PAULINE HOLLAND | MARK BARNES

JACARANDA MATHS QUEST
GENERAL MATHEMATICS 12
FOR QUEENSLAND
UNITS 3&4

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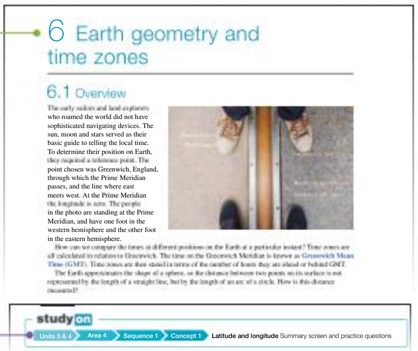
ABOUT THIS RESOURCE

Jacaranda Maths Quest 12 General Mathematics Units 3 & 4 for Queensland is expertly tailored to address comprehensively the intent and structure of the new syllabus. The *Jacaranda Maths Quest for Queensland* series provides easy-to-follow text and is supported by a bank of resources for both teachers and students. At Jacaranda we believe that every student should experience success and build confidence, while those who want to be challenged are supported as they progress to more difficult concepts and questions.

Preparing students for exam success

Chapter openers place mathematics in real-world contexts to drive engagement.

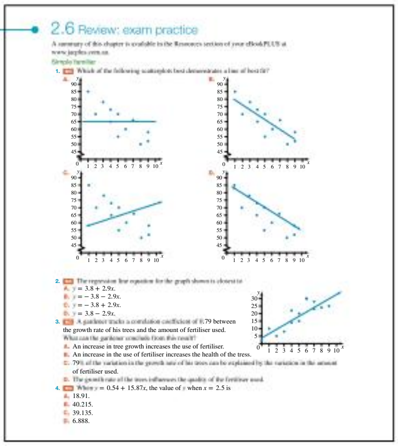


6 Earth geometry and time zones

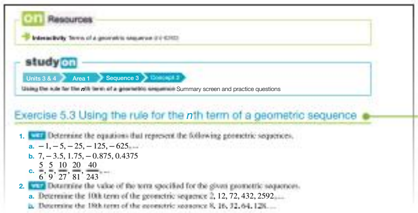
6.1 Overview

FREE access to studyON — our exam, study, revision and practice tool — is included with every title. studyON allows you to revise at the concept, chapter, curriculum topic or unit level.

Every chapter concludes with exam practice questions classified as Simple familiar, Complex familiar and Complex unfamiliar.

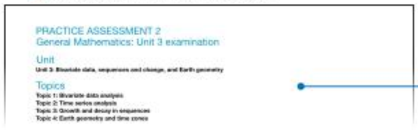


2.6 Review: exam practice



Exercise 5.3 Using the rule for the *n*th term of a geometric sequence

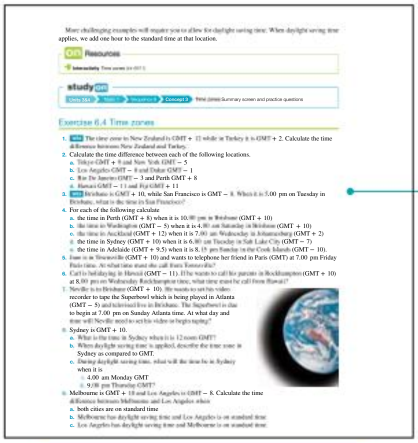
Each subtopic concludes with carefully graded questions.



PRACTICE ASSESSMENT 2

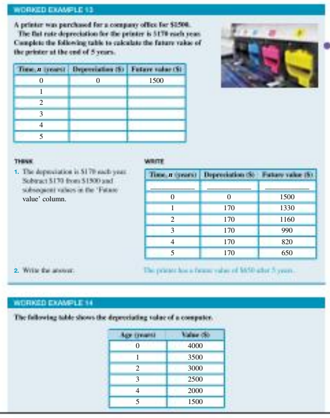
General Mathematics: Unit 3 examination

Two complete sets of practice assessments modelled on QCAA guidelines — a set for student revision and a quarantined set for teachers — are included. Exemplary responses and worked solutions are provided for teachers.



Exercise 6.4 Time zones

Chapter questions and activities are aligned with Marzano and Kenall’s taxonomy of cognitive process — retrieval, comprehension, analysis and knowledge utilisation.



WORKED EXAMPLE 13

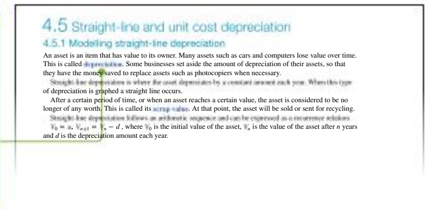
WORKED EXAMPLE 14

Fully worked examples in the Think/Write format provide guidance and are linked to questions.

Features of the *Maths Quest* series

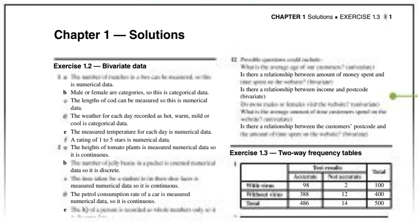
Questions and chapters are sequenced from lower to higher levels of complexity; ideas and concepts are logically developed and questions are carefully graded, allowing every student to achieve success.

An extensive glossary of mathematical terms is provided in print and as a hover-over feature in the eBookPLUS.



4.5 Straight-line and unit cost depreciation

4.5.1 Modelling straight-line depreciation



Chapter 1 — Solutions

Free fully worked solutions are provided, enabling students to get help where they need it, whether at home or in the classroom — help at the point of learning is critical. Answers are provided at the end of each chapter in the print and offline PDF.

eBookPLUS features

A downloadable PDF of the entire chapter of the print text

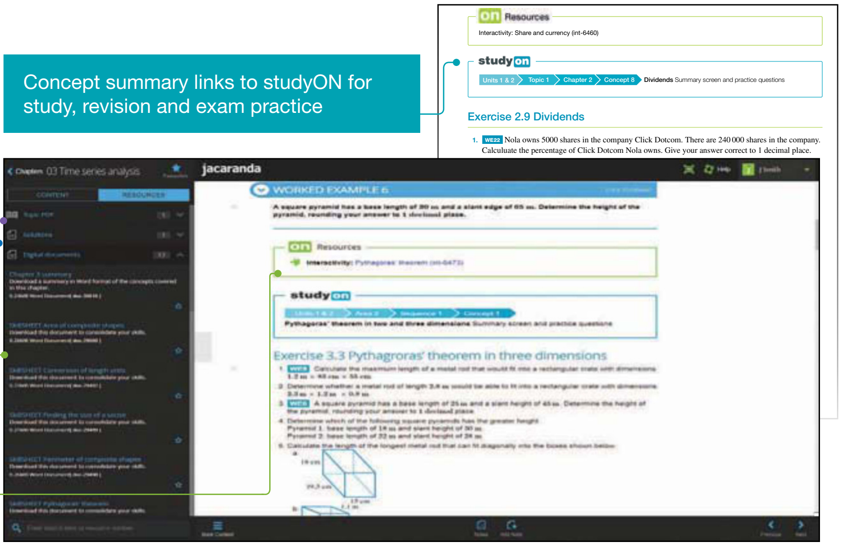
Fully worked solutions for every question

Chapter summaries in downloadable format to assist in study and exam preparation

Digital documents: downloadable SkillSHEETS to support skill development and SpreadSHEETS to explore mathematical relationships and concepts

Interactivities and video eLessons placed at the point of learning to enhance understanding and correct common misconceptions

Concept summary links to studyON for study, revision and exam practice



on Resources

Interactivity: Share and currency (int-6460)

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Units 1 & 2 > Topic 1 > Chapter 2 > Concept 8 **Dividends** Summary screen and practice questions

Exercise 2.9 Dividends

1. WE22 Nola owns 5000 shares in the company Click Dotcom. There are 240 000 shares in the company. Calculate the percentage of Click Dotcom Nola owns. Give your answer correct to 1 decimal place.

on Resources

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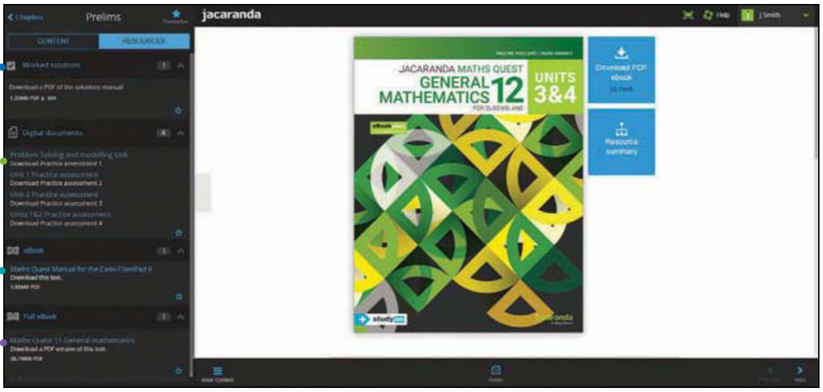
Exercise 3.3 Pythagroras' theorem in three dimensions

In the Prelims section of your eBooKPLUS

A downloadable PDF of the entire solutions manual, containing worked solutions for every question in the text

A set of four practice assessments: a problem solving and modelling task and three examination-style assessments

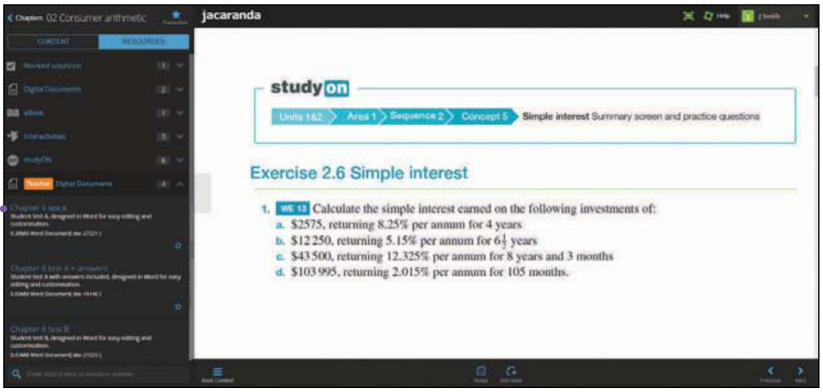
FREE copies of the *Maths Quest Manual for the TI-Nspire CAS calculator* and the *Maths Quest Manual for the Casio Classpad II calculator*



A downloadable PDF of the entire print text

Additional resources for teachers available in the eGuidePLUS

In the Resources tab of every chapter there are two chapter tests in downloadable, customisable Word format with worked solutions.



studyon

Units 1&2 > Area 1 > Sequence 2 > Concept 5 **Simple interest** Summary screen and practice questions

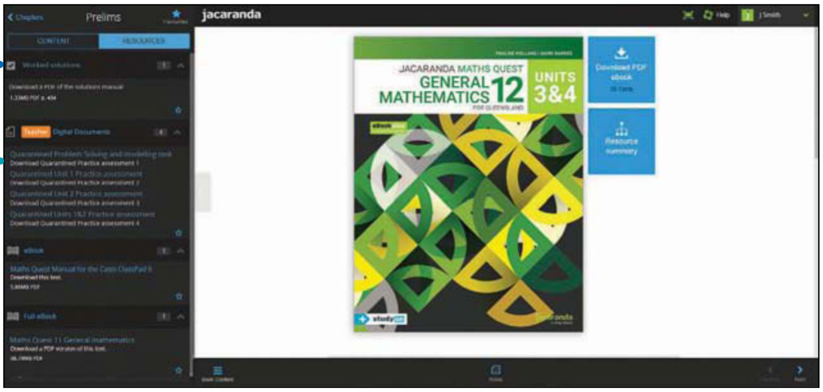
Exercise 2.6 Simple interest

1. WE 12 Calculate the simple interest earned on the following investments of:
   a. \$2575, returning 8.25% per annum for 4 years
   b. \$12 250, returning 5.15% per annum for $6\frac{1}{2}$ years
   c. \$43 500, returning 12.325% per annum for 8 years and 3 months
   d. \$103 995, returning 2.015% per annum for 105 months.

In the Prelims section of the eGuidePLUS

Work programs and Curriculum grids are provided to assist with classroom planning.

Practice assessments: in addition to the four provided in the eBookPLUS, teachers have access to a further four quarantined assessments. Modelled on QCAA guidelines, the problem solving and modelling tasks are provided with exemplary responses while the examination-style assessments include annotated worked solutions. They are downloadable in Word format to allow teachers to customise as they need.



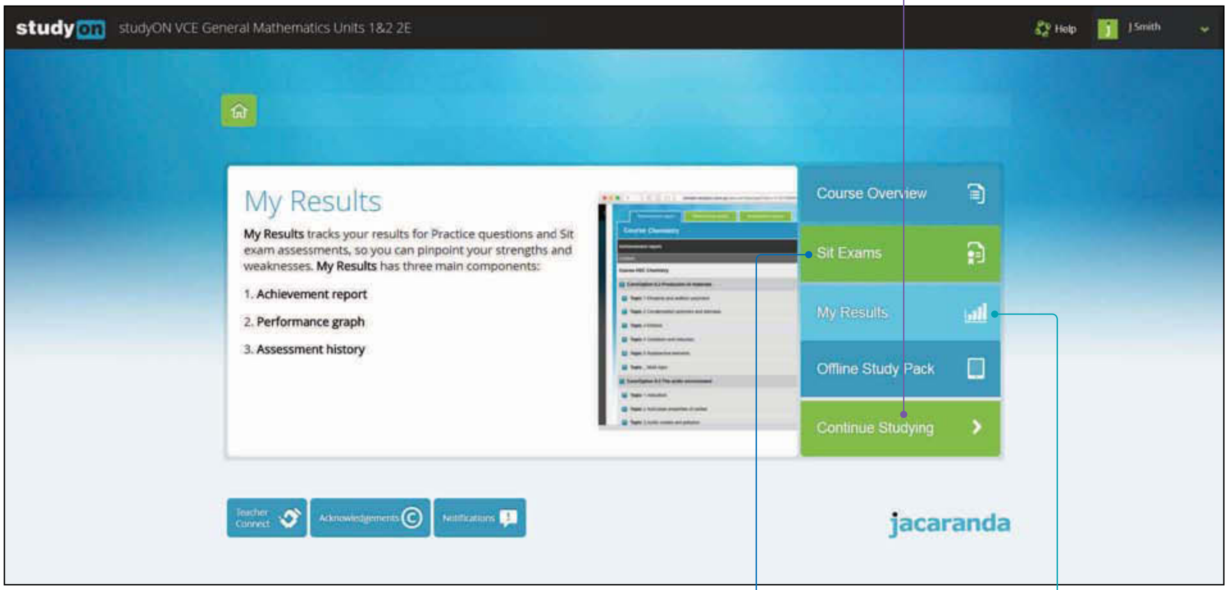
studyON — an invaluable exam preparation tool

studyON provides a complete study solution. An interactive and highly visual online study, revision and exam practice tool, it is designed to help students and teachers maximise exam results.

Concept summary screens and interactivities summarise key concepts and help prevent misconceptions.

Direct links from the eBookPLUS help scaffold students' understanding and study practices.

The studyON question hierarchy allows students in the *Continue Studying* feature to revise across the entire course, or to drill down to concept level for a more granular set of questions.



My Results

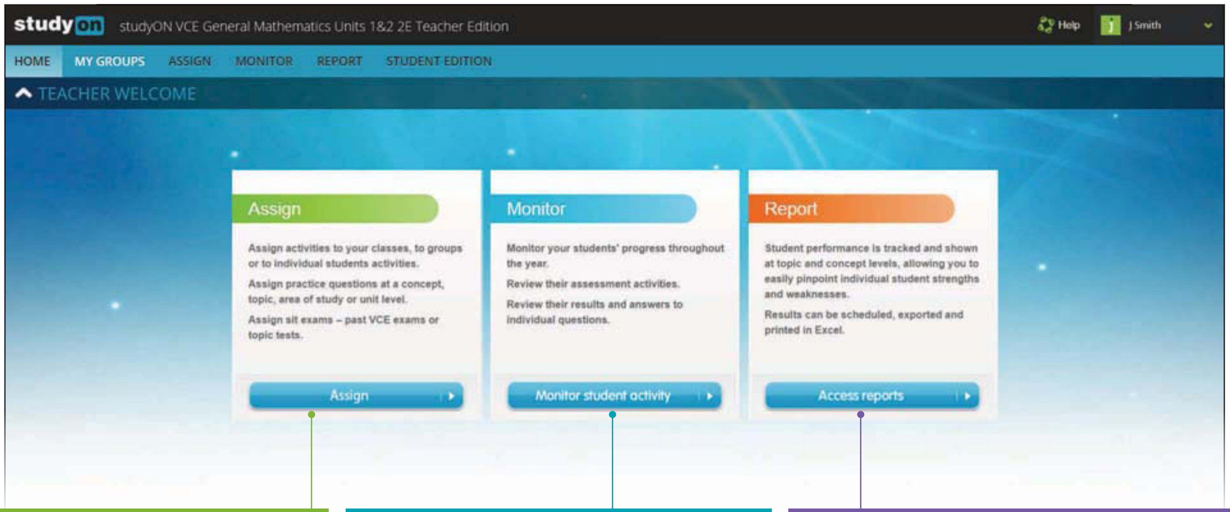
My Results tracks your results for Practice questions and Sit exam assessments, so you can pinpoint your strengths and weaknesses. **My Results** has three main components:

1. Achievement report
2. Performance graph
3. Assessment history

- The Sit Exams feature allows students to sit past VCAA exams (Units 3 & 4) or topic tests (Units 1 & 2) in timed, exam-like situations.
- Students have access to 1000+ past VCAA questions or custom-authored practice questions at a concept, topic or entire course level, and receive immediate feedback for every question.

studyON's built-in progress tracker enables self-diagnosis of strengths and weaknesses at a topic and concept level, so students know exactly what needs extra revision and can sit their exams with confidence.

studyON Teacher edition is a powerful diagnostic tool



Assign activities to your classes, to groups or to individual students activities.

Assign practice questions at a concept, topic, area of study or unit level.

Assign sit exams – past VCE exams or topic tests.

Monitor your students' progress throughout the year.

Review their assessment activities.

Review their results and answers to individual questions.

Student performance is tracked and shown at topic and concept levels, allowing you to easily pinpoint individual student strengths and weaknesses.

Results can be scheduled, exported and printed in Excel.

Enables teachers to assign activities for extra revision and practice, and track progress at an individual, group and classroom level

Allows teachers to monitor students' activities and results to pinpoint strengths and weaknesses. Armed with evidence-based insights, teachers can intervene at the right time.

Alignment with the Jacaranda text helps with planning, and instant feedback saves marking time. Built-in reporting functionality lets teachers easily schedule, export and print reports in Excel.

About eBookPLUS and studyON

Access your online Jacaranda resources anywhere, anytime, from any device in three easy steps:

STEP 1 Go to **www.jacplus.com.au** and create a user account.
STEP 2 Enter your registration code.
STEP 3 Instant access!

eBookplus + **studyon**

eBookPLUS is an electronic version of the textbook, together with a targeted range of supporting multimedia resources.

eBookPLUS features:

- **eBook** — the entire textbook in electronic format
- **Digital documents** designed for easy customisation and editing
- **Interactivities** to reinforce and enhance students' learning
- **eLessons** — engaging video clips and supporting material
- **Weblinks** to relevant support material on the internet

eGuidePLUS features assessment and curriculum material to support teachers.

studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.

studyON features:

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access exam questions** that have been written by experienced examiners for practice at a concept, topic or entire course level, and receive immediate feedback. **From 2020, QCAA questions** will be included with exemplary worked solutions.
- **Sit past QCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 & 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

NEED HELP? Go to **www.jacplus.com.au** and select the Help link.

- Visit the JacarandaPLUS Support Centre at **http://jacplus.desk.com** to access a range of step-by-step user guides, ask questions or search for information.
- **Contact** John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
  Phone: 1800 JAC PLUS (1800 522 7587)

Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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Text

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1 Identifying and describing associations between two variables

1.1 Overview

Recent research by the University of Sydney using data from 80 000 adults over the age of 30 has determined that strength bearing exercise and cardio exercise are equally important in the prevention of premature death, but strength bearing exercise may be more effective when applied to premature death from cancer.

In this study, researchers would have considered the relationships between the many variables that pertained to the 80 000 people in their study. These variables would have included their age, diet and pre-existing health issues. After taking these variables into consideration, mortality rates could be compared based on the type of exercise undertaken by participants in the study.

Detailed studies such as this require researchers with excellent skill sets in data science. Data scientists, engineers and statisticians are needed in every industry, university and government department to help set up data collection infrastructure and to provide in-depth analysis of the data to assist the decision makers within the company or organisation. Some of these roles require programming skills, statistical skills, technical skills, written and oral communication skills or any combination of these. Small and large businesses are also employing consultants to provide timely analysis of available data to grow and sustain their businesses. There is a vast range of careers within the data science field and it is an area that will continue to provide employment opportunities into the future.

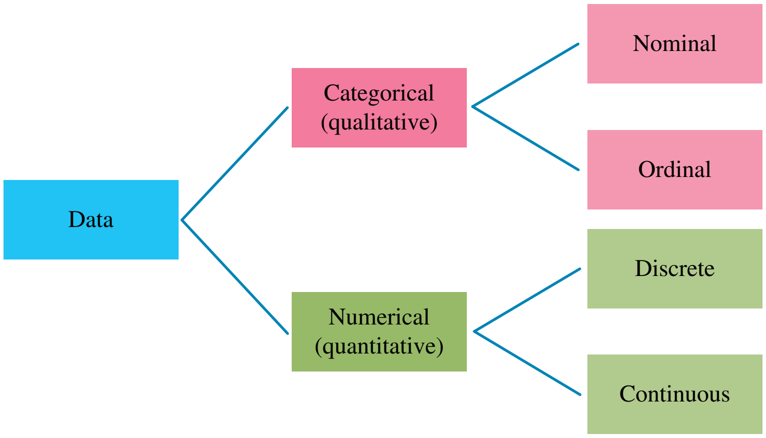
LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.2 Bivariate data

1.2.1 Types of data

Data are classified into two main types. The diagram below distinguishes the types.



Categorical data are data whose values are categories, for example, models of cars or ice-cream flavours. If there is no order associated with the data categories, then they are called **nominal data** (for example, answers to questions about a student's hair colour or method of transport used to travel to school). When order is associated with the data categories, they are called **ordinal data**. A response to a question on a scale (for example, *strongly disagree* to *strongly agree*) would constitute ordinal data (that is, some order is implied). Categorical data is sometimes referred to as **qualitative** data.

Numerical data involve numerical values. **Discrete data** are numerical responses, observations or records that can be counted (usually whole numbers). They take only certain, set values. Examples of discrete data would include the number of children in a family and the number of doors on a car. **Continuous data** may take any value within the range of the data. Here, we often find data arising as the result of taking measurements (for example, a person's height, the daily temperature). Numerical data is sometimes referred to as **quantitative data**.

WORKED EXAMPLE 1

State whether the following examples of data are categorical or numerical.

a. The values of sales recorded at each branch of a fast-food outlet.

b. The breeds of dog that appear at a dog show.

THINK	WRITE
a. The values of sales at each branch can be measured.	**a.** The values of sales are numerical data.
b. The breeds of dog at a show give non-numerical results.	**b.** The breeds of dog are categorical data.

WORKED EXAMPLE 2

State whether each of the following records of numerical data is discrete or continuous.

a. The number of people in each car that passes through a tollgate

b. The mass of a baby at birth

THINK	WRITE
a. The number of people in the car must be a whole number.	a. The data are numerical and discrete.
b. A baby's mass can be measured to various degrees of accuracy.	b. The data are numerical and continuous.

1.2.2 Univariate and bivariate data

In General Mathematics Unit 2, univariate data analysis was covered in detail. The study and analysis of univariate data involves the study of only one variable at a time, for example, the analysis of the heights of 100 basketballers or the favourite television series of 150 first year university students. The study and analysis of **bivariate data** is the study of two variables at the same time to determine if a relationship exists between those variables and how that relationship can be used to make predictions. Examples of bivariate data are studying the relationship between weight and height, gender and voting patterns or blood pressure and age.

WORKED EXAMPLE 3

For the list of variables below, select those that are

a. univariate and categorical data

b. univariate and numerical data

c. bivariate and categorical data

d. bivariate and numerical data

1. **The weekly income of 24 Year 12 students**
2. **The favourite spectator sport of 20 Year 12 students**
3. **The minutes spent on daily exercise and the yearly earnings for 50 commuters on the morning train**
4. **Favourite ice-cream flavor and gender of 30 people in a shopping centre**
5. **The weight and length of 40 new-born babies**
6. **A telephone poll of 1000 voters asked if the voters were going to vote for Liberal, Labor, Green, Other or Not sure at the next election**

▶

THINK	**WRITE**
a. Univariate and categorical data involves only one variable that cannot be measured. Options 1, 2 and 6 involve only one variable and options 2 and 6 are categorical.	**a.** Option 2: The favourite spectator sport of 20 Year 12 students. Option 6: A telephone poll of 1000 voters asked if the voters were going to vote for Liberal, Labor, Green, Other or Not sure at the next election.
b. Univariate and numerical data involves only one variable that can be measured. Option 1 is both univariate and numerical.	**b.** Option 1: The weekly income of 24 Year 12 students.
c. Bivariate and categorical data involves two variables that cannot be measured. Options 3, 4 and 5 involve two variables but only the variables in option 4 cannot be measured.	**c.** Option 4: Favourite ice-cream flavor and gender of 30 people in a shopping centre
d. Bivariate and numerical data involves two variables that can be measured. Options 3 and 5 involve two variables that can be measured.	**d.** Option 3: The minutes spent on daily exercise and the yearly earnings for 50 commuters on the morning train. Option 5: The weight and length of 40 new-born babies

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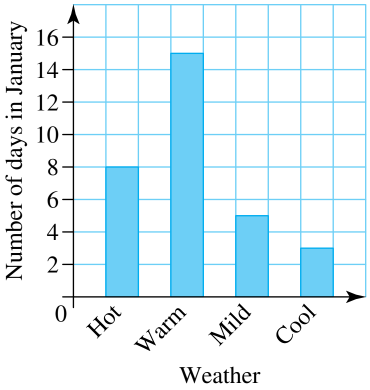
Units 3 & 4 > Area 1 > Sequence 1 > Concept 1 > **Bivariate data** Summary screen and practice questions

Exercise 1.2 Bivariate data

1. **WE1** State whether the data collected in each of the following situations would be categorical or numerical.
   - **a.** The number of matches in each box is counted for a large sample of boxes.
   - **b.** The sex of respondents to a questionnaire is recorded as either M or F.
   - **c.** A fisheries inspector records the lengths of 40 cod.
   - **d.** The occurrence of hot, warm, mild and cool weather for each day in January is recorded.
   - **e.** The actual temperature for each day in January is recorded.
   - **f.** Cinema critics are asked to judge a film by awarding it a rating from one to five stars.
2. **WE2** State whether the numerical data gathered in each of the following situations are discrete or continuous.
   - **a.** The heights of 60 tomato plants at a plant nursery
   - **b.** The number of jelly beans in each of 50 packets
   - **c.** The time taken for each student in a class of six-year-olds to tie his or her shoelaces
   - **d.** The petrol consumption rate of a large sample of cars
   - **e.** The IQ (intelligence quotient) of each student in a class

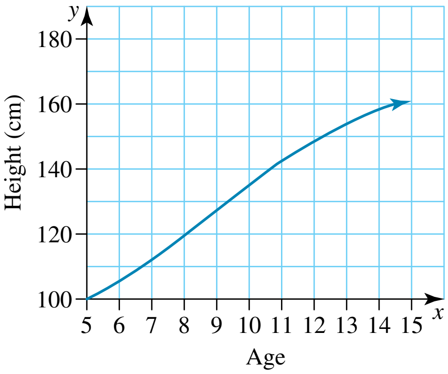
3. For each of the following, state if the data are categorical or numerical.
If numerical, state if the data are discrete or continuous.
   a. The number of students in each class at your school
   b. The teams people support at a football match
   c. The brands of peanut butter sold at a supermarket
   d. The heights of people in your class
   e. The interest rate charged by each bank
   f. A person's pulse rate
4. An opinion poll was conducted. A thousand people were given the statement 'Euthanasia should be legalised'. Each person was offered five responses: strongly agree, agree, unsure, disagree and strongly disagree. Describe the data type in this example.
5. A teacher marks her students' work with a grade A, B, C, D, or E. Describe the data type used.
6. A teacher marks his students' work using a mark out of 100. Describe the data type used.
7. **MC** The number of people who are using a particular bus service are counted over a 2-week period.
The data formed by this survey would best be described as
   A. categorical data.
   B. numerical and discrete data.
   C. numerical and continuous data.
   D. quantitative data.

8. The following graph shows the number of days of each weather type for the Gold Coast in January.



Describe the data shown in this graph.

9. The following graph shows a girl's height each year for 10 years.



Describe the data shown in this graph.

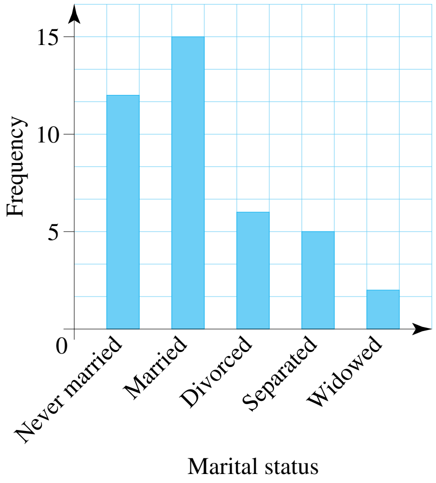
10. **WE3** For the list of variables below select those that are

- **a.** univariate and categorical data
- **b.** univariate and numerical data
- **c.** bivariate and categorical data
- **d.** bivariate and numerical data.
  1. The mark scored on a mathematics test and the hours of mathematics homework completed during the previous week
  2. Water usage and time spent in the shower for 50 students in a hostel
  3. The number of vehicles passing a point on a road during a given time period
  4. A group of customers were asked to respond to the level of service they received in a store as poor, fair, excellent.
  5. The time each customer spends in a supermarket queue on a Saturday morning
  6. Days of the week and choices of public transport

11. **MC** The chart shows the marital status of respondents to a survey.

Choose the option below that best describes this data.

- **A.** univariate and categorical data
- **B.** univariate and numerical data
- **C.** bivariate and categorical data
- **D.** bivariate and numerical



12. An online department store decides to send out a survey to collect some information about their customers.

The survey asks their customers to provide the following information

- Age $(< 18, 18 - 29, 30 - 50, 50+)$
- Gender
- Income
- Postcode

The customers' digital footprints provide records of their purchases, the amount of time the customer spends browsing and the amount of money the customer spends on a weekly basis.

Using this information identify 5 questions that could be answered by analysing this data and state whether this analysis would be univariate or bivariate.

1.3 Two-way frequency tables

1.3.1 Categorical variables and two-way frequency tables.

Two-way frequency tables are a useful means of organising, identifying and describing an association between two categorical variables for a group of individuals or objects.

Consider, for example, data collected on the eye colour of 200 couples. It may be represented in a table such as the one on the following page.

		Female			
		Green	**Brown**	**Blue**	**Total**
Male	**Blue**	11	25	9	45
	Brown	19	51	28	98
	Green	17	27	13	57
	Total	47	103	50	200

Splitting the data into categories based on the eye colour of the **males** and calculating percentages in each category would yield the following results. That is, 11 of the 45 blue-eyed males have female partners who have green eyes. So, $\frac{11}{45} \times 100 = 24.4\%$ of blue-eyed males have female partners with green eyes, and so on.

		Female			
		Green	**Brown**	**Blue**	**Total**
Male	**Blue**	$\frac{11}{45} \times 100 = 24.4\%$	$\frac{25}{45} \times 100 = 55.6\%$	$\frac{9}{45} \times 100 = 20\%$	100%
	Brown	$\frac{19}{98} \times 100 = 19.4\%$	$\frac{51}{98} \times 100 = 52.0\%$	$\frac{28}{98} \times 100 = 28.6\%$	100%
	Green	$\frac{17}{57} \times 100 = 29.8\%$	$\frac{27}{57} \times 100 = 47.4\%$	$\frac{13}{57} \times 100 = 22.8\%$	100%

The following table gives the percentages rounded to the nearest whole number.

		Female			
		Green	**Brown**	**Blue**	**Total**
Male	**Blue**	24%	56%	20%	100%
	Brown	19%	52%	29%	100%
	Green	30%	47%	23%	100%

Splitting the data into categories based on the eye colour of the **females** and calculating percentages in each category would yield the following results. That is, 11 of the 47 green-eyed females have male partners who have blue eyes. So, $\frac{11}{47} \times 100 = 23.4\%$ of green-eyed females have male partners with blue eyes.

		Female		
		Green	**Brown**	**Blue**
Male	**Blue**	$\frac{11}{47} \times 100 = 23.4\%$	$\frac{25}{103} \times 100 = 24.3\%$	$\frac{9}{50} \times 100 = 18\%$
	Brown	$\frac{19}{47} \times 100 = 40.4\%$	$\frac{51}{103} \times 100 = 49.5\%$	$\frac{28}{50} \times 100 = 56\%$
	Green	$\frac{17}{47} \times 100 = 36.2\%$	$\frac{27}{103} \times 100 = 26.2\%$	$\frac{13}{50} \times 100 = 26\%$
	Total	100%	100%	100%

The following table gives the percentages rounded to the nearest whole number.

		Female		
		Green	**Brown**	**Blue**
Male	**Blue**	23%	24%	18%
	Brown	41%	50%	56%
	Green	36%	26%	26%
	Total	100%	100%	100%

It is obvious that the interpretation of the data depends on the reference basis. We may wish to consider those couples where the male is blue-eyed and the female brown-eyed. Note that this represents 25 couples. What if we talk about percentages? Comparing the percentages in the two tables, it can be seen that:

1. 56% of blue-eyed males have female partners with brown eyes (see pink shading on the previous page).
2. 24% of brown-eyed females have male partners with blue eyes (see green shading above).

These percentages have vastly different values, yet they both describe the same set of 25 couples of blue-eyed males and brown-eyed females. It is important, particularly when dealing with two-way frequency tables, to consider the reference basis for percentages.

WORKED EXAMPLE 4

A new test was designed to assess the reading ability of students entering high school.

The results were used to determine if the students' reading level was adequate to cope with high school.

The students' results were then checked against existing records.

Of the 150 adequate readers who sat for the test, 147 of them passed.

Of the 50 inadequate readers who sat for the test, 9 of them passed.

Present this information in a two-way frequency table.

THINK

Draw up the table showing the existing records (adequate readers and inadequate readers) and the results of the new test (passed and did not pass).

WRITE

	New test results		
	Passed	**Did not pass**	**Total**
Adequate readers	147	3	150
Inadequate readers	9	41	50
Total	156	44	200

When information on a test is presented in a two-way frequency table, conclusions can be made about the accuracy of the test. An example of this follows.

WORKED EXAMPLE 5

A batch of sniffer dogs is trained by customs to smell drugs in suitcases. Before the dogs are used at airports they must pass a test by detecting the smell of drugs and selecting the bags. The results of that test are shown in the following two-way-frequency table.

		Test results		
		Detected	**Not detected**	**Total**
Bags	**With drugs**	**24**	**1**	**25**
	Without drugs	**11**	**164**	**175**
	Total	**35**	**165**	**200**

a. **How many bags did the sniffer dogs examine?**
b. **In how many bags did the dogs detect drugs?**
c. **In what percentage of bags without drugs did the dogs incorrectly detect drugs? Write your answer correct to one decimal place.**
d. **Based on the above results, what percentage of the time will the dogs not detect a bag carrying drugs?**

THINK	WRITE
a. The total number is located in the bottom right cell.	a. 200 bags were examined.
b. The total of the detected column.	b. The dogs detected drugs in 35 bags.
c. There were 175 bags without drugs but dogs incorrectly detected drugs in 11 bags. Write this as a percentage, rounding to one decimal place.	c. Percentage incorrectly detected $= \frac{11}{175} \times 100\%$ $= 6.3\%$
d. Of 25 bags with drugs, 1 went undetected. Write this as a percentage.	d. Percentage not detected $= \frac{1}{25} \times 100\%$ $= 4\%$

After studying a two-way frequency table, we should also be able to make judgements about the information given in the tables. In the previous worked example examining 200 bags in total, only one bag out of 25 with drugs went undetected. So, 24 bags were correctly identified. Although the dogs incorrectly detected drugs in 11 bags that did not have drugs, they correctly identified 164 bags not containing drugs. So they still have an overall accuracy of 94% as shown by the calculation $[(24 + 164) \div 200] \times 100\%$.

Many two-way frequency tables will require you to make your own value judgements about the conclusions established. For example, the 94% overall accuracy recorded may be considered ‘very acceptable’.

WORKED EXAMPLE 6

The following data table shows the composition of the employees of a small law firm.

	Full-time	**Part-time**
Female	**4**	**11**
Male	**30**	**5**

a. **Extend the table to show totals in all categories and an overall total.**
b. **Draw a table showing percentages with respect to type of employment (full or part-time).**
c. **Redraw the table showing percentages with respect to the gender of the employee.**
d. **What percentage of females work full time?**
e. **What percentage of full-time workers are female?**

THINK	WRITE
a. Add the numbers in the cells for all the rows and columns and enter the totals. Check that the overall total is consistent for the rows and columns.	**a.**

	Full-time	Part-time	Total
Female	4	11	15
Male	30	5	35
Total	34	16	50

THINK	WRITE
b. Percentages with respect to type of employment are based on totals in columns. The totals in the columns are on the denominator when calculating percentages.	**b.**

	Full-time	Part-time
Female	$\frac{4}{34} \times 100 = 12\%$	$\frac{11}{16} \times 100 = 69\%$
Male	$\frac{30}{34} \times 100 = 88\%$	$\frac{5}{16} \times 100 = 31\%$
Total	100%	100%

THINK	WRITE
c. Percentages with respect to gender are based on totals in rows. The totals in the rows are on the denominator when calculating percentages.	**c.**

	Full-time	Part-time	Total
Female	$\frac{4}{15} \times 100 = 27\%$	$\frac{11}{15} \times 100 = 73\%$	100%
Male	$\frac{30}{35} \times 100 = 86\%$	$\frac{5}{35} \times 100 = 14\%$	100%

THINK	WRITE
d. **1.** This is based on female totals shown in the table in part **c**.	**d.** $\frac{\text{full time}}{\text{female total}} \times 100 = \frac{4}{15} \times 100 = 27\%$
2. Write the answer.	Percentage of females who work full time = 27%.
e. **1.** This is based on full-time totals shown in the table in part **b**.	**e.** $\frac{\text{female}}{\text{full-time total}} \times 100 = \frac{4}{34} \times 100 = 12\%$
2. Write the answer.	Percentage of full-time workers who are female = 12%.

We are constantly bombarded with statistics, some of which are a valid interpretation of data, and some of which are not. On occasions, the misuse of statistics may be unintentional or through ignorance, but there are occasions when misleading figures are quoted intentionally.

WORKED EXAMPLE 7

When discussing the probability of female attendance of 15–19 year olds at an educational institution in a large city, it was claimed that 51% of the females in this age group attended an educational institution. Some of the information is contained in the following two-way frequency table.

15–19 year olds	Male	Female	Total
Attend an educational institution	40 549	42 704	
Do not attend an educational institution			
Total	63 978	61 948	

a. **Complete the two-way frequency table displaying the attendance/non-attendance of 15–19-year-old males and females at an educational institution and the totals.**

b. **Use the two-way frequency table to discuss the validity of the claim that 51% of the females in the group attended an educational institution.**

THINK

a. Complete the two-way frequency table with totals for rows and columns. Subtract the attendance figure from its relevant total to determine the non-attendance figure.

WRITE

a.

15–19 year olds	Male	Female	Total
Attend an educational institution	40 549	42 704	83 253
Do not attend an educational institution	23 429	19 244	42 673
Total	63 978	61 948	125 926

THINK

b. 1. Calculate the percentage of females who attend an educational institution; that is,

$$\frac{\text{no. females at education institute}}{\text{total number of females}} \times 100$$

WRITE

b. The percentage of females who attend an educational institution $= \frac{42\,704}{61\,948} \times 100$

$= 69\%$

69% of females attend an educational institution.

THINK

2. Calculate the percentage of those in an educational institution who are female; that is,

$$\frac{\text{no. females at education institute}}{\text{total number in an educational institution}} \times 100$$

WRITE

The percentage of people in an educational institution who are female $= \frac{42\,704}{82\,253} \times 100$

$= 51\%$

51% of those in an educational institution are female.

THINK

3. Compare these figures with the claim above to determine if the claim is correct.

WRITE

The claim that 51% of females in the group attended an educational institution is not correct. It should have said that 69% of 15–19 year old females attend an educational institution.

on Resources

Digital documents SpreadSHEET Fractions/decimals to percentages (doc-1911)
SpreadSHEET One amount as a percentage of another (doc-1913)

studyon

Units 3 & 4 › Area 1 › Sequence 1 › Concept 2

Two-way frequency tables Summary screen and practice questions

Exercise 1.3 Two-way frequency tables

1. **WE4** A test is developed to test for infection with the flu virus. To test the accuracy, the following 500 people are tested.
   - Of the 100 people who are known to have the flu who are tested, the test returns 98 positive results.
   - Of the 400 people who are known not to be infected with the virus who are tested, 12 false positives are returned.

   Display this information in the following two-way frequency table.

	Test results		
	Accurate	**Not accurate**	**Total**
With virus			
Without virus			
Total			

2. One thousand people take a lie detector test. Of 800 people known to be telling the truth, the lie detector indicates that 23 are lying. Of 200 people known to be lying, the lie detector indicates that 156 are lying. Present this information in a two-way frequency table.
3. **WE5** The following table displays the accuracy gained from a medical test screening for a virus. Note: A positive test indicates that the patient has the virus.

	Test results		
	Accurate	**Not accurate**	**Total**
With virus	45	3	48
Without virus	922	30	952
Total	967	33	1000

   a. How many patients were screened for the virus?
   b. How many positive tests were recorded? (That is, in how many tests was the virus detected?)
   c. What percentage of test results were accurate?

4. **WE6** The following table indicates the results of a radar surveillance system. If the system detects an intruder, an alarm is activated.

	Test results		
	Alarm activated	**Not activated**	**Total**
Intruders	40	8	48
No intruders	4	148	152
Total	44	156	200

a. Over how many nights was the system tested?
b. On how many occasions was the alarm activated?
c. If the alarm is activated, what is the percentage chance that there actually is an intruder?
d. If the alarm was not activated, what is the percentage chance that there was an intruder?
e. What was the percentage of accurate results over the test period?
f. Comment on the overall performance of the radar detection system.

The information below is to be used in questions 5 to 7.
A test for a medical disease does not always produce the correct result. A positive test indicates that the patient has the condition. The following table indicates the results of a trial on a number of patients who were known to either have the disease or known not to have the disease.

	Test results		
	Accurate	**Not accurate**	**Total**
With disease	57	3	60
Without disease	486	54	540
Total	543	57	600

5. **MC** The overall accuracy of the test is
A. 9.5%. B. 90%. C. 90.5%. D. 92.5%.

6. **MC** Based on the table, what is the probability that a patient who has the disease has it detected by the test?
A. 9.5% B. 90% C. 90.5% D. 95%

7. **MC** Which of the following statements is correct?
A. The test has a greater accuracy with positive results than with negative results.
B. The test has a greater accuracy with negative results than with positive results.
C. The test is equally accurate with positive and negative test results.
D. The test is equally inaccurate with positive and negative test results.

8. Airport scanning equipment is tested by scanning 200 pieces of luggage. Prohibited items were placed in 50 bags and the scanning equipment detected 48 of them. The equipment detected prohibited items in five bags that did not have any forbidden items in them.
a. Use the above information to complete the following two-way frequency table.

	Test results		
	Accurate	**Not accurate**	**Total**
Bags with prohibited items			
Bags with no prohibited items			
Total			

b. Use the table to answer the following.
  - **i.** What percentage of bags with prohibited items were detected as having prohibited items?
  - **ii.** What was the percentage of false positives among the bags that had no prohibited items?
  - **iii.** What percentage of the original 50 prohibited items pass through the scanning equipment undetected?
  - **iv.** What is the overall percentage accuracy of the scanning equipment?

9. In some cases it is easier to count numbers in a particular category by considering a different population. In each of the following pairs of proportions, which one would be easier to determine?

a. **i.** Proportion of males who are left-handed.
  ii. Proportion of left-handers who are males.

b. **i.** Proportion of Mathematics A students in your school who are over 16.
  ii. Proportion of over 16 year olds in your school who study Mathematics A.

c. **i.** Proportion of state school students who live in Queensland.
  ii. Proportion of Queensland school students who attend a state school.

10. **WE7** The following two-way frequency table shows the results of a test done on a speed camera. A positive result means that the camera detected the car as speeding.

	Positive	**Negative**	**Total**
Speeding	18	2	20
Not speeding	5	95	100
Total	23	97	

a. How many cars in total were involved in the test?
b. How many cars were speeding?
c. How many cars were recorded as speeding?
d. Based on the above results, what is the percentage chance (probability) that a car that is speeding escapes detection by the camera?

11. The following two-way frequency table shows the results of the tests done on a car alarm system.

	Alarm activated	**Alarm not activated**	**Total**
Car break-in	43	7	50
No break-in	2	298	300
Total	45	305	

a. How many tests were conducted on the alarm?
b. How many times was the alarm activated?
c. Based on the above results, if the alarm is ringing what is the percentage chance (probability) that a break-in is actually taking place?

12. A medical test is designed to detect the presence of a certain gene in unborn babies. The test is conducted on a sample of unborn babies with the following results.

- Of 100 babies with the gene, 95 were detected as having the gene during the test.
- Of 900 babies without the gene, 37 were detected as having the gene during the test.

a. Display the results in a two-way frequency table.
b. What is the percentage probability that, if the gene is present, it is detected?

c. After birth, a baby that is thought to have the gene is chosen at random. What is the percentage probability that the baby does actually have the gene being tested for?

d. What is the percentage probability that a baby chosen at random from the sample was correctly diagnosed?

13. The following table shows information about a particular survey of a sample of men. It shows the percentage of men by age and their preferred type of movie.

	Preferred type of movie		
Age group of men surveyed	**Action**	**Comedy**	**Drama**
19 years and under	9.5%	4.7%	3.0%
20 to 35 years	41.1%	30.3%	20.5%
36 to 59 years	22.4%	30.7%	35.5%
60 years and over	27.0%	34.3%	41.0%

a. Of the men who preferred drama series, what percentage were between 20 to 59 years inclusive?

b. Does the information in the table support the opinion that, for this sample of men, the preferred type of movie is associated with age? Justify your answer with the appropriate percentages.

1.4 Scatterplots

1.4.1 Numerical variables and scatterplots

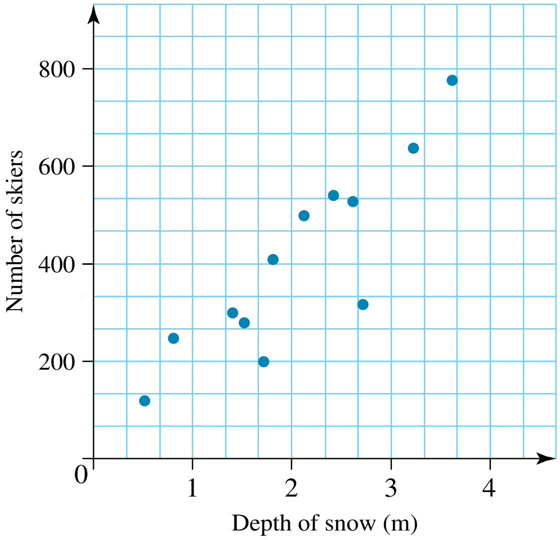
The manager of a ski resort collects the following numerical data over twelve consecutive weekends at his resort.

Depth of snow (m)	0.5	0.8	2.1	3.6	1.4	1.5	1.8	2.7	3.2	2.4	2.6	1.7
Number of skiers	120	250	500	780	300	280	410	320	640	540	530	200

As there are two types of data in this example, they are called bivariate data. For each item (weekend), two variables are considered (depth of snow and number of skiers). When analysing bivariate data, we are interested in examining the relationship between the two variables. In the case of the ski resort data, the manager might be interested in answering the following questions.

1. Are visitor numbers related to depth of snow?
2. If there is a relationship between visitor numbers and depth of snow, is it always true or is it just a guide? In other words, how strong is the relationship?
3. How much confidence could be placed in the prediction?

Scatterplots are a useful means of identifying and describing an association between two numerical variables. The data from the ski resort is used to create the scatterplot on the following page. Each of the data points is represented by a single visible point on the graph.



1.4.2 Explanatory and response variables

When drawing a scatterplot, it is important to choose the correct variable to assign to each of the axes. The convention is to place the **explanatory** (independent) **variable** on the *x*-axis and the **response** (dependent) **variable** on the *y*-axis. The explanatory variable in an experiment or investigation is the variable that is deliberately controlled or adjusted by the investigator. The response variable is the variable that responds to changes in the explanatory variable.

Neither of the variables involved in the ski resort data was controlled directly by the investigator but ‘Number of skiers’ would be considered the response variable because it is likely to change depending on depth of snow. (The snow depth does not depend on numbers of skiers.) As ‘Number of skiers’ is the response variable, we graph it on the *y*-axis and the ‘Depth of snow’ on the *x*-axis. Notice how the scatterplot above, for the ski resort data shows a general upward trend. It is not a perfectly straight line, but it is still clear that a general trend or relationship has formed: as the depth of snow increases, so too does the number of skiers.

WORKED EXAMPLE 8

Identify the explanatory and response variables in each of the following scenarios.

a. Distance walked in an hour and the age of a person

b. The cost of bananas and the average daily temperature in Queensland

THINK	WRITE
a. Consider which variable does not respond to the other. The age of a person will not be changed due to the distance they walk; however, their age could explain the distance they have covered.	**a.** Explanatory variable = age Response variable = distance walked in an hour

b. The cost of bananas is influenced by supply and demand. If the growing season has been affected by higher than expected daily temperatures, the number of bananas produced will be less, therefore increasing the price.

b. Explanatory variable = average daily temperature in Queensland

Response variable = cost of bananas

WORKED EXAMPLE 9

The following table shows the height and mass of ten Year 11 students.

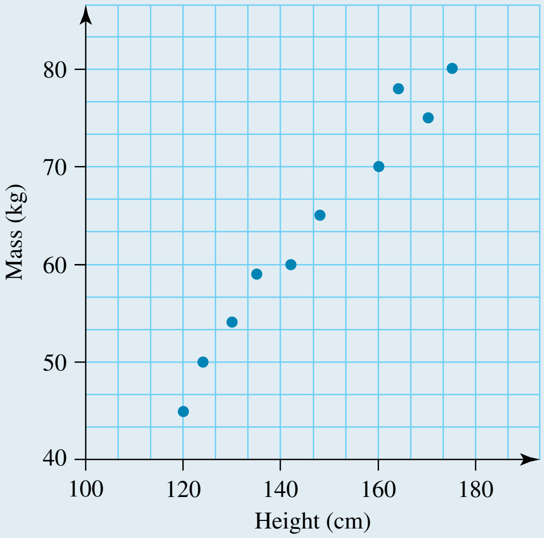
Height (cm)	**120**	**124**	**130**	**135**	**142**	**148**	**160**	**164**	**170**	**175**
Mass (kg)	**45**	**50**	**54**	**59**	**60**	**65**	**70**	**78**	**75**	**80**

Display the data on a scatterplot.

THINK

1. Show the height on the *x*-axis and the mass on the *y*-axis since height is the explanatory variable and mass is the response variable.
2. Plot the point given by each pair.

WRITE



1.4.3 Linear and non-linear relationships

When we are describing the relationships between two variables displayed on a scatterplot, we need to comment on:

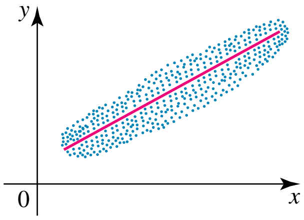
- the form — whether it is linear or non-linear
- the direction — whether it is positive or negative
- the strength — whether it is strong, moderate or weak
- possible outliers.

The form

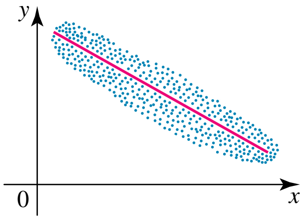
The form of a bivariate data set can be categorised as either linear or non-linear. The points on the scatterplot will produce a linear pattern or a non-linear pattern.

In a linear pattern, the points tend to form a straight line, whereas in a non-linear pattern, the points tend to from a curve.

Linear relationships

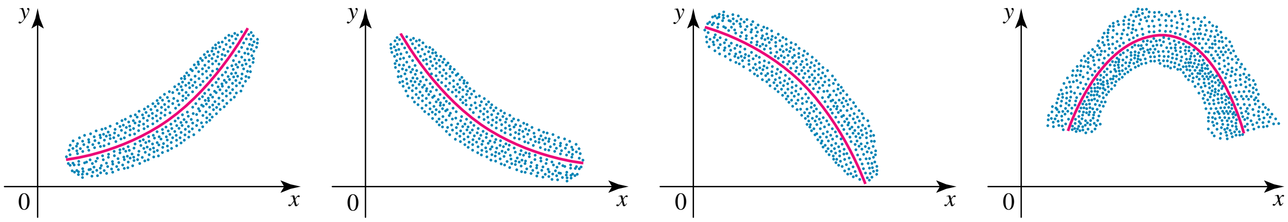


There is a positive relationship between variables; that is, as one variable increases, the other variable also increases.



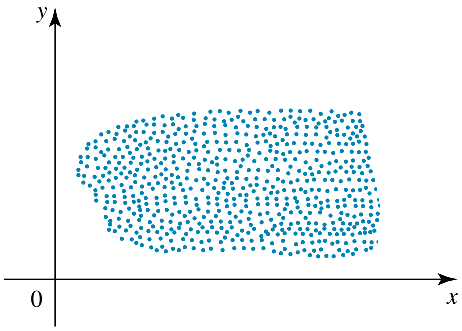
There is a negative relationship between variables; that is, as one variable increases, the other variable decreases.

Non-linear relationships



No relationship

In other cases it may be that there is no relationship at all between the two variables. Such a scatterplot would look like the one shown.



WORKED EXAMPLE 10

The following table shows the length and mass of a dozen eggs.

Length (cm)	**6.2**	**3.9**	**4.5**	**5.8**	**7.2**	**7.6**	**6.1**	**6.7**	**7.3**	**5.1**	**6.0**	**7.3**
Mass (g)	**60**	**15**	**25**	**50**	**95**	**110**	**55**	**75**	**95**	**35**	**54**	**96**

a. **Display this information in a scatterplot.**

b. **Determine if there is any relationship between the length and mass of the eggs and state if the relationship is linear.**

THINK

a. **1.** Display length on the x-axis and mass on the$y$-axis since the length is the explanatory variable and the mass is the response variable.

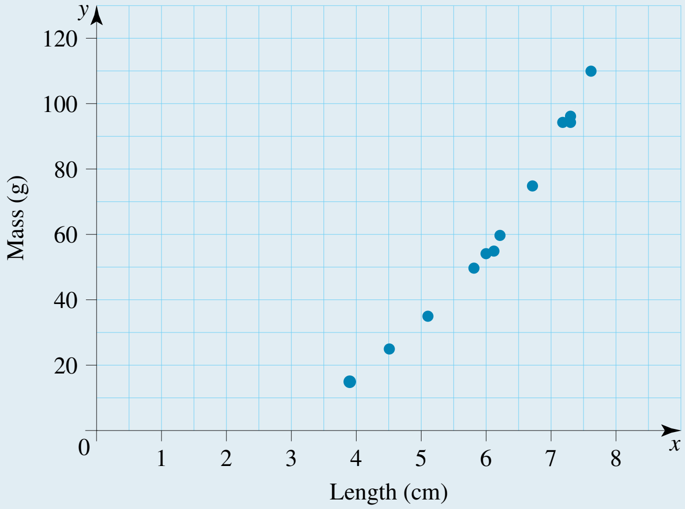
2. Plot the point given by each pair.

b. **1.** Study the scatterplot to see if mass increases as length increases.

2. Study the scatterplot to see if the points seem to approximate a straight line.

WRITE

a.



b. As length increases, so does the mass of the egg.

The points do not approximate a straight line and so the relationship is non-linear.

Linear bivariate data sets can be further described in terms of direction and strength.

- Direction describes whether the plot slopes up or down.
- Strength describes how closely the points form a linear pattern.

The direction

For a linear bivariate data set,

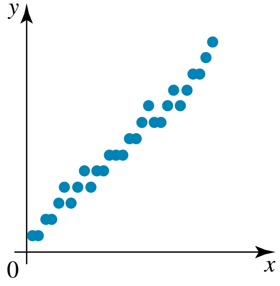
- If the points on the scatterplot slope up to the right, we say that there is a positive association between the variables. This shows that as the explanatory variable increases, the response variable also increases.
- If the scatterplot points slope down to the right, we say that there is a negative association between the two variables. This shows that as the explanatory variable increases, there is a decrease in the response variable.

The strength

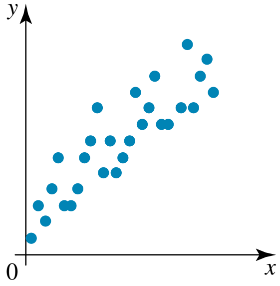
When interpreting a scatterplot, the association or correlation provides an insight into the relationship between the two variables. The association is a measure of the strength of the linear relationship between the two variables.

To measure how strongly the scatterplot points tend to form a straight line, we begin by estimating the strength of the association as strong, moderate or weak based on inspection of the scatterplot.

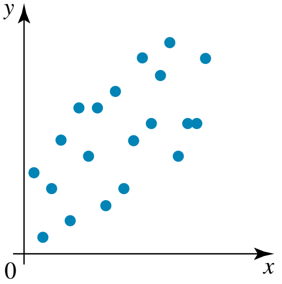
On the following page is a gallery of scatterplots showing various patterns to look for.



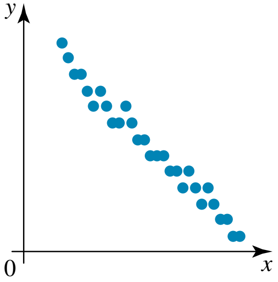
Strong positive correlation



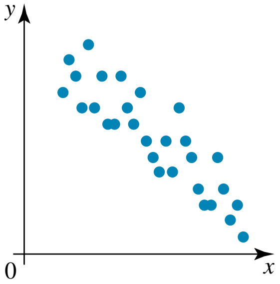
Moderate positive correlation



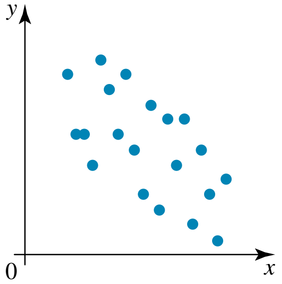
Weak positive correlation



Strong negative correlation



Moderate negative correlation



Weak negative correlation

1.4.4 Graphing a scatterplot using a spreadsheet

WORKED EXAMPLE 11

This table shows the hearing test scores of people of different ages:

Age (years)	**55**	**40**	**35**	**30**	**42**	**48**	**50**	**48**	**32**	**45**	**30**	**56**
Hearing test score	**2.5**	**3.8**	**4.0**	**3.9**	**2.5**	**3.2**	**2.2**	**1.8**	**3.0**	**2.0**	**4.0**	**1.8**

a. **Which is the explanatory variable (put on the x-axis)?**

b. **Using Excel, construct a scatterplot showing hearing test score against age.**

c. **By examining the scatterplot, determine if there is an association between age and hearing loss.**

THINK

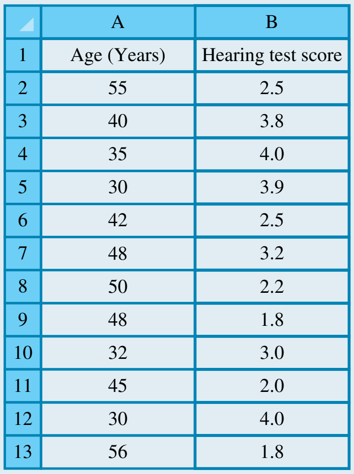
a. The explanatory variable in an experiment or investigation is the variable that is deliberately controlled or adjusted by the investigator. This would be the age of the participants.

WRITE

a. The explanatory variable is 'age'.

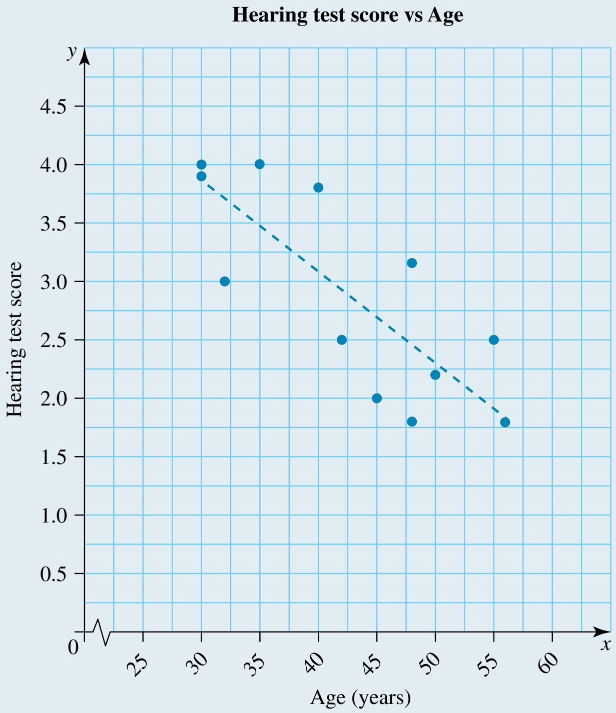
b. 1. Open a new spreadsheet and enter the data for 'Age' in Column A and 'Hearing test score' in Column B.

b.



	A	B
1	Age (Years)	Hearing test score
2	55	2.5
3	40	3.8
4	35	4.0
5	30	3.9
6	42	2.5
7	48	3.2
8	50	2.2
9	48	1.8
10	32	3.0
11	45	2.0
12	30	4.0
13	56	1.8

2. Construct a scatterplot in Excel with age on the horizontal axis and hearing test score on the vertical axis.



c. The points appear to follow a weak negative linear pattern.

c. The scatterplot shows a weak, negative linear pattern which would suggest there is a weak negative association between age and hearing loss, this suggests that as a person ages, their hearing decreases.

on Resources

Interactivities Create scatterplots (int-6497)
Solving linear equations (int-6450)

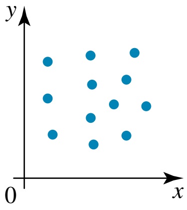
studyon

Units 3 & 4 › Area 1 › Sequence 1 › Concept 3 › **Scatterplots** Summary screen and practice questions

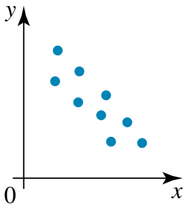
Exercise 1.4 Scatterplots

1. **WE8** Identify the explanatory and response variables in each of the following scenarios.
   a. Age and height
   b. Distance travelled and time taken (at a fixed speed)
   c. Temperature and elevation
   d. Blood alcohol level and reaction time
   e. IQ and results on an academic test
   f. Overtime pay and hours worked
   g. Value of a car and its age
   h. Time taken to travel a given distance and speed of travel
2. For each of the pairs of variables in question 1, state what happens to the value of the second variable when there is an increase in the value of the first variable.
3. For the graphs below, indicate whether the two variables graphed display a positive, negative or no relationship between their values.

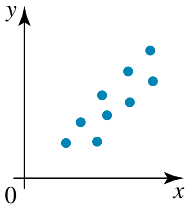
a.



b.



c.



Where appropriate use a spreadsheet to graph the scatterplots in the following questions.

4. **WE9** The following table shows the marks obtained by a group of 10 students in history and geography.

History	36	65	82	72	58	39	58	74	82	66
Geography	45	78	66	72	50	51	61	70	60	88

Display this information on a scatterplot.

5. The following table shows the maximum temperature each day, together with the number of people who attend the cinema that day.

Temperature (°C)	25	33	30	22	15	18	27	22	28	20
Number at cinema	256	184	190	312	458	401	200	357	312	423

Display the information on a scatterplot.

6. The following table shows the wages of 20 people and the amount of money they spend each week on entertainment.

Wages ($)	370	380	500	510	395	430	535	490	495	550
Amount spent on entertainment ($)	55	85	150	75	145	100	130	115	70	150
Wages ($)	810	460	475	520	530	475	610	780	350	460
Amount spent on entertainment ($)	220	50	100	150	140	160	90	130	40	50

Display this information on a scatterplot.

7. **WE10** The following table shows the marks obtained by nine students in English and History.

English	55	20	27	33	73	18	37	51	79
History	72	37	53	74	73	44	59	55	84

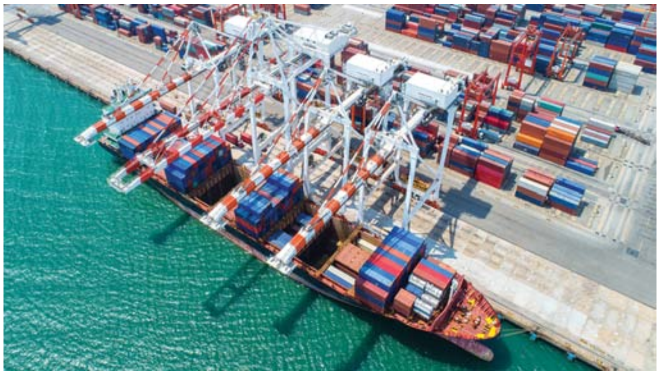
a. Display the information on a scatterplot.
b. Is there any relationship between the marks obtained in English and in History? If there does appear to be a relationship, is the relationship linear?

8. **WE11** The following table shows the daily temperature and the number of hot pies sold at the school canteen.

Temperature (°C)	24	32	28	23	16	14	26	20	29	21
Number of pies sold	56	20	24	60	84	120	70	95	36	63

a. Which is the explanatory variable?
b. Using Excel, construct a scatterplot showing the number of pies sold against temperature.
c. By examining the scatterplot, determine if there appears to be any relationship between the two variables and if the relationship appears to be linear.

9. Container ships arriving on a wharf are unloaded by work teams.



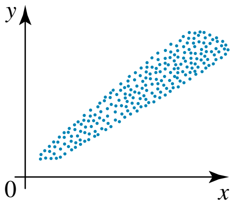
The following table shows the number of people in the work team and the time taken to unload the container ship.

Number in work team	15	18	12	19	22	21	17	16	18	20
Hours taken	20	16	25	15	14	13	18	20	17	14

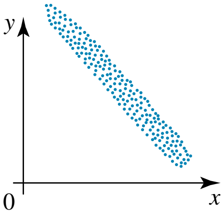
a. Display the information on a scatterplot.
b. Determine if there appears to be a relationship between the number of people in the work team and the time taken to unload the container ship. If there is a relationship, does the relationship appear to be linear?

10. **MC** Which of the following scatterplots does *not* display a linear relationship?

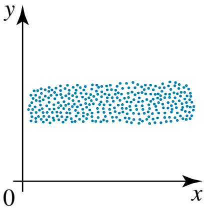
A.



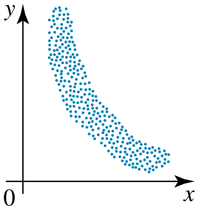
B.



C.

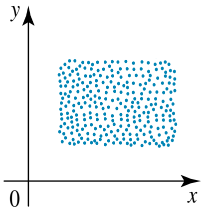


D.

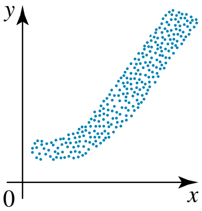


11. **MC** In which of the following is no relationship evident between the variables?

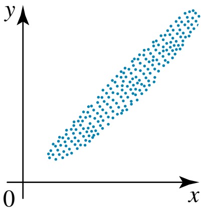
A.



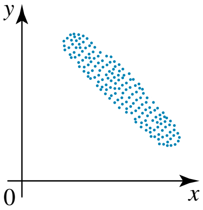
B.



C.

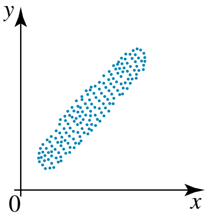


D.

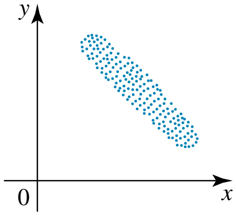


12. Give an example of a situation where the scatterplot may look like the ones below.

a.



b.



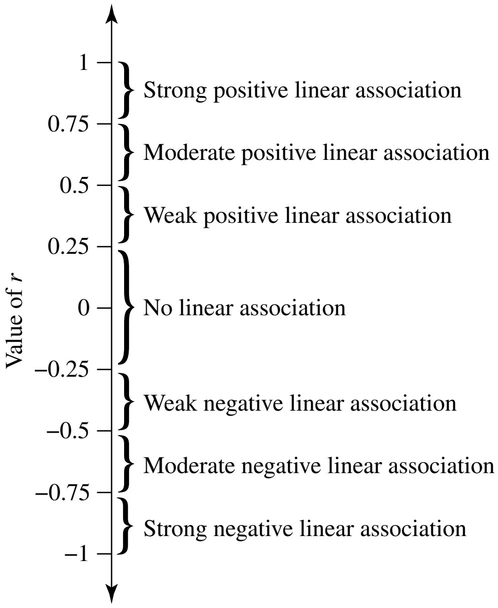
1.5 Pearson's correlation coefficient (r)

1.5.1 Calculating Pearson's correlation coefficient

The strength of a linear relationship can be observed from a scatterplot of the data. However, to determine exactly how strong this relationship is we can use Pearson's correlation coefficient, r, which measures the strength of a linear trend and associates it with a numerical value between -1 and $+1$.

A value of either -1 or $+1$ indicates a perfect linear correlation, while a result closer to zero indicates no correlation between the variables.

The diagram below describes the strength of the association between variables for a given value of the Pearson's correlation coefficient.



Pearson's correlation coefficient is calculated using the formula shown below.

Pearson's correlation coefficient

$$r = \frac{1}{n-1}\sum\left(\frac{x_i-\bar{x}}{s_x}\right)\left(\frac{y_i-\bar{y}}{s_y}\right), \text{ where}$$

n is the numbers of pieces of data in the data set
x_i is an x-value (explanatory variable)
y_i is a y-value (response variable)
s_x is the standard deviation of the x-values
s_y is the standard deviation of the y-values
$\bar{x}$ is the mean of the x-values
$\bar{y}$ is the mean of the y-values.

Note: The mean $\bar{x} = \frac{\sum x_i}{n}$ and standard deviation $s = \sqrt{\frac{\sum(x_i-\bar{x})^2}{n-1}}$ were studied in detail in Unit 2 of General Mathematics.

WORKED EXAMPLE 12

Consider the following set of data.

x	2	3	4	5	6	7	8
y	1.2	2.7	2.3	4.1	3.9	8.4	6.5

The means and standard deviations for these two variables are:

$$\bar{x} = 5,\ \bar{y} = 4.16,\ s_x = 2.16,\ s_y = 2.51$$

Complete the following table and hence calculate Pearson's correlation coefficient (r) and describe the strength of the association between the variables x and y.

$\frac{x_i-\bar{x}}{s_x}$	$\frac{y_i-\bar{y}}{s_y}$	$\left(\frac{x_i-\bar{x}}{s_x}\right)\left(\frac{y_i-\bar{y}}{s_y}\right)$
−1.39	−1.18	1.64
−0.46	−0.74	0.34
0.46	−0.10	−0.05
		$\sum\left(\frac{x_i-\bar{x}}{s_x}\right)\left(\frac{y_i-\bar{y}}{s_y}\right) =$

THINK

WRITE

1. Calculate $\frac{x_i - \bar{x}}{s_x}$ for all the missing values.
Calculate $\frac{y_i - \bar{y}}{s_y}$ for all the missing values.
Calculate $\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$ for all the missing values.

$\frac{x_i - \bar{x}}{s_x}$	$\frac{y_i - \bar{y}}{s_y}$	$\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$
−1.39	−1.18	1.64
−0.93	−0.58	0.54
−0.46	−0.74	0.34
0	−0.02	0
0.46	−0.10	−0.05
0.93	1.69	1.57
1.39	0.93	1.29
		$\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) = 5.33$

2. Write the formula for *r*.

$$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

3. Substitute the values for n and $\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$ into the formula and calculate.

$$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) = \frac{1}{6}(5.33) = 0.89$$

4. Write the answer.

$r = 0.89$
This suggests a strong positive association between the variables *x* and *y*.

Technology can be used to easily calculate Pearson’s correlation coefficient (*r*)

WORKED EXAMPLE 13

A hotel owner wished to determine if there is an association between the average weekly temperature and occupancy rates.

