrow \sec\left(\frac{3\pi}{2} + \theta\right) = \operatorname{cosec}(\theta)$$

WORKED EXAMPLE 2

Use the triangle to find the exact value of:

a $\sin(x)$ **b** $\sec(x)$ **c** $\cot(x)$.



THINK	WRITE
a 1 Use Pythagoras' theorem to find the magnitude of the hypotenuse of the triangle.	**a** $\text{hyp} = \sqrt{4^2 + 2^2}$ $= \sqrt{20}$
2 Express the magnitude of the hypotenuse in its simplest surd form.	$= 2\sqrt{5}$
3 Evaluate $\sin(x)$ by rule.	$\sin(x) = \frac{\text{opp}}{\text{hyp}}$ $= \frac{2}{2\sqrt{5}}$
4 Simplify the ratio.	$= \frac{1}{\sqrt{5}}$ or $\frac{\sqrt{5}}{5}$

b **1** Evaluate $\sec(x)$ by rule.

b $\sec(x) = \frac{\text{hyp}}{\text{adj}}$

$= \frac{2\sqrt{5}}{4}$

2 Simplify the ratio.

$= \frac{\sqrt{5}}{2}$

c **1** Evaluate $\cot(x)$ by rule.

c $\cot(x) = \frac{\text{adj}}{\text{opp}}$

$= \frac{4}{2}$

2 Simplify the ratio.

$= 2$

WORKED EXAMPLE 3

If $\text{cosec}(x) = \frac{4}{3}$ and $0 \le x \le 90°$, find x (to the nearest tenth of a degree).

THINK	WRITE
1 Express the equation $\text{cosec}(x) = \frac{4}{3}$ in terms of $\sin(x)$.	$\text{cosec}(x) = \frac{1}{\sin(x)} = \frac{4}{3}$
2 Write as an equation for $\sin(x)$.	$\sin(x) = \frac{3}{4}$ for $x \in [0°, 90°]$
3 Write the solution.	$x = 48.5904°$
4 Round off the answer to 1 decimal place.	$x = 48.6°$

WORKED EXAMPLE 4

Find the exact value of $\sec(150°)$.

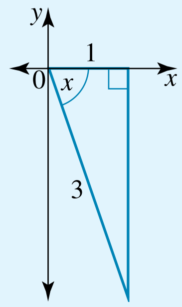
THINK	WRITE
1 Express $\sec(150°)$ in terms of $\cos(150°)$.	$\sec(150°) = \frac{1}{\cos(150°)}$
2 Use symmetry to simplify $\cos(150°)$.	$= \frac{1}{\cos(180 - 30)°}$ $= \frac{1}{-\cos(30°)}$
3 Substitute the exact value for $\cos(30°)$.	$= \frac{1}{-\left(\frac{\sqrt{3}}{2}\right)}$
4 Simplify the ratio.	$= \frac{-2}{\sqrt{3}}$ or $\frac{-2\sqrt{3}}{3}$

WORKED EXAMPLE 5

If $\cos(x) = \frac{1}{3}$ and $\frac{3\pi}{2} < x < 2\pi$, find the exact value of:

a $\sin(x)$

b $\cot(x)$.

THINK	WRITE/DRAW
a 1 Draw a right-angled triangle in the fourth quadrant, showing $\cos(x) = \frac{1}{3}$.	**a** 
2 Use Pythagoras' theorem to calculate the magnitude of the opposite side.	$\text{opp} = \sqrt{3^2 - 1^2}$ $= \sqrt{8}$
3 Express in simplest surd form.	$= 2\sqrt{2}$
4 State the 'signed value' of the opposite side in the fourth quadrant.	$\text{opp} = -2\sqrt{2}$ in the fourth quadrant.
5 Evaluate $\sin(x)$ by rule.	$\sin(x) = \frac{\text{opp}}{\text{hyp}}$ $= \frac{-2\sqrt{2}}{3}$
b 1 Evaluate $\cot(x)$ by rule.	**b** $\cot(x) = \frac{\text{adj}}{\text{opp}}$ $= \frac{1}{-2\sqrt{2}}$
2 Simplify the ratio by rationalising the denominator.	$= \frac{1}{-2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{-\sqrt{2}}{4}$

EXERCISE 10.3 Reciprocal trigonometric functions

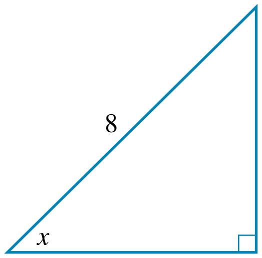
PRACTISE

1 **WE2** Copy and complete the table, using the right-angled triangles on the next page. Give exact values for:

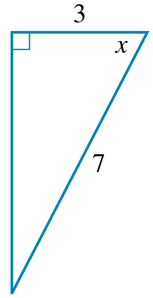
i $\sin(x)$ **ii** $\cos(x)$
iii $\tan(x)$ **iv** $\operatorname{cosec}(x)$
v $\sec(x)$ **vi** $\cot(x)$.

	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\operatorname{cosec}(x)$	$\sec(x)$	$\cot(x)$
a						
b						
c						

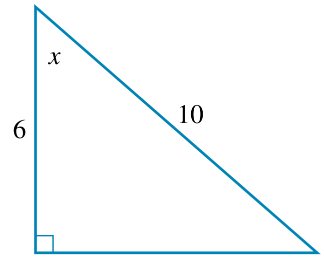
a



b



c

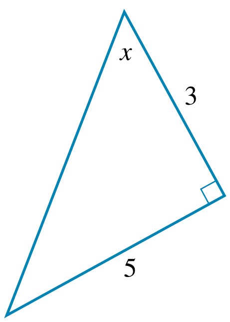


2 Copy and complete the table, using the right-angled triangles below it. Give exact values for:

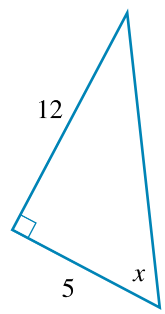
i $\sin(x)$ ii $\cos(x)$
iii $\tan(x)$ iv $\operatorname{cosec}(x)$
v $\sec(x)$ vi $\cot(x)$.

	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\operatorname{cosec}(x)$	$\sec(x)$	$\cot(x)$
a						
b						
c						

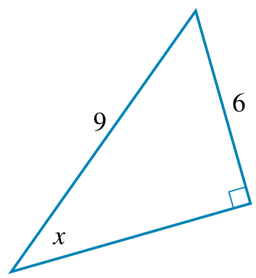
a



b



c



3 WE3 For $0° < a < 90°$, find the value of a to the nearest tenth of a degree.

a $\sin(a) = 0.6$ b $\cos(a) = 0.95$ c $\tan(a) = 1.8$

4 For $0° < a < 90°$, find the value of a to the nearest tenth of a degree.

a $\operatorname{cosec}(a) = 2$ b $\sec(a) = 3.5$ c $\cot(a) = 0.7$

5 WE4 Find the exact value of each of the following.

a $\tan(60°)$ b $\cot(30°)$ c $\cos(-150°)$
d $\sec(45°)$ e $\operatorname{cosec}(30°)$ f $\cot(300°)$

6 Find the exact value of each of the following.

a $\operatorname{cosec}(450°)$ b $\sec(210°)$ c $\cot(-135°)$
d $\operatorname{cosec}(240°)$ e $\sec(120°)$ f $\cot(-150°)$

7 WE5 If $\sin(x) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of:

a $\cos(x)$ b $\tan(x)$ c $\cot(x)$.

8 If $\tan(x) = \frac{5}{4}$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of:

a $\sin(x)$ b $\cos(x)$ c $\sec(x)$.

CONSOLIDATE

9 Choose the correct answer for parts **a** to **d** below given that $\sin(x) = 0.8$ and $90° < x < 180°$.

a $\cos(x)$ is equal to:

A 0.6 **B** 1.67 **C** 1.33 **D** -0.6 **E** -0.8

b $\operatorname{cosec}(x)$ is equal to:

A 1.33 **B** 0.6 **C** -1.25 **D** -1.33 **E** 1.25

c $\cot(x)$ is equal to:

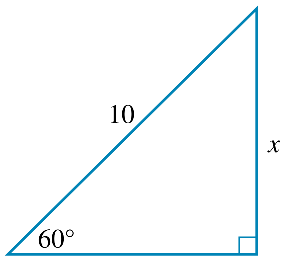
A -1.33 **B** 0.75 **C** -0.75 **D** 1 **E** 1.33

d $\sec(x)$ is equal to:

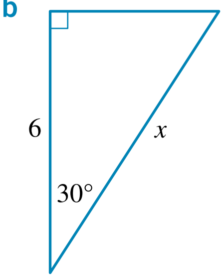
A -1.67 **B** 1.25 **C** 0.75 **D** -1.25 **E** 1.67

10 Find the exact value of the pronumeral shown in each triangle below.

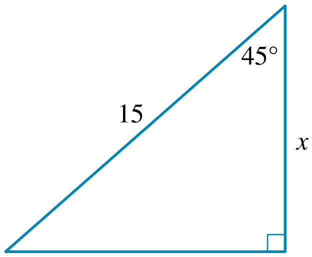
a



b

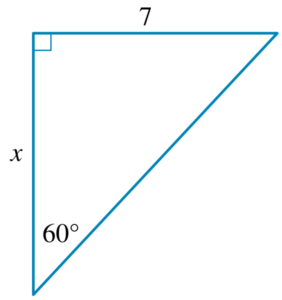


c

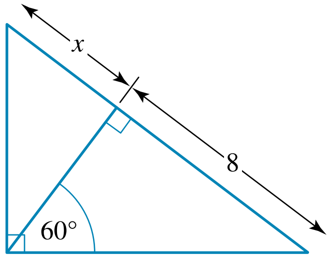


11 Find the exact value of the pronumeral shown in each triangle below.

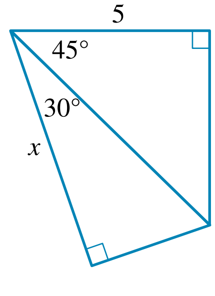
a



b



c



12 If $\cos(x) = \dfrac{\sqrt{2}}{2}$ and $\dfrac{3\pi}{2} < x < 2\pi$, find the exact value of:

a $\sin(x)$ **b** $\cot(x)$ **c** $\sec(x)$.

13 If $\operatorname{cosec}(x) = -1.5$ and $180° < x < 270°$, find the exact value of:

a $\sin(x)$ **b** $\cos(x)$ **c** $\cot(x)$.

14 If $\sec(x) = -3.2$ and $90° < x < 180°$, find the exact value of:

a $\cos(x)$ **b** $\sin(x)$ **c** $\cot(x)$.

15 If $\cot(x) = 0.75$ and $180° < x < 270°$, find the exact value of:

a $\tan(x)$ **b** $\sec(x)$ **c** $\operatorname{cosec}(x)$.

16 If $\tan(x) = 0.5$ and $\pi < x < \dfrac{3\pi}{2}$, find the exact value of:

a $\sec(x)$ **b** $\cos(x)$ **c** $\cot(x)$.

17 If $\cot(x) = \sqrt{2}$, and x is in the first quadrant, find the exact value of:

a $\sin(x)$ **b** $\sec(x)$ **c** $\cos^2(x) + \tan^2(x)$.

18 If $\sin(x) = \dfrac{-\sqrt{3}}{2}$ and $180° < x < 270°$, then find:

a $\cos(x)$ **b** $\cot(x)$ **c** $\sec(x)$.

MASTER

19 Use a CAS calculator to solve the following equations over $[0, 2\pi]$.

a $\operatorname{cosec}(x) = \frac{2\sqrt{3}}{3}$ b $\cot(x) = -1$ c $\sec(x) = 2$.

20 Use CAS technology to simplify the following:

a $\dfrac{\sin\left(\frac{\pi}{2} - x\right)}{\sec(\pi + x)}$ b $\operatorname{cosec}\left(\frac{3\pi}{2}\right) \times \cot(2\pi - x)$

c $\dfrac{\cot\left(\frac{\pi}{2} + x\right) \times \sec\left(\frac{3\pi}{2} - x\right)}{\operatorname{cosec}\left(\frac{\pi}{2} + x\right)}$.

10.4 Graphs of reciprocal trigonometric functions

studyon

Units 1 & 2
AOS 1
Topic 2
Concept 3

Graphs of reciprocal trigonometric functions
Concept summary
Practice questions

The graphs of the sine, cosine and tangent functions are familiar from previous work covered in Mathematical Methods. The graphs of the reciprocal trigonometric functions are obtained by applying the reciprocal-of-ordinates method to the original functions. The reciprocal-of-ordinates method involves estimating or calculating the reciprocal of the y-values for any x-value from the original function.

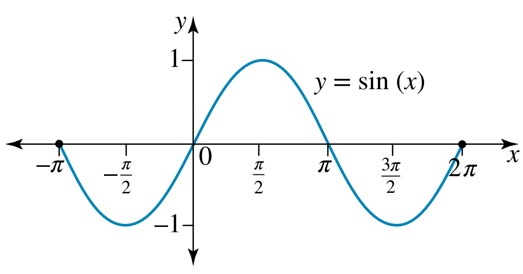
It should be further noted that:

1. as $y \to \infty$, $\frac{1}{y} \to 0$
2. as $y \to 0$, $\frac{1}{y} \to \infty$
3. if $y = 0$, $\frac{1}{y}$ is undefined — that is, reciprocal functions contain vertical asymptotes wherever x-intercepts occur in the original function
4. the expression $\frac{1}{y}$ has the same sign as y
5. if $y = \pm 1$, $\frac{1}{y} = \pm 1$ also
6. if $f(a)$ is a local maximum, then $\frac{1}{f(a)}$ is a local minimum.

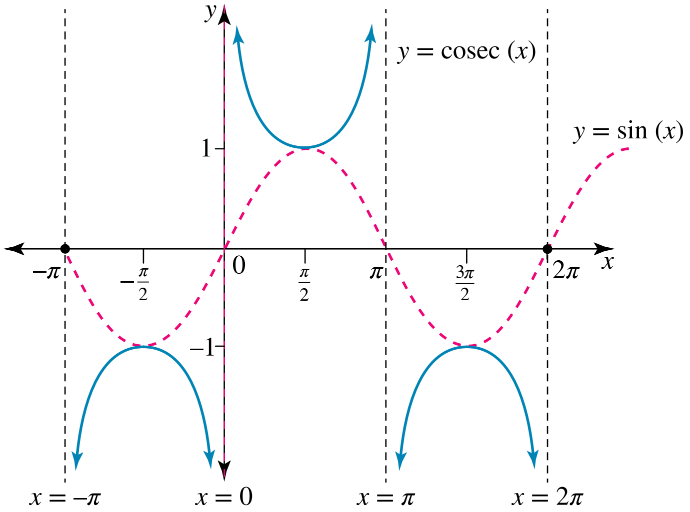
Graph of $y = \operatorname{cosec}(x)$

The graph of $y = \sin(x)$ over $[-\pi, 2\pi]$ is shown in the figure at right. Note that:

1. the graph of $y = \sin(x)$ has turning points at $x = \frac{\pi}{2} + n\pi$, $n \in Z$
2. $\sin(x) = 0$ at $x = n\pi$, $n \in Z$
3. period $= 2\pi$.



The graph of $y = \text{cosec}\,(x)$ over $[-\pi, 2\pi]$ is shown in the figure below.



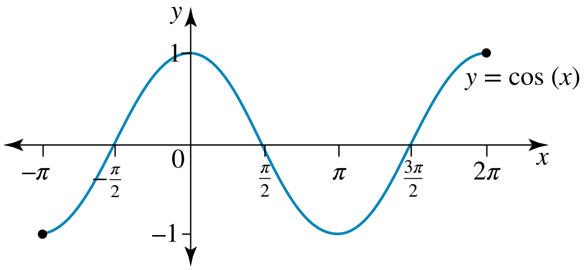
Note that:

1. the turning points are $\left(\frac{-\pi}{2}, -1\right), \left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$
2. the vertical asymptotes are $x = -\pi$, $x = 0$, $x = \pi$ and $x = 2\pi$
3. period $= 2\pi$.

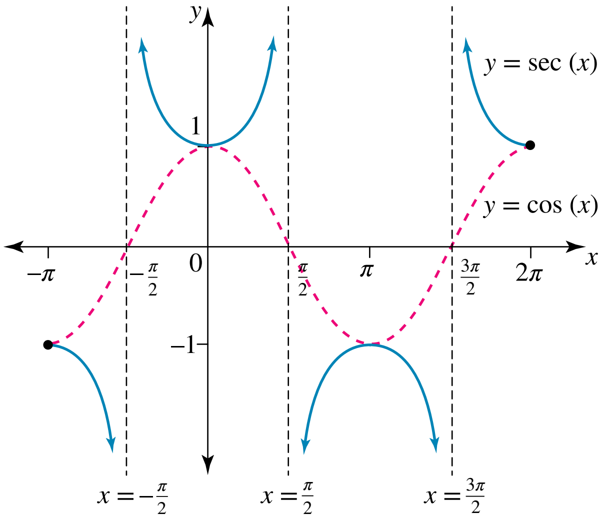
Graph of $y = \sec(x)$

The graph of $y = \cos(x)$ over $[-\pi, 2\pi]$ is shown in the figure at right. Note that:

1. the graph of $y = \cos(x)$ has turning points at $x = n\pi$, $n \in Z$
2. $\cos(x) = 0$ at $x = \frac{\pi}{2} + n\pi$, $n \in Z$
3. period $= 2\pi$.



The graph of $y = \sec(x)$ over $[-\pi, 2\pi]$ is shown in the figure below.



Note that:

1. the turning points are $(-\pi, -1)$, $(0, 1)$, $(\pi, -1)$ and $(2\pi, 1)$
2. the vertical asymptotes are $x = \frac{-\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$
3. period $= 2\pi$.

Note: The turning points of $y = \text{cosec}\,(x)$ and $y = \sec(x)$ are located halfway between consecutive asymptotes.

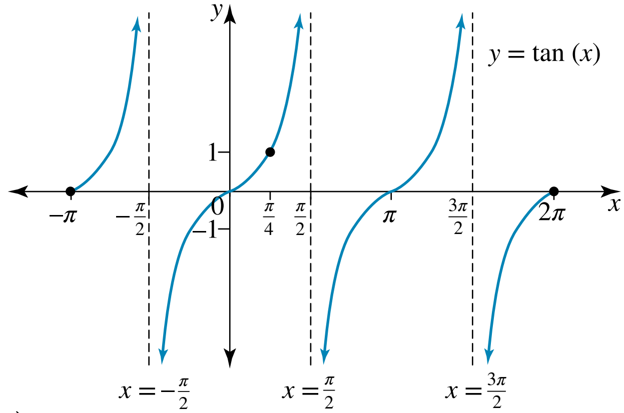
Graph of $y = \cot(x)$

First recall that $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$ and $\cot(x) = \dfrac{1}{\tan(x)}$.

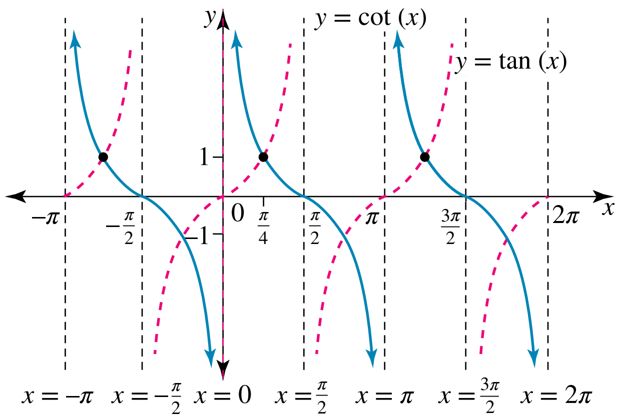
So the graph of $y = \tan(x)$ has vertical asymptotes wherever $\cos(x) = 0$, that is, at $x = \dfrac{\pi}{2} + n\pi,\ n \in Z$.

The graph of $y = \tan(x)$ over $[-\pi, 2\pi]$ is shown in the figure at right. Note that:

1. $\tan(x) = 0$ at $x = n\pi,\ n \in Z$
2. $\tan(x) = 1$ at $x = \dfrac{\pi}{4} + n\pi,\ n \in Z$
3. $\tan(x) = -1$ at $x = -\dfrac{\pi}{4} + n\pi,\ n \in Z$
4. period $= \pi$.



The graph of $y = \cot(x)$ $\left(\text{that is, } \dfrac{1}{\tan(x)}\right)$ over $[-\pi, 2\pi]$ is shown below.



Note that:

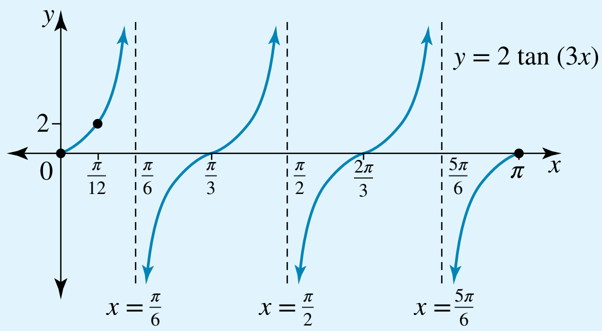
1. the asymptotes are $x = -\pi$, $x = 0$, $x = \pi$ and $x = 2\pi$
2. period $= \pi$.

Note: The translation of the reciprocal trigonometric functions is the same as the original trigonometric functions; for example, the graph of $y = \operatorname{cosec}(x - a) + b$ is the same as the graph of $y = \operatorname{cosec}(x)$ translated a units right and b units up.

WORKED EXAMPLE 6 **Sketch the graph of $y = 2\tan(3x)$ over the domain $[0, \pi]$.**

THINK	WRITE/DRAW
1 The vertical asymptotes occur at $3x = \dfrac{\pi}{2} + n\pi, n \in Z$.	$y = 2\tan(3x)$ Vertical asymptotes at: $3x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}, \ldots$
2 Divide both sides by 3.	$x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \ldots$
3 Only values between 0 and π are valid (the given domain).	$x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ over $[0, \pi]$

4 Sketch a standard tan curve, a little steeper because of the 2 in $2\tan(3x)$, between the asymptotes.



5 Verify the graph using a calculator.

WORKED EXAMPLE 7 Sketch the graph of $y = \operatorname{cosec}\left(x - \frac{\pi}{4}\right)$ over the domain $[0, 2\pi]$.

THINK

WRITE/DRAW

1 The graph is the same as $y = \operatorname{cosec}(x)$ translated $\frac{\pi}{4}$ units right.

$y = \operatorname{cosec}\left(x - \frac{\pi}{4}\right)$

2 The asymptotes are at $x = 0 + \frac{\pi}{4}$, $x = \pi + \frac{\pi}{4}$.

Asymptotes: $x = 0 + \frac{\pi}{4}$ and $x = \pi + \frac{\pi}{4}$

$\Rightarrow x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

3 The y-intercept is $y = \operatorname{cosec}\left(\frac{-\pi}{4}\right)$.

y-intercept: $y = \operatorname{cosec}\left(\frac{-\pi}{4}\right)$

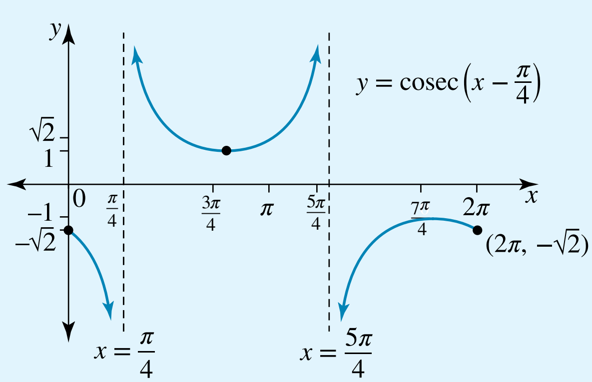
$$= \frac{1}{\sin\left(\frac{-\pi}{4}\right)}$$

$$= \frac{1}{-\left(\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{-2}{\sqrt{2}}$$

$$= -\sqrt{2}$$

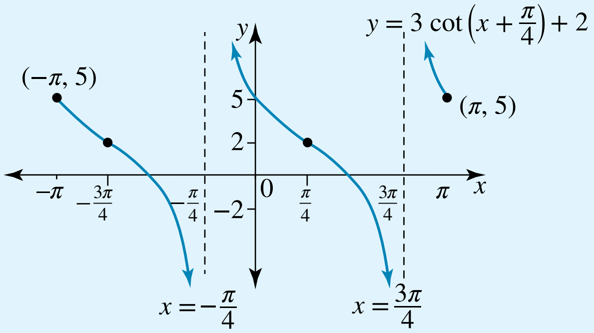
4 Sketch the graph.



WORKED EXAMPLE 8

Sketch the graphs of each of the following functions over the domain $[-\pi, \pi]$.

a $f(x) = 3\cot\left(x + \frac{\pi}{4}\right) + 2$ **b** $f(x) = \sec(2x + \pi)$

THINK	WRITE/DRAW
a 1 Compare the graph to $y = 3\cot(x)$.	**a** $y = 3\cot\left(x + \frac{\pi}{4}\right) + 2$. This graph is the same as $y = 3\cot(x)$ translated $\frac{\pi}{4}$ units left and 2 units up.
2 Locate asymptotes on domain $[-\pi, \pi]$.	Asymptotes: $x + \frac{\pi}{4} = 0, \pi$
3 Subtract $\frac{\pi}{4}$ from both sides to locate the asymptotes exactly.	$x = 0 - \frac{\pi}{4}$ and $x = \pi - \frac{\pi}{4}$ $x = \frac{-\pi}{4}$ and $x = \frac{3\pi}{4}$
4 Find the y-intercept.	y-intercept: $y = 3\cot\left(0 + \frac{\pi}{4}\right) + 2$ $= \frac{3}{\tan\left(\frac{\pi}{4}\right)} + 2$ $= \frac{3}{1} + 2$ $= 5$
5 Sketch the graph.	
6 Verify this graph using a graphics calculator.	
b 1 Remove the factor 2 from the brackets so the translation is obvious.	**b** $y = \sec(2x + \pi)$ $y = \sec 2\left(x + \frac{\pi}{2}\right)$
2 Compare the graph to $y = \sec(2x)$.	This graph is the same as $y = \sec(2x)$ translated $\frac{\pi}{2}$ units left.
3 Locate asymptotes on domain $[-\pi, \pi]$.	Asymptotes: $2x + \pi = \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

4 Subtract π from both sides.

$$2x = \ldots, \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{3\pi}{2}, \ldots$$

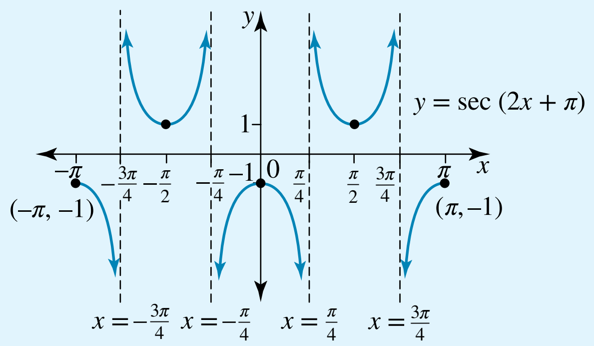
5 Divide by 2 to exactly locate the asymptotes between $-\pi$ and π.

$$x = \frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \text{ over } [-\pi, \pi]$$

6 Find the y-intercept.

$$y\text{-intercept: } y = \sec(0 + \pi) = \frac{1}{\cos\pi} = \frac{1}{-1} = -1$$

7 Sketch the graph.



8 Verify this graph using a graphics calculator.

EXERCISE 10.4 Graphs of reciprocal trigonometric functions

PRACTISE

1 WE6 Sketch the graph of each of the following over the domain $[0, 2\pi]$.

a $y = 2\tan(x)$ b $y = \tan(2x)$ c $y = \tan\left(\frac{x}{2}\right)$

2 Sketch the graph of each of the following over the domain $[0, 2\pi]$.

a $y = 3\sec(x)$ b $y = \frac{1}{2}\cot(x)$ c $y = 4\operatorname{cosec}(x)$

d $y = \operatorname{cosec}(2x)$ e $y = \cot(3x)$ f $y = \sec\left(\frac{x}{2}\right)$

3 WE7 Sketch the graph of each of the following over the domain $[0, 2\pi]$.

a $f(x) = \tan\left(x - \frac{\pi}{4}\right)$ b $f(x) = \operatorname{cosec}\left(x - \frac{\pi}{3}\right)$ c $f(x) = \sec\left(x + \frac{\pi}{2}\right)$

4 Sketch the graph of each of the following over the domain $[0, 2\pi]$.

a $f(x) = \cot\left(x - \frac{\pi}{6}\right)$ b $f(x) = \sec\left(x + \frac{\pi}{6}\right)$

c $f(x) = \operatorname{cosec}\left(x + \frac{2\pi}{3}\right)$ d $f(x) = \cot\left(x - \frac{\pi}{4}\right)$

5 WE8 Sketch the graph of each of the following over the domain $[-\pi, \pi]$.

a $y = \tan\left(x + \frac{\pi}{4}\right) + 1$ b $y = \tan\left(x - \frac{\pi}{2}\right) - 1$ c $y = 2\sec\left(x - \frac{\pi}{6}\right)$

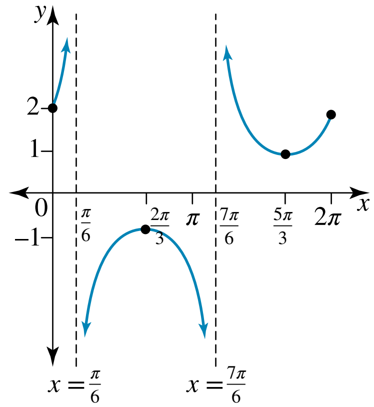
6 Sketch the graph of each of the following over the domain $[-\pi, \pi]$.

a $y = 3\ \text{cosec}\left(x + \frac{\pi}{3}\right)$ b $y = \cot\left(x - \frac{\pi}{4}\right) + 2$ c $y = \sec\left(x + \frac{\pi}{3}\right) - 1$

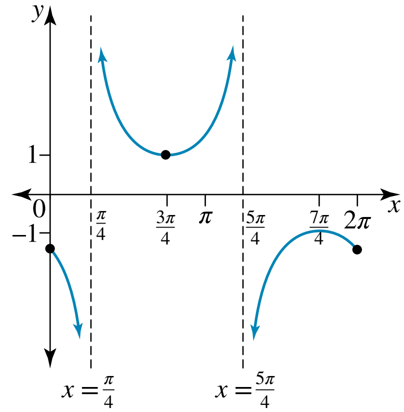
CONSOLIDATE

7 Match each of the following graphs with the correct rule below.

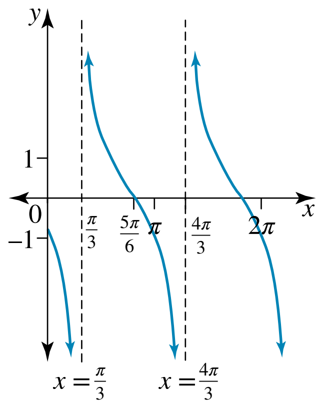
a



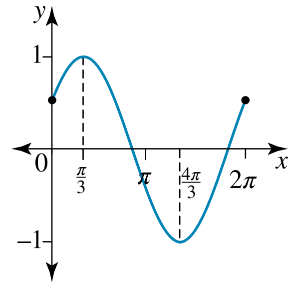
b



c



d



A $y = \text{cosec}\left(x - \frac{\pi}{4}\right)$

B $y = \cos\left(x - \frac{\pi}{3}\right)$

C $y = \sec\left(x + \frac{\pi}{3}\right)$

D $y = \cot\left(x - \frac{\pi}{3}\right)$

8 Consider the function $f(x) = \sqrt{3}\sec\left(2x + \frac{\pi}{2}\right) - 2$ over the domain $[0, \pi]$.

a The function $f(x)$ has x-intercepts:

A $\frac{\pi}{3}, \frac{2\pi}{3}$ B $\frac{2\pi}{3}, \frac{5\pi}{6}$ C $\frac{\pi}{4}$ only D $\frac{\pi}{4}, \frac{3\pi}{4}$ E $\frac{\pi}{6}, \frac{5\pi}{6}$

b The function $f(x)$ has vertical asymptotes where x is equal to:

A $\frac{\pi}{4}, \frac{3\pi}{4}$ B $\frac{\pi}{2}, \frac{3\pi}{2}$ C $0, \frac{\pi}{2}, \pi$ D $\frac{\pi}{3}, \frac{2\pi}{3}$ E $\frac{-\pi}{2}, \frac{\pi}{2}, \pi$

c The function $f(x)$ has turning points given by:

A $\left(\frac{\pi}{4}, 2 + \sqrt{3}\right)$ and $\left(\frac{3\pi}{4}, 2 - \sqrt{3}\right)$

B $\left(\frac{-\pi}{4}, \sqrt{3} - 2\right)$ and $\left(\frac{\pi}{2}, \sqrt{3}\right)$

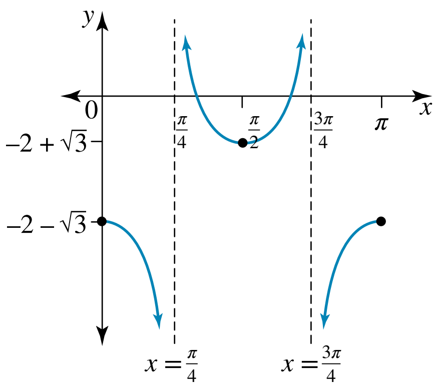
C $(0, -\sqrt{3} - 2)$ and $\left(\frac{\pi}{4}, \sqrt{3} - 2\right)$

D $\left(\frac{\pi}{2}, 2 - \sqrt{3}\right)$

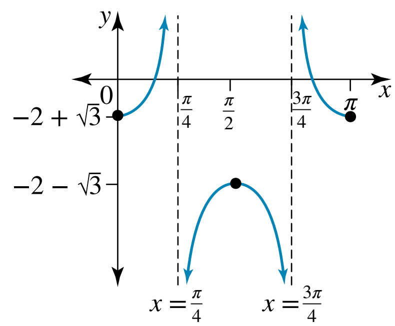
E $\left(\frac{\pi}{4}, -2 - \sqrt{3}\right)$ and $\left(\frac{3\pi}{4}, \sqrt{3} - 2\right)$

d The graph of $f(x)$ is:

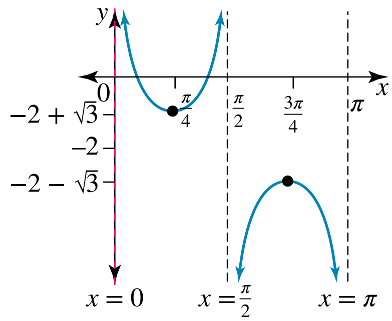
A



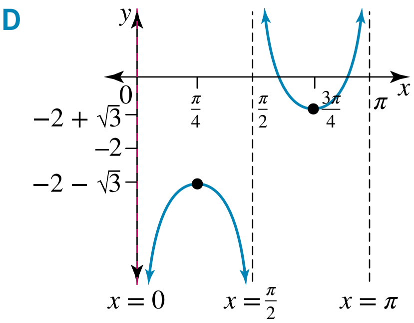
B



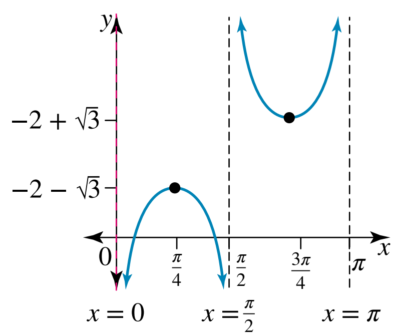
C



D



E



9 Sketch the graph of each of the following over the domain $[-\pi, \pi]$.

a $y = 3\operatorname{cosec}\left(x - \dfrac{\pi}{3}\right) + 2$ **b** $y = 2\cot\left(x - \dfrac{\pi}{3}\right) + 1$ **c** $y = \sec\left(2x - \dfrac{\pi}{2}\right)$

10 Sketch the graph of each of the following over the domain $[-\pi, \pi]$.

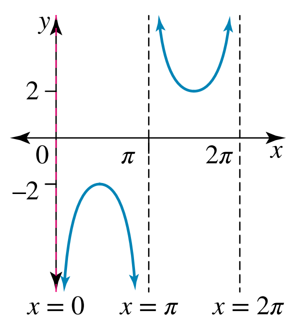
a $y = \operatorname{cosec}\left(3x + \dfrac{\pi}{2}\right) - 1$ **b** $y = -\sec(x)$

11 Sketch the graph of each of the following over the domain $[-\pi, \pi]$.

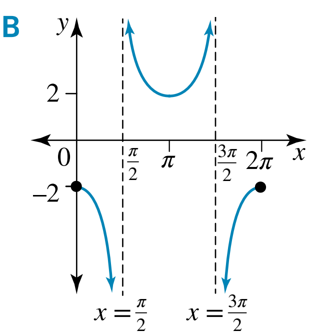
a $y = -\operatorname{cosec}(x)$ **b** $y = -\cot(x)$

12 The graph which best represents $y = 2\operatorname{cosec}\left(x - \dfrac{\pi}{2}\right)$ over $[0, 2\pi]$ is:

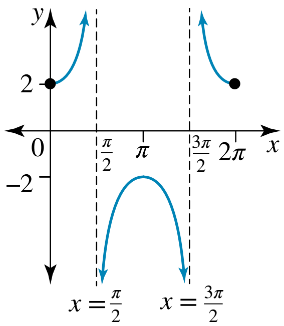
A



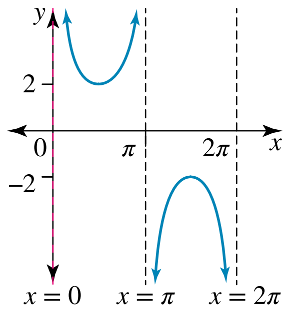
B



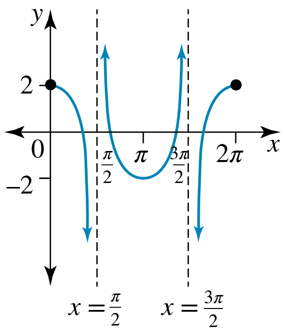
C



D



E



For questions **13** and **14** consider the function $f: [0, \pi] \rightarrow R,\ f(x) = \sec\left(2x - \frac{\pi}{4}\right) - 2$.

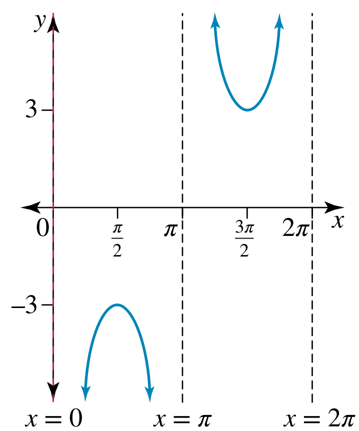
13 $f(x)$ has vertical asymptotes where x is equal to:

A $\frac{\pi}{4}, \frac{3\pi}{4}$
B $\frac{3\pi}{8}, \frac{7\pi}{8}$
C $\frac{\pi}{8}, \frac{5\pi}{8}$
D $\frac{3\pi}{4}$ only
E $0, \pi$

14 $f(x)$ has x-intercepts where x equals:

A $\frac{\pi}{12}, \frac{17\pi}{12}$
B $\frac{\pi}{6}$
C $\frac{7\pi}{24}, \frac{23\pi}{24}$
D $\frac{\pi}{3}, \frac{2\pi}{3}$
E $0, \frac{\pi}{2}$

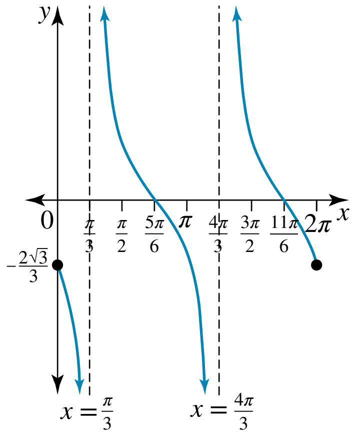
15 The rule for the graph below could be:



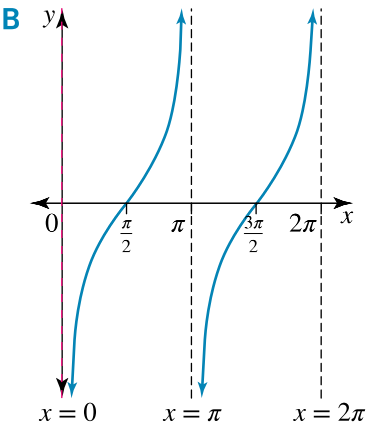
A $y = 3 \operatorname{cosec}\left(x + \frac{\pi}{2}\right)$
B $y = -3 \operatorname{cosec}\left(x + \frac{\pi}{2}\right)$
C $y = -3 \sec\left(x + \frac{\pi}{2}\right)$
D $y = 3 \sec\left(x - \frac{\pi}{2}\right)$
E $y = 3 \sec\left(x + \frac{\pi}{2}\right)$

16 The graph that best represents $y = 2\cot\left(x + \frac{\pi}{3}\right)$ over $[0, 2\pi]$ is:

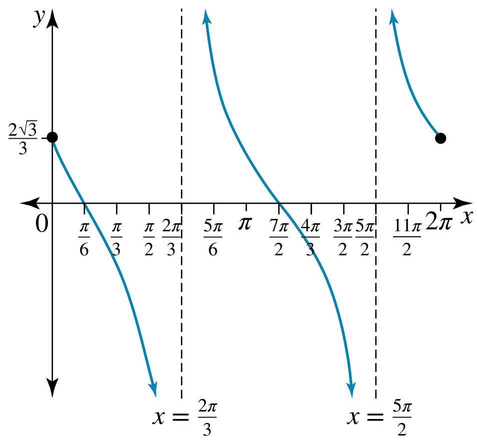
A



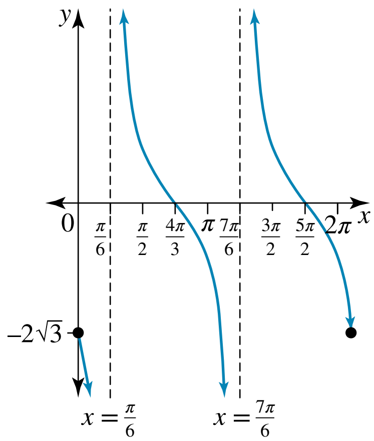
B



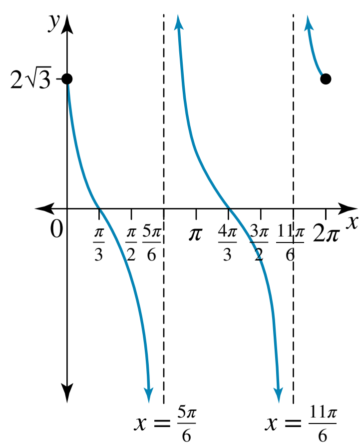
C



D



E



MASTER

17 Using CAS technology, determine the turning points for the function $f(x) = -4\,\text{cosec}\left(2x - \frac{\pi}{3}\right) + 1$ over the domain $[0, \pi]$.

