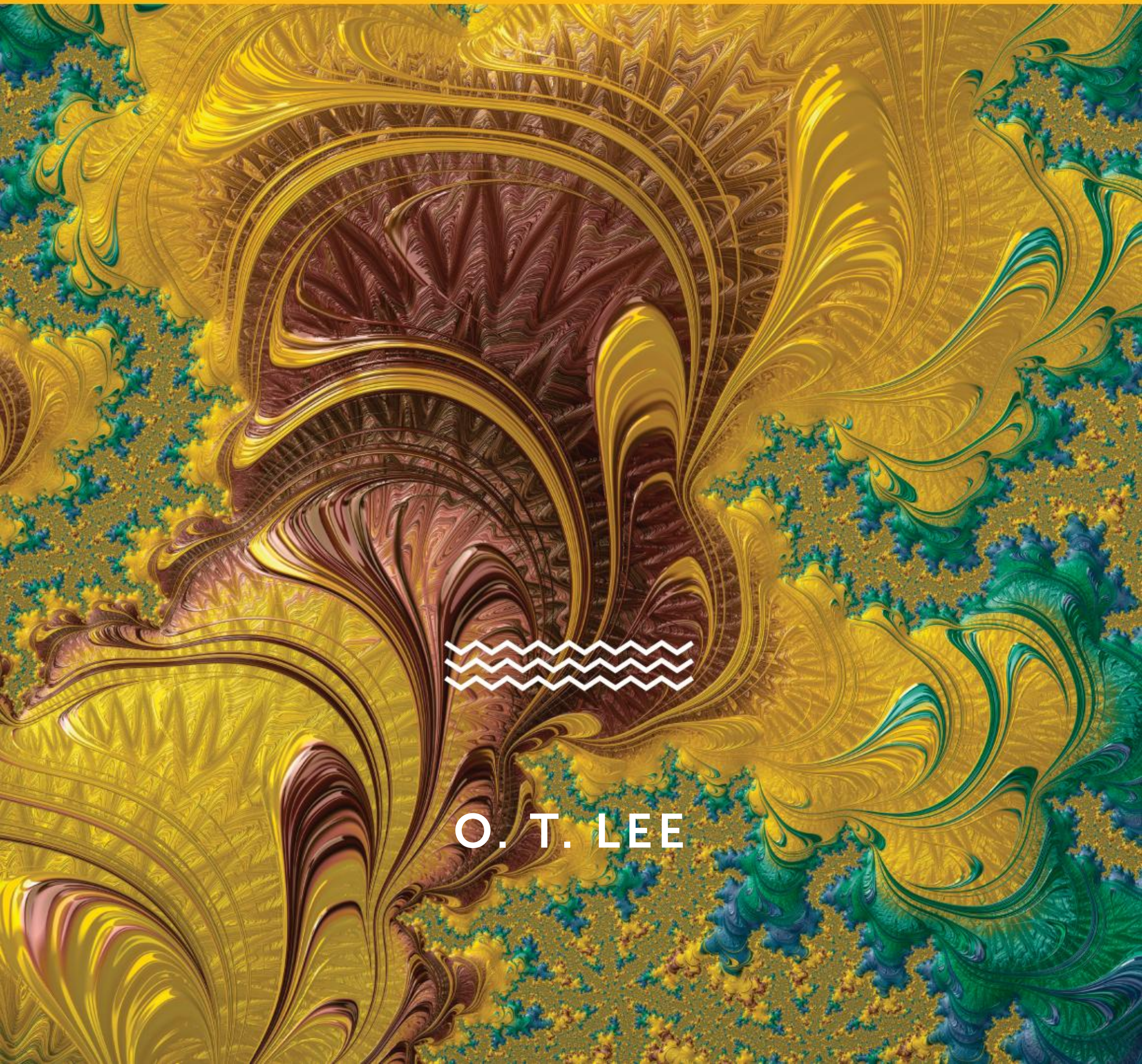
YEAR 12 ATAR COURSE REVISED EDITION

ACADEMIC TASK FORCE

REVISION SERIES

MATHEMATICS METHODS

UNITS 3 & 4



O. T. LEE

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YEAR 12 ATAR COURSE
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SECOND EDITION

O. T. LEE

ACADEMIC GROUP
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About the Author
Dr O. T. Lee is an author of many books which are used extensively in WA schools. Dr Lee is an exceptional, insightful teacher with wide-ranging experience as a WACE marker.

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Mathematics Methods Revision Series Units 3 & 4

Contents

Notes

Mathematics Methods Revision Series
Units 3 & 4

- The Mathematics Methods Revision Series Units 3 & 4 provides a comprehensive set of revision/review questions for the new year 12 Mathematics Methods Units 3 & 4 course.

- The review questions are written at test/examination level for both the Calculator Free and Calculator Assumed Sections and presented in a write-on format in topical order.

- This book exposes students to questions and problems at test/examination level.

- These questions are suitable for end-of-topic reviews and pre-test and pre-examination reviews.

- It is accompanied by a set of fully worked solutions with which students can measure their solutions. These solutions are often not the only solutions but they provide a model for students to work with. Students, interrogate your solutions to understand your errors and your successes. It may sometimes to possible to achieve a correct numerical answer with faulty reasoning!

- Do not memorise solutions. Understand the techniques and processes used in relation to the questions asked.

Notes

Indices and Logarithms

- $a^x \times a^y = a^{x+y}$ $\quad \dfrac{a^x}{a^y} = a^{x-y}$

  $(a^x)^y = a^{xy}$ $\quad a^0 = 1$

  $\dfrac{1}{a^x} = a^{-x}$ $\quad \sqrt[n]{a} = a^{\frac{1}{n}}$

- $a^x = y \Leftrightarrow \log_a y = x$

  $\log_a \mathrm{P} + \log_a \mathrm{Q} = \log_a (\mathrm{P} \times \mathrm{Q})$

  $\log_a \mathrm{P} - \log_a \mathrm{Q} = \log_a \left[\dfrac{\mathrm{P}}{\mathrm{Q}}\right]$

  $\log_a (\mathrm{P}^n) = n \times \log_a \mathrm{P}$

  $\log_a a = 1$

Differentiation

- $f'(x) = \dfrac{d}{dx}f(x) = \lim\limits_{h\to 0}\left[\dfrac{f(x+h)-f(x)}{h}\right]$
- $y = a\,x^n \;\Rightarrow\; y' = n \times a\,x^{n-1}$
- $y = [f(x)]^n \;\Rightarrow\; y' = n \times [f(x)]^{n-1} \times f'(x)$
- $y = u\,v \;\Rightarrow\; y' = u'\,v + u\,v'$
- $y = \dfrac{u}{v} \;\Rightarrow\; y' = \dfrac{v\,u' - u\,v'}{v^2}$
- $y = \sin x \;\Rightarrow\; y' = \cos x$
- $y = \cos x \;\Rightarrow\; y' = -\sin x$
- $y = e^{mx} \;\Rightarrow\; y' = m \times e^{mx}$

  $y = e^{f(x)} \;\Rightarrow\; y' = f'(x) \times e^{f(x)}$

- $y = \ln x \;\Rightarrow\; y' = \dfrac{1}{x}$

  $y = \ln f(x) \;\Rightarrow\; y' = \dfrac{f'(x)}{f(x)}$

Rate of change

- The instantaneous rate of change of Q at time $t = a$ is $Q'(a)$.
- The average rate of change between $t = a$ and $t = b$ is $\dfrac{Q(b)-Q(a)}{b-a}$

Features of graphs

$y = f(x)$	$y = f'(x)$
max point	x-intercept (above – below)
min point	x-intercept (below – above)
inflection point	turning point

Stationary & Inflection Points

- For max point at $x = a$: $y' = 0, y'' < 0$

x	a^-	a	a^+
y'	+	0	–

- For min point at $x = a$: $y' = 0, y'' > 0$

x	a^-	a	a^+
y'	–	0	+

- For horizontal inflection point at $x = a$:
  $y' = y'' = 0$, $y''(a^+)$ and $y''(a^-)$ have opposite signs

x	a^-	a	a^+
y'	$\pm$	0	$\pm$

- For oblique inflection point at $x = a$:
  $y' \neq 0$, $y'' = 0$, $y''(a^+)$ and $y''(a^-)$ have opposite signs

Small Increments

- $\delta y = \dfrac{dy}{dx} \times \delta x$

Marginal Rates

- If $C(x)$ represents the cost of manufacturing x units of a product, then the marginal rate is $\dfrac{dC}{dx}$. This is the cost involved in producing the $(x+1)th$ unit when x units have already been produced.
- If $P(x)$ represents the profit associated with the sale of x units of a product, then the marginal profit is $\dfrac{dP}{dx}$. This is the profit associated with the sale of the $(x+1)th$ unit when x units have already been sold.

Exponential growth and Decay

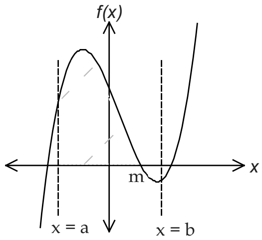
- $\dfrac{dQ}{dt} \propto Q \;\Rightarrow\; \dfrac{dQ}{dt} = kQ \;\Rightarrow\; Q = Q_0 e^{kt}$
  (k is the continuous rate of change)
- Half-life is the time taken for the initial value to be halved (subst. $Q = 0.5\,Q_0$)
- Doubling-time is the time taken for the initial value to be doubled (subst. $Q = 2\,Q_0$)

Integration

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad [n \neq -1]$
- $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1) \times a} + C \quad [n \neq -1]$
- $\int f'(x) \times [f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + C$
  where $n \neq -1$.
- $\int e^{mx} \, dx = \frac{e^{mx}}{m} + C$
- $\int f'(x)\, e^{f(x)} \, dx = e^{f(x)} + C$
- $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$
- $\int \cos(ax+b) \, dx = \frac{\sin(ax+b)}{a} + C$
- $\int \sin(ax+b) \, dx = -\frac{\cos(ax+b)}{a} + C$
- $\int \sec^2(ax+b) \, dx = \frac{\tan(ax+b)}{a} + C$
- $\int_a^b f(x) \, dx = F(b) - F(a) \quad$ where $F'(x) = f(x)$
- $\frac{d}{dx} \int_a^{g(x)} f(t) \, dt = f[g(x)].g'(x)$.

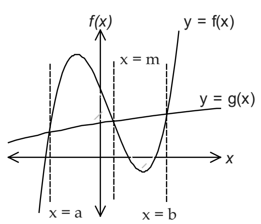
Area

- Area of region trapped between $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is:



$$A = \int_a^b |f(x)| \, dx = \left| \int_a^m f(x) \, dx \right| + \left| \int_m^b f(x) \, dx \right|$$

- Area of region trapped between $y = f(x)$ and $y = g(x)$ as shown is:



$$A = \int_a^b |f(x) - g(x)| \, dx$$

$$= \left| \int_a^m f(x) - g(x) \, dx \right| + \left| \int_m^b g(x) - f(x) \, dx \right|$$

Net Change

- Net change in Q for $a \leq t \leq b$ is $\int_a^b \frac{dQ}{dt} \, dt$

Rectilinear Motion

- Displacement at time t, $x = \int v \, dt$

  Velocity $v = \frac{dx}{dt} = \int a \, dt$

  Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$
- Body returns to origin when $x = 0$.
- Body changes direction when $v = 0$.
- Change in displacement for $a \leq t \leq b = \int_a^b v \, dt$
- Distance travelled for $a \leq t \leq b = \int_a^b |v| \, dt$.

Discrete Random Variables

- If X is a discrete random variable, then its probability distribution function $p(x)$ must satisfy the following 2 conditions:
  - $0 < p(x) < 1$ • $\sum p(x) = 1$ (exact)

  Mean for X, $\mu = E(X) = \sum x \times p(x)$

  Variance $= \sum (x-\mu)^2 \times p(x)$

  $= E(X^2) - [E(X)]^2 = \sum [x^2 \times p(x)] - \mu^2$

The Bernoulli Variable

- A Bernoulli variable assigns "success" the numerical value of 1 and "failure" the numerical value of 0.
- If X is a Bernoulli variable with parameter p,
  - the probability distribution function of X is

    $$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$
  - mean for X, $\mu = E(X) = p$

    variance for X, $\sigma^2 = p(1-p)$

The Binomial Distribution

- The generic definition of a binomial variable is:

  X: No. of successes in n independent Bernoulli trials.
  - The probability of achieving a successful outcome for each trial must be the same for each trial (constant). That is, the trials must be independent.
  - X = 0, 1, 2, 3, ……, n.

    The maximum value for X is n.

- The probability distribution function for a binomial variable X defined over n trials with p as the constant probability of producing a successful outcome for each trial is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $x = 0, 1, 2, 3, \ldots, n$

- If $X \sim B(n, p)$, then:
  the mean of X = np
  the variance of X = $np(1-p)$.

Continuous Random Variables

- The values taken by a continuous random variable are usually "measured" values and stated in the form of an interval. Eg $1 \le X < 5$.
- If X is a continuous random variable, then its probability density function $f(x)$ must satisfy the following 2 conditions:
  - within the defined domain, $f(x) \ge 0$
  - over the defined domain, $\int f(x)\,dx = 1$.
- Interpreted geometrically, within the defined domain:
  - the graph of the pdf is never below the x-axis
  - the area of the region between the graph, the x-axis is exactly 1.
- If $f(x)$ is the probability density function of the continuous random variable X,

then $P(a \le X \le b) = \int_a^b f(x)\,dx$.

Mean for X, $\mu = E(X) = \int_a^b x \times f(x)\,dx$

Variance = $\int_a^b (x-\mu)^2 \times f(x)\,dx$

$= E(X^2) - [E(X)]^2 = \int_a^b x^2 \times f(x)\,dx - \mu^2$

The Uniform Distribution

- The shape of the graph of the pdf of a uniform variable over $a \le x \le b$ is that of a rectangle of width $(b-a)$ and height $\frac{1}{b-a}$.
- If X is uniformly distributed over $a \le x \le b$, then the pdf for X is $f(x) = \frac{1}{b-a}$ for $a \le x \le b$.

Mean for X = $\frac{a+b}{2}$

Variance for X = $\frac{(a-b)^2}{12}$.

The Normal Distribution

- The standard normal variable Z has a mean of 0 and a standard deviation of 1.
- If $X \sim N(\mu, \sigma^2)$ then $\frac{X-\mu}{\sigma} \sim N(0, 1)$.
- If $X \sim N(\mu, \sigma^2)$, the standard score for one of its values x is $\frac{x-\mu}{\sigma}$. The standard score for x defines how many standard deviations x is away from its mean.

Sampling Distribution for sample proportion

- Let the proportion of population with a given attribute be π. Samples of size n are chosen from this population.
  - The sampling distribution of sample proportions of size n has mean π and standard deviation $\sqrt{\frac{\pi(1-\pi)}{n}}$.
  - For $n \ge 30$, the sampling distribution has an approximate normal distribution.

Point & Interval Estimates for π

If the population proportion π is not known, π may be approximated with $\hat{\pi}_0$ the sample proportion of *a* particular sample.

- If the sample size n is large (≥ 30) and $n\hat{\pi}_0 > 5$ and $n\hat{\pi}_0(1-\hat{\pi}_0) > 5$, then the sampling distribution for the sample proportion of size n will be approximately normal with mean $\hat{\pi}_0$ and standard deviation $\sqrt{\frac{\hat{\pi}_0(1-\hat{\pi}_0)}{n}}$.
- Interval Estimates for π.

Confidence Level	Confidence interval
90%	$\hat{\pi} \pm 1.645\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
95%	$\hat{\pi} \pm 1.960\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
99%	$\hat{\pi} \pm 2.576\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
100c %	$\hat{\pi} \pm z_c \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$

- $P(-z_c \le Z \le z_c) = c$

01 Exponential Functions

Calculator Free

1. [3 marks: 1, 2]

   Consider $y = e^{x+1}$.

   (a) State the equation of the horizontal asymptote of this curve.

   (b) Find the point of intersection of this curve with the line $y = \dfrac{1}{e}$.

---

2. [6 marks: 1, 1, 2, 2]

   Consider $y = e^{-2x} - 1$.

   (a) State the equation of the horizontal asymptote of this curve.

   (b) Find the coordinates of the y-intercept of this curve.

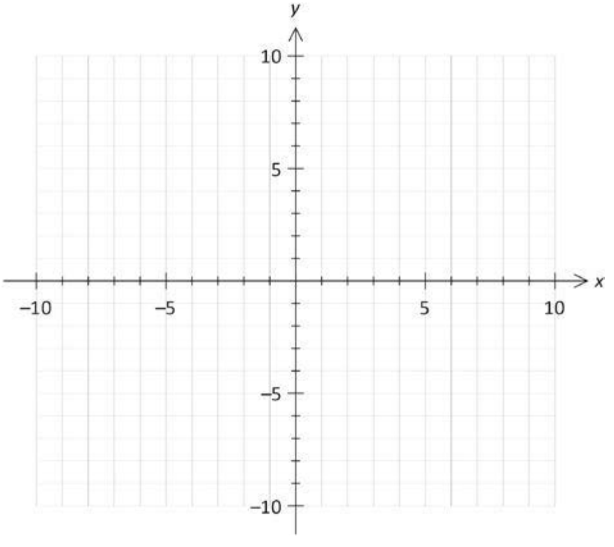
   (c) Find the point of intersection of this curve with the line $y = e^4 - 1$.

   (d) Find the point of intersection of this curve with the curve $y = e^{x-1} - 1$.

Calculator Free

3. [4 marks] [TISC]

Sketch the graph of $y = -e^{-x} - 1$.
Indicate clearly the intercepts and asymptotes where they exist.



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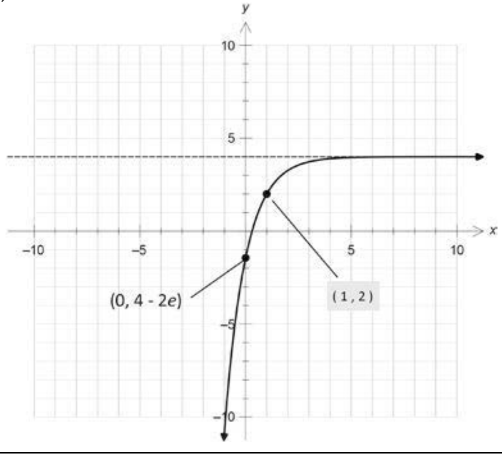
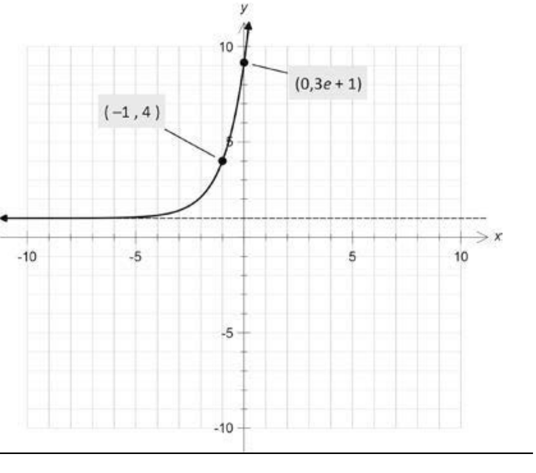
4. [10 marks: 5, 5]

The graphs of $y = Ae^{kx} + B$ are sketched below. Find the values of A, k and B.

(a)



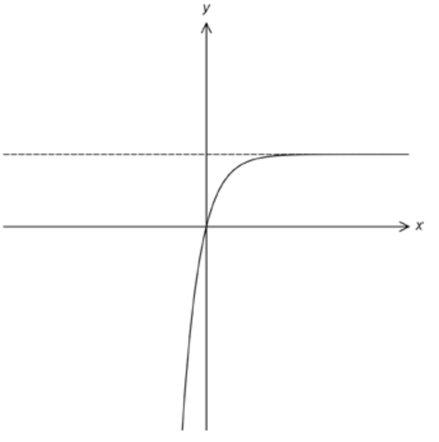
(b)



Calculator Free

5. [6 marks: 3, 3] [TISC]

The accompanying diagram shows the sketch of $y = a + b\,e^{kx}$ where a, b and k are constants.



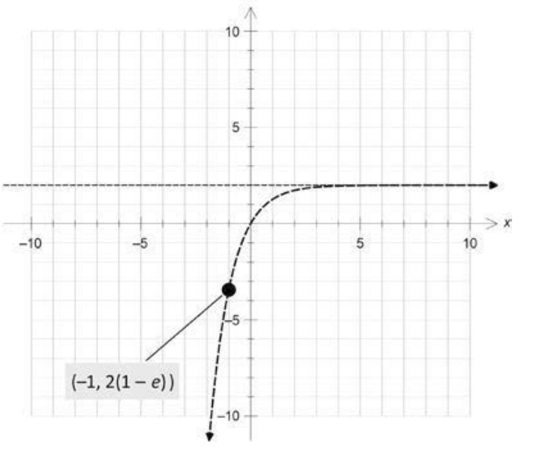
(a) Complete the table below, indicating whether the constants a, b and k have positive or negative values.

Constant	Positive or negative value
a	
b	
k	

(b) Given that the curve passes through the origin and $b = k$, suggest one possible set of numerical values for a, b and k.

6. [6 marks: 3, 3]

The sketch of $y = A\,e^{k\,x} + B$ is given in the accompanying diagram. The curve passes through the point $(-1, 2(1-e))$



(a) Find A, B and k.

(b) On the diagram given above, sketch $y = -A\,e^{-k\,x} - B$
Indicate clearly the y-intercept and asymptotes, if any.

Calculator Assumed

7. [4 marks: 3, 1]

(a) Use your CAS calculator to determine exactly $\lim_{x\to\infty}\left(1+\frac{1}{kx}\right)^x$ for $k = 1, 2$ & 3.

(b) Use your results in (a) to suggest the exact value of $\lim_{x\to\infty}\left(1+\frac{1}{kx}\right)^x$ where k is a positive integer.

---

8. [4 marks: 3, 1]

(a) Determine to 8 decimal places, the value of $\frac{x}{\sqrt[x]{x!}}$ for $x = 50$, 100 and 400.

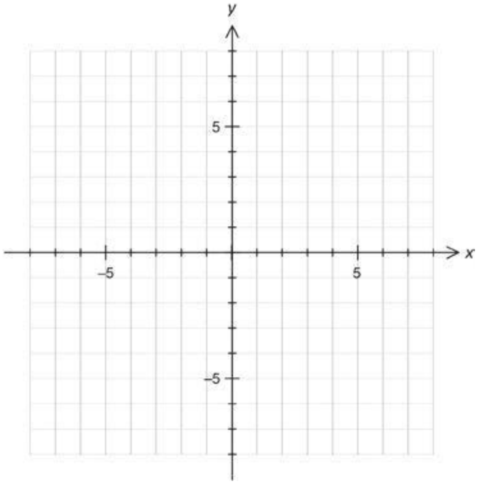
(b) Use your results in (a) to suggest the exact value of $\lim_{x\to\infty}\left(\frac{x}{\sqrt[x]{x!}}\right)$ where $x \in \mathbb{Z}^+$.

02 Logarithms

Calculator Free

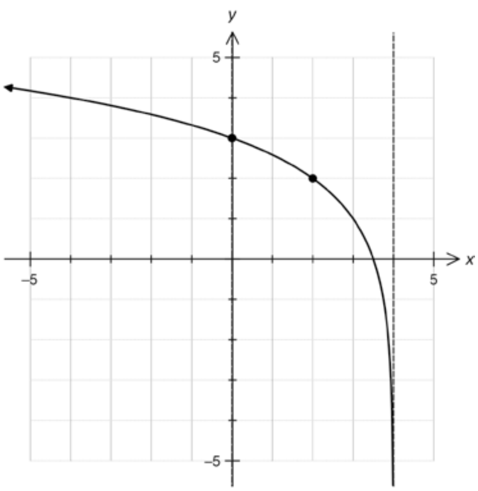
1. [3 marks]

On axes provided below sketch the graph of $y = 2 - \log_2\left(\dfrac{1}{x+2}\right)$. Indicate clearly the asymptotes (if any) and the coordinates of at least two points on this curve.



2. [4 marks]

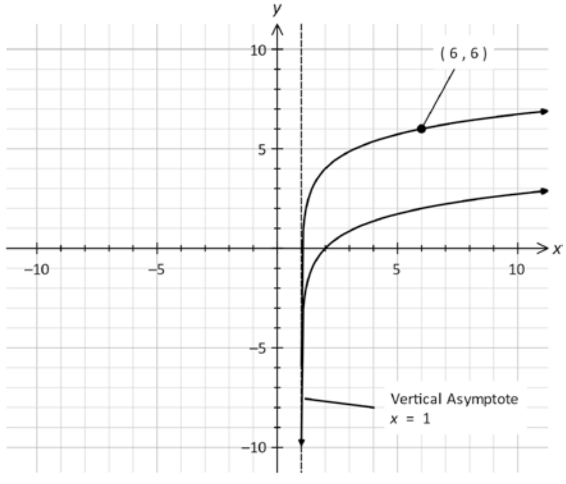
The diagram below shows the graph of $y = a + b\log_c(c - x)$.
Determine the values of a, b and c.



Calculator Free

3. [6 marks: 3, 3]

The accompanying diagram shows the the graphs of $y = 2\ \log_a(x - b)$ and $y = 2\ \log_a(x - b)\ + c$, where $c > 0$.



(a) Determine the values of a, b and c.

(b) On the same set of axes, sketch the graph of $y = -\ \log_a(b - x)$

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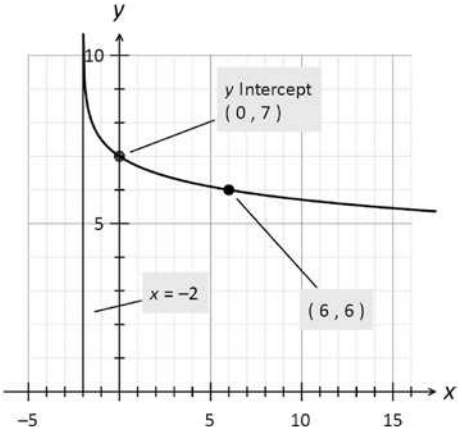
4. [6 marks: 2, 4]

The accompanying diagram shows the graph of $y = a - \ \log_b(cx + d)$ where a, b, c, d and e are real constants.



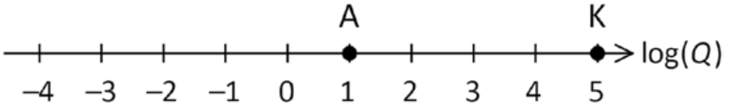
(a) Determine the algebraic relationship between the constants d and e.

(b) The curve passes through the points (0, 7) and (6, 6), find the value of b.

Calculator Free

5. [4 marks: 2, 1, 1] [TISC]

The diagram below shows a logarithmic scale (base 10) for the amount of units of Q contained in a chemical compound. Chemical compound A contains 10 units of Q and is plotted at the point A indicated in the diagram below since log (10) = 1. The point K represents the amount of units of Q in chemical compound K on the same logarithmic scale.



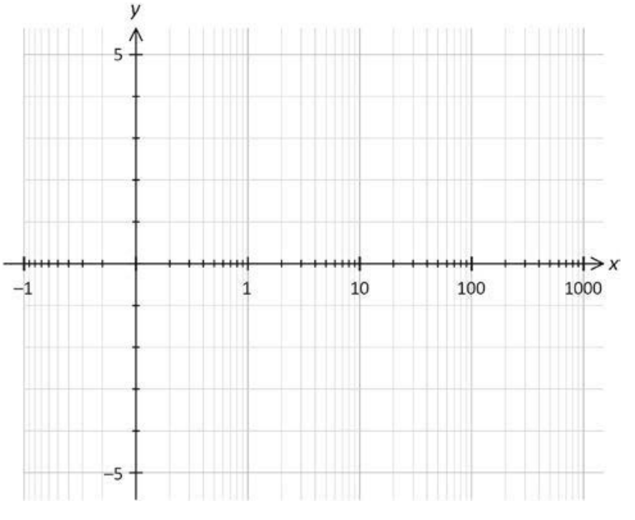
(a) How many times more/less units of Q does compound K have compared to compound A?

(b) On the diagram above, plot and label the point representing compound B which has 100 times more of Q than compound A.

(c) On the diagram above, plot and label the point representing compound C which has 1 000 000 times less of Q than compound K.

6. [3 marks]

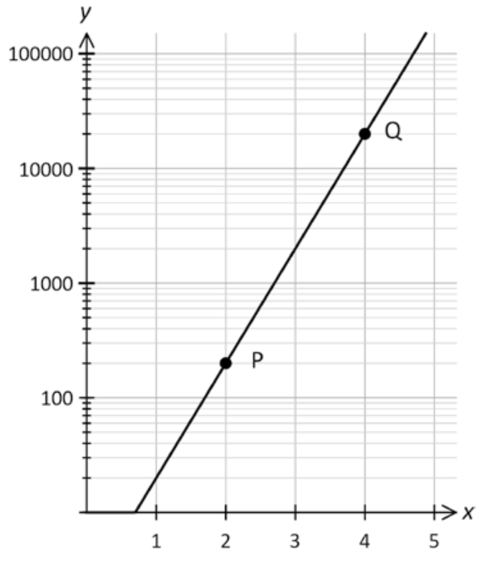
The x-axis in the diagram below is drawn using a logarithmic scale. On the same axes, plot the curve with equation $y = \log(x)$ for $x \geq 1$.



Calculator Free

7. [6 marks: 2, 4]

The accompanying diagram shows the graph of $y = A(10^{kx})$ where the y-axis is in the form of a logarithmic scale. The graph passes through the points P and Q.



(a) State the coordinates of the points P and Q.

(b) Determine the values of A and k.

---

8. [9 marks: 2, 3, 4]

Express in its simplest form :

(a) $\dfrac{\log_4 16}{\log_4 64}$

(b) $2\log 3 + \frac{1}{2}\log 5 - \log 4^{-1}$

(c) $2 + \log_5 4 - \log_5 20$

Calculator Free

9. [5 marks: 2, 3]

(a) Express $-2+\log_3 a+2\log_3 b$ as a single logarithmic term.

(b) Simplify $\dfrac{\log_5(1+2x+x^2)}{\log_5(1+x)}$ where $x > 0$.

---

10. [12 marks: 3, 3, 3, 3]

Given that $p = \log_5 2$ and $q = \log_5 6$, find in terms of p and q :

(a) $\log_5 12$

(b) $\log_5 3$

(c) $\log_5 24$

(d) $\log_5 60$

Calculator Free

11. [9 marks: 3, 3, 3]

Let $p = \log_7 5$ and $q = \log_7 2$.

(a) Find $\log_7 700$ in terms of p and/or q.

(b) Find $\log_7 2.8$ in terms of p and/or q.

(c) Evaluate 7^{p-2q}.

---

12. [7 marks: 3, 4]

Let $p = \log_5 3$ and $q = \log_5 4$.

(a) Find $\log_5 0.15$ in terms of p and/or q.

(b) Evaluate 25^{2p-q}.

Calculator Free

13. [7 marks: 3, 4]

(a) Solve for x in $\log_2 10 + \log_2 x = 2$

(b) Solve for x in $\log_x 4 - \log_x 3 = -2$

14. [6 marks: 3, 3]

Use common logarithms to solve for t where appropriate.

(a) $5^t \times 25^{t-1} = 0.04$

(b) $\dfrac{2^{2t+1}}{2^{1-t}} = 5$

Calculator Free

15. [5 marks]

Use common logarithms to solve for x in $2(3^{2x}) + 5(3^{x}) - 3 = 0$.

16. [6 marks: 2, 4]

(a) Solve for x in $4\log_9(x+1) - 2 = 0$.

(b) Given that $p = \log_2 3$, solve for x in terms of p in the equation $4^{x+2} = 6^{x}$.

Calculator Free

17. [4 marks]

The solution for x in the equation $6^{x-1} = 3^{1+x}$ can be written as $x = \log_2 A$.
Determine the value of A.

---

18. [6 marks: 1, 2, 3]

Consider the equation $5^x = 1\,000$

(a) Solve for x giving your answer in the form $\log_a M$ where a is an appropriate real number and $a \neq e$ and $a \neq 10$.

(b) Use common logarithms to solve for x giving your answer in its simplest form.

(c) Hence, express $\log_5 10^n$ using common logarithms where n is a real number.

Calculator Assumed

19. [5 marks: 1, 2, 2]

The number of decades between two frequencies f_1 and f_2 where $f_2 > f_1$ is defined by $d = \log_{10}\left(\frac{f_2}{f_1}\right)$. The unit of measurement for frequency is Hertz.

(a) Calculate the number of decades between the 3.2 GigaHertz and 5 GigaHertz.

(b) The frequency of signal B is 1 000 times higher than the frequency of signal A. How many decades are there between signal A and signal B.

(b) The frequency of a signal S is 2 decades below 200 MegaHertz. What is the frequency of signal S, f_S ?

---

20. [8 marks: 2, 2, 4]

The Krumbein phi scale ϕ used to compare the size of particles is defined as: $\phi = -\log_2 D$ where D is the diameter of the particle.

(a) Calculate the ϕ number for a particle with size 250 μm. (1 μm = 1×10^{-6} m).

(b) The ϕ number for a bolder is less than –8. Determine the minimum diameter for a particle to be classified as a bolder.

Calculator Assumed

20. (c) How much larger or smaller is a particle A with a ϕ number of –3.4 compared to particle B with ϕ number of 3.4?

---

21. [8 marks: 1, 3, 4] [TISC]

Consider an activity where 1 person in N persons involved in that activity suffer a fatality (death). The safety index of that activity may be defined as $S = \log N$.

(a) Calculate the safety index for activity A if there is 1 death for every 50 000 persons involved in that activity.

(b) Activity B has a safety index of 3.85. Determine with reasons if activity B is safer than activity A.

(c) Activity C recorded 11 deaths in 24 000 persons. The probability of a participant dying in activity D is 0.0075. Use safety indices to determine how many times safer/less safe activity C is compared to activity D.

Calculator Assumed

22. [6 marks: 2, 2, 2] [TISC]

p and q are two terms in a mathematical sequence { …. p, ……, q, …. }.

The number of terms *between* p and q is given by $n = \dfrac{2(\log q - \log p)}{\log 2} - 1$.

(a) The numbers 2 and 64 are part of this sequence of numbers.
Calculate how many terms there are between 2 and 64.

(b) $2\sqrt{2}$ is a term in this sequence.
Find the value of the term that is the 19th term after $2\sqrt{2}$.

(c) Determine an expression for q in terms of p and n.

23. [4 marks]

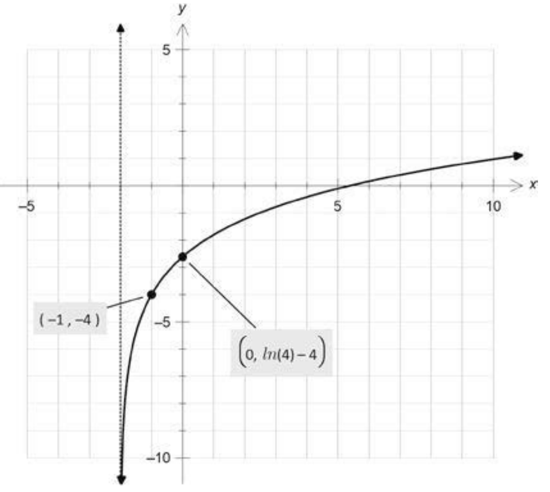
If $\log_2(x+1) = \log_4(x+y)$, find y in terms of x. [Hint: let $\log_2(x+1) = k$]

03 Natural Logarithms

Calculator Free

1. [3 marks]

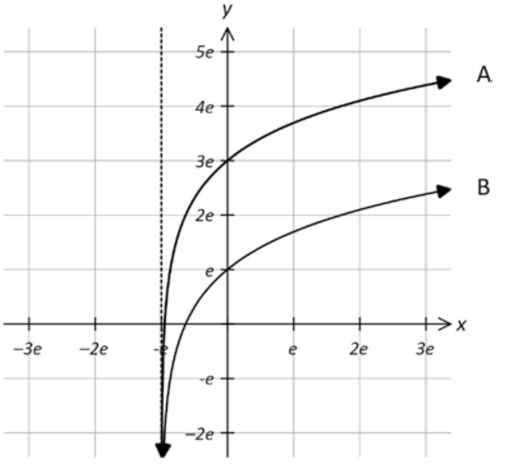
The sketch of $y = k\ ln\ (x + a) + b$ is given in the accompanying diagram. Find a, b and k.



2. [5 marks: 2, 2, 1] [TISC]

The accompanying diagram shows the graphs of $y = k\ ln\ (x + p)$ and $y = q + k\ ln\ (x + p)$, where k, p and q are constants.

(a) Explain clearly why $p = e$.



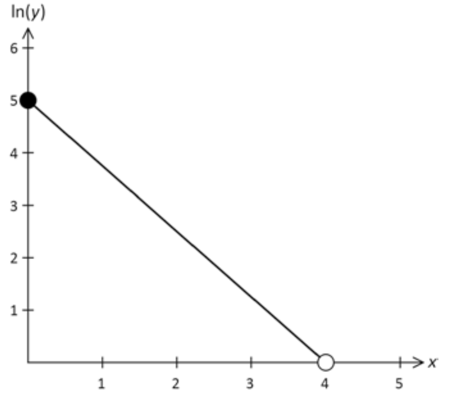
(b) Use the diagram above to explain geometrically why $q = 2e$.

(c) Calculate the value of k.

Calculator Free

3. [5 marks: 2, 3]

The graph of $ln\, y$ against x is given in the accompanying diagram.



(a) Given that $ln\, y = mx + c$, find m and c.

(b) Given that $e^5 \approx 150$, find y in terms of x.

---

4. [7 marks: 1, 2, 4]

Solve *exactly* for x:

(a) $ln\, x = 5$

(b) $ln\,(4x - 2) = -1$

(c) $2(ln\, x)^2 - 5\, ln\, x + 2 = 0$

Calculator Free

5. [7 marks: 3, 4]

(a) Solve *exactly* for x in $100\,e^{-0.02x} = 40$

(b) Solve *exactly* for x in $400\,e^{0.03x} = 500\,e^{0.01x}$

---

6. [9 marks: 2, 4, 3]

(a) Solve *exactly* for t *in* $t\,(e^{t} - 2) = 0$

(b) Solve *exactly* for t *in* $e^{2t} - 3e^{t} + 2 = 0$

(c) Solve *exactly* for t in $2e^{-t} - 2^{t} = 0$

Calculator Free

7. [7 marks: 4, 3]

(a) Solve for t in $e^t - 6e^{-t} + 1 = 0$

(b) Solve for x in the equation $2e^x + e^{0.5x} = 1$.

---

8. [5 marks] [TISC]

Given $2\,ln\,x + ln\,y = ln\,400$ and $ln\,x - ln\,y = ln\left(\frac{5}{2}\right)$,

calculate the values of x and y.

Calculator Assumed

9. [7 marks: 1, 4, 2]

The number of bacteria in a controlled culture is modelled by $P = \dfrac{10\ 000}{10e^{-t} + 40}$

where t is time in hours.

(a) Find the initial population size.

(b) Use logarithms to find *exactly* when the population size will be 240.

(c) Describe the population size for large values of t.

Calculator Assumed

10. [7 marks: 2, 2, 3,]

A translucent plastic sheet reduces the intensity of light that passes through it. The intensity of light after passing through x identical sheets placed adjacent to each other is given by $I = I_0\, e^{kx}$.

(a) Each plastic sheet reduces the intensity of light passing through it by 60%.

(i) Find the value of k correct to four significant figures.

(ii) How many sheets would be required to reduce the intensity of light passing through these sheets by at least 99%?

(c) Five sheets are required to reduce the light intensity by 20%. Find the percentage reduction of light intensity by one sheet.

04 Derivatives: First Principles

Calculator Free

1. [3 marks]

   Use first principles to determine the derivative of $y = 5x^2$ with respect to x.

---

2. [4 marks]

   Use first principles to determine the derivative of $y = \dfrac{1}{x^2}$ with respect to x.

Calculator Free

3. [2 marks]

Use an appropriate derivative to evaluate $\lim_{h \to 0}\left[\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\right]$.

4. [3 marks]

Evaluate $\lim_{h \to 0}\left[\frac{ln(3+h)-ln\,3}{h}\right]$ giving your answer in exact form.
Show clearly how you obtained your answer.

5. [4 marks]

Evaluate $\lim_{h \to 0}\left[\frac{\left(1+\sqrt{5+h}\right)^2-\left(1+\sqrt{5}\right)^2}{h}\right]$ giving your answer in exact form.
Show clearly how you obtained your answer.

05 Differentiation I (Chain Rule)

Calculator Free

1. [10 marks: 1, 2, 2, 2, 3]

Differentiate with respect to x. Leave answers with positive indices.

(a) $4x^3 + 2x^2 - 3x + 5$

(b) $\dfrac{2}{3x^2} - x - 1$

(c) $\dfrac{-1}{2x^3} + 5\sqrt{x}$

(d) $\dfrac{1}{2\sqrt{x}} + \dfrac{5\sqrt{x}}{2}$

(e) $\dfrac{\sqrt{x} - x^4}{3x}$

Calculator Free

2. [2 marks]

   Find the gradient of the curve $y = x^2 + 2\sqrt{x} + 1$ at the point where $x = 1$.

---

3. [5 marks]

   Find the equation of the tangent to the curve $y = \dfrac{x^2 - x^3}{x^4}$ at the point where $x = -1$.

---

4. [5 marks]

   Find the coordinates of the point(s) on the curve $y = \dfrac{1}{x} + x$ with a gradient of 0.

Calculator Free

5. [15 marks: 2, 3, 3, 3, 4]

   Differentiate with respect to x (do not simplify):

   (a) $(1 - 2x^2)^5$

   (b) $\dfrac{1}{(2x+1)^5}$

   (c) $\dfrac{1}{(1+4x)^2} + \dfrac{2}{3(x^2-2)^4}$

   (d) $\dfrac{2}{\sqrt{1-x}} + \sqrt{(2x+1)}$

   (e) $\dfrac{2}{\sqrt[3]{1-3x^2}} - \dfrac{\sqrt{1+x^4}}{3}$

Calculator Free

6. [3 marks]

Find the gradient of the curve $y = (2 - \sqrt{x})^3$ at the point where $x = 4$.

---

7. [5 marks]

Find the equation of the tangent to the curve $y = \left(1 - \dfrac{1}{x}\right)^3$ at the point where $x = -1$.

---

8. [7 marks]

The gradient of the curve with equation $y = \dfrac{1}{ax^2 + bx + 5}$ at the point $(1, \frac{1}{3})$ is 0. Find a and b.

Calculator Assumed

9. [7 marks: 4, 3] [TISC]

Given that $y = \dfrac{4}{(1+x)^3}$ where $x = f(u)$ and $\dfrac{dx}{du} = -1$ for all values of u.

(a) Use the chain rule to determine $\dfrac{dy}{du}$ when $y = \dfrac{1}{2}$.

(b) If $u = g(t)$, use the chain rule to determine $\dfrac{dy}{dt}$ when $x = 0$ and $\dfrac{du}{dt} = 2$.

06 Differentiation II (Product & Quotient Rules)

Calculator Free

1. [12 marks: 2, 3, 3, 4]

Differentiate with respect to x (do not simplify):

(a) $(1 + x^2)(1 - 2x)$

(b) $2x^3(1 - x^2)^4$

(c) $\sqrt{2x}\,(1 + x)^2$

(d) $x^3\sqrt{2 + 3x^2}$

---

2. [12 marks: 2, 3, 3, 4]

Differentiate with respect to t (do not simplify):

(a) $\dfrac{2 - t}{(1 + 2t)}$

Calculator Free

2. (b) $\dfrac{1-4t}{(1-2t)^2}$

(c) $\dfrac{-t}{\sqrt{2t-3}}$

(d) $\sqrt{\dfrac{1+2t}{1-2t}}$

3. [3 marks]

Find the gradient of the tangent to the curve $y = x^2\sqrt{1-x}$ at the point where $x = -3$.

Calculator Free

4. [4 marks]

Find the equation of the tangent to the curve $y = 2x\,(1 + \sqrt{x}\,)^3$ at the point where $x = 1$.

---

5. [5 marks]

A curve has equation $y = (x^2 - 1)(1 + x)^3$. Show that $y' = (5x - 3)(1 + x)^3$.
Hence, find the x-coordinates of the point(s) where the gradient of the curve is 0.

---

6. [7 marks]

Find the coordinates of the point(s) on the curve $y = \dfrac{2x}{1-x}$ at which the tangents are parallel to the line $2y = x - 1$.

Calculator Free

7. [6 marks: 3, 3]

Consider the functions $f(x)$ and $g(x)$ where $f(3) = 1$, $f'(3) = -6$, $g(3) = 2$ and $g'(3) = \frac{1}{4}$.

(a) Given that $h(x) = [f(x)]^3$, determine the value of $h'(x)$ when $x = 3$.

(b) Given that $h(x) = f(x) \times g(x)$, determine the value of $h'(x)$ when $x = 3$.

---

8. [6 marks: 3, 3]

Consider the functions $f(x)$ and $g(x)$ where $f(1) = g(1) = 1$, $f'(1) = 2$ and $g'(1) = -1$.

(a) Given that $h(x) = [f(x) + g(x)]^4$, determine the value of $h'(1)$.

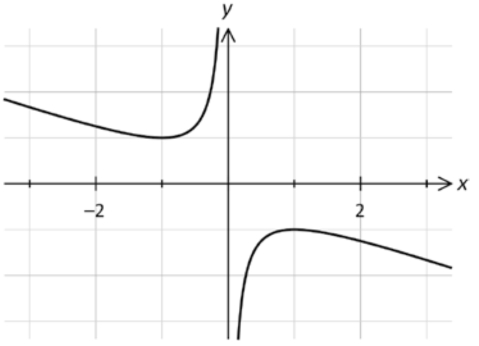
(b) Given that $y = \dfrac{2f(x)}{1 + g(x)}$, determine the value of $h'(1)$.

07 Differentiation III (Graphs)

Calculator Free

1. [4 marks: 2, 2]

The graph of $y = f(x)$ is given in the accompanying diagram.



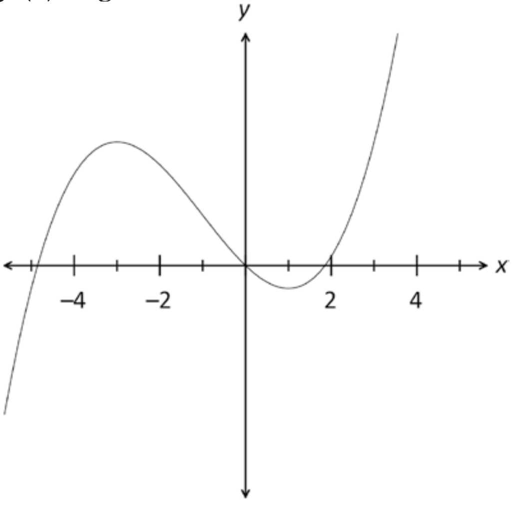
(a) Find the x-coordinate of the point(s) where the gradient of the curve is 0.

(b) For what values of x is the gradient of the curve negative?

---

2. [5 marks: 2, 3]

The graph of $y = f(x)$ is given below.



(a) For what values of x is the gradient positive?

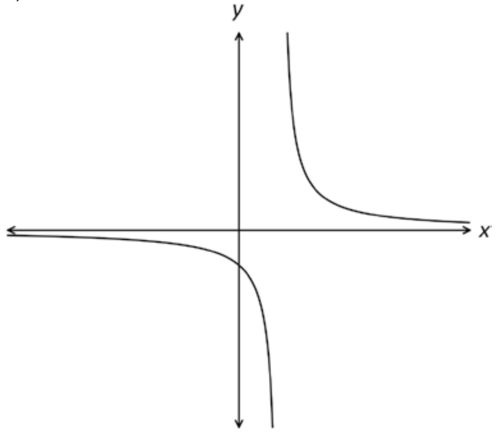
(b) Sketch on the same axes, a possible graph of $y = f'(x)$.

Calculator Free

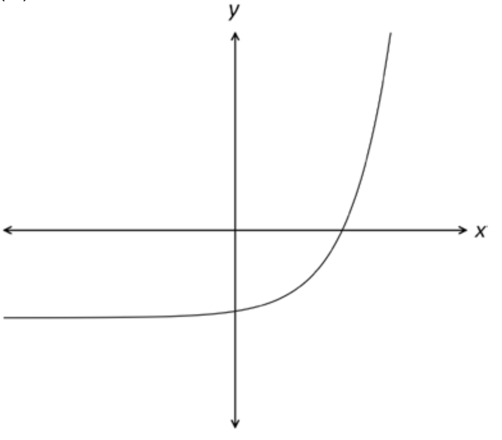
3. [6 marks: 3, 3]

The graph of $y = f(x)$ is given below. Sketch on the same axes, a possible graph of $y = f'(x)$.

(a)

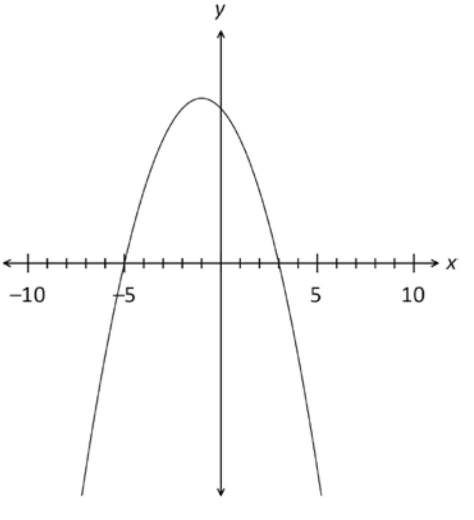


(b)



4. [6 marks: 2, 1, 3]

The graph of $y = f'(x)$ is given in the accompanying diagram.



(a) State the x-coordinate of the point(s) where the gradient of $y = f(x)$ is zero.

(b) State the x-coordinate of the minimum turning point on $y = f(x)$.

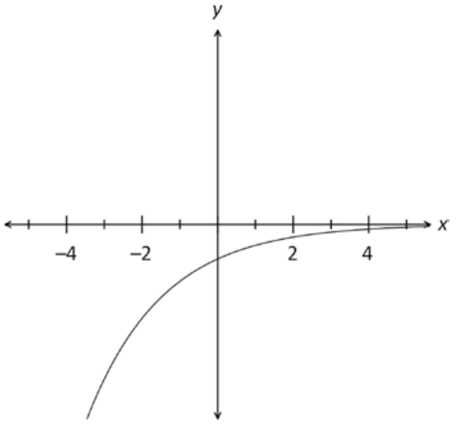
(c) Sketch on the same axes a possible graph of $y = f(x)$.

Calculator Free

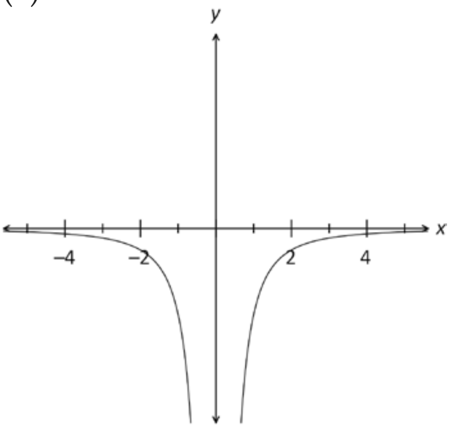
5. [4 marks: 2, 2]

The graph of $y = f\,'(x)$ is given below.
Sketch on the same axes, a possible graph of $y = f\,(x)$.

(a)

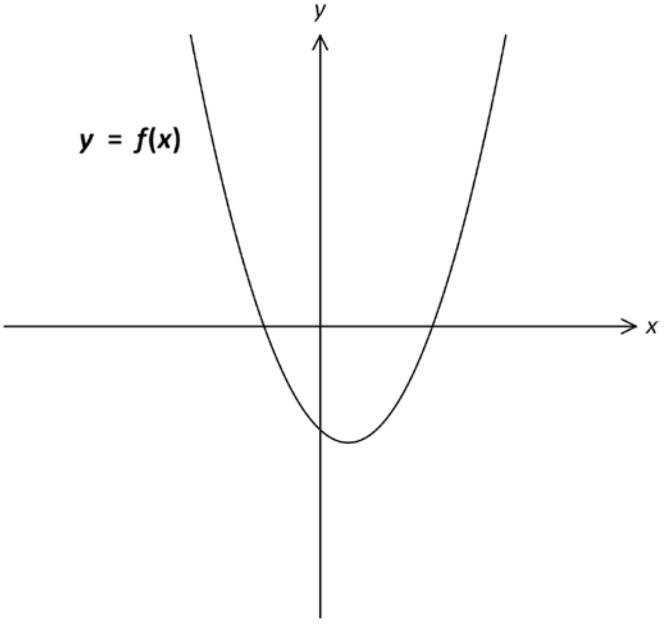


(b)



6. [3 marks]

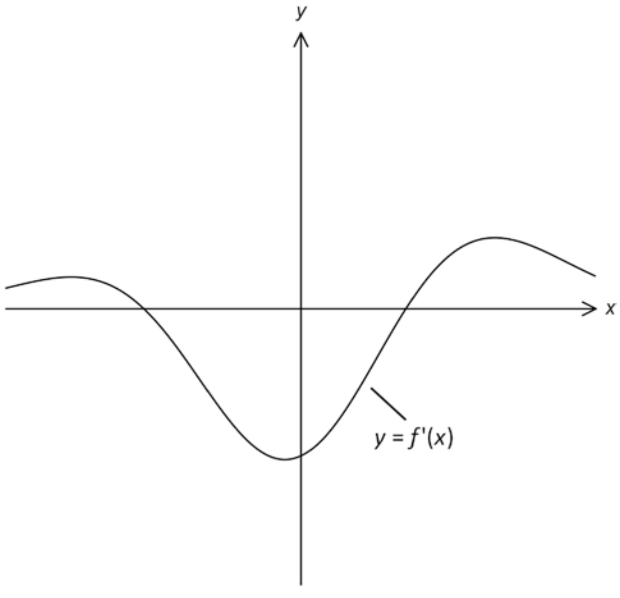
The sketch of $y = f\,(x)$ is given below. On the same set of axes, sketch a possible graph of $y = \int f(x)\ dx$.



Calculator Free

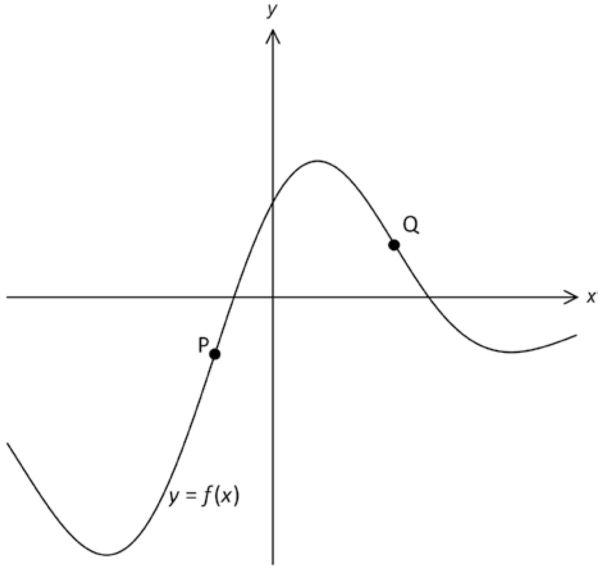
7. [3 marks]

The sketch of $y = f'(x)$ is given below. On the same set of axes, sketch a possible graph of $y = f''(x)$.



8. [4 marks]

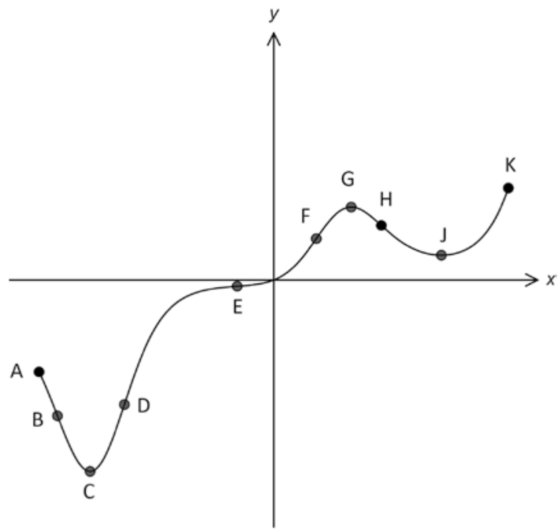
The sketch of $y = f(x)$ is given below. The points P and Q are inflections points on the graph of $y = f(x)$. On the same set of axes, sketch a possible graph of $y = f'(x)$.



Calculator Free

9. [5 marks: 2, 3]

The sketch of $y = f(x)$ is given in the accompanying diagram.



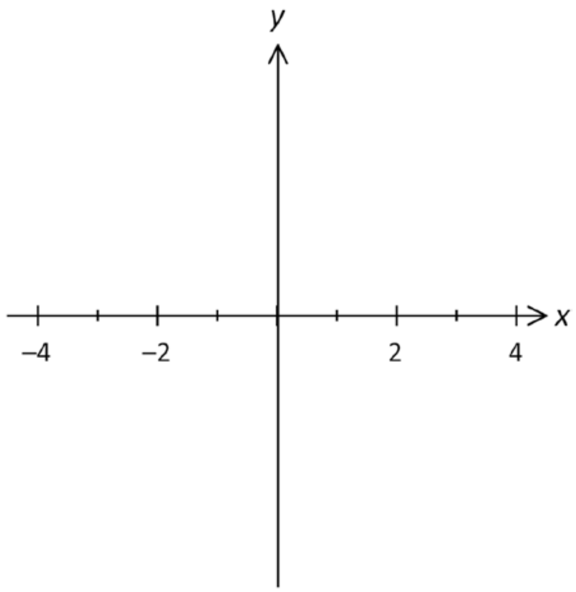
(a) Between which points on the curve is $f'(x) \geq 0$?

(b) Between which points on the curve is $f''(x) \leq 0$?

---

10. [4 marks]

The curve $y = f(x)$ cuts the x-axis at the origin and nowhere else.
Also, $f'(1) = f'(2) = 0$ and $f'(x) < 0$ only for $1 < x < 2$.
Give a possible sketch of $y = f(x)$.



08 Differentiation of Exponential Functions

Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

Find $\dfrac{dy}{dx}$.

(a) $y = 5\,e^{3x}$

(b) $y = \dfrac{4}{5e^{x}}$

(c) $y = e^{x} + \dfrac{1}{2e^{2x}}$

(d) $y = \dfrac{e^{2x} + e^{x}}{e^{3x}}$

(e) $y = e^{2x^2 + 3x}$

Calculator Free

2. [11 marks: 2, 3, 3, 3]

Find $\frac{dy}{dx}$.

(a) $y = (e^2 + e^{2x})^3$

(b) $y = x^4 e^{x^2}$

(c) $y = \frac{e^{2x}}{1+x^2}$

(d) $y = \frac{x^2 - e^{2x}}{2e^x}$

Calculator Free

3. [4 marks]

Find the equation of the tangent to the curve $y = -x\,e^{2x}$ at $x = \frac{-1}{2}$.

4. [5 marks]

Given that $y = x^2\,e^x$, prove that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2e^x\,(1 + x) = 0$

09 Differentiation of Logarithmic Functions

Calculator Free

1. [12 marks: 2, 2, 2, 2, 2, 2]

Differentiate with respect to x:

(a) $y = ln\,(1 + x^3)$

(b) $y = ln\,x^4$

(c) $y = ln\,(1 - 2x)^3$

(d) $y = ln\,\sqrt{x^2 + 2x}$

(e) $y = ln\,(1 + e^{2x})$

(f) $y = \log_2(1 + x)$

Calculator Free

2. [9 marks: 3, 3, 3]

Differentiate with respect to x:

(a) $y = ln\,[(x + 1)^2(2x - 1)^3]$

(b) $y = ln\left[\dfrac{1+2x}{1+x^2}\right]$

(c) $y = ln\,[x^2\sqrt{(3 - 2x)}]$

---

3. [9 marks: 3, 3, 3]

Differentiate with respect to x:

(a) $y = x^2\,ln\,(1 - x^3)$

Calculator Free

3. (b) $y = (1 + e^x)\ ln\ (x^2 + x - 1)$

(c) $y = \dfrac{ln\ (x)}{x}$

4. [4 marks: 2, 2]

(a) Express 7^x in the form e^{ax} where a is a constant.

(b) Hence, determine $\dfrac{d}{dx}(7^x)$

Calculator Free

5. [2 marks]

Find the gradient of the curve $y = ln\,(2x - 5)$ at the point where $x = 3$.

---

6. [4 marks]

Find the coordinates of the point on $y = ln\,(1 + 2x)$ where the gradient of the curve is 2.

---

7. [6 marks]

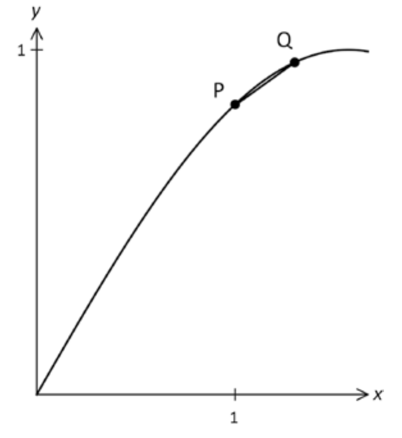
Find the coordinates of the point(s) on the curve $y = x^2\ ln(x)$ where the gradient of the curve is zero.

10 Differentiation of Trigonometric Functions

Calculator Assumed

1. [8 marks: 2, 1, 1, 4] [TISC]

The accompanying diagram shows the graph of $y = \sin x$ where x is measured in radians.
The point P has coordinates $(x, \sin(x))$ and the point Q has coordinates $(x + \delta x, \sin(x + \delta x))$.



(a) Write an expression for the gradient of the line PQ.

(b) For $x = 1$, complete the table below for the given value of δx.

	$\delta x = 0.0001$	$\delta x = 0.000001$
$\dfrac{\sin(x+\delta x)-\sin(x)}{\delta x}$		

(c) For $x = 2$, complete the table below for the given value of δx.

	$\delta x = 0.0001$	$\delta x = 0.000001$
$\dfrac{\sin(x+\delta x)-\sin(x)}{\delta x}$		

(d) Compare the values of $\dfrac{\sin(x+\delta x)-\sin(x)}{\delta x}$ in the tables above with the values of $\cos(x)$ for $x = 1$ and $x = 2$. Explain why these results suggest that the derivative of $\sin(x)$ could be $\cos(x)$.

Calculator Free

2. [12 marks: 1, 2, 2, 3, 2, 2]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \cos \frac{\pi}{4}$

(b) $y = \sin (1 + 2x)$

(c) $y = \sin (1 + \frac{1}{x})$

(d) $y = \cos (1 - 2x)^3$

(e) $y = \tan x^2$

(f) $y = \frac{\sin x + \cos x}{\cos x}$

Calculator Free

3. [13 marks: 2, 3, 3, 3, 2]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = (1 - \cos t)^3$

(b) $y = \tan^3(2t)$

(c) $y = \sin^2(1 + \sqrt{t})$

(d) $y = \sqrt{\cos(e^{2t})}$

(e) $y = e^{\sin 3t}$

Calculator Free

4. [15 marks: 2, 3, 4, 3, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = x^2 \cos(1 - x)$

(b) $y = x \sin x^2$

(c) $y = (1 + 2x)^3 \tan(1 - \sqrt{x})$

(d) $y = x^2 \ln(\cos x)$

(e) $y = \cos 2x \ \sin(1 + 3x)$

Calculator Free

5. [15 marks: 3, 3, 3, 3, 3]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = \dfrac{t^2}{\sin(1+2t)}$

(b) $y = \dfrac{\cos^2 t}{t}$

(c) $y = \dfrac{e^t}{\tan t}$

(d) $y = \dfrac{\sin t}{\cos^2 t}$

(e) $y = \dfrac{e^{\sin t}}{ln \sin t}$

Calculator Free

6. [2 marks]

Find the gradient of the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

7. [4 marks]

Find the coordinates of the point on $y = \sin^2 x$ for $0 \le x \le 2\pi$ where the gradient of the curve is 1.

8. [3 marks]

Find the equation of the tangent to the curve $y = x + \cos x$ at the point $(\pi, \pi - 1)$.

11 Differentiation -Miscellaneous

Calculator Free

1. [8 marks: 2, 3, 3]

(a) Determine $\frac{d}{dx}\left(\sin^3(2x)\right)$.

(b) Determine $\frac{d}{dx}\left((\cos x)\ ln(2+e^{2x})\right)$.

(c) Determine $\frac{d}{dx}\left(\frac{\sqrt{x}}{1-x^2}\right)$. Simplify your answer.

---

2. [8 marks: 2, 3, 3]

(a) Determine $\frac{dy}{dx}$ given that $y = (1+e^{2x})^4$.

Calculator Free

2. (b) Given $f(x) = x^2 \, ln(2 + \sin x)$, determine $f'(x)$.

(c) Differentiate $\dfrac{x}{1 + x^4}$ with respect to x, simplifying your answer.

---

3. [9 marks: 3, 3, 3]

(a) Determine $\dfrac{d^2y}{dx^2}$ given that $\dfrac{dy}{dx} = (x + ln\,(1+x^2)\,)^3$.

(b) Given that $y' = x\,e^{\sin 2x}$, determine y'' .

(c) Given $f'(x) = \dfrac{1}{\cos 2x}$, determine $f''(x)$.

Calculator Assumed

4. [11 marks: 5, 2, 2, 2]

A curve has equation $y = x^3 + 5x^2 + 3x + 2$.

(a) Use calculus to find all points on the curve where the gradient is 0.

(b) Find $\frac{d^2y}{dx^2}$.

(c) Find the x-coordinate of the point on this curve where $\frac{d^2y}{dx^2} = 0$.

(d) Find the gradient of this curve at the point where $\frac{d^2y}{dx^2} = 0$.

---

5. [4 marks]

Use calculus to find the equation of the tangent to the curve $y = x^3\left(1+\frac{1}{\sqrt{x}}\right)^2$ at the point (1, 4).

Calculator Assumed

6. [8 marks: 2, 4]

A curve has equation $y = (x + 1)^2(x - 2)$.

(a) Find the x-intercepts of the curve.

(b) Use derivatives to find the equation of the tangents to this curve at each of the x-intercepts

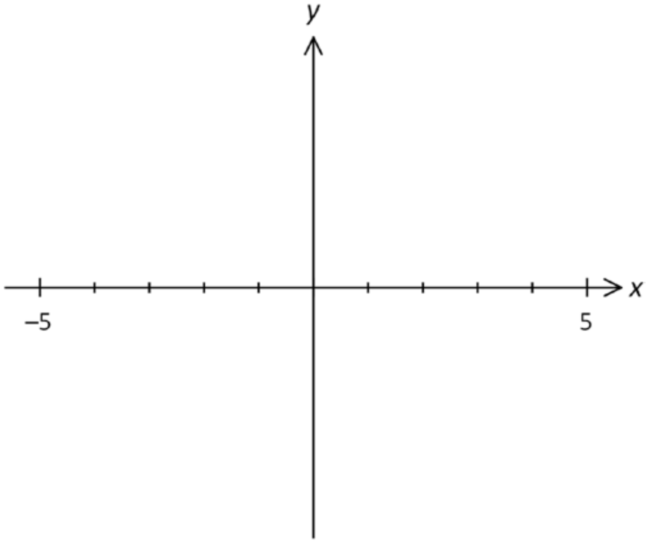
---

7. [5 marks]

The graph of $y = f(x)$ has the following properties:

- $y \geq 0$ for all x
- $\dfrac{dy}{dx} = 0$ for $x = -2, 0, 3$
- $\dfrac{dy}{dx} \geq 0$ for $-2 \leq x \leq 0$ and $x \geq 3$

Sketch a possible graph for $y = f(x)$.



Calculator Assumed

8. [8 marks]

The graph of $y = \dfrac{ax+b}{cx+d}$,where a, b, c and d are non-zero constants, intersects the x-axis at the point where $x = \dfrac{-1}{2}$. It also intersects the y-axis at the point where $y = 1$. At the point it intersects the y-axis, it is also parallel to the line $y = 3x + 10$. Find the values of a, b, c and d.

12 Stationary & Inflection Points

Calculator Free

1. [8 marks]

   Consider the curve with equation $y = x^2 + \frac{1}{x^2}$. Use Calculus to find all the stationary points on this curve. State the nature of each stationary point.

---

2. [7 marks]

   Use Calculus to find the exact coordinates of the turning point(s) on the curve $y = xe^{0.05x}$. State the nature of the turning point(s).

Calculator Free

3. [7 marks]

Consider the curve with equation $y = \sin(x) - x\cos(x)$, for $0 \le x \le \pi$.
Use a calculus method to determine the coordinates of the stationary point on this curve and identify the nature of this point.

---

4. [7 marks]

Use Calculus to determine the coordinates of the point(s) of inflection of the curve with equation $y = x(1 - x)^3$.

Calculator Free

5. [9 marks: 2, 7]

(a) Given that $x^3 - 3x + 2 = (x-1)(x^2 + bx + c)$, determine the values of b and c.

Hence, solve for x in $x^3 - 3x + 2 = 0$.

(b) Consider the curve with equation $y = x^5 - 10x^3 + 20x^2$.
Use a calculus method to determine the coordinates of the inflection point(s) on this curve.

6. [4 marks] [TISC]

Use calculus to explain why the curve $y = x^4$ does not have a point of inflection.

Calculator Free

7. [6 marks]

Consider the curve with equation $y = e^x \sin(x)$ for $0 \le x \le \pi$. Use a calculus method to determine the coordinates of the inflection point(s) on this curve.

---

8. [7 marks: 3, 4] [TISC]

(a) Given that $y = ln\,(1 + x^2)$, show that $\dfrac{d^2y}{dx^2} = \dfrac{2(1-x^2)}{(1+x^2)^2}$.

Calculator Free

8. (b) Show that the curve with equation $y = ln\,(1 + x^2)$ for $x > 0$ has one inflection point. Also show that this inflection point is not a stationary point.

---

9. [10 marks]

A curve has equation $y = ax^3 + bx^2 + cx + d$. The curve has an inflection point at $x = -2$, a turning point at $x = 1$, a y-intercept at $(0, -33)$ and a tangent with equation $y = -24x - 37$ at $x = -1$. Find the values of a, b, c and d. Show clearly how you obtained your answer.

Calculator Free

10. [8 marks: 2, 2, 4]

The points A, B, C, D, E, F and G are points on the graph of a continuous function $y = f(x)$. The table below shows the sign of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at these points.

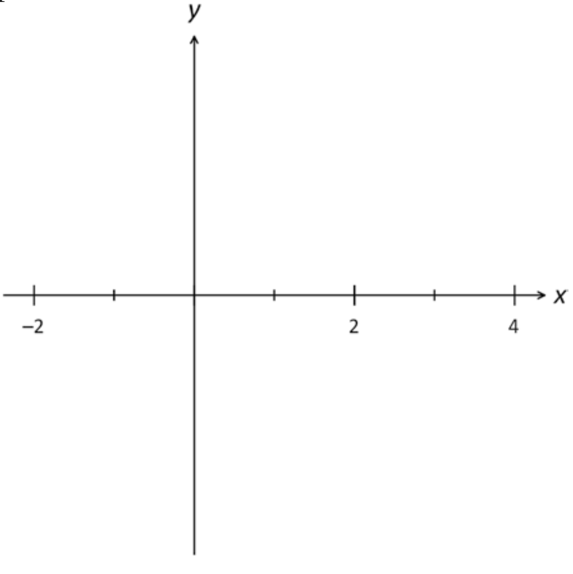
y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ have zero values at only the points indicated in this table.

Point	A	B	C	D	E	F	G
x	–2	–1	0	1	2	3	4
y	–	0	+	+	+	0	–
dy/dx	+	+	0	–	–	0	–
d^2y/dx^2	–	–	–	0	+	0	–

(a) Identify the maximum point on this graph. Justify your answer.

(b) Identify the inflection point(s) if any on this graph.

(c) Sketch this graph.



Calculator Free

11. [7 marks: 5, 2] [TISC]

Consider the curve with equation $y = \dfrac{2x^2 - 1}{1 - x^2}$. This curve has one stationary point.

(a) Find the coordinates of this stationary point.

(b) Use an appropriate test to determine if this stationary point is a minimum point, maximum point or an inflection point.

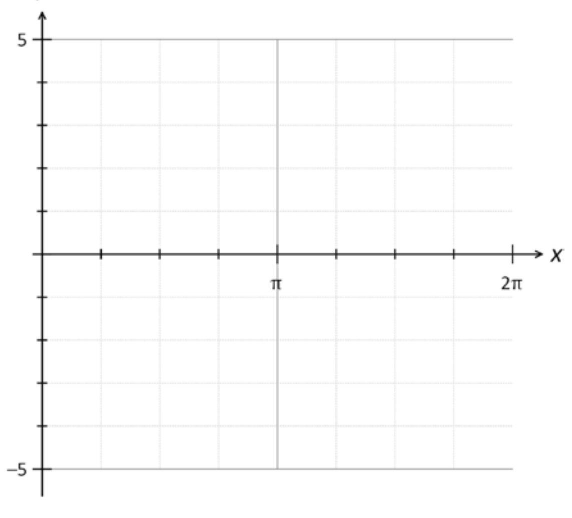
Calculator Assumed

12. [11 marks: 8, 3]

Consider the curve with equation $y = e^{\cos x}$ for $0 \le x \le 2\pi$.

(a) Use a calculus method to determine the coordinates of the stationary points on this curve. Use the second derivative test to determine the nature of these stationary points.

(b) On the axes provided in the accompanying diagram, sketch $y = e^{\cos x}$. Indicate clearly the location of the stationary points.



Calculator Assumed

13. [9 marks: 4, 5] [TISC]

Consider the curve with equation $y = \dfrac{1}{\sqrt{1+e^x}}$.

(a) Use a calculus method to show that the curve has no stationary points.

(b) Given that the curve has one inflection point, use derivatives to determine the coordinates of the inflection point.

Calculator Assumed

14. [7 marks]

Consider the curve with equation $y = ax^4 + bx^3 + cx^2 + d$. The curve has a tangent with equation $y = x - 1$ at the point where $x = 1$. The curve has a inflection points at $x = -2$ and $\frac{1}{2}$. Determine the values of a, b, c and d.

13 Rates of Change

Calculator Assumed

1. [4 marks: 1, 1, 2]

Given that, $A = 2t^2 - 5t + 3$, find:
(a) an expression for the instantaneous rate of change of A with respect to t

(b) the instantaneous rate of change of A with respect to t when $t = 0$

(c) t when the instantaneous rate of change of A with respect to t is 3.

---

2. [6 marks: 3, 2, 2]

Given that, $P = \sqrt{1+t^2}$, find:
(a) the instantaneous rate of change of P with respect to t when $t = 0$ and $t = 4$

(b) the average rate of change of P with respect to t for $0 \le t \le 4$

(c) Compare your answers in (a) and (b) and comment on them.

Calculator Assumed

3. [10 marks: 3, 3, 2, 2]

The concentration of a chemical (g/L) in a swimming pool is given by $C = 2000\,e^{-0.08t}$ where t is the number of *hours* after the chemical is introduced into the pool.

(a) Calculate the average rate of change of the concentration of this chemical in the first ten hours.

(b) Determine the rate of change of the concentration of this chemical at the end of the tenth hour.

(c) Interpret your answers in (a) and (b).

(d) Calculate how long it will take for the rate of decrease of the concentration of the chemical to be less than 10 g/L per hour.

Calculator Assumed

4. [7 marks: 3, 2, 2]

Given that, $Q = 100\, t\, e^{-0.5t}$, find:

(a) using derivatives, the instantaneous rate of change of Q with respect to t when $t = 1$ and $t = 5$

(b) the average change in Q for $1 \le t \le 5$

(c) the value of t when the instantaneous rate of change of Q with respect to t is zero.

---

5. [5 marks: 2, 3]

Given that, $N = \dfrac{2t}{5+t}$ where $t \ge 0$, find, showing the use of derivatives:

(a) the instantaneous rate of change of N with respect to t when $t = 0$

(b) the value of t when the instantaneous rate of change of N with respect to t is 0.1.

Calculator Assumed

6. [5 marks: 2, 1, 2]

Given that, $A = 50(1 + t)^2 e^{-t}$ for $t \geq 0$, find:
(a) an expression for the instantaneous rate of change of A with respect to t

(b) the instantaneous rate of change of A with respect to t when $t = 1$

(c) the average rate of change of A in the first second.

---

7. [7 marks: 1, 2, 2, 2]

The price of a listed share C cents, is modelled by $C = 75\sqrt{1+0.8t}$, $t \geq 0$, where t is the number of years after 2000.

(a) Find the per unit cost in 2000.

(b) Find the average rate of cost rise between 2000 and 2010.

(c) Find using derivatives, the instantaneous rate of cost rise in 2005

(d) Find when the instantaneous rate of cost rise is 10 cents per year.

Calculator Assumed

8. [13 marks: 2, 3, 3, 2, 3]

The water level at a jetty is modelled by $h = 3 + \cos\left(\frac{\pi t}{12}\right)$, $0 \le t \le 24$, where h metres is the depth of water measured from the river bed at time t hours after 6 am.

(a) Find the water level at 7 am and 10 am.

(b) Find the average rate of change of the water depth between 7 am and 10 am.

(c) Find the rate of change of the water depth at 10 am.

(d) Comment on your answers in (b) and (c).

(e) Find when the water level is increasing at a rate of 0.1 metres per hour.

Calculator Assumed

9. [12 marks: 4, 3, 2, 3]

The mass of an object being printed by a 3D-printer is given by $M = ln\ (1 + t^3)$ g for $0 \le t \le 10$ minutes.

(a) Find the average rate of change of mass of the object during the first 5 seconds and the second 5 seconds.

(b) Find the rate of change of mass at $t = 5$ and $t = 10$ minutes.

(c) Comment on your answers in (a) and (b).

(d) Use an analytical method to determine the values of $0 \le t \le 10$ for which the rate of change of mass is decreasing.

14 Optimisation

Calculator Assumed

1. [10 marks: 6, 2, 2]

The cost per hour , \$C, of operating a truck travelling at a constant speed of v kmh^{-1} is modelled by $C = \dfrac{(v-20)^3}{5000} + \dfrac{400}{v} + 200$ where $v > 0$. The truck has a speed limit of 80 kmh^{-1}.

(a) Use Calculus to find the speed the truck should travel on for the hourly cost to be minimized. Find the minimum hourly cost.

(b) Find the difference in hourly cost if the truck were to be travelling at its posted speed limit.

(c) Suggest a reason why the Company that owns this truck may not operate it at the speed that achieves the minimum hourly cost.

Calculator Assumed

2. [10 marks: 3, 3, 4]

The quantity Q (mg) of a drug in a patient's blood stream is given by $Q = 10\,t\,e^{-0.05t}$ where t is the number of *minutes* after the drug is taken.

(a) Calculate the average rate of change of the amount of this drug in the patient's blood stream in first hour.

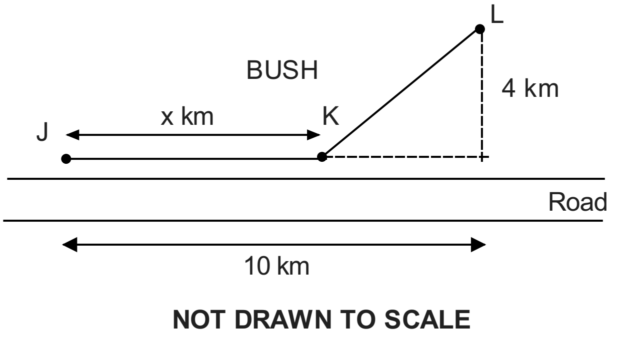
(b) Determine the rate of change of the amount of this drug in the patient's blood stream at the end of the first hour.

(c) Use a calculus method to determine when the rate of change of the amount of drug in the patient's blood stream is a minimum.

Calculator Assumed

3. [6 marks: 1, 2, 3]

A fibre optic cable is to be laid from J to L. It costs $500 per km to lay the cable alongside the road and $800 per km to lay it across the bush. K is a point x km from J along the road. The cable will be laid alongside the road from J to K and across the bush from K to L.



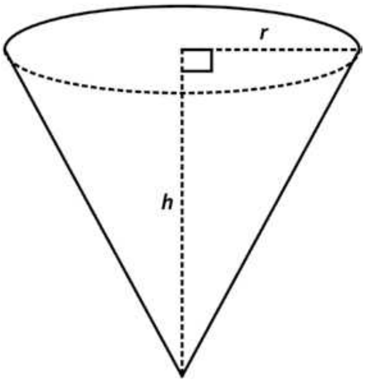
(a) Show that the distance between K and L is $\sqrt{x^2 - 20x + 116}$ km.

(b) Find the total cost for laying the cable from J to L via K (as described).

(c) Use an analytical method to find x so that this cost is minimized.

Calculator Assumed

4. [8 marks: 3, 5]



The accompanying diagram shows an inverted cone of height h cm and base radius r cm.

The volume of the cone is fixed at $\frac{\pi}{3}$ m^3.

The curved surface area of the cone is given by $A = \pi r\sqrt{h^2 + r^2}$.

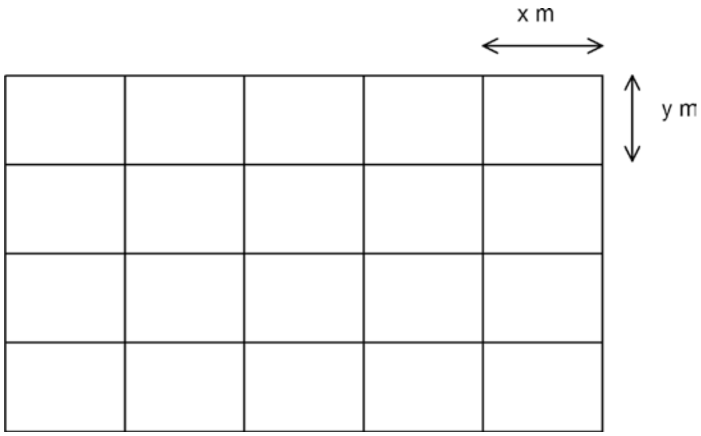
(a) Show that $A = \frac{\pi}{r}\sqrt{1 + r^6}$.

(b) Use a calculus method to find the value of r and h that minimises A.

Calculator Assumed

5. [11 marks: 3, 8]

The rectangular backyard of area 100 m^2 is to be divided into 20 equal rectangles as shown below. The boundaries are to be marked with single length red ribbons.



(a) Find a rule for y in terms of x.

(b) Use a calculus method to find the exact length and width of each rectangle that will minimize the total length of ribbon used.