He recorded this information for 12 weeks.

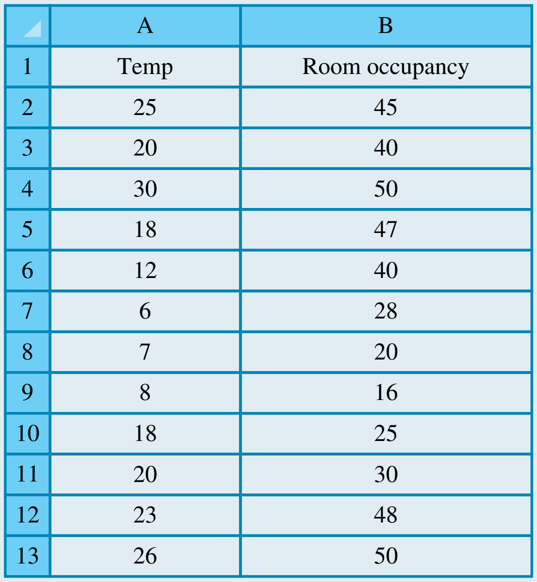
Temp (°C)	**25**	**20**	**30**	**18**	**12**	**6**	**7**	**8**	**18**	**20**	**23**	**26**
Room Occupancy	**45**	**40**	**50**	**47**	**40**	**28**	**20**	**16**	**25**	**30**	**48**	**50**

Using a spreadsheet, calculate *r* for the data below and describe the strength of the association.

THINK

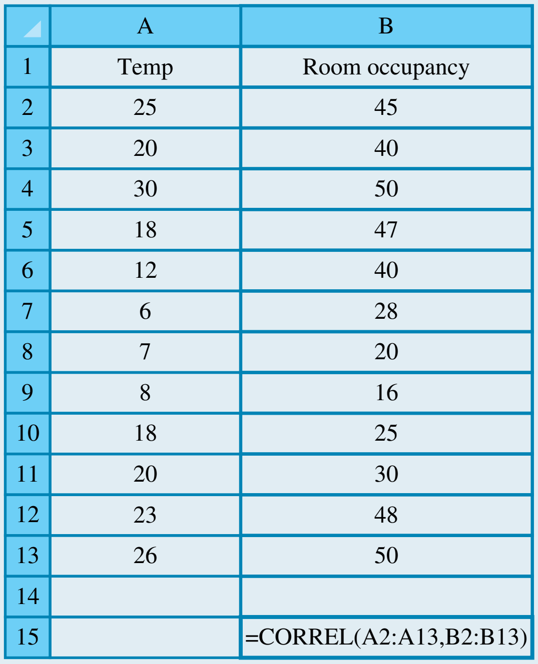
WRITE

1. Enter the data into two columns in a spreadsheet.



	A	B
1	Temp	Room occupancy
2	25	45
3	20	40
4	30	50
5	18	47
6	12	40
7	6	28
8	7	20
9	8	16
10	18	25
11	20	30
12	23	48
13	26	50

2. To calculate the correlation coefficient enter = CORREL(A2:A13, B2:B13) into an empty cell.



	A	B
1	Temp	Room occupancy
2	25	45
3	20	40
4	30	50
5	18	47
6	12	40
7	6	28
8	7	20
9	8	16
10	18	25
11	20	30
12	23	48
13	26	50
14		
15		=CORREL(A2:A13,B2:B13)

3. Press enter and the correlation coefficient will be displayed.	$r = 0.80$ This would suggest there is a strong positive correlation between the average temperature and the room occupancy each week.

on Resources

Interactivity Pearson's product-moment correlation coefficient and the coefficient of determination (int-6251)

studyon

Units 3 & 4 > Area 1 > Sequence 1 > Concept 4

The correlation coefficient (r) Summary screen and practice questions

Exercise 1.5 Pearson's correlation coefficient (*r*)

1. **WE12** Consider the following set of data.

x	1	3	5	7	8	9	11
y	12	11	14	10	17	16	18

The means and standard deviations for these two variables are:

$$\bar{x} = 6.29,\ \bar{y} = 14,\ s_x = 3.50,\ s_y = 3.11$$

Complete the following table and hence calculate Pearson's correlation coefficient (r) and describe the strength of the association between the variables x and y.

$\dfrac{x_i - \bar{x}}{s_x}$	$\dfrac{y_i - \bar{y}}{s_y}$	$\left(\dfrac{x_i - \bar{x}}{s_x}\right)\left(\dfrac{y_i - \bar{y}}{s_y}\right)$
−1.51	−0.64	0.97
−0.37	0	0
0.49	0.96	0.47
		$= \sum\left(\dfrac{x_i - \bar{x}}{s_x}\right)\left(\dfrac{y_i - \bar{y}}{s_y}\right)$

2. Consider the following set of data.

x	1	2	3	4	5	6
y	10	7	8	6	5	4

$$\bar{x} = 3.5,\ \bar{y} = 6.67,\ s_x = 1.87,\ s_y = 2.16$$

Complete the following table and hence calculate Pearson's correlation coefficient (r) and describe the strength of the association between the variables x and y.

$\dfrac{x_i - \bar{x}}{s_x}$	$\dfrac{y_i - \bar{y}}{s_y}$	$\left(\dfrac{x_i - \bar{x}}{s_x}\right)\left(\dfrac{y_i - \bar{y}}{s_y}\right)$
		$= \sum\left(\dfrac{x_i - \bar{x}}{s_x}\right)\left(\dfrac{y_i - \bar{y}}{s_y}\right)$

3. **WE13** A golf club recorded the average weekly temperature and the number of cold drinks sold per week to determine how often they should send the golf cart with cold drinks around the course.

Temp (°C)	36	32	28	26	30	24	19	25	33	35	37	34
No. of cold drinks sold	162	136	122	118	134	121	65	124	140	154	156	148

Using a spreadsheet, calculate r for the data above and describe the strength of the association.

4. Data on 15 people's shoe size and the length of their hair was collected.

Shoe size	6	8	7	8	9	6	7	12	8	9	10	12	7	9	11
Length of hair (cm)	9	14	12	1	7	8	5	22	15	8	18	4	5	9	3

Calculate the Pearson's correlation coefficient, *r*, for this data and comment on the strength of the relationship.

5. A class of Year 12 students was asked to record the amount of time in hours that they spent on a Practice Internal assessment and the mark out of 100 that they received for the assessment.

Time spent (hours)	2	0.5	1.5	2.5	0.25	2	2.5	2.5	2	0.5
Mark	72	52	76	82	36	73	84	80	74	48
Time spent (hours)	0.75	1.5	1	2	3	3.5	1	3	2.5	3
Mark	58	69	62	78	90	94	70	92	88	97

a. Determine the explanatory variable and response variable for this data and draw a scatterplot.
b. Does the scatterplot indicate a relationship between the variables? If so, what sort of relationship?
c. Calculate *r* and comment on the strength of the relationship.

6. A researcher who is investigating the proposition that 'tall mothers have tall sons' measures the heights of twelve mothers and the heights of their adult sons. The results are shown below.

Height of mother (cm)	185	155	171	169	170	175	158	156	168	169	179	173
Height of son (cm)	188	157	172	173	174	180	159	150	172	175	180	190

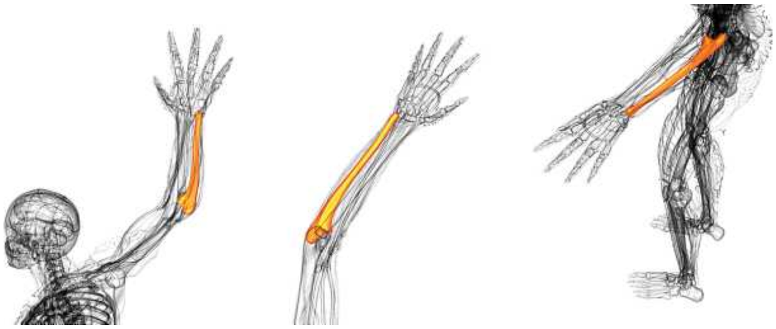
a. Draw a scatterplot of this data.
b. Calculate *r* for this data and use this to comment on the statement 'tall mothers have tall sons'.

7. The 10 am temperature (°C) and rainfall (mm) over two weeks during the month of April in a city is shown below.

Temperature (°C)	5	16	20	10	14	8	18	13	9	17	7	14	4	3
Rainfall (mm)	3	5	8	9	6	7	8	7	6	7	4	10	2	1

a. Which is the explanatory variable (to be put on the *x*-axis)?
b. Construct a scatterplot of rainfall against temperature.
c. Calculate *r* and comment on the strength of the relationship.

8. The heights and ulna (elbow to the centre of wrist bone) lengths of twelve Year 12 students from Clever High are shown below.



Ulna (cm)	26	28	25	25	24	23	25	26	25	27	25	26
Height (cm)	170	174	178	167	166	164	176	177	170	172	183	175

a. Which measurement is the explanatory variable (put on the *x*-axis)?
b. Construct a scatterplot of height against ulna lengths.
c. Calculate *r* and comment on the strength of the relationship.

9. The following table shows the number of Icy-poles sold and the maximum temperature of 10 days recorded by a shopkeeper.

Maximum temperature (°C)	14	10	22	26	30	20	18	12	8	6
Number of icy-poles sold	250	200	365	500	630	420	320	280	220	150

a. Plot the data upon a scatterplot.
b. Calculate *r* and comment on the strength of the relationship.

10. The following information was provided at the beginning of Subtopic 1.4.
The manager of a ski resort collects the following numerical data over 12 consecutive weekends at his resort.

Depth of snow (m)	0.5	0.8	2.1	3.6	1.4	1.5	1.8	2.7	3.2	2.4	2.6	1.7
Number of skiers	120	250	500	780	300	280	410	320	640	540	530	200

a. Draw a scatterplot of the data.
b. Calculate *r* for the data.

c. Use the information from parts **a** and **b** to answer the following.
  - **i.** Are visitor numbers related to depth of snow?
  - **ii.** If there is a relationship between visitor numbers and depth of snow, is it always true or is it just a guide? In other words, how strong is the relationship?
  - **iii.** How much confidence could be placed in the prediction?

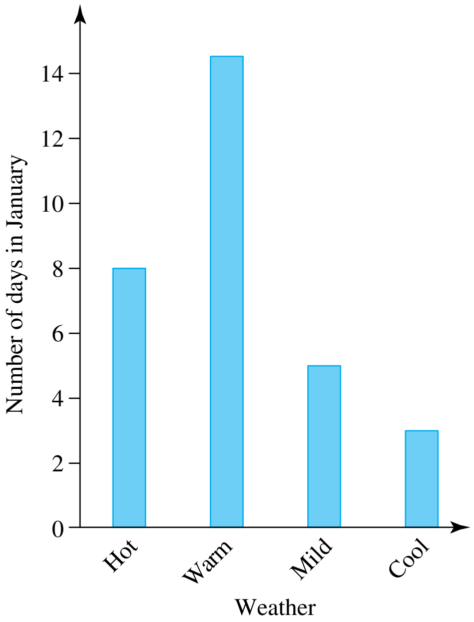
1.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** During a hybridisation experiment a scientist counted the tall and dwarf marigold plants. This is an example of
- **A.** categorical data.
- **B.** nominal data.
- **C.** discrete data.
- **D.** continuous data.

2. **MC** The graph shows the number of days of each weather type for the Gold Coast in January.
Choose the option below that best describes this data.
- **A.** Univariate and numerical data
- **B.** Univariate and categorical data
- **C.** Bivariate and categorical data
- **D.** Bivariate and numerical data



3. **MC** A researcher took 100 blood samples from a group of volunteers. He wanted to determine if there is a relationship between the amount of iron present in the blood and the amount of vitamin C present in the blood.
Choose the option below that describes the best method of displaying his results.
- **A.** Stem and leaf plot
- **B.** Line graph
- **C.** Two-way frequency table
- **D.** Scatterplot

4. MC The following two-way frequency table shows the composition of left- and right-handed students in Years 11 and 12 at a school.

	Left handers	**Right handers**
Year 12	15	105
Year 11	10	90

The percentage of all left-handers who are in Year 11 is closest to

A. 6.8%. **B.** 12.5%. **C.** 14.3%. **D.** 40%.

5. MC Using the data displayed in the two-way frequency table in question 4, the percentage of Year 12 students who are left-handed is closest to

A. 6.8%. **B.** 12.5%. **C.** 14.3%. **D.** 40%.

6. MC From the data displayed in the two-way frequency table in question 4, the difference between the percentage of all right-handers in Year 11 and in Year 12 is

A. 7.7%. **B.** 6.8%. **C.** 40.9%. **D.** 47.7%.

7. MC The following two-way frequency table shows the results of a trial on new metal detectors for aircraft. The metal detector scans a piece of hand luggage and lights up if metal is found.

	Test results		
	Accurate	**Not accurate**	**Total**
With metal	9	1	10
Without metal	87	3	90
Total	96	4	

Based on the above results, the chance of metal going undetected in a piece of hand luggage is

A. 1%. **B.** 10%. **C.** 25%. **D.** 75%.

8. MC A researcher administers different amounts of fertiliser to a number of trial plots of potato crop. She then measures the total mass of potatoes harvested from each plot. When drawing the scatterplot, the researcher should graph

A. mass of harvest on the *x*-axis because it is the explanatory variable, and amount of fertiliser on the *y*-axis because it is the response variable.

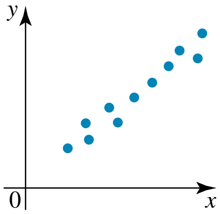
B. mass of harvest on the *y*-axis because it is the explanatory variable, and amount of fertiliser on the *x*-axis because it is the response variable.

C. mass of harvest on the *x*-axis because it is the response variable, and amount of fertiliser on the *y*-axis because it is the explanatory variable.

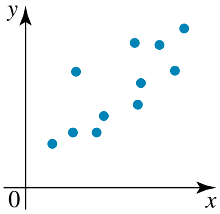
D. mass of harvest on the *y*-axis because it is the response variable, and amount of fertiliser on the *x*-axis because it is the explanatory variable.

9. MC Which of the following graphs best depicts a strong negative relationship between the two variables?

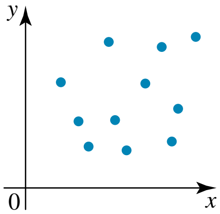
A.



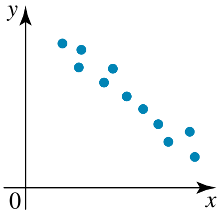
B.

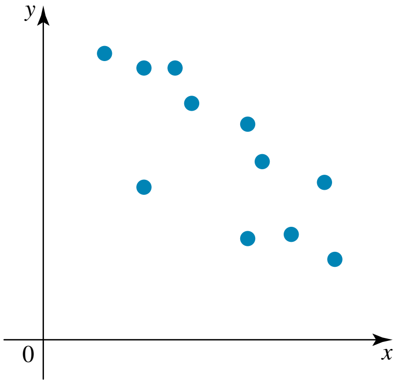


C.



D.





10. **MC** What type of relationship is shown by the graph?
   - **A.** Strong positive relationship
   - **B.** Moderate positive relationship
   - **C.** Moderate negative relationship
   - **D.** Strong negative relationship
11. **MC** What type of correlation does an r-value of 0.64 indicate?
   - **A.** Strong, positive correlation
   - **B.** Strong, negative correlation
   - **C.** Moderate, positive correlation
   - **D.** Moderate, negative correlation
12. **MC** A gardener tracks a correlation coefficient of 0.79 between the growth rate of his trees and the amount of fertiliser used. What can the gardener conclude from this result?
   - **A.** The growth rate of the trees is influenced by the amount of fertiliser used.
   - **B.** An increase in tree growth increases the use of fertiliser.
   - **C.** There is no correlation between the growth rate of the trees and the amount of fertiliser used.
   - **D.** The growth rate of the trees influences the quality of the fertiliser used.

Complex familiar

13. A reading test for people with dyslexia is given and the results are shown in the following two-way frequency table.

	Test results		
	Accurate	**Not accurate**	**Total**
With dyslexia	39	1	40
Without dyslexia	85	5	90
Total	124	6	

   - **a.** How many people were tested?
   - **b.** What percentage of people tested positive for dyslexia?
   - **c.** Based on the above results, if a person with dyslexia takes the test, what is the percentage chance that they will be accurately diagnosed?

14. The following table shows the number of sick days taken by ten employees and relates this to the number of children that they have.

Number of children	1	0	3	2	2	4	6	0	1	2
Number of sick days	5	3	10	8	4	12	12	0	1	5

   - **a.** Show this information on a scatterplot.
   - **b.** Does a relationship appear to exist between the number of sick days taken and the number of children they have? If so, is the relationship linear?

15. The following table shows the number of cars and number of televisions in each household.

Number of cars	1	1	2	2	2	3	1	0	1	2
Number of televisions	2	1	1	2	0	1	4	3	1	1

a. Show this information on a scatterplot.
b. Determine if a relationship appears to exist between the number of televisions in each household and the number of cars they have. If so, is the relationship linear?

16. A potato farmer records the yield in kilograms and the length in metres of 10 commercial potato plots as shown.

Length (m)	10	6	16	2	13	7	4	12	5	8
Yield (kg)	220	250	400	25	500	430	120	350	310	280

a. Identify the explanatory variable and the response variable.
b. Construct a scatterplot to illustrate this data.
c. State the type of relationship shown by the scatterplot.
d. If the scatterplot shows a suitable relationship, calculate r using technology and comment on the strength of the relationship in terms of strength, direction and form.

Complex unfamiliar

17. A medical test screens 200 people for a virus. A positive test result indicates that the patient has the virus.

- Of 50 people known to have the virus, the test produced 48 positive results.
- Of the remainder who were known not to have the virus, the test produced one positive result.

a. Use the above information to complete the following two-way frequency table.

	Test results		
	Accurate	**Not accurate**	**Total**
With virus			
Without virus			
Total			

b. From the completed table in part a, complete the following sentences.
The percentage of people with the virus who returned a positive result was ________.
The probability of returning a positive test when not infected was ___________.

18. The results of a lie detector test are given below.

- Of 80 people who are known to be telling the truth, the lie detector indicates that three are lying.
- Of 20 people known to be lying, the lie detector indicates that 17 are lying.

a. Display this information in a two-way frequency table.
b. For this particular lie detector test, the acceptable % error is 5%. Does this test pass the acceptable level of error?

19. Sixty-seven primary and 47 secondary school students were asked about their attitude towards the number of school holidays which should be given.

They were asked whether there should be more, fewer or the same number. Five primary students and 2 secondary students wanted fewer holidays, 29 primary and 9 secondary students thought that they had enough holidays (that is, they chose the same number) and the rest thought that they needed to be given more holidays.

a. Form a two-way frequency table with reference to primary and secondary school percentages.

b. Use the information in the table to determine if there is an association between the age of the student (primary or secondary) and their view about holidays, use percentages to justify your answer.

20. The following table shows weight and height for a group of men and women.

Height (cm)	150	160	170	180	190	200
Weight (kg)	50	57	65	72	81	90

a. Construct a scatterplot showing this data.

b. If $\overline{x} = 175$, $s_x = 18.71$, $\overline{y} = 69.17$ and $s_y = 14.93$, without the use of technology, calculate the value of r and draw a suitable conclusion about the relationship between heights and weights for this sample.

studyon

Units 3 & 4 Sit exam

Answers

1 Identifying and describing associations between two variables

Exercise 1.2 Bivariate data

1. a. Numerical b. Categorical c. Numerical
   d. Categorical e. Numerical f. Numerical
2. a. Continuous b. Discrete c. Continuous
   d. Continuous e. Discrete
3. a. Numerical and discrete
   b. Categorical
   c. Categorical
   d. Numerical and continuous
   e. Numerical and continuous
   f. Numerical and discrete
4. Categorical, ordinal
5. Categorical, ordinal
6. Numerical and discrete
7. B
8. Categorical
9. Numerical and continuous
10. a. 4 b. 3 and 5 c. 6 d. 1 and 2
11. A
12. Possible questions could include:
    What is the average age of our customers? (univariate)
    Is there a relationship between amount of money spent and time spent on the website? (bivariate)
    Is there a relationship between income and postcode? (bivariate)
    Do more males or females visit the website? (univariate)
    What is the average amount of time customers spend on the website? (univariate)
    Is there a relationship between the customer's postcode and the amount of time spent on the website? (bivariate)

Exercise 1.3 Two-way frequency tables

1.

	Test results		
	Accurate	Not accurate	Total
With virus	98	2	100
Without virus	388	12	400
Total	486	14	500

2.

	Test results		
	Accurate	Not accurate	Total
Telling truth	777	23	800
Telling lies	156	44	200
Total	933	67	1000

3. a. 1000 b. 75 c. 96.7%
4. a. 200 b. 44 c. 90.9% d. 5.1% e. 94%
   f. Sample response: The alarm appears to be reliable in detecting intruders and activating the alarm (80+%).
5. C
6. D
7. A
8. a.

	Test results		
	Accurate	Not accurate	Total
Bags with prohibited items	48	2	50
Bags with no prohibited items	145	5	150
Total	193	7	200

   b. i. 96% ii. 3.3% iii. 4% iv. 96.5%
9. a. ii b. ii c. ii
10. a. 120 b. 20 c. 23
    d. $P(\text{avoids detection}) = \frac{2}{20} \times 100$
    $= 10\%$
11. a. 350
    b. 45
    c. $P(\text{break in taking place}) = \frac{43}{45} \times 100$
    $= 95.6\%$
12. a.

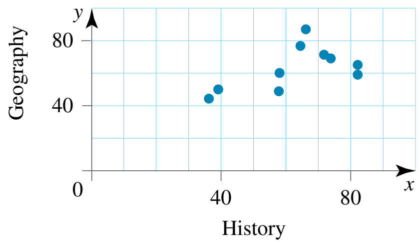
	Detected	Not detected	Total
Gene present	95	5	100
Not present	37	863	900
Total	132	868	1000

    b. $P(\text{gene detected}) = \frac{95}{100} \times 100$
    $= 95\%$
    c. $P(\text{has gene}) = \frac{95}{132} \times 100$
    $= 72.0\%$
    d. $P(\text{correct diagnosis}) = \frac{958}{1000} \times 100$
    $= 95.8\%$
13. a. 56%
    b. Yes, if you consider the 20 to 35-year-old age group, the percentages, 41.1%, 30.3% and 20.5% show a decrease in preferred type from action being the most popular to drama the least popular. This is reversed for the 60 years and over group.

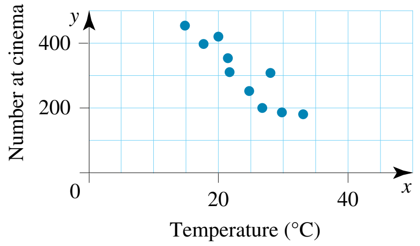
Exercise 1.4 Scatterplots

1. a. Height b. Distance travelled
   c. Temperature d. Reaction time
   e. Test results f. Overtime pay
   g. Value of car h. Travelling time
2. a. Increase b. Increase
   c. Decrease d. Decrease
   e. Increase f. Increase
   g. Decrease h. Decrease
3. a. No relationship b. Negative relationship
   c. Positive relationship

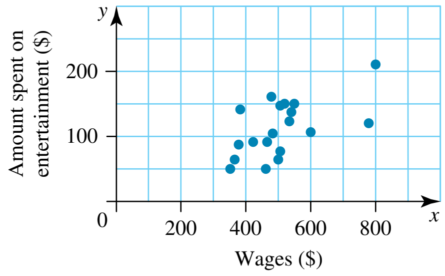
4.



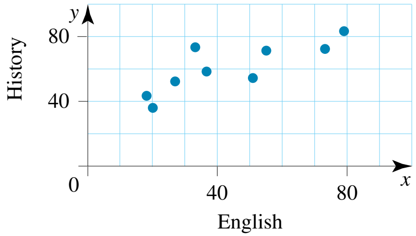
5.



6.



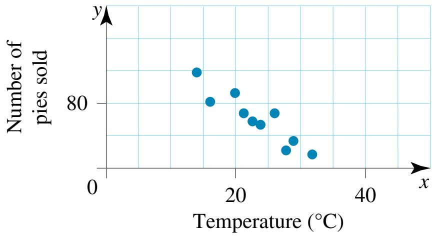
7. a.



b. The greater the English mark, the greater the history mark, generally. However, as the points on the scatterplot do not form a straight line, the relationship is non-linear.

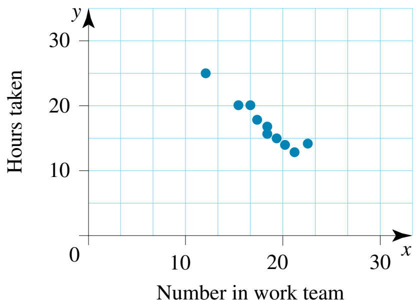
8. a. Temperature

b.



c. The greater the temperature, the fewer pies are sold. The points on the scatterplot approximate a straight line and so the relationship can be said to be linear.

9. a.



b. More workers on the team reduces the amount of time taken to unload the ship and, as the points on the scatterplot approximate a straight line, the relationship is linear.

10. D

11. A

12. Sample response:

a. Hours spent practicing basketball and points scored

b. The density of an object and its volume

Exercise 1.5 Pearson's correlation coefficient (*r*)

1.

$\dfrac{x_i - \bar{x}}{s_x}$	$\dfrac{y_i - \bar{y}}{s_y}$	$\left(\dfrac{x_i - \bar{x}}{s_x}\right)\left(\dfrac{y_i - \bar{y}}{s_y}\right)$
−1.51	−0.64	0.97
−0.94	−0.96	0.90
−0.37	0	0
0.20	−1.29	−0.26
0.49	0.96	0.47
0.77	0.64	0.49
1.35	1.29	1.74

$$\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) = 4.31$$

$r = 0.72$

This suggests a moderate positive association between the variables x and y.

2.

$\dfrac{x_i - \bar{x}}{s_x}$	$\dfrac{y_i - \bar{y}}{s_y}$	$\left(\dfrac{x_i - \bar{x}}{s_x}\right)\left(\dfrac{y_i - \bar{y}}{s_y}\right)$
−1.34	1.54	−2.06
−0.80	0.15	−0.12
−0.27	0.62	−0.17
0.27	−0.31	−0.08
0.80	−0.77	−0.62
1.34	−1.24	−1.66

$$\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) = -4.71$$

$r = -0.94$

This suggests a strong negative association between the variables x and y.

3. $r = 0.9355$

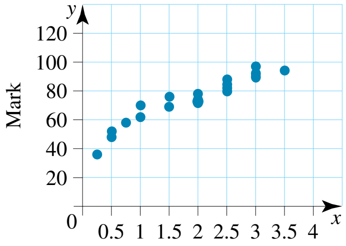
This suggests a strong positive association between the variables *Temperature* and *Number of cold drinks sold*. As the temperature increases so does the number of cold drinks sold.

4. $r = 0.2055$

You would not expect to find a relationship between shoe size and hair length and as $r = 0.2055$, this indicates that there is no relationship between the two variables.

5. a. Time spent – explanatory variable, Mark – response variable

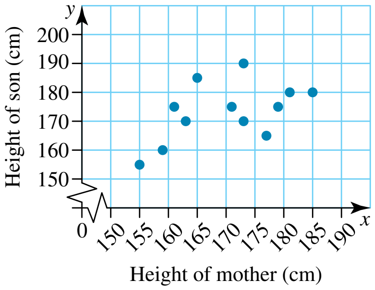
Mark vs Time spent



b. The scatterplot indicates a linear relationship.

c. $r = 0.95$. This indicates a strong positive relationship between the time spent on the assignment and the mark achieved. The more time spent on the assessment, the higher the mark they achieved.

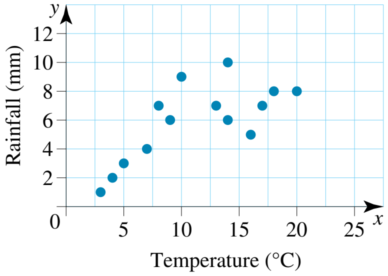
6. a. **Height of son vs Height of mother**



b. $r = 0.92$. This indicates a strong positive relationship between the height of the mother and the height of the son. It can be said that tall mothers tend to have tall sons based on this data.

7. a. Temperature – explanatory variable, rainfall – response variable. (In this case the explanatory and response variables could be interchangeable.)

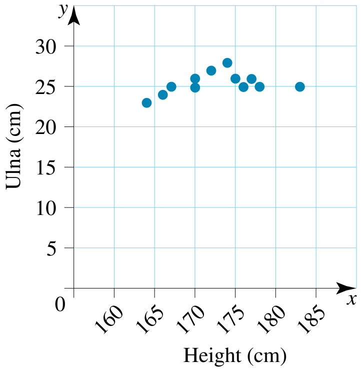
b. **Rainfall vs Temperature**



c. $r = 0.72$. This indicates a moderate positive relationship between the temperature and the amount of rainfall.

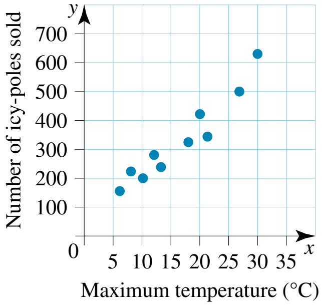
8. a. Height is the explanatory variable

b. **Ulna vs Height**



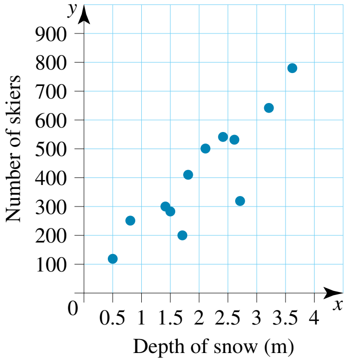
c. $r = 0.38$. This indicates a weak positive relationship between the Year 12 students' height and length of their ulna.

9. a. **Number of icy-poles sold vs Maximum temperature**



b. $r = 0.96$. This indicates a strong positive relationship between the maximum temperature and the number of icy-poles sold. The hotter the day, the more icy-poles sold.

10. a. **Number of skiers vs Depth of snow**



b. $r = 0.8840$

c. **i.** As the depth of snow increases there appears to be greater numbers of skiers overall.

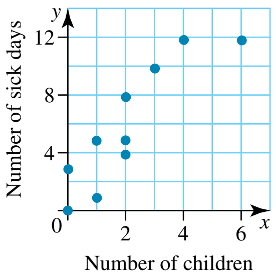
ii. There is a strong positive linear association between the number of skiers at the resort and the depth of snow for those 12 weekends.

iii. Within these times, one could predict more visitors when the snowfalls are heavy.

1.6 Review: exam practice

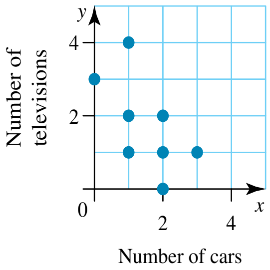
1. C
2. B
3. D
4. D
5. B
6. A
7. B
8. D
9. D
10. C
11. C
12. A
13. **a.** 130
    b. 33.8%
    c. 97.5%

14. a.



b. There appears to be a positive relationship which is linear.

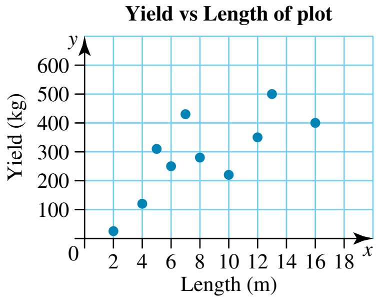
15. a.



b. There is no apparent relationship.

16. a. Explanatory variable – length
Response variable – yield

b.



c. The scatterplot indicates a positive linear relationship.

d. Using Excel, $r = 0.74$. There is a moderate positive relationship between the length of the plot and the yield of potatoes.

17. a.

	Test results		
	Accurate	**Not accurate**	**Total**
With virus	48	2	50
Without virus	149	1	150
Total	197	3	200

b. The percentage of people with the virus who returned a positive result was 96%.
The probability of returning a positive test when not infected was 0.67%.

18. a.

	Test results		
	Accurate	**Not accurate**	**Total**
Telling truth	77	3	80
Telling lies	17	3	20
Total	94	6	100

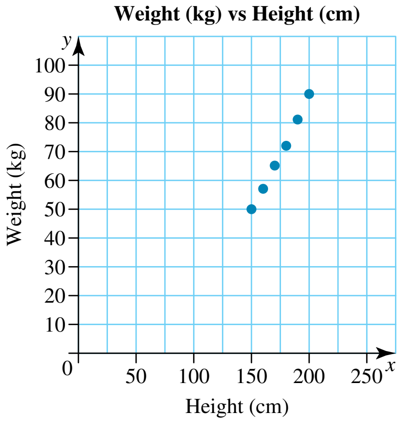
b. No

19. a.

Attitude	Primary	Secondary
Fewer	7.5%	4.3%
Same	43.3%	19.1%
More	49.2%	76.6%
Total	100%	100%

b. Secondary students were much keener on having more holidays than primary students as the percentage of secondary students who desired more holidays was 76.6% compared to primary students who desired more holidays was 49.2%.

20. a.



b. Let height be x and weight be y.

$\frac{x_i - \bar{x}}{s_x}$	$\frac{y_i - \bar{y}}{s_y}$	$\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$
−1.34	−1.28	1.72
−0.80	−0.82	0.66
−0.27	−0.28	0.08
0.27	0.19	0.05
0.80	0.79	0.63
1.34	1.39	1.86
		$\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) = 5$

$$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) = \frac{1}{6-1} \times 5 = 1$$

There is a perfect positive linear correlation between heights and weights for this sample.

2 Fitting a linear model to numerical data

2.1 Overview

A mathematical model uses mathematical concepts and language to demonstrate how something in the real world works. These models can then be used to investigate an aspect of the natural world or society without the researcher being present in the situation. A very simple mathematical model is the use of the formula, $V = l \times w \times h$ which describes the volume of a rectangular box. Rather than creating many boxes, and measuring them to determine their volume, this formula can be used to calculate the volume of a rectangular box of any dimensions. A more complicated model, for example the formula required to send a rocket into space, requires information about the energy needed to break through gravity, which depends on the size of the rocket, the starting and ending points of the journey, and the energy provided by the preferred fuel. A trial and error model to determine the variables would compromise safety and waste valuable resources. Mathematical models are first determined and are used to ensure the greatest chance of success.

A statistical model is a particular type of mathematical model. It is created by collecting data from a sample of the population that is under investigation. It then uses this data to create an idealised way of predicting how any future data will behave. The manufacturer of a new running shoe wants to determine the best tread thickness for maximum support. Rather than spending years measuring tread thicknesses and the extent of running injuries across the world, the researchers would take a sample and use the data from the sample to make predictions and, ultimately, designs.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

2.2 Review of the general equation of a straight line

2.2.1 Linear relationships

When a line is formed by a set of points on a **Cartesian plane**, there is a consistent relationship between the x-coordinates and the y-coordinates of the points on the line. If the line formed is a straight line, there is a consistent linear relationship between the x-coordinates and the y-coordinates of the points on the line. Linear relationships can be identified by using a table of values, plotting points or looking at an equation.

The general form of a linear equation is $y = mx + c$.

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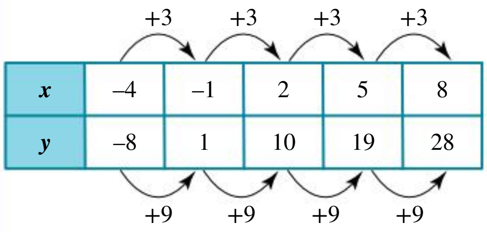
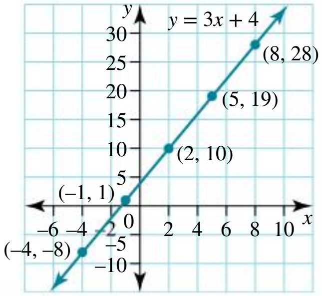
<td> </td>
<td>$y = 3x + 4$</td>
<td> </td>
</tr>
<tr>
<td>Each variable has a constant difference.</td>
<td>The equation can be written in the form $y = mx + c$.</td>
<td>The points form a straight line.</td>
</tr>
</table>

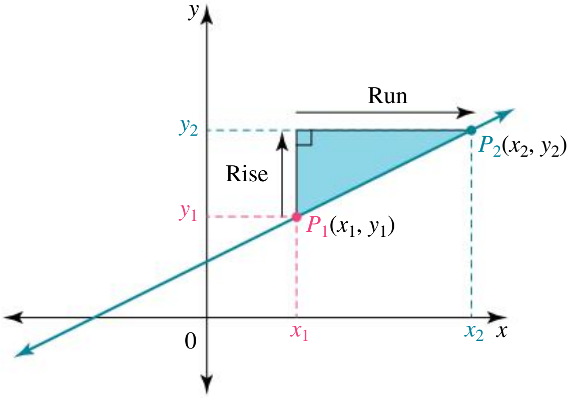
2.2.2 Linear relationships

The **gradient**, m, of the line is a measure of the slope or steepness of the line. The gradient is also a measure of the rate of change in y with respect to x. If the gradient of a line is 3 ($m = 3$), this means that for each increase of 1 in the x-value, there is a corresponding increase of 3 in the y-value.

$m > 0$	$m < 0$	$m = 0$	m is undefined.
$y = mx + c$	$y = mx + c$	Equations: $y = k$ where k is a constant.	Equations: $x = k$ where k is a constant.
			

The gradient of a line can be determined from any two points on the line.

$$\text{Gradient, } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$



Subscripts are used to show which points the coordinates belong to. For example, the x- and y-coordinates of point 1 are (x_1, y_1); the x- and y-coordinates of point 2 are (x_2, y_2).

WORKED EXAMPLE 1

Calculate the gradient of the line that passes through the points $(3, -2)$ and $(-4, 6)$.

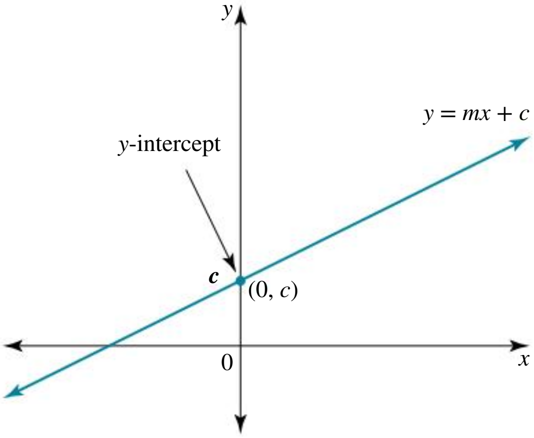
THINK	WRITE
1. Let one point be point 1 and the other point be point 2. Record their *x*- and *y*-coordinates.	**a.** Let: $(x_1, y_1) = (3, -2)$ $(x_2, y_2) = (-4, 6)$
2. Substitute the coordinates into the gradient formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ and simplify.	**b.** $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{6 - -2}{-4 - 3}$ $= \frac{6 + 2}{-4 - 3}$ $= \frac{8}{-7}$ $= -1\frac{1}{7}$

2.2.3 The *y*-intercept

The *y*-intercept, *c*, of the line is the point where the line intersects the *y*-axis (the line $x = 0$).

The value of the *y*-intercept can be determined by any of these methods:

- looking at the graph of the line and determining the point at which the line crosses the *y*-axis
- writing the equation of the line in the form of $y = mx + c$ and identifying the constant value *c*
- substituting $x = 0$ into the equation of the line, since the *y*-axis is also the line $x = 0$ and hence the *x*-coordinate of any *y*-intercept is 0.



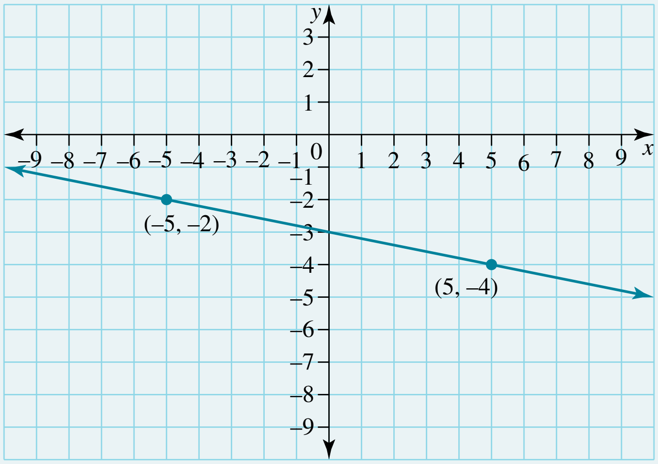
2.2.4 Determining the equation of a line using two points

The coordinates of two points on a line are all that is needed to determine the equation of a line. To determine the equation of the line, the value of both the gradient and the *y*-intercept are required.

- Calculate the gradient using the two points.
- Use the general form of the equation, the coordinates of one point and the gradient to calculate the value of the *y*-intercept.
- Write the equation of the line in the form $y = mx + c$

WORKED EXAMPLE 2

Determine the equation of the line that passes through the points $(-5, -2)$ and $(5, -4)$.



THINK	WRITE
1. To determine the equation of the line, both the gradient and the y-intercept are required. Calculate the gradient first. Let one point be point 1 and the other point be point 2. Record their x- and y-coordinates.	**a.** Let: $(x_1, y_1) = (-5, -2)$ $(x_2, y_2) = (5, -4)$
2. Substitute the coordinates into the gradient formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ and simplify.	**b.** $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{-4 - -2}{5 - -5}$ $= \frac{-4 + 2}{5 + 5}$ $= -\frac{2}{10}$ $= -\frac{1}{5}$
3. Write the general equation of a straight line. Substitute the value of the gradient m, into the formula. To find the value of the y-intercept, c, substitute the coordinates of one of the points into the equation as the values for x and y.	**c.** $y = mx + c$ Let $m = -\frac{1}{5}$ $y = -\frac{1}{5}x + c$ Let $(x, y) = (5, -4)$ $-4 = -\frac{1}{5}(5) + c$ $-4 = -1 + c$ $c = -3$
4. Write the equation of the line.	**d.** $y = -\frac{1}{5}x - 3$ or $y = \frac{-x - 15}{5}$

WORKED EXAMPLE 3

For each of the linear equations below:

i. **state the gradient and y-intercept**

ii. **sketch the graph of the equation.**

a. $y = 4x - 11$

b. $y = -4x$

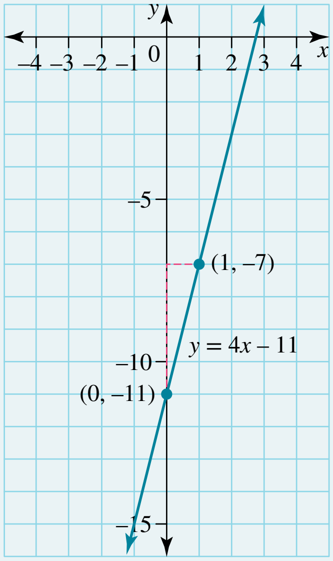
THINK

a. i. Compare the equation given with the general form of a linear equation: $y = mx + c$. The coefficient of x is m (the gradient), and the constant c is the y-intercept.

ii. Construct a set of axes and mark the position of the y-intercept.
The y-intercept is -11, as shown in blue.
The gradient is 4, so $\frac{rise}{run} = \frac{4}{1}$.
From the y-intercept, rise 4 and run 1, then mark in a second point $(1, -7)$, as shown in pink.
The two points can now be connected with a straight line.
Write the equation next to the line.

WRITE

a. For $y = 4x - 11$
Gradient $(m) = 4$
y-intercept $(c) = -11$.



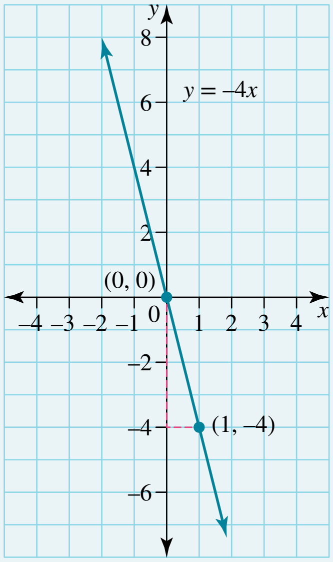
THINK

b. i. Compare the equation given with the general form of a linear equation: $y = mx + c$. Identify m as the gradient and c as the y-intercept.

ii. Construct a set of axes.
Mark in the position of the y-intercept at 0, as shown in blue. The gradient is -4, so $\frac{rise}{run} = -\frac{4}{1}$.
From the y-intercept, rise -4 and run 1, then mark in a second point $(1, -4)$, as shown in pink.
The two points can now be connected with a straight line to form the graph.
Write the equation next to the line.

WRITE

b. For $y = -4x$ or $(y = -4x + 0)$
Gradient $(m) = -4$ and
y-intercept $(c) = 0$.



on Resources

Interactivities: Linear graphs (int-6484)
Equations of straight lines (int-6485)

studyon

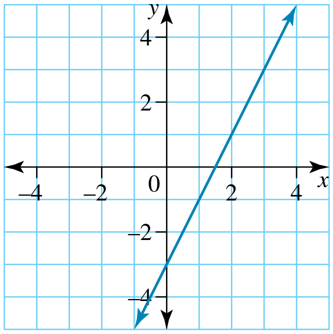
Units 3 & 4 › Area 1 › Sequence 2 › Concept 1

The equation of a straight line Summary screen and practice questions

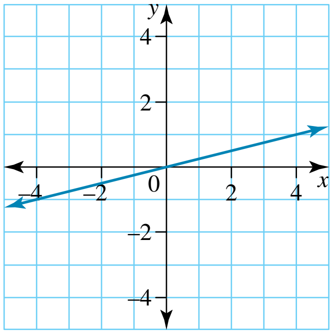
Exercise 2.2 Review of the general equation of a straight line

1. In your own words, explain:
   a. the general form of a straight line
   b. what the letters represent in the general form of a straight line, $y = mx + c$
2. State the value of m and c in each of the following linear relationships.
   a. $y = -5x + 4$
   b. $y = 3x + 11$
   c. $y = -6x$
   d. $y = \frac{5}{2}x + 5$
   e. $y = \frac{2}{3}x - 6$
   f. $y = 2x - 1$
3. Complete the sentences below.
   a. The equation $y = 3x - 1$ has a gradient of ____ units. This means that for every increase of 1 unit in the horizontal direction there is an increase of ___ units in the vertical direction.
   b. The equation $y = -x + 1$ has a gradient of ____ units. This means that for every increase of ___ unit in the horizontal direction there is an increase of ___ units in the vertical direction.
   c. The equation $y = \frac{1}{3}x - 1$ has a gradient of ____ units. This means that for every increase of ___ unit in the horizontal direction there is an increase of ___ units in the vertical direction.
4. **WE1** Calculate the gradient between each pair of points for each of the following.
   a. $(3, 4), (1, 0)$
   b. $(1, 3), (4, 6)$
   c. $(-2, 4), (-4, 2)$
   d. $(6, 3), (2, 5)$
   e. $(3, -1), (-1, 0)$
   f. $(6, 8), (-3, -4)$
   g. $(-2, -3), (-5, 1)$
   h. $(-3, -6), (-10, -4)$
   i. $(2, 5), (-1, 5)$
5. Write the linear relationship for the line with the following properties:
   a. gradient $= 2$ y–intercept $= -2$
   b. gradient $= -2$ y–intercept $= -1$
   c. gradient $= -\frac{3}{8}$ y–intercept $= 2$
   d. gradient $= 0$ y–intercept $= 4$
   e. gradient $= 1$ y–intercept $= 0$
   f. gradient $= -0.25$ y–intercept $= -4$
6. **WE2** Determine the equation of the straight lines that join each of the following pairs of points.
   a. $(3, 4), (-2, -3)$
   b. $(4, 6), (-5, 1)$
   c. $(-2, 4), (-4, 2)$
   d. $(6, 3), (1, 3)$
   e. $(-1, 0), (-10, -4)$
   f. $(-3, -4), (2, 5)$
7. For each of the linear graphs shown:
   i. state the y-intercept
   ii. calculate the gradient
   iii. write a linear relationship to describe the graph.

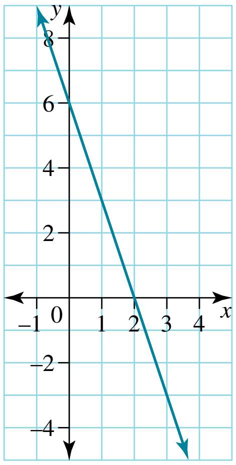
a.



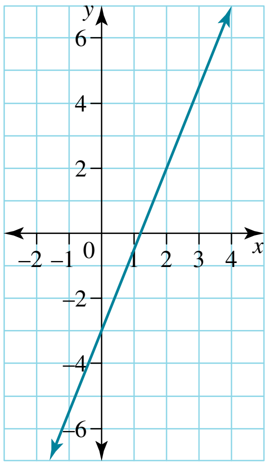
b.



c.



d.



8. **WE3** For each of the linear relationships below:

- **i.** state the gradient and the y-intercept
- **ii.** sketch the graph of the equation.

a. $y = 2x + 5$ **b.** $y = -4x - 3$ **c.** $y = -3x + 5$

d. $y = \frac{1}{2}x - 3$ **e.** $y = -\frac{2}{3}x + 3$ **f.** $y = \frac{4}{5}x$

9. Construct graphs for the linear relationships in question **2**.

10. **a.** If t represents the time in hours and C represents cost (\$), construct a table of values for 0–3 hours for the cost of playing tenpin bowling at a new alley.

Save \$\$\$ with Supa-Bowl!!!
NEW Tenpin Bowling Alley
Shoe rental just \$2 (fixed fee)
Rent a lane for ONLY \$6/hour!

b. Use your table of values to plot a graph of time versus cost. [*Hint:* Ensure your time axis (horizontal axis) extends to 6 hours and your cost axis (vertical axis) extends to \$40.

c. Calculate the gradient.

d. Write a linear equation to describe the relationship between cost and time.

e. Use your linear equation from part **e** to calculate the cost of a 5 hour tournament.

f. Use your graph to check your answer to part **f**.

11. For the equation $y = 4x + 1$, substitute the following values for x to calculate the corresponding values for y.

a. $x = 0$ **b.** $x = -2$ **c.** $x = 6$ **d.** $x = 10$

e. $x = -7$ **f.** $x = -5$ **g.** $x = 4.2$ **h.** $x = \frac{1}{2}$

12. For the equation $y = 4x + 1$, substitute the following values for y to calculate the corresponding values for x.

a. $y = 0$ **b.** $y = 9$ **c.** $y = -11$ **d.** $y = 19$

e. $y = 3$ **f.** $y = 0.6$ **g.** $y = -20$ **h.** $y = 2.5$

13. The equation $C = 3.5g + 55$ represents the cost of hiring online computer games from a gaming club, where C is the cost of hiring g games.

a. How much will it cost for the year to hire:

- **i.** 1 game a week for the year (52 weeks in a year)?
- **ii.** 10 games a month for the year (12 months in a year)?

b. The cost for hire is generated by an annual membership fee plus a small hire fee for each game. How much is:

- **i.** the annual membership fee? ($g = 0$)
- **ii.** the hire fee per game?

c. If a teenager budgets on spending \$300 on games for the year, how many games will this hire?

d. When considering the annual membership fee, what is the real cost of hiring a game, given the number of games from part **c** that the teenager intends to hire?

e. The cost of hiring a new game at a regular online subscription service is \$5. If the teenager budgets on spending \$300, will being a member of the club save money in the long run?

14. **MC** A family dental fund charges an annual membership fee of \$200 and then \$20 per dental appointment.

a. Which of the following equations best represents the cost of the fund in terms of dental appointments, if C is the cost and d is the number of dental appointments in the year?

A. $C = 200$ **B.** $C = 200 + 20d$

C. $C = 20d$ **D.** $C = 200d + 20$

b. What would be the cost over the year for the following families if on average each member of the family visited the dentist 2 times in a year?

i. A family of 2 **ii.** A family of 4 **iii.** A family of 10

c. If the average cost of a dental appointment is \$60 without a dental plan, which of the families in part **b** will have saved money by being in the fund? How much will they have saved?

d. If a family paid \$340 to the dental fund for the year, how many dental appointments did they have?

2.3 Fitting a least-squares line to data

2.3.1 Lines of best fit

In Chapter 1 scatterplots were constructed from raw data and studied to determine if a linear association existed between the two variables.

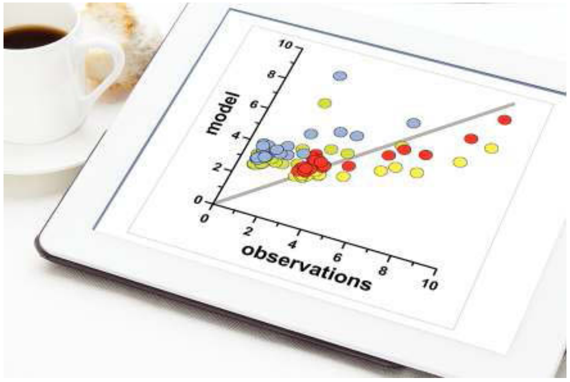
If the points on a scatterplot appear to be distributed in a linear pattern, a straight line can be drawn through the data.

A **line of best fit** is the straight line that is positioned as close as possible to all the data points, that is, the average distance between the data points and the line is minimised. It is used to generalise the relationship between two variables.

There are a number of ways to draw a line of best fit.

This General Mathematics course focuses on the least-squares regression line as a mathematical means of constructing a line of best fit.

2.3.2 The least-squares regression line



Least-squares regression involves an exact mathematical approach to fitting a line of best fit to bivariate data that shows a strong linear relationship.

This line minimises the vertical distances between the data points and the line of best fit. It is called the least-squares regression line because if we took the squares of these vertical distances, this line would represent the smallest possible sum of all these squares.

The equation for the least-squares regression line takes the form: $y = a + bx$, where y is the response (dependent) variable, x is the explanatory (independent) variable, b is the gradient or slope of the line and a is the y-intercept.

Technology can be used to calculate the equation of the least-squares regression line.

The equation of a least-squares regression line

To determine the equation of the least-squares regression line, the following summary data is required:

$\bar{x}$ — the mean of the explanatory variable (x-variable)
$\bar{y}$ — the mean of the response variable (y-variable)
s_x — the standard deviation of the explanatory variable
s_y — the standard deviation of the response variable
r — Pearson's correlation coefficient.

The general form of the least-squares regression line is:

$$y = a + bx$$

where:

the slope of the regression line is $b = \dfrac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = r\dfrac{s_y}{s_x}$

the y-intercept of the regression line is $a = \bar{y} - b\bar{x}$.

WORKED EXAMPLE 4

A study to find a relationship between the height of men and the height of their female partner revealed the following details.

- **Mean height of the men: 180 cm**
- **Mean height of the female partners: 169 cm**
- **Standard deviation of the heights of the men: 5.3 cm**
- **Standard deviation of the heights of the female partners: 4.8 cm**
- **Correlation coefficient: $r = 0.85$**

The form of the least-squares regression line is:

Height of female partner $= b \times$ height of man $+ a$

a. **Which variable is the response variable?**

b. **Calculate the value of b for the regression line (to 2 decimal places).**

c. **Calculate the value of a for the regression line (to 2 decimal places).**

d. **Use the equation of the regression line to predict the height of a female whose male partner is 195 cm tall (to nearest cm).**

THINK	WRITE
a. Recall that the response variable is the subject of the equation in $y = a + bx$ form; that is, y.	**a.** The response variable is the height of the female.
b. 1. The value of b is the gradient of the regression line. Write the formula and state the required values.	**b.** $b = r\dfrac{s_y}{s_x}$ $r = 0.85, s_y = 4.8$ and $s_x = 5.3$
2. Substitute the values into the formula and evaluate b.	$b = 0.85 \times \dfrac{4.8}{5.3}$ $= 0.7698$ $b = 0.77$
c. 1. The value of a is the y-intercept of the regression line. Write the formula and state the required values.	**c.** $a = \bar{y} - b\bar{x}$ $\bar{y} = 169, \bar{x} = 180$ and $b = 0.7698$ (from part **b**)
2. Substitute the values into the formula and evaluate a.	$a = 169 - 0.7698 \times 180$ $= 30.436$ $= 30.44$

d. 1. State the equation of the regression line, using the values calculated from parts **b** and **c**. In this equation, y represents the height of the female and x represents the height of the male.	**d.** $y = 0.77x + 30.44$
2. The height of the male is 195 cm, so substitute $x = 195$ into the equation and evaluate.	$y = 0.77 \times 195 + 30.44$ $= 180.59$
3. Write a statement, rounding your answer to the nearest centimetre.	Using the equation of the regression line found the female's height is predicted to be 181 cm.

2.3.3 Interpreting the intercept and slope

Often data is collected in order to make informed decisions or predictions about a situation. The regression line equation from a scatterplot can be used for this purpose.

Remember that the equation for the regression line is in the form $y = a + bx$, where b is the gradient or slope, a is the y-intercept, and x and y refer to the two variables. Two important pieces of information can be attained from this equation.

1. When the explanatory variable is equal to 0, the value of the response variable is indicated by the y-intercept, a.
2. For each increment of 1 unit of change in the explanatory variable, the change in the response variable is indicated by the value of the slope, b.

WORKED EXAMPLE 5

The following table shows data from Bilbo's Real Estate for house sales in The Shire in November 2020.

House	Number of bedrooms	Number of bathrooms	Size of garage (cars)	Size of land (m^2)	Price (\$)
1	2	1	1	117	730 000
2	4	2	1	630	1 875 000
3	3	1	2	688	1 300 000
4	2	1	1	228	790 000
5	3	1	2	858	1 610 000
6	2	1	1	637	670 000
7	3	1	1	588	1 400 000
8	6	4	1	700	2 060 000
9	2	1	1	93	520 000
10	2	1	1	73	639 000
11	3	1	1	242	720 000
12	1	1	1	112	460 000
13	2	1	1	167	737 000

a. **Using Excel, or other suitable technology, draw a scatterplot of *S* (size of land) against *P* (price of house)**

b. **Determine the least-squares regression line.**

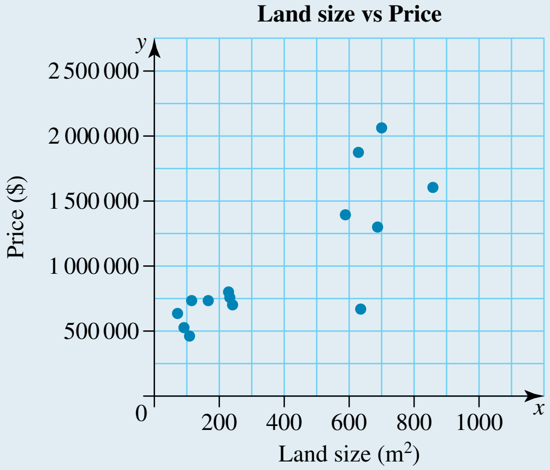
c. **What does the least-squares regression line tell you about property prices in The Shire?**

THINK

WRITE

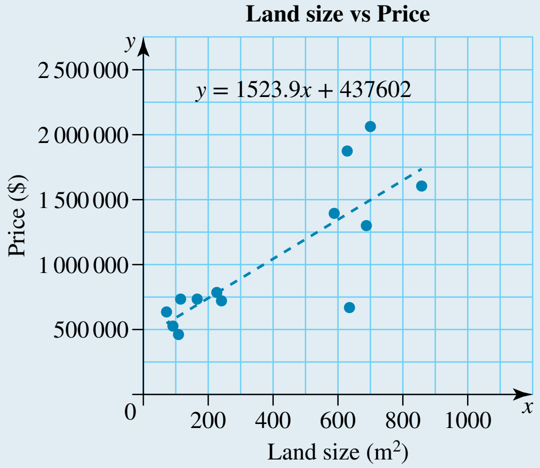
a. Size of land (S) is the explanatory variable. Price of house (P) is the response variable. Plot the table of values using Excel.

a.



b. Determine the regression line using Excel.

b.



The regression line calculated is:
$y = 1523.9x + 437\,602$.
In terms of price and size of land, the equation is:
$P = 1523.9 \times S + 437\,602$.

c. Interpret the least-squares regression line by referring to the slope ($1500) and the *y*-intercept ($437 602).

c. Property prices begin at $437 602 and increase by over $1500 per square metre after that.

WORKED EXAMPLE 6

The least-squares regression equation for a line is $y = 62 - 8x$.

a. **Identify the *y*-intercept.**

b. **For each unit of change in the explanatory variable, by how much does the response variable change?**

c. **What does your answer to part b tell you about the direction of the line?**

THINK	WRITE
a. Consider the equation in the form $y = a + bx$. Identify the value that represents a.	**a.** $y -$ intercept $(a) = 62$
b. The change in the response variable due to the explanatory variable is reflected in the slope. Identify the b value in the equation.	**b.** slope $(b) = -8$
c. A positive b value indicates a line pointing in a positive direction, while a negative b value indicates a line pointing in a negative direction.	**c.** As the b value is negative, the direction of the line is negative.

2.3.4 Interpolation and extrapolation

The regression line can be used to explore data points both inside and outside of the given scatterplot range. When investigating data inside the variable range, the data is being **interpolated**. Data points that lie above or below the scatterplot range can also be used to make predictions. Predictions outside the range of data is **extrapolation**.

The regression equation can be used to make predictions from the data by substituting in a value for either the explanatory variable (x) or the response variable (y) to calculate the value of the other variable.

WORKED EXAMPLE 7

Flowers with a diameter of 5 – 17 cm were measured and the number of petals for each flower was documented. A regression equation of $N = 0.41 + 1.88d$, where N is the number of petals and d is the diameter of the flower (in cm) was established.

a. Identify the explanatory variable.

b. Determine the number of petals that would be expected on a flower with a diameter of 15 cm. Round to the nearest whole number.

c. Is the value found in part b an example of interpolated or extrapolated data?

d. A flower with 35 petals is found. Use the equation to predict the diameter of the flower, correct to 1 decimal place.

e. Is part d an example of interpolated or extrapolated data?

THINK	WRITE
a. Consider the format of the equation. The variable on the right-hand side of the equation will be the explanatory variable.	**a.** Explanatory variable = flower diameter (d)
b. 1. Using the equation, substitute 15 for d and evaluate.	**b.** $N = 0.41 + 1.88d$ $= 0.41 + 1.88 \times 15$ $= 28.61$
2. Round N to the nearest whole value. *Note*: round down as you can't have 0.61 of a petal.	28 petals
c. Consider the data range given in the opening statement.	**c.** 15 cm is inside the data range, so this is interpolation, not extrapolation.

d. **1.** Using the equation, substitute 35 in place of N. **2.** Transpose the equation to solve for d. **3.** Round to 1 decimal place.	**d.** $35 = 0.41 + 1.88d$ $d = \dfrac{35 - 0.41}{1.88}$ $= 18.40$ $= 18.4$ (correct to 1 decimal place)
e. Consider the data range given in the opening statement.	**e.** 18.4 cm is outside the data range, so this is an example of extrapolated data.

2.3.5 Limitations of regression line predictions

When reviewing predictions drawn from a scatterplot, it is necessary to question the reliability of the results. As with any conclusion or prediction, the results rely heavily on the initial data. If the data was collected from a small sample, then the limited information could contain biases or a lack of diversity that would not be present in a larger sample. The more data that can be provided at the start, the more accurate a result will be produced.

The strength of the association between the variables also provides an indication of the reliability of the data. Data that produces no association or a low association would suggest that any conclusions drawn from the data will be unreliable.

When extrapolating data it is assumed that additional data will follow the same pattern as the data already in use. This assumption means extrapolated data is not as reliable as interpolated data.

on Resources

Interactivity: Fitting a straight line using least-squares regression (int-6254)

studyon

Units 3 & 4 › Area 1 › Sequence 2 › Concepts 2, 3 & 4

Fitting a least-squares line to data Summary screen and practice questions
Interpretation of slope and intercept Summary screen and practice questions
Interpolation and extrapolation Summary screen and practice questions

Exercise 2.3 Fitting a least-squares line to data

1. **WE4** The following summary details were calculated from a study to find a relationship between Mathematics exam marks and English exam marks from the results of 120 Year 10 students.

Mean Mathematics exam mark = 64%
Mean English exam mark = 74%
Standard deviation of Mathematics exam mark = 14.5%
Standard deviation of English exam mark = 9.8%
Correlation coefficient, $r = 0.64$.
The form of the least-squares regression line is:
Mathematics exam mark $= m \times$ English exam mark $+ c$

a. Which variable is the response variable (y-variable)?

b. Calculate the value of m for the least-squares regression line.

c. Calculate the value of c for the least-squares regression line.
d. Use the regression line to predict the expected Mathematics exam mark if a student scores 85% in an English exam (to the nearest percentage).

2. Find the least-squares regression equation, given the following summary data.
a. $\bar{x} = 5.6,\ s_x = 1.2, \bar{y} = 110.4,\ s_y = 5.7, r = 0.7$
b. $\bar{x} = 110.4, s_x = 5.7, \bar{y} = 5.6,\ s_y = 1.2, r = -0.7$
c. $\bar{x} = 25, s_x = 4.2, \bar{y} = 10\,200,\ s_y = 250,\ r = 0.88$
d. $\bar{x} = 10, s_x = 1, \bar{y} = 20,\ s_y = 2, r = -0.5$

3. **WE5** Recall from Chapter 1 that a researcher investigating the proposition that 'tall mothers have tall sons' measures the height of 12 mothers and the height of their adult sons. The results are shown below.

Height of mother (cm)	Height of son (cm)
185	188
155	157
171	172
169	173
170	174
175	180
158	159
156	150
168	172
169	175
179	180
173	190

a. Which variable is the response variable?
b. Using Excel or other suitable technology draw a scatterplot.
c. Fit a least-squares regression line to the data and determine the equation of the line of best fit, expressing the equation in terms of height of mother (M) and height of son (S). Give values correct to 4 significant figures.

4. **WE6** The least-squares regression equation for a line is $y = -1.837 + 1.701x$.
a. Identify the y-intercept.
b. For each unit of change in the explanatory variable, by how much does the response variable change?
c. What does your answer to part **b** tell you about the direction of the line?

5. The least-squares regression equation for a line is $y = 105.90 - 1.476x$.
a. Identify the y-intercept.
b. For each unit of change in the explanatory variable, by how much does the response variable change?
c. What does your answer to part **b** tell you about the direction of the line?

6. **WE7** A brand of medication for babies bases the dosage on the age (in months) of the child. The regression equation for this situation is $M = 0.157 + 0.312A$, where M is the amount of medication in mL and A is the age in months.
a. Identify the explanatory variable.
b. Calculate the amount of medication required for a child aged 6 months.
c. Determine the age of a child who requires 2.5 mL of the medication. Give your answer correct to 1 decimal place.

7. A survey of the nightly room rate for Sydney hotels and their proximity to the Sydney Harbour Bridge produced the regression equation $C = 281.92 - 50.471d$, where C is the cost of a room per night in dollars and d is the distance to the bridge in kilometres.
   a. Identify the response variable.
   b. Based on this equation, calculate the cost of a hotel room 2.5 km from the bridge. Give your answer correct to the nearest cent.
   c. Determine the distance of a hotel room from the bridge if the cost of the room was \$115. Give your answer correct to 2 decimal places.
8. An equation for a regression line is $y = 3.2 - 1.56x$. What conclusions about the direction of the regression line can be determined from the equation?
9. a. Use technology to plot the regression line $y = -1.6 + 2.5x$.
   b. Would a data point of $(3, 4)$ be found above or below the regression line?
10. Answer the following questions for the equation $y = 60 - 5x$.
    a. Identify the y-intercept.
    b. For each unit of change in the explanatory variable, by how much does the response variable change?
    c. Is the direction of the data positive or negative?
    d. Calculate the value of y when $x = 40$.
11. Lucy was given the equation $y = -12.9 + 7.32x$ and asked to find the value of x when $y = 15.68$. Her working steps are below:

$$\begin{aligned} \mathbf{y} &= \mathbf{-12.9 + 7.32x} \\ \mathbf{15.68} &= \mathbf{-12.9 + 7.32x} \\ \mathbf{x} &= \mathbf{12.9 + \frac{15.68}{7.32}} \\ \mathbf{x} &= \mathbf{15.04} \end{aligned}$$

    Her teacher indicates her answer is wrong.
    a. Calculate the correct value of x. Give your answer correct to 2 decimal places.
    b. Identify and explain Lucy's error.
12. Answer the following questions for the equation $y = -12 + 25x$.
    a. Identify the y-intercept.
    b. For each unit of change in the explanatory variable, by how much does the response variable change?
    c. Is the direction of the data positive or negative?
    d. Calculate the value of y when $x = 3.5$.
13. Answer the following questions for the equation $I = 0.43 + 1.1s$, where I is the number of insects caught and s is the area of a spider's web in cm^2.
    a. Identify the response variable.
    b. For each unit of change in the explanatory variable, by how much does the response variable change?
    c. Is the direction of the data positive or negative?
    d. Determine how many insects are likely to be caught if the area of the spider's web is $60\,\text{cm}^2$. Give your answer correct to the nearest whole number.
14. A data set produced a positive direction and for each incremental increase in the explanatory variable, the response variable increased by 2.5. If $y = 4$ when $x = 0$, determine the equation for the regression line.
15. Use the data given below and appropriate technology to answer the following questions.

x	10	11	12	13	14	15	16	17	18
y	22	18	20	15	17	11	11	7	9

a. Draw a scatterplot and determine the equation of the least-squares regression line. Give values correct to 4 significant figures.

b. Extrapolate the data to predict the value of y when $x = 23$.

c. What assumptions are made when extrapolating data?

16. While camping a mathematician estimated that number of mosquitoes around the fire $= 10.2 + 0.5 \times$ temperature of the fire (°C).

a. Determine the number of mosquitoes that would be expected if the temperature of the fire was 240°C. Give your answer correct to the nearest whole number.

b. What would be the temperature of the fire if there were only 12 mosquitoes in the area?

c. Identify some factors that could affect the reliability of this equation.

17. Data on 15 people's average monthly income and the amount of money they spend at restaurants was collected.

Average monthly income ($000s)	Money spent at restaurants per month ($)
4.2	620
3.6	395
2.7	185
2.8	150
2.5	130
3.0	220
3.1	245
2.2	100
4.0	400
3.7	380
3.8	200
3.5	360
2.9	175
3.6	350
4.1	600

a. Draw a scatterplot of this data on technology of your choosing.

b. Find the equation of the least-squares regression line in terms of average monthly income in thousands of dollars (I) and money spent at restaurants in dollars (R). Give values correct to 4 significant figures.

c. Predict how much a person who earns $5000 a month might spend at restaurants each month.

d. Explain why part **c** is an example of extrapolation.

e. A person spent $265 eating out last month. Estimate their monthly income, giving your answer to the nearest $10. Is this an example of interpolation or extrapolation?

18. Data on 15 students' marks in Geography and Music assessments were collected.

Geography marks	Music marks
65	91
80	57
72	77
61	89
99	51
54	76
39	62
66	87
78	88
89	64
84	90
73	45
68	60
57	79
60	69

a. Is there an obvious explanatory variable in this situation?

b. Draw a scatterplot of this data on your calculator, using the marks in Geography as the explanatory variable.

c. Find the equation of the line of best fit. Give values correct to 4 significant figures.

d. Based on your equation, if a student received a mark of 85 for Geography, what mark (to the nearest whole number) would you predict they would receive for Music?

e. How confident do you feel about making predictions for this data? Explain your reasons.

f. Calculate Pearson's product–moment correlation coefficient, r for this data. How can you use this value to evaluate the reliability of your data?

19. For three months, Cameron has been wearing an exercise-tracking wristband that records the distance he walks and the number of calories he burns. A graph shows his weekly totals. The regression line equation for the data where y represents the number of calories burned and x represents the distance walked is $y = 14\,301 + 115.02x$.

a. Identify the response variable in this situation.

b. Rewrite the equation in terms of the explanatory and response variables.

c. Using the equation for the regression line, determine the number of calories burned if a person walked 50 km in a week. Is this an example of interpolation or extrapolation? Explain.

d. Due to an injury, in one-week Cameron only walked 10 km. Use the data to determine the number of calories this distance would burn. Is this an example of interpolation or extrapolation? Explain.

e. Pearson's product–moment correlation coefficient for this data is 0.9678. How can you use this value to evaluate the reliability of the data?

f. List at least two other factors that could influence this data set.

20. A phone carrier company used fingerprint biometric technology to collate some results about people using a particular social media application. A graph is drawn that shows their results for people's ages and the number of social media friends they had. The regression line equation for this data set is

$y = -13.613x + 777.84$, where y represents the number of social media friends and x represents the ages of the people.

a. Identify the explanatory variable in this situation.

b. Rewrite the equation in terms of the explanatory and response variables.

c. Pearson's product–moment correlation coefficient for this data set is -0.893. How can you use this value to evaluate the reliability of the data?

d. Using the regression line equation for this data set, determine the number of friends a 35-year-old person is likely to have on this social media application. Is this an example of interpolation or extrapolation? Explain your response.