18 Using CAS technology, sketch the graph of $y = \frac{1}{2}\cot\left(2x - \frac{\pi}{3}\right) + 1$ for $0 < x < 2\pi$.

10.5 Trigonometric identities

An identity is an expression that is true for all values of the variable in its implied domain.

Recall the Pythagorean identity:

$$\sin^2(A) + \cos^2(A) = 1 \qquad [1]$$

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Trigonometric identities
Concept summary
Practice questions

Identities involving the reciprocal trigonometric functions can be derived from this identity as follows.

If we divide both sides of equation [1] by $\cos^2(A)$, we obtain:

$$\frac{\sin^2(A)}{\cos^2(A)} + 1 = \frac{1}{\cos^2(A)}, \cos(A) \neq 0$$

$$\tan^2(A) + 1 = \sec^2(A)$$

$$\text{or} \quad 1 + \tan^2(A) = \sec^2(A), \cos(A) \neq 0 \qquad [2]$$

If we divide both sides of equation [1] by $\sin^2(A)$, we obtain:

$$1 + \frac{\cos^2(A)}{\sin^2(A)} = \frac{1}{\sin^2(A)}, \quad \sin(A) \neq 0$$

$$1 + \cot^2(A) = \text{cosec}^2(A), \ \sin(A) \neq 0 \qquad [3]$$

These identities can be used to simplify or evaluate trigonometric expressions.

WORKED EXAMPLE 9

Simplify the expression: $\cos^2(x) \times [1 + \tan^2(x)]$.

THINK	WRITE
1 Write the expression.	$\cos^2(x) \times [1 + \tan^2(x)]$
2 Simplify the expression in the brackets using identity [2].	$= \cos^2(x) \times [\sec^2(x)]$
3 Express $\sec(x)$ in terms of $\cos(x)$.	$= \cos^2(x)\dfrac{1}{\cos^2(x)}$
4 Simplify by cancelling down the fraction.	$= 1$

WORKED EXAMPLE 10

If $x \in \left[\frac{\pi}{2}, \pi\right]$ and $\cos(x) = \frac{-3}{5}$, find:

a $\sin(x)$ **b** $\cot(x)$ **c** $\sec(x)$.

THINK	WRITE
a 1 Write down identity [1].	a $\sin^2(x) + \cos^2(x) = 1$
2 Substitute $\cos(x) = \frac{-3}{5}$ into identity [1].	If $\cos(x) = \frac{-3}{5}$ $\sin^2(x) + \left(\frac{-3}{5}\right)^2 = 1$ $\sin^2(x) + \frac{9}{25} = 1$
3 Solve for $\sin(x)$.	$\sin^2(x) = \frac{16}{25}$ $\sin(x) = \frac{\pm 4}{5}$
4 Retain the positive solution only as sine is positive in the second quadrant.	$\sin(x) = \frac{4}{5}$, since x is in quadrant 2.

b **1** Use the reciprocal function $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$.

b $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

2 Substitute $\cos(x)$ and $\sin(x)$ into the expression and evaluate.

Note: If we did not know the value of $\cos(x)$, identity [3] could be used.

$= \dfrac{\frac{-3}{5}}{\frac{4}{5}}$

$= \dfrac{-3}{4}$

c Evaluate the reciprocal of $\cos(x)$.

c $\sec(x) = \dfrac{1}{\cos(x)}$

$= \dfrac{1}{-\left(\frac{3}{5}\right)}$

$= \dfrac{-5}{3}$

WORKED EXAMPLE 11

If $\sec(x) = \dfrac{3}{2}$ and $x \in \left[\dfrac{3\pi}{2}, 2\pi\right]$, use the identities to find the exact value of:

a $\tan(x)$ **b** $\sin(x)$ **c** $\operatorname{cosec}(x)$.

THINK

a **1** Write identity [2].

WRITE

a $1 + \tan^2(x) = \sec^2(x)$

2 Substitute $\sec(x) = \dfrac{3}{2}$ in the identity.

If $\sec(x) = \dfrac{3}{2}$

$1 + \tan^2(x) = \left(\dfrac{3}{2}\right)^2$

$= \dfrac{9}{4}$

3 Solve for $\tan(x)$.

$\tan^2(x) = \dfrac{5}{4}$

$\tan(x) = \pm\dfrac{\sqrt{5}}{2}$

4 Since $x \in \left[\dfrac{3\pi}{2}, 2\pi\right]$, x is a fourth-quadrant angle. Retain the negative solution only, as tan is negative in the fourth quadrant.

$\tan(x) = \dfrac{-\sqrt{5}}{2}$ in quadrant 4.

b **1** Evaluate the reciprocal of $\sec(x)$ to obtain $\cos(x)$.

b $\cos(x) = \dfrac{1}{\sec(x)}$

$= \dfrac{2}{3}$

2 Write identity [1].

$\sin^2(x) + \cos^2(x) = 1$

3 Substitute $\cos(x) = \dfrac{2}{3}$ into the identity.

If $\cos(x) = \dfrac{2}{3}$

$\sin^2(x) + \dfrac{4}{9} = 1$

4 Solve for $\sin(x)$.

$\sin^2(x) = \frac{5}{9}$

$\sin(x) = \pm\frac{\sqrt{5}}{3}$

5 Retain the negative solution only, as sine is negative in the fourth quadrant.

$\sin(x) = \frac{-\sqrt{5}}{3}$ in quadrant 4.

c 1 Evaluate the reciprocal of $\sin(x)$.

c $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$

$= \frac{-3}{\sqrt{5}}$

2 Rationalise the denominator.

$= \frac{-3\sqrt{5}}{5}$

WORKED EXAMPLE 12 **If $\cot(x) = \frac{1}{\sqrt{3}}$, solve for x over the interval $[0, 2\pi]$.**

THINK

1 Use the reciprocal of $\cot(x)$ to express the equation in terms of $\tan(x)$.

2 State the solution for x in the first quadrant.

3 Use symmetry to identify the solution in the third quadrant where tan is also positive.

4 State the two solutions in the domain $[0, 2\pi]$.

WRITE

1 $\tan(x) = \frac{1}{\cot(x)}$

$= \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}$

2 $x = \frac{\pi}{3}$ in the first quadrant

3 or $x = \pi + \frac{\pi}{3}$

$= \frac{4\pi}{3}$ in the third quadrant

4 $x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$ over $[0, 2\pi]$

WORKED EXAMPLE 13 **Solve $\operatorname{cosec}(x) = 1.8$ over the interval $0 \le x \le 4\pi$. Give your answer(s) correct to 2 decimal places.**

THINK

1 Use the reciprocal of $\operatorname{cosec}(x)$ to find $\sin(x)$.

2 Write the solution.

3 Round the answers to 2 decimal places.

WRITE

1 $\sin(x) = \frac{1}{\operatorname{cosec}(x)} = \frac{1}{1.8} = 0.5556$

2 Solving $\operatorname{cosec}(x) = 1.8$ over the interval $0 \le x \le 4\pi$ gives
$x = 0.589\,031, 2.552\,56, 6.872\,22, 8.835\,75$

3 $x = 0.59, 2.55, 6.87, 8.84$

EXERCISE 10.5 Trigonometric identities

PRACTISE

1 WE9 Simplify each of the following expressions.

a $\tan^2(x) - \sec^2(x)$ b $\sin^2(x)\cot^2(x)$ c $\dfrac{\tan^2(x)}{1+\tan^2(x)}$

2 Simplify each of the following expressions.

a $\sin^4(x) - \cos^4(x)$ b $\text{cosec}^2(x) - \cot^2(x)$ c $\dfrac{\cos^2(x)}{\sin(x)} + \sin(x)$

3 WE10 If $x \in \left[0, \dfrac{\pi}{2}\right]$ and $\sin(x) = 0.8$, find:

a $\cos(x)$ b $\tan(x)$ c $\sec(x)$ d $\text{cosec}(x)$ e $\cot(x)$.

4 If $x \in \left[\dfrac{3\pi}{2}, 2\pi\right]$ and $\cos(x) = \dfrac{1}{4}$, find:

a $\sin(x)$ b $\cot(x)$ c $\sec(x)$.

5 WE11 If $x \in \left[\dfrac{\pi}{2}, \pi\right]$ and $\sec(x) = -2$, use the identities to find exact values for:

a $\cos(x)$ b $\sin(x)$ c $\tan(x)$ d $\text{cosec}(x)$ e $\cot(x)$.

6 If $x \in \left[\dfrac{3\pi}{2}, 2\pi\right]$ and $\cot(x) = -0.4$, use the identities to find exact values for:

a $\tan(x)$ b $\sin(x)$ c $\cos(x)$ d $\text{cosec}(x)$ e $\sec(x)$.

7 WE12 Solve for x in $\sin(x) = -\dfrac{\sqrt{3}}{2}$ over the interval $[0, 2\pi]$.

8 Solve for x in $\tan(x) = 1$ over the interval $[0, 2\pi]$.

9 WE13 Solve $\text{cosec}(x) = 3.5$ over the interval $[-\pi, \pi]$. Give your answer correct to 2 decimal places.

10 Solve $\sec(x) = -4$ over the interval $[-\pi, \pi]$. Give your answer correct to 2 decimal places.

CONSOLIDATE

11 If $\tan(x) = \dfrac{-4}{3}$ and $\sin(x) > 0$, then the value of $\text{cosec}^2(x)$ is:

A $\dfrac{5}{3}$ B $\dfrac{16}{25}$ C $\dfrac{9}{25}$ D $\dfrac{25}{16}$ E $\dfrac{25}{9}$

12 If $\text{cosec}(x) = 4$ and $x \in \left[\dfrac{\pi}{2}, \pi\right]$, find exact values for:

a $\cos(x)$ b $\tan(x)$ c $\sec(x)$.

13 If $\tan(x) = -0.5$ and $x \in \left[\dfrac{3\pi}{2}, 2\pi\right]$, find exact values for:

a $\sec(x)$ b $\sin(x)$ c $\cos(x)$.

14 If $\cos(x) = -0.9$ and $x \in \left[\pi, \dfrac{3\pi}{2}\right]$, find exact values for:

a $\sin(x)$ b $\cot(x)$ c $\text{cosec}(x)$.

15 Solve for x in each of the following equations over the interval $[0, 2\pi]$.

a $\cos(x) = \dfrac{\sqrt{2}}{2}$ b $\sec(x) = -\sqrt{2}$ c $\text{cosec}(x) = 2$ d $\cot^2(x) = 3$

16 Solve each of the following equations over the interval $[-\pi, \pi]$. Give your answers correct to 2 decimal places.

a $\cot(x) = -0.75$
b $\sec(x) = 2.39$
c $\operatorname{cosec}(x) = -1.1$
d $\cot(x) = 5.4$

17 When simplified, $\dfrac{\tan(x)}{\sec(x)} + \cos(x)$ is equal to:

A $\sec(x)$
B $\sec^2(x)$
C $\operatorname{cosec}(x)$
D $\sin(x) + \cos(x)$
E $\operatorname{cosec}^2(x)$

18 If $\sin(x) = \dfrac{-4}{7}$, then $\cot^2(x)$ is equal to:

A $2\dfrac{1}{16}$
B $\dfrac{\sqrt{33}}{4}$
C $\dfrac{16}{33}$
D $\dfrac{33}{49}$
E $\dfrac{4\sqrt{33}}{33}$

19 If $\sec(x) = 8$, then x is nearest to:

A 7°
B 83°
C 97°
D 173°
E 66°

20 Given that $\sec(x) = \dfrac{5}{4}$, $\cot(x)$ is equal to:

A $\dfrac{4}{5}$ only
B $\dfrac{-3}{4}$ or $\dfrac{3}{4}$
C $\dfrac{-4}{5}$ or $\dfrac{4}{5}$
D $\dfrac{-4}{3}$ or $\dfrac{4}{3}$
E $\dfrac{3}{5}$ only

MASTER

21 Prove the following.

a $\dfrac{\sec(x) \times \operatorname{cosec}(x)}{\operatorname{cosec}^2(x)} = \tan(x)$
b $\dfrac{\tan^2(x) \times \cos(x)}{2\sec(x)} = \dfrac{1}{2}\sin^2(x)$

22 Prove the following.

a $\tan(x) + \sec(x) = \dfrac{\cos(x)}{1 - \sin(x)}$
b $\tan^2(\theta) - \sin^2(\theta) = \tan^2(\theta) \times \sin^2(\theta)$

10.6 Compound- and double-angle formulas

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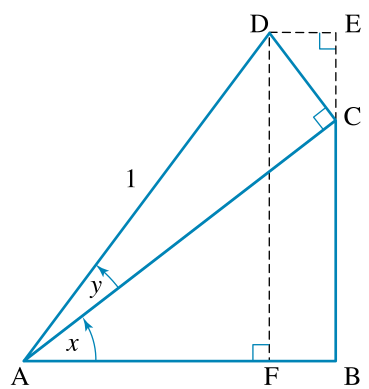
Compound-angle formulas

Consider the right-angled triangles shown in the figure below.

Let AD = 1, ∠BAC = x and ∠DAC = y.



Then
$$\begin{aligned}
AC &= \cos(y)\\
CD &= \sin(y)\\
\angle DCE &= x \text{ (since } \angle BCA = 90° - x \text{ and } \angle ECB = 180°)\\
\Rightarrow DE &= \sin(x)\sin(y)\\
\Rightarrow CE &= \cos(x)\sin(y)\\
\Rightarrow BC &= \sin(x)\cos(y)\\
\Rightarrow AB &= \cos(x)\cos(y)
\end{aligned}$$

Now
$$\begin{aligned}
BE &= \sin(x + y) \text{ (as BE = FD)}\\
&= BC + CE\\
&= \sin(x)\cos(y) + \sin(y)\cos(x)
\end{aligned}$$

$\therefore \quad \sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

and $\quad AF = \cos(x + y)$

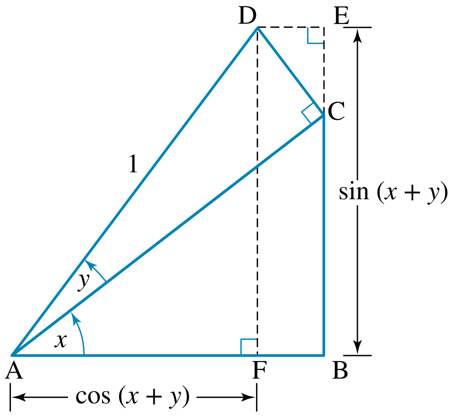
$$= AB - BF$$

$$= AB - DE \text{ (since } DE = BF)$$

$$= \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\therefore \quad \cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Using a similar approach, or by replacing y with $-y$, the following identities can also be derived:



1. $\sin(x - y) = \sin(x)\cos(-y) + \cos(x)\sin(-y)$

   $= \sin(x)\cos(y) - \cos(x)\sin(y)$

   — since $\cos(-y) = \cos(y)$ and $\sin(-y) = -\sin(y)$.

2. $\cos(x - y) = \cos(x)\cos(-y) - \sin(x)\sin(-y)$

   $= \cos(x)\cos(y) + \sin(x)\sin(y)$.

Furthermore:

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)}$$

$$= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$$

Dividing the numerator and denominator by $\cos(x)\cos(y)$, this simplifies to:

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

Similarly:

$$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

Note: These identities can also be derived using a unit circle approach.

In summary, the compound-angle formulas are:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}.$$

WORKED EXAMPLE 14

a Expand, and simplify where possible, each of the following.

i $\sin(x - 2y)$

ii $\cos(x + 30°)$

b Simplify the expression $\sin(2x)\cos(y) + \cos(2x)\sin(y)$.

THINK	WRITE
a i 1 Write the appropriate compound-angle formula.	**a i** $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
2 Substitute $A = x$ and $B = 2y$.	$\sin(x - 2y) = \sin(x)\cos(2y) - \cos(x)\sin(2y)$
ii 1 Write the appropriate compound-angle formula.	**ii** $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
2 Substitute $A = x°$ and $B = 30°$.	$\cos(x + 30°) = \cos(x°)\cos(30°) - \sin(x°)\sin(30°)$
3 Replace $\sin(30°)$ and $\cos(30°)$ with their exact values.	$= \frac{\sqrt{3}}{2}\cos(x°) - \frac{1}{2}\sin(x°)$
b 1 Write the appropriate compound-angle formula.	**b** $\sin(A)\cos(B) + \cos(A)\sin(B) = \sin(A + B)$
2 Substitute $A = 2x$ and $B = y$ to reveal the answer.	$\sin(2x)\cos(y) + \cos(2x)\sin(y) = \sin(2x + y)$

Double-angle formulas

If A replaces B in the compound-angle formula:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

then $$\sin(A + A) = \sin(A)\cos(A) + \cos(A)\sin(A)$$

or $$\sin(2A) = 2\sin(A)\cos(A) \quad [1]$$

Similarly, $$\cos(A + A) = \cos(A)\cos(A) - \sin(A)\sin(A)$$

or $$\cos(2A) = \cos^2(A) - \sin^2(A) \quad [2]$$

Two other forms of $\cos(2A)$ are obtained using the Pythagorean identity.

$$\sin^2(A) + \cos^2(A) = 1$$

that is, $$\cos(2A) = [1 - \sin^2(A)] - \sin^2(A) \quad [3]$$
$$= 1 - 2\sin^2(A)$$

and $$\cos(2A) = \cos^2(A) - [1 - \cos^2(A)] \quad [4]$$
$$= 2\cos^2(A) - 1$$

Finally, $$\tan(A + A) = \frac{\tan(A) + \tan(A)}{1 - \tan(A)\tan(A)}$$

or $$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)} \quad [5]$$

In summary, the double-angle formulas are:

$$\begin{aligned}\sin(2A) &= 2\sin(A)\cos(A)\\ \cos(2A) &= \cos^2(A) - \sin^2(A)\\ &= 1 - 2\sin^2(A)\\ &= 2\cos^2(A) - 1\\ \tan(2A) &= \frac{2\tan(A)}{1-\tan^2(A)}\end{aligned}$$

WORKED EXAMPLE 15

Expand the following using the double-angle formulas.

a $\tan(4x)$

b $\sin\left(\frac{\theta}{2}\right)$

THINK

a 1 Express in double-angle notation as $\tan[2(2x)]$.

2 Expand using the appropriate formula.

b 1 Express in double-angle notation as $\sin\left[2\left(\frac{\theta}{4}\right)\right]$.

2 Expand using the appropriate formula.

WRITE

a $\tan(4x) = \tan[2(2x)]$

$= \frac{2\tan(2x)}{1-\tan^2(2x)}$

b $\sin\left(\frac{\theta}{2}\right) = \sin\left[2\left(\frac{\theta}{4}\right)\right]$

$= 2\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right)$

WORKED EXAMPLE 16

Simplify:

a $\sin(270 - C)°$

b $\sec\left(\frac{\pi}{2} - \theta\right)$.

THINK

a 1 Write the appropriate compound-angle formula.

2 Substitute $A = 270°$ and $B = C°$.

3 Simplify.

b 1 Express in terms of cos using the reciprocal identity $\sec(x) = \frac{1}{\cos(x)}$.

2 Expand the denominator using the appropriate angle formula.

WRITE

a $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

$\sin(270 - C)°$

$= \sin(270°)\cos(C°) - \cos(270°)\sin(C°)$

$= (-1)\cos(C°) - (0)\sin(C°)$

$= -\cos(C°)$

b $\sec\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\cos\left(\frac{\pi}{2} - \theta\right)}$

$= \frac{1}{\cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta)}$

3 Simplify the denominator.

$$= \frac{1}{(0)\cos(\theta) + (1)\sin(\theta)}$$

$$= \frac{1}{\sin(\theta)}$$

4 Express as a reciprocal function.

$$= \operatorname{cosec}(\theta)$$

WORKED EXAMPLE 17 Find the exact value of $\cot\left(\frac{5\pi}{12}\right)$.

THINK

WRITE

1 Express $\frac{5\pi}{12}$ as the sum of $\frac{\pi}{4}$ and $\frac{\pi}{6}$.

$$\cot\left(\frac{5\pi}{12}\right) = \cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

2 Express cot in terms of its reciprocal, $\frac{1}{\tan}$.

$$= \frac{1}{\tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right)}$$

3 Use the appropriate compound-angle formula to expand the denominator.

$$= \frac{1}{\left(\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}\right)}$$

4 Express in simplest fraction form.

$$= \frac{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)}$$

5 Simplify.

$$= \frac{1 - (1)\left(\frac{1}{\sqrt{3}}\right)}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{\left(\frac{\sqrt{3} - 1}{\sqrt{3}}\right)}{\left(\frac{\sqrt{3} + 1}{\sqrt{3}}\right)}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

6 Rationalise the denominator.

$$= \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$

7 Simplify.

$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$

$= \frac{4 - 2\sqrt{3}}{2}$

$= 2 - \sqrt{3}$

WORKED EXAMPLE 18

If $\sin(\theta) = \frac{\sqrt{5}}{3}$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, find the exact values of:

a $\sin(2\theta)$ **b** $\cos(2\theta)$ **c** $\sin\left(\frac{\theta}{2}\right)$.

THINK	WRITE
a 1 Write the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to find $\cos(\theta)$.	**a** $\sin^2(\theta) + \cos^2(\theta) = 1$
2 Substitute $\sin(\theta) = \frac{\sqrt{5}}{3}$ into the identity.	If $\sin(\theta) = \frac{\sqrt{5}}{3}$ $\left(\frac{\sqrt{5}}{3}\right)^2 + \cos^2(\theta) = 1$
3 Solve for $\cos(\theta)$.	$\frac{5}{9} + \cos^2(\theta) = 1$ $\cos^2(\theta) = 1 - \frac{5}{9}$ $= \frac{4}{9}$ $\cos(\theta) = \pm\frac{2}{3}$
4 Retain the positive solution only, as θ is in the first quadrant.	Since θ is in the first quadrant, $\cos(\theta) = \frac{2}{3}$.
5 Expand $\sin(2\theta)$ using the appropriate double-angle formula.	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
6 Substitute $\sin(\theta) = \frac{\sqrt{5}}{3}$ and $\cos(\theta) = \frac{2}{3}$.	$= 2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)$
7 Simplify.	$= \frac{4\sqrt{5}}{9}$
b 1 Expand $\cos(2\theta)$ using the appropriate double-angle formula.	**b** $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
2 Substitute for $\sin(\theta)$ and $\cos(\theta)$.	$= \frac{4}{9} - \frac{5}{9}$
3 Simplify.	$= \frac{-1}{9}$
c 1 Use the alternative double-angle formula $\cos(2A) = 1 - 2\sin^2(A)$.	**c** $\cos(2A) = 1 - 2\sin^2(A)$

2 Replace A with $\frac{\theta}{2}$.

$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

3 Substitute $\cos(\theta) = \frac{2}{3}$.

$\cos(\theta) = \frac{2}{3}$

so $\frac{2}{3} = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

4 Solve for $\sin\left(\frac{\theta}{2}\right)$.

$$2\sin^2\left(\frac{\theta}{2}\right) = 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{6}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\frac{1}{\sqrt{6}}$$

5 Retain the positive solution only, as $\theta \in \left[0, \frac{\pi}{2}\right]$ implies $\frac{\theta}{2} \in \left[0, \frac{\pi}{4}\right]$.

$\sin\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{6}}$ or $\frac{\sqrt{6}}{6}$ since $\frac{\theta}{2}$ is in the first quadrant.

EXERCISE 10.6 Compound- and double-angle formulas

PRACTISE

1 **WE14** Expand each of the following, simplifying if possible.
 a $\sin(2x + y)$ b $\cos(3x - 2y)$ c $\tan(x + 2y)$

2 Expand each of the following, simplifying if possible.
 a $\sin(y - 4x)$ b $\cos(2x + 60)°$ c $\tan(x - 45)°$

3 **WE15** Expand each of the following using the double-angle formula.
 a $\sin(6x)$ b $\cos(4x)$ c $\tan(8x)$

4 Expand each of the following using the double-angle formula.
 a $\sin(A)$ b $\cos\left(\frac{B}{2}\right)$ c $\tan(10A)$

5 **WE16** Simplify each of the following expressions.
 a $\cos(90 - A)°$ b $\sin(270 + A)°$ c $\tan(180 + B)°$

6 Simplify each of the following expressions.
 a $\sin(90 + B)°$ b $\cos(270 - A)°$ c $\tan(360 + A)°$

7 **WE17** Find the exact value of each of the following.
 a $\tan\left(\frac{\pi}{12}\right)$ b $\cos\left(\frac{5\pi}{12}\right)$ c $\sin\left(\frac{7\pi}{12}\right)$

8 Find the exact value of each of the following.
 a $\operatorname{cosec}\left(\frac{5\pi}{12}\right)$ b $\cot\left(\frac{7\pi}{12}\right)$ c $\sec\left(\frac{\pi}{12}\right)$

9 **WE18** If $\cos(x) = -0.6$ and $x \in \left[\frac{\pi}{2}, \pi\right]$, find the values of:
 a $\cos(2x)$ b $\sin(2x)$ c $\tan(2x)$.

10 If $\tan(x) = 2$ and $x \in \left[0, \frac{\pi}{2}\right]$, find the exact values of:

a $\tan(2x)$ b $\sin(2x)$ c $\cos(2x)$.

CONSOLIDATE

11 Use compound- and double-angle formulas to express each of the following in a different form.

a $\cos(3x)\cos(x) + \sin(3x)\sin(x)$
b $\sin(A+B)\cos(A-B) - \cos(A+B)\sin(A-B)$
c $\cos(z)\cos(-z) - \sin(z)\sin(-z)$
d $\tan(x) - \tan(x+y)$
e $\tan(A+B) + \tan(A-B)$

12 Consider the reciprocal function $\operatorname{cosec}(\pi + A)$.

a When expanded, $\operatorname{cosec}(\pi + A)$ is equal to:

A $\dfrac{1}{\sin(\pi)\cos(A)}$ B $\dfrac{1}{\cos(\pi)\cos(A) - \sin(\pi)\sin(A)}$

C $\dfrac{1}{\cos(\pi)\sin(A) - \sin(\pi)\cos(A)}$ D $\dfrac{1}{\sin(\pi)\sin(A) + \cos(\pi)\cos(A)}$

E $\dfrac{1}{\sin(\pi)\cos(A) + \cos(\pi)\sin(A)}$

b The expression $\operatorname{cosec}(\pi + A)$ simplifies to:

A $-\sin(A)$ B $\cos(A)$ C $-\operatorname{cosec}(A)$
D $-\sec(A)$ E $\sec(A)$

13 Simplify each of the following.

a $\sin\left(\frac{\pi}{2} - x\right)$ b $\sec\left(\frac{\pi}{2} + y\right)$ c $\tan\left(\frac{\pi}{2} - A\right)$

d $\cot\left(\frac{3\pi}{2} + A\right)$ e $\operatorname{cosec}\left(\frac{3\pi}{2} - B\right)$ f $\sec\left(\frac{3\pi}{2} + B\right)$

g $\cot\left(\frac{\pi}{2} + y\right)$ h $\operatorname{cosec}(\pi - A)$

14 a The expression $\frac{\pi}{12}$ is equal to:

A $\frac{\pi}{6} - \frac{\pi}{4}$ B $\frac{\pi}{2} - \frac{\pi}{3}$ C $\frac{\pi}{6} + \frac{\pi}{6}$ D $\frac{\pi}{3} - \frac{\pi}{4}$ E $\frac{\pi}{5} + \frac{\pi}{7}$

b The exact value of $\cos\left(\frac{\pi}{12}\right)$ is:

A $\dfrac{\sqrt{2} - \sqrt{6}}{4}$ B $\dfrac{\sqrt{2} + \sqrt{3}}{2}$ C $\dfrac{\sqrt{6} + \sqrt{2}}{4}$

D $\dfrac{\sqrt{2} - \sqrt{3}}{4}$ E $\dfrac{2\sqrt{2} + \sqrt{3}}{4}$

15 If $\sin(x) = \frac{4}{5}$, $x \in \left[0, \frac{\pi}{2}\right]$ and $\tan(y) = \frac{5}{12}$, $y \in \left[\pi, \frac{3\pi}{2}\right]$, find the exact value of each of the following.

a $\tan(x)$ b $\cos(y)$ c $\sec(x)$ d $\operatorname{cosec}(y)$
e $\sin(x+y)$ f $\cos(x-y)$ g $\tan(x-y)$ h $\tan(x+2y)$

16 If $\cos(x) = -0.4,\ x \in \left[\frac{\pi}{2}, \pi\right]$ and $\operatorname{cosec}(y) = 1.25,\ y \in \left[0, \frac{\pi}{2}\right]$, find the value of each of the following, correct to 2 decimal places.

a $\sin(x)$ b $\tan(x + y)$ c $\cos(y)$ d $\cos(x - y)$

17 Simplify each of the following expressions using double-angle formulas.

a $\sin^2(x) - \cos^2(x)$ b $\sin(x)\cos(x)$ c $\dfrac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$

d $\dfrac{\sin^2(x) - \sin^4(x)}{\sin(2x)}$ e $\dfrac{\cos(2x)}{\sin^4(x) - \cos^4(x)}$ f $(\sin x - \cos x)^2$

18 If $\sin(A) = \dfrac{1}{\sqrt{5}}$ and $x \in \left[0, \frac{\pi}{2}\right]$, find the exact values of:

a $\sin(2A)$ b $\cos(2A)$ c $\sin(3A)$ d $\sin^2\left(\frac{A}{2}\right)$.

19 If $\cos(B) = 0.7$ and $x \in \left[0, \frac{\pi}{2}\right]$, then find each of the following, correct to 2 decimal places.

a $\cos\left(\frac{B}{2}\right)$ b $\sin\left(\frac{B}{2}\right)$ c $\tan\left(\frac{B}{2}\right)$ d $\cot\left(\frac{B}{2}\right)$

20 Use the double-angle formulas to find the exact values of:

a $\sin\left(\frac{\pi}{8}\right)$ b $\cos\left(\frac{\pi}{8}\right)$ c $\tan\left(\frac{\pi}{8}\right)$.

MASTER

21 Prove the following identities.

a $\cot(x)\sec(x) = \operatorname{cosec}(x)$

b $[1 + \cot^2(x)][1 - \cos^2(x)] = 1$

c $[1 + \sin(x)][1 - \sin(x)] = \dfrac{1}{\sec^2(x)}$

d $\operatorname{cosec}^2(x) + \sec^2(x) = \operatorname{cosec}^2(x)\sec^2(x)$

22 a Write $\cos(3t)$ as $\cos(2t + t)$ and use compound-angle and double-angle formulas to show that $\cos(3t) = 4\cos^3(t) - 3\cos(t)$. Hence solve $\cos(3t) = \cos(t),\ t \in [0, 2\pi]$ using CAS technology.

b Solve for x in $\sqrt{2}\sin^2(x) = \cos(x),\ x \in [0, 2\pi]$. Give answer to 2 decimal places.

10.7 Other identities

Factorisation identities

From our work with the addition identities, we have:

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

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If we select the two sine identities and add them, we obtain:

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \quad + \\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\ \hline \sin(x+y) + \sin(x-y) &= 2\sin(x)\cos(y) \end{aligned}$$

Now, if we subtract the same two sine identities, we have:

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \quad - \\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\ \hline \sin(x+y) - \sin(x-y) &= 2\cos(x)\sin(y) \end{aligned}$$

Repeating this procedure for the two cosine identities we have, in the case of addition:

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \quad + \\ \cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \\ \hline \cos(x+y) - \cos(x-y) &= 2\cos(x)\cos(y) \end{aligned}$$

And, in the case of subtraction:

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \quad - \\ \cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \\ \hline \cos(x+y) - \cos(x-y) &= -2\sin(x)\sin(y) \end{aligned}$$

In summary, we have the following four results:

$$\sin(x+y) + \sin(x-y) = 2\sin(x)\cos(y)$$

$$\sin(x-y) - \sin(x-y) = 2\cos(x)\sin(y)$$

$$\cos(x+y) + \cos(x-y) = 2\cos(x)\cos(y)$$

$$\cos(x+y) - \cos(x-y) = -2\sin(x)\sin(y)$$

These results enable us to express the sum or difference of two sines and the sum or difference of two cosines, as a product. By writing these results in reflexive form, we have the following factorisation identities.

$$\begin{aligned} 2\sin(x)\cos(y) &= \sin(x+y) + \sin(x-y) \\ 2\cos(x)\sin(y) &= \sin(x+y) - \sin(x-y) \\ 2\cos(x)\cos(y) &= \cos(x+y) + \cos(x-y) \\ -2\sin(x)\sin(y) &= \cos(x+y) - \cos(x-y) \end{aligned}$$

WORKED EXAMPLE 19

Express $2\sin(5x)\cos(2x)$ as a sum or difference.

THINK	WRITE
1 Note the product sin × cos. This indicates a sum of two sines. Write the appropriate identity.	$2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$
2 Replace x with $5x$ and y with $2x$ to write the equation for $2\sin(5x)\cos(2x)$.	$2\sin(5x)\cos(2x) = \sin(5x+2x) + \sin(5x-2x)$
3 Simplify.	$= \sin(7x) + \sin(3x)$

WORKED EXAMPLE 20 Express $\cos(3x)\cos(x)$ as a sum or difference.

THINK	WRITE
1 Note the product cos × cos. This indicates a sum of two cosines. Write the appropriate identity.	$2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$
2 Replace x with $3x$ and y with x to write an equation for $2\cos(3x)\cos(x)$.	$2\cos(3x)\cos(x) = \cos(3x+x) + \cos(3x-x)$
3 Simplify.	$= \cos(4x) + \cos(2x)$
4 Multiply both sides of the equation by $\frac{1}{2}$ to obtain an expression for $\cos(3x)\cos(x)$.	$\cos(3x)\cos(x) = \frac{1}{2}(\cos(4x) + \cos(2x))$

WORKED EXAMPLE 21 Express $5\cos(4x)\sin(2x)$ as a sum or difference.

THINK	WRITE
1 Note the product cos × sin. This indicates a difference of two sines. Write the appropriate identity.	$2\cos(x)\sin(y) = \sin(x+y) - \sin(x-y)$
2 Replace x with $4x$ and y with $2x$ to write an equation for $2\cos(4x)\sin(2x)$.	$2\cos(4x)\sin(2x) = \sin(4x+2x) - \sin(4x-2x)$
3 Simplify.	$= \sin(6x) - \sin(2x)$
4 Multiply both sides of the equation by $\frac{5}{2}$ to obtain an expression for $5\cos(4x)\sin(2x)$.	$5\cos(4x)\sin(2x) = \frac{5}{2}(\sin(6x) - \sin(2x))$

WORKED EXAMPLE 22 Express $\sin(3x)\sin(x)$ as a sum or difference.

THINK	WRITE
1 Note the product sin × sin. This indicates a difference of two cosines. Write the appropriate identity.	$-2\sin(x)\sin(y) = \cos(x+y) - \cos(x-y)$
2 Replace x with $3x$ and y with x to write an equation for $-2\sin(3x)\sin(x)$.	$-2\sin(3x)\sin(x) = \cos(3x+x) - \cos(3x-x)$
3 Simplify.	$= \cos(4x) - \cos(2x)$
4 Multiply both sides of the equation by $-\frac{1}{2}$ to obtain an expression for $\sin(3x)\sin(x)$.	$\sin(3x)\sin(x) = -\frac{1}{2}(\cos(4x) - \cos(2x))$

$r \sin(x \pm \alpha)$ or $r \cos(x \pm \alpha)$

The function $y = a \sin(x) + b \cos(x)$ may be conveniently expressed in one of two forms:

$$y = A \sin(x + \alpha) \quad \text{or} \quad y = A \cos(x + \alpha)$$

given appropriate restrictions on α and by taking $A > 0$. To do this we consider the addition identities as follows:

$$\begin{aligned} A \sin(x + \alpha) &= A \sin(x) \cos(\alpha) + A \cos(x) \sin(\alpha) \\ &= A \cos(\alpha) \sin(x) + A \sin(\alpha) \cos(x) \end{aligned}$$

Note: In the formulas below, α and a represent different angles.

Comparing $y = a \sin(x) + b \cos(x)$ with $y = A \cos(\alpha) \sin(x) + A \sin(\alpha) \cos(x)$ gives:

$$A \cos(\alpha) = a \quad \text{and} \quad A \sin(\alpha) = b$$

So
$$A^2\cos^2(\alpha) + A^2\sin^2(\alpha) = a^2 + b^2$$
$$A^2(\cos^2(\alpha) + \sin^2(\alpha)) = a^2 + b^2$$

By applying the Pythagorean identity:

$$A^2 = a^2 + b^2$$
$$A = \sqrt{a^2 + b^2}, A > 0$$

Also:
$$\frac{A \sin(\alpha)}{A \cos(\alpha)} = \frac{b}{a}$$
$$\Rightarrow \tan(\alpha) = \frac{b}{a}$$

Therefore:
$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

WORKED EXAMPLE 23

Express $2 \cos(x) - 3 \sin(x)$ in the form $A \cos(x - \alpha)$ where $A > 0$ and $0° < \alpha < 360°$.

THINK	WRITE
1 Write the appropriate addition identity.	$A \cos(x - \alpha) = A \cos(x) \cos(\alpha) + A \sin(x) \sin(\alpha)$
2 Compare the given expression with the identity and assign appropriate values for the variables. Apply the Pythagorean identity to find A.	$A \cos(\alpha) = 2 \quad \text{and} \quad A \sin(\alpha) = -3$ $A^2 \cos^2(\alpha) + A^2 \sin^2(\alpha) = (-3)^2 + 2^2$ $\therefore A^2 = 13$ $\Rightarrow A = \sqrt{13}, A > 0$
3 Find α by obtaining a tan relationship. Decide in which quadrant α is located. Remember that $0° < \alpha < 360°$.	$\frac{A \sin(\alpha)}{A \cos(\alpha)} = \frac{-3}{2}$ $\Rightarrow \tan(\alpha) = \frac{-3}{2}$ $\therefore \alpha = \tan^{-1}\left(\frac{-3}{2}\right)$

As $\cos(\alpha) = \frac{2}{\sqrt{3}}$ and $\sin(\alpha) = \frac{-3}{\sqrt{3}}$

then α must lie in the fourth quadrant.

Hence, $\alpha = \tan\left(\frac{-3}{2}\right)$

$\Rightarrow \alpha \approx 360° - 56°19'$

$\therefore \alpha \approx 303°41'$.

4 Write the answer.

So $2\cos(x) - 3\sin(x) = \sqrt{13}\cos(x - 303°41')$.

WORKED EXAMPLE 24

a Express $\sqrt{3}\cos(x) + \sin(x)$ in the form $A\sin(x + \alpha)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Determine the extreme values of the given function.

THINK	WRITE
a 1 Write the appropriate addition identity.	**a** $A\sin(x + \alpha)$ $= A\sin(x)\cos(\alpha) + A\cos(x)\sin(\alpha)$
2 Compare the given expression with the identity and assign appropriate values for the variables.	$A\sin(\alpha) = \sqrt{3}$ and $A\cos(\alpha) = 1$
3 Determine the unknown quantities A and α.	$A^2\cos^2(\alpha) + A^2\sin^2(\alpha) = (\sqrt{3})^2 + 1^2$ $\therefore A^2 = 4$ $\Rightarrow A = 2,\ A > 0$ $\frac{A\sin(\alpha)}{A\cos(\alpha)} = \sqrt{3}$ $\Rightarrow \tan(\alpha) = \sqrt{3}$ $\therefore \alpha = \tan^{-1}(\sqrt{3})$ Since α lies in the first quadrant, $\alpha = \frac{\pi}{3}$.
4 Write the answer.	$\therefore \sqrt{3}\cos(x) + \sin(x) = 2\sin\left(x + \frac{\pi}{3}\right)$.
b The extreme values are the greatest and least values of the function.	**b** The extreme values follow from the fact that $2\sin\left(x + \frac{\pi}{3}\right)$ oscillates between ± 2, since $\left\lvert\sin\left(x + \frac{\pi}{3}\right)\right\rvert \le 1$. The greatest value is 2 and the least value is -2.