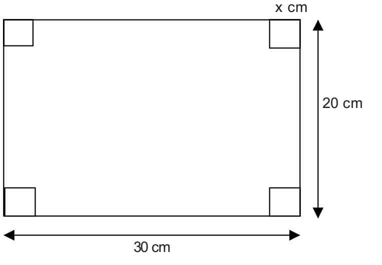
Calculator Assumed

6. [6 marks]

A closed cylindrical can of diameter $2r$ is to have a volume of 50π cm^3. Use Calculus to find the dimensions of this can if it is to have minimum surface area. Ignore the thickness of the material used to make the can.

7. [7 marks]

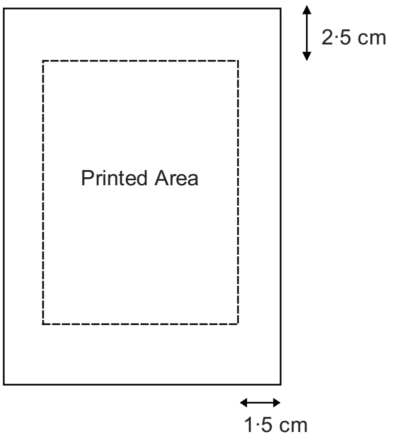
A rectangular sheet of cardboard measuring 20 cm by 30 cm is used to make an open box. A square of width x cm is removed from each corner of the sheet to form the net of the box. The net is then folded up to form the box. Use Calculus to find the dimensions of the box with the largest possible volume. Give this volume.



Calculator Assumed

8. [8 marks]

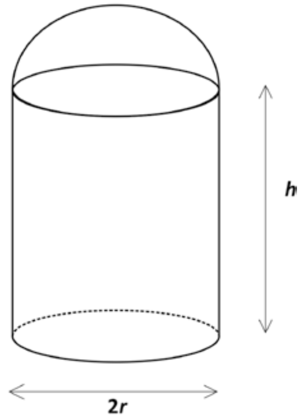
A printed poster using a minimal area of cardboard is to be designed. The printed area must be 500 cm^2. The top and bottom margins must be 2.5 cm each. The left and right margins must be 1.5 cm each. Use Calculus to determine the optimal dimensions of the poster. Give the width, height and total area of the optimal poster.



Calculator Assumed

9. [10 marks: 3, 3, 4] [TISC]

A container consists of a cylinder of height h cm and base radius r cm with a hemispherical cap sitting on top of the cylinder. The container has a fixed volume of 360π cm^3.



(a) Show that $h = \dfrac{360}{r^2} - \dfrac{2r}{3}$.

(b) Show that the total external surface area of the container is given by

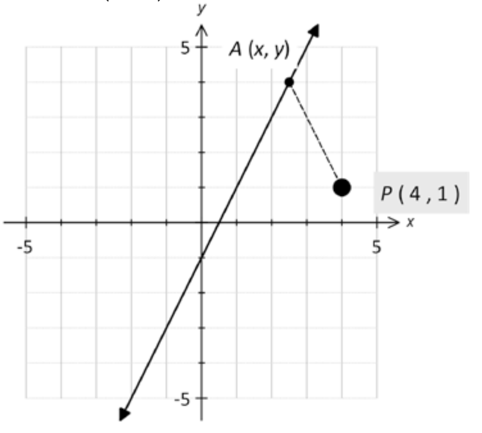
$$S = \frac{5\pi r^2}{3} + \frac{720\pi}{r}.$$

(c) Use a calculus method to determine the minimum surface area of the container. Give your answer in exact form.

Calculator Assumed

10. [8 marks: 2, 6] [TISC]

The point A (x, y) lies on the line with equation $y = 2x - 1$.
The point P has coordinates (4, 1).



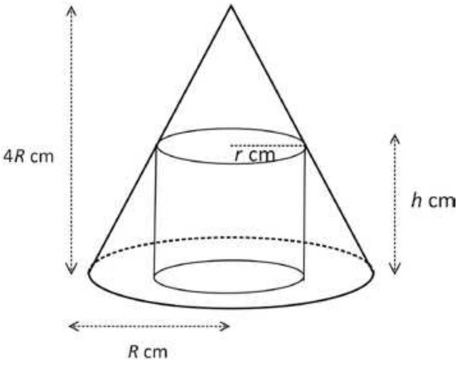
(a) Show that the distance between A and P is given by $s = \sqrt{5x^2 - 16x + 20}$.

(b) Use a calculus/algebraic method to find the shortest distance and the longest distance between the point P and the line $y = 2x - 1$ for $1 \le x \le 5$.

Calculator Assumed

11. [7 marks: 3, 4]

The accompanying diagram shows a closed cylinder of base radius r cm and height h cm trapped within a regular cone of base radius R cm and height $4R$ cm. One flat end of the cylinder is in full contact with the base of the cone and the edge of the other flat end of the cylinder is in full contact with the inner surface of the cone.



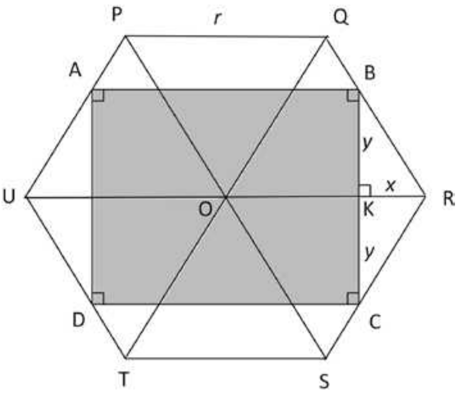
(a) Show that $h = 4(R - r)$. Hence, show that the total surface area of the closed cylinder is given by $S = 8\pi Rr - 6\pi r^2$.

(b) Use a calculus method to determine the optimal total surface area of the closed cylinder in terms of R. Verify that this optimal value is a maximum value.

Calculator Assumed

12. [6 marks: 2, 4] [TISC]

The accompanying diagram shows a regular hexagon PQRSTU with side length r cm. The centre of the hexagon is located at the point O. ABCD is a rectangle. The vertices A, B, C and D lie on the sides of the hexagon.
BC meets UR at right angles at K.
Let KR = x cm and BC = $2y$ cm.



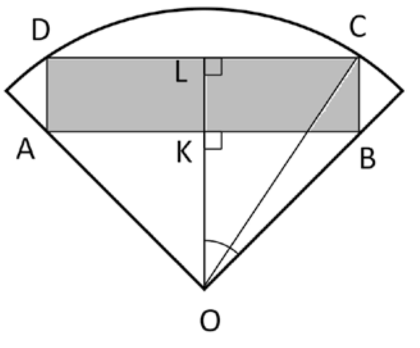
(a) Show that $y = x\sqrt{3}$.

(b) Use a calculus method to calculate in terms of r, the dimensions of rectangle ABCD so that it has a maximum area.

Calculator Assumed

13. [8 marks: 2, 2, 4] [TISC]

The accompanying diagram shows a sector of a circle with radius 10 cm and with $\angle AOB = 90^o$. ABCD is a rectangle with the vertices touching the edges of the sector. K is the foot of the perpendicular from O to AB. OK bisects $\angle AOB$. L is the foot of the perpendicular from O to CD. Let AB = $2x$ cm and AD = y cm.



(a) Determine in terms of x, the length of OK.

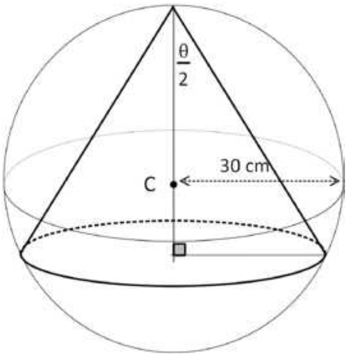
(b) Use ΔOLC to show that $y = \sqrt{100 - x^2} - x$.

(c) Use a calculus method to determine the value for x for which the area of rectangle ABCD is a maximum.
[You are not required to verify that a maximum value has been obtained.]

Calculator Assumed

14. [10 marks: 5, 1, 4]

The accompanying diagram shows a right circular cone of semi-vertical angle $\frac{\theta}{2}$ inscribed within a sphere with centre C of radius 30cm. The vertex of the cone is in contact with the inner surface of the sphere. The circular edge of the base of the cone is in full contact with the inner surface of the sphere.



(a) Show that volume of the cone is given by $V = 9000\pi(\sin^2\theta)(1+\cos\theta)$.

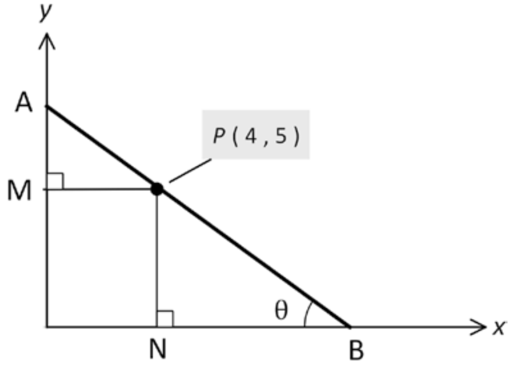
(b) Use an appropriate trigonometric identity to show that V can be rewritten as $V = 9000\pi(1+\cos\theta-\cos^2\theta-\cos^3\theta)$.

(c) Use a calculus method to determine the maximum volume of the cone.

Calculator Assumed

15. [7 marks]

A line passes through the point P with coordinates (4, 5).
The line intersects the positive y-axis at A and the positive x-axis at B.
Let the acute angle between the line segment AB and the positive x-axis be θ.
Also, let L be the length of the line segment AB.



Use a calculus method to determine the minimum value for L.

15 Incremental Change, Rates & Marginal Rates

Calculator Assumed

1. [3 marks]

   Consider $y = \dfrac{x}{1+e^x}$.
   Use the method of incremental change to find the approximate change in y (4 significant places) when x changes from 1.00 to 0.99.

---

2. [7 marks: 4, 3]

   A curve has equation $y = 2x^3 + 3x^2 - 12x$.

   (a) Use the method of small increments to estimate the change in y when x changes from 2.00 to 1.99.

   (b) Use your answer in (a) to estimate the value of y when x changes from 2.00 to 2.01.

Calculator Assumed

3. [4 marks]

The quantity of a substance Q (g) is related to time t (minutes) by the formula $Q = 60t(t + 1)$. Use the method of small changes to estimate the value of Q when t changes from 1 minute by 1 second.

---

4. [8 marks: 5, 3]

(a) Use the method of small changes to find the approximate change in the radius of a spherical balloon corresponding to a change in its volume from 500 cm^3 to 499 cm^3.

(b) Use your answer in (a) to find the approximate change in the surface area of a spherical balloon corresponding to a change in its volume from 500 cm^3 to 499 cm^3.

Calculator Assumed

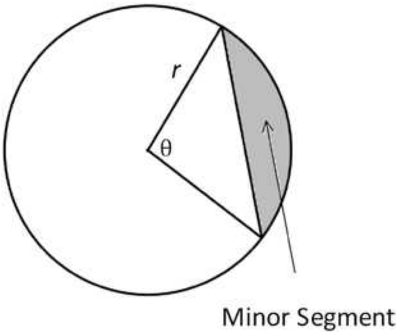
5. [5 marks]

Given that $y = \frac{1}{\sqrt{t}}$, use the method of small increments to find the percentage change in y corresponding to a 1% increase in t.

---

6. [5 marks]

The area of a minor segment associated with a sector of a circle with radius r and with a fixed sector angle of θ radians is given by $A = \frac{1}{2}r^2(\theta - \sin\theta)$. Use the method of small increments to calculate the approximate percentage change in the radius of the sector corresponding to a 1% decrease in the area of the minor segment.



Calculator Assumed

7. [4 marks] [TISC]

Let $A(t) = \int_0^t \sqrt{16 + x^2}\, dx$. Use the incremental formula to calculate the approximate change in A corresponding to a change in t from 3.00 to 3.01.

---

8. [6 marks: 3, 3]

Let $A = \dfrac{1}{\sqrt{x+1}}$ where $x = f(t) \geq 0$ for time $t \geq 0$.

(a) Use the incremental method (method of small changes) to find the approximate change in A when x changes from 3 to 2.99.

(b) Use the chain rule to determine $\dfrac{dA}{dt}$. Hence, calculate the rate of change of A, when $x = 3$, given that when $x = 3$, $\dfrac{dx}{dt} = 4$.

Calculator Assumed

9. [10 marks: 3, 4, 3]

A circle has radius r metres.

(a) The circumference of the circle is increasing at a rate of π metres per hour.

(i) Use the chain rule to find the rate with which the radius of the circle is changing when the radius of the circle is 10 metres.

(ii) Find the rate with which the area of the circle is changing when the radius of the circle is 10 metres.

(b) Use the incremental method (method of small changes) to find the approximate change in the area of the circle when the radius of the circle changed from 10 metres to 10.1 metres.

Calculator Assumed

10. [10 marks: 2, 1, 2, 2, 3]

Mathcom sells each "Template X" for $30. The cost of producing x items is given by $C(x) = \dfrac{80x}{(x+1)} + 0.04x^2 + 500$.

(a) Find an expression for the profit $P(x)$ corresponding to the manufacture and sale of x items.

(b) Find an expression $P'(x)$.

(c) Find $P'(100)$. Interpret this value.

(d) Find the average profit per item associated with the manufacture and sale of 100 items.

(e) Find how many items were manufactured and sold if the profit associated with the sale of the next item is approximately $20.

Calculator Assumed

11. [10 marks: 4, 3, 3]

The revenue $\$R$ million from the sales of x units of a product is given by
$R = 100 - x - \dfrac{2000}{x+5}$.

(a) Use the incremental method to determine the approximate change in revenue when the number of units sold changes from 80 to 82.

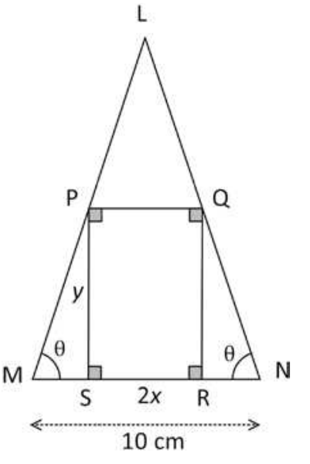
(b) Use the incremental method to determine an expression for marginal revenue when x items are sold.

(c) Determine an expression for the average revenue when x items are sold. Hence, determine the value of x when the marginal revenue and average revenue are equal.

Calculator Assumed

12. [12 marks: 3, 3, 3, 3] [TISC]

The accompanying diagram shows an isosceles triangle LMN with $\angle LMN = \angle LNM = \theta$. MN = 10 cm and $\tan\theta = 3$. Points P and Q lie on LM and LN respectively. Points S and R lie on MN. PQRS form a rectangle with PS = QR = y cm and PQ = SR = $2x$ cm. Let the area of rectangle PQRS be A cm^2.



(a) Show that $A = 2x(15 - 3x)$ cm^2.

(b) Use the method of small increments to determine the increment in A when x increases from 0.5 cm to 0.6 cm.

(c) Given that x increases at a constant rate of 0.1 cm per minute, use the chain rule to calculate the rate of increase in A when $x = 1$ cm.

(d) Calculate the maximum possible area of rectangle PQRS.

16 Exponential Growth & Decay

Calculator Assumed

1. [12 marks: 1, 1, 2, 3, 5]

The amount (A g) of radioactive substance R remaining after t days is given by $A = 800\,e^{-0.04t}$.

(a) What is the initial amount of substance present?

(b) How much is left after 7 days?

(c) How much has decayed after 14 days?

(d) Find the half-life of this radioactive substance.

(e) Find the rate of decay when 100 g of the substance is left.

Calculator Assumed

2. [13 marks: 4, 3, 3, 3]

The instantaneous rate of population growth is proportional to its population size. The population grew from an initial 10 000 to 15 000 in 8 years.

(a) Find an expression for the population size at time t years.

(b) Find doubling time, the time taken for the population to double its size.

(c) Find the instantaneous rate of population growth when $t = 8$.

(d) Find when the instantaneous rate of population growth is 1000 persons/year.

Calculator Assumed

3. [5 marks]

A radioactive substance has a half-life of 50 days. After 20 days, only 30g were left. Assume that the radioactive substance decays exponentially.
Find the initial amount of substance.

---

4. [5 marks] [TISC]

A batch of cattle grain is found to be contaminated with radioactive substance X. The radioactive substance X decays exponentially with a half-life of ten days. The amount of radioactive substance X was found to be ten times the maximum permitted level. How long should be grain be stored before it is fed to the cattle? Justify your answer.

Calculator Assumed

5. [8 marks: 2, 3, 3]

The mass (M mg) of a slow growing tumour grows exponentially according to the formula $M = A e^{kt}$ where t is time in weeks. The mass of the tumour doubles every 100 weeks.

(a) Find k to four significant figures.

(b) A 25 year old patient was diagnosed with a similar tumour. The mass of the tumour was estimated to be 120 mg. The tumour is operable only when it reaches a mass of 200 mg.

(i) When will the tumour be operable?

(ii) Estimate how long the tumour has been growing in the patient's body. Explain clearly how you arrived at your answer.

---

6. [7 marks: 3, 2, 2] [TISC]

50 L of a chemical is accidentally spilled into a swimming pool. The amount of chemical (in L) left in the pool t days after the spill is given by $A = A_0 e^{kt}$.

(a) Determine the value of k (to four significant figures) if 80% of the chemical disappears after 10 days.

Calculator Assumed

6. (b) Find the rate with which the amount of chemical left in the pool is changing when 80% of the chemical has disappeared.

(c) The swimming pool is safe for use if the amount of chemical in the pool is between 1 litre and 5 litres inclusive. To one decimal place, for what value(s) of t will the pool be safe for use?

---

7. [7 marks: 2, 3, 2] [TISC]

The length of a vine (L cm) grows according to the formula $L = 10\ e^{0.03t}$ where $t \geq 10$ days. The vine is trimmed each time its length exceeds 5 metres. Each time it is trimmed to a length of approximately 2 metres.
Assume that the vine grows at the same rate.

(a) Find when the vine is trimmed for the first time.

(b) Find when the vine is trimmed for the second time.

(c) To what length (to the nearest cm) should the vine be trimmed so that it is trimmed once every 60 days.

Calculator Assumed

8. [12 marks: 2, 4, 2, 2, 2]

Two heated objects, P and Q are placed one at each end of a long laboratory bench. The temperature (in degrees Celsius) of P and Q, t minutes after being placed on the bench top are given by $\theta_P = 18 + 75\,e^{-0.09t}$ and $\theta_Q = 18 + 60\,e^{-0.01t}$ respectively.

(a) Find when the two objects have the same temperature.

(b) Find when the two objects are losing heat at the same rate.

(c) Find the time taken for the temperature of each object to reach 25 C.

(d) Which object is losing heat at a faster rate? Justify your answer.

(e) Find the temperature of each body for large values of t.

Calculator Assumed

9. [8 marks: 2, 1, 2, 3]

The instantaneous rate of population growth of a colony of bacteria at any time t is proportional to N, the number of bacteria at time t. The initial number of bacteria is 100 and its initial instantaneous growth rate is 2 bacteria per hour.

(a) Explain why the continuous percentage growth rate is 2% per hour.

(b) State the mathematical expression for N, in terms of time t (hours).

(c) Determine the average growth rate in the first 240 hours.

(d) Determine the rate of change of the rate of population growth at t = 240 hours.

Calculator Assumed

10. [8 marks: 2, 3, 3]

N, the number of bacteria in a culture is modelled by the equation $N = N_0 e^{kt}$, where t is the number of hours after it was cultivated.
The number of bacteria grew by 30% in the first twelve hours.

(a) Find k to four significant figures.

The average rate of growth in the bacterial numbers in the first 10 hours is 0.8 bacteria per hour.

(b) Calculate the value of N_0.

(c) When the number of bacteria reaches 100 000, an anti-bacterial drug is introduced. The doubling time for the bacterial growth is now 69 hours. Determine with reasons if the drug has been effective in reducing the growth in the number of bacteria.

Calculator Assumed

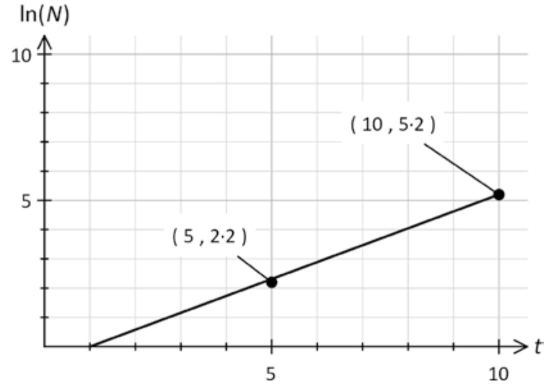
11. [10 marks: 2, 3, 3, 2]

P, the number of people infected by a certain virus at the end of day t is modelled by $P = e^{0.6t}$, (t is time in days).

(a) Calculate the minimum amount of time required for the entire world's population of 7.7 billion to be infected. (1 billion = 1 000 million)

(b) Calculate the total number of people infected by the end of the 20th day. Hence, calculate the number of new people infected on the 20th day.

Let N be the number of new people infected on day t. The accompanying diagram shows the graph of $\ln N$ against t.



(c) Determine the algebraic relationship between N and t in the form $N = N_0 e^{kt}$

(d) Explain clearly why the number of newly infected people each day is always a fixed percentage of the total number of people infected by the end of the day. State this fixed percentage.

17 Anti-Differentiation

Calculator Free

1. [13 marks: 2, 2, 2, 2, 2, 3]

Find the anti-derivative of each of the following:

(a) $\frac{3x^2}{4} + 4x + 5$

(b) $\frac{1}{x^3} - \frac{2}{3x^2}$

(c) $(1 + 4x)^5$

(d) $\frac{1}{2\sqrt{x}}$

(e) $\sqrt{x}(x^2 + 1)$

(f) $(2x - \frac{3}{x})^2$

Calculator Free

2. [17 marks: 3, 2, 3, 3, 3, 3]

Find:

(a) $\int \frac{3}{2}(\sqrt{x}+1)^2 \ dx$

(b) $\int \frac{x^4 + x^3}{x} dx$

(c) $\int \frac{x + 2x^3}{3x^5} dx$

(d) $\int \sqrt{1-2x} \ dx$

(e) $\int \frac{4}{(3x-2)^2} dx$

(f) $\int \frac{4}{3\sqrt{x-1}} dx$

Calculator Free

3. [12 marks: 2, 2, 2, 2, 2, 2]

Find:

(a) $\int 2x(1-x^2)^3 \; dx$

(b) $\int \frac{x}{(1-x^2)^4} dx$

(c) $\int x\sqrt{1-2x^2} \; dx$

(d) $\int \frac{-5x}{\sqrt{1+x^2}} dx$

(e) $\int \frac{1}{x^2}\left(1-\frac{1}{x}\right)^3 dx$

(f) $\int \frac{1}{\sqrt{x}}\left(2+\sqrt{x}\right)^4 dx$

Calculator Free

4. [13 marks: 1, 2, 2, 2, 2, 2, 2]

Find:

(a) $\int e^{-3x}\ dx$

(b) $\int \frac{1}{2e^{2x}}\ dx$

(c) $\int (1+e^{2x})^2\ dx$

(d) $\int x\, e^{-x^2}\ dx$

(e) $\int e^{2x}(1+e^{x})\ dx$

(f) $\int x^2\ e^{2x^3}\ dx$

(g) $\int e^{-x}(1+e^{-x})^3\ dx$

Calculator Free

5. [16 marks: 2, 2, 2, 2, 2, 3, 3]

Determine:

(a) $\int \cos 2x \; dx$

(b) $\int \sin\left(\frac{x}{2}\right) \; dx$

(c) $\int 3\cos(1-4x) \; dx$

(d) $\int x\sin(x^2) \; dx$

(e) $\int 3\cos x\,(\sin x)^4 \; dx$

(f) $\int \frac{2\sin x}{5(\cos x)^3} \; dx$

(g) $\int -3\cos 2x\,\sqrt{1+\sin 2x} \; dx$

Calculator Free

6. [14 marks: 2, 2, 3, 4, 3]

Determine:

(a) $\int \frac{-4}{3+2x}\ dx$

(b) $\int \frac{5x}{1+3x^2}\ dx$

(c) $\int \left(1-\frac{1}{x}\right)^2\ dx$

(d) $\int \frac{(x+2)^2}{x}\ dx$

(e) $\int \frac{1+x+x^2}{1+x^2}\,dx$

Calculator Free

7. [10 marks: 2, 3, 2, 3]

Determine:

(a) $\displaystyle\int \frac{5e^{-x}}{3+4e^{-x}}\ dx$

(b) $\displaystyle\int \tan 2x\ \ dx$

(c) $\displaystyle\int \frac{\cos \pi x}{1+\sin \pi x}\ dx$

(d) $\displaystyle\int \frac{4\sin 2x}{1-3\sin^2 x}\ dx$

Calculator Free

8. [4 marks]

Find $f(x)$ if $f'(x) = 2x^2 - 3x + a$ where a is a constant and $f(0) = -2$ and $f(-1) = -4$.

---

9. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = x - \frac{e^x}{2}$. Find the equation of this curve given that it passes through the point (0, –2).

---

10. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = x(1+0.5x^2)^5$. Find the equation of this curve given that it passes through the point (0, 1)

Calculator Free

11. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = \frac{x}{1+x^2}$. Find the equation of this curve given that it passes through the point (0, 1).

12. [5 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = a + b\cos 2x$. The curve has a stationary point at (0, 0). Find the equation of this curve.

Calculator Free

13. [7 marks: 2, 1, 2, 2] [TISC]

The gradient function of the curve $y = f(x)$ is given by the equation $f'(x) = \dfrac{1}{2\sqrt{x}}$.

(a) Determine the equation of the family of curves $y = f(x)$.

(b) Determine $y = f(x)$ if the curve passes through the point (1, 4).

(c) If another function in the family of curves $y = f(x)$ has a tangent with equation $y = \dfrac{x}{8}$:

(i) Explain why the tangent meets the curve at $x = 16$.

(ii) Determine the equation of the curve $y = f(x)$.

Calculator Free

14. [9 marks: 2, 4, 3]

(a) Given that $y = x\,e^x$, find $\dfrac{dy}{dx}$.

(b) Use your answer in (a) to find $\int x\,e^x\,dx$.

(c) The gradient function of a curve is given by $\dfrac{dy}{dx} = x\,e^x$. Find the equation of this curve given that when $x = 0$, $y = 2$.

---

15. [5 marks: 2, 3]

(a) Determine $\dfrac{d}{dx}\big(x\sin(2x)\big)$.

(b) Hence, or otherwise, determine $\int 2\sin(2x) + 2x\cos(2x)\,dx$.

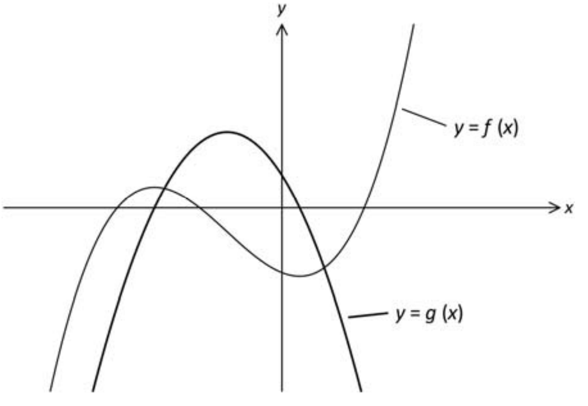
Calculator Free

16. [4 marks]

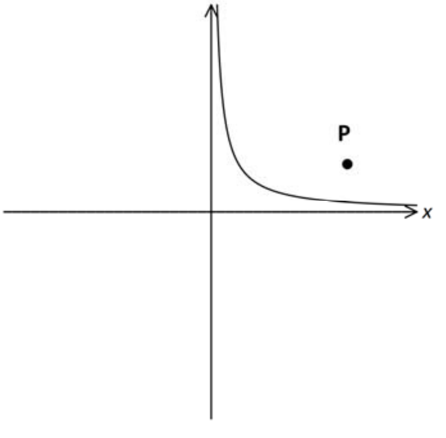
Use the results $\frac{d}{dx}\left(e^x \sin x\right) = e^x \sin x + e^x \cos x$ and $\frac{d}{dx}\left(e^x \cos x\right) = e^x \cos x - e^x \sin x$ to determine $\int e^x \cos x \, dx$.

17. [5 marks: 2, 3] [TISC]

(a) The graphs of the functions $f(x)$ and $g(x)$ are shown in the accompanying diagram. Determine with reasons if $g(x)$ is the derivative of $f(x)$.



(b) The graph of the function $f(x) = \frac{1}{x}$ for $x > 0$ is shown in the accompanying diagram. On the same axes, draw a possible graph of $g(x)$ where $g(x) = \int f(x) \; dx$ given that it passes through the point $\boldsymbol{P}$.



18 The Fundamental Theorem of Calculus

Calculator Free

1. [12 marks: 3, 3, 3, 3]

Evaluate each of the following definite integrals.

(a) $\int_0^1 \frac{3x^2}{4}+\sqrt{x}-2\ dx$

(b) $\int_1^4 \frac{2}{x^3}-\frac{1}{2\sqrt{x}}\ dx$

(c) $\int_0^5 \sqrt{(4+x)^3}\ dx$

(d) $\int_{\frac{-1}{2}}^{0} \frac{-3}{\sqrt{(1-6x)}}\ dx$

Calculator Free

2. [14 marks: 2, 4, 4, 4]

Evaluate each of the following definite integrals.

(a) $\int_0^2 3\,e^{\frac{x}{2}}\;dx$

(b) $\int_0^1 (1+e^x)^2\;dx$

(b) $\int_0^2 \frac{2t^3}{t^4+2}\,dt$

(d) $\int_0^{\frac{1}{3}} \frac{\sin(\pi x)}{\cos(\pi x)}\,dx$.

Calculator Free

3. [13 marks: 2, 2, 2, 2, 2, 3]

Find $\dfrac{dy}{dx}$:

(a) $y = \displaystyle\int_1^x \sqrt{1+t^3}\, dt$

(b) $y = \displaystyle\int_1^{x^2} \frac{1}{\sqrt{1+t^2}}\, dt$

(c) $y = \displaystyle\int_1^{e^x} (1+\sqrt{t})^5\, dt$

(d) $y = \displaystyle\int_1^{e^{2x}} \frac{1}{\sqrt{1+t}}\, dt$

(e) $y = \displaystyle\int_1^{1+x^2} \frac{1}{\sqrt{1+e^t}}\, dt$

(f) $y = \displaystyle\int_{e^{2x}}^{1} e^t\, dt$

Calculator Free

4. [11 marks: 2, 2, 2, 3, 2]

Determine $\frac{dy}{dx}$:

(a) $y = \int_{1}^{\cos x} 1 + t^2 \, dt$

(b) $y = \int_{1}^{\tan x} e^{t^2} \, dt$

(c) $y = \int_{1}^{\sin 2x} \sin^2(1+t) \, dt$

(d) $y = \int_{\cos^2 x}^{0} \tan(\pi + u) \, du$

(e) $y = \int_{0}^{e^{\sin x}} \sqrt{1+u} \, du$

Calculator Free

5. [12 marks: 2, 2, 2, 3, 3]

Determine $\frac{dy}{dx}$:

(a) $y = \int_{1}^{\ln x} \ln(2+3t)\, dt$

(b) $y = \int_{1}^{\ln x} \frac{1}{t}\, dt$

(c) $y = \int_{1}^{\ln \sin(x)} \sqrt{t}\, dt$

(d) $y = \int_{1}^{\ln(1+e^x)} (1-t)^2\, dt$

(e) $y = \int_{x}^{\ln(x)} t^2\, dt$

Calculator Free

6. [4 marks: 2, 2]

Evaluate:

(a) $\displaystyle\int_1^2 \frac{d}{dx}(1+x)\ dx$

(b) $\displaystyle\int_0^2 \frac{d}{dx}\left(\frac{1+x}{1+x^2}\right)\ dx$

7. [5 marks: 2, 3]

(a) Determine $\dfrac{d}{dx}\left(\dfrac{1}{1+x^2}\right)$. Simplify your answer.

(b) Hence, or otherwise, determine $\displaystyle\int_0^1 \frac{x}{(1+x^2)^2}\,dx$.

Calculator Free

8. [4 marks]

A curve has equation given by $y = \int_0^{x^2} \sqrt{1+u^2}\, du$. When $x = 1$, $\dfrac{dx}{dt} = -1$.

Use the chain rule to determine the value of $\dfrac{dy}{dt}$ when $x = 1$.

---

9. [7 marks: 2, 5]

A curve has equation given by $y = \int_0^{x^2} (t-4)^3\, dt$.

(a) Find the gradient of the curve at the point where $x = 1$.

(b) Find x-coordinate of the turning points of the curve.
Identify the nature of these points.

Calculator Assumed

10. [6 marks: 3, 3]

A curve has equation given by $y = \int_{2}^{e^x} 5(t-2)^4 \, dt$.

(a) Find the equation of the tangent to the curve at the point (0, –1).

(b) Find the x-coordinate of the stationary point on this curve. State its nature.

---

11. [4 marks]

The temperature of a body ($^\circ$C) at time t hours is given by $\theta = \int_{0}^{t} (x-4)e^x \, dx$ for $0 \le t \le 7$. The temperature of the body becomes critical when the temperature is either a local maximum or minimum. Determine when the temperature of the body becomes critical and classify the nature of this critical temperature.

Calculator Assumed

12. [10 marks: 3, 5, 2]

Let $F(x) = \int_0^x 3t^2 - 12t + 9 \; dt$ where for $1 \le x \le 5$.

(a) Calculate $F(1)$ and $F(5)$.

(b) Find the local minimum and maximum points on the curve $y = F(t)$.

(c) Determine the global minimum and global maximum values for $F(x)$.

Calculator Assumed

13. [7 marks: 1, 3, 3]

A curve has equation given by $f(x) = \int_0^x 1 - 2e^{-t^2}\, dt$ for $0 \le x \le 3$.

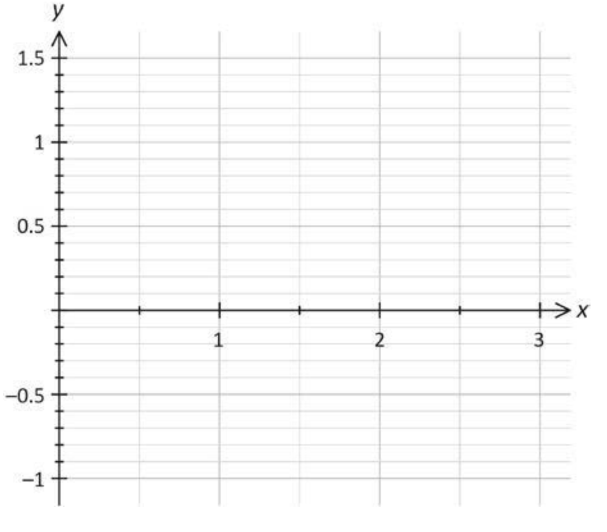
(a) Use your calculator to complete the table below.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	0	−0.4226	−0.4936	−0.2124		0.7283	1.2276

(b) The graph of $g(t) = 1 - 2e^{-t^2}$ for $0 \le t \le 3$ has a root at $t \approx 0.83$.
Explain how this result may be used to determine the coordinates of the stationary point for $f(x)$.

(c) On the axes provided below, sketch the graph of $f(x) = \int_0^x 1 - 2e^{-t^2}\, dt$

for $0 \le x \le 3$. Indicate clearly the stationary point.



Calculator Assumed

14. [8 marks: 2, 3, 3]

A curve has equation given by $f(x) = \int_0^x \sin(t^2)\, dt$ for $0 \le x \le 2$.

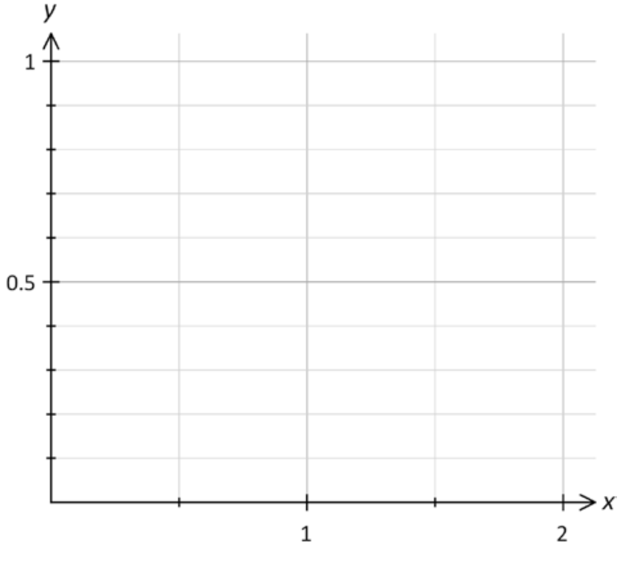
(a) Use your calculator to complete the table below.

x	0	0.5	1	1.5	2
$f(x)$					

(b) Determine the coordinates of the stationary point on the graph of $y = f(x)$ for $0 \le x \le 2$.

(c) On the axes provided below, sketch the graph of $f(x) = \int_0^x \sin(t^2)\, dt$

for $0 \le x \le 2$. Indicate clearly the stationary point on the curve.



Calculator Assumed

15. [9 marks: 2, 2, 2, 3]

The functions $f(x)$ and $g(x)$ are continuous everywhere. It is known that:

$\int_{-5}^{1} f(x)\,dx = -2$, $\int_{-5}^{7} f(x)\,dx = 8$, $\int_{1}^{7} g(x)\,dx = 18$ and $\int_{-5}^{1} f(x) + g(x)\,dx = -20$.

(a) Calculate $\int_{1}^{7} f(x)\,dx$.

(b) Calculate $\int_{-5}^{1} g(x)\,dx$.

(c) Calculate $\int_{-6}^{0} f(x) + g(x)\,dx$

(d) Calculate $\int_{-5}^{7} f(x) - g(x)\,dx$

Calculator Assumed

16. [8 marks: 3, 2, 3]

Let $\int f(x)\,dx = F(x)$. The table below provides the values for $f(x)$ and $F(x)$ for several values of x.

x	–8	–2	0	4	8	10
$f(x)$	–2		0	4		4
$F(x)$	–4	–16	–18	–10	6	14

Use the table above to answer the following questions.

(a) Calculate $\int_{4}^{10} f(-8)\,dx$

(b) Calculate $\int_{-8}^{0} f(x)\,dx$.

(c) Calculate $\int_{-4}^{4} f(2x)\,dx$.

19 Net Change

Calculator Assumed

1. [5 marks: 1, 3, 1]

Given that $\frac{dV}{dt} = (t-2)^2 + 1$, find:

(a) the instantaneous rate of change of V with respect to t when $t = 4$

(b) the net change in V when t changes from $t = 1$ to $t = 4$

(c) the average rate of change of V in the interval $1 \le t \le 4$ seconds.

---

2. [5 marks: 2, 3]

Given that $P = (t-2)(t-4)$ cm where t is time in seconds.

(a) Find using Calculus, the instantaneous rate of change of P when $t = 5$ s.

(c) Find the average rate of change of P over the interval $0 \le t \le 5$ s.

Calculator Assumed

3. [8 marks: 2, 2, 2, 2]

The instantaneous rate with which the mass $M(t)$ kg of a body changes with respect to time t (days) is modelled by $\frac{dM}{dt} = t\sin\left(\frac{\pi t}{12}\right)$ for $t \geq 0$.

(a) Find the net change in M in the first 24 days.

(b) Find the net change in M on the 24th day.

(c) Calculate the instantaneous change in M on the 24th day.

(d) Find the average rate of change for M on the 24th day.

4. [7 marks: 2, 3, 2]

The instantaneous rate with which the number of computers infected with a virus at time t hours is modelled by $\frac{dN}{dt} = \frac{1}{\sqrt{1+0.0001t}}$ where $N(t)$ is the number of computers already infected with the virus.

(a) Find the net change in the number of computers infected with the virus within the first day.

Calculator Assumed

4. (b) How long will it take to infect 1000 computers?

(c) Determine with reasons if this mathematical model is a reasonable model.

---

5. [7 marks: 2, 2, 3]

The instantaneous rate with which the volume of water in a holding tank changes with time, is modelled by $\frac{dV}{dt} = (t-1)(t^2-9)$, for $0 \le t \le 5$, where V is the volume of water in the tank in kL and t is time in hours.

(a) Find the interval of time during which water is flowing out of the tank. Justify your answer.

(b) Find the amount of water that has flowed out of the tank.

(c) Find the amount of water that has flowed into the tank.

Calculator Assumed

6. [8 marks: 2, 3, 3]

The change in altitude of a balloon is modelled by $\frac{dh}{dt} = \frac{1}{2+t}$ where h metres is the altitude of the balloon at time t seconds.

(a) Find the height increase of the balloon in the 5th second.

(b) Find the average rate of height increase in the first 10 seconds.

(c) The initial height of the balloon was 2 metres. Find when the height of the balloon first exceeds 5 metres.

---

7. [8 marks: 2, 2, 2, 2]

The rate of population change of a bacteria culture is modelled by $\frac{dP}{dt} = 100\,e^{-0.01t}$ where t is time in hours.

(a) Find the initial rate of population change.

Calculator Assumed

7. (b) Describe the rate of change for large values of t.

(c) Calculate the net population change in the first 2 000 hours.

(d) Given that the initial population was 100, find the maximum population size. Show clearly how you obtained your answer.

---

8. [6 marks: 3, 3]

The instantaneous rate with which V the amount of fluid in a tank changes with time t hours is modelled by $\dfrac{dV}{dt} = \dfrac{t}{\sqrt{9+t^2}}$ for $t \geq 0$.

(a) Determine an expression for the amount of fluid at time t hours if at $t = 4$ hours, the amount of fluid in the tank is 85 L.

(b) The net change in Q from $t = 5$ to $t = k$ is 10L. Calculate the value of k.

Calculator Assumed

9. [10 marks: 3, 3, 3, 1]

The rate of change of pressure acting on an object is modelled by
$\frac{dP}{dt} = 4\cos\left(\frac{\pi t}{12}\right)$ where P (kilopascals) is the pressure at time t hours.

(a) Find the net change in pressure in the interval $0 \le t \le 12$.

(b) Find the net change in pressure in the interval $6 \le t \le 12$.

(c) The net increase in pressure in the interval $0 \le t \le T$ is 10 kilopascals.
Find T given that $T \le 24$.

(d) Comment on your answer in part (c).

Calculator Assumed

10. [5 marks: 3, 2]

The marginal cost for producing x hundred units of a product is given by $\frac{dC}{dx} = 0.08x$, where $\$C$ is the cost of producing x hundred items of the product.

(a) Find the cost of producing 10 000 of these items if the fixed cost is $2000.

(b) Find the net change in cost if the number of items produced is changed from 1 000 to 2 000. Justify your answer.

---

11. [5 marks: 3, 2]

The marginal profit associated with the sale of x items of a product is given by $\frac{dP}{dx} = -0.000\,81x^2 + 0.4x - 5.4$, where $\$P$ is the profit associated with the sale of x units of this product.

(a) Given that there is a loss of $500 if no items are sold, find the profit associated with the sale of 50 items.

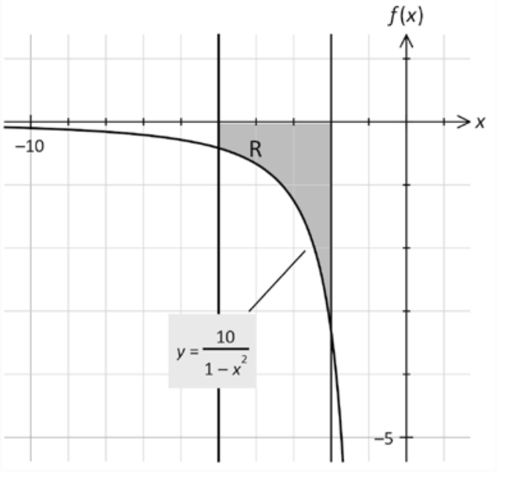
(b) Find the net change in profit if the number of items sold is changed from 50 to 350.

20 Area under a curve & Area Functions

Calculator Assumed

1. [7 marks: 3, 3, 1]

The accompanying diagram shows the graph of $f(x) = \dfrac{10}{1-x^2}$. The shaded region R is trapped between the curve, the axes and the lines $x = -5$ and $x = -2$.



(a) Estimate the area of the region R using three inscribed strips of uniform width, giving your answer to 3 decimal places. Show clearly how you obtained your answer.

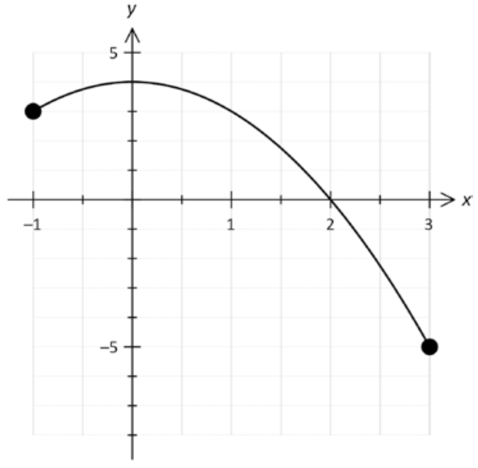
(b) Estimate the area of the region R using three circumscribing strips of uniform width, giving your answer to 3 decimal places. Show clearly how you obtained your answer.

(c) Use your answer in (a) and (b) to provide a more accurate estimate for the area of R.

Calculator Assumed

2. [6 marks: 3, 3]

The graph of $y = 4 - x^2$ is shown in the accompanying diagram. The region R is trapped between this curve and the lines $x = 1$ and $x = 3$. Estimate the area of region R using:



(a) *inscribed* rectangles of uniform width 0.5.

(b) *circumscribed* rectangles of uniform width 0.5.

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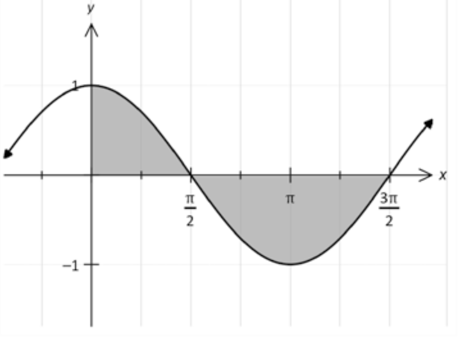
3. [4 marks: 2, 2]

The shaded region in the accompanying diagram, is trapped between the curve $y = \cos(x)$, the x-axis and the y-axis.



(a) Explain why $\int_0^{\frac{3\pi}{2}} \cos(x)\,dx$ does not represent the area of the shaded region.

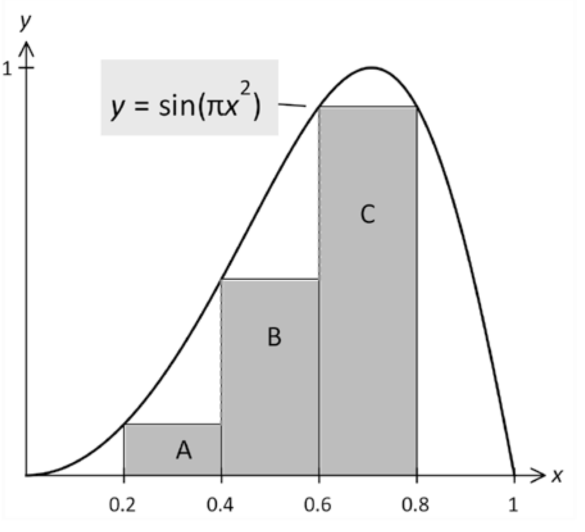
(b) Estimate the area of this shaded region using uniform circumscribing rectangles of width $\frac{\pi}{2}$.

Calculator Assumed

4. [8 marks: 3, 3, 2] [TISC]

Region R is trapped between the curve $y = \sin(\pi x^2)$ for $0 \le x \le 1$ and the x-axis.

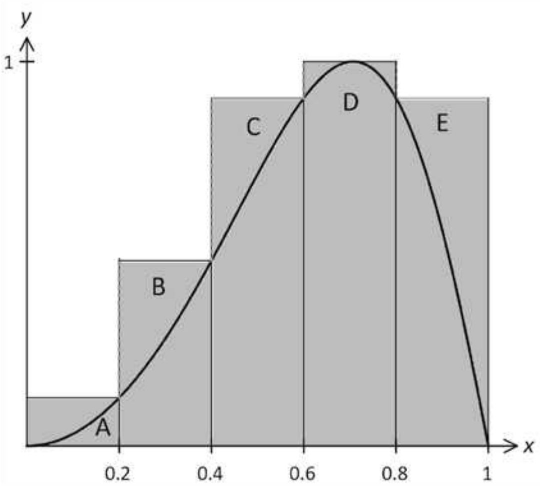
(a) S_1 is an under-estimate of the area of region R using the areas of rectangles A, B and C as shown in the diagram below. Complete the table below to show the calculations required to obtain S_1 to four decimal places.



Rectangle	Area
A	
B	
C	
Total S_1	

Calculator Assumed

4. (b) S_2 is an over-estimate of the area of region R using the areas of rectangles A, B, C, D and E as shown in the diagram below. Complete the table below to show the calculations required to obtain S_2 to four decimal places.



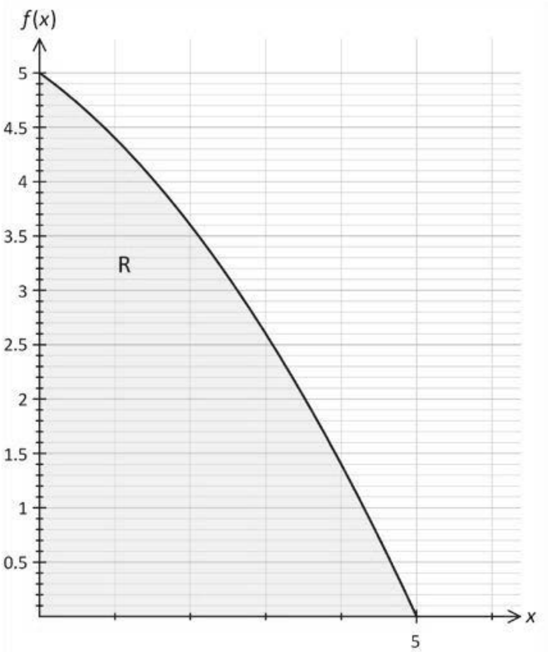
Rectangle	Area
A	
B	
C	
D	
E	
Total S_2	

(c) Explain how you could use your answers in (a) and (b) to give a better estimate for the area of R. Give this better estimate correct to three decimal places.

Calculator Assumed

5. [5 marks: 3, 3]

The diagram below shows the graph of $y = f(x)$. The shaded region R is trapped between the curve, the axes and the lines $x = -5$ and $x = -2$.
The accompanying table shows estimates for A, the area of region R using N uniform circumscribing rectangular strips for various values of N.



N	Estimate for A
2	20.3125
5	
10	15.8125
20	15.203125
50	14.8325
100	14.7081
500	14.6083
1 000	14.5958
5 000	14.5858
10 000	14.5846
15 000	14.5842

(a) Estimate the value for A for $N = 5$.

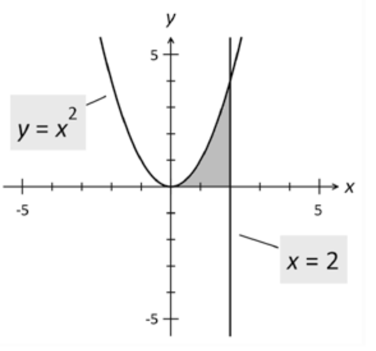
(b) Estimate to two decimal places the value for A for $N = 20\,000$. Justify your answer.

Calculator Free

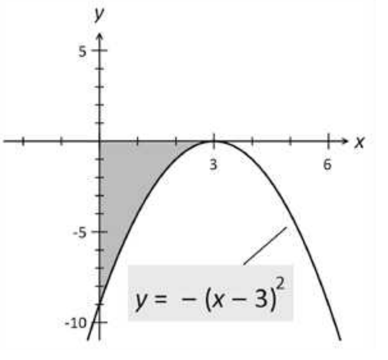
6. [10 marks: 3, 4]

Use calculus to find the exact area of the shaded region:

(a)

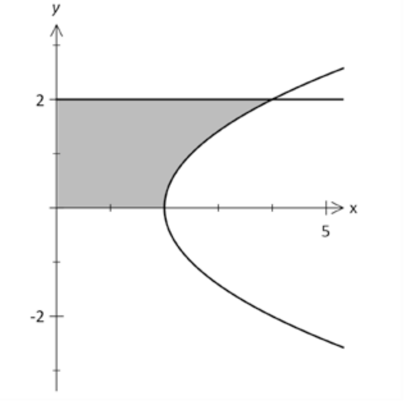


(b)



7. [5 marks]

The shaded region shown in the accompanying diagram is trapped by the x-axis, the y-axis, the line $y = 2$ and the curve with equation $y^2 = 2x - 4$. Find the area of the shaded region.

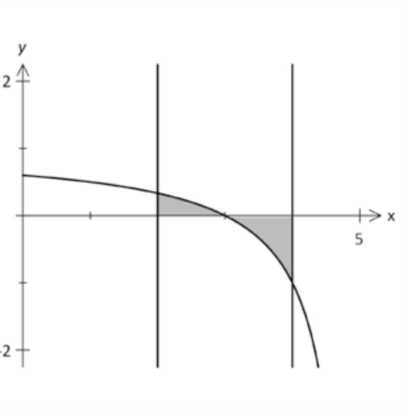


Calculator Free

8. [6 marks]

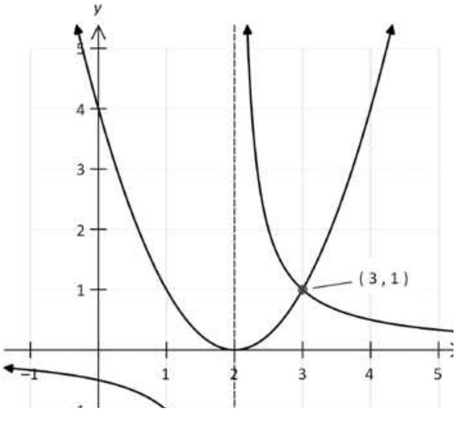
The shaded region shown in the accompanying diagram is trapped by the lines $x = 2$, $x = 4$ and the curve with equation $y = \dfrac{2}{x-5} + 1$.

Find the area of the shaded region.



9. [7 marks: 2, 5]

The accompanying diagram shows the graphs of $y = \dfrac{1}{(x-2)}$ and $y = (x-2)^2$.



(a) Determine with reasons if the area trapped between the curve $y = \dfrac{1}{(x-2)}$, the x-axis and the lines $x = 2$ and $x = 3$ can be calculated using $\displaystyle\int_2^3 \frac{1}{(x-2)}\,dx$.

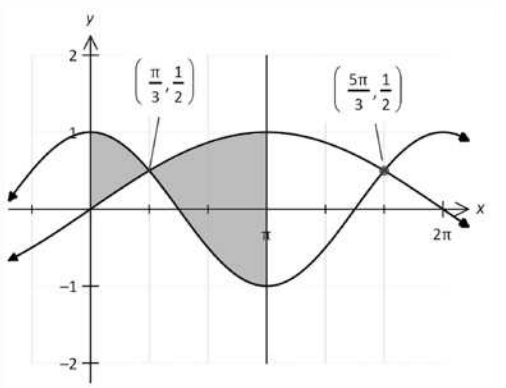
Calculator Free

9. (b) Calculate the area of region trapped between the curves $y = \dfrac{1}{(x-2)}$,

   $y = (x - 2)^2$ and the line $x = 4$.

10. [6 marks]

    The shaded region in the accompanying diagram is trapped between the curves $y = \cos(x)$, $y = \sin(0.5x)$, the x-axis, the y-axis and the line $x = \pi$.
    Calculate the area of the shaded region.



Calculator Free

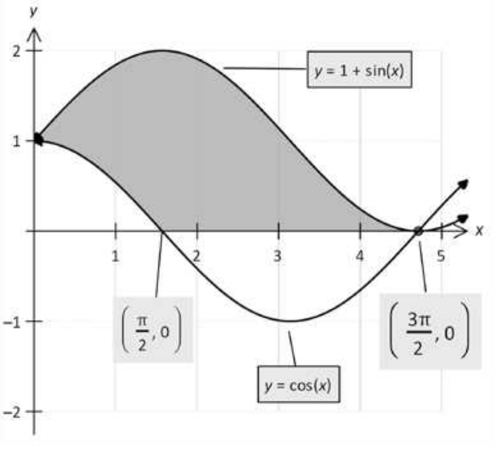
11. [8 marks: 3, 5]

(a) The gradient function of a curve $y = f(x)$ is given by $f'(x) = \cos(x)\sin(x)$.

The curve $y = f(x)$ passes through the point $(\frac{\pi}{3}, 1)$.

Find the equation of the curve $y = f(x)$.

(b) The shaded region shown in the accompanying diagram is trapped by the curve with equation $y = 1 + \sin x$, $y = \cos(x)$ and the x-axis.
Find the area of the shaded region.



Calculator Free

12. [9 marks: 2, 2, 3, 2]

The function $y = f(x)$ is continuous for all real values of x and $f(x) \geq 0$ for $1 \leq x \leq 4$. It is known that $\int_1^4 f(x)\, dx = \text{A}$ and $\int_4^6 f(x)\, dx = -\text{B}$ here A and B are positive real numbers. Find, with reasons, in terms of A and/or B where appropriate:

(a) the area of the region trapped between the curve $y = 2f(x)$, the x-axis and the lines $x = 1$ and $x = 4$.

(b) $\int_1^6 f(x)\, dx$

(c) $\int_4^6 2x - f(x)\, dx$

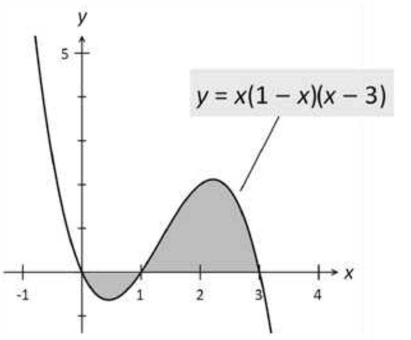
(d) $\int_{-1}^{-4} f(-x)\, dx$

Calculator Assumed

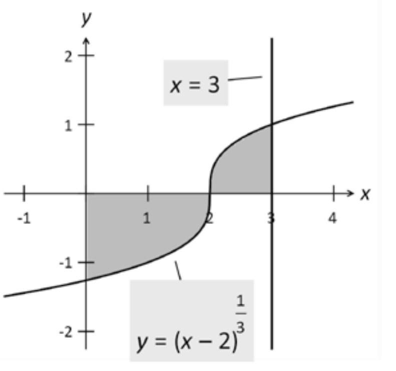
13. [14 marks: 3, 3, 4, 4]

Use an appropriate method to find the area of the shaded region. Show clearly how you obtained your answer.

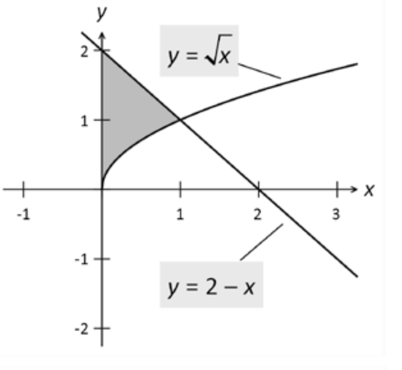
(a)



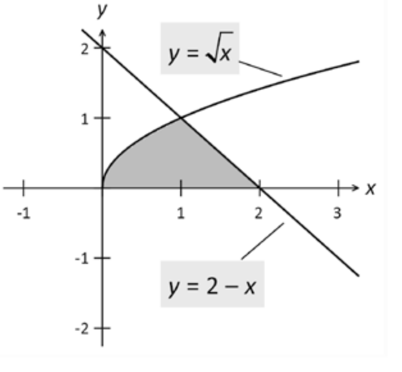
(b)



(c)



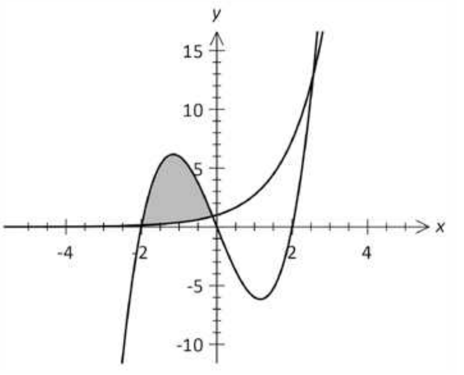
(d)



Calculator Assumed

14. [8 marks: 4, 4]

The accompanying diagram shows the graphs of $y = e^x$ and $y = 2x(x^2 - 4)$.

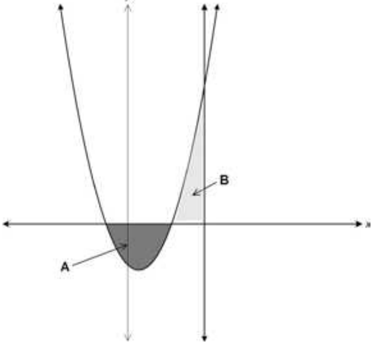


(a) Write an integral that can be used to determine the area of the shaded region. Hence, find the area of this region.

(b) Write but do not evaluate, an expression involving integrals that can be used to determine the area of the region trapped between the two curves, for $x \geq 0$.

15. [6 marks]

The accompanying diagram shows the graph of $y = (x + 1)(x - 2)$. The region bounded by the curve and the x-axis is denoted A. The region bounded by the curve, the positive x-axis and the line $x = k$ is denoted B.
Find k if the area of A = area of B.

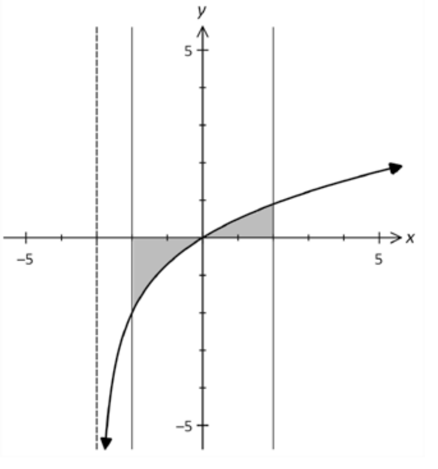


Calculator Assumed

16. [6 marks: 1, 5] [TISC]

(a) Determine $\frac{d}{dx}\left((2x-12)\sqrt{x+3}\right)$.

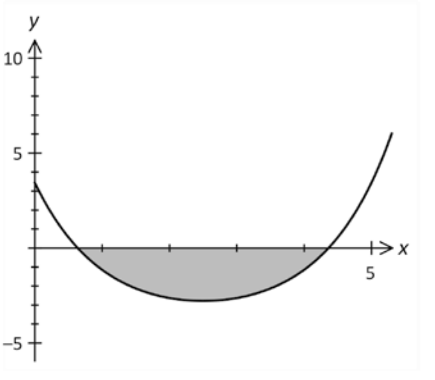
(b) The accompanying diagram shows the graph of $y = \frac{x}{\sqrt{x+3}}$. Region R (shaded) is trapped between the curve, the x-axis and the lines $x = -2$ and $x = 2$.
Show how your answer in (a) can be used to calculate the area of region R. Give your answer correct to 2 decimal places.



Calculator Assumed

17. [12 marks: 2, 5, 5]

A 10-metre portion of an irrigation channel is of uniform cross-section. The cross-section (shaded) is modelled by the equation $y = e^{x-3} + e^{-x+2} - 4$ where x is measured in metres. The top edge of the channel is modelled by the line $y = 0$.



(a) For what values of x is the model of the cross-section valid?

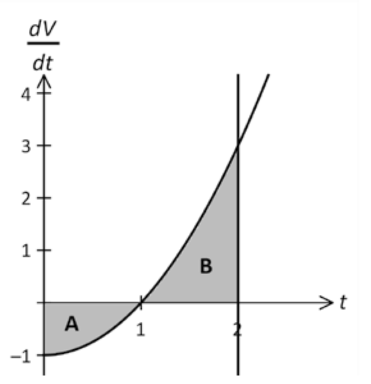
(b) Use Calculus to find the depth of the deepest point of this portion of the channel.

(c) Find the maximum capacity of this portion of the channel. [1 m^3 = 1 kL]

Calculator Assumed

18. [9 marks: 1, 2, 2, 2, 2]

The instantaneous rate with which the amount of fuel, V litres, in a holding tank, changes with respect to time t minutes, is modelled by $\frac{dV}{dt} = t^2 - 1$. The sketch of $\frac{dV}{dt}$ against t is shown in the accompanying diagram.



(a) Explain what happens at $t = 1$ minute.

(b) Find the area of region A and interpret your answer.

(c) Find the area of region B and interpret your answer.

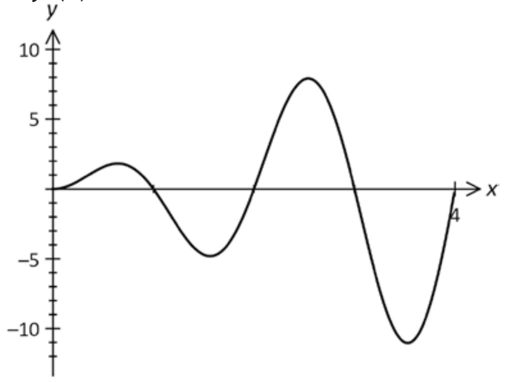
(d) Find the amount of fuel in the tank after 2 minutes, if initially there were 5 litres in the tank.

(e) Use the information in (d) to find the average rate of change of the amount of fluid in the first 2 minutes.

Calculator Assumed

19. [7 marks: 2, 5] [TISC]

The diagram below shows the graph of the function $f(x) = \pi x \sin(\pi x)$ where the domain of the function $f(x)$ is $0 \le x \le 4$.



The function $A(t) = \int_0^t f(x)\,dx$ where $0 \le t \le 4$.

(a) Use your CAS calculator to help complete the table below.

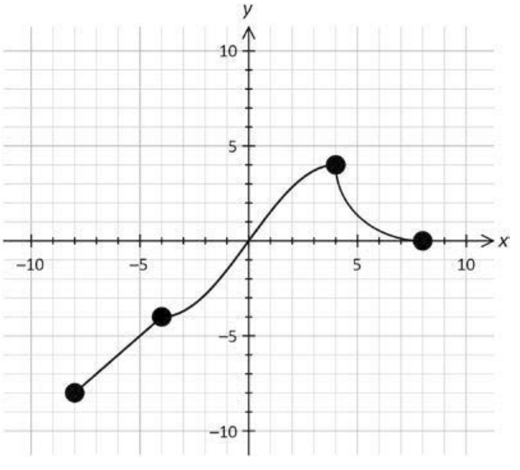
t	0	1	2	3	4
$A(t)$	0				

(b) Use a calculus method to determine the minimum value for $A(t)$ and the corresponding value of t. Show clearly how you obtained your answer.

Calculator Assumed

20. [6 marks: 2, 4]

The accompanying diagram shows the graph of $y = f(x)$ for $-8 \le x \le 8$. The graph consists of the line $y = -x$ for $-8 \le x < -4$, the curve $y = 4\sin\left(\frac{\pi x}{8}\right)$ for $-4 \le x < 4$ and the quarter circle with centre at (8, 4) and radius 4 for $4 \le x \le 8$.



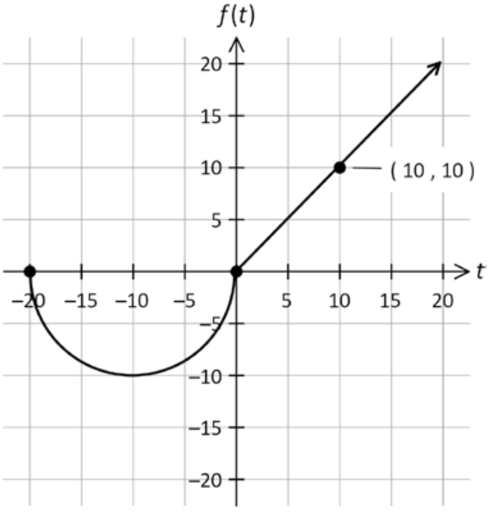
(a) Determine with reasons the value of $\int_{-4}^{4} f(x)\,dx$.

(b) Evaluate $\int_{-8}^{8} f(x)\,dx$.

Calculator Assumed

21. [6 marks] [TISC]

The accompanying diagram shows the graph of $y = f(t)$ which consists of a semi-circle with radius 10 for $-20 \le t \le 0$ and a straight line $y = t$ for $t > 0$.



The function $A(x) = \int_{-20}^{x} f(t)\, dt$.

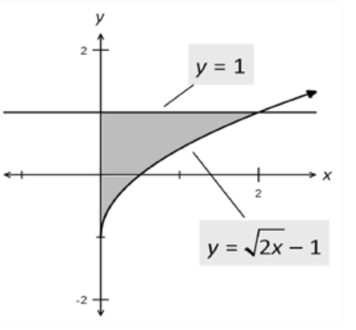
(a) Determine the *exact* minimum value for $A(x)$. Explain clearly how you obtained your answer.

(b) Determine with reasons the *exact root* of the function $A(x)$.

Calculator Assumed

22. [7 marks: 2, 3, 2]

The integral $\int_a^b f(y)\,dy$ represents the sum of the signed areas of the regions trapped between the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$.



Consider the curve with equation $y = \sqrt{2x} - 1$.

(a) Rewrite the equation of the curve with y as the independent variable.

(b) Let R represent the region trapped between the curve with equation $y = \sqrt{2x} - 1$, the y-axis and the line $y = 1$.

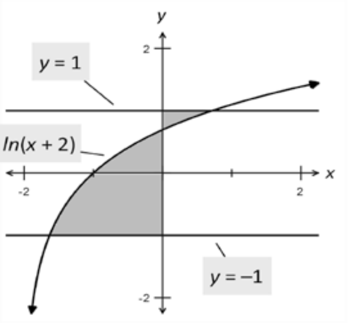
(i) Use your answer in (a) and an integral to express the area of R.

(ii) Use your answer above to determine the exact area of R.

Calculator Assumed

23. [8 marks: 2, 4, 2]

The integral $\int_a^b f(y)\, dy$ represents the sum of the signed areas of the regions trapped between the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$.



Consider the curve with equation $y = \ln(x + 2)$.

(a) Rewrite the equation of the curve with y as the independent variable.

(b) Let R represent the region trapped between the curve with equation $y = \ln(x + 2)$, the y-axis and the lines $y = -1$ and $y = 1$.

(i) Use your answer in (a) and integrals to express the area of R.

(ii) Use your answer above to determine the area of R. Give your answer to two decimal places.

21 Rectilinear Motion

Calculator Assumed

1. [11 marks: 4, 3, 4]

The displacement of a body moving along a straight line is given by $s = -t^3 + at^2 + bt + 3$ metres where t is time in seconds. The initial velocity of the body is 5 ms^{-1}. The body is momentarily at rest when $t = 1$ second.

(a) Find the values of a and b.

(b) Find when the body changes direction.

(c) Find the instantaneous speed at $t = 5$ seconds and the average speed in the first 5 seconds.

Calculator Assumed

2. [7 marks: 1, 3, 1, 2]

The displacement of a body at time t seconds is given by $s = 4t + \dfrac{1}{1+t}$ metres.

(a) Find an expression for the velocity of the body at time t seconds.

(b) Show that the body is never stationary.

(c) Find an expression for the acceleration at time t seconds.

(d) Describe the motion of the body for large values of t.

Calculator Assumed

3. [13 marks: 2, 5, 2, 4]

The displacement (s metres) of particle P, t seconds after passing a fixed point O is given by $s = 10\, t\, e^{-t} - 1$.

(a) Find an expression for the velocity at time t seconds and hence the velocity at $t = 2$ seconds.

(b) Use Calculus to find the maximum displacement of P in the first two seconds.

(c) Find the acceleration of P at maximum displacement.

(d) Find the average speed in the first 2 seconds.

Calculator Assumed

4. [13 marks: 3, 3, 2, 5]

An object P travels along the x-axis. The velocity of the particle t seconds after passing through the origin is given by $v = \frac{1}{1+t} + \frac{1}{(1+t)^2}$ cms^{-1}.

(a) Calculate the magnitude and direction of the acceleration of P as it passes through the origin.

(b) Determine the displacement of P at the end of the fifth second.

(c) Determine if P ever reverses direction after it travels past O.

(d) Determine an expression for D, the distance travelled in the first N seconds. Hence, find N if the average speed of P in the first N seconds is 0.33 cms^{-1}. Justify your answer.

Calculator Assumed

5. [12 marks: 2, 3, 2, 3, 2]

An object P travels in a straight line. The velocity of the particle t seconds after passing a fixed point O is given by $v = 12t^3 - 48t^2 + 60t - 24 \;\; cms^{-1}$.

(a) Calculate the initial acceleration of P.

(b) Determine when P reverses direction.

(c) Determine the displacement of P at the time it reverses direction.

(d) The change in displacement in the first N seconds is 9 metres. Determine the value of N.

(e) Calculate the average speed of P in the first five seconds.

Calculator Assumed

6. [9 marks: 2, 3, 4]

The acceleration (ms^{-2}) of a particle moving along a straight line is given by $a = -4\cos 2t$, where t is time in seconds. At $t = 0$, the velocity of the particle is $0\ ms^{-1}$ and the displacement of the particle is 1 m.

(a) Find an expression for the velocity of the particle at any time t.

(b) Find an expression for the displacement of the particle at any time t.

(c) Find the average speed in the first π seconds.

Calculator Assumed

7. [10 marks: 1, 6, 3]

A particle starts off from a fixed point O with an acceleration (mms^{-2}) of $a = mt - 24$, where t is time in seconds. The particle travels in a straight line and returns to O at $t = 4$ seconds and has a change of displacement of –9 mm in the third second (it moves in the same direction during this time).

(a) Find in terms of m an expression for the velocity of the particle at any time t.

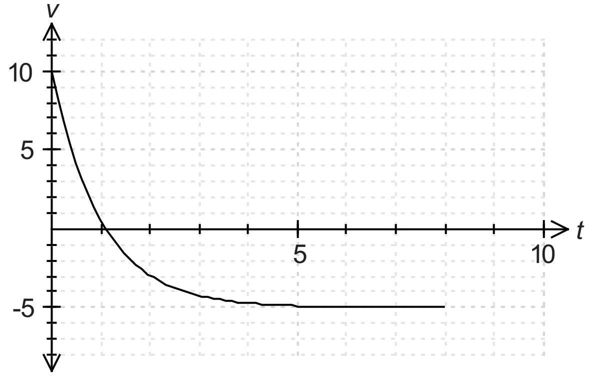
(b) Find the displacement of the particle at any time t.

(c) Find when the particle is at O the third time (if it does).

Calculator Assumed

8. [11 marks: 2, 3, 3, 3]

A particle P moving in a straight line, starts off from a fixed point O with velocity v_0 ms^{-1}. Its velocity at any time t is given by $v = 15e^{-t} - k$ ms^{-1}, where k is a constant. The velocity-time graph of P is given in the accompanying diagram. Velocity v is in ms^{-1} and time t in seconds.



(a) Find v_0 and k.

(b) Find the time (correct to 2 decimal places) when P reverses direction.

(c) Find the displacement of P at the time it reversed its direction.

(d) Find the average speed in the first 8 seconds.

Calculator Assumed

9. [11 marks: 3, 3, 2, 3] [TISC]

The acceleration a ms^{-2}, of a particle P moving in a straight line at time t seconds is given by $a = mt + n$. The average acceleration during the first two seconds is 1 ms^{-2} and the initial velocity of the particle is 4 ms^{-1}. The acceleration of the particle at $t = 2$ is -1 ms^{-2},

(a) Find the change in velocity in the first two seconds.

(b) Show that $v = \frac{1}{2}(1 - n)\, t^2 + nt + 4$.

(c) Show that $v = -t^2 + 3t + 4$.

(d) Find the total distance travelled from the moment the particle starts travelling to before the particle changes direction.

Calculator Assumed

10. [11 marks: 4, 7]

A particle P travels along the x-axis and its acceleration at time t seconds is given by $\frac{d^2x}{dt^2} = pt^2 + qt + r$ cms^{-2} where p, q and r are constants. The particle starts from the point K with coordinates (8, 0) with velocity –12 cms^{-1}. The particle changes direction at the same point when $t = 1$ and $t = 2$ seconds.

(a) Show that its displacement at time t seconds is given by

$$x = \frac{pt^4}{12} + \frac{qt^3}{6} + \frac{rt^2}{2} - 12t + 8.$$

(b) Find the values of p, q and r.

Calculator Assumed

11. [13 marks: 4, 3, 3, 3] [TISC]

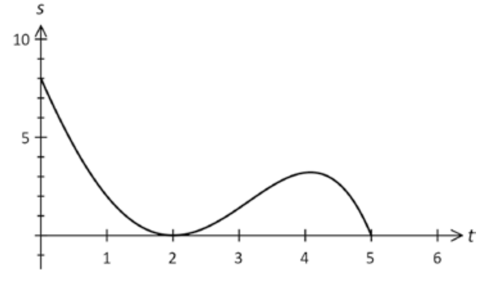
A particle P travels in a straight line. Its displacement s (metres) from a fixed point O at time t seconds is given $s = -0.1t^4 + 0.5t^3 + 1.2t^2 - 7.6t + 8$.
The points A and B are on opposite sides of O.
The particle starts at A and completes its journey at B where OA = OB.

(a) Calculate how long it takes P to travel from A to B.

(b) Determine the total distance travelled by P in travelling from A to B.

(c) The acceleration of P is positive between t_1 and t_2. Determine t_1 and t_2.

(d) The accompanying diagram shows the graph of s against t for $0 \le t \le 5$ seconds. Describe the motion of P in the first two seconds.in terms of its displacement, direction and speed and acceleration.



Calculator Assumed

12. [12 marks: 1, 3, 2, 4, 2] [TISC]

A particle P starts from rest from a fixed point O and travels in a straight line. The velocity of P after t seconds is given by $v = 100(e^{-0.1t} - e^{-1.1t})$ ms^{-1} for $t \geq 0$ seconds.

(a) Show that P travels only in one direction.

(b) Determine using a calculus method when the acceleration of P is zero.

(c) Determine the maximum speed of P in the first two seconds.
Explain how you obtained your answer.

(d) Determine the displacement of P when $t = 99$ seconds and $t = 100$ seconds.

(e) Describe the speed and displacement of P for extremely large values of t.

Calculator Assumed

13. [11 marks: 2, 3, 2, 4]

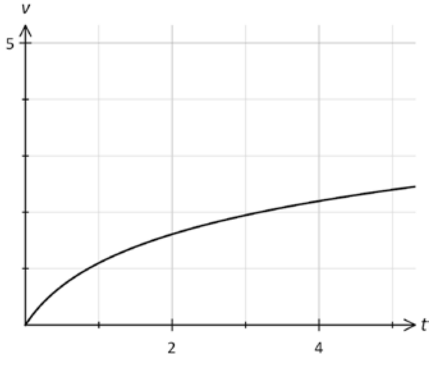
A particle P travels along a straight line.
The velocity of particle P at time t seconds is given by $v = ln(1+2t)$ ms^{-1}.

(a) Calculate the acceleration of P when $t = 2$ seconds.

(b) Use the incremental formula to estimate the increase in the velocity of P when time t increases from 2 seconds by 0.1 seconds.

(c) Write a mathematical expression involving the use of integrals that represents the distance travelled by P in the first two seconds.
DO NOT evaluate this integral.

The accompanying diagram shows the graph of $v = ln(1+2t)$.



(d) Estimate the change in displacement of P between $t = 2$ and $t = 4$ seconds by using two rectangular strips of equal width. Show clearly how you obtained your answer. State whether your answer is an under-estimate or an over-estimate

Calculator Assumed

14. [13 marks: 3, 2, 4, 2, 2] [TISC]

A particle P travels in a straight line with an initial velocity of 10 cms^{-1}.
The acceleration of the particle at time t seconds is given by
$a = 12t^2 - 24t - k \;\; \text{cms}^{-2}$ where k is a constant.

(a) Determine an expression, in terms of k, for the velocity of P at time t seconds.

(b) Determine an expression, in terms of k, for the displacement of P at time t seconds.

(c) The particle P changes direction at $t = 5$ seconds.

(i) Determine the change of displacement in the first second. [4 marks]

(ii) Calculate the distance travelled in the first second.

(iii) Determine with reasons if there is evidence that P undergoes a change in direction before $t = 5$ seconds.

22 Discrete Random Variables I

Calculator Free

1. [4 marks: 2, 2]

Determine with reasons if each of the following functions are probability distribution functions for discrete random variables.

(a) $f(k) = 1/5$ for $k = 0, 1, 2, 3, 4, 5$

(b) $f(k) = k/5$ for $k = -2, -1, 0, 1, 2, 3$

---

2. [8 marks: 3, 3, 2]

The random variable X has probability distribution

$$P(X = x) = \begin{cases} x \times P(X = x+1) & x = 1, 2, 3 \\ k & x = 4 \end{cases}.$$

(a) Find the value of k.

(b) Find $P(X \leq 3 | X > 1)$.

(c) Determine the mean for X.

Calculator Free

3. [10 marks: 1, 3, 3, 3]

The table below describes the cumulative probability distribution function of a discrete random variable X.

x	0	1	2	3	4	5
$P(X \le x)$	$\frac{11}{20}$	$\frac{15}{20}$	$\frac{17}{20}$	$\frac{18}{20}$	$\frac{19}{20}$	k

(a) Determine the value of k.

(b) Calculate $P(X < 4 \mid X \ge 1)$.

(c) Calculate μ, the mean for X.

(d) Calculate the σ^2, the variance for X.

Calculator Free

4. [4 marks: 2, 2] [TISC]

The random variable Y has values Y = 0, 1, 2, 3 & 4. The table below describes the cumulative probability distribution function for Y, where a, c, d, e and f are non-negative constants.

y	0	1	2	3	4
$P(Y \le y)$	a	c	d	e	f

(a) Determine the range of values for a and the value for f.

(b) Determine $P(1 < Y \le 3)$.

5. [9 marks: 2, 2, 5]

The random variable X has probability distribution function $p(x)$ defined by

$p(x) = \dfrac{x+2}{k}$ for $x = -1, 0, 1$ and 2.

(a) Determine the value of k.

(b) Calculate $P(X = 0 \mid X \neq 1)$.

(c) Calculate μ and σ^2, respectively the mean and variance for X.

Calculator Free

6. [6 marks: 3, 3] [TISC]

The accompanying table describes the probability distribution of a discrete random variable X.

X	1	2	3	4	5
$P(X = x)$	0.1	p	q	0.2	0.2

(a) Determine the values of p and q if $P(X \le 2) = 0.25$.

(b) Determine the values of p and q if $P(X \ge 3 \mid X \le 4) = \dfrac{5}{8}$.

7. [9 marks: 3, 3, 3]

Consider the probability distribution function $f(x) = \begin{cases} a & x = -2, -1, 0, 2 \\ b & x = 1 \end{cases}$

where a and b are real numbers.

(a) Determine *all* mathematical constraints for a and b.

(b) Determine E(X) in terms of a.

(c) Determine with reasons the conditions for E(X) to be positive.