2.4 The coefficient of determination and residual plots

2.4.1 The coefficient of determination (r^2)

In Chapter 1, the strength of a linear relationship was determined using Pearson's correlation coefficient (r). **The coefficient of determination**, r^2, is Pearson's correlation coefficient squared.

If a value for r^2 of 0.71 is found, for example, this would indicate that 71% of the variation in the y-variable is explained by the variation in the x-variable and 29% can be explained by other factors.

The coefficient of determination can be calculated using an Excel spreadsheet or other suitable technology.

WORKED EXAMPLE 8

Data was collected on the time it takes students to get to school and their ATAR scores.

Time (mins)	12	35	19	42	33	31	25	46	45	40	14	44	39	31	22
ATAR score	53	75	97	59	87	70	71	66	37	48	94	68	33	59	42

Appropriate technology should be used to answer the following questions.

a. Draw a scatterplot of this data.

b. Determine the equation of the line of best fit.

c. Determine Pearson's correlation coefficient, r, and the coefficient of determination, r^2.

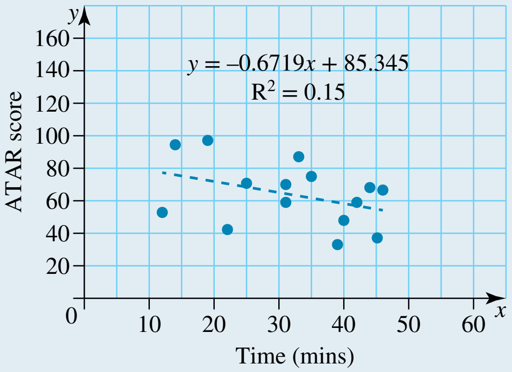
d. Interpret the values of r and r^2.

THINK

a. Using Excel, draw a scatterplot representing the data collected. Time is the explanatory variable and ATAR score is the response variable.

WRITE

a.



b. Write the equation of the line of best fit using appropriate variable names.	**b.** ATAR score = −0.6719 × Time + 85.345
c. Write the value of r^2 and r. r^2 is given by Excel when a line of best fit is added.	**c.** $r^2 = 0.15$ $r = \sqrt{0.15}$ $= -0.39$ (negative gradient on line)
d. Interpret the r value. Interpret the r^2 value	**d.** An r value of −0.39 implies that there is a weak negative linear relationship, which is not a clear indication that an increased distance from school could have a negative effect on your ATAR score. An r^2 value of 0.15 implies that 15% of the variation in the ATAR score can be explained by the variation in the distance students live from their schools and 85% is explained by other factors.

2.4.2 Residual plots and analysis

In the world of statistical modelling, it is often not enough to use a scatterplot to determine if a linear model is appropriate. There may be other underlying patterns that are difficult to recognise simply by studying the scatterplot. A **residual plot** can be constructed using the least-squares regression line to highlight any underlying patterns which would indicate that the data does not fit a linear model.

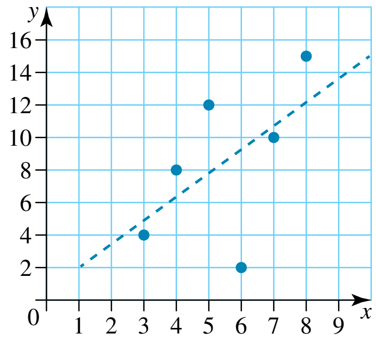
Calculating residuals

Consider the following set of data.

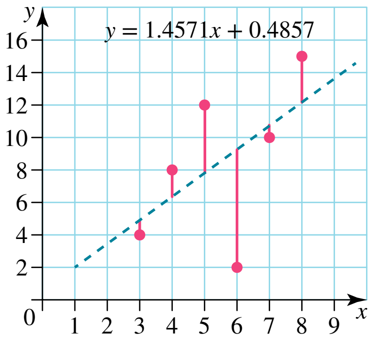
x	3	4	5	6	7	8
y	4	8	12	2	10	15

The following graph shows the data plotted on a scatterplot and a regression line fitted.

For this set of data, the linear regression equation is $y = 1.4571x + 0.4857$



The lengths of the vertical lines joining the data points to the regression line are the **residuals**.



For each value of x (explanatory variable), there is the actual value y (response variable) from the data supplied and there is the predicted value, found from the linear regression equation.

Residual value = actual y-value − predicted y-value

In the data in the previous table for the x-value of 3, the actual y-value is 4 and the predicted y-value using the linear regression equation is:

$$\begin{aligned} y &= 1.4571x + 0.4857 \\ &= 1.4571 \times 3 + 0.4857 \\ &= 4.857 \end{aligned}$$

$$\begin{aligned} \text{Residual value} &= \text{Actual value} - \text{predicted value} \\ &= 4 - 4.857 \\ &= -0.86 \text{ (correct to 2 decimal places)} \end{aligned}$$

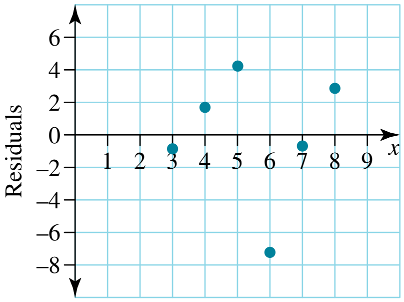
The following table is completed for all the predicted y-values and the residuals.

x	3	4	5	6	7	8
Actual y-value	4	8	12	2	10	15
Predicted y-value	4.86	6.31	7.77	9.23	10.69	12.14
Residual value	−0.86	1.69	4.23	−7.23	−0.69	2.86

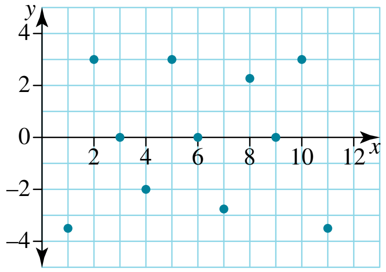
Residual plots

A **residual plot** is a graph of the points obtained by plotting the residuals on the vertical axis and the explanatory variable (x-value) on the horizontal axis. When the points in a residual plot are randomly spread around the horizontal axis, a linear regression model is appropriate for the data; otherwise a non-linear model should be considered.

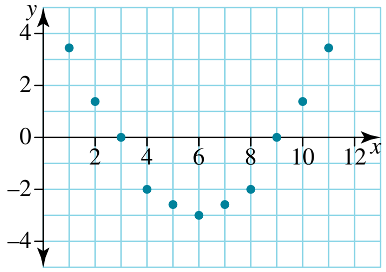
For the data in the previous table a residual plot is completed below.



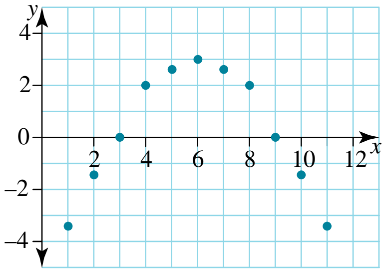
The previous residual plot shows a random pattern, therefore the linear model is seen as a good fit. The following residual plots show some typical patterns for residuals.



Random pattern
Linear model appropriate



Non random
Linear model *not* appropriate



Non-random
Linear model *not* appropriate

If a residual plot showed non-random patterns then a non-linear model for the original data should be investigated. Non-linear models are outside the scope of this course.

WORKED EXAMPLE 9

For the set of data below, the least-squares regression equation is $y = -0.56x + 45.6$. Complete the table.

x	50	55	60	65	70
Actual *y*-value	20	10	15	8	7
Predicted *y*-value					
Residual value					

THINK

1. Calculate the predicted *y*-values:
$y = -0.56x + 45.6.$
$y = -0.56 \times 50 + 45.6 = 17.6$
$y = -0.56 \times 55 + 45.6 = 14.8$
$y = -0.56 \times 60 + 45.6 = 12$
$y = -0.56 \times 65 + 45.6 = 9.2$
$y = -0.56 \times 70 + 45.6 = 6.4$

WRITE

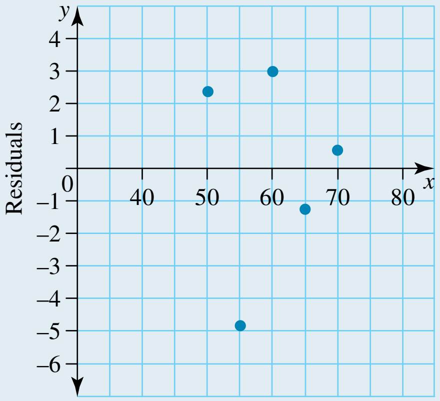
x	50	55	60	65	70
Actual *y*-value	20	10	15	8	7
Predicted *y*-value	17.6	14.8	12	9.2	6.4
Residual value					

2. Calculate the residual values:
Residual value = actual *y*-value – predicted *y*-value
$20 - 17.6 = 2.4$
$10 - 14.8 = -4.8$
$15 - 12 = 3$
$8 - 9.2 = -1.2$
$7 - 6.4 = 0.6$

x	50	55	60	65	70
Actual *y*-value	20	10	15	8	7
Predicted *y*-value	17.6	14.8	12	9.2	6.4
Residual value	2.4	−4.8	3	−1.2	0.6

WORKED EXAMPLE 10

For the set of data in Worked example 9, plot the residuals and interpret the graph.

THINK	**WRITE**
1. Draw a set of axes with the *x* value on the horizontal axis and the residual value on the vertical axis. Plot the points on the axes.	
2. The points do not appear to form a pattern.	As the residuals appear to be randomly spread above and below the *x*-axis, then a linear model is an appropriate model for this set of data.

on Resources

Interactivity: Pearson's product–moment correlation coefficient and the coefficient of determination (int-2451)

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Units 3 & 4 › Area 1 › Sequence 2 › Concepts 5, 6 & 7

The coefficient of determination Summary screen and practice questions
Residuals Summary screen and practice questions
Residual plots Summary screen and practice questions

Exercise 2.4 The coefficient of determination and residual plots

1. **WE8** Data on the daily sales of gumboots and the maximum temperature was collected.

Temperature (°C)	17	16	12	10	14	17	18	22	23	19	17	15	12	15	20
Daily sales (no. of pairs)	2	3	8	16	7	3	2	1	1	2	3	3	12	9	1

Appropriate technology should be used to answer the following questions.

a. Draw a scatterplot of this data.
b. Find the equation of the line of best fit.
c. Find Pearson's product–moment correlation coefficient, *r*, and the coefficient of determination, r^2.
d. Interpret these values.

2. **MC** If Pearson's correlation coefficient, r, is found to be -0.7564, then the coefficient of determination, r^2, will be

A. 0.2436. **B.** -0.7564. **C.** -0.4279. **D.** 0.5721.

3. **MC** If the coefficient of determination, r^2, is found to be 0.5781, the percentage of variation that can be explained by other factors is

A. 57.81%. **B.** 42.19%. **C.** 76.03%. **D.** 23.97%.

4. Calculate the value of the coefficient of determination, r^2, in the following scenarios.
   - **a.** A study was conducted, and it was found that the association between a child's diet and their health is $r = 0.8923$.
   - **b.** The association between global warming and the amount of water in the oceans was found to be $r = 0.9997$.
   - **c.** Interpret these values.

5. **WE9** For the set of data below, the least- squares regression equation is $y = 1.27x + 7.65$. Complete the following table.

x	13	18	23	28	33	38
Actual y-value	25	31	40	36	48	60
Predicted y-value						
Residual value						

6. **WE10** For the set of data in question **5**, plot the residuals and interpret the graph.

7. A woman diagnosed as anaemic has a level of 120 g/L of iron at her initial blood test. She agreed to join a research group to determine how quickly an iron supplement in capsule form, administered daily, would impact on her iron levels.

   Her iron level was measured once a week. The following data were collected.

Week of experiment	1	2	3	4	5	6
Iron level	120	122	130	135	135	140

   - **a.** Using appropriate technology, construct a scatterplot and determine the equation of the least-squares regression line.
   - **b.** Complete the following table.

Week of experiment	1	2	3	4	5	6
Iron level	120	122	130	135	135	140
Predicted iron level						
Residua value						

   - **c.** Draw a residual plot and interpret the plot.
   - **d.** Determine the coefficient of determination using technology and hence calculate Pearson's correlation coefficient.
   - **e.** Using the information obtained in parts **a**, **b**, **c** and **d**, is this a suitable model to use to determine when the woman will reach a healthy iron count of 155 g/L of iron?

8. A farmer's market is held once a month and features local produce, handicrafts and trash and treasure stalls.

The number of local produce and handicraft stalls varies each weekend and is announced prior to the weekend on the website for the market. The number of local produce and handicraft stalls participating, together with the number of visitors to the market each month, is shown in the following table.

Number of stalls	53	34	61	32	61	25
Number of visitors	501	339	611	300	450	333

The least-squares regression line for this data is

$\text{Number of visitors} = 6.64 \times \text{Number of stalls} + 128.11$

a. Using the least-squares regression line equation, complete the following table.

Number of stalls	53	34	61	32	61	25
Number of visitors	501	339	611	300	450	333
Predicted number of visitors						
Residual value						

b. Draw a residual plot and interpret the plot.

c. If the coefficient of determination for this set of data is 0.7681, determine the Pearson correlation coefficient, correct to two decimal places.

d. Using the information obtained in parts a, b and c, is this a suitable model to use to predict the number of visitors based on the number of stalls?

9. The following table represents the costs for transporting a consignment of surfboards from Brisbane factories.

The cost is given in terms of distance from Brisbane. There are two factories which can be used. The data are summarised below.

Distance from Brisbane (km)	10	20	30	40	50	60	70	80
Factory 1 cost ($)	70	70	90	100	110	120	150	180
Factory 2 cost ($)	70	75	80	100	100	115	125	135

a. Using appropriate technology, construct a scatterplot and determine the equation of the least-squares regression line for each factory.

b. Complete the table.

Distance from Brisbane (km)	10	20	30	40	50	60	70	80
Factory 1 cost ($)	70	70	90	100	110	120	150	180
Predicted cost for factory 1								
Residuals for factory 1								
Factory 2 cost ($)	70	75	80	100	100	115	125	135
Predicted cost for factory 2								
Residuals for factory 2								

c. Draw a residual plot for each factory and interpret the plots.

d. Calculate Pearson's correlation coefficient and the coefficient of determination for each factory.

e. Which factory is likely to have the lower cost to transport to a shop in Brisbane?

f. Which factory is likely to have the lower cost to transport to Mytown, 115 kilometres from Brisbane?

10. The following table contains the age and systolic blood pressure (SBP) for a group of volunteers at a University Health Science department.

Age	37	38	40	42	45	48	50	52	53	55
Systolic blood pressure	130	140	132	149	144	157	161	145	165	162

a. A linear regression equation of the form $y = a + bx$ was calculated for this data and the results were $b = 1.61$, $a = 74.35$, $r = 0.84$. Use this information to calculate the predicted systolic blood pressure and the residuals.

Age	37	38	40	42	45	48	50	52	53	55
Systolic blood pressure	130	140	132	149	144	157	161	145	165	162
Predicted systolic blood pressure										
Residual value										

b. Plot the residuals and comment on the likely linearity of the data.

c. Estimate the systolic blood pressure at age 75 and comment on the reliability of this estimation.

11. Consider the following data set.

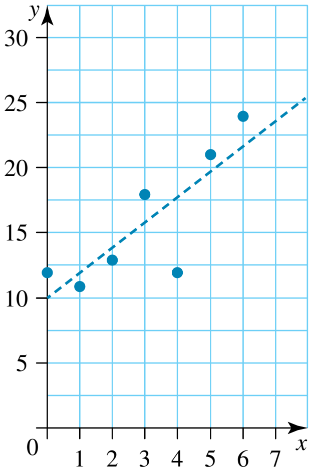
x	1	2	3	4	5	6	7	8	9	10
y	1	5	10	16	26	34	50	62	80	101

a. A linear regression equation of the form $y = a + bx$ was calculated for this data and the results were $a = -21.333$, $b = 10.879$, $r = 0.97$. Use this information to calculate the predicted values of y and the residuals.

b. Plot the residuals and comment on the likely linearity of the data.

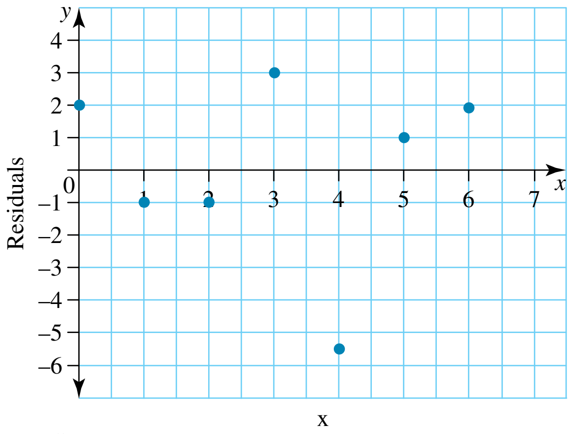
The following information relates to questions 12 and 13.

The following graph shows a least-squares regression line fitted to a set of data.

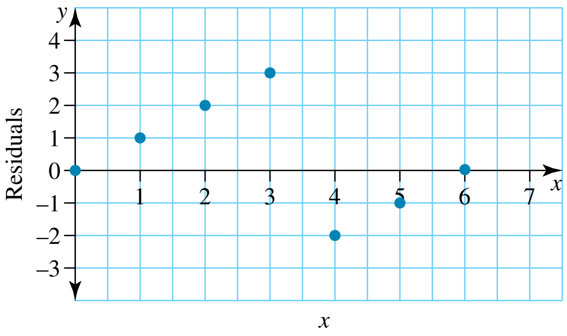


12. From the following options, choose the residual plot that best represents the residuals for this line.

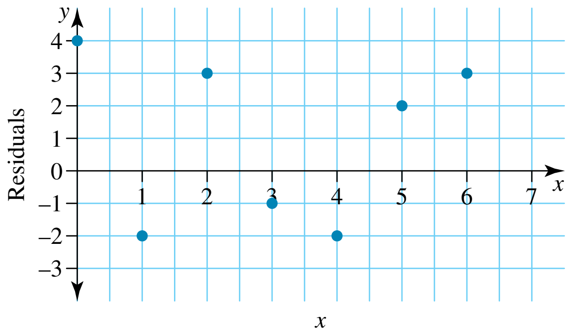
a.



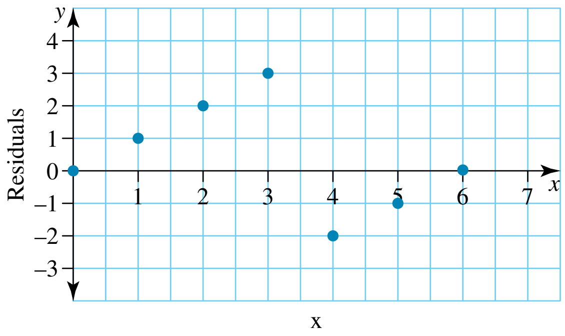
b.



c.



d.



13. **MC** The value of the product–moment correlation coefficient (r) for this data is closest to

A. −0.81.
B. −0.33.
C. 0.33.
D. 0.81.

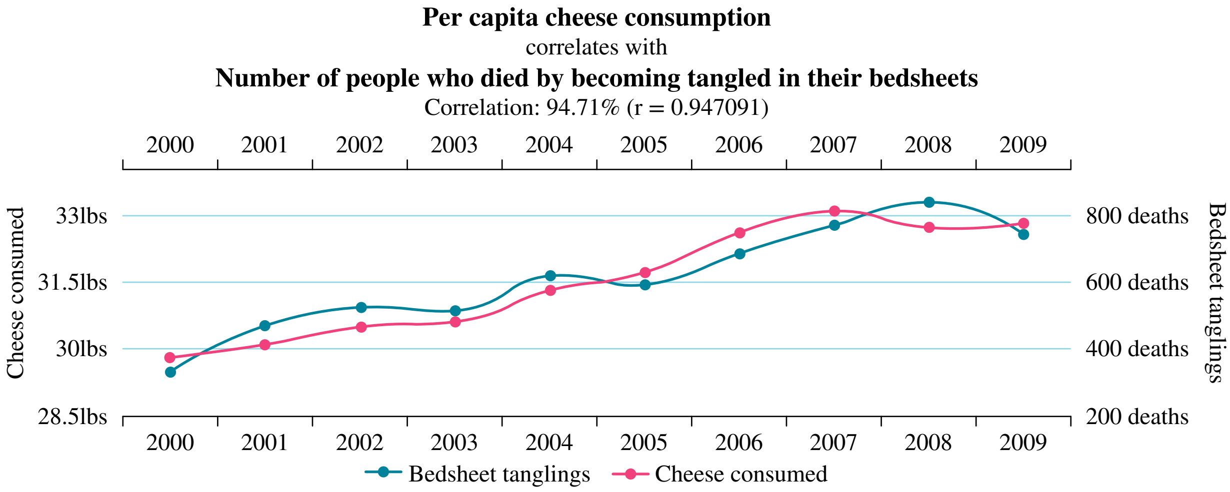
14. The coefficient of determination for a set of data is found to be 0.78.
Complete the following sentence.
78% of the variation in the __________ variable can be explained by the variation in the __________ variable. __________% of the variation can be explained by other factors.

2.5 Association and causation

2.5.1 Observation and association

Spurious correlations (http://www.tylervigen.com/spurious-correlations) is a website and now a book created by Tyler Vigen. The website shows many examples of graphs of data sets that appear to have a relationship. For example, the following graph shows an association of $r = 0.95$ for the Number of people who died by becoming entangled in their bedsheets and per capita cheese consumption from 2000 to 2009.

This is an extreme example of why **association** does not necessarily imply **causation**.



Establishing a high degree of association between two variables can provide a starting point for an investigation but establishing causation requires further **observation** and **experimentation**. For example, a basketball fan noted that there was a high positive association between the height of a basketballer and the number of points he scored. Does this mean that if a basketball team needs players with greater scoring ability then they just should recruit more tall players? An experienced recruiter knows that height is only one advantage for a basketballer; fitness levels, skill level, hand–eye coordination and ability to read the game are all necessary attributes of a player who can consistently score. The recruiter would observe all of these attributes of potential new players and study the statistics of all the variables, before making a decision.

Therefore, a high degree of association between two variables does not mean that there is a causal relationship between the variables, so further investigation is necessary.

One way to establish causation is to conduct experiments where a control group is used. Agricultural scientists researching the effectiveness of a new fertiliser designed to improve the productivity of a new type of tomato plant split the population (the tomato plants) into two groups. One group of plants will be fed the fertiliser in water at regular intervals, the other group of plants are given the same amount of water without the fertiliser and all other variables are the same across both groups of plants. (Other variables could include the type of soil and weather conditions.) A study of the association between administering the fertiliser and the size of the tomato crop can then establish causation. In this example, the administered water and fertiliser is the explanatory variable, and the size of the tomato crop is the response variable.

2.5.2 Common response, confounding and coincidence explanations

In some cases, the association between two variables can be explained by a **common response** which provides the association. For example, a study may show that there is a strong association between the GDP (gross domestic product) of a country and the infant mortality rate. While a larger GDP will not directly lead to a lower infant mortality rate, a common response — the money spent on childhood vaccination programs — provides a direct link to both variables and is more likely to be the underlying cause for the observed association.

In another example, a study of a group of people may provide a strong association between the number of deaths from heart attacks (response variable) and distance from a major hospital (explanatory variable). But there are other factors that can influence the number of deaths, such as diet, age, starting weight and gender. These are called **confounding variables**. If a study does not take confounding variables into account when setting up the study the results cannot be used to show causality.

Sometimes an association between variables can just be a **coincidence**, as in the example in the previous graph, which shows an association between cheese consumption and deaths from becoming entangled in bed sheets.

When observing an association between two variables it can never be stated that the one variable causes a response in the other variable. In professional research many controlled experiments must be carried out to determine and identify a causal relationship.

WORKED EXAMPLE 11

Data showing the average weekly hours of exercise for a group of 8 people and their LDL-cholesterol reading are as given.

Average weekly hours of moderate exercise	7	1	3	8	3	10	6	2
LDL-C	100	125	130	120	128	90	115	135

a. **Construct a scatterplot for the data**

b. **Comment on the association between the number of hours of exercise and the LDL-C reading.**

c. **Calculate *r*.**

d. **Based on the value of *r* obtained in part c, would it be appropriate to conclude that the decrease in the LDL-C reading is caused by the increasing hours of exercise?**

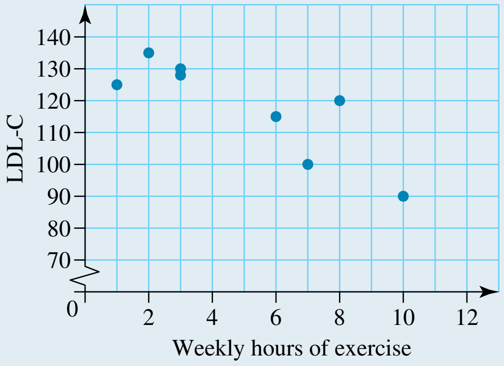
THINK

a. 1. Determine the explanatory and response variables

2. Sketch a scatterplot

WRITE

a. Weekly hours of exercise (explanatory)
LDL-C reading (response)



b. A negative association can be observed.	**b.** From the scatterplot there appears to be a strong negative association between the weekly hours of exercise and the LDL-C reading.
c. Use technology to find the value of *r*.	**c.** $r = -0.85$
d. Association does not imply causation.	**d.** Just because $r = -0.85$, it cannot be stated that increasing weekly hours of exercise will cause a decrease in the LDL-C reading. Other factors may need to be considered. Further investigation is needed before any conclusions can be drawn.

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Units 3 & 4 › Area 1 › Sequence 2 › Concept 8

Association and causation Summary screen and practice questions

Exercise 2.5 Association and causation

1. **WE11** Data showing the number of rose bushes in the gardens of 10 houses and the annual income for each house is given.

Number of roses	4	2	10	5	3	11	6	7
Annual household income ($)	80 000	32 000	120 000	65 000	21 000	122 000	75 000	82 000

   a. Using technology, construct a scatterplot for the data
   b. Comment on the association between the number of rose bushes and the household income.
   c. Calculate *r* using technology.
   d. Based on the value of *r* obtained in part c, would it be appropriate to conclude that the increase in household income is caused by an increase in the number of rose bushes in the garden?

2. Data was collected on the level of aerobic fitness of 100 15-year-olds and the amount of time they spent playing computer games. The correlation coefficient for this data was -0.86.
   What can be said about the association between the level of aerobic fitness and the amount of time spent playing computer games for this group of 15-year-olds?

3. **MC** During the months of August and September there was a strong positive association ($r = 0.93$) between the number of sightings of whales in the Whitsundays each week and the number of fines issued for boating infringements off Airlie Beach.

A strong positive association was also found both between the number of whale sightings ($r = 0.96$), the number of fines issued for boating infringements ($r = 0.82$), and the number of visitors to Airlie Beach.

Using this information, which of the following statements is true?

A. It is just a coincidence that there is a strong positive association between the number of sightings of whales in the Whitsundays and the number of fines issued for boating infringements.

B. During the months of August and September, people are so excited about seeing whales that they forget to follow the rules for driving a boat.

C. Tourists don’t know that there are rules to be followed when at sea.

D. August and September attract larger numbers of visitors and more visitors means more people taking out boats which leads to more boating infringements. The association between the number of whale sightings and the number of boating infringements can be explained by the common response variable, the number of tourists at Airlie Beach.

The following information applies to questions 4 and 5.

A set of data was collected from a large group of professional sportspeople. They were asked the number of hours they trained per week and the amount of money they earned. The results were recorded, and the value of Pearson’s correlation coefficient was found to be 0.87.

4. **MC** Which of the following is NOT true?

A. There is a positive association between the number of hours of training and the amount of money earned.

B. The association between the number of training hours and the amount of money earned can be classified as strong.

C. The linear relationship between the two variables suggests that as the number of training hours increase, so too does the amount of money earned.

D. The increase in the number of training hours causes the increase in the amount of money earned.

5. **MC** Which of the following is NOT true?

A. The coefficient of determination is about 0.76.

B. About 76% of the variation in the number of hours of training can be explained by the variation in the amount of money earned.

C. Other factors such as the type of sport played can affect the amount of money earned.

D. The number of training hours is the major factor in predicting the amount of money earned.

6. It has been found that there is a strong positive association between the number of fire fighters that attend a fire and the amount of damage caused by the fire. Does this mean that more fire fighters cause more fire damage? What could be the common cause that links these variables?
7. There is a strong positive association between the shoe size of a child and their academic knowledge. Does a larger shoe size mean a smarter child? What could be the common cause(s) that links these two variables?
8. A group of university students volunteered to be part of a healthy food supplement trial. Their health was monitored every month for 10 months. A positive association was found between improvements in good health statistics such as cholesterol levels and blood pressure and the consumption of the supplement. Does this mean that the consumption of the supplement causes improvement in health? What other confounding variables could also contribute to this association?

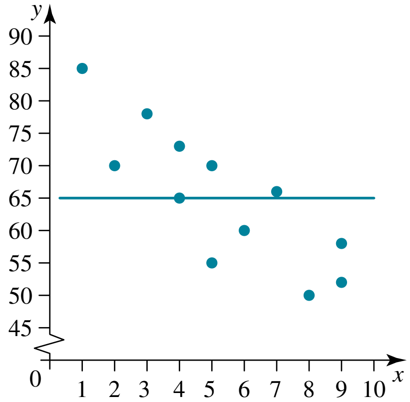
2.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

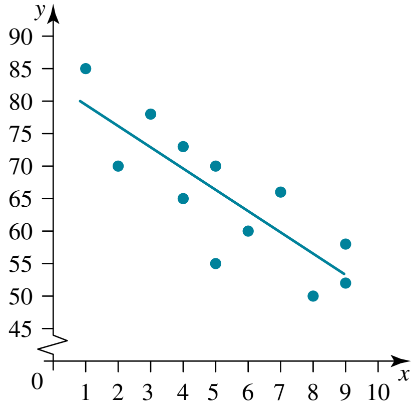
Simple familiar

1. **MC** Which of the following scatterplots best demonstrates a line of best fit?

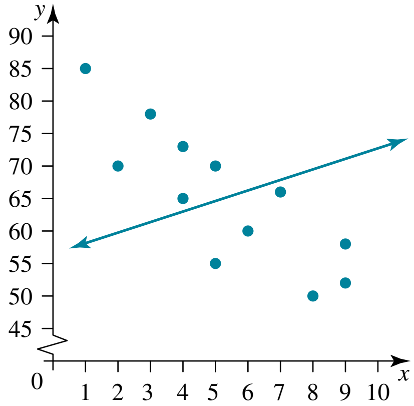
A.



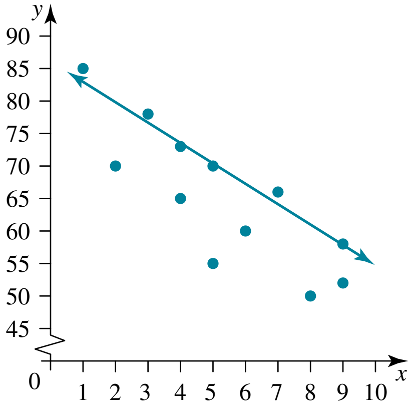
B.



C.

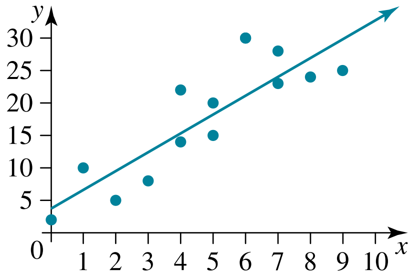


D.



2. **MC** The regression line equation for the graph shown is closest to

A. $y = 3.8 + 2.9x$.
B. $y = -3.8 - 2.9x$.
C. $y = -3.8 + 2.9x$.
D. $y = 3.8 - 2.9x$.



3. **MC** A gardener tracks a correlation coefficient of 0.79 between the growth rate of his trees and the amount of fertiliser used. What can the gardener conclude from this result?

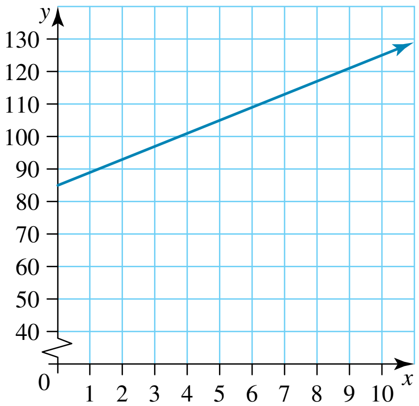
A. An increase in tree growth increases the use of fertiliser.
B. An increase in the use of fertiliser increases the health of the tress.
C. 79% of the variation in the growth rate of his trees can be explained by the variation in the amount of fertiliser used.
D. The growth rate of the trees influences the quality of the fertiliser used.

4. **MC** When $y = 0.54 + 15.87x$, the value of y when $x = 2.5$ is

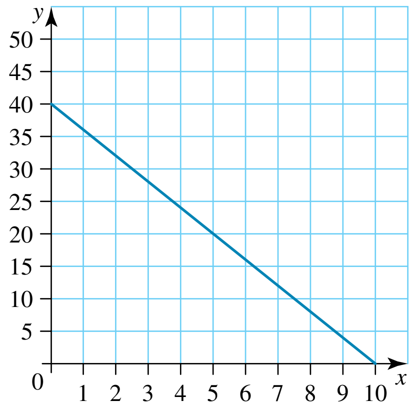
A. 18.91.
B. 40.215.
C. 39.135.
D. 6.888.

5. **MC** The graph for the regression line equation $y = 85 - 4x$ is most likely to be

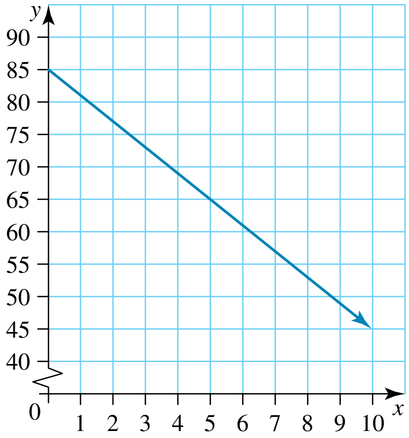
A.



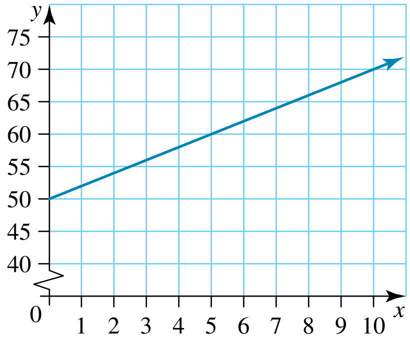
B.



C.



D.



6. **MC** A series of data points recorded a coefficient of determination value of 0.82. Calculate Pearson's correlation coefficient.

A. 82% **B.** 0.18 **C.** 0.67 **D.** 0.91

7. **MC** For the following sample data set, which of the following is an example of interpolating data?

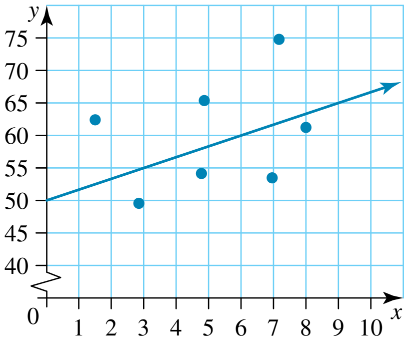
x	1	5	15	25
y	10	16	18	22

A. Finding the value of x when $y = -7$
B. Finding the value of y when $x = 17$
C. Finding the value of x when $y = 27$
D. Finding the value of y when $x = 37$

8. **MC** For the data set $\bar{x} = 11.5, \bar{y} = 16.5, s_x = 10.8, s_y = 5, r = 0.94$ the regression line equation is closest to

A. $y = 10 + x$.
B. $y = 0.435 + 11.496x$.
C. $y = 0.876 + 0.936x$.
D. $y = 11.498 + 0.435x$.

9. **MC** Consider the regression line drawn on the scatterplot shown.



The gradient of the regression line is closest to

A. $\frac{5}{3}$.

B. $-\frac{5}{3}$.

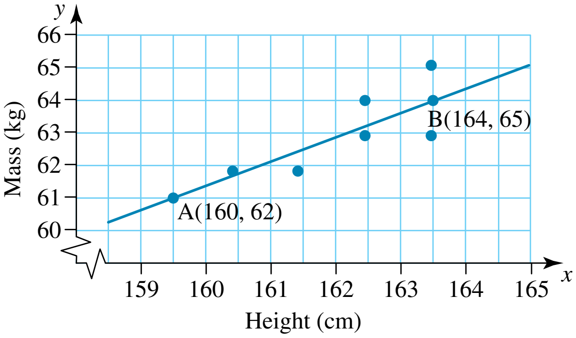
C. $\frac{3}{5}$.

D. $-\frac{3}{5}$.

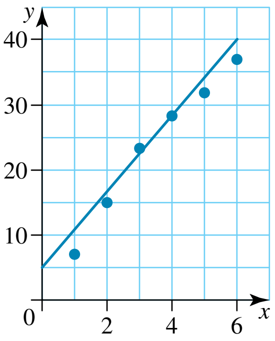
10. **MC** Consider the regression line drawn on the scatterplot shown below.

The equation of the regression line is

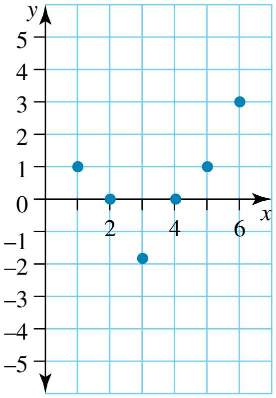
A. height $= 0.75 \times$ mass $- 58$.
B. height $= 0.75 \times$ mass $+ 58$.
C. height $= -0.75 \times$ mass $- 58$.
D. mass $= 0.75 \times$ height $+ 58$



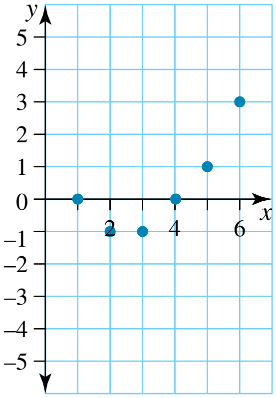
11. **MC** A least-squares regression line is fitted to the 6 points shown in the figure. Which of the following looks most similar to the plot of residuals?



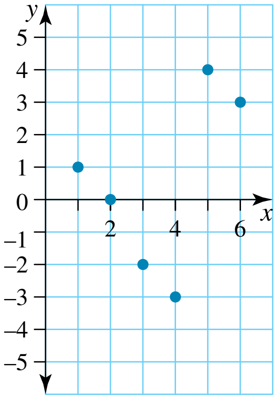
A.



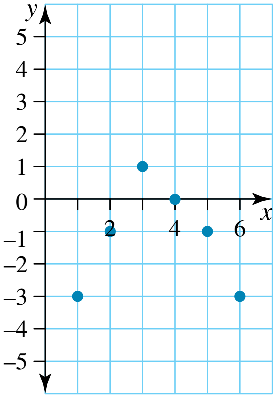
B.



C.



D.



12. **MC** A study of a group of people found that there was a strong positive association ($r = 0.91$) between their blood pressure reading and the number of times they visited the podiatrist. A strong positive association was also found both between the blood pressure reading ($r = 0.95$), the number of times they visited the podiatrist ($r = 0.82$), and their age.

Using this information, which of the following statements is true?

A. High blood pressure is more common as one ages and problems with feet occur more often with ageing. The association between the blood pressure reading and the number of podiatry visits can be explained by the common response variable, the age of the person.
B. It is just a coincidence that there is a strong positive association between the blood pressure reading and the number of podiatrist visits.
C. No-one likes people touching their feet, so their blood pressure goes up when they visit the podiatrist.
D. Whenever a person has a blood pressure reading, they have to go to the podiatrist.

Complex familiar

13. Consider the following data set.

x	1	2	3	4	5	6	7	8	9	10
y	55	40	42	38	35	43	51	40	47	60

a. Using technology, plot the data and fit a least-squares regression line.
b. Using technology determine the coefficient of determination and interpret its value.
c. Calculate the correlation coefficient and explain its meaning.
d. Calculate the predicted y values and the residuals and hence complete the table.

x	1	2	3	4	5	6	7	8	9	10
y	55	40	42	38	35	43	51	40	47	60
Predicted y-value										
Residual value										

e. Construct the residual plot and use it to comment on the appropriateness of the assumption that the relationship between the variables is linear.

14. Identify the explanatory and response variable for each of the following scenarios:
a. In a junior Science class, students plot the time taken to boil various quantities of water.
b. Extra buses are ordered to transport a number of students to the school athletics carnival.

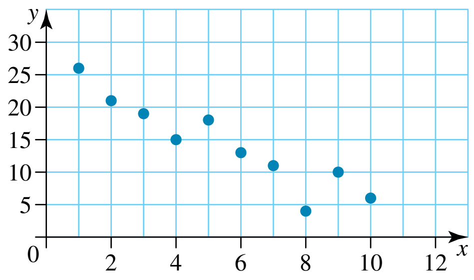
15. A scatterplot is drawn using the following data.

x	10	9	8	7	6	5	4	3	2	1
y	6	10	4	11	13	18	15	19	21	26

The following summary statistics has been obtained for this data.

$$\bar{x} = 5.5, s_x = 3.03, \bar{y} = 14.3, s_y = 6.86, r = -0.93$$

a. Comment on the form, direction and strength of the data.
b. Determine the least-squares regression line for this data.
c. Use your answer from part **b** to calculate the value of x when y is 17. Is this interpolation or extrapolation?



16. Pearson’s correlation coefficient for a scatterplot was found to be -0.7564.

a. Calculate the value of the coefficient of determination.

b. What would these values indicate to you about the strength of relationship between the two variables?

Complex unfamiliar

17. During an interview investigating the link between the sales of healthy snack foods (functional foods) and the increasing consumer demand for these products, an advertising expert made the following comment:

‘There is an association but it’s not causation ... our increasing need for healthy food and our laziness has resulted in mass innovation of functional foods.’

Explain why he might have stated there is no causative link between the sales of healthy foods and laziness.

18. An independent agency test-drove a random sample of current model vehicles and measured their fuel tank capacity against the average fuel consumption. Along with the following scatterplot, a regression equation of $y = 0.1119x + 0.6968$ was established.

a. Identify the response variable in this situation.

b. Rewrite the equation in terms of the explanatory and response variables.

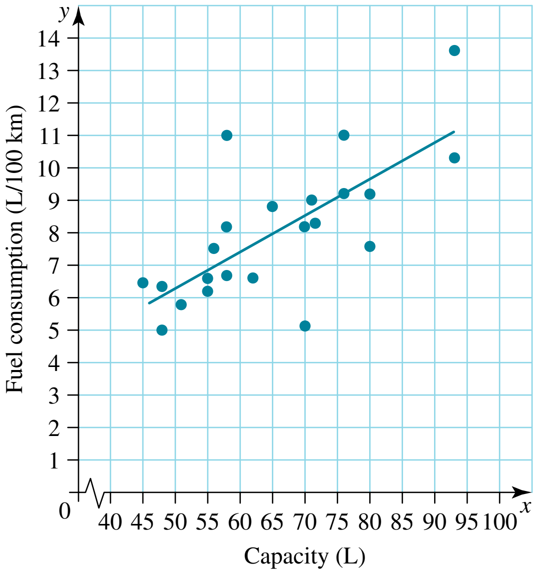
c. It is often said that smaller vehicles are more economical. Determine correct to 2 decimal places the fuel consumption of a vehicle that had a 40-litre fuel tank.

d. Is your answer to part **c** an example of interpolation or extrapolation? Explain your response.

e. Calculate, correct to the nearest whole number, the tank size of a vehicle that had a fuel consumption rate of 10.2 L per 100 km.

f. Pearson’s correlation coefficient for this data is 0.516. How can you use this value to evaluate the reliability of your data?

g. List at least two other factors that could influence the data.



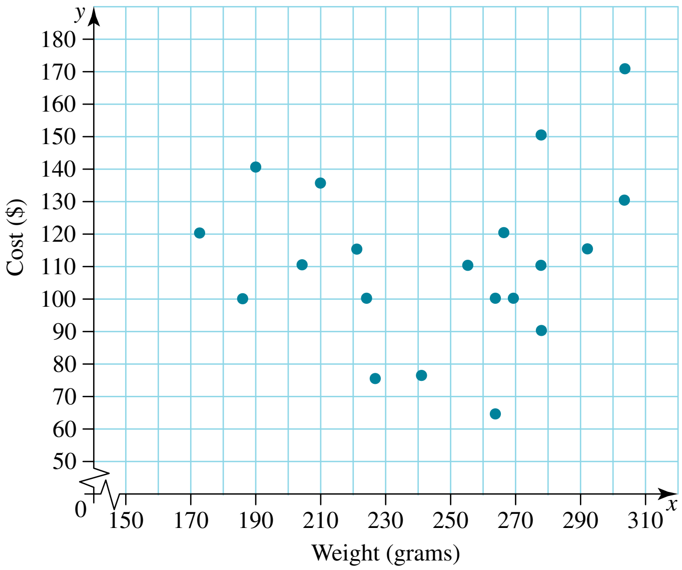
19. The weight of top brand runners was tracked against the recommended retail price, and the results were recorded in the following scatterplot.

a. Identify the explanatory variable for this situation.

b. How would you describe the relationship between these two variables?

c. The coefficient of determination for this data is $r^2 = 0.018\,72$. What conclusions can be established from this result?

d. Identify two external factors that could explain the distribution of the data points.



20. The Bureau of Meteorology records data such as maximum temperatures and solar exposure on a daily and monthly basis. The following data table, for the Botanical Gardens in Melbourne, shows the monthly average amount of solar energy that falls on a horizontal surface and the monthly average maximum temperature. (*Note:* The data values have been rounded to the nearest whole number.)

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Average solar exposure (MJ)	25	21	17	11	8	6	7	10	13	18	21	24
Average max daily temp. (°C)	43	41	34	33	24	19	24	24	28	32	25	40

a. Identify the explanatory and response variables for this situation.
b. Using technology, plot the data on a scatterplot.
c. Describe the trend of the data.
d. Calculate Pearson's coefficient and coefficient of determination for this data. What do these values tell you about the reliability of the data?
e. Plot the regression line for this data and write the equation in terms of the variables.
f. Using your equation, calculate the amount of solar exposure for a monthly maximum temperature of 37°C.
g. Extrapolate the data to find the average maximum temperature expected for a month that recorded an average solar exposure of 3 MJ.
h. Explain why part **g** is an example of extrapolation.

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Units 3 & 4 Sit exam

Answers

Exercise 2.2 Review of the general equation of a straight line

1. a. $y = mx + c$

b. m is the gradient of the graph and c is the y-intercept.

2. a. $m = -5, c = 4$ **b.** $m = 3, c = 11$

c. $m = -6,\ c = 0$ **d.** $m = \frac{5}{2}, c = 5$

e. $m = \frac{2}{3}, c = -6$ **f.** $m = 2, c = -1$

3. Complete the sentences below.

a. The equation $y = 3x - 1$ has a gradient of 3 units. This means that for every increase of 1 unit in the horizontal direction there is an increase of 3 units in the vertical direction.

b. The equation $y = -x + 1$ has a gradient of -1 unit. This means that for every increase of 1 unit in the horizontal direction there is an increase of -1 unit in the vertical direction.

c. The equation $y = \frac{1}{3}x - 1$ has a gradient of $\frac{1}{3}$ units. This means that for every increase of 3 units in the horizontal direction there is an increase of 1 unit in the vertical direction.

4. a. 2 **b.** 1 **c.** 1

d. $-\frac{1}{2}$ **e.** $-\frac{1}{4}$ **f.** $\frac{4}{3}$

g. $-\frac{4}{3}$ **h.** $-\frac{2}{7}$ **i.** 0

5. a. $y = 2x - 2$ **b.** $y = -2x - 1$

c. $y = -\frac{3}{8}x + 2$ **d.** $y = 4$

e. $y = x$ **f.** $y = -0.25x - 4$

6. a. $y = \frac{7x}{5} - \frac{1}{5}$ or $5y = 7x - 1$

b. $y = \frac{5x}{9} + \frac{34}{9}$ or $9y = 5x + 34$

c. $y = x + 6$

d. $y = 3$

e. $y = \frac{4x}{9} + \frac{4}{9}$ or $9y = 4x + 4$

f. $y = \frac{9x}{5} + \frac{7}{5}$ or $5y = 9x + 7$

7. a. i. $(0, -3)$

ii. $m = 2$

iii. $y = 2x - 3$

b. i. $(0, 0)$

ii. $m = \frac{1}{4}$

iii. $y = \frac{1}{4}x$

c. i. $(0, 6)$

ii. $m = -3$

iii. $y = -3x + 6$

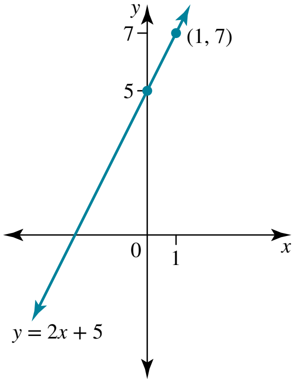
d. i. $(0, -3)$

ii. $m = \frac{5}{2}$

iii. $y = \frac{5}{2}x - 3$

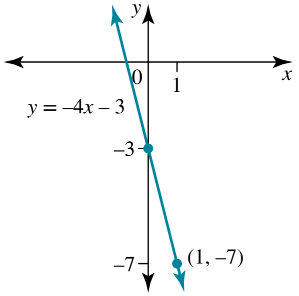
8. a. i. $m = 2, (0, 5)$

ii.



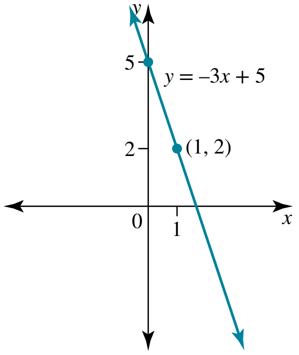
b. i. $m = -4, (0, -3)$

ii.



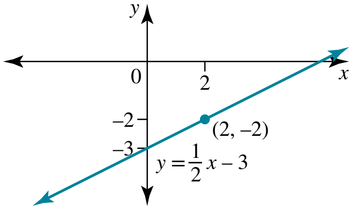
c. i. $m = -3, (0, 5)$

ii.



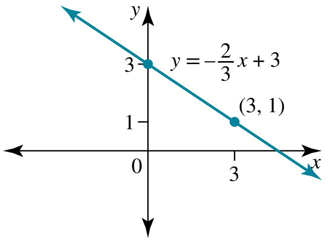
d. i. $m = \frac{1}{2}, (0, -3)$

ii.



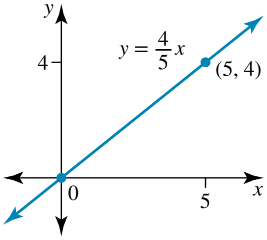
e. i. $m = -\frac{2}{3}, (0, 3)$

ii.

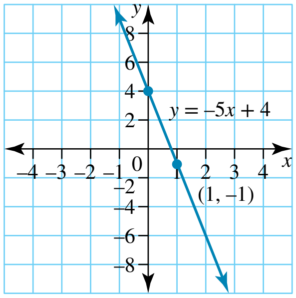


f. **i.** $m = \frac{4}{5}, (0, 0)$

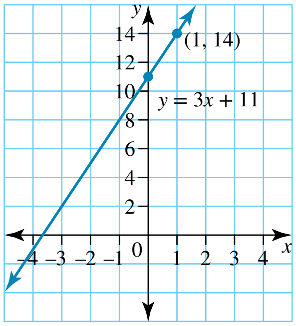
ii.



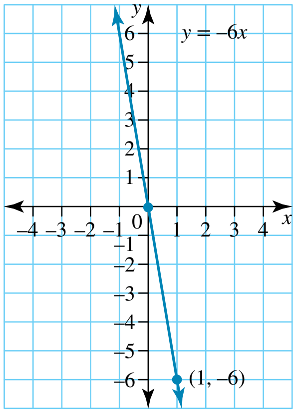
9. a.



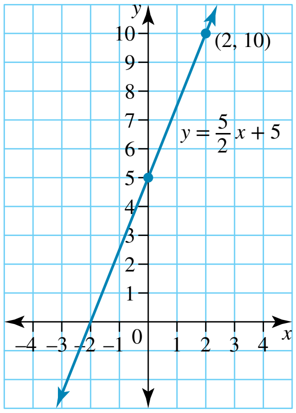
b.



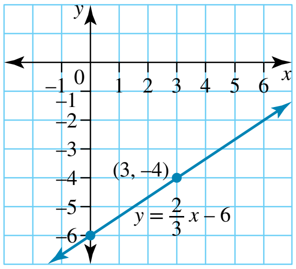
c.



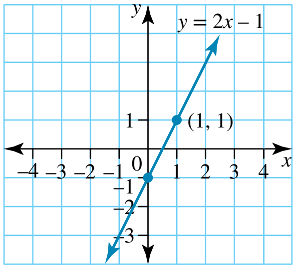
d.



e.



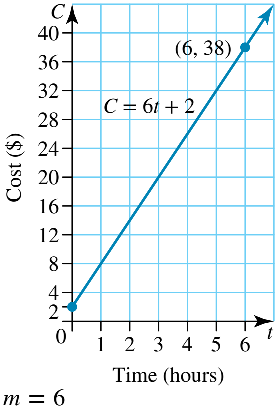
f.



10. a.

t	1	2	3
C	8	14	20

b.



c. $m = 6$

d. $C = 6t + 2$

e. $32

f. Answers will vary.

11. **a.** $y = 1$ **b.** $y = -7$ **c.** $y = 25$

d. $y = 41$ **e.** $y = -27$ **f.** $y = -19$

g. $y = 17.8$ **h.** $y = 3$

12. **a.** $x = -\frac{1}{4}$ **b.** $x = 2$ **c.** $x = -3$

d. $x = \frac{18}{4}$ **e.** $x = \frac{1}{2}$ **f.** $x = -0.1$

g. $x = -\frac{21}{4}$ **h.** $x = 0.375$

13. **a.** **i.** $237

ii. $475

b. **i.** $55

ii. $3.50

c. 70

d. $4.29

e. Yes

14. **a.** B

b. **i.** $280

ii. $360

iii. $600

c. **ii.** save $120 and

iii. save $600

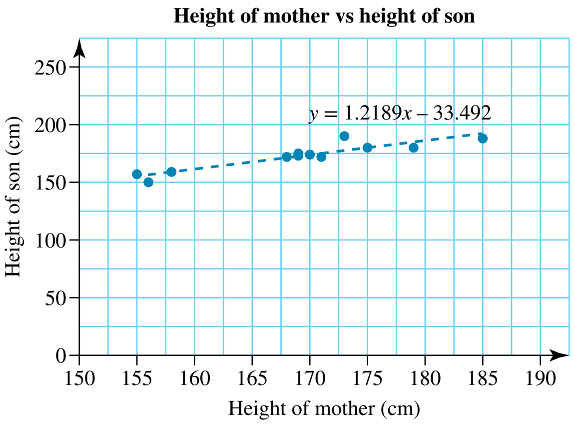
d. 7

Exercise 2.3 Fitting a least-squares line to data

1. a. The Mathematics exam mark
b. 0.95
c. −6.30
d. 74%

2. a. $y = 91.78 + 3.33x$
b. $y = -0.15x + 21.87$
c. $y = 52.38x + 8890.48$
d. $y = 30 - x$

3. a. The response variable is the height of the son
b.



c. $S = -33.49 + 1.219M$

4. a. −1.837 **b.** 1.701
c. Positive direction

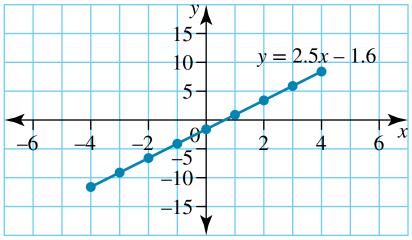
5. a. 105.9 **b.** −1.476
c. Negative direction

6. a. Age in months **b.** 2.029 mL
c. 7.5 months old

7. a. Cost per night **b.** \$155.74
c. 3.31 km

8. The trend line is negative.

9. a.



b. Below the regression line

10. a. 60 **b.** −5
c. Negative **d.** $y = -140$

11. a. 3.90
b. Lucy incorrectly transposed the 12.9. She should have moved this first before dividing by 7.32.

12. a. −12 **b.** 25
c. Positive **d.** 75.5

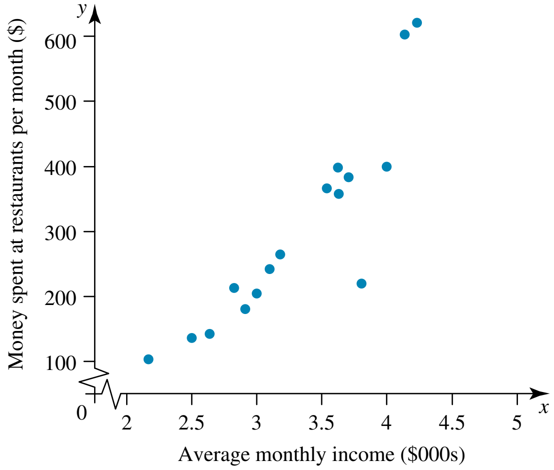
13. a. Number of insects caught **b.** 1.1
c. Positive **d.** 66

14. $y = 4 + 2.5x$

15. a. $y = -1.783x + 39.41$. See the scatterplot in the worked solutions.
b. $y = -1.599$
c. When extrapolating data, it is assumed that the data will continue to follow the same trend line.

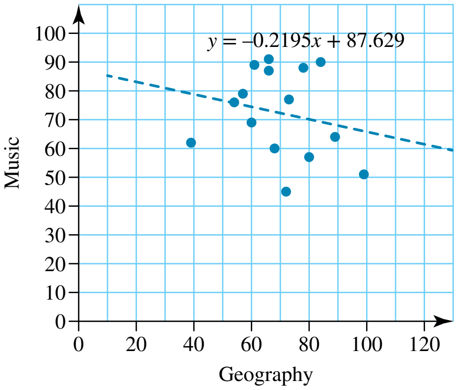
16. a. No of mosquitoes $= 10.2 + 0.5 \times$ temperature of fire
$= 10.2 + 0.5 \times 240$
$= 130.2$
The number of mosquitoes is 130 (correct to the nearest whole number)
b. $t = 3.6$°C
c. Mosquitoes are in hibernation in the cooler months of the year, so once the temperature drops below a certain level this model would not be appropriate. Also, the location of the fire, air temperature, wind conditions, proximity to water, etc. would impact on the mosquito population.

17. a.



b. $R = -459.8 + 229.5I$
c. \$ 687.70
d. Part **c** asks you to predict outside of the original data set range.
e. \$ 3160, interpolation

18. a. There is no obvious explanatory variable.
b.



c. $M = 87.63 - 0.2195G$
d. 69
e. Not very confident. The graph does not indicate a strong correlation between the two variables.

f. $r = -0.2172$. The value of r is low, showing that the line of best fit is poor and not a reliable predictor

19. a. Calories burned

b. Calories burned = 14301 + 115.02 × distance walked

c. 20 052. Interpolation, as this data is inside the original range.

d. 15 451.2. Extrapolation, as the explanatory variable provided is outside the original data range.

e. An r value of 0.9678 indicates a very strong positive linear relationship, indicating the relationship between the two variables is very strong and can be used to draw conclusions.

f. Examples: speed of walking, difficulty of walking surface, foods eaten.

20. a. Age of the people

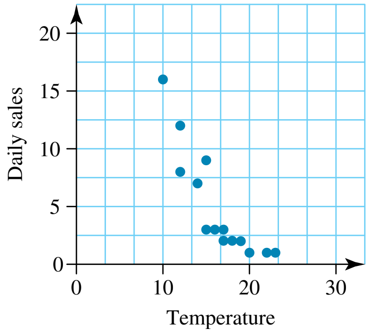
b. Number of social media friends = $-13.613 \times age + 777.84$

c. Since $r = -0.893$, the relationship between the two variables is a strong negative linear relationship.

d. 301; this is interpolation as the predicted value is within the given data values.

Exercise 2.4 The coefficient of determination and residual plots

1. a.



b. Daily sales = 22.50 − 1.07 × Temperature

c. $r^2 = 0.74$, $r = -0.86$

d. An r value of -0.86 implies that there is a strong negative linear relationship, which is an indication that an increase in temperature could mean a decrease in daily sales numbers of gumboots.

An r^2 value of 0.74 implies that 74% of the variation in the daily sales of gumboots can be explained by the variation in the temperature. 26% is explained by other factors.

2. D

3. B

4. a. 0.7962

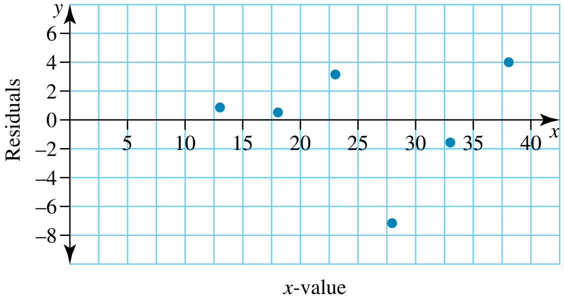
b. 0.9994

c. 79.62 of the variation in a child's health can be explained by the variation in their diet. 20.38% is explained by other factors.

99.94% of the variation in the amount of water in the oceans can be explained by global warming, 0.06% can be explained by other factors.

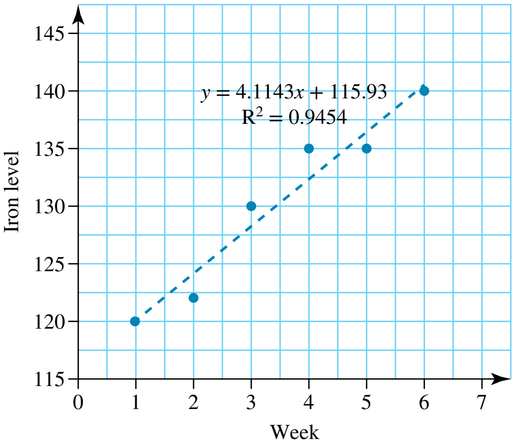
5. * See the table at the bottom of the page.

6.



As the residuals appear to be randomly spread above and below the x-axis, this linear model is an appropriate model for this set of data.

7. a.



Iron level = 4.11 × Week + 115.93

b. ** See the table at the bottom of the page.

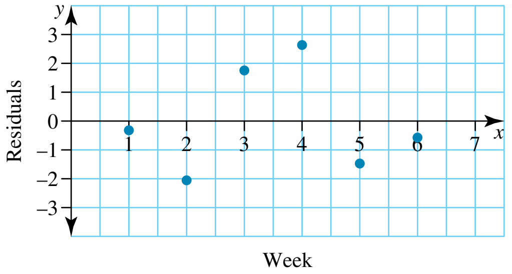
***5.**

x	13	18	23	28	33	38
Actual y-value	25	31	40	36	48	60
Predicted y-value	24.16	30.51	36.86	43.21	49.56	55.91
Residual value	0.84	0.49	3.14	−7.21	−1.56	4.09

****7. b.**

Week of experiment	1	2	3	4	5	6
Iron level	120	122	130	135	135	140
Predicted iron level	120.04	124.15	128.26	132.37	136.48	140.59
Residual value	−0.04	−2.15	1.74	2.63	−1.48	−0.59

c.



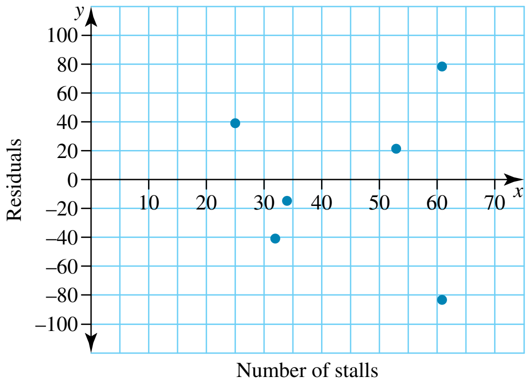
The residuals appear to be scattered randomly about the horizontal axis, so this suggests that the linear model is a suitable model for this data.

d. $r^2 = 0.9454,\ r = 0.9723$

e. The iron level reaches 155 g/L by week 9.5, so according to this model the iron level will reach 155 g/L in week 10. A study of the initial scatterplot of the data suggests that the data shows a linear trend. The residual plot shows no pattern about the horizontal axis and Pearson's product–moment correlation coefficient of 0.9723 indicates a strong positive linear correlation, so the model is suitable to make predictions. However, the iron level of 155 g/L goes outside the range of the given data, so this needs to be considered when making the prediction.

8. a. * See the table at the bottom of the page.

b.



The residuals are scattered randomly above and below the horizontal axis, so this suggests that the linear model is suitable.

c. $r = 0.88$

d. Yes

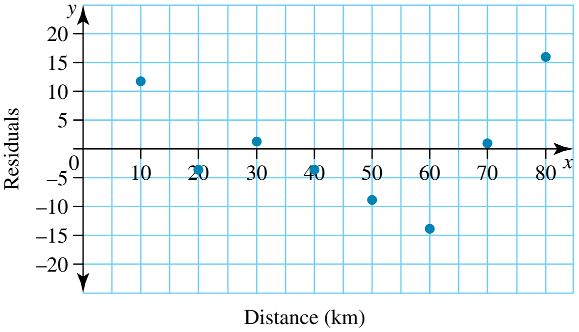
9. a. Cost for factory1 = 1.51 × Distance from Brisbane + 43.21

Cost for factory 2 = 0.96 × Distance from Brisbane + 56.61

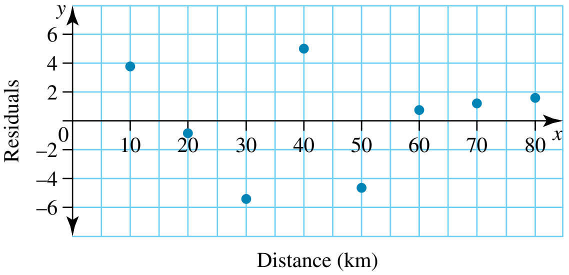
Refer to the worked solutions for the scatterplot.

b. ** See the table at the bottom of the page.

c. Factory 1



Factory 2



The residual plots for both factories appear to be randomly distributed about the horizontal axes, so the linear model is suitable for predictions.

d. Factory 1

$r^2 = 0.9332,\ r = 0.9660$

Factory 2

$r^2 = 0.9763,\ r = 0.9881$

---

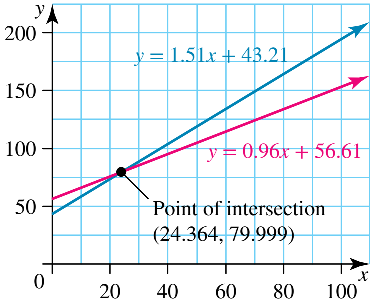
*8. a.

Number of stalls	53	34	61	32	61	25
Number of visitors	501	339	611	300	450	333
Predicted number of visitors	480	354	533	341	533	294
Residual value	21	−15	78	−41	−83	39

**9. b.

Distance from Brisbane (km)	10	20	30	40	50	60	70	80
Factory 1 cost ($)	70	70	90	100	110	120	150	180
Predicted cost for factory 1	58.31	73.41	88.51	103.61	118.71	133.81	148.91	164.01
Residuals for factory 1	11.69	−3.41	1.49	−3.61	−8.71	−13.81	1.09	15.99
Factory 2 cost ($)	70	75	80	100	100	115	125	135
Predicted cost for factory 2	66.21	75.81	85.41	95.01	104.61	114.21	123.81	133.41
Residuals for factory 2	3.79	−0.81	−5.41	4.99	−4.61	0.79	1.19	1.59

e.



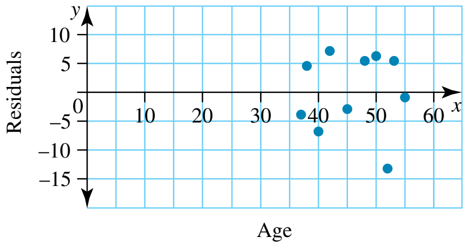
Using the above linear graphs, based on the regression models, if the shop is in Brisbane it is more cost effective to have the surfboards delivered from factory 1.

f. If the shop is more than 24.36 km from Brisbane, as in the case of Mytown, then it is more cost effective to get the surfboards delivered from factory 2.

10. a. $SBP = 74.35 + 1.61 \times Age$

*See the table at the bottom of the page.

b.



The dots are scattered randomly around the horizontal axis, so the linear model is suitable for this data.

c. $SBP = 74.35 + 1.61 \times Age$

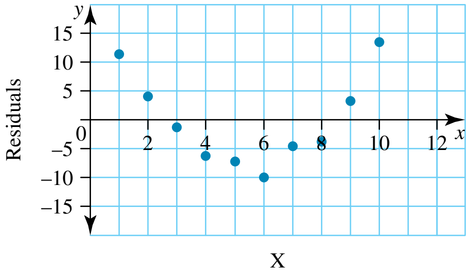
$= 74.35 + 1.61 \times 75$

$= 195.10$

Extrapolating outside the range of data given gives a predicted systolic blood pressure for someone aged 75, which could not be considered reliable.

11. a. **See the table at the bottom of the page.

b.



The residuals form a pattern around the horizontal axis, therefore the linear model is not a good model for this set of data.

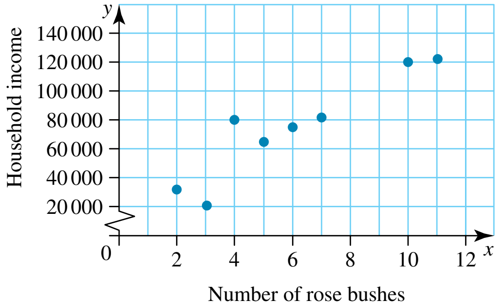
12. A

13. D

14. 78% of the variation in the **response** variable can be explained by the variation in the **explanatory** variable. **22%** of the variation can be explained by other factors.

Exercise 2.5 Association and causation

1. a.



b. From the scatterplot there appears to be a strong positive association between the number of rose bushes in a home garden and the annual household income.

c. $r = 0.93$

d. Just because $r = 0.93$, it cannot be stated that increasing the number of rose bushes in a home garden will increase the annual household income. Other factors may need to be considered. Further investigation is needed before any conclusions can be drawn.

***10. a.**

Age	37	38	40	42	45	48	50	52	53	55
Systolic blood pressure	130	140	132	149	144	157	161	145	165	162
Predicted systolic blood pressure	133.92	135.53	138.75	141.97	146.8	151.63	154.85	158.07	159.68	162.9
Residual value	−3.92	4.47	−6.75	7.03	−2.8	5.37	6.15	−13.07	5.32	−0.9

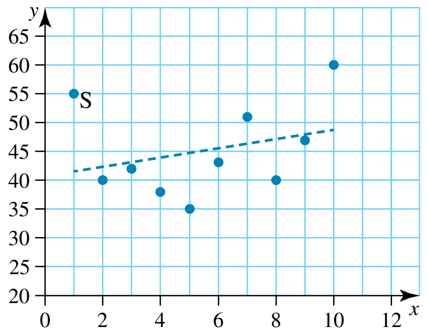
****11. a.**

x	1	2	3	4	5	6	7	8	9	10
y	1	5	10	16	26	34	50	62	80	101
Predicted	−10.454	0.425	11.304	22.183	33.062	43.941	54.82	65.699	76.578	87.457
Residuals	11.454	4.575	−1.304	−6.183	−7.062	−9.941	−4.82	−3.699	3.422	13.543

2. There is a negative association of −0.86 between the amount of time spent playing computer games and the level of aerobic fitness.
3. D
4. D
5. B
6. No. More fire fighters are called to larger fires, so the damage is due to the size of the fire.
7. No. Larger shoe sizes are associated with age and so is greater academic knowledge. Age is the common cause.
8. It cannot be said that the consumption of the supplement causes improvement in health. Upon taking the supplement, the students may have also implemented other lifestyle changes, such as giving up smoking, lower alcohol consumption, eating a healthier diet or exercising more. There is no information about the gender or the age of the participants which also may have had an impact on the health improvements.

2.6 Review: exam practice

1. B
2. A
3. C
4. B
5. C
6. D
7. B
8. D
9. C
10. A
11. D
12. A
13. a.



Equation of least-squares regression line

$$y = 0.7939x + 40.733$$

b. $r^2 = 0.0901$

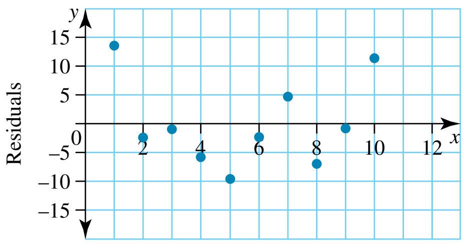
9% of the variation in y can be explained by the variation in x.

c. $r = 0.30$

There is a weak positive linear association between the x and y variables.

d. *See the table at the bottom of the page.

e.