EXERCISE 10.7 Other identities

PRACTISE

Express the following as a sum or difference for questions **1** to **8**.

1 WE19 $2\sin(3x)\cos(x)$

2 $2\sin(2x)\cos(3x)$

3 WE20 $2\cos(2x)\cos(x)$

4 $\cos(x)\cos(4x)$

5 WE21 $4\cos(4x)\sin(3x)$

6 $2\cos(2x)\sin(x)$

7 WE22 $-\sin(5x)\sin(3x)$

8 $\sin(2x)\sin(4x)$

9 WE23 Express $\cos(x) - \sin(x)$ in the form $A\cos(x - \alpha)$ where $A > 0$ and $0° < \alpha < 360°$.

10 Express $\sin(x) + \cos(x)$ in the form $A\sin(x + \alpha)$ where $A > 0$ and $0° < \alpha < 90°$.

11 WE24 **a** Express $\sqrt{3}\cos(x) - 2\sin(x)$ in the form $A\sin(x - \alpha)$ where $A > 0$; and $0 < \alpha < 2\pi$.

b Find the maximum and minimum values of $\sqrt{3}\cos(x) - 2\sin(x)$.

12 **a** Express $4\sin(x) - 3\cos(x)$ in the form $A\sin(x - \alpha)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Find the maximum and minimum values of $4\sin(x) - 3\cos(x)$.

CONSOLIDATE

Express the following as a sum or difference for questions **13** to **20**.

13 $2\cos(4x)\sin(x)$

14 $-2\sin(3x)\sin(2x)$

15 $\cos(x + 30°)\cos(30°)$

16 $2\sin(a + b)\cos(a - b)$

17 $a\cos(3b)\cos(3b)$

18 $5\cos(2x - y)\sin(x + 2y)$

19 $\sin\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y)$

20 $\sin(x - y)\sin(x + y)$

21 **a** Express $\cos(x) - 2\sin(x)$ in the form $A\cos(x + \alpha)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Express $\cos(x) - \sqrt{3}\sin(x)$ in the form $A\sin(x - \alpha)$ where $A > 0$ and $180° < \alpha < 270°$.

c Express $\sin(x) + \sqrt{2}\cos(x)$ in the form $A\cos(x - \alpha)$ where $A > 0$ and $0 < \alpha < 360°$.

22 Express each of the following products as a sum or difference.

a $2\sin(4x)\cos(2y)$ **b** $a\sin^2(\pi x)\cos(\pi x)$

MASTER

23 Solve $\dfrac{\cos(5x) - \cos(x)}{\sin(5x) + \sin(x)} = 0$.

24 If $8\cos(x) - 15\sin(x) = r\cos(x + \alpha)$:

a find the values of r and α where $r > 0$ and $0° < \alpha < 360°$

b use CAS technology to determine the least value of x for which $8\cos(x) - 15\sin(x) = 9$ over $0° < x < 360°$.

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10.8 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

studyON

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

studyON

Units 1 & 2 — **Circular functions**

Sit topic test

10 Answers

EXERCISE 10.2

1 a 1.6 m

b i 1 m

ii 0.7 m

c 3.254 s

2 a 40 m

b $\frac{5\pi}{4}$ s

c $\frac{15\pi}{4}$ s

3 a i 1 kg

ii 6 days

b $W = \cos\left(\frac{\pi t}{3}\right) + 3$

4 a 110 beats/min

b i 50 beats/min

ii 60 min

c $H = 50\sin\left(\frac{\pi t}{30}\right) + 110$

5 a 26 °C at 2 pm

b i 18 °C

ii 22 °C

iii $(18 - 4\sqrt{3})$ °C

6 a i 12 mm

ii $\frac{1}{10}$ s

b 10

c $-3\sqrt{2(\sqrt{5}+5)}$ mm; if the displacement is positive to the right, then the string is $3\sqrt{2(\sqrt{5}+5)}$ mm to the left (or vice versa).

d $\frac{1}{120}$ s

7 a 60 s

b 50 m

c 100 m

d 314.16 m

e i 50 m

ii 75 m

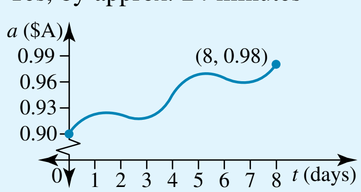
f 25 s and 35 s

8 a i 6 m

ii 3 m

b Yes, by approx. 24 minutes

9 a



b $t = 4.248$ (3 d.p.), ∴ on the 5th day

c $t = 6.580$; $t = 7$

d (1.206, 0.931) and (5.266, 0.971)

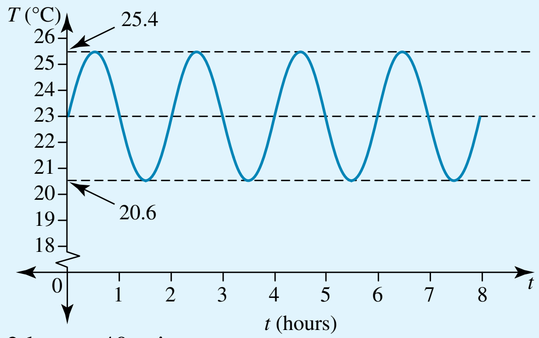
e \$0.98 when $t = 8$

10 a 25.4 °C, 20.6 °C

b i 23 °C

ii 25.4 °C

c



d 2 hours, 40 minutes

11 a 40 cm

b 80 cm

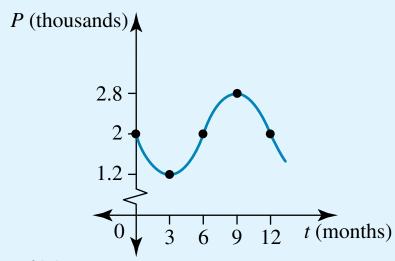
c $0.3\sin\left(\frac{\pi}{16}x\right)$

12 a i 2800 ii 1200

iii 2000

b i 12 months ii 800 rabbits

c



d 1600

e 4 months

f Approx. $5\frac{1}{2}$ months

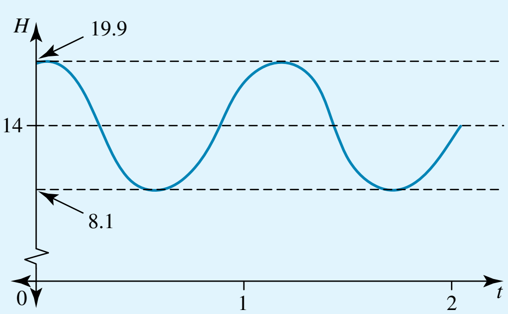
13 a i 19.9 cm

ii 8.1 cm

b i 11.742 cm

ii 8.1 cm

c



d See **c**.

e $\frac{4}{7}$ s

f 52.5 times

g $H = 14 + 5.9\cos\left(\frac{8\pi t}{5}\right)$

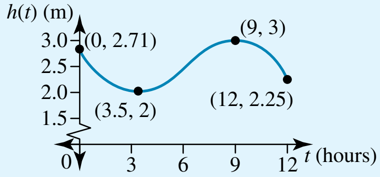
14 a 2.71 m

b 11 hours

c i 3 m at 9 pm

ii 2 m at 3:30 pm

d



e 1:40 pm and 5:20 pm

f 3 h 40 min

EXERCISE 10.3

1

	sin(x)	cos(x)	tan(x)	cosec(x)	sec(x)	cot(x)
a	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
b	$\frac{2\sqrt{10}}{7}$	$\frac{3}{7}$	$\frac{2\sqrt{10}}{3}$	$\frac{7\sqrt{10}}{20}$	$\frac{7}{3}$	$\frac{3\sqrt{10}}{20}$
c	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{3}{4}$

2

	sin(x)	cos(x)	tan(x)	cosec(x)	sec(x)	cot(x)
a	$\frac{5\sqrt{34}}{34}$	$\frac{3\sqrt{34}}{34}$	$\frac{5}{3}$	$\frac{\sqrt{34}}{5}$	$\frac{\sqrt{34}}{3}$	$\frac{3}{5}$
b	$\frac{12}{13}$	$\frac{5}{13}$	$\frac{12}{5}$	$\frac{13}{12}$	$\frac{13}{5}$	$\frac{5}{12}$
c	$\frac{2}{3}$	$\frac{\sqrt{5}}{3}$	$\frac{2\sqrt{5}}{5}$	$\frac{3}{2}$	$\frac{3\sqrt{5}}{5}$	$\frac{\sqrt{5}}{2}$

3 a 36.9°

b 18.2°

c 60.9°

4 a 30.0°

b 73.4°

c 55.0°

5 a $\sqrt{3}$

b $\sqrt{3}$

c $\frac{-\sqrt{3}}{2}$

d $\sqrt{2}$

e 2

f $\frac{-\sqrt{3}}{3}$

6 a 1

b $\frac{-2\sqrt{3}}{3}$

c 1

d $\frac{-2\sqrt{3}}{3}$

e −2

f $\sqrt{3}$

7 a $\frac{-1}{2}$

b $-\sqrt{3}$

c $\frac{-\sqrt{3}}{3}$

8 a $\frac{-5}{\sqrt{41}}$

b $\frac{-4}{\sqrt{41}}$

c $\frac{-\sqrt{41}}{4}$

9 a D

b E

c C

d A

10 a $5\sqrt{3}$

b $4\sqrt{3}$

c $\frac{15\sqrt{2}}{2}$

11 a $\frac{7\sqrt{3}}{3}$

b $\frac{8}{3}$

c $\frac{5\sqrt{6}}{2}$

12 a $\frac{-\sqrt{2}}{2}$

b −1

c $\sqrt{2}$

13 a $\frac{-2}{3}$

b $\frac{-\sqrt{5}}{3}$

c $\frac{\sqrt{5}}{2}$

14 a $\frac{-5}{16}$

b $\frac{\sqrt{231}}{16}$

c $\frac{-5\sqrt{231}}{231}$

15 a $\frac{4}{3}$

b $-\frac{5}{3}$

c $-\frac{5}{4}$

16 a $\frac{-\sqrt{5}}{2}$

b $\frac{-2}{\sqrt{5}}$

c 2

17 a $\frac{\sqrt{3}}{3}$

b $\frac{\sqrt{6}}{2}$

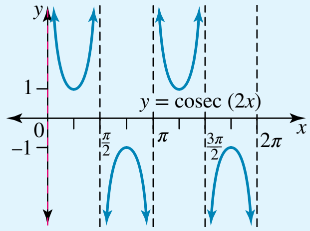
c $\frac{7}{6}$

18 a $\frac{-1}{2}$ **b** $\frac{-1}{\sqrt{3}}$ **c** -2

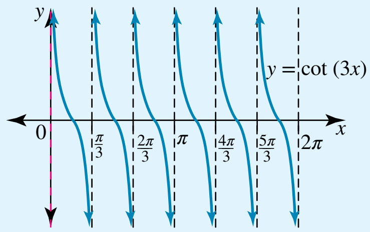
19 a $\frac{\pi}{3}, \frac{2\pi}{3}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{5\pi}{3}$

20 a $-\cos^2 x$ **b** $\frac{1}{\sin(x)}$ **c** 1

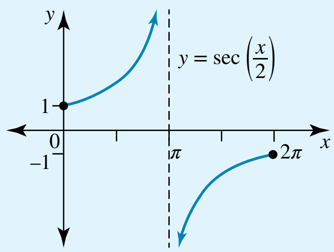
d



e

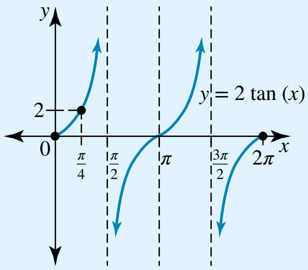


f

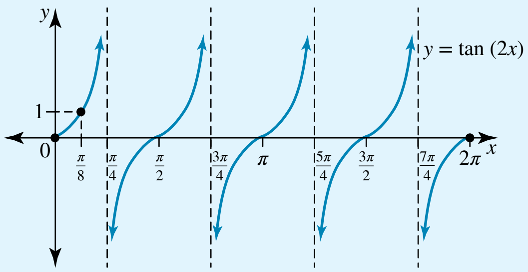


EXERCISE 10.4

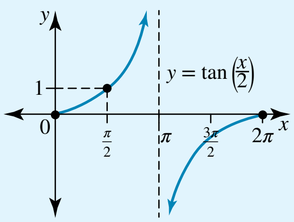
1 a



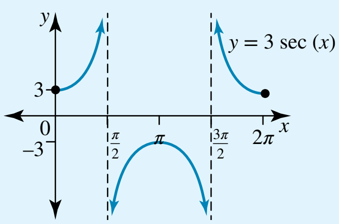
b



c



2 a



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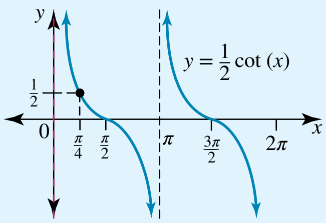
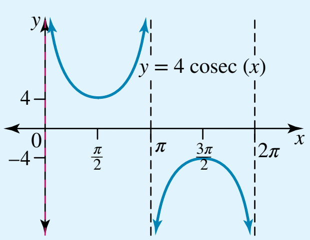
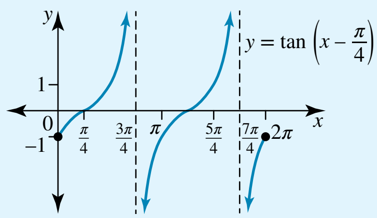


c



3 a



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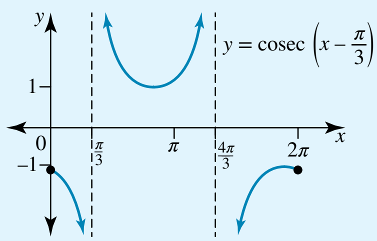

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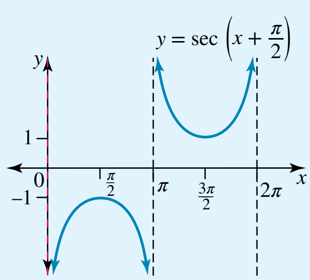
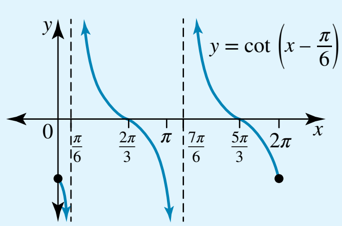
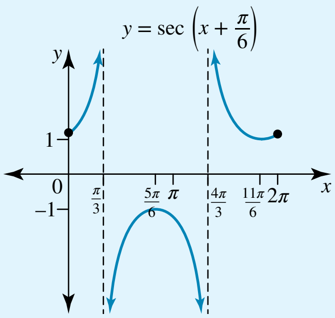


4 a



b



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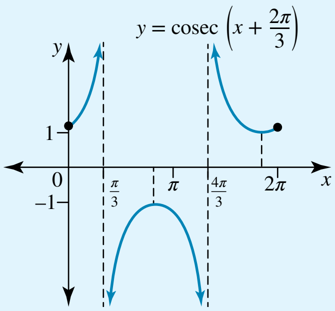
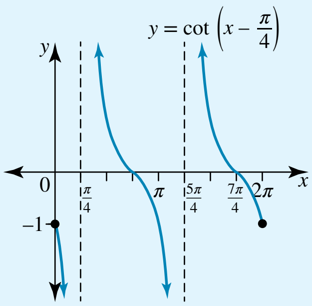
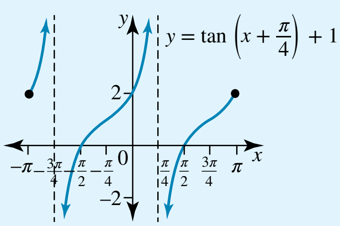


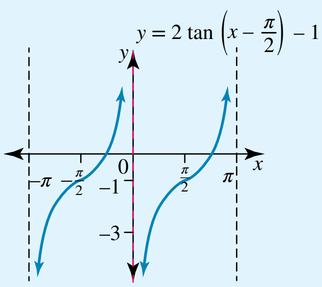
d



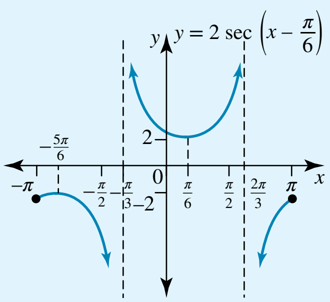
5 a



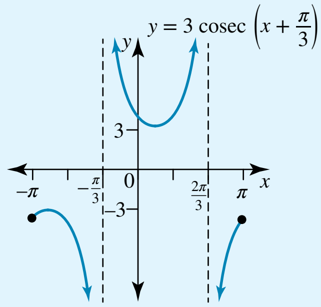
b



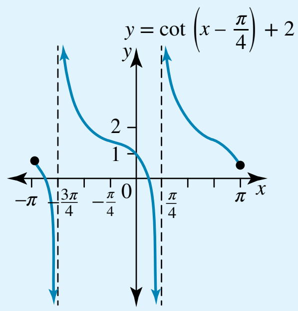
c



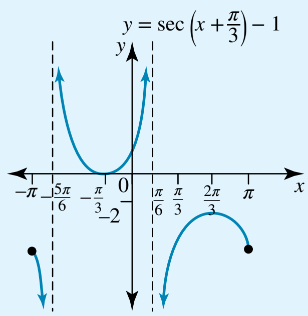
6 a



b



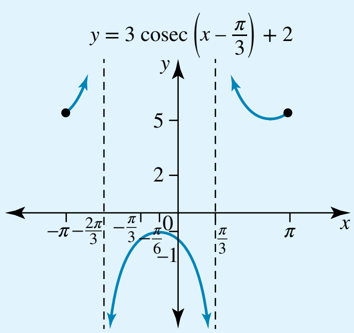
c



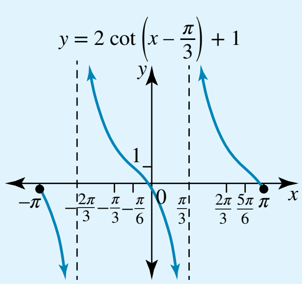
7 a C **b** A **c** D **d** B

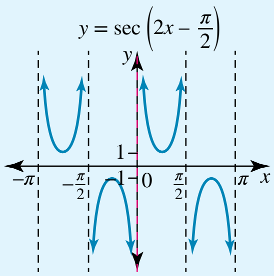
8 a B **b** C **c** E **d** D

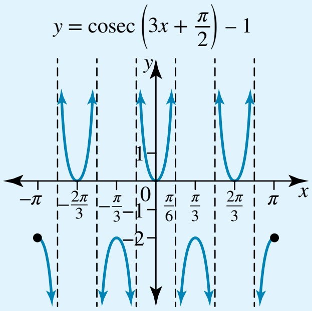
9 a



b



c


10 a


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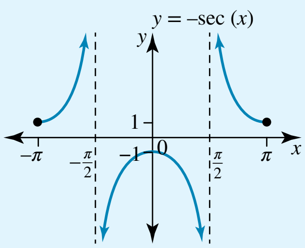
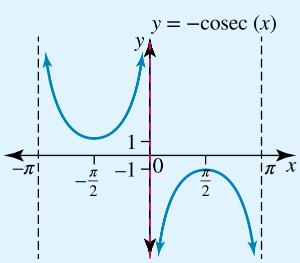
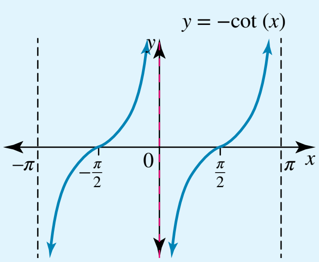


11 a


b


12 B

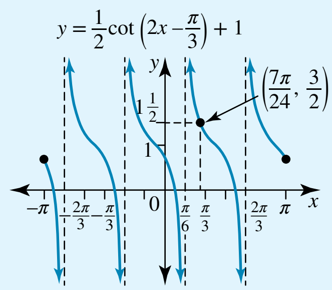
13 B

14 C

15 E

16 C

17 $\left(\frac{5\pi}{12}, -3\right)\left(\frac{11\pi}{12}, 5\right)$

18


EXERCISE 10.5

1 a -1

b $\cos^2(x)$

c $\sin^2(x)$

2 a $\sin^2(x) - \cos^2(x)$

b 1

c $\text{cosec}(x)$

3 a 0.6

b $\frac{4}{3}$

c $\frac{5}{3}$

d 1.25

e 0.75

4 a $\frac{-\sqrt{15}}{4}$

b $\frac{1}{\sqrt{15}}$

c 4

5 a $\frac{-1}{2}$

b $\frac{\sqrt{3}}{2}$

c $-\sqrt{3}$

d $\frac{2\sqrt{3}}{3}$

e $\frac{-\sqrt{3}}{3}$

6 a -2.5

b $\frac{-5\sqrt{29}}{29}$

c $\frac{2\sqrt{29}}{29}$

d $\frac{-\sqrt{29}}{5}$

e $\frac{\sqrt{29}}{2}$

7 $x = \frac{4\pi}{3}, \frac{5\pi}{3}$

8 $x = \frac{\pi}{4}, \frac{5\pi}{4}$

9 $x = 0.29, 2.86$

10 $x = 1.82, -1.82$

11 D

12 a $\frac{-\sqrt{15}}{4}$

b $\frac{-\sqrt{15}}{15}$

c $\frac{-4\sqrt{15}}{15}$

13 a $\frac{\sqrt{5}}{2}$

b $\frac{-\sqrt{5}}{5}$

c $\frac{2\sqrt{5}}{5}$

14 a $\frac{-\sqrt{19}}{10}$

b $\frac{9\sqrt{19}}{19}$

c $\frac{-10\sqrt{19}}{19}$

15 a $x = \frac{\pi}{4}, \frac{7\pi}{4}$

b $x = \frac{3\pi}{4}, \frac{5\pi}{4}$

c $x = \frac{\pi}{6}, \frac{5\pi}{6}$

d $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

16 a $x = -0.93, 2.21$

b $x = -1.14, 1.14$

c $x = -1.14, -2.00$

d $x = -2.96, 0.18$

17 D

18 A

19 B

20 D

21 Answers will vary.

22 Answers will vary.

EXERCISE 10.6

1 a $\sin(2x)\cos(y) + \cos(2x)\sin(y)$

b $\cos(3x)\cos(2y) + \sin(3x)\sin(2y)$

c $\frac{\tan(x) + \tan(2y)}{1 - \tan(x)\tan(2y)}$

2 a $\sin(y)\cos(4x) - \cos(y)\sin(4x)$

b $\frac{1}{2}\cos(2x) - \left(\frac{\sqrt{3}}{2}\right)\sin(2x)$

c $\frac{\tan(x) - 1}{1 + \tan(x)}$

3 a $2\sin(3x)\cos(3x)$

b $\cos^2(2x) - \sin^2(2x)$

c $\frac{2\tan(4x)}{1 - \tan^2(4x)}$

4 a $2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$

b $\cos^2\left(\frac{B}{4}\right) - \sin^2\left(\frac{B}{4}\right)$

c $\frac{2\tan(5A)}{1 - \tan^2(5A)}$

5 a $\sin(A)$ b $-\cos(A)$ c $\tan(B)$

6 a $\cos(B)$ b $-\sin(A)$ c $\tan(A)$

7 a $2 - \sqrt{3}$

b $\frac{\sqrt{6} - \sqrt{2}}{4}$

c $\frac{\sqrt{6} + \sqrt{2}}{4}$

8 a $\sqrt{6} - \sqrt{2}$ b $\sqrt{3} - 2$ c $\sqrt{6} - \sqrt{2}$

9 a -0.28 b -0.96 c $\frac{24}{7}$

10 a $\frac{-4}{3}$ b $\frac{4}{5}$ c $\frac{-3}{5}$

11 a $\cos(2x)$

b $\sin(2B)$

c 1

d $-\frac{\tan(y)(\tan^2 x + 1)}{1 - \tan(x)\tan(y)}$

e $\frac{2\tan(A)(1 + \tan^2 B)}{1 - \tan^2(A)\tan^2(B)}$

12 a E

b C

13 a $\cos(x)$ b $-\text{cosec}(y)$ c $\cot(A)$

d $-\tan(A)$ e $\sec(B)$ f $\text{cosec}(B)$

g $-\tan(y)$ h $\text{cosec}(A)$

14 a D

b C

15 a $\frac{4}{3}$

b $\frac{-12}{13}$

c $\frac{5}{3}$

d $\frac{-13}{5}$

e $\frac{-63}{65}$

f $-\frac{56}{65}$

g $\frac{33}{56}$

h $\frac{-836}{123}$

16 a 0.92

b -0.24

c 0.6

d 0.49

17 a $-\cos(2x)$

b $\frac{1}{2}\sin(2x)$

c $\tan(x)$

d $\frac{1}{4}\sin(2x)$

e -1

f $1-\sin(2x)$

18 a $\frac{4}{5}$

b $\frac{3}{5}$

c $\dfrac{11\sqrt{5}}{25}$

d $\dfrac{5-2\sqrt{5}}{10}$

19 a 0.92 **b** 0.39

c 0.42 **d** 2.38

20 a $\dfrac{\sqrt{2-\sqrt{2}}}{2}$

b $\dfrac{\sqrt{2+\sqrt{2}}}{2}$

c $\sqrt{3-2\sqrt{2}}$

21 a $\cot(x)\sec(x)=\operatorname{cosec}(x)$

$$\begin{aligned}\cot(x)\ \sec(x)&=\frac{1}{\tan(x)}\times\frac{1}{\cos(x)}\\&=\frac{\cos(x)}{\sin(x)}\times\frac{1}{\cos(x)}\\&=\frac{1}{\sin(x)}\\&=\operatorname{cosec}(x)\end{aligned}$$

b $[1+\cot^2(x)]\,[1-\cos^2(x)]=1$

$1-\cos^2(x)=\sin^2(x)$

$\therefore [1+\cot^2(x)]\sin^2(x)$

$=\sin^2(x)+\sin^2(x)\cot^2(x)$

$=\sin^2(x)+\sin^2(x)\dfrac{\cos^2(x)}{\sin^2(x)}$

$=\sin^2(x)+\cos^2(x)$

$=1$

c $[1+\sin(x)][1-\sin(x)]=\dfrac{1}{\sec^2(x)}$

$=(1-\sin^2(x))$

$=\cos^2(x)$

$\dfrac{1}{\cos^2(x)}=\sec^2(x)$

$\therefore \dfrac{1}{\sec^2(x)}=\cos^2(x)$

So, $[1+\sin(x)][1-\sin(x)]=\dfrac{1}{\sec^2(x)}$

d $\operatorname{cosec}^2(x)+\sec^2(x)=\operatorname{cosec}^2(x)\sec^2(x)$

$$\begin{aligned}\operatorname{cosec}^2(x)+\sec^2(x)&=\frac{1}{\sin^2(x)}+\frac{1}{\cos^2(x)}\\&=\frac{\cos^2(x)+\sin^2(x)}{\sin^2(x)\cos^2(x)}\\&=\frac{1}{\sin^2(x)\cos^2(x)}\\&=\frac{1}{\sin^2(x)}\times\frac{1}{\cos^2(x)}\\&=\operatorname{cosec}^2(x)\sec^2(x)\end{aligned}$$

22 a $t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi$

b $x=0.79, 5.50$

EXERCISE 10.7

1 $\sin(4x)+\sin(2x)$

2 $\sin(5x)-\sin(x)$

3 $\cos(3x)+\cos(x)$

4 $\frac{1}{2}\cos(5x)+\frac{1}{2}\cos(3x)$

5 $2[\sin(7x)-\sin(x)]$

6 $\sin(3x)-\sin(x)$

7 $\frac{1}{2}[\cos(8x)-\cos(2x)]$

8 $\frac{1}{2}\cos(2x)-\frac{1}{2}\cos(6x)$

9 $\sqrt{2}\cos(x-315°)$

10 $\sqrt{2}\sin(x+45°)$

11 a $\sqrt{7}\sin(x-3.855)$

b The maximum is $\sqrt{7}$ and the minimum $-\sqrt{7}$.

12 a $5\sin(x-0.644)$

b Maximum 5, minimum -5

13 $\sin(5x)-\sin(3x)$

14 $\cos(5x)-\cos(x)$

15 $\frac{1}{2}[\cos(x+60°)+\cos(x)]$

16 $\sin(2a)+\sin(2b)$

17 $\dfrac{a}{2}[\cos(6b)+1]$

18 $\frac{5}{2}[\sin(3x+y)-\sin(x-3y)]$

19 $\frac{1}{2}(\sin x+\sin y)$

20 $-\frac{1}{2}[\cos(2x)-\cos(2y)]$

21 a $\sqrt{5}\cos(x+1.107)$

b $2\sin(x-210°)$

c $\sqrt{3}\cos(x-35°16')$

22 a $\sin(4x+2y)+\sin(4x-2y)$

b $\dfrac{a}{4}(\cos\pi x-\cos 3\pi x)$

23 $x=\pm\dfrac{n\pi}{2},\ n\in Z$

24 a $r=17$ and $\alpha=61°56'$

b $x=240°2'$

11

Linear and non-linear relationships

11.1 Kick off with CAS

Exploring polar graphs with CAS

The polar coordinate system specifies each point in the plane by a pair of numbers, (r, θ). This pair of numbers represents the distance from a fixed point, r, and the angle, θ, from a fixed direction.

The polar coordinate system will be studied in more detail in this topic.

1 Using CAS technology, sketch graphs of the following polar equations.

a $r = 4\sin(2\theta), 0 \le \theta \le 2\pi$

b $r = 3\sin(2\theta), 0 \le \theta \le 2\pi$

c $r = 5\sin(4\theta), 0 \le \theta \le 2\pi$

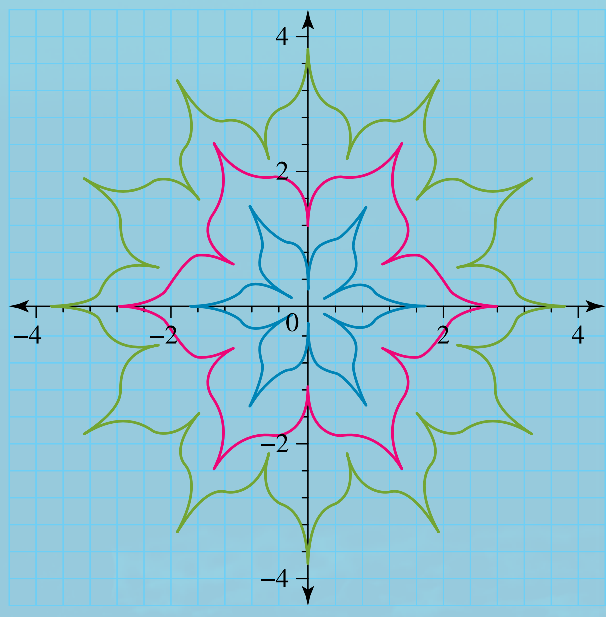
d $r = 8\sin(5\theta), 0 \le \theta \le 2\pi$

If $r = a\sin(n\theta)$, where $a \neq 0$ and $n > 1$, comment on the effect of changing a and n on the graphs of the above equations.

2 Using CAS technology, sketch the graph of the following equation.

$$r = 2 + \frac{|\cos(3\theta)| + \left(0.25 - \left|\cos\left(3\theta + \frac{\pi}{2}\right)\right|\right) \times 2}{2 + \left|\cos\left(6\theta + \frac{\pi}{2}\right)\right| \times 8}$$

3 Determine the equations of the two other graphs shown in the figure below by changing the values in the equation in question 2.



Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

11.2 Reciprocal graphs

This technique involves sketching the graph of $y = \dfrac{1}{f(x)}$ from the graph of $y = f(x)$.

1. When $f(x) \to 0$, $y = \dfrac{1}{f(x)} \to \infty$, so the graph of $y = \dfrac{1}{f(x)}$ approaches the vertical asymptote(s).
2. Therefore, the graph of $y = \dfrac{1}{f(x)}$ will have vertical asymptotes at the x-intercepts of $y = f(x)$.
3. When $f(x) \to \infty$, $\dfrac{1}{f(x)} \to 0$, so the graph of $y = \dfrac{1}{f(x)}$ approaches the horizontal asymptote (the x-axis in this case).
4. These graphs also have common points:
   (a) When $f(x) = \pm 1$, $\dfrac{1}{f(x)} = \pm 1$. The graphs are in the same quadrant.
   (b) When $f(x) < 0$, $\dfrac{1}{f(x)} < 0$.
   (c) When $f(x) > 0$, $\dfrac{1}{f(x)} > 0$.
5. The x-intercepts of $f(x)$ determine the equations of the vertical asymptotes for the reciprocal of the functions.
6. The minimum turning point of $f(x)$ gives the maximum turning point of the reciprocal function.
7. The maximum turning point of $f(x)$ gives the minimum turning point of the reciprocal function.

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Sketch graphs of reciprocal functions
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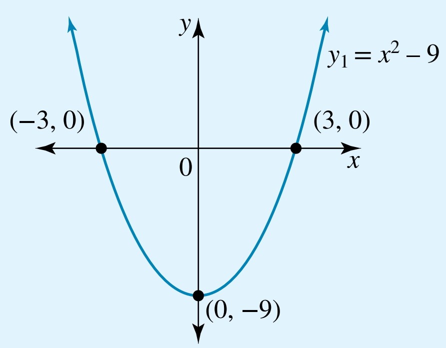
Practice questions

Note: If $y = \dfrac{-1}{f(x)}$ then:

- for $f(x) = 1$, $y = -1$ and for $f(x) = -1$, $y = 1$
- for $f(x) < 0$, $y > 0$ and for $f(x) > 0$, $y < 0$.

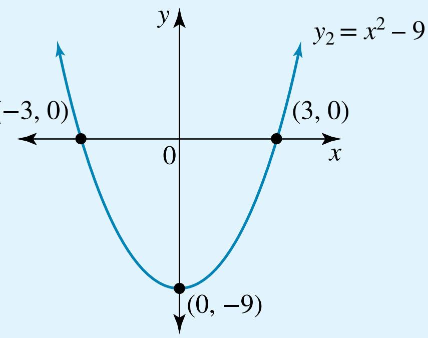
WORKED EXAMPLE 1

Sketch the graph of the function $y = \dfrac{1}{x^2 - 9}$, $x \neq \pm 3$ from the given graph of $y_1 = x^2 - 9$.



THINK | **WRITE/DRAW**

1 Sketch the graph of the function $y_1 = x^2 - 9$ as given.



2 Work out the asymptotes for $y = \dfrac{1}{x^2 - 9}$.

The x-intercepts of $y_1 = x^2 - 9$ are $x = \pm 3$.

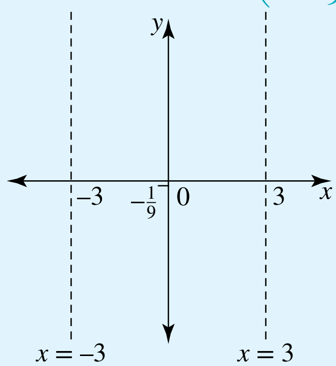
The vertical asymptotes for $y = \dfrac{1}{x^2 - 9}$ are $x = \pm 3$.

The horizontal asymptote is $y = 0$.

3 For the y-intercept, $x = 0$.

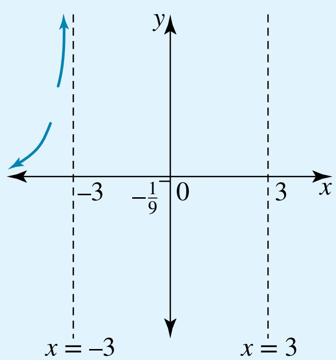
The y-intercept is $\left(0, -\dfrac{1}{9}\right)$.

4 Draw a diagram based on the information gathered so far.



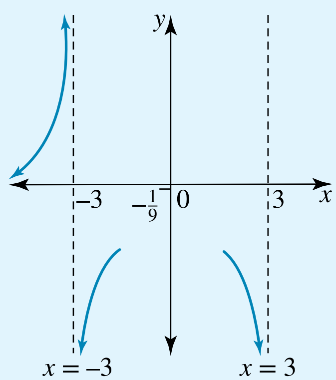
5 As $x \to -\infty$, y approaches 0 from the positive direction ($y \to 0^+$).

As x approaches −3 from the negative direction ($x \to -3^-$), $y \to \infty$.

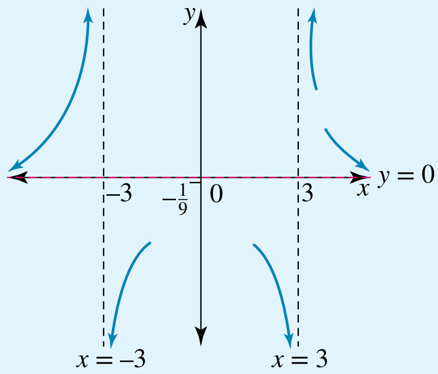


6 As $x \to -3^+$, $y \to -\infty$.

As $x \to +3^-$, $y \to -\infty$.

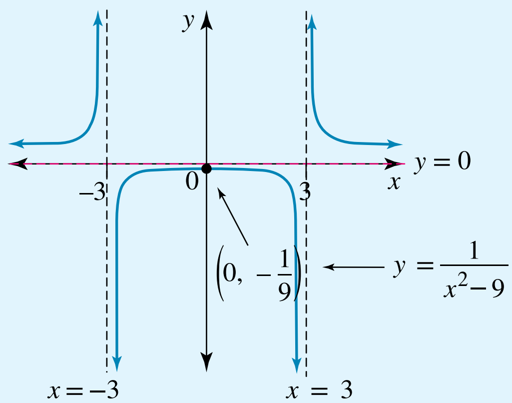


7 As $x \to +3^+$, $y \to \infty$.
As $x \to \infty$, $y \to 0^+$.



8 To determine the shape of the graph near the y-intercept, evaluate the value of y when x is ± 1 and ± 2.

9 Sketch the graph of $y = \dfrac{1}{x^2 - 9}$.



The following examples show a different approach to sketching reciprocal functions.

WORKED EXAMPLE 2

Sketch the graphs of $f(x)$ and $g(x)$ on the same set of axes where:
$f(x) = x^2 - 2x - 3$ and $g(x) = \dfrac{1}{x^2 - 2x - 3}$, $x \neq 3, -1$.

THINK	WRITE/DRAW
1 The graph of $f(x)$ is an upright parabola, as $a = 1$.	
2 Calculate the x-intercepts.	x-intercepts: $x^2 - 2x - 3 = 0$ so $(x - 3)(x + 1) = 0$ so $x - 3 = 0$ or $x + 1 = 0$ and $x = 3$ or $x = -1$
3 State the coordinates of the x-intercepts.	The x-intercepts are $(3, 0)$ and $(-1, 0)$.
4 Calculate the y-intercepts.	y-intercept: $f(0) = -3$ The y-intercept is $(0, -3)$.
5 The vertex or turning point x-coordinate is halfway between the x-intercepts.	Turning point: $x = \dfrac{-1 + 3}{2} = 1$

6 Substitute to find the y-value of the turning point.

$$f(1) = (1)^2 - 2(1) - 3$$
$$= 1 - 2 - 3$$
$$= -4$$

The turning point is $(1, -4)$.

7 Sketch the graph of $f(x) = x^2 - 2x - 3$.

See below.

8 Use the above to determine important features for $g(x) = \dfrac{1}{x^2 - 2x - 3}$.

$$g(x) = \frac{1}{(x^2 - 2x - 3)}$$
$$= \frac{1}{(x - 3)\ (x + 1)}$$

9 Vertical asymptotes occur where $f(x)$ has its x-intercepts.

Vertical asymptotes:
$x = 3$ and $x = -1$

as $x \to \pm\infty$, $\left|\dfrac{1}{x^2 - 2x - 3}\right| \to 0^+$, and so $g(x) \to 0^+$.

10 Find the horizontal asymptotes.

The horizontal asymptote is $g(x) = 0$.

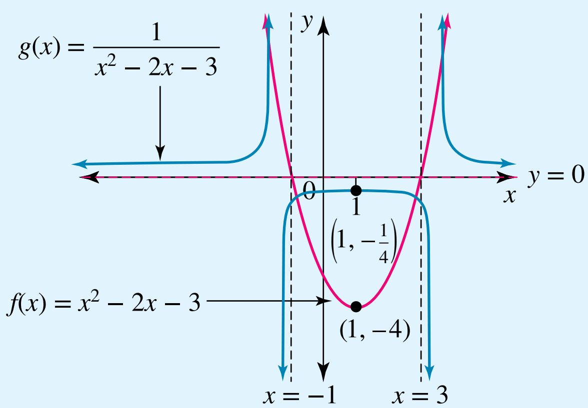
11 The reciprocal of the turning point for $f(x)$ is a turning point for $g(x)$.