Calculator Assumed

8. [6 marks]

Verify that the function $f(x) = \dfrac{\dbinom{15}{x}\dbinom{5}{5-x}}{\dbinom{20}{5}}$ for $x = 0, 1, 2, 3, 4, 5$

may be used as the probability distribution function of a discrete random variable X. Determine the exact mean and variance for X.

---

9. [9 marks: 3, 3, 3]

A random variable X has probability distribution $P(X = k) = \begin{cases} \dfrac{x}{4} & x = 1, 2 \\ \dfrac{x}{k} & x = 3, 4, 5 \end{cases}$.

(a) Find the value of k .

Calculator Assumed

9. (b) Determine the mean and variance of X.

(c) Find $P(X \le 4 \mid X > 2)$.

---

10. [8 marks: 3, 5]

A discrete random variable X has cumulative probability distribution function given by $P(X \le x) = \dfrac{x^3 + 3x^2 + 2x}{k}$ for $x = 1, 2, 3, 4, 5$.

(a) Determine the value of k.

(b) Calculate the mean and variance for X.

Calculator Assumed

11. [7 marks: 2, 2, 3] [TISC]

(a) The table below shows the values taken by a function $f(x)$.

x	–1	0	1	0.5
$f(x)$	0.2	0.6	0.1	0.1

Peter argues that $f(x)$ cannot be a probability distribution function of a discrete random variable as x has a negative value. Comment on his answer.

(b) The table below shows the values taken by a function $f(x)$.

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.2	0.5	a	b	0.1

(i) Under what conditions can $f(x)$ represent the probability distribution function of a discrete random variable?

(ii) If $f(x)$ is the probability distribution function of a discrete random variable X, find the values of a and b given that P(X = 1.0) = 2 × P(X = 1.5).

Calculator Assumed

12. [8 marks: 2, 3, 3]

The discrete random variable X has mean 4 and standard deviation 1.

(a) Calculate the mean and standard deviation for the random variable $Y = 5 + 2X$.

(b) Calculate the mean and standard deviation for the random variable W if $X = 7 + 3W$.

(c) Each value of X is decreased by 20% and the result increased by 20 units. Determine the mean and variance of the new variable.

---

13. [6 marks]

The random variable X has mean 100 and standard deviation 4. The random variable $Y = aX + b$. Find a and b if the mean and standard deviation for Y are 90 and 6 respectively.

Calculator Assumed

14. [9 marks: 3, 6]

The probability distribution function for a random variable X is given by $p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$. The random variable Y = 10 – 2X.

(a) Calculate P(2 ≤ Y ≤ 6).

(b) Calculate the mean and variance for Y.

Calculator Assumed

15. [9 marks: 1, 2, 3, 3] [TISC]

X is a discrete random variable with probability distribution function
$P(X = x) = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$.
Y is discrete random variable with probability distribution function
$P(Y = y) = \frac{1}{10}$ for $y = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
It is known that the variables X and Y are independent.

(a) Find $P(X \leq 4)$.

(b) Find $P(X = 5$ and $Y = 5)$.

(c) Find $P(X = 5$ or $Y = 5)$.

(d) Find $P(X + Y = 12)$. Show clearly how you obtained your answer.

23 Discrete Random Variables II

Calculator Free

1. [7 marks: 2, 2, 1, 2]

The table below shows the projected returns for every $100 000 in an investment scheme and the accompanying probabilities.

Returns	–$20 000	–$10 000	k	$20 000	$50 000
Probability	0.01	p	0.2	0.27	0.02

(a) Determine the value of p.

(b) Find the mean return per $100 000 in terms of k.

(c) Find the mean profit per $100 000 if k = $5 000.

(d) Find the value(s) of k if the mean profit per $100 000 must exceed $5000.

2. [14 marks: 1, 5, 3, 5]

At an agricultural fair, a games stall operator offers prizes worth $20, $5, and $1 for one attempt at a particular game. The probabilities of winning these prizes are respectively 0.001, 0.01 and 0.5.

(a) Find the probability of not winning a prize.

Calculator Assumed

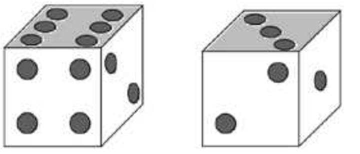
2. (b) If each game costs \$1, find the expected profit per game and the accompanying standard deviation for the games stall operator.

(c) Vegas played 50 games at \$2 for a game. Find her expected profit/loss.

(d) The games stall operator made a profit of \$193 from 100 games. How much did he charge per game?

Calculator Assumed

3. [10 marks: 3, 5, 2]



In a game of dice, a fair six sided dice is rolled twice and S the sum of the dots displayed on the upper-most faces recorded. The organiser of the game pays out \$6 for sums that exceed 10, \$2 for sums that are between 7 and 10 inclusive and \$0 otherwise. Define the random variable T as the profit per game for the organiser.

(a) Calculate $P(S > 10)$.

(b) Determine the expected value for T if it costs \$2 to play each game.

(c) How much should the organiser charge per game to break even (no loss)? Justify your answer.

Calculator Assumed

4. [12 marks: 6, 6]

A random number generator generates whole numbers randomly between 1 and n. Define X: the number generated by the random number generator.

(a) Given that the mean number generated is 5 000, determine the standard deviation for X.

(b) Find the mean for X if the variance for X is 20 750.

Calculator Assumed

5. [8 marks: 4, 2, 2]

A committee of five students is to be selected from a group of five female students and five male students.
Define X: Number of female students in this committee.

(a) Find the probability distribution for X.

(b) Find the probability that there are at least as many females than males in the committee.

(c) Find the expected number of females in the committee.

Calculator Assumed

6. [10 marks: 1, 1, 2, 2, 4]

It is known that 0.5% of USB Drives are defective. USB drives are randomly picked from a large carton.

(a) Find the probability that the second USB drive picked is defective.

(b) Find the probability that the first defective drive is the 3rd drive picked.

(c) Find the probability the first defective drive is the 11th drive picked.

(d) Write an expression for the probability that first defective USB drive is the nth drive picked.

(e) Define X: No of USB drives that need to be selected to pick the first defective drive. Determine with reasons if X is a discrete random variable with an appropriate probability distribution function.

Calculator Assumed

7. [10 marks: 2, 2, 6]

It is known that 65% of students at a certain college are foreign born. Students are randomly chosen from this college.

(a) Find the probability that the second foreign born student is the third student selected.

(b) Find the probability that 4 students need to be picked before picking the second foreign born student.

(c) Define X: No of students that need to be selected before picking the second foreign born student. Determine with reasons if X is a discrete random variable with an appropriate probability distribution function..

---

8. [6 marks: 3, 3]

It is estimated that 9% of Australians have O^- type blood.

(a) Determine the probability that the 8^{th} Australian randomly selected is the second person who has O^- type blood.

Calculator Assumed

8. (b) A group of 20 Australians comprise 3 adults with O^- type blood.
Five persons are chosen from this group. Calculate the probability that exactly one of the five persons chosen has O^- type blood.

---

9. [6 marks: 3, 3]

60% of adults in a certain country are lactose intolerant. Five adults are randomly chosen from this country. Define X as the fifth adult chosen is the *n*th person chosen who is lactose intolerant.

(a) Explain why $P(X = n) = \left({}^4C_{n-1} \times 0.4^{5-n} \times 0.6^{n-1}\right) \times 0.6$ for $n = 1, 2, 3, 4, 5$.

(b) Determine with reasons if $P(X = n) = \left({}^4C_{n-1} \times 0.4^{5-n} \times 0.6^{n-1}\right) \times 0.6$

for $n = 1, 2, 3, 4, 5$ can be classified as a probability distribution of a discrete random variable.

Calculator Assumed

10. [9 marks: 4, 5]

It is known that $100p\%$ of homes in a certain city have broadband internet connections.

(a) Define X: No. of homes with broadband internet connections out of n homes selected. Determine with reasons if X is a discrete random variable.

(b) Define X: Out of 10 homes selected, the only home with broadband internet connection is the xth home selected, where $x \leq 10$. For $p = 0.2$, determine with reasons if X is a discrete random variable.

Calculator Assumed

11. [8 marks: 4, 4]

A box contains 5 red and 15 green balls.

(a) Four balls are chosen *with* replacement from this box.
Define X: No. of red balls chosen.
Determine with reasons if X is a discrete random variable with a clearly stated probability distribution function..

(b) Four balls are chosen *without* replacement from this box.
Define X: No. of red balls chosen.
Determine with reasons if X is a discrete random variable with a clearly stated probability distribution function..

Calculator Assumed

12. [9 marks: 2, 2, 3, 2]

The number of items per order for an online store is modelled by the random variable X with probability distribution $P(X = x) = \frac{-1}{\ln 0.4} \times \frac{0.6^x}{x}$ for $x = 1, 2, 3, \ldots$

(a) Calculate the probability that an order contains not more than 3 items.

(b) Calculate the probability that an order of not more than 3 items has exactly one item.

(c) 99% of orders have no more than n items.
Determine the value of n. Justify your answer.

(d) Calculate the mean for X.

Calculator Assumed

13. [8 marks: 3, 2, 3]

Define the random variable X: Number of accidents at a traffic intersection in one week. The probability distribution $P(X = x) = \dfrac{e^{-2}2^x}{x!}$ for $x = 0, 1, 2, 3, \ldots$

(a) Calculate the probability that within any given week there are at least 2 accidents.

(b) Calculate the probability that in any week with at least two accidents, there are exactly 2 accidents.

(c) More than 30% of weeks have at least n accidents.
Determine the value of n. Justify your answer.

24 The Binomial Distribution

Calculator Free

1. [7 marks: 1, 2, 2, 2]

An eight sided die has faces numbered 1, 2, 3, 4, 5, 6, 7 or 8.

(a) The die is rolled once. Define the random variable

$$X = \begin{cases} 1 & \text{if die lands on a face with numbers 1, 3 or 5} \\ 0 & \text{if die lands on a face with numbers 2, 4, 6, 7 or 8} \end{cases}.$$

(i) Identify the name of the probability distribution for X and state the associated parameters.

(ii) Determine the mean and variance of X.

(b) The die is rolled 40 times and the random variable Y is defined as,
Y : Number of times the die lands on faces with numbers 1, 3 or 5.

(i) State the mathematical expression for the probability distribution for Y.

(ii) Determine the mean and variance for Y.

Calculator Free

2. [8 marks: 2, 2, 2, 2]

40% of students in a certain school are short-sighted.

Five students from this school are randomly selected. The random variable

$$S_i = \begin{cases} 1 & \text{if student } i \text{ is short-sighted (success)} \\ 0 & \text{if student } i \text{ is not short-sighted (failure)} \end{cases} \quad \text{for } i = 1, 2, 3, 4, 5.$$

(a) State the probability distribution function for S_i.

(b) Determine the mean and variance for S_i.

Define the random variable $X = S_1 + S_2 + S_3 + S_4 + S_5$.

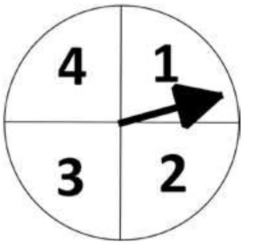
(c) State the probability distribution function for X.

(d) State the mean and variance for X.

Calculator Assumed

3. [13 marks: 3, 3, 4, 3]

The accompanying diagram shows a circle divided into four equal sized sectors numbered 1, 2, 3 and 4 respectively and a spinning pointer. The pointer is spun and will come to rest in one of the 4 sectors. It is spun again if it comes to rest on the line dividing the sectors.



(a) Define the random variable

$$X = \begin{cases} 1 & \text{if pointer stops in sector labelled 1} \\ 0 & \text{if pointer does not stop in sector labelled 1} \end{cases}.$$

Determine the mean and variance of X.

(b) The pointer is spun 20 times and the random variable Y is defined as,
Y : Number of times the pointer stops in sector labelled 1.
Show the use of a *combinatorial method* to determine the probability that:

(i) the counter stops in the sector labelled 1 exactly four times.

(ii) the counter stops in the sector labelled 1 at least 2 times.

(iii) the counter stops in the sector labelled 1 only on the last spin.

Calculator Assumed

4. [12 marks: 4, 3, 5]

10 students in a class of 25 are short-sighted. 5 students are randomly selected from this class. The random variable

$$S_i = \begin{cases} 1 & \text{if student } i \text{ is short-sighted (success)} \\ 0 & \text{if student } i \text{ is not short-sighted (failure)} \end{cases} \quad \text{for } i = 1, 2, 3, 4, 5.$$

Define the random variable $X = S_1 + S_2 + S_3 + S_4 + S_5$.

(a) Use the variables S_i to verify that X is not a binomial variable.

(b) Determine the probability distribution function for X.

(c) Calculate the mean and variance for X.

Calculator Free

5. [3 marks] [TISC]

X is a Binomial variable with parameters n and $p = \frac{1}{2}$.

Find in terms of n, P(X > 1).

---

6. [6 marks: 1, 2, 2, 1] [TISC]

The probability that John is late for school on any school day is 0.4 and it is independent of other days. For a school week of five days, write expressions, but *do not evaluate*, for:

(a) the probability that he is late every day

(b) the probability that he is late on exactly two days

(c) the probability that he is late at least once

(d) the mean number of days he will be late

Calculator Assumed

7. [8 marks: 3, 2, 3]

On average, 20% of teachers in a particular state have previously been treated for work related depression. The mathematics department in a particular school in this state has 11 staff members (all teachers!).

(a) Find the expected number of depressed teachers and its expected standard deviation. Justify your answer.

(b) Find the probability that there are no more than five staff members in this department that have previously been treated for work related stress.

(c) Find the probability that there are exactly two staff members who have previously been treated for work related stress given that there are no more than five of them.

Calculator Assumed

8. [12 marks: 3, 1, 3, 2, 3]

It is known that the probability of finding a defective biro in a large consignment of biros is 0.05. These biros are packed and sold in packs of 12 biros each.

(a) Find the expected number of defective biros in any given pack and its associated standard deviation.

(b) Find the probability that a randomly chosen pack has exactly two defective biros.

(c) Given that a randomly chosen pack has no more than 4 defective biros, find the probability that the pack has at least two defective biros.

(d) Find the probability that there were no more than 4 defective biros each in two randomly chosen packs.

(e) Find the probability that in a box of 50 such packs (of 12 biros each), no more than 10 of these packs had exactly two defective biros each.

Calculator Assumed

9. [11 marks: 4, 4, 3] [TISC]

60% of students in a school own at least one Apple® device.

(a) Ten students were randomly chosen. Find the probability that more than six of these students own at least one Apple® device. Show how you obtained your answer.

(b) n students were randomly chosen. Let X: Number of students out of n students who own at least one Apple® device. The mean for X is μ and the accompanying standard deviation is $\frac{4\sqrt{15}}{5}$. Find n and μ.

(c) n students were randomly chosen. The probability that all of these students chosen owned at least one Apple® device is 0.01. Find n.

Calculator Assumed

10. [16 marks: 3, 3, 3, 4, 3]

20% of residents in a suburb owned dogs.

(a) In a group of 30 randomly chosen residents, calculate the probability that no more than 5 of these residents owned dogs.

(b) In a group of n residents, the mean number of residents owning dogs is 40. Calculate the associated standard deviation.

(c) The probability that n residents need to be chosen before the first resident owning a cat is chosen is 0.01374. Calculate n.

(d) Calculate the probability that sixth resident picked is the third resident picked who owned a dog.

(e) In a group of 20 residents, 4 of these residents owned dogs.
5 residents were chosen from this group of residents.
Calculate the probability that exactly one resident in this group of 5 residents owned a dog.

Calculator Assumed

11. [10 marks: 2, 1, 3, 4] [TISC]

It is estimated that 35% of students at a school come from a Non-English Speaking Background.
Clearly stating the probability distribution you are using, estimate the probability that, in a class of twenty students from this school,

(a) exactly five of them are from a Non-English Speaking Background.

(b) more than ten of them are from a Non-English Speaking Background.

In five classes of 20 students each,

(c) estimate the probability that at least two of these classes each contain more than ten students from Non-English Speaking backgrounds.

A sample of n students from this school was chosen.
The probability that at least one of the students selected is from a Non-English Speaking Background is greater than 0.99.

(d) Find the minimum value of n. Show all working.

Calculator Assumed

12. [14 marks: 4, 5, 5]

It is known that the probability of finding a defective biro in a large consignment of biros is p .

(a) Given that $p = 0.05$, find the minimum number of biros that need to be selected so that the probability that:

(i) at least one of them is defective is greater than 90%

(ii) more than one of them is defective is greater than 90%.

(b) In a sample of 25 biros, what is the maximum value for p so that the probability that there are no more than two defective biros in the sample exceeds 0.5%? Justify your answer.

Calculator Assumed

13. [13 marks: 2, 2, 5, 4]

Police know that from long experience, on a particular stretch of road, 1 car in every 10 will exceed the speed limit. A radar trap is set on this stretch of road.

(a) Find the probability that the police will find that the first 5 cars will be within the limit and the sixth will be speeding.

(b) Find the probability that of 6 cars passing this radar trap, exactly one will be speeding.

(c) n cars passed this speed trap. The probability that more than 2 cars were speeding was more than 95%. Find the least value of n.

(d) On another stretch of road, $100p$ cars out of 100 will have speeds exceeding the speed limit. In a sample of 20 cars, find the range of values for p such that the probability of less than 1 speeding car is exceeds 1%.

Calculator Assumed

14. [12 marks: 4, 2, 2, 4]

X is a binomial variable with parameters n and p.

(a) Find n and p if the mean for X and its standard deviation are $\frac{85}{4}$ and $\frac{\sqrt{51}}{4}$ respectively.

(b) For $n = 100$, $p = 0.8$, in each instance write algebraic expressions for evaluating each of the following probabilities before evaluating them.

(i) $P(X = 80)$

(ii) $P(79 \le X \le 81)$.

(iii) $P(X \ge 79 \mid X \le 81)$

Calculator Assumed

15. [9 marks: 5, 3]

It is estimated that a out of every 100 learners are successful in obtaining their driver's licence in their first attempt. On a given day there were n learners having their first attempt in getting their driver's licence.

(a) Given that the expected number of successful learners in the sample is $\frac{112}{5}$ and its variance is $\frac{112}{25}$. Find a and n.

(b) Find the most likely number of successful learners in the sample. Justify your answer.

Calculator Assumed

16. [8 marks: 5, 3]

A box contains b blue and w white discs.

(a) n discs are randomly chosen from this box *with* replacement.
Define X: No. of white discs selected.
X is a discrete random variable. Find the probability distribution for X. Determine the expected number of white discs selected.

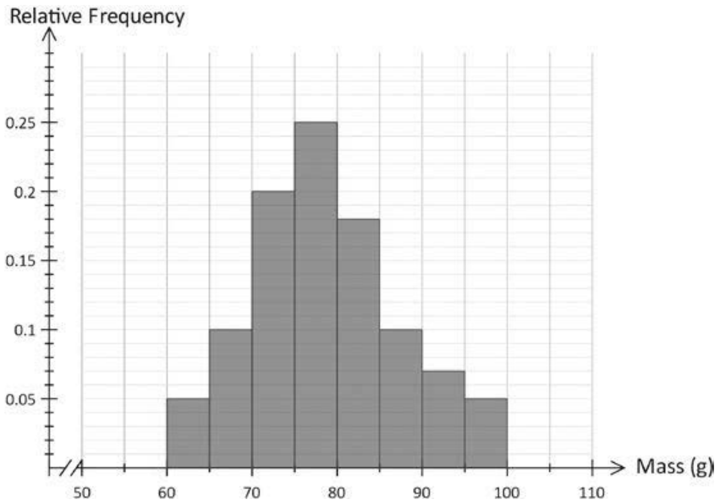
(b) n discs are randomly chosen from this box *without* replacement.
Define X: No. of white discs selected.
X is a discrete random variable. Find the probability distribution for X.

25 Continuous Random Variables

Calculator Assumed

1. [7 marks: 2, 3, 2] [TISC]

The histogram below shows the relative frequencies for the mass W of apples in a crate. Use this histogram to answer the following questions.



(a) Would it be correct to claim that approximately 75% of apples in the crate have masses between 70g and 90g? Give a reason for your answer.

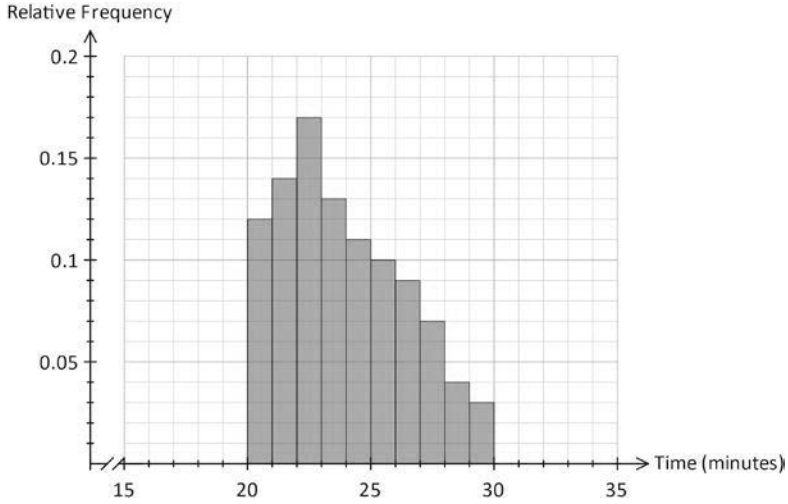
(b) Calculate the probability that a randomly selected apple from the crate has mass less than 90 g given that it has mass of at least 70 g.

(c) 85% of all apples in the crate have mass greater than k g. Calculate the value of k. Show how you obtained your answer.

Calculator Assumed

2. [9 marks: 1, 3, 3, 2]

The random variable T is defined as the time (minutes) required to complete a task. Use the relative frequency histogram for T drawn below to answer the following questions.



(a) Estimate $P(23 \leq T \leq 26)$.

(b) Estimate $P(T \geq 23 \mid T \leq 26)$

(c) Use an appropriate method to calculate the mean and standard deviation for T. Explain the method you used.

(d) Determine with reasons if the median for T is more likely to be less than or greater than the mean for T.

Calculator Free

3. [4 marks: 2, 2]

Determine with reasons if each of the following functions are probability density functions for continuous random variables:

(a) $f(x) = \begin{cases} \dfrac{2}{x} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

(b) $g(x) = \dfrac{x}{15}$ for $1 \le x \le 5$.

---

4. [6 marks]

The random variable X has probability density function $f(x) = \dfrac{x}{2}$ for $0 \le x \le 2$.

Calculate the variance σ^2 for X using the formula $\sigma^2 = \left(\int x^2 f(x)\, dx\right) - \mu^2$ where μ is the mean for X.

Calculator Free

5. [6 marks]

The random variable X has mean $\frac{2}{3}$ and probability density function $f(x) = kx$ for $0 \le x \le a$. Determine the values of k and a.

6. [4 marks] [TISC]

$F(x) = \int_0^x k(1-x^2)\,dx$ for $0 \le x \le 1$, is the cumulative distribution function for a continuous random variable. Find k.

Calculator Free

7. [9 marks: 3, 4, 2] [TISC]

The random variable X has probability density function $f(x) = \dfrac{2}{x^2}$ for $a \le x \le 2a$, where a is a constant.

(a) Calculate the value of a.

(b) The random variable Y = 2X.

(i) Determine $\overline{Y}$.

(ii) If the variance for Y is v, state in terms of v, the standard deviation for X.

Calculator Assumed

8. [6 marks: 3, 3]

A random variable X has cumulative probability distribution given by
$P(X \le x) = \dfrac{(x+5)(x-1)}{16}$ for $a \le x \le b$ where a and b are both positive real numbers.

(a) Determine with reasons the values of a and b.

(b) Determine $f(x)$, the probability density function for X.

---

9. [7 marks: 4, 3]

The random variable X has probability density function $f(x) = 4x^3$ for $0 \le x \le 1$.

(a) Determine μ and σ^2, respectively the mean and variance for X.

(b) Find m the median of X.

Calculator Assumed

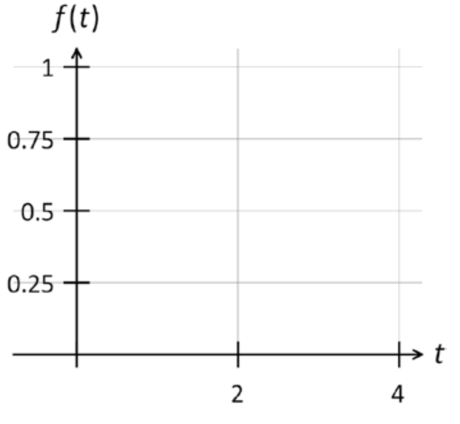
10. [10 marks: 3, 3, 2, 2]

The probability density function for a continuous random variable T is given by

$$f(t) = \begin{cases} mt & 0 \le t \le 2 \\ \frac{1}{4} & 2 < t < 4 \end{cases}$$

(a) Find the value of m .

(b) Sketch the graph of the probability density function of T.



(c) Find P(T ≤ 1).

(d) Find the median of T.

Calculator Assumed

11. [12 marks: 3, 2, 3, 4]

The probability density function of a random variable X is given by
$f(x) = x^2 + ax$ for $0 < x < 1$.

(a) Find the value of a .

(b) Find P(X > 0.5).

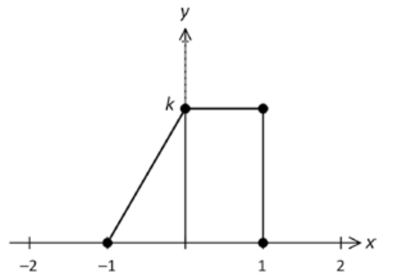
(c) Find the value of k if $P(X \le k) = 0.9$.

(d) Calculate the mean and variance for X.

Calculator Assumed

12. [9 marks: 2, 2, 3, 2] [TISC]

The graph of the probability density function of a continuous random variable X is shown in the accompanying diagram.



(a) Find the value of $\boldsymbol{k}$.

(b) Find $P(X < 0.5)$.

(c) Find $P(X > 0 | X < 0.5)$.

(d) Find $\boldsymbol{a}$ if $P(X > \boldsymbol{a}) = \frac{1}{2}$.

Calculator Assumed

13. [8 marks: 2, 3, 3] [TISC]

The probability density function of a continuous random variable X is given by $f(x) = k\left(\sqrt{4-x}\right)$ for $0 \le x \le 4$ where k is a real constant.

(a) Show that $k = \dfrac{3}{16}$.

(b) Find $P(X > 1 \mid X < 3)$.

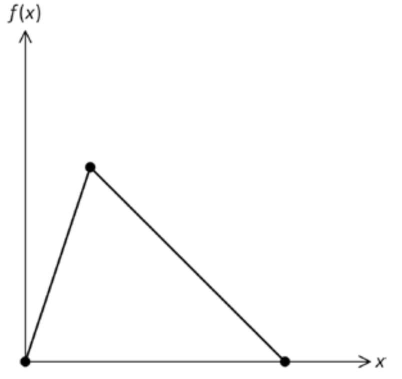
(c) Find the median for X accurate to 2 decimal places. Justify your answer.

Calculator Assumed

14. [9 marks: 2, 2, 2, 3] [TISC]

The probability density function of a continuous random variable X is given by

$$f(x)=\begin{cases}0.5x & 0\le x<1\\ \dfrac{-x}{6}+\dfrac{2}{3} & 1\le x\le k\end{cases}$$ where k is a real number. The sketch of $y=f(x)$ is given in the accompanying diagram.



(a) Show that $k = 4$.

(b) Find $P(X > 0.5)$.

(c) Find $P(X \le 2 \mid X > 0.5)$.

(d) Find m, the median value of x, such that $P(X < m) = 0.5$.

Calculator Assumed

15. [9 marks: 2, 3, 4] [TISC]

The random variable X has probability density function
$f(x) = \dfrac{9}{(x+1)^3}$ for $1 \le x \le 5$.

(a) Calculate P(X ≥ 2).

(b) Given that $P(X \ge k \mid X \ge 2) = \dfrac{3}{16}$, calculate the value of k.

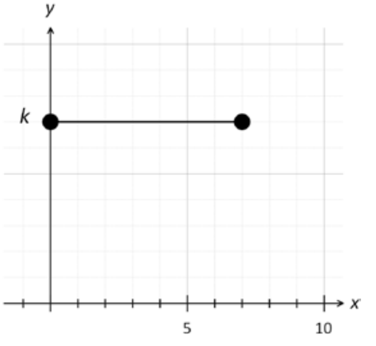
(c) Calculate the exact variance for X.

26 The Uniform Distribution

Calculator Free

1. [10 marks: 1, 2, 3, 4]

The sketch of the probability density function of a continuous random variable X is given in the accompanying diagram.



(a) Find k.

(b) State the probability density function of the variable.

(c) Find $P(X > 2 \mid X < 5)$.

(d) Find μ and σ are respectively the mean and standard deviation for X.

Calculator Free

2. [10 marks: 2, 2, 3, 3]

The random variable X is uniformly distributed in the interval $2 \le x \le k$.

(a) Find in terms of k, the probability distribution function for X.

(b) Sketch the probability distribution function for X.

(c) Find $P(X \le 10 \mid X \ge 4)$.

(d) It is known that $P(X \le 5) = 0.25$. Find the median for X. Justify your answer.

---

3. [4 marks]

The probability density function of X is given by $f(x) = 0.1$ for $a \le x \le b$.

Given that $P(X > 7 \mid X \le 8) = \dfrac{1}{3}$, find a and b.

Calculator Assumed

4 [9 marks: 2, 1, 2, 4]

The length of time Daniel is late to class may be modelled by a uniform distribution with a minimum late time of 5 minutes and a maximum late time of 25 minutes. Define T: The length of time Daniel is late to class.

(a) Write the probability density function for T.

(b) Find the probability that Daniel is exactly 15 minutes late.

(c) Find the probability the Daniel is no more than 20 minutes late given that he is at least 10 minutes late.

(d) On a school week of five days, find the probability that Daniel is late by at least 15 minutes at least once.

Calculator Assumed

5. [8 marks: 2, 2, 4]

The length of the red cycle of a set of traffic lights is 90 seconds. Assume that vehicles arrive at the traffic lights randomly and independently of each other. Define the random variable T as the waiting time at the traffic lights.

(a) Describe the probability density function of T.

(b) Find the probability that a motorist has to wait less than 30 seconds.

(c) Calculate $P(\mu - \sigma < T < \mu + \sigma)$ where μ and σ are respectively the mean and standard deviation for T.

---

6. [9 marks: 2, 2, 2, 3]

An automatic filling machine fills and packs 1 kg packs of sugar. The machine can fill any pack with any amount of between 0 and 10 grams (inclusive) of extra sugar. Define M as the extra mass of sugar (in grams) fed into each bag.

(a) Find the probability density function for M.

(b) Find $P(3 \leq M \leq 7)$.

Calculator Assumed

6. (c) Find the probability of obtaining a 1 kg pack of sugar with mass of between 1.003 and 1.007 kg.

(d) The probability of obtaining a 1 kg pack of sugar with a mass of no more than α kg is 0.75. Find α.

---

7. [6 marks: 3, 3]

The random variable X is uniformly distributed in the interval $a \le x \le b$ where a and b are real constants. The mean and variance for X are respectively 7 and $\dfrac{(b-a)^2}{12}$.

(a) Determine with reasons, the values of a and b if $P(X \le 8) = 0.75$.

(b) Determine with reasons, the values of a and b if the variance for X is 3.

Calculator Assumed

8. [9 marks: 4, 5] [TISC]

A random variable X is distributed uniformly over the interval $a \le x \le b$.
The mean and variance for X are respectively 12 and 12.

(a) Show that the probability distribution function for X can be written as

$$f(x) = \frac{1}{2b-24}; \quad 24 - b \le x \le b.$$

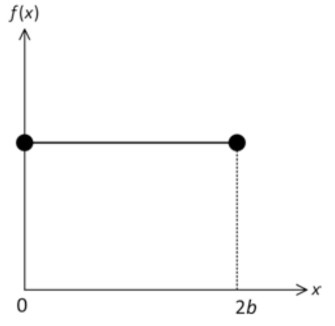
(b) Hence or otherwise, determine the probability distribution function for X.

(5 marks)

Calculator Assumed

9. [10 marks: 2, 3, 2, 3] [TISC]

A continuous random variable X is distributed uniformly over the interval $0 \le x \le 2b$. The graph of the probability density function for X is shown in the accompanying diagram.



(a) State $f(x)$, the probability distribution function for X. arks]

(b) Given that $E(X) = b$, write an integral which when evaluated will give the variance of X. Hence, determine the variance for X.

(c) Let $b = 5$.

(i) Calculate $P(X \le 4)$.

(ii) The random variable Y is defined as $Y = \begin{cases} 1 & \text{if } X \le 4 \\ 0 & \text{if } X > 4 \end{cases}$.

Calculate the mean and variance for Y.

27 The Normal Distribution

Calculator Free

1. [3 marks: 1, 1, 1]

   X is a normal variable with mean 100 and standard deviation 20.
   Given that $P(X \geq 150) = a$, determine in terms of a:

   (a) $P(X \leq 150)$

   (b) $P(50 \leq X \leq 150)$

   (c) $P(100 \leq X \leq 150)$

---

2. [11 marks: 2, 3, 3, 3]

   Use the empirical rule described below to answer the questions that follow.

   68% of values lie within one standard deviation of the mean
   95% of values lie within two standard deviations of the mean
   99.7% of values lie within three standard deviations of the mean.

   X is a normal variable with mean 100 and standard deviation 10. Estimate:

   (a) $P(80 < X < 100)$

   (b) $P(X \geq 80)$

Calculator Free

2. (c) $P(X \geq 80 \mid X \leq 100)$

(d) $P(X \leq 120 \mid X \geq 100)$

---

3. [8 marks: 2, 3, 3]

The length of nails produced by a factory is normally distributed with mean 10 mm and standard deviation 0.5 mm. Use the empirical rule given below to answer the following questions.

68% of values lie within one standard deviation of the mean
95% of values lie within two standard deviations of the mean
99.7% of values lie within three standard deviations of the mean.

(a) Calculate the probability that a randomly chosen nail produced in this factory measures more than 9 mm long.

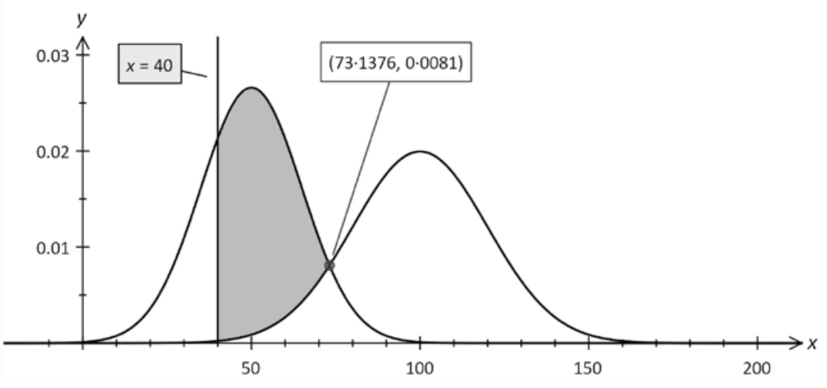
(b) Determine the probability that a nail with length no more than 10 mm has length in excess of 9 mm.

(c) Determine the 97.5^{th} percentile length.

Calculator Assumed

4. [9 marks: 2, 2, 2, 3]

The diagram below shows the graph of $y = f(x)$ and $y = g(x)$ where $f(x)$ is the probability density function of a normal random variable with mean 50 and standard deviation 15 and $g(x)$ is the probability density function of a normal random variable with mean 100 and standard deviation 20. The two graphs intersect at the point (73.1376, 0.0081). The region R is trapped between the line with equation $x = 40$ and the two curves.



(a) Determine with reasons if each of the following functions may be used as probability density functions of random variables.

(i) $y = 2f(x)$

(ii) $y = g(x) - 1$

(iii) $y = f(x - 20)$

(b) Calculate the area of region R.

Calculator Assumed

5. [7 marks: 2, 2, 3]

The speed of vehicles passing a school zone each school-day is normally distributed with a mean of 35 km/h and a standard deviation of 3 km/h.

(a) Find the probability that a vehicle passing the school zone travels with a speed in excess of the mean speed by at least 2 standard deviations.

(b) 25% of vehicles passing the school zone travel in excess of a km/h. Find a.

(c) On a certain morning, 30 vehicles were noted passing through the school zone. Find the probability that no more than 15 were travelling in excess of 35 km/h.

---

6. [5 marks]

The length of sleep that Emily gets each night is a normal variable with mean μ minutes and standard deviation σ minutes. The probability that Emily sleeps more than 8 and a-half hours is 0.01313. The probability that Emily sleeps less than 8 hours is 0.13326. Calculate the values of μ and σ.

Calculator Assumed

7. [10 marks: 1, 1, 3, 2, 3] [TISC]

The life-span of a light bulb manufactured by GloWest is normally distributed with a mean of 800 hours and a standard deviation of 120 hours.

(a) Find the probability that a randomly chosen light bulb manufactured by GloWest has a life-span :

(i) of exactly 800 hours

(ii) that exceeds 700 hours

(iii) that is less than 900 hours given that it exceeds 700 hours.

(b) Find the lifespan exceeded by 95% of all globes manufactured by GloWest.

(c) Jan needs to calculate the probability that in the batch of 500 light bulbs from GloWest, there are at least 400 light bulbs with life-spans that exceed 700 hours. State what probability distribution(s) Jan should use and the values of the corresponding parameters. Calculate this probability.

Calculator Assumed

8. [9 marks: 1, 2, 3, 3] [TISC]

Mr Green owns an environmentally friendly car called the eco-car. The fully charged battery of an eco-car, allows the driver to travel a certain distance, D km, before the battery needs recharging. This distance D is a normal variable with mean 120 km and standard deviation 12 km.

(a) Find the probability that the car will travel exactly 100 km before the battery needs recharging.

Mr Green needs to drive from the town where he lives to visit his mother 150 km away. He starts off with a fully charged battery.

(b) What is the probability that Mr Green will be able to get to his mother without having to recharge the car battery along the way. Show clearly the probability you calculated.

(c) Given that Mr Green was not able to get to his mother without having to recharge the car battery along the way, what is the probably that he got to within 10 km of his mother. Show clearly the probabilities you calculated.

(d) Mr Green attached a solar-powered booster to his car battery so that the mean for D is now μ with the standard deviation remaining at 12 km. Find μ given that the probability that he will be able to reach his mother on a fully charged battery without recharging along the way is 10%. Show clearly the different stages of your calculations.

Calculator Assumed

9. [10 marks: 1, 3, 3, 3] [TISC]

The annual rainfall (in mm) over a dam catchment area may be considered a normal variable with mean 900 mm and standard deviation 30 mm.

(a) Find the probability of the catchment area receiving an annual rainfall of less than 850 mm.

(b) Find the probability of the catchment area receiving an annual rainfall of no more than 850 mm given that it received no more than 900 mm.

Water Restrictions are imposed in this area in the year following two consecutive years where the annual rainfall is less than 850 mm. Assume that the amount of annual rainfall is independent from year to year.
In 2007 there were no Water Restrictions imposed in this area.

(c) Use the given information to find the probability of Water Restrictions being imposed in 2010. Justify your answer.

(d) Use the given information to find the probability of Water Restrictions being imposed in 2010 or 2011. Justify your answer

Calculator Assumed

10. [12 marks: 2, 3, 3, 4]

A poultry farm supplies processed chickens to a Fast Food Store. The mass of chickens supplied is normally distributed with mean 1.8 kg and standard deviation 100 g.

(a) Find the probability that a randomly chosen chicken has mass more than 2.0 kg.

(b) Chickens that have mass less than 1.5 kg are rejected. In a delivery of 1 000 chickens, how many will be rejected? Justify your answer.

(c) A sample of fifty chickens were selected. Determine the probablity that no more than two of these chickens have mass more than 2.0 kg.

(d) To improve the consistency of the mass of the chickens supplied, the poultry farm wishes to reduce the probability of a chicken with mass more than 2.0 kg to 0.005. Keeping the mean mass unchanged, what should the new standard deviation for the mass be? Give your answer to the nearest g.

Calculator Assumed

11. [13 marks: 2, 2, 4, 5]

Janine and John run a fish farm. The mass (when mature) of a species of fish bred in the farm is normally distributed with a mean of 1.8 kg and a standard deviation of 100 g. In a certain season, 10 000 fish were harvested.

(a) Find the 95th percentile mass of this species of fish. Show clearly how you obtained your answer.

(b) Estimate the number of fish with mass within 100g of the mean weight.

When the fish reaches maturity, the harvested fish are sold at the following prices.

Mass, m kg	Price/kg
$m \geq 2.0$	\$30
$1.8 \leq m < 2.0$	\$40
$1.6 \leq m < 1.8$	\$25
$m < 1.6$	\$10

(c) Estimate the revenue received from the sales of fish with mass in excess of 1.9 kg.

(d) Estimate the total revenue from this season's harvest.

Calculator Assumed

12. [11 marks: 1, 2, 6, 2] [TISC]

(a) The life span of a native toad (Species A) is normally distributed with mean 58 months and standard deviation 3 months.

(i) Find the probability of a toad having a life span less than 56 months.

(ii) A sample of 50 toads was selected. Find the probability no more than 10 of these toads have life spans less than 56 months.

(b) The life span of a related species of toad (Species B) may be modelled by a normal distribution with mean μ months and standard deviation σ months. The 75th percentile life span is 63 months, while the 20th percentile lifespan is 56 months. Find to one decimal place, μ and σ.

(c) A toad is captured and is found to have a life span less than 56 months. Determine with reasons, whether this toad is more likely to be of Species A or B.

Calculator Assumed

13. [9 marks: 2, 1, 2, 4] [TISC]

The 8 am bus arrives each week day at bus stop C anytime between 7.58 am and 8.08 am. If the bus arrives at C before 8 am, it cannot leave until 8 am.
The bus is late if it arrives after 8.00 am.

(a) State the probability density function for L.

(b) Find the probability that the bus arrives early at bus stop C.

(c) Find the probability that the bus is no more than 5 minutes late given that it is late.

Debbie lives near bus stop C. The time she takes to walk to the bus stop C is a normal variable with mean 150 seconds and standard deviation 15 seconds. She leaves her home each day at 7.57 am.

(d) Find the probability that Debbie misses the bus given that the bus is early. Show clearly how you arrived at your answer.

Calculator Assumed

14. [11 marks: 2, 2, 3, 4] [TISC]

Let the variable R denote the Body Mass Index for residents in a certain country. R is assumed to be normally distributed with mean 22 and standard deviation 3. The table below classifies the residents into categories of "Underweight", "Healthy", "Overweight" or "Obese" based on their Body Mass Index.

BMI	Classification
$R < 18.5$	Underweight
$18.5 \le R < 25$	Healthy weight range
$25 \le R < 30$	Overweight
$R \ge 30$	Obese

(a) A resident is randomly chosen from this country.
Calculate the probability that this resident is overweight.

(b) Calculate the probability that a randomly chosen resident is either overweight or obese.

(c) A resident is randomly chosen from this country. Given that this resident is not underweight, what is the probability that this resident is overweight?

(d) 70% of residents have a BMI greater than k.
For $Z \sim N(0, 1)$, $P(Z > -0.5244) \approx 0.7$.
Show how this result may be used to calculate the value of k.

Calculator Assumed

15. [8 marks: 2, 2, 4] [TISC]

The mass of mangoes sold at a supermarket is normally distributed with a mean of 300 g and a standard deviation of 8 g.

(a) Find the probability that a randomly chosen mango has a mass that is less than 310 g.

(b) Calculate the probability that a randomly chosen mango with a mass less than 310 g has a mass of at least 290 g.

(c) Due to customer complaints, the supermarket now sells mangoes with a mass that is normally distributed with a mean of 300 g and a standard deviation of σ g. 98% of mangoes sold must now have mass that differ from the mean mass by no more than 4 g. Calculate the value of σ, showing the use of the *standard normal distribution*.

Calculator Assumed

16. [12 marks: 2, 3, 7]

In region R, the mean height of adult women is 173 cm with standard deviation 12 cm.

(a) Determine the proportion of adult females that are able to walk through doorways of heights 200 cm without having to bend their bodies.

(b) What height should doorways be if 99.5% of adult females should be able to walk through the doorways without having to bend their bodies?

(c) In region R, the mean height of adult men is μ cm with standard deviation σ cm. Only 95.7% adult men in region R can walk through doorways of heights 210 cm without bending their bodies. Of these, 12.1% would not be able to walk through doorways of heights 200 cm without bending their bodies. Calculate the values of μ and σ.

Calculator Assumed

17. [8 marks: 1, 2, 5]

The driver reaction time, T, to a hazard on the road is normally distributed with mean 2 seconds and standard deviation 0.4 seconds.

(a) Calculate the probability that a driver:

(i) chosen at random will have a reaction time of greater than 2.5 seconds

(ii) chosen from those with reaction times of at least 1.5 seconds has a reaction greater than 2.5 seconds.

(b) The probability that a driver randomly chosen from those with reaction times of less than *k* seconds has reaction time of at least 1.5 seconds is 0.8341. Calculate the value of *k*.

28 Sampling & Sample Proportion

Calculator Free

1. [6 marks: 1, 2, 1, 2]

A company employs 4 000 permanent and 1 000 casual staff, spread over several work sites. A stratified sample of 100 employees is to be formed.

(a) How many of each category of staff (permanent of casual) should be in the sample?

(b) Describe clearly how the permanent staff in this sample may be selected.

(c) Due to an economic downturn which affected the company's earnings, the company's owners proposed that all employees take a pay reduction of 20%. One of the company's managers, Adam was given the task of determining the level of support among the employees for this proposal.

(i) Adam visited one of their worksites and in a meeting with the employees he requested that those who opposed the proposal to stand up. Discuss the type of bias present in this type of sampling.

(ii) Discuss with reasons if the Adam could obtain a more accurate result by repeating the above procedure on all the company's worksites.

Calculator Free

2. [4 marks: 2, 2]

A co-educational college has years ten, eleven and twelve students in the ratio 5 : 4 : 3. The views of students in a stratified random sample of 60 students regarding a possible change to the college starting time is to be considered.

(a) Describe how you would create such a sample.

(b) Suggest two other factors that should be considered in the making of such a random sample.

---

3. [4 marks: 2, 1, 1] [TISC]

A coffee manufacturer receives complaints that the 1 kg packets of coffee it sells are underweight. Each working day (8 hours), the manufacturer produces 5 000 packets. The packets are shipped in cartons of 50 packets each.

(a) A manager takes one carton of 50 packets produced one morning; weighs each packet and finds that none of the packets are underweight. The manager then concludes that the complaints are untrue. Give two reasons why the manager could be wrong.

(b) A second manager selects 50 cartons from those produced that day and picks and weighs one packet from each of the 50 cartons. The manager finds that all these packets had coffee of the correct weight and concludes that the complaints are untrue.

(i) In what way is this method of forming a sample better than the method used in part (a)?

(ii) Give one reason why the second manager could be wrong.

Calculator Free

4. [5 marks: 2, 3] [TISC]

Each train in a city rail service has six carriages. On average, each day the rail service runs 10 000 train trips. Passengers travelling on a train must have a valid train ticket.

(a) A ticket inspector selects a train at random. He then randomly selects one of the train's six carriages and checks all the passengers in that carriage for valid tickets. Of the twenty passengers in the chosen carriage, five did not have valid train tickets. Explain why it is *not* appropriate to conclude that 25% of all train passengers travel without valid train tickets.

(b) Another ticket inspector selected a train at random and checked five randomly selected passengers from each of the six carriages. Of the thirty passengers checked, six did not have valid tickets. Determine with reasons, if it would be appropriate to conclude that 20% of all train passengers travel without valid tickets.

---

5. [4 marks: 2, 2] [TISC]

(a) To determine the most popular sports activity in Perth, James randomly interviews 50 persons at a basketball game and 50 persons at a netball game. Give two reasons why James' sample may be biased.

(b) A study is being designed to determine the level of support for banning students bringing mobile phones to schools. A systematic random sample of 500 persons is to be formed. The sample is to consist of students and teachers. Discuss if this is adequate to form a fair and unbiased systematic random sample.

Calculator Free

6. [3 marks: 1, 2]

A researcher mails out survey forms asking the question "Should the Australian Federal Parliamentary term be extended from three years to four years?" The question was to be answered with a YES or a NO.

- One survey form was delivered to each of 10 000 letter boxes in four randomly chosen suburbs in Australia. Completed survey forms are to be returned using the postal reply paid envelopes provided.
- 2 450 correctly completed forms were returned to the researcher.
- 1 981 of the survey forms returned were answered with a YES.

(a) Determine an estimate for the proportion of respondents that answered YES.

(b) Discuss two possible sources of bias inherent in this survey

---

7. [7 marks: 4, 3]

It is known that 10% of adult residents in a state are fluent in at least two languages. 200 samples each with 64 adult residents were randomly chosen and the proportions of those fluent in at least two languages calculated.

(a) Describe the <u>*sampling distribution*</u> of the sample proportions of 64 adults fluent in at least two languages, stating its mean and standard deviation.

(b) Describe the <u>*frequency distribution*</u> of the 200 sample proportions of adults fluent in at least two languages, stating its mean and standard deviation.

Calculator Free

8. [6 marks: 2, 2, 2] [TISC]

The table below shows three relative frequency histograms.
Match with reasons, each of the relative frequency histogram to one of the distributions listed below.

A. Distribution of observations from a uniform distribution.

B. Distribution of sample proportions $\hat{p}$.

C. Distribution of $\dfrac{\hat{p}-p}{\sqrt{\dfrac{p(1-p)}{n}}}$ where p is the population proportion and $\hat{p}$ represents sample proportions.

<table>
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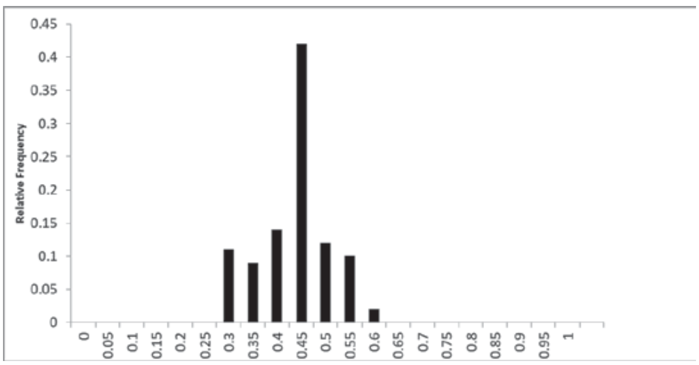
<tr><td>
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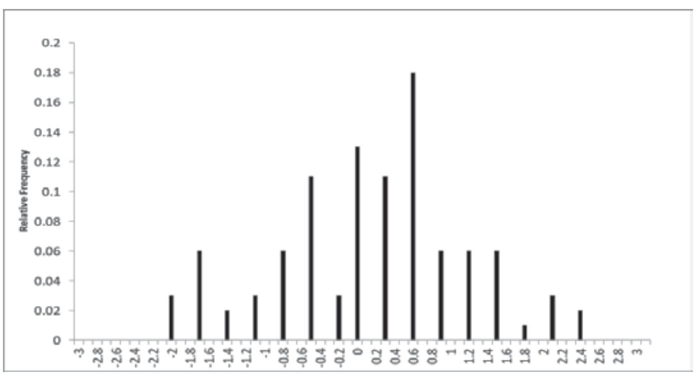
<tr><td>
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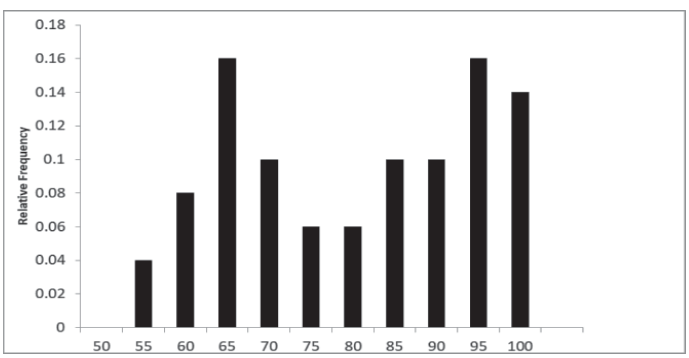
<tr><td>
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Calculator Free

9. [8 marks: 1, 4, 3]

The mass of sugar dispensed by an automatic sugar dispenser is uniformly distributed over the interval 1.5 g to 2.5 g.

(a) Calculate the probability that in any use of the dispenser, the mass of sugar dispensed exceeds 2.4g.

The dispenser was used 36 times and the proportion of times the mass of sugar dispensed exceed 2.4 g recorded. This was repeated 100 times so that a collection of 100 sample proportions was obtained.

(b) Describe the *sampling distribution* of sample proportions of size 36 for the mass of sugar dispensed exceeding 2.5 g, stating its mean and standard deviation.

(c) Describe the *frequency distribution* of the 100 sample proportions of the mass of sugar dispensed exceeding 2.5 g, stating its mean and standard deviation.

Calculator Assumed

10. [9 marks: 2, 3, 1, 3]

The waiting time at a Transport Licensing Centre is uniformly distributed over the interval 5 to 25 minutes.

(a) Find the probability that the waiting time for any customer is no more than 10 minutes.

In a review conducted on the queuing system used, the waiting times of samples of 50 customers each, were recorded.

(b) Describe the sampling distribution (size 50) of the proportion of customers with waiting times of no more than 10 minutes.

(c) Find the probability that a randomly chosen sample has a sample proportion of customers with waiting times of no more than 10 minutes that exceeds 0.3.

(d) 40 samples each comprising 50 customers were chosen. Determine with reasons, the expected number of samples with sample proportions of customers with waiting times of no more than 10 minutes that exceeds 0.3.

Calculator Assumed

11. [9 marks: 2, 3, 2, 2]

The mass of sugar in a 1 kg pack is normally distributed with mean 998 g with standard deviation 1 g.

(a) Find the probability that the mass of sugar in a randomly chosen pack exceeds 1 kg. Give your answer to 3 significant figures.

Samples of size n packs, where $n > 50$, are selected and the proportion of packs with sugar mass exceeding 1 kg recorded.

(b) Describe the sampling distribution (size $n > 50$) of the proportion of packs with sugar mass exceeding 1 kg.

(c) For $n = 100$, calculate the probability that a randomly chosen sample has a sample proportion of packs with sugar mass exceeding 1kg of between 0.02 and 0.03.

(d) Determine the value of n if the standard deviation of the sampling distribution (size $n > 50$) of the proportion of packs with sugar mass exceeding 1 kg is not to exceed 0.01.

Calculator Assumed

12. [9 marks: 4, 3, 2]

An unbiased six-sided die is rolled 80 times. This is repeated 150 times to form 150 samples each consisting of 80 rolls of the die. Event S is defined as the roll of the die producing a six.

(a) Calculate the probability that a randomly chosen sample has a sample proportion of event S that exceeds 15%.

(b) Estimate with reasons, the expected number of samples with sample proportions of event S that exceeds 15%.

(c) In a separate experiment, the same die was rolled n times. Find n if the standard deviation of the sampling distribution of sample proportion of event E is not to exceed 0.04.

Calculator Assumed

13. [8 marks: 3, 5]

60% of vehicles arriving at a school entrance are classified as sport utility vehicles (SUVs).

(a) Calculate the probability that in a random sample of 50 cars arriving at the school entrance, exactly 30 are SUVs.

(b) Samples each comprising 50 vehicles arriving at the school entrance were taken and the number of samples with exactly 30 SUVs recorded. For a randomly chosen sample of 50 vehicles, estimate the probability that the sample proportion of exactly 30 SUVs does not exceed 12% given that it exceeds 11%.

---

14. [3 marks]

It is known that p % of high school students carry school bags with masses exceeding 15 kg. Samples of 100 students are chosen. The sampling distribution for the sample proportion of students with school bags exceeding 15 kg has standard deviation $\frac{\sqrt{91}}{200}$. Find p.

Calculator Assumed

15. [9 marks: 2, 4, 3]

NyRopes manufactures high tensile nylon ropes. The continuous random variable X describes the rope length (m) between two consecutive kinks in the rope. The probability density function of X is given by $f(x) = 0.01\, e^{-0.01x}$, where $x > 0$.

(a) Find the probability that a randomly chosen piece of rope has a rope length of at least 50 m between consecutive kinks.

Samples of 30 coils of nylon ropes were examined and $\hat{\pi}$ the proportion of ropes with rope length of at least 50 m between consecutive kinks recorded.

(b) Calculate the probability that a random sample of 30 coils of nylon ropes has a $\hat{\pi}$ value between 0.6 and 0.7.

(c) Determine the minimum number of coils of nylon rope per sample required so that the standard deviation for $\hat{\pi}$ is less than 0.08.

Calculator Assumed

16. [7 marks: 3, 2, 2]

The angle of scatter, S, of a particle after it collides with an uneven surface has probability density function given by $f(s) = \cos s$ for $0 \le s \le \frac{\pi}{2}$.

(a) Show that half of all particles colliding with the surface are scattered by more than 30^{o}.

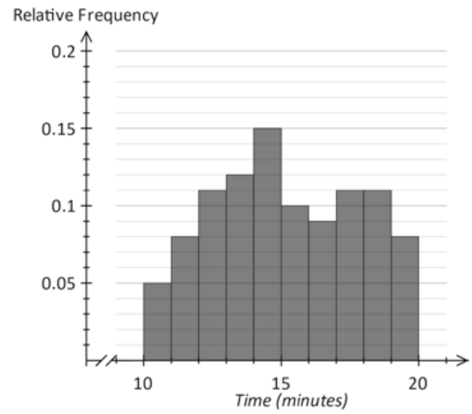
(b) Describe $\hat{p}$ the sampling distribution of sample proportions of particles scattered by more than 30^{o} in samples of 250 particles colliding with the surface.

(c) Calculate the probability that the sample proportion of collisions with scatter angles exceeding 30^{o} in samples of 250 collisions does not exceed 0.48.

Calculator Assumed

17. [7 marks: 1, 2, 1, 1, 2]

The *continuous* random variable T is defined as the time (minutes) John takes to travel to work. The relative histogram in the accompanying diagram shows the travel times that John takes for each of 100 days. Use this relative frequency histogram for T to answer the following questions.



(a) Estimate the probability that John takes no less than 12 minutes but no more than 15 minutes to get to work.

(b) On those occasions when John's travel times are least 12 minutes, calculate the probability that John's travel times do not exceed 15 minutes

(c) It is known that T is a continuous uniform variable over $10 \le t \le 20$.

(i) Calculate the true value for the proportion of travel times that are between 12 and 15 minutes inclusive.

(ii) Why is the answer in part (a) is different from the answer in part (b) (i)?

(iii) In the axes provided below, draw an approximate relative frequency histogram for John's travel times over 1 000 days.

Calculator Assumed

18. [7 marks: 2, 1, 1, 3] [TISC]

It is believed that 60% of residents in a city ride their bicycles to work.
N samples of 100 residents each were surveyed. Let $\hat{p}$ be the proportion of those in a sample of 100 residents that ride their bicycles to work.

(a) State the name of the probability distribution of $\hat{p}$ and the associated parameters.

(b) Calculate the probability that a randomly chosen sample of 100 residents will have a sample proportion of at least 65%.

(c) Let $N = 200$.

(i) How many of these samples are expected to have $\hat{p}$ values of at least 65%?

(ii) Calculate the probability that no more than forty of these samples with have $\hat{p}$ values of at least 65%. [3 marks]

Calculator Assumed

19. [11 marks: 3, 1, 2, 2, 1, 2]

In a certain country 0.9% of all vehicles are electric. Samples of 500 vehicles were selected. Let the random variable X: Number of vehicles in a sample that are electric. Let $\hat{p}$ be the proportion of vehicles in a sample that are electric.

(a) State the name and the mean and standard deviation of the probability distribution for X.

(b) Calculate the probability that in a randomly chosen sample of 500 vehicles at least 1.0 % of the vehicles are electric.

Let the random variable Y = 0.002X.

(c) State the mean and standard deviation of Y.

(d) Explain why Y may be approximated with a normal distribution.

(e) Use the normal distribution to calculate the probability that in a randomly chosen sample of 500 vehicles the proportion of cars that are electric is at least 1.0 %.

(f) Which of the answers; (b) or (e) is more accurate? Why?

29 Point & Interval Estimates for *p*

Calculator Free

1. [7 marks: 1, 3, 3]

In a sample of 400 students from Perth, 200 indicated that they had never visited Albany, a town 400 km south of Perth. Let p be the proportion of students from Perth that have never visited Albany.

(a) Determine a point estimate for p.

(b) Given that $P(-2 \leq Z \leq 2) = 0.954$ where $Z \sim N(0, 1)$, calculate a 95.4% confidence interval for p.

(c) In a second sample of 100 students from Perth, 54 students indicated that they had never visited Albany. Use your answer in (b) to determine if the second sample was statistically different from the first sample.

Calculator Assumed

2. [9 marks: 1, 3, 3, 2]

The mass (nearest g) of 30 eggs from an egg farm is listed below.

65	66	64	68	65	67	66	64	69	65
66	68	65	67	66	64	70	68	65	67
66	67	63	65	67	68	64	68	67	66

(a) Use this sample to estimate p, the proportion of eggs with mass above 66 g.

(b) Use this sample to provide a 95% confidence interval for p, the proportion of eggs with mass above 66 kg.

(c) In a second sample, 27 eggs out of 60 had mass above 66 kg.
Use the confidence interval in (b) to determine if eggs in the second sample have mass that are statistically different from those of the first sample.

(d) A third sample of eggs has a 95% confidence interval for p as $0.58 \le p \le 0.89$.
Determine with reasons if the mass of the third sample of eggs are statistically different from those of the first sample.

Calculator Assumed

3. [12 marks: 1, 3, 3, 5]

The accompanying table shows the number of students achieving x correct responses in a standardised test consisting of 60 questions. Let p be the true proportion of students achieving no more than 20 correct responses.

Number of correct responses, n	No. of Students
$1 \le t \le 10$	7
$11 \le t \le 20$	12
$21 \le t \le 30$	18
$31 \le t \le 40$	20
$41 \le t \le 50$	10
$51 \le t \le 60$	3

(a) Use this data to find a point estimate for p.

(b) Use this sample to provide a 90% confidence interval for p.

(c) Use your answer in (a) to find the size of the next sample if the error margin for a 90% confidence interval for p, is no more than 0.05.

(d) A third sample consisting of 100 students provided a confidence interval of $0.17 \le p \le 0.33$. Find the point estimate for p in this sample and the level of confidence for this interval.

Calculator Assumed

4. [8 marks: 2, 5, 1]

To estimate the true proportion π of the residents of a certain city that agree that the international airport serving the city should be relocated, samples were taken and the proportion of those in agreement calculated.

(a) In one sample of 200 residents, the 99% confidence for π was $0.78 \le \pi \le 0.92$. How many in this sample were in agreement with the proposal?

(b) A second sample of 500 residents, the 90% confidence interval for π was $0.77 \le \pi \le 0.83$. Determine if the second sample is statistically different from the first sample.

(c) If 100 samples of 50 residents each were selected, and the associated 99% confidence intervals for π calculated in the same manner. How many of these confidence intervals would be expected to contain π?

Calculator Assumed

5. [8 marks: 3, 1, 1, 3]

To estimate the true proportion p of the adults in a certain state that suffer from hay fever (allergic rhinitis), samples were taken and the proportion of those suffering from hay fever calculated.

A sample of 500 adults was taken and 105 were found to suffer from hay fever.

(a) Calculate a 95% confidence interval for p.

(b) Calculate a 99% confidence interval for p.

(c) Comment on width of the confidence intervals in (a) and (b).

In a second sample of 500 adults, 126 were found to suffer from hay fever.

(d) Using your previously calculated confidence intervals, determine with reasons if the results of the second sample is statistically different from that of the first.

Calculator Assumed

6. [8 marks: 3, 2, 3]

To estimate the true proportion π of the 1 kg packets of sugar that are under the advertised weight, samples of 1 kg packets of sugar were examined.

In a sample of 50 packets of sugar 3 were found to be underweight.

(a) Use this sample to calculate a 90% confidence interval for π.

An additional 450 packets were added to the 50 packets to form a larger sample of 500 packets and 28 were found to be underweight.

(b) Use the larger sample to calculate a 90% confidence interval for π.

(c) Determine with reasons which of the two confidence intervals would provide a statistically more reliable interval estimate for π.

Calculator Assumed

7. [8 marks: 2, 3, 1, 2]

In a certain country the proportion of residents with type A blood is $p = 0.38$
Samples of 1000 residents are selected and the sample proportion $\hat{p}$ calculated.

(a) State the sampling distribution for $\hat{p}$.

(b) Determine the interval $0.38 - k \le \hat{p} \le 0.38 + k$ such that $P(-k \le \hat{p} \le k) = 0.95$.

In a sample of 1000 residents taken only from residents of ethnic group G, 312 were found with type A blood.

(c) Calculate a point estimate for the proportion of residents from G with type A blood.

(d) Determine with reasons if the proportion of residents from G with type A blood is significantly different from the overall population.

Calculator Assumed

8. [11 marks: 3, 1, 4, 3]

Let the proportion of people who are ambidextrous (those who can use both left and right hands equally well) be p.

(a) In a random sample of 1000 persons, 12 were found to be ambidextrous. Use this sample to provide a 95% confidence interval for p.

(b) A 90% confidence interval for p is $0.004088 \le p \le 0.013912$. Determine with reasons if it is mathematically correct to say that there is a 90% chance that p lies within this interval.

(c) A sample of 700 students had a confidence interval of $0.002258 \le p \le 0.014885$. Determine the confidence level associated with this interval.

(d) One hundred 90% confidence intervals for p was calculated. Determine the probability that ninety of these intervals contain p.

Calculator Assumed

9. [7 marks: 2, 2, 3]

Let the proportion of international students in a sample of 1 000 students from Australian universities be p. For this sample, let e be the margin of error associated with a 90% confidence interval for π, the true proportion of international students in Australian universities.

(a) Write a mathematical expression for e in terms of p.

(b) Determine the maximum value for e (to three significant figures).

(c) For a sample of 1000, determine with reasons if $0.343 \le p \le 0.397$ could be a possible 90% confidence interval for p.

Calculator Assumed

10. [10 marks: 5, 1, 2, 2]

Let the proportion of residents in a country who suffer from a particular genetic abnormality be p.

(a) A confidence interval for p calculated from sample A consisting of 400 residents is $0.098\,36 \le p \le 0.181\,64$. Determine the confidence level of this interval.

(b) Sample B consists of N residents. The proportion of those who suffer from this abnormality in sample B is the same as the sample proportion in sample A. A 99% confidence interval is calculated from sample B.

(i) Write an expression for the margin of error in the 99% confidence interval for sample B, in terms of N.

(ii) The margin of error in the 99% confidence interval from sample B is k times the margin of error for the 99% confidence interval from sample A. Calculate N in terms of k.

(iii) Calculate the possible range of values for k, if $N \ge 100$.

Calculator Assumed

11. [12 marks: 3, 4, 3, 2] [TISC]

Let the proportion of Year Twelve students who speak at least two languages fluently be p.

(a) In a sample of 50 Year Twelve students, 12 students spoke at least two languages fluently.

(i) Use this sample to determine the margin of error for a 90% confidence interval for p.

(ii) For this sample, what would be the new confidence level if the margin of error in part (i) were to be halved?

(b) A second sample of 100 students had the same proportion of students able to speak two languages fluently as the sample in part (a). A 90% confidence interval is calculated using the second sample. Calculate the ratio of the margin of error for this 90% confidence interval to the margin of error for the 90% confidence interval in part (a).

(c) Is it true that the probability of p falling within a confidence interval is greater for a confidence interval with a higher confidence level? Explain.

Calculator Assumed

12. [10 marks 3, 2, 2, 3]

Heterochromia is a condition where the colour of a person's irises are different. Let p be the proportion of people who have heterochromia. Samples of 500 persons were taken and $\hat{p}$ the proportion of persons in each sample with heterochromia calculated. Sample A consists of 500 randomly chosen people and 8 were found to suffer from heterochromia.

(a) Estimate the probability that a randomly chosen sample of 500 adults will have a $\hat{p}$ value less than 0.01. Justify your answer.

(b) Use sample A to calculate a 95% confidence interval for p.

(c) Sample B has another 500 randomly chosen persons and had a 95% confidence interval for p as $0.016\,57 \le p \le 0.047\,43$. Determine with reasons if it is more likely that samples A and B come from the same population or it is more likely that samples A and B come from different populations.

(d) Assume that $p = 0.011$. Define the random variable $Y = \dfrac{\hat{p} - p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}$.

Calculate $P(Y \ge 0.9)$. Justify your answer

Fully Worked Solutions

01 Exponential Functions

Calculator Free

1. [3 marks: 1, 2]

Consider $y = e^{x+1}$.

(a) State the equation of the horizontal asymptote of this curve.

$y = 0$ ✓

(b) Find the point of intersection of this curve with the line $y = \frac{1}{e}$.

$e^{x+1} = e^{-1}$ ✓
$x + 1 = -1$
$x = -2$
Hence, $(-2,\ e^{-1})$. ✓

---

2. [6 marks: 1, 1, 2, 2]

Consider $y = e^{-2x} - 1$.

(a) State the equation of the horizontal asymptote of this curve.

$y = -1$ ✓

(b) Find the coordinates of the y-intercept of this curve.

$(0, 0)$ ✓

(c) Find the point of intersection of this curve with the line $y = e^4 - 1$.

$e^4 - 1 = e^{-2x} - 1$ ✓
$x = -2$
Hence, $(-2, e^4 - 1)$ ✓

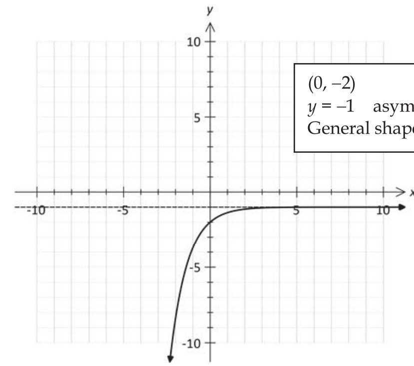
(d) Find the point of intersection of this curve with the curve $y = e^{x-1} - 1$.

$e^{x-1} - 1 = e^{-2x} - 1$
$x - 1 = -2x$
$x = \frac{1}{3}$ ✓
Hence, $(\frac{1}{3}, e^{-2/3} - 1)$ ✓

Calculator Free

3. [4 marks] [TISC]

Sketch the graph of $y = -e^{-x} - 1$.
Indicate clearly the intercepts and asymptotes where they exist.



$(0, -2)$ ✓
$y = -1$ asymptote ✓
General shape ✓✓

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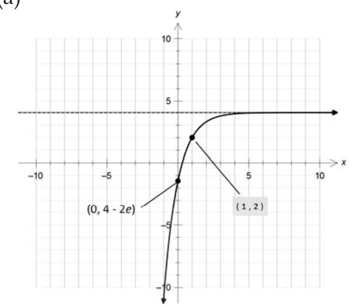
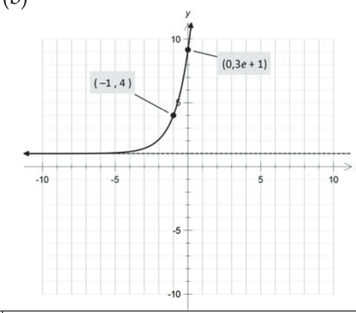
4. [10 marks: 5, 5]

The graphs of $y = Ae^{kx} + B$ are sketched below. Find the values of A, k and B.

(a)



(b)

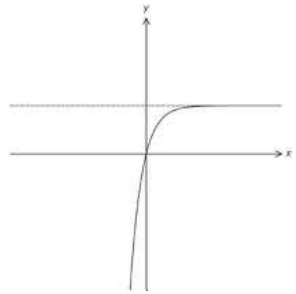


(a)	(b)
$B = 4$ ✓	$B = 1$ ✓
$(0, 4 - 2e) \Rightarrow 4 - 2e = Ae + 4$ ✓ $A = -2$ ✓	$(0, 3e + 1) \Rightarrow 3e + 1 = A + 1$ ✓ $A = 3e$ ✓
$(1, 2) \Rightarrow 2 = -2e^{-k+1} + 4$ ✓ $k = 1$ ✓	$(-1, 4) \Rightarrow 4 = 3e^{-k+1} + 1$ ✓ $k = 1$ ✓

Calculator Free

5. [6 marks: 3, 3] [TISC]

The accompanying diagram shows the sketch of $y = a + b e^{kx}$ where a, b and k are constants.



(a) Complete the table below, indicating whether the constants a, b and k have positive or negative values.

Constant	Positive or negative value
a	positive ✓
b	negative ✓
k	negative ✓

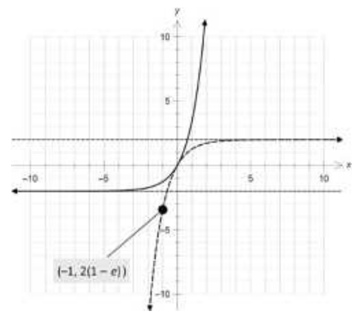
(b) Given that the curve passes through the origin and $b = k$, suggest one possible set of numerical values for a, b and k.

> $(0, 0) \Rightarrow 0 = a + b \Rightarrow b = -a$
>
> Hence, one possible set is $a = 1$, $b = -1$, $k = -1$
> $a > 0$ ✓ $b = -a$ ✓ $k = -a$ ✓

---

6. [6 marks: 3, 3]

The sketch of $y = A e^{kx} + B$ is given in the accompanying diagram. The curve passes through the point $(-1, 2(1 - e))$



(a) Find A, B and k.

> $A = -2$ ✓
> $B = 2$ ✓
> $k = -1$ ✓

(b) On the diagram given above, sketch $y = -A e^{-kx} - B$
Indicate clearly the y-intercept and asymptotes, if any.

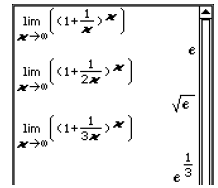
> y-intercept $(0, 0)$ ✓ Asymptote $y = -2$ ✓
> Correct shape ✓

Calculator Assumed

7. [4 marks: 3, 1]

(a) Use your CAS calculator to determine exactly $\lim_{x\to\infty}\left(1 + \frac{1}{kx}\right)^x$ for $k = 1, 2$ & 3.

> $k = 1,\ \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$ ✓
>
> $k = 2,\ \lim_{x\to\infty}\left(1 + \frac{1}{2x}\right)^x = e^{1/2}$ ✓
>
> $k = 3,\ \lim_{x\to\infty}\left(1 + \frac{1}{3x}\right)^x = e^{1/3}$ ✓



(b) Use your results in (a) to suggest the exact value of $\lim_{x\to\infty}\left(1 + \frac{1}{kx}\right)^x$ where k is a positive integer.

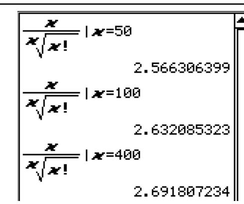
> $\lim_{x\to\infty}\left(1 + \frac{1}{kx}\right)^x = e^{1/k}$ ✓

---

8. [4 marks: 3, 1]

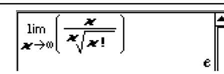
(a) Determine to 8 decimal places, the value of $\frac{x}{\sqrt[x]{x!}}$ for $x = 50$, 100 and 400.

> $x = 50$ $\frac{x}{\sqrt[x]{x!}} \approx 2.566\,306\,40$ ✓
>
> $x = 100$ $\frac{x}{\sqrt[x]{x!}} \approx 2.632\,085\,32$ ✓
>
> $x = 400$ $\frac{x}{\sqrt[x]{x!}} \approx 2.691\,807\,23$ ✓



(b) Use your results in (a) to suggest the exact value of $\lim_{x\to\infty}\left(\frac{x}{\sqrt[x]{x!}}\right)$ where $x \in \mathbb{Z}^+$.

> $\lim_{x\to\infty}\left(\frac{x}{\sqrt[x]{x!}}\right) = e$ ✓

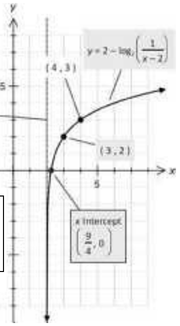


02 Logarithms

Calculator Free

1. [3 marks]

On axes provided below sketch the graph of $y = 2 - \log_2\left(\frac{1}{x+2}\right)$. Indicate clearly the asymptotes (if any) and the coordinates of at least two points on this curve.

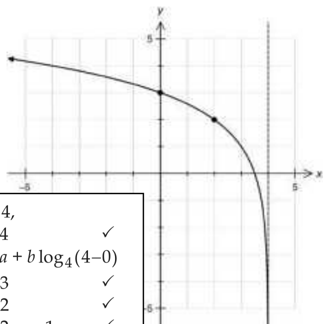


✓ Draws and labels vertical asymptote $x = 2$.
✓ Labels first point.
✓ Labels second point.

---

2. [4 marks]

The diagram below shows the graph of $y = a + b\log_c(c - x)$.
Determine the values of a, b and c.

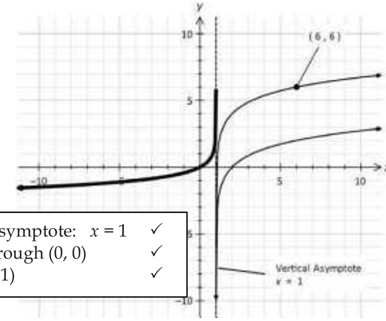


Vertical asymptote is	$x = 4$,	
hence,	$c = 4$	✓
(0, 3):	$3 = a + b\log_4(4-0)$	
	$a + b = 3$	✓
(2, 2):	$a + 0.5b = 2$	✓
	$b = 2, a = 1$	✓

Calculator Free

3. [6 marks: 3, 3]

The accompanying diagram shows the the graphs of $y = 2\log_a(x - b)$ and $y = 2\log_a(x - b) + c$, where $c > 0$.



(a) Determine the values of a, b and c.

$a = 5$ ✓
$b = 1$ ✓
$c = 4$ ✓

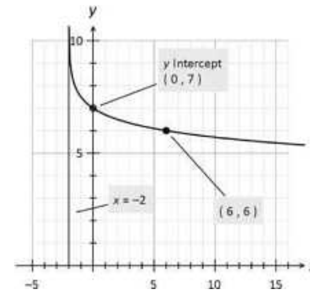
(b) Vertical asymptote: $x = 1$ ✓
Passes through (0, 0) ✓
and (–4, –1) ✓

(b) On the same set of axes, sketch the graph of $y = -\log_a(b - x)$

---

4. [6 marks: 2, 4]

The accompanying diagram shows the graph of $y = a - \log_b(cx + d)$ where a, b, c, d and e are real constants.



(a) Determine the algebraic relationship between the constants d and e.

Vertical asymptote for $y = a - \log_b(cx+d)$ is $x = \frac{-d}{c}$. ✓
But from graph, vertical asymptote is $x = -2$.
Hence, $\frac{-d}{c} = -2 \Rightarrow d = 2c$ ✓

(b) The curve passes through the points (0, 7) and (6, 6), find the value of b.

$x = 0, y = 7 \Rightarrow 7 = a - \log_b(2c)$
$x = 6, y = 6 \Rightarrow 6 = a - \log_b(8c)$ ✓
$1 = \log_b(8c) - \log_b(2c)$ ✓
$1 = \log_b\left(\frac{8c}{2c}\right)$
$1 = \log_b 4$ ✓
$b = 4$ ✓

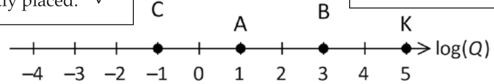
Calculator Free

5. [4 marks: 2, 1, 1] [TISC]

The diagram below shows a logarithmic scale (base 10) for the amount of units of Q contained in a chemical compound. Chemical compound A contains 10 units of Q and is plotted at the point A indicated in the diagram below since log (10) = 1. The point K represents the amount of units of Q in chemical compound K on the same logarithmic scale.

(c) C correctly placed. ✓

(b) B correctly placed. ✓



(a) How many times more/less units of Q does compound K have compared to compound A?

K has 10^4 = 10 000 times more of Q than A. ✓✓

(b) On the diagram above, plot and label the point representing compound B which has 100 times more of Q than compound A.

(c) On the diagram above, plot and label the point representing compound C which has 1 000 000 times less of Q than compound K.

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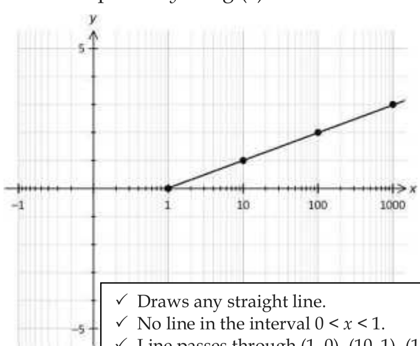
6. [3 marks]

The x-axis in the diagram below is drawn using a logarithmic scale. On the same axes, plot the curve with equation $y = \log(x)$ for $x \geq 1$.

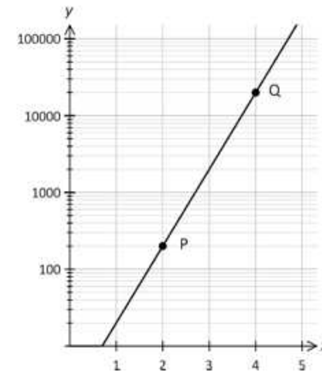


✓ Draws any straight line.
✓ No line in the interval $0 < x < 1$.
✓ Line passes through (1, 0), (10, 1), (100, 2) and (1000,3).

Calculator Free

7. [6 marks: 2, 4]

The accompanying diagram shows the graph of $y = A(10^{kx})$ where the y-axis is in the form of a logarithmic scale. The graph passes through the points P and Q.



(a) State the coordinates of the points P and Q.

P(2, 200) ✓
Q(4, 20 000) ✓

(b) Determine the values of A and k.