The residuals do not appear to be randomly scattered about the horizontal axis. The first point on the left is at about 13, then the points fluctuate above and below the axis, finishing at about 11 on the right. Given the value of r is 0.30, indicating a weak association, it is not appropriate to assume that the relationship between these variables is linear.

14. a. Amount of water is the explanatory variable and time is the response variable.
 b. Number of students is the explanatory variable and number of buses is the response variable.
15. a. The data shows a strong, negative linear relationship. The correlation coefficient of −0.93 also indicates a strong, negative linear relationship.
 b. $y = 25.91 - 2.11x$
 c. 4.22, interpolation
16. a. $r^2 = 0.5721$
 b. A strong, negative, linear association
17. Although there appears to be a link between the laziness of people and the increase in sales of healthy foods, there are also many other possible factors besides laziness; for example, people are very time poor, unsure of what constitutes healthy food, and are lacking confidence and the skills to cook for themselves. Based on this observation alone, the cause of an increase in sales of healthy foods cannot be concluded to be due to laziness.
18. a. Fuel consumption
 b. Fuel consumption = 0.1119 × Capacity + 0.6968
 c. 5.17 L/100 km
 d. This is an example of extrapolation as 40 L capacity is outside the range of data given.
 e. 85 L
 f. This value indicates a moderate relationship between the variables. Therefore, the data can be used, but other factors should also be considered.
 g. Fuel consumption can be influenced by the size of the engine, the size of the vehicle, the type of driving (city, country) and the age of the vehicle.
19. a. Weight (grams)
 b. No correlation
 c. This supports the view that there is no correlation between the variables. Based on this value, no conclusions can be made from the data.
 d. Various answers are possible, e.g. popularity of the shoe or desired profits.

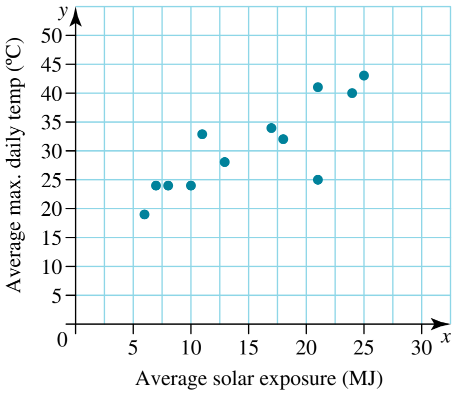
---

*13. d.

x	1	2	3	4	5	6	7	8	9	10
y	55	40	42	38	35	43	51	40	47	60
Predicted y value	41.53	42.32	43.11	43.91	44.70	45.50	46.29	47.08	47.88	48.67
Residual value	13.47	−2.32	−1.11	−5.91	−9.70	−2.50	4.71	−7.08	−0.88	11.33

20. **a.** Explanatory variable = average solar exposure
Response variable = maximum daily temperature

b.

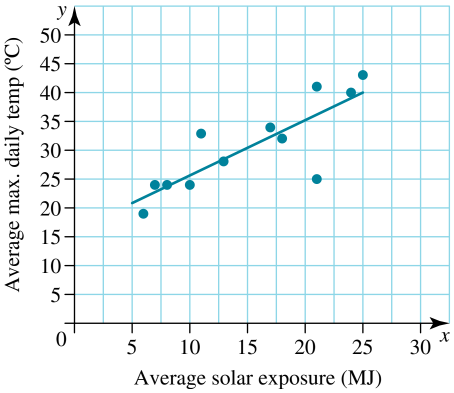


c. Strong positive correlation

d. $r = 0.8242$; $r^2 = 0.6793$

These values indicate a strong relationship between the two variables. The coefficient of determination suggests that nearly 70% of the maximum daily temperature is due to the amount of solar exposure.

e.



$$\text{Maximum daily temperature} = 16.232 + 0.9515 \times \text{average solar exposure}$$

f. 22 MJ

g. 19°C

h. An average solar exposure of 3 MJ is outside the original data set.

REVISION UNIT 3 Bivariate data, sequences and change, and Earth geometry

TOPIC 1 Bivariate data analysis

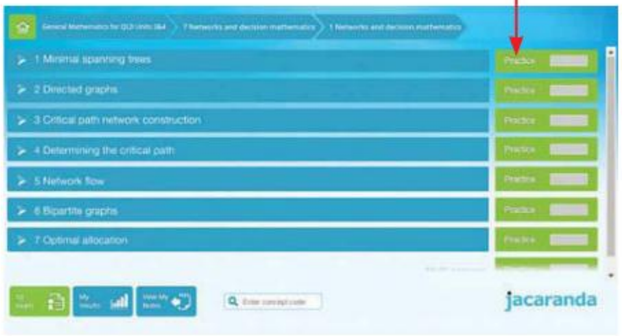
- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *General Mathematics for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level *OR* select **Practice** to answer all practice questions available for each area.

General Mathematics for QLD Units 3&4

1 Bivariate data analysis
2 Time series analysis
3 Growth and decay in sequences
4 Earth geometry and time zones
5 Loans, investments and annuities
6 Graphs and networks
7 Networks and decision mathematics

- Select **Practice** at the sequence level to access all questions in the sequence.

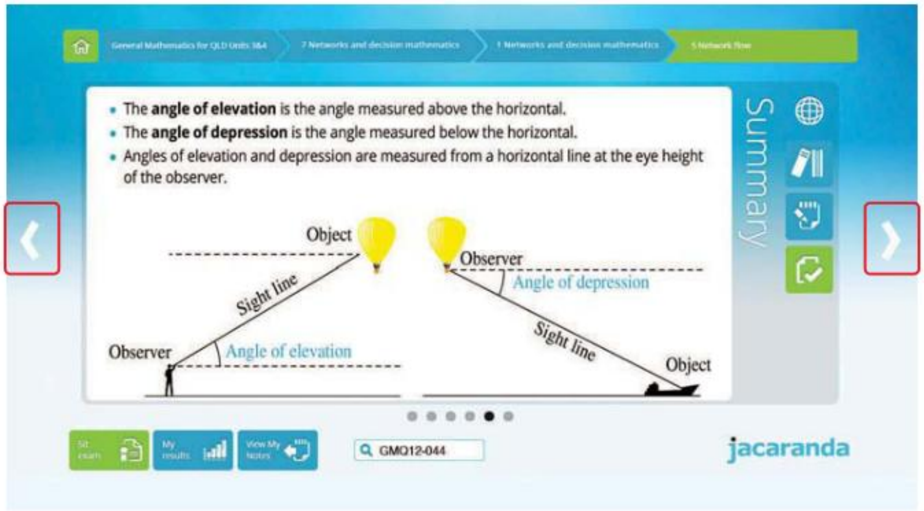


1 Minimal spanning trees
2 Directed graphs
3 Critical path network construction
4 Determining the critical path
5 Network flow
6 Bipartite graphs
7 Optimal allocation

- At **sequence level**, drill down to concept level.

7 Networks and decision mathematics
1 Networks and decision mathematics

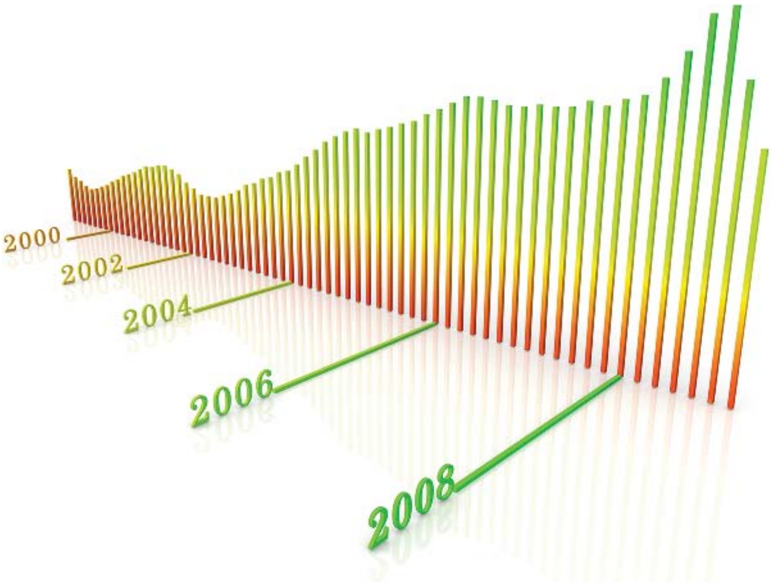
- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



- The **angle of elevation** is the angle measured above the horizontal.
- The **angle of depression** is the angle measured below the horizontal.
- Angles of elevation and depression are measured from a horizontal line at the eye height of the observer.

3 Time series analysis

3.1 Overview



Modelling using time series data can be used for forecasting and monitoring predicted sales figures, population growth, mortality rates and long-term health outcomes, endangered species, weather patterns, food consumption and production, economic growth, interest rate fluctuations, and future consumer and voting trends. This information can be used to improve people's lives and ensure that our planet continues to thrive. It can also be used to influence choices and monitor behaviour. The more information that government, political lobby groups, business and other organisations and interest groups can obtain, the more the information can be used in both positive and negative ways.

Those who study time series data are interested in trends and patterns in the data. These may be short-term or long-term trends and patterns. Climate scientists at NASA have used time series data and other technologies to determine that the Earth's average surface temperature has increased by 1.1 degrees Celsius since the late 19th century, with 16 of the 17 warmest years on record occurring since 2001.

Every action that connects a person to a website leaves a digital footprint. This footprint can then be used to track an individual's political ideology, friends, consumer habits, credit rating, personal hobbies, location and interests. This may have a significant impact on an individual's ability to obtain employment, purchase a house or a car, have a relationship and travel overseas. It is important that individuals are aware and informed regarding the collection of their own personal data and the ways in which data can be used to manipulate choices and hence patterns of behaviour.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

3.2 Constructing and describing time series plots

3.2.1 Time series data

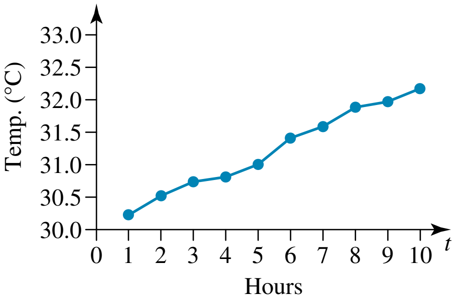
Time series data are any data that have time as the explanatory variable.

The data can be graphed, and the graphs then used to determine if a **trend** is present in the data. A trend is a long-term upward or downward direction. Identifying a trend can assist in making predictions about the future.

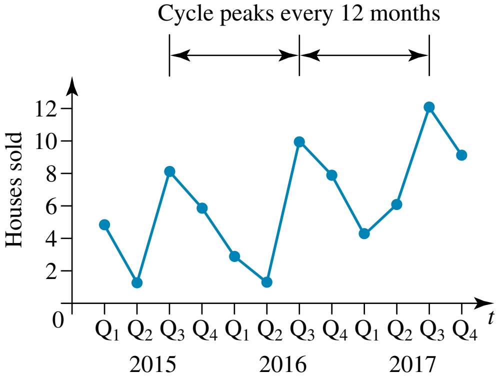
The points on a time series graph are usually joined.

3.2.2 Types of trends

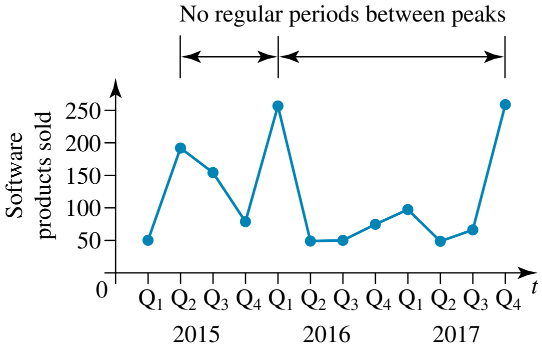
A **general upward or downward trend** is a graph that overall goes up or down. The following graph is an illustration of an overall upward trend.



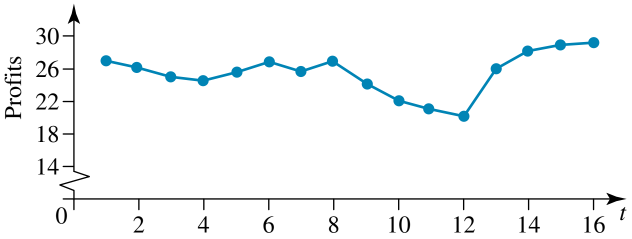
A **seasonal pattern** displays fluctuations that are systematic and repeat at the same time each week, month or quarter and usually last less than one year. They are calendar related. The following graph illustrates that the peak selling time for houses is spring (Q3).



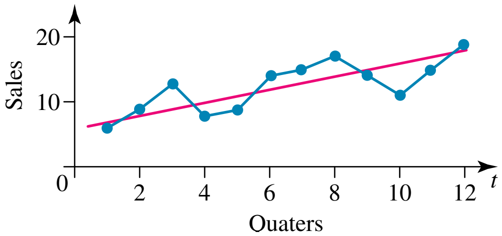
A **cyclical pattern** displays fluctuations that repeat but will usually take longer than a year to repeat. An example of this is shown in the following graph, which depicts software products sold.



Irregular fluctuations do not show any clear pattern. They are short-term fluctuations, unsystematic and random. They are usually caused by unpredictable events such as drought or an economic downturn, as illustrated in the following graph.



Trends can work in combinations; for example, you can have a seasonal pattern with an upward trend.



Note: Sometimes one or more points may lie outside the overall trend. These could be one-off unanticipated events called outliers.

WORKED EXAMPLE 1

Draw a scatterplot of the following data that show the sales revenues ($) for the first 8 years of a business selling homewares online.

Describe any trends present in the data.

Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Sales($)	**132 600**	**143 800**	**137 600**	**139 800**	**141 300**	**144 500**	**147 700**	**146 400**

THINK	**WRITE**
1. Identify the explanatory and the response variables. As this is times series data, the explanatory variable is the year.	Explanatory variable — Year Response variable — Sales ($)
2. Draw the scatterplot.	
3. Examine the graph to identify any trends present.	The second data point is possibly an outlier, but overall the data appears to be following an upward trend.

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Units 3 & 4 > Area 2 > Sequence 1 > Concepts 1 & 2

Time series plots Summary screen and practice questions
Trends in time series data Summary screen and practice questions

Exercise 3.2 Constructing and describing time series plots

1. Describe the different types of trends in your own words: upward and downward trends, seasonal patterns, cyclical patterns and irregular fluctuations.

For questions 2–4, what type of trend would you expect to see in the scenarios given?

2. **MC** The average price of houses in Queensland over the past 30 years recorded yearly.
   A. Upward **B.** Downward **C.** Cyclical **D.** Seasonal
3. **MC** The amount of rain per month over the past 5 years.
   A. Upward **B.** Downward **C.** Cyclical **D.** Seasonal
4. **MC** The sale of snow ski equipment at a shop in the village at Mt Buller.
   A. Upward **B.** Downward
   C. Cyclical **D.** Seasonal

5. **WE1** Draw a scatterplot of the following data that show the number of people who watch a news broadcast on a given day of the week over a two week period and describe any trends present in the data.

Day	M	T	W	Th	F	S	Su	M	T	W	Th	F	S	Su
Number of views (000 000s)	1.20	1.18	1.16	1.18	0.90	0.75	1.00	1.21	1.23	1.19	1.16	0.95	0.68	0.98

6. Identify whether the following trends are likely to be seasonal, cyclic or random:
   a. the amount of rainfall, per month, in north Queensland
   b. the number of soldiers in the Australian army, measured annually
   c. the number of people living in Australia, measured annually
   d. the share price of BHP, measured monthly
   e. the number of seats held by the Liberal Party in Federal Parliament.

7. The following table shows the share price of a new technology company taken at the close of business on a Friday for eight weeks.

Week	1	2	3	4	5	6	7	8
Price	5.50	10.80	16.20	21.40	26.20	30.80	34.40	38.00

   a. Plot a time series graph for this data.
   b. Describe the trend in the graph.

8. The following table shows the number of hot chocolate drinks sold at a café each month.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Number of drinks sold	100	92	122	372	469	680	675	702	643	589	437	234

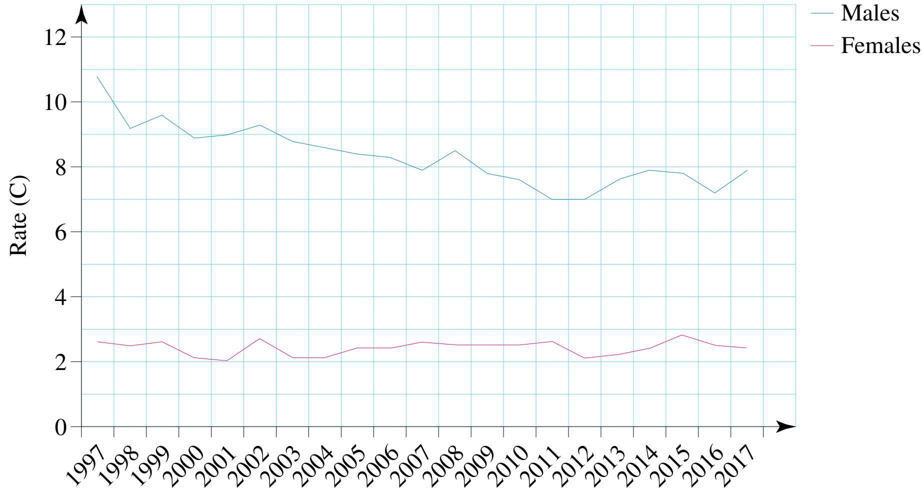
a. Plot a time series graph for this data.
b. Describe the trend in the graph.

9. The following table shows the number of pairs of bathers sold by a shop in four and a half years.

Season	Summer 14	Autumn 15	Winter 15	Spring 15	Summer 15	Autumn 16	Winter 16	Spring 16	Summer 16	Autumn 17	Winter 17	Spring 17	Summer 17	Autumn 18	Winter 18	Spring 18	Summer 18	Autumn 19
Number of pairs of bathers sold	312	182	126	289	308	232	162	296	345	233	180	266	455	321	233	388	516	409

a. Plot a time series graph for this data.
b. Describe the trend in the graph.

10. The following time series graph shows the standardised death rates for alcohol-induced deaths over the 10-year period from 1997–2017 (a standardised death rate is a statistical measure of the death rate of a population).



Source: Australian Bureau of Statistics.

a. Describe any trend in the rate of male alcohol induced deaths for the 20 year period 1997–2017.
b. Describe any trend in the rate of female alcohol induced deaths for the 20 year period 1997–2017.
c. What is the change in the difference in the rate of alcohol induced deaths for males and females from 1997 to 2017? Comment on this change.

3.3 Fitting a least-squares line to time series data

3.3.1 Using technology to fit a least-squares regression line

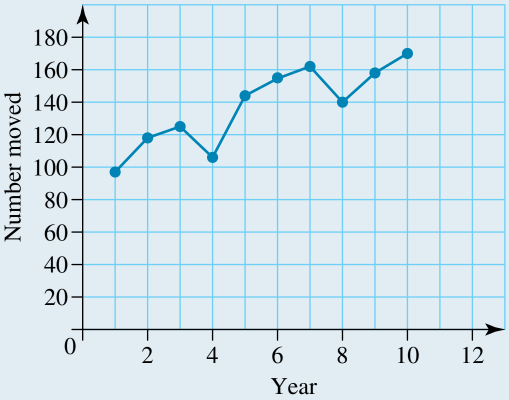
A least-squares regression line can be used to model long-term trends in time series data. This is called forecasting.

WORKED EXAMPLE 2

Data was recorded about the number of families who have moved from Brisbane to Maroochydore over the past 10 years.

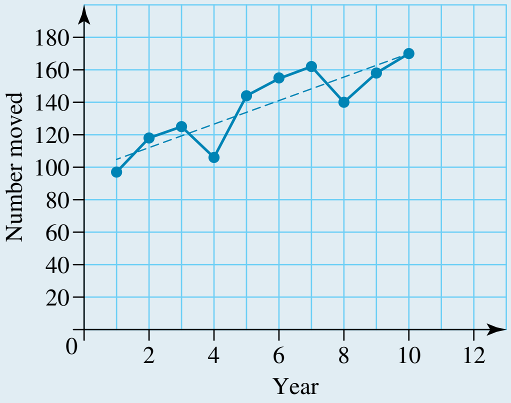
a. **Using appropriate technology, construct a graph of the time series data.**
b. **Describe any trends present.**
c. **Draw a least-squares regression line and determine its equation.**
d. **Use this line to predict how many people will move in 2030. Comment on this answer.**

Year	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**
Number moved	**97**	**118**	**125**	**106**	**144**	**155**	**162**	**140**	**158**	**170**

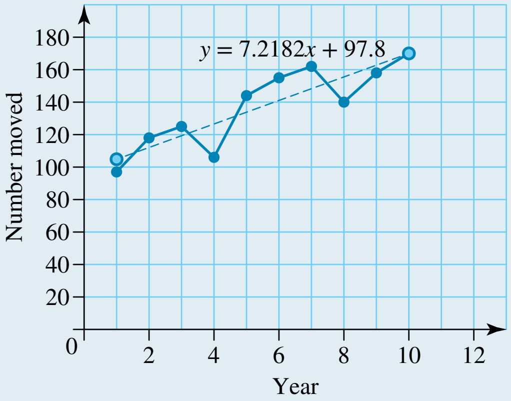
THINK	**WRITE**
a. Using a spreadsheet, enter the data then draw a scatterplot. *Note:* The values for the years have been entered as 1, 2, 3 etc. We do this because it is easier to work with smaller numbers and convert back to the year when required.	**a.** 
b. Describe any trends present.	**b.** There appears to be an overall upward trend in the 10 years.

c. 1. To add the least-squares regression line, select the symbol for chart elements in the top right corner of the graph and tick the box next to trendline.

c.



2. To include the least-squares regression equation, place the cursor on the least-squares line and left click, under the Format Trendline panel, then tick the box next to Display Equation on chart.



3. Write the equation with correct variable names.

$y = 7.2182x + 97.8$

Number moved = 97.8 + 7.2182 × Year

d. Substitute Year = 22 for 2030.

d. Number moved = 97.8 + 7.2182 × Year
= 97.8 + 7.2182 × 22
= 256.6

256 people will move in 2030.

Comment on the result

This linear model predicts that approximately 256 families will move from Brisbane to Maroochydore in 2030. This answer assumes a continuation of the upward trend and does not allow for issues such as a change in the economy, a change in house prices or a natural disaster.

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Units 3 & 4 > Area 2 > Sequence 1 > Concept 3

Fitting a least-squares line to time series data Summary screen and practice questions

Exercise 3.3 Fitting a least-squares line to time series data

1. **WE2** The following data represents the body temperature of a patient with appendicitis, taken every hour.

Hour	1	2	3	4	5	6	7	8	9
Temp (°C)	37.2	37.5	37.8	37.9	38.0	38.5	38.6	39.0	39.2

   a. Using appropriate technology, construct a graph of the time series data.
   b. Describe any trends present.
   c. Draw a least-squares regression line and determine its equation using technology.
   d. Use this line to predict the temperature of the patient after 12 hours. Comment on the answer.

2. Using technology, determine the least-squares regression line for the following data. What type of trend is best shown by these data?

t	1	2	3	4	5	6	7	8	9	10	11
y	6	9	13	8	9	14	15	17	14	11	15

3. The monthly share prices of a recently privatised telephone company were recorded as follows.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Price ($)	2.50	2.70	3.00	3.20	3.60	3.70	3.90	4.20

   Graph the data (let 1 = Jan., 2 = Feb. and so on) and fit a least-squares regression line to the data. Use this line to predict the share price in January of the following year. Comment on the feasibility of the predicted share price.

4. Plot the following monthly sales data for umbrellas.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales	100	90	60	35	20	10	5	10	15	40	70	95

   Determine the equation of a least-squares regression line. Discuss the type of trend best reflected by the data and the limitations of your trend line.

5. Consider the following data, which represent the price of oranges over a 19-week period.

Weeks	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Price (cents)	40	45	50	45	40	45	60	54	65	60	70	60	65	75	70	65	75	80	80

The summary statistics for this data are as follows (where n = week number, $\overline{n}$ = average of week number, p = price, $\overline{p}$ = average price, s_n = standard deviation of the number of weeks, s_p = standard deviation of the price and r = Pearson's correlation coefficient).

$$\overline{n} = 10, \quad s_n = 5.63, \quad \overline{p} = 60.21, \quad s_p = 13.20, \quad r = 0.93$$

a. Determine the equation of the least-squares regression line for the data.
b. Predict the price in week 25 to the nearest cent.

6. The following table represents the quarterly sales figures (in thousands) of a popular software product. Using technology plot the data and fit a least-squares regression line. Discuss the type of trend best reflected by these data.

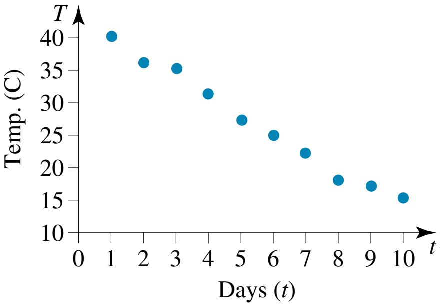
Quarter	Q1-2017	Q2-2017	Q3-2017	Q4-2017	Q1-2018	Q2-2018	Q3-2018	Q4-2018	Q1-2019	Q2-2019	Q3-2019	Q4-2019
Sales	120	135	150	145	140	120	100	110	120	140	190	220

7. The number of employees at the Comnatpac Bank was recorded over a 10-month period. Plot and fit a trend line to the data. What would you say about the trend?

Month	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Number of employees	6100	5700	5400	5200	4800	4400	4200	4000	3700	3300

8. The summary statistics for the following time series data plotted are

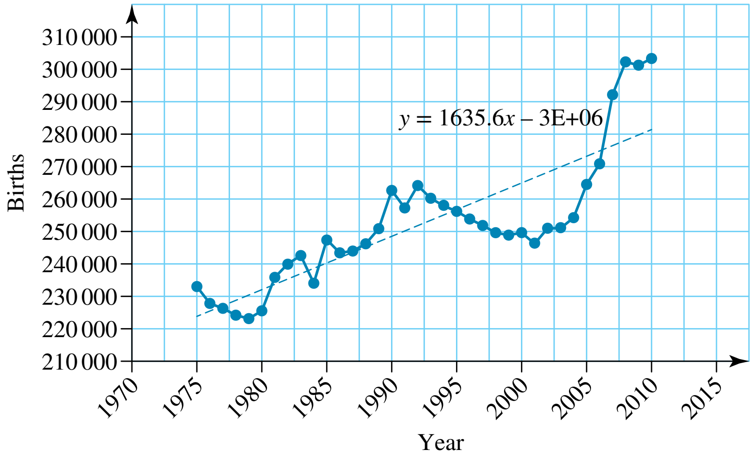
$$\overline{t} = 5.5, \quad s_t = 3.03, \quad \overline{T} = 26.4, \quad s_T = 8.44, \quad r = -0.98.$$



Determine the least-squares regression line for this data.

9. A software company determines that there has been a steady upward trend in the company’s share prices in $ for the previous 10 years. The equation that models the long-term trend in these share prices is: $y = 2.45 + 0.53x$
   a. If the explanatory variable is ‘number of years’ and the response variable is ‘price’, rewrite the equation using the appropriately named variables.
   b. For this linear equation, what do the gradient and y-intercept represent?
   c. Predict the price of shares for this company in 5 years. How reliable is this prediction?

10. The number of births in Australia from 1975 until 2010 is recorded in the following time series graph.



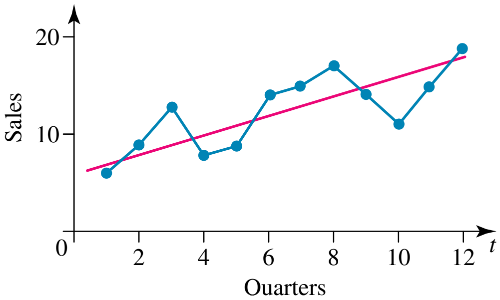
   a. Between 1975 and 1980, there was a decrease in the number of births in Australia. In what year was the lowest number of births recorded?
   b. The trend line equation for this time series plot is: Number of births $= -3\,000\,000 + 1635.6 \times$ Year
      i. Using this trend line, predict the number of births in 1990.
      ii. If the actual number of births in 1990 is 262 648, calculate the residual value for 1990.
   c. Interpret the gradient for this trend line.

3.4 Smoothing time series data using simple moving averages

3.4.1 Smoothing time series

In section 3.2.2 the following graph was used to illustrate an example of time series data that demonstrated an overall upward trend but also showed irregular fluctuations.

By using a **smoothing process**, irregular and cyclic fluctuations can be removed and a clearer picture of any underlying trends emerges.



3.4.2 Moving averages

There are several ways of smoothing data, but the focus for this course is on the **moving average**.

Calculating moving averages involves taking n data points at a time and creating a moving average table.

3.4.3 Three-point and five-point moving averages

WORKED EXAMPLE 3

The following set of data gives the number of cups of coffee sold by a coffee cart over a 15-day period.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Number of sales	**430**	**550**	**490**	**710**	**730**	**790**	**880**	**800**	**950**	**820**	**990**	**880**	**1000**	**990**	**1050**

a. **Calculate the 3-point and 5-point moving averages for this data.**

b. **Graph the original data, the 3-point moving average and the 5-point moving average on the same set of axes. Comment on the smoothing effect.**

THINK

a. 1. Create a table with headings as shown. To calculate the 3-point moving average, the average of the sales figures for the first three days is calculated and the value placed next to the second day. This process is then continued.

Note: All values are rounded to the nearest whole number.

$$\frac{430 + 550 + 490}{3} = 490$$

$$\frac{550 + 490 + 710}{3} = 583$$

$$\frac{490 + 710 + 730}{3} = 643$$

$$\frac{710 + 730 + 790}{3} = 743$$

$$\frac{730 + 790 + 880}{3} = 800$$

$$\frac{790 + 880 + 800}{3} = 823$$

$$\frac{880 + 800 + 950}{3} = 877$$

$$\frac{800 + 950 + 820}{3} = 857$$

$$\frac{950 + 820 + 990}{3} = 920$$

WRITE

a.

Day	**Number of sales**	**3-point moving average**	**5-point moving average**
1	430		
2	550	490	
3	490	583	582
4	710	643	654
5	730	743	720
6	790	800	782
7	880	823	830
8	800	877	848
9	950	857	888
10	820	920	888
11	990	897	928
12	880	957	936
13	1000	957	982
14	990	1013	
15	1050		

$$\frac{820 + 990 + 880}{3} = 897$$

$$\frac{990 + 880 + 1000}{3} = 957$$

$$\frac{880 + 1000 + 990}{3} = 957$$

$$\frac{1000 + 990 + 1050}{3} = 1013$$

2. To calculate the 5-point moving average, the average of the sales figures for the first 5 days is calculated and the value placed next to the third day. This process is then continued.
Note: All values are rounded to the nearest whole number.

$$\frac{430 + 550 + 490 + 710 + 730}{5} = 582$$

$$\frac{550 + 490 + 710 + 730 + 790}{5} = 654$$

$$\frac{490 + 710 + 730 + 790 + 880}{5} = 720$$

$$\frac{710 + 730 + 790 + 880 + 800}{5} = 782$$

$$\frac{730 + 790 + 880 + 800 + 950}{5} = 830$$

$$\frac{790 + 880 + 800 + 950 + 820}{5} = 848$$

$$\frac{880 + 800 + 950 + 820 + 990}{5} = 888$$

$$\frac{800 + 950 + 820 + 990 + 880}{5} = 888$$

$$\frac{950 + 820 + 990 + 880 + 1000}{5} = 928$$

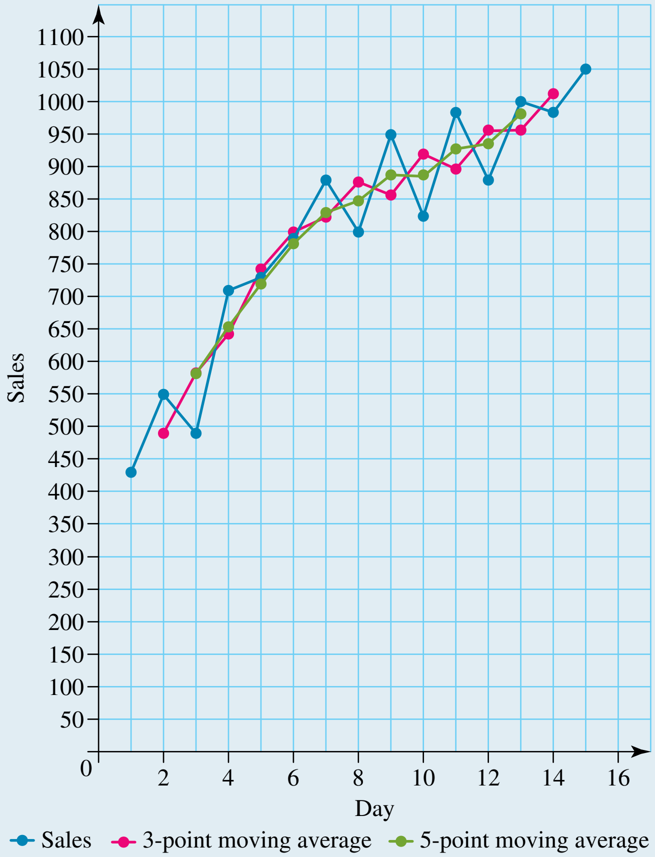
$$\frac{820 + 990 + 880 + 1000 + 990}{5} = 936$$

$$\frac{990 + 880 + 1000 + 990 + 1050}{5} = 982$$

Day	Sales	3-point moving average	5-point moving average
1	430		
2	550	490	
3	490	583	582
4	710	643	654
5	730	743	720
6	790	800	782
7	880	823	830
8	800	877	848
9	950	857	888
10	820	920	888
11	990	897	928
12	880	957	936
13	1000	957	982
14	990	1013	
15	1050		

b. 1. Graph the original data, the 3-point moving average data and the 5-point moving average data on the same set of axes.

b.



2. Comment on the smoothing effect.

The 3-point moving average smooths out the data and hence the upward trend in sales can be seen more clearly. The smoothing effect of the 5-point moving average is greater than the 3-point moving average.

3.4.4 Even numbered moving averages

When calculating odd numbered moving averages, the average point found corresponds to a value on the horizontal axis.

When calculating even numbered moving averages, there is no corresponding point on the horizontal axis. To solve this problem a process called **centring** must be used.

WORKED EXAMPLE 4

Use the same set of data as in Worked example 3.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Number of sales	**430**	**550**	**490**	**710**	**730**	**790**	**880**	**800**	**950**	**820**	**990**	**880**	**1000**	**990**	**1050**

a. **Calculate the 4-point moving average for this data.**

b. **Graph the original data and the 4-point centred moving average on the same set of axes. Comment on the smoothing effect.**

THINK

a. 1. Create a table with headings as shown. To calculate the 4-point moving average, the average of the sales figures for the first four days is calculated and the value placed between the second and third day. This process is then continued.

Note: All values are rounded to the nearest whole number.

$$\frac{430 + 550 + 490 + 710}{4} = 545$$

$$\frac{550 + 490 + 710 + 730}{4} = 620$$

$$\frac{490 + 710 + 730 + 790}{4} = 680$$

$$\frac{710 + 730 + 790 + 880}{4} = 778$$

$$\frac{730 + 790 + 880 + 800}{4} = 800$$

$$\frac{790 + 880 + 800 + 950}{4} = 855$$

$$\frac{880 + 800 + 950 + 820}{4} = 863$$

$$\frac{800 + 950 + 820 + 990}{4} = 890$$

$$\frac{950 + 820 + 990 + 880}{4} = 910$$

$$\frac{820 + 990 + 880 + 1000}{4} = 923$$

$$\frac{990 + 880 + 1000 + 990}{4} = 965$$

$$\frac{880 + 1000 + 990 + 1050}{4} = 980$$

To calculate the centred 4-point moving average, average the two values as shown in the table.

Note: All values are rounded to the nearest whole number.

$$\frac{545 + 620}{2} = 583$$

$$\frac{620 + 680}{2} = 650$$

$$\frac{680 + 778}{2} = 729$$

$$\frac{778 + 800}{2} = 789$$

$$\frac{800 + 855}{2} = 828$$

WRITE

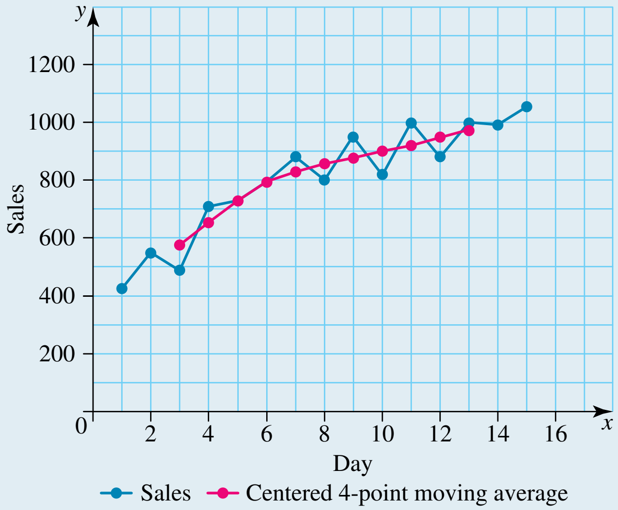
a.

Day	Sales	4-point moving average	Centred 4-point moving average
1	430		
2	550		
		545	
3	490		583
		620	
4	710		650
		680	
5	730		729
		778	
6	790		789
		800	
7	880		828
		855	
8	800		859
		863	
9	950		877
		890	
10	820		900
		910	
11	990		917
		923	
12	880		944
		965	
13	1000		973
		980	
14	990		
15	1050		

$$\frac{855 + 863}{2} = 859$$
$$\frac{863 + 890}{2} = 877$$
$$\frac{890 + 910}{2} = 900$$
$$\frac{910 + 923}{2} = 917$$
$$\frac{923 + 965}{2} = 944$$
$$\frac{965 + 980}{2} = 973$$

b. **1.** Graph the original data and the 4-point centred moving average data on the same set of axes.

b.



2. Comment on the smoothing effect.

The centred 4-point moving average smooths out the data and hence the upward trend in sales can be seen more clearly.

3.4.5 Using a spreadsheet to smooth time series data

A spreadsheet is an efficient way to smooth time series data.

WORKED EXAMPLE 5

Consider the data from the previous worked example.

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Sales	**430**	**550**	**490**	**710**	**730**	**790**	**880**	**800**	**950**	**820**	**990**	**880**	**1000**	**990**	**1050**

a. **Use a spreadsheet to calculate the 3-point and 5-point moving averages for this data.**

b. **Using technology, fit a least-squares line to the smoothed 3-point moving average data and use this line to predict the sales figure on day 10.**

c. **Based on the least-squares regression line from part b, calculate the residual value for day 10.**

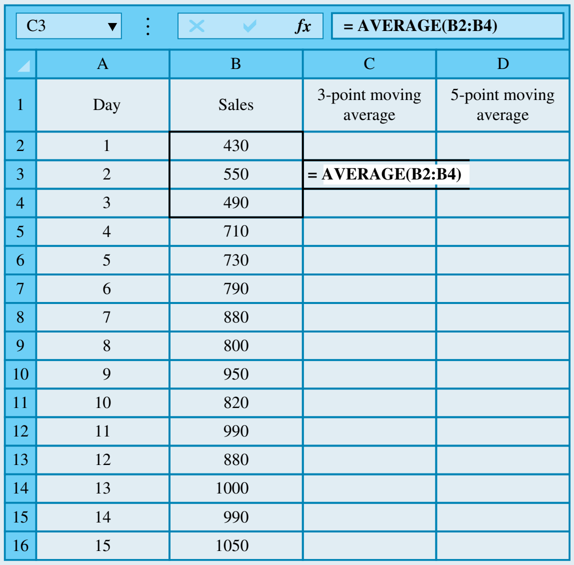
THINK

a. 1. Create a spreadsheet with the headings as shown and enter the Day and Sales data in the first and second columns.

Place the cursor in cell C3 and enter the formula
= AVERAGE(B2:B4)
Then press Enter.

WRITE

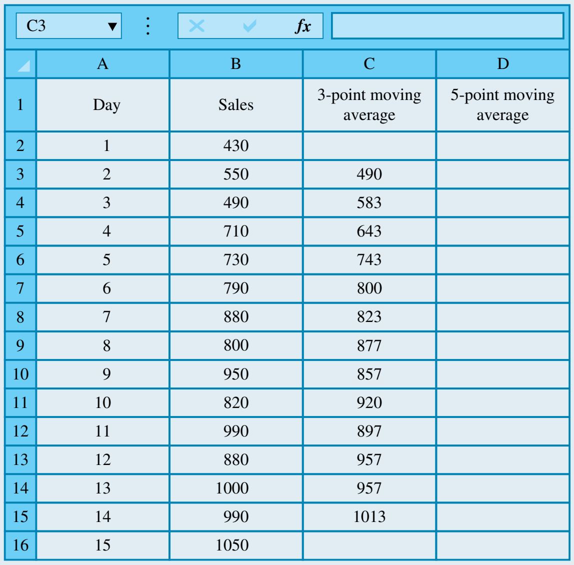
a.



C3 · fx · = AVERAGE(B2:B4)

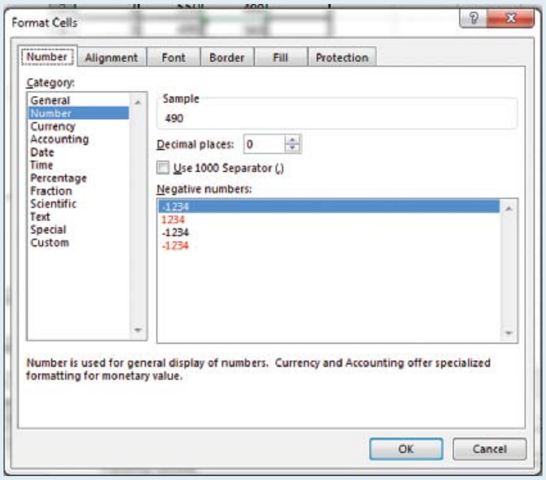
	A	B	C	D
1	Day	Sales	3-point moving average	5-point moving average
2	1	430		
3	2	550	= AVERAGE(B2:B4)	
4	3	490		
5	4	710		
6	5	730		
7	6	790		
8	7	880		
9	8	800		
10	9	950		
11	10	820		
12	11	990		
13	12	880		
14	13	1000		
15	14	990		
16	15	1050		

2. Place the cursor on the small box on the bottom right corner of cell C3 and fill down the column.
Note: To round the moving averages to whole numbers select:
Format cells
Number and change the number of decimal places to 0.



C3 · fx

	A	B	C	D
1	Day	Sales	3-point moving average	5-point moving average
2	1	430		
3	2	550	490	
4	3	490	583	
5	4	710	643	
6	5	730	743	
7	6	790	800	
8	7	880	823	
9	8	800	877	
10	9	950	857	
11	10	820	920	
12	11	990	897	
13	12	880	957	
14	13	1000	957	
15	14	990	1013	
16	15	1050		



Category: Number

Sample: 490

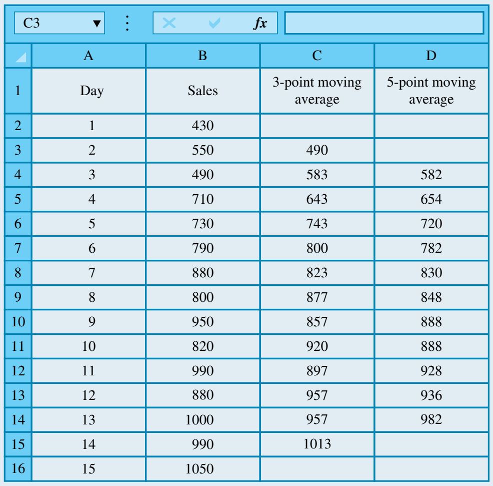
Decimal places: 0

☐ Use 1000 Separator (,)

Negative numbers:
-1234
1234
-1234
-1234

Number is used for general display of numbers. Currency and Accounting offer specialized formatting for monetary value.

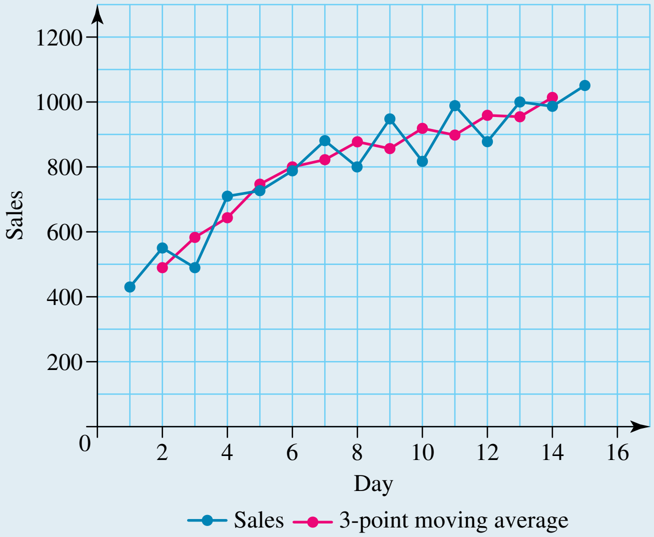
3. Repeat the process for calculating the 5-point moving average. The formula entered into cell D4 is:
= AVERAGE(B2:B6)



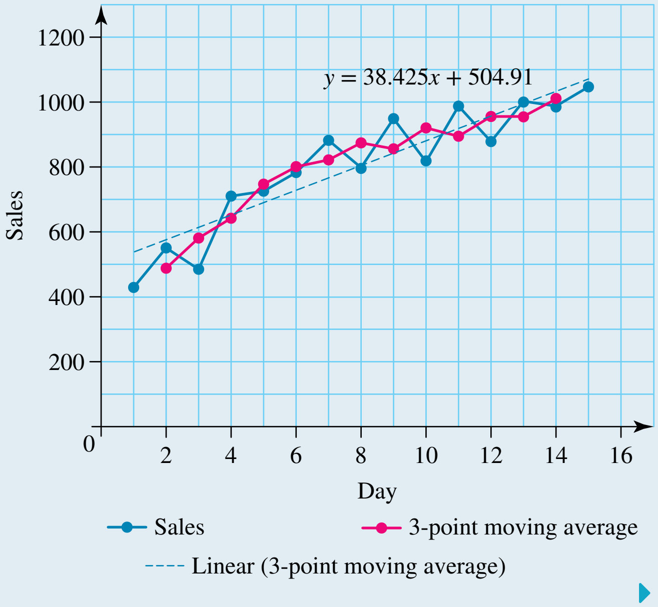
	A	B	C	D
1	Day	Sales	3-point moving average	5-point moving average
2	1	430		
3	2	550	490	
4	3	490	583	582
5	4	710	643	654
6	5	730	743	720
7	6	790	800	782
8	7	880	823	830
9	8	800	877	848
10	9	950	857	888
11	10	820	920	888
12	11	990	897	928
13	12	880	957	936
14	13	1000	957	982
15	14	990	1013	
16	15	1050		

b. 1. Using the spreadsheet created in part **a**, graph the original data and the 3-point moving average data.

b.



2. Add the trendline to the graph for the 3-point moving average data



3. Write the equation with the correct variable names.	Sales = 504.91 + 38.425 × Day
4. Substitute Day = 10 into the equation and calculate the amount of sales.	$\text{Sales} = 504.91 + 38.425 \times \text{Day}$ $= 504.91 + 38.425 \times 10$ $= 889.16$ Sales = 889 (rounded to the nearest whole number)
c. Residual value = Actual value − Predicted value	**c.** Actual value for Day 10 = 820. Residual value = 820 − 889 = −69

on Resources

Interactivity: Smoothing: moving mean method (int-6255)

studyon

Units 3 & 4 › Area 2 › Sequence 1 › Concept 4

Moving average smoothing Summary screen and practice questions

Exercise 3.4 Smoothing time series data using simple moving averages

1. **WE3** The following set of data gives the number of houses sold by auction over a 12-month period for a real estate business.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Number of houses sold	25	20	42	37	21	50	43	85	72	55	77	102

a. Calculate the 3-point and 5-point moving averages for this data. Round all values to the nearest whole number.

b. Graph the original data, the 3-point moving average and the 5-point moving average on the same set of axes. Comment on the smoothing effect.

2. Complete the following table.

Day	Sales ($)	3-point moving average	5-point moving average
1	16		
2	24	19	
3	18		31
4	31	38	
5	66	63	
6	92		81
7	100		92
8	115	101	
9	88	92	
10	72		

3. Complete the following sentence.

The process of smoothing time series data removes ______________ and gives a clearer picture of the underlying ___________.

4. **WE4** Use the same set of data as in question 1.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Number of houses sold	25	20	42	37	21	50	43	85	72	55	77	102

a. Calculate the 4-point centred moving average for this data.

b. Graph the original data and the 4-point centred moving average on the same set of axes. Comment on the smoothing effect.

5. Complete the following table.

Day	Sales ($)	4-point moving average	Centred 4-point moving average
1	16		
2	24		
3	18		28.5
		34.75	
4	31		
		51.75	
5	66		62
6	92		
		93.25	
7	100		
8	115		

6. **WE5** Consider the same set of data as for question 1.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Number of houses sold	25	20	42	37	21	50	43	85	72	55	77	102

a. Use a spreadsheet to calculate the 3-point and 5-point moving averages for this data.
b. Using technology, fit a least-squares line to the smoothed 3-point moving average data and use this line to predict the number of houses sold in month 10.
c. Based on the least-squares regression line from part b, calculate the residual value for day 10.

7. The total number of people using a new city car park every day was recorded each day for 10 days and the results are in the following table.

Day	1	2	3	4	5	6	7	8	9	10
Number of people	360	320	309	339	371	330	335	354	398	420

a. Using technology, calculate the 3-point and 5-point moving averages. Round values to the nearest whole number.
b. Graph the original data, the 3-point moving average and the 5-point moving average, on the same set of axes.
c. Is the 3-point or 5-point moving average more effective?

8. A record of the maximum temperature recorded at the Richmond post office on 31 December for 12 years is recorded in the following table.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Maximum temperature °C	37.7	36.0	36.8	33.1	34.8	37.0	37.2	43.6	40.4	36.2	38.4	41.9

Source: http://www.bom.gov.au/climate/change/acorn/sat/data/acorn.sat.maxT.030045.daily.txt

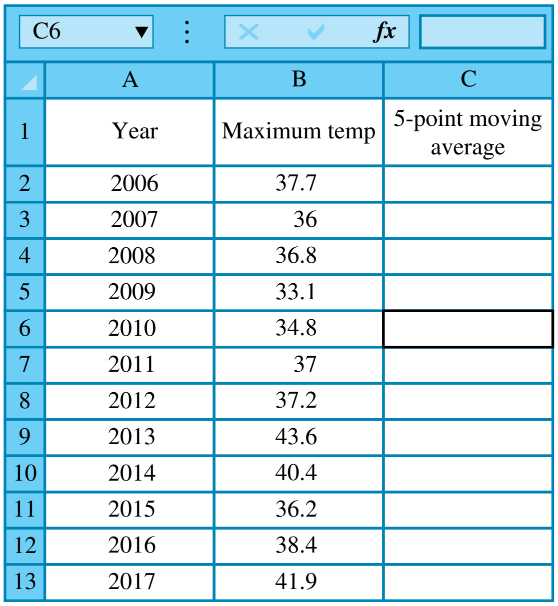
a. Using technology, calculate the 3-point moving average, rounding each value to the nearest whole number and graph the original data and the smoothed data on the same axes.
b. The equation of the least-squares regression line for the smoothed data is of the form

$$\text{Maximum temperature} = a + b \times \text{Year}$$

Using technology, determine the values of a and b.

c. Use the equation from part b to predict the maximum temperature on 31 December 2025.

9. **MC** Consider the following portion of an Excel spreadsheet.



	A	B	C
1	Year	Maximum temp	5-point moving average
2	2006	37.7	
3	2007	36	
4	2008	36.8	
5	2009	33.1	
6	2010	34.8	
7	2011	37	
8	2012	37.2	
9	2013	43.6	
10	2014	40.4	
11	2015	36.2	
12	2016	38.4	
13	2017	41.9	

The formula that needs to be typed in cell C6, to calculate the 5-point moving average for the year 2010 is:

A. = AVERAGE(B2:B6)

B. = AVERAGE(B5:B7)

C. = AVERAGE(B4:B8)

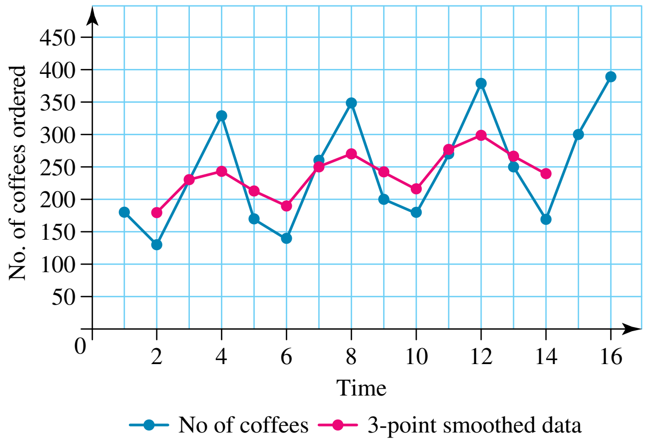
D. = AVERAGE(C4:C8)

The information below refers to questions 10, 11 and 12.

A café keeps a record of the number of takeaway coffees ordered for each of the four seasons for four years. The data are summarised in the following table.

	Spring	**Summer**	**Autumn**	**Winter**
2016	180	130	230	330
2017	170	140	260	350
2018	200	180	270	380
2019	250	170	300	390

The graph of the original data and the 3-point smoothed data is shown.



10. **MC** Autumn 2018 refers to the point in time of

- **A.** 10.
- **B.** 11.
- **C.** 12.
- **D.** 13.

11. **MC** The original data could be described as

- **A.** upward trending with irregular fluctuations.
- **B.** upward trending with seasonal fluctuations.
- **C.** no trend but seasonality present.
- **D.** irregular fluctuations only.

12. **MC** The last data value for the smoothed data has been left off the graph. When calculated it would be closest to

- **A.** 276.
- **B.** 280.
- **C.** 283.
- **D.** 287.

3.5 Seasonal indices and deseasonalising data

3.5.1 Deseasonalising data

In the last section we studied using smoothing techniques to remove fluctuations and make overall trends more visible.

Much of our economic and financial data are subject to seasonal fluctuations. For example, when the weather changes from warm to cold, people may need to outfit themselves or their growing families with new clothes. Car sales frequently increase at the end of the financial year as car salespeople increase their efforts to sell and hence increase their bonus. More houses are put on the market during the spring to ensure gardens are looking their best.

The process of **deseasonalising data** removes the distraction of seasonality from time series data.

3.5.2 Seasonal indices

Seasonal indices tell us how a particular season, usually a day, month or quarter, compares to the average season.

The average value of a **seasonal index** is 1. Therefore, if the seasonal index for a monthly sales figure was 1.3, this means that for that month the sales figure was 30% higher than the average. A seasonal index of 0.7 for the monthly sales figure means for that month the sales figure was 30% lower than the average.

The seasonal index

$$\text{Seasonal index} = \frac{\text{data value}}{\text{seasonal average}}$$

The following steps need to be followed to calculate the seasonal indices.

Step 1: Calculate the average for each year.

Step 2: Divide each value for that year by the average for the year.

Step 3: Calculate the average for the season (day/month/quarter) from the new table of data.

The total sum of the seasonal indices is equal to the number of seasons.

Examples:

If there are four seasons (spring, summer, autumn, winter) or quarters, then the total sum of the seasonal indices is 4.

If there are 12 months, then the total sum of the seasonal indices is 12.

WORKED EXAMPLE 6

The following set of data gives the number of new homes built by a company each quarter for 3 years.

Quarter	2017	2018	2019
1	14	14	14
2	15	15	16
3	17	17	16
4	14	15	14

a. **Calculate the seasonal indices for this data.**
b. **Calculate the seasonally adjusted data.**
c. **Graph the original and seasonally adjusted data.**
d. **Comment on the graph.**

THINK	WRITE
a. **1.** Calculate the averages for each year.	**a.** 2017: $\frac{14+15+17+14}{4} = 15$ 2018: $\frac{14+15+17+15}{4} = 15.25$ 2019: $\frac{14+16+16+14}{4} = 15$
2. Divide each value for that year by the average for that year.	

Quarter	2017	2018	2019
1	$\frac{14}{15} = 0.9333$	$\frac{14}{15.25} = 0.9180$	$\frac{14}{15} = 0.9333$
2	$\frac{15}{15} = 1$	$\frac{15}{15.25} = 0.9836$	$\frac{16}{15} = 1.0667$
3	$\frac{17}{15} = 1.1333$	$\frac{17}{15.25} = 1.1148$	$\frac{16}{15} = 1.0667$
4	$\frac{14}{15} = 0.9333$	$\frac{15}{15.25} = 0.9836$	$\frac{14}{15} = 0.9333$

3. Calculate the average for the season (quarter). These are the seasonal indices.

Quarter 1: $\dfrac{0.9333 + 0.9180 + 0.9333}{3} = 0.9282$

Quarter 2: $\dfrac{1 + 0.9836 + 1.0667}{3} = 1.0168$

Quarter 3: $\dfrac{1.1333 + 1.1148 + 1.0667}{3} = 1.1049$

Quarter 4: $\dfrac{0.9333 + 0.9836 + 0.9333}{3} = 0.9501$

4. Write the answer.

Quarter	**1**	**2**	**3**	**4**
Seasonal index	0.9282	1.0168	1.1049	0.9501

b. 1. Divide each figure in the original table by its seasonal index.

b.

Quarter	**2017**	**2018**	**2019**
1	$\dfrac{14}{0.9282} = 15.083$	$\dfrac{14}{0.9282} = 15.083$	$\dfrac{14}{0.9282} = 15.083$
2	$\dfrac{15}{1.0168} = 14.752$	$\dfrac{15}{1.0168} = 14.752$	$\dfrac{16}{1.0168} = 15.736$
3	$\dfrac{17}{1.1049} = 15.386$	$\dfrac{17}{1.1049} = 15.386$	$\dfrac{16}{1.1049} = 14.481$
4	$\dfrac{14}{0.9501} = 14.735$	$\dfrac{15}{0.9501} = 15.788$	$\dfrac{14}{0.9501} = 14.735$

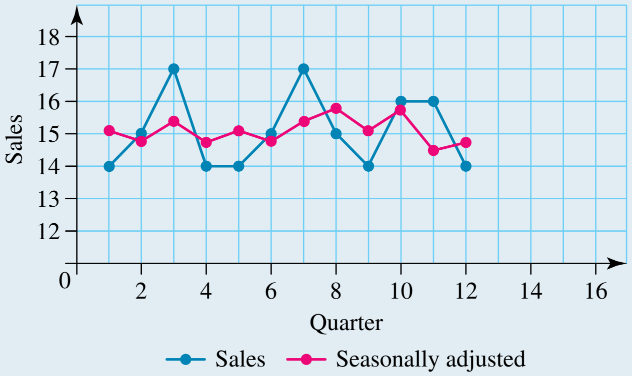
2. Write the answer.

The seasonally adjusted data are

Quarter	**2017**	**2018**	**2019**
1	15.083	15.083	15.083
2	14.752	14.752	15.736
3	15.386	15.386	14.481
4	14.735	15.788	14.735

c. Graph the original time series data and the seasonally adjusted data on the same axes.

c.



d. Describe the trend.

d. The seasonally adjusted data has removed the obvious seasonal fluctuations. There appears to be a very slight downward trend within the seasonally adjusted graph.

3.5.3 Using a spreadsheet to deseasonalise time series data

Calculating seasonal indices and deseasonalising data by hand can be a very time-consuming process. Using a spreadsheet is a far more efficient means of deseasonalising large amounts of time series data.

WORKED EXAMPLE 7

The following set of data gives the population of Queensland (in '000 000s) from the first quarter of 2010 to the last quarter of 2017.

Quarter	2010	2011	2012	2013	2014	2015	2016	2017
1	4.388	4.458	4.546	4.634	4.705	4.764	4.827	4.907
2	4.405	4.477	4.569	4.653	4.720	4.778	4.845	4.928
3	4.421	4.498	4.592	4.671	4.735	4.792	4.866	4.949
4	4.437	4.519	4.611	4.685	4.747	4.805	4.883	4.964

Source: http://www.qgso.qld.gov.au/products/tables/erp-components-change-no/index.php

Use a spreadsheet to answer the following questions.

a. **Calculate the seasonal indices for this data.**

b. **Calculate the seasonally adjusted data.**

c. **Graph the original and seasonally adjusted data on the same axes.**

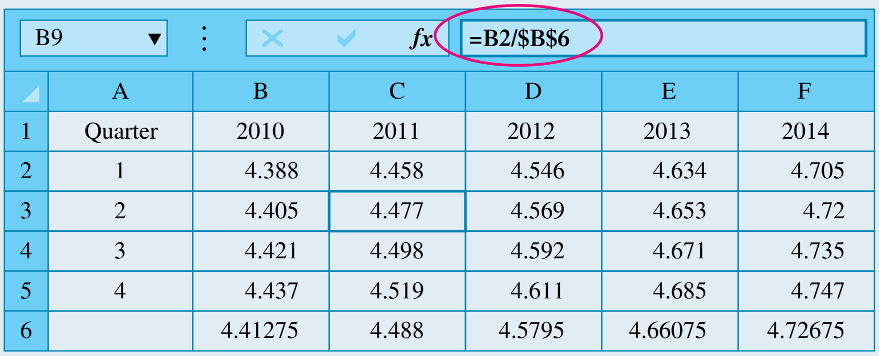
d. **Comment on the graph.**

THINK

a. **1.** Enter the data into a new spreadsheet. To calculate the average for each year:
Place the cursor in Cell B6, type =AVERAGE (B2:B5), then press Enter.
Repeat process to calculate the average for each year.

WRITE

a.

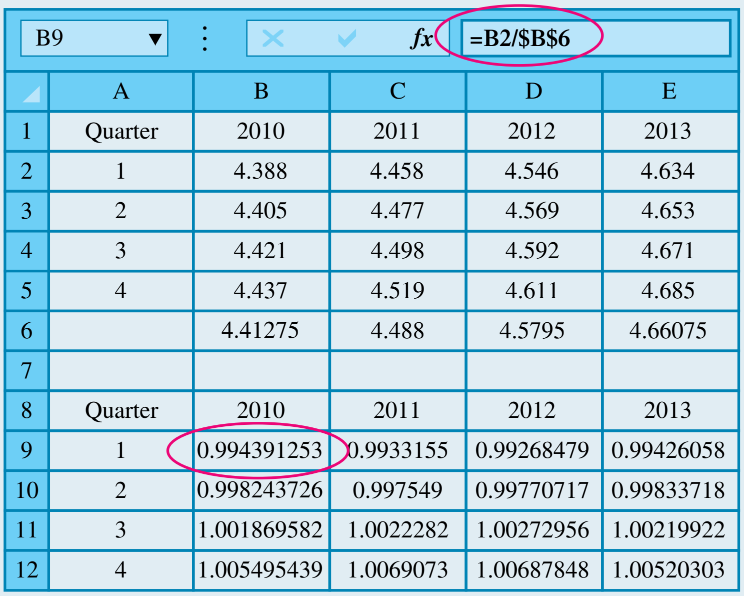


B9

=B2/$B$6

	A	B	C	D	E	F
1	Quarter	2010	2011	2012	2013	2014
2	1	4.388	4.458	4.546	4.634	4.705
3	2	4.405	4.477	4.569	4.653	4.72
4	3	4.421	4.498	4.592	4.671	4.735
5	4	4.437	4.519	4.611	4.685	4.747
6		4.41275	4.488	4.5795	4.66075	4.72675

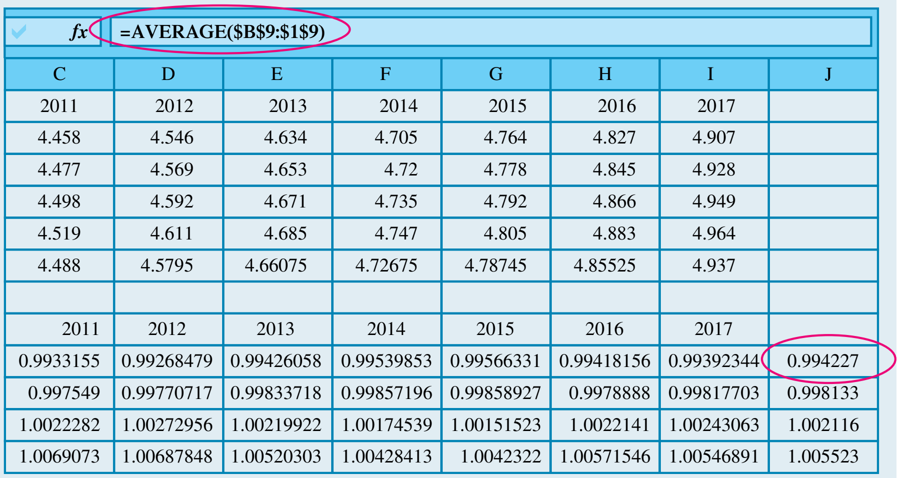
2. Copy the completed table with the averages and paste below. To divide each value for that year by the average for that year, place the cursor in Cell B9, type = B2/$B$6, then press Enter. Fill down the column for 2010 and repeat the process for the other years. Note: Typing $B$6 ensures that the numerical value of 4.412575 from Cell B6 is used.



B9 ▼ ⋮ ✕ ✓ *fx* =B2/$B$6

	A	B	C	D	E
1	Quarter	2010	2011	2012	2013
2	1	4.388	4.458	4.546	4.634
3	2	4.405	4.477	4.569	4.653
4	3	4.421	4.498	4.592	4.671
5	4	4.437	4.519	4.611	4.685
6		4.41275	4.488	4.5795	4.66075
7					
8	Quarter	2010	2011	2012	2013
9	1	0.994391253	0.9933155	0.99268479	0.99426058
10	2	0.998243726	0.997549	0.99770717	0.99833718
11	3	1.001869582	1.0022282	1.00272956	1.00219922
12	4	1.005495439	1.0069073	1.00687848	1.00520303

3. To calculate the seasonal indices or average for the season (quarter), place the cursor in Cell J9, type = AVERAGE ($B$9:$I$9), then press Enter. Repeat this process for each quarter.



✓ *fx* =AVERAGE($B$9:$1$9)

C	D	E	F	G	H	I	J
2011	2012	2013	2014	2015	2016	2017	
4.458	4.546	4.634	4.705	4.764	4.827	4.907	
4.477	4.569	4.653	4.72	4.778	4.845	4.928	
4.498	4.592	4.671	4.735	4.792	4.866	4.949	
4.519	4.611	4.685	4.747	4.805	4.883	4.964	
4.488	4.5795	4.66075	4.72675	4.78745	4.85525	4.937	
2011	2012	2013	2014	2015	2016	2017	
0.9933155	0.99268479	0.99426058	0.99539853	0.99566331	0.99418156	0.99392344	0.994227
0.997549	0.99770717	0.99833718	0.99857196	0.99858927	0.9978888	0.99817703	0.998133
1.0022282	1.00272956	1.00219922	1.00174539	1.00151523	1.0022141	1.00243063	1.002116
1.0069073	1.00687848	1.00520303	1.00428413	1.0042322	1.00571546	1.00546891	1.005523

4. Write the answer.

Seasonal indices are:

Quarter	**1**	**2**	**3**	**4**
Seasonal index	0.9943	0.9981	1.0021	1.0055

b. 1. To divide each figure in the original table by its seasonal index, copy the original table of data, then place the cursor in Cell B15 and type, = B2/$J$9, then press Enter. Repeat this process for the other quarters.

b.

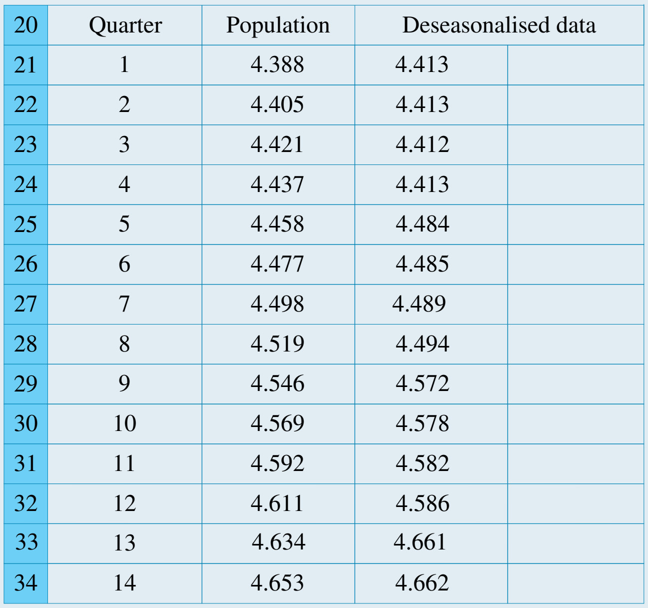
	A	B	C	D
1	Quarter	2010	2011	2012
2	1	4.388	4.458	4.546
3	2	4.405	4.477	4.569
4	3	4.421	4.498	4.592
5	4	4.437	4.519	4.611
6		4.41275	4.488	4.5795
7				
8	Quarter	2010	2011	2012
9	1	0.994391253	0.9933155	0.99268479
10	2	0.998243726	0.997549	0.99770717
11	3	1.001869582	1.0022282	1.00272956
12	4	1.005495439	1.0069073	1.00687848
13				
14	Quarter	2010	2011	2012
15	1	4.413	4.484	4.572
16	2	4.413	4.485	4.578
17	3	4.412	4.489	4.582
18	4	4.413	4.494	4.586

2. Write the answer.

Quarter	2010	2011	2012	2013	2014	2015	2016	2017
1	4.413	4.484	4.572	4.661	4.732	4.792	4.855	4.935
2	4.413	4.485	4.578	4.662	4.729	4.787	4.854	4.937
3	4.412	4.489	4.582	4.661	4.725	4.782	4.856	4.939
4	4.413	4.494	4.586	4.659	4.721	4.779	4.856	4.937

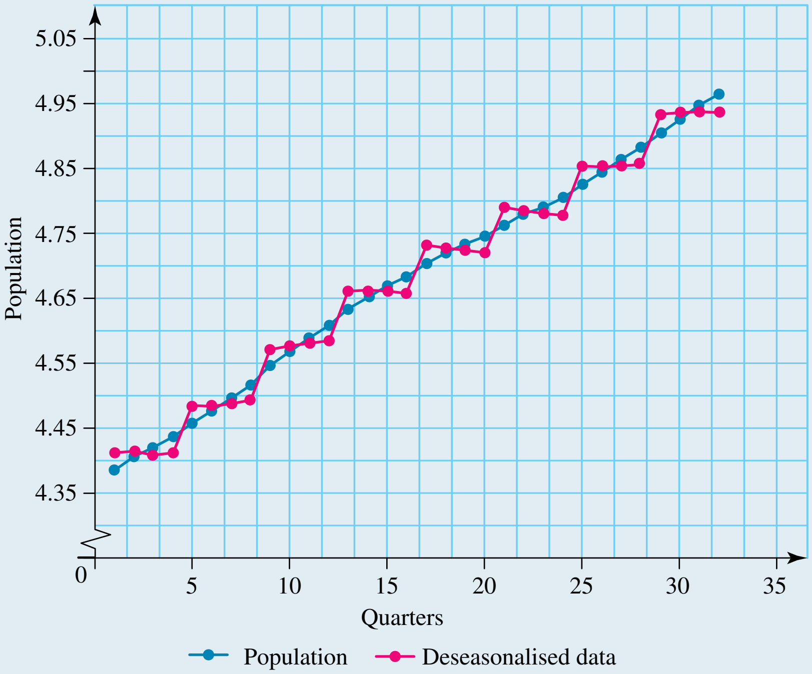
c. Create a table with headings as shown. Number the quarters as 1, 2, 3, 4, 5, 6, ... 32. Copy the original data into Column 2, with each set of quarters pasted below the previous four quarters and the deseasonalised data into Column 3. Note: Copy the deseasonalised data and paste as 'values' into the table.

c.



20	Quarter	Population	Deseasonalised data	
21	1	4.388	4.413	
22	2	4.405	4.413	
23	3	4.421	4.412	
24	4	4.437	4.413	
25	5	4.458	4.484	
26	6	4.477	4.485	
27	7	4.498	4.489	
28	8	4.519	4.494	
29	9	4.546	4.572	
30	10	4.569	4.578	
31	11	4.592	4.582	
32	12	4.611	4.586	
33	13	4.634	4.661	
34	14	4.653	4.662	

Use Excel to create a graph with both the original data and the deseasonalised data on the same axes.



d. Describe the trend.

d. There is a very clear upward trend to both the original data and the seasonally adjusted data. The seasonally adjusted data has removed the obvious seasonal fluctuations.