The reciprocal of the turning point $(1, -4)$ is $\left(1, -\dfrac{1}{4}\right)$.

12 $f(x)$ and $g(x)$ intercept at $y = 1$ and $y = -1$.

13 As $g(x) = \dfrac{1}{f(x)}$, the graphs of $f(x)$ and $g(x)$ are in the same quadrants.

14 Sketch the graph of $g(x)$ on the same axes as $f(x)$.



WORKED EXAMPLE 3

Sketch the graphs of $f(x)$ and $g(x)$ on the same set of axes where:
$f(x) = -(x + 3)^2$ and $g(x) = \dfrac{-1}{(x + 3)^2}$, $x \neq -3$.

THINK	WRITE/DRAW
1 Work out important features for $f(x) = -(x + 3)^2$. This is a parabola reflected in the x-axis, as $a = -1$. Calculate the x-intercept(s).	x-intercepts: $-(x + 3)^2 = 0$; $(x + 3)^2 = 0$; $x + 3 = 0$; $x = -3$

2 State the coordinates of the x-intercept. — The x-intercept is $(-3, 0)$.

3 Calculate the y-intercept. — The y-intercept:

$$\begin{aligned} f(0) &= -(0 + 3)^2 \\ &= -(9) \\ &= -9 \end{aligned}$$

The y-intercept is $(0, -9)$.

4 As the graph touches the x-axis at $(-3, 0)$, it must also turn at this point. Hence, $(-3, 0)$ is the turning point. — The turning point is $(-3, 0)$.

5 Sketch the graph of $f(x)$. — See below.

6 Use the above to determine important features for $g(x) = \dfrac{-1}{(x + 3)^2}$. — For $g(x) = \dfrac{-1}{(x + 3)^2}$:

7 Vertical asymptotes occur where $f(x)$ has its x-intercepts. — Vertical asymptote: $x = -3$

8 Find the horizontal asymptotes. — as $x \to \pm\infty$, $\left|\dfrac{-1}{(x + 3)^2}\right| \to 0$, and so $g(x) \to 0$.

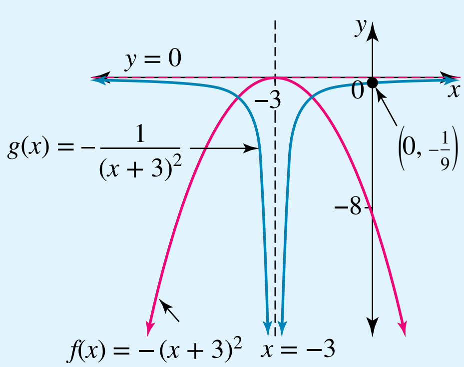
9 We cannot take the reciprocal of the turning point for $f(x)$ as the reciprocal of 0 is not defined — it was worked out in step 7 above that this was the vertical asymptote. — The horizontal asymptote is $g(x) = 0$.

10 The y-intercept of $g(x)$ is the reciprocal of the y-intercept of $f(x)$. — The y-intercept is $\left(0, -\dfrac{1}{9}\right)$.

11 Since $g(x) = \dfrac{1}{f(x)}$, then $g(x) = 1$ or -1 when $f(x) = 1$ or -1.

12 As $g(x) = \dfrac{1}{f(x)}$, the graphs of $f(x)$ and $g(x)$ are in the same quadrants.

13 Sketch the graph of $g(x)$ on the same axes as $f(x)$.



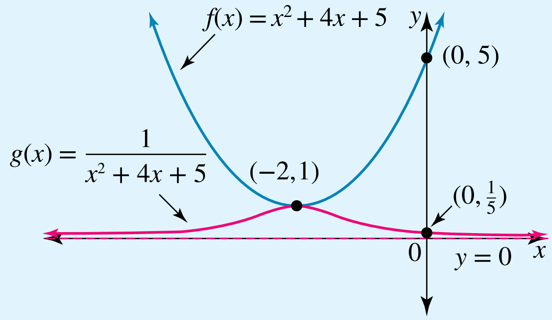
WORKED EXAMPLE 4 Sketch the graphs of $f(x)$ and $g(x)$ on the same set of axes where:

$f(x) = x^2 + 4x + 5$ and $g(x) = \dfrac{1}{x^2 + 4x + 5}$.

THINK	WRITE/DRAW
Work out important features for $f(x) = x^2 + 4x + 5$.	
1 This is an upright parabola, as $a = 1$.	
2 Find the x-value of the turning point by solving $\frac{dy}{dx} = 0$ or $f'(x) = 0$.	$f'(x) = 2x + 4$ For $f'(x) = 0$, $2x + 4 = 0$ $2x = -4$ $x = -2$
3 Evaluate $f(x)$ when $x = -2$.	$f(-2) = (-2)^2 + 4(-2) + 5$ $= 4 - 8 + 5$ $= 1$ The turning point is $(-2, 1)$.
4 As the parabola is upright and turns at $(-2, 1)$ it is completely above the x-axis and hence there is no x-intercept.	There is no x-intercept.
5 Calculate the y-intercept.	y-intercept: $f(0) = 5$ The y-intercept is $(0, 5)$. By symmetry $(-4, 5)$ is also on the curve.
6 Sketch the graph of $f(x)$.	See below.
7 Use the above to determine important features for $g(x) = \frac{1}{x^2 + 4x + 5}$.	
8 Since there are no x-intercepts for $f(x)$, $g(x)$ has no vertical asymptotes.	There are no vertical asymptotes.
9 Find the horizontal asymptote.	as $x \to \pm\infty$, $\left\lvert \frac{1}{x^2 + 4x + 5} \right\rvert \to 0$, and so $g(x) \to 0$. The horizontal asymptote is $f(x) = 0$.
10 The vertex of $g(x)$ is the reciprocal of the vertex of $f(x)$.	The vertex is $(-2, 1)$.
11 The y-intercept for $g(x)$ is the reciprocal of the y-intercept of $f(x)$.	The y-intercept is $\left(0, \frac{1}{5}\right)$.

12 Since $g(x) = \dfrac{1}{f(x)}$ the graphs of $f(x)$ and $g(x)$ are in the same quadrants.

13 Sketch the graph of $g(x)$ on the same axes as $f(x)$.

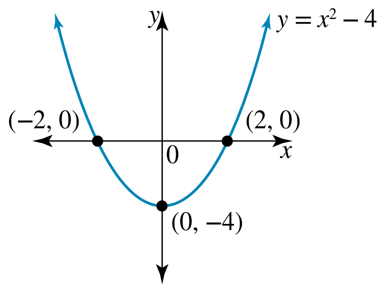


EXERCISE 11.2 Reciprocal graphs

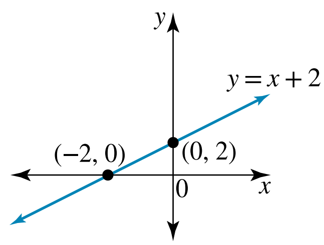
PRACTISE

1 **WE1** Sketch the graph of each of the following functions from the given graph.

a $y = \dfrac{1}{x^2 - 4}, x \neq \pm 2$

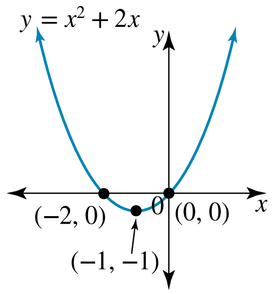


b $y = \dfrac{1}{x + 2}, x \neq -2$

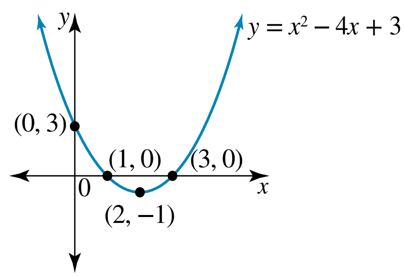


2 Sketch the graph of each of the following functions from the given graph.

a $y = \dfrac{1}{x^2 + 2x}, x \neq 0, -2$



b $y = \dfrac{1}{x^2 - 4x + 3}, x \neq 3, 1$



For questions 3–8, sketch the graph of the functions $f(x)$ and $g(x)$ on the same set of axes. Show all asymptotes with equations and turning points.

3 **WE2** $f(x) = x - 4,\ g(x) = \dfrac{1}{x - 4}, x \neq 4$

4 $f(x) = x^2 - 4x,\ g(x) = \dfrac{1}{x^2 - 4x}, x \neq 0, 4$

5 **WE3** $f(x) = (x - 4)^2,\ g(x) = \dfrac{1}{(x - 4)^2}, x \neq 4$

6 $f(x) = (x + 3)^2,\ g(x) = \dfrac{1}{(x + 3)^2},\ x \neq -3$

7 WE4 $f(x) = x^2 + 2,\ g(x) = \dfrac{1}{x^2 + 2}$

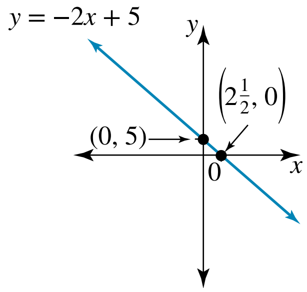
8 $f(x) = x^2 + 2x + 4,\ g(x) = \dfrac{1}{x^2 + 2x + 4}$

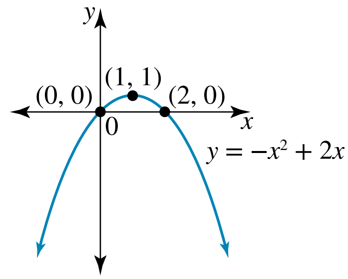
CONSOLIDATE

9 Sketch the graph of each of the following functions from the given graph.

a $y = \dfrac{1}{-2x + 5},\ x \neq \dfrac{5}{2}$

b $y = \dfrac{1}{-x^2 + 2x},\ x \neq 0, 2$

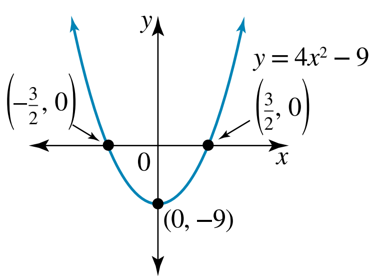


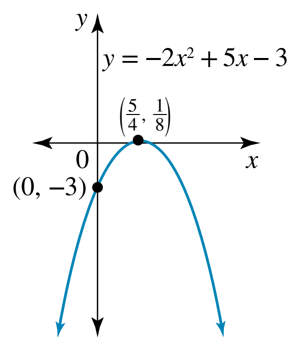


10 Sketch the graph of each of the following functions from the given graph.

a $y = \dfrac{1}{4x^2 - 9},\ x \neq \pm\dfrac{3}{2}$

b $y = \dfrac{1}{-2x^2 + 5x - 3},\ x \neq \dfrac{3}{2}, 1$





For questions **11–16**, sketch the graphs of $f(x)$ and $g(x)$ on the same set of axes. Show all asymptotes with equations and turning points.

11 $f(x) = 3 - x,\ g(x) = \dfrac{1}{3 - x},\ x \neq 3$

12 $f(x) = x^2 + 3x + 2,\ g(x) = \dfrac{1}{x^2 + 3x + 2},\ x \neq -1, -2$

13 $f(x) = 3x + x^2,\ g(x) = \dfrac{1}{3x + x^2},\ x \neq -3, 0$

14 $f(x) = 3x^2 - 8x - 3,\ g(x) = \dfrac{1}{3x^2 - 8x - 3},\ x \neq \dfrac{-1}{3}, 3$

15 $f(x) = -x^2 + 4x - 4,\ g(x) = \dfrac{1}{-x^2 + 4x - 4},\ x \neq 2$

16 $f(x) = x^2 + x + \dfrac{1}{4},\ g(x) = \dfrac{1}{x^2 + x + \frac{1}{4}},\ x \neq \dfrac{-1}{2}$

17 Consider the function $f(x) = \dfrac{1}{x^2 + 4x + 3}$.

a $f(x)$ has asymptotes with equations:

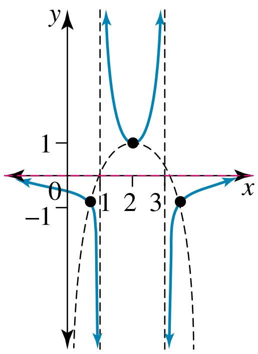
A $x = -1, x = 3$ and $y = 0$
B $x = -1, x = -3$ and $y = 0$
C $x = 1, x = 3$ and $y = 0$
D $x = 1$ and $x = 3$ only
E $x = 0$ and $y = \dfrac{1}{3}$ only

b The y-intercept and turning point are respectively:

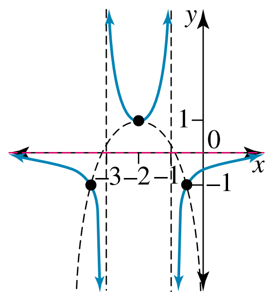
A (0, 3) and (2, 1)
B (0, 3) and (2, −1)
C $\left(0, \dfrac{1}{3}\right)$ and (−2, 1)
D $\left(0, \dfrac{1}{3}\right)$ and (−2, −1)
E $\left(0, \dfrac{1}{3}\right)$ and (2, −1)

18 The graph of $f(x)$ in question **17** is best represented by:

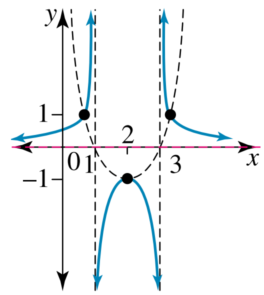
A



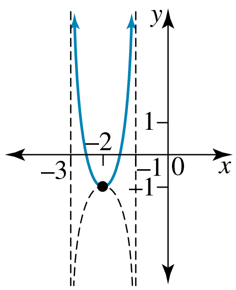
B



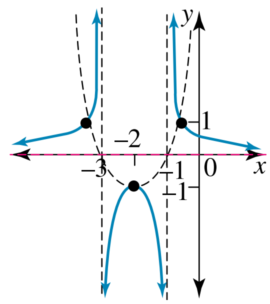
C



D



E



MASTER

19 A box in the shape of a rectangular prism has a base of length x cm and width $(4 - x)$ cm.

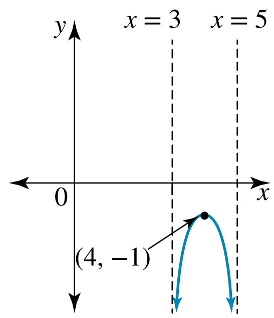
a Express the area of the base, A cm^2, in terms of x.

b If the volume of the box is fixed at 1 cm^3, express the height, h cm, in terms of x.

c Determine the height of the box when the length of the base is 3.95 cm.

d Sketch the graph of h against x.

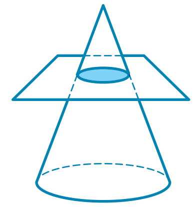
e Find the minimum height of the box and the dimensions in this case.



20 The graph shown has the form $y = \dfrac{a}{x^2 + bx + c}$. Find the values of a, b and c.

11.3 The circle and the ellipse

The circle



The circle belongs to the family of conics. That is, a circle is a curve produced by the intersection of a plane with a cone.

A circle is the path or locus traced out by a point at a constant distance (the radius) from a fixed point (the centre).

Consider the circles shown below. The first circle has its centre at the origin and radius r.

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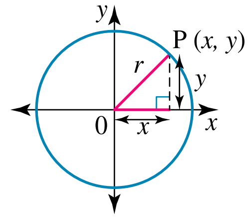
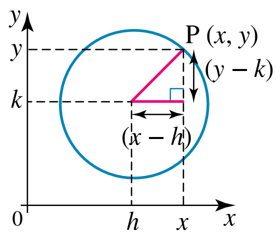
Let P (x, y) be a point on the circle. By Pythagoras' theorem, $x^2 + y^2 = r^2$.

The equation of a circle with centre (0, 0) and radius r is: $x^2 + y^2 = r^2$.



If the circle is translated h units to the right, parallel to the x-axis, and k units upwards, parallel to the y-axis, then:

the equation of a circle with centre (h, k) and radius r is: $(x - h)^2 + (y - k)^2 = r^2$.



Domain $[h - r, h + r]$
Range $[k - r, k + r]$

WORKED EXAMPLE 5 Sketch the graph $4x^2 + 4y^2 = 25$. Stating the centre, radius, domain and range.

THINK

1 Express the equation in standard form by dividing both sides by 4.

2 State the coordinates of the centre.

3 Find the length of the radius by taking the square root of both sides.

4 Sketch the graph.

5 State the domain and range.

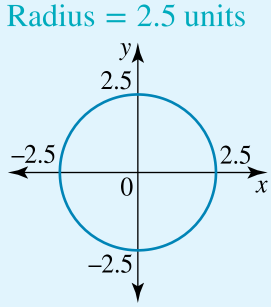
WRITE/DRAW

$$x^2 + y^2 = r^2$$
$$4x^2 + 4y^2 = 25$$
$$x^2 + y^2 = \frac{25}{4}$$
$$x^2 + y^2 = \left(\frac{5}{2}\right)^2$$

Centre (0, 0)

$$r^2 = \left(\frac{5}{2}\right)^2$$

Radius = 2.5 units



Domain [−2.5, 2.5]
Range [−2.5, 2.5]

WORKED EXAMPLE 6 Sketch the graph of $(x - 2)^2 + (y + 3)^2 = 16$, clearly showing the centre, radius, domain and range.

THINK

1 Express the equation in standard form by expressing 16 as 4^2.

2 State the coordinates of the centre.

3 State the length of the radius.

4 Sketch the graph.

5 State the domain and range.

WRITE/DRAW

$(x - h)^2 + (y - k)^2 = r^2$

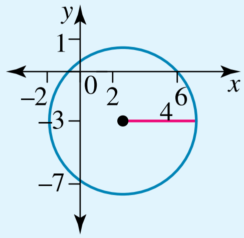
$(x - 2)^2 + (y + 3)^2 = 4^2$

Centre $(2, -3)$

$r^2 = 4^2$

$r = 4$

Radius = 4 units



Domain $[-2, 6]$
Range $[-7, 1]$

WORKED EXAMPLE 7 Sketch the graph of the circle $x^2 + 2x + y^2 - 6y + 6 = 0$, stating the domain and range.

THINK

1 Express the equation in standard form using the 'completing the square' method twice.

2 State the coordinates of the centre.

3 State the length of the radius.

4 Sketch the graph.

5 State the domain and range.

WRITE/DRAW

$$(x - h)^2 + (y - k)^2 = r^2$$

$$x^2 + 2x + y^2 - 6y + 6 = 0$$

$$(x^2 + 2x + 1) - 1 + (y^2 - 6y + 9) - 9 + 6 = 0$$

$$(x + 1)^2 + (y - 3)^2 - 4 = 0$$

$$(x + 1)^2 + (y - 3)^2 = 4$$

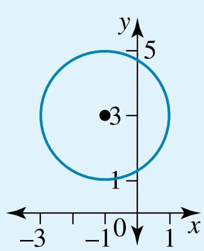
$$(x + 1)^2 + (y - 3)^2 = 2^2$$

Centre $(-1, 3)$

$r^2 = 2^2$

$r = 2$

Radius = 2 units



Domain $[-3, 1]$
Range $[1, 5]$

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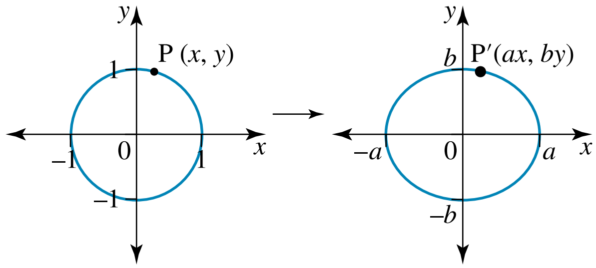
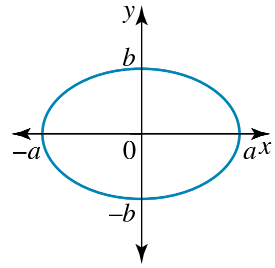
Graphs of the ellipse

If a circle with Cartesian equation $x^2 + y^2 = 1$ is dilated by a factor a from the y-axis and by a factor b from the x-axis then all points P (x, y) on the circle become the points P′(ax, by) as shown at right. The basic equation of an ellipse is:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Its graph is shaped like an elongated circle — see graph at right.



This ellipse:

1. is centred at (0, 0)
2. has vertices at $(-a, 0)$, $(a, 0)$ (found by letting $y = 0$ and solving), and $(0, -b)$ and $(0, b)$ (found by letting $x = 0$ and solving).

If this curve were shifted h units to the right and k units up, then the centre would move to (h, k) and its equation would become:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Note: If $a = b$ then the equation becomes $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1$ and can be rearranged to $(x-h)^2 + (y-k)^2 = a^2$ (by multiplying both sides by a^2). This is the equation of a circle.

For an ellipse in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ we can deduce the following, which will help us to sketch the ellipse:
1. (h, k) are the coordinates of the centre of the ellipse.
2. The vertices are $(-a + h, k)$, $(a + h, k)$, $(h, -b + k)$, $(h, b + k)$.

Notes

1. a is half the length of the **major axis**, the length of the semi-major axis (axis of symmetry parallel to the x-axis if $a > b$ or parallel to the y-axis if $a < b$).
2. b is half the length of the **minor axis**, the length of the semi-minor axis (axis of symmetry parallel to the y-axis if $a > b$ or parallel to the x-axis if $a < b$).
3. a and b are lengths and so are positive values.

WORKED EXAMPLE 8

Sketch the graph of $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$, showing x- and y-intercepts and stating the domain and range.

THINK	WRITE/DRAW
1 Compare $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$ with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.	$h = 1$, $k = 2$ and so the centre is (1, 2). $a^2 = 25$ $b^2 = 9$ $a = 5$ $b = 3$
2 The major axis is parallel to the x-axis as $a > b$.	

3 The extreme points (vertices) parallel to the x-axis for the ellipse are:

$(-a + h, k)$ $(a + h, k)$

Vertices are: $(-5 + 1, 2)$ $= (-4, 2)$; $(5 + 1, 2)$ $= (6, 2)$

So domain is $[-4, 6]$

4 The extreme points (vertices) parallel to the y-axis for the ellipse are:

$(h, -b + k)$ $(h, b + k)$

Vertices are: $(1, -3 + 2)$ $= (1, -1)$; $(1, 3 + 2)$ $= (1, 5)$

So range is $[-1, 5]$

5 Find the x- and y-intercepts.

x-intercepts:

$$y = 0 \Rightarrow \frac{(x-1)^2}{25} + \frac{4}{9} = 1$$

$$\Rightarrow 9x^2 - 18x - 116 = 0$$

$$x = \frac{3 \pm 5\sqrt{5}}{3}$$

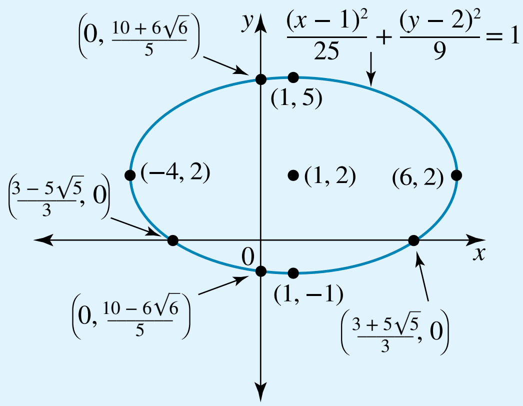
y-intercepts:

$$x = 0 \Rightarrow \frac{1}{25} + \frac{(y-2)^2}{9} = 1$$

$$\Rightarrow 25y^2 - 100y - 116 = 0$$

$$y = \frac{10 \pm 6\sqrt{6}}{5}$$

6 Sketch the graph of the ellipse.



WORKED EXAMPLE 9

Sketch the graph of $\frac{(x-2)^2}{9} + \frac{(y+4)^2}{16} = 1$, showing x- and y-intercepts and stating the domain and range.

THINK | WRITE/DRAW

1 Compare $\frac{(x-2)^2}{9} + \frac{(y+4)^2}{16} = 1$ with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

$h = 2, k = -4$. So the centre is $(2, -4)$.

$a^2 = 9$ $b^2 = 16$

$a = 3$ $b = 4$

2 The major axis is parallel to the y-axis as $b > a$.

3 The extreme points (vertices) parallel to the x-axis for the ellipse are:

$(-a + h, k)$ $(a + h, k)$

Vertices are: $(-3 + 2, -4)$ $= (-1, -4)$; $(3 + 2, -4)$ $= (5, -4)$

So domain is $[-1, 5]$

4 The extreme points (vertices) parallel to the y-axis for the ellipse are:
$(h, -b + k)$ $\quad$ $(h, b + k)$

Vertices are: $(2, -4 - 4) = (2, -8)$ $\quad$ $(2, 4 - 4) = (2, 0)$

So range is $[-8, 0]$

5 Find the x- and y-intercepts.

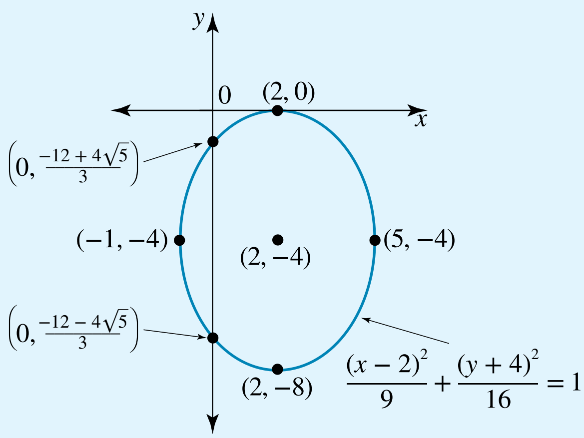
x-intercept:

$$y = 0 \Rightarrow \frac{(x-2)^2}{9} + \frac{16}{16} = 1$$
$$\Rightarrow (x-2)^2 = 0$$
$$x = 2$$

y-intercepts:

$$x = 0 \Rightarrow \frac{4}{9} + \frac{(y+4)^2}{16} = 1$$
$$\Rightarrow 9y^2 + 72y + 64 = 0$$
$$y = \frac{-12 \pm 4\sqrt{5}}{3}$$

6 Sketch the graph of the ellipse.



WORKED EXAMPLE 10 Sketch the graph of $5x^2 + 9(y - 2)^2 = 45$.

THINK / **WRITE/DRAW**

1 Rearrange and simplify by dividing both sides by 45 to make the RHS = 1.

$$\frac{5x^2}{45} + \frac{9(y-2)^2}{45} = \frac{45}{45}$$

2 Simplify by cancelling.

$$\frac{x^2}{9} + \frac{(y-2)^2}{5} = 1$$

3 Compare $\frac{x^2}{9} + \frac{(y-2)^2}{5} = 1$ with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

$h = 0$, $k = 2$ and so the centre is $(0, 2)$.

$a^2 = 9$ $\quad$ $b^2 = 5$ as $a, b > 0$

$a = 3$ $\quad$ $b = \sqrt{5}$

4 Major axis is parallel to the x-axis as $a > b$.

5 The extreme points (vertices) parallel to the x-axis for the ellipse are:
$(-a + h, k)$ $\quad$ $(a + h, k)$

Vertices are: $(-3 + 0, 2) = (-3, 2)$ $\quad$ $(3 + 0, 2) = (3, 2)$

Domain is $[-3, 3]$

6 The extreme points (vertices) parallel to the y-axis for the ellipse are:

$(h, -b + k)$ $\quad$ $(h, b + k)$

Vertices are: $(0, -\sqrt{5} + 2)$ or $(0, 2 - \sqrt{5})$ $\approx (0, -0.24)$ $\quad$ $(0, \sqrt{5} + 2)$ $(0, 2 + \sqrt{5})$ $\approx (0, 4.24)$

Range is $[2 - \sqrt{5}, 2 + \sqrt{5}]$

7 Find the x- and y-intercepts.

x-intercepts:

$y = 0 \Rightarrow 5x^2 + 36 = 45$

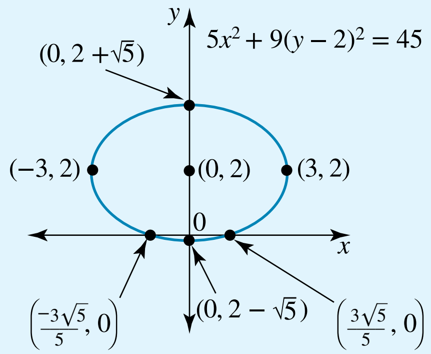
$$x = \frac{\pm 3\sqrt{5}}{5}$$

y-intercepts:

$x = 0 \Rightarrow 9(y - 2)^2 = 45$

$$y = 2 \pm \sqrt{5}$$

8 Sketch the graph of the ellipse.



WORKED EXAMPLE 11

Sketch the graph of the relation described by the rule: $\mathbf{25x^2 + 150x + 4y^2 - 8y + 129 = 0}$.

THINK | **WRITE/DRAW**

1 Factorise, complete the square and simplify. Then divide both sides by 100 to make the RHS = 1.

$25x^2 + 150x + 4y^2 - 8y + 129 = 0$

$25(x^2 + 6x + 9) - 225 + 4(y^2 - 2y + 1) - 4 + 129$

$= 25(x + 3)^2 + 4(y - 1)^2 = 100$

$$\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{25} = 1;\text{ ellipse}$$

2 Compare $\dfrac{(x + 3)^2}{4} + \dfrac{(y - 1)^2}{25} = 1$ with $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

$h = -3, k = 1$

So the centre is $(-3, 1)$.

$a^2 = 4$ $\quad$ $b^2 = 25$

$a = 2$ $\quad$ $b = 5$ as $a, b > 0$

3 The major axis is parallel to the y-axis as $b > a$.

4 The extreme points (vertices) parallel to the x-axis for the ellipse are:

$(-a + h, k)$ $\quad$ $(a + h, k)$

Vertices are: $(-2 - 3, 1) = (-5, 1)$ $\quad$ $(2 - 3, 1) = (-1, 1)$

Domain is $[-5, -1]$

5 The extreme points (vertices) parallel to the y-axis for the ellipse are:

$(h, -b + k)$ $\quad$ $(h, b + k)$

Vertices are: $(-3, -5 + 1) = (-3, -4)$ $\quad$ $(-3, 5 + 1) = (-3, 6)$

Range is $[-4, 6]$

6 Find the x- and y-intercepts.

x-intercepts:

$$y = 0 \Rightarrow 25x^2 + 125x + 129 = 0$$

$$x = \frac{-125 \pm \sqrt{125^2 - 12\,900}}{50}$$

$$= \frac{-15 \pm 4\sqrt{6}}{5}$$

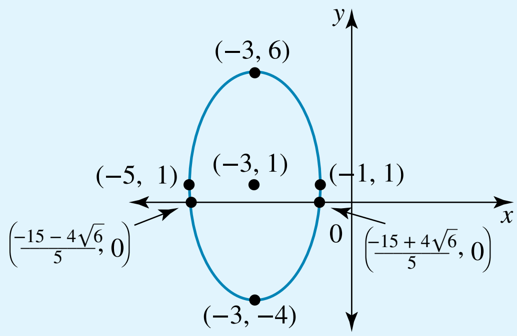
y-intercepts:

$$x = 0 \Rightarrow 4y^2 - 8y + 129 = 0$$

$$\Rightarrow \Delta < 0$$

No y-intercepts.

7 Sketch the graph of the ellipse.



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Locus

A **locus** (plural *loci*) is the path traced by a point that moves according to a condition or rule.

A locus can be defined by a description, for example, '6 units from the x-axis', or by an equation, for example, '$y = x^2 + 4x + 1$'.

WORKED EXAMPLE 12

State the equation(s) of the locus of a point P (x, y) such that:

a P is 3 units from the y-axis

b P is 4 units from the x-axis.

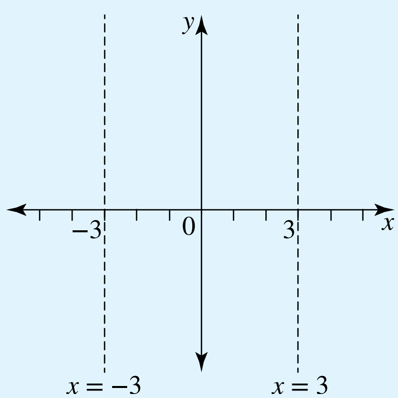
THINK

a 1 Since the point P has to be 3 units from the y-axis, this could be three units in the positive or negative direction.

2 Draw a diagram with points 3 units from the y-axis. These points create two vertical lines.

WRITE/DRAW

a

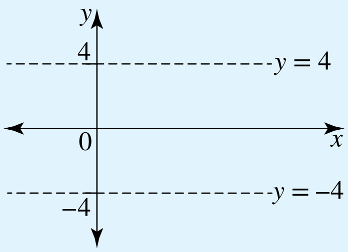


3 Since the lines are vertical, going through 3 and −3, the equations of the loci are $x = -3$ and $x = 3$.

The equations of the loci are: $x = -3$ and $x = 3$.

b 1 Since the point P has to be 4 units from the x-axis, this could be 4 units in the positive or negative direction.

b



2 Draw a diagram with points 4 units from the x-axis. These points create two horizontal lines.

3 Since the lines are horizontal, going through 4 and –4, the equations of the loci are $y = -4$ and $y = 4$.

The equations of the loci are: $y = -4$ and $y = 4$.

Locus from a point

A set of points **equidistant** (the same distance) from a point is represented by a circle.

The equation for a locus that describes a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

The equation for a locus that describes a circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

WORKED EXAMPLE 13

State the equation of each of the following loci.

a **A set of points 3 units from the origin**

b **A set of points 2 units from the point (2, −3)**

THINK	WRITE
a 1 The locus is a set of points equidistant from one particular point, so it is a circle. The set of points is about the origin. Write the equation of a circle with centre (0, 0).	**a** $x^2 + y^2 = r^2$
2 The points are 3 units from the origin, so the radius is 3.	$r = 3$
3 Substitute $r = 3$ into the equation of a circle with centre (0, 0).	$x^2 + y^2 = 3^2$ $x^2 + y^2 = 9$
4 Write the equation of the locus.	The equation of the locus that is a set of points 3 units from the origin is $x^2 + y^2 = 9$.

b 1 The locus is a set of points equidistant from one point, so it is a circle. The locus is about (2, –3), that is, not the origin. Use the formula: $(x - h)^2 + (y - k)^2 = r^2$.

b $(x - h)^2 + (y - k)^2 = r^2$

2 The locus is 2 units from the point (2, –3), so the radius (r) is 2, $h = 2$ and $k = -3$.

$r = 2,\ h = 2,\ k = -3$

3 Substitute $r = 2$, $h = 2$ and $k = -3$ into the formula.

$(x - 2)^2 + (y + 3)^2 = 2^2$

$(x - 2)^2 + (y + 3)^2 = 4$

4 Write the equation of the locus.

The equation of the locus that is a set of points 2 units from the point (2, –3) is $(x - 2)^2 + (y + 3)^2 = 4$.

Distance between two points

The distance between two points (x_1, y_1) and (x_2, y_2) is given by the equation:

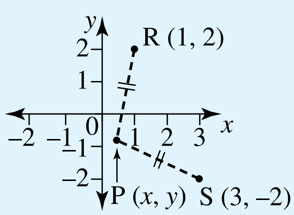
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

WORKED EXAMPLE 14

Find the equation of the locus of P (x, y) given that P is equidistant from R (1, 2) and S (3, –2).

THINK / **WRITE/DRAW**

1 Draw a diagram to show points S and R and the moving point P.



2 Since point P is equidistant from points R and S (PR = PS), the distance formula can be used to derive the equation. Write the formula for the distance between two points.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

3 Identify (x_1, y_1) and (x_2, y_2) for the points S and P.

For S and P: $(x_1, y_1) = (3, -2)$

$(x_2, y_2) = (x, y)$

4 Substitute the known information into the formula.

$d_{PS} = \sqrt{(x - 3)^2 + (y + 2)^2}$

5 Repeat steps 3 and 4 for the points R and P.

For R and P: $(x_1, y_1) = (1, 2)$

$(x_2, y_2) = (x, y)$

$d_{PR} = \sqrt{(x - 1)^2 + (y - 2)^2}$

6 As PS = PR, equate the two expressions.

$\sqrt{(x - 3)^2 + (y + 2)^2} = \sqrt{(x - 1)^2 + (y - 2)^2}$

7 Simplify by first squaring both sides. | $(x-3)^2+(y+2)^2=(x-1)^2+(y-2)^2$

8 Expand both sides. | $x^2-6x+9+y^2+4y+4$
$=x^2-2x+1+y^2-4y+4$

9 Move all terms to the left and simplify to find the equation of the locus of P (x, y). | $x^2-6x+9+y^2+4y+4-x^2$
$+2x-1-y^2+4y-4=0$
$8y-4x+8=0$
$2y-x+2=0$
(or $2y=x-2$, so $y=\frac{1}{2}x-1$)

10 Write the equation of the locus. | The equation of the locus is $y=\frac{1}{2}x-1$.

EXERCISE 11.3 The circle and the ellipse

PRACTISE

1 WE5 Sketch the graphs of the following, stating the centre and radius of each.
a $x^2+y^2=49$
b $x^2+y^2=4^2$

2 Sketch the graphs of the following, stating the centre and radius of each.
a $x^2+y^2=36$
b $x^2+y^2=81$

3 WE6 Sketch the graphs of the following, clearly showing the centre and the radius.
a $(x-1)^2+(y-2)^2=5^2$
b $(x+2)^2+(y+3)^2=6^2$

4 Sketch the graphs of the following, clearly showing the centre and the radius.
a $(x+3)^2+(y-1)^2=49$
b $(x-4)^2+(y+5)^2=64$
c $x^2+(y+3)^2=4$
d $(x-5)^2+y^2=100$

5 WE7 Sketch the graphs of the following circles.
a $x^2+4x+y^2+8y+16=0$
b $x^2-10x+y^2-2y+10=0$

6 Sketch the graphs of the following circles.
a $x^2-14x+y^2+6y+9=0$
b $x^2+8x+y^2-12y-12=0$
c $x^2+y^2-18y-19=0$
d $2x^2-4x+2y^2+8y-8=0$

7 WE8 Sketch the following ellipses.
a $\frac{(x-1)^2}{9}+\frac{(y+2)^2}{4}=1$
b $\frac{(x+5)^2}{25}+\frac{(y-2)^2}{16}=1$

8 Sketch the following ellipses.
a $\frac{(x+5)^2}{49}+\frac{(y+1)^2}{25}=1$
b $\frac{(x-2)^2}{169}+\frac{(y-3)^2}{25}=1$
c $\frac{(x-5)^2}{36}+y^2=1$
d $x^2+9(y+2)^2=9$

9 WE9 Sketch the graph of $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1$.

10 Sketch the graph of $\frac{x^2}{9}+\frac{(y+3)^2}{16}=1$.

11 WE10 Sketch the graph of $9(x-5)^2+16(y+1)^2=144$.

12 Sketch the graph of $16x^2+25y^2=400$.

13 **WE11** Sketch the graph of $9x^2 - 72x + y^2 - 4y + 112 = 0$.

14 Sketch the graph of $9x^2 + 16y^2 + 32y - 128 = 0$.

15 **WE12** State the equation of the locus of a point P (x, y) such that:

a P is 3 units from the y-axis
b P is 5 units from the x-axis.

16 State the equation of the locus of a point P (x, y) such that:

a P is 2 units from $x = 2$
b P is 4 units from $y = -1$.