P(2, 200): $\Rightarrow 200 = A(10^{2k})$ ✓
Q(4, 20 000): $\Rightarrow 20\,000 = A(10^{4k})$ ✓
$A = 2, k = 1$ ✓✓

---

8. [9 marks: 2, 3, 4]

Express in its simplest form :

(a) $\dfrac{\log_4 16}{\log_4 64}$

$\dfrac{\log_4 16}{\log_4 64} \equiv \dfrac{2\log_4 4}{3\log_4 4} \equiv \dfrac{2}{3}$ ✓✓

(b) $2\log 3 + \frac{1}{2}\log 5 - \log 4^{-1}$

$2\log 3 + \frac{1}{2}\log 5 - \log 4^{-1} \equiv \log 3^2 + \log 5^{1/2} + \log 4$ ✓
$\equiv \log(9 \times 4 \times \sqrt{5})$ ✓
$\equiv \log(36\sqrt{5})$ ✓

(c) $2 + \log_5 4 - \log_5 20$

$2 + \log_5 4 - \log_5 20 \equiv \log_5 25 + \log_5 4 - \log_5 20$ ✓✓
$\equiv \log_5 \dfrac{25 \times 4}{20}$ ✓
$\equiv \log_5 5$
$\equiv 1$ ✓

Calculator Free

9. [5 marks: 2, 3]

(a) Express $-2+\log_3 a+2\log_3 b$ as a single logarithmic term.

$$-2+\log_3 a+2\log_3 b = \log_3 3^{-2}+\log_3 a+\log_3 b^2 \quad ✓$$
$$= \log_3\left(\frac{ab^2}{9}\right) \quad ✓$$

(b) Simplify $\dfrac{\log_5(1+2x+x^2)}{\log_5(1+x)}$ where $x > 0$.

$$\frac{\log_5(1+2x+x^2)}{\log_5(1+x)} = \frac{\log_5(1+x)^2}{\log_5(1+x)} \quad ✓$$
$$= \frac{2\log_5(1+x)}{\log_5(1+x)} \quad ✓$$
$$= 2 \quad ✓$$

---

10. [12 marks: 3, 3, 3, 3]

Given that $p = \log_5 2$ and $q = \log_5 6$, find in terms of p and q:

(a) $\log_5 12$

$$\log_5 12 = \log_5(2\times 6) = \log_5 2+\log_5 6 \quad ✓✓$$
$$= p+q \quad ✓$$

(b) $\log_5 3$

$$\log_5 3 = \log_5\left(\tfrac{6}{2}\right) = \log_5 6-\log_5 2 \quad ✓✓$$
$$= q-p \quad ✓$$

(c) $\log_5 24$

$$\log_5 24 = \log_5(6\times 4)$$
$$= \log_5 6+2\log_5 2 \quad ✓✓$$
$$= q+2p \quad ✓$$

(d) $\log_5 60$

$$\log_5 60 = \log_5(2\times 5\times 6)$$
$$= \log_5 2+\log_5 5+\log_5 6 \quad ✓✓$$
$$= 1+p+q \quad ✓$$

Calculator Free

11. [9 marks: 3, 3, 3]

Let $p = \log_7 5$ and $q = \log_7 2$.

(a) Find $\log_7 700$ in terms of p and/or q.

$$\log_7 700 = \log_7(7\times 2^2\times 5^2) \quad ✓$$
$$= \log_7 7 + 2\log_7 2 + 2\log_7 5 \quad ✓$$
$$= 1+2p+2q \quad ✓$$

(b) Find $\log_7 2.8$ in terms of p and/or q.

$$\log_7 2.8 = \log_7\left(\frac{7\times 2}{5}\right) \quad ✓$$
$$= \log_7 7 + \log_7 2 - \log_7 5 \quad ✓$$
$$= 1+q-p \quad ✓$$

(c) Evaluate 7^{p-2q}.

$$p-2q = \log_7 5 - 2\log_7 2 = \log_7\frac{5}{4} \quad ✓$$
$$\Rightarrow\quad 7^{p-2q} = 7^{\log_7\left(\frac{5}{4}\right)} \quad ✓$$
$$= \frac{5}{4} \quad ✓$$

---

12. [7 marks: 3, 4]

Let $p = \log_5 3$ and $q = \log_5 4$.

(a) Find $\log_5 0.15$ in terms of p and/or q.

$$\log_5 0.15 = \log_5\left(\frac{3}{4\times 5}\right) \quad ✓$$
$$= \log_5 3 - \log_5 4 - \log_5 5 \quad ✓$$
$$= p-q-1 \quad ✓$$

(b) Evaluate 25^{2p-q}.

$$25^{2p-q} = 5^{4p-2q} \quad ✓$$
$$= 5^{4\log_5 3-2\log_5 4} \quad ✓$$
$$= \frac{5^{\log_5 3^4}}{5^{\log_5 4^2}} \quad ✓$$
$$= \frac{81}{16} \quad ✓$$

Calculator Free

13. [7 marks: 3, 4]

(a) Solve for x in $\log_2 10 + \log_2 x = 2$

Rewrite equation as:	$\log_2 10x = 2$	✓
	$10x = 2^2$	✓
	$x = \frac{2}{5}$	✓

(b) Solve for x in $\log_x 4 - \log_x 3 = -2$

Rewrite equation as:	$\log_x \frac{4}{3} = -2$	✓
	$\frac{4}{3} = x^{-2}$	✓
	$x = \pm \frac{\sqrt{3}}{2}$	✓
But $x > 0$, hence, $x = \frac{\sqrt{3}}{2}$		✓

14. [6 marks: 3, 3]

Use common logarithms to solve for t where appropriate.

(a) $5^t \times 25^{t-1} = 0.04$

Rewrite equation as:	$5^t \times (5^2)^{t-1} = 5^{-2}$	✓
	$5^{t+2t-2} = 5^{-2}$	
Hence,	$3t - 2 = -2 \Rightarrow t = 0$	✓✓

(b) $\frac{2^{2t+1}}{2^{1-t}} = 5$

Rewrite equation as:	$2^{2t+1-(1-t)} = 5$	✓
	$2^{3t} = 5$	
Hence,	$3t \log 2 = \log 5$	✓
	$\Rightarrow t = \frac{\log 5}{3 \log 2}$	✓

Calculator Free

15. [5 marks]

Use common logarithms to solve for x in $2(3^{2x}) + 5(3^x) - 3 = 0$.

Rewrite equation as: $2(3^x)^2 + 5(3^x) - 3 = 0$ ✓

Let $y = 3^x$.

Hence, equation becomes:

$2(y)^2 + 5(y) - 3 = 0$ ✓

$(2y-1)(y+3) = 0$

Hence, $y = \frac{1}{2}$ or -3 ✓

Therefore: $3^x = \frac{1}{2}$ or $3^x = -3$

For $3^x = \frac{1}{2}$, $x \log 3 = \log \frac{1}{2} \Rightarrow x = -\frac{\log 2}{\log 3}$ ✓

For $3^x = -3$, there is no solution for x. ✓

Hence, solution is $x = -\frac{\log 2}{\log 3}$.

16. [6 marks: 2, 4]

(a) Solve for x in $4\log_9(x+1) - 2 = 0$.

$4\log_9(x+1) = 2$

$\log_9(x+1) = \frac{1}{2}$ ✓

$x + 1 = 9^{\frac{1}{2}}$

$\Rightarrow x = 2$ ✓

(b) Given that $p = \log_2 3$, solve for x in terms of p in the equation $4^{x+2} = 6^x$.

$(x+2)\log_2 4 = x \log_2 6$ ✓

$2(x+2) = x\log_2(3\times2)$

$2x + 4 = x\log_2 3 + x$ ✓

$x(\log_2 3 - 1) = 4$

$x = \frac{4}{\log_2 3 - 1}$ ✓

$= \frac{4}{p-1}$ ✓

Calculator Free

17. [4 marks]

The solution for x in the equation $6^{x-1} = 3^{1+x}$ can be written as $x = \log_2 A$. Determine the value of A.

$$\begin{aligned}(x-1)\log_2 6 &= (1+x)\log_2 3 \quad ✓\\ x(\log_2 6 - \log_2 3) &= \log_2 3 + \log_2 6\\ x &= \frac{\log_2 18}{\log_2 2} \quad ✓\\ &= \log_2 18 \quad ✓\\ A &= 18 \quad ✓\end{aligned}$$

---

18. [6 marks: 1, 2, 3]

Consider the equation $5^x = 1\,000$

(a) Solve for x giving your answer in the form $\log_a M$ where a is an appropriate real number and $a \neq e$ and $a \neq 10$.

$x = \log_5 1000$ ✓

(b) Use common logarithms to solve for x giving your answer in its simplest form.

$$\begin{aligned}x\log_{10} 5 &= \log_{10} 1000 \quad ✓\\ x &= \frac{3}{\log_{10} 5} \quad ✓\end{aligned}$$

(c) Hence, express $\log_5 10^n$ using common logarithms where n is a real number.

From (a): $x = \log_5 1000 = \log_5 10^3$ ✓

From (b): $x = \dfrac{3}{\log_{10} 5}$

Hence: $\log_5 10^3 = \dfrac{3}{\log_{10} 5}$ ✓

$\Rightarrow \log_5 10^n = \dfrac{n}{\log_{10} 5}$ ✓

Calculator Assumed

19. [5 marks: 1, 2, 2]

The number of decades between two frequencies f_1 and f_2 where $f_2 > f_1$ is defined by $d = \log_{10}\left(\dfrac{f_2}{f_1}\right)$. The unit of measurement for frequency is Hertz.

(a) Calculate the number of decades between the 3.2 GigaHertz and 5 GigaHertz.

$d = \log_{10}\left(\dfrac{5}{3.2}\right) = 0.1938$ ✓

(b) The frequency of signal B is 1 000 times higher than the frequency of signal A. How many decades are there between signal A and signal B.

$d = \log_{10}\left(\dfrac{1000 f_A}{f_A}\right) = 3$ ✓✓

(b) The frequency of a signal S is 2 decades below 200 MegaHertz. What is the frequency of signal S, f_S?

$\log_{10}\left(\dfrac{200}{f_S}\right) = 2$ ✓

$f_S = 2$ MegaHertz ✓

---

20. [8 marks: 2, 2, 4]

The Krumbein phi scale ϕ used to compare the size of particles is defined as: $\phi = -\log_2 D$ where D is the diameter of the particle.

(a) Calculate the ϕ number for a particle with size 250 μm. ($1\ \mu\text{m} = 1 \times 10^{-6}$ m).

$250\ \mu\text{m} = 2.5 \times 10^{-4}\ \text{m} = 0.25\ \text{mm}$ ✓

$\phi = -\log_2 0.25 = 4$ ✓

(b) The ϕ number for a bolder is less than –8. Determine the minimum diameter for a particle to be classified as a bolder.

$-\log_2 D < -8$ ✓

$D > 2^8$ mm

$D > 256$ mm ✓

Calculator Assumed

20. (c) How much larger or smaller is a particle A with a ϕ number of –3.4 compared to particle B with ϕ number of 3.4?

> $-\log_2 D_A = -3.4 \quad \Rightarrow \quad D_A = 2^{3.4}$
> $-\log_2 D_B = 3.4 \quad \Rightarrow \quad D_B = 2^{-3.4}$ ✓
> $\frac{D_A}{D_B} = \frac{2^{3.4}}{2^{-3.4}}$ ✓
> $= 2^{6.8} \approx 111$ ✓
> Hence, A is about 111 times larger than B. ✓

---

21. [8 marks: 1, 3, 4] [TISC]

Consider an activity where 1 person in N persons involved in that activity suffer a fatality (death). The safety index of that activity may be defined as $S = \log N$.

(a) Calculate the safety index for activity A if there is 1 death for every 50 000 persons involved in that activity.

> $S = \log 50\,000 = 4.69897 \approx 4.70$ ✓

(b) Activity B has a safety index of 3.85. Determine with reasons if activity B is safer than activity A.

> For activity B: $N_B = 10^{3.85} \approx 7\,079$.
> Hence, there is one death for every 7079 persons. ✓
> For activity A: $N_A = 50\,000$.
> Since $N_A >> N_B$, ✓
> activity B is *not* safer than activity A. ✓

(c) Activity C recorded 11 deaths in 24 000 persons. The probability of a participant dying in activity D is 0.0075. Use safety indices to determine how many times safer/less safe activity C is compared to activity D.

> $S_C = \log\left(\frac{24000}{11}\right) = 3.33882$ ✓
> $S_D = \log\left(\frac{1}{0.0075}\right) = 2.12494$ ✓
> $S_C - S_D = 1.2139$ ✓
> $10^{1.2139} = 16.3644 \approx 16$
> Hence, C is about 16 times less safe than D. ✓

Calculator Assumed

22. [6 marks: 2, 2, 2] [TISC]

p and q are two terms in a mathematical sequence { …. p, ……, q, …. }.

The number of terms *between* p and q is given by $n = \frac{2(\log q - \log p)}{\log 2} - 1$.

(a) The numbers 2 and 64 are part of this sequence of numbers. Calculate how many terms there are between 2 and 64.

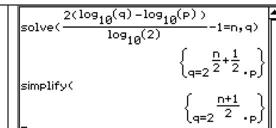
> $n = \frac{2(\log 64 - \log 2)}{\log 2} - 1$ ✓
> $= 10 - 1 = 9$ ✓

(b) $2\sqrt{2}$ is a term in this sequence.
Find the value of the term that is the 19th term after $2\sqrt{2}$.

> $\frac{2(\log q - \log 2\sqrt{2})}{\log 2} - 1 = 18$ ✓
> $q = 2048$ (or 2^{11}) ✓

(c) Determine an expression for q in terms of p and n.

> $n = \frac{2(\log q - \log p)}{\log 2} - 1$
> Solve on CAS calculator:
> $q = p \times \sqrt{2^{n+1}}$ ✓✓



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23. [4 marks]

If $\log_2(x+1) = \log_4(x+y)$, find y in terms of x. [Hint: let $\log_2(x+1) = k$]

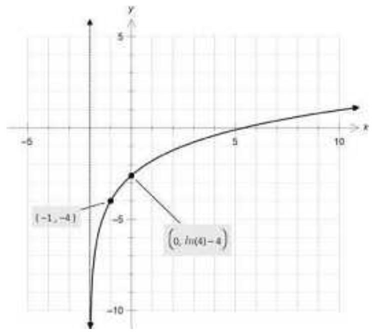
> Let $\log_2(x+1) = k$
> $x + 1 = 2^k$ ✓
> $\log_4(x+y) = k$
> $x + y = 4^k$ ✓
> $= 2^{2k}$
> $= (2^k)^2$ ✓
> $y = (x+1)^2$ ✓

03 Natural Logarithms

Calculator Free

1. [3 marks]

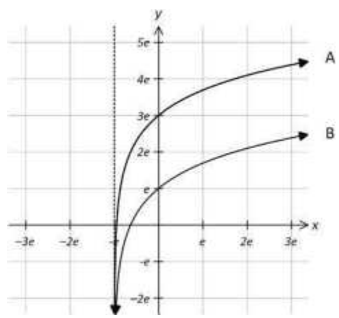
The sketch of $y = k\ ln\ (x + a) + b$ is given in the accompanying diagram. Find a, b and k.



Vertical asymptote is	$x = -2$.	
Hence:	$a = 2$	✓
(–1, –4):	$b = -4$	✓
(0, $ln(4) - 4$):	$ln(4) - 4 = k\ ln(2) - 4$	
	$k = 2$	✓

2. [5 marks: 2, 2, 1] [TISC]

The accompanying diagram shows the graphs of $y = k\ ln\ (x + p)$ and $y = q + k\ ln\ (x + p)$, where k, p and q are constants.



(a) Explain clearly why $p = e$.

Vertical asymptote for $y = q + k\ ln\ (x + p)$,
has equation $x = -p$. ✓
From graph, VA is $x = -e$. ✓
Hence, $p = e$.

(b) Use the diagram above to explain geometrically why $q = 2e$.

From diagram given, graph B is translated $2e$ units upwards to give graph A. ✓
From equations given, graph B is translated q units upwards to give graph A. ✓
Hence, $q = 2e$

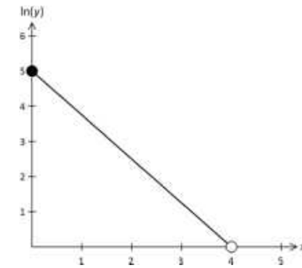
(c) Calculate the value of k.

$y = k\ ln\ (x + p)$
$x = 0, y = e \Rightarrow e = k\ ln\ e$
$k = e$ ✓

Calculator Free

3. [5 marks: 2, 3]

The graph of $ln\ y$ against x is given in the accompanying diagram.



(a) Given that $ln\ y = mx + c$, find m and c.

Gradient $m = -\frac{5}{4}$ ✓
Intercept $c = 5$ ✓

(b) Given that $e^5 \approx 150$, find y in terms of x.

$ln\ y = -1.25x + 5$
$y = e^{-1.25x+5}$ ✓
$= e^5 e^{-1.25x} \approx 150\, e^{-1.25x}$ ✓✓

4. [7 marks: 1, 2, 4]

Solve *exactly* for x:

(a) $ln\ x = 5$

$x = e^5$ ✓

(b) $ln\ (4x - 2) = -1$

$4x - 2 = e^{-1}$ ✓
$\Rightarrow\ x = \frac{1}{4}\ (2 + \frac{1}{e})$ ✓

(c) $2(ln\ x)^2 - 5\ ln\ x + 2 = 0$

$(2\ ln\ x - 1)(ln\ x - 2) = 0$ ✓
$\Rightarrow ln\ x = \frac{1}{2}$ or $ln\ x = 2$ ✓
Hence, $x = e^{\frac{1}{2}}$ or e^2 ✓✓

Calculator Free

5. [7 marks: 3, 4]

(a) Solve *exactly* for x in $100\,e^{-0.02x} = 40$

$$e^{-0.02x} = \frac{2}{5} \quad ✓$$
$$\Rightarrow -0.02x = \ln\frac{2}{5}$$
$$\text{Hence,} \quad x = -50\ln\frac{2}{5} \quad ✓✓$$

(b) Solve *exactly* for x in $400\,e^{0.03x} = 500\,e^{0.01x}$

$$\frac{e^{0.03x}}{e^{0.01x}} = \frac{500}{400} \quad ✓$$
$$e^{0.02x} = \frac{5}{4} \quad ✓$$
$$0.02x = \ln\frac{5}{4} \quad ✓$$
$$\Rightarrow x = 50\ln\frac{5}{4} \quad ✓$$

6. [9 marks: 2, 4, 3]

(a) Solve *exactly* for t in $t\,(e^t - 2) = 0$

$t = 0$ or $e^t = 2 \Rightarrow t = 0$ or $\ln 2$ ✓✓

(b) Solve *exactly* for t in $e^{2t} - 3e^t + 2 = 0$

Rewrite as $(e^t)^2 - 3e^t + 2 = 0$ ✓
Let $y = e^t$.
Hence, equation becomes: $y^2 - 3y + 2 = 0$
$y = 1$ or 2 ✓
Hence, $e^t = 1$ or $e^t = 2 \Rightarrow t = 0$ or $\ln 2$ ✓✓

(c) Solve *exactly* for t in $2e^{-t} - 2^t = 0$

$$2e^{-t} = 2^t$$
$$2 = (2e)^t \quad ✓$$
$$\ln(2) = t\ln(2e) \quad ✓$$
$$t = \frac{\ln(2)}{\ln(2e)} \quad ✓$$

Calculator Free

7. [7 marks: 4, 3]

(a) Solve for t in $e^t - 6e^{-t} + 1 = 0$

Rewrite as $(e^t) - \frac{6}{e^t} + 1 = 0$
$\times e^t$ $(e^t)^2 + e^t - 6 = 0$ ✓
Let $y = e^t$. $\Rightarrow y^2 + y - 6 = 0$
$y = 2$ or -3 ✓
Hence, $e^t = 2$ or $e^t = -3 \Rightarrow t = \ln 2$ (reject $e^t = -3$ as $e^t > 0$) ✓✓

(b) Solve for x in the equation $2e^x + e^{0.5x} = 1$.

$$2(e^{0.5x})^2 + e^{0.5x} - 1 = 0 \quad ✓$$
$$(2e^{0.5x} - 1)(e^{0.5x} + 1) = 0 \quad ✓$$
$$e^{0.5x} = 0.5 \text{ or } e^{0.5x} = -1$$
$$x = -2\ln(2) \quad ✓$$

8. [5 marks] [TISC]

Given $2\ln x + \ln y = \ln 400$ and $\ln x - \ln y = \ln\left(\frac{5}{2}\right)$,

calculate the values of x and y.

$$2\ln x + \ln y = \ln 400 \Rightarrow \ln x^2 y = \ln 400$$
$$x^2 y = 400 \quad ✓$$
$$\ln x - \ln y = \ln\left(\frac{5}{2}\right) \Rightarrow \ln\left(\frac{x}{y}\right) = \ln\left(\frac{5}{2}\right)$$
$$\frac{x}{y} = \frac{5}{2} \quad ✓$$
$$y = \frac{2x}{5}$$
$$\text{Hence:} \quad x^2 \times \frac{2x}{5} = 400 \quad ✓$$
$$x = 10,\ y = 4 \quad ✓✓$$

Calculator Assumed

9. [7 marks: 1, 4, 2]

The number of bacteria in a controlled culture is modelled by $P = \dfrac{10\ 000}{10e^{-t} + 40}$

where t is time in hours.

(a) Find the initial population size.

$$P(0) = \frac{10\ 000}{10 + 40} = 200 \quad ✓$$

(b) Use logarithms to find *exactly* when the population size will be 240.

$$\frac{10\ 000}{10e^{-t} + 40} = 240$$

$$10000 = 240\,(10e^{-t} + 40) \quad ✓$$

$$10e^{-t} + 40 = \frac{125}{3} \quad ✓$$

$$e^{-t} = \frac{1}{6} \quad \Rightarrow \quad t = \ln 6 \quad ✓✓$$

(c) Describe the population size for large values of t.

As $t \to \infty$ $\quad P \to \dfrac{10\ 000}{0 + 40} = 250$ ✓

Hence, for large values of t, the number of bacteria never exceeds 250 . ✓

Calculator Assumed

10. [7 marks: 2, 2, 3,]

A translucent plastic sheet reduces the intensity of light that passes through it. The intensity of light after passing through x identical sheets placed adjacent to each other is given by $I = I_0\, e^{kx}$.

(a) Each plastic sheet reduces the intensity of light passing through it by 60%.

(i) Find the value of k correct to four significant figures.

$0.4I_0 = I_0 e^{k}$ ✓

$k = \ln 0.4 = -0.9163$ ✓

(ii) How many sheets would be required to reduce the intensity of light passing through these sheets by at least 99%?

$0.01I_0 < I_0 e^{-0.9163x}$ ✓

$x > 5.026$

Hence, at least 6 sheets. ✓

(c) Five sheets are required to reduce the light intensity by 20%. Find the percentage reduction of light intensity by one sheet.

$0.8I_0 = I_0 e^{5k}$ ✓

$k = -0.04463$

When $x = 1$, $I = I_0\, e^{-0.04463}$ ✓

$= 0.9564I_0$

Hence, % reduction per sheet = 4.4% ✓

04 Derivatives: First Principles

Calculator Free

1. [3 marks]

Use first principles to determine the derivative of $y = 5x^2$ with respect to x.

Let $f(x) = 5x^2 = 5 \times x^2$

$$\frac{dy}{dx} = 5 \times \lim_{h\to 0}\left[\frac{(x+h)^2 - x^2}{h}\right] \quad ✓$$

$$= 5 \times \lim_{h\to 0}\left[\frac{x^2 + 2xh + h^2 - x^2}{h}\right] \quad ✓$$

$$= 5 \times 2x = 10x \quad ✓$$

---

2. [4 marks]

Use first principles to determine the derivative of $y = \frac{1}{x^2}$ with respect to x.

Let $f(x) = \frac{1}{x^2}$.

$$\frac{dy}{dx} = \lim_{h\to 0}\left[\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\left(\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right)}{h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\frac{\left(\frac{-2xh-h^2}{x^2(x+h)^2}\right)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2xh-h^2}{x^2(x+h)^2 h}\right] \quad ✓$$

$$= \lim_{h\to 0}\left[\frac{-2x-h}{x^2(x+h)^2}\right] \quad ✓$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3} \quad ✓$$

Calculator Free

3. [2 marks]

Use an appropriate derivative to evaluate $\lim_{h\to 0}\left[\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\right]$.

$$\lim_{h\to 0}\left[\frac{\sqrt{2(x+h)}-\sqrt{2x}}{h}\right] = \frac{d}{dx}\left(\sqrt{2x}\right) = \frac{1}{\sqrt{2x}} \quad ✓✓$$

---

4. [3 marks]

Evaluate $\lim_{h\to 0}\left[\frac{ln(3+h)-ln\,3}{h}\right]$ giving your answer in exact form.
Show clearly how you obtained your answer.

$$\lim_{h\to 0}\left[\frac{ln(3+h)-ln\,3}{h}\right] = \frac{d}{dx}ln\,x\Big|_{x=3} \quad ✓$$

$$= \frac{1}{x}\Big|_{x=3} \quad ✓$$

$$= \frac{1}{3} \quad ✓$$

---

5. [4 marks]

Evaluate $\lim_{h\to 0}\left[\frac{\left(1+\sqrt{5+h}\right)^2-\left(1+\sqrt{5}\right)^2}{h}\right]$ giving your answer in exact form.
Show clearly how you obtained your answer.

$$\lim_{h\to 0}\left[\frac{\left(1+\sqrt{5+h}\right)^2-\left(1+\sqrt{5}\right)^2}{h}\right] = \frac{d}{dx}\left(1+\sqrt{x}\right)^2\Big|_{x=5} \quad ✓$$

$$= 2\left(1+\sqrt{x}\right)\times\frac{1}{2\sqrt{x}}\Big|_{x=5} \quad ✓$$

$$= \left(1+\sqrt{5}\right)\times\frac{1}{\sqrt{5}} \quad ✓$$

$$= \frac{1+\sqrt{5}}{\sqrt{5}} \quad ✓$$

05 Differentiation I (Chain Rule)

Calculator Free

1. [10 marks: 1, 2, 2, 2, 3]

Differentiate with respect to x. Leave answers with positive indices.

(a) $4x^3 + 2x^2 - 3x + 5$

Derivative = $12x^2 + 4x - 3$ ✓

(b) $\dfrac{2}{3x^2} - x - 1$

Derivative = $\dfrac{-4}{3x^3} - 1$ ✓✓

(c) $\dfrac{-1}{2x^3} + 5\sqrt{x}$

Derivative = $\dfrac{3}{2x^4} + \dfrac{5}{2\sqrt{x}}$ ✓✓

(d) $\dfrac{1}{2\sqrt{x}} + \dfrac{5\sqrt{x}}{2}$

Derivative = $\dfrac{-1}{4x^{3/2}} + \dfrac{5}{4\sqrt{x}}$ ✓✓

(e) $\dfrac{\sqrt{x} - x^4}{3x}$

$\dfrac{\sqrt{x}-x^4}{3x} = \dfrac{\sqrt{x}}{3x} - \dfrac{x^4}{3x} = \dfrac{x^{-1/2}}{3} - \dfrac{x^3}{3}$ ✓✓

Derivative = $\dfrac{-1}{6\sqrt{x^3}} - x^2$

Calculator Free

2. [2 marks]

Find the gradient of the curve $y = x^2 + 2\sqrt{x} + 1$ at the point where $x = 1$.

$y' = 2x + \dfrac{1}{\sqrt{x}}$ ✓

$y'(1) = 3$ ✓

---

3. [5 marks]

Find the equation of the tangent to the curve $y = \dfrac{x^2 - x^3}{x^4}$ at the point where $x = -1$.

$y = \dfrac{x^2 - x^3}{x^4} = x^{-2} - x^{-1}$ ✓

$y' = \dfrac{-2}{x^3} + \dfrac{1}{x^2}$ ✓

$y'(-1) = 3$ ✓

When $x = -1$, $y = 2$.

Equation of tangent is $y - 2 = 3(x + 1)$

$y = 3x + 5$ ✓✓

---

4. [5 marks]

Find the coordinates of the point(s) on the curve $y = \dfrac{1}{x} + x$ with a gradient of 0.

$y' = \dfrac{-1}{x^2} + 1$ ✓

Gradient = 0 $\Rightarrow \dfrac{-1}{x^2} + 1 = 0$ ✓

$x^2 = 1$

$\Rightarrow x = -1$ or 1. ✓

Hence, (–1, –2) and (1, 2). ✓✓

Calculator Free

5. [15 marks: 2, 3, 3, 3, 4]

Differentiate with respect to x (do not simplify):

(a) $(1-2x^2)^5$

Derivative = $5 \times (1-2x^2)^4 \times (-4x)$ ✓✓

(b) $\dfrac{1}{(2x+1)^5}$

Derivative = $(-5) \times (2x+1)^{-6} \times (2)$ ✓ ✓ ✓

(c) $\dfrac{1}{(1+4x)^2} + \dfrac{2}{3(x^2-2)^4}$

Derivative = $(-2) \times (1+4x)^{-3} \times (4)$ ✓

$+ (-4) \times \frac{2}{3}(x^2-2)^{-5} \times 2x$ ✓✓

(d) $\dfrac{2}{\sqrt{1-x}} + \sqrt{(2x+1)}$

Derivative = $\frac{-1}{2} \times 2(1-x)^{-3/2} \times (-1) + \frac{1}{2} \times (2x+1)^{-1/2} \times (2)$ ✓✓ ✓

(e) $\dfrac{2}{\sqrt[3]{1-3x^2}} - \dfrac{\sqrt{1+x^4}}{3}$

Derivative = $\frac{-1}{3} \times 2(1-3x^2)^{-4/3} \times (-6x)$ ✓✓

$-\frac{1}{2} \times \frac{1}{3}(1+x^4)^{-1/2} \times (4x^3)$ ✓✓

Calculator Free

6. [3 marks]

Find the gradient of the curve $y = (2-\sqrt{x})^3$ at the point where $x = 4$.

$y' = 3 \times (2-\sqrt{x})^2 \times \left(\frac{-1}{2\sqrt{x}}\right)$ ✓✓

$y'(4) = 0$ ✓

---

7. [5 marks]

Find the equation of the tangent to the curve $y = \left(1-\frac{1}{x}\right)^3$ at the point where $x = -1$.

$y' = 3 \times \left(1-\frac{1}{x}\right)^2 \times \left(\frac{1}{x^2}\right)$ ✓✓

$y'(-1) = 12$ ✓

When $x = -1$, $y = 8$.

Equation of tangent is $y - 8 = 12(x+1)$

$y = 12x + 20$. ✓✓

---

8. [7 marks]

The gradient of the curve with equation $y = \dfrac{1}{ax^2+bx+5}$ at the point $(1, \frac{1}{3})$ is 0.

Find a and b.

Rewrite equation as $y = (ax^2+bx+5)^{-1}$.

Gradient function $y' = -(ax^2+bx+5)^{-2} \times (2ax+b)$

$= \dfrac{-(2ax+b)}{(ax^2+bx+5)^2}$ ✓

On the curve, when $x = 1$, $y = \frac{1}{3}$. $\Rightarrow \frac{1}{3} = \frac{1}{a+b+5}$ ✓

$a + b + 5 = 3 \Rightarrow a + b = -2$ (I) ✓

Gradient at $x = 1$ is 0. $\Rightarrow 0 = \dfrac{-(2a+b)}{(a+b+5)^2}$ ✓

$2a + b = 0$ (II) ✓

Solve I and II simultaneously, $a = 2$, $b = -4$. ✓✓

Calculator Assumed

9. [7 marks: 4, 3] [TISC]

Given that $y = \dfrac{4}{(1+x)^3}$ where $x = f(u)$ and $\dfrac{dx}{du} = -1$ for all values of u.

(a) Use the chain rule to determine $\dfrac{dy}{du}$ when $y = \dfrac{1}{2}$.

$\dfrac{dy}{dx} = \dfrac{-12}{(1+x)^4}$ ✓

But:

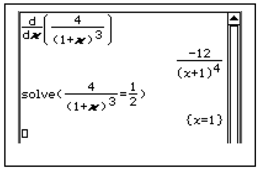
$\dfrac{dy}{du} = \dfrac{dy}{dx} \times \dfrac{dx}{du}$

$= \dfrac{-12}{(1+x)^4} \times \dfrac{dx}{du}$ ✓

When $y = \dfrac{1}{2}$, $x = 1$. ✓

Hence, $\dfrac{dy}{du} = \dfrac{-12}{(1+1)^4} \times -1$

$= \dfrac{3}{4}$ ✓



(b) If $u = g(t)$, use the chain rule to determine $\dfrac{dy}{dt}$ when $x = 0$ and $\dfrac{du}{dt} = 2$.

$\dfrac{dy}{dt} = \dfrac{dy}{du} \times \dfrac{du}{dt}$ ✓

When $x = 0$, $\dfrac{dy}{du} = 12$. ✓

Hence, $\dfrac{dy}{dt} = 12 \times 2 = 24$ ✓

OR

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{du} \times \dfrac{du}{dt}$ ✓

$= \dfrac{-12}{(1+x)^4} \times -1 \times \dfrac{du}{dt}$ ✓

When $x = 0$:

$\dfrac{dy}{dt} = \dfrac{-12}{(1+0)^4} \times -1 \times 2 = 24$ ✓

06 Differentiation II (Product & Quotient Rules)

Calculator Free

1. [12 marks: 2, 3, 3, 4]

Differentiate with respect to x (do not simplify):

(a) $(1 + x^2)(1 - 2x)$

Derivative $= 2x(1 - 2x) + (1 + x^2)(-2)$ ✓✓

(b) $2x^3(1 - x^2)^4$

Derivative $= 6x^2(1 - x^2)^4 + 2x^3 \times 4(1 - x^2)^3 \times (-2x)$ ✓ ✓ ✓

(c) $\sqrt{2x}\,(1 + x)^2$

Derivative $= \dfrac{\sqrt{2}}{2\sqrt{x}} \times (1 + x)^2 + \sqrt{2x} \times 2 \times (1 + x)$ ✓ ✓ ✓

(d) $x^3\sqrt{2 + 3x^2}$

Derivative $= 3x^2 \times \sqrt{2 + 3x^2} + x^3 \times \dfrac{1}{2} \times \dfrac{6x}{\sqrt{2 + 3x^2}}$ ✓ ✓✓ ✓

2. [12 marks: 2, 3, 3, 4]

Differentiate with respect to t (do not simplify):

(a) $\dfrac{2 - t}{(1 + 2t)}$

Derivative $= \dfrac{(-1)(1 + 2t) - (2 - t)(2)}{(1 + 2t)^2}$ ✓✓

Calculator Free

2. (b) $\dfrac{1-4t}{(1-2t)^2}$

Derivative = $\dfrac{(-4)(1-2t)^2 - (1-4t)\times 2\,(1-2t)\times(-2)}{(1-2t)^4}$ ✓ ✓ ✓

(c) $\dfrac{-t}{\sqrt{2t-3}}$

Derivative = $\dfrac{-1\times\sqrt{2t-3} - (-t)\times\frac{1}{2}(2t-3)^{-1/2}\times 2}{2t-3}$ ✓ ✓ ✓

(d) $\sqrt{\dfrac{1+2t}{1-2t}}$

$\sqrt{\dfrac{1+2t}{1-2t}} = \dfrac{(1+2t)^{1/2}}{(1-2t)^{1/2}}$ ✓

Derivative

$= \dfrac{\frac{1}{2}\times 2\times(1+2t)^{-1/2}(1-2t)^{1/2} - (1+2t)^{1/2}\times\frac{1}{2}\times(-2)\times(1-2t)^{-1/2}}{(1-2t)}$ ✓ ✓ ✓

---

3. [3 marks]

Find the gradient of the tangent to the curve $y = x^2\sqrt{1-x}$ at the point where $x = -3$.

$y' = 2x\times(1-x)^{1/2} + x^2\times\frac{1}{2}\times(1-x)^{-1/2}\times(-1)$ ✓✓

$y'(-3) = \dfrac{-57}{4}$ ✓

Calculator Free

4. [4 marks]

Find the equation of the tangent to the curve $y = 2x\,(1+\sqrt{x})^3$ at the point where $x = 1$.

$y' = 2\times(1+\sqrt{x})^3 + 2x\times 3\times(1+\sqrt{x})^2\times\dfrac{1}{2\sqrt{x}}$ ✓✓

$y'(1) = 28$ ✓

When $x = 1$, $y = 16$.

Equation of tangent is $y - 16 = 28(x-1)$

$y = 28x - 12$ ✓

---

5. [5 marks]

A curve has equation $y = (x^2-1)(1+x)^3$. Show that $y' = (5x-3)(1+x)^3$. Hence, find the x-coordinates of the point(s) where the gradient of the curve is 0.

$y' = 2x\,(1+x)^3 + (x^2-1)\times 3\,(1+x)^2$ ✓

$= 2x\,(1+x)^3 + 3(x^2-1)(1+x)^2$

$= (1+x)^2\,[2x\,(1+x) + 3\,(x^2-1)]$ ✓

$= (1+x)^2\,[5x^2+2x-3]$

$= (5x-3)(1+x)^3$.

When $y' = 0$; $(5x-3)(1+x)^3 = 0$ ✓

$x = -1, \frac{3}{5}$ ✓✓

---

6. [7 marks]

Find the coordinates of the point(s) on the curve $y = \dfrac{2x}{1-x}$ at which the tangents are parallel to the line $2y = x - 1$.

$y' = \dfrac{2(1-x) - 2x(-1)}{(1-x)^2} = \dfrac{2}{(1-x)^2}$ ✓✓

Line $2y = x - 1$ has gradient $\frac{1}{2}$. ✓

Hence, $\dfrac{2}{(1-x)^2} = \dfrac{1}{2}$ ✓

$\Rightarrow (1-x)^2 = 4 \Rightarrow x = -1$ or 3 ✓

Therefore, $(-1, -1)$ and $(3, -3)$. ✓✓

Calculator Free

7. [6 marks: 3, 3]

Consider the functions $f(x)$ and $g(x)$ where $f(3) = 1$, $f'(3) = -6$, $g(3) = 2$ and $g'(3) = \frac{1}{4}$.

(a) Given that $h(x) = [f(x)]^3$, determine the value of $h'(x)$ when $x = 3$.

Using the chain rule: $h'(x) = 3[f(x)]^2 \times f'(x)$ ✓
$h'(3) = 3[f(3)]^2 \times f'(3)$ ✓
$= 3 \times 1^2 \times (-6) = -18$ ✓

(b) Given that $h(x) = f(x) \times g(x)$, determine the value of $h'(x)$ when $x = 3$.

$h'(x) = f(x) \times g'(x) + f'(x) \times g(x)$ ✓
$h'(3) = f(3) \times g'(3) + f'(3) \times g(3)$ ✓
$= 1 \times \frac{1}{4} + (-6) \times 2 = -\frac{47}{4}$ ✓

8. [6 marks: 3, 3]

Consider the functions $f(x)$ and $g(x)$ where $f(1) = g(1) = 1$, $f'(1) = 2$ and $g'(1) = -1$.

(a) Given that $h(x) = [f(x) + g(x)]^4$, determine the value of $h'(1)$.

$h'(x) = 4[f(x)+g(x)]^3 \times (f'(x) + g'(x))$ ✓
$h'(1) = 4[f(1)+g(1)]^3 \times (f'(1) + g'(1))$ ✓
$= 4[1 + 1]^3 \times (2 - 1) = 32$ ✓

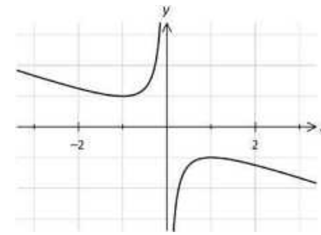
(b) Given that $y = \dfrac{2f(x)}{1 + g(x)}$, determine the value of $h'(1)$.

$h'(x) = \dfrac{(1+g(x)) \times 2f'(x) - 2f(x) \times g'(x)}{(1+g(x))^2}$ ✓
$h'(1) = \dfrac{(1+g(1)) \times 2f'(1) - 2f(1) \times g'(1)}{(1+g(1))^2}$ ✓
$= \dfrac{(1+1) \times 2(2) - 2(1) \times (-1)}{(1+1)^2} = \dfrac{5}{2}$ ✓

07 Differentiation III (Graphs)

Calculator Free

1. [4 marks: 2, 2]



The graph of $y = f(x)$ is given in the accompanying diagram.

(a) Find the x-coordinate of the point(s) where the gradient of the curve is 0.

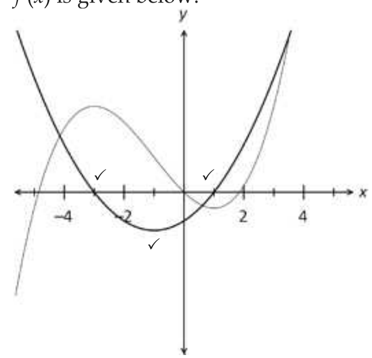
$x = -1, 1$ ✓✓

(b) For what values of x is the gradient of the curve negative?

$x < -1, x > 1$ ✓✓

2. [5 marks: 2, 3]

The graph of $y = f(x)$ is given below.



(a) For what values of x is the gradient positive?

$x < -3, x > 1$ ✓✓

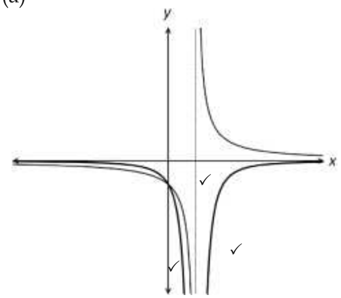
(b) Sketch on the same axes, a possible graph of $y = f'(x)$.

Calculator Free

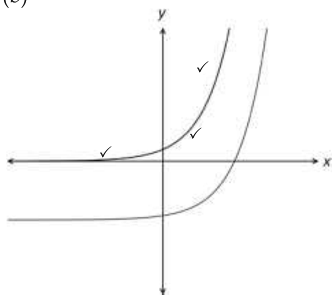
3. [6 marks: 3, 3]

The graph of $y = f(x)$ is given below. Sketch on the same axes, a possible graph of $y = f'(x)$.

(a)

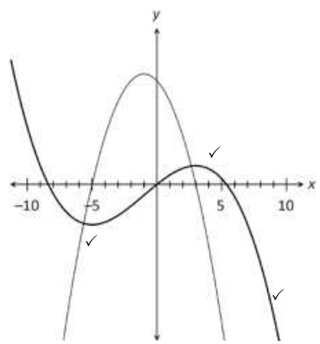


(b)



4. [6 marks: 2, 1, 3]

The graph of $y = f'(x)$ is given in the accompanying diagram.



(a) State the x-coordinate of the point(s) where the gradient of $y = f(x)$ is zero.

$x = -5, 3$ ✓✓

(b) State the x-coordinate of the minimum turning point on $y = f(x)$.

$x = -5$ ✓

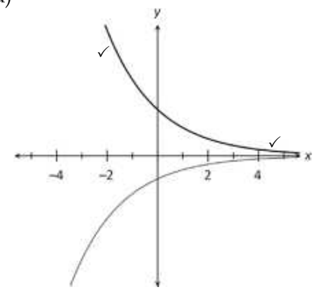
(c) Sketch on the same axes a possible graph of $y = f(x)$.

Calculator Free

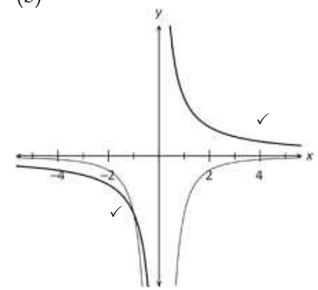
5. [4 marks: 2, 2]

The graph of $y = f'(x)$ is given below.
Sketch on the same axes, a possible graph of $y = f(x)$.

(a)

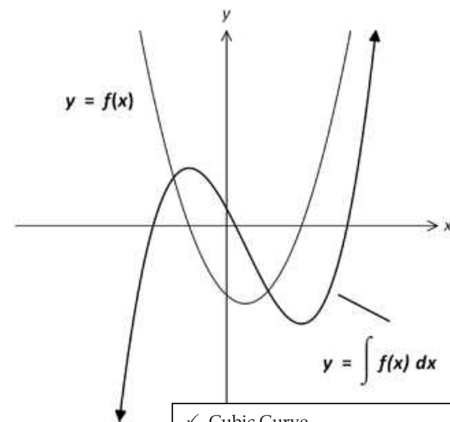


(b)



6. [3 marks]

The sketch of $y = f(x)$ is given below. On the same set of axes, sketch a possible graph of $y = \int f(x)\, dx$.

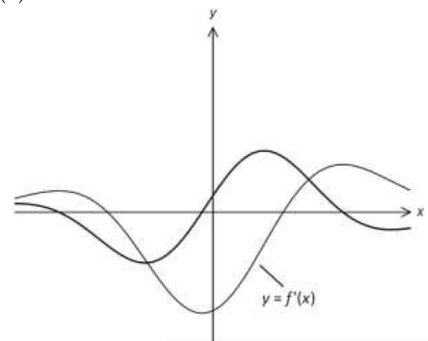


✓ Cubic Curve
✓ Max coincides with negative root of $y = f(x)$.
✓ Min coincides with positive root of $y = f(x)$.

Calculator Free

7. [3 marks]

The sketch of $y = f'(x)$ is given below. On the same set of axes, sketch a possible graph of $y = f''(x)$.

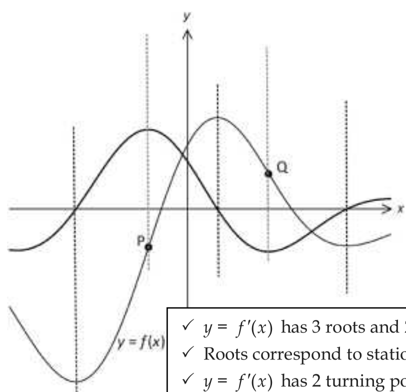


- ✓ $y = f''(x)$ has 2 negative and 1 positive roots.
- ✓ Positive root directly underneath max point in Q1.
- ✓ Other roots coincide with turning points.

---

8. [4 marks]

The sketch of $y = f(x)$ is given below. The points P and Q are inflections points on the graph of $y = f(x)$. On the same set of axes, sketch a possible graph of $y = f'(x)$.



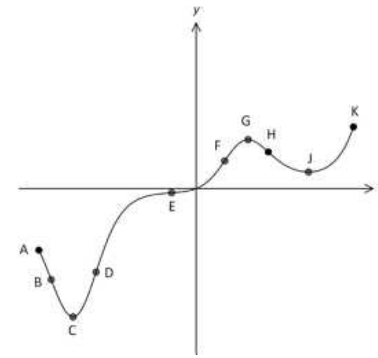
- ✓ $y = f'(x)$ has 3 roots and 2 turning points.
- ✓ Roots correspond to stationary points on $y = f(x)$.
- ✓ $y = f'(x)$ has 2 turning points.
- ✓ Turning points correspond to points P and Q.

Mathematics Methods Units 3 & 4 Revision Series *07 Differentiation III*

Calculator Free

9. [5 marks: 2, 3]

The sketch of $y = f(x)$ is given in the accompanying diagram.



(a) Between which points on the curve is $f'(x) \geq 0$?

Between C and G inclusive. ✓
Between J and K inclusive. ✓

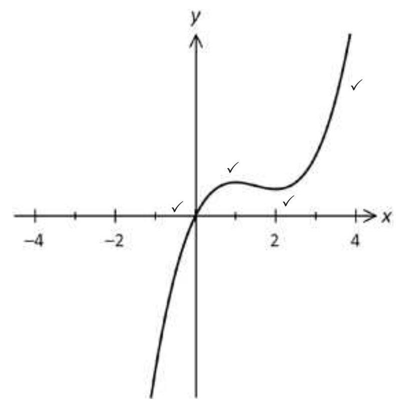
(b) Between which points on the curve is $f''(x) \leq 0$?

Between A and B inclusive. ✓
Between D and E inclusive. ✓
Between F and H inclusive. ✓

---

10. [4 marks]

The curve $y = f(x)$ cuts the x-axis at the origin and nowhere else.
Also, $f'(1) = f'(2) = 0$ and $f'(x) < 0$ only for $1 < x < 2$.
Give a possible sketch of $y = f(x)$.



08 Differentiation of Exponential Functions

Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

Find $\frac{dy}{dx}$.

(a) $y = 5e^{3x}$

$$\frac{dy}{dx} = 15e^{3x} \quad ✓$$

(b) $y = \frac{4}{5e^x}$

$$y = \frac{4}{5}e^{-x} \quad ✓$$
$$\frac{dy}{dx} = \frac{-4}{5}e^{-x} \quad ✓$$

(c) $y = e^x + \frac{1}{2e^{2x}}$

$$y = e^x + \frac{1}{2}e^{-2x} \quad ✓$$
$$\frac{dy}{dx} = e^x + -e^{-2x} \quad ✓$$

(d) $y = \frac{e^{2x} + e^x}{e^{3x}}$

$$y = e^{-x} + e^{-2x} \quad ✓$$
$$\frac{dy}{dx} = -e^{-x} - 2e^{-2x} \quad ✓$$

(e) $y = e^{2x^2 + 3x}$

$$\frac{dy}{dx} = (4x+3)e^{2x^2+3x} \quad ✓✓$$

Calculator Free

2. [11 marks: 2, 3, 3, 3]

Find $\frac{dy}{dx}$.

(a) $y = (e^2 + e^{2x})^3$

$$\frac{dy}{dx} = 3(e^2 + e^{2x})^2 \times 2e^{2x} \quad ✓✓$$

(b) $y = x^4 e^{x^2}$

$$\frac{dy}{dx} = 4x^3 e^{x^2} + x^4 \times \left(2x\, e^{x^2}\right)$$
✓ ✓ ✓

(c) $y = \frac{e^{2x}}{1+x^2}$

$$\frac{dy}{dx} = \frac{(1+x^2)(2e^{2x}) - 2xe^{2x}}{(1+x^2)^2}$$
✓ ✓ ✓

(d) $y = \frac{x^2 - e^{2x}}{2e^x}$

$$\frac{dy}{dx} = \frac{2e^x(2x - 2e^{2x}) - 2e^x(x^2 - e^{2x})}{(2e^x)^2}$$
✓ ✓ ✓

Calculator Free

3. [4 marks]

Find the equation of the tangent to the curve $y = -x\,e^{2x}$ at $x = \frac{-1}{2}$.

$\frac{dy}{dx} = -e^{2x} + -2x\,e^{2x}$ ✓

When $x = \frac{-1}{2}$, $y = \frac{1}{2e}$ and $\frac{dy}{dx} = 0$. ✓✓

Hence, equation of tangent is $y = \frac{1}{2e}$. ✓

4. [5 marks]

Given that $y = x^2 e^x$, prove that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2e^x(1 + x) = 0$

$y = x^2 e^x$

$\frac{dy}{dx} = 2x\,e^x + x^2 e^x$ ✓

$\frac{d^2y}{dx^2} = (2e^x + 2x\,e^x) + (2x\,e^x + x^2 e^x)$ ✓

$= 4x\,e^x + x^2 e^x + 2e^x$ ✓

LHS $= (4x\,e^x + x^2 e^x + 2e^x) - (2x\,e^x + x^2 e^x) - 2e^x(1 + x)$ ✓✓

$= 0$

$=$ RHS

09 Differentiation of Logarithmic Functions

Calculator Free

1. [12 marks: 2, 2, 2, 2, 2, 2]

Differentiate with respect to x:

(a) $y = \ln(1 + x^3)$

$y' = \frac{3x^2}{1 + x^3}$ ✓✓

(b) $y = \ln x^4$

Rewrite as $y = 4\ln x$

$y' = \frac{4}{x}$ ✓✓

(c) $y = \ln(1 - 2x)^3$

Rewrite as $y = 3\ln(1 - 2x)$

$y' = 3\left[\frac{-2}{1 - 2x}\right]$ ✓✓

$= \frac{-6}{1 - 2x}$

(d) $y = \ln\sqrt{x^2 + 2x}$

Rewrite as $y = ½\ln(x^2 + 2x)$

$y' = \frac{1}{2}\left[\frac{2x + 2}{x^2 + 2x}\right]$ ✓✓

$= \frac{x + 1}{x^2 + 2x}$

(e) $y = \ln(1 + e^{2x})$

$y' = \frac{2e^{2x}}{1 + e^{2x}}$ ✓✓

(f) $y = \log_2(1 + x)$

Rewrite as $y = \frac{\ln(1+x)}{\ln 2}$ ✓

$y' = \frac{1}{\ln 2\,(1 + x)}$ ✓

Calculator Free

2. [9 marks: 3, 3, 3]

Differentiate with respect to x:

(a) $y = \ln[(x+1)^2(2x-1)^3]$

Rewrite as $y = \ln(x+1)^2 + \ln(2x-1)^3$

$y = 2\ln(x+1) + 3\ln(2x-1)$ ✓

$y' = \frac{2}{x+1} + 3\left[\frac{2}{2x-1}\right]$ ✓✓

$= \frac{2}{x+1} + \frac{6}{2x-1}$

(b) $y = \ln\left[\frac{1+2x}{1+x^2}\right]$

Rewrite as $y = \ln(1+2x) - \ln(1+x^2)$ ✓

$y' = \frac{2}{1+2x} - \frac{2x}{1+x^2}$ ✓✓

(c) $y = \ln[x^2\sqrt{(3-2x)}]$

Rewrite as $y = \ln x^2 + \ln\sqrt{(3-2x)}$

$y = 2\ln x + \frac{1}{2}\ln(3-2x)$ ✓

$y' = \frac{2}{x} + \frac{1}{2}\left[\frac{-2}{3-2x}\right]$ ✓✓

$= \frac{2}{x} - \frac{1}{3-2x}$

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3. [9 marks: 3, 3, 3]

Differentiate with respect to x:

(a) $y = x^2\ln(1-x^3)$

$y' = 2x\ln(1-x^3) + x^2\left[\frac{-3x^2}{1-x^3}\right]$ ✓✓✓

$= 2x\ln(1-x^3) - \frac{3x^4}{1-x^3}$

Calculator Free

3. (b) $y = (1+e^x)\ln(x^2+x-1)$

$y' = e^x\ln(x^2+x-1) + (1+e^x)\left[\frac{2x+1}{x^2+x-1}\right]$ ✓✓✓

(c) $y = \frac{\ln(x)}{x}$

$y' = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}$ ✓✓✓

$= \frac{1-\ln x}{x^2}$

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4. [4 marks: 2, 2]

(a) Express 7^x in the form e^{ax} where a is a constant.

$7 = e^{\ln(7)}$ ✓

Hence: $7^x = \left[e^{\ln(7)}\right]^x$

$= e^{x\ln(7)}$ ✓

(b) Hence, determine $\frac{d}{dx}(7^x)$

$\frac{d}{dx}(7^x) = \frac{d}{dx}\left[e^{x\ln(7)}\right]$

$= e^{x\ln(7)} \times \ln(7)$ ✓

$= 7^x\ln(7)$ ✓

Calculator Free

5. [2 marks]

Find the gradient of the curve $y = ln\,(2x - 5)$ at the point where $x = 3$.

> Gradient function $y' = \frac{2}{2x-5}$ ✓
>
> When $x = 3$, $y' = 2$. ✓

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6. [4 marks]

Find the coordinates of the point on $y = ln\,(1 + 2x)$ where the gradient of the curve is 2.

> Gradient function $y' = \frac{2}{1+2x}$. ✓
>
> Gradient of curve = 2 $\Rightarrow$ $y' = 2$.
>
> Hence, $\frac{2}{1+2x} = 2 \Rightarrow 1 + 2x = 1 \Rightarrow x = 0$ ✓✓
>
> When $x = 0$, $y = 0$. Hence, the required point is (0, 0). ✓

---

7. [6 marks]

Find the coordinates of the point(s) on the curve $y = x^2\,ln(x)$ where the gradient of the curve is zero.

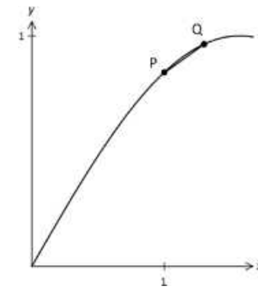
> Gradient function $y' = 2x\,ln\,x + x^2 \times \frac{1}{x}$ ✓
>
> $= 2x\,ln\,x + x$
>
> $= x\,(2\,ln\,x + 1)$ ✓
>
> When gradient = 0, $y' = 0$.
>
> $\Rightarrow x = 0$ or $ln\,x = -\frac{1}{2}$ ✓✓
>
> Reject $x = 0$ as $x > 0$. Hence, $x = e^{-\frac{1}{2}}$ ✓
>
> Hence, required point is $(e^{-\frac{1}{2}}, -\frac{1}{2e})$. ✓

10 Differentiation of Trigonometric Functions

Calculator Assumed

1. [8 marks: 2, 1, 1, 4] [TISC]

The accompanying diagram shows the graph of $y = \sin x$ where x is measured in radians.
The point P has coordinates $(x, \sin(x))$ and the point Q has coordinates $(x + \delta x, \sin(x + \delta x))$.



(a) Write an expression for the gradient of the line PQ.

> $m_{PQ} = \frac{\sin(x+\delta x) - \sin(x)}{\delta x}$ ✓✓

(b) For $x = 1$, complete the table below for the given value of δx.

	$\delta x = 0.0001$	$\delta x = 0.000001$
$\frac{\sin(x+\delta x)-\sin(x)}{\delta x}$	0.540 260 2314	0.540 302 27

✓ Correct values

(c) For $x = 2$, complete the table below for the given value of δx.

	$\delta x = 0.0001$	$\delta x = 0.000001$
$\frac{\sin(x+\delta x)-\sin(x)}{\delta x}$	– 0.416 192 300 72	– 0.416 147 291

✓ Correct values

(d) Compare the values of $\frac{\sin(x+\delta x)-\sin(x)}{\delta x}$ in the tables above with the values of cos (x) for $x = 1$ and $x = 2$. Explain why these results suggest that the derivative of sin (x) could be cos (x).

> $\frac{\sin(1+0.000001)-\sin(1)}{0.000001} \approx 0.540\,302\,27$
>
> $\cos(1) = 0.5403023059$
>
> $\Rightarrow \frac{\sin(1+0.000001)-\sin(1)}{0.000001} \approx \cos(1)$
>
> Similarly, $\frac{\sin(2+0.000001)-\sin(2)}{0.000001} \approx \cos(2)$ ✓
>
> Hence, results seem to suggest that for small values of δx:
>
> $\frac{\sin(x+\delta x)-\sin(x)}{\delta x} \approx \cos(x)$ ✓
>
> But $\frac{\sin(x+\delta x)-\sin(x)}{\delta x} \approx$ gradient of the curve $y = \sin(x)$. ✓
>
> Hence, derivative of sin (x) could be cos (x). ✓

Calculator Free

2. [12 marks: 1, 2, 2, 3, 2, 2]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = \cos \frac{\pi}{4}$

$\frac{dy}{dx} = 0$ ✓

(b) $y = \sin(1 + 2x)$

$\frac{dy}{dx} = 2\cos(1 + 2x)$ ✓✓

(c) $y = \sin(1 + \frac{1}{x})$

$\frac{dy}{dx} = \left(-\frac{1}{x^2}\right)\cos(1 + \frac{1}{x})$ ✓✓

(d) $y = \cos(1 - 2x)^3$

$\frac{dy}{dx} = -\sin(1 - 2x)^3 \times 3(1 - 2x)^2 \times (-2)$ ✓✓✓

(e) $y = \tan x^2$

$\frac{dy}{dx} = 2x\sec^2 x^2$ or $\frac{2x}{\cos^2(x^2)}$ ✓✓

(f) $y = \frac{\sin x + \cos x}{\cos x}$

Rewrite $y = \tan x + 1$ ✓

$\frac{dy}{dx} = \sec^2 x$ or $\frac{1}{\cos^2(x)}$ ✓

Calculator Free

3. [13 marks: 2, 3, 3, 3, 2]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = (1 - \cos t)^3$

$\frac{dy}{dt} = 3(1 - \cos t)^2 \times \sin t$ ✓✓

(b) $y = \tan^3(2t)$

$\frac{dy}{dt} = 3 \times [\tan^2(2t)] \times [\sec^2(2t)] \times 2$
✓ ✓ ✓

(c) $y = \sin^2(1 + \sqrt{t})$

$\frac{dy}{dt} = 2\sin(1 + \sqrt{t}) \times \cos(1 + \sqrt{t}) \times \frac{1}{2\sqrt{t}}$
✓ ✓ ✓

(d) $y = \sqrt{\cos(e^{2t})}$

$\frac{dy}{dt} = \frac{-\sin(e^{2t}) \times (2e^{2t})}{2\sqrt{\cos(e^{2t})}}$
✓ ✓ ✓

(e) $y = e^{\sin 3t}$

$\frac{dy}{dt} = e^{\sin 3t} \times 3\cos 3t$
✓ ✓

Calculator Free

4. [15 marks: 2, 3, 4, 3, 3]

Find $\frac{dy}{dx}$ for each of the following. You do not need to simplify your answer.

(a) $y = x^2 \cos(1-x)$

$$\frac{dy}{dx} = 2x\cos(1-x) + x^2\sin(1-x) \quad \checkmark\checkmark$$

(b) $y = x\sin x^2$

$$\frac{dy}{dx} = \sin x^2 + x\cos x^2 \times 2x \quad \checkmark \checkmark \checkmark$$

(c) $y = (1+2x)^3 \tan(1-\sqrt{x})$

$$\frac{dy}{dx} = 6(1+2x)^2\tan(1-\sqrt{x}) + (1+2x)^3[\sec^2(1-\sqrt{x})] \times -\frac{1}{2\sqrt{x}} \quad \checkmark \checkmark \checkmark \checkmark$$

(d) $y = x^2 \ln(\cos x)$

$$\frac{dy}{dx} = 2x\ln(\cos x) + x^2 \times \frac{-\sin x}{\cos x} \quad \checkmark \checkmark \checkmark$$

(e) $y = \cos 2x \sin(1+3x)$

$$\frac{dy}{dx} = -2\sin 2x \sin(1+3x) + \cos 2x\cos(1+3x) \times 3 \quad \checkmark \checkmark \checkmark$$

Calculator Free

5. [15 marks: 3, 3, 3, 3, 3]

Find $\frac{dy}{dt}$ for each of the following. You do not need to simplify your answer.

(a) $y = \frac{t^2}{\sin(1+2t)}$

$$\frac{dy}{dt} = \frac{2t\sin(1+2t) - 2t^2\cos(1+2t)}{\sin^2(1+2t)} \quad \checkmark \checkmark \checkmark$$

(b) $y = \frac{\cos^2 t}{t}$

$$\frac{dy}{dt} = \frac{-2t\cos t\sin t - \cos^2 t}{t^2} \quad \checkmark \checkmark \checkmark$$

(c) $y = \frac{e^t}{\tan t}$

$$\frac{dy}{dt} = \frac{e^t\tan t - e^t\sec^2 t}{\tan^2 t} \quad \checkmark \checkmark \checkmark$$

(d) $y = \frac{\sin t}{\cos^2 t}$

$$\frac{dy}{dt} = \frac{\cos^3 t - \sin t \times 2\cos t \times (-\sin t)}{\cos^4 t} \quad \checkmark \checkmark \checkmark$$

(e) $y = \frac{e^{\sin t}}{\ln \sin t}$

$$\frac{dy}{dt} = \frac{\cos t\, e^{\sin t}(\ln \sin t) - e^{\sin t} \times \frac{\cos t}{\sin t}}{(\ln \sin t)^2} \quad \checkmark \checkmark \checkmark$$

Calculator Free

6. [2 marks]

Find the gradient of the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

> Gradient function $y' = \frac{1}{\cos^2 x}$ ✓
>
> When $x = \frac{\pi}{4}$, $y' = 2$. ✓

7. [4 marks]

Find the coordinates of the point on $y = \sin^2 x$ for $0 \le x \le 2\pi$ where the gradient of the curve is 1.

> Gradient function $y' = 2 \sin x \cos x$. ✓
> Gradient of curve = 1 $\Rightarrow y' = 1$.
> Hence, $2 \sin x \cos x = 1 \Rightarrow \sin 2x = 1$
> $2x = \frac{\pi}{2}, \frac{5\pi}{2}$ ✓
> $x = \frac{\pi}{4}, \frac{5\pi}{4}$ ✓
> When $x = \frac{\pi}{4}, y = \frac{1}{2}$. When $x = = \frac{5\pi}{4}, y = \frac{1}{2}$.
> Hence: $(\frac{\pi}{4}, \frac{1}{2})$ and $(\frac{5\pi}{4}, \frac{1}{2})$. ✓

8. [3 marks]

Find the equation of the tangent to the curve $y = x + \cos x$ at the point $(\pi, \pi - 1)$.

> Gradient function $y' = 1 + \sin x$ ✓
> At the point $(\pi, \pi - 1)$, $y' = 1$ ✓
>
> Hence, equation of tangent is
> $y - (\pi - 1) = (x - \pi)$
> $y = x - 1$ ✓

11 Differentiation -Miscellaneous

Calculator Free

1. [8 marks: 2, 3, 3]

(a) Determine $\frac{d}{dx}\left(\sin^3(2x)\right)$.

> $\frac{d}{dx}\left(\sin^3(2x)\right) = 3\sin^2(2x) \times \cos(2x) \times 2$ ✓ ✓

(b) Determine $\frac{d}{dx}\left((\cos x)\ ln(2 + e^{2x})\right)$.

> $\frac{d}{dx}\left((\cos x)\ ln(2+e^{2x})\right) = (-\sin x)\ ln(2 + e^{2x}) + (\cos x) \times \left(\frac{2e^{2x}}{2+e^{2x}}\right)$ ✓ ✓✓

(c) Determine $\frac{d}{dx}\left(\frac{\sqrt{x}}{1-x^2}\right)$. Simplify your answer.

> $\frac{d}{dx}\left(\frac{\sqrt{x}}{1-x^2}\right) = \frac{(1-x^2)\left(\frac{1}{2\sqrt{x}}\right)+2x\sqrt{x}}{(1-x^2)}$ ✓✓
>
> $= \frac{3x^2+1}{2\sqrt{x}\,(1-x^2)^2}$ ✓

2. [8 marks: 2, 3, 3]

(a) Determine $\frac{dy}{dx}$ given that $y = (1 + e^{2x})^4$.

> $\frac{dy}{dx} = 4\,(1+e^{2x})^3 \times (2e^{2x})$ ✓ ✓

Calculator Free

2. (b) Given $f(x) = x^2 \ln(2+\sin x)$, determine $f'(x)$.

$f'(x) = 2x\ln(2+\sin x) + x^2 \times \left(\frac{\cos x}{2+\sin x}\right)$ ✓ ✓✓

(c) Differentiate $\frac{x}{1+x^4}$ with respect to x, simplifying your answer.

$\frac{d}{dx}\left(\frac{x}{1+x^4}\right) = \frac{(1+x^4)\times 1 - x\times 4x^3}{(1+x^4)^2}$ ✓✓

$= \frac{1-3x^4}{(1+x^4)^2}$ ✓

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3. [9 marks: 3, 3, 3]

(a) Determine $\frac{d^2y}{dx^2}$ given that $\frac{dy}{dx} = (x + \ln(1+x^2))^3$.

$\frac{d^2y}{dx^2} = 3(x+\ln(1+x^2))^2 \times \left(1+\frac{2x}{1+x^2}\right)$ ✓ ✓✓

(b) Given that $y' = xe^{\sin 2x}$, determine y''.

$y'' = e^{\sin 2x} + 2xe^{\sin 2x}\cos 2x$ ✓ ✓✓

(c) Given $f'(x) = \frac{1}{\cos 2x}$, determine $f''(x)$.

$f'(x) = (\cos 2x)^{-1}$ ✓

$f''(x) = -1 \times (\cos 2x)^{-2} \times (-\sin 2x) \times (2)$ ✓ ✓

Calculator Assumed

4. [11 marks: 5, 2, 2, 2]

A curve has equation $y = x^3 + 5x^2 + 3x + 2$.

(a) Use calculus to find all points on the curve where the gradient is 0.

$y' = 3x^2 + 10x + 3$ ✓

When $y' = 0$, $3x^2 + 10x + 3 = 0$ ✓

$(3x+1)(x+3) = 0$

$x = \frac{-1}{3}, -3$. ✓

Hence, $(\frac{-1}{3}, \frac{41}{27})$ and $(-3, 11)$. ✓✓

(b) Find $\frac{d^2y}{dx^2}$.

$y'' = 6x + 10$ ✓✓

(c) Find the x-coordinate of the point on this curve where $\frac{d^2y}{dx^2} = 0$.

$6x + 10 = 0$ ✓

$x = \frac{-5}{3}$ ✓

(d) Find the gradient of this curve at the point where $\frac{d^2y}{dx^2} = 0$.

$y' = 3x^2 + 10x + 3$

When $x = \frac{-5}{3}$, $y' = \frac{-16}{3}$. ✓✓

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5. [4 marks]

Use calculus to find the equation of the tangent to the curve $y = x^3\left(1+\frac{1}{\sqrt{x}}\right)^2$ at the point (1, 4).

$y' = 3x^2\left(1+\frac{1}{\sqrt{x}}\right)^2 + x^3\left[2\times\frac{-1}{2x^{3/2}}\times\left(1+\frac{1}{\sqrt{x}}\right)\right]$ ✓✓

$y'(1) = 10$ ✓

Hence, equation of tangent is $y - 4 = 10(x-1)$

$y = 10x - 6$. ✓

Calculator Assumed

6. [8 marks: 2, 4]

A curve has equation $y = (x+1)^2(x-2)$.

(a) Find the x-intercepts of the curve.

> $x = -1, 2$ ✓✓

(b) Use derivatives to find the equation of the tangents to this curve at each of the x-intercepts

> $y' = 2(x+1)(x-2) + (x+1)^2$ ✓
> $y'(-1) = 0$. $\Rightarrow$ Equation of tangent is $y = 0$. ✓
>
> $y'(2) = 9$. ✓
> Hence, equation of tangent is $y - 0 = 9(x-2)$
> $y = 9x - 18$. ✓

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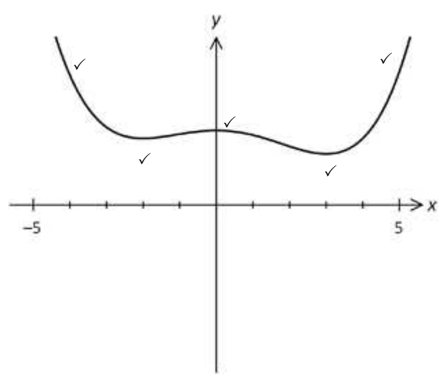
7. [5 marks]

The graph of $y = f(x)$ has the following properties:

- $y \ge 0$ for all x
- $\frac{dy}{dx} = 0$ for $x = -2, 0, 3$
- $\frac{dy}{dx} \ge 0$ for $-2 \le x \le 0$ and $x \ge 3$

Sketch a possible graph for $y = f(x)$.



Calculator Assumed

8. [8 marks]

The graph of $y = \frac{ax+b}{cx+d}$,where a, b, c and d are non-zero constants, intersects the x-axis at the point where $x = \frac{-1}{2}$. It also intersects the y-axis at the point where $y = 1$. At the point it intersects the y-axis, it is also parallel to the line $y = 3x + 10$. Find the values of a, b, c and d.

> $y = \frac{ax+b}{cx+d}$
> When $y = 0$, $x = \frac{-1}{2}$.
> $\Rightarrow \frac{-a}{2} + b = 0 \quad \Rightarrow a = 2b$ ✓✓
>
> When $x = 0$, $y = 1$.
> $\Rightarrow \frac{b}{d} = 1 \quad \Rightarrow d = b$ ✓
>
> Hence, $y = \frac{2bx+b}{cx+b}$.
> $y' = \frac{(2b)(cx+b) - (2bx+b)(c)}{(cx+b)^2}$ ✓
> When $x = 0$, $y' = 3$.
> $\Rightarrow \frac{2b^2 - bc}{b^2} = 3 \quad \Rightarrow b(b+c) = 0$ ✓
> $\Rightarrow c = -b$ (Reject $b = 0$) ✓
>
> Hence, $y = \frac{2bx+b}{-bx+b} = \frac{2x+1}{-x+1}$
>
> Therefore, $a = 2k$, $b = k$, $c = -k$ and $d = k$ where k is a real number. ✓✓

12 Stationary & Inflection Points

Calculator Free

1. [8 marks]

Consider the curve with equation $y = x^2 + \frac{1}{x^2}$. Use Calculus to find all the stationary points on this curve. State the nature of each stationary point.

Gradient function $y' = 2x - 2x^{-3}$ ✓
At stationary points, $y' = 0. \Rightarrow 2x - 2x^{-3} = 0$ ✓
$x = \frac{1}{x^3}$
$x^4 = 1$
Hence, $x = \pm 1.$ ✓

$y'' = 2 + 6x^{-4}$ ✓

When $x = 1$, $y = 2$ and $y'' > 0$. ✓
Hence (1, 2) is a minimum point. ✓

When $x = -1$, $y = 2$ and $y'' > 0$. ✓
Hence (–1, 2) is a minimum point. ✓

2. [7 marks]

Use Calculus to find the exact coordinates of the turning point(s) on the curve $y = xe^{0.05x}$. State the nature of the turning point(s).

Gradient function $y' = e^{0.05x} + 0.05x\, e^{0.05x}$ ✓
$= e^{0.05x}(1 + 0.05x)$ ✓

For turning points, $y' = 0. \Rightarrow 1 + 0.05x = 0$ ✓
$x = -20$ ✓

When $x = -20$, $y = -\frac{20}{e}$. ✓

	20^-	20	20^+
y'	–	0	+

✓

Hence, $(-20, -\frac{20}{e})$ is a minimum point. ✓

Calculator Free

3. [7 marks]

Consider the curve with equation $y = \sin(x) - x\cos(x)$, for $0 \le x \le \pi$.
Use a calculus method to determine the coordinates of the stationary point on this curve and identify the nature of this point.

$\frac{dy}{dx} = \cos(x) - x\cos(x) + x\sin(x)$ ✓
$= x\sin(x)$
$\frac{dy}{dx} = 0 \Rightarrow x\sin(x) = 0$ ✓
$x = 0, \pi$ ✓
$\frac{d^2y}{dx^2} = x\cos(x) + \sin(x)$ ✓
$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0 \quad \left.\frac{d^2y}{dx^2}\right|_{x=0^-} < 0$ and $\left.\frac{d^2y}{dx^2}\right|_{x=0^+} > 0$ ✓
Hence, horizontal inflection point at (0, 0). ✓

$\left.\frac{d^2y}{dx^2}\right|_{x=\pi} < 0 \Rightarrow$ Maximum Point at (π, π). ✓

4. [7 marks]

Use Calculus to determine the coordinates of the point(s) of inflection of the curve with equation $y = x(1 - x)^3$.

Gradient function $y' = (1 - x)^3 + x \times 3(1 - x)^2 \times -1$
$= (1 - x)^2(1 - 4x)$ ✓

Second derivative $y'' = 2(1 - x) \times (-1)(1 - 4x) + (1 - x)^2 \times -4$
$= -2(1 - x)(3 - 6x)$ ✓

For inflection points; $y'' = 0 \Rightarrow x = 1, \frac{1}{2}$ ✓✓

$y'' = -12x^2 + 18x - 6 \Rightarrow y''' = -24x + 18$ ✓
When $x = 1$, $y''' \ne 0$. Also, when $x = \frac{1}{2}$, $y''' \ne 0$. ✓
Hence, (1, 0) and $(\frac{1}{2}, \frac{1}{16})$ are inflection points. ✓

Calculator Free

5. [9 marks]

(a) Given that $x^3 - 3x + 2 = (x-1)(x^2 + bx + c)$, determine the values of b and c.
Hence, solve for x in $x^3 - 3x + 2 = 0$.

By Inspection $b = 1$ and $c = -2$ ✓
$(x-1)(x^2 + x - 2) = 0$
$(x-1)(x+2)(x-1) = 0$
$x = -2, 1$ ✓

(b) Consider the curve with equation $y = x^5 - 10x^3 + 20x^2$.
Use a calculus method to determine the coordinates of the inflection point(s) on this curve.

$\frac{dy}{dx} = 5x^4 - 30x^2 + 40x$ ✓
$\frac{d^2y}{dx^2} = 20x^3 - 60x + 40 = 20(x^3 - 3x + 2)$ ✓
For inflection points: $x^3 - 3x + 2 = 0$
$x = -2, 1$ ✓

For $x = 1$, $\left.\frac{d^2y}{dx^2}\right|_{x=1^-} > 0$ and $\left.\frac{d^2y}{dx^2}\right|_{x=1^+} > 0$. ✓
Hence, $x = 1$ does not yield an inflection point. ✓

For $x = 2$, $\left.\frac{d^2y}{dx^2}\right|_{x=2^-} < 0$ and $\left.\frac{d^2y}{dx^2}\right|_{x=2^+} > 0$. ✓
Hence, (–2, 128) is an inflection point. ✓

6. [4 marks] [TISC]

Use calculus to explain why the curve $y = x^4$ does not have a point of inflection.

$\frac{dy}{dx} = 4x^3$ $\quad \frac{d^2y}{dx^2} = 12x^2$ ✓
For inflection point: $\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0$ ✓
But $\left.\frac{d^2y}{dx^2}\right|_{x=0^-} > 0$ and $\left.\frac{d^2y}{dx^2}\right|_{x=0^+} > 0$. ✓
Hence, there is no change of curvature in the neighbourhood of $x = 0$. ✓
Hence, $x = 0$ does not correspond to an inflection point.

Calculator Free

7. [6 marks]

Consider the curve with equation $y = e^x \sin(x)$ for $0 \le x \le \pi$. Use a calculus method to determine the coordinates of the inflection point(s) on this curve.

$\frac{dy}{dx} = e^x \sin(x) + e^x \cos(x)$ ✓
$\frac{d^2y}{dx^2} = e^x \sin(x) + e^x \cos(x) + e^x \cos(x) - e^x \sin(x)$ ✓
$= 2e^x \cos(x)$
For inflection points: $\frac{d^2y}{dx^2} = 0 \Rightarrow 2e^x \cos(x) = 0$ ✓
$x = \frac{\pi}{2}$ ✓
$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{2}^-} > 0$ and $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{2}^+} < 0$. ✓
Hence, $(\frac{\pi}{2}, e^{\frac{\pi}{2}})$ is an inflection point. ✓

8. [7 marks: 3, 4] [TISC]

(a) Given that $y = ln\,(1 + x^2)$, show that $\frac{d^2y}{dx^2} = \frac{2(1-x^2)}{(1+x^2)^2}$.

$\frac{dy}{dx} = \frac{2x}{1+x^2}$ ✓
$\frac{d^2y}{dx^2} = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2}$ ✓✓
$= \frac{2(1-x^2)}{(1+x^2)^2}$

Calculator Free

8. (b) Show that the curve with equation $y = ln\,(1 + x^2)$ for $x > 0$ has one inflection point. Also show that this inflection point is not a stationary point.

$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{2(1-x^2)}{(1+x^2)^2} = 0$ ✓

$x = 1 \quad (x > 0)$ ✓

$\left.\frac{d^2y}{dx^2}\right|_{x=1^-} > 0$ and $\left.\frac{d^2y}{dx^2}\right|_{x=1^+} < 0$

Hence, there is a change of curvature at $x = 1$. ✓
Therefore, there is an inflection point at $x = 1$.

Since $\left.\frac{dy}{dx}\right|_{x=1} = 1 \neq 0$, ✓
inflection point at $x = 1$ is not a stationary point.

9. [10 marks]

A curve has equation $y = ax^3 + bx^2 + cx + d$. The curve has an inflection point at $x = -2$, a turning point at $x = 1$, a y-intercept at $(0, -33)$ and a tangent with equation $y = -24x - 37$ at $x = -1$. Find the values of a, b, c and d. Show clearly how you obtained your answer.

y-intercept at $(0, -33) \Rightarrow d = -33$ ✓

$\frac{dy}{dx} = 3ax^2 + 2bx + c$ ✓

$\frac{d^2y}{dx^2} = 6ax + 2b$ ✓

Inflection Point at $x = -2 \Rightarrow -12a + 2b = 0$ ✓
$b = 6a$

Hence, $\frac{dy}{dx} = 3ax^2 + 12ax + c$ ✓

Turning Point at $x = 1 \Rightarrow 15a + c = 0$ ✓
Gradient at $x = 1$ is $-24 \Rightarrow -9a + c = -24$ ✓
$a = 1$ ✓
$c = -15$ ✓
$b = 6$ ✓

Calculator Free

10. [8 marks: 2, 2, 4]

The points A, B, C, D, E, F and G are points on the graph of a continuous function $y = f(x)$. The table below shows the sign of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at these points.

y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ have zero values at only the points indicated in this table.

Point	A	B	C	D	E	F	G
x	–2	–1	0	1	2	3	4
y	–	0	+	+	+	0	–
dy/dx	+	+	0	–	–	0	–
d^2y/dx^2	–	–	–	0	+	0	–

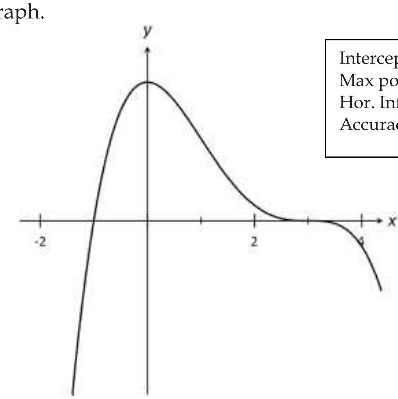
(a) Identify the maximum point on this graph. Justify your answer.

Maximum point is C. ✓
At C, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$. ✓

(b) Identify the inflection point(s) if any on this graph.

Inflection Points: D & F. ✓✓

(c) Sketch this graph.



Intercepts: ✓
Max point: ✓
Hor. Inflection Point: ✓
Accuracy: ✓

Calculator Free

11. [7 marks: 5, 2] [TISC]

Consider the curve with equation $y = \dfrac{2x^2-1}{1-x^2}$. This curve has one stationary point.

(a) Find the coordinates of this stationary point.

$$\frac{dy}{dx} = \frac{(1-x^2)(4x) - (2x^2-1)(-2x)}{(1-x^2)^2} \quad ✓✓$$
$$= \frac{4x-4x^3+4x^3-2x}{(1-x^2)^2}$$
$$= \frac{2x}{(1-x^2)^2}$$

For stationary points: $\frac{dy}{dx} = 0$.

$$\frac{2x}{(1-x^2)^2} = 0 \quad ✓$$
$$x = 0 \quad ✓$$

Hence, $(0, -1)$. ✓

(b) Use an appropriate test to determine if this stationary point is a minimum point, maximum point or an inflection point.

Use the sign test:
$\left.\frac{dy}{dx}\right|_{x=0^-} < 0$ and $\left.\frac{dy}{dx}\right|_{x=0^+} > 0$. ✓
Therefore, stationary point is a minimum point. ✓

Calculator Assumed

12. [11 marks: 8, 3]

Consider the curve with equation $y = e^{\cos x}$ for $0 \le x \le 2\pi$.

(a) Use a calculus method to determine the coordinates of the stationary points on this curve. Use the second derivative test to determine the nature of these stationary points.

$\frac{dy}{dx} = -\sin x\, e^{\cos x}$ ✓

$\frac{dy}{dx} = 0 \Rightarrow -\sin x\, e^{\cos x} = 0$ ✓

$x = 0, \pi, 2\pi$ ✓

Hence, stationary points are:

$(0, e)$, $(\pi, \frac{1}{e})$ and $(2\pi, e)$ ✓

$\frac{d^2y}{dx^2} = \sin^2 x\, e^{\cos x} - \cos x\, e^{\cos x}$ ✓

For $(0, e)$, $\frac{d^2y}{dx^2} = -e < 0$.