3.5.4 Fitting a least-squares regression line to deseasonalised time series data

If the deseasonalised data shows a linear upward or downward trend, a least-squares regression line can be fitted.

If the regression line is used to make predictions or forecasts it is important to remember that the data are deseasonalised and hence any value obtained must be reseasonalised using the seasonal index value for that season.

Reseasonalised value = deseasonalised value × seasonal index

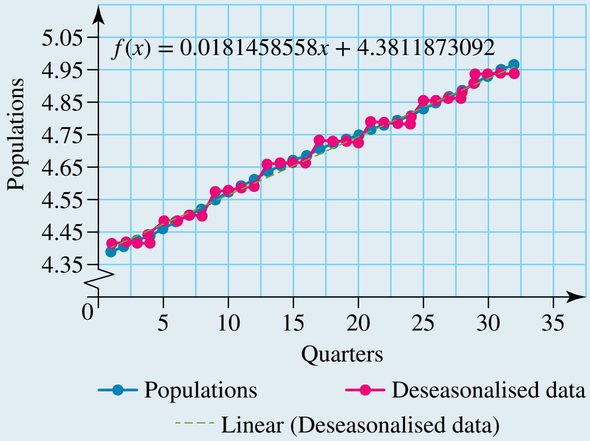
WORKED EXAMPLE 8

Use the data in Worked example 7.

a. Using technology determine the equation of a least-squares regression line for the seasonally adjusted data.

b. Use the equation of the line to predict the population at the end of 2020.

c. Comment on the reliability of this prediction.

THINK	WRITE
a. 1. Using the Excel spreadsheet, fit a least-squares regression line to the graph.	**a.** 
2. Write the equation with appropriately named variables.	Population = 4.3812 + 0.0181 × Quarter
b. 1. The end of 2020 will be Quarter 44.	**b.** When Quarter = 44 Population = 4.3812 + 0.0181 × 44 = 5.178
2. Value must be reseasonalised.	Seasonal index for Quarter 4 is 1.0055. Population for the end of 2020 is 5.178 × 1.0055 = 5.206479
3. Write the answer.	The population for the end of 2020 is predicted to be 5 206 479.
c. Comment on the prediction.	**c.** The end of 2020 is outside the range of the data and many other issues may have affected population growth in that time, including immigration rates and policies, unemployment levels and other economic issues.

on Resources

Interactivity Seasonal adjustment (int-6257)

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Units 3 & 4 › Area 2 › Sequence 1 › Concepts 5 & 6

Seasonal indices Summary screen and practice questions
Deseasonalising data Summary screen and practice questions

Exercise 3.5 Seasonal indices and deseasonalising data

1. **WE6** The following set of data gives the number of new cars sold by a car salesman each quarter for 3 years.

Quarter	2017	2018	2019
1	40	44	46
2	28	25	32
3	31	28	33
4	23	21	25

   a. Calculate the seasonal indices for this data, correct to 4 decimal places
   b. Calculate the seasonally adjusted data.
   c. Graph the original and seasonally adjusted data.
   d. Comment on the graph.

2. **MC** Three of the quarterly seasonal indices for a sales company are given in the following table.

Quarter	1	2	3	4
Seasonal indices	1.4	0.70		0.88

   The seasonal index for the third quarter must be
   A. 3.00. B. 1.02. C. 1.00. D. 1.92.

3. **WE8** The following table below gives the number of sick days taken per month by the staff of a large company.

Month	2016	2017	2018	2019
Jan	26	29	33	31
Feb	25	28	30	35
Mar	29	33	38	29
Apr	31	35	37	33
May	32	36	41	37
June	35	40	42	39
July	36	43	44	42
Aug	40	45	47	44
Sept	34	38	43	46
Oct	31	25	38	39
Nov	29	33	36	45
Dec	25	30	32	48

   Use a spreadsheet to answer the following questions.
   a. Calculate the seasonal indices for this data, correct to 4 decimal places.
   b. Calculate the seasonally adjusted data, correct to the nearest whole number.
   c. Graph the original and seasonally adjusted data on the same axes.
   d. Comment on the graph.

4. **WE8** Use the data in question 3 above to answer the following.
   a. Using technology, determine the equation of a least-squares regression line for the seasonally adjusted data.
   b. Use the equation of the line to predict the number of sick days taken in June 2020 to the nearest whole number.
   c. Comment on the reliability of this prediction.

Use the following information to answer questions 5, 6 and 7.

Monthly sales figures for a company between January 2016 and December 2018 were used to calculate the following seasonal indices.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Seasonal indices	1.22	0.79	1.01	1.23	0.89	0.80	1.02	1.10	0.94	1.12		1.11

5. MC The missing seasonal index for November is
A. 0.77. **B.** 1.77. **C.** 0.88. **D.** 1.88.

6. MC If the original sales figure for June 2017 was \$187 600, the deseasonalised value for June 2017 would be closest to
A. \$150 080. **B.** \$174 600. **C.** \$188 000. **D.** \$234 500.

7. MC The deseasonalised sales figure for March 2016 was \$166 280. The original sales figure for March 2016 would be closest to
A. \$165 000. **B.** \$166 000. **C.** \$168 000. **D.** \$170 000.

8. The following table shows the amount of sparkling water in (000 000s litres) produced by a new factory over a two year period.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
2018	14.6	16.0	15.7	12.6	9.5	6.3	5.1	6.8	9.2	10.9	12.5	17
2019	14.1	15.8	16.1	13.1	8.7	5.6	4.5	4.5	7.9	9.9	9.9	12.4

Technology can be used to answer the following questions.

a. Calculate the seasonal indices, correct to 4 decimal places.
b. Calculate the seasonally adjusted data, correct to one decimal place.
c. Graph the seasonally adjusted data on the same axes as the original data.
d. If a trend line fitted to the deseasonalised data is of the form

$$\text{Amount produced} = a + b \times \text{Month}$$, determine the values of a and b.

e. Use the trend line to predict the number of litres of sparkling water produced in February in 2020.
f. Use the seasonal indices to convert the answer to part e into a predicted actual value.

9. MC The seasonal index for the unemployment rate for the month of August is 1.15. This means that
A. the unemployment rate is up by 15% in August.
B. the unemployment rate for August is 15% higher than the monthly average.
C. the unemployment rate is 15 times higher in August.
D. the unemployment rate is 1.15 times higher in August.

10. Table 1 provides the data for the number of items shipped from a warehouse in each quarter between 2017 and 2019. The other tables show a partially completed deseasonalised data process. Complete the tables.

Table 1 Original data

Year	**Quarter 1**	**Quarter 2**	**Quarter 3**	**Quarter 4**	**Averages**
2017	7590	8970	7922	8899	8345
2018	5840	6757	6111	7140	
2019	6010	7893	8200	8344	7612

Table 2 Quarterly indices

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2017		1.0749	0.9493	1.0664
2018	0.9037	1.0457	0.9457	
2019	0.7896	1.0369	1.0773	1.0962
Seasonal indices	0.8676		0.9907	1.0892

Table 3 Deseasonalised data (rounded to the nearest whole number)

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2017		8523	7996	8171
2018	6731	6420	6168	6556
2019	6927	7499	8277	

11. **MC** The equation for the trend line fitted to the deseasonalised data in question **10** is

$$\text{Number of items shipped} = 8033.2 - 86.178 \times \text{Quarter}.$$

Using this equation, the predicted number of items shipped in the second quarter of 2020 will be closest to

A. 6517.
B. 6600.
C. 6753.
D. 6827.

12. The following table provides the data for the quarterly sales of houses in a suburb for one year.

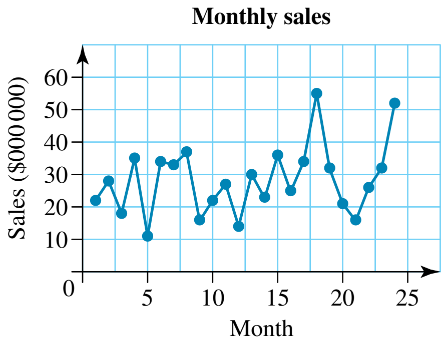
Quarter	March quarter	June quarter	Sept quarter	Dec quarter
Sales	**120**	**145**	**155**	**134**

a. Calculate the seasonal indices for each of the four quarters.
b. Calculate the sum of the seasonal indices.

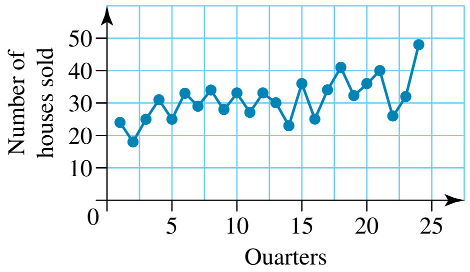
3.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar



1. **MC** The sales figures for a company over a 2 year period are recorded in the graph.
   The graph can be described as showing
   A. irregular fluctuations.
   B. upward trend with seasonal fluctuations.
   C. seasonal fluctuations.
   D. no trend.



2. **MC** A least-squares regression line has been fitted to the time series in the graph.
   Its equation is most likely to be
   A. Number of houses sold = 5 + 0.5 × Quarter.
   B. Number of houses sold = 24 − 0.5 × Quarter.
   C. Number of houses sold = 24 + 0.5 × Quarter.
   D. Number of houses sold = −24 + 0.5 × Quarter.
3. **MC** Smoothing processes are used on time series data to
   A. smooth out irregular fluctuation.
   B. determine if there is an upward or downward trend.
   C. to determine if there are any patterns in the data.
   D. to remove any seasonal trends.

Use the following information to answer questions 4–7.
The mortality rate from a particular type of cancer following the discovery of a new drug is shown in the following table.

Year	**1**	**2**	**3**	**4**	**5**	**6**
Rate	20.5	19.6	15.0	12.3	9.5	8.1

Summary statistics:
Year: Mean = 3.5, Standard deviation = 2
Rate: Mean = 14.2, Standard deviation = 5
$r = -0.9868$

4. **MC** The 3-point moving average for year 4, correct to 2 decimal places is
   A. 15.63. **B.** 12.30. **C.** 12.27. **D.** 9.97.
5. **MC** The centred 4-point moving average for year 3, correct to 2 decimal places, is
   A. 16.85. **B.** 14.10. **C.** 13.45. **D.** 15.48.
6. **MC** The equation of the least-squares regression line that best fits this data is closest to
   A. Rate = 24 − 3 × Year. **B.** Rate = 24 + 3 × Year.
   C. Rate = 3 − 24 × Year. **D.** Rate = 3 + 24 × Year.
7. **MC** Using the regression equation found in question 6, the mortality rate in year 8 is closest to
   A. 195. **B.** 48. **C.** 20. **D.** 0.

8. **MC** The deseasonalising process on time series data is used to
A. smooth out irregular fluctuation.
B. determine if there is an upward or downward trend.
C. determine if there are any patterns in the data.
D. remove any seasonal trends.

9. **MC** The following table shows the seasonal indices from a time series.

Quarter	**1**	**2**	**3**	**4**
Index	0.98	0.81		1.02

The value of the seasonal index for quarter 3 is
A. 102.
B. 0.98.
C. 1.19.
D. 1.00.

10. **MC** Using the data from question 9, a seasonally adjusted value for quarter 1 of 2020, when the original value was 445, is closest to
A. 436.
B. 454.
C. 484.
D. 650.

Use the following information to answer questions 11 and 12.

The following table gives the number of children (000s) visiting a newly opened indoor play centre.

	Summer	**Autumn**	**Winter**	**Spring**	**Average**
2017	25.3	20.6	36.3	16.3	24.625
2018	27.2	21.8	35.0	19.2	25.8
2019	25.7	19.2	37.1	20.3	25.575

11. **MC** The seasonal index for summer is closest to

A. $\left(\frac{25.3}{24.625}+\frac{27.2}{25.8}+\frac{25.7}{25.575}\right)$.

B. $\frac{1}{3}\left(\frac{24.625}{25.3}+\frac{25.8}{27.2}+\frac{25.575}{25.7}\right)$.

C. $\frac{25.3+27.2+25.7}{3}$.

D. $\frac{1}{3}\left(\frac{25.3}{24.625}+\frac{27.2}{25.8}+\frac{25.7}{25.575}\right)$.

12. **MC** The seasonally adjusted figure for summer 2017, correct to one decimal place, is
A. 24.6.
B. 25.
C. 27.2.
D. 25.3.

Complex familiar

13. The following table gives the number of new bicycles purchased in a large city for 10 consecutive years. Complete the table by filling in the shaded cells.

Year	Number of bicycles sold	3-point moving average	2-point centred moving average
2011	7434		
2012	5680		6479
2013	7123	7123	7123
2014	8567	8308	
2015	9234		8932
2016	8690	9164	
2017	9567	9615	9603
2018	10 588		10 761
2019	12 300		

14. The following table gives the sales of takeaway lunches sold each month in a café in a shopping centre. Use technology to answer the following questions.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Sales ($)	4566	3125	4100	2890	3244	2533	3499	2105	2567	1345	2889	1400

a. Smooth the data using 3-point moving averages.

b. Plot the original data and the smoothed data on the same set of axes and comment on the effect of smoothing.

c. If the equation of the trendline is of the form $y = ax + b$, determine the values of a and b, correct to the nearest whole number.

d. Interpret the value of b.

e. Using the equation calculated in part **c**, predict the number of takeaway lunches sold in May. How different is this value to the actual number of lunches sold in May?

15. The following table gives the seasonal sales of gumboots in a garden supplies store for the years 2018 and 2019.

Season	Winter	Spring	Summer	Autumn
2018	710	530	450	780
2019	820	678	546	420

To calculate the seasonal indices for the data in the previous table, complete the following tables, filling in the shaded cells.

Season	Winter	Spring	Summer	Autumn	Averages
2018	710	530	450	780	
2019	820	678	546	420	617

Season	Winter	Spring	Summer	Autumn
2018	1.15	0.86	0.73	1.26
2019		1.10		0.68
Seasonal indices	1.24		0.81	0.97

16. Complete the following table with the deseasonalised data from question 15. Round the deseasonalised values to the nearest whole number.

Season	Winter	Spring	Summer	Autumn
2018				
2019				

Complex unfamiliar

17. The coach of a cyclist has been keeping records of her time trials each quarter for 3 years and has recorded the information in the following table.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Time (mins)	52.10	46.01	51.44	50.43	50.42	50.42	49.80	49.40	46.35	48.45	49.43	48.13

a. Graph the data.
b. Smooth the data using 3-point moving average
c. Graph the 3-point moving averages data.
d. Comment on any trends in the athlete's performance.

Use technology to complete the following questions.

18. The following table shows the amount of sugar produced (tonnes) by a sugar factory over two years.

Month	Year 1	Year 2
Jan	7051	3942
Feb	5936	4092
Mar	4334	4511
Apr	3447	4712
May	2501	4826
June	5244	5820
July	3540	3844
Aug	3221	4208
Sept	3142	4621
Oct	3587	4911
Nov	4410	5631
Dec	5311	6152

a. Plot the data as a time series plot and comment on the type of trend that exists. Justify your choice.
b. How many seasons are there?
c. Calculate the average monthly production for each of the years, giving the answers correct to the nearest whole number.
d. Calculate the seasonal indices for each month.

19. Use the results from question 18 to answer the following questions.
a. Complete the table of deseasonalised sugar production figures, rounding the values to the nearest whole number.
b. Determine the equation of the trend line using technology and the least-squares method and interpret the values of the gradient and the *y*-intercept.
c. Using the trend line, predict the deseasonalised production figures for
   i. January, year 3
   ii. June, year 3.
d. Using the deseasonalised values from part c, calculate the actual expected future production for each year. Comment on the reliability of the predictions.

20. The following data provides the monthly rainfall data for the Alderley weather station in Brisbane for 2015 and 2016.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
2015	230.2	276.8	75.2	182.2	263.2	32.8	7.8	32.4	68.2	52.2	114.2	43.8
2016	107.8	120.2	121.0	10.8	27.0	311.6	24.8	33.2	99.8	24.0	53.0	121.4

a. Plot the data points and comment on any trends.

b. Calculate 3-point and 5-point moving averages. Compare the results of these two methods by plotting the smoothed data.

c. Use the data from the 5-point moving average smoothing and fit a straight line using the least-squares method. Interpret the *y*-intercept and gradient.

d. Calculate the seasonal indices from the original data. Then, seasonally adjust the data.

e. Take the seasonally adjusted data from part **d** and fit a trend line using the least-squares method. Comment on this result.

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Units 3 & 4 Sit exam

Answers

3 Time series analysis

Exercise 3.2 Constructing and describing time series plots

1. **i.** General upward and downward trends show an overall long term upward or downward pattern for the data.
  ii. Seasonal patterns have peaks and troughs that repeat at the same time each season. They generally last less than one year.
  iii. Cyclical patterns have peaks and troughs that are random and generally last longer than one year. There is no pattern to their fluctuations.
  iv. Irregular fluctuations have their points distributed randomly and show no pattern.

2. A

3. D

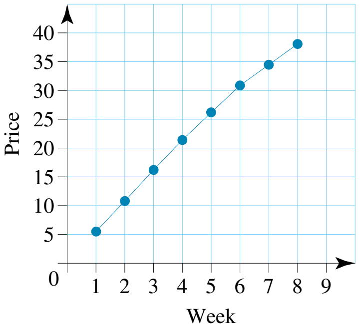
4. D

5. Seasonal trend. Fewer people watch the news on Friday, Saturday and Sunday.

6. **a.** Seasonal
  b. Irregular fluctuations
  c. Upward trend
  d. Irregular fluctuations
  e. Cyclical

7. **a.**

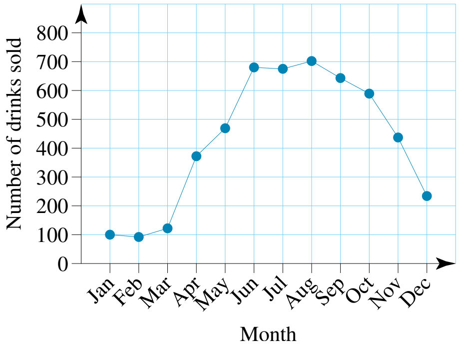
Share price



  b. There is a clear upward trend.

8. **a.**

No. of drinks sold per month



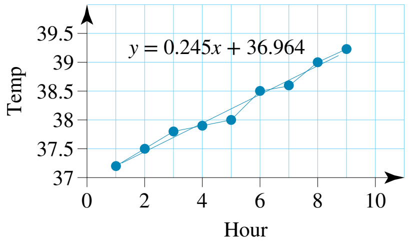
  b. The data appears to be seasonal, but more years of data are needed to determine if the seasonal nature continues.

9. **a.** *See the figure at the bottom of the page.
  b. The data shows a seasonal upward trend.

10. **a.** There is an overall downward trend of alcohol-induced deaths for males from 1997–2017, starting at a peak of about 10.8 in 1997 with a rate lower than 8 in 2017.
  b. There does not appear to be any overall trend in the alcohol-induced death rate for females over this 20-year period.
  c. The difference in 1997 is approximately 8.2.
  The difference in 2017 is 5.50.
  The change in the difference is 2.7. The difference in the rate of alcohol-induced deaths for males and females has decreased over this 20-year period.

Exercise 3.3 Fitting a least-squares line to time series data

1. **a.**



  b. There appears to be an overall upward trend for the first 9 hours.

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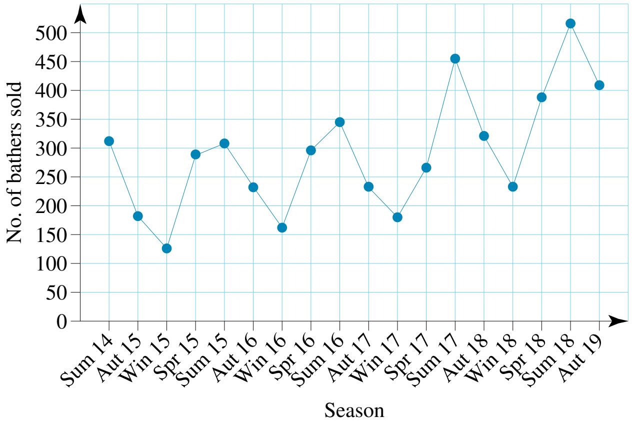
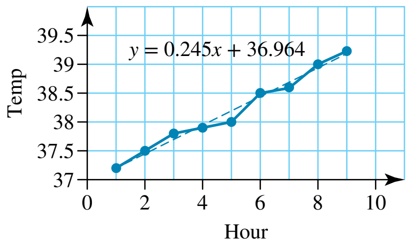
*9. a.



c.

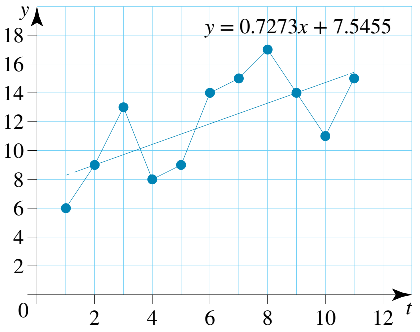


$$y = 36.964 + 0.245x$$

$$\text{Temperature} = 36.964 + 0.245 \times \text{hour}$$

d. Temperature = 39.9°C. It is unlikely that the temperature would be allowed to rise indefinitely as this could seriously impact the health of the patient. The model could be useful in the short term to monitor the infected appendix.

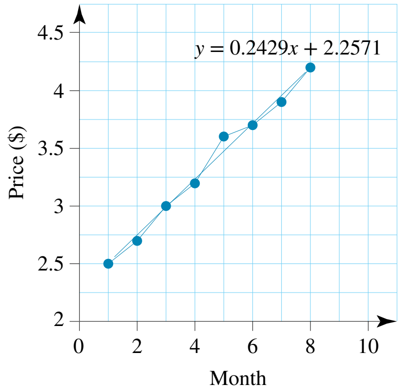
2.



$$y = 7.55 + 0.727t$$

This data shows an upward trend.

3.

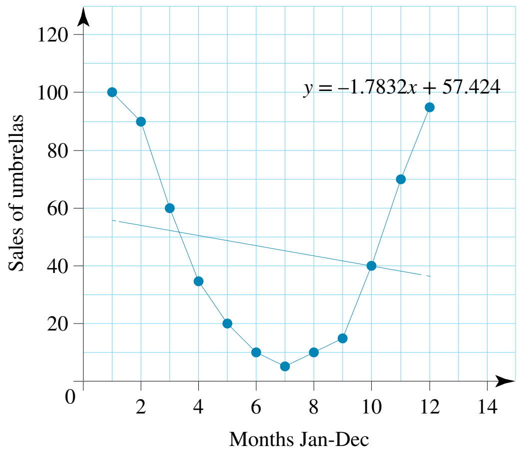


January of the following year is Month = 13.

$$\begin{aligned}\text{Price} &= 2.2571 + 0.2429 \times \text{Month}\\ &= 2.2571 + 0.2429 \times 13\\ &= \$5.41\end{aligned}$$

This time series shows an overall upward trend but January of the next year is 5 months further on than the given data and only 8 months of data has been given. Given the fluctuating nature of the share market one could not rely upon this prediction.

4.



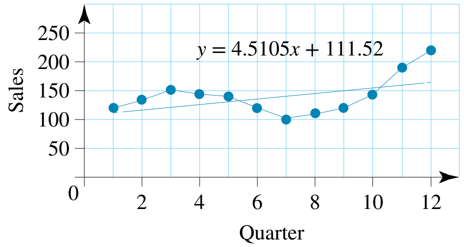
$$\text{Sales of umbrellas} = 57.424 - 1.783 \times \text{Month}$$

The graph shows a seasonal pattern with a very slight downward trend. The sales of umbrellas are linked to the weather, so seasonality is expected. This is only 12 months of data so more data would be needed before any ongoing trend could be assumed. A linear model is not suitable for this data.

5. a. Price = 38.41 + 2.18 × Week

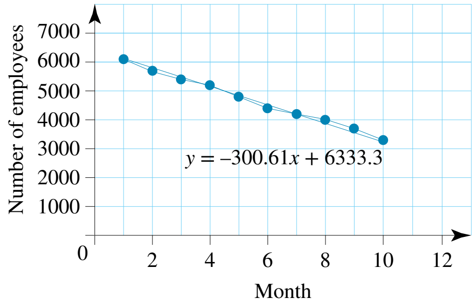
b. 93 cents

6.



The equation of the least squares line is
Sales = 111.52 + 4.5 × quarter
This data reflects an upward trend with random fluctuations.

7.



This scatterplot shows a downward trend.

8. Temp = 41.42 − 2.73 × Days

9. a. Price = 2.45 + 0.53 × number of years

b. The y-intercept of 2.45 is the price of shares when the company first started to track the share prices, that is, year 0. The gradient of 0.53 means that for every increase of one year, the share price increases by $0.53.

c. $5.10

Share prices are dependent on many things, such as demand and world share market prices and 5 years is a long time in economic terms. Therefore, a predicted price of $5.10 in 5 years is not very reliable.

10. a. 1979

b. i. 254 844

ii. 262648 − 254844 = 7804

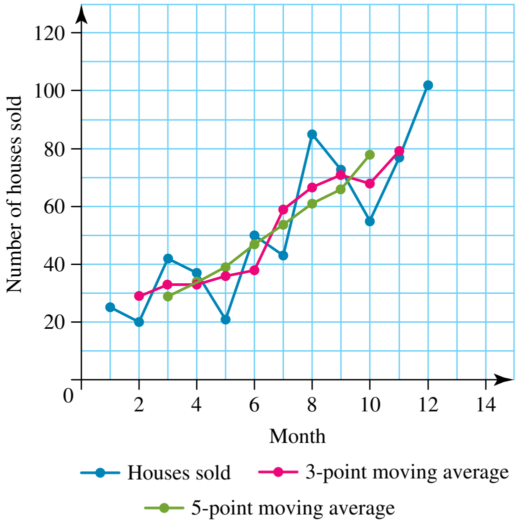
c. For every increase of one year, the number of births increases by 1635 (rounded down to the nearest whole number).

Exercise 3.4 Smoothing time series data using simple moving averages

1. a.

Month	Houses sold	3-point moving average	5-point moving average
1	25		
2	20	29	
3	42	33	29
4	37	33	34
5	21	36	39
6	50	38	47
7	43	59	54
8	85	67	61
9	72	71	66
10	55	68	78
11	77	78	
12	102		

b.



The 5-point moving average smooths out more of the irregular fluctuations but 2 data points from each end are lost.

2.

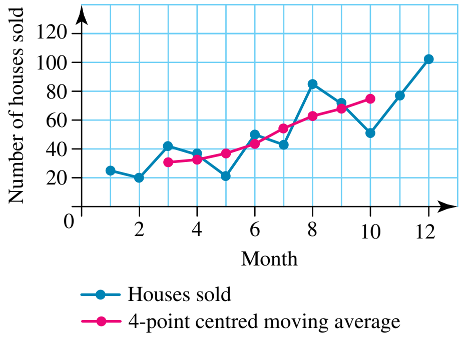
Day	Sales($)	3-point moving average	5-point-moving average
1	16		
2	24	19	
3	18	24	31
4	31	38	46
5	66	63	61
6	92	86	81
7	100	102	92
8	115	101	93
9	88	92	
10	72		

3. The process of smoothing time series data removes fluctuations and gives a clearer picture of the underlying trend.

4. a.

Month	Houses sold	4-point moving average	4-point centred moving average
1	25		
2	20		
		31	
3	42		30.5
		30	
4	37		33.75
		37.5	
5	21		37.625
		37.75	
6	50		43.75
		49.75	
7	43		56.125
		62.5	
8	85		63.125
		63.75	
9	72		68
		72.25	
10	55		74.375
		76.5	
11	77		
12	102		

b.

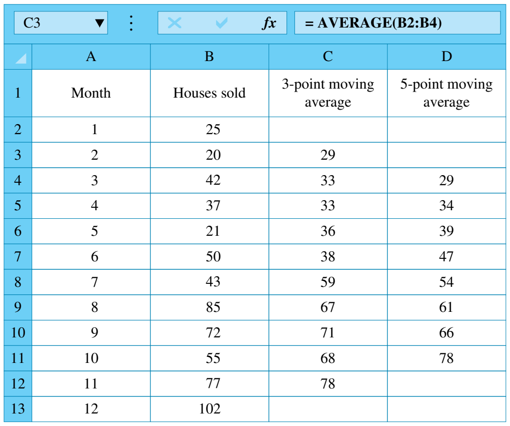


The 4-point centred moving average has smoothed out the irregular fluctuations and the upward trend of the data can be seen clearly.

5.

Day	Sales ($)	4-point moving average	Centred 4-point moving average
1	16		
2	24		
		22.25	
3	18		28.5
		34.75	
4	31		43.25
		51.75	
5	66		62
		72.25	
6	92		82.75
		93.25	
7	100		
8	115		

6. **a.**



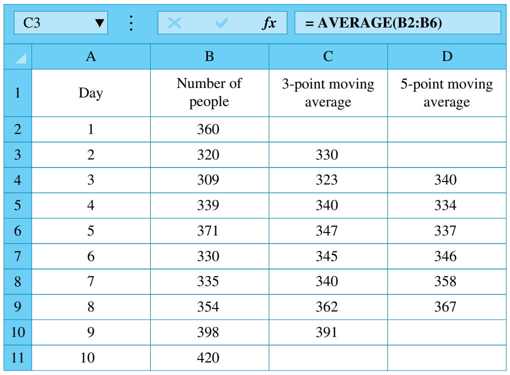
C3 ▼ ⋮ ✕ ✓ *fx* **= AVERAGE(B2:B4)**

	A	B	C	D
1	Month	Houses sold	3-point moving average	5-point moving average
2	1	25		
3	2	20	29	
4	3	42	33	29
5	4	37	33	34
6	5	21	36	39
7	6	50	38	47
8	7	43	59	54
9	8	85	67	61
10	9	72	71	66
11	10	55	68	78
12	11	77	78	
13	12	102		

b. Number of houses sold = 12.36 + 5.98 × Month
In month 10, 72 houses will be sold (to the nearest whole number).

c. Residual $= -17$

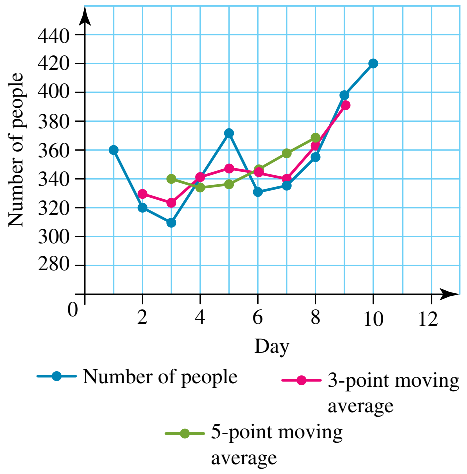
7. **a.**



C3 ▼ ⋮ ✕ ✓ *fx* **= AVERAGE(B2:B6)**

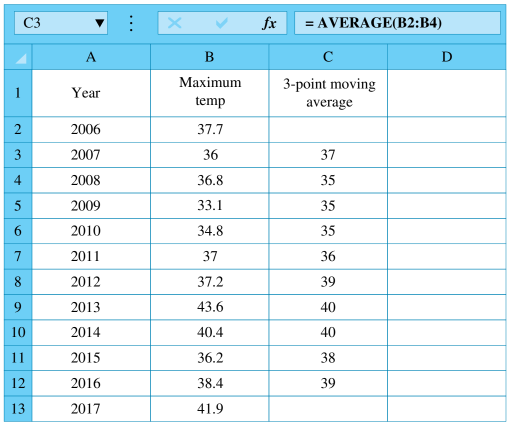
	A	B	C	D
1	Day	Number of people	3-point moving average	5-point moving average
2	1	360		
3	2	320	330	
4	3	309	323	340
5	4	339	340	334
6	5	371	347	337
7	6	330	345	346
8	7	335	340	358
9	8	354	362	367
10	9	398	391	
11	10	420		

b.



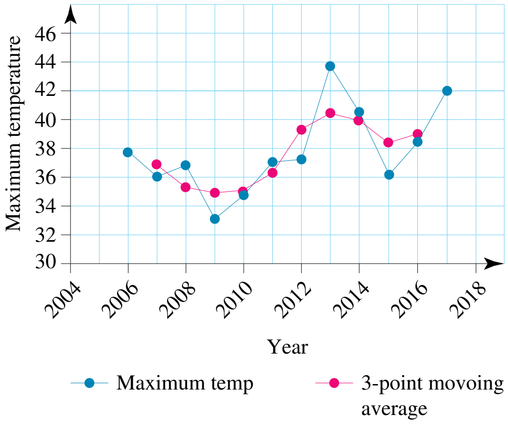
c. The 5-point moving average graph is smoother.

8. **a.**



C3 ▼ ⋮ ✕ ✓ *fx* **= AVERAGE(B2:B4)**

	A	B	C	D
1	Year	Maximum temp	3-point moving average	
2	2006	37.7		
3	2007	36	37	
4	2008	36.8	35	
5	2009	33.1	35	
6	2010	34.8	35	
7	2011	37	36	
8	2012	37.2	39	
9	2013	43.6	40	
10	2014	40.4	40	
11	2015	36.2	38	
12	2016	38.4	39	
13	2017	41.9		



b. a = −990.17, b = 0.51

c. Max temp = 42.6º C

9. C

10. B

11. B

12. D

Exercise 3.5 Seasonal indices and deseasonalising data

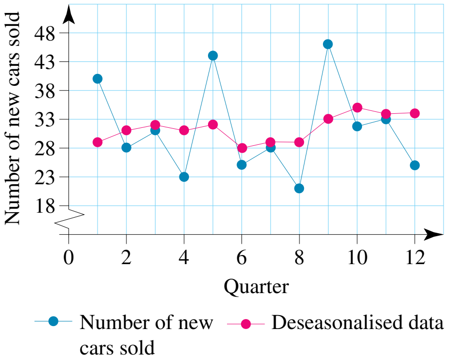
1. a.

Quarter	1	2	3	4
Seasonal indices	1.3853	0.9022	0.9787	0.7338

b.

Quarter	2017	2018	2019
1	29	32	33
2	31	28	35
3	32	29	34
4	31	29	34

c.



d. Deseasonalising the data has smoothed out the fluctuations caused by the seasonal nature of the data.

2. B

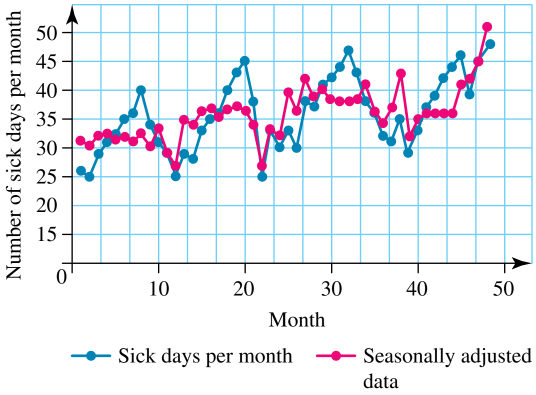
3. a.

Month	
Jan	0.8322
Feb	0.8231
Mar	0.9050
Apr	0.9547
May	1.0216
June	1.0940
July	1.1560
Aug	1.2349
Sept	1.1229
Oct	0.9273
Nov	0.9945
Dec	0.9339

b.

Month	2016	2017	2018	2019
Jan	31	35	40	37
Feb	30	34	36	43
Mar	32	36	42	32
Apr	32	37	39	35
May	31	35	40	36
June	32	37	38	36
July	31	37	38	36
Aug	32	36	38	36
Sept	30	34	38	41
Oct	33	27	41	42
Nov	29	33	36	45
Dec	27	32	34	51

c.



d. Although the deseasonalising has removed much of the seasonal fluctuations, there does seem to be some underlying cyclic pattern. There is a very slight upward trend.

4. a. Sick days per month $= 30.13 + 0.2269 \times$ Month, where January 2016 $=$ Month 1

b. 42 days (to the nearest day)

c. Given that there is only a very slight upward trend to this data, a linear model to predict the number of sick days will not be reliable.

5. A

6. D

7. C

8. a. See the table at the bottom of the page.*

---

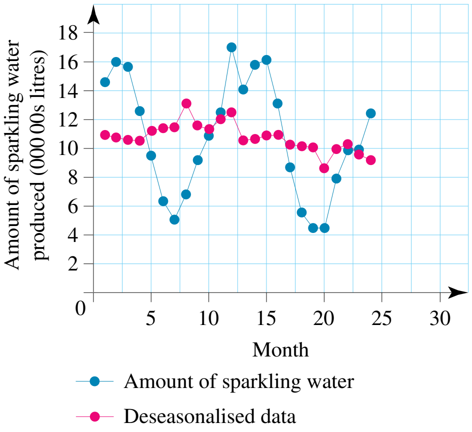
*8. a.

Month	Jan	Feb	Mar	Apr	May	Jun
Seasonal indices	1.3338	1.4787	1.4802	1.1967	0.8446	0.5518
Month	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal indices	0.4451	0.5200	0.7922	0.9651	1.0356	1.3562

b.

Month	Jan	Feb	Mar	Apr	May	Jun
2018	10.9	10.8	10.6	10.5	11.2	11.4
2019	10.6	10.7	10.9	10.9	10.3	10.1
Month	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
2018	11.5	13.1	11.6	11.3	12.1	12.5
2019	10.1	8.7	10.0	10.3	9.6	9.1

c.



d. $a = 11.830$, $b = -0.0836$

e. Month = 26,
Amount of sparkling water produced = 9 656 400 litres

f. 9 656 400 × 1.4787 = 14 278 918.68 litres

9. B

10. Table 1 Original data

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4	Averages
2017	7590	8970	7922	8899	8345
2018	5840	6757	6111	7140	6462
2019	6010	7893	8200	8344	7612

Table 2

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2017	0.9095	1.0749	0.9493	1.0664
2018	0.9037	1.0457	0.9457	1.1049
2019	0.7896	1.0369	1.0773	1.0962
Seasonal indices	0.8676	1.0525	0.9907	1.0892

Table 3 Deseasonalised data (Rounded to the nearest whole number)

Year	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2017	8748	8523	7996	8171
2018	6731	6420	6168	6556
2019	6927	7499	8277	7661

11. D

12. a.

Quarter	March quarter	June quarter	Sept quarter	Dec quarter
Sales	0.8664	1.0469	1.1191	0.9675

b. 4

3.6 Review: exam practice

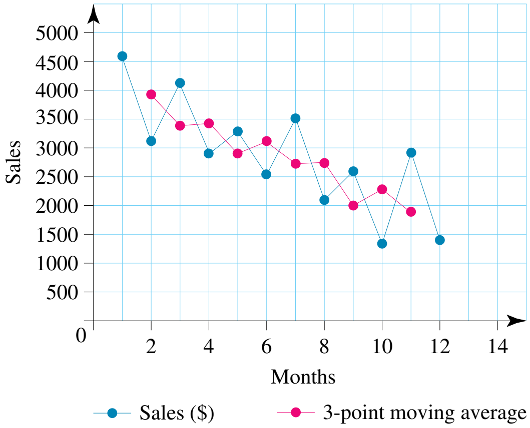
1. B
2. C
3. A
4. C
5. D
6. A
7. D
8. D
9. C
10. B
11. D
12. A
13.

Year	Number of bicycles sold	3-point moving average	2-point moving average	2-point centred moving average
2011	7434			
			6557	
2012	5680	6746		6479
			6402	
2013	7123	7123		7123
			7845	
2014	8567	8308		8373
			8901	
2015	9234	8830		8932
			8962	
2016	8690	9164		9046
			9129	
2017	9567	9615		9603
			10078	
2018	10588	10818		10761
			11444	
2019	12300			

14. a.

Month	Sales($)	3-point moving average
Jan	4566	
Feb	3125	3930
Mar	4100	3372
Apr	2890	3411
May	3244	2889
June	2533	3092
July	3499	2712
Aug	2105	2724
Sept	2567	2006
Oct	1345	2267
Nov	2889	1878
Dec	1400	

b.



Smoothing has removed the irregular fluctuations and hence the downward trend can be seen more clearly.

c. $a = 4172, \ b = -207$

d. For every increase of one month, the sales of takeaway lunches decreases by 207.

e. $y = 4172 - 207x$, when $x = 5$, $y = 3137$. The actual value is 3244. The difference is 107.

15.

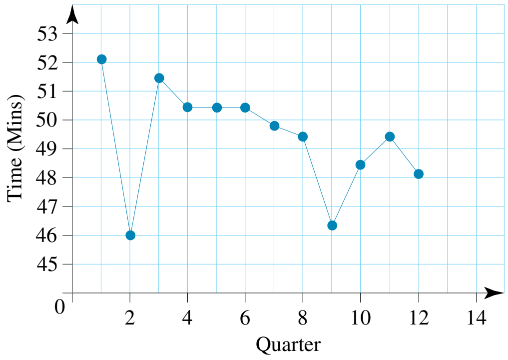
Season	Winter	Spring	Summer	Autumn	Averages
2018	710	530	450	780	618
2019	820	678	546	420	617

Season	Winter	Spring	Summer	Autumn
2018	1.15	0.86	0.73	1.26
2019	1.33	1.10	0.88	0.68
Seasonal indices	1.24	0.98	0.81	0.97

16.

Season	Winter	Spring	Summer	Autumn
2018	572	541	557	802
2019	661	692	676	432

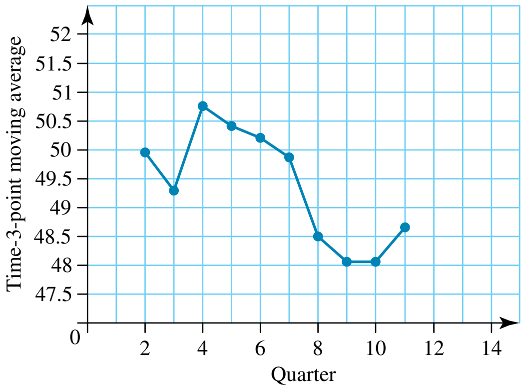
17. a.



b.

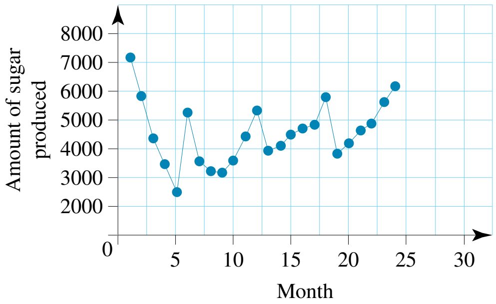
Quarter	Time (mins)	3-point moving averages
1	52.1	
2	46.01	49.85
3	51.44	49.29
4	50.43	50.76
5	50.42	50.42
6	50.42	50.21
7	49.8	49.87
8	49.4	48.52
9	46.35	48.07
10	48.45	48.08
11	49.43	48.67
12	48.13	

c.



d. There is an overall downward trend.

18. a.



The data are seasonal.

b. 5

c. Year 1: 4310, Year 2: 4773

d.

Month	Seasonal indices
Jan	1.2309
Feb	1.1173
Mar	0.9753
Apr	0.8935
May	0.7957
June	1.2180
July	0.8134
Aug	0.8145
Sept	0.8486
Oct	0.9306
Nov	1.1015
Dec	1.2606

19. a.

Month	Year1	Year2
Jan	5728	3203
Feb	5313	3662
Mar	4444	4625
Apr	3858	5274
May	3143	6065
June	4305	4778
July	4352	4726
Aug	3955	5166
Sept	3703	5445
Oct	3584	5277
Nov	4004	5112
Dec	4213	4880

b. Let p = sugar production in tonnes and m = Month

$$p = 4048 + 39.75 \times m$$

The gradient is 39.75, therefore for every increase of one month, sugar production increases by 39.75 tonnes. The y-intercept is 4048, so at month 0 or when the data was first collected, the production of sugar was 4048 tonnes.

c. **i.** January year 3 is month 25, P = 5042 tonnes (to the nearest tonne)

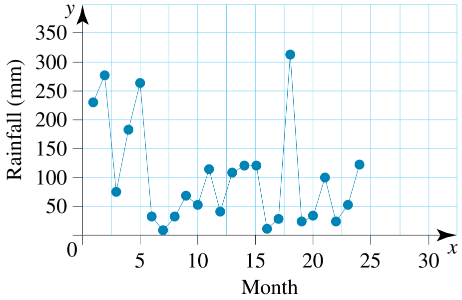
ii. June year 3 is month 30, P = 5241 tonnes (to the nearest tonne)

d. **i.** 6206 tonnes

ii. 6384 tonnes

These data values are outside the given data range and hence making predictions based outside the range could mean the results are not very reliable.

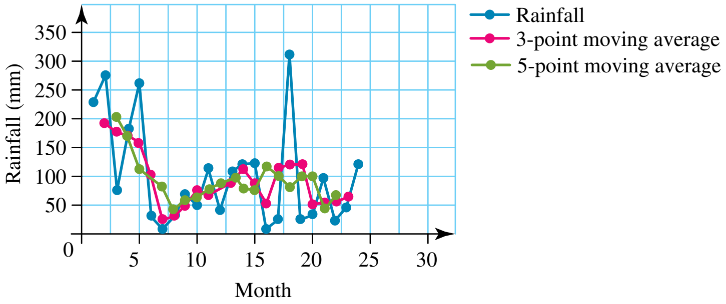
20. a.



There is an overall downward trend with an outlier in June 2016.

b.

Month	Rainfall	3-point moving average	5-point moving average
1	230.2		
2	276.8	194.1	
3	75.2	178.1	205.5
4	182.2	173.5	166.0
5	263.2	159.4	112.2
6	32.8	101.3	103.7
7	7.8	24.3	80.9
8	32.4	36.1	38.7
9	68.2	50.9	55.0
10	52.2	78.2	62.2
11	114.2	70.1	77.2
12	43.8	88.6	87.6
13	107.8	90.6	101.4
14	120.2	116.3	80.7
15	121	84.0	77.4
16	10.8	52.9	118.1
17	27	116.5	99.0
18	311.6	121.1	81.5
19	24.8	123.2	99.3
20	33.2	52.6	98.7
21	99.8	52.3	47.0
22	24	58.9	66.3
23	53	66.1	
24	121.4		



The moving average graphs ensure the irregular fluctuations are smoothed. The 5-point moving averages graph is a smoother curve than the 3-point.

c. Rainfall = 128.99 − 2.886 × month

The gradient of −2.886 means for every increase of one month, the rainfall decreases by 2.886 mm.

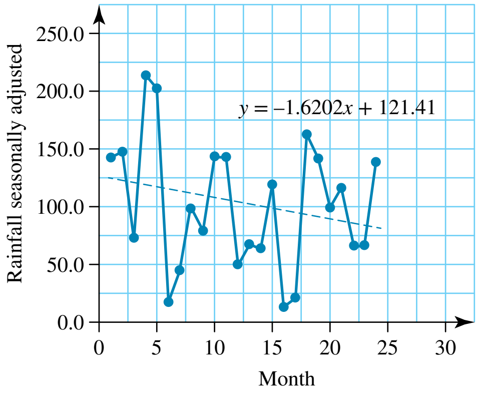
The y-intercept of 128.99 means that at the commencement of data collection, that is, month 0, the rainfall was 128.99 mm.

d. *See the table at the bottom of the page.

Deseasonalised data

Month	Jan	Feb	Mar	Apr	May	Jun
2015	142.5	146.6	74.0	213.3	202.7	17.1
2016	66.8	63.7	119.1	12.6	20.8	162.7
Month	**July**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
2015	44.6	98.2	78.9	143.5	143.0	49.7
2016	141.7	100.6	115.4	66.0	66.4	137.8

e.



Rainfall = 121.41 − 1.6202 × month

The seasonally adjusted data has smaller fluctuations about the least-squares regression line.

***20. d.**

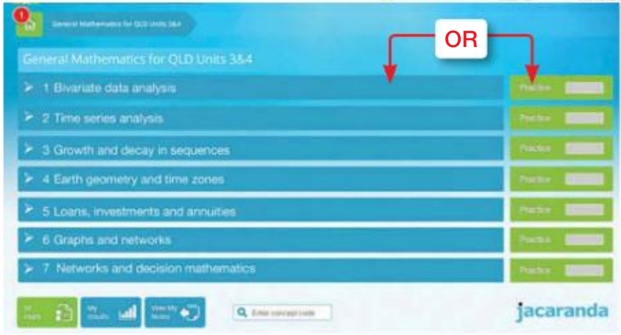
Month	Jan	Feb	Mar	Apr	May	Jun
Seasonal indices	1.614908	1.888212	1.015606	0.854193	1.29879	1.915517
Month	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Seasonal indices	0.175034	0.329859	0.864535	0.363666	0.798418	0.881261

REVISION UNIT 3 Bivariate data, sequences and change, and Earth geometry

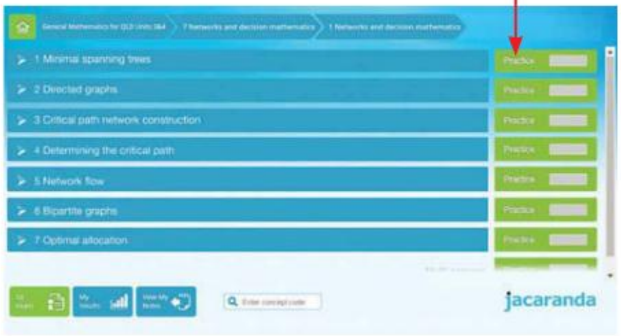
TOPIC 2 Time series analysis

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *General Mathematics for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level *OR* select **Practice** to answer all practice questions available for each area.

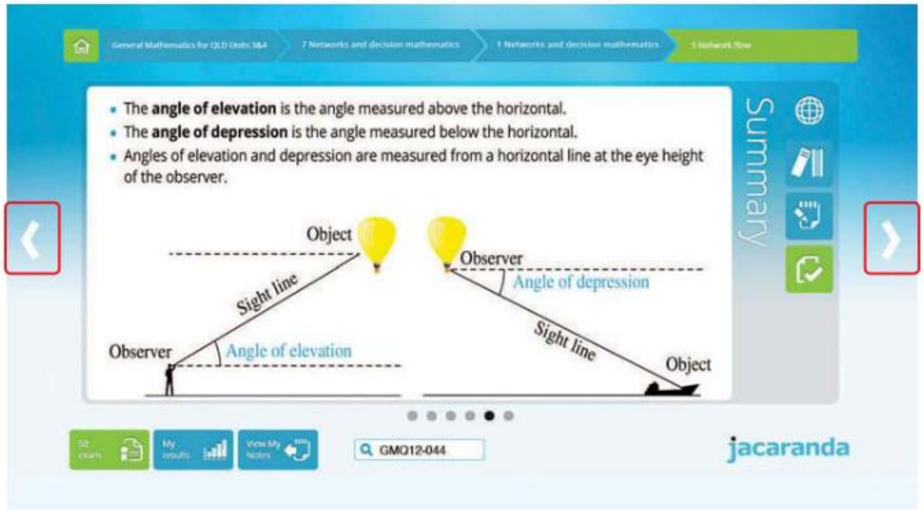


- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level.



- The **angle of elevation** is the angle measured above the horizontal.
- The **angle of depression** is the angle measured below the horizontal.
- Angles of elevation and depression are measured from a horizontal line at the eye height of the observer.

4 The arithmetic sequence

4.1 Overview

Many mathematical sequences are found in the world of nature, science, finance and art, to name a few.

One of the most famous mathematical sequences is the Fibonacci sequence. The Fibonacci series gets its name from Leonardo of Pisa, who lived in the twelfth century and later came to be known as Leonardo Fibonacci. Although it is sometimes claimed that he discovered the Fibonacci series, there is evidence of the series in ancient Sanskrit texts which were written many centuries before Leonardo. In his book *Liber Abaci*, published in 1202, he posed the question, 'if we start with a new pair of rabbits from birth, how many pairs will there be in one year?' He made some assumptions: there would be no deaths and each pair would reach maturity at one month and produce another pair after one month. In January, a new pair of rabbits would be born (1), reaching maturity in February (1) and breeding, producing a new pair in March (2). They would then breed again and produce a new pair in April (3), and another pair in May. Meanwhile, the rabbits born in March would reach maturity in April so in May two new pairs of bunnies would be produced, meaning a total of 5 pairs. Now the rabbits born in January, March, and April would all be adding new pairs, bringing June's total to 8 pairs.

The series would continue, with each new pair coming to maturity and one month later giving birth to another pair to be added to the whole. Over the months, the rabbit pair expansion would look like this:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots.$$

If you look carefully at this sequence of numbers, starting with 1, 1, each successive term in the Fibonacci sequence is found by adding the previous two terms together.

The Fibonacci sequence is found in many places in nature. The spirals of pine cones follow a Fibonacci series, as do the petals of flowers and the pattern of seeds within a sunflower.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.2 Using recursion to generate an arithmetic sequence

4.2.1 An arithmetic sequence

When we walk on a beach or track, each footprint is the same distance from the previous footprint. If the length of a person's foot is 20 cm and the distance between each footprint is 50 cm, then beginning from the first step on the sand the distance travelled after each step would look like this:

$$20,\ 70,\ 120,\ 220,\ 270\ldots$$

This is called an arithmetic sequence as each term is found by adding a certain number to the previous term. In this case 50 is added to each successive term.

An arithmetic sequence is a sequence in which the difference between any two successive terms in the sequence is the same. In an arithmetic sequence, the next term in the sequence can be found by adding or subtracting a fixed value.

Consider the sequence $5, 9, 13, 17, 21\ldots$. This is an arithmetic sequence, as each term is obtained by adding 4 (a fixed value) to the preceding term.

Now consider the sequence $1, 3, 6, 10, 15$. This is not an arithmetic sequence, as each term does not increase by the same constant value.

4.2.2 The common difference

In an arithmetic sequence, the first term is referred to as t_1, and the nth term is t_n, where n is the term number. The successive terms are referred to as $t_1,\ t_2,\ t_3,\ t_4,\ \ldots\ldots\ t_n$

The difference between two consecutive terms in an arithmetic sequence is known as the **common difference**. If the common difference is positive, the sequence is increasing. If the common difference is negative, the sequence is decreasing.

The common difference is referred to as d, so for an arithmetic sequence:

$$\begin{aligned} t_2 - t_1 &= d \\ t_3 - t_2 &= d \\ t_4 - t_3 &= d \\ &. \\ &. \\ &. \\ t_{n+1} - t_n &= d \end{aligned}$$

WORKED EXAMPLE 1

Determine which of the following sequences are arithmetic sequences, and for those sequences which are arithmetic, state the values of t_1 and d.

a. $\mathbf{2, 5, 8, 11, 14, \ldots}$

b. $\mathbf{4, -1, -6, -11, -16, \ldots}$

c. $\mathbf{3, 5, 9, 17, 33, \ldots}$

THINK	WRITE
a. 1. Calculate the difference between consecutive terms of the sequence.	**a.** $t_2 - t_1 = 5 - 2$ $= 3$ $t_3 - t_2 = 8 - 5$ $= 3$ $t_4 - t_3 = 11 - 8$ $= 3$ $t_5 - t_4 = 14 - 11$ $= 3$
2. If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is t_1 and the common difference is d.	The common differences are constant, so the sequence is arithmetic. $t_1 = 2$ and $d = 3$
b. 1. Calculate the difference between consecutive terms of the sequence.	**b.** $t_2 - t_1 = -1 - 4$ $= -5$ $t_3 - t_2 = -6 - -1$ $= -6 + 1$ $= -5$ $t_4 - t_3 = -11 - -6$ $= -11 + 6$ $= -5$ $t_5 - t_4 = -16 - -11$ $= -16 + 11$ $= -5$
2. If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is t_1 and the common difference is d.	The common differences are constant, so the sequence is arithmetic. $t_1 = 4$ and $d = -5$
c. 1. Calculate the difference between consecutive terms of the sequence.	**c.** $t_2 - t_1 = 5 - 3$ $= 2$ $t_3 - t_2 = 9 - 5$ $= 4$ $t_4 - t_3 = 17 - 9$ $= 8$ $t_5 - t_4 = 33 - 17$ $= 16$
2. If the differences between consecutive terms are constant, then the sequence is arithmetic.	The common differences are not constant, so the sequence is not arithmetic.

4.2.3 The recurrence method of generating an arithmetic sequence

If we know the first term (t_1) and the common difference (d) of an arithmetic sequence, then all the terms of the sequence can be generated.

Consider an arithmetic sequence with first term, 3 and common difference, 2.

The sequence will be $3,\ 5,\ 7,\ 9,\ 11,\ \ldots.$

The recurrence relation for this sequence is: $t_1 = 3,\ t_{n+1} = t_n + 2$

This is known as the **recurrence method** of generating an arithmetic sequence.

Term number	**Term**
1	t_1
2	$t_1 + d$
3	$t_2 + d$
4	$t_3 + d$
n	$t_{n-1} + d$
$n + 1$	$t_n + d$

Each successive term can be found by adding the common difference to the previous term.

Recursive definition of an arithmetic sequence:

$t_1 = a, t_{n+1} = t_n + d$, where t_1 is the first term and d is the common difference.

WORKED EXAMPLE 2

Generate the first 5 terms of the following arithmetic sequences.

a. $\mathbf{t_1 = 4, t_{n+1} = t_n - 1}$ **b.** $\mathbf{t_1 = -10, t_{n+1} = t_n + 6}$ **c.** $\mathbf{t_1 = 50, t_{n+1} = t_n - 100}$

THINK	**WRITE**
a. The first term is 4 and the common difference is -1.	**a.** The first 5 terms are: $4, 3, 2, 1, 0$
b. The first term is -10 and the common difference is 6.	**b.** The first 5 terms are: $-10, -4, 2, 8, 14$
c. The first term is 50 and the common difference is -100.	**c.** The first 5 terms are $50, -50, -150, -250, -350$

4.2.4 Tabular and graphical displays of arithmetic sequences

Tables of values

When we draw a graph of an arithmetic sequence, it helps to first draw a table of values for the sequence. The top row of the table displays the term number of the sequence, n, and the bottom of the table displays the term value t_n.

Term number (n)	1	2	3	...	n
Term value (t_n)					

The data from the table of values can then be used to identify the points to plot the points of the graph of the sequence.

Graphs of arithmetic sequences

When we draw a graph of an arithmetic sequence, the term number, n is the explanatory variable, so it appears on the x-axis of the graph. The term value, t_n is the response variable, so it appears on the y-axis of the graph.

Because there is a common difference between the terms of an arithmetic sequence, the relation between the terms is a linear relation. This means that when the terms of an arithmetic sequence are graphed, the points form a straight line. When the graph has a positive gradient/slope, the common difference is a positive number and the relation can be described as a **model of linear growth**. When the graph has a negative gradient/slope, the common difference is a negative number and the relation can be described as a **model of linear decay**.

When we draw a graph of an arithmetic sequence, we can extend the straight line to determine values of terms in the sequence that haven't yet been determined.

WORKED EXAMPLE 3

An arithmetic sequence is described by the recurrence relation

$$t_1 = 7, t_{n+1} = t_n + 2$$

a. **Create a table of values showing the term number and term value for the first 5 terms of the sequence.**

b. **Plot the graph of the sequence.**

c. **Use your graph of the sequence to determine the value of the 12th term of the sequence.**

THINK

a. **1.** Set up a table with the term number in the top row and the term value in the bottom row.

2. Use the recurrence relation to determine the first 5 values of the sequence

3. Complete the table with the calculated values.

b. **1.** Use the table of values to identify the points to be plotted.

2. Plot the points on the graph.

WRITE

a.

Term number	1	2	3	4	5
Term value					

$t_1 = 7$

$t_2 = 7 + 2 = 9$

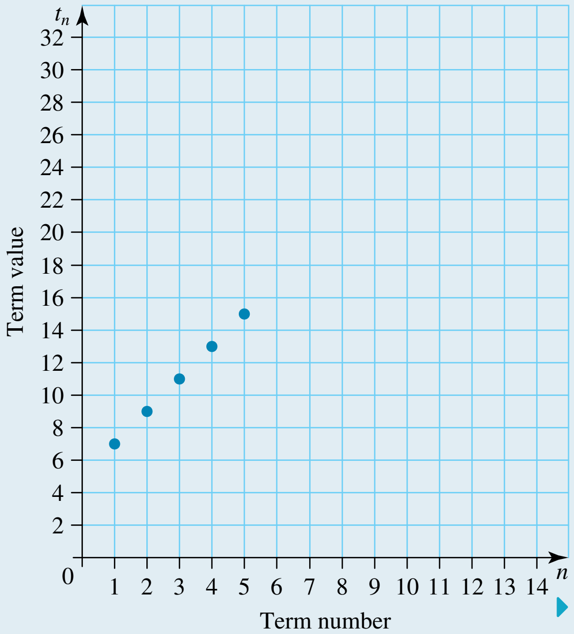
$t_3 = 9 + 2 = 11$

$t_4 = 11 + 2 = 13$

$t_5 = 13 + 2 = 15$

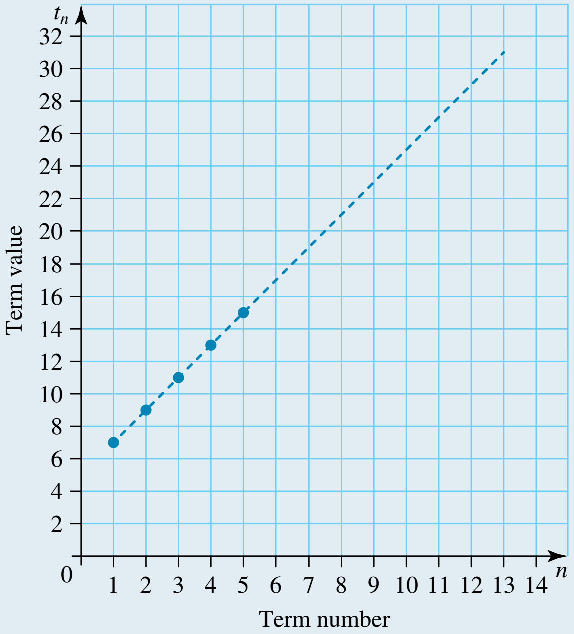
Term number	1	2	3	4	5
Term value	7	9	11	13	15

b. The points to be plotted are $(1, 7), (2, 9), (3, 11), (4, 13)$ and $(5, 15)$.

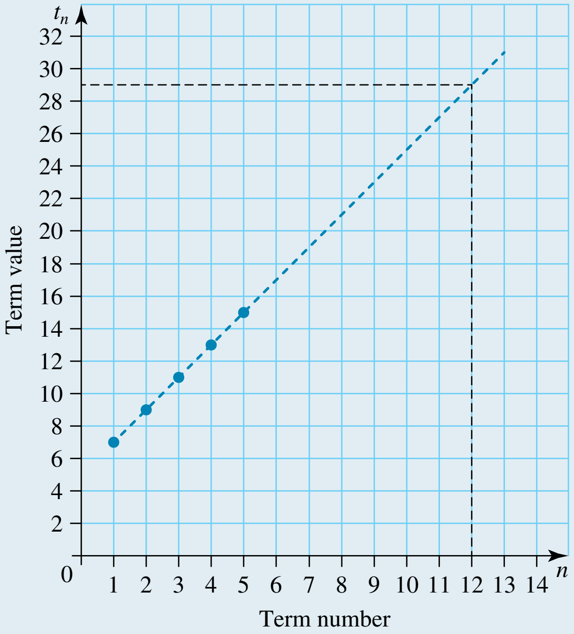


c. 1. Join the points with a straight line and extend the line to cover future values of the sequence.

c.



2. Read the required value from the graph (when $n = 12$).



3. Write the answer.

The 12th term of the sequence is 29.

4.2.4 Using a scientific calculator to generate terms of an arithmetic sequence

A scientific calculator can be used to generate the terms of an arithmetic sequence.

WORKED EXAMPLE 4

An arithmetic sequence is described by the recurrence relation $t_1 = -8$, $t_{n+1} = t_n + 1.5$. Use a scientific calculator to generate the first 6 terms of this sequence.

THINK	WRITE
1. Type the first term of the sequence into the calculator screen and then press Enter or =. 2. Type +1.5, then press Enter or = 5 times.	-8 -6.5 -5 -3.5 -2 -0.5
3. Write the answer.	The first 6 terms are $-8, -6.5, -5, -3.5, -2, -0.5$

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Units 3 & 4 › Area 3 › Sequence 1 › Concepts 1 & 2

Arithmetic sequences Summary screen and practice questions
Generating an arithmetic sequence using recursion Summary screen and practice questions

Exercise 4.2 Using recursion to generate an arithmetic sequence

1. **WE1** Determine which of the following sequences are arithmetic sequences, and for those sequences which are arithmetic, state the values of t_1 and d.
   a. $23, 68, 113, 158, 203, \ldots\ldots$ b. $3, 8, 23, 68, 203, \ldots\ldots$ c. $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots\ldots$
2. Calculate the values of the unknown in the following arithmetic sequences.
   a. $13, -12, -37, f, -87, \ldots\ldots$ b. $2.5, j, 8.9, 12.1, k, \ldots\ldots$ c. $p, q, r, \frac{9}{2}, \frac{25}{4}, \ldots\ldots$
3. **MC** Which of the following sequences is NOT an arithmetic sequence?
   A. $4, -5, -14, -23, -32, \ldots\ldots$ B. $13, 20, 27, 34, 41, \ldots\ldots.$
   C. $21, 31, 41, 51, 61, \ldots\ldots.$ D. $1, 1, 2, 3, 5, 8, 13, 23, \ldots\ldots\ldots$
4. State which of the following situations are arithmetic sequences.
   a. A teacher hands out 2 lollies to the first student, 4 lollies to the second student, 6 lollies to the third student and 8 lollies to the fourth student.
   b. The sequence of numbers after rolling a die 8 times.
   c. The number of layers of paper each time a large sheet of paper is folded in half.
   d. The house numbers on the same side of a street on a newspaper delivery route.
   e. The cumulative total of the number of seats in the first 10 rows in a regular cinema (for example, with 8 seats in each row, so there are 8 seats after the first row, 16 seats after the first 2 rows, and so on).

5. For those arithmetic sequences found in question 4, where appropriate information is given, write down the value of t_1 and d.
6. **WE2** Generate the first 5 terms for the arithmetic sequences defined by the following recurrence relations.
   a. $t_1 = -3, t_{n+1} = t_n + 2$ b. $t_1 = 5, t_{n+1} = t_n - 8$ c. $t_1 = -30, t_{n+1} = t_n + 60$

7. Write the following arithmetic sequences as recurrence relations.

a. $2, 4, 6, 8, 10, \ldots\ldots\ldots$ **b.** $-6, -10, -14, -18, \ldots\ldots\ldots$ **c.** $10, 20, 30, 40, 50, \ldots\ldots\ldots$

8. Tanya is given $50 by her grandparents on her birthday. She decides to save $5 of her weekly pocket money each week to add to the gift of $50.

a. Write Tanya's savings as a recurrence relation.

b. Is this a model of linear growth or linear decay?

9. Daniel has been given some toy dinosaurs. At the end of each year, he adds the same number of dinosaurs to his collection. His collection can be modelled using the recurrence relation

$$t_1 = 6, t_{n+1} = t_n + 3.$$

a. How many dinosaurs was Daniel given?

b. How many dinosaurs did Daniel add to his collection each year?

c. How many dinosaurs did Daniel have at the end of 4 years?

10. **WE3** An arithmetic sequence is described by the recurrence relation

$$t_1 = 5, t_{n+1} = t_n + 10.$$

a. Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.

b. Plot the graph of the sequence.

c. Use your graph of the sequence to determine the 9th term of the sequence.

11. An arithmetic sequence is defined by the recurrence relation $t_1 = 6.4, t_{n+1} = t_n + 1.6$

a. Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.

b. Plot the graph of the sequence.

c. Use your graph of the sequence to determine the 13th term of the sequence.

12. **WE4** Use your calculator to generate the first 6 terms of the arithmetic sequences defined by the following recurrence relations.

a. $t_1 = -3.6, t_{n+1} = t_n + 5.2$ **b.** $t_1 = -0.4, t_{n+1} = t_n - 0.8$

13. Spreadsheets can be used to generate sequences that are not arithmetic by using recursion. Use a spreadsheet to generate the first 10 terms of the following sequences.

a. $t_1 = 2.13, t_{n+1} = 2t_n + 1.1$ **b.** $t_1 = -1.3, t_{n+1} = \frac{1}{2}t_n - \frac{1}{3}$

14. An ice shelf is shrinking at a rate of 1200 km^2 per year. When measurements of the ice shelf began, the area of the shelf was 37 000 km^2.

a. Create a recurrence relation to express the area of the ice shelf after *n* years.

b. Use your relation to determine the area of the ice shelf after each of the first 6 years.

c. Plot a graph showing the area of the shrinking ice shelf over time.

d. Is this a model of linear growth or decay?

15. Kane's starting salary at his job was $68 000. At the end of his first year his salary increased by 2.5% and then increased by that same amount each year.

a. Write a recurrence relation to reflect Kane's salary in his *n*th year in the job.

b. How much will he earn at the end of his 4th year in the job?

c. Is this recurrence relation a model of linear growth or decay?

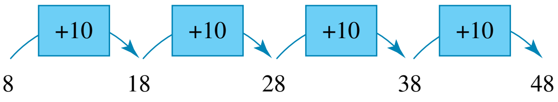
16. The population of an isolated country town is decreasing by 250 people every year.
At the end of 2018 the population was 3750.
  a. Set up a recurrence relation to show the population of the town in n years.
  b. If the population continues to decrease by the same number, at the end of what year will the population of the town be 0? (You may use a spreadsheet to determine the answer to this question.)

4.3 Using the rule for the *n*th term of an arithmetic sequence to make predictions

4.3.1 The pattern of the terms in an arithmetic sequence

Recursion is one way of determining the value of the terms in an arithmetic sequence but calculating t_{100} using recursion is a very tedious process.

Consider the arithmetic sequence for which $t_1 = 8$ and $d = 10$.



$$\begin{array}{llll}
 & t_1 = 8 & t_1 = a & \\
 & t_2 = 8 + 10 & t_2 = a + d & t_2 = a + 1d \\
\text{Now,} & t_3 = 8 + 10 + 10 & t_3 = a + d + d & t_3 = a + 2d \\
 & t_4 = 8 + 10 + 10 + 10 & t_4 = a + d + d + d & t_4 = a + 3d \\
 & t_5 = 8 + 10 + 10 + 10 + 10 & t_5 = a + d + d + d + d & t_5 = a + 4d
\end{array}$$

We notice a pattern emerging. That pattern can be described by the equation

$$t_n = 8 + (n - 1) \times 10$$

where n represents the number of the term.
For example, if $n = 4$, then the fourth term is:

$$\begin{aligned}
t_4 &= 8 + (4 - 1) \times 10 \\
t_4 &= 8 + 3 \times 10 \\
t_4 &= 38
\end{aligned}$$

Therefore, the 4th term is 38.
We can generalise this rule for all arithmetic sequences.

$$t_n = t_1 + (n - 1)d$$

where t_n is the nth term
t_1 is the first term
d is the common difference
Therefore, if we know or can determine the values of t_1 and d, we can construct the rule for the sequence.

WORKED EXAMPLE 5

Determine the rules that represent the following arithmetic sequences.

a. **3, 6, 9, 12, 15, . . .**

b. **40, 33, 26, 19, 12, . . .**

THINK	WRITE
a. 1. Determine the values of t_1 and d.	**a.** $t_1 = 3$ $d = t_2 - t_1$ $= 6 - 3$ $= 3$
2. Substitute the values for t_1 and d into the formula for arithmetic sequences.	$t_n = t_1 + (n - 1)d$ $= 3 + (n - 1) \times 3$ $= 3 + 3(n - 1)$ $= 3 + 3n - 3$ $= 3n$
b. 1. Determine the values of t_1 and d.	**b.** $t_1 = 40$ $d = t_2 - t_1$ $= 33 - 40$ $= -7$
2. Substitute the values for t_1 and d into the formula for arithmetic sequences.	$t_n = a + (n - 1)d$ $= 40 + (n - 1) \times -7$ $= 40 - 7(n - 1)$ $= 40 - 7n + 7$ $= 47 - 7n$

4.3.2 Predicting future values of an arithmetic sequence

This rule enables us to predict any term of an arithmetic sequence provided we know the values of t_1 and d.

WORKED EXAMPLE 6

Determine the 20th term of the following arithmetic sequence.

5, 40, 75, 110, 145, . . .

THINK	WRITE
1. Determine the value of t_1.	$t_1 = 5$
2. Determine the value of d. You need to calculate only one difference as the question states that it is an arithmetic sequence.	$d = t_2 - t_1$ $= 40 - 5$ $= 35$
3. Use the rule $t_n = t_1 + (n - 1)d$ where n is 20 for the 20th term.	$t_{20} = 5 + (20 - 1) \times 35$ $= 5 + 19 \times 35$ $= 670$
4. Write the answer.	The 20th term is 670.

If we are given only two terms of an arithmetic sequence, we are able to use the rule $t_n = t_1 + (n - 1)d$ to set up two simultaneous equations to determine the value of t_1 and d and hence write down the rule for the arithmetic sequence.