17 **WE13** State the equation of each of the following loci.

a A set of points 2 units from (0, 0)
b A set of points 5 units from (0, 0)

18 State the equation of each of the following loci.

a A set of points 2 units from (2, 3)
b A set of points 5 units from (2, –1)

19 **WE14** Find the equation of the locus of P (x, y) given that P is equidistant from R and S in each of the following.

a R (0, 2) and S (4, 2)
b R (1, 5) and S (1, –1)

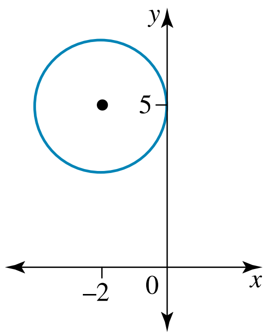
20 Find the equation of the locus of P (x, y) given that P is equidistant from R and S in each of the following.

a R (–1, –1) and S (3, –1)
b R (0, 3) and S (0, 5)

CONSOLIDATE

21 The graph of $(x - 2)^2 + (y + 5)^2 = 4$ is:

A



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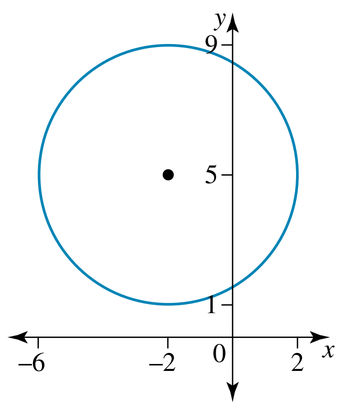
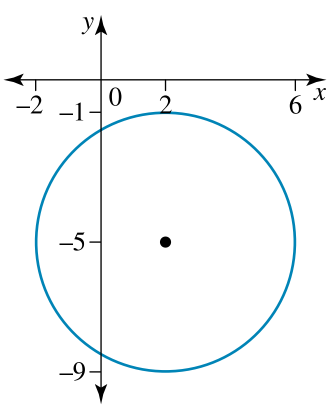
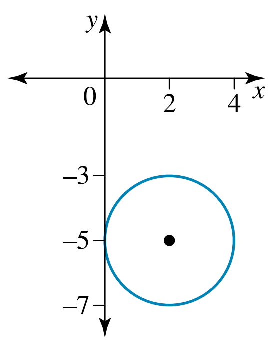


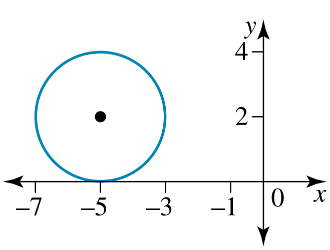
C



D



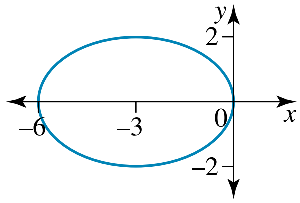
E



22 The centre and radius of the circle $(x + 1)^2 + (y - 3)^2 = 4$ is:

A (1, –3), 4 **B** (–1, 3), 2 **C** (3, –1), 4 **D** (1, –3), 2 **E** (–1, 3), 16

23 The equation of the ellipse at right is:



A $\frac{(x+3)^2}{9} + (y-2)^2 = 1$ **B** $\frac{(x+3)^2}{9} + \frac{y^2}{2} = 1$

C $\frac{(x+3)^2}{9} + \frac{y^2}{4} = 1$ **D** $\frac{(x-3)^2}{9} + y^2 = 1$

E $\frac{(x+3)^2}{9} - \frac{y^2}{4} = 1$

24 The centre of a circle with the equation $(x - 2)^2 + (y + 3)^2 = 18$ is:

A (–2, 3) **B** (0, 0) **C** (–2, –3) **D** (2, –3) **E** (2, 3)

25 The radius of the circle $(x - 2)^2 + y^2 = 9$ is:

A 2 **B** 0 **C** 3 **D** 9 **E** 81

26 Determine the equation of the locus of a point P (x, y) such that:

a P is 10 units from the line $x = 1$ **b** P is 3 units from the line $y = 4$

c P is 2 units from the line $x = 5$ **d** P is 7 units from the line $y = -7$.

27 For a circle with the equation $(x + 4)^2 + (y - 3)^2 = 24$, the centre and radius are respectively:

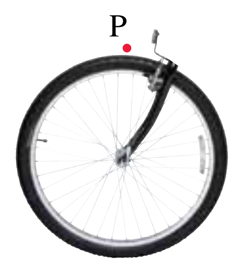
A (4, –3), 24 **B** (–4, 3), 24 **C** (3, –4), $2\sqrt{6}$

D (–4, 3), $2\sqrt{6}$ **E** (–4, 3), $6\sqrt{2}$

28 The distance between (7, 3) and (1, m) is $2\sqrt{10}$. Determine the value of m.

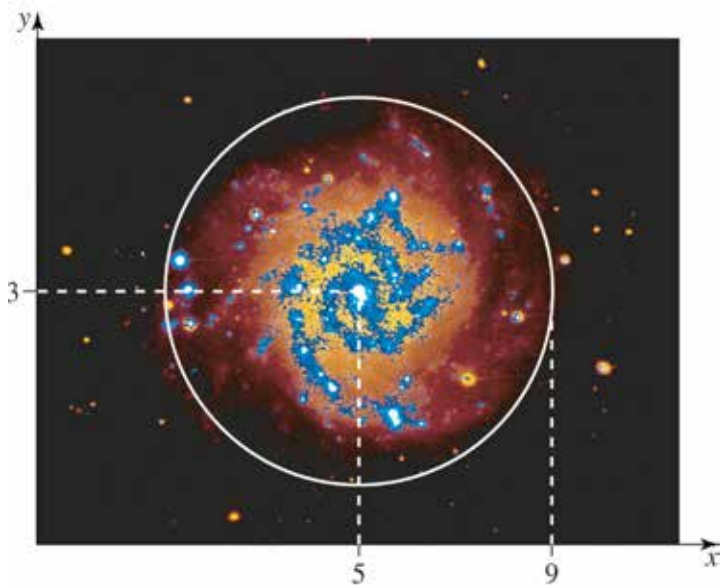
29 A point moves so that its perpendicular distance from the x-axis is equal to its distance from the point (3, 1). Determine the equation of the locus.

30 The photograph at right shows a bicycle wheel. Consider a point P on the edge of the wheel. Describe the locus of the point as it moves along the road.

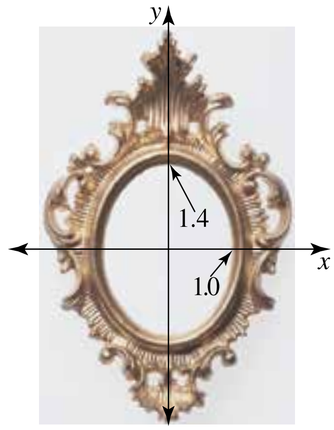


MASTER

31 Find the equation representing the outer edge of the galaxy as shown in the photo below, using the astronomical units provided.



32 In order to program a gemstone cutting machine, a jeweller requires an equation for the edge of the stone based on the coordinate system shown at right (centred at the stone's centre). What is the equation required?



11.4 The hyperbola

studyon

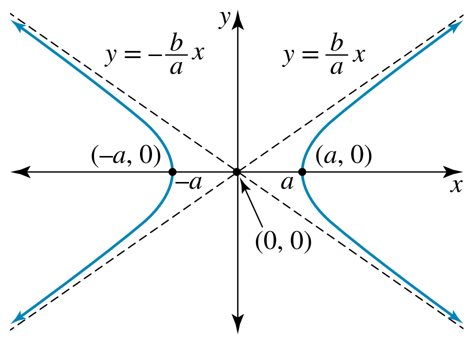
Unit 1 & 2
AOS 5
Topic 3
Concept 5

The hyperbola
Concept summary
Practice questions

Graphs of hyperbolas

Hyperbolas have the following important characteristics.

1. The basic equation of a hyperbola centred at (0, 0) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
2. If this curve were shifted h units to the right and k units up, then the centre would move to (h, k) and its equation would become $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.



3. The basic form of a hyperbola centred at (0, 0) is shown above right.
   The vertices for this curve are at $(-a, 0)$ and $(a, 0)$ and the two asymptotes are given by $y = \frac{-b}{a}x$ and $y = \frac{b}{a}x$.

When the hyperbola is not centred at (0, 0):

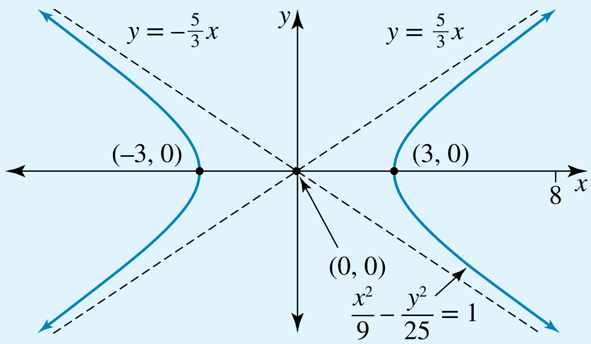
1. **For the curve of the function $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, the points on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are moved h units to the right and k units up (or x has been replaced with $(x - h)$, and y replaced with $(y - k)$).**
2. **Therefore, the vertices are $(-a + h, k)$ and $(a + h, k)$, and the centre is at (h, k).**
3. **The asymptotes are at $y - k = \frac{b}{a}(x - h)$ and $y - k = \frac{-b}{a}(x - h)$ or $y = \frac{b}{a}(x - h) + k$ and $y = \frac{-b}{a}(x - h) + k$.**

To draw sketches of hyperbolic relations:

1. Rearrange the equation into the appropriate general form and determine the values of a and b.
2. Write down the coordinates of the centre.
3. State the coordinates of the vertices.
4. Write down the equations of the asymptotes.
5. Sketch a hyperbolic graph that fits the above information.

WORKED EXAMPLE 15

Sketch the graph of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

THINK	WRITE/DRAW
1 The equation is in the correct form, so read off the values of a, b, h and k.	As $h = 0$, $k = 0$, there are no translations. $a^2 = 9$ $\quad b^2 = 25$ $a = 3$ $\quad b = 5$
2 Write the coordinates of the centre.	The centre is at (0, 0).
3 Write the coordinates of the vertices.	The vertices are (−3, 0) and (3, 0).
4 Write the equations of the asymptotes.	The asymptotes are $y = \frac{-5}{3}x$ and $y = \frac{5}{3}x$.
5 Draw the asymptotes, plot the vertices and centre, and then sketch the hyperbola.	

WORKED EXAMPLE 16

Sketch the graph of the hyperbola with the equation $\frac{(x-3)^2}{16} - \frac{(y-2)^2}{9} = 1$.

THINK	WRITE/DRAW
1 The equation is in the correct form, so read off the values of a, b, h and k.	$h = 3$, $k = 2$ $a^2 = 16$ $\quad b^2 = 9$ $a = 4$ $\quad b = 3$
2 Write the coordinates of the centre.	The centre is (3, 2).
3 Write the coordinates of the vertices.	The vertices are (−4 + 3, 2) and (4 + 3, 2) or (−1, 2) and (7, 2).
4 Write the equations of the asymptotes.	The asymptotes: $y - 2 = \frac{-3}{4}(x-3)$ $\quad y - 2 = \frac{3}{4}(x-3)$ $4(y-2) = -3(x-3)$ $\quad 4(y-2) = 3(x-3)$ $4y - 8 = -3x + 9$ $\quad 4y - 8 = 3x - 9$ $4y + 3x = 17$ $\quad 4y - 3x = -1$
5 For each asymptote find the x- and y-intercepts.	For $4y + 3x = 17$ $\quad$ For $4y - 3x = -1$ $x = 0$, $4y = 17$ $\quad x = 0$, $4y = -1$ $y = \frac{17}{4}$ $\quad y = -\frac{1}{4}$ $\left(0, \frac{17}{4}\right)$ $\quad \left(0, -\frac{1}{4}\right)$ $y = 0$, $3x = 17$ $\quad y = 0$, $-3x = -1$

6 The x- and y-intercepts for $4y - 3x = -1$ are too close to each other so use one of these points, say $\left(\frac{1}{3}, 0\right)$, and the centre to sketch this line — as both asymptotes intersect here.

$x = \dfrac{17}{3}$ $\quad$ $x = \dfrac{1}{3}$

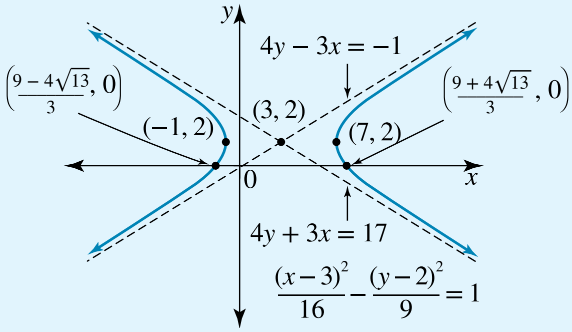
$\left(\dfrac{17}{3}, 0\right)$ $\quad$ $\left(\dfrac{1}{3}, 0\right)$

7 Find the x-intercepts of the function.

$y = 0$: $\dfrac{(x-3)^2}{16} - \dfrac{(-2)^2}{9} = 1$

$x = \dfrac{9 \pm 4\sqrt{13}}{3}$

8 Plot the vertices and centre, and then sketch the hyperbola.



WORKED EXAMPLE 17 **Sketch the graph of the hyperbola with equation $6x^2 - 9(y-2)^2 = 54$.**

THINK / **WRITE/DRAW**

1 Rearrange the equation by dividing both sides by 54 to make the RHS = 1.

$6x^2 - 9(y-2)^2 = 54$

$\dfrac{6x^2}{54} - \dfrac{9(y-2)^2}{54} = \dfrac{54}{54}$

2 Simplify by cancelling.

$\dfrac{x^2}{9} - \dfrac{(y-2)^2}{6} = 1$

3 Read off the values of h and k.

Work out values of a and b.

$h = 0$, $k = 2$, translation of 2 units up

$a^2 = 9 \quad b^2 = 6$

$a = 3 \quad b = \sqrt{6}$ as a and $b > 0$

4 Write the coordinates of the centre.

The centre is at (0, 2).

5 Write the coordinates of the vertices.

The vertices are: $(-3 + 0, 2)$ and $(3 + 0, 2)$ or $(-3, 2)$ and $(3, 2)$.

6 Write the equations of the asymptotes.

The asymptotes are:

$y - 2 = \dfrac{-\sqrt{6}}{3}(x - 0)$ and $\quad y - 2 = \dfrac{\sqrt{6}}{3}x$

$3(y-2) = -\sqrt{6}x \quad 3(y-2) = \sqrt{6}x$

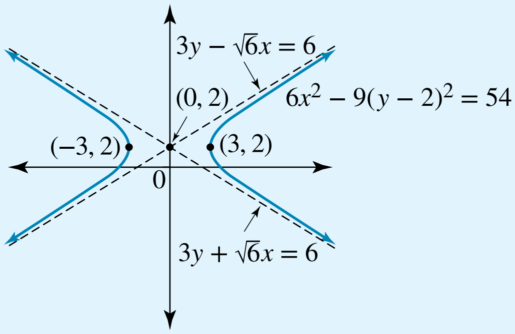
$3y - 6 = -\sqrt{6}x \quad 3y - 6 = \sqrt{6}x$

$3y + \sqrt{6}x = 6 \quad 3y - \sqrt{6}x = 6$

7 Write the x- and y-intercepts for the asymptotes.

Intercepts for $3y + \sqrt{6}x = 6$ are $(\sqrt{6}, 0)$ and $(0, 2)$.
Intercepts for $3y - \sqrt{6}x = 6$ are $(-\sqrt{6}, 0)$ and $(0, 2)$.

8 Draw the asymptotes, plot the vertices and centre, and then sketch the hyperbola.



EXERCISE 11.4 The hyperbola

PRACTISE

1 WE15 Sketch the following hyperbolas, showing the coordinates of the centre, vertices and asymptotes.

a $\frac{x^2}{16} - \frac{y^2}{9} = 1$ b $\frac{x^2}{144} - \frac{y^2}{25} = 1$ c $\frac{x^2}{4} - \frac{y^2}{4} = 1$

2 Sketch the following hyperbolas, showing the coordinates of the centre, vertices and asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{9} = 1$ b $\frac{x^2}{64} - \frac{y^2}{36} = 1$ c $4x^2 - 9y^2 = 36$

3 WE16 Sketch the following hyperbolas.

a $\frac{(x-1)^2}{16} - \frac{y^2}{9} = 1$ b $\frac{(x+3)^2}{144} - \frac{y^2}{25} = 1$ c $\frac{x^2}{9} - \frac{(y+2)^2}{9} = 1$

4 Sketch the following hyperbolas.

a $x^2 - (y-3)^2 = 4$ b $\frac{(x+1)^2}{64} - \frac{(y-2)^2}{36} = 1$

c $4(x-5)^2 - 9(y+3)^2 = 36$

5 WE17 Sketch the graph of the hyperbola with equation:

a $25x^2 - 6y^2 = 400$ b $9x^2 - 16y^2 = 144$.

6 Sketch the graph of the hyperbola with equation:

a $x^2 - y^2 = 25$ b $9x^2 - 25y^2 = 225$.

CONSOLIDATE

7 Given that a hyperbola has vertex points of (4, 0) and (–4, 0), with asymptotes of $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$, the equation of the hyperbola is:

A $\frac{x^2}{4} - \frac{y^2}{2} = 1$ B $\frac{x^2}{4} + \frac{y^2}{2} = 1$ C $\frac{x^2}{64} - \frac{y^2}{16} = 1$

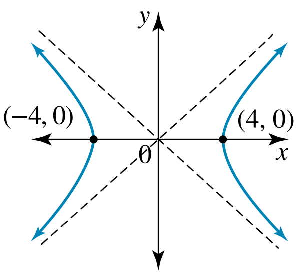
D $\frac{x^2}{16} - \frac{y^2}{4} = 1$ E $\frac{y^2}{2} - \frac{x^2}{4} = 1$

8 The equation of the graph at right is:

A $\frac{x^2}{64} - \frac{y^2}{9} = 1$ B $\frac{x^2}{4} - \frac{y^2}{12} = 1$

C $\frac{x^2}{16} - \frac{y^2}{9} = 1$ D $\frac{x^2}{16} + \frac{y^2}{9} = 1$

E $3x^2 - 4y^2 = 12$



For questions **9–12** below, choose the correct alternative for the equations of the asymptotes and the coordinates of the vertices.

9 $\frac{x^2}{81} - \frac{y^2}{64} = 1$

A $y = \frac{-8}{9}x, y = \frac{8}{9}x, (-9, 0), (9, 0)$

B $y = \frac{-9}{8}x, y = \frac{9}{8}x, (-9, 0), (9, 0)$

C $y = \frac{-8}{9}x, y = \frac{8}{9}x, (-8, 0), (8, 0)$

D $y = \frac{-9}{8}x, y = \frac{9}{8}x, (-8, 0), (8, 0)$

E $y = \frac{-8}{9}x, y = \frac{8}{9}x, (-9, 8), (9, 8)$

10 $\frac{x^2}{36} - \frac{(y-3)^2}{144} = 1$

A $2y = x + 6, 2y = 6 - x, (-6, 3), (6, 3)$

B $y = 2x + 3, y = 3 - 2x, (-6, 3), (6, 3)$

C $y = 2x + 3, y = 3 - 2x, (0, -3), (0, 3)$

D $2y = x + 6, 2y = 6 - x, (-12, 3), (12, 3)$

E $y = 2x + 3, y = 3 - 2x, (-12, 3), (12, 3)$

11 $\frac{(x+3)^2}{9} - \frac{y^2}{4} = 1$

A $2y = 3x - 9, 2y = 9 - 3x, (-6, 0), (0, 0)$

B $3y + 2x - 6 = 0, 3y - 2x + 6 = 0, (-5, 0), (-1, 0)$

C $2y = 3x - 9, 2y = 9 - 3x, (-5, 0), (-1, 0)$

D $2y = 3x - 9, 2y = 9 - 3x, (-3, 3), (-3, -3)$

E $3y + 2x + 6 = 0, 3y - 2x - 6 = 0, (-6, 0), (0, 0)$

12 $\frac{(x-2)^2}{16} - \frac{(y-1)^2}{4} = 1$

A $2y = x, 2y = 4 - x, (0, 1), (4, 1)$

B $y = 2x - 3, y = 5 - 2x, (-2, 1), (6, 1)$

C $2y = x, 2y = 4 - x, (-2, 1), (6, 1)$

D $y = 2x - 3, y = 5 - 2x, (0, 1), (4, 1)$

E $2y = x, 2y = 4 - x, (2, -1), (2, 3)$

13 Sketch the graphs of the hyperbolas with the following equations.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{x^2}{4} - \frac{y^2}{9} = 1$

c $\frac{x^2}{9} - \frac{y^2}{16} = 1$

d $\frac{x^2}{16} - \frac{y^2}{4} = 1$

14 Sketch the graphs of the hyperbolas with the following equations.

a $\frac{(x-2)^2}{9} - \frac{(y-3)^2}{4} = 1$

b $\frac{(x-2)^2}{25} - \frac{(y-3)^2}{4} = 1$

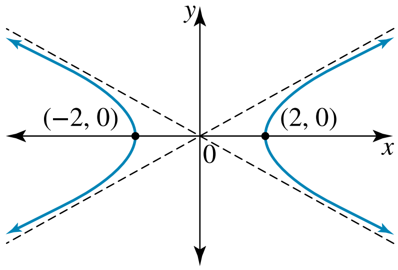
c $\frac{(x-2)^2}{9} - \frac{(y-2)^2}{25} = 1$

d $\frac{x^2}{9} - \frac{(y-1)^2}{25} = 1$

e $\frac{(x-3)^2}{25} - \frac{(y-3)^2}{4} = 1$

f $\frac{(x-1)^2}{25} - \frac{(y+1)^2}{9} = 1$

15 The rule representing the graph shown at right is:



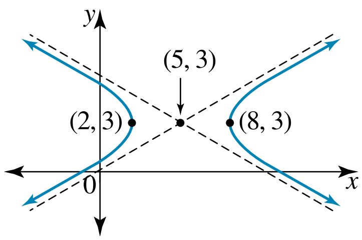
A $x^2 + \frac{y^2}{4} = 1$ **B** $\frac{x^2}{2} - y^2 = 1$

C $\frac{(x + 2)^2}{4} - y^2 = 1$ **D** $\frac{x^2}{4} - y^2 = 1$

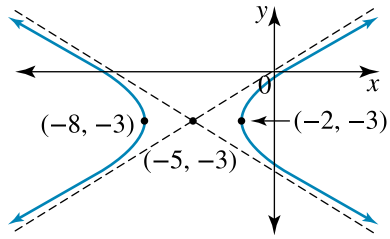
E $\frac{x^2}{4} + y^2 = 1$

16 The graph which best represents the relation $\frac{(x - 5)^2}{9} - \frac{(y - 3)^2}{4} = 1$ is:

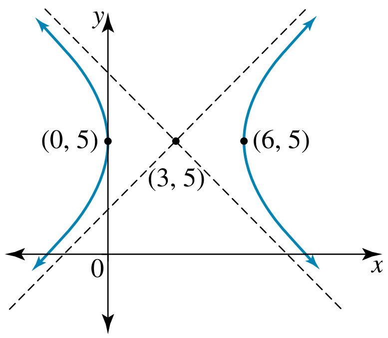
A



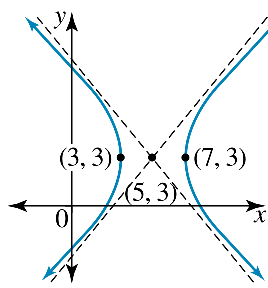
B



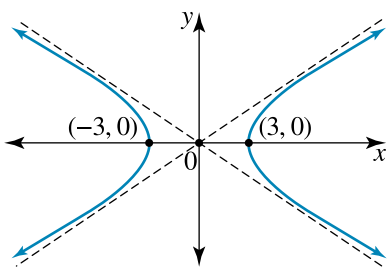
C



D



E



MASTER

17 Sketch the graph of the hyperbola with the equation:

a $16(x - 2)^2 - 9(y + 5)^2 = 144$

b $25(x - 3)^2 - 9(y + 2)^2 = 225$

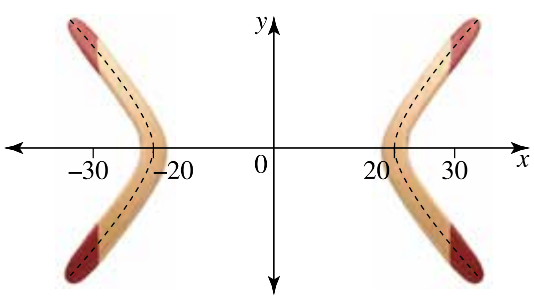
c $36(x - 4)^2 - 4(y - 2)^2 = 144$

d $9x^2 - 16(y + 1)^2 = 144$

e $\frac{(y - 3)^2}{4} - \frac{(x - 3)^2}{25} = 1$

f $\frac{(y + 1)^2}{9} - \frac{(x - 1)^2}{25} = 1$

18 A boomerang manufacturer's specifications for a particular model of boomerang appear at right. Find an equation for the dashed curve drawn through the boomerangs if the equation of one asymptote is $y = \frac{\sqrt{5}}{2}x$.



11.5 Polar coordinates, equations and graphs

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Polar coordinates
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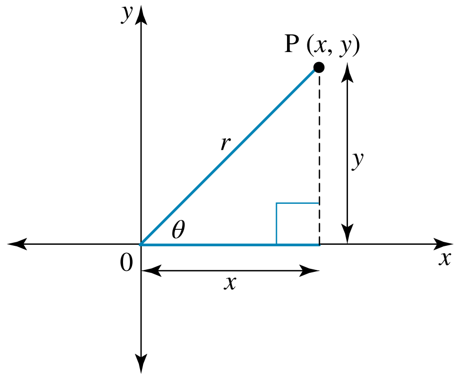
Polar coordinates

In the Cartesian coordinate system, a point, P, is located using (x, y) coordinates. The same point can be located by stating the distance of the point from the origin, the radius, r, and the angle, θ, it makes with the positive x-direction. These are known as **polar coordinates**. We write the polar coordinates of point P as $[r, \theta]$.

Note: θ may be given in degrees or radians.



WORKED EXAMPLE 18

Plot the following polar coordinates.

a $[2, 60°]$ **b** $\left[-3, \frac{2\pi}{3}\right]$

THINK

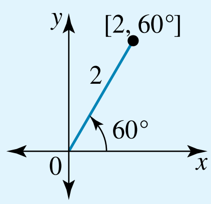
a 1 Draw the positive x-direction.

2 Rotate 60° anticlockwise.

3 Extend the line 2 units.

WRITE/DRAW

a



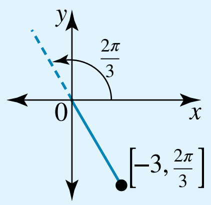
b 1 Draw the positive x-direction.

2 Rotate $\frac{2\pi}{3}$ anticlockwise.

3 Extend the line 3 units in the opposite direction.

Note: $\left[-3, \frac{2\pi}{3}\right]$ is the same as $\left[3, \frac{5\pi}{3}\right]$. Why? Can you find another set of coordinates for the same point?

b



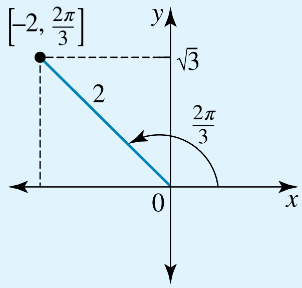
From the initial polar coordinates diagram, by trigonometry:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Hence, we can convert polar coordinates to Cartesian coordinates.

WORKED EXAMPLE 19 Convert $\left[2, \frac{2\pi}{3}\right]$ to Cartesian coordinates.

THINK	WRITE
1 Find the x-coordinate.	$x = r\cos(\theta)$ $= 2\cos\left(\frac{2\pi}{3}\right)$ $= 2 \times -\frac{1}{2}$ $x = -1$
2 Find the y-coordinate.	$y = r\sin(\theta)$ $= 2\sin\left(\frac{2\pi}{3}\right)$ $= 2 \times \frac{\sqrt{3}}{2}$ $y = \sqrt{3}$
3 State the Cartesian coordinates.	Hence, the Cartesian coordinates are $(-1, \sqrt{3})$.



From the initial polar coordinates diagram, by trigonometry:

$$\tan(\theta) = \frac{y}{x}$$

By applying Pythagoras' theorem:

$$r = \sqrt{x^2 + y^2}$$

Hence, we can convert Cartesian coordinates to polar coordinates.

WORKED EXAMPLE 20 Convert $(3, -4)$ to polar coordinates.

THINK	WRITE
1 Find r.	$r = \sqrt{x^2 + y^2}$ $= \sqrt{3^2 + (-4)^2}$ $= \sqrt{9 + 16}$ $= \sqrt{25}$ $r = 5$
2 Find θ.	$\tan(\theta) = \frac{y}{x}$ $= -\frac{4}{3}$ (θ in 4th quadrant)

Note that, as $\tan(\theta)$ is negative, θ is in the fourth quadrant.

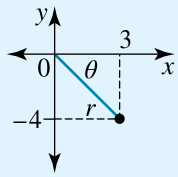
$\theta = \tan^{-1}\left(-\frac{4}{3}\right)$

so $\theta = -53°8'$ or $(360 - 53°8')$

$\theta = -53°8'$ or $306°52'$

3 State the polar coordinates (choose $\theta = 306°52'$ in this case).

Hence, the polar coordinates are $[5, 306°52']$.



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Polar equations and graphs

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Polar equations

A polar equation is an equation written in terms of r and/or θ.

Using the conversions for x and y into polar coordinates,

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

we can change Cartesian equations into polar equations.

WORKED EXAMPLE 21

Convert the following Cartesian equations into polar equations.

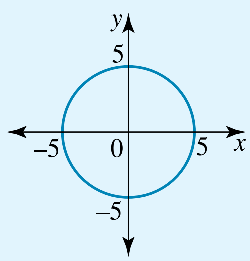
a $x^2 + y^2 = 25$

b $y = 2x$

c $2x - 3y = 5$

d $x^2 + y^2 + 6x - 8y = 0$

e $\frac{x^2}{16} + \frac{y^2}{9} - 1$

THINK	WRITE
a 1 Substitute the polar expressions for x and y.	a $x^2 + y^2 = 25$ Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$ $(r\cos(\theta))^2 + (r\sin(\theta))^2 = 25$
2 Expand and simplify. (Use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$.)	$(r^2\cos^2(\theta) + r^2\sin^2(\theta)) = 25$ $r^2(\cos^2(\theta) + \sin^2(\theta)) = 25$ $r^2 = 25$ $r = 5$
3 Alternatively, because $x^2 + y^2 = 25$ represents a circle of radius 5 units, the polar equation must be $r = 5$.	

b 1 Substitute the polar expressions for x and y.

b $y = 2x$

Since $x = r\cos(\theta)$

and $y = r\sin(\theta)$

then $r\sin(\theta) = 2r\cos(\theta)$

2 Divide both sides by $r\cos(\theta)$ and recall the identity $\dfrac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$.

$$\frac{\sin(\theta)}{\cos(\theta)} = 2$$

$$\tan(\theta) = 2 \text{ since } \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

3 Isolate θ.

$$\theta = \tan^{-1}(2)$$

$$\theta = 63°26' \text{ or } 1.107^c$$

c 1 Substitute the polar expressions for x and y.

c $2x - 3y = 5$

Since $x = r\cos(\theta)$

and $y = r\sin(\theta)$

then $2r\cos(\theta) - 3r\sin(\theta) = 5$

2 Simplify.

$$r(2\cos(\theta) - 3\sin(\theta)) = 5$$

$$r = \frac{5}{2\cos(\theta) - 3\sin(\theta)}$$

d 1 Substitute the polar expressions for x and y.

d $x^2 + y^2 + 6x - 8y = 0$

Since $x = r\cos(\theta)$

and $y = r\sin(\theta)$

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) + 6r\cos(\theta) - 8r\sin(\theta) = 0$$

$$r^2(\cos^2(\theta) + \sin^2(\theta)) + r(6\cos(\theta) - 8\sin(\theta)) = 0$$

2 Note that $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$r^2 + r(6\cos(\theta) - 8\sin(\theta)) = 0$$

3 Divide both sides by r.

$$r + 6\cos(\theta) - 8\sin(\theta) = 0$$

Hence, $r = 8\sin(\theta) - 6\cos(\theta)$.

e 1 Substitute the polar expressions for x and y.

e $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

Since $x = r\cos(\theta)$

and $y = r\sin(\theta)$

$$\frac{(r\cos(\theta))^2}{16} + \frac{(r\sin(\theta))^2}{9} = 1$$

2 Simplify.

$$\frac{r^2\cos^2(\theta)}{16} + \frac{r^2\sin^2(\theta)}{9} = 1$$

$$\frac{9r^2\cos^2(\theta) + 16r^2\sin^2(\theta)}{144} = 1$$

$$r^2(9\cos^2(\theta) + 16\sin^2(\theta)) = 144$$

$$r^2 = \frac{144}{9\cos^2(\theta) + 16\sin^2(\theta)}$$

Similarly, polar equations can be changed to Cartesian form.

WORKED EXAMPLE 22

Convert the following polar equations into Cartesian equations.

a $r = 4\cos(\theta)$

b $\tan(\theta) = 2$

c $r = \dfrac{2}{1 + \sin(\theta)}$

THINK	WRITE
a 1 Find r^2 by multiplying both sides of the equation by r.	**a** $r = 4\cos(\theta)$ $r^2 = 4r\cos(\theta)$
2 Substitute the Cartesian expressions for r and θ.	Since $r^2 = x^2 + y^2$ and $x = r\cos(\theta)$ then $x^2 + y^2 = 4x$
3 Simplify by 'completing the square'. *Note:* This is the equation of a circle of radius 2 units and centre (2, 0).	$x^2 - 4x + y^2 = 0$ $x^2 - 4x + 4 - 4 + y^2 = 0$ $(x-2)^2 + y^2 = 4$
b 1 Substitute $\tan(\theta) = \dfrac{y}{x}$.	**b** $\tan(\theta) = 2$ As $\tan(\theta) = \dfrac{y}{x}$, $\dfrac{y}{x} = 2$
2 Simplify by making y the subject.	$y = 2x$
c 1 Simplify the equation by multiplying both sides of the equation by $(1 + \sin(\theta))$.	**c** $r = \dfrac{2}{1 + \sin(\theta)}$ $r(1 + \sin(\theta)) = 2$ $r + r\sin(\theta) = 2$
2 Substitute the Cartesian expressions for r and θ.	Since $y = r\sin(\theta)$ $r + y = 2$
3 Make r the subject.	$r = 2 - y$
4 Find r^2 by squaring both sides.	$r^2 = (2 - y)^2$
5 Substitute for r^2.	Since $r^2 = x^2 + y^2$ $x^2 + y^2 = (2 - y)^2$
6 Expand and simplify. *Note:* This is the equation of a parabola.	$= 4 - 4y + y^2$ $x^2 = 4 - 4y$ $4y = 4 - x^2$ $y = 1 - \dfrac{x^2}{4}$

Polar graphs

Polar equations can be graphed using polar coordinates. This is often a better alternative than converting polar equations to the sometimes more complicated Cartesian equation form.

When using polar equations, θ is assumed to be measured in radians.

WORKED EXAMPLE 23 Sketch the graph of $r = \theta$ for $0 < \theta < 4\pi$ using a calculator.

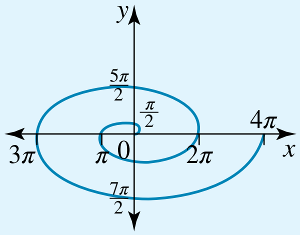
THINK

On a calculator, complete the entry line as:

$$\begin{cases} r1(\theta) = \theta \\ 0 \leq \theta \leq 4\pi \ \theta step = 0.13 \end{cases}$$

The increment step of $\frac{\pi}{6}$ (0.13) was used as a default.

WRITE/DRAW



WORKED EXAMPLE 24 Sketch the graph of $r = 8$ for $0 \leq \theta \leq 2\pi$.

THINK

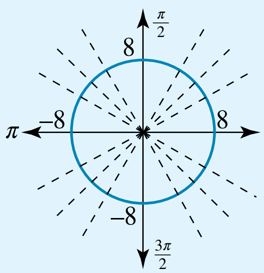
1 Construct a table of values for $0 \leq \theta \leq 2\pi$ and find the corresponding r values.

WRITE/DRAW

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
r	8	8	8	8	8	8

θ	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	8	8	8	8	8	8	8

2 Sketch the graph, using a protractor and ruler to plot each of the points from the table. Remember r is the distance from the centre (the origin).



WORKED EXAMPLE 25 Sketch the graph of $r = 2 \sin(\theta)$ for $0 \leq \theta \leq 2\pi$ using a calculator.

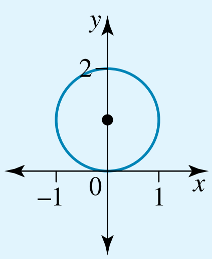
THINK

On a calculator, complete the entry line as:

$$\begin{cases} r1(\theta) = 2\sin(\theta) \\ 0 \leq \theta \leq 2\pi \ \theta\text{step} = \frac{\pi}{12} \end{cases}$$

Zoom out to see the whole circle as needed.

WRITE/DRAW



EXERCISE 11.5 Polar coordinates, equations and graphs

PRACTISE

1 WE18 Plot the following polar coordinates on the same graph.

a $[2, 0°]$ b $[5, 180°]$
c $[0.5, 270°]$ d $[3, 90°]$

2 Plot the following polar coordinates on the same graph.

a $\left[1, \frac{\pi}{3}\right]$ b $\left[1, -\frac{\pi}{3}\right]$ c $\left[1, \frac{2\pi}{3}\right]$ d $\left[1, \frac{5\pi}{3}\right]$

3 WE19 Convert the following to Cartesian coordinates.

a $[2, 45°]$ b $[5, 30°]$ c $[3, 60°]$
d $[2.7, 90°]$ e $[1.5, 120°]$ f $[12, 210°]$

4 Convert the following to Cartesian coordinates.

a $\left[2.6, \frac{\pi}{2}\right]$ b $[7.8, \pi]$ c $\left[10, \frac{\pi}{3}\right]$
d $\left[9.1, \frac{5\pi}{3}\right]$ e $[25, 5.8^c]$ f $[1.6, 4^c]$

5 WE20 Convert the following Cartesian coordinates into polar coordinates. Express θ in parts **a**–**e** in degrees and part **f** in radians.

a $(5, 0)$ b $(0, 4.3)$ c $(-30, 0)$
d $(0, -9)$ e $(6, 6)$ f $(-4, 3)$

6 Convert the following Cartesian coordinates into polar coordinates.

a $(-5, -12)$ b $(6, -8)$ c $(-1, \sqrt{3})$
d $(-2, -2)$ e $(2\sqrt{3}, -2)$ f $(5, 6)$

7 WE21 Convert the following Cartesian equations into polar equations.

a $x = 3$ b $y = 2$ c $x^2 + y^2 = 9$ d $x^2 + y^2 = 36$

8 Convert the following Cartesian equations into polar equations.

a $y = 5x$ b $y = x$ c $3x - 4y = 1$ d $5x + y = 7$

9 WE22 Convert the following polar equations into Cartesian equations.

a $r = 2$ b $r = 5$ c $r = 6\sin(\theta)$ d $r = 2\cos(\theta)$

10 Convert the following polar equations into Cartesian equations.

a $\tan(\theta) = 3$ b $\tan(\theta) = -4$ c $\theta = \frac{\pi}{4}$ d $\theta = \frac{3\pi}{4}$

11 WE23 Using a calculator or other method, sketch the graph of each of the following polar equations for $0 \le \theta \le 4\pi$.

a $r = \theta$ b $r = -\theta$

12 Using a calculator or other method, sketch the graph of each of the following polar equations for $0 \le \theta \le 4\pi$.

a $r = 2\theta$ b $r = -\frac{1}{2}\theta$

13 WE24 Using a calculator or other method, sketch the graph of each of the following polar equations for $0 \le \theta \le 2\pi$.

a $r = 2$ b $r = 4$

14 Using a calculator or other method, sketch the graph of each of the following polar equations for $0 \le \theta \le 2\pi$.

a $r = 1.5$ b $r = -2$

15 WE25 Using a calculator or other method, sketch the graph of each of the following polar equations for $0 \le \theta \le 2\pi$.

a $r = \sin(\theta)$

b $r = 1.5 \sin(\theta)$

16 Using a calculator or other method, sketch the graph of each of the following polar equations for $0 \le \theta \le 2\pi$.

a $r = -3 \sin(\theta)$

b $r = 4 \sin(\theta)$

CONSOLIDATE

17 Locate each of the following points on the same graph.

a $[3, 45°]$

b $[4, 100°]$

c $[1, 300°]$

d $[2.5, -30°]$

18 Convert the following Cartesian equations into polar equations.

a $x^2 + y^2 - 10x + 6y = 0$

b $x^2 + y^2 + 6x + 8y = 0$

c $x^2 + y^2 - 12y = 0$

d $x^2 + y^2 - 2x = 0$

e $\frac{x^2}{9} + \frac{y^2}{4} = 1$

f $\frac{x^2}{4} + \frac{y^2}{25} = 1$

19 Convert the following polar equations into Cartesian equations.

a $r \cos(\theta) = 4$

b $r \sin(\theta) = -1$

c $r = 4 \sin(\theta) - 2 \cos(\theta)$

d $r = 6 \sin(\theta) + 8 \cos(\theta)$

e $r = \frac{3}{1 + \sin(\theta)}$

f $r = \frac{4}{1 - \cos(\theta)}$

20 Using a calculator or other method, sketch the following polar equations for $0 \le \theta \le 2\pi$.

a $r = \cos(\theta)$

b $r = 2 \cos(\theta)$

c $r = 3 \cos(\theta)$

d $r = -4 \cos(\theta)$

e Hence, predict the graph of $r = -0.5 \cos(\theta)$. Check your prediction on the calculator.

21 Using a calculator or other method, sketch $r = 3 \sin(\theta) + 4 \cos(\theta)$ for $0 \le \theta \le 2\pi$.

a Comment on the shape of the curve.

b State:

i the y-intercepts

ii the x-intercepts

iii the length of the diameter

iv the length of the radius

v the coordinates of the centre

vi the Cartesian equation of the curve.

22 Using a calculator or other method, sketch $r = 5 \sin(\theta) + 12 \cos(\theta)$ for $0 \le \theta \le 2\pi$.

a Comment on the shape of the curve.

b State:

i the y-intercept(s)

ii the x-intercept(s)

iii the length of the diameter

iv the length of the radius

v the coordinates of the centre

vi the Cartesian equation of the curve.

23 Using a calculator or other method, plot the graph of $r = 1 - \cos(\theta)$ for $0 \le \theta \le 2\pi$. Hence, investigate the graphs:

a $r = \cos(\theta) - 1$
b $r = 1 - \sin(\theta)$
c $r = 4(\sin(\theta) - 1)$.