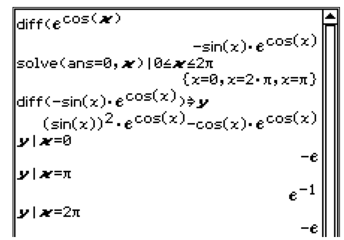
Hence, $(0, e)$ is a maximum point. ✓

For $(\pi, \frac{1}{e})$, $\frac{d^2y}{dx^2} = \frac{1}{e} > 0$.

Hence, $(\pi, \frac{1}{e})$ is a minimum point. ✓

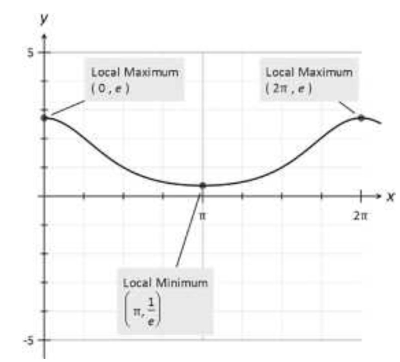
For $(2\pi, e)$, $\frac{d^2y}{dx^2} = -e < 0$.

Hence, $(2\pi, e)$ is a maximum point. ✓



(b) On the axes provided in the accompanying diagram, sketch $y = e^{\cos x}$. Indicate clearly the location of the stationary points.

Correct shape ✓
Min. Point ✓
Max. Point ✓



Calculator Assumed

13. [9 marks: 4, 5] [TISC]

Consider the curve with equation $y = \dfrac{1}{\sqrt{1+e^x}}$.

(a) Use a calculus method to show that the curve has no stationary points.

$$\frac{dy}{dx} = \frac{-e^x}{2(e^x+1)^{\frac{3}{2}}} \quad ✓✓$$

For stationary points, $\dfrac{dy}{dx} = 0$.

$$\Rightarrow \frac{-e^x}{2(e^x+1)^{\frac{3}{2}}} = 0 \quad ✓$$

$$e^x = 0$$

But $e^x > 0$ for all real x. ✓

Hence, curve does not have any stationary points.

(b) Given that the curve has one inflection point, use derivatives to determine the coordinates of the inflection point.

$$\frac{d^2y}{dx^2} = \frac{(e^x-2)e^x}{4(e^x+1)^{\frac{5}{2}}} \quad ✓✓$$

For inflection points, $\dfrac{d^2y}{dx^2} = 0$.

$$\Rightarrow \frac{(e^x-2)e^x}{4(e^x+1)^{\frac{5}{2}}} = 0 \quad ✓$$

$$x = 0.69315 \quad ✓$$

Hence, inflection point has coordinates (0.69, 0.58) ✓

$\frac{d}{dx}\left(\frac{1}{\sqrt{1+e^x}}\right)$ → $\dfrac{-e^x}{2\cdot(e^x+1)^{\frac{3}{2}}}$

$\frac{d}{dx}$(ans) → $\dfrac{3\cdot(e^x+1)^{\frac{3}{2}}\cdot e^{2\cdot x}-2\cdot(e^x+1)^{\frac{5}{2}}\cdot e^x}{4\cdot(e^x+1)^4}$

simplify(ans → $\dfrac{(e^x-2)\cdot e^x}{4\cdot(e^x+1)^{\frac{5}{2}}}$

solve(ans=0 → {x=0.6931471806}

$\frac{1}{\sqrt{1+e^x}}|x=0.69315$ → 0.5773497266

Calculator Assumed

14. [7 marks]

Consider the curve with equation $y = ax^4 + bx^3 + cx^2 + d$. The curve has a tangent with equation $y = x - 1$ at the point where $x = 1$. The curve has a inflection points at $x = -2$ and $\dfrac{1}{2}$. Determine the values of a, b, c and d.

Tangent $y = x - 1$ at $x = 1$:

$\Rightarrow$ When $x = 1$, $y = 0$

$a + b + c + d = 0$ (1) ✓

$\dfrac{dy}{dx} = 4ax^3 + 3bx^2 + 2cx$ ✓

$\left.\dfrac{dy}{dx}\right|_{x=1} = 1 \Rightarrow 4a + 3b + 2c = 1$ (2) ✓

$\dfrac{d^2y}{dx^2} = 12ax^2 + 6bx + 2c$ ✓

$\left.\dfrac{d^2y}{dx^2}\right|_{x=-2} = 0$

$\Rightarrow 48a - 12b + 2c = 0$ (3) ✓

$\left.\dfrac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = 0$

$\Rightarrow 3a + 3b + 2c = 0$ (4) ✓

Solve (1), (2), (3) & (4):

$a = 1, b = 3, c = -6, d = 2$ ✓

{a+b+c+d=0, 4a+3b+2c=1, 48a−12b+2c=0, 3a+3b+2c=0} | a,b,c,d → {a=1, b=3, c=−6, d=2}

13 Rates of Change

Calculator Assumed

1. [4 marks: 1, 1, 2]

Given that, $A = 2t^2 - 5t + 3$, find:

(a) an expression for the instantaneous rate of change of A with respect to t

$\frac{dA}{dt} = 4t - 5$ ✓

(b) the instantaneous rate of change of A with respect to t when $t = 0$

When $t = 0$, $\frac{dA}{dt} = -5$ ✓

(c) t when the instantaneous rate of change of A with respect to t is 3.

When $\frac{dA}{dt} = 3$, $4t - 5 = 3$ ✓

$\Rightarrow t = 2$ ✓

---

2. [6 marks: 3, 2, 2]

Given that, $P = \sqrt{1+t^2}$, find:

(a) the instantaneous rate of change of P with respect to t when $t = 0$ and $t = 4$

$P'(t) = \frac{t}{\sqrt{1+t^2}}$ ✓

Hence, $P'(0) = 0$ ✓

and $P'(4) = 0.9701$ ✓

(b) the average rate of change of P with respect to t for $0 \le t \le 4$

Average Rate of change $= \frac{P(4) - P(0)}{4}$ ✓

$= 0.7808$ ✓

(c) Compare your answers in (a) and (b) and comment on them.

The rates in (a) refer to the rate at a particular instant ✓
in time whereas the average rate refers to the rate across
the time interval. ✓

Calculator Assumed

3. [10 marks: 3, 3, 2, 2]

The concentration of a chemical (g/L) in a swimming pool is given by $C = 2000\,e^{-0.08t}$ where t is the number of *hours* after the chemical is introduced into the pool.

(a) Calculate the average rate of change of the concentration of this chemical in the first ten hours.

$C(0) = 2000 \quad C(10) = 898.66$ ✓

Average change $= \frac{898.66 - 2000}{10} \approx -110.1$ ✓

Decrease of 110.1 g/L per hour. ✓

(b) Determine the rate of change of the concentration of this chemical at the end of the tenth hour.

$\frac{dC}{dt} = -160e^{-0.08t}$ ✓

$\left.\frac{dC}{dt}\right|_{t=10} = -71.9$ ✓

Decrease of 71.9 g/L per hour. ✓

(c) Interpret your answers in (a) and (b).

The answer in (a) indicates that within the first ten hours,
the concentration of chemical decreases by 110.1 g/L. ✓
Whereas, the answer in (b) indicates that at the particular time
of the end of the tenth hour, the concentration of chemical decreases by 71.9 g/L. ✓

(d) Calculate how long it will take for the rate of decrease of the concentration of the chemical to be less than 10 g/L per hour.

$160e^{-0.08t} < 10$ ✓

$t > 34.66$ hours. ✓

That is, at least 34.7 hours.

Calculator Assumed

4. [7 marks: 3, 2, 2]

Given that, $Q = 100\,t\,e^{-0.5t}$, find:
(a) using derivatives, the instantaneous rate of change of Q with respect to t when $t = 1$ and $t = 5$

$Q'(t) = 100\,e^{-0.5t} - 50\,t\,e^{-0.5t}$, ✓
Hence, $Q'(1) = 30.3265$ ✓
and $Q'(5) = -12.3127$ ✓

(b) the average change in Q for $1 \le t \le 5$

Average Rate of change $= \dfrac{Q(5) - Q(1)}{4}$ ✓
$= -4.90$ ✓

(c) the value of t when the instantaneous rate of change of Q with respect to t is zero.

$100\,e^{-0.5t} - 50\,t\,e^{-0.5t} = 0$ ✓
$t = 2$ ✓

---

5. [5 marks: 2, 3]

Given that, $N = \dfrac{2t}{5+t}$ where $t \ge 0$, find, showing the use of derivatives:
(a) the instantaneous rate of change of N with respect to t when $t = 0$

$N'(t) = 10/(5+t)^2$, ✓
Hence, $N'(0) = 0.4$. ✓

(b) the value of t when the instantaneous rate of change of N with respect to t is 0.1.

When $N'(t) = 0.1 \Rightarrow 10/(5+t)^2 = 0.1$ ✓
$(5+t)^2 = 100$ ✓
$\Rightarrow t = 5$ (reject -15) ✓

Calculator Assumed

6. [5 marks: 2, 1, 2]

Given that, $A = 50(1+t)^2 e^{-t}$ for $t \ge 0$, find:
(a) an expression for the instantaneous rate of change of A with respect to t

$A'(t) = -(50\,t^2 - 50)\,e^{-t}$ ✓✓

(b) the instantaneous rate of change of A with respect to t when $t = 1$

$A'(1) = 0$ ✓

(c) the average rate of change of A in the first second.

Average Rate of change $= \dfrac{A(1) - A(0)}{1}$ ✓
$= 23.58$ ✓

---

7. [7 marks: 1, 2, 2, 2]

The price of a listed share C cents, is modelled by $C = 75\sqrt{1+0.8t}$, $t \ge 0$, where t is the number of years after 2000.

(a) Find the per unit cost in 2000.

$C(0) = 75$ ✓

(b) Find the average rate of cost rise between 2000 and 2010.

Average Rate of change $= \dfrac{C(10) - C(0)}{10}$ ✓
$= 15$ ✓

(c) Find using derivatives, the instantaneous rate of cost rise in 2005

$C'(t) = \dfrac{30\sqrt{5}}{\sqrt{4t+5}}$ ✓
Hence, $C'(5) = 6\sqrt{5}$. ✓

(d) Find when the instantaneous rate of cost rise is 10 cents per year.

$C'(t) = 10 \Rightarrow \dfrac{30\sqrt{5}}{\sqrt{4t+5}} = 10$ ✓
Hence, $t = 10$. ✓

Calculator Assumed

8. [13 marks: 2, 3, 3, 2, 3]

The water level at a jetty is modelled by $h = 3 + \cos\left(\frac{\pi t}{12}\right)$, $0 \le t \le 24$, where h metres is the depth of water measured from the river bed at time t hours after 6 am.

(a) Find the water level at 7 am and 10 am.

At 7 am, $t = 1$: $h(1) = 3.9659 \approx 3.97$ m ✓
At 10 am, $t = 4$: $h(4) = 3.5$ m ✓

(b) Find the average rate of change of the water depth between 7 am and 10 am.

$$\text{Average rate of change} = \frac{h(4) - h(1)}{4 - 1} = \frac{3.5 - 3.9659}{3} \checkmark = -0.1553 \checkmark$$

Water depth is decreasing at a rate of 0.15 m/hr ✓

(c) Find the rate of change of the water depth at 10 am.

$$\frac{dh}{dt} = -\frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right) \checkmark$$

At 10 am, $t = 4$: $\frac{dh}{dt} = -0.2267$. ✓

Water depth is decreasing at a rate of 0.23 m/hr ✓

(d) Comment on your answers in (b) and (c).

Answer in (c) describes the rate of change specifically at 10 am while answer in (d) describes the rate of change between 7 am and 10 am. ✓✓

(e) Find when the water level is increasing at a rate of 0.1 metres per hour.

$$\frac{dh}{dt} = -\frac{\pi}{12}\sin\left(\frac{\pi t}{12}\right) = 0.1$$

$t = 13.4971, 22.5029$ ✓

That is at 7.30 pm and 4.30 am ✓✓

Calculator Assumed

9. [12 marks: 4, 3, 2, 3]

The mass of an object being printed by a 3D-printer is given by $M = \ln(1 + t^3)$ g for $0 \le t \le 10$ minutes.

(a) Find the average rate of change of mass of the object during the first 5 seconds and the second 5 seconds.

1st 5 seconds: $\text{Average rate of change} = \frac{M(5) - M(0)}{5 - 0} = \frac{4.8363 - 0}{5} \checkmark = 0.9673 \approx 0.97$ g/minute ✓

2nd 5 seconds: $\text{Average rate of change} = \frac{M(10) - M(5)}{10 - 5} = \frac{6.9088 - 4.8363}{5} \checkmark = 0.4145 \approx 0.42$ g/minute ✓

(b) Find the rate of change of mass at $t = 5$ and $t = 10$ minutes.

$$\frac{dM}{dt} = \frac{3t^2}{1 + t^3} \checkmark$$

For $t = 5$: $\frac{dM}{dt} = 0.5952 \approx 0.60$ ✓

For $t = 10$: $\frac{dM}{dt} = 0.2997 \approx 0.30$ ✓

(c) Comment on your answers in (a) and (b).

The average rate of change and instantaneous rates of change are declining for increasing values of t. ✓✓

(d) Use an analytical method to determine the values of $0 \le t \le 10$ for which the rate of change of mass is decreasing.

$$\frac{d^2M}{dt^2} = \frac{6t - 3t^4}{(1 + t^3)^2} \checkmark$$

$$\frac{6t - 3t^4}{(1 + t^3)^2} < 0 \checkmark$$

$1.26 \le t \le 10$ ✓

```
diff(3·x^2/(x^3+1))
                    -(3·x^4-6·x)/(x^3+1)^2
solve(-(3·x^4-6·x)/(x^3+1)^2<0,x)|0≤x≤10
                    {x>1.25992105,x<0 and x≠-1}
```

14 Optimisation

Calculator Assumed

1. [10 marks: 6, 2, 2]

The cost per hour , \$C, of operating a truck travelling at a constant speed of v kmh^{-1} is modelled by $C = \frac{(v-20)^3}{5000} + \frac{400}{v} + 200$ where $v > 0$. The truck has a speed limit of 80 kmh^{-1}.

(a) Use Calculus to find the speed the truck should travel on for the hourly cost to be minimized. Find the minimum hourly cost.

> $C = \frac{(v-20)^3}{5000} + \frac{400}{v} + 200$
> $C'(v) = \frac{3(v-20)^2}{5000} - \frac{400}{v^2}$ ✓
> $C'(v) = 0 \Rightarrow v = 40.2737$ (reject -20.2737) ✓
> $C''(v) = \frac{6(v-20)}{5000} - \frac{800}{v^3}$ ✓
> $C''(40.2737) > 0$. ✓
> Hence, C has a local minimum when $v = 40.3$ kmh^{-1}. ✓
> Minimum cost $C(40.2737) = \$211.60$. ✓

(b) Find the difference in hourly cost if the truck were to be travelling at its posted speed limit.

> C(40.2737) = \$211.60.
> C(80) = \$ 248.20 ✓
> Hence, difference in cost = \$36.60 ✓

(c) Suggest a reason why the Company that owns this truck may not operate it at the speed that achieves the minimum hourly cost.

> Optimum speed is half the maximum speed; hence jobs will take twice as long. In that same time, twice as many jobs may be completed, possibly generating greater profits. ✓✓

Calculator Assumed

2. [10 marks: 3, 3, 4]

The quantity Q (mg) of a drug in a patient's blood stream is given by $Q = 10te^{-0.05t}$ where t is the number of *minutes* after the drug is taken.

(a) Calculate the average rate of change of the amount of this drug in the patient's blood stream in first hour.

> $Q(0) = 0 \quad Q(60) = 29.8722$ ✓
> Average change $= \frac{29.8722}{60} \approx 0.49787$ ✓
> Amount of drug increases by 0.50 mg/minute. ✓

(b) Determine the rate of change of the amount of this drug in the patient's blood stream at the end of the first hour.

> $\frac{dQ}{dt} = -0.5(t-20)e^{-0.05t}$ ✓
> $\left.\frac{dQ}{dt}\right|_{t=60} = -0.9957$ ✓
> Amount of drug decreases by 1.00mg/minute. ✓

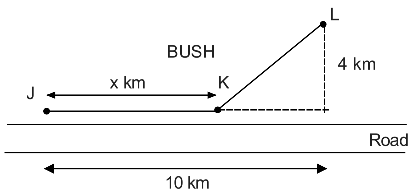
(c) Use a calculus method to determine when the rate of change of the amount of drug in the patient's blood stream is a minimum.

> $\frac{dQ}{dt} = -0.5(t-20)e^{-0.05t}$
> $\frac{d^2Q}{dt^2} = 0.025(t-40)e^{-0.05t}$ ✓
> $\frac{d^2Q}{dt^2} = 0 \Rightarrow t = 40$ ✓✓
> $\left.\frac{d^3Q}{dt^3}\right|_{t=40} > 0$.
> Hence, rate of change is a minimum at $t = 40$ hours. ✓

Calculator Assumed

3. [6 marks: 1, 2, 3]

A fibre optic cable is to be laid from J to L. It costs \$500 per km to lay the cable alongside the road and \$800 per km to lay it across the bush. K is a point x km from J along the road. The cable will be laid alongside the road from J to K and across the bush from K to L.



NOT DRAWN TO SCALE

(a) Show that the distance between K and L is $\sqrt{x^2 - 20x + 116}$ km.

$$KL = \sqrt{(10-x)^2 + (4)^2} \quad ✓$$
$$= \sqrt{x^2 - 20x + 116}$$

(b) Find the total cost for laying the cable from J to L via K (as described).

$$\text{Cost } C = 500x + 800\sqrt{x^2 - 20x + 116} \quad ✓✓$$

(c) Use an analytical method to find x so that this cost is minimized.

$$C'(x) = 500 + \frac{800(x-10)}{\sqrt{x^2 - 20x + 116}} \quad ✓$$

$C'(x) = 0 \Rightarrow x = 6.7974$ ✓

When $x = 6.7974$:

x	6.7974^-	6.7974	6.7974^+
$C'(x)$	–	0	+

✓

Hence C is minimised when $x = 6.80$ km.

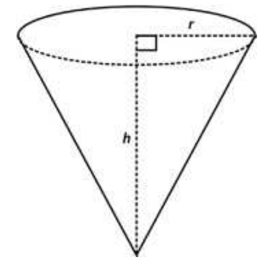
Calculator Assumed

4. [8 marks: 3, 5]

The accompanying diagram shows an inverted cone of height h cm and base radius r cm.

The volume of the cone is fixed at $\frac{\pi}{3}$ m^3.

The curved surface area of the cone is given by $A = \pi r\sqrt{h^2 + r^2}$.



(a) Show that $A = \frac{\pi}{r}\sqrt{1 + r^6}$.

Volume of cone $\frac{1}{3}\pi r^2 h = \frac{\pi}{3}$ ✓

$h = \frac{1}{r^2}$ ✓

Hence, $A = \pi r\sqrt{h^2 + r^2}$

$= \pi r\sqrt{\left(\frac{1}{r^2}\right)^2 + r^2}$ ✓

$= \frac{\pi\sqrt{1+r^6}}{r}$

(b) Use a calculus method to find the value of r and h that minimises A.

$A = \frac{\pi\sqrt{1+r^6}}{r}$

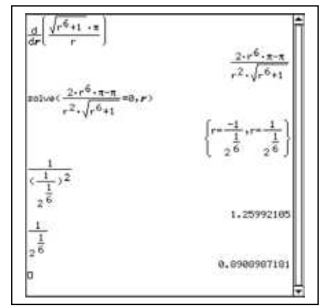
$\frac{dA}{dr} = \pi\left(\frac{2r^6 - 1}{r^2\sqrt{1+r^6}}\right)$ ✓✓

For max/min values: $\frac{dA}{dr} = 0$

$\pi\left(\frac{2r^6 - 1}{r^2\sqrt{1+r^6}}\right) = 0$ ✓

$r = \frac{1}{2^{\frac{1}{6}}} = 0.89$ m ✓

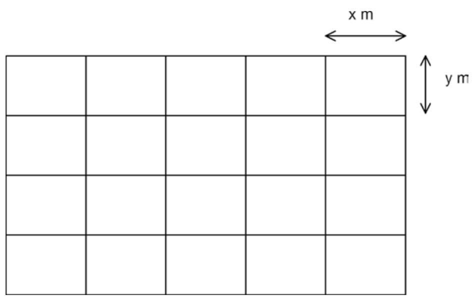
$h = 2^{\frac{1}{3}} = 1.26$ m ✓



Calculator Assumed

5. [11 marks: 3, 8]

The rectangular backyard of area 100 m^2 is to be divided into 20 equal rectangles as shown below. The boundaries are to be marked with single length red ribbons.



(a) Find a rule for y in terms of x.

Total Area covered = $20xy$
Hence $20xy = 100$ ✓✓
$y = \frac{5}{x}$ ✓

(b) Use a calculus method to find the exact length and width of each rectangle that will minimize the total length of ribbon used.

Total Perimeter $P = 25x + 24y$ ✓
$P = 25x + \frac{120}{x}$ ✓
$\frac{dP}{dx} = 25 - \frac{120}{x^2}$ ✓
For max/min values: $25 - \frac{120}{x^2} = 0$ ✓
$x = \frac{2\sqrt{30}}{5}$ reject $-\frac{2\sqrt{30}}{5}$ ✓
$\frac{d^2P}{dx^2} = \frac{240}{x^3}$
When $x = \frac{2\sqrt{30}}{5}$, $\frac{d^2P}{dx^2} > 0$. ✓
Hence, P is minimised when $x = \frac{2\sqrt{30}}{5}$ and $y = \frac{5\sqrt{30}}{12}$ ✓✓

Calculator Assumed

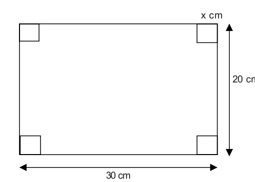
6. [6 marks]

A closed cylindrical can of diameter $2r$ is to have a volume of 50π cm^3. Use Calculus to find the dimensions of this can if it is to have minimum surface area. Ignore the thickness of the material used to make the can.

Let height of cylinder be h.
Volume $\pi \times r^2 \times h = 50\pi \Rightarrow h = \frac{50}{r^2}$ ✓
Surface Area $A = 2(\pi r^2) + 2\pi rh$
$= 2\pi r^2 + 2\pi r \times \frac{50}{r^2}$
$= 2\pi r^2 + \frac{100\pi}{r}$ ✓
$A'(x) = 4\pi r - \frac{100\pi}{r^2}$ ✓
$A'(x) = 0 \Rightarrow x = 2.9240$ ✓
$A''(x) = 4\pi + \frac{200}{r^3}$, $A''(2.9240) > 0$. ✓
Hence, A is minimised when $x = 2.9240$
Cylinder has base radius = 2.9 cm and height = 5.8 cm. ✓

7. [7 marks]

A rectangular sheet of cardboard measuring 20 cm by 30 cm is used to make an open box. A square of width x cm is removed from each corner of the sheet to form the net of the box. The net is then folded up to form the box. Use Calculus to find the dimensions of the box with the largest possible volume. Give this volume.

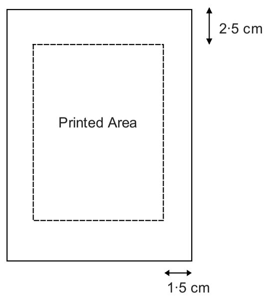


Length of box = $30 - 2x$
Width of box = $20 - 2x$
Height of box = x ✓
Hence, volume $V = x(20 - 2x)(30 - 2x)$
$= 4x^3 - 100x^2 + 600x$ ✓
$V'(x) = 12x^2 - 200x + 600$
$V'(x) = 0 \Rightarrow x = 3.9237$ (reject 12.7429 as this makes the width < 0) ✓
$V''(x) = 24x - 200$, $V''(3.9237) < 0$. ✓
Hence, V has a maximum when height = 3.93 cm, width = 12.15 cm and length = 22.14 cm ✓✓
Maximum value for $V = 3.93 \times 12.15 \times 22.14 \approx 1057$ cm^3 ✓

Calculator Assumed

8. [8 marks]

A printed poster using a minimal area of cardboard is to be designed. The printed area must be 500 cm^2. The top and bottom margins must be 2.5 cm each. The left and right margins must be 1.5 cm each. Use Calculus to determine the optimal dimensions of the poster. Give the width, height and total area of the optimal poster.



Let width of printed area = x
Let length of printed area = y

Area of printed area $A = xy$
But $xy = 500$. $\Rightarrow y = 500/x$ ✓

Width of poster = $x + (2 \times 1.5) = x + 3$ ✓
Length of poster = $y + (2 \times 2.5) = 500/x + 5$ ✓

Area of poster, $A(x) = (x + 3)(500/x + 5)$
$= 515 + 5x + 1500/x$ ✓

$A'(x) = 5 - 1500/x^2$ ✓
$A'(x) = 0 \Rightarrow 5x^2 = 1500$
$x = 17.3205$

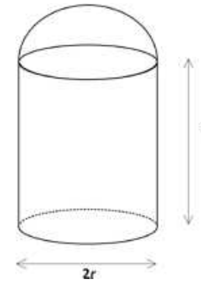
$A''(x) = 3000/x^3$, $A''(17.3205) > 0$. ✓

Hence, A has a minimum
when width = 17.32 + 3 = 20.32 cm
and length = 28.87 + 5 = 33.87 cm. ✓
Minimum value for A = 20.32 × 33.87 ≈ 688 cm^2 ✓

Calculator Assumed

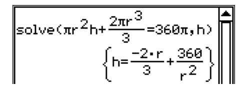
9. [10 marks: 3, 3, 4] [TISC]

A container consists of a cylinder of height h cm and base radius r cm with a hemispherical cap sitting on top of the cylinder. The container has a fixed volume of 360π cm^3.



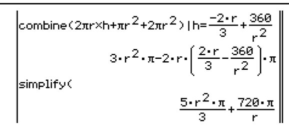
(a) Show that $h = \frac{360}{r^2} - \frac{2r}{3}$.

Volume $= \pi r^2 h + \frac{1}{2} \times \frac{4\pi r^3}{3}$ ✓
$= \pi r^2 h + \frac{2\pi r^3}{3}$
But Volume = 360π.
Hence: $\pi r^2 h + \frac{2\pi r^3}{3} = 360\pi$ ✓✓
$h = \frac{360}{r^2} - \frac{2r}{3}$



(b) Show that the total external surface area of the container is given by
$$S = \frac{5\pi r^2}{3} + \frac{720\pi}{r}.$$

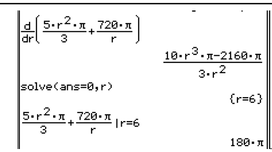
$S = 2\pi rh + \pi r^2 + \frac{1}{2} \times 4\pi r^2$ ✓
$= 3\pi r^2 + 2\pi r \times \left(\frac{360}{r^2} - \frac{2r}{3}\right)$ ✓✓
$= \frac{5\pi r^2}{3} + \frac{720\pi}{r}$.



(c) Use a calculus method to determine the minimum surface area of the container. Give your answer in exact form.

$S = \frac{5\pi r^2}{3} + \frac{720\pi}{r}$
$\frac{ds}{dr} = \frac{10\pi r}{3} - \frac{720\pi}{r^2}$ ✓
$\frac{ds}{dr} = 0 \Rightarrow r = 6$ ✓✓

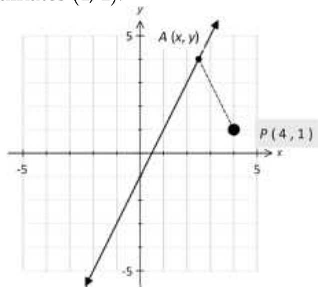
Hence: $S_{min} = 180\pi$ ✓



Calculator Assumed

10. [8 marks: 2, 6] [TISC]

The point A (x, y) lies on the line with equation $y = 2x - 1$.
The point P has coordinates (4, 1).



(a) Show that the distance between A and P is given by $s = \sqrt{5x^2 - 16x + 20}$.

Distance $s = \sqrt{(x-4)^2 + (y-1)^2}$ ✓
But $y = 2x - 1$.
Hence $s = \sqrt{(x-4)^2 + (2x-1-1)^2}$ ✓
$= \sqrt{5x^2 - 16x + 20}$

(b) Use a calculus/algebraic method to find the shortest distance and the longest distance between the point P and the line $y = 2x - 1$ for $1 \le x \le 5$.

Distance $s = \sqrt{5x^2 - 16x + 20}$

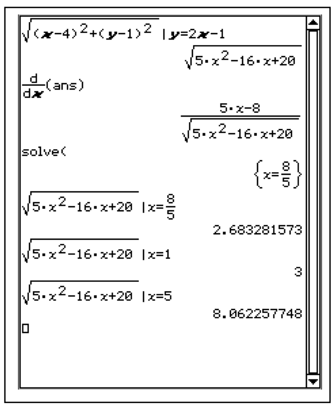
$\frac{ds}{dt} = \frac{5x - 8}{\sqrt{5x^2 - 16x + 20}}$

$\frac{ds}{dt} = 0 \Rightarrow x = \frac{8}{5}$

$s\left(\frac{8}{5}\right) = 2.68$

$s(1) = 3$
$s(5) = 8.06$
Hence, shortest distance = 2.68
longest distance = 8.06

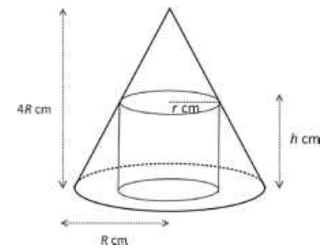
- Obtain derivative ✓
- Set derivative = 0 ✓
- Relative extremum ✓
- Test end points ✓
- Correct answers ✓✓



Calculator Assumed

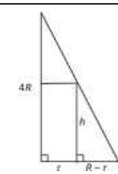
11. [7 marks: 3, 4]

The accompanying diagram shows a closed cylinder of base radius r cm and height h cm trapped within a regular cone of base radius R cm and height $4R$ cm. One flat end of the cylinder is in full contact with the base of the cone and the edge of the other flat end of the cylinder is in full contact with the inner surface of the cone.



(a) Show that $h = 4(R - r)$. Hence, show that the total surface area of the closed cylinder is given by $S = 8\pi Rr - 6\pi r^2$.

Using similar triangles:
$\frac{h}{R-r} = \frac{4R}{R}$ ✓✓
$h = 4(R - r)$
$S = 2\pi rh + 2\pi r^2$
$= 2\pi r \times 4(R - r) + 2\pi r^2$ ✓
$= 8\pi Rr - 6\pi r^2$



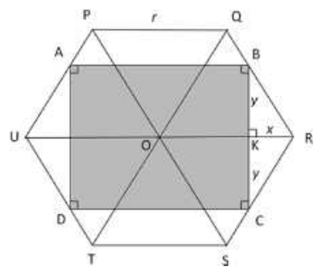
(b) Use a calculus method to determine the optimal total surface area of the closed cylinder in terms of R. Verify that this optimal value is a maximum value.

$S = 8\pi Rr - 6\pi r^2$
$\frac{dS}{dr} = 8\pi R - 12\pi r$ ✓
$\frac{dS}{dr} = 0 \Rightarrow r = \frac{2R}{3}$ ✓
$\frac{d^2S}{dr^2} = -12\pi < 0$
Hence, S is maximised when $r = \frac{2R}{3}$. ✓
$S_{max} = \frac{8\pi R^2}{3}$ ✓

Calculator Assumed

12. [6 marks: 2, 4] [TISC]

The accompanying diagram shows a regular hexagon PQRSTU with side length r cm. The centre of the hexagon is located at the point O. ABCD is a rectangle. The vertices A, B, C and D lie on the sides of the hexagon.
BC meets UR at right angles at K.
Let KR = x cm and BC = $2y$ cm.



(a) Show that $y = x\sqrt{3}$.

In ΔBRK: ∠BRK = 60° ✓
Hence: $y = x \tan 60°$ ✓
$= x\sqrt{3}$

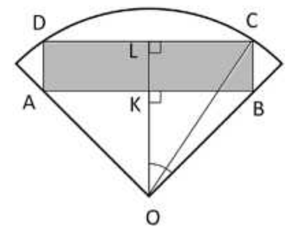
(b) Use a calculus method to calculate in terms of r, the dimensions of rectangle ABCD so that it has a maximum area.

Length of rectangle = $2r - 2x$
Height of rectangle = $2y = 2x\sqrt{3}$
.Area of ABCD = $2x\sqrt{3}(2r-2x)$ ✓
$= (4r\sqrt{3})x - (4\sqrt{3})x^2$
$\frac{d(Area)}{dy} = 4r\sqrt{3} - (8\sqrt{3})x$ ✓
$\frac{d(Area)}{dy} = 0 \Rightarrow x = \frac{r}{2}$ ✓
$\frac{d^2(Area)}{dy^2} = -8\sqrt{3} < 0$
Hence, Area is maximised when:
Length = r cm
Height = $\frac{2r\sqrt{3}}{3}$ cm ✓

Calculator Assumed

13. [8 marks: 2, 2, 4] [TISC]

The accompanying diagram shows a sector of a circle with radius 10 cm and with ∠AOB = 90°. ABCD is a rectangle with the vertices touching the edges of the sector. K is the foot of the perpendicular from O to AB. OK bisects ∠AOB. L is the foot of the perpendicular from O to CD. Let AB = $2x$ cm and AD = y cm.



(a) Determine in terms of x, the length of OK.

In ΔOKB: ∠BOK = 45° and KB = x. ✓
Hence: OK = x ✓

(b) Use ΔOLC to show that $y = \sqrt{100 - x^2} - x$.

In ΔOLC: $OL^2 + LC^2 = 10^2$ ✓
$(x + y)^2 + x^2 = 100$ ✓
Using CAS: $y = \sqrt{100 - x^2} - x$

```
solve((x+y)^2+x^2=100,y
  {y=-x-√(-x^2+100) ,y=-x+√(-x^2+100)}
```

(c) Use a calculus method to determine the value for x for which the area of rectangle ABCD is a maximum.
[You are not required to verify that a maximum value has been obtained.]

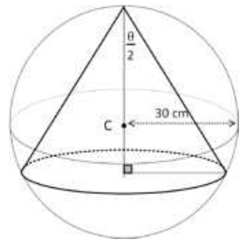
Area $A = 2x \times \sqrt{100-x^2} - x$ ✓
$\frac{dA}{dx} = \frac{-\left(4x^2-200+4x\sqrt{100-x^2}\right)}{\sqrt{100-x^2}}$ ✓
$\frac{dA}{dx} = 0 \Rightarrow x = 3.83$ cm ✓✓

```
d/dx(2x(√(100-x^2) -x))
        -(4·x^2+4·x·√(-x^2+100) -200)
        ------------------------------
                √(-x^2+100)
solve(
    {x=3.826834324,x=-9.238795325}
```

Calculator Assumed

14. [10 marks: 5, 1, 4]

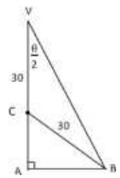
The accompanying diagram shows a right circular cone of semi-vertical angle $\frac{\theta}{2}$ inscribed within a sphere with centre C of radius 30cm. The vertex of the cone is in contact with the inner surface of the sphere. The circular edge of the base of the cone is in full contact with the inner surface of the sphere.



(a) Show that volume of the cone is given by $V = 9000\pi(\sin^2\theta)(1+\cos\theta)$.

Diagram. ✓
$\angle ACB = 2\theta$ ✓
$AC = 30\cos\theta$
$AB = 30\sin\theta$ ✓

Height of cone = $30(1 + \cos\theta)$
Base radius of cone = $30\sin\theta$ ✓
Hence:
$V = \frac{1}{3}\pi(30\sin\theta)^2 30(1+\cos\theta)$
$= 9000\pi(\sin^2\theta)(1+\cos\theta)$ ✓



(b) Use an appropriate trigonometric identity to show that V can be rewritten as $V = 9000\pi(1+\cos\theta-\cos^2\theta-\cos^3\theta)$.

$V = 9000\pi(1-\cos^2\theta)(1+\cos\theta)$ ✓
$= 9000\pi(1+\cos\theta-\cos^2\theta-\cos^3\theta)$

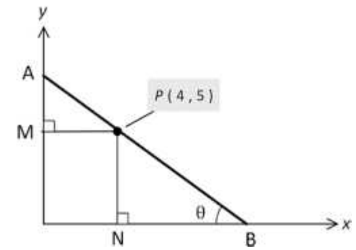
(c) Use a calculus method to determine the maximum volume of the cone.

$V = 9000\pi(1+\cos\theta-\cos^2\theta-\cos^3\theta)$
$\frac{dV}{d\theta} = 9000\pi(-\sin\theta+2\cos\theta\sin\theta+3\cos^2\theta\sin\theta)$ ✓
$\frac{dV}{d\theta} = 0 \Rightarrow 9000\pi\sin\theta(-1+2\cos\theta+3\cos^2\theta) = 0$ ✓
$9000\pi\sin\theta(3\cos\theta-1)(\cos\theta+1) = 0$
But $0 < \theta < \frac{\pi}{2}$: $\theta = \cos^{-1}\left(\frac{1}{3}\right)$ ✓
Hence, $V_{max} = 9000\pi(1+\frac{1}{3}-\frac{1}{9}-\frac{1}{27})$
$= \frac{32000\pi}{3} \approx 33510.3$ ✓

Calculator Assumed

15. [7 marks]

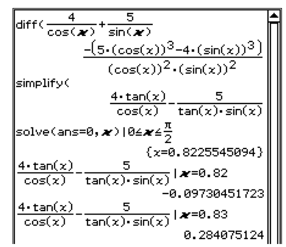
A line passes through the point P with coordinates (4, 5).
The line intersects the positive y-axis at A and the positive x-axis at B.
Let the acute angle between the line segment AB and the positive x-axis be θ.
Also, let L be the length of the line segment AB.



Use a calculus method to determine the minimum value for L.

In ΔPAM: $PA = \frac{PM}{\cos\theta} = \frac{4}{\cos\theta}$ ✓
In ΔPNB: $PB = \frac{PN}{\sin\theta} = \frac{5}{\sin\theta}$ ✓
Hence: $L = \frac{4}{\cos\theta} + \frac{5}{\sin\theta}$. ✓
$\frac{dL}{d\theta} = \frac{4\tan\theta}{\cos\theta} - \frac{5}{\tan\theta\sin\theta}$ ✓
$\frac{dL}{d\theta} = 0 \Rightarrow \theta = 0.8226$ ✓
$\left.\frac{dL}{d\theta}\right|_{\theta=0.8226^-} < 0$ and $\left.\frac{dL}{d\theta}\right|_{\theta=0.8226^+} > 0$ ✓

Hence L is minimum when $\theta = 0.8226$
Minimum value of $L = 12.7016$ ✓



15 Incremental Change, Rates & Marginal Rates

Calculator Assumed

1. [3 marks]

Consider $y = \dfrac{x}{1+e^x}$.

Use the method of incremental change to find the approximate change in y (4 significant places) when x changes from 1.00 to 0.99.

$\dfrac{dy}{dx} = \dfrac{1+e^x - xe^x}{(1+e^x)^2}$ ✓

$\delta y \approx \dfrac{1+e^x - xe^x}{(1+e^x)^2} \times \delta x$ ✓

$x = 1, \delta x = -0.01 \Rightarrow \delta y \approx -0.000\,7233.$ ✓

---

2. [7 marks: 4, 3]

A curve has equation $y = 2x^3 + 3x^2 - 12x$.

(a) Use the method of small increments to estimate the change in y when x changes from 2.00 to 1.99.

$\dfrac{dy}{dx} = 6x^2 + 6x - 12$ ✓

$\delta y \approx (6x^2 + 6x - 12) \times \delta x$ ✓

When x changes from 2.00 to 1.99, $\delta x = -0.01$. ✓

$\delta y \approx 24 \times -0.01 \approx -0.24$ ✓

(b) Use your answer in (a) to estimate the value of y when x changes from 2.00 to 2.01.

When x changes from 2.00 to 2.01, $\delta x = 0.01$.

$\delta y \approx 24 \times 0.01 \approx 0.24$ ✓

When $x = 2$, $y = 4$.

When $x = 2.01$:

$y = 4 + \delta y$ ✓

$\approx 4 + 0.24 \approx 4.24$ ✓

Calculator Assumed

3. [4 marks]

The quantity of a substance Q (g) is related to time t (minutes) by the formula $Q = 60t(t+1)$. Use the method of small changes to estimate the value of Q when t changes from 1 minute by 1 second.

$\dfrac{dQ}{dt} = 120t + 60$ ✓

$\delta Q \approx \dfrac{dQ}{dt} \times \delta t \Rightarrow \delta Q \approx (120t + 60) \times \delta t$ ✓

$t = 1, \delta t = \pm\dfrac{1}{60}$: $\delta Q \approx 180 \times \pm\dfrac{1}{60} \approx \pm 3$ ✓

$Q(1) = 120.$

Hence, $Q(1 \pm \frac{1}{60}) \approx 120 \pm 3 \approx 117$ or 123 ✓

---

4. [8 marks: 5, 3]

(a) Use the method of small changes to find the approximate change in the radius of a spherical balloon corresponding to a change in its volume from 500 cm³ to 499 cm³.

Volume $V = \dfrac{4}{3}\pi r^3$.

$\dfrac{dV}{dr} = 4\pi r^2$. ✓

$\delta V \approx 4\pi r^2\, \delta r$. ✓

When $V = 500$, $\delta V = -1$, $r = \sqrt[3]{\dfrac{375}{\pi}}$. ✓

Hence, $-1 = 4\pi\left(\sqrt[3]{\dfrac{375}{\pi}}\right)^2 \times \delta r$ ✓

$\Rightarrow \delta r \approx -0.0033$ cm. ✓

(b) Use your answer in (a) to find the approximate change in the surface area of a spherical balloon corresponding to a change in its volume from 500 cm³ to 499 cm³.

Surface area $A = 4\pi r^2$.

$\dfrac{dA}{dr} = 8\pi r$. ✓

$\delta A \approx 8\pi r\, \delta r$. ✓

$r = \sqrt[3]{\dfrac{375}{\pi}}$ and $\delta r = -0.0033$

$\Rightarrow \delta A \approx -0.41$ cm^2 ✓

Calculator Assumed

5. [5 marks]

Given that $y = \frac{1}{\sqrt{t}}$, use the method of small increments to find the percentage change in y corresponding to a 1% increase in t.

$\frac{dy}{dt} = \frac{-1}{2t^{\frac{3}{2}}}$ ✓

$\delta y \approx \frac{dy}{dt} \times \delta t \;\Rightarrow\; \delta y \approx \frac{-1}{2t^{\frac{3}{2}}} \times \delta t$ ✓

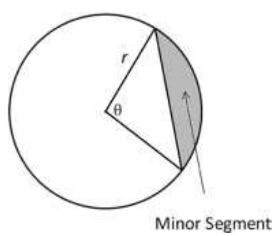
$\delta t = 0.01t \qquad \delta y \approx \frac{-0.005}{t^{\frac{1}{2}}}$ ✓

$\approx -0.005\,y$ ✓

Hence, y decreases by 0.5%. ✓

6. [5 marks]

The area of a minor segment associated with a sector of a circle with radius r and with a fixed sector angle of θ radians is given by $A = \frac{1}{2}r^2(\theta - \sin\theta)$. Use the method of small increments to calculate the approximate percentage change in the radius of the sector corresponding to a 1% decrease in the area of the minor segment.



$\frac{dA}{dr} = r(\theta - \sin\theta)$ ✓

$\delta A \approx r(\theta - \sin\theta) \times \delta r$ ✓

$\frac{\delta A}{A} \approx \frac{r(\theta-\sin\theta)\delta r}{\left[\frac{1}{2}r^2(\theta-\sin\theta)\right]}$ ✓

$\frac{\delta A}{A} \approx \frac{2\,\delta r}{r}$ ✓

But $\frac{\delta A}{A} = -0.01$: $\Rightarrow \frac{\delta r}{r} \approx -0.005$

Hence, the radius decreases by 0.5%. ✓

Calculator Assumed

7. [4 marks] [TISC]

Let $A(t) = \int_0^t \sqrt{16+x^2}\,dx$. Use the incremental formula to calculate the approximate change in A corresponding to a change in t from 3.00 to 3.01.

$\frac{dA}{dt} = \sqrt{16+t^2}$ ✓

$\delta A \approx \sqrt{16+t^2} \times \delta t$ ✓

When t changes from 3.00 to 3.01:

$\delta A \approx \sqrt{16+9} \times 0.01$ ✓

≈ 0.05 ✓

8. [6 marks: 3, 3]

Let $A = \frac{1}{\sqrt{x+1}}$ where $x = f(t) \geq 0$ for time $t \geq 0$.

(a) Use the incremental method (method of small changes) to find the approximate change in A when x changes from 3 to 2.99.

$\frac{dA}{dx} = \frac{-1}{2(x+1)^{\frac{3}{2}}}$ ✓

$\delta A \approx \frac{-1}{2(x+1)^{\frac{3}{2}}} \times \delta x$ ✓

$\approx (-\frac{1}{16}) \times (-0.01) \approx 0.000\,625$ ✓

(b) Use the chain rule to determine $\frac{dA}{dt}$. Hence, calculate the rate of change of A, when $x = 3$, given that when $x = 3$, $\frac{dx}{dt} = 4$.

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} \;\Rightarrow\; \frac{dA}{dt} = \frac{-1}{2(x+1)^{\frac{3}{2}}} \times \frac{dx}{dt}$ ✓

When $x = 3$, $\frac{dx}{dt} = 4$.

Hence: $\frac{dA}{dt} = (-\frac{1}{16}) \times 4 = -\frac{1}{4}$ ✓✓

Calculator Assumed

9. [10 marks: 3, 4, 3]

A circle has radius r metres.

(a) The circumference of the circle is increasing at a rate of π metres per hour.

(i) Use the chain rule to find the rate with which the radius of the circle is changing when the radius of the circle is 10 metres.

Circumference of circle $P = 2\pi r$

$\frac{dP}{dt} = \frac{dP}{dr} \times \frac{dr}{dt} \Rightarrow \frac{dP}{dt} = 2\pi \frac{dr}{dt}$ ✓

But $\frac{dP}{dt} = \pi$ $\quad \pi = 2\pi \frac{dr}{dt}$ ✓

$\frac{dr}{dt} = \frac{1}{2}$ metres per hour. ✓

(ii) Find the rate with which the area of the circle is changing when the radius of the circle is 10 metres.

Area of circle $A = \pi r^2$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ ✓

From (a), when $r = 10$, $\frac{dr}{dt} = \frac{1}{2}$ ✓

Hence: $\frac{dA}{dt} = 2\pi \times 10 \times \frac{1}{2}$ ✓

$= 10\pi$ metres^2 per hour. ✓

(b) Use the incremental method (method of small changes) to find the approximate change in the area of the circle when the radius of the circle changed from 10 metres to 10.1 metres.

Area of circle $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$ ✓

$\delta A = 2\pi r\, \delta r$ ✓

$= 2\pi \times 10 \times 0.1$

$= 2\pi$ metres^2 ✓

Calculator Assumed

10. [10 marks: 2, 1, 2, 2, 3]

Mathcom sells each "Template X" for \$30. The cost of producing x items is given by $C(x) = \frac{80x}{(x+1)} + 0.04x^2 + 500$.

(a) Find an expression for the profit $P(x)$ corresponding to the manufacture and sale of x items.

Profit = Revenue – Cost

$= 30x - [\frac{80x}{(x+1)} + 0.04x^2 + 500]$ ✓

$= -0.04x^2 + 30x - \frac{80x}{(x+1)} - 500$ ✓

(b) Find an expression $P'(x)$.

$P'(x) = -0.08x + 30 - \frac{80}{(x+1)^2}$ ✓

(c) Find $P'(100)$. Interpret this value.

$P'(100) = \$21.99$ ✓

This is the profit associated with the sale of the 101st item. ✓

(d) Find the average profit per item associated with the manufacture and sale of 100 items.

Average profit $= \frac{P(100)}{100}$ ✓

$= \frac{2\,020.79}{100} = \$20.21$ ✓

(e) Find how many items were manufactured and sold if the profit associated with the sale of the next item is approximately \$20.

$-0.08x + 30 - \frac{80}{(x+1)^2} = 20$ ✓

$x = -3.8$ (reject), 1.8, and 124.9 ✓

Reject $x = 1.8$ as P(1) < 0 and P(2) < 0.

P(125) – P(124) ≈ \$20.03

Hence, 125 items. ✓

Calculator Assumed

11. [10 marks: 4, 3, 3]

The revenue \$$R$ million from the sales of x units of a product is given by $R = 100 - x - \dfrac{2000}{x+5}$.

(a) Use the incremental method to determine the approximate change in revenue when the number of units sold changes from 80 to 82.

$\dfrac{dR}{dx} = -1 + \dfrac{2000}{(x+5)^2}$ ✓

$\delta R \approx \dfrac{dR}{dx} \times \delta x \Rightarrow \delta R \approx -1 + \dfrac{2000}{(x+5)^2} \times \delta x$ ✓

$\delta x = 2$ $\quad \delta R \approx \left(-1 + \dfrac{2000}{(80+5)^2}\right) \times 2$

≈ -1.446366782 ✓

Hence, revenue decreases by \$1 446 366.78. ✓

(b) Use the incremental method to determine an expression for marginal revenue when x items are sold.

$\delta R \approx \left(-1 + \dfrac{2000}{(x+5)^2}\right) \times \delta x$ ✓

For marginal revenue, $\delta x = 1$ ✓

Marginal Revenue $\approx \left(-1 + \dfrac{2000}{(x+5)^2}\right)$ ✓

(c) Determine an expression for the average revenue when x items are sold. Hence, determine the value of x when the marginal revenue and average revenue are equal.

Average revenue $= \dfrac{\left(100 - x - \dfrac{2000}{x+5}\right)}{x}$ ✓

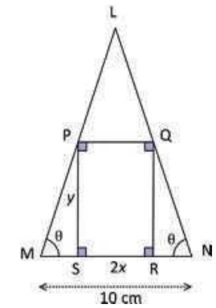
$\dfrac{\left(100 - x - \dfrac{2000}{x+5}\right)}{x} = \left(-1 + \dfrac{2000}{(x+5)^2}\right)$ ✓

$x = 32.3 \approx 32$ ✓

Calculator Assumed

12. [12 marks: 3, 3, 3, 3] [TISC]

The accompanying diagram shows an isosceles triangle LMN with ∠LMN = ∠LNM = θ. MN = 10 cm and tan θ = 3. Points P and Q lie on LM and LN respectively. Points S and R lie on MN. PQRS form a rectangle with PS = QR = y cm and PQ = SR = $2x$ cm. Let the area of rectangle PQRS be A cm².



(a) Show that $A = 2x(15 - 3x)$ cm².

Area $A = 2xy$ ✓

In ΔPMS: $\tan\theta = \dfrac{y}{5-x}$ ✓

But $\tan\theta = 3 \Rightarrow y = 15 - 3x$. ✓

Hence: $A = 2x(15 - 3x)$

(b) Use the method of small increments to determine the increment in A when x increases from 0.5 cm to 0.6 cm.

$\dfrac{dA}{dx} = 30 - 12x$ ✓

$\delta A \approx (30 - 12x) \times \delta x$ ✓

$\delta A|_{x=0.5} \approx (30 - 6) \times 0.1 \approx 2.4 \text{ cm}^2$ ✓

(c) Given that x increases at a constant rate of 0.1 cm per minute, use the chain rule to calculate the rate of increase in A when $x = 1$ cm.

$\dfrac{dA}{dt} = \dfrac{dA}{dx} \times \dfrac{dx}{dt} \Rightarrow \dfrac{dA}{dt} = (30 - 12x) \times \dfrac{dx}{dt}$ ✓✓

Hence: $\left.\dfrac{dA}{dt}\right|_{x=1} = 1.8 \text{ cm}^2/\text{minute}$ ✓

(d) Calculate the maximum possible area of rectangle PQRS.

$\dfrac{dA}{dx} = 30 - 12x = 0$ ✓ $\Rightarrow x = 2.5$ ✓ $\dfrac{d^2A}{dx^2} = -12 < 0$ Hence, Max $A = 37.5 \text{ cm}^2$. ✓	$A = 2x(15 - 3x)$ Graph is a maximum parabola. ✓ LOS $x = 2.5$ ✓ Hence, Max $A = 37.5 \text{ cm}^2$. ✓

16 Exponential Growth & Decay

Calculator Assumed

1. [12 marks: 1, 1, 2, 3, 5]

The amount (A g) of radioactive substance R remaining after t days is given by $A = 800\,e^{-0.04t}$.

(a) What is the initial amount of substance present?

800 g ✓

(b) How much is left after 7 days?

$A(7) = 800\,e^{-0.04(7)} = 604.6$ g ✓

(c) How much has decayed after 14 days?

$A(14) = 800\,e^{-0.04(14)} = 456.97$ ✓
Hence, amount left = 800 − 456.97 = 343.0 g ✓

(d) Find the half-life of this radioactive substance.

When $A = 400$, $800\,e^{-0.04t} = 400$ ✓✓
$e^{-0.04t} = 0.5$
$t = 17.33$ ✓
Hence, the half-life is 17.33 days.

(e) Find the rate of decay when 100 g of the substance is left.

Rate of decay $\frac{dA}{dt} = -32\,e^{-0.04t}$. ✓

When $A = 100$, $800\,e^{-0.04t} = 100$ ✓
$e^{-0.04t} = \frac{1}{8}$ ✓

Hence, when $A = 100$, $\frac{dA}{dt} = -32 \times \frac{1}{8}$ ✓
$= -4$ g/day ✓

Calculator Assumed

2. [13 marks: 4, 3, 3, 3]

The instantaneous rate of population growth is proportional to its population size. The population grew from an initial 10 000 to 15 000 in 8 years.

(a) Find an expression for the population size at time t years.

$\frac{dP}{dt} = kP \Rightarrow P = 10\,000\,e^{kt}$ ✓
When $t = 8$, $P = 15\,000$.
$\Rightarrow 10\,000\,e^{8k} = 15\,000$ ✓
$k = 0.05068$ ✓

Hence, $P = 10\,000\,e^{0.05068t}$ ✓

(b) Find doubling time, the time taken for the population to double its size.

When $P = 20\,000$,
$10\,000\,e^{0.05068t} = 20\,000$ ✓✓
$t = 13.68$

Hence, the doubling-time is 13.68 years. ✓

(c) Find the instantaneous rate of population growth when $t = 8$.

Rate of decay $\frac{dP}{dt} = 506.83\,e^{0.05068t}$. ✓
When $t = 8$, $\frac{dP}{dt} = 506.83\,e^{0.05068(8)}$
≈ 760 persons per year ✓✓

(d) Find when the instantaneous rate of population growth is 1000 persons/year.

When $\frac{dP}{dt} = 1000$,
$506.83\,e^{0.05068t} = 1000$ ✓✓
$t = 13.4$ years ✓

Calculator Assumed

3. [5 marks]

A radioactive substance has a half-life of 50 days. After 20 days, only 30g were left. Assume that the radioactive substance decays exponentially.
Find the initial amount of substance.

> Let $A = A_0 e^{kt}$
> When $t = 50$, $A = 0.5A_0$. ✓
> $\Rightarrow A_0 e^{50k} = 0.5 A_0$
> $e^{50k} = 0.5$ ✓
> $k = -0.01386$ ✓
>
> Hence $A = A_0 e^{-0.01386t}$
>
> When $t = 20$, $A = 30$.
> $\Rightarrow A_0 e^{-0.01386(20)} = 30$ ✓
> $A_0 = 39.6$ g ✓

4. [5 marks] [TISC]

A batch of cattle grain is found to be contaminated with radioactive substance X. The radioactive substance X decays exponentially with a half-life of ten days. The amount of radioactive substance X was found to be ten times the maximum permitted level. How long should be grain be stored before it is fed to the cattle? Justify your answer.

> Let $A(t)$: Amount of substance X left after t days.
> Then, $A(t) = A(0)\, e^{kt}$
> Half-life of 10 days $\Rightarrow e^{10k} = 0.5$ ✓
> $k = -0.069315$ ✓
>
> Let maximum permitted level be M.
> $\Rightarrow A(0) = 10M$ ✓
> $10M e^{-0.069315t} < M$ ✓
> $t > 33.2$ days
> Hence, at least 34 days. ✓

Calculator Assumed

5. [8 marks: 2, 3, 3]

The mass (M mg) of a slow growing tumour grows exponentially according to the formula $M = A e^{kt}$ where t is time in weeks. The mass of the tumour doubles every 100 weeks.

(a) Find k to four significant figures.

> $2 = e^{100k}$ ✓
> $k = 0.006931$ ✓

(b) A 25 year old patient was diagnosed with a similar tumour. The mass of the tumour was estimated to be 120 mg. The tumour is operable only when it reaches a mass of 200 mg.

(i) When will the tumour be operable?

> $200 = 120 e^{0.006931t}$ ✓✓
> $t = 73.7$ weeks
> That is after another ≈ 74 weeks ✓

(ii) Estimate how long the tumour has been growing in the patient's body. Explain clearly how you arrived at your answer.

> For $t = 25$ years = 1300 weeks:
> $120 = A e^{0.006931(1300)}$ ✓
> $A \approx 0.015$ mg ✓
> Hence, the patient was probably born with it! ✓

6. [7 marks: 3, 2, 2] [TISC]

50 L of a chemical is accidentally spilled into a swimming pool. The amount of chemical (in L) left in the pool t days after the spill is given by $A = A_0 e^{kt}$.

(a) Determine the value of k (to four significant figures) if 80% of the chemical disappears after 10 days.

> $A_0 = 50$ litres ✓
> When $t = 10$, $A = 50 \times 0.2 = 10$
> $10 = 50 e^{10k}$ ✓
> $k = -0.1609438$
> $= -0.1609$ ✓

Calculator Assumed

6. (b) Find the rate with which the amount of chemical left in the pool is changing when 80% of the chemical has disappeared.

> Safe when $1 \le A \le 5$.
> Hence, $1 \le 50\,e^{-0.1609t} \le 5$ ✓
> $\Rightarrow 14.3 \le t \le 24.3$ ✓

(c) The swimming pool is safe for use if the amount of chemical in the pool is between 1 litre and 5 litres inclusive. To one decimal place, for what value(s) of t will the pool be safe for use?

> $\frac{dA}{dt} = -0.1609A$ ✓
> When $A = 10$, $\frac{dA}{dt} = -0.1609 \times 10 = -1.6$ litres/day ✓

---

7. [7 marks: 2, 3, 2] [TISC]

The length of a vine (L cm) grows according to the formula $L = 10\,e^{0.03t}$ where $t \ge 10$ days. The vine is trimmed each time its length exceeds 5 metres. Each time it is trimmed to a length of approximately 2 metres. Assume that the vine grows at the same rate.

(a) Find when the vine is trimmed for the first time.

> $10\,e^{0.03t} = 500$ ✓
> $t = 130.4 \approx 130$ days ✓

(b) Find when the vine is trimmed for the second time.

> New formula $L = 200\,e^{0.03t}$ ✓
> $200\,e^{0.03t} = 500$ ✓
> $t = 30.5 \approx 31$ days ✓

(c) To what length (to the nearest cm) should the vine be trimmed so that it is trimmed once every 60 days.

> New formula $L = A\,e^{0.03t}$
> When $t = 60$, $L = 500$
> $A\,e^{0.03\times 60} = 500$ ✓
> $A = 82.6 \approx 83$ cm ✓

Calculator Assumed

8. [12 marks: 2, 4, 2, 2, 2]

Two heated objects, P and Q are placed one at each end of a long laboratory bench. The temperature (in degrees Celsius) of P and Q, t minutes after being placed on the bench top are given by $\theta_P = 18 + 75\,e^{-0.09t}$ and $\theta_Q = 18 + 60\,e^{-0.01t}$ respectively.

(a) Find when the two objects have the same temperature.

> $18 + 75\,e^{-0.09t} = 18 + 60\,e^{-0.01t}$ ✓
> $t = 2.79$ minutes ✓

(b) Find when the two objects are losing heat at the same rate.

> For P: $\frac{d\theta}{dt} = -6.75\,e^{-0.09t}$ ✓
> For Q: $\frac{d\theta}{dt} = -0.6\,e^{-0.01t}$ ✓
> When $-6.75\,e^{-0.09t} = -0.6\,e^{-0.01t}$ ✓
> $t = 30.25$ minutes ✓

(c) Find the time taken for the temperature of each object to reach 25 C.

> For P: $25 = 18 + 75\,e^{-0.09t} \Rightarrow t = 26.4$ minutes ✓
> For Q: $25 = 18 + 60\,e^{-0.01t} \Rightarrow t = 214.8$ minutes ✓

(d) Which object is losing heat at a faster rate? Justify your answer.

> P is losing heat at a faster rate as it reaches 25° C more quickly than Q. ✓✓

(e) Find the temperature of each body for large values of t.

> For P: As $t \to \infty$, its temperature approaches 18°C. ✓
> For Q: As $t \to \infty$, its temperature approaches 18°C. ✓

Calculator Assumed

9. [8 marks: 2, 1, 2, 3]

The instantaneous rate of population growth of a colony of bacteria at any time t is proportional to N, the number of bacteria at time t. The initial number of bacteria is 100 and its initial instantaneous growth rate is 2 bacteria per hour.

(a) Explain why the continuous percentage growth rate is 2% per hour.

$$\frac{dN}{dt} = kN \quad ✓$$
$$\Rightarrow 2 = k \times 100 \quad ✓$$
$$k = 0.02$$

(b) State the mathematical expression for N, in terms of time t (hours).

$$N = 100\,e^{0.02t} \quad ✓$$

(c) Determine the average growth rate in the first 240 hours.

$N(240) = 12\,151$ ✓

$$\text{Average growth rate} = \frac{12\,151 - 100}{240}$$
$= 50.2$ bacterium/hour ✓

(d) Determine the rate of change of the rate of population growth at $t = 240$ hours.

$$N = 100\,e^{0.02t}$$
$$\frac{dN}{dt} = 2\,e^{0.02t} \quad ✓$$
$$\frac{d^2N}{dt^2} = 0.04\,e^{0.02t} \quad ✓$$
$$\left.\frac{d^2N}{dt^2}\right|_{t=240} = 4.9 \text{ bacteria/hr/hr.} \quad ✓$$

Calculator Assumed

10. [8 marks: 2, 3, 3]

N, the number of bacteria in a culture is modelled by the equation $N = N_0\,e^{kt}$, where t is the number of hours after it was cultivated.
The number of bacteria grew by 30% in the first twelve hours.

(a) Find k to four significant figures.

$$1.3 = e^{12k} \quad ✓$$
$$k = 0.021863 \approx 0.02186 \quad ✓$$

The average rate of growth in the bacterial numbers in the first 10 hours is 0.8 bacteria per hour.

(b) Calculate the value of N_0.

Total growth in 10 hours is $10 \times 0.8 = 8$.
$\Rightarrow$ when $t = 10$, $N = N_0 + 8$. ✓

$$N_0 + 8 = N_0\,e^{0.02186(10)} \quad ✓$$
$$N_0 \approx 33 \quad \text{(Accept 32)} \quad ✓$$

(c) When the number of bacteria reaches 100 000, an anti-bacterial drug is introduced. The doubling time for the bacterial growth is now 69 hours. Determine with reasons if the drug has been effective in reducing the growth in the number of bacteria.

Let W: No. of viruses after drug is used.

$$W = 100\,000\,e^{mt}$$
$$200\,000 = 100\,000\,e^{69m} \quad ✓$$
$$m = 0.0100$$

Hence, new growth rate is 1% ✓
which is less the original growth rate of 2.2%.
Hence, Yes. ✓

Calculator Assumed

11. [10 marks: 2, 3, 3, 2]

P, the number of people infected by a certain virus at the end of day t is modelled by $P = e^{0.6t}$, (t is time in days).

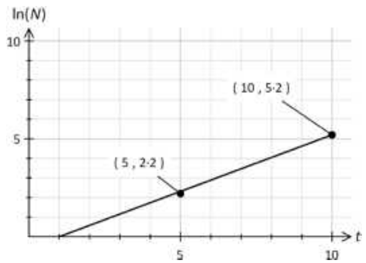
(a) Calculate the minimum amount of time required for the entire world's population of 7.7 billion to be infected. (1 billion = 1 000 million)

$7.7 \times 10^9 = e^{0.6t}$ ✓
$t = 37.9$ days ✓

(b) Calculate the total number of people infected by the end of the 20th day. Hence, calculate the number of new people infected on the 20th day.

$P(20) = 162\,754.8 \approx 162\,754$ ✓
$P(19) = 89\,321.7 \approx 89\,321$ ✓
Hence, newly infected on 20th day $\approx 73\,433$ ✓

Let N be the number of new people infected on day t. The accompanying diagram shows the graph of $ln\ N$ against t.



(c) Determine the algebraic relationship between N and t in the form $N = N_0 e^{kt}$

Equation of line is $ln\ N = 0.6t - 0.8$ ✓
$N = e^{0.6t-0.8}$ ✓
$= 0.45\,e^{0.6t}$ ✓

(d) Explain clearly why the number of newly infected people each day is always a fixed percentage of the total number of people infected by the end of the day. State this fixed percentage.

$N = 0.45\,e^{0.6t}$
$P = e^{0.6t}$
Hence, $N = 0.45P$ ✓
That is, the newly infected people each day is always 45% of the total number of people infected by the end of the day. ✓

17 Anti-Differentiation

Calculator Free

1. [13 marks: 2, 2, 2, 2, 2, 3]

Find the anti-derivative of each of the following:

(a) $\dfrac{3x^2}{4} + 4x + 5$

Anti-derivative = $\dfrac{x^3}{4} + 2x^2 + 5x + C$ ✓✓

(b) $\dfrac{1}{x^3} - \dfrac{2}{3x^2}$

Anti-derivative = $\dfrac{-1}{2x^2} + \dfrac{2}{3x} + C$ ✓✓

(c) $(1 + 4x)^5$

Anti-derivative = $\dfrac{(1+4x)^6}{24} + C$ ✓✓

(d) $\dfrac{1}{2\sqrt{x}}$

$\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}x^{-1/2}$ ✓
Anti-derivative = $\sqrt{x} + C$ ✓

(e) $\sqrt{x}(x^2 + 1)$

$\sqrt{x}(x^2+1) = x^{5/2} + x^{1/2}$ ✓
Anti-derivative = $\dfrac{2}{7}x^{7/2} + \dfrac{2}{3}x^{3/2} + C$ ✓

(f) $(2x - \dfrac{3}{x})^2$

$(2x - \dfrac{3}{x})^2 = 4x^2 - 12 + 9x^{-2}$ ✓
Anti-derivative = $\dfrac{4x^3}{3} - 12x - \dfrac{9}{x} + C$ ✓✓

Calculator Free

2. [17 marks: 3, 2, 3, 3, 3, 3]

Find:

(a) $\int \frac{3}{2}(\sqrt{x}+1)^2 \; dx$

$$\int \frac{3}{2}(\sqrt{x}+1)^2 \, dx = \frac{3}{2}\int x + 2x^{\frac{1}{2}} + 1 \; dx \quad ✓$$
$$= \frac{3}{2}\left(\frac{x^2}{2} + \frac{4x^{3/2}}{3} + x\right) + C \quad ✓✓$$

(b) $\int \frac{x^4+x^3}{x} dx$

$$\int \frac{x^4+x^3}{x} dx = \int x^3 + x^2 \, dx \quad ✓$$
$$= \frac{x^4}{4} + \frac{x^3}{3} + C \quad ✓$$

(c) $\int \frac{x+2x^3}{3x^5} dx$

$$\int \frac{x+2x^3}{3x^5} dx = \int \frac{x^{-4}}{3} + \frac{2x^{-2}}{3} dx \quad ✓$$
$$= \frac{-1}{9x^3} - \frac{2}{3x} + C \quad ✓✓$$

(d) $\int \sqrt{1-2x} \, dx$

$$\int \sqrt{1-2x} \; dx = \int (1-2x)^{\frac{1}{2}} \; dx \quad ✓$$
$$= \frac{-(1-2x)^{3/2}}{3} + C \quad ✓✓$$

(e) $\int \frac{4}{(3x-2)^2} dx$

$$\int \frac{4}{(3x-2)^2} dx = \int 4(3x-2)^{-2} \; dx \quad ✓$$
$$= \frac{-4}{3(3x-2)} + C \quad ✓✓$$

(f) $\int \frac{4}{3\sqrt{x-1}} dx$

$$\int \frac{4}{3\sqrt{x-1}} dx = \int \frac{4(x-1)^{-1/2}}{3} dx \quad ✓$$
$$= \frac{8(x-1)^{1/2}}{3} + C \quad ✓✓$$

Calculator Free

3. [12 marks: 2, 2, 2, 2, 2, 2]

Find:

(a) $\int 2x(1-x^2)^3 \; dx$

$$\int 2x(1-x^2)^3 \; dx = -\int -2x(1-x^2)^3 \; dx$$
$$= \frac{-(1-x^2)^4}{4} + C \quad ✓✓$$

(b) $\int \frac{x}{(1-x^2)^4} dx$

$$\int \frac{x}{(1-x^2)^4} dx = -\frac{1}{2}\int -2x(1-x^2)^{-4} \; dx$$
$$= \frac{1}{6(1-x^2)^3} + C \quad ✓✓$$

(c) $\int x\sqrt{1-2x^2} \; dx$

$$\int x\sqrt{1-2x^2} \; dx = \frac{-1}{4}\int -4x(1-2x^2)^{1/2} \; dx$$
$$= \frac{-(1-2x^2)^{3/2}}{6} + C \quad ✓✓$$

(d) $\int \frac{-5x}{\sqrt{1+x^2}} dx$

$$\int \frac{-5x}{\sqrt{1+x^2}} dx = -\frac{5}{2}\int 2x(1+x^2)^{-1/2} \; dx$$
$$= -5(1+x^2)^{1/2} + C \quad ✓✓$$

(e) $\int \frac{1}{x^2}\left(1-\frac{1}{x}\right)^3 dx$

$$\int \frac{1}{x^2}\left(1-\frac{1}{x}\right)^3 dx = \frac{1}{4}\left(1-\frac{1}{x}\right)^4 + C \quad ✓✓$$

(f) $\int \frac{1}{\sqrt{x}}\left(2+\sqrt{x}\right)^4 dx$

$$\int \frac{1}{\sqrt{x}}\left(2+\sqrt{x}\right)^4 dx = 2\int \frac{1}{2\sqrt{x}}\left(2+\sqrt{x}\right)^4 dx$$
$$= \frac{2\left(2+\sqrt{x}\right)^5}{5} + C \quad ✓✓$$

Calculator Free

4. [13 marks: 1, 2, 2, 2, 2, 2, 2]

Find:

(a) $\int e^{-3x}\ dx$

$$\int e^{-3x}\ dx = \frac{e^{-3x}}{-3} + C \quad ✓$$

(b) $\int \frac{1}{2e^{2x}}\ dx$

$$\int \frac{e^{-2x}}{2}\ dx = \frac{e^{-2x}}{-4} + C \quad ✓✓$$

(c) $\int (1+e^{2x})^2\ dx$

$$\int (1+e^{2x})^2\ dx = \int 1+2e^{2x}+e^{4x}\ dx \quad ✓$$
$$= x+e^{2x}+\frac{e^{4x}}{4} + C \quad ✓$$

(d) $\int x\,e^{-x^2}\ dx$

$$\int x\,e^{-x^2}\ dx = -\frac{1}{2}\int -2x\,e^{-x^2}\ dx \quad ✓$$
$$= -\frac{1}{2}e^{-x^2} + C \quad ✓$$

(e) $\int e^{2x}(1+e^x)\,dx$

$$\int e^{2x}(1+e^x)\ dx = \int e^{2x}+e^{3x}\ dx \quad ✓$$
$$= \frac{e^{2x}}{2} + \frac{e^{3x}}{3} + C \quad ✓$$

(f) $\int x^2\,e^{2x^3}\ dx$

$$\int x^2 e^{2x^3}\ dx = \frac{1}{6}\int 6x^2 e^{2x^3}\ dx \quad ✓$$
$$= \frac{1}{6}e^{2x^3} + C \quad ✓$$

(g) $\int e^{-x}(1+e^{-x})^3\ dx$

$$\int e^{-x}(1+e^{-x})^3\ dx = -\int -e^{-x}(1+e^{-x})^3\ dx \quad ✓$$
$$= \frac{-(1+e^{-x})^4}{4} + C \quad ✓$$

Calculator Free

5. [16 marks: 2, 2, 2, 2, 2, 3, 3]

Determine:

(a) $\int \cos 2x\ dx$

$$\int \cos 2x\ dx = \frac{\sin 2x}{2} + C \quad ✓✓$$

(b) $\int \sin\left(\frac{x}{2}\right)\ dx$

$$\int \sin\left(\frac{x}{2}\right)\ dx = -2\cos\left(\frac{x}{2}\right) + C \quad ✓✓$$

(c) $\int 3\cos(1-4x)\ dx$

$$\int 3\cos(1-4x)\ dx = \frac{3\sin(1-4x)}{-4} + C \quad ✓✓$$

(d) $\int x\sin(x^2)\ dx$

$$\int x\sin(x^2)\ dx = \frac{-\cos(x^2)}{2} + C \quad ✓✓$$

(e) $\int 3\cos x(\sin x)^4\ dx$

$$\int 3\cos x(\sin x)^4\ dx = \frac{3(\sin x)^5}{5} + C \quad ✓✓$$

(f) $\int \frac{2\sin x}{5(\cos x)^3}\ dx$

$$\int \frac{2\sin x}{5(\cos x)^3}\ dx = \frac{2}{5}\times\frac{(\cos x)^{-2}}{2} + C \quad ✓✓✓$$

(g) $\int -3\cos 2x\sqrt{1+\sin 2x}\ dx$

$$\int -3\cos 2x\sqrt{1+\sin 2x}\ dx = \frac{-3}{2}\times\frac{(1+\sin 2x)^{\frac{3}{2}}}{\frac{3}{2}} + C \quad ✓✓✓$$

Calculator Free

6. [14 marks: 2, 2, 3, 4, 3]

Determine:

(a) $\int \frac{-4}{3+2x}\ dx$

$$\int \frac{-4}{3+2x}\ dx = -2\int \frac{2}{3+2x}\ dx \quad ✓$$
$$= -2\ ln\ |3+2x| + C \quad ✓$$

(b) $\int \frac{5x}{1+3x^2}\ dx$

$$\int \frac{5x}{1+3x^2}\ dx = \frac{5}{6}\int \frac{6x}{1+3x^2}\ dx \quad ✓$$
$$= \frac{5}{6}\ ln\ |1+3x^2| + C \quad ✓$$

(c) $\int \left(1-\frac{1}{x}\right)^2\ dx$

$$\int \left(1-\frac{1}{x}\right)^2\ dx = \int 1-\frac{2}{x}+\frac{1}{x^2}\ dx \quad ✓$$
$$= x - 2\ ln\ |x| - \frac{1}{x} + C \quad ✓✓$$

(d) $\int \frac{(x+2)^2}{x}\ dx$

$$\int \frac{(x+2)^2}{x}\ dx = \int \frac{(x^2+4x+4)}{x}\ dx \quad ✓$$
$$= \int x+4+\frac{4}{x}\ dx \quad ✓$$
$$= \frac{x^2}{2} + 4x + 4\ ln\ |x| + C \quad ✓✓$$

(e) $\int \frac{1+x+x^2}{1+x^2}\ dx$

$$\int \frac{1+x+x^2}{1+x^2}\ dx = \int \frac{1+x^2+x}{1+x^2}\ dx$$
$$= \int 1+\frac{x}{1+x^2}\ dx \quad ✓$$
$$= x + \frac{1}{2}\ ln(1+x^2) \quad ✓✓$$

Calculator Free

7. [10 marks: 2, 3, 2, 3]

Determine:

(a) $\int \frac{5e^{-x}}{3+4e^{-x}}\ dx$

$$\int \frac{5e^{-x}}{3+4e^{-x}}\ dx = \frac{5}{-4}\int \frac{-4e^{-x}}{3+4e^{-x}}\ dx \quad ✓$$
$$= -\frac{5}{4}\ ln\ |3+4e^{-x}| + C \quad ✓$$

(b) $\int \tan 2x\ dx$

$$\int \tan 2x\ dx = \int \frac{\sin 2x}{\cos 2x}\ dx \quad ✓$$
$$= \frac{1}{-2}\int \frac{-2\sin 2x}{\cos 2x}\ dx \quad ✓$$
$$= -\frac{1}{2}\ ln\ |\cos 2x| + C \quad ✓$$

(c) $\int \frac{\cos \pi x}{1+\sin \pi x}\ dx$

$$\int \frac{\cos \pi x}{1+\sin \pi x}\ dx = \frac{1}{\pi}\int \frac{\pi \cos \pi x}{1+\sin \pi x}\ dx \quad ✓$$
$$= \frac{1}{\pi}\ ln\ |1+\sin \pi x| + C \quad ✓$$

(d) $\int \frac{4\sin 2x}{1-3\sin^2 x}\ dx$

$$\int \frac{4\sin 2x}{1-3\sin^2 x}\ dx = \int \frac{8\sin x\cos x}{1-3\sin^2 x}\ dx \quad ✓$$
$$= \frac{8}{-6}\int \frac{-6\sin x\cos x}{1-3\sin^2 x}\ dx \quad ✓$$
$$= -\frac{4}{3}\ ln\ |1-3\sin^2 x| + C \quad ✓$$

Calculator Free

8. [4 marks]

Find $f(x)$ if $f'(x) = 2x^2 - 3x + a$ where a is a constant and $f(0) = -2$ and $f(-1) = -4$.

$f(x) = \int 2x^2 - 3x + a \, dx$

$= \frac{2x^3}{3} - \frac{3x^2}{2} + ax + C$ ✓

$f(0) = -2 \Rightarrow -2 = C$ ✓

Hence, $f(x) = \frac{2x^3}{3} - \frac{3x^2}{2} + ax - 2$

$f(-1) = -4 \Rightarrow \frac{-2}{3} - \frac{3}{2} - a - 2 = -4 \Rightarrow a = \frac{-1}{6}$ ✓

Hence, $f(x) = \frac{2x^3}{3} - \frac{3x^2}{2} - \frac{x}{6} - 2$ ✓

---

9. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = x - \frac{e^x}{2}$. Find the equation of this curve given that it passes through the point (0, –2).

$y = \int x - \frac{e^x}{2} \, dx$

$= \frac{x^2}{2} - \frac{e^x}{2} + C$ ✓

$x = 0, y = -2 \Rightarrow -2 = -\frac{1}{2} + C \Rightarrow C = -\frac{3}{2}$ ✓✓

Hence, $y = \frac{x^2}{2} - \frac{e^x}{2} - \frac{3}{2}$ ✓

---

10. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = x(1+0.5x^2)^5$. Find the equation of this curve given that it passes through the point (0, 1)

$y = \int x(1+0.5x^2)^5 \, dx$

$= \frac{(1+0.5x^2)^6}{6} + C$ ✓✓

$x = 0, y = 1 \Rightarrow 1 = \frac{1}{6} + C \Rightarrow C = \frac{5}{6}$ ✓

Hence, $y = \frac{(1+0.5x^2)^6}{6} + \frac{5}{6}$. ✓

Calculator Free

11. [4 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = \frac{x}{1+x^2}$. Find the equation of this curve given that it passes through the point (0, 1).

$y = \int \frac{x}{1+x^2} \, dx$

$= \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$ ✓

$= \frac{1}{2} \ln|1+x^2| + C$ ✓

When $x = 0, y = 1 \Rightarrow C = 1$ ✓

Hence, $y = \frac{1}{2} \ln|1+x^2| + 1$ ✓

---

12. [5 marks]

The gradient function of a curve is given by $\frac{dy}{dx} = a + b\cos 2x$. The curve has a stationary point at (0, 0). Find the equation of this curve.

Stationary point at (0, 0):

$\Rightarrow a + b\cos 0 = 0$ ✓

$b = -a$ ✓

$y = \int a - a\cos 2x \, dx$

$= ax - \frac{a}{2}\sin 2x + C$ ✓

When $x = 0, y = 0: \Rightarrow C = 0$

Hence, $y = ax - \frac{a}{2}\sin 2x$ for $a \neq 0, a \in \mathbb{R}$ ✓✓

Calculator Free

13. [7 marks: 2, 1, 2, 2] [TISC]

The gradient function of the curve $y = f(x)$ is given by the equation $f'(x) = \frac{1}{2\sqrt{x}}$.

(a) Determine the equation of the family of curves $y = f(x)$.

$f(x) = \int \frac{1}{2\sqrt{x}}\,dx$ ✓
$= \sqrt{x} + C$ ✓

(b) Determine $y = f(x)$ if the curve passes through the point (1, 4).

$f(x) = \sqrt{x} + C$
$f(1) = 4 \Rightarrow C = 3$.
Hence: $f(x) = \sqrt{x} + 3$ ✓

(c) If another function in the family of curves $y = f(x)$ has a tangent with equation $y = \frac{x}{8}$:

(i) Explain why the tangent meets the curve at $x = 16$.

Gradient of tangent $= \frac{1}{8}$ ✓
But gradient function is $\frac{1}{2\sqrt{x}}$
Hence: $\frac{1}{2\sqrt{x}} = \frac{1}{8}$ ✓
$x = 16$

(ii) Determine the equation of the curve $y = f(x)$.

Point of contact between tangent and curve is (16, 2).
$f(x) = \sqrt{x} + C$
$f(16) = 2 \Rightarrow C = -2$ ✓
Hence: $f(x) = \sqrt{x} - 2$ ✓

Calculator Free

14. [9 marks: 2, 4, 3]

(a) Given that $y = x\,e^x$, find $\frac{dy}{dx}$.

$\frac{dy}{dx} = x\,e^x + e^x$ ✓✓

(b) Use your answer in (a) to find $\int x\,e^x\,dx$.

Reversing the result in (a):
$\int x\,e^x + e^x\,dx = x\,e^x + C$ ✓
$\int x\,e^x\,dx + \int e^x\,dx = x\,e^x + C$ ✓
$\int x\,e^x\,dx + e^x = x\,e^x + C$ ✓
Hence, $\int x\,e^x\,dx = x\,e^x - e^x + C$ ✓

(c) The gradient function of a curve is given by $\frac{dy}{dx} = x\,e^x$. Find the equation of this curve given that when $x = 0$, $y = 2$.

$y = x\,e^x - e^x + C$ ✓
$x = 0, y = 2 \Rightarrow 2 = -1 + C \Rightarrow C = 3$ ✓
Hence, $y = x\,e^x - e^x + 3$ ✓

---

15. [5 marks: 2, 3]

(a) Determine $\frac{d}{dx}(x\sin(2x))$.

$\frac{d}{dx}(x\sin(2x)) = \sin(2x) + 2x\cos(2x)$ ✓✓

(b) Hence, or otherwise, determine $\int 2\sin(2x) + 2x\cos(2x)\,dx$.