WORKED EXAMPLE 7

The third term of an arithmetic sequence is -1 and the fifth term is 11.

a. Write down the rule for the arithmetic sequence.

b. Determine the 50th term of the sequence.

THINK	WRITE
a. 1. We know that $t_3 = -1$ and that $t_n = t_1 + (n-1)d$.	**a.** $t_3 = t_1 + 2d = -1$
2. We know that $t_5 = 11$ and that $t_n = t_1 + (n-1)d$.	$t_5 = t_1 + 4d = 11$
3. Solve the 2 equations simultaneously using the elimination technique. Eliminate t_1, by subtracting equation [1] from equation [2].	$t_1 + 2d = -1$ [1] $t_1 + 4d = 11$ [2] $2d = 12$ [2] − [1] $d = 6$
4. Evaluate t_1 by substituting $d = 6$ into either of the two equations.	Substituting $d = 6$ into [1]: $t_1 + 12 = -1$ $t_1 = -13$
5. To determine the rule, substitute values for a and d into $t_n = t_1 + (n-1)d$.	$t_n = -13 + (n-1) \times 6$ $t_n = -13 + 6n - 6$ $t_n = -19 + 6n$
b. 1. To determine the 50th term or t_{50}, substitute $n = 50$ into the rule.	**b.** $t_n = -19 + 6n$ $t_{50} = -19 + 6 \times 50$ $= -19 + 300$ $= 281$
2. Write your answer.	The 50th term is 281.

on Resources

Interactivities Terms of an arithmetic sequence (int-6261)

Exercise 4.3 Using the rule for the *n*th term of an arithmetic sequence to make predictions

1. **WE5** Determine the equations that represent the following arithmetic sequences.
   - a. $-1, 3, 7, 11, 15, \ldots$
   - b. $1.5, -2, -5.5, -9, -12.5$
   - c. $\frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \frac{23}{2}, \ldots$
2. Determine the first five terms of the arithmetic sequences defined by the following recurrence relations.
   - a. $t_n = 5 + 3(n-1)$
   - b. $t_n = -1 - 7(n-1)$
   - c. $t_n = \frac{1}{3} + \frac{2}{3}(n-1)$

3. **WE6** For each of the arithmetic sequences given, determine

- **a.** the 25th term of the sequence 2, 7, 12, 17, 22, …
- **b.** the 30th term of the sequence 0, 100, 200, 300, 400, …
- **c.** the 33rd term of the sequence 5, −2, −9, −16, −23, …

4. **MC** The 41st term of the arithmetic sequence −4.3, −2.1, 0.1, 2.3, 4.5, … is

A. 83.7. **B.** 85.9. **C.** 92.3. **D.** 172.4.

5. **WE7** Evaluate the following.

- **a.** The 2nd term of an arithmetic sequence is 13 and the 5th term is 31. What is the 17th term of this sequence?
- **b.** The 2nd term of an arithmetic sequence is −23 and the 5th term is 277. What is the 20th term of this sequence?
- **c.** The 2nd term of an arithmetic sequence is 0 and the 6th term is −8. What is the 32nd term of this sequence?
- **d.** The 3rd term of an arithmetic sequence is 5 and the 7th term is −19. What is the 40th term of this sequence?
- **e.** The 4th term of an arithmetic sequence is 2 and the 9th term is −33. What is the 26th term of this sequence?

6. **MC** The 2nd term of an arithmetic sequence is −2 and the 6th term is 2. The 27th term of this sequence is

A. 20. **B.** 21. **C.** 22. **D.** 23.

7. **WE4**

- **a.** Determine the 20th term of the sequence 85, 72, 59, 46, 33, …
- **b.** Determine the first value of the arithmetic sequence in which $t_{70} = 500$ and $d = -43$.

8.

- **a.** Determine the common difference of the arithmetic sequence that has a first term of −32 and an 8th term of 304.
- **b.** An arithmetic sequence has a first term of 5 and a common difference of 40. Which term number has a value of 85?
- **c.** An arithmetic sequence has a first term of 40 and a common difference of 12. Which term number has a value of 196?

9. For the following arithmetic sequences:

- **a.** 4, 13, 22, 31, … which term, t_n, will be equal to 58?
- **b.** 9, 4.5, 0, … which term, t_n, will be equal to −18?
- **c.** −60, −49, −38, … which term, t_n, will be the first to be greater than 10?
- **d.** 100, 87, 74, … which term, t_n, will be the first to be less than 58?

10. A batsman made 23 runs in his first innings, 33 in his second and 43 in his third. If he continued to add 10 runs each innings, write down a rule for the number of runs he would have made in his *n*th innings.

11. In a vineyard, rows of wire fences are built to support the vines. The length of the fence in row 1 is 40 m, the length of the fence in row 2 is 43 m, and the length of the fence in row 3 is 46 m. If the lengths of the fences continue in this pattern, write down a rule for the length of a fence in row number *n*.

12. The first fence post in a fence is 12 m from the road, the next is 15.5 m from the road and the next is 19 m from the road. The rest of the fence posts are spaced in this pattern.

a. Write down a rule for the distance of fence post n from the road.

b. If 100 posts are to be erected, how far will the last post be from the road?

13. A taxi company charges a flag fall of \$20 and \$0.90/km.

a. Write the rule for calculating the cost (C) of catching a taxi for n km.

b. If the distance from the airport to the city is 15 km, what is the total charge for a taxi ride?

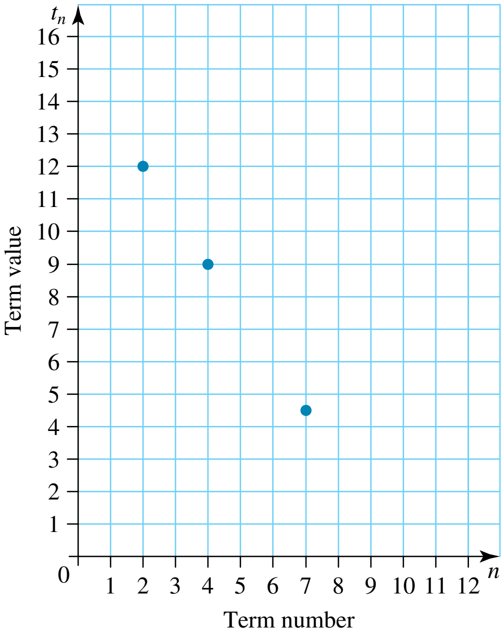
14. Three consecutive terms of an arithmetic sequence are $x - 5, x + 4$ and $2x - 7$. Determine the value of x.

15. The graph shows some points of an arithmetic sequence.

a. What is the common difference between consecutive terms?

b. What is the value of the first term of the sequence?

c. What is the value of the 12th term of the sequence?



16. Sketch the graph of $t_n = t_1 + (n - 1)d$, where $t_1 = 15$ and $d = 25$, for the first 10 terms.

17. An employee starts a new job with a \$60 000 salary in the first year and the promise of a pay rise of \$2500 at the end of each year.

a. How much will her salary be in the 6th year?

b. How long will it take for her salary to reach \$85 000?

4.4 Simple interest and other applications of arithmetic sequences

4.4.1 Modelling a simple interest loan or investment using recurrence relation

When money is borrowed or invested, and interest is calculated as a percentage of the initial investment or loan, this is known as **simple interest**.

Consider an investment of \$5000 at a simple interest rate of 5% p.a., invested for 5 years.

Simple interest

$$\begin{aligned}\text{Simple interest} &= \text{principal} \times \frac{\text{rate}}{100} \times \text{time} \\ I &= Pin\end{aligned}$$

where

I is the total interest

P is the principal or initial amount borrowed or invested

i is $\dfrac{\text{interest rate}}{100}$

n is the number of periods (time)

As 5% of \$5000 $= \dfrac{5}{100} \times 5000 = \$250$, the simple interest paid each year is \$250.

The following table summarises the investment over the 5 years.

Note: For financial calculations, V is used in place of t.

n	V_n	**Interest**	V_{n+1}
0	5000	250	5250
1	5250	250	5500
2	5500	250	5750
3	5750	250	6000
4	6000	250	6250

The investment at the end of each of the first 5 years will be
Year 1: $5000 + 250 = 5000 + 250 \times 1 = \$5250$
Year 2: $5000 + 250 + 250 = 5000 + 250 \times 2 = \$5500$
Year 3: $5000 + 250 + 250 + 250 = 5000 + 250 \times 3 = \$5750$
Year 4: $5000 + 250 + 250 + 250 + 250 = 5000 + 250 \times 4 = \$6000$
Year 5: $5000 + 250 + 250 + 250 + 250 + 250 = 5000 + 250 \times 5 = \$6250$
The terms of this investment follow an arithmetic sequence:

$$\$5000, \$5250, \$5500, \$5750, \$6000, \$6250$$

where $V_0 = \$5000$ and $d = \$250$.
The recurrence relation for a simple interest loan or investment is:
$V_0 = a, V_{n+1} = V_n + d$
where V_0 is the initial amount and d is the interest added each year.
Note: Graphs will be drawn with broken lines to indicate that the data is discrete, but a loan or investment period may end at any time.

WORKED EXAMPLE 8

$645 is invested in a simple interest account for 5 years at 5% p.a.

a. Set up a recurrence relation for this investment.

b. Using the recurrence relation from part a, complete the following table to determine the value of the investment at the end of each of the 5 years.

n	V_n	Interest	V_{n+1}
0			
1			
2			
3			
4			

c. Sketch the graph of the investment for the first 5 years.

d. Using the graph from part c, determine the total amount of the investment at the end of 8 years, correct to the nearest $100.

THINK	WRITE
a. 1. Calculate the amount of interest earned per year.	**a.** Interest $= \frac{5}{100} \times 645 = \$32.25$
2. Write the values of V_0 and d.	$V_0 = 645$ $d = 32.25$
3. Write the recurrence equation	$V_0 = 645, V_{n+1} = V_n + 32.25$

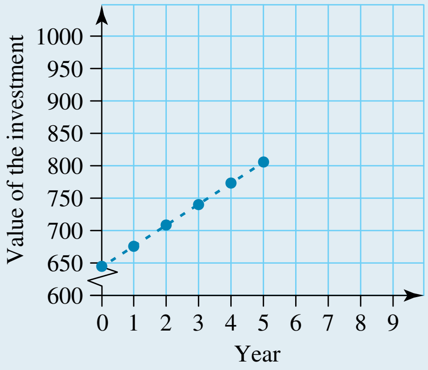
b. Complete the table using the recurrence relation.

b.

n	V_n	Interest	V_{n+1}
0	645	32.25	645 + 32.25 = \$677.25
1	677.25	32.25	677.25 + 32.25 = \$709.50
2	709.50	32.25	709.50 + 32.25 = \$741.75
3	741.75	32.25	741.75 + 32.25 = \$774
4	774	32.25	774 + 32.25 = \$806.25

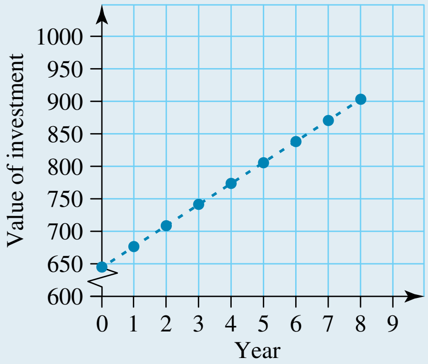
c. Draw a set of axes with the term number (year) on the horizonal axes and the value of the investment on the vertical axes. Plot the points (0, 645), (1, 677.25), (2, 709.50), (3, 741.75), (4, 774), (5, 806.25)

c.



d. 1. Extend the graph from part c to $n = 8$.

d.



2. Read the value of investment from the graph at $n = 8$.

Value of investment after 8 years is \$900 to the closest \$100.

4.4.2. Modelling a simple interest loan or investment using a rule

The total value of a simple interest loan or investment can also be calculated using the rule:

$$A = P + I$$

where
A is the total amount at the end of the investment or loan period
P is the initial amount borrowed or invested

$I = Pin$ where I is the total interest, P is the principal or initial amount borrowed or invested, i is $\frac{\text{interest rate}}{100}$ and n is the number of periods (time).

WORKED EXAMPLE 9

Calculate the amount of simple interest, *I*, earned and the total amount, *A*, at the end of the term, if:

a. $12 000 is invested for 5 years at 9.5% p.a.

b. $2500 is invested for 3 months at 4.5% p.a.

THINK	WRITE
a. 1. Write down the formula for simple interest.	**a.** $I = Pin$
2. Write down the known values of the variables.	$P = \$12\,000$ $i = \frac{9.5}{100}$ $n = 5$ years
3. Substitute the values into the given formula.	$I = 12000 \times \frac{9.5}{100} \times 5$ $= \frac{570\,000}{100}$
4. Evaluate.	$= 5700$
5. Answer the question and include the appropriate unit.	The amount of interest earned is $5700.
6. Write down the formula for the total amount.	$A = P + I$
7. Substitute the values for P and I.	$= 12\,000 + 5\,700$
8. Evaluate.	$= 17\,700$
9. Answer the question and include the appropriate unit.	The total amount at the end of the term is $17 700.
b. 1. Write down the formula for simple interest.	**b.** $I = Pin$
2. Write down the known values of the variables. *Note:* T must be expressed in years, so divide 3 months by 12 months.	$P = \$2500$ $i = \frac{4.5}{100}$ $T = 3 \text{ months} = \frac{3}{12}$ or 0.25 years
3. Substitute the values into the given formula.	$I = 2500 \times \frac{4.5}{100} \times 0.25$ $= \frac{2812.5}{100}$
4. Evaluate and round off the answer to 2 decimal places.	$= 28.13$
5. Answer the question and include the appropriate unit.	The amount of interest earned is $28.13.
6. Write down the formula for the total amount.	$A = P + I$

▶

7. Substitute the values for P and I.	$= 2500 + 28.13$
8. Evaluate.	$= 2528.13$
9. Answer the question and include the appropriate unit.	The total amount at the end of the term is \$2528.13.

4.4.3 Calculating *P*, *i* and *n* using the simple interest rule

In many cases we may wish to determine the principal, interest rate or period of a loan. In these situations, it is necessary to rearrange the simple interest rule.

Rearranged simple interest rule $I = Pin$

Rearranged simple interest

To determine the principal $P = \dfrac{I}{in}$

To determine the interest rate $i = \dfrac{I}{Pn}$

To determine the period of the loan or investment $n = \dfrac{I}{Pi}$

WORKED EXAMPLE 10

A bank offers 9% p.a. simple interest on an investment. At the end of 4 years the interest earned was \$215. How much was invested?

THINK	WRITE
1. Write the simple interest formula.	$I = Pin$
2. List the values of I, i and n.	$I = 215,\ i = \dfrac{9}{100} = 0.09,\ n = 4$
3. Substitute into the formula.	$I = Pin$ $215 = P \times 0.09 \times 4$
4. Make P the subject by dividing both sides by (0.09×4)	$P = \dfrac{215}{0.09 \times 4}$
5. Use a calculator to evaluate.	$P = 597.22$
6. Write your answer.	The amount invested was \$597.22.

WORKED EXAMPLE 11

When \$720 is invested for 36 months it earns \$205.20 simple interest. Calculate the yearly interest rate.

THINK	WRITE
1. Write the simple interest formula with i as the subject.	$I = Pin$ $i = \dfrac{I}{Pn}$

2. List the values of P, I, and n. n must be expressed in years.	$P = 720,\ I = 205.20, n = 36$ months (3 years)
3. Substitute into the formula	$i = \frac{I}{Pn}$ $= \frac{205.20}{720 \times 3}$ $= 0.095$
4. Multiply i by 100 to obtain the yearly interest rate. Use a calculator to evaluate.	Interest rate $= 0.095 \times 100 = 9.5\%$
5. Write your answer.	The interest rate offered was 9.5% per annum.

WORKED EXAMPLE 12

Determine how long it will take, to the nearest year, to earn \$86.70 in interest when \$255 is invested at 8.5% p.a.

THINK	WRITE
1. Write the simple interest formula with n as the subject.	$I = Pin$ $n = \frac{I}{Pi}$
2. List the values of P, I, and i.	$P = 255, I = 86.70, i = \frac{8.5}{100} = 0.085$
3. Substitute into the formula	$n = \frac{I}{Pi}$ $= \frac{86.70}{255 \times 0.085}$ $= 4$
4. Write your answer.	The period of the investment is 4 years.

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Units 3 & 4 › Area 3 › Sequence 1 › Concept 3

Simple interest and other applications Summary screen and practice questions

Exercise 4.4 Simple interest and other applications of arithmetic sequences

1. **WE8** \$415 is invested in a simple interest account for 4 years at 7% p.a.
   a. Set up a recurrence relation for this investment.
   b. Using the recurrence relation from part a, complete the following table to determine the value of the investment at the end of each of the 4 years.

n	V_n	Interest	V_{n+1}
0			
1			
2			
3			

   c. Sketch the graph of the investment for the first 4 years.
   d. Using the graph from part c, determine the total amount of the investment at the end of 7 years, correct to the nearest \$100.
2. Savings of \$20 000 are to be invested at 8% p.a. simple interest.
   a. Copy and complete the following table to calculate the value of the investment at the end of 5 years.

n	V_n	Interest	V_{n+1}
0			
1			
2			
3			
4			

   b. Draw a graph of the value of the investment.
   c. Determine the gradient of the linear graph drawn in part b and interpret the value of the gradient.
   d. Use you graph or some other method to determine the amount of interest that would have been earned after 10 years.
3. **WE9** Calculate the amount of simple interest, I, earned and the total amount, A, at the end of the term if
   a. \$750 is invested for 3 years at 12% p.a.
   b. \$20 000 is invested for 3 years and 6 months at 11% p.a.
4. Robyn wishes to purchase a new dress worth \$350 to wear to the school formal. If she borrows the total amount from the bank and pays it off over 3 years at 11% p.a. simple interest, what is the total amount Robyn must pay back to the bank?
5. The Sharks Building Society offers loans at 8% p.a. simple interest for a period of 18 months. Andrew borrows \$2000 from Sharks to buy Monique an engagement ring. Calculate the amount of interest Andrew is to pay over the 18 months.

6. Silvio invested the $1500 he won in Lotto with an insurance company bond that pays 12% p.a. simple interest provided he keeps the bond for 5 years. What is Silvio's total return from the bond at the end of the 5 years?
7. The insurance company that Silvio used in the previous question allows people to withdraw part or all the money early. If this happens the insurance company will only pay 6% p.a. simple interest on the amount which is withdrawn over the period it was invested in the bond. The part which is left in the bond receives the original agreed interest. Silvio needed $700 for repairs to his car 2 years after he had invested the money but left the rest in for the full 5 years. How much interest did he earn from the bond in total?
8. **WE10** A bank offers 7% p.a. simple interest on an investment. At the end of 4 years the interest earned was $1232. How much was invested?
9. Lisa invested in a loan that offered 9% p.a. simple interest. After 3 years the investment had earned $576. What was Lisa's initial investment?
10. **WE11** When $1500 is borrowed for 2 months the simple interest charged is $20. Calculate the yearly interest rate.
11. **WE12** An amount of $6000 was invested at 4.5% p.a. Determine how long it will take, to the nearest month, to earn $630 in interest.
12. Jack Kahn put some money away for 5 years in a bank account which paid 3% p.a. interest. He found from his bank statement that he had earned $66. How much did Jack invest?
13. James needed to earn $225. He invested $2500 in an account earning simple interest at a rate of 4.5% p.a. How many months will it take James to achieve his aim?
14. Carol has $3000 to invest. Her aim is to earn $450 in interest at a rate of 5% p.a. Over what term would she invest?
15. Alisha has $8900 that she is able to invest. She has a goal of earning at least $1100 in 2 years or less. Do any of the following investments satisfy Alisha's goal?
    a. 10% p.a. for 15 months
    b. 4.25% earning $1200
    c. After 100 weeks a final payout of $10 500
    d. After 2 years at 7.2% p.a.
16. Jill and John decide to borrow money to improve their yacht, but cannot agree which loan is the better value. They would like to borrow $2550. Jill goes to the Big-4 Bank and determines that they will lend her the money at 11% p.a. simple interest for 3 years. John determines that the Friendly Building Society will lend the $2550 to them at 1% per month simple interest for the 3 years.
    a. Which institution offers the better rate over the 3 years?
    b. Explain why.

17. Sue and Harry invested $14 500 in state government bonds at 8.65% p.a. The investment is for 10 years and the interest is paid semi-annually (that is, every six months). Calculate how much interest:
    a. they receive every payment
    b. will be received in total.
18. Anna invested $85 000 in Ski International debentures. She earns 7.25% p.a. which is paid quarterly for one year.
    a. Calculate how much interest
        i. she receives quarterly
        ii. will be received in total, over a year.
    b. Would Anna receive the same amount of interest over a 3 year period if it were paid annually rather than quarterly?

19. Mrs Williams invested \$60 000 in government bonds at 7.49% p.a. with interest paid semi-annually (that is, every 6 months).

- **a.** How much interest is she paid each 6 months?
- **b.** How much interest is she paid over 3 years?
- **c.** How long would the money need to be invested to earn a total of \$33 705 in interest?

20. A graph is drawn to show the growth of an investment of \$6000 at 4.8% p.a. simple interest for each year for 5 years.

- **a.** Explain why this will be a linear graph.
- **b.** Without drawing the graph, calculate the value of the gradient.
- **c.** Is this situation a linear model of growth or decay?

21. Mark has \$5500 to invest at i. 3% p.a. ii . 3.5% p.a. iii. 3.75% p.a.
For each of the interest rate options complete the recurrence relations below.

- **a.** **i.** $V_0 = 5500, V_{n+1} = V_n + ?$
  - **ii.** $V_0 = ?, V_{n+1} = V_n + 192.50$
  - **iii.** $V_0 = ?, V_{n+1} = V_n + ?$
- **b.** Complete the following table.

Year	**0**	**1**	**2**	**3**	**4**	**5**
3%	\$5500		\$5830		\$6160	\$6325
3.5%		\$5692.50	\$5885	\$6077.50		\$6462.50
3.75%	\$5500	\$5706.25		\$6118.75	\$6325	

- **c.** Sketch the three graphs of the investment options on the same axes.
- **d.** Compare and comment on the gradient of the graphs.

4.5 Straight-line and unit cost depreciation

4.5.1 Modelling straight-line depreciation

An asset is an item that has value to its owner. Many assets such as cars and computers lose value over time. This is called **depreciation**. Some businesses set aside the amount of depreciation of their assets, so that they have the money saved to replace assets such as photocopiers when necessary.

Straight-line depreciation is where the asset depreciates by a constant amount each year. When this type of depreciation is graphed, a straight line occurs.

After a certain period of time, or when an asset reaches a certain value, the asset is considered to be no longer of any worth. This is called its **scrap value**. At that point, the asset will be sold or sent for recycling.

Straight-line depreciation follows an arithmetic sequence and can be expressed as a recurrence relation:
$V_0 = a,\ V_{n+1} = V_n - d$, where V_0 is the initial value of the asset, V_n is the value of the asset after n years and d is the depreciation amount each year.

WORKED EXAMPLE 13

A printer was purchased for a company office for $1500.

The flat rate depreciation for the printer is $170 each year. Complete the following table to calculate the future value of the printer at the end of 5 years.

Time, *n* (years)	Depreciation ($)	Future value ($)
0	0	1500
1		
2		
3		
4		
5		

THINK

1. The depreciation is $170 each year. Subtract $170 from $1500 and subsequent values in the ‘Future value’ column.

WRITE

Time, *n* (years)	Depreciation ($)	Future value ($)
0	0	1500
1	170	1330
2	170	1160
3	170	990
4	170	820
5	170	650

2. Write the answer.

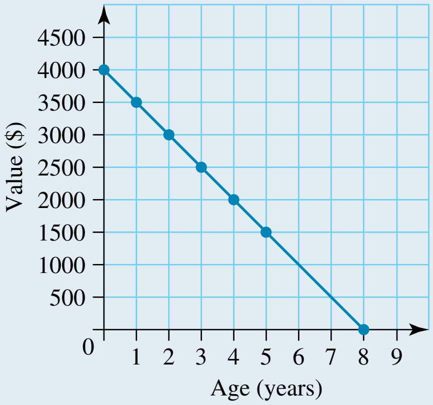
The printer has a future value of $650 after 5 years.

WORKED EXAMPLE 14

The following table shows the depreciating value of a computer.

Age (years)	Value ($)
0	4000
1	3500
2	3000
3	2500
4	2000
5	1500

a. **Graph the value against time.**
b. **Is this a linear model of growth or decay?**
c. **Determine the gradient of the graph and interpret the gradient.**

THINK	WRITE
a. 1. Draw a set of axes with age on the horizontal axes and value on the vertical. **2.** Plot each point given by the table. **3.** Join all points to graph the function.	**a.** 
b. The graph is a straight line, sloping down from left to right, so a linear model of decay.	**b.** A linear model of decay.
c. The gradient is the amount the computer is depreciating each year.	**c.** Gradient = 3500 − 4000 = −500 For every increase of 1 year, the computer decreases in value by $500.

4.5.2 Calculating straight-line depreciation using a rule

The future value of an asset can be calculated using a linear equation.

Future value

$$V_n = V_0 - nd,$$

where

V_n is the future value of the asset after n years,
V_0 is the initial value of the asset,
d is the amount of depreciation each year.

WORKED EXAMPLE 15

A plumber purchases new equipment for a total of $80 000.

The value of the equipment is depreciated by $7500 per year. The equipment is considered to have a scrap value of $10 000.

a. Calculate the future value of the equipment after 5 years.

b. Calculate the number of years before the equipment reaches its scrap value.

THINK	WRITE
a. 1. Write the formula.	**a.** $V_n = V_0 - nd$
2. Substitute for V_0, d and n.	$V_n = 80\,000 - 5 \times 7500$ $= 42\,500$

3. Write the answer.

After 5 years, the equipment will be worth \$42 500.

b. 1. Write the formula.

2. Substitute for V_n, V_0 and d.

3. Solve the equation to determine the value of n.

Write the answer.

b. $V_n = V_0 - nd$

$10\,000 = 80\,000 - 7\,500n$

$7500n = 70000$

$n = 9\frac{1}{3}$

The equipment reaches its scrap value after $9\frac{1}{3}$ years.

4.5.3 Calculating unit cost depreciation using a recurrence relation

Many assets depreciate based on their use rather than their age. A car that has a reading of 160 648 km on its odometer will not be worth the same as a car the same age and make but with an odometer reading of 60 000 km.

Unit cost depreciation follows an arithmetic sequence and can be expressed as a recurrence relation:

$V_0 = a, V_{n+1} = V_n - d$, where V_0 is the initial value of the asset, V_n is the value of the asset after n outputs and d is the depreciation amount per output.

WORKED EXAMPLE 16

A photocopier is purchased for an office for \$13 000. It depreciates at a rate of 19 cents per copy.

a. Set up a recurrence relation for the depreciation of the photocopier.

b. Generate a depreciation schedule for the photocopier after it has printed 10000, 20000, 30000, 40000 and 50000 copies.

THINK

a. 1. Write the formula for the recurrence relation.

2. Write the values for V_0 and d.

Substitute into the formula.

b. The depreciation amount for 10 000 pages is: $0.19 \times 10\,000 = \$1900$.
Draw the depreciation schedule and complete the table.

WRITE

a. $V_0 = a, V_{n+1} = V_n - d$

$V_0 = 13000, d = 0.19$

$V_0 = 13000, V_{n+1} = V_n - 0.19$

b.

Pages printed	Future value (V_n)
0	13000
10000	11 100
20000	9200
30000	7300
40000	5400
50000	3500

4.5.4 Calculating unit cost depreciation using a rule

Unit cost depreciation can be calculated using the rule:

Unit cost depreciation

$$V_n = V_0 - nd,$$

where

V_n is the future value of the asset after n outputs,

V_0 is the initial value of the asset,

d is the amount of depreciation per output.

WORKED EXAMPLE 17

A truck is purchased for \$65 000 and depreciates at a rate of 25 cents per kilometre driven. In the first year, the truck is driven for 45 215 km. The scrap value of the truck is \$15 000.

a. Calculate the depreciation for the first year.

b. Calculate the number of kilometres on the odometer for the truck to reach its scrap value.

THINK	WRITE
a. 1. Depreciation = Number of km × rate	**a.** Depreciation $= 45\,215 \times 0.25$ $= \$11\,303.75$
2. Write the answer.	In the first year, the truck depreciates by \$11 303.75
b. 1. Determine the decrease in value from purchase to scrap value.	**b.** Total depreciation $= 65\,000 - 15\,000$ $= \$50\,000$
2. Number of km to scrap value $= \dfrac{\text{Total depreciation}}{\text{Rate}}$	Number of km to scrap value $= \dfrac{\text{Total depreciation}}{\text{Rate}}$ $= \dfrac{50\,000}{0.25}$ $= 200\,000$
3. Write the answer.	The truck will reach its scrap value after 200 000 km.

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Straight-line and unit cost depreciation Summary screen and practice questions

Exercise 4.5 Straight-line and unit cost depreciation

1. **WE13** New exercise equipment was purchased for a gym for \$25 000.

   The flat rate depreciation for the equipment is \$1750 each year. Complete the following table to calculate the future value of the equipment at the end of 5 years.

Time, *n* (years)	**Depreciation ($)**	**Future value ($)**
0	0	25 000
1		
2		
3		
4		
5		

2. A car that is purchased for $45 000 depreciates by $5000 each year.
   a. Write the recurrence relation for this situation.
   b. Using the recurrence relation, complete the depreciation schedule below.

Time, *n* (years)	**Depreciation ($)**	**Future value ($)**
0	0	45 000
1		
2		
3		
4		

3. **WE14** The following table shows the depreciating value of a tractor.

Age (years)	**Value ($)**
0	100 000
1	90 000
2	80 000
3	70 000
4	60 000
5	50 000

   a. Graph the value against age.
   b. Is this a linear model of growth or decay?
   c. Determine the gradient of the graph and interpret the gradient.
4. **WE15** An airline purchases an airplane for $60 million. The airplane depreciates by $4.5 million each year. The airplane is considered to have a scrap value of $24 million.
   a. Calculate the future value of the airplane after 5 years.
   b. Calculate the number of years before the airplane reaches its scrap value.

5. The rule $V_n = 50\,000 - 6000n$ gives the value, V_n, of a car when it is n years old.
   a. Plot a graph of V_n against n, for $n = 0, 1, 2, 3, 4, 5$.
   b. Use your graph to determine after how many years (to the nearest year) the car reaches a scrap value of $0.
6. Calculate the future value
   a. after 5 years of a computer that is purchased for $5000 and depreciates by $800 per year.
   b. after 6 years of a semi-trailer that is purchased for $750 000 and depreciates by $90 000 each year.

7. **WE16** A laundry purchased an industrial dryer for $15 000. It depreciates at a rate of 55 cents per use.
   a. Set up a recurrence relation for the depreciation of the dryer.
   b. Generate a depreciation schedule for the value of the dryer after it has been used 500, 1000, 1500, 2000 and 2500 times.
8. **WE17** A taxi is purchased for $44 000 and depreciates at a rate of 32 cents per kilometre driven.

   In the first year, the taxi is driven for 82 000 km. The scrap value of the taxi is $8000.
   a. Calculate the depreciation for that year.
   b. Calculate the number of km on the odometer for the taxi to reach its scrap value.

9. Calculate the length of time in years for each of the following items to depreciate to the value given
   a. A computer purchased for $5600 to depreciate to less than $1000 at $900 per year
   b. An electric guitar purchased for $1200 to depreciate to less than $500 at $150 per year
   c. An entertainment unit purchased for $6000 to become worthless at $750 per year
   d. Office equipment purchased for $12 000 to depreciate to less than $2500 at $1500 per year
10. A motor vehicle depreciates from $40 000 to $15 000 in 10 years. Assuming that it is depreciating in a straight line, calculate the annual amount of depreciation.
11. Calculate the annual amount of depreciation in an asset that depreciates
    a. from $20 000 to $4000 in 4 years
    b. from $175 000 to $50 000 in 10 years
    c. from $430 000 to $299 500 in 9 years.
12. A computer purchased for $3600 is written off in 4 years. Calculate the annual amount of depreciation.
13. A car that is 5 years old has an insured value of $12 500. If the car is depreciating at a rate of $2500 per year, calculate its purchase price.
14. Calculate the purchase price of each of the following assets given that:
    a. after 5 years the value is $50 000 and is depreciating at $12 000 per year
    b. after 15 years the value is $4000 and is depreciating at $1500 per year
    c. after 25 years the value is $200 and is depreciating at $50 per year.
15. An asset that depreciates at $6500 per year is written off after 12 years. Calculate the purchase price of that asset.

16. A tradesman's utility was purchased new for \$48 000 and its scrap value is \$6000.

The utility needs to be replaced after it has done 120 000 km.

a. Determine the depreciation rate (\$/km).
b. Calculate the amount of depreciation in a year if 15 555 km were travelled
c. Write the equation to determine the value of the utility after travelling n km.
d. Calculate the future value of the utility after it has travelled 70 000 km.
e. Complete the following depreciation schedule.

n (km)	Future value (V_n)
0	48 000
30 000	
60 000	
90 000	
120 000	6000

17. A printer was purchased for \$34 000 and depreciates at a rate of \$21.20 per million pages printed. In its first year, 295 million pages were printed and in its second year, 381 million pages were printed.

Calculate
a. the depreciation for the first year.
b. the future value at the end of the second year.

4.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** Which of the following sequences is generated by the following recurrence relation?

$$t_1 = -5, t_{n+1} = t_n - 6$$

A. $-5, -6, -7, -8, \ldots\ldots\ldots$
B. $-6, -1, 4, 9, \ldots\ldots\ldots$
C. $-5, -11, -17, -23, \ldots\ldots\ldots$
D. $-5, 1, 7, 13, \ldots\ldots\ldots$

2. **MC** The population of kangaroos in a national park is increasing by 250 kangaroos each year.

If the initial population of kangaroos was 2300, which of the following recurrence relation fits this situation?

A. $t_1 = 250, t_{n+1} = t_n + 2300$
B. $t_n = 250, t_{n+1} = t_0 + 2300$
C. $t_1 = 2300, t_{n+1} = t_n - 250$
D. $t_1 = 2300, t_{n+1} = t_n + 250$

3. **MC** The following table shows the first 6 terms of an arithmetic sequence.

n	t_n
1	2
2	9
3	
4	23
5	
6	37

The two missing values are

A. 16, 30.
B. 15, 29.
C. 16, 29.
D. 15, 30.

4. **MC** For the arithmetic sequence, $-1, 1, 3, 5, 7, \ldots$. the value of t_1, the value of d and the rule for the sequence are given respectively as

A. $t_1 = -1, d = 2, t_n = -3 + 2n$.
B. $t_1 = -1, d = 2, t_n = -3 - n$.
C. $t_1 = 1, d = -1, t_n = -3 + 2n$.
D. $t_1 = 2, d = -1, t_n = -3 + 2n$.

5. **MC** The 43rd term of the arithmetic sequence $-7, 2, 11, 20, 29, \ldots$ is

A. -327.
B. -243.
C. 371.
D. 380.

6. **MC** The 3rd term of an arithmetic sequence is 3.1 and the 7th term is -1.3. The value of the 31st term is

A. -153.7.
B. -27.7.
C. 28.9.
D. 38.3.

7. **MC** If Wai Keet had \$63 in his bank account and earned 9% p.a. over 3 years, the simple interest earned would be

A. \$17.01.
B. \$5.67.
C. \$22.68.
D. \$80.01.

8. **MC** Maclay invested \$160 in a bank for 6 years, earning 8% simple interest each year. At the end of 6 years, he will receive in total

A. \$76.80.
B. \$768.
C. \$928.
D. \$236.80.

9. **MC** A loan of \$1000 is taken over 5 years. The total amount repaid for this loan is \$1800. The simple interest rate per year on this loan is closest to

A. 5%.
B. 9%.
C. 16%.
D. 18%.

10. **MC** Which of the following tables shows a straight-line depreciation?

A.

Age (years)	Value ($)
New (0)	4000
1	3600
2	3240
3	2916
4	2624
5	2362

B.

Age (years)	Value ($)
New (0)	4000
1	3600
2	3200
3	2800
4	2400
5	2000

C.

Age (years)	Value ($)
New (0)	4000
1	3600
2	3300
3	3100
4	3000
5	2950

D.

Age (years)	Value ($)
New (0)	4000
1	3000
2	2500
3	1500
4	1000
5	500

11. **MC** A restaurant purchased a new industrial grade dishwasher for $8500

It depreciates according to a straight-line model at $400 each year. How many years until it reaches a scrap value of $200?

A. 18 **B.** 19 **C.** 20 **D.** 21

12. **MC** A photocopier was purchased for $5400. It depreciates at a rate of $122.20 per million pages copied. If 8 million pages were printed in the first year, the value of the copier at the end of that year would be:

A. $4422.40. **B.** $3123.56. **C.** $2154.89. **D.** $2322.44.

Complex familiar

13. Julian has $25 000 to invest at 5% p.a., 6% p.a. or 8% p.a.

a. Complete the following table to show the interest that he would earn over 5 years.

No. of years	1	2	3	4	5
Interest (5%)					
Interest (6%)					
Interest (8%)					

b. Graph the information in the table on the same set of axes.

14. Proceeds from the church fete were \$3000 in 2010. In 2011 the proceeds were \$3400 and in 2012 they were \$3800. If they continued in this pattern, what were the proceeds in 2019?

15. The following table shows the depreciating value of a yacht.

Age (years)	Value (\$)
0	200 000
1	180 000
2	160 000
3	140 000
4	120 000
5	100 000

    a. Draw a graph of the value of the yacht against its age.
    b. Write a rule for the value (V_n) of the yacht at n years of age.

16. Calculate the future value of an asset
    a. after 6 years, that was purchased for \$4000 and depreciates by \$450 each year.
    b. after 10 years, that was purchased for \$75 000 and depreciates by \$6000 each year.
    c. after 9 years, that was purchased for \$640 000 and depreciates by \$45 000 each year.

Complex unfamiliar

17. An arithmetic sequence can be modelled using the recurrence relation shown below.

$$t_1 = 1535, t_{n+1} = t_n - 354$$

    a. List the second and third term in the sequence.
    b. What is the value of d, the common difference?
    c. If the rule for calculating t_n after n years is in the form, $t_n = a + bn$, what are the values of a and b?

18. Simone invested \$114 000 in a savings account. She earns 7.25% p.a. simple interest which is paid monthly for one year.
    a. Calculate how much interest
        i. she receives monthly
        ii. will be received in total, over a year.
    b. Would Simone receive the same amount of interest over a 3 year period if it were paid annually rather than monthly?

19. Alisha has \$8900 to invest. She has a goal of earning at least \$1100 in 2 years or less. Do any of the following investments satisfy Alisha's goal?
    a. 10% p.a. for 15 months
    b. 4% p.a. earning \$1200
    c. After 100 weeks a final payout of \$10 500
    d. After 2 years at 7.2% p.a.

20. a. A movie projector is purchased by a cinema for \$30 000. The projector depreciates by \$2500 each year. Calculate the length of time it takes for the projector to be written off.
    b. A camera that was purchased new for \$1500 has a future value of \$500 four years later. Calculate the annual amount of depreciation on the camera.

studyon

Units 3 & 4 Sit exam

Answers

4 The arithmetic sequence

Exercise 4.2 Using recursion to generate an arithmetic sequence

1. a. Arithmetic; $t_1 = 23, d = 45$

b. Not arithmetic

c. Arithmetic: $t_1 = \frac{1}{2}, d = \frac{1}{4}$

2. a. $f = -62$

b. $j = 5.7, k = 15.3$

c. $p = -\frac{3}{4},\ q = 1,\ r = \frac{11}{4}$

3. D

4. a, d, e

5. a. $t_1 = 2, d = 2$

b, c not arithmetic sequences

d. $a =$ not specified, $d = 2$

e. $a = 8, d = 8$

6. a. $-3, -1, 1, 3, 5$

b. $5, -3, -11, -19, -27$

c. $-30, 30, 90, 150, 210$

7. a. $t_1 = 2, t_{n+1} = t_n + 2$

b. $t_1 = -6, t_{n+1} = t_n - 4$

c. $t_1 = 10, t_{n+1} = t_n + 10$

8. a. $t_1 = 50, t_{n+1} = t_n + 5$

b. Growth

9. a. 6

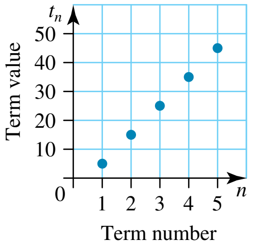
b. 3

c. 18

10. a.

Term number	1	2	3	4	5
Term value	5	15	25	35	45

b.

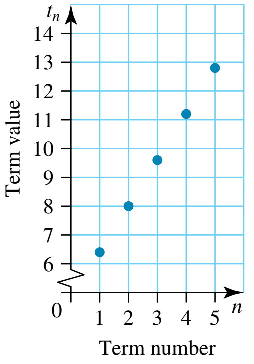


c. 85

11. a.

Term number	1	2	3	4	5
Term value	6.4	8	9.6	11.2	12.8

b.



c. 25.6

12. a. $-3.6, 1.6, 6.8, 12, 17.2$

b. $-0.4, -1.2, -2, -2.8, -3.6, -4.4$

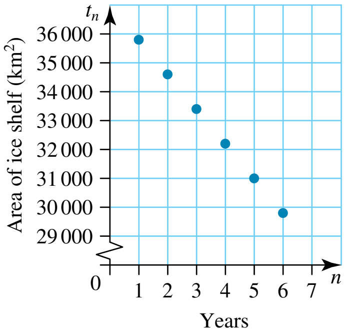
13. a. $2.13, 5.36, 11.82, 24.74, 50.58, 102.26, 205.62, 412.34, 825.78, 1652.66$

b. $-1.3, -0.9833, -0.825, -0.7458, -0.7063, -0.6865, -0.6766, -0.6716, -0.6691, -0.6679$

14. a. $t_{n+1} = t_n - 1200,\ t_1 = 35\,800$

b. $35\,800\ \text{km}^2, 34\,600\ \text{km}^2, 33\,400\ \text{km}^2, 32\,200\ \text{km}^2, 31\,000\ \text{km}^2, 29\,800\ \text{km}^2$

c.



d. Decay

15. a. $t_1 = 68\,000, t_{n+1} = t_n + 1700$

b. \$74 800

c. Growth

16. a. $t_1 = 3750, t_{n+1} = t_n - 250$

b. 2033

Exercise 4.3 Using the rule for the *n*th term of an arithmetic sequence to make predictions

1. a. $t_n = -1 + 4(n-1) = 4n - 5$

b. $t_n = 1.5 - 3.5(n-1) = 5 - 3.5n$

c. $t_n = \frac{7}{2} + 2(n-1) = 2n + \frac{3}{2}$

2. a. $5, 8, 11, 14, 17$

b. $-1, -8, -15, -22, -29$

c. $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, 3$

3. a. 122 **b.** 2900 **c.** -219

4. A

5. a. 103 **b.** 1777 **c.** -60

d. -217 **e.** -152

6. D

7. a. -162 **b.** 3467

8. a. 48 **b.** 3rd term **c.** 14th term

9. a. 7th

b. 7th

c. 8th

d. 5th

10. $t_n = 13 + 10n$

11. $t_n = 37 + 3n$

12. a. $t_n = 8.5 + 3.5n$

b. 358.5 m

13. a. $C = 20 + 0.9n$

b. \$33.50

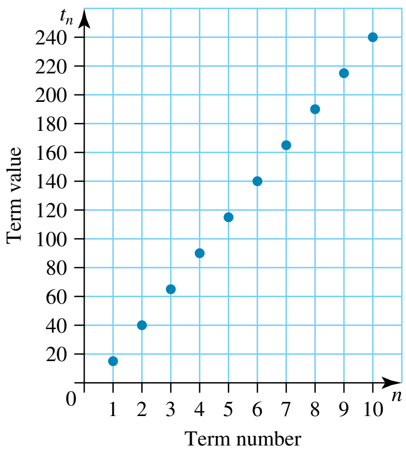
14. $x = 20$

15. a. -1.5

b. 13.5

c. -3

16.



17. a. \$72 500 **b.** 10 years

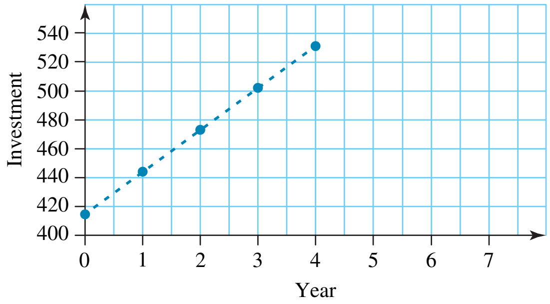
Exercise 4.4 Simple interest and other applications of arithmetic sequences

1. a. $V_0 = 415, V_{n+1} = V_n + 29.05$

b.

n	V_n	Interest	V_{n+1}
0	415	29.05	415 + 29.05 = \$444.05
1	444.05	29.05	444.05 + 29.05 = \$473.10
2	473.10	29.05	473.10 + 29.05 = \$502.15
3	502.15	29.05	502.15 + 29.05 = \$531.20

c.

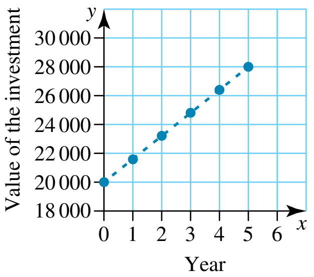


d. \$618.35

2. a.

n	V_n	Interest	V_{n+1}
0	20 000	1600	21 600
1	21 600	1600	23 200
2	23 200	1600	24 800
3	24 800	1600	26 400
4	26 400	1600	28 000

b.



c. 1600
For every increase of one year, the investment increases by \$1600.

d. \$36 000

3. a. $I = \$270, A = \$1020$
b. $I = \$7700, A = \$27\,700$

4. \$465.50

5. \$240

6. \$2400

7. \$564

8. \$4400

9. \$2133.33

10. 8%

11. 28 months

12. \$440

13. 24 months

14. 3 years

15. a. Yes (\$1112.50)
b. No
c. Yes (\$1600 in 23 months)
d. Yes (\$1281.60)

16. a. The Big-4 Bank offers the better rate
b. The Big-4 Bank charges 11% p.a. for a loan while the Friendly Building Society charges 12% (= 12 × 1% per month)

17. a. \$627.13 **b.** \$12 542.50

18. a. i. \$1540.63
ii. \$6162.50
b. Yes

19. a. \$2247 **b.** \$13 482 **c.** $7\frac{1}{2}$ years

20. a. Simple interest follows an arithmetic sequence with the same amount being added to each successive term. A linear graph follows the same pattern.
b. $\frac{4.8}{100} \times 6000 = 288$. The gradient is \$288.
c. As the interest of \$288 is added to the initial investment each year, this is a linear model of growth.

21. a. i. $V_0 = 5500, V_{n+1} = V_n + 165$
ii. $V_0 = 5500, V_{n+1} = V_n + 192.50$
iii. $V_0 = 5500, V_{n+1} = V_n + 206.25$

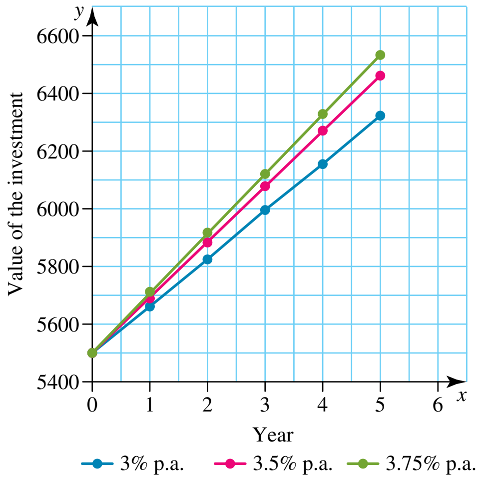
b. * See the table at the bottom of the page.

---

*__21. b.__

Year	0	1	2	3	4	5
3%	5500	5665	5830	5995	6160	6325
3.5%	5500	5692.50	5885	6077.50	6270	6462.50
3.75%	5500	5706.25	5912.50	6118.75	6325	6531.25

c.



d. The gradients of the graphs are: \$165, \$192.50 and \$206.25, so as the gradients increase, the graphs get steeper.

Exercise 4.5 Straight-line and unit cost depreciation

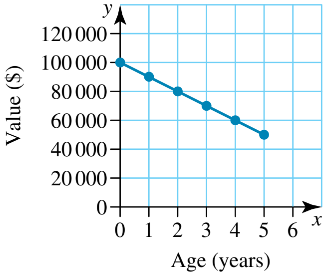
1.

Time, *n* (years)	Depreciation (\$)	Future value (\$)
0	0	25 000
1	1750	23 250
2	1750	21 500
3	1750	19 750
4	1750	18 000
5	1750	16 250

2. a. $V_0 = 45\,000, V_{n+1} = V_n - 5000$

b. * See the table at the bottom of the page.

3. a.



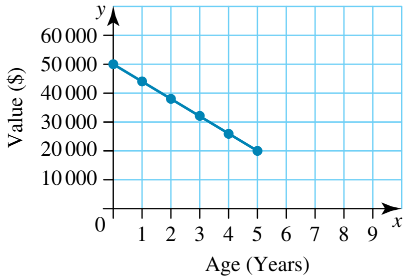
b. This is a linear model of decay.

c. The gradient of the graph is −\$10 000. For every increase of one year in age, the tractor decreases by \$10 000 in value.

4. a. \$37.5 million

b. 8 years

5. a.



b. 8 years

6. a. \$1000

b. \$210 000

7. a. $V_0 = 15\,000, V_{n+1} = V_n - 0.55$

b.

Number of uses	Value
0	15 000
500	14 725
1000	14 450
1500	14 175
2000	13 900
2500	13 625

8. a. \$26 240

b. 112 500 km

9. a. 6 years

b. 5 years

c. 8 years

d. 7 years

10. \$2500/year

11. a. \$4000/year

b. \$12 500/year

c. \$14 500/year

12. \$900/year

13. \$25 000

14. a. \$110 000

b. \$26 500

c. \$1450

15. \$78 000

16. a. \$0.35/km

b. \$5444.25

***2. b.**

Time, *n* (years)	Depreciation (\$)	Future value (\$)
0	0	45 000
1	5000	40 000
2	5000	35 000
3	5000	30 000
4	5000	25 000

c. $V = 48000 - 0.35n$

d. $23 500

e.

n (km)	Future value (V_n)
0	48 000
30 000	37 500
60 000	27 000
90 000	16 500
120 000	6000

17. a. $6254

b. $19 668.80

Exercise 4.6 Review: exam practice

1. C

2. D

3. A

4. A

5. C

6. B

7. A

8. D

9. C

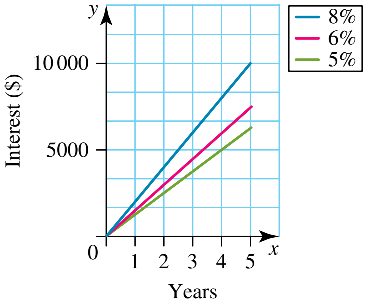
10. B

11. D

12. A

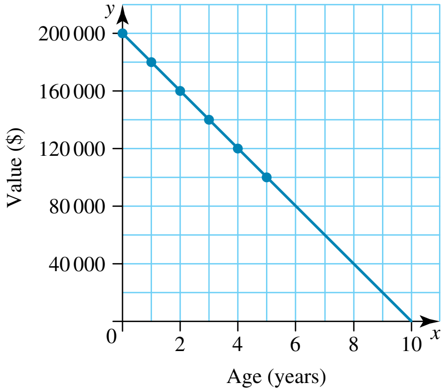
13. a. * See the table at the bottom of the page.

b.



14. $6600

15. a.



b. $V_n = 2 00 000 - 20 000n$

16. a. $1300

b. $15 000

c. $235 000

17. a. 1181, 827

b. −354

c. a = 1889, b = −354

18. a. i. $688.75

ii. $8265

b. Yes

19. a. Yes ($1112.50)

b. No

c. Yes ($1600 in 23 months)

d. Yes ($1281.60)

20. a. 12 years

b. $250

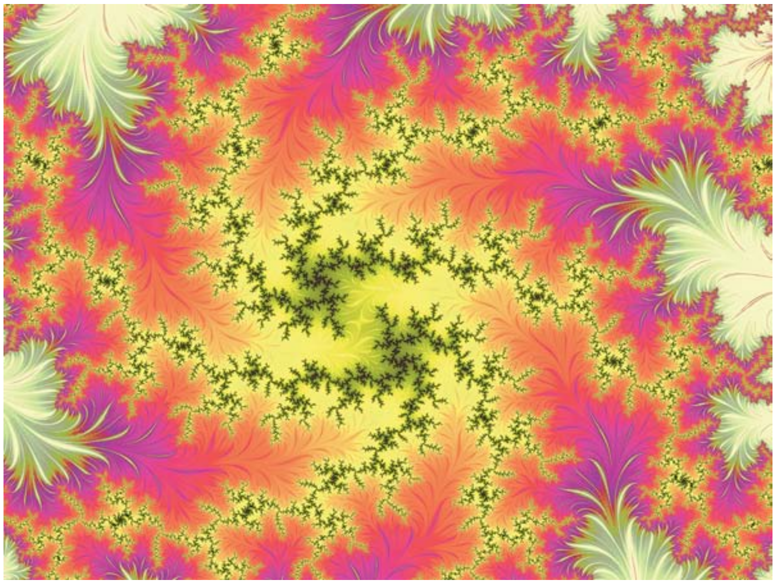
---

*13.

No. of years	1	2	3	4	5
Interest (5%)	$1250	$2500	$3750	$5000	$6250
Interest (6%)	$1500	$3000	$4500	$6000	$7500
Interest (8%)	$2000	$4000	$6000	$8000	$10 000

5 The geometric sequence

5.1 Overview



Nature exhibits many patterns that are not composed of regular shapes such as rectangles or circles, but are made up of intricate patterns that are repeated over and over. A fern leaf, for example, is made up of many smaller leaves that all have the same shape as the bigger leaf. The smaller leaves, in turn, are made up of even smaller identical leaves. Snowflakes are composed similarly, with each snow flake made up of smaller snowflakes. A head of cauliflower has large clusters of smaller identical clusters.

Discovered in 1980 by Benoit Mandelbrot, the Mandelbrot Set is one of the most intricate and beautiful geometrical patterns in mathematics. The Mandelbrot Set is an image that captures many of the qualities people find fascinating about mathematics. Although it is generated by repeating a simple formula, its patterns are infinitely complex. If you select any portion of the Mandelbrot Set and magnify it, you will see that no detail is lost — the magnified shape is as intricate and even contains parts that look like copies of the original. This notion of 'worlds within worlds' appeals to the philosopher in all of us.

Benoit Mandelbrot joined the IBM Thomas J. Watson Research Centre in 1958 and worked there for 32 years. He was one of the first people to use the power of computer graphics to generate these patterns and create extraordinary fractal geometric images by using rules recursively.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

5.2 Using recursion to generate a geometric sequence

5.2.1 A geometric sequence

A scientist observed that the cells in a bacterial infection doubled every hour. There were 150 infected cells when the scientist started her observations. She recorded the number of infected cells for 6 hours and collected the following data:

$$150, 300, 600, 1200, 2400, 4800, 9600, \ldots$$

This is an example of a geometric sequence as each term is found by multiplying the previous term by the same number. In this case each term is multiplied by 2 to obtain the next term.

A **geometric sequence** is a sequence in which the ratio between any successive two terms in the sequence is the same. The next term can be found by multiplying by a fixed value.

Consider the sequence $1, 3, 9, 27, 81, \ldots$ This is a geometric sequence because each term is obtained by multiplying the previous term by 3.

5.2.2 The common ratio

In a geometric sequence, the first term is t_1, and the nth term is t_n, where n is the term number. The successive terms are referred to as $t_1, t_2, t_3, t_4, \ldots t_n$.

The ratio between two consecutive terms in a geometric sequence is known as the common ratio. To calculate the common ratio, r, of a geometric sequence you need to calculate the ratio of successive terms, namely, $\frac{t_2}{t_1}$. You could alternatively calculate $\frac{t_3}{t_2}$ or $\frac{t_4}{t_3}$ and so on.

WORKED EXAMPLE 1

Which of the following are geometric sequences?

a. **2, 10, 50, 250, 1250, …**

b. **−2, − 6, 18, 54, − 162, …**

THINK	WRITE
a. **1.** Write the sequence.	**a.** $2, 10, 50, 250, 1250, \ldots$
2. Calculate the ratio of $\frac{t_2}{t_1}$.	$\frac{t_2}{t_1} = \frac{10}{2}$ $= 5$
3. Calculate the ratio of $\frac{t_3}{t_2}$.	$\frac{t_3}{t_2} = \frac{50}{10}$ $= 5$
4. Calculate the ratio of $\frac{t_4}{t_3}$.	$\frac{t_4}{t_3} = \frac{250}{50}$ $= 5$
5. Calculate the ratio of $\frac{t_5}{t_4}$.	$\frac{t_5}{t_4} = \frac{1250}{50}$ $= 5$
6. Check that all ratios are the same.	There is a common ratio of 5. This is a geometric sequence.

b. 1. Write the sequence.	**b.** $-2, -6, 18, 54, -162, ...$
2. Calculate the ratio of $\frac{t_2}{t_1}$.	$\frac{t_2}{t_1} = \frac{-6}{-2}$ $= +3$
3. Calculate the ratio of $\frac{t_3}{t_2}$.	$\frac{t_3}{t_2} = \frac{18}{-6}$ $= -3$
4. There is no need to check any further as the two ratios are not the same.	There is no common ratio. This is not a geometric sequence.

5.2.3 Using a recurrence relation to generate a geometric sequence

If we know the first term (t_1) and the common ratio (r) of a geometric sequence, then all the terms of the sequence can be generated.

Consider a geometric sequence with first term, 3 and common ratio, 4.

The sequence will be $3, 12, 48, 192, 768, ...$

This is known as the **recursive method** of generating a geometric sequence.

Term number	Term
1	t_1
2	$t_1 \times r$
3	$t_2 \times r$
4	$t_3 \times r$
n	$t_{n-1} \times r$
$n + 1$	$t_n \times r$

Each successive term can be found by multiplying the previous term by r.

Recursive definition of a geometric sequence

$t_1 = a, t_{n+1} = t_n \times r$, where t_1 is the first term and r is the common ratio.

WORKED EXAMPLE 2

Generate the first five terms for the geometric sequences defined by the following recurrence relations.

a. $t_1 = 10,\ t_{n+1} = t_n \times 3$

b. $t_1 = -4,\ t_{n+1} = t_n \times 2$

c. $t_1 = 6,\ t_{n+1} = t_n \times 1.01$

THINK	**WRITE**
a. The first term is 10 and the common ratio is 3.	**a.** The first 5 terms are: 10, 30, 90, 270, 810,
b. The first term is −4 and the common ratio is 2.	**b.** The first 5 terms are: −4, −8, −16, −32, −64
c. The first term is 6 and the common ratio is 1.01.	**c.** 6, 6.06, 6.1206, 6.18181, 6.24362

5.2.4 Tabular and graphical displays of geometric sequences

When we draw a graph of a geometric sequence, it helps to first draw a table of values for the sequence. The top row of the table displays the term number of the sequence, n and the bottom of the table displays the term value, t_n.

Term number (n)	1	2	3	...	n
Term value (t_n)					

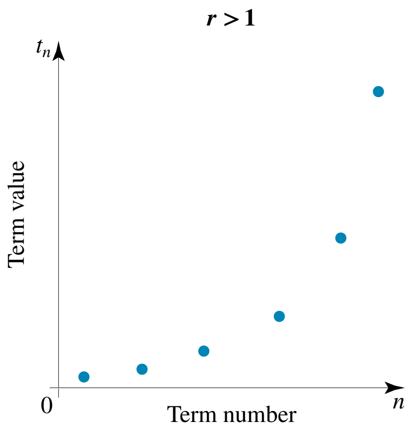
The data from the table of values can then be used to identify the points to plot the graph of the sequence.

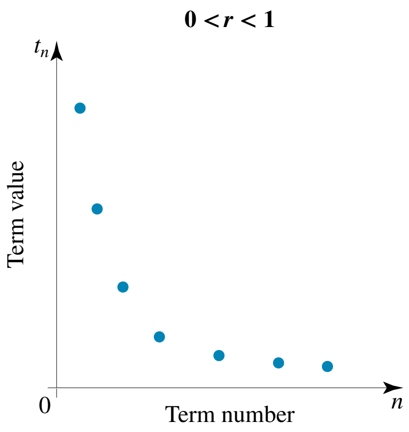
Graphs of geometrical sequences

When we draw a graph of a geometrical sequence, the term number, n, is the independent variable, so it appears on the x-axis of the graph. The term value, t_n, is the dependent value, so it appears on the y-axis of the graph.

The shape of the graph of a geometrical sequence depends on the value of r.

- When $r > 1$, the values of the terms increase at an exponential rate.
- When $0 < r < 1$, the values of the terms decrease at an exponential rate and converge towards 0.

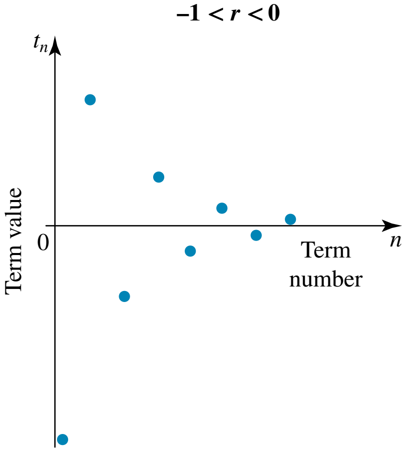




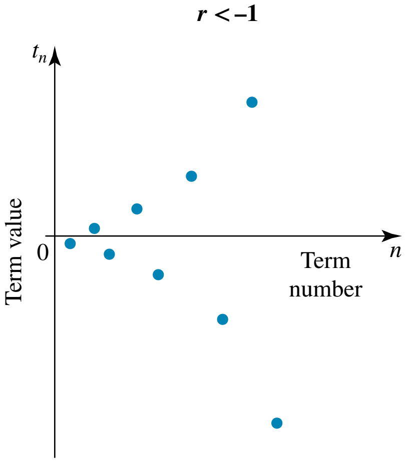
In the first graph, the term values are increasing so the relationship can be described as **a model of exponential growth**.

In the second graph, the term values are decreasing so the relationship can be described as a **model of exponential decay.**

- When $-1 < r < 0$, the values of the terms oscillate on either side of 0 but converge towards 0.



- When $r < -1$, the values of the terms oscillate on either side of 0 and move away from the starting value at an exponential rate.



WORKED EXAMPLE 3

A geometric sequence is described by the recurrence relation

$$t_1 = 5, t_{n+1} = t_n \times 2$$

a. **Create a table of values showing the term number and term value for the first 5 terms of the sequence.**

b. **Plot the graph of the sequence.**

THINK

a. **1.** Set up a table with the term number in the top row and the term value in the bottom row.

WRITE

a.

Term number	1	2	3	4	5
Term value					

2. Use the recursion relation to determine the first 5 values of the sequence

$t_1 = 5$
$t_2 = 5 \times 2 = 10$
$t_3 = 10 \times 2 = 20$
$t_4 = 20 \times 2 = 40$
$t_5 = 40 \times 2 = 80$

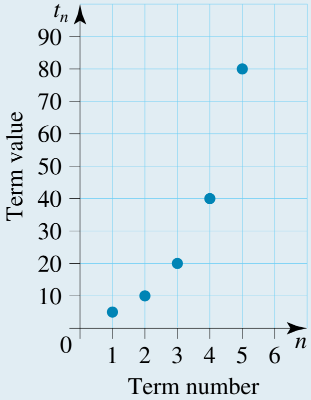
3. Complete the table with the calculated values.

Term number	1	2	3	4	5
Term value	5	10	20	40	80

b. 1. Use the table of values to identify the points to be plotted.

b. The points to be plotted are $(1, 5), (2, 10), (3, 20), (4, 40)$ and $(5, 80)$.

2. Plot the points on the graph.



5.2.5 Using a scientific calculator to generate terms of a geometric sequence

A scientific calculator can be used to generate the terms of a geometric sequence.

WORKED EXAMPLE 4

A geometric sequence is described by the recurrence relation $t_1 = -8$, $t_{n+1} = t_n \times 0.80$. Use a scientific calculator to generate the first 6 terms of this sequence.

THINK

a. 1. Type the first term of the sequence into the calculator screen and then press Enter.

2. Type × 0.80, then press Enter 5 times.

WRITE

a. -8

```
−8
−6.4
−5.12
−4.096
−3.2768
−2.62144
```

3. Write the answer.	The first 6 terms are $-8, -6.4, -5.12, -4.096, -3.2768, -2.62144$

5.2.6 Using geometric recurrence relations to model practical situations

As we have already illustrated, geometric sequences are sequences that involve geometric growth or decay. Examples include compound interest loans or investments, the reducing height of a bouncing ball and the number of bacteria in a culture after n periods of time.

If percentages are involved in generating a sequence of numbers, this can result in a geometric sequence.

When there is a percentage increase of x per cent between terms, the value of the common ratio, r, will be $\left(1 + \frac{x}{100}\right)$.

Similarly, when there is a percentage decrease of x per cent between terms, the value of the common ratio, r, will be $\left(1 - \frac{x}{100}\right)$.

WORKED EXAMPLE 5

According to the International Federation of Tennis, a tennis ball must meet certain bounce regulations. The test involves dropping a ball vertically from a height of 254 cm and then measuring the rebound height. To meet the regulations, the ball must rebound 135 to 147 cm high, just over half the original distance.

Janine decided to test the ball bounce theory. She dropped a ball from a height of 200 cm. She found that it bounced back up to 108 cm, with the second rebound reaching 58.32 cm and the third rebound reaching 31.49 cm.

- **a.** **Set up a recurrence relation to model the bounce height of the ball.**
- **b.** **Use your relation from part a to estimate the height of the 4th and 5th rebounds, giving your answers correct to 2 decimal places.**
- **c.** **Sketch the graph of the number of bounces against the height of each bounce.**

THINK	WRITE/DRAW
a. 1. List the known information.	**a.** 1st bounce: 108 cm 2nd bounce: 58.32 cm 3rd bounce: 31.49 cm
2. Check if there is a common ratio between consecutive terms. If so, this situation can be modelled using a geometric sequence.	$\frac{t_2}{t_1} = \frac{58.32}{108}$ $\quad \frac{t_3}{t_2} = \frac{31.49}{58.32}$ $= 0.54 \quad\quad = 0.539\ldots$ ≈ 0.54 There is a common ratio between consecutive terms of 0.54.

3. Set up the equation to represent the geometric sequence.

$a = 108,\ r = 0.54$

$t_{n+1} = r\,t_n,\ t_1 = a$

$t_{n+1} = 0.54\ t_n,\ t_1 = 108$

b. 1. Use the formula from part **a** to determine the height of the 4th rebound ($n = 4$).

b.

$$\begin{aligned} t_3 &= 31.49 \\ t_{n+1} &= 0.54t_n \\ t_4 &= 0.54t_3 \\ &= 0.54 \times 31.49 \\ &= 17.0046 \\ &= 17.00 \text{ (correct to 2 decimal places)} \end{aligned}$$

2. Use the formula from part **a** to determine the height of the 5th rebound ($n = 5$).

$$\begin{aligned} t_{n+1} &= 0.54t_n \\ t_5 &= 0.54t_4 \\ &= 0.54 \times 17.00 \\ &= 9.18 \end{aligned}$$

3. Write the answer.

The estimated height of the 4th rebound is 17.00 cm, and the estimated height of the 5th rebound is 9.18 cm.

c. 1. Draw up a table showing the bounce number against the rebound height.

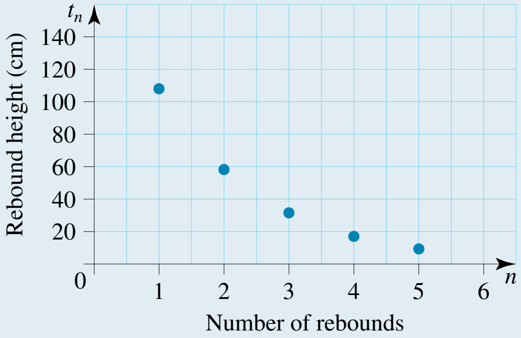
c.

Bounce number	1	2	3	4	5
Rebound height (cm)	108	58.32	31.49	17.00	9.18

2. Identify the points to be plotted on the graph.

The points to be plotted are (1, 108), (2, 58.32), (3, 31.49), (4, 17.00) and (5, 9.18).

3. Plot the points on the graph.



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Interactivity Geometric sequences (int-6259)

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Units 3 & 4 > Area 3 > Sequence 2 > Concepts 1 & 2

Geometric sequences Summary screen and practice questions

Generating a geometric sequence using recursion Summary screen and practice questions

Exercise 5.2 Using recursion to generate a geometric sequence

1. **WE1** Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of t_1 and r.
   - a. $3, 6, 12, 24, 48, ...$
   - b. $\frac{1}{2}, \frac{5}{4}, \frac{25}{8}, \frac{125}{16}, ...$
   - c. $9, 6, 3, 0, -3, ...$
   - d. $\frac{1}{2}, \frac{1}{5}, \frac{2}{25}, \frac{4}{125}, ...$
2. Determine the missing values in the following geometric sequences.
   - a. $1, 6, c, 216, 1296$
   - b. $3, g, h, -24, 48$
   - c. $p, q, s, 300, 1500$
3. **WE2** Generate the first 5 terms for the following geometric sequences.
   - a. $t_1 = 2,\ t_{n+1} = t_n \times 5$
   - b. $t_1 = -3,\ t_{n+1} = t_n \times 4$
   - c. $t_1 = 1000,\ t_{n+1} = t_n \times 1.12$
4. **WE3** A geometric sequence is described by the recurrence relation

$$t_1 = 10,\ t_{n+1} = t_n \times 5$$

   - a. Create a table of values showing the term number and term value for the first 5 terms of the sequence.
   - b. Plot the graph of the sequence from the table of values in part **a**.
5. A geometric sequence is described by the recurrence relation

$$t_1 = 64,\ t_{n+1} = t_n \times \frac{1}{2}.$$

   - a. Create a table of values showing the term number and term value for the first 5 terms of the sequence.
   - b. Plot the graph of the sequence from the table of values in part **a**.
6. A geometric sequence is described by the recurrence relation

$$t_1 = 1.5,\ t_{n+1} = t_n \times 3.$$

   - a. Create a table of values showing the term number and term value for the first 5 terms of the sequence.
   - b. Plot the graph of the sequence from the table of values in part **a**.
7. **WE4** A geometric sequence is described by the recurrence relation $t_1 = 10\,000,\ t_{n+1} = t_n \times 1.02$. Use a scientific calculator to generate the first 6 terms of this sequence.
8. Set up a recurrence relation to represent the geometric sequence $2.5, -7.5, 22.5, -67.5, 202.5, ...$
9. A geometric sequence is represented by the recurrence relation $t_{n+1} = -3.5t_n,\ t_1 = -4$. Determine the first five terms of the sequence.
10. **WE5** Eric decided to test the rebound height of a tennis ball. He dropped a ball from a height of 300 cm and found that it bounced back up to 165 cm, with the second rebound reaching 90.75 cm, and the third rebound reaching 49.91 cm.
    - a. Set up a recurrence relation to model the bounce height of the ball.
    - b. Use your relation from part **a** to estimate the height of the 4th and 5th rebounds, giving your answers correct to 2 decimal places.
    - c. Sketch the graph of the number of bounces against the height of the bounce.
11. Rosanna decided to test the ball rebound height of a basketball. She dropped the basketball from a height of 500 cm and noted that each successive rebound was two-fifths of the previous height.
    - a. Set up a recurrence relation to model the bounce height of the ball.
    - b. Use your relation to estimate the heights of the first 5 rebounds, correct to 2 decimal places.
    - c. Sketch the graph of the first 5 bounces against the rebound height.
12. A bouncing ball rebounds to 70% of its previous height.
    - a. From how high would the ball have to be dropped for the 10th bounce to reach 50 cm in height? Give your answer correct to 1 decimal place.
    - b. Define a recurrence relation to determine the height of the ball after n bounces.