24 Using a calculator, plot the graph of $r = 1 + 2\sin(\theta)$ for $0 \le \theta \le 2\pi$. Hence, investigate the following graphs of the type $r = b + a\sin(\theta)$ where $a > b$.

a $r = 1 + 3\sin(\theta)$
b $r = 1 + 6\sin(\theta)$
c $r = 2 + 6\sin(\theta)$
d $r = 1 + 2\cos(\theta)$
e $r = 1 - 2\sin(\theta)$
f Hence, predict the graph of $r = 1 - 3\cos(\theta)$.

25 Using a calculator, plot the graph $r = \sin(2\theta)$ for $0 \le \theta \le 2\pi$. Hence, investigate the following graphs.

a $r = \sin(3\theta)$
b $r = \sin(4\theta)$
c $r = \cos(2\theta)$
d $r = 2\sin(3\theta)$
e $r = -2\sin(3\theta)$
f Hence, predict the graph of $r = 4\cos(3\theta)$.

26 Compare and comment on the graphs of the polar equations $r = 2 - 4\sin(\theta)$ and $r = -2 - 4\sin(\theta)$.

MASTER

27 Sketch the graph of the 'Spiral of Archimedes', $r = \theta$ for $0 < \theta \le 2\pi$.

28 Use technology to investigate graphs of the following polar equations. Replace a and b with actual values (such as 1, 2, 3, 4 and so on) and hence comment on the effect of a and b on each graph.

	Equations	Name of graph
a	$r = a + b\sin(\theta)$	Limacon
b	$r = a\sin(b\theta)$	Four leaf rose
c	$r = 2a(1 + \cos(\theta))$	Cardioid
d	$r = \dfrac{a}{\theta}$	Hyperbolic spiral
e	$r = \dfrac{a\cos(2\theta)}{\cos(\theta)}$	Right strophoid
f	$r^2 = a^2\cos(2\theta)$ You may need to plot two graphs here, namely $r = a\sqrt{\cos(2\theta)}$ and $r = -a\sqrt{\cos(2\theta)}$.	Lemniscate of Bernoulli
g	$r = \dfrac{2a\,\theta}{\pi\sin(\theta)}$	Quadratrix of Hippias

11.6 Parametric equations

Parametric equations of a curve express the coordinates of the point of a curve as functions of a variable called a parameter.

By using parametric equations, these curves can be interpreted as graphs of functions.

For example, the graph of an ellipse in the Cartesian plane cannot be expressed as a function $y = f(x)$, since some values of x are associated with more than one value of y (a relation).

By viewing the curve as the path traced out by the point P[$x(t)$, $y(t)$] as the independent variable, t, ranges through some interval, $x(t)$ and $y(t)$ are expressed as functions of the single variable, t.

Simple algebraic techniques can be used to convert a set of parametric equations into Cartesian form.

The following worked example shows how to eliminate the parameter and form a single Cartesian equation involving only x and y.

WORKED EXAMPLE 26

Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = t, y = t + 2$

b $x = t + 1, y = t^2$.

THINK	WRITE
a 1 Write the parametric equations and number them.	**a** $x = t$ [1] $y = t + 2$ [2]
2 Substitute equation [1] into equation [2]. That is, replace t with x in the second equation. This produces a Cartesian equation of a straight line.	Substituting [1] into [2] gives $y = x + 2$.
b 1 Write the parametric equations and number them.	**b** $x = t + 1$ [1] $y = t^2$ [2]
2 Rearrange equation [1] to write t in terms of x.	Equation [1] becomes $t = x - 1$.
3 Substitute this expression for t into equation [2]. This produces a Cartesian equation of a parabola.	Substituting $t = x - 1$ into [2] gives $y = (x - 1)^2$.

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Let us now look at some examples that involve another parameter. When parametric equations involve trigonometric functions, we often need to make use of standard trigonometric identities to eliminate the parameter. For example, consider Pythagorean identities, addition identities, double-angle identities and factorisation identities.

1. The Pythagorean identities
   $\sin^2(\theta) + \cos^2(\theta) = 1$
   $\sec^2(\theta) = 1 + \tan^2(\theta)$
   $\operatorname{cosec}^2(\theta) = 1 + \cot^2(\theta)$
2. The addition identities
   $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
   $\cos(x \pm y) = \cos(x)\cos(y) \pm \sin(x)\sin(y)$
   $$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \pm \tan(x)\tan(y)}$$
3. The double-angle identities
   $\sin(2x) = 2\sin(x)\cos(x)$
   $\cos(2x) = \cos^2(x) - \sin^2(x)$
   $\cos(2x) = 2\cos^2(x) - 1$

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

4. The factorisation identities

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

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AOS 5
Topic 3
Concept 9

Parametric equations of circles, ellipses and hyperbolas
Concept summary
Practice questions

WORKED EXAMPLE 27

Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = 3\cos(\theta)$, $y = 2\sin(\theta)$

b $x = \sin(\theta)$, $y = \cos(2\theta)$.

THINK	WRITE
a 1 Write the parametric equations and number them.	**a** $x = 3\cos(\theta)$ [1] $y = 2\sin(\theta)$ [2]
2 Square equation [1] and then rearrange it to isolate the trigonometric function. Number the equation.	Squaring [1] gives $x^2 = 9\cos^2(\theta)$ or $\frac{x^2}{9} = \cos^2(\theta)$ [3]
3 Square equation [2] and then rearrange it to isolate the trigonometric function. Number the equation.	Squaring [2] gives $y^2 = 4\sin^2(\theta)$ or $\frac{y^2}{4} = \sin^2(\theta)$ [4]
4 Add equations [3] and [4].	Adding [3] and [4] gives $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2(\theta) + \sin^2(\theta)$
5 Use the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to simplify. This produces a Cartesian equation of an ellipse.	$\frac{x^2}{9} + \frac{y^2}{4} = 1$
b 1 Write the parametric equations and number them.	**b** $x = \sin(\theta)$ [1] $y = \cos(2\theta)$ [2]
2 Use the double-angle identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ to rewrite equation [2]. Number the equation.	Since $\cos(2\theta) = 1 - \sin^2(\theta)$, [2] becomes $y = 1 - 2\sin^2(\theta)$ [3]
3 Substitute equation [1] into equation [3]. This produces a Cartesian equation of a parabola.	Substituting [1] into [3] gives $y = 1 - 2x^2$.

EXERCISE 11.6 Parametric equations

PRACTISE

1 **WE26** Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = t,\ y = t - 7$

b $x = t + 1,\ y = t^2 + 4$.

2 Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = t,\ y = \dfrac{1}{t}$

b $x = t^3,\ y = t^2$.

3 **WE27** Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = \sin(\theta),\ y = \cos(\theta)$

b $x = 4\cos(\theta),\ y = 2\sin(\theta)$.

4 Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = \cos(\theta),\ y = \sin(\theta)$

b $x = 2\sec(\theta),\ y = \tan(\theta)$.

CONSOLIDATE

For questions **5** to **14**, find the Cartesian equation of the curve with the given parametric equation.

5 $x = 2t + 3,\ y = 4t^2 - 9$

6 $x = t^2 + t,\ y = t^2 - t$

7 $x = \dfrac{2t}{1 + t^2},\ y = \dfrac{1 - t^2}{1 + t^2}$.

8 $x = -\sec(\theta),\ y = \tan(\theta)$

9 $x = \cos(2\theta),\ y = \sin(\theta)$

10 $x = t - \sin(t),\ y = 3 - 2\cos(t)$

11 $x = \cos(\theta),\ y = \sin(2\theta)$

12 $x = \sec^2(\theta) - 1,\ y = \tan(\theta)$

13 $x = \cos(2\theta),\ y = \sin(2\theta)$

14 $x = 2\cos^3(\theta),\ y = 2\sin^3(\theta)$.

MASTER

15 a Find the Cartesian equation of the curve whose parametric equations are given by:

$$x = t\cos(t),\ y = t\sin(t),\ t > 0.$$

b The curve described by the parametric equations in part **a** is called a spiral. Using technology, sketch the graph of this spiral for $0 < t < 4\pi$.

16 Lissajous figures are sometimes used in graphic design as logos or as screen savers. The general parametric equations to describe a Lissajous figure are:

$$x = A\sin(at + c),\ y = B\sin(bt).$$

Using technology, sketch the graph of a Lissajous figure for $A = 4$, $B = 4$, $a = 1,\ b = 3$ and $c = \dfrac{\pi}{2}$.

Do you recognise this shape as a well-known logo?

Try some different values for A, B, a, b and c.

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11.7 Review

www.jacplus.com.au

The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.

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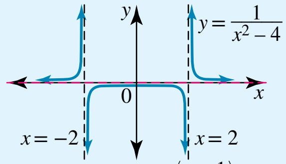
Units 1 & 2 — **Linear and non-linear relationships**

Sit topic test

11 Answers

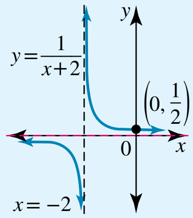
EXERCISE 11.2

1 a

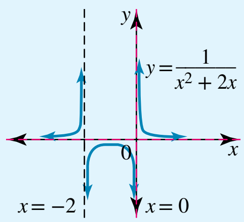


Turning point $\left(0, -\frac{1}{4}\right)$

b

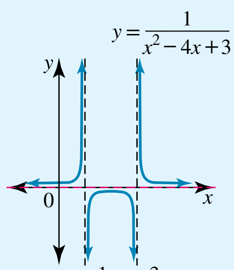


2 a



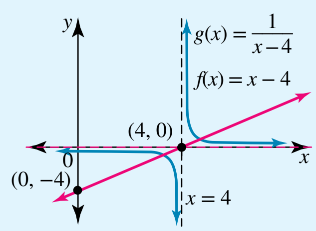
Turning point $(-1, -1)$

b

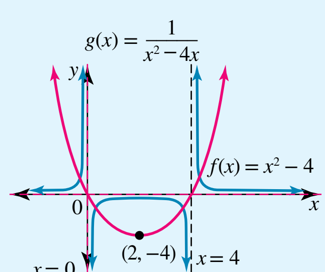


$x = 1$ $x = 3$
Turning point $(2, -1)$

3

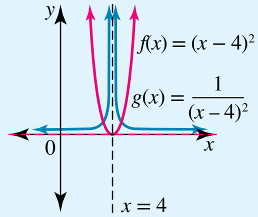


4



Turning point $\left(2, -\frac{1}{4}\right)$

5



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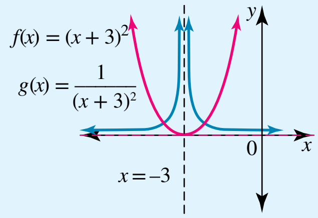

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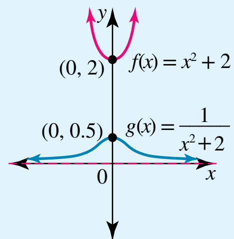
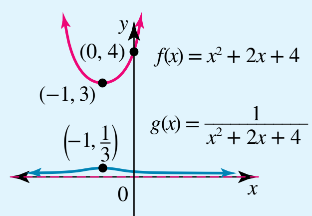


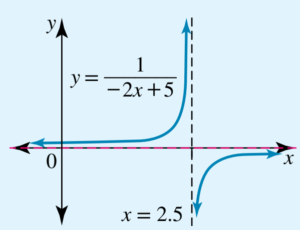
Turning point $\left(0, \frac{1}{2}\right)$

8

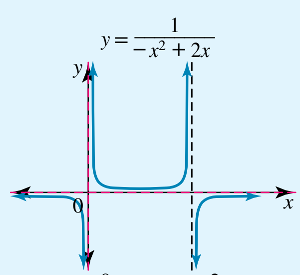


Turning point $\left(-1, \frac{1}{3}\right)$

9 a

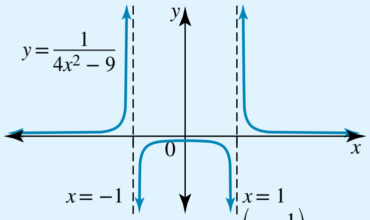


b



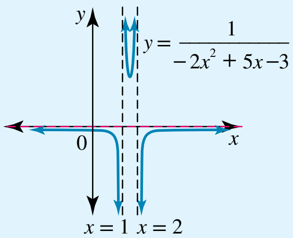
$x = 0$ $x = 2$
Turning point $(1, 1)$

10 a



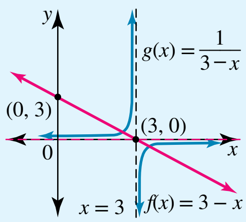
Turning point $\left(0, -\frac{1}{9}\right)$

b

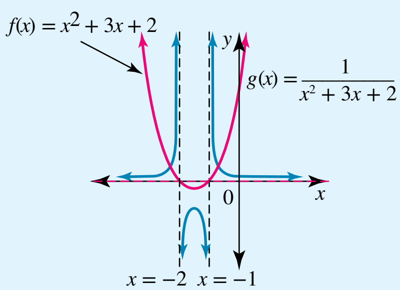


Turning point $\left(\frac{5}{4}, 8\right)$

11

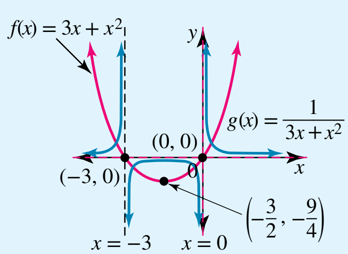


12



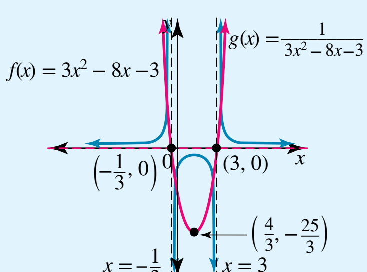
Turning point $\left(-\frac{3}{2}, -4\right)$

13



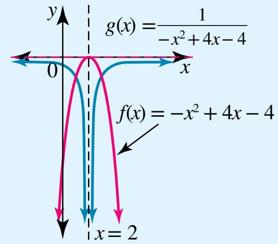
Turning point $\left(-\frac{3}{2}, -\frac{4}{9}\right)$

14

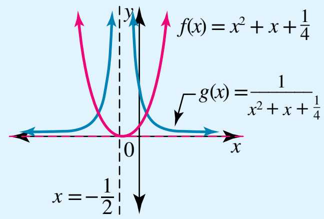


Turning point $\left(\frac{4}{3}, -\frac{3}{25}\right)$

15



16



17 a B

b D

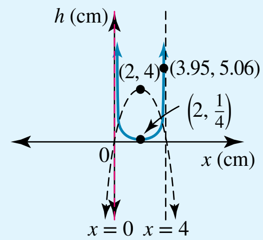
18 E

19 a $A = 4x - x^2$

b $h = \frac{1}{(4x - x^2)}$

c ~5.06 cm

d

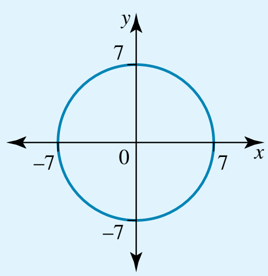


e Min. $h = 0.25$ cm when length = width = 2 cm

20 $a = 1$, $b = -8$, $c = 15$

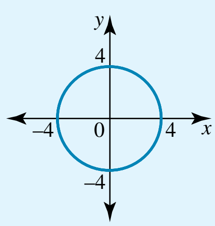
EXERCISE 11.3

1 a



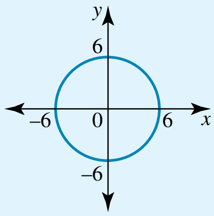
Centre (0, 0); radius 7

b



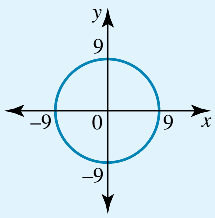
Centre (0, 0); radius 4

2 a



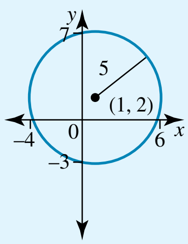
Centre (0, 0); radius 6

b

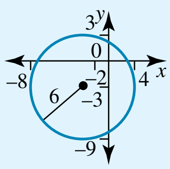


Centre (0, 0); radius 9

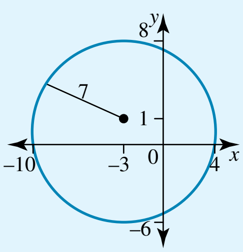
3 a



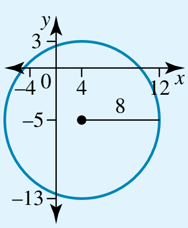
b



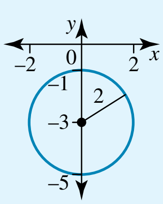
4 a



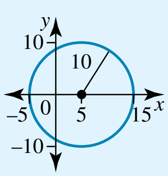
b



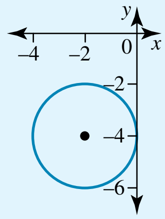
c



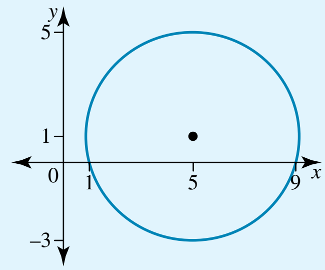
d



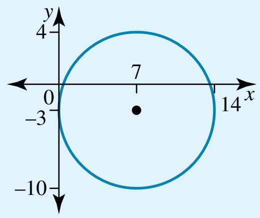
5 a $(x + 2)^2 + (y + 4)^2 = 2^2$



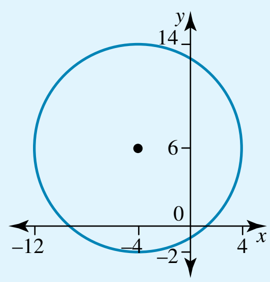
b $(x - 5)^2 + (y - 1)^2 = 4^2$



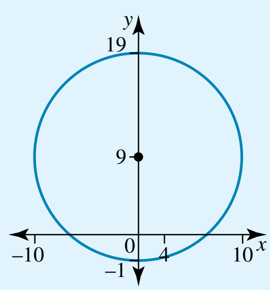
6 a $(x - 7)^2 + (y + 3)^2 = 7^2$



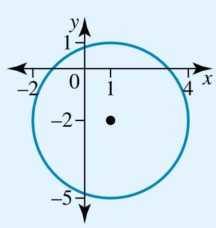
b $(x + 4)^2 + (y - 6)^2 = 8^2$



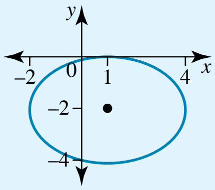
c $x^2 + (y - 9)^2 = 10^2$



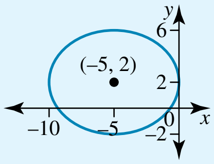
d $(x - 1)^2 + (y + 2)^2 = 3^2$



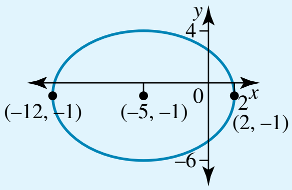
7 a



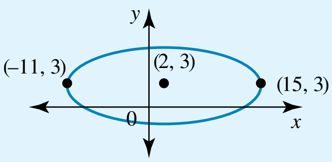
b



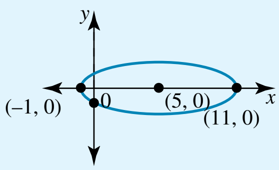
8 a



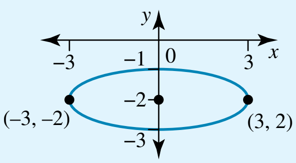
b



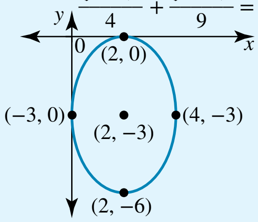
c



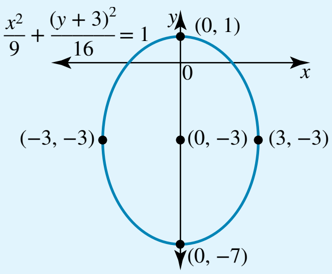
d



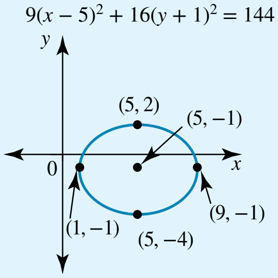
9 $\frac{(y-2)^2}{4} + \frac{(y+3)^2}{9} = 1$



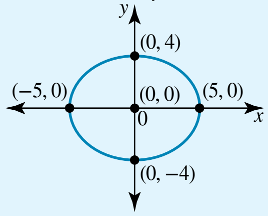
10 $\frac{x^2}{9} + \frac{(y+3)^2}{16} = 1$



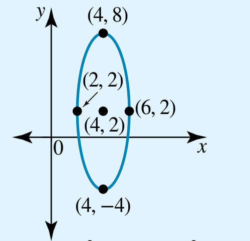
11 $9(x-5)^2 + 16(y+1)^2 = 144$



12 $16x^2 + 25y^2 = 400$

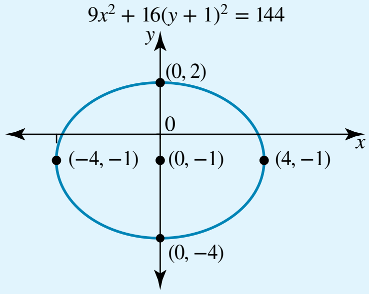


13



$36(x-4)^2 + 4(y-2)^2 = 144$

14 $9x^2 + 16(y+1)^2 = 144$



15 a $x = 3$ or $x = -3$

b $y = 5$ or $y = -5$

16 a $x = 4$ or $x = 0$

b $y = 3$ or $y = -5$

17 a $x^2 + y^2 = 4$

b $x^2 + y^2 = 25$

18 a $(x-2)^2 + (y-3)^2 = 4$

b $(x-2)^2 + (y+1)^2 = 25$

19 a $x = 2$

b $y = 2$

20 a $x = 1$

b $y = 4$

21 D

22 B

23 C

24 D

25 C

26 a $x = 11$ or $x = -9$

b $y = 7$ or $y = 1$

c $x = 3$ or $x = 7$

d $y = 0$ or $y = -14$

27 D

28 $m = 1$ or $m = 5$

29 $y = \frac{1}{2}x^2 - 3x + 5$

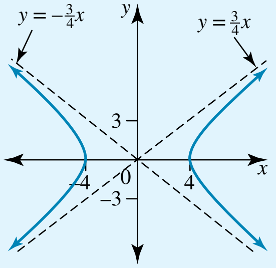
30 Circular

31 $(x-5)^2 + (y-3)^2 = 16$

32 $x^2 + \frac{y^2}{1.4^2} = 1$

EXERCISE 11.4

1 a



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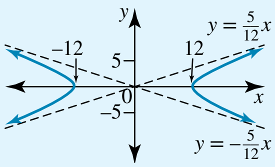

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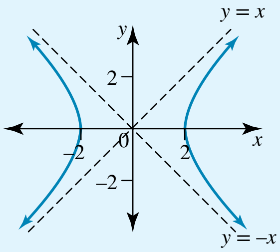
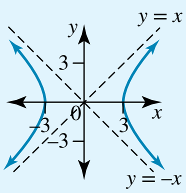


2 a



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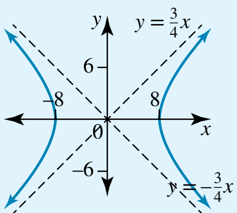

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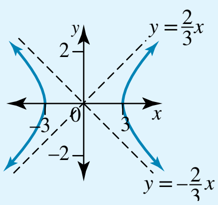
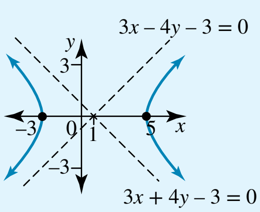


3 a



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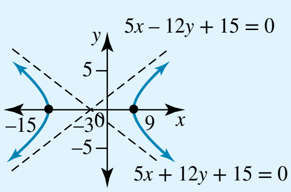

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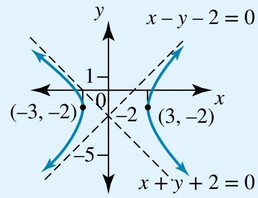
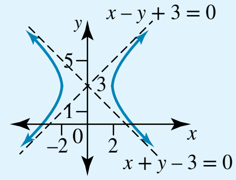


4 a



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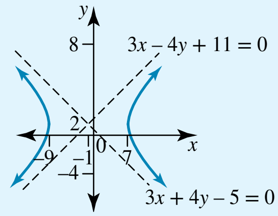

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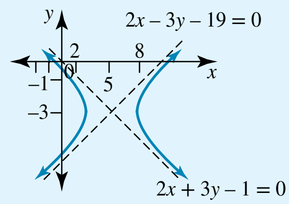
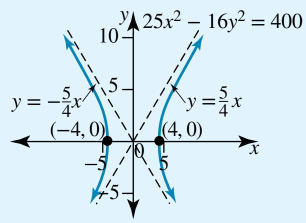


5 a



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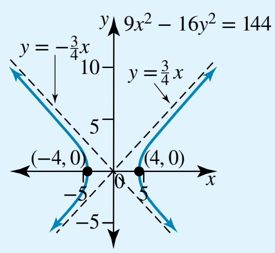
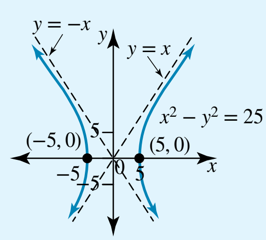
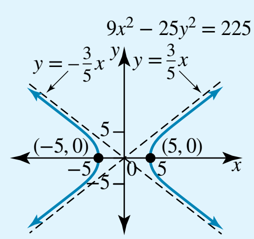


6 a



b



7 D

8 C

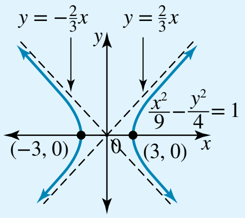
9 A

10 B

11 E

12 C

13 a



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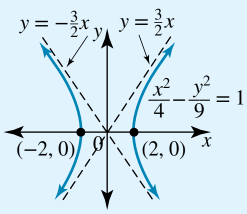

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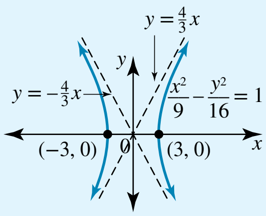
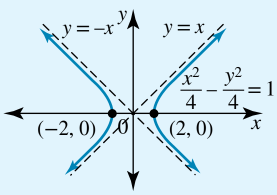
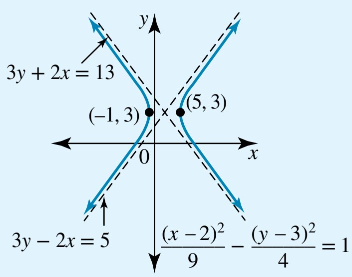


d



14 a



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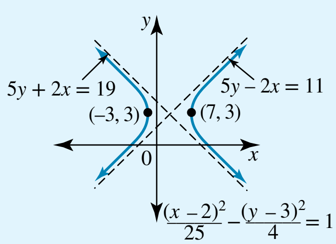

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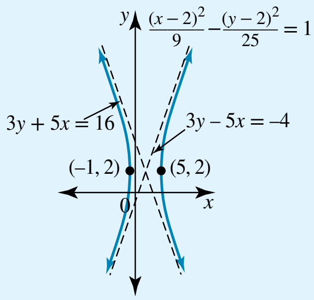

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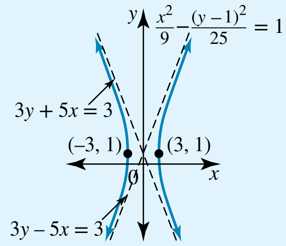
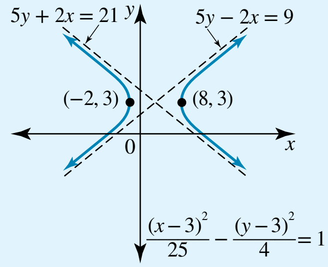
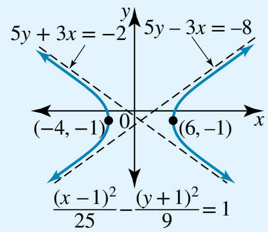


e



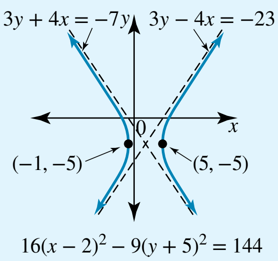
f



15 D

16 A

17 a



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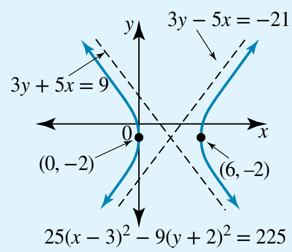

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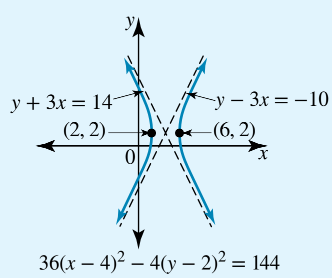

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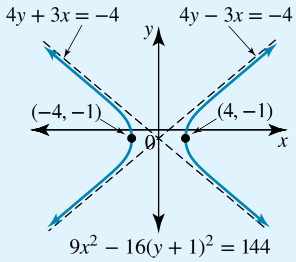
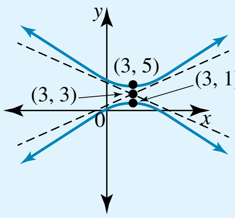
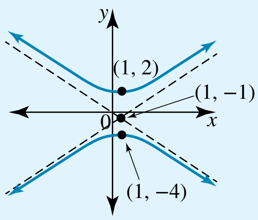


e



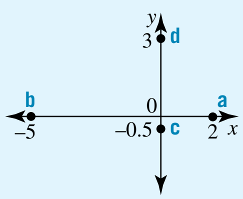
f



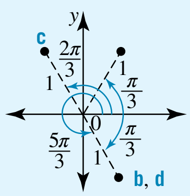
18 $\frac{x^2}{400} - \frac{y^2}{500} = 1$

EXERCISE 11.5

1



2



3 a $(\sqrt{2}, \sqrt{2})$ **b** $\left(\frac{5\sqrt{3}}{2}, 2\frac{1}{2}\right)$

c $\left(1.5, \frac{3\sqrt{3}}{2}\right)$ **d** $(0, 2.7)$

e $\left(-0.75, \frac{3\sqrt{3}}{4}\right)$ **f** $(-6\sqrt{3}, -6)$

4 a $(0, 2.6)$ **b** $(-7.8, 0)$

c $(5, 5\sqrt{3})$ **d** $\left(4.55, \frac{-91\sqrt{3}}{20}\right)$

e $(22.14, -11.62)$ **f** $(-1.05, -1.21)$

5 a $[5, 0°]$ **b** $[4.3, 90°]$

c $[30, 180°]$ **d** $[9, 270°]$

e $[6\sqrt{2}, 45°]$ **f** $[5, 2.5^c]$

6 a $[13, 4.32^c]$ **b** $[10, 5.36^c]$

c $\left[2, \frac{2\pi}{3}\right]$ **d** $\left[2\sqrt{2}, \frac{5\pi}{4}\right]$

e $\left[4, \frac{11\pi}{6}\right]$ **f** $[7.81, 0.88^c]$

7 a $r = \frac{3}{\cos(\theta)}$ **b** $r = \frac{2}{\sin(\theta)}$

c $r = 3$ **d** $r = 6$

8 a $\theta = 78°41'$ **b** $\theta = \frac{\pi}{4}$

c $r = \frac{1}{3\cos(\theta) - 4\sin(\theta)}$ **d** $r = \frac{7}{5\cos(\theta) + \sin(\theta)}$

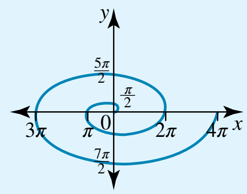
9 a $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 25$

c $x^2 + (y-3)^2 = 9$ **d** $(x-1)^2 + y^2 = 1$

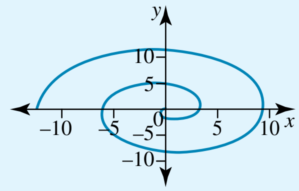
10 a $y = 3x$ **b** $y = -4x$

c $y = x$ **d** $y = -x$

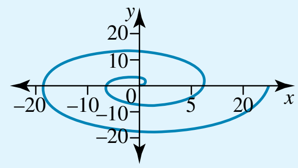
11 a



b



12 a



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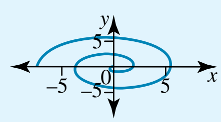
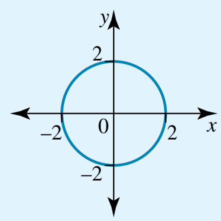


13 a



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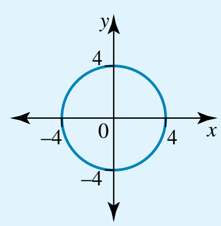
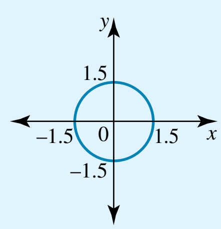


14 a



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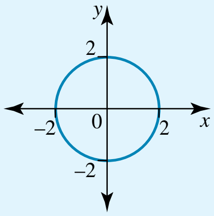
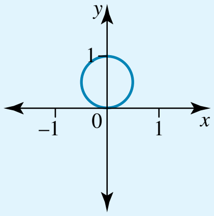


15 a



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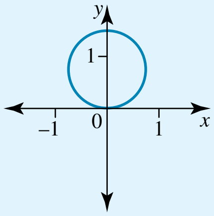
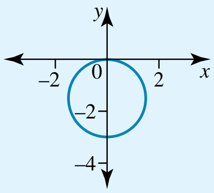
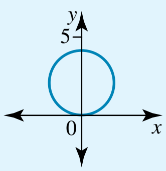


16 a



b



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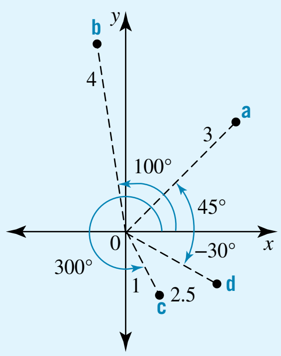


18 a $r = 10\cos(\theta) - 6\sin(\theta)$

b $r = -6\cos(\theta) - 8\sin(\theta)$

c $r = 12\sin(\theta)$

d $r = 2\cos(\theta)$

e $r^2 = \dfrac{36}{4\cos^2(\theta) + 9\sin^2(\theta)}$

f $r^2 = \dfrac{100}{25\cos^2(\theta) + 4\sin^2(\theta)}$

19 a $x = 4$

b $y = -1$

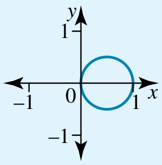
c $x^2 + y^2 + 2x - 4y = 0$

d $x^2 + y^2 - 8x - 6y = 0$

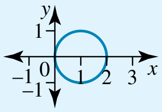
e $y = \dfrac{9 - x^2}{6}$

f $x = \dfrac{y^2 - 16}{8}$

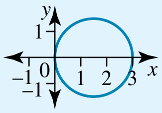
20 a



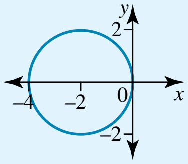
b



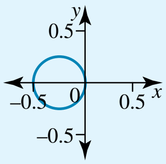
c



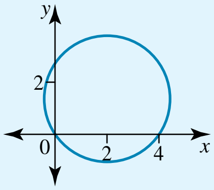
d



e



21 a



A circle

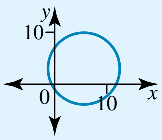
b **i** 0, 3 **ii** 0, 4

iii 5 **iv** 2.5

v (2, 1.5)

vi $(x - 2)^2 + (y - 1.5)^2 = 6.25$

22 a



b A circle

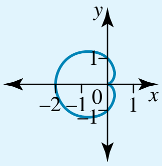
i 0, 5 **ii** 0, 12

iii 13 **iv** 6.5

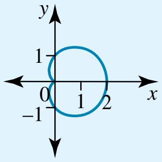
v (6, 2.5)

vi $(x - 6)^2 + (y - 2.5)^2 = 42.25$

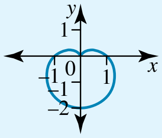
23



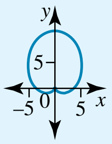
a



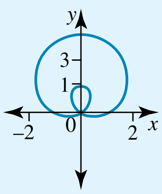
b



c



24



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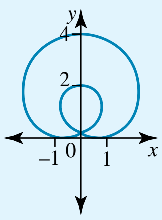

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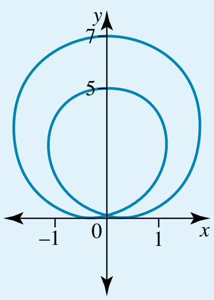

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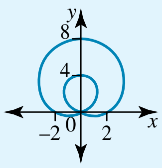

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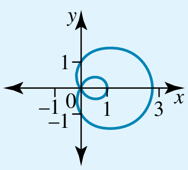

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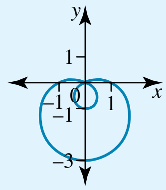

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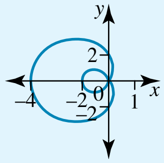

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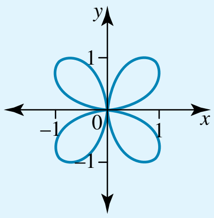
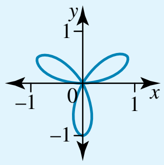


a



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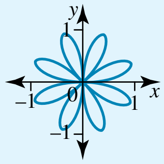

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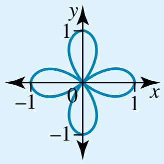

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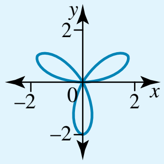

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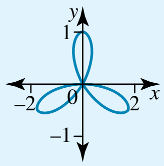

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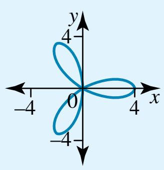
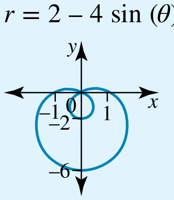
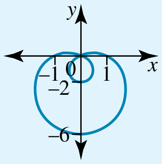


26 $r = 2 - 4\sin(\theta)$

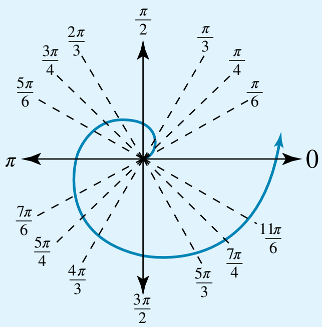


$r = -2 - 4\sin(\theta)$

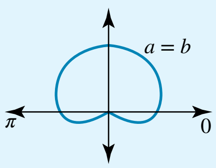


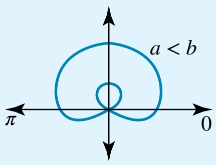
The graphs are identical.

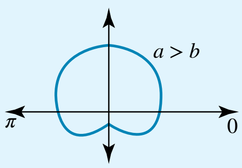
27



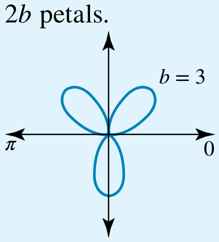
28 a If $a = b$, a cardioid results. If $a < b$, the limacon contains an inner loop. If $a > b$, there is no inner loop (a dimple results).

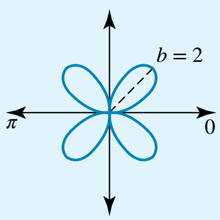




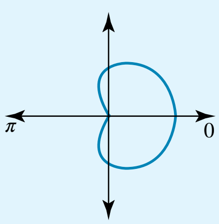


b a is the radius of each petal, and b is the number of petals if b is odd. If b is even, then there will be $2b$ petals.

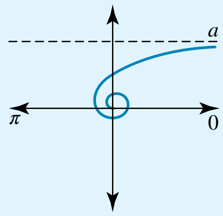




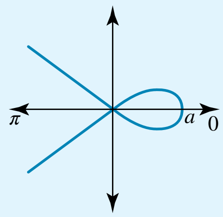
c Increasing the a value increases the size of the resulting cardioid in all directions.



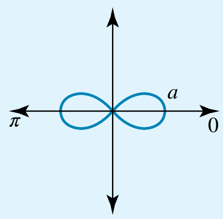
d The graph creates a spiral about the centre from the chosen a value. There is a horizontal asymptote at the a value.



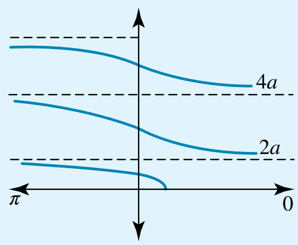
e The loop passes through the horizontal axis at the chosen a value.



f The value of a is the length of one petal (along the horizontal axis).



g The a value helps to determine the position of the horizontal asymptote.