$\int 2\sin(2x) + 2x\cos(2x)\,dx = \int \sin(2x) + [\sin(2x) + 2x\cos(2x)]\,dx$ ✓
$= \frac{-\cos(2x)}{2} + x\sin 2x + C$ ✓✓

Calculator Free

16. [4 marks]

Use the results $\frac{d}{dx}\left(e^x \sin x\right) = e^x \sin x + e^x \cos x$ and $\frac{d}{dx}\left(e^x \cos x\right) = e^x \cos x - e^x \sin x$ to determine $\int e^x \cos x \, dx$.

$\frac{d}{dx}\left(e^x \sin x\right) = e^x \sin x + e^x \cos x \Rightarrow \int e^x \sin x + e^x \cos x \, dx = e^x \sin x$

$\int e^x \sin x \, dx + \int e^x \cos x \, dx = e^x \sin x$ (1) ✓

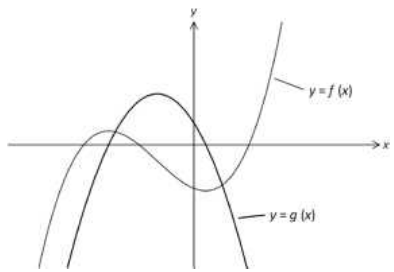
Similarly: $\int e^x \cos x \, dx - \int e^x \sin x \, dx = e^x \cos x$ (2) ✓

(1) + (2): $2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$ ✓

$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C$ ✓

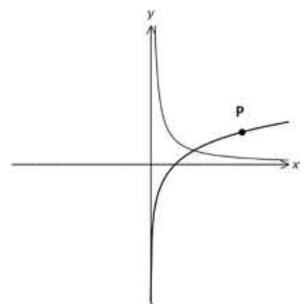
17. [5 marks: 2, 3] [TISC]

(a) The graphs of the functions $f(x)$ and $g(x)$ are shown in the accompanying diagram. Determine with reasons if $g(x)$ is the derivative of $f(x)$.



No. ✓
Slope of $g(x)$ in the neighbourhood of the root corresponding to the maximum point on $f(x)$ changes from negative to positive instead of from positive to negative. ✓

(b) The graph of the function $f(x) = \frac{1}{x}$ for $x > 0$ is shown in the accompanying diagram. On the same axes, draw a possible graph of $g(x)$ where $g(x) = \int f(x) \, dx$ given that it passes through the point $\boldsymbol{P}$.



✓ $g(x)$ has vertical asymptote $x = 0$.
✓ $g(x)$ crosses point P.
✓ Domain for $g(x) > 0$.

18 The Fundamental Theorem of Calculus

Calculator Free

1. [12 marks: 3, 3, 3, 3]

Evaluate each of the following definite integrals.

(a) $\int_0^1 \frac{3x^2}{4} + \sqrt{x} - 2 \, dx$

$\int_0^1 \frac{3x^2}{4} + \sqrt{x} - 2 \, dx = \left[\frac{x^3}{4} + \frac{2x^{3/2}}{3} - 2x\right]_0^1$ ✓✓

$= -\frac{13}{12} - 0 = -\frac{13}{12}$ ✓

(b) $\int_1^4 \frac{2}{x^3} - \frac{1}{2\sqrt{x}} \, dx$

$\int_1^4 2x^{-3} - \frac{x^{-1/2}}{2} \, dx = \left[\frac{-1}{x^2} - x^{1/2}\right]_1^4$ ✓✓

$= -\frac{33}{16} - (-2) = -\frac{1}{16}$ ✓

(c) $\int_0^5 \sqrt{(4+x)^3} \, dx$

$\int_0^5 (4+x)^{3/2} \, dx = \left[\frac{2(4+x)^{5/2}}{5}\right]_0^5$ ✓✓

$= \frac{422}{5}$ ✓

(d) $\int_{\frac{-1}{2}}^0 \frac{-3}{\sqrt{(1-6x)}} \, dx$

$\int_{\frac{-1}{2}}^0 -3(1-6x)^{-1/2} \, dx = \left[(1-6x)^{1/2}\right]_{-1/2}^0$ ✓✓

$= -1$ ✓

Calculator Free

2. [14 marks: 2, 4, 4, 4]

Evaluate each of the following definite integrals.

(a) $\int_0^2 3e^{\frac{x}{2}}\,dx$

$$\int_0^2 3e^{\frac{x}{2}}\,dx = [6e^{\frac{x}{2}}]_0^2 \quad ✓$$
$$= 6e - 6 \quad ✓$$

(b) $\int_0^1 (1+e^x)^2\,dx$

$$\int_0^1 (1+e^x)^2\,dx = \int_0^1 (1+2e^x+e^{2x})\,dx \quad ✓$$
$$= \left[x+2e^x+\frac{e^{2x}}{2}\right]_0^1 \quad ✓$$
$$= \left(1+2e+\frac{e^2}{2}\right)-\frac{5}{2} = 2e+\frac{e^2}{2}-\frac{3}{2} \quad ✓✓$$

(b) $\int_0^2 \frac{2t^3}{t^4+2}\,dt$

$$\int_0^2 \frac{2t^3}{t^4+2}\,dt = \frac{1}{2}\int_0^2 \frac{2t^3}{t^4+2}\,dt \quad ✓$$
$$= \frac{1}{2}\left[ln(x^4+2)\right]_0^2 \quad ✓$$
$$= \frac{1}{2}[ln(18)-ln(2)] = ln(3) \quad ✓✓$$

(d) $\int_0^{\frac{1}{3}} \frac{\sin(\pi x)}{\cos(\pi x)}\,dx$.

$$\int_0^{\frac{1}{3}} \frac{\sin(\pi x)}{\cos(\pi x)}\,dx = \frac{-1}{\pi}\int_0^{\frac{1}{3}} \frac{-\pi\sin(\pi x)}{\cos(\pi x)}\,dx \quad ✓$$
$$= \frac{-1}{\pi}\left[ln(\cos(\pi x))\right]_0^{\frac{1}{3}} \quad ✓$$
$$= \frac{-1}{\pi}ln(\frac{1}{2}) = \frac{ln(2)}{\pi} \quad ✓✓$$

Calculator Free

3. [13 marks: 2, 2, 2, 2, 2, 3]

Find $\frac{dy}{dx}$:

(a) $y = \int_1^x \sqrt{1+t^3}\,dt$

$$\frac{dy}{dx} = \sqrt{1+x^3} \quad ✓✓$$

(b) $y = \int_1^{x^2} \frac{1}{\sqrt{1+t^2}}\,dt$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1+x^4}} \quad ✓✓$$

(c) $y = \int_1^{e^x} (1+\sqrt{t})^5\,dt$

$$\frac{dy}{dx} = \left(1+\sqrt{e^x}\right)^5 e^x \quad ✓✓$$

(d) $y = \int_1^{e^{2x}} \frac{1}{\sqrt{1+t}}\,dt$

$$\frac{dy}{dx} = \frac{2e^{2x}}{\sqrt{1+e^{2x}}} \quad ✓✓$$

(e) $y = \int_1^{1+x^2} \frac{1}{\sqrt{1+e^t}}\,dt$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1+e^{1+x^2}}} \quad ✓✓$$

(f) $y = \int_{e^{2x}}^1 e^t\,dt$

$$\frac{dy}{dx} = -e^{e^{2x}} \times 2e^{2x} \quad ✓✓✓$$

Calculator Free

4. [11 marks: 2, 2, 2, 3, 2]

Determine $\frac{dy}{dx}$:

(a) $y = \int_{1}^{\cos x} 1 + t^2 \, dt$

$\frac{dy}{dx} = (-\sin x)(1 + \cos^2 x)$ ✓✓

(b) $y = \int_{1}^{\tan x} e^{t^2} \, dt$

$\frac{dy}{dx} = (\sec^2 x) e^{\tan^2 x}$ ✓✓

(c) $y = \int_{1}^{\sin 2x} \sin^2(1+t) \, dt$

$\frac{dy}{dx} = (2\cos 2x) \sin^2(1 + \sin 2x)$ ✓✓

(d) $y = \int_{\cos^2 x}^{0} \tan(\pi + u) \, du$

$\frac{dy}{dx} = (2 \sin x \cos x) \tan(\pi + \cos^2 x)$ ✓✓✓

(e) $y = \int_{0}^{e^{\sin x}} \sqrt{1+u} \, du$

$\frac{dy}{dx} = (\cos x) e^{\sin x} \sqrt{1 + e^{\sin x}}$ ✓✓

Calculator Free

5. [12 marks: 2, 2, 2, 3, 3]

Determine $\frac{dy}{dx}$:

(a) $y = \int_{1}^{ln\, x} ln(2+3t) \, dt$

$\frac{dy}{dx} = \frac{ln[2 + 3\,ln(x)]}{x}$ ✓✓

(b) $y = \int_{1}^{ln\, x} \frac{1}{t} \, dt$

$\frac{dy}{dx} = \frac{1}{x\,ln(x)}$ ✓✓

(c) $y = \int_{1}^{ln \sin(x)} \sqrt{t} \, dt$

$\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)} \times \sqrt{ln \sin(x)}$ ✓✓

(d) $y = \int_{1}^{ln(1+e^x)} (1-t)^2 \, dt$

$\frac{dy}{dx} = \frac{e^x}{1+e^x} \times [1 - (ln(1 + e^x)]^2$ ✓✓✓

(e) $y = \int_{x}^{ln(x)} t^2 \, dt$

$\frac{dy}{dx} = \frac{[ln(x)]^2}{x} - x^2$ ✓✓✓

Calculator Free

6. [4 marks: 2, 2]

Evaluate:

(a) $\int_1^2 \frac{d}{dx}(1+x)\ dx$

$$\int_1^2 \frac{d}{dx}(1+x)\ dx = [1+x]_1^2 \quad ✓$$
$$= 3 - 2 = 1 \quad ✓$$

(b) $\int_0^2 \frac{d}{dx}\left(\frac{1+x}{1+x^2}\right) dx$

$$\int_0^2 \frac{d}{dx}\left(\frac{1+x}{1+x^2}\right) dx = \left[\frac{1+x}{1+x^2}\right]_0^2 \quad ✓$$
$$= \frac{3}{5} - 1 = -\frac{2}{5} \quad ✓$$

---

7. [5 marks: 2, 3]

(a) Determine $\frac{d}{dx}\left(\frac{1}{1+x^2}\right)$. Simplify your answer.

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{0-2x}{(1+x^2)^2} \quad ✓$$
$$= \frac{-2x}{(1+x^2)^2} \quad ✓$$

(b) Hence, or otherwise, determine $\int_0^1 \frac{x}{(1+x^2)^2}\, dx$.

Using the result from (a):

$$\int \frac{-2x}{(1+x^2)^2}\, dx = \frac{1}{1+x^2} + C \quad ✓$$
$$\Rightarrow \int_0^1 \frac{x}{(1+x^2)^2}\, dx = \frac{-1}{2}\left[\frac{1}{1+x^2}\right]_0^1 \quad ✓$$
$$= \frac{1}{4} \quad ✓$$

Calculator Free

8. [4 marks]

A curve has equation given by $y = \int_0^{x^2} \sqrt{1+u^2}\, du$. When $x = 1$, $\frac{dx}{dt} = -1$.

Use the chain rule to determine the value of $\frac{dy}{dt}$ when $x = 1$.

$$\frac{dy}{dx} = 2x\sqrt{1+x^4} \quad ✓✓$$
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = 2x\sqrt{1+x^4} \times \frac{dx}{dt} \quad ✓$$

When $x = 1$,
$$\frac{dy}{dt} = 2\sqrt{2} \times -1$$
$$= -2\sqrt{2} \quad ✓$$

---

9. [7 marks: 2, 5]

A curve has equation given by $y = \int_0^{x^2} (t-4)^3\, dt$.

(a) Find the gradient of the curve at the point where $x = 1$.

$$\frac{dy}{dx} = (x^2-4)^3 \times 2x \quad ✓$$

When $x = 1$: $\frac{dy}{dx} = -54$ ✓

(b) Find x-coordinate of the turning points of the curve. Identify the nature of these points.

For turning points $\frac{dy}{dx} = 0$: $\Rightarrow$ $x = 0, \pm 2$ ✓✓

$\left.\frac{dy}{dx}\right|_{x=0^-} > 0$ $\left.\frac{dy}{dx}\right|_{x=0^+} < 0$.
Hence there is a maximum point at $x = 0$. ✓

$\left.\frac{dy}{dx}\right|_{x=2^-} < 0$ $\left.\frac{dy}{dx}\right|_{x=2^+} > 0$.
Hence there is a minimum point at $x = 2$. ✓

$\left.\frac{dy}{dx}\right|_{x=-2^-} < 0$ $\left.\frac{dy}{dx}\right|_{x=-2^+} > 0$.
Hence there is a minimum point at $x = -2$. ✓

Calculator Assumed

10. [6 marks: 3, 3]

A curve has equation given by $y = \int_2^{e^x} 5(t-2)^4 \, dt$.

(a) Find the equation of the tangent to the curve at the point (0, –1).

$\frac{dy}{dx} = 5e^x(e^x-2)^4$ ✓

When $x = 0$, $\frac{dy}{dx} = 5$ ✓

Hence, tangent has equation $y = 5x - 1$ ✓

$\frac{d}{dx}\left(\int_2^{e^x} 5(t-2)^4 dt\right)$ $5\cdot(e^x-2)^4\cdot e^x$

ans | $x=0$ 5

(b) Find the x-coordinate of the stationary point on this curve. State its nature.

$\frac{dy}{dx} = 5e^x(e^x-2)^4$

$\frac{dy}{dx} = 0 \Rightarrow x = \ln 2 \approx 0.6931$ ✓

$\left.\frac{dy}{dx}\right|_{x=0.6931^-} > 0$ $\left.\frac{dy}{dx}\right|_{x=0.6931^+} > 0$ ✓

Hence, $x = \ln 2 \approx 0.6931$ is the x-coordinate of a horizontal inflection point. ✓

---

11. [4 marks]

The temperature of a body ($^\circ$C) at time t hours is given by $\theta = \int_0^t (x-4)e^x \, dx$ for $0 \le t \le 7$. The temperature of the body becomes critical when the temperature is either a local maximum or minimum. Determine when the temperature of the body becomes critical and classify the nature of this critical temperature.

$\frac{d\theta}{dt} = 0 \Rightarrow t = 4$ ✓

$\frac{d^2\theta}{dt^2} = e^t + (t-4)e^t$ ✓

$\left.\frac{d^2\theta}{dt^2}\right|_{t=4} = e^4 > 0$ ✓

Hence, temperature of the body is a local minimum at $t = 4$ hours. ✓

Calculator Assumed

12. [10 marks: 3, 5, 2]

Let $F(x) = \int_0^x 3t^2 - 12t + 9 \, dt$ where for $1 \le x \le 5$.

(a) Calculate $F(1)$ and $F(5)$.

$F(x) = \int_0^x 3t^2 - 12t + 9 \, dt = x^3 - 6x^2 + 9x$ ✓

$F(1) = 4$ ✓

$F(5) = 20$ ✓

(b) Find the local minimum and maximum points on the curve $y = F(t)$.

$F'(x) = 3x^2 - 12x + 9$ ✓

$F'(x) = 0 \Rightarrow x = 1, 3$ ✓

$F''(x) = 6x - 12$ ✓

When $x = 1$, $F(1) = 4$ and $F''(1) = -6 < 0$

Hence. (1, 4) is a local maximum point. ✓

When $x = 3$, $F(3) = 0$ and $F''(1) = 6 > 0$

Hence. (3, 0) is a local minimum point. ✓

(c) Determine the global minimum and global maximum values for $F(x)$.

End points: $F(1) = 4$
$F(5) = 20$

Local Minimum (3, 0)
Local Maximum (1, 4).

Hence, global minimum for $F(x)$ is 0 ✓
global maximum for $F(x)$ is 20. ✓

Calculator Assumed

13. [7 marks: 1, 3, 3]

A curve has equation given by $f(x) = \int_0^x 1 - 2e^{-t^2}\,dt$ for $0 \le x \le 3$.

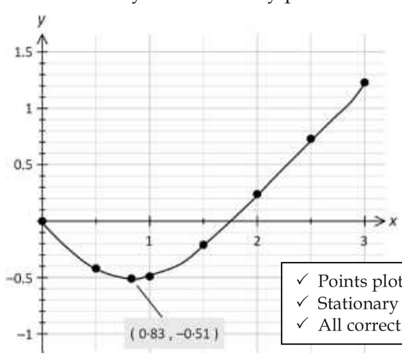
(a) Use your calculator to complete the table below. ✓ Correct value.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	0	–0.4226	–0.4936	–0.2124	0.2358	0.7283	1.2276

(b) The graph of $g(t) = 1 - 2e^{-t^2}$ for $0 \le t \le 3$ has a root at $t \approx 0.83$.
Explain how this result may be used to determine the coordinates of the stationary point for $f(x)$.

By the FTC: $f'(x) = 1 - 2e^{-x^2}$. ✓
Hence, $f'(x) = g(x)$.
For stationary points $f'(x) = 0$.
Since $g(0.83) = 0$, $f'(0.83) = 0$. ✓
Therefore, $f(x)$ has a stationary point at $x = 0.83$.
$f(0.83) = \int_0^{0.83} 1 - 2e^{-t^2}\,dt \approx -0.51$.
Hence, stationary point is at (0.83, –0.51). ✓

(c) On the axes provided below, sketch the graph of $f(x) = \int_0^x 1 - 2e^{-t^2}\,dt$ for $0 \le x \le 3$. Indicate clearly the stationary point.



✓ Points plotted correctly.
✓ Stationary point indicated.
✓ All correct.

Calculator Assumed

14. [8 marks: 2, 3, 3]

A curve has equation given by $f(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 2$.

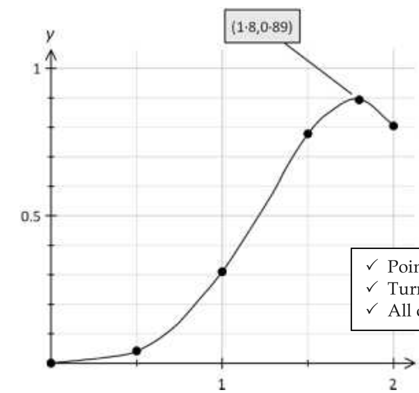
(a) Use your calculator to complete the table below. ✓✓ –1 per error

x	0	0.5	1	1.5	2
$f(x)$	0	0.04148	0.3103	0.7782	0.8048

(b) Determine the coordinates of the stationary point on the graph of $y = f(x)$ for $0 \le x \le 2$.

By the FTC: $f'(x) = \sin(x^2)$ ✓
For stationary points: $f'(x) = 0 \Rightarrow x = 1.77$ ✓
$f(x) = \int_0^{1.77} \sin(t^2)\,dt \approx 0.89$.
Hence, stationary point is at (1.77, 0.89). ✓

(c) On the axes provided below, sketch the graph of $f(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 2$. Indicate clearly the stationary point on the curve.



✓ Points plotted correctly.
✓ Turning point indicated.
✓ All correct.

Calculator Assumed

15. [9 marks: 2, 2, 2, 3]

The functions $f(x)$ and $g(x)$ are continuous everywhere. It is known that:
$\int_{-5}^{1} f(x)\,dx = -2$, $\int_{-5}^{7} f(x)\,dx = 8$, $\int_{1}^{7} g(x)\,dx = 18$ and $\int_{-5}^{1} f(x)+g(x)\,dx = -20$.

(a) Calculate $\int_{1}^{7} f(x)\,dx$.

$$\int_{-5}^{7} f(x)\,dx = \int_{-5}^{1} f(x)\,dx + \int_{1}^{7} f(x)\,dx \quad ✓$$
$$\int_{1}^{7} f(x)\,dx = 8 + 2 = 10 \quad ✓$$

(b) Calculate $\int_{-5}^{1} g(x)\,dx$.

$$\int_{-5}^{1} f(x)+g(x)\,dx = \int_{-5}^{1} f(x)\,dx + \int_{-5}^{1} g(x)\,dx \quad ✓$$
$$\int_{-5}^{1} g(x)\,dx = -20 + 2 = -18 \quad ✓$$

(c) Calculate $\int_{-6}^{0} f(x)+g(x)\,dx$

$$\int_{-6}^{0} f(x+1)+g(x+1)\,dx = \int_{-5}^{1} f(x)+g(x)\,dx \quad ✓$$
$$= -20 \quad ✓$$

(d) Calculate $\int_{-5}^{7} f(x)-g(x)\,dx$

$$\int_{-5}^{7} f(x)-g(x)\,dx = \int_{-5}^{7} f(x)\,dx - \int_{-5}^{7} g(x)\,dx \quad ✓$$
$$= 8 - \left[\int_{-5}^{1} g(x)\,dx + \int_{1}^{7} g(x)\,dx\right] \quad ✓$$
$$= 8 - [-18 + 18] = 8 \quad ✓$$

Calculator Assumed

16. [8 marks: 3, 2, 3]

Let $\int f(x)\,dx = F(x)$. The table below provides the values for $f(x)$ and $F(x)$ for several values of x.

x	–8	–2	0	4	8	10
$f(x)$	–2		0	4		4
$F(x)$	–4	–16	–18	–10	6	14

Use the table above to answer the following questions.

(a) Calculate $\int_{4}^{10} f(-8)\,dx$

$$\int_{4}^{10} f(-8)\,dx = \int_{4}^{10} -2\,dx \quad ✓$$
$$= [-2x]_4^{10} \quad ✓$$
$$= -12 \quad ✓$$

(b) Calculate $\int_{-8}^{0} f(x)\,dx$.

$$\int_{-8}^{0} f(x)\,dx = F(0) - F(-8) \quad ✓$$
$$= -18 - (-4) = -14 \quad ✓$$

(c) Calculate $\int_{-4}^{4} f(2x)\,dx$.

$$\int_{-4}^{4} f(2x)\,dx = \frac{1}{2}\int_{-8}^{8} f(x)\,dx \quad ✓$$
$$= \frac{1}{2}\left[F(8) - F(-8)\right] \quad ✓$$
$$= \frac{1}{2}\left[6-(-4)\right] = 5 \quad ✓$$

19 Net Change

Calculator Assumed

1. [5 marks: 1, 3, 1]

Given that $\frac{dV}{dt} = (t-2)^2 + 1$, find:

(a) the instantaneous rate of change of V with respect to t when $t = 4$

> When $t = 4$, $\frac{dV}{dt} = 5$ ✓

(b) the net change in V when t changes from $t = 1$ to $t = 4$

> Net Change $= \int_1^4 (t-2)^2 + 1 \, dt$ ✓✓
> $= 6.$ ✓

(c) the average rate of change of V in the interval $1 \le t \le 4$ seconds.

> Average Change $= \frac{6}{3} = 2$ ✓

---

2. [5 marks: 2, 3]

Given that $P = (t-2)(t-4)$ cm where t is time in seconds.

(a) Find using Calculus, the instantaneous rate of change of P when $t = 5$ s.

> $\frac{dP}{dt} = 2t - 6$ ✓
> When $t = 5$, $\frac{dP}{dt} = 4 \text{ cms}^{-1}$. ✓

(c) Find the average rate of change of P over the interval $0 \le t \le 5$ s.

> Net Change $= \int_0^5 2t - 6 \, dt$ ✓
> $= -5$ cm. ✓
> Hence, average rate of change $= -5/5 = -1 \text{ cms}^{-1}$. ✓

Calculator Assumed

3. [8 marks: 2, 2, 2, 2]

The instantaneous rate with which the mass $M(t)$ kg of a body changes with respect to time t (days) is modelled by $\frac{dM}{dt} = t\sin\left(\frac{\pi t}{12}\right)$ for $t \ge 0$.

(a) Find the net change in M in the first 24 days.

> $\Delta M = \int_0^{24} t\sin\left(\frac{\pi t}{12}\right) dt$ ✓
> $= -91.6732 \approx -91.67$ kg ✓

(b) Find the net change in M on the 24th day.

> $\Delta M = \int_{23}^{24} t\sin\left(\frac{\pi t}{12}\right) dt$ ✓
> $= -3.0370 \approx -3.04$ kg ✓

(c) Calculate the instantaneous change in M on the 24th day.

> $\left.\frac{dM}{dt}\right|_{t=24} = 0$ ✓✓

(d) Find the average rate of change for M on the 24th day.

> $\frac{\Delta M}{t} = \frac{-3.0370}{1}$ ✓
> ≈ -3.04 kg ✓

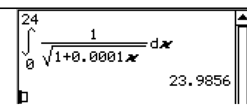
---

4. [7 marks: 2, 3, 2]

The instantaneous rate with which the number of computers infected with a virus at time t hours is modelled by $\frac{dN}{dt} = \frac{1}{\sqrt{1+0.0001t}}$ where $N(t)$ is the number of computers already infected with the virus.

(a) Find the net change in the number of computers infected with the virus within the first day.

> $\Delta N = \int_0^{24} \frac{1}{\sqrt{1+0.0001t}} dt$ ✓
> ≈ 24 computers ✓



Calculator Assumed

4. (b) How long will it take to infect 1000 computers?

$$\int_0^k \frac{1}{\sqrt{1+0.0001t}}\,dt = 1000 \quad ✓✓$$

$k = 1025$ hours ✓

```
solve(∫[0,t] 1/√(1+0.0001x) dx = 1000, t)
                                  {t=1025}
```

(c) Determine with reasons if this mathematical model is a reasonable model.

No. ✓
The number of computers infected increases without limit! ✓

---

5. [7 marks: 2, 2, 3]

The instantaneous rate with which the volume of water in a holding tank changes with time, is modelled by $\frac{dV}{dt} = (t-1)(t^2-9)$, for $0 \le t \le 5$, where V is the volume of water in the tank in kL and t is time in hours.

(a) Find the interval of time during which water is flowing out of the tank. Justify your answer.

$\frac{dV}{dt} < 0$ ✓ for $1 < t < 3$
Hence, the interval $1 < t < 3$ hours. ✓

```
solve((t-1)(t^2-9)<0, t)|t≥0
                         {1<t<3}
```

(b) Find the amount of water that has flowed out of the tank.

Amount of water that has flowed out

$$= \left| \int_1^3 (t-1)(t^2-9)\,dt \right| \quad ✓$$

$$= \frac{20}{3}\text{ kL} \quad ✓$$

(c) Find the amount of water that has flowed into the tank.

Amount of water that has flowed in

$$= \int_0^1 (t-1)(t^2-9)\,dt + \int_3^5 (t-1)(t^2-9)\,dt \quad ✓✓$$

$$= \frac{53}{12} + \frac{148}{3} = \frac{215}{4}\text{ kL} \quad ✓$$

Calculator Assumed

6. [8 marks: 2, 3, 3]

The change in altitude of a balloon is modelled by $\frac{dh}{dt} = \frac{1}{2+t}$ where h metres is the altitude of the balloon at time t seconds.

(a) Find the height increase of the balloon in the 5th second.

$$\Delta h = \int_4^5 \frac{1}{2+t}\,dt \quad ✓$$

$= 0.1542 \approx 0.15$ metres ✓

(b) Find the average rate of height increase in the first 10 seconds.

$$\Delta h = \int_0^{10} \frac{1}{2+t}\,dt \quad ✓$$

$= 1.7918$ ✓

Average height increase $= \frac{1.7918}{10} \approx 0.18$ metres/second ✓

(c) The initial height of the balloon was 2 metres. Find when the height of the balloon first exceeds 5 metres.

$\Delta h = 3$ ✓

$$\int_0^T \frac{1}{2+t}\,dt = 3 \quad ✓$$

$T = 38.17$ (reject -42.17)
Hence, after 38.2 seconds. ✓

```
solve(∫[0,x] 1/(2+t) dt = 3, x)
{x=-42.17107385,x=38.17107385}
```

---

7. [8 marks: 2, 2, 2, 2]

The rate of population change of a bacteria culture is modelled by $\frac{dP}{dt} = 100\,e^{-0.01t}$ where t is time in hours.

(a) Find the initial rate of population change.

$\left.\frac{dP}{dt}\right|_{t=0} = 100$ ✓
Initial rate of change = increasing at rate 100 per hour. ✓

Calculator Assumed

7. (b) Describe the rate of change for large values of t.

As $t \to \infty$, $\frac{dP}{dt} \to 0$ ✓
For large values of t, the population size stops growing. ✓

(c) Calculate the net population change in the first 2 000 hours.

$$\Delta P = \int_0^{2000} 100e^{-0.01t}\,dt \approx 10\,000$$ ✓ ✓

(d) Given that the initial population was 100, find the maximum population size. Show clearly how you obtained your answer.

Maximum population size = Initial size + ΔP(for large t) ✓
= 100 + 10 000
= 10 100 ✓

---

8. [6 marks: 3, 3]

The instantaneous rate with which V the amount of fluid in a tank changes with time t hours is modelled by $\frac{dV}{dt} = \frac{t}{\sqrt{9+t^2}}$ for $t \geq 0$.

(a) Determine an expression for the amount of fluid at time t hours if at $t = 4$ hours, the amount of fluid in the tank is 85 L.

$V = \int \frac{t}{\sqrt{9+t^2}}\,dt$ ✓
$= \sqrt{9+t^2} + C$ ✓
$V(4) = 85 \Rightarrow C = 80$
Hence: $V = \sqrt{9+t^2} + 80$ ✓

(b) The net change in Q from $t = 5$ to $t = k$ is 10L. Calculate the value of k.

$\int_5^k \frac{t}{\sqrt{9+t^2}}\,dt = 10$ ✓
$\left[\sqrt{9+t^2}\right]_5^k = 10$ ✓
$\sqrt{9+k^2} - \sqrt{34} = 10$
$k = 15.5$ hours ✓

Calculator Assumed

9. [10 marks: 3, 3, 3, 1]

The rate of change of pressure acting on an object is modelled by $\frac{dP}{dt} = 4\cos\left(\frac{\pi t}{12}\right)$ where P (kilopascals) is the pressure at time t hours.

(a) Find the net change in pressure in the interval $0 \leq t \leq 12$.

$\Delta P = \int_0^{12} 4\cos\left(\frac{\pi t}{12}\right) dt$ ✓
$= 0$ ✓
Net change is zero ✓

(b) Find the net change in pressure in the interval $6 \leq t \leq 12$.

$\Delta P = \int_6^{12} 4\cos\left(\frac{\pi t}{12}\right) dt$ ✓
$= -15.2789$ ✓
Net change is a decrease in pressure of 15.3 kilopascals. ✓

(c) The net increase in pressure in the interval $0 \leq t \leq T$ is 10 kilopascals. Find T given that $T \leq 24$.

$\Delta P = \int_0^{T} 4\cos\left(\frac{\pi t}{12}\right) dt = 10$ ✓
$T = 2.7254$ or 9.2746 ✓✓

```
solve(∫₀ˣ 4cos(πt/12)dt = 10, x) | 0≤x≤24
                {x=9.2746, x=2.7254}
```

(d) Comment on your answer in part (c).

Net change for is 10 kilopascals
for $0 \leq t \leq 2.73$ hours and $0 \leq t \leq 9.27$ hours.
Hence, for $2.73 \leq t \leq 9.27$ hours the net change is 0. ✓

Calculator Assumed

10. [5 marks: 3, 2]

The marginal cost for producing x hundred units of a product is given by $\frac{dC}{dx} = 0.08x$, where \$$C$ is the cost of producing x hundred items of the product.

(a) Find the cost of producing 10 000 of these items if the fixed cost is \$2000.

$C = \int 0.08x\, dx = 0.04x^2 + K$ ✓
$C(0) = 2000 \Rightarrow K = 2000$ ✓
Hence, $C = 0.04x^2 + 2000$
$\Rightarrow C(100) = \$2400$ ✓

(b) Find the net change in cost if the number of items produced is changed from 1 000 to 2 000. Justify your answer.

Net change in cost $= \int_{10}^{20} 0.08x\, dx$ ✓
$= \$12$ ✓

---

11. [5 marks: 3, 2]

The marginal profit associated with the sale of x items of a product is given by $\frac{dP}{dx} = -0.000\,81x^2 + 0.4x - 5.4$, where \$$P$ is the profit associated with the sale of x units of this product.

(a) Given that there is a loss of \$500 if no items are sold, find the profit associated with the sale of 50 items.

$P = \int -0.000\,81x^2 + 0.4x - 5.4\, dx$
$= -0.000\,27x^3 + 0.2x^2 - 5.4x + K$ ✓
$P(0) = -500 \Rightarrow K = -500$ ✓
Hence, $P = -0.000\,27x^3 + 0.2x^2 - 5.4x - 500$
$\Rightarrow P(50) = -\$303.75$ ✓

(b) Find the net change in profit if the number of items sold is changed from 50 to 350.

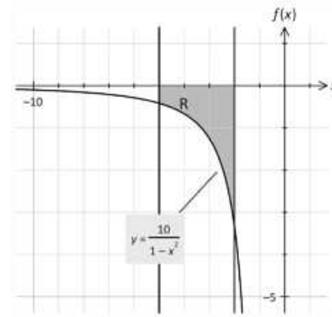
Net Change in $P = \int_{50}^{350} -0.000\,81x^2 + 0.4x - 5.4\, dx$ ✓
$= \$10\,837.50$ ✓

20 Area under a curve & Area Functions

Calculator Assumed

1. [7 marks: 3, 3, 1]

The accompanying diagram shows the graph of $f(x) = \frac{10}{1-x^2}$. The shaded region R is trapped between the curve, the axes and the lines $x = -5$ and $x = -2$.



(a) Estimate the area of the region R using three inscribed strips of uniform width, giving your answer to 3 decimal places. Show clearly how you obtained your answer.

Area $= 1 \times (-f(-5) - f(-4) - f(-3))$ ✓
$= 0.4167 + 0.6667 + 1.25$ ✓
$= 2.3333$
≈ 2.333 ✓

	A	B
1	x	-f(x)
2	-5	0.4167
3	-4	0.6667
4	-3	1.25
5	sum	2.3333

seq$\left(\frac{-10}{1-x^2}, x, -5, -3, 1\right)$　$\left\{\frac{5}{12}, \frac{2}{3}, \frac{5}{4}\right\}$
sum(　$\frac{7}{3}$

(b) Estimate the area of the region R using three circumscribing strips of uniform width, giving your answer to 3 decimal places. Show clearly how you obtained your answer.

Area $= 1 \times (-f(-4) - f(-3) - f(-2))$ ✓
$= 0.6667 + 1.25 + 3.3333$ ✓
$= 5.250$ ✓

	A	B
1	x	-f(x)
2	-4	0.6667
3	-3	1.25
4	-2	3.3333
5	sum	5.25

seq$\left(\frac{-10}{1-x^2}, x, -4, -2, 1\right)$　$\left\{\frac{2}{3}, \frac{5}{4}, \frac{10}{3}\right\}$
sum(　$\frac{21}{4}$

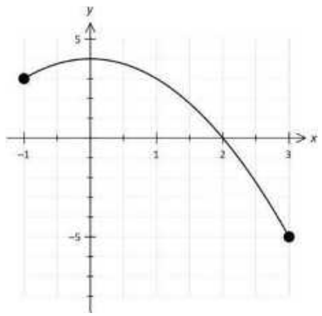
(c) Use your answer in (a) and (b) to provide a more accurate estimate for the area of R.

Area $\approx \frac{1}{2}\left(\frac{7}{3} + \frac{21}{4}\right) = \frac{91}{24} = 3.792$ ✓

Calculator Assumed

2. [6 marks: 3, 3]

The graph of $y = 4 - x^2$ is shown in the accompanying diagram. The region R is trapped between this curve and the lines $x = 1$ and $x = 3$. Estimate the area of region R using:



(a) *inscribed* rectangles of uniform width 0.5.

> Area = $0.5 \times (4 - 1.5^2) + 0.5 \times (4 - 2.5^2)$ ✓✓
> $= 0.875 + 1.125 = 2$ ✓

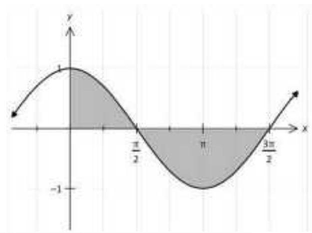
(b) *circumscribed* rectangles of uniform width 0.5.

> Area = $0.5 \times (4 - 1^2) + 0.5 \times (4 - 1.5^2)$ ✓
> $+ 0.5 \times (4 - 2.5^2) + 0.5 \times (4 - 3^2)$ ✓
> $= 2.375 + 3.625 = 6$ ✓

---

3. [4 marks: 2, 2]

The shaded region in the accompanying diagram, is trapped between the curve $y = \cos(x)$, the x-axis and the y-axis.



(a) Explain why $\int_0^{\frac{3\pi}{2}} \cos(x)\,dx$ does not represent the area of the shaded region.

> $\int_0^{3\pi/2} \cos(x)\,dx$ represents the area of the shaded region only if $y \geq 0$
> over the interval $0 \leq x \leq \frac{3\pi}{2}$. But for $\frac{\pi}{2} < x < \frac{3\pi}{2}$, $y < 0$. ✓✓

(b) Estimate the area of this shaded region using uniform circumscribing rectangles of width $\frac{\pi}{2}$.

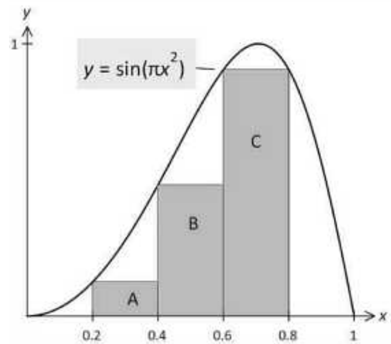
> Area $\approx (\frac{\pi}{2} \times 1) \times 3$ ✓
> $\approx \frac{3\pi}{2}$ ✓

Calculator Assumed

4. [8 marks: 3, 3, 2] [TISC]

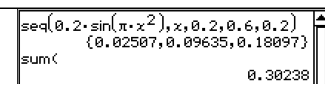
Region R is trapped between the curve $y = \sin(\pi x^2)$ for $0 \leq x \leq 1$ and the x-axis.

(a) S_1 is an under-estimate of the area of region R using the areas of rectangles A, B and C as shown in the diagram below. Complete the table below to show the calculations required to obtain S_1 to four decimal places.



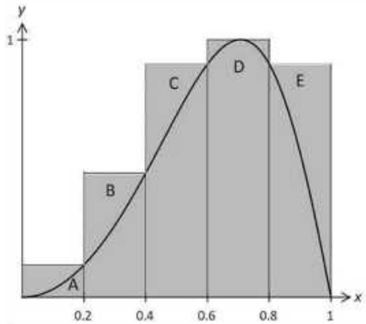
Rectangle	Area
A	$0.2 \times \sin(\pi \times 0.2^2) = 0.02507$
B	$0.2 \times \sin(\pi \times 0.4^2) = 0.09635$
C	$0.2 \times \sin(\pi \times 0.6^2) = 0.18097$
Total S_1	$0.30238 \approx 0.3024$

> ✓ Shows use of Area = 0.2 × height.
> ✓ Area of at least one rectangle correct.
> ✓ S_1 correct to 4 decimal places.



Calculator Assumed

4. (b) S_2 is an over-estimate of the area of region R using the areas of rectangles A, B, C, D and E as shown in the diagram below. Complete the table below to show the calculations required to obtain S_2 to four decimal places.



Rectangle	Area
A	$0.2 \times \sin(\pi \times 0.2^2) = 0.025067$
B	$0.2 \times \sin(\pi \times 0.4^2) = 0.096351$
C	$0.2 \times \sin(\pi \times 0.6^2) = 0.180965$
D	$0.2 \times 1 = 0.2$
E	$0.2 \times \sin(\pi \times 0.8^2) = 0.180965$
Total S_2	$0.683348 \approx 0.6833$

✓ Area of D correct.
✓ Area of at least two other rectangles correct.
✓ S_2 correct to 4 dp.

```
seq(0.2·sin(π·x²),x,0.2,0.8,0.2)
 {0.02507,0.09635,0.18097,0.18097}
sum(
                              0.48335
ans+0.2
                              0.68335
```

(c) Explain how you could use your answers in (a) and (b) to give a better estimate for the area of R. Give this better estimate correct to three decimal places.

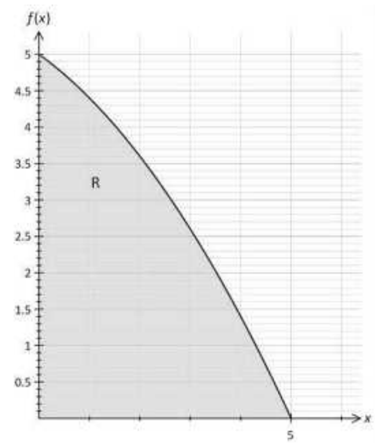
Better estimate is the mean of S_1 and S_2. ✓

$$\text{Mean} = \frac{0.3024+0.6834}{2}$$
$$= 0.4929 \approx 0.493$$ ✓

Calculator Assumed

5. [5 marks: 3, 3]

The diagram below shows the graph of $y = f(x)$. The shaded region R is trapped between the curve, the axes and the lines $x = -5$ and $x = -2$.
The accompanying table shows estimates for A, the area of region R using N uniform circumscribing rectangular strips for various values of N.



N	Estimate for A
2	20.3125
5	
10	15.8125
20	15.203125
50	14.8325
100	14.7081
500	14.6083
1 000	14.5958
5 000	14.5858
10 000	14.5846
15 000	14.5842

(a) Estimate the value for A for $N = 5$.

For $N = 5$, width of strip = 1 unit ✓
$A \approx [f(0) + f(1) + f(2) + f(3) + f(4)] \times 1$
$\approx [5 + 4.4 + 3.6 + 2.6 + 1.4] \times 1$ ✓
≈ 17 ✓

(b) Estimate to two decimal places the value for A for $N = 20\,000$. Justify your answer.

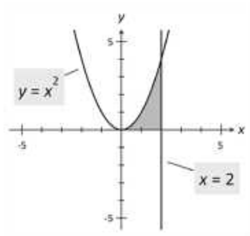
$A \approx 14.58$ ✓
For $N = 15\,000$,
$A = 14.5841 \approx 14.58$ to 2 DP ✓
As estimates are dropping,
for $N = 20\,000$, value of A will be 14.58 or 14.57.
But the value of A dropped by only 0.0004 over 5 000 extra strips.
Hence, it will be unlikely to be 14.57. ✓

Calculator Free

6. [10 marks: 3, 4]

Use calculus to find the exact area of the shaded region:

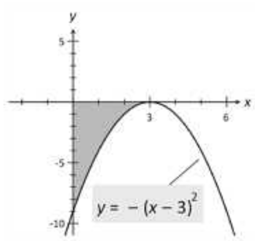
(a)



Area = $\int_0^2 x^2\,dx$ ✓

$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$ ✓✓

(b)



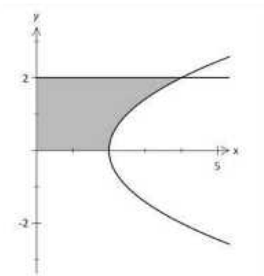
Area = $-\int_0^3 -(x-3)^2\,dx$ ✓✓

$= \left[\frac{(x-3)^3}{3}\right]_0^3$ ✓

$= 9$ ✓

7. [5 marks]

The shaded region shown in the accompanying diagram is trapped by the x-axis, the y-axis, the line $y = 2$ and the curve with equation $y^2 = 2x - 4$. Find the area of the shaded region.



$y = 2$ and $y^2 = 2x - 4$ intersect at:
$4 = 2x - 4 \Rightarrow x = 4$ ✓

Area = $4 \times 2 - \int_2^4 \sqrt{2x-4}\,dx$ ✓✓

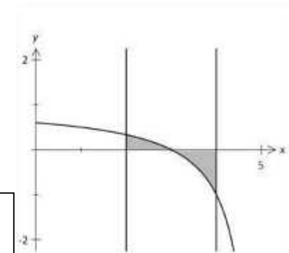
$= 8 - \left[\frac{2(2x-4)^{\frac{3}{2}}}{3(2)}\right]_2^4$ ✓

$= 8 - \frac{1}{3}\left[(8-4)^{\frac{3}{2}}\right] = 8 - \frac{8}{3} = \frac{16}{3}$ ✓

Calculator Free

8. [6 marks]

The shaded region shown in the accompanying diagram is trapped by the lines $x = 2$, $x = 4$ and the curve with equation $y = \frac{2}{x-5} + 1$.
Find the area of the shaded region.



The given curve cuts the x-axis at $x = 3$. ✓

Area = $\int_2^3 \frac{2}{x-5} + 1\,dx - \int_3^4 \frac{2}{x-5} + 1\,dx$ ✓✓

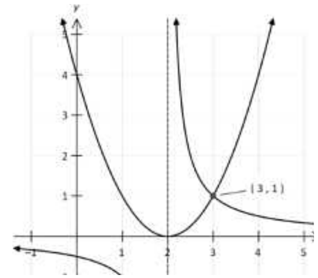
$= \left[2\ln(x-5) + x\right]_2^3 - \left[2\ln(x-5) + x\right]_3^4$ ✓

$= 1 + \ln\frac{4}{9} - (1 - 2\ln 2)$ ✓

$= 2\ln\frac{4}{3}$ ✓

9. [7 marks: 2, 5]

The accompanying diagram shows the graphs of $y = \frac{1}{(x-2)}$ and $y = (x-2)^2$.



(a) Determine with reasons if the area trapped between the curve $y = \frac{1}{(x-2)}$, the x-axis and the lines $x = 2$ and $x = 3$ can be calculated using $\int_2^3 \frac{1}{(x-2)}\,dx$.

NO! ✓
$y = \frac{1}{(x-2)}$ has an asymptote $x = 2$ ✓

Calculator Free

9. (b) Calculate the area of region trapped between the curves $y = \dfrac{1}{(x-2)}$, $y = (x-2)^2$ and the line $x = 4$.

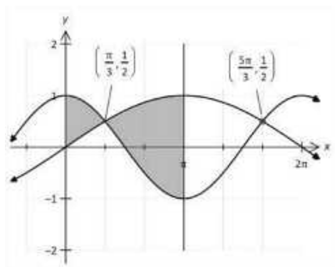
$$\text{Area} = \int_3^4 (x-2)^2 \, dx - \int_3^4 \frac{1}{x-2} \, dx \quad ✓✓$$

$$= \left[\frac{(x-2)^3}{3}\right]_3^4 - \left[ln(x-2)\right]_3^4 \quad ✓✓$$

$$= \frac{7}{3} - ln\,2 \quad ✓$$

10. [6 marks]

The shaded region in the accompanying diagram is trapped between the curves $y = \cos(x)$, $y = \sin(0.5x)$, the x-axis, the y-axis and the line $x = \pi$.
Calculate the area of the shaded region.



$$\text{Area} = \int_0^{\frac{\pi}{3}} \cos(x) - \sin(0.5x) \, dx + \int_{\frac{\pi}{3}}^{\pi} \sin(0.5x) - \cos(x) \, dx \quad ✓✓$$

$$= \left[\sin(x) + 2\cos(0.5x)\right]_0^{\frac{\pi}{3}} + \left[-\sin(x) - 2\cos(0.5x)\right]_{\frac{\pi}{3}}^{\pi} \quad ✓$$

$$= \left(\frac{\sqrt{3}}{2} + \sqrt{3}\right) - 2 + 0 - \left(-\frac{\sqrt{3}}{2} - \sqrt{3}\right) \quad ✓✓$$

$$= 3\sqrt{3} - 2 \quad ✓$$

Calculator Free

11. [8 marks: 3, 5]

(a) The gradient function of a curve $y = f(x)$ is given by $f'(x) = \cos(x)\sin(x)$.

The curve $y = f(x)$ passes through the point $(\frac{\pi}{3}, 1)$.

Find the equation of the curve $y = f(x)$.

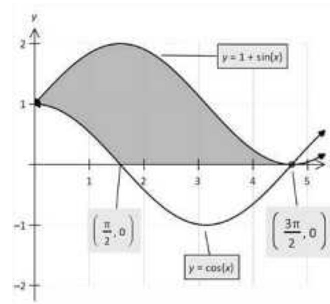
$$y = \int \cos(x)\sin(x)\, dx$$

$$= \frac{\sin^2(x)}{2} + C \quad ✓$$

When $x = \frac{\pi}{3}$, $y = 1$: $1 = \frac{3}{8} + C \Rightarrow C = \frac{-5}{8}$ ✓

Hence, $y = \dfrac{\sin^2(x)}{2} - \dfrac{5}{8}$ ✓

(b) The shaded region shown in the accompanying diagram is trapped by the curve with equation $y = 1 + \sin x$, $y = \cos(x)$ and the x-axis.
Find the area of the shaded region.



$$\text{Area} = \int_0^{\frac{3\pi}{2}} 1 + \sin(x) \, dx - \int_0^{\frac{\pi}{2}} \cos(x) \, dx \quad ✓✓$$

$$= \left[x - \cos(x)\right]_0^{\frac{3\pi}{2}} - \left[\sin(x)\right]_0^{\frac{\pi}{2}} \quad ✓✓$$

$$= \left(\frac{3\pi}{2} + 1\right) - 1$$

$$= \frac{3\pi}{2} \quad ✓$$

Calculator Free

12. [9 marks: 2, 2, 3, 2]

The function $y = f(x)$ is continuous for all real values of x and $f(x) \ge 0$ for $1 \le x \le 4$. It is known that $\int_1^4 f(x)\,dx = \text{A}$ and $\int_4^6 f(x)\,dx = -\text{B}$

here A and B are positive real numbers. Find, with reasons, in terms of A and/or B where appropriate:

(a) the area of the region trapped between the curve $y = 2f(x)$, the x-axis and the lines $x = 1$ and $x = 4$.

$$\text{Area} = \int_1^4 2f(x)\,dx \quad ✓$$
$$= 2\int_1^4 f(x)\,dx$$
$$= 2\text{A} \quad ✓$$

(b) $\int_1^6 f(x)\,dx$

$$\int_1^6 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad ✓$$
$$= \text{A} - \text{B} \quad ✓$$

(c) $\int_4^6 2x - f(x)\,dx$

$$\int_4^6 2x - f(x)\,dx = \int_4^6 2x\,dx - \int_4^6 f(x)\,dx \quad ✓$$
$$= \left[x^2\right]_4^6 - (-\text{B}) \quad ✓$$
$$= 20 + \text{B} \quad ✓$$

(d) $\int_{-1}^{-4} f(-x)\,dx$

$$\int_{-1}^{-4} f(-x)\,dx = -\int_{-4}^{-1} f(-x)\,dx \quad ✓$$
$$= -\int_1^4 f(x)\,dx$$
$$= -\text{A} \quad ✓$$

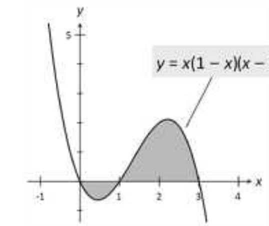
Calculator Assumed

13. [14 marks: 3, 3, 4, 4]

Use an appropriate method to find the area of the shaded region. Show clearly how you obtained your answer.

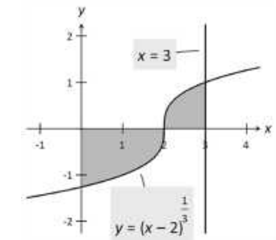
(a)

$$\text{Area} = \int_0^3 \left| x(1-x)(x-3) \right| dx \quad ✓✓$$
$$= \frac{37}{12} \text{ (or 3.08)} \quad ✓$$



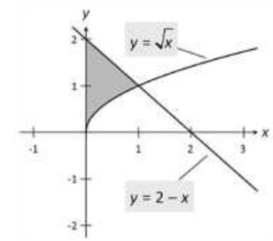
(b)

$$\text{Area} = \int_0^3 \left| (x-2)^{\frac{1}{3}} \right| dx \quad ✓✓$$
$$= 2.64 \quad ✓$$



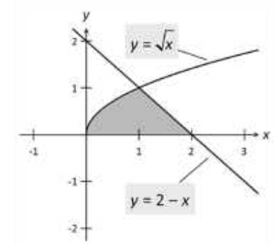
(c)

$$\text{Area} = \int_0^1 (2-x) - x^{\frac{1}{2}}\,dx \quad ✓✓✓$$
$$= \frac{5}{6} \quad ✓$$



(d)

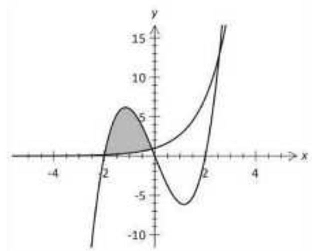
$$\text{Area} = \left[\int_0^1 x^{\frac{1}{2}}\,dx\right] + \left[\frac{1}{2} \times 1 \times 1\right] \quad ✓✓$$
$$= \frac{2}{3} + \frac{1}{2} = \frac{7}{6} \quad ✓✓$$



Calculator Assumed

14. [8 marks: 4, 4]

The accompanying diagram shows the graphs of $y = e^x$ and $y = 2x(x^2 - 4)$.



(a) Write an integral that can be used to determine the area of the shaded region. Hence, find the area of this region.

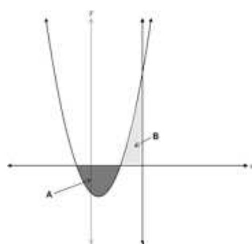
> For $x \le 0$, the two curves intersect at $x = -1.99141$ and $x = -0.112097$ ✓
>
> Area = $\int_{-1.99141}^{-0.112097} 2x(x^2 - 4) - e^x \, dx$ ✓✓
>
> $= 7.19178 \approx 7.19$ units^2 ✓

(b) Write but do not evaluate, an expression involving integrals that can be used to determine the area of the region trapped between the two curves, for $x \ge 0$.

> For $x \ge 0$, the two curves intersect at $x = 2.552433$ and $x = 5.890915$ ✓✓
>
> Required Area = $\int_0^{2.552433} e^x - 2x(x^2 - 4)\, dx + \int_{2.552433}^{5.890915} 2x(x^2-4) - e^x \, dx$
> ✓ ✓

15. [6 marks]

The accompanying diagram shows the graph of $y = (x + 1)(x - 2)$. The region bounded by the curve and the x-axis is denoted A. The region bounded by the curve, the positive x-axis and the line $x = k$ is denoted B.
Find k if the area of A = area of B.



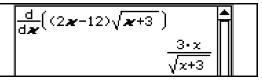
> Area of A = $-\int_{-1}^{2} (x+1)(x-2)\, dx = \frac{9}{2}$ ✓✓
>
> Area of B = $\int_2^k (x+1)(x-2)\, dx$
>
> $= \frac{k^3}{3} - \frac{k^2}{2} - 2k + \frac{10}{3}$ ✓✓
>
> Hence, $\frac{k^3}{3} - \frac{k^2}{2} - 2k + \frac{10}{3} = \frac{9}{2}$ ✓
>
> $k = \frac{7}{2}$ (reject $k = -1$) ✓

Calculator Assumed

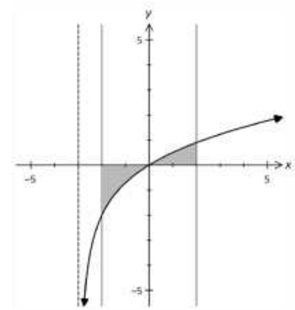
16. [6 marks: 1, 5] [TISC]

(a) Determine $\frac{d}{dx}\left((2x-12)\sqrt{x+3}\right)$.

> $\frac{d}{dx}\left((2x-12)\sqrt{x+3}\right) = \frac{3x}{\sqrt{x+3}}$. ✓



(b) The accompanying diagram shows the graph of $y = \frac{x}{\sqrt{x+3}}$. Region R (shaded) is trapped between the curve, the x-axis and the lines $x = -2$ and $x = 2$.
Show how your answer in (a) can be used to calculate the area of region R. Give your answer correct to 2 decimal places.

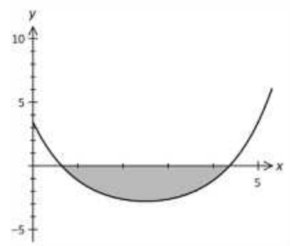


> Area = $-\int_{-2}^{0} \frac{x}{\sqrt{x+3}}\, dx + \int_0^2 \frac{x}{\sqrt{x+3}}\, dx$. ✓✓
>
> Since $\frac{d}{dx}\left((2x-12)\sqrt{x+3}\right) = \frac{3x}{\sqrt{x+3}}$
>
> $\int \frac{3x}{\sqrt{x+3}}\, dx = (2x-12)\sqrt{x+3} + C$
>
> $\int \frac{x}{\sqrt{x+3}}\, dx = \frac{1}{3}(2x-12)\sqrt{x+3} + D$ ✓
>
> Hence:
>
> Area = $\frac{-1}{3}\left[(2x-12)\sqrt{x+3}\right]_{-2}^{0} + \frac{1}{3}\left[(2x-12)\sqrt{x+3}\right]_0^2$ ✓
>
> $= \left(4\sqrt{3} - \frac{16}{3}\right) + \left(\frac{-8\sqrt{5}}{3} + 4\sqrt{3}\right)$
>
> $= 2.5602 \approx 2.56$ ✓

Calculator Assumed

17. [12 marks: 2, 5, 5]

A 10-metre portion of an irrigation channel is of uniform cross-section. The cross-section (shaded) is modelled by the equation $y = e^{x-3} + e^{-x+2} - 4$ where x is measured in metres. The top edge of the channel is modelled by the line $y = 0$.



(a) For what values of x is the model of the cross-section valid?

> Model is valid for $y \le 0$.
> Hence, $0.6375 \le x \le 4.3625$ ✓✓

(b) Use Calculus to find the depth of the deepest point of this portion of the channel.

> $y' = e^{x-3} - e^{-x+2}$ ✓
>
> At the deepest point, $y' = 0$.
> $\Rightarrow e^{x-3} - e^{-x+2} = 0$ ✓
> $\Rightarrow x = 2.5$ ✓
>
> $y'' = e^{x-3} + e^{-x+2}$
> When $x = 2.5$, $y'' > 0$. ✓
>
> Hence, channel is deepest when $x = 2.5$ m
> with depth $y = e^{-0.5} + e^{-0.5} - 4 = 2.79$ metres. ✓

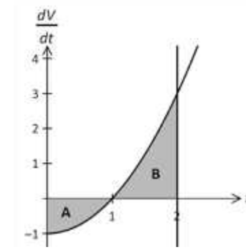
(c) Find the maximum capacity of this portion of the channel. [$1 \text{ m}^3 = 1$ kL]

> Cross-sectional area = $\int_{0.6375}^{4.3625} \left| e^{x-3} + e^{-x+2} - 4 \right| dt$ ✓✓
> $= 7.2765 \text{ m}^2$ ✓
>
> Hence, maximum capacity = 7.2765×10 ✓
> $= 72.765 \text{ m}^3$
> $= 72.77$ kL ✓

Calculator Assumed

18. [9 marks: 1, 2, 2, 2, 2]

The instantaneous rate with which the amount of fuel, V litres, in a holding tank, changes with respect to time t minutes, is modelled by $\frac{dV}{dt} = t^2 - 1$. The sketch of $\frac{dV}{dt}$ against t is shown in the accompanying diagram.



(a) Explain what happens at $t = 1$ minute.

> The amount of fuel in the tank is at minimum level (locally). ✓

(b) Find the area of region A and interpret your answer.

> Area of A = $-\int_0^1 t^2 - 1 \, dt = 2/3$ ✓
> This is the amount of fuel (L) drawn out from the tank in the first minute. ✓

(c) Find the area of region B and interpret your answer.

> Area of A = $\int_1^2 t^2 - 1 \, dt = 4/3$ ✓
> This is the amount of fuel (L) entering the tank in the second minute. ✓

(d) Find the amount of fuel in the tank after 2 minutes, if initially there were 5 litres in the tank.

> Amount of fuel = $5 - 2/3 + 4/3$ ✓
> $= 17/3$ L ✓

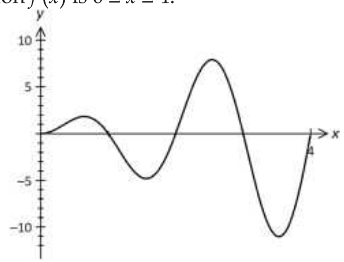
(e) Use the information in (d) to find the average rate of change of the amount of fluid in the first 2 minutes.

> Average rate of change = $\frac{17/3 - 5}{2}$ ✓
> $= 1/3$ L /minute ✓

Calculator Assumed

19. [7 marks: 2, 5] [TISC]

The diagram below shows the graph of the function $f(x) = \pi x \sin(\pi x)$ where the domain of the function $f(x)$ is $0 \le x \le 4$.



The function $A(t) = \int_0^t f(x)\,dx$ where $0 \le t \le 4$.

(a) Use your CAS calculator to help complete the table below.

t	0	1	2	3	4
$A(t)$	0	1	−2	3	−4

✓ $A(1)$ correct. ✓ All correct.

(b) Use a calculus method to determine the minimum value for $A(t)$ and the corresponding value of t. Show clearly how you obtained your answer.

$\frac{dA}{dt} = \frac{d}{dt}\left(\int_0^t \pi x \sin(\pi x)\,dx\right)$ ✓

$= \pi t \sin(\pi t)$ ✓

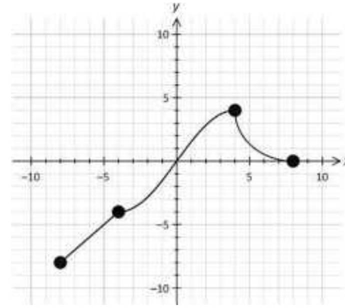
$\frac{dA}{dt} = 0 \Rightarrow t = 0, 1, 2, 3, 4$ ✓

From table in (a):
minimum value for $A(t) = -4$ ✓
when $t = 4$ ✓

Calculator Assumed

20. [6 marks: 2, 4]

The accompanying diagram shows the graph of $y = f(x)$ for $-8 \le x \le 8$. The graph consists of the line $y = -x$ for $-8 \le x < -4$, the curve $y = 4\sin\left(\frac{\pi x}{8}\right)$ for $-4 \le x < 4$ and the quarter circle with centre at (8, 4) and radius 4 for $4 \le x \le 8$.



(a) Determine with reasons the value of $\int_{-4}^{4} f(x)\,dx$.

$y = 4\sin\left(\frac{\pi x}{8}\right)$ for $-4 \le x < 4$ is symmetrical about the origin. ✓

Hence sum of signed areas $\int_{-4}^{4} f(x)\,dx = 0$. ✓

(b) Evaluate $\int_{-8}^{8} f(x)\,dx$.

$\int_{-8}^{8} f(x)\,dx = \int_{-8}^{-4} f(x)\,dx + \int_{-4}^{4} f(x)\,dx + \int_{4}^{8} f(x)\,dx$ ✓

$\int_{-8}^{-4} f(x)\,dx = -\frac{1}{2}(8+4)\times 4 = -24$ ✓

From (a): $\int_{-4}^{4} f(x)\,dx = 0$

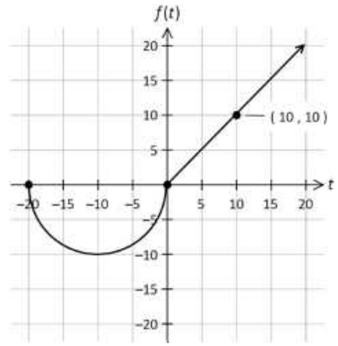
$\int_{4}^{8} f(x)\,dx = 4 \times 4 - \frac{\pi\times 4^2}{4} = 16 - 4\pi$ ✓

Hence: $\int_{-8}^{8} f(x)\,dx = -24 + 0 + 16 - 4\pi$ ✓

Calculator Assumed

21. [6 marks] [TISC]

The accompanying diagram shows the graph of $y = f(t)$ which consists of a semi-circle with radius 10 for $-20 \le t \le 0$ and a straight line $y = t$ for $t > 0$.



The function $A(x) = \int_{-20}^{x} f(t)\,dt$.

(a) Determine the <u>*exact*</u> minimum value for $A(x)$. Explain clearly how you obtained your answer.

> Using the FTC, $\frac{dA}{dx} = f(x)$.
>
> For min point $\frac{dA}{dx} = f(x) = 0$. ✓
>
> From given graph $f(-20) = f(0) = 0$.
> Hence, minimum value $= A(0)$ ✓
>
> $= \int_{-20}^{0} f(t)\,dx$
>
> $= -\frac{\pi \times 10^2}{2} = -50\pi$ ✓

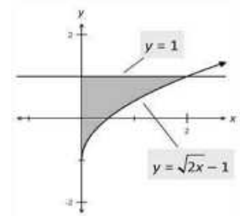
(b) Determine with reasons the *exact root* of the function $A(x)$.

> For A(k) = 0: $\int_{-20}^{k} f(t)\,dx = 0$
>
> $\int_{-20}^{0} f(t)\,dx + \int_{0}^{k} f(t)\,dx = 0$ ✓
>
> $\Rightarrow \int_{0}^{k} f(t)\,dx = 50\pi$ where $k > 0$. ✓
>
> $\frac{1}{2} \times k \times k = 50\pi \Rightarrow k = 10\sqrt{\pi}$ ✓

Calculator Assumed

22. [7 marks: 2, 3, 2]

The integral $\int_{a}^{b} f(y)\,dy$ represents the sum of the signed areas of the regions trapped between the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$.



Consider the curve with equation $y = \sqrt{2x} - 1$.

(a) Rewrite the equation of the curve with y as the independent variable.

> $\sqrt{2x} = y + 1$
>
> $x = \frac{1}{2}(y+1)^2$ ✓✓

(b) Let R represent the region trapped between the curve with equation $y = \sqrt{2x} - 1$, the y-axis and the line $y = 1$.

(i) Use your answer in (a) and an integral to express the area of R.

> $y = \sqrt{2x} + 1$ intersects the y-axis at $y = -1$. ✓
>
> Area $= \int_{-1}^{1} \frac{1}{2}(y+1)^2\,dy$ ✓✓

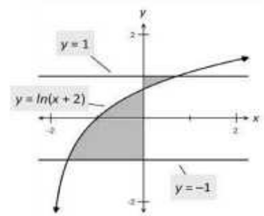
(ii) Use your answer above to determine the exact area of R.

> Area $= \int_{-1}^{1} \frac{1}{2}(y+1)^2\,dy$
>
> $= \frac{4}{3}$ ✓✓
>
> $\int_{-1}^{1} \frac{(y+1)^2}{2}\,dy$ $\frac{4}{3}$

Calculator Assumed

23. [8 marks: 2, 4, 2]

The integral $\int_a^b f(y)\,dy$ represents the sum of the signed areas of the regions trapped between the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$.



Consider the curve with equation $y = ln\,(x + 2)$.

(a) Rewrite the equation of the curve with y as the independent variable.

$$ln\,(x+2) = y$$
$$x + 2 = e^y \quad ✓$$
$$x = e^y - 2 \quad ✓$$

(b) Let R represent the region trapped between the curve with equation $y = ln\,(x + 2)$, the y-axis and the lines $y = -1$ and $y = 1$.

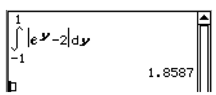
(i) Use your answer in (a) and integrals to express the area of R.

$y = ln\,(x + 2)$ intersects the y-axis at $y = ln\,2$. ✓

$$\text{Area} = -\int_{-1}^{ln 2} e^y - 2\ dy + \int_{ln 2}^{1} e^y - 2\ dy \quad ✓✓✓$$

(ii) Use your answer above to determine the area of R. Give your answer to two decimal places.

$$\text{Area} = -\int_{-1}^{ln 2} e^y - 2\ dy + \int_{ln 2}^{1} e^y - 2\ dy$$
$$= 1.7542 + 0.1046 \quad ✓$$
$$\approx 1.86 \quad ✓$$



21 Rectilinear Motion

Calculator Assumed

1. [11 marks: 4, 3, 4]

The displacement of a body moving along a straight line is given by $s = -t^3 + at^2 + bt + 3$ metres where t is time in seconds. The initial velocity of the body is 5 ms^{-1}. The body is momentarily at rest when $t = 1$ second.

(a) Find the values of a and b.

Velocity $v = \frac{ds}{dt} = -3t^2 + 2at + b$ ✓

When $t = 0$, $v = 5$. $\Rightarrow b = 5$ ✓

Hence, $v = -3t^2 + 2at + 5$

Body is at rest when $t = 1$. $\Rightarrow -3 + 2a + 5 = 0$ ✓

Hence, $a = -1$ ✓

(b) Find when the body changes direction.

When body changes direction,
$v = 0$;
$\Rightarrow -3t^2 - 2t + 5 = 0$ ✓
$t = 1$ second (reject $\frac{-5}{3}$) ✓

$v(1^-) > 0$ and $v(1^+) < 0$, hence $t = 1$ second ✓

(c) Find the instantaneous speed at $t = 5$ seconds and the average speed in the first 5 seconds.

Instantaneous speed at $t = 5$,
$v(5) = -80\ ms^{-1}$. ✓

Distance travelled in the first 5 seconds
$= \int_0^5 \left|-3t^2 - 2t + 5\right| dt = 131$ m ✓✓

Hence, average speed = 26.2 ms^{-1}. ✓

Calculator Assumed

2. [7 marks: 1, 3, 1, 2]

The displacement of a body at time t seconds is given by $s = 4t + \frac{1}{1+t}$ metres.

(a) Find an expression for the velocity of the body at time t seconds.

> Velocity $v = \frac{ds}{dt} = 4 - \frac{1}{(1+t)^2}$ ✓

(b) Show that the body is never stationary.

> For the body to be stationary,
> $4 - \frac{1}{(1+t)^2} = 0$ ✓
> $t = -\frac{1}{2}$ or $-\frac{3}{2}$ ✓
> But time $t \geq 0$. ✓
> Hence, the body is never stationary.

(c) Find an expression for the acceleration at time t seconds.

> Acceleration $a = \frac{dv}{dt} = \frac{2}{(1+t)^3}$ ✓

(d) Describe the motion of the body for large values of t.

> $v = 4 - \frac{1}{(1+t)^2}$.
> For large values of t, $\frac{1}{(1+t)^2} \to 0$. ✓
> Hence, $v \to 4$.
> That is, for large values of t the body moves with a constant velocity of 4 ms^{-1}. ✓

Calculator Assumed

3. [13 marks: 2, 5, 2, 4]

The displacement (s metres) of particle P, t seconds after passing a fixed point O is given by $s = 10\,t\,e^{-t} - 1$.

(a) Find an expression for the velocity at time t seconds and hence the velocity at $t = 2$ seconds.

> Velocity $v = \frac{ds}{dt} = 10\,e^{-t}(1-t)$ ✓
> $v(2) = -1.35$ ms^{-1}. ✓

(b) Use Calculus to find the maximum displacement of P in the first two seconds.

> For local maximum: $v = \frac{ds}{dt} = 0$. ✓
> $\Rightarrow 1 - t = 0 \Rightarrow t = 1$ ✓
> At $t = 1$;

	1^-	1	1^+
v	+	0	–

> Hence, local maximum occurs at $t = 1$. ✓
> s(0) = –1, s(2) = 1.71, s(1) = 2.68 ✓
> Hence, maximum displacement for P (in the first 2 seconds) is 2.68 m ✓

(c) Find the acceleration of P at maximum displacement.

> Acceleration $a = \frac{dv}{dt} = -20\,e^{-t} + 10\,t\,e^{-t}$ ✓
> When $t = 1$, $a = -3.68$ ms^{-2} ✓

(d) Find the average speed in the first 2 seconds.

> Distance travelled in the first 2 seconds
> $= \int_0^2 \left|10e^{-t}(1-t)\right| dt = 4.6509$ m ✓✓✓
> Hence, average speed = 2.33 ms^{-1}. ✓

Calculator Assumed

4. [13 marks: 3, 3, 2, 5]

An object P travels along the x-axis. The velocity of the particle t seconds after passing through the origin is given by $v = \frac{1}{1+t} + \frac{1}{(1+t)^2}$ cms^{-1}.

(a) Calculate the magnitude and direction of the acceleration of P as it passes through the origin.

> Acceleration $a = \frac{d}{dt}\left(\frac{1}{1+t} + \frac{1}{(1+t)^2}\right) = \frac{-1}{(1+t)^2} - \frac{2}{(1+t)^3}$ ✓
> $a(0) = -3$ cms^{-2}
> Hence, magnitude = 3 ✓
> Direction: Direction of the negative axis. ✓

(b) Determine the displacement of P at the end of the fifth second.

> $s = \int \frac{1}{1+t} + \frac{1}{(1+t)^2}\, dt$ ✓
> $= ln\,(1+t) - \frac{1}{1+t} + C$ ✓
> $s(0) = 0 \Rightarrow C = 1$
> $s(5) = 2.625$ cm. ✓

(c) Determine if P ever reverses direction after it travels past O.

> When P reverses direction $v = 0$
> $\frac{1}{1+t} + \frac{1}{(1+t)^2} = 0$ ✓
> $\Rightarrow t = -2.$
> Hence, No! ✓

(d) Determine an expression for D, the distance travelled in the first N seconds. Hence, find N if the average speed of P in the first N seconds is 0.33 cms^{-1}. Justify your answer.

> There is no change in direction after passing through O. ✓
> $\Rightarrow$ Total distance travelled in first N seconds:
> $D = \int_0^N \frac{1}{1+t} + \frac{1}{(1+t)^2}\, dt$ ✓
> $= ln\,(1+N) - \frac{1}{1+N} + 1$ ✓
> Average speed $= \frac{ln(1+N) - \frac{1}{1+N} + 1}{N} = 0.33$ ✓
> $N = 10$ ✓

Calculator Assumed

5. [12 marks: 2, 3, 2, 3, 2]

An object P travels in a straight line. The velocity of the particle t seconds after passing a fixed point O is given by $v = 12t^3 - 48t^2 + 60t - 24$ cms^{-1}.

(a) Calculate the initial acceleration of P.

> Acceleration $a = \frac{d}{dt}(12t^3 - 48t^2 + 60t - 24) = 36t^2 - 96t + 60$ ✓
> $a(0) = 60$ cms^{-2} ✓

(b) Determine when P reverses direction.

> $v = 0 \Rightarrow 12t^3 - 48t^2 + 60t - 24 = 0$
> $\Rightarrow t = 1, 2$ ✓
> $a(1) = 0$
> $a(2) = 12 \neq 0$ ✓
> Hence, reverses direction at $t = 2$ seconds. ✓

(c) Determine the displacement of P at the time it reverses direction.

> $s(0) = 0$
> $s = \int 12t^3 - 48t^2 + 60t - 24\, dt$
> $= 3t^4 - 16t^3 + 30t^2 - 24t$ ✓
> $s(2) = -8$ cm. ✓

(d) The change in displacement in the first N seconds is 9 metres. Determine the value of N.

> $s(N) - s(0) = 900$ ✓
> $3N^4 - 16N^3 + 30N^2 - 24N = 900$ ✓
> $N = 5.55$ (reject -2.87 as $N \in \mathbb{Z}^+$) ✓

(e) Calculate the average speed of P in the first five seconds.

> Total distance travelled in first 5 seconds
> $D = \int_0^5 \left| 12t^3 - 48t^2 + 60t - 24 \right| dt$ ✓
> $= 521$
> Average speed $= \frac{521}{5} = 104.2$ cms^{-2}. ✓

Calculator Assumed

6. [9 marks: 2, 3, 4]

The acceleration (ms^{-2}) of a particle moving along a straight line is given by $a = -4\cos 2t$, where t is time in seconds. At $t = 0$, the velocity of the particle is $0\ ms^{-1}$ and the displacement of the particle is 1 m.

(a) Find an expression for the velocity of the particle at any time t.

$v = \int -4\cos 2t\ dt$
$= -2\sin 2t + C$ ✓

When $t = 0$, $v = 0$. $\Rightarrow C = 0$
Hence, $v = -2\sin 2t$. ✓

(b) Find an expression for the displacement of the particle at any time t.

Displacement $x = \int -2\sin 2t\ dt$
$= \cos 2t + K$ ✓

When $t = 0$, $x = 1$. $\Rightarrow K = 0$ ✓
Hence, $x = \cos 2t$. ✓

(c) Find the average speed in the first π seconds.

Distance travelled $= \int_0^{\pi} |-2\sin 2t|\ dt$ ✓✓
$= 4$ m ✓
Average speed $= \frac{4}{\pi} \approx 1.27\ ms^{-1}$ ✓

Calculator Assumed

7. [10 marks: 1, 6, 3]

A particle starts off from a fixed point O with an acceleration (mms^{-2}) of $a = mt - 24$, where t is time in seconds. The particle travels in a straight line and returns to O at $t = 4$ seconds and has a change of displacement of –9 mm in the third second (it moves in the same direction during this time).

(a) Find in terms of m an expression for the velocity of the particle at any time t.

Velocity $v = \int mt - 24\ dt$
$= \frac{mt^2}{2} - 24t + C$ ✓

(b) Find the displacement of the particle at any time t.

Displacement $x = \frac{mt^3}{6} - 12t^2 + Ct + K$ ✓

When $t = 0$, $x = 0$. $\Rightarrow K = 0$.
Hence, $x = \frac{mt^3}{6} - 12t^2 + Ct$ ✓

When $t = 4$, $x = 0$. $\Rightarrow \frac{64m}{6} + 4C = 192$ (I) ✓

Also, $x(3) - x(2) = -9$. $\Rightarrow \frac{19m}{6} + C = 51$ (II) ✓

Solve I and II simultaneously, $m = 6$, $C = 32$

Hence, $x = t^3 - 12t^2 + 32t$ ✓✓

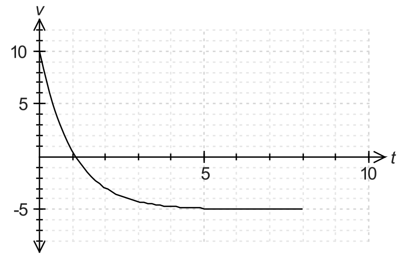
(c) Find when the particle is at O the third time (if it does).

When it is at O, $x = 0$.
$\Rightarrow t^3 - 12t^2 + 32t = 0$ ✓
$t = 0, 4, 8$ ✓
Hence, the particle is at O for the third time at t = 8 seconds. ✓

Calculator Assumed

8. [11 marks: 2, 3, 3, 3]

A particle P moving in a straight line, starts off from a fixed point O with velocity v_0 ms^{-1}. Its velocity at any time t is given by $v = 15e^{-t} - k$ ms^{-1}, where k is a constant. The velocity-time graph of P is given in the accompanying diagram. Velocity v is in ms^{-1} and time t in seconds.



(a) Find v_0 and k.

$v_0 = 10$ ms^{-1} ✓
Hence, $k = 5$. ✓

(b) Find the time (correct to 2 decimal places) when P reverses direction.

$v = 15e^{-t} - 5$
When it reverses direction, $v = 0$.
$\Rightarrow v = 15e^{-t} - 5 = 0$ ✓
$t = 1.10$ seconds. ✓
$v(1.1^-) > 0$ and $v(1.1^+) < 0$, hence $t = 1.1$ seconds ✓

(c) Find the displacement of P at the time it reversed its direction.

Displacement $x = \int 15e^{-t} - 5 \ dt$
$= -15e^{-t} - 5t + C$ ✓

When $t = 0$, $x = 0$. $\Rightarrow C = 15$
Hence, $x = -15e^{-t} - 5t + 15$ ✓

At $t = 1.10$, $x = 4.51$ m ✓

(d) Find the average speed in the first 8 seconds.

Distance travelled $= \int_0^8 \left|15e^{-t} - 5\right| dt$ ✓
$= 34.0189$ m ✓
Hence, average speed $= \dfrac{34.0189}{8}$
$= 4.25$ ms^{-1} ✓

Calculator Assumed

9. [11 marks: 3, 3, 2, 3] [TISC]

The acceleration a ms^{-2}, of a particle P moving in a straight line at time t seconds is given by $a = mt + n$. The average acceleration during the first two seconds is 1 ms^{-2} and the initial velocity of the particle is 4 ms^{-1}. The acceleration of the particle at $t = 2$ is –1 ms^{-2},

(a) Find the change in velocity in the first two seconds.

$v = \int mt + n \ dt = \dfrac{mt^2}{2} + nt + k$ ✓
$v(0) = k$
$v(2) = 2m + 2n + k$ ✓
Change in velocity $= v(2) - v(0)$
$= 2(m + n)$ ✓

(b) Show that $v = \dfrac{1}{2}(1 - n)\,t^2 + nt + 4$.

Average acceleration $= \dfrac{2(m+n)}{2-0}$ ✓
$\Rightarrow m + n = 1$
$m = 1 - n$ ✓
Hence, $v = \dfrac{(1-n)\,t^2}{2} + nt + k$
$v(0) = 4 \Rightarrow k = 4$.
$v = \dfrac{(1-n)\,t^2}{2} + nt + 4$ ✓

(c) Show that $v = -t^2 + 3t + 4$.

$t = 2$, $a = -1 \Rightarrow 2m + n = -1$
But from (b) $m + n = 1$
Hence, $m = -2$, $n = 3$ ✓✓
Hence, $v = -t^2 + 3t + 4$

(d) Find the total distance travelled from the moment the particle starts travelling to before the particle changes direction.

$v = -t^2 + 3t + 4$
Particle changes direction when $v = 0$:
$t = 4$ (reject $t = -1$) ✓
Distance travelled $= \int_0^4 -t^2 + 3t + 4 \ dt$ ✓
$= \dfrac{56}{3}$ metres. ✓

Calculator Assumed

10. [11 marks: 4, 7]

A particle P travels along the x-axis and its acceleration at time t seconds is given by $\frac{d^2x}{dt^2} = pt^2 + qt + r \text{ cms}^{-2}$ where p, q and r are constants. The particle starts from the point K with coordinates (8, 0) with velocity -12 cms^{-1}. The particle changes direction at the same point when $t = 1$ and $t = 2$ seconds.

(a) Show that its displacement at time t seconds is given by

$$x = \frac{pt^4}{12} + \frac{qt^3}{6} + \frac{rt^2}{2} - 12t + 8.$$

$$\frac{d^2x}{dt^2} = pt^2 + qt + r$$

$$\frac{dx}{dt} = \frac{pt^3}{3} + \frac{qt^2}{2} + rt - 12 \quad ✓✓$$

$$x = \frac{pt^4}{12} + \frac{qt^3}{6} + \frac{rt^2}{2} - 12t + 8 \quad ✓✓$$

(b) Find the values of p, q and r.