5.3 Using the rule for the *n*th term of a geometric sequence

5.3.1 Equations representing geometric sequences

Consider the finite geometric sequence of five terms for which $t_1 = 3$ and $r = 4$.

$$3 \xrightarrow{\times 4} 12 \xrightarrow{\times 4} 48 \xrightarrow{\times 4} 192 \xrightarrow{\times 4} 768$$

$$\begin{aligned} t_1 &= 3 \\ t_2 &= 3 \times 4 \\ \text{Now, } t_3 &= 3 \times 4 \times 4 \\ t_4 &= 3 \times 4 \times 4 \times 4 \\ t_5 &= 3 \times 4 \times 4 \times 4 \times 4 \end{aligned}$$

$$\begin{aligned} t_1 &= t_1 \\ t_2 &= t_1 \times r \\ t_3 &= t_1 \times r \times r \\ t_4 &= t_1 \times r \times r \times r \\ t_5 &= t_1 \times r \times r \times r \times r \\ t_2 &= t_1 \times r^1 \\ t_3 &= t_1 \times r^2 \\ t_4 &= t_1 \times r^3 \\ t_5 &= t_1 \times r^4 \quad \text{and so on ...} \end{aligned}$$

We notice a pattern emerging. That pattern can be described by the equation:

$$t_n = 3 \times 4^{n-1}.$$

For example, if $n = 5$, $t_5 = 3 \times 4^4$.

The general rule for a geometric sequence

$$t_n = t_1 r^{n-1}$$

where t_n is the nth term
t_1 is the first term
r is the common ratio.

WORKED EXAMPLE 6

Determine the equations that represent the following geometric sequences.

a. **7, 28, 112, 448, 1792, …**

b. $\mathbf{8, -4, 2, -1, \frac{1}{2}, \ldots}$

THINK	**WRITE**
a. 1. Determine the values of t_1 and r.	**a.** $t_1 = 7$ $r = \frac{t_2}{t_1}$ $= \frac{28}{7}$ $= 4$
2. Substitute the values for t_1 and r into the formula for geometric sequences.	$t_n = t_1 r^{n-1}$ $= 7 \times 4^{n-1}$
b. 1. Determine the values of t_1 and r.	**b.** $t_1 = 8$ $r = \frac{t_2}{t_1}$ $= \frac{-4}{8}$ $= -\frac{1}{2}$
2. Substitute the values for t_1 and r into the formula for geometric sequences.	$t_n = t_1 r^{n-1}$ $= 8 \times \left(-\frac{1}{2}\right)^{n-1}$

5.3.2 Calculating the terms of a geometric sequence

After an equation has been set up to represent a geometric sequence, we can use this equation to determine any term in the sequence. Simply substitute the value of n into the equation to determine the value of that term.

WORKED EXAMPLE 7

Determine the 12th term of the geometric sequence

$$2,\ 10,\ 50,\ 250,\ 1250,\ \ldots$$

THINK	**WRITE**
1. Determine the value of t_1.	$t_1 = 2$
2. It has been stated that it is a geometric sequence, so determine the value of r.	$r = \frac{10}{2}$ $= 5$
3. Use the rule $t_n = t_1 \times r^{n-1}$ to determine the 12th term.	$t_n = t_1 \times r^{n-1}$ $t_n = 2 \times 5^{n-1}$ $t_{12} = 2 \times 5^{12-1}$ $= 97\,656\,250$
4. Write your answer.	The value of the 12th term is 97 656 250.

WORKED EXAMPLE 8

The 2nd term of a geometric sequence is 8 and the 5th is 512. Determine the 10th term of this sequence.

THINK	WRITE
1. We know that $t_2 = 8$ and that $t_n = t_1 \times r^{n-1}$.	$t_n = t_1 \times r^{n-1}$ $t_2 = t_1 \times r^1$ $= 8$
2. We know that $t_5 = 512$ and that $t_n = t_1 \times r^{n-1}$.	$t_5 = t_1 \times r^4$ $= 512$
3. Solve the 2 equations simultaneously by eliminating t_1, to determine r. Divide equation 2 by equation 1.	$t_1 \times r^1 = 8 \quad [1]$ $t_1 \times r^4 = 512 \quad [2]$ $\dfrac{t_1 \times r^4}{t_1 \times r} = \dfrac{512}{8} \quad [2] \div [1]$ $r^3 = 64$ $r = 4$
4. To determine t_1, substitute the value of r.	Substituting $r = 4$ into equation [1]: $t_1 \times 4 = 8$ $t_1 = 2$
5. Write down the rule.	$t_n = 2 \times 4^{n-1}$
6. To determine the 10th term, let $n = 10$.	$t_{10} = 2 \times 4^9$ $= 524\,288$
7. Write your answer.	The 10th term in the sequence is 524 288.

Note: In Worked example 8 to solve the equation $r^3 = 64$ for r, use the $\sqrt[3]{x}$ button on your scientific calculator.

To solve an equation $r^n = x$, for r, use the $\sqrt[y]{x}$ on your scientific calculator.

WORKED EXAMPLE 9

The first three terms of a geometric sequence are 2, 6 and 18.

Which numbered term would be the first to exceed 1 000 000 in this sequence?

THINK	WRITE
1. Determine the rule for the sequence.	$a = 2$ and $r = 3$ $t_n = 2 \times 3^{n-1}$
2. Set up the equation to be solved.	$2 \times 3^{n-1} = 1\,000\,000$ $3^{n-1} = 5\,000\,00$

3. Try various values of n. With $n = 8$, value is less than 500 000. With $n = 15$, value is greater than 500 000. With $n = 12$, value is too small. With $n = 14$, value is too large. With $n = 13$, value just exceeds 500 000.

Let $n = 8$, $3^7 = 2187$ (too small)
Let $n = 15$, $3^{14} = 4\,782\,969$ (too large)
Let $n = 12$, $3^{11} = 177\,147$ (too small)
Let $n = 14$, $3^{13} = 1\,594\,323$ (too large)
Let $n = 13$, $3^{12} = 531\,441$
The 13th term is the required term.

on Resources

Interactivity Terms of a geometric sequence (int-6260)

Exercise 5.3 Using the rule for the *n*th term of a geometric sequence

1. **WE7** Determine the equations that represent the following geometric sequences.
- **a.** $-1, -5, -25, -125, -625, \ldots$
- **b.** $7, -3.5, 1.75, -0.875, 0.4375$
- **c.** $\frac{5}{6}, \frac{5}{9}, \frac{10}{27}, \frac{20}{81}, \frac{40}{243}, \ldots$

2. **WE7** Determine the value of the term specified for the given geometric sequences.
- **a.** Determine the 10th term of the geometric sequence $2, 12, 72, 432, 2592, \ldots$
- **b.** Determine the 18th term of the geometric sequence $8, 16, 32, 64, 128, \ldots$
- **c.** Determine the 11th term of the geometric sequence $5, 15, 45, 135, 405, \ldots$
- **d.** Determine the 10th term of the geometric sequence $2.3, 2.76, 3.312, 3.9744, \ldots$
- **e.** Determine the 9th term of the geometric sequence $-2, -8, -32, -128, -512, \ldots$

3. **WE8** Determine the value of the term specified in each of the following geometric sequences.
- **a.** The 2nd term of a geometric sequence is 6 and the 5th term is 162. Determine the 10th term.
- **b.** The 2nd term of a geometric sequence is 6 and the 5th term is 48. Determine the 12th term.
- **c.** The 2nd term of a geometric sequence is 2 and the 5th term is 16. Determine the 16th term.
- **d.** The 4th term of a geometric sequence is -32 and the 7th term is -256. Determine the 14th term.
- **e.** The 4th term of a geometric sequence is -192 and the 7th term is $-12\,288$. Determine the 12th term.
- **f.** The 3rd term of a geometric sequence is 36 and the 6th term is -972. Determine the 12th term.

4. **WE9** Evaluate the following.
- **a.** The first three terms of a geometric sequence are 5, 12.5 and 31.25. Which term would be the first to exceed 50 000?
- **b.** The first three terms of a geometric sequence are 3.2, 9.6 and 28.8. Which term would be the first to exceed 1 000 000?
- **c.** The first three terms of a geometric sequence are 5.1, 20.4 and 81.6. Which term would be the first to exceed 100 000?
- **d.** The first three terms of a geometric sequence are 4.3, 9.46 and 20.812. Which term would be the first to exceed 500 000?

5. a. Determine the 15th term of the geometric sequence with $t_1 = 4$ and $r = 3$.
   b. A geometric sequence has a first term of 2 and a 12th term of 97 656 250. Determine the common ratio between consecutive terms of the sequence.
   c. Determine the first term of a geometric series with a common ratio of $-\frac{1}{2}$ and a 6th term of 13.125.
6. a. Determine the 11th term of the geometric sequence with a first value of 1.2 and a common ratio of 4.
   b. A geometric sequence has a first term of -1.5 and a 10th term of 768. Determine the common ratio between consecutive terms of the sequence.
   c. Determine the first term of a geometric series with a common ratio of 0.4 and a 6th term of 6.5536.
7. On the first day Jenny hears a rumour. On the second day, she tells two friends. On the third day, each of these two friends tell two of their own friends, and so on.
   a. Write the geometric sequence for the first five days of the above real-life situation.
   b. Determine the value of r.
   c. How many people are told of the rumour on the 12th day?

8. Decay of radioactive material is modelled as a geometric sequence where $r = \frac{1}{2}$. If there are 20 million radioactive atoms, write the first 7 terms of the sequence.
9. The number of cells of a micro-organism, after each process of cell division, can be summarised as follows.

$$1, 2, 4, 8, 16$$

   If the number of cells after each division continues to follow a geometric sequence, determine
   a. a rule for the number of cells after n divisions
   b. the number of cells after 12 divisions.
10. The takings at a new cinema each month are recorded.

   If the takings each month continue to follow a geometric sequence, determine
   a. a rule for the takings in month n
   b. the takings in month 9.

Month number	Takings
1	\$10 000
2	\$8500
3	\$7225

11. A financial company advertises that the value of money invested in the company doubles every 6 years. If Simon invests \$100, how much will he have in 24 years?
12. The first term of a geometric sequence is 1.2, while the common ratio is 1.1. What would be the first term to exceed 20?

13. A small town is renowned for spreading rumours. All of its citizens are aware in a short time of any new rumours. The spread of the rumour can be summarised in the following table.

Day	Number of citizens in the know
1	1
2	6
3	36

If the number of citizens who have been told the rumour each day continues to follow a geometric sequence, determine

a. a rule for the number of citizens in day *n*

b. the number of citizens told of the rumour by day 5

c. on which day all 4230 citizens will know of the rumour.

14. A ranger records the distance from a sand dune to the water's edge at low tide over a number of years.

Year number	Distance from water's edge (m)
1	60
2	64.8
3	69.984

If the distance from the water's edge each year continues to follow a geometric sequence, determine

a. a rule for the distance from the water's edge in year *n*

b. the distance from the water's edge in year 6 (correct to 2 decimal places)

c. in which year the distance from the water's edge will exceed 100 m.

5.4 Compound interest and other applications of geometric sequences

5.4.1 Growth and decay

Growth and decay of discrete variables is found in many real-life situations. Some examples are increasing or decreasing populations and increase or decrease in financial investments.

WORKED EXAMPLE 10

A city produced 100 tonnes of rubbish in the year 2018. Forecasts suggest that this may increase by 2% each year. If these forecasts are true,

a. what will be the city's rubbish output in 2022?

b. in which year will the amount of rubbish reach 120 tonnes?

THINK	WRITE/DISPLAY
a. 1. This is an example of a geometric sequence. Determine the first term, t_1.	**a.** $t_1 = 100$
2. Determine the common ratio, r. The amount of rubbish increases by 2%, that is, the original amount plus an extra 2%. Note that $r \neq 0.02$.	Increase by 2% 100% + 2% = 102% $r = 1.02$
3. Determine which term is represented by the amount of rubbish for the year 2022.	Year 2018 is the first term, so $n = 1$. Year 2019 is the second term, so $n = 2$. Year 2022 is the fifth term, so $n = 5$.
4. Use $t_n = t_1 \times r^{n-1}$ to Determine the amount of rubbish collected in the fifth year.	$t_5 = 100 \times 1.02^{5-1}$ $= 100 \times 1.0824$ $= 108.24$
5. Write your response.	The amount of rubbish produced in the fifth year, or 2022, will be 108.24 tonnes.
b. 1. Use $t_n = t_1 \times r^{n-1}$ and $t_n = 120$.	**b.** $100\,(1.02)^{n-1} = 120$ $(1.02)^{n-1} = 1.2$
2. Try various values of n.	Let $n = 10,\ (1.02)^9 = 1.195$ Let $n = 11,\ (1.02)^{10} = 1.22$
3. Write your answer.	During the 11th year, that is, during 2028, the rubbish will have exceeded 120 tonnes.

WORKED EXAMPLE 11

A computer system decreases in value each year by 15% of the previous year's value. Determine an expression for the value of the computer, which shall be referred to as V_n, after n years. Its initial purchase price is given as $V_1 = \$12\,000$.

THINK	WRITE
1. This is a geometric sequence since there is a 15% decrease on the previous year's value. Determine t_1 and r. *Note*: Since this is a decreasing value, r is a value less than 1.	$t_1 = 12\,000$ $r = 100\% - 15\%$ $= 85\%$ $= 0.85$
2. We want an expression for the value after n years. Use $t_n = t_1 \times r^{n-1}$ which gives the value of the nth term. Use V_n instead of t_n.	$t_n = t_1 \times r^{n-1}$ $V_n = 12\,000 \times (0.85)^{n-1}$
3. Write your answer.	The value of the computer is given by the expression $V_n = 12\,000\ (0.85)^{n-1}$.

5.4.2 Calculating compound interest

Consider the case where a bank pays compound interest of 5% per annum on an amount of \$20 000. The amount is invested for 4 years and interest is calculated yearly.

Compound interest receives its name because the interest which is earned is paid back into the account so that the next time the interest is calculated, it is calculated on an increased amount. There is a compounding effect on the money in the account.

If we calculated the amount in the account mentioned on the previous page each year, we would have the following amounts.

Start	\$20 000
At the end of year 1	$\$20\,000 \times 1.05 = \$20\,000 \times 1.05^1 = \$21\,000$
At the end of year 2	$\$20\,000 \times 1.05 \times 1.05 = \$20\,000 \times 1.05^2 = \$22\,050$
At the end of year 3	$\$20\,000 \times 1.05 \times 1.05 \times 1.05 = \$20\,000 \times 1.05^3 = \$23\,152.50$
At the end of year 4	$\$20\,000 \times 1.05 \times 1.05 \times 1.05 \times 1.05 = \$20\,000 \times 1.05^4 = \$24\,310.13$

The amounts $20\,000, 21\,000, 22\,050, 23\,152.50, 24\,310.13, ...$ form a geometric sequence, where each successive term is found by multiplying the previous term by 1.05.

This can be written as the recurrence relation

$$A_0 = 20\,000, A_{n+1} = 1.05 \times A_n.$$

For compound interest we use A_0 as the initial value because in banking terms, A_1 would represent the amount in the account after the first year of interest has been added on.

Compound interest

$$A = P(1 + i)^n$$

where $i = \dfrac{r}{100}$

A = total amount in the account
P = initial amount invested or principal
r = interest rate per period (that is, per year, quarter, month etc)
n = number of periods during the investment.

WORKED EXAMPLE 12

Helen inherits \$60 000 and invests it for 3 years in an account that pays compound interest of 8% per annum compounding every 6 months.

a. What will be the amount in Helen's account at the end of 3 years?

b. How much will Helen receive in interest over the 3 year period?

THINK	WRITE
a. 1. This is an example of compound interest. Use $A = P(1+i)^n$, where $1 + i = 1 + \dfrac{r}{100}$. Interest is calculated each 6 months so, over 3 years, there are 6 periods: $n = 6$. Interest is 8% per year or 4% per 6 months. So, $r = 4\%$.	a. $P = 60\,000$ $n = 6$ half years $r = 4\%$ per half year

		So, $1 + i = 1 + \frac{4}{100}$ $= 1.04$ $A = P(1 + i)^n$ $= 60\,000\,(1.04)^6$ $= 75\,919.14$
	2. Write your answer.	At the end of 3 years, Helen will have a total amount of $75 919.14.
b.	**1.** Interest equals the amount in the account at the end of 3 years, less the amount in the account at the start of the investment.	**b.** Interest = Total amount − Principal = $75 919.14 − $60 000 = $15 919.14
	2. Write your answer.	Amount of interest earned over 3 years is $15 919.14.

WORKED EXAMPLE 13

Jim invests $16 000 in a bank account that earns compound interest at the rate of 12% per annum compounding every quarter.

At the end of the investment, there is $25 616.52 in the account.

For how many years did Jim have his money invested?

THINK	**WRITE/DISPLAY**
1. We know the value of A, P, r. We need to determine n using the compound interest formula.	$A = 25\,616.52$ and so $1 + i = 1 + \frac{3}{100}$ $P = 16\,000$ $\quad = 1.03$ $r = \frac{12}{4}$ $= 3\%$ per quarter Now, $A = P(1 + i)^n$ So, $25\,616.52 = 16\,000\,(1.03)^n$ $1.601 = 1.03^n$
2. Try some different values of n.	Let $n = 5$ $\quad 1.03^5 = 1.159$ Let $n = 10$ $\quad 1.03^{10} = 1.344$ Let $n = 15$ $\quad 1.03^{15} = 1.558$ Let $n = 16$ $\quad 1.03^{16} = 1.605$
3. Write your answer.	It will take: Time = 16 periods = 16 × 3 months = 48 months = 4 years

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Units 3 & 4 > Area 3 > Sequence 2 > Concept 3

Applications of geometric sequences Summary screen and practice questions

Exercise 5.4 Compound interest and other applications of geometric sequences

1. **WE10** A farmer harvests 4 tonnes of lucerne in his first year of production. In his business plan, he has estimated an annual increase of 6% on his lucerne harvest.
   a. According to this plan, how many tonnes of lucerne should he harvest in his 7th year of production?
   b. In which year will his harvest reach 10 tonnes?
2. A taxi driver estimates that the cost of keeping her taxi on the road increases by 4.5% each year. If the cost of keeping her taxi on the road in her first year of owning a taxi was $1800
   a. what was the cost in the 5th year?
   b. during which year did costs exceed $2500?
3. **WE11** The population of a town is decreasing by 10% each year. Write an expression for the population of the town, which will be referred to as P_n. The population in the first year, P_1, was 10 000.
4. **WE12** $13 000 is invested in an account which earns compound interest of 8%, compounding quarterly.
   a. After 5 years, how much is in the account?
   b. How much interest was earned in that period?
5. The population of the newly established town of Alansford in its first year was 6000. It is predicted that the town's population will increase by 10% each year. If this were to be the case, determine
   a. the population of the town in its 10th year
   b. in which year the population of Alansford would reach 25 000.
6. The promoters of Fleago flea treatment assert that continued application of the treatment will reduce the number of fleas on a dog by 15% each week. At the end of week 1, Fido the dog has 200 fleas left on him and his owner continues to apply the treatment.
   a. How many fleas would Fido be expected to have on him at the end of the 4th week?
   b. How many weeks would Fido have to wait before the number of fleas on him had dropped to less than 50?

7. Young saplings should increase in height by 9% each year under optimum conditions. If a batch of saplings that have been planted out measure 2.2 metres in their first year
   a. how high should they be in their 4th year?
   b. in which year should they exceed 5 metres in height?
8. A number of timber beams support a ramp. The first of the beams is 0.8 metres long and each successive beam is 3% longer than the previous one.
   a. How long will the 7th beam in the line be?
   b. Which beam will be the first to exceed 2 metres in length?
9. **MC** A colony of ants is studied and the population of the colony in week 1 of the study is 800. If the population of the colony is expected to increase at the rate of 2% each week, then the week in which the number of ants would exceed 1000 would be closest to
   A. 6. **B.** 10. **C.** 13. **D.** 26.
10. A company exported $300 000 worth of manufactured goods in its first year of production. According to the business plan of the company, this amount should increase each year by 7.5%.
    a. How much would the company be expected to export in its 5th year?
    b. In which year would exports exceed $500 000?
    c. What is the total amount exported by the company in its first 7 years of operation?
11. Country football crowds have been decreasing by 3% each year since records of crowd attendance were kept. If the number of people attending in the first year that records were kept was 63 000 in a season:
    a. how many people attended in the 5th year?
    b. when did the number of people attending in a year drop below 50 000?

12. $10 000 is invested in an account that earns compound interest of 10% per annum. Determine the amount in the account after 5 years if the interest is compounded
   a. yearly
   b. every 6 months
   c. quarterly
   d. monthly.

13. $20 000 is invested in an account earning compound interest of 10% per annum compounding quarterly. What is the amount in the account after
   a. 1 year?
   b. 3 years?
   c. 5 years?
   d. 10 years?

14. $7000 is invested in an account that earns compound interest of 6% per annum compounding monthly. After 3 years, how much is in the account?

15. **WE13** In an account earning compound interest of 8% per annum compounding quarterly, an amount of $6000 is invested. When the account is closed, there is $7609.45 in the account. For how many years was the account open?

16. Sue earns 12% interest per annum compounding quarterly on her investment of $40 000. For how many years would this investment need to operate for the amount to rise to $50 670.80?

17. Helena receives $15 627.12 after closing an investment account that earned compound interest of 9% per annum compounding every 6 months. If Helena originally deposited $12 000 in the account, for how long was it in the account?

18. Todd receives $66 277.33 after having invested an inheritance of $60 000 in an account earning compound interest of 12% per annum compounding monthly. For how long did Todd have the money invested?

5.5 Reducing balance depreciation

5.5.1 Graphs and tables of reducing balance depreciation

In chapter 4 we studied the straight-line method of depreciation.

The other method of depreciation used is the declining or reducing balance method of depreciation.

In this method, the value of the item depreciates each year by a percentage of its current value. Under such depreciation, the value of the item never actually becomes zero.

This type of depreciation is an example of exponential decay. A graph depicting the value over time is non-linear, showing a downward-falling curve which never actually reaches a zero value.

WORKED EXAMPLE 14

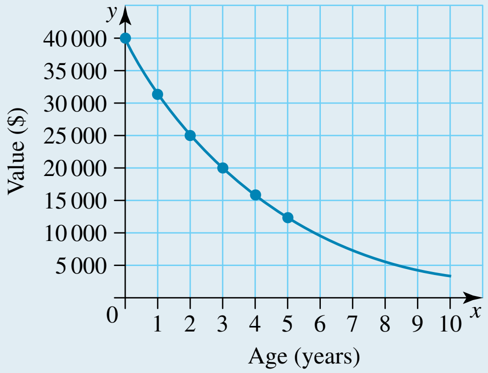
The following table shows the value of a car that is purchased new for $ 40 000. Plot the points on a set of axes and graph the depreciation of the car. Use the graph to estimate the value of the car after 10 years.

Age of car (years)	Value ($)
New (0)	40 000
1	32 000
2	25 600
3	20 480
4	16 384
5	13 107

THINK

1. Draw a set of axes with age on the horizontal axis and value on the vertical.

WRITE



2. Plot the points from the table.
3. Join the points with a smooth curve.
4. Estimate the value after 10 years from the graph you have drawn.

From the graph, the approximate value of the car after 10 years is $ 4000.

5.5.2 Calculating reducing balance depreciation using a recurrence relation

Consider the case of a car purchased new for $30 000, which depreciates at the rate of 20% p.a. Each year the future value (V_n) of the car is 80% of its value at the end of the previous year.

Recurrence relation

$$V_0 = 30\,000, V_{n+1} = V_n \times 0.80$$

After 1 year: $V_1 = 30\,000 \times 0.80 = 24\,000$
After 2 years: $V_2 = 24\,000 \times 0.80 = 19\,200$
After 3 years: $V_3 = 19\,200 \times 0.80 = 15\,360$

Reducing balance depreciation

$$V_0 = \text{Purchase price}, V_{n+1} = \left(1 - \frac{r}{100}\right) V_n, \text{ where}$$

V_n is the value of the asset after n depreciating periods and r is the depreciation rate.

WORKED EXAMPLE 15

A small truck that was purchased for $ 45 000 depreciates at a rate of 25% p.a.

a. Set up a recurrence relation to describe this situation.

b. By calculating the value at the end of each year, determine the future value of the truck after 4 years.

THINK

a. 1. Write the rule for a reducing balance recurrence relation

WRITE

a. $V_0 = \text{Purchase price}, V_{n+1} = \left(1 - \frac{r}{100}\right) V_n$

2. Substitute the values for V_0 and r.	$V_0 = \$45\,000,\ V_{n+1} = \left(1 - \frac{25}{100}\right) V_n$
3. Simplify	$V_0 = \$45\,000,\ V_{n+1} = 0.75V_n$
b. 1. Determine the value after 1 year by calculating $0.75 \times 45\,000$	**b.** After 1 year: $V_1 = 0.75 \times 45\,000 = \$33\,750$
2. Determine the value after 2 years by calculating $0.75 \times 33\,750$	After 2 years: $V_2 = 0.75 \times 33\,750 = \$25\,312.50$
3. Determine the value after 3 years by calculating $0.75 \times 25\,312.50$	After 3 years: $V_3 = 0.75 \times 25\,312.50 = \$18\,984.38$
4. Determine the value after 4 years by calculating $0.75 \times 18\,984.38$	After 4 years: $V_4 = 0.75 \times 18\,984.38 = \$14\,238.28$
5. Write the answer.	The value of the truck after 4 years is \$14 238.28

5.5.3 Calculating reducing balance depreciation using a rule

Future value (V_n) under a reducing balance

$$V_n = V_0(1 - i)^n$$

where

V_n is the future value

V_0 is the purchase price

$i = \frac{r}{100}$, r is the annual percentage depreciation

n is the number of years.

WORKED EXAMPLE 16

The purchase price of a yacht is \$15 000. The value of the yacht depreciates by 10% p.a. Calculate (correct to the nearest \$1) the future value of the yacht after 8 years.

THINK	WRITE
1. Write the formula.	$V_n = V_o\,(1 - i)^n$
2. Substitute values for V_o, i and n.	$= 15\,000\left(1 - \frac{10}{100}\right)^8$ $= \$15\,000 \times 0.9^8$
3. Calculate the future value.	$= \$6457.01$

To calculate the amount by which the asset has depreciated, we subtract the future value from the purchase price.

WORKED EXAMPLE 17

The purchase price of a computer for a music studio is $ 40 000. The computer depreciates by 12% p.a. Calculate the amount by which the computer depreciates in 10 years.

THINK

1. Write the formula.
2. Substitute the value of V_o, i and n.
3. Calculate the value of V_n.
4. Calculate the amount of depreciation by subtracting the future value from the purchase price.

WRITE

$$V_n = V_o(1 - i)^n$$

$$= 40\,000\left(1 - \frac{12}{100}\right)^{10}$$

$$= \$40\,000 \times 0.88^{10}$$

$$= \$11\,140.04$$

$$\text{Depreciation} = \$40\,000 - \$11\,140.04$$

$$= \$28\,859.96$$

Exercise 5.5 Reducing balance depreciation

1. **WE14** The following table shows the declining value of a new motor scooter.
   a. Plot the points shown by the table and draw a graph of the value of the motor scooter against age.
   b. Use your graph to estimate the value of the motor scooter after 8 years.

Age (years)	Value ($)
New (0)	20 000
1	15 000
2	11 250
3	8500
4	6250
5	4750

2. The following table shows the declining value of a semitrailer.
   - a. Plot the points as given in the table and then draw a curve of best fit to graph the depreciation of the semitrailer.
   - b. Use your graph to estimate the value of the semitrailer after 10 years.
   - c. After what number of years will the value of the semitrailer fall below \$50 000?

Age (years)	**Value (\$)**
New (0)	6 00 000
1	4 20 000
2	2 95 000
3	2 05 000
4	1 45 000
5	1 00 000

3. a. A gymnasium values its equipment at \$200 000. Each year the value of the equipment depreciates by 20% of the value of the previous year. Calculate the value of the equipment after:
   - i. 1 year
   - ii. 2 years
   - iii. 3 years
   - iv. 4 years

   b. Plot these points on a set of axes and draw a graph of the value of the equipment against its age.
4. **WE15** The purchase price of a forklift is \$50 000. The value of the forklift depreciates by 20% p.a.
   - a. Set up a recurrence relation to describe this situation.
   - b. By calculating the value at the end of each year, determine the future value of the truck after 4 years.
5. A trailer is purchased for \$5000. The value of the trailer depreciates by 15% each year.
   - a. Use a recurrence relation to calculate the future value of the trailer after 5 years (to the nearest \$10).
   - b. Calculate the amount by which the trailer depreciates
     - i. in the first year.
     - ii. in the fifth year.
6. A company purchases a mainframe computer for \$3 000 000. The value of the computer depreciates by 15% p.a. By calculating the value at the end of each year, determine the number of years that it takes for the future value of the mainframe to fall below \$1 000 000.
7. **WE16** Use the declining balance depreciation formula to calculate the future value after 7 years of a power generator purchased for \$800 000 that depreciates at a rate of 10% p.a. (Give your answer correct to the nearest \$1000.)
8. Calculate the future value of an asset (correct to the nearest \$100) with a purchase price of
   - a. \$10 000 that depreciates at 10% p.a. for 5 years
   - b. \$250 000 that depreciates at 15% p.a. for 8 years
   - c. \$5000 that depreciates at 25% p.a. for 5 years
   - d. \$2.2 million that depreciates at 30% p.a. for 10 years
   - e. \$50 000 that depreciates at 40% p.a. for 5 years.
9. A plumber has tools and equipment valued at \$ 18 000. If the value of the equipment depreciates by 30% each year, calculate the value of the equipment after 3 years.
10. **WE17** A yacht is valued at \$950 000. The value of the yacht depreciates by 22% p.a. Calculate the amount that the yacht will depreciate in value over the first 5 years (correct to the nearest \$1000).
11. A new car is purchased for \$35 000. The owner plans to keep the car for 5 years then trade the car in on another new car. The estimate is that the value of the car will depreciate by 16% p.a. Calculate:
    - a. the amount the owner can expect as a trade-in for the car in 5 years (correct to the nearest \$100)
    - b. the amount by which the car will depreciate in 5 years.

12. **MC** A shop owner purchases fittings for her store that cost \$120 000. Three years later, the shop owner is asked to value the fittings for insurance. If the shop owner allows for depreciation of 15% p.a. on the fittings, which of the following calculations will give the correct estimate of their value?

A. $120\,000 \times 0.85^3$ **B.** $120\,000 \times 0.15^3$ **C.** $120\,000 \times 1.15^3$ **D.** $120\,000 \times 0.55$

13. **MC** A computer purchased for \$3000 will depreciate by 25% p.a. The future value of the computer after 4 years will be closest to

A. \$0. **B.** \$10. **C.** \$950. **D.** \$1000.

14. An electrician purchases tools of trade for \$8000. Each year the electrician is entitled to a tax deduction for the depreciation of this equipment. If the rate of depreciation allowed is 33% p.a., calculate
    - **a.** the value of the equipment at the end of one year (correct to the nearest \$1)
    - **b.** the tax deduction allowed in the first year
    - **c.** the value of the equipment at the end of two years (correct to the nearest \$1)
    - **d.** the tax deduction allowed in the second year.

15. An accountant purchased a computer for \$6000. The value of the computer depreciates by 33% p.a. When the value of the computer falls below \$1000, it is written off and a new one is purchased. How many years will it take for the computer to be written off?

5.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** Which of these is a geometric sequence?

A. $\{2, 4, 6, ...\}$ **B.** $\{2, 4, 16, ...\}$
C. $\{2, 20, 40, ...\}$ **D.** $\{2, 4, 8, ...\}$

2. **MC** A geometric sequence is described with the recurrence relation

$$t_1 = 5, t_{n+1} = 3 \times t_n.$$

Which of the following sequences is described by this recurrence relation?

A. 3, 15, 75, 225, 1125, ... **B.** 5, 15, 45, 135, 405, ...
C. 5, 8, 11, 14, 17, ... **D.** 5, 9, 11, 15, ...

3. **MC** The 10th term of the geometric sequence 4, 12, 36, ... is

A. 78 732. **B.** 177 147. **C.** 236 196. **D.** 786 432.

4. **MC** The 3 consecutive terms of a geometric sequence are 5, y, 20. The value of y is

A. 5. **B.** 12.5. **C.** 20. **D.** 10.

5. **MC** What is the common ratio of the geometric sequence $\frac{1}{8}, \frac{1}{2}, 2, ...$?

A. $\frac{1}{8}$ **B.** $\frac{1}{2}$ **C.** 2 **D.** 4

6. **MC** What is the rule for the geometric sequence 7, 21, 63, ...?

A. $7 \times 3^{n-1}$ **B.** 7×3^n **C.** 7×3 **D.** $3 \times 7^{n-1}$

7. **MC** The 4th term of a geometric sequence is 3 and the 7th term is 24. Which is the first term of this sequence?

A. $\frac{1}{8}$ **B.** $\frac{3}{8}$ **C.** 2 **D.** 8

8. **MC** A 500 g packet of chocolate costs \$2.50 at the beginning of the year. Assuming inflation averages 2.8% per annum over each of the next 3 years, how much will the chocolate cost in three years?

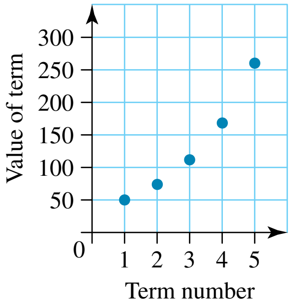
A. \$2.64 **B.** \$2.72 **C.** \$2.79 **D.** \$2.87

9. **MC** An investment of \$10 000 at the rate of 8% per annum, compounded quarterly, will reach \$14 800 in close to

A. 2 years. **B.** 3 years. **C.** 4 years. **D.** 5 years.

10. **MC** The first five terms of a sequence are plotted on the graph. The sequence could be described by which of the following?

A. Arithmetic sequence with $t_1 = 50$ and $d = 25$
B. Arithmetic sequence with $t_1 = 50$ and $d = 0.5$
C. Geometric sequence with $t_1 = 50$ and $r = 0.5$
D. Geometric sequence with $t_1 = 50$ and $r = 1.5$



11. **MC** A new engraving machine was bought for \$30 000. If depreciation is calculated at a rate of 27% reducing balance, the value at the end of 5 years is

A. \$ 21 900. **B.** \$ 15 987.
C. \$ 11 670.11. **D.** \$ 6219.21.

12. **MC** The Australian Tax Office allows a 30% p.a. tax deduction for depreciation on a \$4850 computer system. The allowable tax deduction in the second year would be

A. \$ 1018.50. **B.** \$ 1455. **C.** \$ 1663.55. **D.** \$ 2376.50.

Complex familiar

13. For each of the following sequences, state whether or not they are a geometric sequence. If they are, state the value of t_1 and r.

a. $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \ldots$

b. $-700, -70, -7, 7, 70, \ldots$

14. The amount of garbage (in tonnes) collected in a particular area by the local council each year is recorded over 3 successive years.

Year number	Amount of garbage (tonnes)
1	7.2
2	8.28
3	9.522

If the amount collected each year were to continue to follow a geometric sequence

a. write a rule for the amount of garbage, t_n, which would be collected in the area in year n.

b. how much garbage would be collected in the 8th year? (Answer correct to 2 decimal places.)

c. in which year would the amount of garbage collected exceed 30 tonnes?

15. Anya invests \$25 000 in an account earning compound interest of 10% per annum compounding quarterly.

a. Determine the amount in the account after 3 years.

b. Determine how long it would take to have \$40 965.41 in her account.

16. The purchase price of a mobile home is \$40 000. The value of the mobile home depreciates by 15% p.a. By calculating the value of the mobile home at the end of each year, determine the future value of the mobile home after 4 years. (Give your answer correct to the nearest \$1.)

Complex unfamiliar

17. The batteries in a toy robot are running down. The toy robot marches 50 cm in the first minute, 30 cm in the second minute, 18 cm in the next and so on. What is the total distance the toy robot has marched in the first 8 minutes?

18. An amount of $5000 is invested for 3 years and earns interest. Consider the following two cases:

   - **i.** simple interest of 10% per annum
   - **ii.** compound interest of 10% per annum compounding yearly.

- **a.** If the investment is earning simple interest, calculate the amount in the account at the end of each of the 3 years.
- **b.** If the investment is earning compound interest, calculate the amount in the account at the end of each of the 3 years.
- **c.** On the same set of axes, plot points showing the amount in each account at the end of each of the 3 years.

19. A company has office equipment that is valued at $100 000. The value of the equipment can be depreciated at $10 000 each year or by 15% p.a.

- **a.** Draw a table that will show the future value of the office equipment for the first 10 years under both methods of depreciation. Give values correct to the nearest $50.
- **b.** Draw a graph of the depreciating value of the equipment under both methods of depreciation.

20. A personal computer is purchased for $4500. A tax deduction for depreciation of the computer is allowed at the rate of 33% p.a. When the value of the computer falls below $1000, the computer can be written off. Copy and complete the following table, rounding off answers to the nearest whole number.

Year	Future value ($)	Tax deduction ($)
1		
2		
3		
4		
5		

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Answers

5 The geometric sequence

Exercise 5.2 Using recursion to generate a geometric sequence

1. a. Geometric; $t_1 = 3, r = 2$

b. Geometric; $t_1 = \frac{1}{2}, r = 2\frac{1}{2}$

c. Not geometric

d. Geometric; $t_1 = \frac{1}{2}, r = \frac{2}{5}$

2. a. $c = 36$

b. $g = -6, h = 12$

c. $p = 2.4, q = 12, s = 60$

3. a. 2, 10, 50, 250, 1250

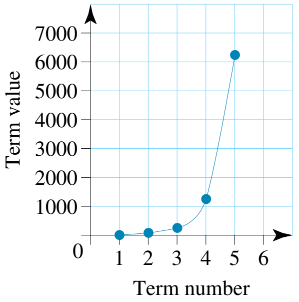
b. $-3, -12, -48, -192, -768, -3072$

c. 1000, 1120, 1254.4, 1404.93, 1573.52

4. a.

Term number	1	2	3	4	5
Term value	10	50	250	1250	6250

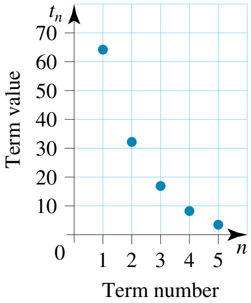
b.



5. a.

Term number	1	2	3	4	5
Term value	64	32	16	8	4

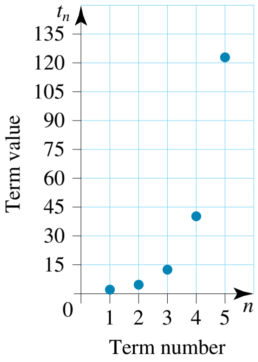
b.



6. a.

Term number	1	2	3	4	5
Term value	1.5	4.5	13.5	40.5	121.5

b.



7. 10 000, 10 200, 10 404, 10 612.08, 10 824.32, 11 040.81

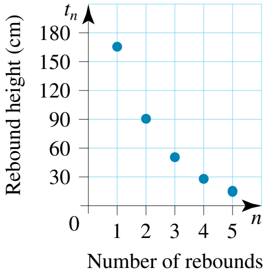
8. $t_{n+1} = -3t_n, t_1 = 2.5$

9. $-4, 14, -49, 171.5, -600.25$

10. a. $t_{n+1} = 0.55t_n, t_1 = 165$

b. 4th rebound: 27.45 cm, 5th rebound: 15.10 cm

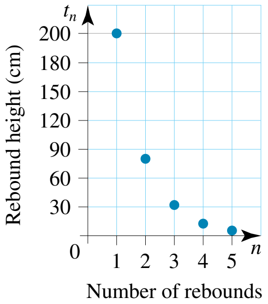
c.



11. a. $t_{n+1} = \frac{2}{5}t_n, t_1 = 200$

b. 1st rebound: 200 cm, 2nd rebound: 80 cm, 3rd rebound: 32 cm, 4th rebound: 12.8 cm, 5th rebound: 5.12 cm

c.



12. a. 17.7 metres

b. $t_{n+1} = 0.7t_n, t_1 = 12.39$

Exercise 5.3 Using the rule for the *n*th term of a geometric sequence

1. a. $t_n = -1 \times 5^{n-1}$ **b.** $t_n = 7 \times (-0.5)^{n-1}$

c. $t_n = \frac{5}{6} \times \left(\frac{2}{3}\right)^{n-1}$

2. a. 20 155 392 **b.** 1 048 576

c. 295 245 **d.** 11.867 494 81

e. $-131\,072$

3. a. 39 366 **b.** 6144 **c.** 32 768

d. $-32\,768$ **e.** $-12\,582\,912$ **f.** $-708\,588$

4. a. 12th **b.** 13th **c.** 9th **d.** 16th

5. a. 19 131 876 **b.** 5

c. -420

6. a. 1 258 291.2 **b.** -2

c. 640

7. a. 1, 2, 4, 8, 16 **b.** 2

c. 2048

8. 20 million, 10 million, 5 million, $2\frac{1}{2}$ million, $1\frac{1}{4}$ million, 625 000, 312 500

9. a. $t_n = 2^{n-1}$

b. 2048

10. a. $t_n = 10\,000 \times 0.85^{n-1}$

b. \$2724.91

11. \$1600

12. 31st term

13. a. $t_n = 6^{n-1}$ **b.** 1296 **c.** 6

14. a. $t_n = 60 \times 1.08^{n-1}$ **b.** 88.16 m

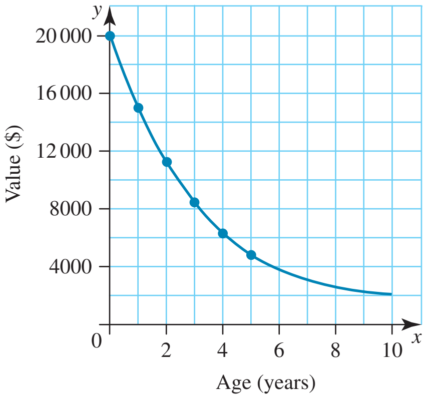
c. 8th year

Exercise 5.4 Compound interest and other applications of geometric sequences

1. **a.** 5.67 **b.** 17th year
2. **a.** \$2146.53 **b.** Year 9
3. $P_n = 10\,000 \times (0.9)^{n-1}$
4. **a.** \$19 317.32 **b.** \$6317.32
5. **a.** 14 147 **b.** 16th year
6. **a.** 123 **b.** 10 weeks
7. **a.** 2.85 m **b.** Year 11
8. **a.** 0.96 m **b.** 32nd beam
9. C
10. **a.** \$400 640.74 **b.** Year 9
c. \$2 636 196.56
11. **a.** 55 773 **b.** 9th year
12. **a.** \$16 105.10 **b.** \$16 288.95
c. \$16 386.16 **d.** \$16 453.09
13. **a.** \$22 076.26 **b.** \$26 897.78
c. \$32 772.33 **d.** \$53 701.28
14. \$8376.76
15. 3 years
16. 2 years
17. 3 years
18. 10 months

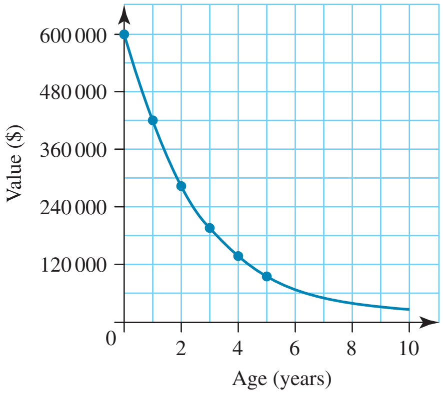
Exercise 5.5 Reducing balance depreciation

1. **a.**



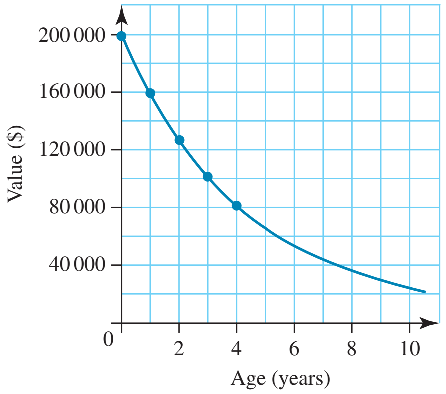
b. Approximately \$2000

2. **a.**



b. \$30 000
c. 7
3. **a.** **i.** \$160 000
ii. \$128 000
iii. \$102 400
iv. \$81 920

b.

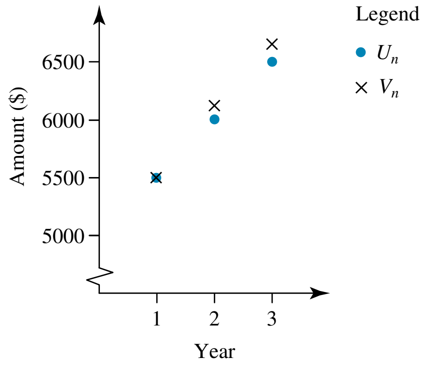


4. **a.** $V_0 = 50\,000,\ V_{n+1} = 0.80 \times V_n$
b. \$20 480
5. **a.** \$2220
b. **i.** \$750
ii. \$391.50
6. 7 years
7. \$383 000
8. **a.** \$5900 **b.** \$68 100 **c.** \$1200
d. \$62 100 **e.** \$3900
9. \$6174
10. \$676 000
11. **a.** \$14 600
b. \$20 400
12. A
13. C
14. **a.** \$5360 **b.** \$2640
c. \$3591 **d.** \$1769
15. 5 years

5.6 Review: exam practice

1. D
2. B
3. A
4. D
5. D
6. A
7. B
8. B
9. D
10. D
11. D
12. A
13. **a.** Yes, $t_1 = 5, r = \frac{1}{2}$
b. No
14. **a.** $t_n = 7.2 \times 1.15^{n-1}$
b. 19.15 tonnes
c. Year 12
15. **a.** \$33 622.22 **b.** 5 years
16. \$20 880
17. 122.90 cm
18. **a.** \$5500, \$ 6000, \$ 6500
b. \$5500, \$ 6050, \$ 6655

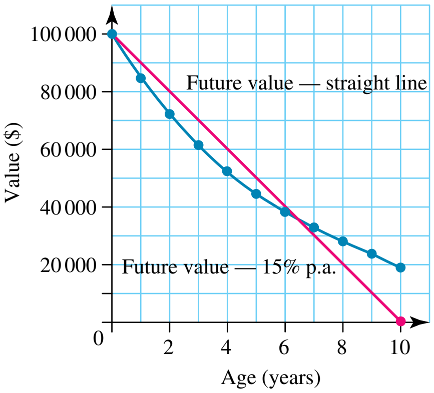
c.



19. a.

Age (years)	Future value — straight line ($)	Future value —15% p.a. ($)
New (0)	100 000	100 000
1	90 000	85 000
2	80 000	72 250
3	70 000	61 400
4	60 000	52 200
5	50 000	44 350
6	40 000	37 700
7	30 000	32 050
8	20 000	27 250
9	10 000	23 150
10	0	19 700

b.



20.

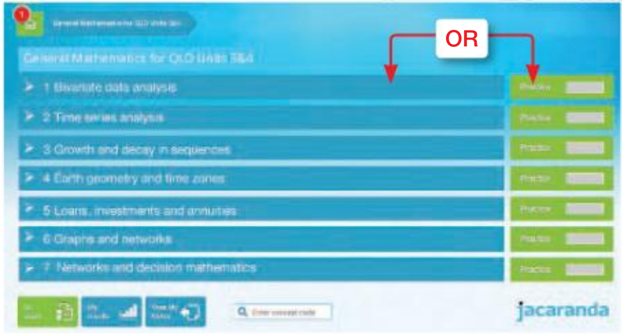
Year	Future value ($)	Tax deduction ($)
1	3015	1485
2	2020	995
3	1353	667
4	907	446
5	0	907

REVISION UNIT 3 Bivariate data, sequences and change, and Earth geometry

TOPIC 3 Growth and decay in sequences

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *General Mathematics for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level *OR* select **Practice** to answer all practice questions available for each area.



General Mathematics for QLD Units 3&4

1 Bivariate data analysis

2 Time series analysis

3 Growth and decay in sequences

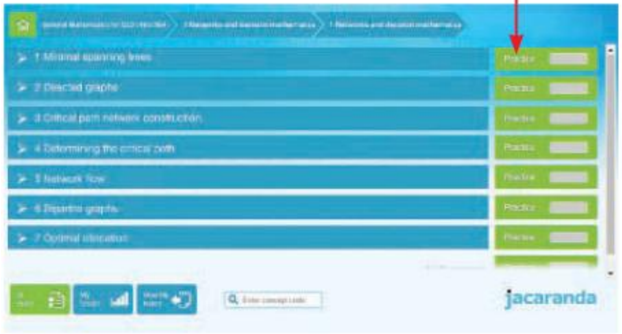
4 Earth geometry and time zones

5 Loans, investments and annuities

6 Graphs and networks

7 Networks and decision mathematics

- Select **Practice** at the sequence level to access all questions in the sequence.



1 Minimal spanning trees

2 Directed graphs

3 Critical path network construction

4 Determining the critical path

5 Network flow

6 Bipartite graphs

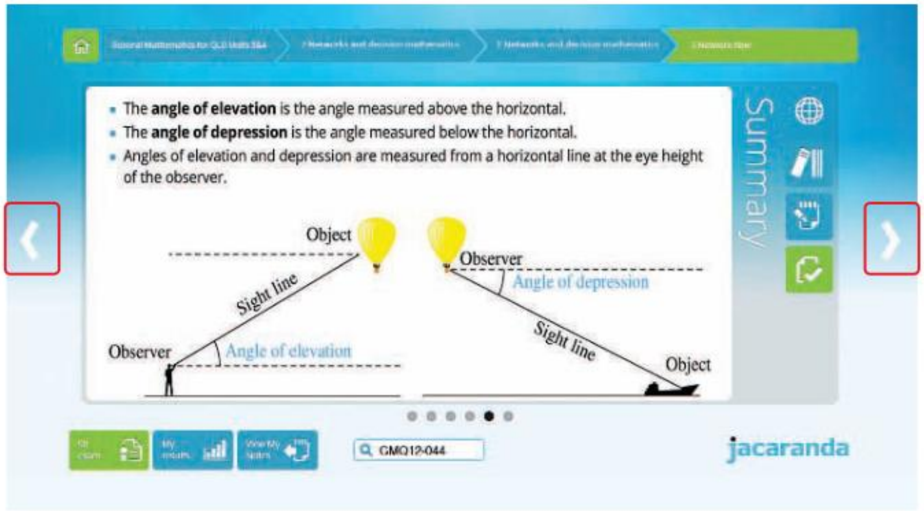
7 Optimal allocation

- At **sequence level**, drill down to concept level.

7 Networks and decision mathematics

1 Networks and decision mathematics

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



- The **angle of elevation** is the angle measured above the horizontal.
- The **angle of depression** is the angle measured below the horizontal.
- Angles of elevation and depression are measured from a horizontal line at the eye height of the observer.

PRACTICE ASSESSMENT 1

General Mathematics: Problem solving and modelling task

Unit

Unit 3: Bivariate data, sequences and change, and Earth geometry

Topic

Topic 1: Bivariate data analysis
Topic 2: Time series analysis

Conditions

Duration	Mode	Individual/group	Other
4 weeks	Written report, up to 10 pages (maximum 2000 words) excluding appendix	Individual	–

Resources permitted
The use of technology is required, for example: • computer • internet • spreadsheet program • calculator • other software/technology.

Milestones		
Week 4		
Week 5		
Week 6		
Week 7 (assessment submission)		
Criterion	**Marks allocated**	**Result**
Formulate *Assessment objectives 1, 2, 5	4	
Solve *Assessment objectives 1, 6	7	
Evaluate and verify *Assessment objectives 4, 5	5	
Communicate *Assessment objective 3	4	
Total	20	
Scaffolding	Please refer to the flow chart on page 14 of the *General Mathematics General Senior Syllabus 2019 v1.2*, Brisbane, describing an appropriate approach to problem solving and modelling.	

For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

Context

On Tuesday 7 August 2018, Australia's population hit 25 000 000.

The Australian Bureau of Statistics has a population clock that estimates Australia's population according to the following statistics:

- one birth every 1 minute and 40 seconds,
- one death every 3 minutes and 16 seconds,
- one person arriving to live in Australia every 57 seconds,
- one Australian resident leaving Australia to live overseas every 1 minute and 49 seconds, leading to
- an overall total population increase of one person every 1 minute and 15 seconds.

These assumptions are consistent with figures released in Australian Demographic Statistics, December Quarter 2017 (cat. no. 3101.0).

Source: http://www.abs.gov.au/ausstats/abs%40.nsf/94713ad445ff1425ca25682000192af2/1647509ef7e25faaca2568a900154b63?OpenDocument

Social researchers are suggesting that if this trend continues, the population of Australia could double in 50 years. An increase in population of this size can bring many advantages but also many challenges.

Task

Investigate the trend in population growth in Australia over the last 25–30 years and make future predictions based on this trend using data from the Australian Bureau of Statistics website. You will use a sample of 10 consecutive years of data of population growth from the ABS website as your raw data. This sample should be of a different 10 years to that of your fellow students.

(Table 1 which can be found at the following website has all the data required: http://www.abs.gov.au/AUSSTATS/abs@.nsf/DetailsPage/3101.0Jun%202018?OpenDocument)

Produce a report based on the results of your mathematical research

You must use:

- the approach to problem solving and mathematical modelling provided
- different data and assumptions compared to other students in your class and school.

You will have four weeks to complete the assessment, including three hours of class time.

To complete this task, you must:

- Use the problem solving and mathematical modelling approach outlined in the General Mathematics syllabus and in the flow chart on the following pages to develop your response.
- Respond with a range of understanding and skills, such as using mathematical language, appropriate calculations, tables of data, graphs and diagrams.
- Provide a unique response that highlights the real-life application of mathematics.
- Develop a written report that can be read and interpreted independently of the instrument task sheet.
- Use both analytic procedures and technology.

Approach to problem solving and modelling

Formulate

In this task you will investigate the trend in the population growth of Australia for the last 25–30 years and use this trend to make predictions regarding Australia's future population growth for the next 20–30 years. Your report should consider:

- the overall population growth of the last 25–30 years using the data
- a model for making predictions
- the reliability of the model
- different ways to interpret the data.

Design a detailed plan, identifying the mathematical procedures required to solve this problem. Remember to state the necessary assumptions, variables and observations. You must also explain how you will make use of technology.

Solve

The spreadsheet from the ABS site has data from 1981 until 2018. You will need to decide what information you need to save as your raw data and hence copy the necessary data into a new spreadsheet and save it as your raw data. Consider any necessary assumptions, variables and observations in your calculations. You will make further refinements as necessary. The report should include:

- appropriate graphs, labelled correctly
- techniques for analysing data that you have studied throughout Unit 3 and discussion about the limitations of these techniques if necessary
- appendices containing your raw data.

You must use technology efficiently and show detailed calculations demonstrating the procedures used to draw any conclusions about Australia's population growth.

Is it solved?

Evaluate and verify

Evaluate the reasonableness of your original solution.

Based on your predictions, consider the advantages and challenges faced by successive Australian governments. Look at the strengths and limitations of your research and make any necessary changes, for example, world events that may impact on the future population. Justify and explain all procedures you have used and decisions you have made. Considering the original task, how valid is your solution?

Is the solution verified?

Communicate

Once you have completed all necessary calculations, you should consider how you have communicated all aspects of your report. Communicate using appropriate language that refers to the calculations and tables included in previous sections. Your response should be coherently and concisely organised.

Ensure you have:

- used mathematical, statistical and everyday language
- considered the strengths and limitations of your solution
- drawn conclusions by discussing your results
- included recommendations.

6 Earth geometry and time zones

6.1 Overview

The early sailors and land explorers who roamed the world did not have sophisticated navigating devices. The sun, moon and stars served as their basic guide to telling the local time. To determine their position on Earth, they required a reference point. The point chosen was Greenwich, England, through which the Prime Meridian passes, and the line where east meets west. At the Prime Meridian the longitude is zero. The people in the photo are standing at the Prime Meridian, and have one foot in the western hemisphere and the other foot in the eastern hemisphere.

How can we compare the times at different positions on the Earth at a particular instant? Time zones are all calculated in relation to Greenwich. The time on the Greenwich Meridian is known as **Greenwich Mean Time (GMT)**. Time zones are then stated in terms of the number of hours they are ahead or behind GMT.

The Earth approximates the shape of a sphere, so the distance between two points on its surface is not represented by the length of a straight line, but by the length of an arc of a circle. How is this distance measured?

LEARNING SEQUENCE

6.1 Overview
6.2 Latitude and longitude
6.3 Distances on the Earth's surface
6.4 Time zones
6.5 Review: exam practice

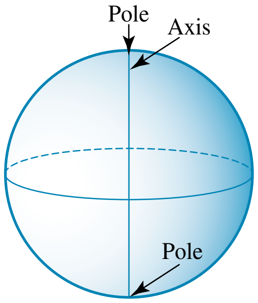
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au

6.2 Latitude and longitude

6.2.1 Position on Earth

The shape of the Earth approximates a sphere. This means that, in measuring the distance between two places on its surface, we are really measuring the distance along the arc of a circle. To measure the distance around the equator, we are actually finding the length of the circumference of a circle. To calculate-distances between points on the Earth's surface, we must first consider the location of points.

6.2.2 Great circles and small circles



Consider the sphere on the right.

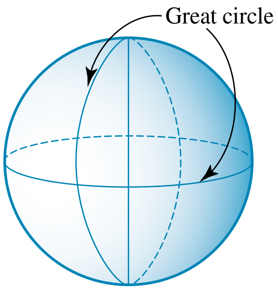
The axis of the sphere is a diameter of that sphere. The ends of the axis are called the poles.

If we draw any direct line around the sphere whose plane contains the centre of the sphere, a **great circle** is formed. A great circle is the largest possible circle that can be drawn around the sphere.

The length of a great circle is found using the formulas for the circumference of a circle:

Length of great circles

$C = \pi D$, where D is the diameter of the sphere
or $C = 2\pi r$, where r is the radius of the sphere.

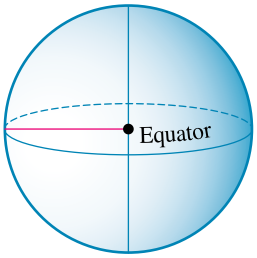


WORKED EXAMPLE 1

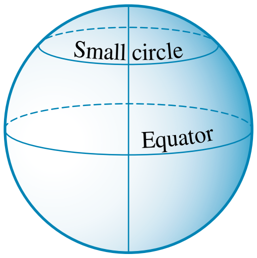
Calculate the length of a great circle on a sphere with a radius of 40 cm. Give your answer correct to the nearest centimetre.

THINK	WRITE
1. Write the formula.	$C = 2\pi r$
2. Substitute the radius of the sphere.	$= 2 \times \pi \times 40$
3. Calculate the length of the great circle.	≈ 251 cm

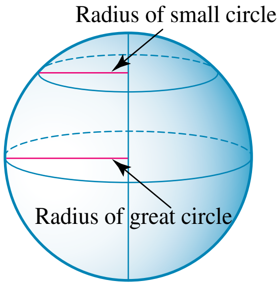
Now consider a circle drawn perpendicular to the axis of the sphere. Only one circle, called the equator, will be a great circle. The centre of the equator will be the centre of the sphere, as shown in the following diagram.



Other circles on the sphere will be smaller than a great circle and are called **small circles** or **circles of latitude.**

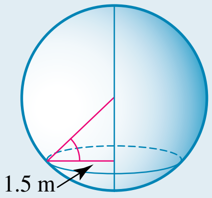


To calculate the length around a small circle, we need to know the small circle's radius. The small circle will have a radius smaller than that of the great circle, as shown in the figure shown.



WORKED EXAMPLE 2

Calculate the length of the small circle shown correct to 1 decimal place.



THINK	WRITE
1. Write the formula for the circumference of a circle.	$C = 2\pi r$
2. The radius of the small circle is given as 1.5 m. Substitute for *r*.	$= 2 \times \pi \times 1.5$
3. Calculate the circumference.	≈ 9.4 m

6.2.3 Latitude and longitude

As the Earth is a sphere, great circles and small-circles on the surface of the Earth are used to locate points on the surface.

Consider the axis of the Earth to be the diameter joining the North Pole and the South Pole. The only great circle that is perpendicular to this axis is the equator. The angular distance from the centre of the Earth either north or south of the equator is the **latitude**.

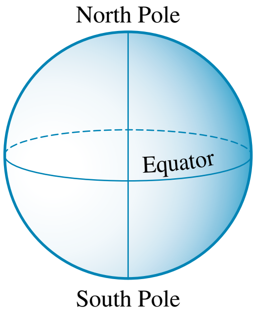
Small circles parallel to the equator are called **parallels of latitude**. These small circles are used to describe how far north or south of the equator a place is located. For example, Brisbane lies close to the small circle 27°S. This means that Brisbane subtends a 27° angle at the centre of the Earth and is south of the equator.

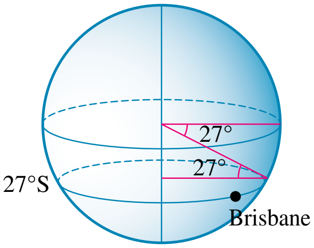
The maximum latitude for any point on the Earth is 90°N or 90°S. The North and South Poles lie at these points.

For latitude, the equator is the line of reference for all measurements.

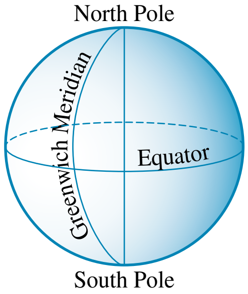
To locate a place on the globe in an east–west direction, the line of reference is the **Greenwich Meridian**. The Greenwich Meridian is half a great circle running from the North to the South Pole.

The Greenwich Meridian is named after Greenwich, a suburb of London.





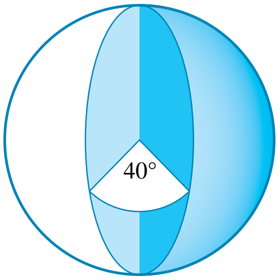
All other places on the globe can be described as being east or west of the Greenwich Meridian. The half great circle passing through the poles on which a place lies is called a **meridian of longitude**.



Each meridian of longitude is identified by the angle measured from the centre of the Earth between it and the Greenwich Meridian and by whether it is east or west of Greenwich.

The meridian of longitude opposite the Greenwich Meridian is the International Date Line. The **International Date Line** has a longitude 180° either east or west (see the map on the next page). On either side of the International Date Line the day changes. (This is explained in more detail later in the chapter.)

For the convenience of some small island nations and Russia, the International Date Line is bent so as not to pass through them.



World maps or globes are drawn with both parallels of latitude and meridians of longitude shown, which can be used to locate cities. Each point is then given a pair of coordinates: the parallel of latitude it lies on, followed by the meridian of longitude. For example, the coordinates of Brisbane are 27°S, 153°E.

WORKED EXAMPLE 3

Identify the major cities closest to each of the following locations using the map on the next page.

a. **30°S, 30°E** **b.** **30°N, 120°E** **c.** **45°N, 75°W**

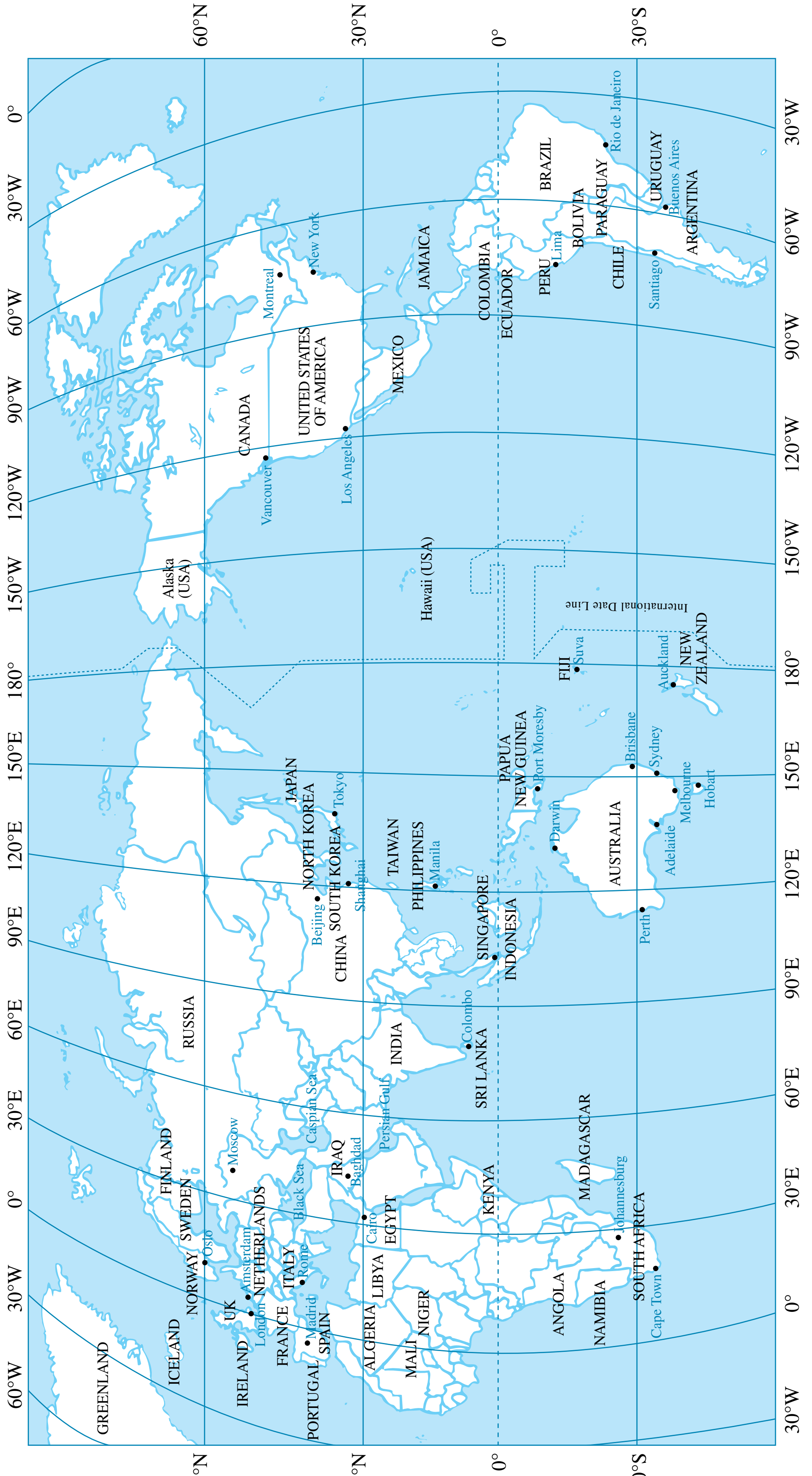
THINK	WRITE
a. Look for the city closest to the intersection of the 30°S parallel of latitude and the 30°E meridian of longitude.	**a.** Johannesburg
b. Look for the city closest to the intersection of the 30°N parallel of latitude and the 120°E meridian of longitude.	**b.** Shanghai
c. Look for the city closest to the intersection of the 45°N parallel of latitude and the 75°W meridian of longitude.	**c.** Montreal

WORKED EXAMPLE 4

Write down the approximate coordinates of each of the following cities using the map on the next page.

a. **Singapore** **b.** **Perth** **c.** **Los Angeles**

THINK	WRITE
a. **1.** Use the parallels of latitude drawn to estimate the latitude. **2.** Use the meridians of longitude drawn to estimate the longitude.	**a.** 1°N, 104°E
b. **1.** Use the parallels of latitude drawn to estimate the latitude. **2.** Use the meridians of longitude drawn to estimate the longitude.	**b.** 32°S, 115°E
c. **1.** Use the parallels of latitude drawn to estimate the latitude. **2.** Use the meridians of longitude drawn to estimate the longitude.	**c.** 35°N, 118°W


60°N
30°N
0°
30°S
60°W
30°W
0°
30°E
60°E
90°E
120°E
150°E
180°
150°W
120°W
90°W
60°W
30°W
0°
International Date Line
GREENLAND
ICELAND
IRELAND
UK
London
NORWAY
Oslo
SWEDEN
FINLAND
Moscow
RUSSIA
Amsterdam
NETHERLANDS
FRANCE
PORTUGAL
Madrid
SPAIN
ITALY
Rome
Black Sea
Caspian Sea
IRAQ
Baghdad
Persian Gulf
ALGERIA
LIBYA
Cairo
EGYPT
MALI
NIGER
KENYA
ANGOLA
NAMIBIA
SOUTH AFRICA
Cape Town
Johannesburg
MADAGASCAR
INDIA
Colombo
SRI LANKA
CHINA
Beijing
NORTH KOREA
SOUTH KOREA
Shanghai
JAPAN
Tokyo
TAIWAN
PHILIPPINES
Manila
SINGAPORE
INDONESIA
PAPUA NEW GUINEA
Port Moresby
Darwin
AUSTRALIA
Perth
Adelaide
Melbourne
Hobart
Sydney
Brisbane
FIJI
Suva
Auckland
NEW ZEALAND
Alaska (USA)
Hawaii (USA)
CANADA
Vancouver
UNITED STATES OF AMERICA
Los Angeles
Montreal
New York
MEXICO
JAMAICA
COLOMBIA
ECUADOR
PERU
Lima
BRAZIL
BOLIVIA
PARAGUAY
CHILE
Santiago
URUGUAY
Buenos Aires
ARGENTINA
Rio de Janeiro

6.2.4 Satellite navigation systems

For this activity you will need a GPS device available on most smart phones and web access.

GPS (global positioning system) is a satellite navigation system that can be accessed by users on the land, at sea, or in the air. Information provided by the system includes the position on the Earth's surface (latitude and longitude), altitude, speed, direction of travel and time.

1. Your school may possess a GPS device or you may have a smart phone. Move around your school grounds and note the change in latitude and longitude readings that the device indicates. See whether you can use the device to map a route around your school.
2. The website www.gps.gov gives information about global positioning systems and their many applications. Access this site (and any other sites of interest to you) to research one area in which the systems have been used to great benefit.
3. Another website of interest is Google Earth (www.google.com/earth). This site allows you to visit any place on the Earth, giving its latitude and longitude. There are many other features such as:
   - providing travel directions between two places
   - giving a satellite view of your home
   - showing the stars and planets beyond Earth.

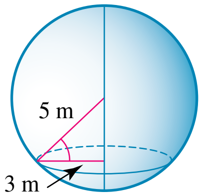
   Visit the website and investigate one feature that particularly interests you.
4. Following your research, record your findings in the form of a presentation to your fellow class members.

studyon

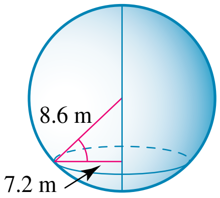
Units 3 & 4 > Area 4 > Sequence 1 > Concept 1 > **Latitude and longitude** Summary screen and practice questions

Exercise 6.2 Latitude and longitude

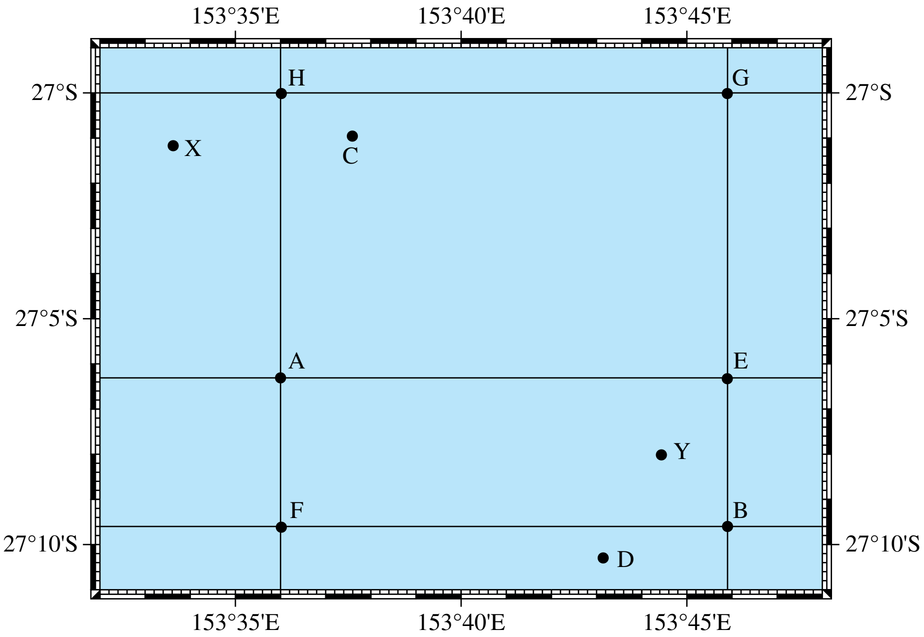
1. **WE1** Calculate the length of a great circle on a sphere with a radius of 60 cm. Give your answer rounded to the nearest centimetre.
2. Calculate the length of a great circle on a sphere with a radius of 275 cm. Give your answer rounded to two decimal places in metres.
3. **WE2** Calculate the length of the circumference of the small circle shown, rounded to 1 decimal place.



4. Calculate the length of the circumference of the small circle shown, rounded to 2 decimal places.



Use the following grid to answer questions 5 and 6.