EXERCISE 11.6

1 a $y = x - 7$

b $y = x^2 - 2x + 5$

2 a $xy = 1$

b $y = x^{\frac{2}{3}}$

3 a $x^2 + y^2 = 1$

b $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

4 a $x^2 + y^2 = 1$

b $\dfrac{x^2}{4} - y^2 = 1$

5 $y = x^2 - 6x$

6 $(x - y)^2 = 2(x + y)$

7 $x^2 + y^2 = 1$

8 $x^2 - y^2 = 1$

9 $x = 1 - 2y^2$

10 $x = \cos^{-1}\left(\frac{3 - y}{2}\right) - \frac{\sqrt{4 - (3 - y)^2}}{2}$

11 $y = 2x\sqrt{1 - x^2}$

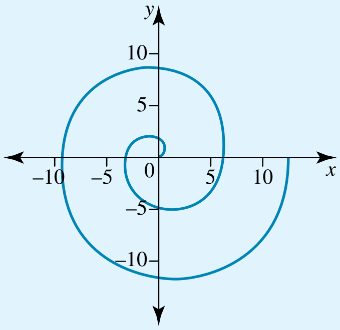
12 $x = y^2$

13 $x^2 + y^2 = 1$

14 $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2^{\frac{2}{3}}$

15 a $x^2 + y^2 = \left(\tan^{-1}\frac{y}{x}\right)^2$

b



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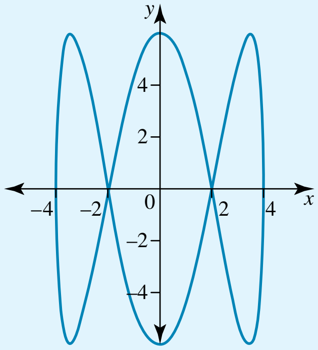


12

Transformations

12.1 Kick off with CAS

Transformations of functions

Function notation can be a useful means to describe a transformation.

1 Define the function $f(x) = 2\sqrt{x + 4}$.

Sketch the graphs of the following, describing the shape, orientation, and the end point and/or intercepts:

a $f(x)$

b $-f(x)$

c $f(-x)$

d $f(x + 5)$

e $3f(x)$

f $f\left(\frac{x}{2}\right)$

g $f(x) - 2$.

2 Define the function $g(x) = \sin(2x), 0 \leq x \leq 2\pi$.

Sketch the graphs of the following, describing the shape, orientation, and the end point and/or intercepts:

a $g(x)$

b $-g(x)$

c $g(-x)$

d $g\left(x - \frac{\pi}{4}\right)$

e $-2g(x)$

f $g(2x)$

g $g(x) + 3$.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

12.2 Translations of points and graphs

Introduction to transformations

Under a transformation of the Cartesian plane, each point (x, y) maps onto its image point (x', y').

studyon

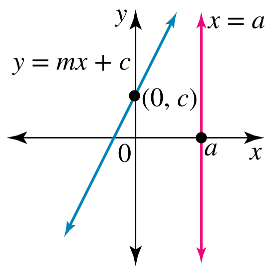
Units 1 & 2
AOS 1
Topic 3
Concept 1

Translations of points and graphs
Concept summary
Practice questions

In this topic, the transformation T_r will be defined by the rule $(x, y) \rightarrow (x', y')$, and the following transformations will be considered: translations, reflections and dilations. For each transformation and combination of transformations, points and their images as well as rules and their image rules will be considered.

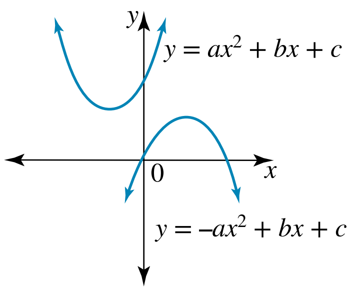
Some simple relations

The following relations with the given rules and properties will be dealt with in this topic.

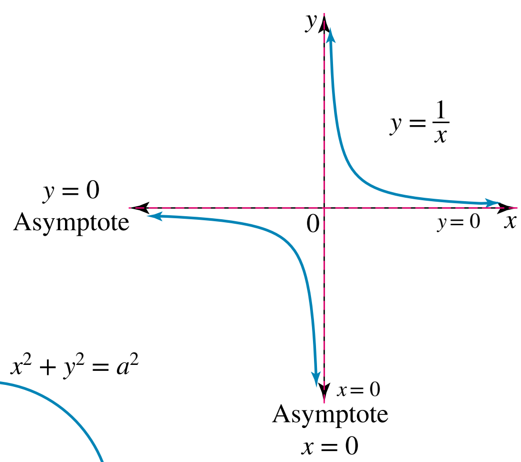


1. Linear (straight lines)
   - $y = mx + c$, where m is the gradient and $(0, c)$ is the y-intercept
   - $x = a$, a vertical line with the gradient undefined

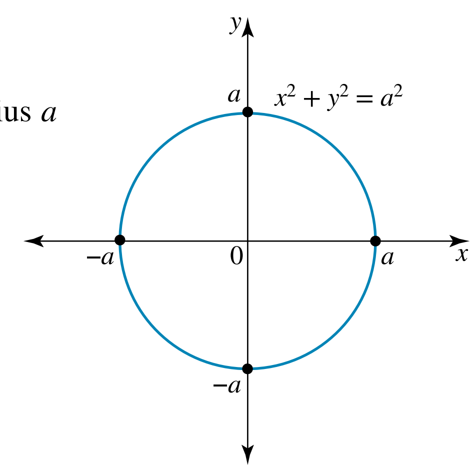
2. Quadratic (parabolas)
   - $y = ax^2 + bx + c$
   - x-intercepts (if they exist) are found by solving $ax^2 + bx + c = 0$.
   - $(0, c)$ is the y-intercept.
   - The equation of the axis of symmetry is $x = \dfrac{-b}{2a}$.
   - The turning point has an x-coordinate of $x = \dfrac{-b}{2a}$ and the y-coordinate is found by substitution of the x-coordinate into the rule.
   - If $a > 0$ the shape is ∪ (upright) and if $a < 0$ the shape is ∩ (inverted).



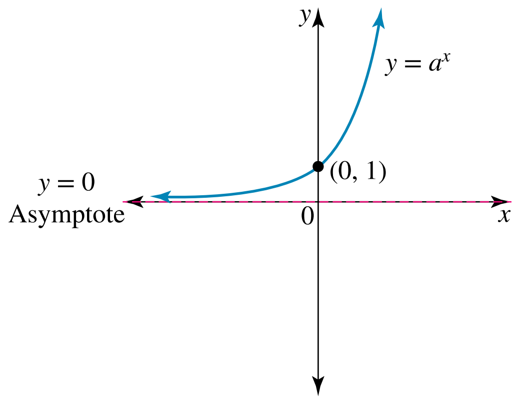
3. Inverse (hyperbolas)
   - $y = \dfrac{1}{x}$
   - Horizontal asymptote $y = 0$, the x-axis
   - Vertical asymptote $x = 0$, the y-axis



4. Circle
   - $x^2 + y^2 = a^2$
   - Centre $(0, 0)$ and radius a



5. Exponential
   - $y = a^x$, $a > 1$
   - Horizontal asymptote $y = 0$
   - Common point of (0, 1) for any a value



Translations

Under a translation given by $T_{h,\,k}$, $(x, y) \rightarrow (x + h, y + k)$, $x' = x + h$, $y' = y + k$:

h is a translation in the x-direction (horizontally)
k is a translation in the y-direction (vertically).

$T_{1,\,2}$ represents a translation of 1 unit to the right and 2 units up for all points in a plane.

$T_{-1,\,-2}$ represents a translation of 1 unit to the left and 2 units down for all points in a plane.

Points under translations

WORKED EXAMPLE 1

The point A (3, 1) maps onto A′ under the translation $T_{1,\,2}$.

Find the coordinates of A′.

THINK	WRITE
The translation $T_{1,\,2}$ indicates that A is moved 1 unit to the right and 2 units up. Write this using mapping notation to find the coordinates of A′.	$(x, y) \rightarrow (x + 1, y + 2)$ $(3, 1) \rightarrow (4, 3)$ A′ (4, 3)

WORKED EXAMPLE 2

Find a translation that maps A (3, −1) onto A′ (2, 3).

THINK	WRITE
1 Use mapping notation to determine the translations.	$(3, -1) \rightarrow (2, 3)$ $\Rightarrow (3, -1) \rightarrow (3 - 1, -1 + 4)$
2 State the answer.	A is translated 1 unit to the left and 4 units up, that is, $T_{-1,\,4}$.

WORKED EXAMPLE 3 **A translation is defined by the rule $(x, y) \to (x - 2, y + 3)$. If the image point is A′ (5, 6), find the coordinates of the original point A.**

THINK	WRITE
1 State the image equations. 5 is the image of x under translation of −2 units. 6 is the image of y under translation of 3 units.	$x' = x - 2$ $5 = x - 2$ $y' = y + 3$ $6 = y + 3$
2 Solve for x and y.	$x = 7, y = 3$
3 State the answer.	If the image point is A′ (5, 6) and the translation is defined by the rule $(x, y) \to (x - 2, y + 3)$, the original point is A (7, 3).

Rules under translations

WORKED EXAMPLE 4 **Find the image rule for each of the following, given the original rule and translation.**

a $y = x, T_{-2, -3}$

b $y = 2x^2, T_{-4, 5}$

c $y = f(x), T_{h, k}$

THINK	WRITE
a 1 State the image equations.	**a** $x' = x - 2$ $y' = y - 3$
2 Find x and y in terms of x' and y'.	$x = x' + 2$ $y = y' + 3$
3 Substitute into $y = x$.	$y = x$ $\Rightarrow y' + 3 = x' + 2$ $y' = x' - 1$
4 Express the answer without using the primes.	Given $y = x$ under the translation $T_{-2, -3}$, the equation of the image (or image rule) is $y = x - 1$.
b 1 State the image equations.	**b** $x' = x - 4$ $y' = y + 5$
2 Find x and y in terms of x' and y'.	$x = x' + 4$ $y = y' - 5$
3 Substitute into $y = 2x^2$.	$y = 2x^2$ $\Rightarrow y' - 5 = 2(x' + 4)^2$ $y' = 2(x' + 4)^2 + 5$

4 Express the answer without using the primes. *Note:* In the first form of the answer, the turning point is (−4, 5), which was the answer expected as (0, 0) → (−4, 5).	Given $y = 2x^2$ under the translation $T_{-4,\,5}$, the equation of the image (or image rule) is $y = 2(x + 4)^2 + 5$ or $y = 2x^2 + 16x + 37$.
c 1 State the image equations.	**c** $x' = x + h$ $y' = y + k$
2 Find x and y in terms of x' and y'.	$x = x' - h$ $y = y' - k$
3 Substitute into $y = f(x)$.	$y' - k = f(x' - h)$ $\Rightarrow y' = f(x' - h) + k$
4 Express the answer without using the primes.	Given $y = f(x)$ under the translation $T_{h,\,k}$, the equation of the image (or image rule) is $y = f(x - h) + k$.

WORKED EXAMPLE 5

Given the rule and its image rule under a translation, state a possible translation and its abbreviated version in the form $T_{a,\,b}$.

a $y = x$, $y = x + 1$

b $y = x^2$, $y = (x - 2)^2 + 1$

c $y = x^2 + 1$, $y = x^2 + 2x - 4$

THINK	WRITE
a 1 The original rule is $y = x$ and its image is $y = x + 1$. We need a point on each graph for comparison. Substitute a value for x, say $x = 0$, into both rules.	**a** $y = x$ When $x = 0$, $y = 0$ $\Rightarrow (0, 0)$ $y = x + 1$ When $x = 0$, $y = 0 + 1$ $\Rightarrow (0, 1)$ $\Rightarrow (0, 0) \rightarrow (0, 1)$
2 State the required translations in the form $T_{a,\,b}$. That is, no translation on the x-axis and translation of 1 unit up on the y-axis.	$T_{0,\,1}$: no translation on the x-axis and translation of 1 unit up on the y-axis
b 1 The original rule is $y = x^2$ and its image is $y = (x - 2)^2 + 1$. We need a point on each graph for comparison. Determine the turning points of each equation.	**b** $y = x^2$ has a turning point at (0, 0). $y = (x - 2)^2 + 1$ has a turning point of (2, 1). $\Rightarrow (0, 0) \rightarrow (2, 1)$
2 State the required translations in the form $T_{a,\,b}$. That is, translation of 2 units to the right on the x-axis and translation of 1 unit up on the y-axis.	$T_{2,\,1}$: translation of 2 units to the right on the x-axis and translation of 1 unit up on the y-axis

c **1** The original rule is $y = x^2 + 1$ and its image is $y = x^2 + 2x - 4$. We need a point on each graph for comparison. Determine the turning points for each equation.

c $y = x^2 + 1$ has a turning point at (0, 1).

$y = x^2 + 2x - 4$

$= (x + 1)^2 - 5$

The turning point is at (−1, −5).

$\Rightarrow (0, 1) \rightarrow (-1, -5)$

2 State the required translations in the form $T_{a,\, b}$. That is, translation of 1 unit left on the x-axis and translation of 6 units down on the y-axis.

$T_{-1,\, -6}$: translation of 1 unit left on the x-axis and translation of 6 units down on the y-axis

EXERCISE 12.2 Translations of points and graphs

PRACTISE

1 WE1 The point A (1, 2) maps onto A′ under the translation $T_{3,\, 4}$. Find the coordinates of A′.

2 The point (1, −2) maps onto A′ under the translation $T_{-3,\, 4.}$ Find the coordinates of A′.

3 WE2 Find a translation that maps A (2, 3) onto A′ (4, 5).

4 Find a translation that maps A (2, −3) onto A′ (4, −2).

5 WE3 A translation is defined by the rule $(x, y) \rightarrow (x + 2, y + 3)$. If the image point is A′ (4, 1), find the coordinates of the original point.

6 A translation is defined by the rule $(x, y) \rightarrow (x - 1, y + 2)$. If the image point is A′ (−2, 1), find the coordinates of the original point A.

7 WE4 Find the image rule for $y = x$ given the translation $T_{1,\, -3}$.

8 Find the image rule for $y = 2x$ given the translation $T_{1,\, -2}$.

9 WE5 Given the rule and its image rule under a translation, state a possible translation.

$y = 2x$, $y = 2x + 3$

10 Given a rule and its image rule under a translation, state a possible translation.

a $y = -x^2$, $y = -(x + 4)^2 - 3$ **b** $y = x^2 + 2x + 1$, $y = x^2 + 4x - 1$

CONSOLIDATE

11 Given the coordinates of A and the translation, find the coordinates of the image, A′.

a A (−1, 2), $T_{3,\, -4}$ **b** A (−1, −2), $T_{-3,\, -4}$

12 Given a point and its image, state the possible translation in words as well as using $T_{a,\, b}$ notation.

a A (−1, −4), A′ (−3, 1) **b** A (−2, −4), A′ (−1, −7)

13 Given the translation and the coordinates of the image, A′, find the coordinates of A.

a $T_{3,\, -2}$, A′ (−2, −3) **b** $T_{-2,\, -3}$, A′ (−5, −2)

14 Find the image rule for each of the following, given the original rule and translation.

For parts **a** to **c**, sketch each graph on the same axes as the original rule and its image.

a $y = -x$, $T_{2,\,-2}$

b $y = 2x + 1$, $T_{-1,\,-2}$

c $y = x^2$, $T_{-1,\,2}$

d $y = 2x^2$, $T_{2,\,-1}$

e $y = x^2 + 2$, $T_{-3,\,4}$

f $y = -2x^2$, $T_{-2,\,3}$

g $y = f(x)$, $T_{-3,\,2}$

h $y = f(x)$, $T_{-1,\,-2}$

i $y = -2x^2$, $T_{h,\,k}$

j $y = -3x$, $T_{h,\,k}$

15 Under $T_{2,\,-1}$, the image of (2, 3) and the point whose image is (2, 3) are:

A (4, 2), (0, 4)
B (0, 4), (4, 2)
C (4, 2), (4, 2)
D (4, 2), (6, 1)
E (0, 4), (0, 4)

16 Under $T_{-2,\,1}$, the image equation of $y = x^2$ and the equation whose image equation is $y = x^2$ are:

A $y = (x + 2)^2 + 1$, $y = (x - 2)^2 - 1$
B $y = (x - 2)^2 - 1$, $y = (x + 2)^2 + 1$
C $y = (x - 2)^2 + 1$, $y = (x + 2)^2 - 1$
D $y = (x + 2)^2 - 1$, $y = (x - 2)^2 + 1$
E $y = (x + 2)^2 + 1$, $y = (x + 2)^2 - 1$

17 The translations that map (3, 4) onto (2, −4) and $y = x^2$ onto $y = (x - 2)^2 + 2$ are, respectively:

A $T_{-1,\,-8}$ and $T_{2,\,1}$
B $T_{1,\,8}$ and $T_{2,\,1}$
C $T_{-1,\,-8}$ and $T_{2,\,2}$
D $T_{1,\,8}$ and $T_{2,\,2}$
E $T_{-1,\,-8}$ and $T_{-2,\,1}$

18 The translations that map $y = 2^x$ onto $y = 2^{x-1} + 1$ and $y = \frac{1}{x}$ onto $y = \frac{1}{x+1} - 1$ are, respectively:

A $T_{-1,\,-1}$ and $T_{1,\,1}$
B $T_{1,\,1}$ and $T_{-1,\,-1}$
C $T_{-1,\,1}$ and $T_{1,\,-1}$
D $T_{1,\,-1}$ and $T_{-1,\,1}$
E $T_{-1,\,-1}$ and $T_{-1,\,1}$

19 Sketch, on the same set of axes, $x^2 + y^2 = 4$ and its image under the translation $T_{1,\,-1}$ (translated 1 unit in the positive x-direction and 1 unit in the negative y-direction).

20 Sketch, on the same set of axes, $y = \frac{1}{x}$ and its image under the translation $T_{2,\,-1}$ (2 units right and 1 unit down). State the equations of the asymptotes.

MASTER

21 Given a rule, its image rule under a translation and a point to consider, state a possible translation.

a $x^2 + y^2 = 9$, $(x - 1)^2 + (y + 3)^2 = 9$. Consider the centre (h, k).

b $(x + 1)^2 + (y - 2)^2 = 4$, $x^2 + y^2 - 2x + 2y - 2 = 0$. Consider the centre (h, k). Complete the square on both x and y for the image circle.

c $y = \frac{1}{x}$, $y = \frac{1}{x - 1} + 2$. Consider the point of intersection of the asymptotes.

22 Using the property that under a translation $T_{h,\,k}$, $y = f(x) \rightarrow y - k = f(x - h)$, state a possible translation in words and in the form $T_{a,\,b}$, given a rule and its image rule.

a $y = 2^x$, $y = 2^x + 3$

b $y = 2^x$, $y = 2^{x+1} + 7$

c $y = 2^x + 1$, $y = 2^{x+3} - 5$

d $y = 2^{x+3} - 4$, $y = 2^{x-2} + 1$

12.3 Reflections and dilations

Reflections of points and graphs

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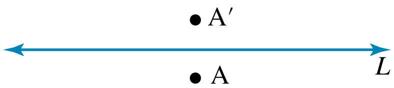
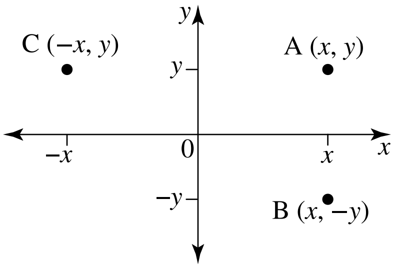
On the diagram, A′ is the image of the point A under reflection in the line L. L is called the mediator.



$M_{y=0}$ represents reflection in the line $y = 0$, the x-axis.

$M_{x=0}$ represents reflection in the line $x = 0$, the y-axis.

To determine the rule for each reflection, consider the diagram at right.



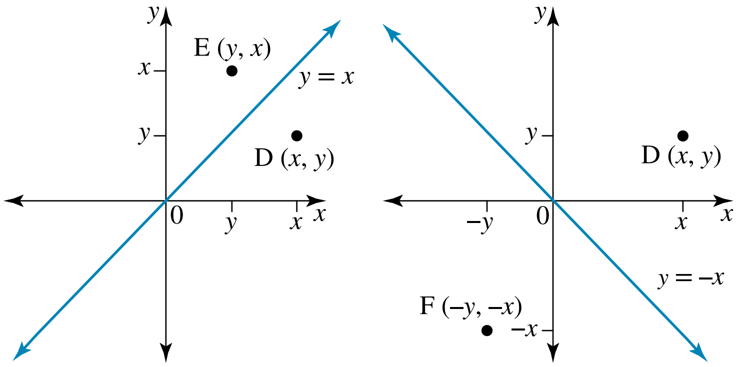
$M_{y=0} : (x, y) \rightarrow (x, -y)$ gives point B.

$M_{x=0} : (x, y) \rightarrow (-x, y)$ gives point C.

Similarly, for reflection in the lines $y = x$ and $y = -x$, consider the diagrams below.

$M_{y=x} : (x, y) \rightarrow (y, x)$ gives point E.

$M_{y=-x} : (x, y) \rightarrow (-y, -x)$ gives point F.



Note: M is used to represent reflections because a reflection is a *mirror* image about its specified axis or line.

WORKED EXAMPLE 6

Find the coordinates of the image of (2, −3) under the following reflections.

a $M_{y=0}$ **(reflection in the x-axis)**

b $M_{x=0}$ **(reflection in the y-axis)**

c $M_{y=x}$ **(reflection in the line $y = x$)**

d $M_{y=-x}$ **(reflection in the line $y = -x$)**

THINK	WRITE
a A reflection in the x-axis, $M_{y=0}$, means $(x, y) \rightarrow (x, -y)$.	**a** $(x, y) \rightarrow (x, -y)$ $(2, -3) \rightarrow (2, 3)$
b A reflection in the y-axis, $M_{x=0}$, means $(x, y) \rightarrow (-x, y)$.	**b** $(x, y) \rightarrow (-x, y)$ $(2, -3) \rightarrow (-2, -3)$
c A reflection in the line $y = x$, $M_{y=x}$, means $(x, y) \rightarrow (y, x)$.	**c** $(x, y) \rightarrow (y, x)$ $(2, -3) \rightarrow (-3, 2)$
d A reflection in the line $y = -x$, $M_{y=-x}$, means $(x, y) \rightarrow (-y, -x)$.	**d** $(x, y) \rightarrow (-y, -x)$ $(2, -3) \rightarrow (3, -2)$

WORKED EXAMPLE 7

Find the image equation of $y = 2x + 1$ under the following reflections.

a $M_{y=0}$ **(reflection in the x-axis)**

b $M_{y=-x}$ **(reflection in the line $y = -x$)**

THINK	WRITE
a 1 A reflection in the x-axis, $M_{y=0}$, means $(x, y) \to (x, -y)$.	**a** $x' = x$ $y' = -y$
2 Transpose to make x and y the subjects.	$x = x' \qquad y = -y'$
3 Substitute to find the image equations.	$y = 2x + 1$ $\Rightarrow -y' = 2x' + 1$
4 State the image equation without the primes.	$y = -2x - 1$
b 1 A reflection in the line $y = -x$, $M_{y=-x}$, means $(x, y) \to (-y, -x)$.	**b** $x' = -y$ $y' = -x$
2 Transpose to make x and y the subjects.	$x = -y' \qquad y = -x'$
3 Substitute to find the image equations. Make y' the subject.	$y = 2x + 1$ $\Rightarrow -x' = -2y' + 1$ $\Rightarrow \quad y' = \frac{1}{2}x' + \frac{1}{2}$
4 State the image equation without the primes.	$y = \frac{1}{2}x + \frac{1}{2}$

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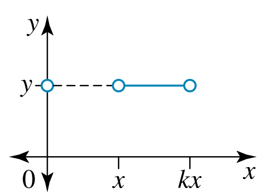
Dilations from axes

A dilation of factor k from the y-axis (or parallel to the x-axis) is represented by $D_{k,\,1}$, with a rule $(x, y) \to (kx, y)$. As shown on the diagram at right, the x-coordinate has been transformed by a dilation factor k, $k \in R^+$, while the y-coordinate is unchanged.



Similarly, $D_{1,\,k}$, represents a dilation of factor k from the x-axis (or parallel to the y-axis), with a rule $(x, y) \to (x, ky)$.

$D_{2,\,3}$, represents a dilation of factor 2 from the y-axis (or parallel to the x-axis) and a dilation of factor 3 from the x-axis, with a rule $(x, y) \to (2x, 3y)$.

WORKED EXAMPLE 8

Find the image of $(4, -2)$ under the following dilations.

a $D_{2,\,1}$ **(dilation factor of 2 from the y-axis)**

b $D_{1,\,\frac{1}{2}}$ **(dilation factor of $\frac{1}{2}$ from the x-axis)**

c $D_{\frac{1}{2},\,3}$ **(dilation factor of $\frac{1}{2}$ from the y-axis and 3 from the x-axis)**

THINK	WRITE
a A dilation factor of 2 from the y-axis, $D_{2,\,1}$, means $(x, y) \to (2x, y)$.	**a** $x' = 2x$ $y' = y$ $(x, y) \to (2x, y)$ $(4, -2) \to (8, -2)$

b A dilation factor of $\frac{1}{2}$ from the x-axis, $D_{1,\frac{1}{2}}$, means $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$.

b $x' = x$

$y' = \frac{1}{2}y$

$(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$

$(4, -2) \rightarrow (4, -1)$

c A dilation factor of $\frac{1}{2}$ from the y-axis and 3 from the x-axis, $D_{\frac{1}{2}, 3}$, means $(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$.

c $x' = \frac{1}{2}x$

$y' = 3y$

$(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$

$(4, -2) \rightarrow (2, -6)$

WORKED EXAMPLE 9

Find the image equation of $y = x^2$ under the following dilations.

a **$D_{2, 1}$ (dilation factor of 2 parallel to the x-axis or dilation factor of 2 from the y-axis)**

b **$D_{\frac{1}{2}, 3}$ (dilation factor of $\frac{1}{2}$ parallel to the x-axis and 3 parallel to the y-axis)**

THINK

a 1 A dilation factor of 2 parallel to the x-axis (i.e. from the y-axis), $D_{2, 1}$, means $(x, y) \rightarrow (2x, y)$.

2 Transpose to make x and y the subjects.

3 Substitute to find the image equations.

4 State the image equation without the primes.

b 1 A dilation factor of $\frac{1}{2}$ parallel to the x-axis and 3 parallel to the y-axis (i.e. from the x-axis), $D_{\frac{1}{2}, 3}$, means $(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$.

2 Transpose to make x and y the subjects.

3 Substitute to find the image equations.

4 State the image equation without the primes.

WRITE

a $x' = 2x$

$y' = y$

$x = \frac{1}{2}x'$

$y = y'$

$y = x^2$

$\Rightarrow y' = \left(\frac{x'}{2}\right)^2$

The image equation is $y = \frac{1}{4}x^2$.

b $x' = \frac{1}{2}x$

$y' = 3y$

$x = 2x'$

$y = \frac{1}{3}y'$

$y = x^2$

$\Rightarrow \frac{1}{3}y' = (2x')^2$

The image equation is $y = 12x^2$.

WORKED EXAMPLE 10

Find the original point if the image point under $D_{2,\,3}$ (dilation factor of 2 from the y-axis and 3 from the x-axis) is $(6, -9)$.

THINK	WRITE
1 A dilation factor of 2 from the y-axis and 3 from the x-axis, $D_{2,\,3}$, means $(x, y) \rightarrow (2x, 3y)$.	$x' = 2x$ $y' = 3y$
2 Transpose to make x and y the subjects. This means that the original point will have an x-value $\frac{1}{2}$ that of the image point and a y-value $\frac{1}{3}$ that of the image point.	$x = \frac{1}{2}x'$ $y = \frac{1}{3}y'$
3 Multiply the x-value of the image point by 2 and the y-value of the image point by 3 to find the original point.	$(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{3}y\right)$ $(6, -9) \rightarrow (3, -3)$ The original point is $(3, -3)$.

WORKED EXAMPLE 11

Find the original equation if the image equation under the dilation $D_{3,\,1}$ (dilation factor of 3 parallel to the x-axis) is $y = 3x^2 + 1$.

THINK	WRITE
1 A dilation factor of 3 parallel to the x-axis (i.e. from the y-axis), $D_{3,\,1}$, means $(x, y) \rightarrow (3x, y)$.	$x' = 3x$ $y' = y$
2 The image equation is $y = 3x^2 + 1$. Reintroduce the primes and substitute the values for x' and y' to find the original equation.	$y' = 3x'^2 + 1$ $\Rightarrow y = 3(3x)^2 + 1$
3 Simplify and state the original equation.	The original equation is $y = 27x^2 + 1$.

EXERCISE 12.3 Reflections and dilations

PRACTISE

1 WE6 Find the coordinates of the image of $(-2, 4)$ under the following reflections.
 a $M_{y=0}$ (reflection in the x-axis)
 b $M_{x=0}$ (reflection in the y-axis)
 c $M_{y=x}$ (reflection in the line $y = x$)
 d $M_{y=-x}$ (reflection in the line $y = -x$)

2 Find the coordinates of the image of $(-1, -2)$ under the following reflections.
 a $M_{y=0}$ (reflection in the x-axis)
 b $M_{x=0}$ (reflection in the y-axis)
 c $M_{y=x}$ (reflection in the line $y = x$)
 d $M_{y=-x}$ (reflection in the line $y = -x$)

3 WE7 Find the image equation of $y = 3x$ under the following reflections.
 a $M_{y=0}$ (reflection in the x-axis)
 b $M_{y=x}$ (reflection in the line $y = x$)

4 Find the image of equation $y = 2x + 3$ under the following reflections.
 a $M_{y=0}$ (reflection in the x-axis)
 b $M_{y=x}$ (reflection in the line $y = x$)

5 WE8 Find the image of (2, 3) under the following dilations.

a $D_{2,1}$ (dilation factor of 2 from the y-axis)

b $D_{1,\frac{1}{2}}$ (dilation factor of $\frac{1}{2}$ from the x-axis)

c $D_{\frac{1}{2},3}$ (dilation factor of $\frac{1}{2}$ from the y-axis and 3 from the x-axis)

6 Find the image of (−2, 5) under the following dilations.

a $D_{1,3}$ (dilation factor of 3 from the x-axis)

b $D_{3,1}$ (dilation factor of 3 from the y-axis)

c $D_{2,2}$ (dilation factor of 2 from the x-axis and 2 from the y-axis)

7 WE9 Find the image of equation $y = x^2$ under the following dilations.

a $D_{1,2}$ (dilation factor of 2 parallel to the y-axis)

b $D_{3,\frac{1}{2}}$ (dilation factor of 3 parallel to the x-axis and $\frac{1}{2}$ parallel to the y-axis)

8 Find the image of equation $y = x + 1$ under the following dilations.

a $D_{1,2}$ (dilation factor of 2 parallel to the y-axis)

b $D_{2,3}$ (dilation factor of 2 parallel to the x-axis and 3 parallel to the y-axis)

9 WE10 Find the original point if the image point under $D_{1,2}$ (dilation factor of 2 from the x-axis) is (3, 5).

10 Find the original point if the image point under $D_{2,2}$ (dilation factor of 2 from the x-axis and 2 from the y-axis) is (–2, 3).

11 WE11 Find the original equation if the image equation under the dilation $D_{2,1}$ (dilation factor of 2 parallel to the x-axis) is $y = x$.

12 Find the original equation if the image equation under the dilation $D_{1,2}$ (dilation factor of 2 parallel to the y-axis) is $y = x^2$.

CONSOLIDATE

13 Find the image rules of the given rules under the following reflections. In each case, sketch, on the same set of axes, the graphs of the relations before and after the reflection.

a $y = -2x$, $M_{y=-x}$

b $y = 2x^2$, $M_{y=0}$

c $y = -3x^2$, $M_{x=0}$

d $y = \frac{-1}{4}x^2$, $M_{y=x}$

e $y = x^2 + 1$, $M_{y=-x}$

f $y = \frac{1}{x}$, $M_{x=0}$

g $y = \frac{1}{x+1}$, $M_{y=x}$

h $y = \frac{1}{x} + 1$, $M_{x=0}$

i $y = \frac{1}{x+1} + 1$, $M_{y=x}$

j $y = \frac{2}{x-1} + 3$, $M_{y=-x}$

k $y = 2^x$, $M_{y=0}$

l $y = 2^x + 3$, $M_{x=0}$

m $y = \sqrt{x}$, $M_{y=x}$

n $y = -\sqrt{x} + 2$, $M_{y=-x}$

14 Under $M_{y=0}$, the image of (2, 3) and the point whose image is (2, 3) are, respectively:

A (2, −3) and (2, −3)

B (2, −3) and (−2, 3)

C (−2, 3) and (−2, 3)

D (−2, 3) and (2, −3)

E (−2, −3) and (−2, −3)

15 Under $M_{y=x}$, the image equations of $y = -\sqrt{x}$ and $y = \frac{1}{x-1}$ are, respectively:

A $y = x^2 (x \geq 0)$ and $y = \frac{1}{x} + 1$

B $y = x^2 (x \leq 0)$ and $y = \frac{1}{x} + 1$

C $y = x^2 (x \geq 0)$ and $y = \frac{1}{x} - 1$

D $y = x^2 (x \leq 0)$ and $y = \frac{1}{x+1}$

E $y = x^2 (x \leq 0)$ and $y = \frac{1}{x} - 1$

16 A translation that maps $(-2, 3)$ onto $(4, 1)$ also maps $y = x^2$ onto its image. Find the equation of the image.

17 Find the coordinates of the image point for the given point under the given dilation.

a $(4, -3), D_{2,\,3}$
b $(-2, -6), D_{\frac{1}{2},\,2}$
c $(-6, 4), D_{\frac{2}{3},\,\frac{3}{4}}$

18 For the given equation under the given dilation, find the image equation.

a $y = x, D_{2,\,1}$
b $y = x + 1, D_{2,\,1}$
c $y = 0, D_{1,\,2}$
d $y = 0, D_{1,\,4}$
e $x = 2, D_{2,\,1}$
f $x = 2, D_{1,\,5}$
g $x = 0, D_{2,\,4}$
h $y = x^2, D_{2,\,1}$
i $y = x^2 - 1, D_{2,\,3}$
j $y = 2^x, D_{1,\,2}$
k $y = 2^x, D_{2,\,3}$
l $y = \frac{1}{x}, D_{1,\,2}$
m $y = \frac{1}{x}, D_{2,\,2}$
n $y = (x - 1)^2, D_{2,\,3}$
o $y = (x + 2)^2 + 1, D_{2,\,2}$
p $y = 3(x - 1)^2 + 2, D_{3,\,2}$
q $y = 2^{-x}, D_{2,\,3}$
r $y = f(x), D_{a,\,b}$

19 Under $D_{3,\,2}$ (dilation factor of 3 from the y-axis and 2 from the x-axis), the image of $(3, -2)$ and the point whose image is $(3, -2)$ are, respectively:

A $(9, -4)$ and $(-1, 1)$
B $(9, -4)$ and $(1, -1)$
C $(-1, 1)$ and $(9, -4)$
D $(1, -1)$ and $(9, -4)$
E $(9, 4)$ and $(1, 1)$

20 Under $D_{2,\,3}$ (dilation factor of 2 parallel to the x-axis and 3 parallel to the y-axis), the image rule of $y = f(x)$ and the rule whose image rule is $y = f(x)$ are, respectively:

A $y = 3f\left(\frac{x}{2}\right)$ and $y = \frac{1}{3}f(2x)$
B $y = \frac{1}{3}f(2x)$ and $y = 3f\left(\frac{x}{2}\right)$
C $y = \frac{1}{3}f\left(\frac{x}{2}\right)$ and $y = 3f(2x)$
D $y = 3f(2x)$ and $y = \frac{1}{3}f\left(\frac{x}{2}\right)$
E $y = 3f\left(\frac{x}{2}\right)$ and $y = 3f(2x)$

21 Find the coordinates of the original point, given the image point under the given dilation.

a $(3, 5), D_{2,\,1}$ (dilation factor of 2 parallel to the x-axis)
b $(-2, -3), D_{2,\,3}$ (dilation factor of 2 parallel to the x-axis and 3 parallel to the y-axis)

22 Find the original equation, given the image equation under the given dilation.

a $y = 2^x, D_{2,\,2}$ (dilation factor of 2 from the x- and y-axes)
b $y = \frac{1}{x}, D_{2,\,3}$ (dilation factor of 2 parallel to the x-axis and 3 parallel to the y-axis)

MASTER

23 Find the image rule for the given rule under the given reflection. Sketch the graphs of the given rule and the image rule on the same set of axes.

a $y = 2x, M_{x=0}$
b $y = 2x^2, M_{y=0}$
c $y = 4x^2, M_{y=x}$
d $y = \frac{1}{x+1}, M_{y=-x}$
e $y = 2^x + 1, M_{y=0}$
f $x^2 + (y + 1)^2 = 9, M_{y=0}$
g $y = \sqrt{x}, M_{y=x}$
h $y = -\sqrt{x}, M_{x=0}$

24 Find the coordinates of the original point, given the coordinates of its image and the dilation.

a $(2, 3), D_{2,\,1}$
b $(-6, 3), D_{1,\,3}$
c $(-4, 4), D_{\frac{1}{2},\,\frac{1}{4}}$

12.4 Successive transformations

Finding the final image rule under successive transformations

WORKED EXAMPLE 12

Under the translation $T_{-2,\,3}$ followed by the reflection $M_{y\,=\,-x}$:

a find the coordinates of the final image of the point (2, −3)

b find the final image equation of $y = x^2$.