$v(1) = 0$

$$\Rightarrow \frac{p}{3} + \frac{q}{2} + r - 12 = 0 \qquad \text{I} \quad ✓$$

$v(2) = 0$

$$\Rightarrow \frac{8p}{3} + 2q + 2r - 12 = 0 \qquad \text{II} \quad ✓$$

$x(1) = x(2)$

$$\Rightarrow \frac{p}{12} + \frac{q}{6} + \frac{r}{2} - 12 + 8 = \frac{16p}{12} + \frac{8q}{6} + 2r - 24 + 8 \qquad \text{III} \quad ✓✓$$

$$p = 12,\ q = -36,\ r = 26 \quad ✓✓✓$$

$$\left\{\begin{array}{l} \frac{x}{3}+\frac{y}{2}+z-12=0 \\ \frac{8x}{3}+2y+2z-12=0 \\ \frac{x}{12}+\frac{y}{6}+\frac{z}{2}-12+8=\frac{16x}{12}+\frac{8y}{6}+2z-24+8 \end{array}\right| x, \quad \{x=12, y=-36, z=26\}$$

Calculator Assumed

11. [13 marks: 4, 3, 3, 3] [TISC]

A particle P travels in a straight line. Its displacement s (metres) from a fixed point O at time t seconds is given $s = -0.1t^4 + 0.5t^3 + 1.2t^2 - 7.6t + 8$.
The points A and B are on opposite sides of O.
The particle starts at A and completes its journey at B where OA = OB.

(a) Calculate how long it takes P to travel from A to B.

$t = 0 \Rightarrow s = 8$ ✓
At B, $s = -8$ ✓
$\Rightarrow -0.1t^4+0.5t^3+1.2t^2-7.6t+8 = -8$ ✓
$t = 5.6303$
Hence, 5.6 seconds. ✓

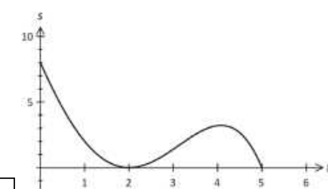
(b) Determine the total distance travelled by P in travelling from A to B.

$v = \frac{ds}{dt} = -0.4t^3+1.5t^2+2.4t-7.6$ ✓
Total distance travelled
$= \int_0^{5.6303} \left| -0.4t^3+1.5t^2+2.4t-7.6 \right| dt$ ✓
$\approx 22.4318\ m$ ✓

(c) The acceleration of P is positive between t_1 and t_2. Determine t_1 and t_2.

$a = \frac{dv}{dt} = -1.2t^2 + 3t + 2.4$ ✓
$a > 0 \Rightarrow -0.6375 < t < 3.1375$ ✓
But $t \geq 0$: $\Rightarrow 0 \leq t < 3.1375$
Hence: $t_1 = 0$ and $t_2 = 3.1375$ ✓

(d) The accompanying diagram shows the graph of s against t for $0 \leq t \leq 5$ seconds. Describe the motion of P in the first two seconds.in terms of its displacement, direction and speed and acceleration.



P is moving towards the origin from $s = 8$ and reaches O at $t = 2$ seconds. ✓
Speed is decreasing from 7.6 ms^{-1} at A to 0 ms^{-1} at O. ✓
P undergoes positive acceleration. ✓

Calculator Assumed

12. [12 marks: 1, 3, 2, 4, 2] [TISC]

A particle P starts from rest from a fixed point O and travels in a straight line. The velocity of P after t seconds is given by $v = 100(e^{-0.1t} - e^{-1.1t})$ ms^{-1} for $t \geq 0$ seconds.

(a) Show that P travels only in one direction.

> $v = 0 \Rightarrow t = 0$.
> Hence, there is no change in direction and P travels in only one direction. ✓

(b) Determine using a calculus method when the acceleration of P is zero.

> $\frac{dv}{dt} = -10e^{-0.1t} + 11e^{-1.1t}$ ✓
> $\frac{dv}{dt} = 0 \Rightarrow t = \ln 11 \approx 2.3979$ seconds ✓✓

(c) Determine the maximum speed of P in the first two seconds. Explain how you obtained your answer.

> For $t \geq 0$, $v \geq 0$
> From (b), v is max at $t \approx 2.4$ seconds which is outside the first 2 seconds. ✓
>
> $v(0) = 0$ & $v(2) = 70.7928$
> Hence, maximum velocity ≈ 70.79 ms^{-1} ✓

(d) Determine the displacement of P when $t = 99$ seconds and $t = 100$ seconds.

> $s(t) = \int 100(e^{-0.1t} - e^{-1.1t})\,dt$
> $= -1000e^{-0.1t} + \frac{100e^{-1.1t}}{1.1} + C$ ✓
> $s(0) = 0 \Rightarrow C = \frac{10000}{11}$
> $\Rightarrow s(t) = -1000e^{-0.1t} + \frac{100e^{-1.1t}}{1.1} + \frac{10000}{11}$ ✓
> $s(99) = 909.0407$ ✓
> $s(100) = 909.0455$ ✓

(e) Describe the speed and displacement of P for extremely large values of t.

> As $t \to \infty$, $v \to 0$ ms^{-1} ✓
> As $t \to \infty$, $s \to \frac{10000}{11} \approx 909.1$ m ✓
> That is, P comes to a stop at $\frac{10000}{11}$.

Calculator Assumed

13. [11 marks: 2, 3, 2, 4]

A particle P travels along a straight line. The velocity of particle P at time t seconds is given by $v = \ln(1+2t)$ ms^{-1}.

(a) Calculate the acceleration of P when $t = 2$ seconds.

> $a(t) = \frac{d}{dt}\ln(1+2t) = \frac{2}{1+2t}$ ✓
> $a(2) = 0.4$ ms^{-2} ✓

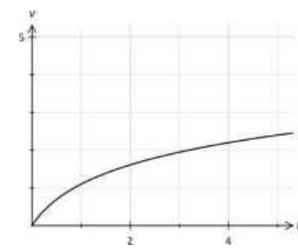
(b) Use the incremental formula to estimate the increase in the velocity of P when time t increases from 2 seconds by 0.1 seconds.

> $\delta v \approx \frac{2}{1+2t} \times \delta t$ ✓
> $\approx 0.4 \times 0.1$ ✓
> ≈ 0.04 ✓

(c) Write a mathematical expression involving the use of integrals that represents the distance travelled by P in the first two seconds. DO NOT evaluate this integral.

> Distance travelled
> $= \int_0^2 |\ln(1+2t)|\,dt$ ✓✓

The accompanying diagram shows the graph of $v = \ln(1+2t)$.



(d) Estimate the change in displacement of P between $t = 2$ and $t = 4$ seconds by using two rectangular strips of equal width. Show clearly how you obtained your answer. State whether your answer is an under-estimate or an over-estimate

> Estimate $= (v(2) \times 1) + (v(3) \times 1)$ ✓
> $= \ln 5 + \ln 7$ ✓
> ≈ 3.5553 ✓
> Answer is an under-estimate. ✓
>
> OR
>
> Estimate $= (v(3) \times 1) + (v(4) \times 1)$ ✓
> $= \ln 7 + \ln 9$ ✓
> ≈ 4.1431 ✓
> Answer is an over-estimate. ✓

Calculator Assumed

14. [13 marks: 3, 2, 4, 2, 2] [TISC]

A particle P travels in a straight line with an initial velocity of 10 cms^{-1}.
The acceleration of the particle at time t seconds is given by
$a = 12t^2 - 24t - k$ cms^{-2} where k is a constant.

(a) Determine an expression, in terms of k, for the velocity of P at time t seconds.

> $v(t) = \int 12t^2 - 24t - k\ dt$ ✓
> $= 4t^3 - 12t^2 - kt + C$ ✓
> $v(0) = 10 \Rightarrow C = 10$
> Hence: $v(t) = 4t^3 - 12t^2 - kt + 10$ ✓

(b) Determine an expression, in terms of k, for the displacement of P at time t seconds.

> $s(t) = \int 4t^3 - 12t^2 - kt + 10\ dt$ ✓
> $= t^4 - 4t^3 - \frac{kt^2}{2} + 10t + D$ ✓

(c) The particle P changes direction at $t = 5$ seconds.

(i) Determine the change of displacement in the first second. [4 marks]

> $v(5) = 0 \Rightarrow -5k + 210 = 0$ ✓
> $k = 42$ ✓
> $s(t) = t^4 - 4t^3 - 21t^2 + 10t + D$ ✓
> $s(1) - s(0) = (-14 + D) - D = -14$ cm. ✓
> OR Change in displacement $= \int_0^1 4t^3 - 12t^2 - 42t + 10\ dt$ ✓
> $= -14$ ✓

(ii) Calculate the distance travelled in the first second.

> Distance travelled $= \int_0^1 \left| 4t^3 - 12t^2 - 21t + 10 \right| dt$ ✓
> $= 16.2878$ cm ✓

(iii) Determine with reasons if there is evidence that P undergoes a change in direction before $t = 5$ seconds.

> Yes. ✓
> Distance travelled in first second $\neq$ |Change in displacement in first second|. ✓

22 Discrete Random Variables I

Calculator Free

1. [4 marks: 2, 2]

Determine with reasons if each of the following functions are probability distribution functions for discrete random variables.

(a) $f(k) = 1/5$ for $k = 0, 1, 2, 3, 4, 5$

> $\sum f(k) = 6/5 \neq 1$. ✓
> Hence, $f(k)$ is not a pdf. ✓

(b) $f(k) = k/5$ for $k = -2, -1, 0, 1, 2, 3$

> $f(-2) = -2/5 < 0$. ✓
> Hence, $f(k)$ is not a pdf. ✓

---

2. [8 marks: 3, 3, 2]

The random variable X has probability distribution

$$P(X = x) = \begin{cases} x \times P(X = x+1) & x = 1, 2, 3 \\ k & x = 4 \end{cases}.$$

(a) Find the value of k.

x	1	2	3	4
$P(X = n)$	$6k$	$6k$	$3k$	k

> Hence, $6k + 6k + 3k + k = 1$ ✓✓
> $k = \frac{1}{16}$ ✓

(b) Find $P(X \le 3 | X > 1)$.

> $P(X \le 3 | X > 1) = \frac{P(1 < X \le 3)}{P(X > 1)}$ ✓✓
> $= \frac{9k}{10k} = \frac{9}{10}$ ✓

(c) Determine the mean for X.

> Mean for X $= 6k + 12k + 9k + 4k$ ✓
> $= 31k = \frac{31}{16}$ ✓

Calculator Free

3. [10 marks: 1, 3, 3, 3]

The table below describes the cumulative probability distribution function of a discrete random variable X.

x	0	1	2	3	4	5
$P(X \le x)$	$\frac{11}{20}$	$\frac{15}{20}$	$\frac{17}{20}$	$\frac{18}{20}$	$\frac{19}{20}$	k

(a) Determine the value of k.

$k = 1$ ✓

(b) Calculate $P(X < 4 | X \ge 1)$.

$$P(X<4|X\ge 1) = \frac{P(1\le X<4)}{P(X\ge 1)} \quad ✓$$
$$= \frac{P(X\le 3)-P(X\le 0)}{1-P(X=0)} = \frac{\frac{18}{20}-\frac{11}{20}}{\frac{9}{20}} \quad ✓$$
$$= \frac{7}{9} \quad ✓$$

(c) Calculate μ, the mean for X.

$$\mu = \sum x\,p(x)$$
$$= 0 + 1\times\frac{4}{20} + 2\times\frac{2}{20} + 3\times\frac{1}{20} + 4\times\frac{1}{20} + 5\times\frac{1}{20} \quad ✓✓$$
$$= 1 \quad ✓$$

(d) Calculate the σ^2, the variance for X.

$$\sigma^2 = \sum (x-1)^2\,p(x) \quad ✓$$
$$= 1\times\frac{11}{20} + 0 + 1\times\frac{2}{20} + 4\times\frac{1}{20} + 9\times\frac{1}{20} + 16\times\frac{1}{20} \quad ✓$$
$$= \frac{42}{20} \quad ✓$$

Calculator Free

4. [4 marks: 2, 2] [TISC]

The random variable Y has values Y = 0, 1, 2, 3 & 4. The table below describes the cumulative probability distribution function for Y, where *a*, *c*, *d*, *e* and *f* are non-negative constants.

y	0	1	2	3	4
$P(Y \le y)$	a	c	d	e	f

(a) Determine the range of values for a and the value for f.

$P(Y\le 0) = a \Rightarrow 0 \le a \le c$ ✓
$P(Y\le 4) = 1 \Rightarrow f = 1.$ ✓

(b) Determine $P(1 < Y \le 3)$.

$P(1<Y\le 3) = P(Y\le 3) - P(Y\le 1)$ ✓
$= e - c$ ✓

5. [9 marks: 2, 2, 5]

The random variable X has probability distribution function $p(x)$ defined by

$p(x) = \frac{x+2}{k}$ for $x = -1, 0, 1$ and 2.

(a) Determine the value of k.

$$\sum p(x) = \frac{1+2+3+4}{k} = 1 \quad ✓$$
$$k = 10 \quad ✓$$

(b) Calculate $P(X = 0 | X \ne 1)$.

$$P(X=0|X\ne 1) = \frac{P(X=0)}{P(X=-1)+P(X=0)+P(X=2)} = \frac{2/10}{7/10} \quad ✓$$
$$= \frac{2}{7} \quad ✓$$

(c) Calculate μ and σ^2, respectively the mean and variance for X.

$$\mu = \sum x\,p(x) = \frac{-1\times1}{10} + \frac{0\times2}{10} + \frac{1\times3}{10} + \frac{2\times4}{10} \quad ✓$$
$$= 1 \quad ✓$$
$$\sigma^2 = \sum (x-1)^2\,p(x) = \frac{4\times1}{10} + \frac{1\times2}{10} + \frac{0\times3}{10} + \frac{1\times4}{10} \quad ✓✓$$
$$= 1 \quad ✓$$

Calculator Free

6. [6 marks: 3, 3] [TISC]

The accompanying table describes the probability distribution of a discrete random variable X.

X	1	2	3	4	5
P(X = x)	0.1	p	q	0.2	0.2

(a) Determine the values of p and q if $P(X \le 2) = 0.25$.

$P(X \le 2) = 0.1 + p = 0.25$
$p = 0.15$ ✓
$\sum P(X=x) = 1 \Rightarrow 0.1 + 0.15 + q + 0.2 + 0.2 = 1$ ✓
$q = 0.35$ ✓

(b) Determine the values of p and q if $P(X \ge 3 \mid X \le 4) = \frac{5}{8}$.

$P(X \ge 3 \mid X \le 4) = \frac{5}{8} \Rightarrow \frac{P(3 \le X \le 4)}{P(X \le 4)} = \frac{5}{8}$
$\frac{q + 0.2}{0.8} = \frac{5}{8}$ ✓
$q = 0.3$ ✓
$\sum P(X=x) = 1 \Rightarrow 0.1 + p + 0.3 + 0.2 + 0.2 = 1$
$p = 0.2$ ✓

7. [9 marks: 3, 3, 3]

Consider the probability distribution function $f(x) = \begin{cases} a & x = -2, -1, 0, 2 \\ b & x = 1 \end{cases}$

where a and b are real numbers.

(a) Determine *all* mathematical constraints for a and b.

$4a + b = 1$ ✓ and $a \ge 0$ ✓ and $b \ge 0$ ✓

(b) Determine E(X) in terms of a.

$E(X) = -2a - a + 0 + b + 2a$ ✓
$= -a + b = -a + (1 - 4a)$ ✓
$= 1 - 5a$ ✓

(c) Determine with reasons the conditions for E(X) to be positive.

For $E(X) > 0$, $0 < 5a < 1 \Rightarrow 0 < a < \frac{1}{5}$ ✓
$a = \frac{1-b}{4} \Rightarrow 0 < \frac{1-b}{4} < \frac{1}{5}$ ✓
$\frac{1}{5} < b < 1$ ✓

Calculator Assumed

8. [6 marks]

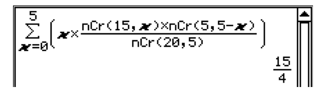
Verify that the function $f(x) = \frac{\binom{15}{x}\binom{5}{5-x}}{\binom{20}{5}}$ for $x = 0, 1, 2, 3, 4, 5$

may be used as the probability distribution function of a discrete random variable X. Determine the exact mean and variance for X.

$\sum f(k) = 1.$ ✓
Also $0 \le f(x) \le 1$ for all x. ✓
Hence, $f(k)$ is a pdf. ✓

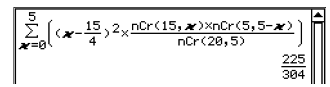
Mean $E(X) = \sum_{x=0}^{x=5} x \times \frac{\binom{15}{x}\binom{5}{5-x}}{\binom{20}{5}}$

$= \frac{15}{4}$ ✓



$Var(X) = \sum_{x=0}^{x=5} (x - \frac{15}{4})^2 \times \frac{\binom{15}{x}\binom{5}{5-x}}{\binom{20}{5}}$ ✓

$= \frac{225}{304}$ ✓



9. [9 marks: 3, 3, 3]

A random variable X has probability distribution $P(X = k) = \begin{cases} \frac{x}{4} & x = 1, 2 \\ \frac{x}{k} & x = 3, 4, 5 \end{cases}$.

(a) Find the value of k.

x	1	2	3	4	5
P(X = k)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{k}$	$\frac{4}{k}$	$\frac{5}{k}$

Hence, $\frac{1}{4} + \frac{2}{4} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} = 1$ ✓✓

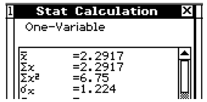
$k = 48.$ ✓

Calculator Assumed

9. (b) Determine the mean and variance of X.

Mean $E(X) = \frac{55}{24} \approx 2.2917$ ✓

$Var(X) = 1.224^2 \approx 1.4982$ ✓✓



(c) Find $P(X \le 4 \mid X > 2)$.

$$P(X \le 4 \mid X > 2) = \frac{P(2 < X \le 4)}{P(X > 2)} \quad ✓✓$$

$$= \frac{7/48}{1/4} = \frac{7}{12} \quad ✓$$

---

10. [8 marks: 3, 5]

A discrete random variable X has cumulative probability distribution function given by $P(X \le x) = \frac{x^3 + 3x^2 + 2x}{k}$ for $x = 1, 2, 3, 4, 5$.

(a) Determine the value of k.

$P(X \le 5) = 1$ ✓

$\Rightarrow \frac{5^3 + 3(5^2) + 2(5)}{k} = 1$ ✓

$k = 210$ ✓

(b) Calculate the mean and variance for X.

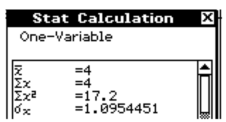
x	1	2	3	4	5
$P(X = x)$	$\frac{1}{35}$	$\frac{3}{35}$	$\frac{6}{35}$	$\frac{10}{35}$	$\frac{15}{35}$

✓✓ Distribution table

Mean $E(X) = 4$ ✓

$\sigma = 1.095445$ ✓

Variance $= 1.095445^2 = 1.2$ ✓



Calculator Assumed

11. [7 marks: 2, 2, 3] [TISC]

(a) The table below shows the values taken by a function $f(x)$.

x	–1	0	1	0.5
$f(x)$	0.2	0.6	0.1	0.1

Peter argues that $f(x)$ cannot be a probability distribution function of a discrete random variable as x has a negative value. Comment on his answer.

x can be negative but $f(x)$ cannot be negative.
$\sum f(k) = 1$. ✓
Also $0 \le f(x) \le 1$ for all x. ✓
Hence, $f(k)$ is a pdf, that is, Peter is wrong.

(b) The table below shows the values taken by a function $f(x)$.

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.2	0.5	a	b	0.1

(i) Under what conditions can $f(x)$ represent the probability distribution function of a discrete random variable?

$a + b = 0.2$ ✓
and $0 < a < 1$ and $0 < b < 1$ ✓

(ii) If $f(x)$ is the probability distribution function of a discrete random variable X, find the values of a and b given that $P(X = 1.0) = 2 \times P(X = 1.5)$.

$a = 2b$ ✓
But $a + b = 0.2$
Hence, $3b = 0.2$
$b = \frac{1}{15}$ ✓
$a = \frac{2}{15}$ ✓

Calculator Assumed

12. [8 marks: 2, 3, 3]

The discrete random variable X has mean 4 and standard deviation 1.

(a) Calculate the mean and standard deviation for the random variable $Y = 5 + 2X$.

$$E(Y) = 5 + 2E(X) = 5 + 2 \times 4 = 13 \quad ✓$$
$$STD(Y) = |2|\,STD(X) = 2 \times 1 = 2 \quad ✓$$

(b) Calculate the mean and standard deviation for the random variable W if $X = 7 + 3W$.

$$E(X) = 7 + 3E(W) \quad ✓$$
$$E(W) = \frac{4-7}{3} = -1 \quad ✓$$
$$STD(W) = \left|\frac{-1}{3}\right| STD(X) = \frac{1}{3} \quad ✓$$

(c) Each value of X is decreased by 20% and the result increased by 20 units. Determine the mean and variance of the new variable.

Let $U = 0.8X + 20$ ✓
$$E(U) = 0.8 \times 4 + 20 = 23.2 \quad ✓$$
$$VAR(U) = 0.8^2 \times 1 = 0.64 \quad ✓$$

---

13. [6 marks]

The random variable X has mean 100 and standard deviation 4. The random variable $Y = aX + b$. Find a and b if the mean and standard deviation for Y are 90 and 6 respectively.

$$Var(Y) = a^2 Var(X) \Rightarrow 36 = 16a^2 \quad ✓$$
$$a = \pm\frac{3}{2} \quad ✓$$

For $a = \frac{3}{2}$: $E(Y) = \frac{3}{2}E(X) + b$ ✓
$\Rightarrow 150 + b = 90$
$b = -60$ ✓

For $a = -\frac{3}{2}$: $\Rightarrow -150 + b = 90$ ✓
$b = 240$ ✓

Calculator Assumed

14. [9 marks: 3, 6]

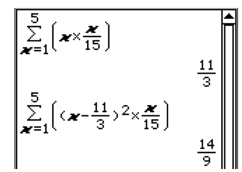
The probability distribution function for a random variable X is given by $p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$. The random variable $Y = 10 - 2X$.

(a) Calculate $P(2 \le Y \le 6)$.

$$P(2 \le Y \le 6) = P(2 \le 10 - 2X \le 6) = P(2 \le X \le 4) \quad ✓✓$$
$$= \frac{2}{15} + \frac{3}{15} + \frac{4}{15} = \frac{3}{5} \quad ✓$$

(b) Calculate the mean and variance for Y.

$$E(X) = \sum_{x=1}^{x=5}\left(x \times \frac{x}{15}\right) = \frac{11}{3} \quad ✓$$
$$VAR(X) = \sum_{x=1}^{x=5}\left((x - \frac{11}{3})^2 \times \frac{x}{15}\right) = \frac{14}{9} \quad ✓$$
$$E(Y) = E(10 - 2X) = 10 - 2E(X) \quad ✓$$
$$= 10 - 2 \times \frac{11}{3} = \frac{8}{3} \quad ✓$$
$$Var(Y) = Var(10 - 2X) = 4Var(X) \quad ✓$$
$$= 4 \times \frac{14}{9} = \frac{56}{9} \quad ✓$$



Calculator Assumed

15. [9 marks: 1, 2, 3, 3] [TISC]

X is a discrete random variable with probability distribution function
$P(X = x) = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$.
Y is discrete random variable with probability distribution function
$P(Y = y) = \frac{1}{10}$ for $y = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
It is known that the variables X and Y are independent.

(a) Find $P(X \le 4)$.

$P(X \le 4) = 4 \times \frac{1}{6} = \frac{2}{3}$ ✓

(b) Find P(X = 5 and Y = 5).

$P(X = 5 \text{ and } Y = 5) = \frac{1}{6} \times \frac{1}{10}$ ✓
$= \frac{1}{60}$ ✓

(c) Find P(X = 5 or Y = 5).

$P(X = 5 \text{ or } Y = 5) = P(X = 5) + P(Y = 5) - P(X = 5 \text{ and } Y = 5)$
$= \frac{1}{6} + \frac{1}{10} - \frac{1}{60}$ ✓✓
$= \frac{1}{4}$ ✓

(d) Find P(X + Y = 12). Show clearly how you obtained your answer.

X	Y	Prob.
6	6	1/60
5	7	1/60
4	8	1/60
3	9	1/60
2	10	1/60

✓✓

Hence, $P(X + Y = 12) = \frac{1}{60} \times 5$
$= \frac{1}{12}$ ✓

23 Discrete Random Variables II

Calculator Free

1. [7 marks: 2, 2, 1, 2]

The table below shows the projected returns for every $100 000 in an investment scheme and the accompanying probabilities.

Returns	–$20 000	–$10 000	k	$20 000	$50 000
Probability	0.01	p	0.2	0.27	0.02

(a) Determine the value of p.

$0.01 + p + 0.2 + 0.27 + 0.02 = 1$ ✓
$p = 0.5$ ✓

(b) Find the mean return per $100 000 in terms of k.

$E(X) = -20\,000 \times 0.01 + -10\,000 \times 0.5 + 0.2k + 20\,000 \times 0.27 + 50\,000 \times 0.02$
$= 1\,200 + 0.2k$ ✓✓

(c) Find the mean profit per $100 000 if k = $5 000.

Mean profit = $1\,200 + 0.2 \times 5000$
= $2 200. ✓

(d) Find the value(s) of k if the mean profit per $100 000 must exceed $5000.

$E(X) > 5\,000$
$1\,200 + 0.2k > 5\,000$ ✓
$k > 19\,000$ ✓

2. [14 marks: 1, 5, 3, 5]

At an agricultural fair, a games stall operator offers prizes worth $20, $5, and $1 for one attempt at a particular game. The probabilities of winning these prizes are respectively 0.001, 0.01 and 0.5.

(a) Find the probability of not winning a prize.

P(not winning a prize) = 1 – 0.001 – 0.01 – 0.5
= 0.489 ✓

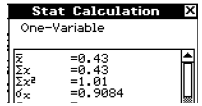
Calculator Assumed

2. (b) If each game costs $1, find the expected profit per game and the accompanying standard deviation for the games stall operator.

Let X: Profit per game for the operator. ✓
Hence, the probability distribution for X is:

x	–19	–4	0	1
P(X = x)	0.001	0.01	0.5	0.489

✓✓



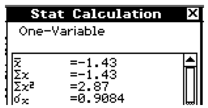
Expected profit = $0.43 ✓
Standard deviation = $0.91 ✓

(c) Vegas played 50 games at $2 for a game. Find her expected profit/loss.

Let X: Profit per game for the Vegas. ✓
Hence, the probability distribution for X is:

x	18	3	–1	–2
P(X = x)	0.001	0.01	0.5	0.489

✓



Expected profit per game = –$1.43
Total Loss = 1.43 × 50 = $71.5 ✓

(d) The games stall operator made a profit of $193 from 100 games. How much did he charge per game?

Let the price per game be $*k*.
Let X: Profit per game for the operator.
Hence, the probability distribution for X is:

x	k – 20	k – 5	k – 1	k
P(X = x)	0.001	0.01	0.5	0.489

✓

Given $E(X) = \frac{193}{100} = 1.93$. ✓

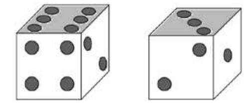
But $E(X) = 0.001(k - 20) + 0.01(k - 5) + 0.5(k - 1) + 0.489k$ ✓

Hence, $0.001(k - 20) + 0.01(k - 5) + 0.5(k - 1) + 0.489k = 1.93$ ✓
$\Rightarrow k = 2.50$

Hence, cost per game = $2.50. ✓

Calculator Assumed

3. [10 marks: 3, 5, 2]



In a game of dice, a fair six sided dice is rolled twice and *S* the sum of the dots displayed on the upper-most faces recorded. The organiser of the game pays out $6 for sums that exceed 10, $2 for sums that are between 7 and 10 inclusive and $0 otherwise. Define the random variable *T* as the profit per game for the organiser.

(a) Calculate P(S > 10).

S	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

✓✓

$P(S > 10) = \frac{3}{36}$ ✓

(b) Determine the expected value for *T* if it costs $2 to play each game.

s	$s < 7$	$7 \le s \le 10$	$s > 10$
P($S = s$)	$\frac{15}{36}$	$\frac{18}{36}$	$\frac{3}{36}$
t	2	0	–4
P($T = t$)	$\frac{15}{36}$	$\frac{18}{36}$	$\frac{3}{36}$

✓ ✓ ✓

$\Rightarrow E(T) = \frac{30-12}{36} = \$0.50$ ✓✓

(c) How much should the organiser charge per game to break even (no loss)? Justify your answer.

Charge of $2 brings a profit of $0.50
which is $1.50 less than the charge of $2. ✓
Hence, charge of $1.50 will be the break-even charge. ✓

Calculator Assumed

4. [12 marks: 6, 6]

A random number generator generates whole numbers randomly between 1 and n. Define X: the number generated by the random number generator.

(a) Given that the mean number generated is 5 000, determine the standard deviation for X.

Probability distribution function for X is $p(x) = \frac{1}{n}$ ✓

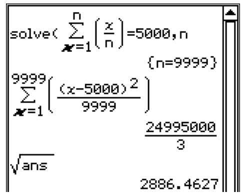
Hence: $\sum_{x=1}^{x=n}\left(\frac{x}{n}\right) = 5\,000$ ✓

$n = 9\,999$ ✓

$\text{Var}(X) = \sum_{x=1}^{x=9999}\left(\frac{(x-5000)^2}{9999}\right)$ ✓

$= \frac{24\,995\,000}{3}$ ✓

Standard deviation for X ≈ 2886.46 ✓



(b) Find the mean for X if the variance for X is 20 750.

Probability distribution function for X is $p(x) = \frac{1}{n}$

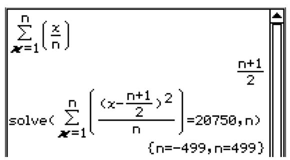
Hence: $E(X) = \sum_{x=1}^{x=n}\left(\frac{x}{n}\right)$ ✓

$= \frac{n+1}{2}$ ✓

$\text{Var}(X) = \sum_{x=1}^{x=n}\left(\frac{(x-\frac{n+1}{2})^2}{n}\right) = 20\,750$ ✓✓

$\Rightarrow n = 499$ ✓

Therefore, $E(X) = \frac{499+1}{2} = 250$ ✓



Calculator Assumed

5. [8 marks: 4, 2, 2]

A committee of five students is to be selected from a group of five female students and five male students.
Define X: Number of female students in this committee.

(a) Find the probability distribution for X.

X = 0, 1, 2, 3, 4, 5 ✓

$P(X = 0) = P(0 \text{ female } \& \text{ 5 males}) = \frac{\binom{5}{0}\binom{5}{5}}{\binom{10}{5}} = \frac{1}{252}$

$P(X = 1) = P(1 \text{ female } \& \text{ 4 males}) = \frac{\binom{5}{1}\binom{5}{4}}{\binom{10}{5}} = \frac{25}{252}$

$P(X = 2) = P(2 \text{ females } \& \text{ 3 males}) = \frac{\binom{5}{2}\binom{5}{3}}{\binom{10}{5}} = \frac{100}{252}$

$P(X = 3) = P(3 \text{ females } \& \text{ 2 males}) = \frac{\binom{5}{3}\binom{5}{2}}{\binom{10}{5}} = \frac{100}{252}$

$P(X = 4) = P(4 \text{ females } \& \text{ 1 male}) = \frac{\binom{5}{4}\binom{5}{1}}{\binom{10}{5}} = \frac{25}{252}$

$P(X = 5) = P(5 \text{ females } \& \text{ 0 male}) = \frac{\binom{5}{5}\binom{5}{0}}{\binom{10}{5}} = \frac{1}{252}$

✓✓✓ –1 per error

(b) Find the probability that there are at least as many females than males in the committee.

$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

$= \frac{100}{252} + \frac{25}{252} + \frac{1}{252}$ ✓

$= \frac{126}{252} = \frac{1}{2}$ ✓

(c) Find the expected number of females in the committee.

Expected number of females = $E(X) = \frac{5}{2}$ ✓✓

Calculator Assumed

6. [10 marks: 1, 1, 2, 2, 4]

It is known that 0.5% of USB Drives are defective. USB drives are randomly picked from a large carton.

(a) Find the probability that the second USB drive picked is defective.

> Prob. = 0.005 ✓

(b) Find the probability that the first defective drive is the 3rd drive picked.

> Prob. = $0.995^2 \times 0.005 = 0.004950$ ✓

(c) Find the probability the first defective drive is the 11th drive picked.

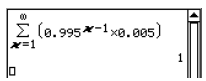
> Prob. = $0.995^{10} \times 0.005 = 0.004756$ ✓✓

(d) Write an expression for the probability that first defective USB drive is the *n*th drive picked.

> Prob. = $0.995^{n-1} \times 0.005$ ✓✓

(e) Define X: No of USB drives that need to be selected to pick the first defective drive. Determine with reasons if X is a discrete random variable with an appropriate probability distribution function.

> X = 1, 2, 3, …… ✓
> $P(X = x) = 0.995^{x-1} \times 0.005$
> Clearly $0 < P(X = x) < 1$ for all x. ✓
> $\sum_{1}^{\infty} 0.995^{x-1} \times 0.005 = 1.$ ✓
> Hence, X is a discrete random variable. ✓



Calculator Assumed

7. [10 marks: 2, 2, 6]

It is known that 65% of students at a certain college are foreign born. Students are randomly chosen from this college.

(a) Find the probability that the second foreign born student is the third student selected.

> Prob. = $(2 \times 0.65 \times 0.35) \times 0.65$ ✓
> $= 0.29575$ ✓

(b) Find the probability that 4 students need to be picked before picking the second foreign born student.

> Prob. = $(3 \times 0.65 \times 0.35^2) \times 0.65$ ✓
> $= 0.1553$ ✓

(c) Define X: No of students that need to be selected before picking the second foreign born student. Determine with reasons if X is a discrete random variable with an appropriate probability distribution function..

> X = 2, 3, 4, 5, …… ✓
> $P(X = x) = (x - 1) \times 0.65 \times 0.35^{x-2} \times 0.65$ ✓✓
> Clearly $0 < P(X = x) < 1$ for all x. ✓
> $\sum_{1}^{\infty} (x-1) \times 0.35^{x-2} \times 0.65^2 = 1.$ ✓
> Hence, X is a discrete random variable. ✓

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8. [6 marks: 3, 3]

It is estimated that 9% of Australians have O^- type blood.

(a) Determine the probability that the 8th Australian randomly selected is the second person who has O^- type blood.

> Prob. = $\left[\binom{7}{1} 0.09^1 \times 0.91^6\right] \times 0.09$ ✓✓
> $= 00.032198$ ✓

Calculator Assumed

8. (b) A group of 20 Australians comprise 3 adults with O^- type blood. Five persons are chosen from this group. Calculate the probability that exactly one of the five persons chosen has O^- type blood.

> Prob. $= \dfrac{{}^3C_1 \times {}^{17}C_4}{{}^{20}C_5}$ ✓✓
> $= 0.460526$ ✓

---

9. [6 marks: 3, 3]

60% of adults in a certain country are lactose intolerant. Five adults are randomly chosen from this country. Define X as the fifth adult chosen is the nth person chosen who is lactose intolerant.

(a) Explain why $P(X = n) = \left({}^4C_{n-1} \times 0.4^{5-n} \times 0.6^{n-1}\right) \times 0.6$ for $n = 1, 2, 3, 4, 5$.

> The fifth adult is the nth person from 5 who is lactose intolerant implies that within the first four, $(n-1)$ adults are lactose intolerant.
>
> The probability that any $(n-1)$ from 4 is lactose intolerant is $0.4^{4-(n-1)} \times 0.6^{n-1} = 0.4^{5-n} \times 0.6^{n-1}$. ✓
> Number of ways of choosing $(n-1)$ from 4 is ${}^4C_{n-1}$.
> Hence, P($n-1$ from 4 is lactose intolerant)
> $= {}^4C_{n-1}\, 0.4^{5-n} \times 0.6^{n-1}$ ✓
> But P(5th is lactose intolerant) = 0.6 ✓
> Hence: $P(X = n) = \left({}^4C_{n-1} \times 0.4^{5-n} \times 0.6^{n-1}\right) \times 0.6$

(b) Determine with reasons if $P(X = n) = \left({}^4C_{n-1} \times 0.4^{5-n} \times 0.6^{n-1}\right) \times 0.6$ for $n = 1, 2, 3, 4, 5$ can be classified as a probability distribution of a discrete random variable.

n	1	2	3	4	5
$P(X = n)$	0.01536	0.09216	0.20736	0.20736	0.07776

✓

Hence, $\sum P(X=n) = 0.6 \neq 1$ ✓
Therefore, NO! ✓

```
 5
 Σ (nCr(4,n-1)×0.4^(5-n)×0.6^n)
n=1
                                         0.6
seq(nCr(4,n-1)×0.4^(5-n)×0.6^n,n,1,5
 {0.01536,0.09216,0.20736,0.20736,0.07776}
sum(
                                         0.6
```


Calculator Assumed

10. [9 marks: 4, 5]

It is known that $100p$% of homes in a certain city have broadband internet connections.

(a) Define X: No. of homes with broadband internet connections out of n homes selected. Determine with reasons if X is a discrete random variable.

> $X = 0, 1, 2, 3, \ldots, n$ ✓
> $P(X = x) = {}^nC_x \times p^x \times (1-p)^{n-x}$
>
> Clearly $0 < P(X = x) < 1$ for all x. ✓
> $\sum_0^n {}^nC_x \times p^x \times (1-p)^{n-x} = 1$. ✓
>
> Hence, X is a discrete random variable. ✓

(b) Define X: Out of 10 homes selected, the only home with broadband internet connection is the xth home selected, where $x \le 10$. For $p = 0.2$, determine with reasons if X is a discrete random variable.

> $X = 1, 2, 3, \ldots, 10$ ✓
> $P(X = x) = 0.2 \times 0.8^9$ ✓
> Clearly $0 < P(X = x) < 1$ for all x. ✓
> $0.2 \times 0.8^9 \times 10 = 0.2684 \neq 1$ ✓
>
> Hence, X is *not* a discrete random variable. ✓

Calculator Assumed

11. [8 marks: 4, 4]

A box contains 5 red and 15 green balls.

(a) Four balls are chosen *with* replacement from this box.
Define X: No. of red balls chosen.
Determine with reasons if X is a discrete random variable with a clearly stated probability distribution function..

> X = 0, 1, 2, 3, 4.
>
> $P(X = x) = {}^4C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x}$ ✓
>
> Clearly $0 < P(X = x) < 1$ for all x. ✓
>
> $\sum_0^4 {}^4C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x} = 1.$ ✓
>
> Hence, X is a discrete random variable. ✓

(b) Four balls are chosen *without* replacement from this box.
Define X: No. of red balls chosen.
Determine with reasons if X is a discrete random variable with a clearly stated probability distribution function..

> X = 0, 1, 2, 3, 4.
>
> $P(X = x) = \frac{{}^5C_x\,{}^{15}C_{4-x}}{{}^{20}C_4}$ ✓
>
> Clearly $0 < P(X = x) < 1$ for all x. ✓
>
> $\sum_0^4 \left[\frac{{}^5C_x\,{}^{15}C_{4-x}}{{}^{20}C_4}\right] = 1.$ ✓
>
> Hence, X is a discrete random variable. ✓

Calculator Assumed

12. [9 marks: 2, 2, 3, 2]

The number of items per order for an online store is modelled by the random variable X with probability distribution $P(X = x) = \frac{-1}{ln\,0.4} \times \frac{0.6^x}{x}$ for $x = 1, 2, 3, \ldots$

(a) Calculate the probability that an order contains not more than 3 items.

> $P(X \le 3) = \frac{-1}{ln\,0.4} \times \left(\frac{0.6^1}{1} + \frac{0.6^2}{2} + \frac{0.63}{3}\right)$ ✓
>
> $= 0.654\,814 + 0.196\,444 + 0.078\,578$
>
> $= 0.929\,836 \approx 0.929\,8$ ✓

(b) Calculate the probability that an order of not more than 3 items has exactly one item.

> $P(X = 1 \mid X \le 3) = \frac{0.654\,814}{0.929\,836}$ ✓
>
> $= 0.7042$ ✓

(c) 99% of orders have no more than n items.
Determine the value of n. Justify your answer.

> $P(X \le 5) = 0.9822$ ✓
>
> $P(X \le 6) = 0.9907$ ✓
>
> Hence, $n = 6$ ✓
>
> $\sum_{x=1}^{5}\left(\frac{-1}{\ln(0.4)} \times \frac{0.6^x}{x}\right)$ 0.982168616
>
> $\sum_{x=1}^{6}\left(\frac{-1}{\ln(0.4)} \times \frac{0.6^x}{x}\right)$ 0.9906550055

(d) Calculate the mean for X.

> $E(X) = \sum_{x=1}^{x=\infty}\left(\left(\frac{-1}{ln\,0.4} \times \frac{0.6^x}{x}\right) \times x\right)$ ✓
>
> $= 1.6370 \approx 1.6$ ✓

Calculator Assumed

13. [8 marks: 3, 2, 3]

Define the random variable X: Number of accidents at a traffic intersection in one week. The probability distribution $P(X = x) = \dfrac{e^{-2}2^x}{x!}$ for $x = 0, 1, 2, 3, \ldots$

(a) Calculate the probability that within any given week there are at least 2 accidents.

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \quad ✓ \\ &= 1 - e^{-2} - 2e^{-2} \quad ✓ \\ &\approx 0.59399 \quad ✓ \end{aligned}$$

(b) Calculate the probability that in any week with at least two accidents, there are exactly 2 accidents.

$$\begin{aligned} P(X = 2 \mid X \geq 2) &= \frac{\left(\frac{e^{-2}2^2}{2!}\right)}{0.59399} \quad ✓ \\ &= 0.4557 \quad ✓ \end{aligned}$$

(c) More than 30% of weeks have at least n accidents. Determine the value of n. Justify your answer.

$$\begin{aligned} P(X \geq 3) &= 1 - e^{-2} - 2e^{-2} - 2e^{-2} \\ &\approx 0.3233 \quad ✓ \\ P(X \geq 4) &= 1 - e^{-2} - 2e^{-2} - 2e^{-2} - \frac{4e^{-2}}{3} \\ &\approx 0.1429 \quad ✓ \end{aligned}$$

Hence, $n = 3$ ✓

24 The Binomial Distribution

Calculator Free

1. [7 marks: 1, 2, 2, 2]

An eight sided die has faces numbered 1, 2, 3, 4, 5, 6, 7 or 8.

(a) The die is rolled once. Define the random variable

$$X = \begin{cases} 1 & \text{if die lands on a face with numbers 1, 3 or 5} \\ 0 & \text{if die lands on a face with numbers 2, 4, 6, 7 or 8} \end{cases}$$

(i) Identify the name of the probability distribution for X and state the associated parameters.

$X \sim \text{Bernoulli}\left(p = \frac{3}{8}\right)$ ✓

(ii) Determine the mean and variance of X.

$E(X) = \frac{3}{8}$ ✓

$Var(X) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$ ✓

(b) The die is rolled 40 times and the random variable Y is defined as,
Y : Number of times the die lands on faces with numbers 1, 3 or 5.

(i) State the mathematical expression for the probability distribution for Y.

$P(Y = y) = {}^{40}C_y \times \left(\frac{3}{8}\right)^y \times \left(\frac{5}{8}\right)^{40-y}$ ✓ for $y = 0, 1, 2, 3, \ldots, 40$ ✓

(ii) Determine the mean and variance for Y.

$E(Y) = 40 \times \frac{3}{8} = 15$ ✓

$VAR(Y) = 40 \times \frac{3}{8} \times \frac{5}{8} = \frac{75}{8}$ ✓

Calculator Free

2. [8 marks: 2, 2, 2, 2]

40% of students in a certain school are short-sighted.

Five students from this school are randomly selected. The random variable

$$S_i = \begin{cases} 1 & \text{if student } i \text{ is short-sighted (success)} \\ 0 & \text{if student } i \text{ is not short-sighted (failure)} \end{cases} \quad \text{for } i = 1, 2, 3, 4, 5.$$

(a) State the probability distribution function for S_i.

$$P(S_i = n) = \begin{cases} 0.4 & n=1 \\ 0.6 & n=0 \end{cases} \quad ✓✓$$

(b) Determine the mean and variance for S_i.

$E(S_i) = 0.4$ ✓
$Var(S_i) = 0.4 \times 0.6 = 0.24$ ✓

Define the random variable $X = S_1 + S_2 + S_3 + S_4 + S_5$.

(c) State the probability distribution function for X.

$P(X = x) = {}^5C_x (0.4)^x (0.6)^{5-x}$ for $x = 0, 1, 2, 3, 4, 5$ ✓✓

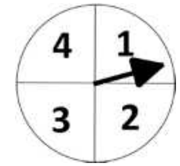
(d) State the mean and variance for X.

$E(X) = 5 \times 0.4 = 2$ ✓
$Var(X) = 5 \times 0.4 \times 0.6 = 1.2$ ✓

Calculator Assumed

3. [13 marks: 3, 3, 4, 3]

The accompanying diagram shows a circle divided into four equal sized sectors numbered 1, 2, 3 and 4 respectively and a spinning pointer. The pointer is spun and will come to rest in one of the 4 sectors. It is spun again if it comes to rest on the line dividing the sectors.



(a) Define the random variable

$$X = \begin{cases} 1 & \text{if pointer stops in sector labelled 1} \\ 0 & \text{if pointer does not stop in sector labelled 1} \end{cases}.$$

Determine the mean and variance of X.

$X \sim \text{Bernoulli}\ (p = \frac{1}{4})$ ✓
$E(X) = \frac{1}{4}$ ✓
$Var(X) = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$ ✓

(b) The pointer is spun 20 times and the random variable Y is defined as,
Y : Number of times the pointer stops in sector labelled 1.
Show the use of a *combinatorial method* to determine the probability that:

(i) the counter stops in the sector labelled 1 exactly four times.

$P(Y = 4) = {}^{20}C_4 \times \left(\frac{1}{4}\right)^4 \times \left(\frac{3}{4}\right)^{16}$ ✓✓
$= 0.1896$ ✓

(ii) the counter stops in the sector labelled 1 at least 2 times.

$P(Y \geq 2) = 1 - P(X = 0) - P(X = 1)$ ✓
$= 1 - \left(\frac{3}{4}\right)^{20} - {}^{20}C_1 \times \left(\frac{1}{4}\right)^1 \times \left(\frac{3}{4}\right)^{19}$ ✓✓
$= 1 - 0.00317 - 0.02114 = 0.9757$ ✓

(iii) the counter stops in the sector labelled 1 only on the last spin.

$\text{Probability} = \left(\frac{3}{4}\right)^{19} \times \left(\frac{1}{4}\right)^1 \times 1$ ✓✓
$= 0.001057$ ✓

Calculator Assumed

4. [12 marks: 4, 3, 5]

10 students in a class of 25 are short-sighted. 5 students are randomly selected from this class. The random variable

$$S_i = \begin{cases} 1 & \text{if student } i \text{ is short-sighted (success)} \\ 0 & \text{if student } i \text{ is not short-sighted (failure)} \end{cases} \quad \text{for } i = 1, 2, 3, 4, 5.$$

Define the random variable $X = S_1 + S_2 + S_3 + S_4 + S_5$.

(a) Use the variables S_i to verify that X is not a binomial variable.

$P(S_1 = 1) = \frac{10}{25}$.

$P(S_2 = 1 \mid S_1 = 1) = \frac{9}{24}$

$P(S_2 = 1 \mid S_1 = 0) = \frac{10}{24}$ ✓

Hence, $P(S_2 = 1 \mid S_1 = 1) \neq P(S_2 = 1 \mid S_1 = 0)$ ✓

That is, the variables S_1 and S_2 are not independent. ✓

Hence, $X = S_1 + S_2 + S_3 + S_4 + S_5$
is not the sum of independent Bernoulli variables. ✓
Therefore X cannot be a binomial variable.

(b) Determine the probability distribution function for X.

$$P(X = x) = \frac{{}^{10}C_x \; {}^{15}C_{5-x}}{{}^{25}C_5} \quad \text{for } x = 0, 1, 2, 3, 4, 5$$

✓✓ ✓

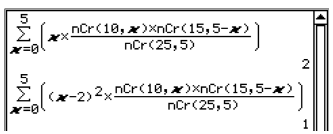
(c) Calculate the mean and variance for X.

$$E(X) = \sum_{x=0}^{x=5} \left(x \times \frac{{}^{10}C_x \; {}^{15}C_{5-x}}{{}^{25}C_5} \right)$$ ✓

$= 2$ ✓

$$Var(X) = \sum_{x=0}^{x=5} \left((x-2)^2 \times \frac{{}^{10}C_x \; {}^{15}C_{5-x}}{{}^{25}C_5} \right)$$ ✓✓

$= 1$ ✓



Calculator Free

5. [3 marks] [TISC]

X is a Binomial variable with parameters n and $p = \frac{1}{2}$.

Find in terms of n, $P(X > 1)$.

$P(X > 1) = 1 - P(X = 0) - P(X = 1)$ ✓

$= 1 - \left(\frac{1}{2}\right)^n - \binom{n}{1}\left(\frac{1}{2}\right)^n$ ✓

$= 1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n$ ✓

$= 1 - (n+1)\left(\frac{1}{2}\right)^n$.

---

6. [6 marks: 1, 2, 2, 1] [TISC]

The probability that John is late for school on any school day is 0.4 and it is independent of other days. For a school week of five days, write expressions, but *do not evaluate*, for:

(a) the probability that he is late every day

P(late every day) = 0.4^5 ✓

(b) the probability that he is late on exactly two days

P(late exactly two days) = $\binom{5}{2} \times 0.4^2 \times 0.6^3$ ✓✓

(c) the probability that he is late at least once

P(late at least once) = $1 - 0.6^5$ ✓✓

(d) the mean number of days he will be late

Mean = $5 \times 0.4 = 2$ days ✓

Calculator Assumed

7. [8 marks: 3, 2, 3]

On average, 20% of teachers in a particular state have previously been treated for work related depression. The mathematics department in a particular school in this state has 11 staff members (all teachers!).

(a) Find the expected number of depressed teachers and its expected standard deviation. Justify your answer.

> Let X: No. of depressed teachers out of 11
> $X \sim B(n = 11, p = 0.2)$ ✓
> Expected No. $= np = 11 \times 0.2 = 2.2$ ✓
> Std. Deviation $= \sqrt{(npq)} = 1.3266$ ✓

(b) Find the probability that there are no more than five staff members in this department that have previously been treated for work related stress.

> $P(X \le 5) = 0.9883$ ✓✓

(c) Find the probability that there are exactly two staff members who have previously been treated for work related stress given that there are no more than five of them.

> $P(X = 2 \mid X \le 5) = \dfrac{P(X = 2 \cap X \le 5)}{P(X \le 5)}$
> $= \dfrac{0.29528}{0.98835}$ ✓✓
> $= 0.2988$ ✓

Calculator Assumed

8. [12 marks: 3, 1, 3, 2, 3]

It is known that the probability of finding a defective biro in a large consignment of biros is 0.05. These biros are packed and sold in packs of 12 biros each.

(a) Find the expected number of defective biros in any given pack and its associated standard deviation.

> Let X: No. of defective biros out of 12.
> $X \sim B(12, 0.05)$ ✓
> Expected No. $= 12 \times 0.05 = 0.6$ ✓
> Std. Deviation $= 0.7550$ ✓

(b) Find the probability that a randomly chosen pack has exactly two defective biros.

> $P(X = 2) = 0.09879$ ✓

(c) Given that a randomly chosen pack has no more than 4 defective biros, find the probability that the pack has at least two defective biros.

> $P(X \ge 2 \mid X \le 4) = \dfrac{P(2 \le X \le 4)}{P(X \le 4)}$
> $= \dfrac{0.11818}{0.99982}$ ✓✓
> $= 0.1182$ ✓

(d) Find the probability that there were no more than 4 defective biros each in two randomly chosen packs.

> $P(X \le 4) \times P(X \le 4) = 0.99982^2$ ✓
> $= 0.9996$ ✓

(e) Find the probability that in a box of 50 such packs (of 12 biros each), no more than 10 of these packs had exactly two defective biros each.

> Let X: No. of packs out of 50 with 2 defective biros.
> $X \sim B(50, 0.09879)$ ✓✓
> $P(X \le 10) = 0.9914$ ✓

Calculator Assumed

9. [11 marks: 4, 4, 3] [TISC]

60% of students in a school own at least one Apple® device.

(a) Ten students were randomly chosen. Find the probability that more than six of these students own at least one Apple® device. Show how you obtained your answer.

> Define X: No. of students out of 10 who own at least one Apple® device.
> $X \sim B(n = 10, p = 0.6)$
> $P(X > 6) = P(7 \le X \le 10) = 0.38228$
>
> - Random variable used declared in words ✓
> - Distribution identified as binomial with correct parameters ✓
> - $P(X \ge 7)$ or $P(7 \le X \le 10)$ ✓
> - Correct probability ✓

(b) n students were randomly chosen. Let X: Number of students out of n students who own at least one Apple® device. The mean for X is μ and the accompanying standard deviation is $\frac{4\sqrt{15}}{5}$. Find n and μ.

> Define X: No. of students out of n who own at least one Apple® device.
> $X \sim B(n, p = 0.6)$ ✓
> Standard deviation for X = $\sqrt{n \times 0.6 \times 0.4} = \frac{4\sqrt{15}}{5}$ ✓
> $n = 40$ ✓
> Mean $\mu = 40 \times 0.6 = 24$ ✓

(c) n students were randomly chosen. The probability that all of these students chosen owned at least one Apple® device is 0.01. Find n.

> Define X: No. of students out of n who own at least one Apple® device.
> $X \sim B(n, p = 0.6)$ ✓
> $P(X = n) = 0.6^n = 0.01$ ✓
> $n = 9$ ✓
> (accept $n = 10$)

Calculator Assumed

10. [16 marks: 3, 3, 3, 4, 3]

20% of residents in a suburb owned dogs.

(a) In a group of 30 randomly chosen residents, calculate the probability that no more than 5 of these residents owned dogs.

> Let X: No. of residents with dogs ✓
> $X \sim B(n = 30, p = 0.2)$ ✓
> $P(X \le 5) = 0.42751 \approx 0.0.4275$ ✓

(b) In a group of n residents, the mean number of residents owning dogs is 40. Calculate the associated standard deviation.

> $\sigma = \sqrt{np(1-p)}$
> But $np = 40$ and $(1 - p) = 0.8$ ✓
> Hence, $\sigma = \sqrt{40 \times 0.8}$ ✓
> $= 5.6569$ ✓

(c) The probability that n residents need to be chosen before the first resident owning a cat is chosen is 0.01374. Calculate n.

> $0.8^{n-1} \times 0.2 = 0.01374$ ✓✓
> $n = 13.0012 \approx 13$ ✓

(d) Calculate the probability that sixth resident picked is the third resident picked who owned a dog.

> ✓ ✓ ✓
> Prob. = $\left({}^5C_2 \times 0.2^2 \times 0.8^3\right) \times 0.2$
> $= 0.04096$ ✓

(e) In a group of 20 residents, 4 of these residents owned dogs. 5 residents were chosen from this group of residents. Calculate the probability that exactly one resident in this group of 5 residents owned a dog.

> Prob. = $\frac{{}^4C_1 \times {}^{16}C_4}{{}^{20}C_5}$ ✓✓
> $= 0.469556 \approx 0.4696$ ✓

Calculator Assumed

11. [10 marks: 2, 1, 3, 4] [TISC]

It is estimated that 35% of students at a school come from a Non-English Speaking Background.
Clearly stating the probability distribution you are using, estimate the probability that, in a class of twenty students from this school,

(a) exactly five of them are from a Non-English Speaking Background.

Let X: No. of NESB students out of 20.
X ~ B(20, 0.35) ✓
P(X = 5) = 0.12720 ✓

(b) more than ten of them are from a Non-English Speaking Background.

P(X > 10) = 0.05317 ✓

In five classes of 20 students each,

(c) estimate the probability that at least two of these classes each contain more than ten students from Non-English Speaking backgrounds.

Let Y: No. of classes out of 5 classes with more than 10 NESB students each.
Y ~ B(5, 0.05317) ✓✓
P(Y ≥ 2) = 0.02538 ✓

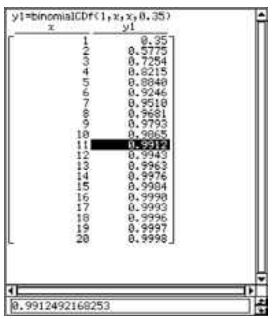
A sample of n students from this school was chosen.
The probability that at least one of the students selected is from a Non-English Speaking Background is greater than 0.99.

(d) Find the minimum value of n. Show all working.

Let W: No. of NESB students out of n.
W ~ B(n, 0.35)
P(W ≥ 1) > 0.99 ✓
$\Rightarrow$ P(W = 0) < 0.01
$^{n}C_0 \times 0.35^0 \times 0.65^n < 0.01$

$0.65^n < 0.01$ ✓
$n > 10.69$ ✓
$n = 11$ ✓

OR From CAS calculator table
$n = 11$



Calculator Assumed

12. [14 marks: 4, 5, 5]

It is known that the probability of finding a defective biro in a large consignment of biros is p .

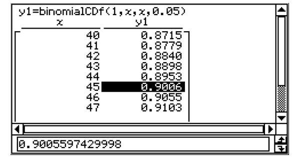
(a) Given that $p = 0.05$, find the minimum number of biros that need to be selected so that the probability that:

(i) at least one of them is defective is greater than 90%

Let X: No. of defective biros out of n.
X ~ B(n, 0.05) ✓

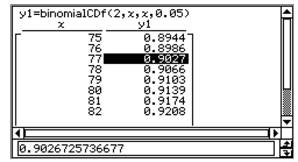
P(X ≥ 1) > 0.9
1 – P(X = 0) > 0.9 $\Rightarrow$ P(X = 0) < 0.1
$^{n}C_0 \times 0.05^0 \times 0.95^n < 0.1$ ✓

$0.95^n < 0.1$
$\Rightarrow n > 44.9$ ✓
Hence, $n = 45$. ✓



(ii) more than one of them is defective is greater than 90%.

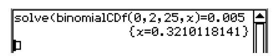
P(X > 1) > 0.9
1 – P(X = 0) – P(X =1) > 0.9 ✓
$\Rightarrow$ P(X = 0) + P(X = 1) < 0.1
$^{n}C_0 \times 0.05^0 \times 0.95^n + {^{n}C_1} \times 0.05 \times 0.95^{n-1} < 0.1$ ✓
$0.95^n + (0.05n \times 0.95^{n-1}) < 0.1$ ✓
Solve graphically, $\Rightarrow n > 76.3$ ✓
Hence, $n = 77$. ✓



(b) In a sample of 25 biros, what is the maximum value for p so that the probability that there are no more than two defective biros in the sample exceeds 0.5%? Justify your answer.

Let X: No. of defective biros out of 25.
X ~ B(25, p) ✓
P(X ≤ 2) > 0.005
P(X = 0) + P(X = 1) + P(X = 2) > 0.005 ✓
$^{25}C_0 \times p^0 \times (1-p)^{25} + {^{25}C_1} \times p \times (1-p)^{24} + {^{25}C_2} \times p^2 \times (1-p)^{23} > 0.005$
$(1-p)^{25} + 25p(1-p)^{24} + 300p^2(1-p)^{23} > 0.005$ ✓
Solve graphically, $\Rightarrow 0 < p < 0.3210$ ✓✓

Or solve on CAS calculator:
$0 < p < 0.3210$



Calculator Assumed

13. [13 marks: 2, 2, 5, 4]

Police know that from long experience, on a particular stretch of road, 1 car in every 10 will exceed the speed limit. A radar trap is set on this stretch of road.

(a) Find the probability that the police will find that the first 5 cars will be within the limit and the sixth will be speeding.

Prob. = $0.9^5 \times 0.1 = 0.059049$ ✓✓

(b) Find the probability that of 6 cars passing this radar trap, exactly one will be speeding.

Prob. = ${}^6C_1 \times 0.9^5 \times 0.1 = 0.3543$ ✓✓

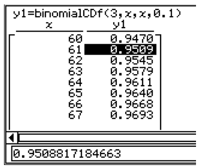
(c) n cars passed this speed trap. The probability that more than 2 cars were speeding was more than 95%. Find the least value of n.

Let X: No. of speeding cars out of n.
$X \sim B(n, 0.1)$ ✓

$P(X > 2) > 0.95$
$P(X = 0) + P(X = 1) + P(X = 2) < 0.05$
${}^nC_0 \times 0.1^0 \times 0.9^n + {}^nC_1 \times 0.1 \times 0.9^{n-1} + {}^nC_2 \times 0.1^2 \times 0.9^{n-2} < 0.05$ ✓

$0.9^n + (0.1n \times 0.9^{n-1}) + (0.01 \times \frac{n(n-1)}{2} \times 0.9^{n-2}) < 0.05$ ✓

Solve graphically, $\Rightarrow n > 60.8$ ✓
Hence, $n = 61$. ✓



(d) On another stretch of road, $100p$ cars out of 100 will have speeds exceeding the speed limit. In a sample of 20 cars, find the range of values for p such that the probability of less than 1 speeding car is exceeds 1%.

Let X: No. of speeding cars out of 20.
$X \sim B(20, p)$ ✓

$P(X < 1) > 0.01$
$P(X = 0) > 0.01$
${}^{20}C_0 \times p^0 \times (1-p)^{20} > 0.01$ ✓
$(1-p)^{20} > 0.01$ ✓
Solve graphically, $\Rightarrow 0 < p < 0.2057$ ✓

Calculator Assumed

14. [12 marks: 4, 2, 2, 4]

X is a binomial variable with parameters n and p.

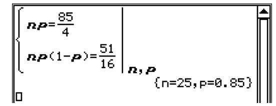
(a) Find n and p if the mean for X and its standard deviation are $\frac{85}{4}$ and $\frac{\sqrt{51}}{4}$ respectively.

$X \sim B(n, p)$.

$np = \frac{85}{4}$ I ✓

$np(1-p) = \frac{51}{16}$ II ✓

Solve simultaneously:
$\Rightarrow n = 25, p = 0.85$ ✓✓



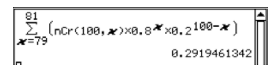
(b) For $n = 100$, $p = 0.8$, in each instance write algebraic expressions for evaluating each of the following probabilities before evaluating them.

(i) $P(X = 80)$

Prob. = ${}^{100}C_{80} \times 0.8^{80} \times 0.2^{20}$ ✓
$= 0.0993002$ ✓

(ii) $P(79 \le X \le 81)$.

$P(79 \le X \le 81) = \sum_{x=79}^{x=81} {}^{100}C_x \times 0.8^x \times 0.2^{100-x}$ ✓
$= 0.29195$ ✓



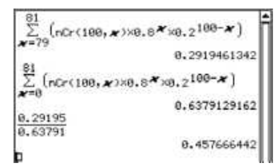
(iii) $P(X \ge 79 \mid X \le 81)$

$P(X \ge 79 \mid X \le 81) = \frac{P(79 \le X \le 81)}{P(X \le 81)}$

$= \frac{\sum_{x=79}^{x=81} {}^{100}C_x \times 0.8^x \times 0.2^{100-x}}{\sum_{x=0}^{x=81} {}^{100}C_x \times 0.8^x \times 0.2^{100-x}}$ ✓✓

$= \frac{0.29195}{0.63791}$ ✓

$= 0.4577$ ✓



Calculator Assumed

15. [9 marks: 5, 3]

It is estimated that a out of every 100 learners are successful in obtaining their driver's licence in their first attempt. On a given day there were n learners having their first attempt in getting their driver's licence.

(a) Given that the expected number of successful learners in the sample is $\frac{112}{5}$ and its variance is $\frac{112}{25}$. Find a and n.

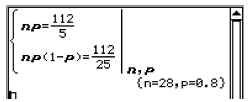
Let: X: No. of successful learners out of n.
$X \sim B(n, p)$.

$np = \frac{112}{5}$ I ✓

$np(1-p) = \frac{112}{25}$ II ✓

Solve simultaneously:
$\Rightarrow n = 28, p = 0.8$ ✓✓

Hence, $a = 80$. ✓



(b) Find the most likely number of successful learners in the sample. Justify your answer.

Let: X: No. of successful learners out of n.
$X \sim B(28, 0.8)$.

Expected number = $\frac{112}{5} = 22.4$ ✓

$P(X = 22) = 0.17791$
$P(X = 23) = 0.18565$
$P(X = 24) = 0.15470$ ✓

Hence, most likely no. = 23. ✓

Calculator Assumed

16. [8 marks: 5, 3]

A box contains b blue and w white discs.

(a) n discs are randomly chosen from this box *with* replacement.
Define X: No. of white discs selected.
X is a discrete random variable. Find the probability distribution for X. Determine the expected number of white discs selected.

Probability Distribution for X:

$P(X = x) = {}^{n}C_x \left(\frac{w}{b+w}\right)^x \left(\frac{b}{b+w}\right)^{n-x}$ ✓✓

where $x = 0, 1, 2, 3, \ldots n$. ✓

Clearly, X is a Binomial Variable with parameters n & $\left(\frac{w}{b+w}\right)$ ✓

Hence, expected No. = $n\left(\frac{w}{b+w}\right)$ ✓

(b) n discs are randomly chosen from this box *without* replacement.
Define X: No. of white discs selected.
X is a discrete random variable. Find the probability distribution for X.

Probability Distribution for X:

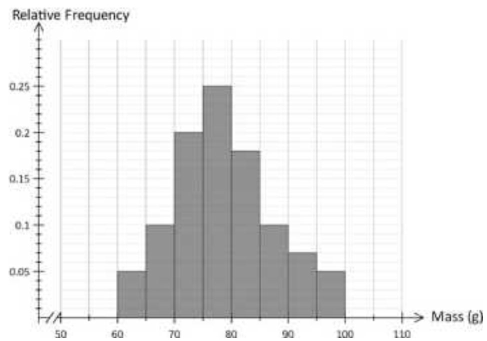
$P(X = x) = \frac{{}^{w}C_x \, {}^{b}C_{n-x}}{{}^{b+w}C_n}$ ✓✓ where $x = 0, 1, 2, 3, \ldots n$. ✓

25 Continuous Random Variables

Calculator Assumed

1. [7 marks: 2, 3, 2] [TISC]

The histogram below shows the relative frequencies for the mass W of apples in a crate. Use this histogram to answer the following questions.



(a) Would it be correct to claim that approximately 75% of apples in the crate have masses between 70g and 90g? Give a reason for your answer.

$P(70 < W < 90) \approx 0.2 + 0.25 + 0.18 + 0.1$
≈ 0.73 ✓
Hence, Yes as 73% is close to 75%! ✓

(b) Calculate the probability that a randomly selected apple from the crate has mass less than 90 g given that it has mass of at least 70 g.

$P(W < 90 \mid W \geq 70) \approx \dfrac{P(70 \leq W < 90)}{P(W \geq 70)}$ ✓

$\approx \dfrac{0.73}{0.85} \approx 0.8588$ ✓✓

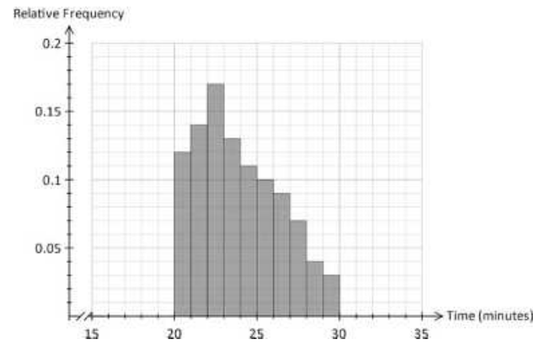
(c) 85% of all apples in the crate have mass greater than k g. Calculate the value of k. Show how you obtained your answer.

$P(W > k) = 0.85$
From histogram $P(W < 70) = 0.15$ ✓
$\Rightarrow P(W > 70) = 0.85$
Hence, $k = 70$ ✓

Calculator Assumed

2. [9 marks: 1, 3, 3, 2]

The random variable T is defined as the time (minutes) required to complete a task. Use the relative frequency histogram for T drawn below to answer the following questions.



(a) Estimate $P(23 \leq T \leq 26)$.

$P(23 \leq T \leq 26) = 0.13 + 0.11 + 0.10$
$= 0.34$ ✓

(b) Estimate $P(T \geq 23 \mid T \leq 26)$

$P(T \geq 23 \mid T \leq 26) = \dfrac{P(23 \leq T \leq 26)}{P(X \leq 26)}$ ✓

$= \dfrac{0.34}{0.34+0.17+0.14+0.12}$ ✓

$= \dfrac{0.34}{0.77} = \dfrac{34}{77}$ ✓

(c) Use an appropriate method to calculate the mean and standard deviation for T. Explain the method you used.

Assign the relative frequency to the midpoint of the interval (column). ✓
Mean ≈ 23.93 ✓
Standard deviation ≈ 2.4789 ✓

(d) Determine with reasons if the median for T is more likely to be less than or greater than the mean for T.

Median for X < Mean for X. ✓
Histogram is skewed right. ✓

Calculator Free

3. [4 marks: 2, 2]

Determine with reasons if each of the following functions are probability density functions for continuous random variables:

(a) $f(x) = \begin{cases} \frac{2}{x} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

$\int_0^2 \frac{2}{x}\,dx$ is undefined. ✓
Hence, $f(x)$ is not a pdf. ✓

(b) $g(x) = \frac{x}{15}$ for $1 \le x \le 5$.

$\int_1^5 \frac{x}{15}\,dx = \left[\frac{x^2}{30}\right]_1^5$
$= \frac{25-1}{30} = \frac{24}{30} \neq 1$ ✓
Hence, No! ✓

---

4. [6 marks]

The random variable X has probability density function $f(x) = \frac{x}{2}$ for $0 \le x \le 2$.

Calculate the variance σ^2 for X using the formula $\sigma^2 = \left(\int x^2 f(x)\,dx\right) - \mu^2$ where μ is the mean for X.

$\mu = \int_0^2 x \times \left(\frac{x}{2}\right) dx$ ✓
$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$ ✓✓
$\sigma^2 = \int_0^2 x^2 \times \left(\frac{x}{2}\right) dx - \left(\frac{4}{3}\right)^2$ ✓
$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 - \frac{16}{9}$ ✓
$= 2 - \frac{16}{9} = \frac{2}{9}$ ✓

Calculator Free

5. [6 marks]

The random variable X has mean $\frac{2}{3}$ and probability density function $f(x) = kx$ for $0 \le x \le a$. Determine the values of k and a.

$\int_0^a kx\,dx = 1$ ✓
$\left[\frac{kx^2}{2}\right]_0^a = 1$
$ka^2 = 2$ I ✓
$E(X) = \int_0^a x \times kx\,dx = \frac{2}{3}$ ✓
$\left[\frac{kx^3}{3}\right]_0^a = \frac{2}{3}$
$ka^3 = 2$ II ✓
Substitute I into II: $a = 1, k = 2$ ✓✓

---

6. [4 marks] [TISC]

$F(x) = \int_0^x k(1-x^2)\,dx$ for $0 \le x \le 1$, is the cumulative distribution function for a continuous random variable. Find k.

$\int_0^1 k(1-x^2)\,dx = 1$ ✓
$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$ ✓
$k\left[1 - \frac{1}{3}\right] = 1$ ✓
$k = \frac{3}{2}$ ✓

Calculator Free

7. [9 marks: 3, 4, 2]   [TISC]

The random variable X has probability density function $f(x) = \dfrac{2}{x^2}$ for $a \le x \le 2a$, where a is a constant.

(a) Calculate the value of a.

$$\int_a^{2a} 2x^{-2}\,dx = 1 \quad ✓$$
$$-2\left[\frac{1}{x}\right]_a^{2a} = 1 \quad ✓$$
$$-2\left(\frac{1}{2a} - \frac{1}{a}\right) = 1 \;\Rightarrow\; a = 1 \quad ✓$$

(b) The random variable Y = 2X.

(i) Determine $\overline{Y}$.

$$\text{Mean for X} = \int_1^2 x \times \frac{2}{x^2}\,dx$$
$$= \int_1^2 \frac{2}{x}\,dx \quad ✓$$
$$= 2\left[ln(x)\right]_1^2 \quad ✓$$
$$= 2\,ln\,(2) \quad ✓$$
$$\text{Hence: } \overline{Y} = 4\,ln\,(2 \quad ✓$$

(ii) If the variance for Y is v, state in terms of v, the standard deviation for X.

$$\text{Var(Y)} = 4\,\text{Var(X)} \quad ✓$$
$$\Rightarrow \text{Var(X)} = \frac{v}{4}$$
$$\text{Hence, STD(X)} = \frac{\sqrt{v}}{2} \quad ✓$$

Calculator Assumed

8. [6 marks: 3, 3]

A random variable X has cumulative probability distribution given by $P(X \le x) = \dfrac{(x+5)(x-1)}{16}$ for $a \le x \le b$ where a and b are both positive real numbers.

(a) Determine with reasons the values of a and b.

$$P(X \le 1) = 0 \;\Rightarrow\; a = 1 \quad ✓$$
$$P(X \le b) = 1 \;\Rightarrow\; \frac{(b+5)(b-1)}{16} = 1$$
$$b^2 + 4b - 21 = 0 \quad ✓$$
$$(b+7)(b-3) = 0$$
$$b = 3 \quad (\text{reject } -7 \text{ as } b > 1) \quad ✓$$

(b) Determine $f(x)$, the probability density function for X.

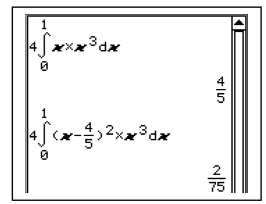
$$f(x) = \frac{d}{dx}\left(\frac{(x+5)(x-1)}{16}\right) \quad ✓$$
$$= \frac{x+2}{8} \quad ✓$$
Domain for $f(x)$ is $1 \le x \le 3$. ✓

9. [7 marks: 4, 3]

The random variable X has probability density function $f(x) = 4x^3$ for $0 \le x \le 1$.

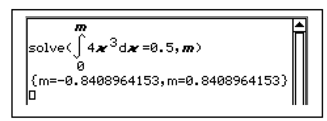
(a) Determine μ and σ^2, respectively the mean and variance for X.

$$\mu = 4\int_0^1 x \times x^3\,dx \quad ✓$$
$$= \frac{4}{5} \quad ✓$$
$$\sigma^2 = 4\int_0^1 (x - \tfrac{4}{5})^2 \times x^3\,dx \quad ✓$$
$$= \frac{2}{75} \quad ✓$$



(b) Find m the median of X.

$$P(X \le m) = 0.5 \;\Rightarrow\; 4\int_0^m x^3\,dx = 0.5 \quad ✓$$
$$m^4 = 0.5 \quad ✓$$
$$\Rightarrow m = 0.8409 \quad ✓$$



Calculator Assumed

10. [10 marks: 3, 3, 2, 2]

The probability density function for a continuous random variable T is given by

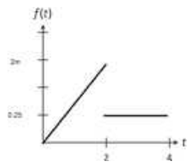
$$f(t) = \begin{cases} mt & 0 \le t \le 2 \\ \frac{1}{4} & 2 < t < 4 \end{cases}$$

(a) Find the value of m.

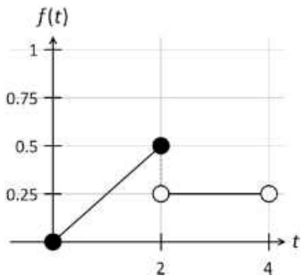
Area under curve = 1

$\frac{1}{2} \times 2 \times 2m + 2 \times \frac{1}{4} = 1$ ✓✓

$m = \frac{1}{4}$ ✓



(b) Sketch the graph of the probability density function of T.



✓ Correct for $0 \le t \le 2$.
✓ Correct for $2 < t < 4$.
✓ End points.

(c) Find $P(T \le 1)$.

$P(T \le 1) = \frac{1}{2} \times 1 \times \frac{1}{4}$ ✓

$= \frac{1}{8}$ ✓

(d) Find the median of T.

Let median be m.
$\Rightarrow P(X \le m) = 0.5$
But $P(X \le 2) = 0.5$ ✓
Hence, median of T = 2. ✓

Calculator Assumed

11. [12 marks: 3, 2, 3, 4]

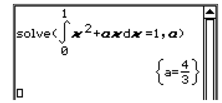
The probability density function of a random variable X is given by $f(x) = x^2 + ax$ for $0 < x < 1$.

(a) Find the value of a.

$\int_0^1 x^2\,dx + a\int_0^1 x\,dx = 1$ ✓

$\frac{1}{3} + \frac{a}{2} = 1$ ✓

$\Rightarrow a = \frac{4}{3}$ ✓



(b) Find $P(X > 0.5)$.

$P(X > 0.5) = \int_{0.5}^{1} x^2 + \frac{4x}{3}\,dx$ ✓

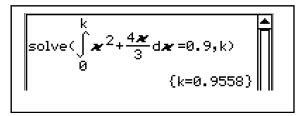
$= \frac{19}{24}$ ✓

(c) Find the value of k if $P(X \le k) = 0.9$.

$\int_0^k x^2 + \frac{4x}{3}\,dx = 0.9$ ✓

$\frac{k^3}{3} + \frac{2k^2}{3} = 0.9$ ✓

$\Rightarrow k = 0.9558$ ✓



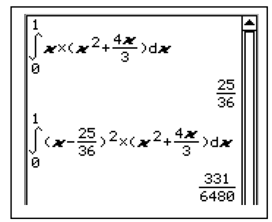
(d) Calculate the mean and variance for X.

$E(X) = \int_0^1 x \times \left(x^2 + \frac{4x}{3}\right) dx$ ✓

$= \frac{25}{36}$ ✓

$Var(X) = \int_0^1 \left(x - \frac{25}{36}\right)^2 \times \left(x^2 + \frac{4x}{3}\right) dx$ ✓

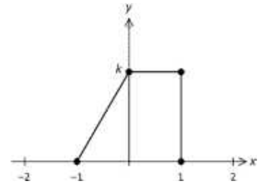
$= \frac{331}{6480}$ ✓



Calculator Assumed

12. [9 marks: 2, 2, 3, 2] [TISC]

The graph of the probability density function of a continuous random variable X is shown in the accompanying diagram.



(a) Find the value of k.

> Area under curve = 1
> Hence, $(k \times 1) + \frac{1}{2} \times 1 \times k = 1$ ✓
> $\Rightarrow k = \frac{2}{3}$ ✓

(b) Find P(X < 0.5).

> $P(X < 0.5) = 1 - \frac{1}{2} \times \frac{2}{3} = \frac{2}{3}$ ✓✓

(c) Find $P(X > 0 \mid X < 0.5)$.

> $P(X > 0 \mid X < 0.5) = \frac{P(X > 0 \cap X < 0.5)}{P(X < 0.5)}$
> $= \frac{P(0 < X < 0.5)}{P(X < 0.5)}$ ✓
> $= \frac{1/3}{2/3} = \frac{1}{2}$ ✓✓

(d) Find a if $P(X > a) = \frac{1}{2}$.

> Area of rectangle to the right of the line x = a is $(1 - a) \times \frac{2}{3}$
> Hence, $(1 - a) \times \frac{2}{3} = \frac{1}{2}$ ✓
> $\Rightarrow a = \frac{1}{4}$ ✓

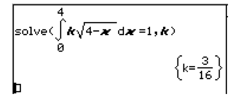
Calculator Assumed

13. [8 marks: 2, 3, 3] [TISC]

The probability density function of a continuous random variable X is given by $f(x) = k\left(\sqrt{4-x}\right)$ for $0 \le x \le 4$ where k is a real constant.

(a) Show that $k = \frac{3}{16}$.

> $k\int_0^4 \sqrt{4-x}\,dx = 1$ ✓
> $k \times \frac{16}{3} = 1$ ✓
> $\Rightarrow k = \frac{3}{16}$

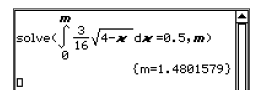


(b) Find $P(X > 1 \mid X < 3)$.

> $P(X > 1 \mid X < 3) = \frac{P(X > 1 \cap X < 3)}{P(X < 3)}$
> $= \frac{P(1 < X < 3)}{P(X < 3)}$ ✓
> $= \frac{0.5245}{0.875}$ ✓
> $= 0.5995$ ✓

(c) Find the median for X accurate to 2 decimal places. Justify your answer.

> Let median be m.
> $\Rightarrow P(X \le m) = 0.5$
> $\frac{3}{16}\int_0^m \sqrt{4-x}\,dx = 0.5$ ✓
> $\frac{3}{16} \times \left[\frac{2(4-x)^{3/2}}{-3}\right]_0^m = 0.5$ ✓
> $\frac{3}{16} \times \left[\frac{2(4-m)^{3/2}}{-3} - \frac{2(4)^{3/2}}{-3}\right] = 0.5$
> $\Rightarrow m = 1.48$ ✓

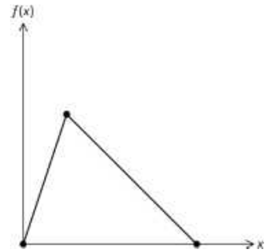


Calculator Assumed

14. [9 marks: 2, 2, 2, 3] [TISC]

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 0.5x & 0 \le x < 1 \\ \dfrac{-x}{6} + \dfrac{2}{3} & 1 \le x \le k \end{cases}$$ where k is a real number. The sketch of $y = f(x)$ is given in the accompanying diagram.



(a) Show that $k = 4$.

$f(1) = \frac{1}{2}$ ✓

Area under curve = 1.

$\Rightarrow \frac{1}{2} \times (\frac{1}{2} \times k) = 1$ ✓

$k = 4$

(b) Find P(X > 0.5).

$P(X > 0.5) = 1 - \frac{1}{2} \times (\frac{1}{2} \times \frac{1}{4})$ ✓

$= \frac{15}{16}$ ✓

(c) Find $P(X \le 2 \mid X > 0.5)$.

$P(X \le 2 \mid X > 0.5) = \frac{P(0.5 < X \le 2)}{P(X > 0.5)}$

$= \frac{1 - \frac{1}{2}(\frac{1}{2} \times \frac{1}{4}) - \frac{1}{2}(2 \times \frac{1}{3})}{\frac{15}{16}}$ ✓

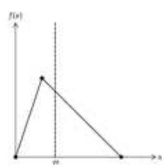
$= \frac{29}{45}$ ✓

(d) Find m, the median value of x, such that $P(X < m) = 0.5$.

$P(X < m) = 0.5$

$\frac{1}{2} \times (4 - m)(\frac{2}{3} - \frac{m}{6}) = \frac{1}{2}$ ✓✓

$m = 1.5505$ ✓



Calculator Assumed

15. [9 marks: 2, 3, 4] [TISC]

The random variable X has probability density function

$f(x) = \frac{9}{(x+1)^3}$ for $1 \le x \le 5$.

(a) Calculate $P(X \ge 2)$.

$P(X \ge 2) = \int_2^5 \frac{9}{(x+1)^3}\,dx$ ✓

$= \frac{3}{8}$ ✓

(b) Given that $P(X \ge k \mid X \ge 2) = \frac{3}{16}$, calculate the value of k.