THINK	WRITE
a 1 Determine a single rule that replaces the successive transformations.	**a** $(x, y) \xrightarrow{T_{-2,3}} (x-2, y+3) \xrightarrow{M_{y=-x}} (-y-3, -x+2)$
2 Use this rule to state the image point.	$(x, y) \to (-y-3, -x+2)$ $(2, -3) \to (-(-3)-3, -2+2)$ $(2, -3) \to (0, 0)$
b 1 Set up the image equations.	**b** $x' = -y - 3$ $y' = -x + 2$
2 Transpose to make x and y the subjects.	$y = -x' - 3$ $x = -y' + 2$
3 Substitute the values of x and y to find the image equation.	$y = x^2$ $\Rightarrow -x - 3 = (-y + 2)^2$
4 Transpose to make y the subject of the image equation.	As $(-y+2)^2 = (y-2)^2$ therefore the image equation is $(y-2)^2 = -x-3$ or $y = 2 \pm \sqrt{-x-3}$

WORKED EXAMPLE 13

Find a transformation for each of the following.

a $y = f(x) \to y = 2f(x)$

b $y = f(x) \to y = f(2x)$

c $y = f(2x) \to y = f(2x - 2)$

d $y = f(2x) \to y = -f(2x)$

e $y = f(-x) \to y = f(-x + 2) - 2$

THINK	WRITE
a $x' = x,\ y' = 2y$	**a** $D_{1,\,2}$ (dilation factor of 2 from the x-axis)
b $2x' = x$ or $x' = \frac{1}{2}x,\ y' = y$	**b** $D_{\frac{1}{2},\,1}$ (dilation factor of $\frac{1}{2}$ from the y-axis)
c $2x = 2x' - 2$ or $x' = x + 1,\ y' = y$	**c** $T_{1,\,0}$ (translation 1 unit right)
d $x' = x,\ y' = -y$	**d** $M_{y\,=\,0}$ (reflection about the x-axis)
e $-x' + 2 = -x$ or $x' = x + 2,\ y' = y - 2$	**e** $T_{2,\,-2}$ (translation 2 units right and 2 units down)

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EXERCISE 12.4 Successive transformations

PRACTISE

1 WE12 Find the final image of (−2, 3) under the following successive transformations.

a $D_{2,\,3}$, then $T_{2,\,4}$

b $M_{x\,=\,0}$, then $T_{1,\,-2}$

c $T_{-2,\,-3}$, then $D_{2,\,3}$

d $T_{1,\,-2}$, then $D_{2,\,2}$, then $M_{y\,=\,-x}$

2 Find the final image equation for $y = \dfrac{1}{x}$ under the following successive transformations.

a $T_{2,\,4}$, then $D_{2,\,3}$

b $T_{1,\,-2}$, then $M_{x\,=\,0}$

c $D_{2,\,3}$, then $T_{-2,\,-3}$

d $D_{2,\,2}$, then $T_{1,\,-2}$, then $M_{y\,=\,-x}$

3 WE13 Find a transformation for each of the following.

a $y = f(x) \rightarrow y = -f(x)$

b $y = -f(x) \rightarrow y = -f(-x)$

4 Find a transformation for each of the following.

a $y = -f(-x) \rightarrow y = -f(-x + 1)$

b $y = -f(-x + 1) \rightarrow y = -2\,f(-x + 1)$

CONSOLIDATE

5 Find the rule of a single transformation that replaces the following successive abbreviated transformations, given the original is (x, y).

a $D_{2,\,1}$, then $T_{3,\,4}$

b $T_{3,\,4}$, then $D_{2,\,1}$

c $M_{y\,=\,x}$, then $T_{3,\,4}$

d $T_{3,\,4}$, then $M_{y\,=\,x}$

e $D_{2,\,3}$, then $T_{3,\,4}$

f $T_{3,\,4}$, then $D_{2,\,3}$

g $T_{3,\,4}$, then $T_{-4,\,3}$

h $D_{2,\,1}$, then $D_{3,\,2}$

i $M_{y\,=\,x}$, then $M_{x\,=\,0}$

j $M_{y\,=\,x}$, then $M_{y\,=\,-x}$

k $T_{2,\,-3}$, then $D_{2,\,3}$, then $M_{y\,=\,0}$

l $M_{y\,=\,0}$, then $T_{2,\,-3}$, then $D_{3,\,2}$

m $D_{3,\,2}$, then $M_{y\,=\,0}$, then $T_{2,\,-3}$

6 Under a transformation T_{r1}, $(x, y) \rightarrow (2x + 1, -3y + 2)$. Under a transformation T_{r2}, $(x, y) \rightarrow (-2y + 2, -3x + 2)$. Find the rule of a single transformation that replaces the following successive transformations.

a T_{r1}, then T_{r2}

b T_{r2}, then T_{r1}

c T_{r1}, then T_{r1}

d T_{r2}, then T_{r2}

7 Under successive transformations, $y = f(x) \rightarrow y = f(2x) \rightarrow y = f(-2x) \rightarrow y = f(-2x - 2)$. The transformations, in order, are:

A $D_{2,\,1}$, $M_{y\,=\,0}$, $T_{-2,\,0}$

B $D_{\frac{1}{2},\,1}$, $M_{x\,=\,0}$, $T_{-2,\,0}$

C $D_{\frac{1}{2},\,1}$, $M_{x\,=\,0}$, $T_{-1,\,0}$

D $D_{\frac{1}{2},\,1}$, $M_{y\,=\,0}$, $T_{1,\,0}$

E $D_{\frac{1}{2},\,1}$, $M_{x\,=\,0}$, $T_{1,\,0}$

8 Under the sequence of transformations $T_{1,\,2}$, $M_{y\,=\,x}$, $D_{2,\,3}$, the final image rule for $y = \dfrac{1}{x}$ is:

A $y = \dfrac{6}{x - 4} - 6$

B $y = \dfrac{6}{x + 4} - 3$

C $y = \dfrac{6}{x - 4} + 3$

D $y = \dfrac{1}{3(x + 4)} - 1$

E $y = \dfrac{1}{3(x + 4)} + 1$

9 Find the transformation for each of the following.

a $y = x^2 \rightarrow y = (x - 2)^2 + 3$

b $y = (x - 1)^2 \rightarrow y = 4(x - 1)^2$

c $y = 2x^2 \rightarrow y = -2x^2$

d $y = 3^x \rightarrow y = 3^{-x}$

e $y = 3^{-x} \rightarrow y = 3^{-x + 4}$

f $y = 3^{2x} \rightarrow y = 3^{2x + 1} - 2$

g $y = \frac{1}{x} \to y = \frac{-1}{x}$

h $y = \frac{1}{2x} \to y = \frac{1}{2x - 5}$

i $y = \frac{5}{x + 1} \to y = \frac{5}{x + 2}$

j $x^2 + y^2 = 1 \to \frac{x^2}{4} + \frac{y^2}{25} = 1$

k $\frac{x^2}{4} + \frac{y^2}{25} = 1 \to x^2 + y^2 = 1$

l $x^2 - y^2 = 1 \to (x + 4)^2 - (y - 3)^2 = 1$

m $\frac{x^2}{9} - \frac{y^2}{16} = 1 \to x^2 - y^2 = 1$

n $y = 4x^2 \to y = (2x - 1)^2$

o $-x^2 + y^2 = 4 \to \frac{-x^2}{9} + \frac{y^2}{16} = 1$

10 Find a transformation for each of the following.

a $y = -2f(-2x + 1) \to y = -2f(2(-x + 1))$

b $y = -2f(2(-x + 1)) \to y = -2f(-2x + 5) + 4$

11 State the successive transformations, starting with the first equation and finishing with the second. There is more than one correct order for the transformations.

a $y = x \to y = -2x + 4$

b $y = x^2 \to y = -2(x + 1)^2 - 3$

c $y = \frac{1}{x} \to y = \frac{2}{x - 3} + 4$

d $y = x^2 \to y = (2x + 3)^2$

e $y = 2^x \to y = -2^{-4x + 3}$

f $y = 2^x \to y = 3(2^{3x - 5})$

g $y = f(x) \to y = -4f(2x - 1) + 3$

h $y = f(x) \to y = 2f(-2x - 1) - 4$

i $y = x^6 \to y = -(-2x + 4)^6 + 7$

12 Sketch the graphs of the following using successive transformations, starting with the first equation.

a $y = x^2, y = -2(x - 2)^2 - 1$

b $y = x^2, y = (2x - 1)^2$

c $y = \frac{1}{x}, y = \frac{-2}{2x - 3} + 4$

d $y = 2^x \to y = -2^{x + 3} + 4$

e $y = 2^x \to y = 3(2^{2x + 3})$

f $y = 2^x \to y = -3(2^{2x + 3}) + 1$

13 Under successive transformations, $y = \sqrt{x} \xrightarrow{1} y = \sqrt{2x} \xrightarrow{2} y = \sqrt{2x + 1}$. The transformations 1 and 2 are:

A $D_{2,\,1}$ and $T_{-1,\,0}$

B $D_{\frac{1}{2},\,1}$ and $T_{-\frac{1}{2},\,0}$

C $D_{\frac{1}{2},\,1}$ and $T_{\frac{1}{2},\,0}$

D $D_{2,\,1}$ and $T_{-\frac{1}{2},\,0}$

E $D_{1,\,2}$ and $T_{-1,\,0}$

14 Under successive transformations, $y = -f(-x) \xrightarrow{1} y = -2f(-2x) \xrightarrow{2}$ $y = -2f(-2x + 2) + 3$. The transformations 1 and 2 are:

A $D_{2,\,2}$ and $T_{1,\,3}$

B $D_{\frac{1}{2},\,2}$ and $T_{-1,\,3}$

C $D_{\frac{1}{2},\,2}$ and $T_{1,\,3}$

D $D_{2,\,1}$ and $T_{-1,\,0}$

E $D_{2,\,\frac{1}{2}}$ and $T_{-1,\,3}$

MASTER

15 Find the transformation for each of the successive transformations.

a $y = 3^x \xrightarrow{1} y = 3^{-x} \xrightarrow{2} y = -3^{-x} \xrightarrow{3}$

$y = -7(3^{-x}) \xrightarrow{4} y = -7(3^{-2x}) \xrightarrow{5} y = -7(3^{-2x+2})$

b $y = \dfrac{1}{x} \xrightarrow{1} y = \dfrac{-1}{x} \xrightarrow{2} y = \dfrac{-1}{2x} \xrightarrow{3} y = \dfrac{-1}{2x+1} - 1$

c $y = f(x) \xrightarrow{1} y = 2f(x) \xrightarrow{2} y = -2f(x) \xrightarrow{3}$

$y = -2f(3x) \xrightarrow{4} y = -2f(-3x) \xrightarrow{5} y = -2f(-3x+2) + 2$

16 Starting with the first equation, graph the second equation using successive transformations.

a $y = x \rightarrow y = -2x + 3$

b $y = x^2 \rightarrow y = -2(x-1)^2 + 6$

c $y = \sqrt{x} \rightarrow y = -2\sqrt{2x-1} + 2$

d $y = 3^x \rightarrow y = -2(3^{-2x+2})$

e $x^2 + y^2 = 1, \dfrac{(x+1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$

f $x^2 - y^2 = 1, \dfrac{x^2}{4} - \dfrac{(y-2)^2}{9} = 1$

12.5 Matrices and transformations

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Units 1 & 2
AOS 1
Topic 3
Concept 5

Matrix transformations
Concept summary
Practice questions

Matrices can be used to describe transformations (translation, reflection, rotation and dilation) of points or curves in the x–y plane. The point (x', y') is the image of the point (x, y) following a sequence of transformations. Operations on matrices were studied in Mathematical Methods Units 1 and 2.

Translation

A point P (x, y) is moved to the point P′ (x', y') by translating a units in the positive direction of the x-axis (right) and b units in the positive direction of the y-axis (up). A horizontal translation of a units and a vertical translation of b units can be represented by:

$$(x', y') = (x + a, y + b) \text{ or } T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}.$$

WORKED EXAMPLE 14

The point (2, −3) is translated by the matrix $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$. Find the new coordinates of the point.

THINK	WRITE
1 Use the formula $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$
2 Add the matrices to find the image point (x, y).	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ $(x', y') = (-2, -1)$

Other transformations by matrix multiplication

It is useful to summarise the effects of various transformations as set out in the table below. Here we deal with (2×2) matrices that cause a change in the position of a point. These can be done on calculators or by hand, using the usual matrix multiplication techniques.

Matrix	What it does	Calculation	Change
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	Nothing	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$	$(x', y') = (x, y)$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = x$	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$	$(x', y') = (y, x)$
$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$	$(x', y') = (-y, -x)$
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the x-axis	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$	$(x', y') = (x, -y)$
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection in the y-axis	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$	$(x', y') = (-x, y)$
$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	Dilation by factor k from the y-axis	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$	$(x', y') = (kx, y)$
$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	Dilation by factor k from the x-axis	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$	$(x', y') = (x, ky)$
$\begin{bmatrix} \cos(A) & -\sin(A) \\ \sin(A) & \cos(A) \end{bmatrix}$	Anticlockwise rotation of A degrees about the origin	$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(A) & -\sin(A) \\ \sin(A) & \cos(A) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$	$x' = x\cos(A) - y\sin(A)$ $y' = x\sin(A) + y\cos(A)$

Invariance

Invariant properties of shapes and graphs are those features that do not change under transformations.

Under translations, lengths, areas, angles and orientation remain the same; that is, they are invariant.

Under rotations and reflections, angles and area properties are invariant but orientation changes and the determinants of the 2×2 rotation and reflection matrices are either ± 1.

Under dilations, orientation is invariant, but lengths, areas and angles all change. The determinants of the 2×2 dilation matrices are $\pm k$.

WORKED EXAMPLE 15

The point $(-5, 1)$ is translated by the matrix $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and then reflected in the line $y = x$. Find its new coordinates.

THINK	WRITE
1 Write the point $(-5, 1)$ in matrix form.	$(-5, 1) = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$
2 Use the formula $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$ to translate the point.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix}$
3 Add the matrices to find the image point (x', y').	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$ $(x', y') = (-8, 6)$
4 The reflection matrix in $y = x$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ from the table. Use the formula $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ to reflect the point.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -8 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$
5 Write the answer.	The new point is $(6, -8)$.

WORKED EXAMPLE 16

Find the image of the point $(3, 8)$ under 30° anticlockwise rotation about the origin.

THINK	WRITE
1 Write the point $(3, 8)$ in matrix form.	$(3, 8) = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$
2 Use the rotation formula $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(A) & -\sin(A) \\ \sin(A) & \cos(A) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, where A is 30°.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix}$
3 Evaluate using exact values.	$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix}$
4 Multiply the matrices.	$= \begin{bmatrix} \frac{3\sqrt{3}}{2} - 4 \\ 4\sqrt{3} + \frac{3}{2} \end{bmatrix}$
5 Write the answer.	The new point is $\left(\frac{3\sqrt{3}}{2} - 4, 4\sqrt{3} + \frac{3}{2}\right)$. As a decimal approximation, the point is $(-1.40, 8.43)$.

EXERCISE 12.5 Matrices and transformations

PRACTISE

1 **WE14** The point (1, 4) is translated by the matrix $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Find the new coordinates of the point.

2 The point (−3, −2) is translated by the matrix $\begin{bmatrix} -1 \\ -5 \end{bmatrix}$. Find the new coordinates of the point.

3 **WE15** The point (4, 2) is translated by the matrix $\begin{bmatrix} -3 \\ -5 \end{bmatrix}$ and then reflected in the line $y = x$. Find its new coordinates.

4 The point (0, –3) is translated by the matrix $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and then reflected in the line $y = x$. Find its new coordinates.

5 **WE16** Find the image of the point (–2, 5) under 45° anticlockwise rotation about the origin.

6 Find the image of the point (–1, –2) under 60° anticlockwise rotation about the origin.

CONSOLIDATE

7 Find the new coordinates of Q $(m, -n)$ after the following transformations.

a Reflection by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ followed by reflection in the line $y = x$

b Reflection in the line $y = -x$ followed by a translation of $\begin{bmatrix} 2n \\ -3m \end{bmatrix}$

8 Two transformations as shown are applied to the point M (x, y).

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

a Find the single (2 × 2) matrix that can be used to describe the combined transformations.

b If M is the point (−1, 0), find its new coordinates after the transformations.

9 Write the matrices for the following rotations about the origin.

a 90° clockwise
b 180° clockwise
c 270° clockwise
d 360° clockwise
e 90° anticlockwise
f 45° anticlockwise
g 120° clockwise
h 150° anticlockwise
i 300° clockwise
j 120° anticlockwise

10 Consider the point (−3, −5).

a The image of this point after a reflection through the line $y = x$ followed by a rotation of 180° anticlockwise is:

A (−5, −3)
B (−3, 5)
C (−5, 3)
D (5, 3)
E (3, 5)

b The image of the point (−3, −5) after a rotation of 270° clockwise followed by a translation of $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ is:

A (1, −3)
B (9, −5)
C (7, 1)
D (−1, 1)
E (6, 4)

c The image of the point (−3, −5) after a reflection in the y-axis, followed by a reflection in the x-axis and then an anticlockwise rotation of 45° is:

A $(-\sqrt{2}, -2\sqrt{2})$
B $(-\sqrt{2}, -4\sqrt{2})$
C $(2\sqrt{2}, -2\sqrt{2})$
D $(-4\sqrt{2}, \sqrt{2})$
E $(-\sqrt{2}, 4\sqrt{2})$

11 The point (x, y) is dilated by a factor of 3 from the x-axis, then reflected in the line $y = x$ and then translated 2 units to the left and 1 unit down. The coordinates of the final point are:

A $(3x - 2, y - 1)$ **B** $(3x + 2, y - 1)$ **C** $(y - 2, 3x + 1)$
D $(-2 + 3y, -1 + x)$ **E** $(y + 2, 3x + 1)$

12 a Combine a dilation of factor 2 from the x-axis and factor 3 from the y-axis to find the new coordinates of the square (0, 0), (1, 0), (1, 1), (0, 1).

b Write down the single (2×2) matrix for the combined dilations.

c Write down the single matrix that would represent dilations of factor n from both the x-axis and the y-axis.

13 The point P (a, b) is subjected to a sequence of transformations as follows:

- a dilation of 2 away from the y-axis
- a translation of 5 to the left and 3 downwards
- a reflection in the x-axis
- a reflection in the line $y = x$.

Find the new coordinates of P after all of the above transformations have been completed in the given order, using matrix methods.

14 Find the images of each of the following points under each of the anticlockwise rotations, **i–v**.

a A (4, 0) **b** B (5, 4) **c** C (−3, 6) **d** D (0, 9)

i 30° **ii** 45° **iii** 90°
iv 180° **v** −60°

15 Find the image of the point (7, −2) after a rotation of 270° anticlockwise followed by a translation of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

16 The triangle ABC, with vertices A (2, −1), B (−4, 0) and C (5, 2), is rotated by a clockwise rotation of 45° (i.e. an anticlockwise rotation of −45°).

a Find the new coordinates (A′, B′ and C′) of the rotated triangle.

b Compare the area of triangle A′B′C′ to that of triangle ABC.

MASTER

17 Consider the point A (a, b).

a Is an anticlockwise rotation of 30° equivalent to a reflection in the line $y = x$ followed by a clockwise rotation of 60°? Give reasons for your answer.

b Is a reflection in the x-axis followed by an anticlockwise rotation of 45° equivalent to a transformation by the identity matrix? Give reasons for your answer.

c Is an anticlockwise rotation of 90° equivalent to a reflection in the line $y = x$ followed by a reflection in the line $x = 0$? Give reasons for your answer.

18 The point P (3, 4) is transformed to become the point P′$(2\sqrt{3}, \sqrt{13})$.

a Prove that this transformation represents an anticlockwise rotation of θ° about the origin.

b Calculate the value of θ in degrees.

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12.6 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions using the most appropriate methods
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Units 1 & 2 **Transformations**

Sit topic test

12 Answers

EXERCISE 12.2

1 A' (4, 6)

2 A' (−2, 2)

3 $A_{2,\,2}$

Translated 2 right and 2 upward

4 $T_{2,\,1}$

Translated 2 right and 1 upward

5 A (2, −2)

6 A (−1, −1)

7 $y = x - 4$

8 $y = 2x - 4$

9 $T_{0,\,3}$

10 a $T_{-4,\,-3}$

b $T_{-1,\,-5}$

11 a A' (2, −2)

b A' (−4, −6)

12 a $T_{-2,\,5}$

Translated 2 left and 5 upward

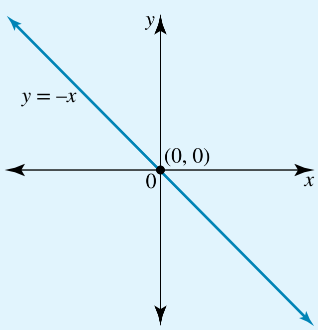
b $T_{1,\,-3}$

Translated 1 right and 3 downward

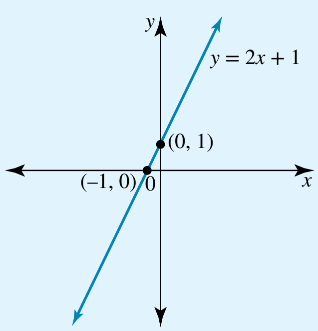
13 a A (−5, −1)

b A (−3, 1)

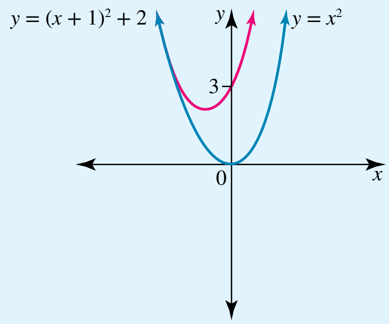
14 a $y = -x$



b $y = 2x + 1$



c $y = (x + 1)^2 + 2$



d $y = 2(x - 2)^2 - 1$

e $y = (x + 3)^2 + 6$

f $y = -2(x + 2)^2 + 3$

g $y = f(x + 3) + 2$

h $y = f(x + 1) - 2$

i $y = -2(x - h)^2 + k$

j $y = -3(x - h) + k$

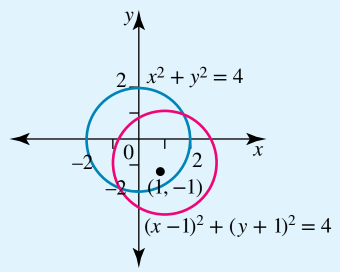
15 A

16 A

17 C

18 B

19



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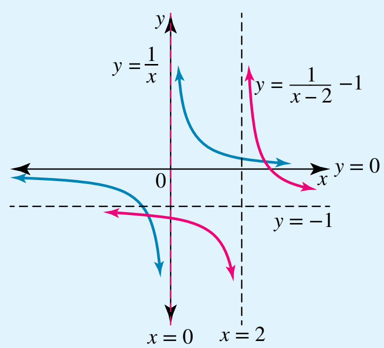


Original asymptotes: $x = 0$, $y = 0$

Image asymptotes: $x = 2$, $y = -1$

21 a $T_{1,\,-3}$

b $T_{2,\,-3}$

c $T_{1,\,2}$

22 a Translated up 3

$T_{0,\,3}$

b Translated up 7 and left 1

$T_{-1,\,7}$

c Translated down 6 and left 3

$T_{-3,\,-6}$

d Translated up 5 and right 5

$T_{5,\,5}$

EXERCISE 12.3

1 a $(-2, -4)$

b $(2, 4)$

c $(4, -2)$

d $(-4, 2)$

2 a $(-1, 2)$

b $(1, -2)$

c $(-2, -1)$

d $(2, 1)$

3 a $y = -3x$

b $y = \frac{1}{3}x$

4 a $y = -2x - 3$

b $y = \frac{1}{2}(x - 3)$

5 a $(4, 3)$

b $\left(2, \frac{3}{2}\right)$

c $(1, 9)$

6 a $(-2, 15)$

b $(-6, 5)$

c $(-4, 10)$

7 a $y = 2x^2$

b $y = \dfrac{x^2}{18}$

8 a $y = 2x + 2$

b $y = \dfrac{3x}{2} + 3$

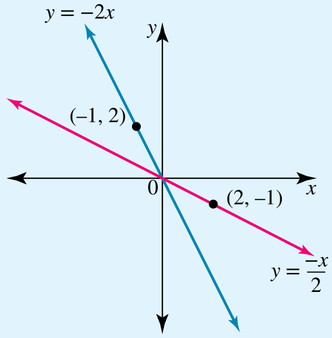
9 $\left(3, \frac{5}{2}\right)$

10 $\left(-1, \frac{3}{2}\right)$

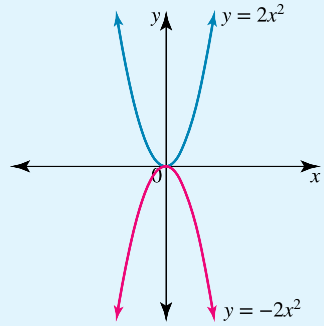
11 $y = 2x$

12 $y = \dfrac{x^2}{2}$

13 a



b



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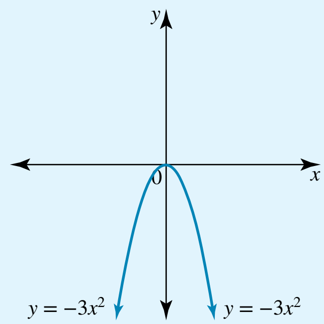
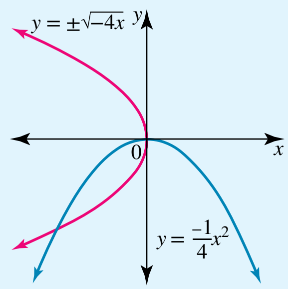
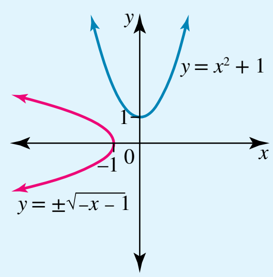


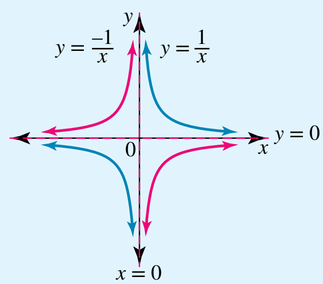
d



e



f



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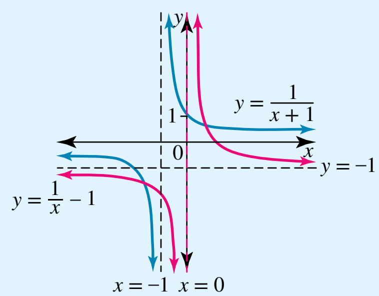

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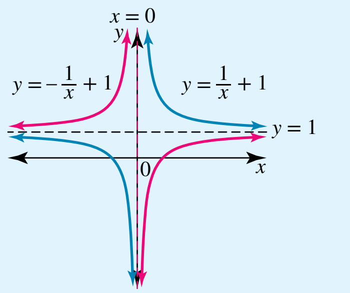

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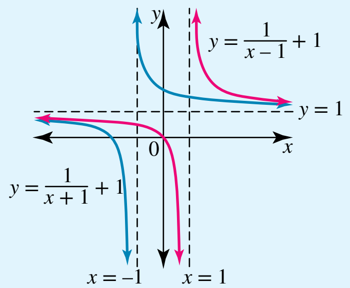

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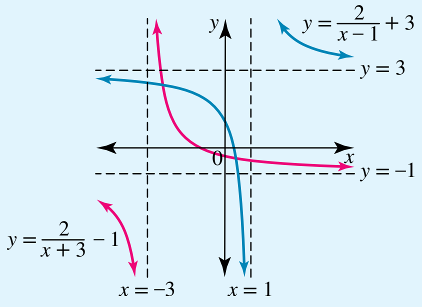

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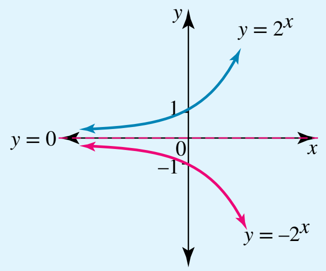

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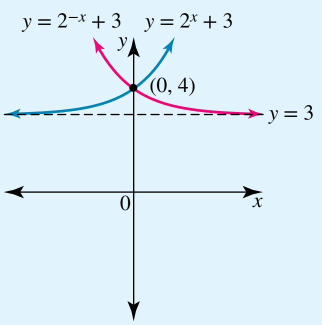

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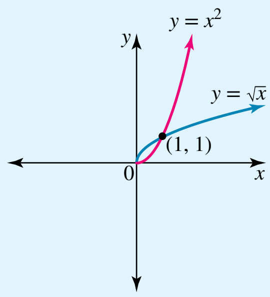

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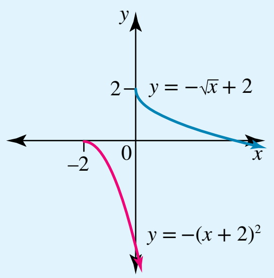


14 A

15 B

16 $y = (x - 6)^2 - 2$

17 **a** $(8, -9)$

b $(-1, -12)$

c $(-4, 3)$

18 **a** $y = \frac{x}{2}$

b $y = \frac{x}{2} + 1$

c $y = 0$

d $y = 0$

e $x = 4$

f $x = 2$

g $x = 0$

h $y = \frac{x^2}{4}$

i $y = \frac{3x^2}{4} - 3$

j $y = 2(2^x)$

k $y = 3 \times 2^{\frac{x}{2}}$

l $y = \frac{2}{x}$

m $y = \frac{4}{x}$

n $y = \frac{3}{4}(x - 2)^2$

o $y = \frac{1}{2}(x + 4)^2 + 2$

p $y = \frac{2}{3}(x - 3)^2 + 4$

q $y = 3 \times 2^{-\frac{x}{2}}$

r $y = bf\left(\frac{x}{a}\right)$

19 B

20 A

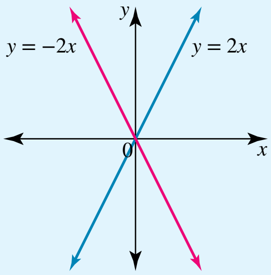
21 **a** $\left(\frac{3}{2}, 5\right)$

b $(-1, -1)$

22 **a** $y = \frac{1}{2} \times 2^{2x}$

b $y = \frac{1}{6x}$

23 a



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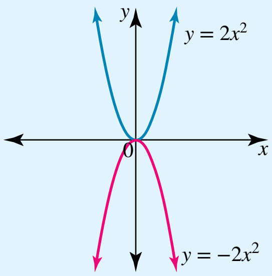

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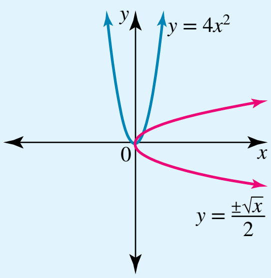

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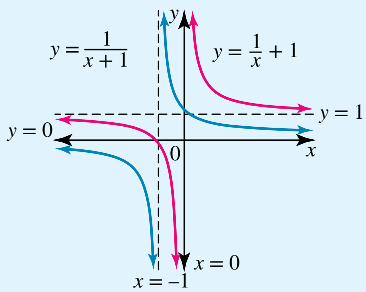

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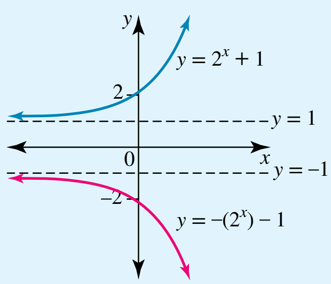

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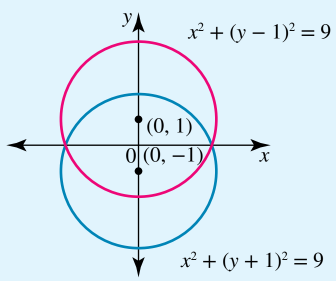

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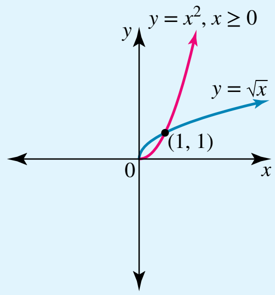

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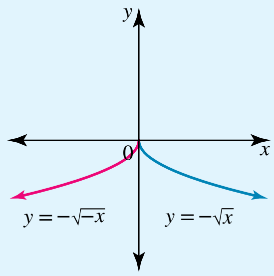


24 a (1, 3)

b (−6, 1)

c (−8, 16)

EXERCISE 12.4

1 a (−2, 13)

b (3, 1)

c (−8, 0)

d (−2, 2)

2 a $y = \dfrac{6}{x - 4} + 12$

b $y = \dfrac{-1}{x + 1} - 2$

c $y = \dfrac{6}{x + 2} - 3$

d $y = \dfrac{4}{x - 2} - 1$

3 a $M_{y = 0}$; Reflected about x-axis

b $M_{x = 0}$; Reflected about y-axis

4 a $T_{1,\,0}$; Translated 1 right

b $D_{1,\,2}$; Dilation factor of 2 from x-axis

5 a $(2x + 3, y + 4)$

b $(2x + 6, y + 4)$

c $(y + 3, x + 4)$

d $(y + 4, x + 3)$

e $(2x + 3, 3y + 4)$

f $(2x + 6, 3y + 12)$

g $(x - 1, y + 7)$

h $(6x, 2y)$

i $(-y, x)$

j $(-x, -y)$

k $(2x + 4, -3y + 9)$

l $(3x + 6, -2y - 6)$

m $(3x + 2, -2y - 3)$

6 a $(6y - 2, -6x - 1)$

b $(-4y + 5, 9x - 4)$

c $(4x + 3, 9y - 4)$

d $(6x - 2, 6y - 4)$

7 C

8 C

9 a $T_{2,3}$; Translated 2 right and 3 upward

b $D_{1,4}$; Dilation factor 4 from x-axis

c $M_{y=0}$; Reflected about x-axis

d $M_{x=0}$; Reflected about y-axis

e $T_{4,0}$; Translated 4 right

f $T_{-\frac{1}{2},-2}$; Translated $\frac{1}{2}$ left and 2 downward

g $M_{y=0}$; Reflected about x-axis

h $T_{\frac{5}{2},0}$; Translated $\frac{5}{2}$ right

i $T_{-1,0}$; Translated 1 left

j $D_{2,5}$; Dilation factor 2 from y-axis and 5 from x-axis

k $D_{\frac{1}{2},\frac{1}{5}}$; Dilation factor $\frac{1}{2}$ from y-axis and $\frac{1}{5}$ from x-axis

l $T_{-4,3}$; Translated 4 left and 3 upward

m $D_{\frac{1}{3},\frac{1}{4}}$; Dilation factor of $\frac{1}{3}$ from y-axis and $\frac{1}{4}$ from x-axis

n $T_{\frac{1}{2},0}$; Translated $\frac{1}{2}$ right

o $D_{\frac{3}{2},2}$; Dilation factor of $\frac{3}{2}$ from y-axis and 2 from x-axis

10 a $T_{\frac{1}{2},0}$; Translated $\frac{1}{2}$ right

b $T_{\frac{3}{2},4}$; Translated $\frac{3}{2}$ right and 4 upward

11 a $M_{y=0}$, $D_{\frac{1}{2},1}$, $T_{0,4}$

Reflected about x-axis, dilation factor of $\frac{1}{2}$ from y-axis and translated 4 upward

b $M_{y=0}$, $D_{1,2}$, $T_{-1,-3}$

Reflected about x-axis, dilation factor of 2 from x-axis and translated 1 left and 3 downward

c $T_{3,4}$, $D_{1,2}$

Translated 3 right and 4 upward, dilation factor of 2 from x-axis

d $D_{\frac{1}{2},1}$, $T_{-\frac{3}{2},0}$

Dilation factor of $\frac{1}{2}$ from y-axis and translated $\frac{3}{2}$ left

e $M_{x=0,\,y=0}$, $D_{\frac{1}{4},1}$, $T_{\frac{3}{4},0}$

Reflected about the x and y-axis, dilation factor of $\frac{1}{4}$ from y-axis and translated $\frac{3}{4}$ right

f $D_{\frac{1}{3},3}$, $T_{\frac{5}{3},0}$

Dilation factor of $\frac{1}{3}$ from y-axis and 3 from x-axis; translated $\frac{5}{3}$ right

g $M_{y=0}$, $D_{\frac{1}{2},4}$, $T_{\frac{1}{2},3}$

Reflected about x-axis and dilation $\frac{1}{2}$ from y-axis and 4 from x-axis; translated by $\frac{1}{2}$ right and 3 upward

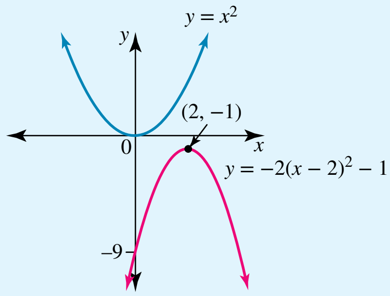
h $M_{x=0}$, $D_{\frac{1}{2},2}$, $T_{-\frac{1}{2},-4}$

Reflected about y-axis, dilation factor of $\frac{1}{2}$ from y-axis and 2 from x-axis; translated $\frac{1}{2}$ left and 4 downward

i $M_{y=0,\,x=0}$, $D_{\frac{1}{2},1}$, $T_{2,7}$

Reflected about x and y-axis; dilation factor of $\frac{1}{2}$ from y-axis and translated 2 right and 7 upward

12 a



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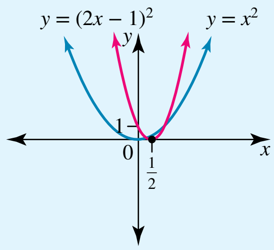

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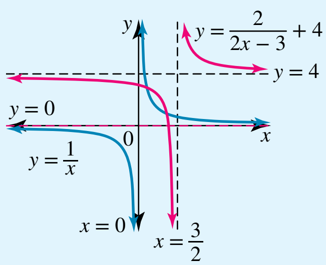

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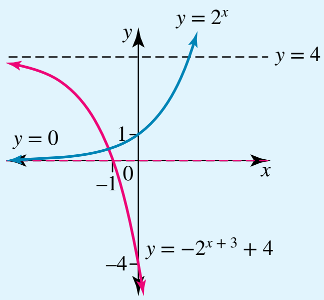

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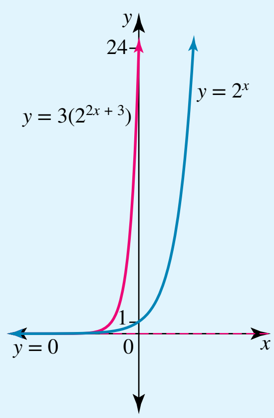

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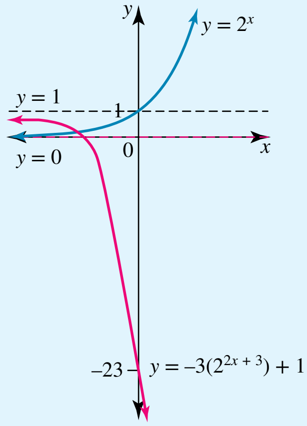


13 B

14 C

15 a 1. $M_{x=0}$ Reflection in the y-axis

2. $M_{y=0}$ Reflection in the x-axis

3. $D_{1,7}$ Dilation factor of 7 from x-axis (parallel to y-axis)

4. $D_{\frac{1}{2},1}$ Dilation factor of $\frac{1}{2}$ from y-axis (parallel to x-axis)

5. $T_{1,0}$ Translation 1 right (positive x-direction)

b 1. $M_{x=0}$ or $M_{y=0}$ Reflection in the y-axis (or x-axis)

2. $D_{\frac{1}{2},1}$ Dilation factor of $\frac{1}{2}$ from y-axis (parallel to x-axis)

3. $T_{-\frac{1}{2},-1}$ Translation $\frac{1}{2}$ unit left and 1 unit down

c 1. $D_{1,2}$ Dilation factor of 2 from x-axis (parallel to y-axis)

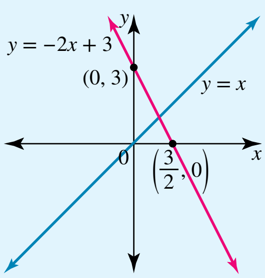
2. $M_{y=0}$ Reflection in the x-axis

3. $D_{\frac{1}{3},1}$ Dilation factor of $\frac{1}{3}$ from y-axis (parallel to x-axis)

4. $M_{x=0}$ Reflection in the y-axis

5. $T_{\frac{2}{3},2}$ Translation $\frac{2}{3}$ of a unit right and 2 units up

16 a



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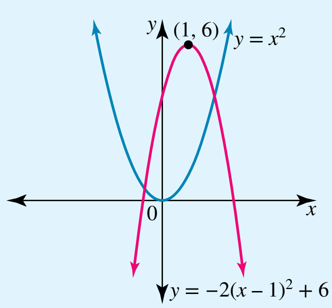

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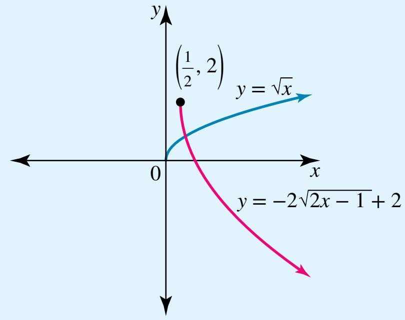

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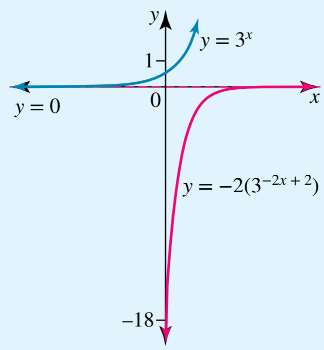

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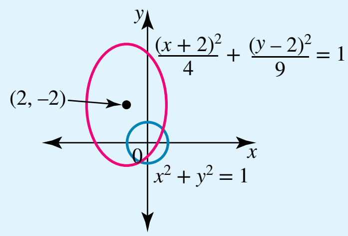

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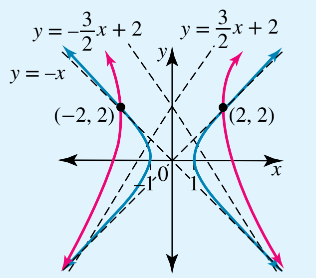

EXERCISE 12.5

1 (3, 1)

2 (−4, −7)

3 (−3, 1)

4 (−4, 2)

5 $\left(\frac{-7\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

6 $\left(\frac{-1}{2} + \sqrt{3}, \frac{-\sqrt{3}}{2} - 1\right)$

7 a $(m, -n)$

b $(3n, -4m)$

8 a $\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$

b (0, −1)

9 a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

c $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

e $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

f $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

g $\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$

h $\frac{1}{2}\begin{bmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{bmatrix}$

i $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

j $\frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

10 a D

b B

c E

11 D

12 a (0, 0), (3, 0), (3, 2), (0, 2)

b $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

c $\begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix}$

13 $(3 - b, 2a - 5)$

14 a i $(2\sqrt{3}, 2)$

ii $(2\sqrt{2}, 2\sqrt{2})$

iii (0, 4)

iv (−4, 0)

v $(2, -2\sqrt{3})$

b i $\left(\frac{5\sqrt{3}}{2} - 2, \frac{5}{2} + 2\sqrt{3}\right)$

ii $\left(\frac{1}{\sqrt{2}}, \frac{9}{\sqrt{2}}\right)$

iii (−4, 5)

iv (−5, −4)

v $\left(\frac{5}{2} + 2\sqrt{3}, \frac{-5\sqrt{3}}{2} + 2\right)$

c i $\left(\frac{-3\sqrt{3}}{2} - 3, \frac{-3}{2} + 3\sqrt{3}\right)$

ii $\left(\frac{-9}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

iii (−6, −3)

iv (3, −6)

v $\left(\frac{-3}{2} + 3\sqrt{3}, \frac{3\sqrt{3}}{2} + 3\right)$

d i $\left(-\frac{9}{2}, \frac{9\sqrt{3}}{2}\right)$

ii $\left(-\frac{9}{\sqrt{2}}, \frac{9}{\sqrt{2}}\right)$

iii (−9, 0)

iv (0, −9)

v $\left(\frac{9\sqrt{3}}{2}, \frac{9}{2}\right)$

15 (0, −3)

16 a $A' = \left(\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right), B' = (-2\sqrt{2}, 2\sqrt{2}), C' = \left(\frac{7}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$

b Areas are the same.

17 a No. Single matrix ≠ combined matrices.

b No. Under the identity matrix nothing moves. Only if $\tan^{-1}\left(\frac{b}{a}\right) = 22.5°$ would this be true.

c Yes. An anticlockwise rotation of 90° $(a, b) \rightarrow (-b, a)$.
A reflection in $y = x$ followed by $x = 0$
$(a, b) \rightarrow (-b, a)$

18 a OP = 5 = OP′. The transformation could be a rotation (in this case anticlockwise) about the origin. (It could also be a translation, but that is not the question.)

b 80.72°

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