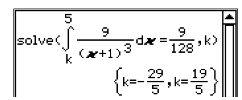
$P(X \ge k \mid X \ge 2) = \frac{P(X \ge k)}{P(X \ge 2)}$ ✓

$P(X \ge k) = \frac{3}{16} \times P(X \ge 2)$

$= \frac{9}{128}$ ✓

$\int_k^5 \frac{9}{(x+1)^3}\,dx = \frac{9}{128}$

$k = \frac{19}{5} = 3.8$ ✓



(c) Calculate the exact variance for X.

Mean $\mu = \int_2^5 \frac{9}{(x+1)^3} \times x\,dx$ ✓

$= 2$ ✓

Variance $= \int_2^5 \frac{9}{(x+1)^3} \times (x-2)^2\,dx$ ✓

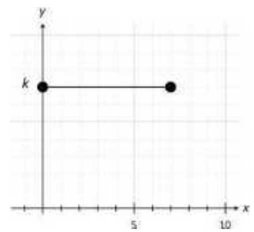
$= 9\,(ln\,3 - 1)$ ✓

26 The Uniform Distribution

Calculator Free

1. [10 marks: 1, 2, 3, 4]

The sketch of the probability density function of a continuous random variable X is given in the accompanying diagram.



(a) Find k.

Area under curve $7 \times k = 1$

$\Rightarrow \quad k = \frac{1}{7}$ ✓

(b) State the probability density function of the variable.

$f(x) = \frac{1}{7} \quad 0 \le x \le 7$. ✓✓

(c) Find $P(X > 2 | X < 5)$.

$P(X > 2 \mid X < 5) = \frac{P(2 < X < 5)}{P(X < 5)}$ ✓✓

$= \frac{3}{5}$ ✓

(d) Find μ and σ are respectively the mean and standard deviation for X.

$\mu = \frac{7}{2}$ ✓	$\mu = \frac{7}{2}$ ✓
$E(X^2) = \int_0^7 \frac{x^2}{7}\,dx$	$\sigma^2 = \int_0^7 \frac{(x-3.5)^2}{7}\,dx$
$= \frac{1}{7}\left[\frac{x^3}{3}\right]_0^7 = \frac{49}{3}$ ✓	$= \frac{1}{21}\left[(x-3.5)^3\right]_0^7$
$\sigma^2 = \frac{49}{3} - \left(\frac{7}{2}\right)^2 = \frac{49}{12}$ ✓	$= \frac{1}{21}\left[\left(\frac{7}{2}\right)^3 + \left(\frac{7}{2}\right)^3\right]$ ✓
$\sigma = \frac{7\sqrt{3}}{6}$ ✓	$= \frac{49}{12}$ ✓
	$\sigma = \frac{7\sqrt{3}}{6}$ ✓

Calculator Free

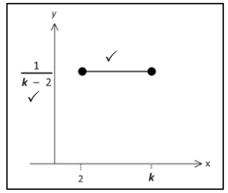
2. [10 marks: 2, 2, 3, 3]

The random variable X is uniformly distributed in the interval $2 \le x \le k$.

(a) Find in terms of k, the probability distribution function for X.

$f(x) = \frac{1}{k-2} \quad 2 \le x \le k$. ✓✓

(b) Sketch the probability distribution function for X.



(c) Find $P(X \le 10 | X \ge 4)$.

$P(X \le 10 \mid X \ge 4) = \frac{P(4 \le X \le 10)}{P(X \ge 4)}$ ✓

$= \frac{6}{k-4}$ ✓✓

(d) It is known that $P(X \le 5) = 0.25$. Find the median for X. Justify your answer.

$P(X \le 5) = 0.25 \quad \Rightarrow \quad \frac{5-2}{k-2} = \frac{1}{4}$ ✓

$k = 14$ ✓

Hence, median = 8 ✓

---

3. [4 marks]

The probability density function of X is given by $f(x) = 0.1$ for $a \le x \le b$.

Given that $P(X > 7 | X \le 8) = \frac{1}{3}$, find a and b.

$P(X > 7 | X \le 8) = \frac{1}{3} \quad \Rightarrow \quad \frac{P(7 < X \le 8)}{P(a < X \le 8)} = \frac{1}{3}$ ✓

$\frac{1 \times 0.1}{(8-a) \times 0.1} = \frac{1}{3}$ ✓

$\Rightarrow \quad a = 5$ ✓

$b = 15$ ✓

Calculator Assumed

4 [9 marks: 2, 1, 2, 4]

The length of time Daniel is late to class may be modelled by a uniform distribution with a minimum late time of 5 minutes and a maximum late time of 25 minutes. Define T: The length of time Daniel is late to class.

(a) Write the probability density function for T.

$f(t) = \frac{1}{20} \quad 5 \le t \le 25$ ✓✓

(b) Find the probability that Daniel is exactly 15 minutes late.

Prob. = 0 ✓

(c) Find the probability the Daniel is no more than 20 minutes late given that he is at least 10 minutes late.

$P(T \le 20 \mid T \ge 10) = \frac{10}{15}$ ✓✓

(d) On a school week of five days, find the probability that Daniel is late by at least 15 minutes at least once.

X: No. of days late by ≥ 15 minutes out of 5. ✓

$P(\text{late} \ge 15 \text{ minutes}) = P(T \ge 15) = \frac{1}{2}$ ✓

Hence, $X \sim B(5, \frac{1}{2})$. ✓

$P(X \ge 1) = 1 - P(X = 0)$

$= 1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$ ✓

Calculator Assumed

5. [8 marks: 2, 2, 4]

The length of the red cycle of a set of traffic lights is 90 seconds. Assume that vehicles arrive at the traffic lights randomly and independently of each other. Define the random variable T as the waiting time at the traffic lights.

(a) Describe the probability density function of T.

$f(t) = \frac{1}{90} \quad 0 \le t \le 90$. ✓✓

(b) Find the probability that a motorist has to wait less than 30 seconds.

$P(T < 30) = \frac{1}{3}$ ✓✓

(c) Calculate $P(\mu - \sigma < T < \mu + \sigma)$ where μ and σ are respectively the mean and standard deviation for T.

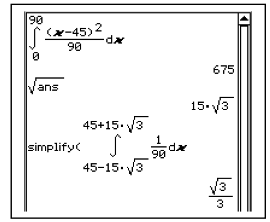
$\mu = 45$ ✓

$\sigma^2 = \int_0^{90} (t-45)^2 \times \frac{1}{90}\, dt$

$= 675$ ✓

$\sigma = 15\sqrt{3}$ ✓

$P(45 - 15\sqrt{3} < T < 45 + 15\sqrt{3}) = \frac{\sqrt{3}}{3}$ ✓



6. [9 marks: 2, 2, 2, 3]

An automatic filling machine fills and packs 1 kg packs of sugar. The machine can fill any pack with any amount of between 0 and 10 grams (inclusive) of extra sugar. Define M as the extra mass of sugar (in grams) fed into each bag.

(a) Find the probability density function for M.

$f(m) = \frac{1}{10} \quad 0 \le t \le 10$. ✓✓

(b) Find $P(3 \le M \le 7)$.

$P(3 \le M \le 7) = \frac{4}{10} = \frac{2}{5}$ ✓✓

Calculator Assumed

6. (c) Find the probability of obtaining a 1 kg pack of sugar with mass of between 1.003 and 1.007 kg.

Required Prob. = $P(3 \le M \le 7) = \frac{2}{5}$ ✓✓

(d) The probability of obtaining a 1 kg pack of sugar with a mass of no more than α kg is 0.75. Find α.

Let the excess be k. $P(M \le k) = 0.75$

Hence, $\frac{k}{10} = 0.75$ ✓

$k = 7.5$ g ✓

Therefore, $\alpha = 1.0075$ kg. ✓

---

7. [6 marks: 3, 3]

The random variable X is uniformly distributed in the interval $a \le x \le b$ where a and b are real constants. The mean and variance for X are respectively 7 and $\frac{(b-a)^2}{12}$.

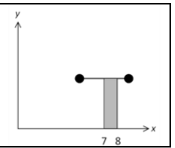
(a) Determine with reasons, the values of a and b if $P(X \le 8) = 0.75$.

$P(X \le 7) = 0.5$ and $P(X \le 8) = 0.75$

$\Rightarrow P(7 \le X \le 8) = 0.25$ ✓

Hence, pdf for X $f(x) = \frac{1}{4}$ for $a \le x \le b$. ✓

Hence: $a = 5$ and $b = 9$ ✓



(b) Determine with reasons, the values of a and b if the variance for X is 3.

$\frac{a+b}{2} = 7 \Rightarrow b = 14 - a$ ✓

$\frac{(b-a)^2}{12} = 3 \Rightarrow (14-2a)^2 = 36$ ✓

$\Rightarrow a = 4$ or 10

Hence: $a = 4, b = 10$ ✓

Reject $a = 10, b = 4$ as $a < b$.

Calculator Assumed

8. [9 marks: 4, 5] [TISC]

A random variable X is distributed uniformly over the interval $a \le x \le b$. The mean and variance for X are respectively 12 and 12.

(a) Show that the probability distribution function for X can be written as

$$f(x) = \frac{1}{2b-24}; \quad 24 - b \le x \le b.$$

$f(x) = \frac{1}{b-a}$ $a \le x \le b$ ✓

$E(X) = 12 \Rightarrow \frac{b+a}{2} = 12$ ✓

$a + b = 24$

$a = 24 - b$ ✓

Hence: $f(x) = \frac{1}{2b-24}$ $24 - b \le x \le b$ ✓

(b) Hence or otherwise, determine the probability distribution function for X. (5 marks)

$Var(X) = 12$

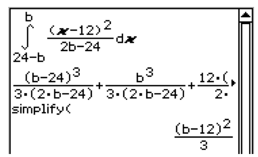
$\Rightarrow \int_{24-b}^{b} \frac{1}{2b-24}(x-12)^2\,dx = 12$ ✓

$\frac{(b-12)^2}{3} = 12$ ✓

$b = 6$ or 18. ✓

Reject $b = 6$ as $24 - b < b$. ✓

Hence: $f(x) = \frac{1}{12}$ $6 \le x \le 18$ ✓



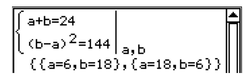
$E(X) = 12 \Rightarrow a + b = 24$ (1) ✓

$Var(X) = \frac{(b-a)^2}{12} = 12$ (2) ✓

$a = 6, b = 18$ or $a = 18, b = 6$ ✓

But $a < b$, hence reject $a = 18, b = 6$ ✓

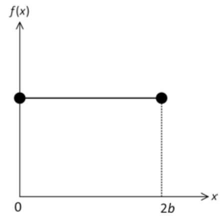
Hence: $f(x) = \frac{1}{12}$ $6 \le x \le 18$ ✓



Calculator Assumed

9. [10 marks: 2, 3, 2, 3] [TISC]

A continuous random variable X is distributed uniformly over the interval $0 \le x \le 2b$. The graph of the probability density function for X is shown in the accompanying diagram.



(a) State $f(x)$, the probability distribution function for X. [2 marks]

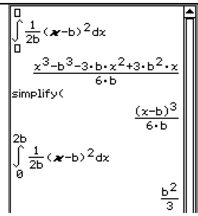
$f(x) = \frac{1}{2b} \quad 0 \le x \le 2b$ ✓ ✓

(b) Given that E(X) = b, write an integral which when evaluated will give the variance of X. Hence, determine the variance for X.

$\text{Var}(X) = \int_0^{2b} \frac{1}{2b}(x-b)^2\, dx$ ✓

$= \frac{1}{2b}\left[\frac{(x-b)^3}{3}\right]_0^{2b}$ ✓

$= \frac{b^2}{3}$ ✓



(c) Let $b = 5$.

(i) Calculate $P(X \le 4)$.

$f(x) = \frac{1}{10} \quad 0 \le x \le 10$ ✓

$P(X \le 4) = 0.4$ ✓

(ii) The random variable Y is defined as $Y = \begin{cases} 1 & \text{if } X \le 4 \\ 0 & \text{if } X > 4 \end{cases}$.

Calculate the mean and variance for Y.

Y is a Bernoulli variable with $p = 0.4$. ✓ Hence: E(Y) = 0.4 ✓ Var(Y) = 0.4 × 0.6 = 0.24 ✓	$E(Y) = 1 \times P(X \le 4) + 0 \times P(X > 4)$ $= 1 \times 0.4 = 0.4$ $\text{Var}(Y) = (1-0.4)^2 \times 0.4 + (0-0.4)^2 \times 0.6$ $= 0.24$

27 The Normal Distribution

Calculator Free

1. [3 marks: 1, 1, 1]

X is a normal variable with mean 100 and standard deviation 20.
Given that $P(X \ge 150) = a$, determine in terms of a:

(a) $P(X \le 150)$

$P(X \le 150) = 1 - a$ ✓

(b) $P(50 \le X \le 150)$

$P(50 \le X \le 150) = 1 - 2a$ ✓

(c) $P(100 \le X \le 150)$

$P(100 \le X \le 150) = 0.5 - a$ ✓

2. [11 marks: 2, 3, 3, 3]

Use the empirical rule described below to answer the questions that follow.

68% of values lie within one standard deviation of the mean
95% of values lie within two standard deviations of the mean
99.7% of values lie within three standard deviations of the mean.

X is a normal variable with mean 100 and standard deviation 10. Estimate:

(a) $P(80 < X < 100)$

$P(80 < X < 100) \approx \frac{1}{2} \times 0.95$ ✓

≈ 0.475 ✓

(b) $P(X \ge 80)$

$P(X \ge 80) \approx \frac{1}{2} + \frac{1}{2} \times 0.95$ ✓✓

$\approx 0.5 + 0.475$

≈ 0.975 ✓

Calculator Free

2. (c) $P(X \geq 80 \mid X \leq 100)$

$$P(X \geq 80 \mid X \leq 100) \approx \frac{\left(\frac{1}{2} \times 0.95\right)}{\frac{1}{2}} \quad ✓✓$$
$$\approx 0.95 \quad ✓$$

(d) $P(X \leq 120 \mid X \geq 100)$

$$P(X \leq 120 \mid X \geq 100) \approx \frac{\left(\frac{1}{2} \times 0.95\right)}{\frac{1}{2}} \quad ✓✓$$
$$\approx 0.95 \quad ✓$$

---

3. [8 marks: 2, 3, 3]

The length of nails produced by a factory is normally distributed with mean 10 mm and standard deviation 0.5 mm. Use the empirical rule given below to answer the following questions.

68% of values lie within one standard deviation of the mean
95% of values lie within two standard deviations of the mean
99.7% of values lie within three standard deviations of the mean.

(a) Calculate the probability that a randomly chosen nail produced in this factory measures more than 9 mm long.

$$P(X > 9) = P(Z > -2) \quad ✓$$
$$= 1 - 0.025 = 0.975 \quad ✓$$

(b) Determine the probability that a nail with length no more than 10 mm has length in excess of 9 mm.

$$P(X > 9 \mid X < 10) = P(Z > -2 \mid Z < 0) \quad ✓$$
$$= \frac{P(-2 < Z < 0)}{P(Z < 0)}$$
$$= \frac{0.475}{0.5} = 0.95 \quad ✓✓$$

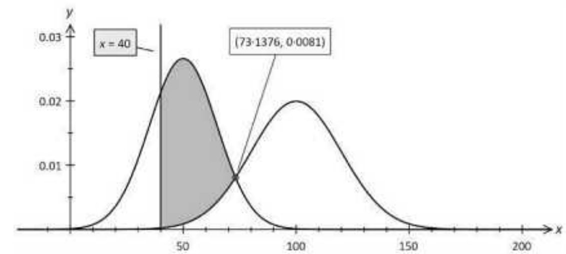
(c) Determine the 97.5th percentile length.

$P(X < k) = 0.975$ ✓
$P(Z < 2) = 0.5 + 0.475$ ✓
Hence $k = 10 + 2(0.5) = 11$ cm ✓

Calculator Assumed

4. [9 marks: 2, 2, 2, 3]

The diagram below shows the graph of $y = f(x)$ and $y = g(x)$ where $f(x)$ is the probability density function of a normal random variable with mean 50 and standard deviation 15 and $g(x)$ is the probability density function of a normal random variable with mean 100 and standard deviation 20. The two graphs intersect at the point (73.1376, 0.0081). The region R is trapped between the line with equation $x = 40$ and the two curves.



(a) Determine with reasons if each of the following functions may be used as probability density functions of random variables.

(i) $y = 2f(x)$

No. ✓ Reason: $\int_{-\infty}^{\infty} 2f(x)\,dx = 2 \neq 1$ ✓

(ii) $y = g(x) - 1$

No. ✓ Reason: $y(73.1376) = -0.99 < 0$ ✓

(iii) $y = f(x - 20)$

Yes. ✓
Reason: $f(x - 20) > 0$ for $-\infty < x < \infty$
and $\int_{-\infty}^{\infty} f(x-20)\,dx = 1$ ✓

(b) Calculate the area of region R.

Let $X \sim N(50, 15^2)$ and $Y \sim N(100, 20^2)$.
Area of R = $P(40 \leq X \leq 73.1376) - P(Y \leq 73.1376)$ ✓
$= 0.686032 - 0.596416$ ✓
$= 0.596416$ ✓

Calculator Assumed

5. [7 marks: 2, 2, 3]

The speed of vehicles passing a school zone each school-day is normally distributed with a mean of 35 km/h and a standard deviation of 3 km/h.

(a) Find the probability that a vehicle passing the school zone travels with a speed in excess of the mean speed by at least 2 standard deviations.

$X \sim N(35, 3^2)$
$P(X > 35 + 2\times 3) = P(X > 41)$ ✓
$= 0.02275$ ✓

(b) 25% of vehicles passing the school zone travel in excess of *a* km/h. Find *a*.

$P(X > a) = 0.25$ ✓
$a = 37.0$ km/h ✓

```
invNormCDf("R",0.25,3,35)
                37.02346925
solve(normCDf(a,∞,3,35)=0.25,a)
              {a=37.02346925}
```

(c) On a certain morning, 30 vehicles were noted passing through the school zone. Find the probability that no more than 15 were travelling in excess of 35 km/h.

X: No. of vehicles out of 30 with speeds > 35 km/h ✓
$X \sim B(30, 0.5)$ ✓
$P(X \le 15) = 0.57223$ ✓

---

6. [5 marks]

The length of sleep that Emily gets each night is a normal variable with mean μ minutes and standard deviation σ minutes. The probability that Emily sleeps more than 8 and a-half hours is 0.01313. The probability that Emily sleeps less than 8 hours is 0.13326. Calculate the values of μ and σ.

Define L: Length of sleep in minutes $\Rightarrow L \sim N(\mu, \sigma^2)$
Given: $P(L \ge 510) = 0.01313$
$\Rightarrow P(Z \ge \frac{510-\mu}{\sigma}) = 0.01313$ ✓
$\frac{510-\mu}{\sigma} = 2.22234$ I ✓
Given: $P(L \le 480) = 0.13326$
$\Rightarrow P(Z \le \frac{480-\mu}{\sigma}) = 0.13326$ ✓
$\frac{480-\mu}{\sigma} = -1.11111$ II ✓
Solve I and II simultaneously: $\mu \approx 490 \quad \sigma = 9$ ✓

Calculator Assumed

7. [10 marks: 1, 1, 3, 2, 3] [TISC]

The life-span of a light bulb manufactured by GloWest is normally distributed with a mean of 800 hours and a standard deviation of 120 hours.

(a) Find the probability that a randomly chosen light bulb manufactured by GloWest has a life-span :

(i) of exactly 800 hours

$X \sim N(800, 120^2)$
$P(X = 800) = 0$ ✓

(ii) that exceeds 700 hours

$P(X > 700) = 0.7977$ ✓

(iii) that is less than 900 hours given that it exceeds 700 hours.

$P(X < 900 \mid X > 700) = \frac{P(700 < X < 900)}{P(X > 700)}$
$= \frac{0.5953}{0.7977}$ ✓✓
$= 0.7463$ ✓

(b) Find the lifespan exceeded by 95% of all globes manufactured by GloWest.

$P(X > k) = 0.95 \Rightarrow k = 602.6$ hours ✓✓

(c) Jan needs to calculate the probability that in the batch of 500 light bulbs from GloWest, there are at least 400 light bulbs with life-spans that exceed 700 hours. State what probability distribution(s) Jan should use and the values of the corresponding parameters. Calculate this probability.

Y: No. of bulbs with life-spans that exceed 700 hours out of 5 00 bulbs.
$Y \sim B(500, 0.7977)$ ✓✓

$P(Y \ge 400) = 0.4755$ ✓

Calculator Assumed

8. [9 marks: 1, 2, 3, 3] [TISC]

Mr Green owns an environmentally friendly car called the eco-car. The fully charged battery of an eco-car, allows the driver to travel a certain distance, D km, before the battery needs recharging. This distance D is a normal variable with mean 120 km and standard deviation 12 km.

(a) Find the probability that the car will travel exactly 100 km before the battery needs recharging.

$D \sim N(120, 12^2)$
$P(D = 100) = 0$ ✓

Mr Green needs to drive from the town where he lives to visit his mother 150 km away. He starts off with a fully charged battery.

(b) What is the probability that Mr Green will be able to get to his mother without having to recharge the car battery along the way. Show clearly the probability you calculated.

$P(D > 150) = 0.0062$ ✓✓

(c) Given that Mr Green was not able to get to his mother without having to recharge the car battery along the way, what is the probably that he got to within 10 km of his mother. Show clearly the probabilities you calculated.

$$P(D > 140 \mid D < 150) = \frac{P(140 < D < 150)}{P(D < 150)} = \frac{0.04158}{0.99379} = 0.04184$$
✓✓ ✓

(d) Mr Green attached a solar-powered booster to his car battery so that the mean for D is now μ with the standard deviation remaining at 12 km. Find μ given that the probability that he will be able to reach his mother on a fully charged battery without recharging along the way is 10%. Show clearly the different stages of your calculations.

$P(D > 150) = 0.1$

$P(Z > \frac{150-\mu}{12}) = 0.1$ ✓

$\frac{150-\mu}{12} = 1.28155$ ✓

$\Rightarrow \mu = 134.6$ km ✓

```
solve(normCDf(150,∞,12,x)=0.1,x)
                    {x=134.6213812}
```


Calculator Assumed

9. [10 marks: 1, 3, 3, 3] [TISC]

The annual rainfall (in mm) over a dam catchment area may be considered a normal variable with mean 900 mm and standard deviation 30 mm.

(a) Find the probability of the catchment area receiving an annual rainfall of less than 850 mm.

$X \sim N(900, 30^2)$
$P(X < 850) = 0.04779$ ✓

(b) Find the probability of the catchment area receiving an annual rainfall of no more than 850 mm given that it received no more than 900 mm.

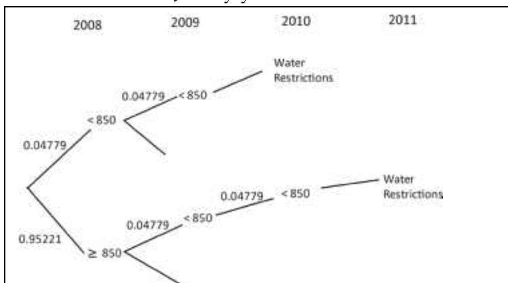
$$P(X \le 850 \mid X \le 900) = \frac{P(X \le 850)}{P(X \le 900)} = \frac{0.04779}{0.5} = 0.09558$$
✓✓ ✓

Water Restrictions are imposed in this area in the year following two consecutive years where the annual rainfall is less than 850 mm. Assume that the amount of annual rainfall is independent from year to year.
In 2007 there were no Water Restrictions imposed in this area.

(c) Use the given information to find the probability of Water Restrictions being imposed in 2010. Justify your answer.

Required Prob. $= 0.04779^2 = 0.00228$ ✓✓ ✓

(d) Use the given information to find the probability of Water Restrictions being imposed in 2010 or 2011. Justify your answer



Required Prob. $= 0.04779^2 + 0.95221 \times 0.04779^2 = 0.00446$ ✓✓ ✓

Calculator Assumed

10. [12 marks: 2, 3, 3, 4]

A poultry farm supplies processed chickens to a Fast Food Store. The mass of chickens supplied is normally distributed with mean 1.8 kg and standard deviation 100 g.

(a) Find the probability that a randomly chosen chicken has mass more than 2.0 kg.

Let X: mass of chicken
$X \sim N(1.8, 0.1^2)$ ✓
$P(X > 2) = 0.02275$ ✓

(b) Chickens that have mass less than 1.5 kg are rejected. In a delivery of 1 000 chickens, how many will be rejected? Justify your answer.

$P(X < 1.5) = 0.0013499$ ✓

Hence, no. rejected $= 0.0013499 \times 1\,000$ ✓
$= 1.35$
≈ 1 (accept 2) ✓

(c) A sample of fifty chickens were selected. Determine the probablity that no more than two of these chickens have mass more than 2.0 kg.

Let Y: No. of chickens with mass > 2.0 kg out of 50 ✓
P(mass > 2.0 kg) = 0.02275 (constant)
$Y \sim B(50, 0.02275)$ ✓
$P(Y \le 2) = 0.8948$ ✓

(d) To improve the consistency of the mass of the chickens supplied, the poultry farm wishes to reduce the probability of a chicken with mass more than 2.0 kg to 0.005. Keeping the mean mass unchanged, what should the new standard deviation for the mass be? Give your answer to the nearest g.

Let W: mass of chicken
$W \sim N(1.8, \sigma^2)$
$P(X > 2) = 0.005$
$P(Z > \frac{2-1.8}{\sigma}) = 0.005$ ✓✓
Hence, $\frac{2-1.8}{\sigma} = 2.575829$ ✓
$\sigma = 0.07764$ kg ≈ 78 g ✓

```
solve(normCDf(2,∞,x,1.8)=0.005,x)
                    {x=0.07764489663}
```


Calculator Assumed

11. [13 marks: 2, 2, 4, 5]

Janine and John run a fish farm. The mass (when mature) of a species of fish bred in the farm is normally distributed with a mean of 1.8 kg and a standard deviation of 100 g. In a certain season, 10 000 fish were harvested.

(a) Find the 95th percentile mass of this species of fish. Show clearly how you obtained your answer.

$X \sim N(1.8, 0.1^2)$
$P(X \le k) = 0.95$ ✓
$\Rightarrow k = 1.9644854 = 1.964$ kg ✓

(b) Estimate the number of fish with mass within 100g of the mean weight.

$P(1.7 \le X \le 1.9) = 0.6826894$ ✓

$N = 10\,000 \times 0.6826894 = 6827$ ✓

When the fish reaches maturity, the harvested fish are sold at the following prices.

Mass, m kg	Price/kg
$m \ge 2.0$	\$30
$1.8 \le m < 2.0$	\$40
$1.6 \le m < 1.8$	\$25
$m < 1.6$	\$10

(c) Estimate the revenue received from the sales of fish with mass in excess of 1.9 kg.

$P(1.9 < X < 2) = 0.1359051$ ✓
$P(X \ge 2) = 0.0227501$ ✓
Revenue = 1359 × \$40 + 228 × \$30 ✓
= \$61 200 ✓

(d) Estimate the total revenue from this season's harvest.

$P(X \ge 2.0) = 0.0227501$ ⇒ Revenue = 228 × \$30 = \$6840 ✓
$P(1.8 \le X < 2.0) = 0.4772498$ ⇒ Revenue = 4772 × \$40 = \$190 880 ✓
$P(1.6 \le X < 1.8) = 0.4772498$ ⇒ Revenue = 4772 × \$25 = \$119 300 ✓
$P(X < 1.6) = 0.0227501$ ⇒ Revenue = 228 × \$10 = \$2280 ✓

Hence, total Revenue = \$319 300 ✓

Calculator Assumed

12. [11 marks: 1, 2, 6, 2] [TISC]

(a) The life span of a native toad (Species A) is normally distributed with mean 58 months and standard deviation 3 months.

(i) Find the probability of a toad having a life span less than 56 months.

> $X \sim N(58, 3^2)$
> $P(X \le 56) = 0.2525$ ✓

(ii) A sample of 50 toads was selected. Find the probability no more than 10 of these toads have life spans less than 56 months.

> Y: No. of toads with life spans less than 56 months out of 50
> $Y \sim B(n = 50, p = 0.2525)$ ✓
>
> $P(Y \le 10) = 0.2493$ ✓

(b) The life span of a related species of toad (Species B) may be modelled by a normal distribution with mean μ months and standard deviation σ months. The 75th percentile life span is 63 months, while the 20th percentile lifespan is 56 months. Find to one decimal place, μ and σ.

> Let $W \sim N(\mu, \sigma^2)$
> $P(W \le 63) = 0.75$
> $P(Z \le \frac{63-\mu}{\sigma}) = 0.75$ ✓
> $\Rightarrow \frac{63-\mu}{\sigma} = 0.6745$ I ✓
>
> $P(W \le 56) = 0.20$
> $P(Z \le \frac{56-\mu}{\sigma}) = 0.20$ ✓
> $\Rightarrow \frac{56-\mu}{\sigma} = -0.8416$ II ✓
>
> Solve I & II: $\mu = 59.9$ months ✓
> $\sigma = 4.6$ months. ✓

(c) A toad is captured and is found to have a life span less than 56 months. Determine with reasons, whether this toad is more likely to be of Species A or B.

> For Species A, $P(X \le 56) = 0.2525$.
> For Species B, $P(W \le 56) = 0.20$. ✓
>
> Hence, this toad is more likely to be of Species A. ✓

Calculator Assumed

13. [9 marks: 2, 1, 2, 4] [TISC]

The 8 am bus arrives each week day at bus stop C anytime between 7.58 am and 8.08 am. If the bus arrives at C before 8 am, it cannot leave until 8 am. The bus is late if it arrives after 8.00 am.

(a) State the probability density function for *L*.

> *L* is uniformly distributed in the interval $-2 \le L \le 8$.
> Hence probability density function is:
> $f(L) = 0.1 \qquad -2 \le L \le 8.$ ✓✓

(b) Find the probability that the bus arrives early at bus stop C.

> $P(-2 \le L < 0) = 0.2$ ✓

(c) Find the probability that the bus is no more than 5 minutes late given that it is late.

> $P(L \le 5 \mid L > 0) = \frac{P(0 < L \le 5)}{P(L > 0)}$ ✓
> $= \frac{5}{8}$ ✓

Debbie lives near bus stop C. The time she takes to walk to the bus stop C is a normal variable with mean 150 seconds and standard deviation 15 seconds. She leaves her home each day at 7.57 am.

(d) Find the probability that Debbie misses the bus given that the bus is early. Show clearly how you arrived at your answer.

> Let W: Time Debbie takes to walk to the bus stop.
> $W \sim N(150, 15^2)$ ✓
> P(Debbie arrives at C after 8 am) = $P(W \ge 180)$
> $= 0.02275$ ✓
> As the event "Debbie misses bus" and the event "bus is early" are independent, ✓
> Hence, P(Debbie misses bus | bus is early) = P(Debbie misses bus)
> $= 0.02275$ ✓

Calculator Assumed

14. [11 marks: 2, 2, 3, 4] [TISC]

Let the variable R denote the Body Mass Index for residents in a certain country. R is assumed to be normally distributed with mean 22 and standard deviation 3. The table below classifies the residents into categories of "Underweight", "Healthy", "Overweight" or "Obese" based on their Body Mass Index.

BMI	Classification
$R < 18.5$	Underweight
$18.5 \le R < 25$	Healthy weight range
$25 \le R < 30$	Overweight
$R \ge 30$	Obese

(a) A resident is randomly chosen from this country. Calculate the probability that this resident is overweight.

$P(25 \le R < 30) = 0.154\,825$ ✓✓

(b) Calculate the probability that a randomly chosen resident is either overweight or obese.

$P(R \ge 25) = 0.158\,655$ ✓✓

(c) A resident is randomly chosen from this country. Given that this resident is not underweight, what is the probability that this resident is overweight?

$$P(25 \le R < 30 \mid R > 18.5) = \frac{P(25 \le R > 30)}{P(R > 18.5)} \quad ✓$$
$$= \frac{0.154825}{0.878327} \quad ✓$$
$$= 0.17627 \quad ✓$$

(d) 70% of residents have a BMI greater than k. For $Z \sim N(0, 1)$, $P(Z > -0.5244) \approx 0.7$. Show how this result may be used to calculate the value of k.

$P(R > k) = 0.7$ ✓

$\Rightarrow P(Z > \frac{k-22}{3}) = 0.7$ ✓

But $P(Z > -0.5244) \approx 0.7$

$\Rightarrow \frac{k-22}{3} \approx -0.5244$ ✓

$k \approx 20.4$ ✓

Calculator Assumed

15. [8 marks: 2, 2, 4] [TISC]

The mass of mangoes sold at a supermarket is normally distributed with a mean of 300 g and a standard deviation of 8 g.

(a) Find the probability that a randomly chosen mango has a mass that is less than 310 g.

Let M: Mass of mango.

$M \sim N(300, 8^2)$ ✓

$P(H < 310) = 0.894\,350$ ✓

(b) Calculate the probability that a randomly chosen mango with a mass less than 310 g has a mass of at least 290 g.

$$P(M \ge 290 \mid M \le 310) = \frac{P(290 \le M \le 310)}{P(M \le 310)}$$
$$= \frac{0.788700}{0.894350} \quad ✓$$
$$\approx 0.881\,87 \quad ✓$$

(c) Due to customer complaints, the supermarket now sells mangoes with a mass that is normally distributed with a mean of 300 g and a standard deviation of σ g. 98% of mangoes sold must now have mass that differ from the mean mass by no more than 4 g. Calculate the value of σ, showing the use of the <u>*standard normal distribution*</u>.

$P(296 \le M \le 304) = 0.98$

$P\left(\frac{296-300}{\sigma} \le Z \le \frac{304-300}{\sigma}\right) = 0.98$ ✓

$P\left(\frac{-4}{\sigma} \le Z \le \frac{4}{\sigma}\right) = 0.98$ ✓

$\frac{4}{\sigma} = 2.326\,348$ ✓

$\Rightarrow \sigma \approx 1.72$ g ✓

```
invNormCDf("C",0.98,1,0)
                -2.326347874
   4
2.326348
                 1.719433206
```

Calculator Assumed

16. [12 marks: 2, 3, 7]

In region R, the mean height of adult women is 173 cm with standard deviation 12 cm.

(a) Determine the proportion of adult females that are able to walk through doorways of heights 200 cm without having to bend their bodies.

Let H: height of adult females

$X \sim N(173, 12^2)$ ✓

$P(X \le 200) = 0.987776 \approx 0.9878$ ✓

(b) What height should doorways be if 99.5% of adult females should be able to walk through the doorways without having to bend their bodies?

$P(X \le k) = 0.995$ ✓

$k = 203.9$ cm ✓

Hence, no more than 203.9 cm ✓

(c) In region R, the mean height of adult men is μ cm with standard deviation σ cm. Only 95.7% adult men in region R can walk through doorways of heights 210 cm without bending their bodies. Of these, 12.1% would not be able to walk through doorways of heights 200 cm without bending their bodies. Calculate the values of μ and σ.

Define Y: Height of adult males

$Y \sim N(\mu, \sigma^2)$

Given: $P(Y \le 210) = 0.957$

$\Rightarrow P(Z \le \frac{210-\mu}{\sigma}) = 0.957$ ✓

$\frac{210-\mu}{\sigma} = 1.716886$ I ✓

Given: $P(Y \ge 200 \mid Y \le 210) = 0.121$

$\Rightarrow P(Y \le 200 \mid Y \le 210) = 1 - 0.121 = 0.879$ ✓

$P(Y \le 200) = 0.879 \times P(Y \le 210)$ ✓

$= 0.8412$

$\Rightarrow P(Z \le \frac{200-\mu}{\sigma}) = 0.8412$ ✓

$\frac{200-\mu}{\sigma} = 0.9994$ II ✓

Solve I and II simultaneously:

$\mu = 186.07 \quad \sigma = 13.94$ ✓

Calculator Assumed

17. [8 marks: 1, 2, 5]

The driver reaction time, T, to a hazard on the road is normally distributed with mean 2 seconds and standard deviation 0.4 seconds.

(a) Calculate the probability that a driver:

(i) chosen at random will have a reaction time of greater than 2.5 seconds

$P(T \ge 2.5) = 0.1056$ ✓

(ii) chosen from those with reaction times of at least 1.5 seconds has a reaction greater than 2.5 seconds.

$P(T \ge 2.5 \mid T \ge 1.5) = \frac{0.1056798}{0.8943502}$ ✓

≈ 0.1181 ✓

(b) The probability that a driver randomly chosen from those with reaction times of less than *k* seconds has reaction time of at least 1.5 seconds is 0.8341. Calculate the value of *k*.

$P(T \ge 1.5 \mid T < k) = \frac{P(1.5 \le T < k)}{P(T < k)}$ ✓

$= 0.8341$

$P(1.5 \le T < k) = 0.8341\, P(T < k)$ ✓

$P(T < k) - P(T < 1.5) = 0.8341\, P(T < k)$ ✓

$P(T < k) = \frac{P(T < 1.5)}{0.1659}$

$= \frac{0.105650}{0.1659}$

≈ 0.6368 ✓

$\Rightarrow k = 2.139967 \approx 2.14$ ✓

```
solve(normCDf(1.5,x,0.4,2)=0.8341×normCDf(-∞,x,0.4,2)
                                          {x=2.139997228}
```

28 Sampling & Sample Proportion

Calculator Free

1. [6 marks: 1, 2, 1, 2]

A company employs 4 000 permanent and 1 000 casual staff, spread over several work sites. A stratified sample of 100 employees is to be formed.

(a) How many of each category of staff (permanent of casual) should be in the sample?

> Permanent: 80 Casual: 20 ✓

(b) Describe clearly how the permanent staff in this sample may be selected.

> - Each permanent staff is given a unique number from 1 to 4 000. ✓
> - Draw 80 numbers form this set by hand or electronically. ✓
>   The sample is to consist of permanent staff members whose numbers are drawn.

(c) Due to an economic downturn which affected the company's earnings, the company's owners proposed that all employees take a pay reduction of 20%. One of the company's managers, Adam was given the task of determining the level of support among the employees for this proposal.

(i) Adam visited one of their worksites and in a meeting with the employees he requested that those who opposed the proposal to stand up. Discuss the type of bias present in this type of sampling.

> Acquiescence or Fear bias. ✓

(ii) Discuss with reasons if the Adam could obtain a more accurate result by repeating the above procedure on all the company's worksites.

> No. ✓ Bias does not disappear when sample sizes are increased. ✓

Calculator Free

2. [4 marks: 2, 2]

A co-educational college has years ten, eleven and twelve students in the ratio 5 : 4 : 3. The views of students in a stratified random sample of 60 students regarding a possible change to the college starting time is to be considered.

(a) Describe how you would create such a sample.

> Sample is to comprise 25 year ten, 20 year eleven and 15 year twelve students. ✓
>
> Students in each year group to be drawn using a simple random sample method. ✓

(b) Suggest two other factors that should be considered in the making of such a random sample.

> Additional strata: Gender ✓
> Transport mode to college ✓

---

3. [4 marks: 2, 1, 1] [TISC]

A coffee manufacturer receives complaints that the 1 kg packets of coffee it sells are underweight. Each working day (8 hours), the manufacturer produces 5 000 packets. The packets are shipped in cartons of 50 packets each.

(a) A manager takes one carton of 50 packets produced one morning; weighs each packet and finds that none of the packets are underweight. The manager then concludes that the complaints are untrue. Give two reasons why the manager could be wrong.

> - Under-representation in terms of daily production. ✓
>   All packets from same carton.
> - Under-representation in terms of overall production. ✓
>   (Packets from other days not examined)

(b) A second manager selects 50 cartons from those produced that day and picks and weighs one packet from each of the 50 cartons. The manager finds that all these packets had coffee of the correct weight and concludes that the complaints are untrue.

(i) In what way is this method of forming a sample better than the method used in part (a)?

> Better representation from packets produced that morning. ✓

(ii) Give one reason why the second manager could be wrong.

> "Faulty" packages produced on other days. ✓

Calculator Free

4. [5 marks: 2, 3] [TISC]

Each train in a city rail service has six carriages. On average, each day the rail service runs 10 000 train trips. Passengers travelling on a train must have a valid train ticket.

(a) A ticket inspector selects a train at random. He then randomly selects one of the train's six carriages and checks all the passengers in that carriage for valid tickets. Of the twenty passengers in the chosen carriage, five did not have valid train tickets. Explain why it is *not* appropriate to conclude that 25% of all train passengers travel without valid train tickets.

> Sample is not random, as it is only from one carriage from one train ✓
> and the sample size of 20 is too small. ✓

(b) Another ticket inspector selected a train at random and checked five randomly selected passengers from each of the six carriages. Of the thirty passengers checked, six did not have valid tickets. Determine with reasons, if it would be appropriate to conclude that 20% of all train passengers travel without valid tickets.

> Not appropriate. ✓
> Only one route and at one time of the day was sampled. ✓
> Sample size of 30 is still too small. ✓

---

5. [4 marks: 2, 2] [TISC]

(a) To determine the most popular sports activity in Perth, James randomly interviews 50 persons at a basketball game and 50 persons at a netball game. Give two reasons why James' sample may be biased.

> - Sample members confined to supporters of only two sports. ✓ (selection bias)
> - Persons with other interests not included. ✓ (under-coverage bias)

(b) A study is being designed to determine the level of support for banning students bringing mobile phones to schools. A systematic random sample of 500 persons is to be formed. The sample is to consist of students and teachers. Discuss if this is adequate to form a fair and unbiased systematic random sample.

> Not adequate.
> More categories needed: parents ✓
> Need sub-categories: Age levels, gender, socio-economic background. ✓

Calculator Free

6. [3 marks: 1, 2]

A researcher mails out survey forms asking the question "Should the Australian Federal Parliamentary term be extended from three years to four years?" The question was to be answered with a YES or a NO.

- One survey form was delivered to each of 10 000 letter boxes in four randomly chosen suburbs in Australia. Completed survey forms are to be returned using the postal reply paid envelopes provided.
- 2 450 correctly completed forms were returned to the researcher.
- 1 981 of the survey forms returned were answered with a YES.

(a) Determine an estimate for the proportion of respondents that answered YES.

> Estimate = $\frac{1981}{2450}$ ✓

(b) Discuss two possible sources of bias inherent in this survey

> Responses more likely to be from older residents who are familiar with "envelopes and mail". ✓
> Responses more likely to be from urban populace as only 4 suburbs were chosen for the 10 000 forms distributed. ✓

---

7. [7 marks: 4, 3]

It is known that 10% of adult residents in a state are fluent in at least two languages. 200 samples each with 64 adult residents were randomly chosen and the proportions of those fluent in at least two languages calculated.

(a) Describe the *sampling distribution* of the sample proportions of 64 adults fluent in at least two languages, stating its mean and standard deviation.

> As sample size $n = 64 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean = $\frac{1}{10}$ ✓ and standard deviation = $\sqrt{\frac{\frac{1}{10}\times\frac{9}{10}}{64}}$ ✓ = $\frac{3}{80}$ ✓

(b) Describe the *frequency distribution* of the 200 sample proportions of adults fluent in at least two languages, stating its mean and standard deviation.

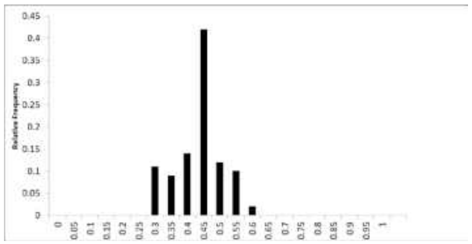
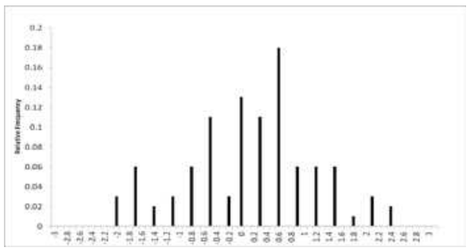
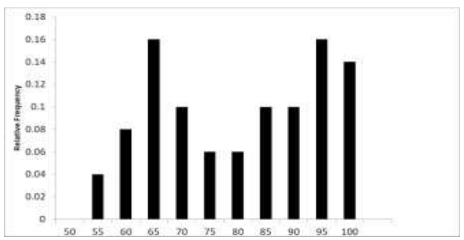
> As the number of samples N = 200 is large,
> frequency distribution tends towards its sampling distribution.
> Hence, it is approximately normal ✓
> with mean = $\frac{1}{10}$ and standard deviation = $\frac{3}{80}$ ✓✓

Calculator Free

8. [6 marks: 2, 2, 2] [TISC]

The table below shows three relative frequency histograms.
Match with reasons, each of the relative frequency histogram to one of the distributions listed below.

A. Distribution of observations from a uniform distribution.
B. Distribution of sample proportions $\hat{p}$.
C. Distribution of $\dfrac{\hat{p}-p}{\sqrt{\dfrac{p(1-p)}{n}}}$ where p is the population proportion and $\hat{p}$ represents sample proportions.

Relative Frequency Histogram	Distribution
	✓ Distribution of sample proportions. Reason. ✓ The horizontal axis ranges from 0 to 1 and graph is approximately "bell-shaped".
	✓ Distribution of $\dfrac{\hat{p}-p}{\sqrt{\dfrac{p(1-p)}{n}}}$. Reason: ✓ The horizontal axis ranges from –3 to 3 representing z-scores and the graph is approximately "bell-shaped".
	✓ Distribution of observations from a uniform distribution. Reason: ✓ The horizontal axis ranges from 50 to 100 and the graph is not "bell-shaped".

Calculator Free

9. [8 marks: 1, 4, 3]

The mass of sugar dispensed by an automatic sugar dispenser is uniformly distributed over the interval 1.5 g to 2.5 g.

(a) Calculate the probability that in any use of the dispenser, the mass of sugar dispensed exceeds 2.4g.

P(mass of sugar > 2.4) = $\dfrac{0.1}{1} = \dfrac{1}{10}$ ✓

The dispenser was used 36 times and the proportion of times the mass of sugar dispensed exceed 2.4 g recorded. This was repeated 100 times so that a collection of 100 sample proportions was obtained.

(b) Describe the <u>*sampling distribution*</u> of sample proportions of size 36 for the mass of sugar dispensed exceeding 2.5 g, stating its mean and standard deviation.

As sample size $n = 36 > 30$,
sample proportion $\hat{\pi}$ is approximately normally distributed ✓
with mean = $\dfrac{1}{10}$ ✓
and standard deviation = $\sqrt{\dfrac{\dfrac{1}{10}\times\dfrac{9}{10}}{36}} = \dfrac{1}{20}$ ✓✓

(c) Describe the <u>*frequency distribution*</u> of the 100 sample proportions of the mass of sugar dispensed exceeding 2.5 g, stating its mean and standard deviation.

As the number of samples N = 100 is large,
frequency distribution tends towards its sampling distribution.
Hence, it is approximately normal ✓
with mean = $\dfrac{1}{10}$ ✓
and standard deviation = $\dfrac{1}{20}$ ✓

Calculator Assumed

10. [9 marks: 2, 3, 1, 3]

The waiting time at a Transport Licensing Centre is uniformly distributed over the interval 5 to 25 minutes.

(a) Find the probability that the waiting time for any customer is no more than 10 minutes.

> X: Waiting time for customers.
> X is uniformly distributed in the interval $5 \le x \le 25$.
> $P(X \le 10) = \frac{5}{20} = \frac{1}{4}$ ✓✓

In a review conducted on the queuing system used, the waiting times of samples of 50 customers each, were recorded.

(b) Describe the sampling distribution (size 50) of the proportion of customers with waiting times of no more than 10 minutes.

> As sample size $n = 50 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = \frac{1}{4}$ ✓
> and standard deviation $= \sqrt{\frac{\frac{1}{4} \times \frac{3}{4}}{50}} = \frac{\sqrt{6}}{40}$. ✓

(c) Find the probability that a randomly chosen sample has a sample proportion of customers with waiting times of no more than 10 minutes that exceeds 0.3.

> $\hat{\pi} \sim N(\frac{1}{4}, \left(\frac{\sqrt{6}}{40}\right)^2)$.
> $P(\hat{\pi} \ge 0.3) = 0.2071$ ✓

(d) 40 samples each comprising 50 customers were chosen. Determine with reasons, the expected number of samples with sample proportions of customers with waiting times of no more than 10 minutes that exceeds 0.3.

> As the number of samples N = 40 is large, ✓
> frequency distribution of sample proportion
> tends towards its sampling distribution.
> Hence, frequency distribution is approximately $N(\frac{1}{4}, \left(\frac{\sqrt{6}}{40}\right)^2)$. ✓
> Therefore, expected number $\approx 0.2071 \times 40 \approx 8$ ✓

Calculator Assumed

11. [9 marks: 2, 3, 2, 2]

The mass of sugar in a 1 kg pack is normally distributed with mean 998 g with standard deviation 1 g.

(a) Find the probability that the mass of sugar in a randomly chosen pack exceeds 1 kg. Give your answer to 3 significant figures.

> Let W: mass of sugar in a 1 kg pack
> $W \sim N(998, 1^2)$. ✓
> $P(W > 1000) = 0.02275 \approx 0.0228$ ✓

Samples of size n packs, where $n > 50$, are selected and the proportion of packs with sugar mass exceeding 1 kg recorded.

(b) Describe the sampling distribution (size $n > 50$) of the proportion of packs with sugar mass exceeding 1 kg.

> As sample size $n > 50$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = 0.0228$ ✓
> and standard deviation $= \sqrt{\frac{0.02275 \times (1 - 0.02275)}{n}} = \sqrt{\frac{0.0222}{n}}$. ✓

(c) For $n = 100$, calculate the probability that a randomly chosen sample has a sample proportion of packs with sugar mass exceeding 1kg of between 0.02 and 0.03.

> $\hat{\pi} \sim N(0.0228, \frac{0.0222}{100})$. ✓
> $P(0.02 \le \hat{\pi} \le 0.03) = 0.260$ ✓

(d) Determine the value of n if the standard deviation of the sampling distribution (size $n > 50$) of the proportion of packs with sugar mass exceeding 1 kg is not to exceed 0.01.

> $\sqrt{\frac{0.0222}{n}} \le 0.01$ ✓
> $n \ge 222$ ✓

Calculator Assumed

12. [9 marks: 4, 3, 2]

An unbiased six-sided die is rolled 80 times. This is repeated 150 times to form 150 samples each consisting of 80 rolls of the die. Event S is defined as the roll of the die producing a six.

(a) Calculate the probability that a randomly chosen sample has a sample proportion of event S that exceeds 15%.

> P(S occurring) = $\frac{1}{6}$
> As sample size $n = 50 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = \frac{1}{6}$ ✓
> and standard deviation = $\sqrt{\dfrac{\frac{1}{6}\times\left(1-\frac{1}{6}\right)}{80}} = \frac{1}{24}$. ✓
> Hence, $P(\hat{\pi} > 0.15) = 0.6554$ ✓

(b) Estimate with reasons, the expected number of samples with sample proportions of event S that exceeds 15%.

> As the number of samples N = 150 is large, ✓
> frequency distribution of sample proportion tends towards its sampling distribution.
> Hence, frequency distribution is approximately $N(\frac{1}{6}, \left(\frac{1}{24}\right)^2)$. ✓
> Therefore, expected number ≈ 0.6554 × 150 ≈ 98 ✓

(c) In a separate experiment, the same die was rolled n times. Find n if the standard deviation of the sampling distribution of sample proportion of event E is not to exceed 0.04.

> $\sqrt{\dfrac{\frac{1}{6}\times\left(1-\frac{1}{6}\right)}{n}} \le 0.04$ ✓
> $n \ge 86.8$
> Integer $n \ge 87$ ✓

Calculator Assumed

13. [8 marks: 3, 5]

60% of vehicles arriving at a school entrance are classified as sport utility vehicles (SUVs).

(a) Calculate the probability that in a random sample of 50 cars arriving at the school entrance, exactly 30 are SUVs.

> P(30 SUVs out of 50) = $\binom{50}{30}\times 0.6^{30}\times 0.4^{20}$ ✓✓
> $= 0.11456 \approx 0.1146$ ✓

(b) Samples each comprising 50 vehicles arriving at the school entrance were taken and the number of samples with exactly 30 SUVs recorded. For a randomly chosen sample of 50 vehicles, estimate the probability that the sample proportion of exactly 30 SUVs does not exceed 12% given that it exceeds 11%.

> As sample size $n = 50 > 30$,
> sample proportion $\hat{\pi}$ is approximately normally distributed ✓
> with mean $\mu = 0.11456$ ✓
> and standard deviation = $\sqrt{\dfrac{0.11456\times(1-0.11456)}{50}} = 0.04504$ ✓
> Hence, $P(\hat{\pi} \le 0.12 \mid \hat{\pi} \ge 0.11) = \dfrac{P(0.11 \le \hat{\pi} \le 0.12)}{P(\hat{\pi} \ge 0.11)}$ ✓
> $= \dfrac{0.08839}{0.54032}$
> ≈ 0.1636 ✓

14. [3 marks]

It is known that p % of high school students carry school bags with masses exceeding 15 kg. Samples of 100 students are chosen. The sampling distribution for the sample proportion of students with school bags exceeding 15 kg has standard deviation $\frac{\sqrt{91}}{200}$. Find p.

> $\sqrt{\dfrac{p(1-p)}{100}} = \dfrac{\sqrt{91}}{200}$ ✓
> $p = 0.35$ or 0.65 ✓✓
>
> solve($\sqrt{\frac{x(1-x)}{100}} = \frac{\sqrt{91}}{200}$, x)
> {x=0.35, x=0.65}

Calculator Assumed

15. [9 marks: 2, 4, 3]

NyRopes manufactures high tensile nylon ropes. The continuous random variable X describes the rope length (m) between two consecutive kinks in the rope. The probability density function of X is given by $f(x) = 0.01\,e^{-0.01x}$, where $x > 0$.

(a) Find the probability that a randomly chosen piece of rope has a rope length of at least 50 m between consecutive kinks.

$$P(X \ge 50) = \int_{50}^{\infty} 0.01\,e^{-0.01x}\,dx \quad ✓$$
$$= 0.60653 \approx 0.6065 \quad ✓$$

Samples of 30 coils of nylon ropes were examined and $\hat{\pi}$ the proportion of ropes with rope length of at least 50 m between consecutive kinks recorded.

(b) Calculate the probability that a random sample of 30 coils of nylon ropes has a $\hat{\pi}$ value between 0.6 and 0.7.

As sample size $n = 30$,
sample proportion $\hat{\pi}$ is approximately normally distributed ✓
with mean $\mu = 0.60653$ ✓

and standard deviation $= \sqrt{\dfrac{0.60653 \times (1 - 0.60653)}{30}} = 0.08919$ ✓

Hence, $P(0.6 < \hat{\pi} \le 0.7) = 0.38186 \approx 0.3819$ ✓

(c) Determine the minimum number of coils of nylon rope per sample required so that the standard deviation for $\hat{\pi}$ is less than 0.08.

$$\sqrt{\frac{0.60653 \times (1 - 0.60653)}{n}} \le 0.08 \quad ✓$$
$$n \ge 37.29 \quad ✓$$

Hence, minimum number is 38. ✓

Calculator Assumed

16. [7 marks: 3, 2, 2]

The angle of scatter, S, of a particle after it collides with an uneven surface has probability density function given by $f(s) = \cos s$ for $0 \le s \le \frac{\pi}{2}$.

(a) Show that half of all particles colliding with the surface are scattered by more than 30°.

$$P\left(S > \frac{\pi}{6}\right) = 1 - \int_{0}^{\frac{\pi}{6}} \cos(s)\,ds \quad ✓$$
$$= 1 - \left[\sin(s)\right]_0^{\frac{\pi}{6}} \quad ✓$$
$$= 0.5 \quad ✓$$

Hence, 50% of all particles are scattered by more than 30°.

(b) Describe $\hat{p}$ the sampling distribution of sample proportions of particles scattered by more than 30° in samples of 250 particles colliding with the surface.

Mean for sample proportions = 0.5
Standard deviation for sample proportions

$$= \sqrt{\frac{0.5(1-0.5)}{250}} = 0.03162 \quad ✓$$

As sample size $n = 250 \ge 30$;
Sampling distribution is approximately normal. ✓

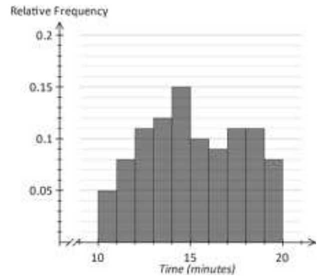
(c) Calculate the probability that the sample proportion of collisions with scatter angles exceeding 30° in samples of 250 collisions does not exceed 0.48.

$\hat{p} \sim N(0.5,\ 0.03162^2)$ ✓
$P(\hat{p} \le 0.48) = 0.2635$ ✓

Calculator Assumed

17. [7 marks: 1, 2, 1, 1, 2]

The *continuous* random variable T is defined as the time (minutes) John takes to travel to work. The relative histogram in the accompanying diagram shows the travel times that John takes for each of 100 days. Use this relative frequency histogram for T to answer the following questions.



(a) Estimate the probability that John takes no less than 12 minutes but no more than 15 minutes to get to work.

> $P(12 \le T \le 15) = 0.11 + 0.12 + 0.15$
> $= 0.38$ ✓

(b) On those occasions when John's travel times are least 12 minutes, calculate the probability that John's travel times do not exceed 15 minutes

> $P(T \le 15 \mid T \ge 12) = \dfrac{P(12 \le T \le 15)}{P(T \ge 12)}$
> $= \dfrac{0.38}{1-0.05-0.08} = \dfrac{38}{87}$ ✓✓

(c) It is known that T is a continuous uniform variable over $10 \le t \le 20$.

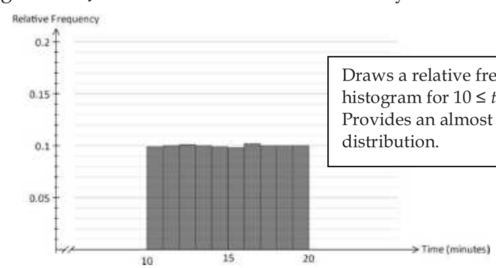
(i) Calculate the true value for the proportion of travel times that are between 12 and 15 minutes inclusive.

> $P(12 \le T \le 15) = \dfrac{3}{10}$ ✓

(ii) Why is the answer in part (a) is different from the answer in part (b) (i)?

> Random variability of samples. ✓

(iii) In the axes provided below, draw an approximate relative frequency histogram for John's travel times over 1 000 days.



> Draws a relative frequency histogram for $10 \le t \le 20$. ✓
> Provides an almost rectangular distribution. ✓

Calculator Assumed

18. [7 marks: 2, 1, 1, 3] [TISC]

It is believed that 60% of residents in a city ride their bicycles to work. *N* samples of 100 residents each were surveyed. Let $\hat{p}$ be the proportion of those in a sample of 100 residents that ride their bicycles to work.

(a) State the name of the probability distribution of $\hat{p}$ and the associated parameters.

> As sample size $n = 100$ is large
> $\hat{p}$ has an approximate normal distribution with mean 0.6 ✓
> and standard deviation $\sqrt{\dfrac{0.6 \times 0.4}{100}} \approx 0.048\,989\,8$ ✓

(b) Calculate the probability that a randomly chosen sample of 100 residents will have a sample proportion of at least 65%.

> $P(\hat{p} \ge 0.65) = 0.153717$ ✓

(c) Let $N = 200$.

(i) How many of these samples are expected to have $\hat{p}$ values of at least 65%?

> Number $= 200 \times 0.153717 \approx 30$ (accept 31) ✓

(ii) Calculate the probability that no more than forty of these samples with have $\hat{p}$ values of at least 65%. [3 marks]

> Define X: Number of samples with $\hat{p} \ge 0.65$. ✓
> $X \sim B(200, 0.153717)$ ✓
> $P(X \le 40) = 0.968645$ ✓

Calculator Assumed

19. [11 marks: 3, 1, 2, 2, 1, 2]

In a certain country 0.9% of all vehicles are electric. Samples of 500 vehicles were selected. Let the random variable X: Number of vehicles in a sample that are electric. Let $\hat{p}$ be the proportion of vehicles in a sample that are electric.

(a) State the name and the mean and standard deviation of the probability distribution for X.

> $X \sim B(n = 500, p = 0.009)$ ✓
> Mean = 4.5 ✓
> Standard deviation = 2.111 753 ✓

(b) Calculate the probability that in a randomly chosen sample of 500 vehicles at least 1.0 % of the vehicles are electric.

> $X \sim B(n = 500, p = 0.009)$
> $P(X \geq 5) = 0.468\ 325$ ✓

Let the random variable Y = 0.002X.

(c) State the mean and standard deviation of Y.

> $E(Y) = 0.002 \times 4.5 = 0.009$ ✓
> $STD(Y) \approx 0.002 \times 2.111\ 753 \approx 0.004\ 223\ 5$. ✓

(d) Explain why Y may be approximated with a normal distribution.

> $Y = 0.002X = \frac{X}{500}$
> Hence. Y is the variable representing the sample proportion $\hat{p}$. ✓
> As sample size 500 is large $Y \equiv \hat{p}$ has an approximate normal distribution. ✓

(e) Use the normal distribution to calculate the probability that in a randomly chosen sample of 500 vehicles the proportion of cars that are electric is at least 1.0 %.

> $\hat{p} \sim N(0.009, 0.004\ 223\ 5^2)$
> $P(\hat{p} \geq 0.01) = 0.406\ 417$ ✓

(f) Which of the answers; (b) or (e) is more accurate? Why?

> Answer in (b) is more accurate. ✓
> An exact distribution is used in (b)
> while an approximate distribution in used in (e). ✓

29 Point & Interval Estimates for *p*

Calculator Free

1. [7 marks: 1, 3, 3]

In a sample of 400 students from Perth, 200 indicated that they had never visited Albany, a town 400 km south of Perth. Let *p* be the proportion of students from Perth that have never visited Albany.

(a) Determine a point estimate for *p*.

> $\hat{p} = \frac{200}{400} = 0.5$ ✓

(b) Given that $P(-2 \leq Z \leq 2) = 0.954$ where $Z \sim N(0, 1)$, calculate a 95.4% confidence interval for *p*.

> CI: $0.5 \pm 2 \times \sqrt{\frac{0.5 \times 0.5}{400}}$ ✓✓
> $0.5 \pm 2 \times \frac{0.5}{20}$
> $0.45 \leq p \leq 0.55$ ✓

(c) In a second sample of 100 students from Perth, 54 students indicated that they had never visited Albany. Use your answer in (b) to determine if the second sample was statistically different from the first sample.

> Sample proportion = 0.54 ✓
> 0.54 falls within the 95.4% CI. ✓
>
> Hence, there is no evidence to suggest that they are statistically different. ✓

Calculator Assumed

2. [9 marks: 1, 3, 3, 2]

The mass (nearest g) of 30 eggs from an egg farm is listed below.

65	66	64	68	65	67	66	64	69	65
66	68	65	67	66	64	70	68	65	67
66	67	63	65	67	68	64	68	67	66

(a) Use this sample to estimate p, the proportion of eggs with mass above 66 g.

> Sample proportion $\hat{p} = \frac{13}{30}$ ✓

(b) Use this sample to provide a 95% confidence interval for p, the proportion of eggs with mass above 66 kg.

> 95% confidence interval for p
>
> $\frac{13}{30} \pm 1.96 \times \sqrt{\frac{\frac{13}{30}\times\left(1-\frac{13}{30}\right)}{30}}$ ✓✓
>
> $\Rightarrow\ 0.26 \le p \le 0.61$ ✓

(c) In a second sample, 27 eggs out of 60 had mass above 66 kg.
Use the confidence interval in (b) to determine if eggs in the second sample have mass that are statistically different from those of the first sample.

> Sample proportion of second sample = $\frac{27}{60} = 0.45$ ✓
>
> This lies within the 95% confidence interval for p from the first sample. ✓
>
> Hence, it is not statistically different. ✓

(d) A third sample of eggs has a 95% confidence interval for p as $0.58 \le p \le 0.89$.
Determine with reasons if the mass of the third sample of eggs are statistically different from those of the first sample.

> Sample proportion for sample 3 = $\frac{0.58+0.89}{2} = 0.735$
>
> This is outside the 95% confidence interval for sample 1. ✓
>
> Hence, the two samples are statistically different. ✓

Calculator Assumed

3. [12 marks: 1, 3, 3, 5]

The accompanying table shows the number of students achieving x correct responses in a standardised test consisting of 60 questions.
Let p be the true proportion of students achieving no more than 20 correct responses.

Number of correct responses, n	No. of Students
$1 \le t \le 10$	7
$11 \le t \le 20$	12
$21 \le t \le 30$	18
$31 \le t \le 40$	20
$41 \le t \le 50$	10
$51 \le t \le 60$	3

(a) Use this data to find a point estimate for p.

> Sample proportion $\hat{p} = \frac{19}{70}$ ✓

(b) Use this sample to provide a 90% confidence interval for p.

> 90% confidence interval for p
>
> $\frac{19}{70} \pm 1.645 \times \sqrt{\frac{\frac{19}{70}\times\left(1-\frac{19}{70}\right)}{70}}$ ✓✓
>
> $\Rightarrow\ 0.18 \le p \le 0.36$ ✓

(c) Use your answer in (a) to find the size of the next sample if the error margin for a 90% confidence interval for p, is no more than 0.05.

> $1.645 \times \sqrt{\frac{\frac{19}{70}\times\left(1-\frac{19}{70}\right)}{n}} \le 0.05$ ✓
>
> $\Rightarrow\ n \ge 214.1$ ✓
>
> Integer $n \ge 215$ ✓

(d) A third sample consisting of 100 students provided a confidence interval of $0.17 \le p \le 0.33$. Find the point estimate for p in this sample and the level of confidence for this interval.

> Sample proportion $\hat{p} = \frac{0.17+0.33}{2}$
>
> $= 0.25$ ✓
>
> Error = 0.33 – 0.25 = 0.08 ✓
>
> Hence, $z \times \sqrt{\frac{0.25\times(1-0.25)}{100}} = 0.08$ ✓
>
> $z = 1.84752$ ✓
>
> $P(-1.84752 \le z \le 1.84752) = 0.9353$
>
> Therefore, level of confidence ≈ 93.5% ✓

Calculator Assumed

4. [8 marks: 2, 5, 1]

To estimate the true proportion π of the residents of a certain city that agree that the international airport serving the city should be relocated, samples were taken and the proportion of those in agreement calculated.

(a) In one sample of 200 residents, the 99% confidence for π was $0.78 \le \pi \le 0.92$. How many in this sample were in agreement with the proposal?

> Sample proportion $\hat{\pi} = \frac{0.78 + 0.92}{2}$
> $= 0.85$ ✓
> Hence, number in agreement $= 0.85 \times 200 = 170$ ✓

(b) A second sample of 500 residents, the 90% confidence interval for π was $0.77 \le \pi \le 0.83$. Determine if the second sample is statistically different from the first sample.

> Sample proportion $\hat{\pi} = \frac{0.77 + 0.83}{2}$
> $= 0.80$ ✓
>
> 99% Confidence Interval for π is:
> $0.80 \pm 2.576 \times \sqrt{\frac{0.8 \times (1 - 0.8)}{500}}$ ✓
> $\Rightarrow 0.75 \le \pi \le 0.85$ ✓
> The 99% confidence interval for the second sample lies entirely within the 99% confidence interval for the first sample. ✓
> Hence, the two samples are not statistically different. ✓

(c) If 100 samples of 50 residents each were selected, and the associated 99% confidence intervals for π calculated in the same manner. How many of these confidence intervals would be expected to contain π?

> Expected number $= 100 \times 0.99$
> $= 99$ ✓

Calculator Assumed

5. [8 marks: 3, 1, 1, 3]

To estimate the true proportion p of the adults in a certain state that suffer from hay fever (allergic rhinitis), samples were taken and the proportion of those suffering from hay fever calculated.

A sample of 500 adults was taken and 105 were found to suffer from hay fever.

(a) Calculate a 95% confidence interval for p.

> 95% confidence interval for p
> $\frac{105}{500} \pm 1.96 \times \sqrt{\frac{\frac{105}{500} \times \left(1 - \frac{105}{500}\right)}{500}}$ ✓✓
> $\Rightarrow 0.17 \le p \le 0.25$ ✓

(b) Calculate a 99% confidence interval for p.

> 99% confidence interval for p
> $\frac{105}{500} \pm 2.576 \times \sqrt{\frac{\frac{105}{500} \times \left(1 - \frac{105}{500}\right)}{500}}$
> $\Rightarrow 0.16 \le p \le 0.26$ ✓

(c) Comment on width of the confidence intervals in (a) and (b).

> The higher the confidence level, the wider the interval. ✓

In a second sample of 500 adults, 126 were found to suffer from hay fever.

(d) Using your previously calculated confidence intervals, determine with reasons if the results of the second sample is statistically different from that of the first.

> Sample proportion for 2nd sample $\hat{p} = \frac{126}{500} = 0.252$ ✓
> This is outside the 95% confidence interval for p but within the 99% confidence interval for p. ✓
>
> Hence, not statistically different if the 99% confidence interval is used but statistically different if the 95% confidence interval is used. ✓

Calculator Assumed

6. [8 marks: 3, 2, 3]

To estimate the true proportion π of the 1 kg packets of sugar that are under the advertised weight, samples of 1 kg packets of sugar were examined.

In a sample of 50 packets of sugar 3 were found to be underweight.

(a) Use this sample to calculate a 90% confidence interval for π.

> 90% confidence interval for π
>
> $$\frac{3}{50} \pm 1.645 \times \sqrt{\frac{\frac{3}{50}\times\left(1-\frac{3}{50}\right)}{50}} \quad ✓✓$$
>
> $\Rightarrow 0.0048 \le \pi \le 0.1152$ ✓

An additional 450 packets were added to the 50 packets to form a larger sample of 500 packets and 28 were found to be underweight.

(b) Use the larger sample to calculate a 90% confidence interval for π.

> 90% confidence interval for π
>
> $$\frac{28}{500} \pm 1.645 \times \sqrt{\frac{\frac{28}{500}\times\left(1-\frac{28}{500}\right)}{500}}$$
>
> $\Rightarrow 0.0391 \le \pi \le 0.0729$ ✓✓

(c) Determine with reasons which of the two confidence intervals would provide a statistically more reliable interval estimate for π.

> The larger the value of n, the closer the sampling distribution for $\hat{\pi}$ is to the normal distribution. ✓
> Hence, the second confidence interval would be more reliable as the size of the second sample is much larger. ✓✓

Calculator Assumed

7. [8 marks: 2, 3, 1, 2]

In a certain country the proportion of residents with type A blood is $p = 0.38$ Samples of 1000 residents are selected and the sample proportion $\hat{p}$ calculated.

(a) State the sampling distribution for $\hat{p}$.

> As sample size $n = 1000$ is large
>
> $\hat{p} \sim N\left(0.38, \left(\frac{0.38\times(1-0.38)}{1000}\right)\right)$. ✓
>
> $\hat{p} \sim N(0.38, 0.0002356)$ ✓

(b) Determine the interval $0.38 - k \le \hat{p} \le 0.38 + k$ such that $P(-k \le \hat{p} \le k) = 0.95$.

> $\hat{p} \sim N(0.38, 0.0002356)$
>
> $P(0.38 - k \le \hat{p} \le 0.38 + k) = 0.95$
>
> $\Rightarrow k = 0.030084$ ✓
>
> Hence, required interval is:
>
> $0.35 \le \hat{p} \le 0.41$ ✓✓
>
> ```
> solve(normCDf(0.38-x,0.38+x,√0.0002356,0.38)=0.95,x)
>                                     {x=0.03008401067}
> ```

In a sample of 1000 residents taken only from residents of ethnic group G, 312 were found with type A blood.

(c) Calculate a point estimate for the proportion of residents from G with type A blood.

> Sample proportion $\hat{p} = \frac{312}{1000}$
>
> $= 0.312$ ✓

(d) Determine with reasons if the proportion of residents from G with type A blood is significantly different from the overall population.

> Significantly different ✓
> as $\hat{p}$ for G = 0.312 is which is outside ✓
> the 95% variation interval for $\hat{p}$.

Calculator Assumed

8. [11 marks: 3, 1, 4, 3]

Let the proportion of people who are ambidextrous (those who can use both left and right hands equally well) be p.

(a) In a random sample of 1000 persons, 12 were found to be ambidextrous. Use this sample to provide a 95% confidence interval for p.

$$\frac{12}{1000} \pm 1.96 \times \sqrt{\frac{\frac{12}{1000} \times \left(1 - \frac{12}{1000}\right)}{1000}} \quad ✓✓$$

$$\Rightarrow \quad 0.005251 \le p \le 0.018748 \quad ✓$$

(b) A 90% confidence interval for p is $0.004088 \le p \le 0.013912$. Determine with reasons if it is mathematically correct to say that there is a 90% chance that p lies within this interval.

No.
The probability that p lies within any given interval is either 0 or 1. ✓

(c) A sample of 700 students had a confidence interval of $0.002258 \le p \le 0.014885$. Determine the confidence level associated with this interval.

$$\hat{p} = \frac{0.002258+0.014885}{2} = 0.008572 \quad ✓$$

$$e = \frac{0.014885-0.002258}{2} = 0.0063135 \quad ✓$$

$$z \times \sqrt{\frac{0.008572(1-0.008572)}{700}} = 0.0063135 \quad ✓$$

$$z = 1.812008$$

$$P(-1.812008 \le Z \le 1.812008) = 0.9300 \quad ✓$$

Hence, approx. 93%.

(d) One hundred 90% confidence intervals for p was calculated. Determine the probability that ninety of these intervals contain p.

Define X: No. of confidence intervals that contain p. ✓
$X \sim B(n = 100, p = 0.9)$ ✓
$P(X = 90) = 0.131865$ ✓

Calculator Assumed

9. [7 marks: 2, 2, 3]

Let the proportion of international students in a sample of 1 000 students from Australian universities be p. For this sample, let e be the margin of error associated with a 90% confidence interval for π, the true proportion of international students in Australian universities.

(a) Write a mathematical expression for e in terms of p.

$$\text{Error } e = 1.645 \times \sqrt{\frac{p(1-p)}{1000}} \quad ✓✓$$

(b) Determine the maximum value for e (to three significant figures).

The quadratic expression $p(1-p)$ is maximised when $p = 0.5$. ✓

Hence, max. for $e = 1.645 \times \sqrt{\frac{0.5(1-0.5)}{1000}}$

$= 0.0260$ ✓

```
fMax(1.645×√(x×(1-x)/1000),x)
{MaxValue=0.02600973375,x=0.5}
```

(c) For a sample of 1000, determine with reasons if $0.343 \le p \le 0.397$ could be a possible 90% confidence interval for p.

Margin of error $= \frac{0.397 - 0.343}{2} = 0.027$ ✓
For a sample of 500, max error for a 90% confidence interval = 0.0260 ✓
Since, margin of error for this interval exceeds the max. possible error, it is not possible to be a 90% CI. ✓

Calculator Assumed

10. [10 marks: 5, 1, 2, 2]

Let the proportion of residents in a country who suffer from a particular genetic abnormality be p.

(a) A confidence interval for p calculated from sample A consisting of 400 residents is $0.098\,36 \le p \le 0.181\,64$. Determine the confidence level of this interval.

> Sample proportion $= \dfrac{0.09836+0.18164}{2} = 0.14$ ✓
> Margin of error $= \left|\dfrac{0.09836-0.18164}{2}\right| = 0.04164$ ✓
> $z \times \sqrt{\dfrac{0.14\times(1-0.14)}{400}} = 0.04164$ ✓
> $z = 2.4$ ✓
> $P(-2.4 \le Z \le 2.4) = 0.983\,605$
> Hence, 98.4% confidence level. ✓

(b) Sample B consists of N residents. The proportion of those who suffer from this abnormality in sample B is the same as the sample proportion in sample A. A 99% confidence interval is calculated from sample B.

(i) Write an expression for the margin of error in the 99% confidence interval for sample B, in terms of N.

> Margin of error $= 2.576 \times \sqrt{\dfrac{0.14\times(1-0.14)}{N}}$ ✓

(ii) The margin of error in the 99% confidence interval from sample B is k times the margin of error for the 99% confidence interval from sample A. Calculate N in terms of k.

> $2.576 \times \sqrt{\dfrac{0.14\times(1-0.14)}{N}} = k \times 2.576 \times \sqrt{\dfrac{0.14\times(1-0.14)}{400}}$ ✓
> $N = \dfrac{400}{k^2}$ ✓

(iii) Calculate the possible range of values for k, if $N \ge 100$.

> $\dfrac{400}{k^2} \ge 100$
> $0 < k \le 2$ ✓✓

Calculator Assumed

11. [12 marks: 3, 4, 3, 2] [TISC]

Let the proportion of Year Twelve students who speak at least two languages fluently be p.

(a) In a sample of 50 Year Twelve students, 12 students spoke at least two languages fluently.

(i) Use this sample to determine the margin of error for a 90% confidence interval for p.

> ME $= 1.645 \times \sqrt{\dfrac{0.24\times(1-0.24)}{50}}$ ✓✓
> $= 0.099\,356$ ✓

(ii) For this sample, what would be the new confidence level if the margin of error in part (i) were to be halved?

> New ME $= 0.099\,356 \times 0.5 = 0.049\,678$ ✓
> $\Rightarrow \quad z \times \sqrt{\dfrac{0.24\times(1-0.24)}{50}} = 0.049\,678$ ✓
> $z = 0.822501$ ✓
> $P(-0.822501 < Z < 0.822501) = 0.5892$
> New Confidence Level ≈ 58.9 % ✓

(b) A second sample of 100 students had the same proportion of students able to speak two languages fluently as the sample in part (a). A 90% confidence interval is calculated using the second sample. Calculate the ratio of the margin of error for this 90% confidence interval to the margin of error for the 90% confidence interval in part (a).

> ME for second sample $= 1.645 \times \sqrt{\dfrac{0.24\times(1-0.24)}{100}}$ ✓
> Ratio $= \dfrac{1.645\times\sqrt{\dfrac{0.24\times(1-0.24)}{100}}}{1.645\times\sqrt{\dfrac{0.24\times(1-0.24)}{50}}}$ ✓
> $= \dfrac{1}{\sqrt{2}} \approx 0.707\,107$ ✓

(c) Is it true that the probability of p falling within a confidence interval is greater for a confidence interval with a higher confidence level? Explain.

> Not True. ✓
> Confidence intervals do not deal with probabilities of p falling within particular intervals. ✓

Calculator Assumed

12. [10 marks 3, 2, 2, 3]

Heterochromia is a condition where the colour of a person's irises are different. Let p be the proportion of people who have heterochromia. Samples of 500 persons were taken and $\hat{p}$ the proportion of persons in each sample with heterochromia calculated. Sample A consists of 500 randomly chosen people and 8 were found to suffer from heterochromia.

(a) Estimate the probability that a randomly chosen sample of 500 adults will have a $\hat{p}$ value less than 0.01. Justify your answer.

> As sample size $n = 500$ is large:
> $\hat{p} \sim N(\mu = 0.016, \sigma^2 = \frac{0.016 \times (1 - 0.016)}{500})$ ✓✓
> $P(\hat{p} < 0.01) = 0.14246$ ✓

(b) Use sample A to calculate a 95% confidence interval for p.

> $0.016 \pm 1.96 \times \sqrt{\frac{0.016 \times (1 - 0.016)}{500}}$ ✓
> $0.005\,002 \le p \le 0.026\,998$ ✓

(c) Sample B has another 500 randomly chosen persons and had a 95% confidence interval for p as $0.016\,57 \le p \le 0.047\,43$. Determine with reasons if it is more likely that samples A and B come from the same population or it is more likely that samples A and B come from different populations.

> Samples A and B are more likely to come from different populations. ✓
> Sample proportion for B = $\frac{0.01657 + 0.04743}{2} = 0.032$
> which is outside the confidence interval for A. ✓

(d) Assume that $p = 0.011$. Define the random variable $Y = \dfrac{\hat{p} - p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}$.

Calculate $P(Y \ge 0.9)$. Justify your answer

> As n is large, $Y = \dfrac{\hat{p} - p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}} \sim N(0, 1)$. ✓✓
> $P(Y \ge 0.9) = 0.184060$ ✓

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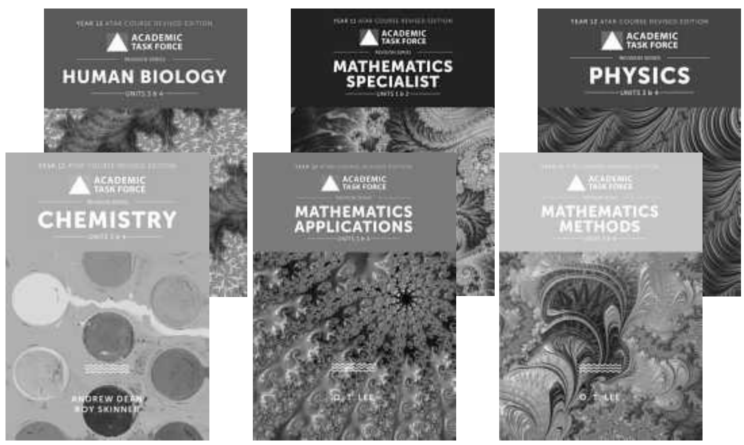

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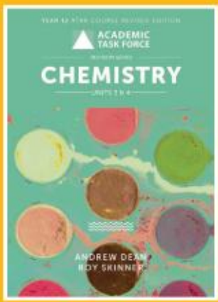

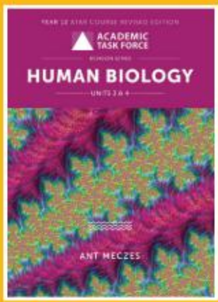

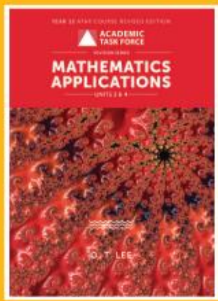

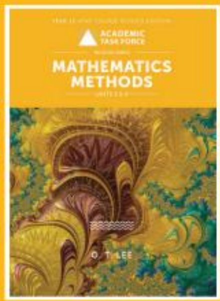

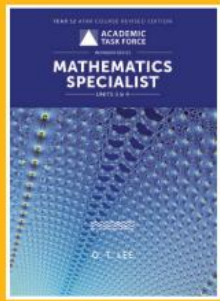

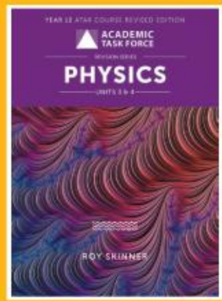

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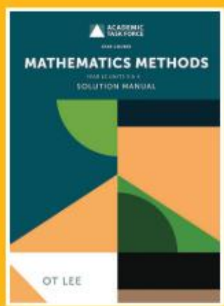

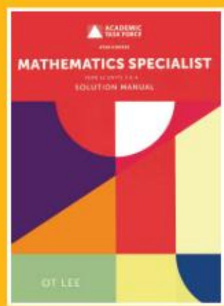

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