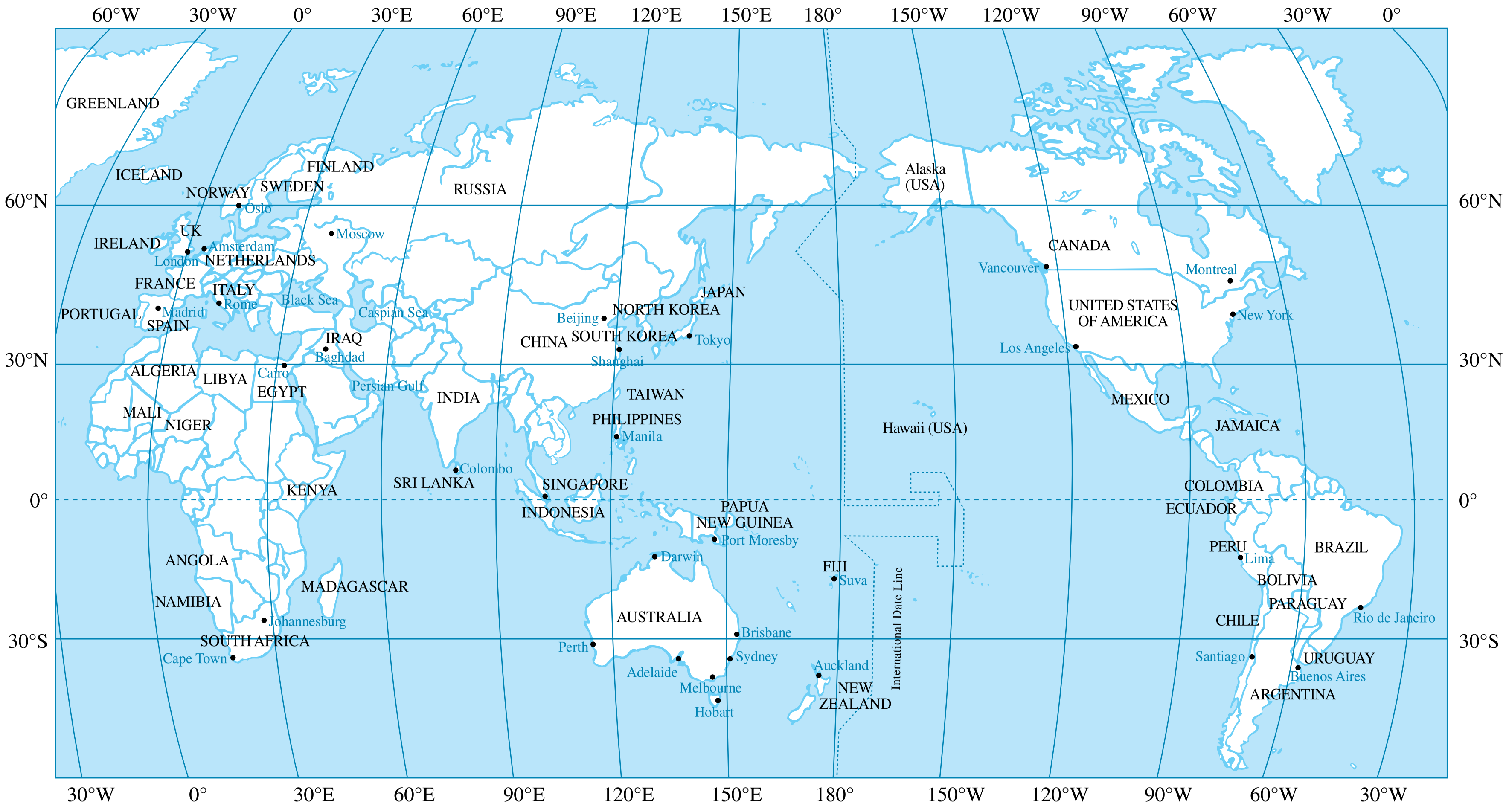
5. Give the position of point E on the grid.
6. **WE4** Use the grid to give the positions of points
   a. F
   b. G
   c. H
   d. C
   e. X
   f. Y.
7. Draw a grid as shown, but without the points, then plot the following positions.
   a. $(27°S, 153°45'E)$
   b. $(27°6.3'S, 153°40'E)$
   c. $(27°7.4'S, 153°39.8'E)$
   d. $(27°10'S, 153°42.7'E)$

8. The following image shows a portion of a map of the Whitsunday Group. Name the feature at
   a. (20°5′S, 148°54.7′E)
   b. (20°5.1′S, 148°53.4′E)
   c. (20°3.8′S, 148°57.8′E).

9. Use the map of the Whitsunday Passage above to give the position of
   a. Double Rock (off the north-east coast of Hook Island)
   b. Langford Island (to the south of Hayman Island)
   c. the entrance to Macona Inlet.

10. **WE3** Using the following world map, write down the name of the city closest to each of the following pairs of coordinates.

a. 30°N, 30°E	b. 30°N, 120°E
c. 15°S, 135°E	d. 45°N, 75°W
e. 50°N, 0°	f. 37°S, 175°E
g. 35°N, 140°E	h. 40°N, 115°E
i. $22\frac{1}{2}$°S, 43°W	j. 60°N, 11°E

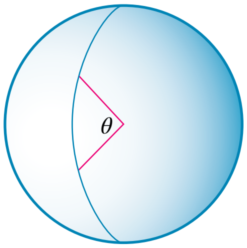

60°W
30°W
0°
30°E
60°E
90°E
120°E
150°E
180°
150°W
120°W
90°W
60°W
30°W
0°
60°N
30°N
0°
30°S
GREENLAND
ICELAND
NORWAY
Oslo
SWEDEN
FINLAND
RUSSIA
Moscow
IRELAND
UK
London
Amsterdam
NETHERLANDS
FRANCE
ITALY
Rome
PORTUGAL
Madrid
SPAIN
Black Sea
Caspian Sea
IRAQ
Baghdad
ALGERIA
LIBYA
Cairo
EGYPT
Persian Gulf
MALI
NIGER
INDIA
CHINA
Beijing
NORTH KOREA
SOUTH KOREA
Shanghai
JAPAN
Tokyo
TAIWAN
PHILIPPINES
Manila
Colombo
SRI LANKA
KENYA
SINGAPORE
INDONESIA
PAPUA
NEW GUINEA
Port Moresby
Darwin
ANGOLA
MADAGASCAR
NAMIBIA
Johannesburg
SOUTH AFRICA
Cape Town
AUSTRALIA
Brisbane
Perth
Sydney
Adelaide
Melbourne
Hobart
FIJI
Suva
Auckland
NEW
ZEALAND
International Date Line
Alaska
(USA)
CANADA
Vancouver
Montreal
UNITED STATES
OF AMERICA
New York
Los Angeles
MEXICO
Hawaii (USA)
JAMAICA
COLOMBIA
ECUADOR
PERU
Lima
BRAZIL
BOLIVIA
PARAGUAY
Rio de Janeiro
CHILE
Santiago
URUGUAY
Buenos Aires
ARGENTINA
30°W
0°
30°E
60°E
90°E
120°E
150°E
180°
150°W
120°W
90°W
60°W
30°W

11. **WE4** Using the previous world map, show the approximate latitude and longitude of each of the following major cities or islands.
    - a. Melbourne
    - b. New York
    - c. Jamaica
    - d. Johannesburg
    - e. Rome
    - f. Buenos Aires
    - g. Baghdad
    - h. Moscow
    - i. Singapore
    - j. Suva
12. Draw a sphere to represent the shape of the Earth.
    - a. Mark the equator, Tropic of Cancer, Tropic of Capricorn, North and South Poles.
    - b. Give the position of each of these lines of latitude in degrees.
13. Draw another sphere. On it mark lines representing
    - a. 0° longitude
    - b. 20°E longitude
    - c. 50°W longitude.
14. Draw one more sphere. On it mark
    - a. the radius and circumference of the equator
    - b. the radius and circumference of the Tropic of Capricorn.

6.3 Distances on the Earth's surface

In this section, we shall consider distances between points on the same great circle, or the same small circle.

6.3.1 Points on the same great circle



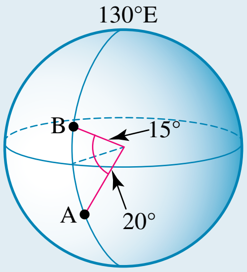
A meridian is a circle of longitude passing through a given place on the Earth's surface and the hemispherical poles. All meridians of longitude are half great circles. The equator is also a great circle. This means that measuring the distance between two points on the same meridian of longitude, or between two points on the equator, involves calculating the length of an arc of a great circle.

Consider a meridian of longitude on the Earth's surface with two points on it. The angular distance between them will be the difference between their latitudes.

The angular distance is calculated by subtracting the latitudes of points if both are on the same side of the equator and adding the latitudes if on opposite sides of the equator.

WORKED EXAMPLE 5

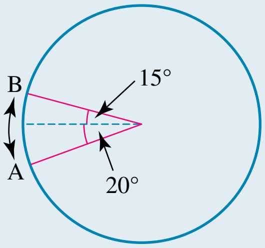
The coordinates of A are (20°S, 130°E) and the coordinates of B are (15°N, 130°E). Find the angular distance between them.



THINK

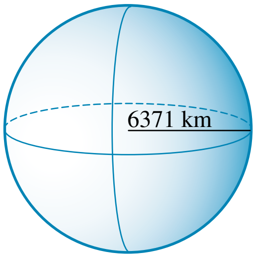
1. Draw a circle with the important information.

WRITE



2. A and B are on opposite sides of the equator on the same meridian (130°E), so add the latitudes.	Angular distance $= 20° + 15°$ $= 35°$

The radius of the Earth is approximately 6371 km.
Calculating the circumference of a great circle on the Earth, we find



$$\begin{aligned} C &= 2\pi r \\ &= 2 \times \pi \times 6371\text{ km} \\ &= 40\,030\text{ km}. \end{aligned}$$

The distance around a great circle on the Earth is 40 030 km.
This also represents an angular distance of 360°.
360° is equivalent to a distance of 40 030 km; that is

$$\begin{aligned} 360° &= 40\,030\text{ km} \\ \text{therefore} \qquad 1° &= 111.2\text{ km}. \end{aligned}$$

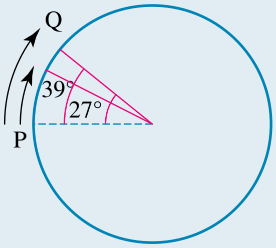
Therefore, we have calculated that an angular distance of 1° on a great circle represents a distance of 111.2 km on the surface of the Earth.

To calculate the distance (in kms) between two places on Earth, on the same meridian, use the formula

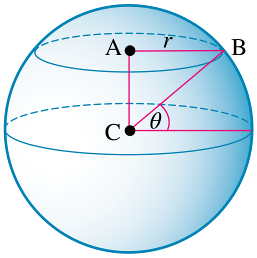
$$D = 111.2 \times angular\,distance.$$

WORKED EXAMPLE 6

P and Q are two points on the Earth's surface with coordinates (27°N, 30°W) and (39°N, 30°W) respectively. Calculate the distance between P and Q (to the nearest kilometre).

THINK	WRITE
1. P and Q are on the same great circle.	
2. Calculate the angular distance, PQ. In the same hemisphere, subtract the angles.	Angular distance $= 39° - 27°$ $= 12°$
3. Convert the angular distance to kilometres using $D = 111.2 \times angular\,distance$.	Distance $= 111.2 \times 12$ $= 1334$ km

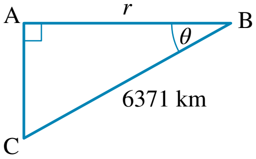
6.3.2 Points on the same small circle



Points on the same parallel of latitude (other than the equator) lie on the circumference of a small circle. To find the distance between two such points, we need to calculate the length of the arc of the small circle. This means that we need to know the radius of the small circle.

Consider the diagram at right, showing a parallel of latitude at an angular distance θ from the equator. (This could be a parallel of latitude north or south of the equator. Here it is shown north of the equator.)

To calculate the radius (r) of this small circle, we need to apply trigonometry.



Angle ABC is equal to θ because of alternate angles on parallel lines. (The equator's radius is parallel to the radius of the parallel of latitude.)

The radius of the Earth is 6371 km:

$$\begin{aligned}\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ &= \frac{r}{6371}\\ \therefore r &= 6371\cos\theta.\end{aligned}$$

To calculate the circumference of the small circle:

$$\begin{aligned}C &= 2\pi r\\ C &= 2\times\pi\times 6371\cos\theta\text{ km}\\ &= 40\,030\ \cos\theta\text{ km}.\end{aligned}$$

The circumference of the circle represents an angular distance of 360°, so

$$\begin{aligned}360^\circ &= 40\,030\ \cos\theta\ \text{ km}\\ \therefore 1^\circ &= 111.2\ \cos\theta\ \text{ km}.\end{aligned}$$

It is not necessary to reproduce the previous working in calculating distances between points on the same parallel of latitude. If we find the angular distance between the two points, we can then apply the relationship that every degree of separation on a small circle is equivalent to a distance of 111.2 cos θ km where θ is the angular distance of the parallel of latitude from the equator.

On a small circle:

$$1^\circ = 111.2\cos\theta\text{ km, where }\theta = \text{degree of latitude.}$$

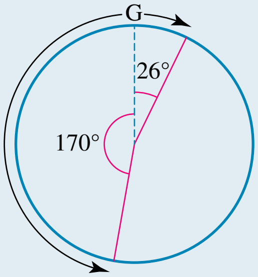
We must bear in mind that, in determining the distance between two points on the Earth's surface, we require the *shortest* distance. The shortest distance lies on the arc of the great circle joining the points. This means that if the angular distance between two points is greater than 180°, a shorter distance would be obtained by measuring the distance on the minor arc, rather than the major arc.

To calculate the distance (in kms) between two places on Earth on the same parallel of latitude, use the formula

$$D = 111.2\cos\theta\times \textit{angular distance}\text{, where }\theta\text{ is the degree of latitude.}$$

WORKED EXAMPLE 7

Find the distance (to the nearest km) between two places whose positions are (40°N, 170°W) and (40°N, 26°E).

THINK	WRITE
1. These points are on the same parallel of latitude (40°N) so both lie on a small circle.	
2. Calculate the angular distance between points. They are on opposite sides of the Greenwich Meridian, so add degrees.	$\text{Angular distance} = 170° + 26°$ $= 196°$
3. This angular distance is $> 180°$, so the shorter distance is required.	$\text{Shortest angular distance} = 360° - 196°$ $= 164°$
4. Calculate for distance on small circles using the formula $D = 111.2 \cos \theta \times \textit{angular distance}$.	$\text{Distance} = 111.2 \cos 40° \times 164$ $= 13\,970 \text{ km}$

studyon

Units 3 & 4 > Area 4 > Sequence 1 > Concepts 2 & 3

Distances on the same meridian Summary screen and practice questions
Distances along a parallel of latitude Summary screen and practice questions

Exercise 6.3 Distances on the Earth's surface

1. **WE5** Two points, A and B, on the Earth's surface are at (30°N, 25°W) and (20°S, 25°W). Calculate the angular distance between A and B.
2. In each of the following, calculate the angular distance between the pairs of points given.
   a. (70°N, 150°E) and (30°N, 150°E)
   b. (25°N, 40°W) and (15°S, 40°W)
   c. (64°N, 0°) and (7°S, 0°)
   d. (42°S, 97°W) and (21°S, 97°W)
   e. (0°, 60°E) and (0°, 20°W)
3. The city of Durban is at approximately (30°S, 30°E) while Cairo is at (30°N, 30°E). What is the angular distance between Durban and Cairo?
4. **WE6** P and Q are two points on the Earth's surface with coordinates (45°N, 10°W) and (15°N, 10°W) respectively. Calculate the distance between P and Q to the nearest kilometre.
5. Calculate the distance between each of the following points to the nearest kilometre.
   a. A (10°N, 45°E) and B (25°S, 45°E)
   b. C (75°N, 86°W) and D (60°S, 86°W)
   c. E (46°S, 52°W) and F (7°S, 52°W)
   d. G (34°N, 172°E) and H (62°S, 172°E)

6. The city of Osaka is at (37°N, 135°E) while Alice Springs is at (23°S, 135°E). Calculate the distance between Osaka and Alice Springs, to the nearest kilometre.
7. The Tropic of Cancer is at latitude $23\frac{1}{2}$°N while the Tropic of Capricorn is at latitude $23\frac{1}{2}$°S. Calculate the distance between these two tropics along the same great circle in kilometres (correct to the nearest km).
8. M and N are two points on the Earth's surface with coordinates (56°N, 122°W) and (3°S, 122°W). Calculate the distance, MN, correct to the nearest 100 km.
9. Calculate the distance between each of the following points, correct to the nearest kilometre.
   a. P (85°S, 89°E) and Q (46°S, 89°E)
   b. R (24°N, 0°) and S (12°S, 0°)
   c. T (34°S, 17°W) and U (0°, 17°W)
10. **MC** Perth is at approximately (31°S, 115°E) while Hong Kong is approximately at (22°N, 115°E). The distance between Perth and Hong Kong is approximately
    A. 19 km.
    B. 98 km.
    C. 1000 km.
    D. 5890 km.
11. **MC** Rachel is a flight navigator. She is responsible for calculating the distance between Stockholm (60°N, 18°E) and Budapest (47°N, 18°E). Rachel's answer would be closest to
    A. 1375 km.
    B. 1446 km.
    C. 1452 km.
    D. 11 890 km.
12. Quito (0°, 78°W) and Kampala (0°, 32°E) are two cities on the equator.
    a. Calculate the angular distance between Quito and Kampala.
    b. Calculate the shortest distance between them, to the nearest 100 kilometres.
13. Calculate the distance between the North Pole and the South Pole in kilometres.
14. The city of Kingston is approximately at (18°N, 76°W). Ottawa is at approximately (46°N, 76°W).
    a. Calculate the angular distance between Kingston and Ottawa.
    b. Calculate the distance between Kingston and Ottawa, to the nearest kilometre.
15. Find the distance (in km) between two places on the equator which have a difference in their longitudes of 35°.
16. Two places on the equator have an angular distance of 200°. What is the shortest distance between these two places (to the nearest kilometre)?
17. Find the distance in kilometres from each of the following places to the closer pole. Indicate which pole (North or South) is the closer.
    a. Warwick (28°S, 152°E)
    b. Vancouver (49°N, 123°W)
    c. St Moritz (46°N, 10°E)
    d. Thursday Island (10°S, 142°E)
18. **WE7** Find the shortest distance (in km) between the following places.
    a. 40°S, 130°E and 40°S, 159°E
    b. 70°N, 15°E and 70°N, 100°E
    c. 50°S, 66°W and 50°S, 106°W
    d. 80°S, 67°W and 80°S, 89°W
    e. 20°S, 150°E and 20°S, 54°W
    f. 30°N, 28°E and 30°N, 39°W

6.4 Time zones

As the Earth rotates, different parts of the globe are experiencing day and night at the same instant. This means that each meridian of longitude on the Earth's surface should have a different time of day. To simplify this, the Earth is divided into time zones.

6.4.1 Greenwich Mean Time

Time zones are all calculated in relation to Greenwich. The time on the Greenwich Meridian is known as **Greenwich Mean Time (GMT)**. Time zones are then stated in terms of the number of hours they are ahead or behind GMT. All places with longitudes east of Greenwich are ahead of GMT while all places with longitudes west of Greenwich are behind GMT.

For example, Eastern Standard Time in Australia is GMT + 10, meaning that Brisbane is 10 hours ahead of GMT. When GMT is 12 noon, EST is 10.00 pm.

The International Date Line is 12 hours ahead of Greenwich when travelling east and 12 hours behind when travelling west, so this totals 24 hours, or one day. Therefore, the day is always different on either side of the International Date Line.

The time difference between two places is calculated by subtracting the comparative time with GMT.

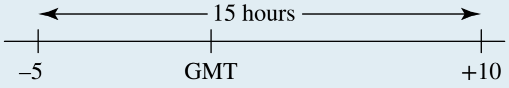
WORKED EXAMPLE 8

Brisbane is GMT + 10 while New York is GMT − 5. Calculate the time difference between Brisbane and New York.

THINK

1. Subtract the comparative times.

WRITE



$$\text{Time difference} = 10 - (-5)$$
$$= 15$$

2. State the difference and which city is ahead in time.

Brisbane is 15 hours ahead of New York.

Once we have calculated the time difference, we are able to calculate the time in one place given the time in another. To calculate the time in a city which is further ahead of GMT we add time, or to calculate the time in a city further behind GMT we subtract time.

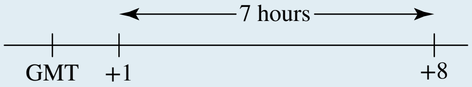
WORKED EXAMPLE 9

Perth is GMT + 8 while Cape Town is GMT + 1. When it is 11.00 am in Cape Town, what is the time in Perth?

THINK

1. Calculate the time difference and state which city is ahead.

WRITE



$$\text{Time difference} = 8 - 1$$
$$= 7 \text{ hours}$$

Perth is 7 hours ahead of Cape Town.

2. Add the time difference to the time in Cape Town to calculate the time in Perth.

$$\text{Time in Perth} = 11.00 \text{ am} + 7 \text{ hours}$$
$$= 6.00 \text{ pm, same day}$$

The time as calculated by the longitude is called the *standard time*. Time zones are calculated to approximate all the standard times within a region.

As there are 24 hours in a day and 360° of longitude (180°E and 180°W), we can calculate that:

$$1 \text{ hour} = 15^\circ \text{ of longitude}$$
$$1^\circ = 4 \text{ minutes}$$

We are now able to compare the time in various cities, given the longitude of each.

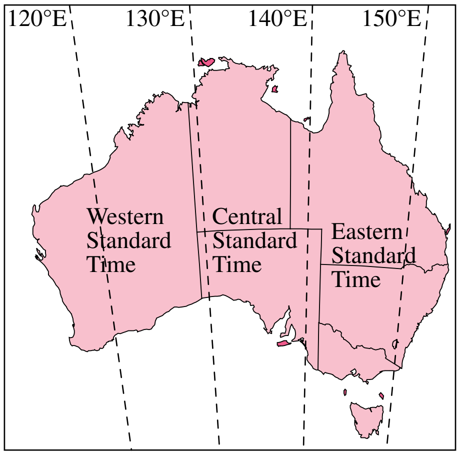
WORKED EXAMPLE 10

Calculate the time in Los Angeles (34°N, 120°W) when it is 8.00 am on Wednesday in Sydney(33°S, 150°E).

THINK	WRITE
1. Calculate the difference in longitudes.	$\text{Longitude difference} = 150° + 120°$ $= 270°$
2. Convert this angular distance into hours using; 1° ≡ 4 minutes.	$\text{Time difference} = 270 \times 4$ $= 1080 \text{ minutes}$ $= 18 \text{ hours}$
3. Subtract the time difference from the time in Sydney.	$\text{Time in Los Angeles} = 8.00 \text{ am Wednesday} - 18 \text{ hours}$ $= 2.00 \text{ pm Tuesday}$

It is important to note that, for convenience, places that have almost the same longitude have the same time. An example of this is Australia's time zones where all of Queensland, New South Wales, Victoria and Tasmania are in the same standard time zone although there is a difference of 12° in longitude between the easternmost and westernmost points in this zone.

These calculations can then be used to calculate the arrival and departure times for domestic and international travel.



WORKED EXAMPLE 11

A plane leaves London (50°N, 0°) at 9.00 am Sunday, London time and flies to Sydney (33°S, 150°E). The flight takes 20 hours. Calculate the time in Sydney when the plane arrives.

THINK	WRITE
1. Calculate the longitude difference between Sydney and London.	$\text{Longitude difference} = 150° - 0°$ $= 150°$
2. Use 1° = 4 minutes to calculate the time difference.	$\text{Time difference} = 150 \times 4$ $= 600 \text{ minutes}$ $= 10 \text{ hours}$
3. Calculate the time in Sydney when the plane is departing London by adding the time difference.	When the plane leaves London at 9.00 am (London time) $\text{Time in Sydney} = 9.00 \text{ am Sunday} + 10 \text{ hours}$ $= 7.00 \text{ pm Sunday}$
4. Add the flying time to calculate the time when the plane lands.	$\text{Plane arrives at } 7.00 \text{ pm Sunday} + 20 \text{ hours}$ $= 3.00 \text{ pm Monday}$

More challenging examples will require you to allow for daylight saving time. When daylight saving time applies, we add one hour to the standard time at that location.

on Resources

Interactivity Time zones (int-6911)

studyon

Units 3 & 4 > Area 4 > Sequence 1 > Concepts 4 & 5

Time zones Summary screen and practice questions
Time zone calculations Summary screen and practice questions

Exercise 6.4 Time zones

1. **WE8** The time zone in New Zealand is GMT + 12 while in Turkey it is GMT + 2. Calculate the time difference between New Zealand and Turkey.
2. Calculate the time difference between each of the following locations.
   a. Tokyo GMT + 9 and New York GMT − 5
   b. Los Angeles GMT − 8 and Dakar GMT − 1
   c. Rio De Janeiro GMT − 3 and Perth GMT + 8
   d. Hawaii GMT − 11 and Fiji GMT + 11
3. **WE9** Brisbane is GMT + 10, while San Francisco is GMT − 8. When it is 5.00 pm on Tuesday in Brisbane, what is the time in San Francisco?
4. For each of the following calculate
   a. the time in Perth (GMT + 8) when it is 10.00 pm in Brisbane (GMT + 10)
   b. the time in Washington (GMT − 5) when it is 4.00 am Saturday in Brisbane (GMT + 10)
   c. the time in Auckland (GMT + 12) when it is 7.00 am Wednesday in Johannesburg (GMT + 2)
   d. the time in Sydney (GMT + 10) when it is 6.00 am Tuesday in Salt Lake City (GMT − 7)
   e. the time in Adelaide (GMT + 9.5) when it is 8.15 pm Sunday in the Cook Islands (GMT − 10).
5. Jane is in Townsville (GMT + 10) and wants to telephone her friend in Paris (GMT + 1) at 7.00 pm Friday Paris time. At what time must she call from Townsville?
6. Carl is holidaying in Hawaii (GMT − 11). If he wants to call his parents in Rockhampton (GMT + 10) at 8.00 pm on Wednesday, Rockhampton time, what time must he call from Hawaii?
7. Neville is in Brisbane (GMT + 10). He wants to set his video recorder to tape the Superbowl which is being played in Atlanta (GMT − 5) and televised live in Brisbane. The Superbowl is due to begin at 7.00 pm on Sunday, Atlanta time. At what day and time will Neville need to set his video to begin taping?

8. Sydney is GMT + 10.
   a. What is the time in Sydney when it is 12 noon GMT?
   b. When daylight saving time is applied, describe the time zone in Sydney as compared to GMT.
   c. During daylight saving time, what will the time be in Sydney when it is
      i. 4.00 am Monday GMT
      ii. 9.00 pm Thursday GMT?
9. Melbourne is GMT + 10 and Los Angeles is GMT − 8. Calculate the time difference between Melbourne and Los Angeles when
   a. both cities are on standard time
   b. Melbourne has daylight saving time and Los Angeles is on standard time
   c. Los Angeles has daylight saving time and Melbourne is on standard time.

10. **WE10** Beijing is at approximately (40°N, 120°E). Rome is at approximately (40°N, 15°E). Calculate the difference in standard time between Beijing and Rome.

11. Calculate the time difference in standard time between each of the following cities.
    - a. Mumbai (19°N, 73°E) and Casablanca (23°N, 82°W)
    - b. Tokyo (36°N, 140°E) and Adelaide (23°S, 134°E)
    - c. Miami (26°N, 80°W) and Seattle (47°N, 122°W)

12. **MC** At a point on the Earth's surface, the coordinates are (45°N, 135°W). The standard time at this point would be
    A. GMT − 3. **B.** GMT + 3. **C.** GMT − 9. **D.** GMT + 9.

13. **MC** It is 11.00 am Tuesday at a point X, with coordinates (32°S, 90°W). At point Y, with coordinates (51°N, 120°E), what is the time if daylight saving time applies at Y?
    - **A.** 9.00 pm Monday
    - **B.** 10.00 pm Monday
    - **C.** 11.00 pm Tuesday
    - **D.** 2.00 am Wednesday

14. **WE11** A plane leaves Sydney (32°S, 150°E) at 2.00 pm on Tuesday. If it is an 18 hour flight to Los Angeles (33°N, 120°W), at what time will the plane touch down in Los Angeles?

15. A plane leaves Perth (32°S, 120°E) on an 8 hour flight to Cape Town (33°S, 15°E) at 3.00 pm Wednesday.
    - a. At what time will the plane arrive in Cape Town?
    - b. The return flight leaves Cape Town at 5.00 pm Saturday. At what time will it arrive in Perth if the flight time is still 8 hours?

16. A flight leaves Melbourne (40°S, 150°E) at 5.00 pm Tuesday on an 18 hour flight to Frankfurt (50°N, 15°E). Calculate the time of arrival in Frankfurt if it is
    - a. daylight saving time in Melbourne only
    - b. daylight saving time in Frankfurt only.

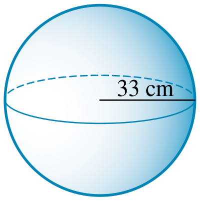
6.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

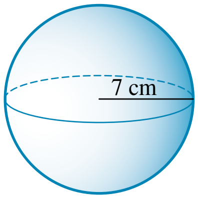
Simple familiar

1. Calculate the circumference of each of the following spheres, rounded to 1 decimal place.

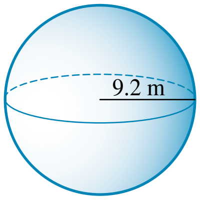
a.



b.



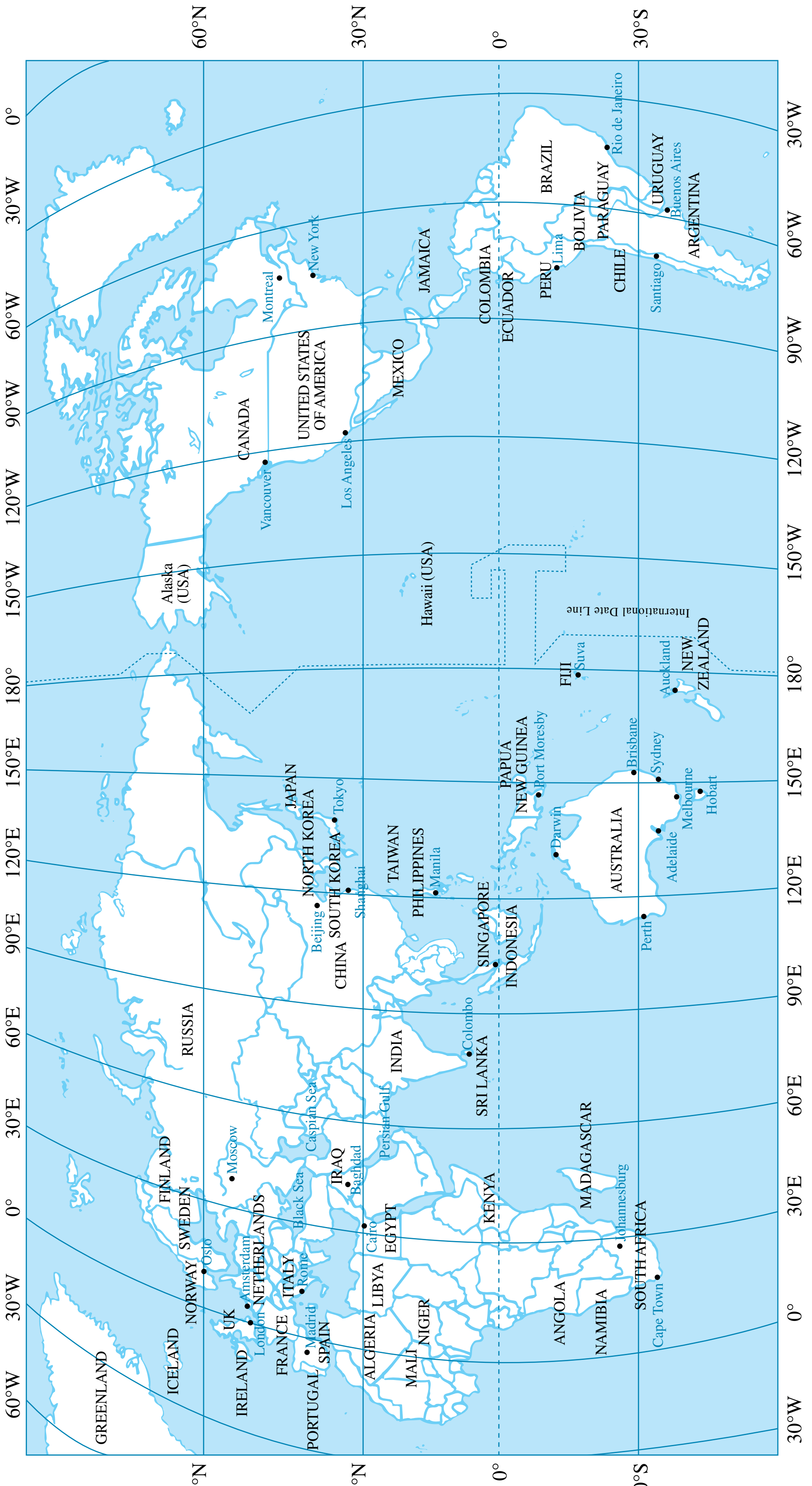
c.



2. Calculate the circumference of a great circle that lies on the surface of a sphere with a radius of 9 km. (Give your answer rounded to 1 decimal place.)

3. A sphere has a diameter of 45 cm. Calculate the distance on a great circle between the poles on this sphere. Give your answer rounded to the nearest centimetre.

4. Use the following world map to identify the cities at each of the following locations.
    - a. (14°N, 121°E)
    - b. (12°S, 76°W)
    - c. (33°S, 71°W)


60°N
30°N
0°
30°S
60°W
30°W
0°
30°E
60°E
90°E
120°E
150°E
180°
150°W
120°W
90°W
GREENLAND
ICELAND
IRELAND
UK
London
NORWAY
Oslo
SWEDEN
FINLAND
Moscow
RUSSIA
Amsterdam
NETHERLANDS
FRANCE
PORTUGAL
Madrid
SPAIN
ITALY
Rome
Black Sea
Caspian Sea
IRAQ
Baghdad
Persian Gulf
ALGERIA
LIBYA
Cairo
EGYPT
MALI
NIGER
KENYA
ANGOLA
NAMIBIA
SOUTH AFRICA
Cape Town
Johannesburg
MADAGASCAR
INDIA
Colombo
SRI LANKA
CHINA
Beijing
NORTH KOREA
SOUTH KOREA
Shanghai
JAPAN
Tokyo
TAIWAN
PHILIPPINES
Manila
SINGAPORE
INDONESIA
PAPUA
NEW GUINEA
Port Moresby
Darwin
AUSTRALIA
Perth
Adelaide
Melbourne
Hobart
Sydney
Brisbane
FIJI
Suva
Auckland
NEW
ZEALAND
International Date Line
Hawaii (USA)
Alaska
(USA)
CANADA
Vancouver
Los Angeles
UNITED STATES
OF AMERICA
Montreal
New York
MEXICO
JAMAICA
COLOMBIA
ECUADOR
PERU
Lima
BRAZIL
BOLIVIA
PARAGUAY
CHILE
Santiago
URUGUAY
Buenos Aires
ARGENTINA
Rio de Janeiro

5. Use the previous world map to give the approximate coordinates of each of the following cities.

a. Madrid **b.** Singapore **c.** Hobart

6. The points X and Y on the Earth's surface have coordinates (32°N, 120°E) and (26°S, 120°E). Calculate the angular distance between X and Y.

7. A ship gives its coordinates as (56°N, 14°W) and is sailing to a port at (40°N, 14°W).

a. Calculate the angular distance through which the ship must sail to reach the port.

b. Calculate the distance the ship must sail, to the nearest kilometre.

8. Santiago has approximate coordinates (33°S, 70°W) while Santo Domingo has approximate coordinates (18°N, 70°W). Calculate the distance between Santiago and Santo Domingo, to the nearest kilometre.

9. Calculate the shortest distance between the following points (to the nearest kilometre).

a. X (40°N, 120°E) and Y (40°N, 70°W)

b. A (23°S, 0°E) and B (23°S, 180°E)

10. Calculate the time difference between each of the following pairs of cities.

a. Sydney (GMT + 10) and Istanbul (GMT + 2)

b. Perth (GMT + 8) and New York (GMT − 3)

c. Ottawa (GMT − 5) and Fiji (GMT + 12)

11. In Dhahran (GMT + 4) the time is 10.00 pm on Wednesday. Calculate the time in Tokyo (GMT + 9).

12. Ann is on a skiing holiday in Winnipeg, Canada (GMT − 6). She needs to call her parents at 7.30 pm on Tuesday night, Brisbane time (GMT + 10). At what time should she make the call in Winnipeg?

Complex familiar

13. Kingston, Jamaica is at approximately (18°N, 75°W) while Oslo, Norway is at approximately (60°N, 15°E). Calculate the time

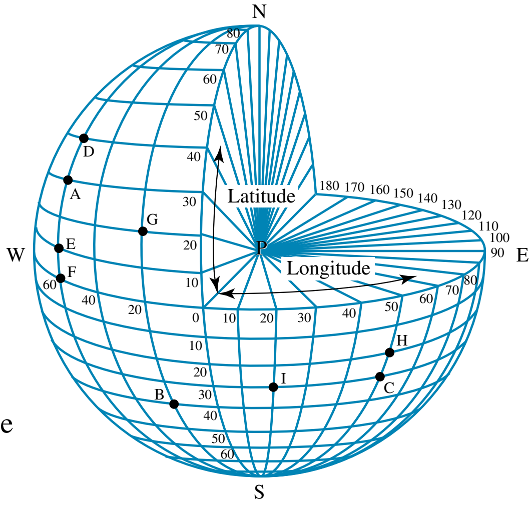
a. in Oslo when it is 5.00 am in Kingston

b. in Kingston when it is 5.00 pm in Oslo.

14. A plane is flying from Munich (48°N, 15°E) to New York (41°N, 75°W). The flight departs Munich at 6.00 pm and takes 7 hours. Calculate the time of arrival in New York.

15. The diagram at right represents the Earth.

- **a.** Give the position of A, B, C and D.
- **b.** Name 3 meridians.
- **c.** Name a point on the equator.
- **d.** If P is the centre of the Earth, give 4 radii.



16. Use the following map of the Whitsunday Group to give the position of

- **a.** Dolphin Point on Hayman Island
- **b.** the entrance to Nara Inlet.

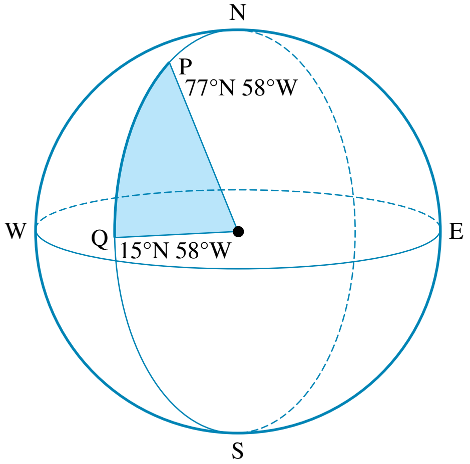
Complex unfamiliar

17. Calculate the distances travelled in the following situations.

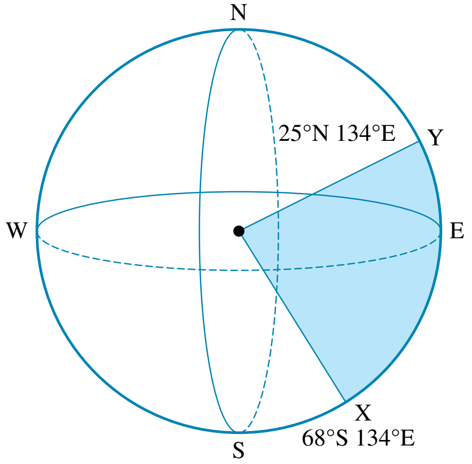
- **a.** An aeroplane travels along the equator from a point of longitude 39°W to 102°E.
- **b.** A ship travels from a point P:11°N, 19°W to point Q:18°S, 19°W.

18. Calculate the great circle distances shown in these figures. Give your answer to the nearest kilometre.

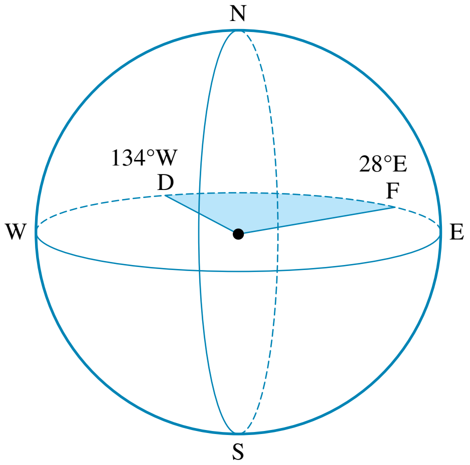
a.



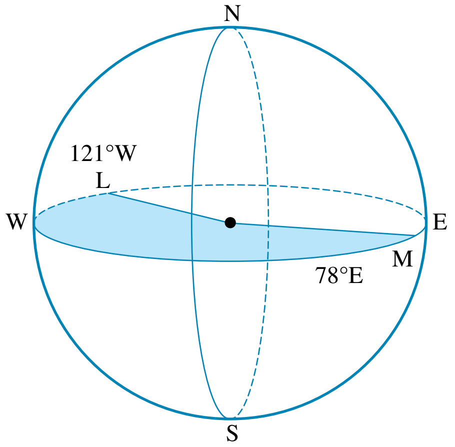
b.



c.



d.



19. The cities of Mecca in Saudi Arabia and Rostov-On-Don in Russia are 2872 km apart. They both lie on the same meridian. What is the angle formed at the centre of the Earth by the two cities?

20. The towns of Yarrawonga and Griffith lie on the same meridian and are 190 km apart. If Griffith lies on a latitude of 34.3°S, find the latitude of Yarrawonga.

studyon

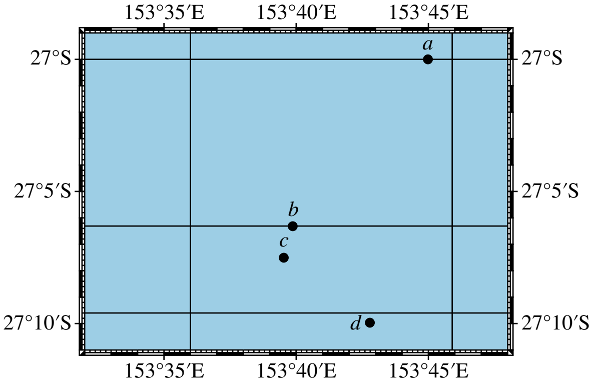
Units 3 & 4 Sit exam

Answers

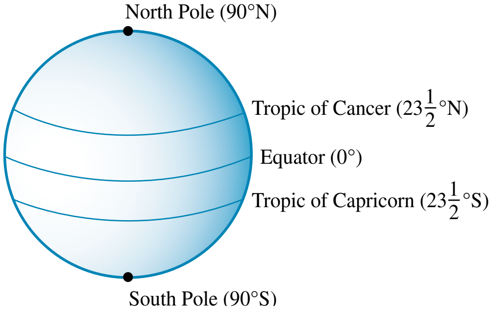
6 Earth geometry and time zones

Exercise 6.2 Latitude and longitude

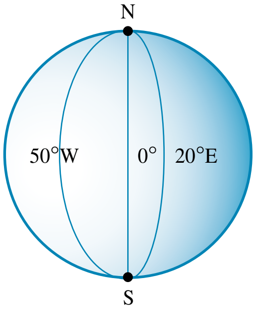
1. 377 cm
2. 17.28 m
3. 18.8 m
4. 45.24 m
5. (27°6.3′S, 153°45.9′E)
6. a. 27°9.6′S, 153°36′E
   b. 27°S, 153°45.9′E
   c. 27°S, 153°36′E
   d. 27°0.9′S, 153°37.6′E
   e. 27°1.1′S, 153°33.6′E
   f. 27°8′S, 153°44.5′E
7.



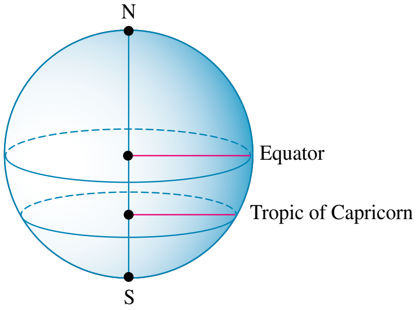
8. a. Mt Sydney
   b. Black Island
   c. Pinnacle Point
9. a. 20°4.3′S, 148°58.3′E
   b. 20°4.8′S, 148°52.2′E
   c. 20°10.5′S, 148°55′E
10. a. 30°N, 30°E Cairo
    b. 30°N, 120°E Shanghai
    c. 15°S, 135°E Darwin
    d. 45°N, 75°W Montreal
    e. 50°N, 0° London
    f. 37°S, 175°E Auckland
    g. 35°N, 140°E Tokyo
    h. 40°N, 115°E Beijing
    i. $22\frac{1}{2}$°S, 43°W Rio de Janeiro
    j. 60°N, 11°E Oslo
11. a. Melbourne (38°S, 145°E)
    b. New York (40°N, 75°W)
    c. Jamaica (18°N, 76°W)
    d. Johannesburg (26°S, 28°E)
    e. Rome (42°N, 12°E)
    f. Buenos Aires (35°S, 57°W)
    g. Baghdad (33°N, 44°E)
    h. Moscow (55°N, 40°E)
    i. Singapore (1°N, 104°E)
    j. Suva (18°S, 178°E)
12.



13.



14.



Exercise 6.3 Distances on the Earth's surface

1. 50°
2. a. 40°
   b. 40°
   c. 71°
   d. 21°
   e. 80°
3. 60°
4. 3336 km
5. a. 3892 km
   b. 15 012 km
   c. 4337 km
   d. 10 675 km
6. 6672 km
7. 5226 km
8. 6600 km
9. a. 4337 km
   b. 4003 km
   c. 3781 km
10. D
11. B

12. **a.** 110°
  b. 12 200 km

13. 20 016 km

14. **a.** 28° **b.** 3114 km

15. 3892 km

16. 17 792 km

17. **a.** 6894 km to South Pole **b.** 4559 km to North Pole
  c. 4893 km to North Pole **d.** 8896 km to South Pole

18. **a.** 2470 km **b.** 3233 km **c.** 2859 km
  d. 425 km **e.** 16 301 km **f.** 6452 km

Exercise 6.4 Time zones

1. 10 h

2. **a.** 14 h **b.** 7 h
  c. 11 h **d.** 22 h

3. 11.00 pm Monday

4. **a.** 8.00 pm
  b. 1.00 pm Friday
  c. 5.00 pm Wednesday
  d. 11.00 pm Tuesday
  e. 3.45 pm Monday

5. 4.00 am Saturday

6. 11.00 pm Tuesday

7. 10.00 am Monday

8. **a.** 10.00 pm, same day
  b. GMT + 11
  c. **i.** 3.00 pm Monday
    ii. 8.00 am Friday

9. **a.** 18 h
  b. 19 h
  c. 17 h

10. 7 h

11. **a.** 10 h 20 min
  b. 24 min
  c. 2 h 48 min

12. C

13. D

14. 2.00 pm Tuesday LAX local time

15. **a.** 4.00 pm Wednesday
  b. 8.00 am Sunday

16. **a.** 1.00 am Wednesday
  b. 3.00 am Wednesday

6.5 Review: exam practice

1. **a.** 207.3 cm
  b. 44.0 cm
  c. 57.8 m

2. 56.5 km

3. 71 cm

4. **a.** Manila
  b. Lima
  c. Santiago

5. **a.** (41°N, 3°W)
  b. (1°N, 104°E)
  c. (43°S, 147°E)

6. 58°

7. **a.** 16°
  b. 1779 km

8. 5671 km

9. **a.** 14 481 km
  b. 18 425 km

10. **a.** 8 h
  b. 11 h
  c. 17 h

11. 3.00 am Thursday

12. 3.30 am Tuesday

13. **a.** 11.00 am the same day
  b. 11.00 am the same day

14. 7.00 pm the same day

15. **a.** A 30°N, 60°W; B 40°S, 20°W;
  C 30°S, 50°E; D 40°N, 60°W
  b. NDS, NGS, NHS or any line of longitude (for example, 40°W)
  c. F
  d. PG, PN, PH, PC etc.

16. **a.** 20°2.2′S, 148°52.7′E
  b. 20°10′S, 148°53.7′E

17. **a.** 15 679 km **b.** 3225 km

18. **a.** 6894 km **b.** 10 341 km
  c. 18 014 km **d.** 22 128 km

19. 25.83°

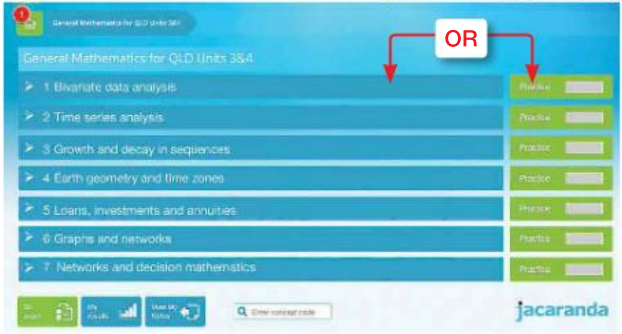
20. 36.0° S

REVISION UNIT 3 Bivariate data, sequences and change, and Earth geometry

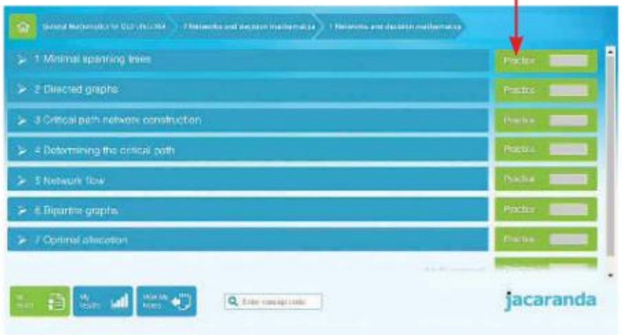
TOPIC 4 Earth geometry and time zones

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *General Mathematics for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.

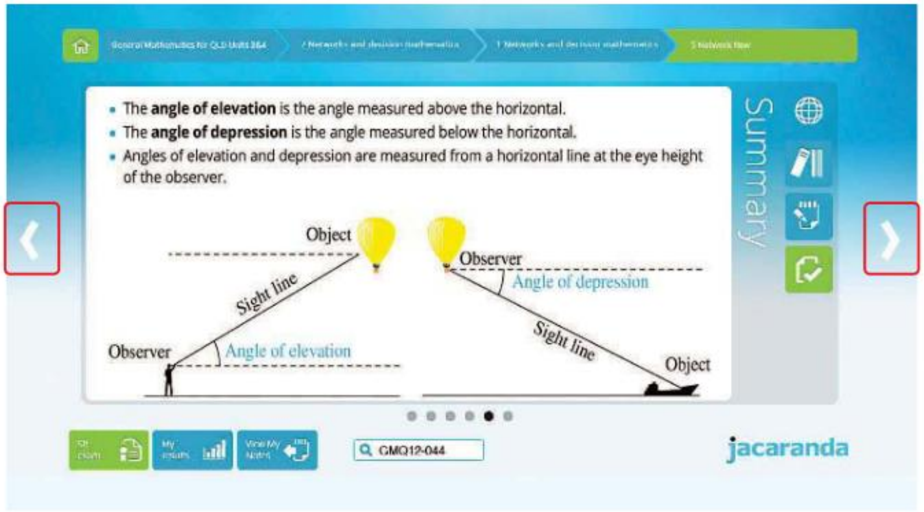


- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



- The **angle of elevation** is the angle measured above the horizontal.
- The **angle of depression** is the angle measured below the horizontal.
- Angles of elevation and depression are measured from a horizontal line at the eye height of the observer.

PRACTICE ASSESSMENT 2

General Mathematics: Unit 3 examination

Unit

Unit 3: Bivariate data, sequences and change, and Earth geometry

Topic

Topic 1: Bivariate data analysis
Topic 2: Time series analysis
Topic 3: Growth and decay in sequences
Topic 4: Earth geometry and time zones

Conditions

<table>
<tr><th>Response Type</th><th>Duration</th><th>Reading</th></tr>
<tr><td>Short response</td><td>120 minutes</td><td>5 minutes</td></tr>
<tr><th>Resources</th><th colspan="2">Instructions</th></tr>
<tr><td>Only the QCAA formula sheet must be provided:
Notes not permitted
Scientific calculator permitted</td><td colspan="2">Show all working.
Write responses using a black or blue pen.
Unless otherwise instructed, give answers to two decimal places.</td></tr>
<tr><th colspan="3">Other</th></tr>
<tr><td colspan="3">Combined paper consisting of simple familiar, complex familiar and complex unfamiliar questions</td></tr>
</table>

Criterion	Marks allocated	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5, and 6	75	

* © State of Queensland (Queensland Curriculum & Assessment Authority), *General Mathematics General Senior Syllabus 2019 v1.2,* Brisbane.
For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

Part A: Simple familiar — total marks 45

Question 1 (7 marks)

A survey was conducted by a company commissioned to determine how voters would vote at the next election based on their gender. The results are compiled in the following incomplete table.

Preferred party	Female	Male	Total
Labor	25	A	47
Liberal	B	C	D
Total	51	E	92

a. Determine the values of A, B, C, D and E.

b. Calculate the percentage of females surveyed who indicated that they would vote Labor at the next election. (Give your answer correct to two decimal places.)

Question 2 (4 marks)

A study to determine the relationship between the height of a group of male basketballers and the length of their arm span revealed the following details.

- Mean height of the men: 250 cm
- Mean length of the arm span: 185 cm
- Standard deviation of the height of the men: 5.3 cm
- Standard deviation of the arm span: 4.8 cm
- Correction coefficient, $r = 0.87$

The form of the least-squares regression line is to be:

$$\text{Arm span} = b \times \text{height} + a$$

a. Which variable is the response variable?

b. Calculate the value of b for the regression line (to 2 decimal places).

c. Calculate the value of a for the regression line (to 2 decimal places).

d. Use the equation of the regression line to predict the arm span of a basketballer whose height is 196 cm tall (to nearest cm).

Question 3 (5 marks)

An online homewares company determines that there has been a steady upward trend in the number of online sales made for the previous 12 months. The equation that models the long-term trend in these sales figures is:
$y = 2100 + 120x$

a. If the explanatory variable is 'number of months' and the response variable is 'sales', rewrite the equation using the appropriately named variables.

b. For this linear equation what do the gradient and y-intercept represent?

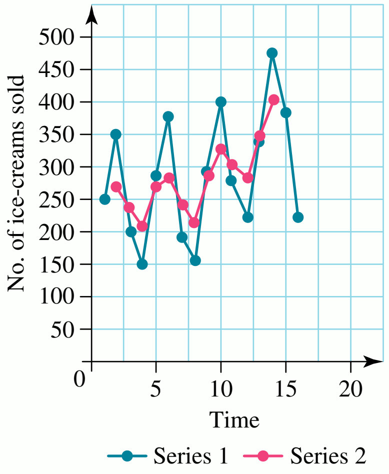
c. Predict the number of sales for this company in 6 months. How reliable is this prediction?

Question 4 (3 marks)

An ice-cream van keeps a record of the number of ice-creams ordered for each of the four seasons for four years. The data is summarised in the following table.

Year	Spring	Summer	Autumn	Winter
2016	250	350	200	150
2017	280	370	190	160
2018	290	400	280	220
2019	340	470	380	220

The following graph shows the original data and the 3-point smoothed data.



a. Which time point represents spring 2019?

b. Which of the following descriptions best describes the data?

A. Upward trending with irregular fluctuations

B. Upward trending with seasonal fluctuations

C. No trend but seasonality present

D. Irregular fluctuations only

c. The last data value for the smoothed data has been left off the graph. Calculate the value of this last smoothed point, giving your answer to the nearest whole number.

Question 5 (3 marks)

Luca is saving some of the money he earns working at a bakery on Saturday mornings. He already has $\$1200$ in his savings account and each week adds another $\$75$ to the account.

a. Write Luca's savings, L_n after n weeks, as a recursive relationship.

b. How much money will be in Luca's account after 6 weeks?

c. Is this a model of linear growth or decay?

Question 6 (2 marks)

A geometric sequence is described by the recursive relation:
$t_1 = -3.02, t_{n+1} = t_n \times 2.6$. Use a scientific calculator to generate the first 5 terms of this sequence.

Question 7 (3 marks)

The following table shows the depreciating value of a truck.

Age (years)	Value ($)
0	160 000
1	140 000
2	120 000
3	100 000
4	80 000
5	60 000

a. Draw a graph of the value of the truck against its age.

b. Write a rule for the value (V_n) of the truck at n years of age.

Question 8 (3 marks)

The population of a town is decreasing by 12% each year. If the population of the town at the beginning of the first year is $32\,000$:

a. write an expression for the population, P_n, of the town after n years.

b. calculate the population of the town after 5 years.

Question 9 (3 marks)

A luxury car that was purchased for $\$95\,000$ depreciates at a rate of 20% p.a.

a. Set up a recurrence relation to describe this situation.

b. By calculating the value at the end of each year, determine the future value of the car after 4 years.

Question 10 (6 marks)

Calculate the distance on the Earth's surface between each of the following points, correct to the nearest kilometre.

a. $\mathrm{M}\,(35°\mathrm{S}, 72°\mathrm{E})$ and $\mathrm{N}\,(54°\mathrm{S}, 72°\mathrm{E})$

b. $\mathrm{P}\,(28°\mathrm{N}, 0°)$ and $\mathrm{Q}\,(18°\mathrm{S}, 0°)$

c. $\mathrm{R}\,(0°, 17°\mathrm{W})$ and $\mathrm{S}\,(42°\mathrm{S}, 17°\mathrm{W})$

Question 11 (6 marks)

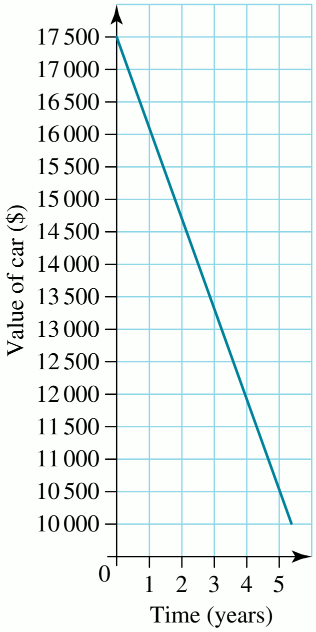
a. An aircraft flies from city A, $30°\mathrm{N}, 75°\mathrm{W}$, to city B, $30°\mathrm{N}, 90°\mathrm{W}$. How far, correct to the nearest kilometre, has it flown along the parallel of latitude?

b. From city B, it then flies to city C, $42°\mathrm{N}, 90°\mathrm{W}$ Calculate the total distance, correct to the nearest kilometre, travelled by the aircraft.

Part B: Complex familiar (total marks — 15)

Question 12 (6 marks)

Amanda's car depreciates in value each year with straight-line depreciation. The following graph shows the value of the car over time.



a. Using the graph, what was the value of Amanda's car after 5 years?

b. Show that the straight line depreciation percentage is 8%.

c. After how many years will the value of Amanda's car's first fall below $\$8000$?

d. If Amanda's car depreciates by a set percentage of 8% each year, explain why after the first year her car's value will be the same as with the straight-line depreciation, but that in future years the straight-line depreciation will decrease the car value by more than the set percentage depreciation.

Question 13 (6 marks)

Nathan runs a large nursery, where he grows and sells various native Australian as well as imported plants. In early spring Nathan plants the seeds of various herbs to be sold when they reach a particular height. At a certain time a coriander seedling is 1.5 cm high and grows at a rate of 2.2 cm per week.

a. Write a recurrence relation to describe this situation.

b. Use the recurrence relation to generate the sequence of the first 5 terms.

c. The coriander seedlings can be sold when they are over 15 cm tall. After how many weeks can Nathan sell these seedlings?

Question 14 (3 marks)

In his nursery Nathan grows and sells bonsai trees. The price of a bonsai tree depends on its age and is given by the equation:

$P_n = 22 \times (1.3)^{n-1}$, $n = 1, 2, 3, ...$, where P is the price (in dollars) and n is the age (in years).

a. Determine the selling price of a 3-year-old bonsai tree.

b. What is the minimum age of a bonsai tree if it costs over $\$100$?

Part C: Complex unfamiliar — total marks 15

Question 15 (5 marks)

38 women and 42 men were asked whether they preferred to attend classes or personal training sessions in their fitness club. Out of those surveyed, 24 women preferred classes, while 30 men preferred personal training sessions.

To investigate the relationship between gender and training option preference, a two-way table can be constructed.

a. Explain why a two-way table is an appropriate means of displaying the data.

b. Name the explanatory and response variables.

c. Construct a contingency (two-way) table showing both frequencies and percentage frequencies correct to 2 decimal places.

d. Does the data support the statement, 'There is a relationship between the variables gender and training option preference'? Justify your answer.

Question 16 (10 marks)

A group of Year 12 students were asked how many hours they spent studying for the most recent General Maths test and the mark they obtained on that test. It was then established that the mean study time was 4.5 hours with a standard deviation of 1.2 hours. The average test mark was 63.2, while the standard deviation of the marks was 8.1. The value of Pearson's correlation coefficient was found to be 0.892.

To investigate the relationship between time spent on studying for the test and the test marks, it was decided that the equation of the least-squares regression line must be calculated.

a. What assumption has been made prior to calculating the equation of the least-squares regression line?

b. Which variable, study time or test mark, is the response variable?

c. Determine the equation of the least-squares regression line. Write the equation using appropriate variables, giving all coefficients correct to 2 decimal places.

d. Calculate the value of the coefficient of determination correct to 4 decimal places. Interpret the value of the coefficient.

e. Use the equation of the least-squares regression line to predict the test mark for Nathan, who spent 5 hours studying for the test.

f. If in fact Nathan's test mark was 92, calculate the residual and explain its meaning

7 Compound interest loans and investments

7.1 Overview

Most people will at some stage borrow and invest money. When a person enters the workforce, they may want or need a car or a house or other goods and do not have the money to purchase these items outright. They then borrow the money, believing that they have the means to service the loan and pay the money back over a specified time. Investing money helps it grow independently which is how people can plan for home loan deposits and retirement. Given the likelihood that every individual will need to wrestle with decisions pertaining to borrowing and investing money throughout their adult life, it is essential that everyone has a competent understanding of financial mathematics. This will ensure that all their options have been thoroughly investigated and that their decision making is therefore informed.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au

7.2 Modelling a compound interest loan or investment using a recurrence relation

7.2.1 Recurrence relations

In chapter 5, compound interest was described as a recurrence relation, where the amount invested or borrowed follows a geometric sequence for each consecutive payment period.

The amount

Let the starting amount be A_n. Then the amount at the start of the next compounding period is A_{n+1}. The recurrence relation rule for this sequence will be:

$$A_{n+1} = rA_n$$

where A_n is the value of the investment or loan after n compounding periods
A_{n+1} is the value of the investment or loan after a time period after A_n
r is $1 + \dfrac{\text{interest rate per period}}{100}$.

Consider an investment of \$2500 invested for 3 years at an interest rate of 5.5% p.a. compounded annually. Use the recurrence relation $A_{n+1} = 1.055A_n$, $A_0 = 2500$, to calculate the amount in the account at the end of each year.

A_n (\$)	Calculation	A_{n+1} (\$)
2500	2500×1.055	2637.50
2637.50	2637.50×1.055	2782.56
2782.56	2782.56×1.055	2935.60

WORKED EXAMPLE 1

A person invests \$5000 in a financial institution at 6% per annum compounded annually.

a. Using A_0 to represent the initial investment, write this investment as a recurrence relation.

b. Complete the table to show the growth of the investment over 3 years.

A_n (\$)	Calculation	A_{n+1} (\$)
5000		

c. How much money was in the account at the end of 3 years?

d. How much interest was made in 3 years?

THINK	WRITE
a. 1. Write the recurrence relation rule.	a. $A_{n+1} = rA_n$

2. Write the rule for r.

$r = 1 + \frac{\text{interest rate per period}}{100} = 1 + \frac{6}{100} = 1.06$

3. Substitute the values for A_0 and r.

$A_{n+1} = 1.06\,A_n,\ A_0 = 5000$

b. Apply the recurrence relation rule.

b.

A_n (\$)	Calculation	A_{n+1} (\$)
5000	1.06 × 5000	5300
5300	1.06 × 5300	5618
5618	1.06 × 5618	5955.08

c. Write the answer.

c. There was \$5955.08 in the account at the end of 3 years.

d. 1. Subtract the initial investment from the total amount at the end of 3 years.

d. Interest = Total amount − Initial investment

$5955.08 - 5000 = 955.08$

2. Write the answer.

The interest is \$955.08

7.2.2 The effect of changing the compounding period

WORKED EXAMPLE 2

The person in Worked example 1 invests \$5000 in a different financial institution at 6% per annum compounded monthly.

a. Using A_0 to represent the initial investment, write this investment as a recurrence relation.

b. Using your calculator and the iterative method, calculate the amount in the account after 1 year.

c. How much interest was made in 1 year?

d. How much more interest was made in one year when the interest was compounded monthly instead of each year?

THINK	WRITE
a. 1. Write the recurrence relation rule.	**a.** $A_{n+1} = rA_n$
2. Write the rule for r.	The interest rate per period is $\frac{6}{12} = 0.5\%$. $r = 1 + \frac{\text{interest rate per period}}{100} = 1 + \frac{0.5}{100} = 1.005$
3. Substitute the values for A_0 and r.	$A_{n+1} = 1.005\,A_n,\ A_0 = 5000$
b. On your calculator screen, type 5000, then press Enter. Type × 1.005, then press Enter 12 times.	**b.** 5000 5000 × 1.005 5025 5025 × 1.005 5050.13 5050.13 × 1.005 5075.38 ⋮ ⋮ 5281.98 × 1.005 5308.39
Write the answer.	There was \$5308.39 in the account at the end of the first year.

c. 1. Subtract the initial investment from the total amount at the end of 1 year.	**c.** Interest = Total amount − initial investment $5308.39 - 5000 = 308.39$
2. Write the answer.	The interest is $308.39
d. In Worked example 1, the interest after 1 year was $300	**d.** When the interest was compounded monthly instead of annually, the amount of interest increased by $308.39 − $300 = $8.39.

Units 3 & 4 > Area 5 > Sequence 1 > Concept 1

Modelling a compound interest loan or investment using recursion Summary screen and practice questions

Exercise 7.2 Modelling a compound interest loan or investment using a recurrence relation

1. A person invests $8500 in a financial institution at 12% per annum compounded annually.

a. Using A_0, to represent the initial investment, write this investment as a recurrence relation.

b. Complete the following table to show the growth of the investment over 3 years.

A_n ($)	Calculation	A_{n+1} ($)
8500		

c. How much money was in the account at the end of the 3 years?

d. How much interest was made in 3 years?

2. The same person as in question **1** invests $8500 in a different financial institution at 12% per annum compounded monthly.

a. Using A_0 to represent the initial investment, write this investment as a recurrence relation.

b. Using your calculator and the iterative method, calculate the amount in the account after 1 year.

c. How much interest was made in 1 year?

d. How much more interest was made in one year when the interest was compounded monthly instead of each year?

3. Sally invested $2575 in a savings account at an interest rate of 8.25% p.a. compounded every 6 months.

a. Using A_0, to represent the initial investment, write this investment as a recurrence relation.

b. What is the total value of Sally's investment at the end of 3 years?

4. Su Fen was left an inheritance by her grandmother of $34 000. She decided to invest it for 18 months in a bank account that offered 8% p.a. compounded monthly.

a. Using A_0 to represent the initial investment, write this investment as a recurrence relation.

b. How much interest did Su Fen earn on her investment in 5 months?

5. **MC** A person invested \$1500 at 5% p.a. compounded annually. The amount of interest, correct to the nearest cent, earned on the investment at the end of 3 years is closest to

A. \$225. **B.** \$236. **C.** \$1575. **D.** \$1736.

The following information relates to questions 6–10.

Josh borrowed money from a finance company that offered a 'No deposit, no payment' for two years option. The interest accrued was compounded annually. The amount he owed after n years, A_n, can be modelled by the recurrence relation:

$$A_{n+1} = 1.08A_n, \; A_0 = 6000$$

6. **MC** The annual percentage compound interest rate for this account is

A. 1.08%. **B.** 8%. **C.** 80%. **D.** 92%.

7. **MC** The interest earned in the first year is

A. \$480. **B.** \$6480. **C.** \$4800. **D.** \$5520.

8. **MC** The balance at the end of the first year is

A. \$480. **B.** \$6480. **C.** \$4800. **D.** \$5520.

9. **MC** The principal at the start of the second year is

A. \$6000. **B.** \$5480. **C.** \$480. **D.** \$6480.

10. **MC** The interest earned during the second year is closest to

A. \$600. **B.** \$480. **C.** \$518. **D.** \$318.

11. **MC** If \$11000 is invested for 5 years at 7% p.a. with interest compounding annually and the amount in the account at the end of the third year is \$13475.47, then the interest earned in the fourth year is closest to

A. \$943. **B.** \$625. **C.** \$650. **D.** \$677.

12. Julia invested \$10000 for 4 years at an interest rate of 6% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

A_n (\$)	Interest (\$)	A_{n+1} (\$)
10000	A% of 10 000 = 600	10 600
B	6% of C = 636	D
11236	6% of 11 236 = 674.16	11 910.16
11910.16	6% of 11 910.16 = 714.61	E
F	6% of 12 624.77 = 757.49	13 382.26

13. Owen invested \$9500 for 4 years at an interest rate of 5.4% p.a. with interest compounded annually. Complete the table by calculating the values A, B, C, D, E and F.

A_n (\$)	Interest (\$)	A_{n+1} (\$)
9500	5.4% of 9500 = 513	10 013
10013	5.4% of 10 013 = A	B
C	5.4% of 10 553.70 = 569.90	11 123.60
11123.60	5.4% of 11 123.60 = 600.67	D
E	5.4% of 11 724.27 = 633.11	F

14. Ahmed invested \$2700 for 4 years at an interest rate of 4.5% p.a. with interest compounded monthly. Complete the table shown to determine the value of the investment at the end of 5 months.

A_n (\$)	A_{n+1} (\$)
2700	$1.00375 \times 2700 = 2710.13$
2710.13	

15. Owen has some medical bills, totalling \$300, that must be paid but he has no access to any money or credit. He decides to apply for a payday loan from a lending company, PayNow. The company charges 10% interest per week. Owen intends to pay back the loan and interest accrued when he receives his pay in 4 weeks.

a. Using A_0, A_n and A_{n+1}, write a recurrence relation that models this situation where n is the number of weeks after the due date.

b. How much interest will Owen need to pay if he pays the full amount on his next pay day?

16. A person invests \$2000 in a financial institution at 4.5% p.a. compounded quarterly. Calculate the value of the investment after 2 years. Write your answer correct to the nearest cent.

7.3 Effective annual rate of interest

7.3.1 Effective annual rate of interest

A financial institution may advertise a **nominal interest rate** at 3.5% p.a. compounding monthly. In reality, the investor would not receive 3.5% per year on the money invested because the interest is compounding. The actual percentage return on the investment is more than 3.5% per year because the interest calculated each consecutive period is based on a higher principal. The **effective interest rate** or actual interest rate represents the actual percentage return per year on an investment. It could also be considered as the simple interest rate that would produce the same return as the nominated compound interest rate.

The same applies to the cost of a loan. Money offered at 5% p.a. compounding quarterly will be at a higher interest rate than 5% due to the compounding factor.

Effective interest rate

$$i_{effective} = \left(1 + \frac{i}{n}\right)^n - 1$$

where $i_{effective}$ is the effective annual interest rate
i is the nominal rate, as a decimal
n is the number of compounding periods per year.

WORKED EXAMPLE 3

An interest rate is quoted at 3.5% p.a. compounding monthly. What is the effective interest rate? Give your answer correct to two decimal places.

THINK	**WRITE**
1. Write the effective interest rate formula.	$i_{effective} = \left(1 + \frac{i}{n}\right)^n - 1$
2. Write the values for *i* and *n*.	$i = 0.035, n = 12$
3. Substitute the values into the formula and calculate.	$i_{effective} = \left(1 + \frac{0.035}{12}\right)^{12} - 1 = 0.035567$
4. Write the answer.	The effective interest rate is 3.56%.

The interest rate in Worked example 3 is 3.5% p.a. compounding monthly, which is equivalent to a simple interest rate of 3.56% p.a. The best way to compare interest rates is to determine the effective interest rate in each case.

WORKED EXAMPLE 4

Jack has the choice of investing money in three different funds.

1. **3.9% compounding quarterly**
2. **3.95% p.a. simple interest**
3. **3.85% p.a. compounding daily.**

Which investment would provide him with the greatest return?

THINK	**WRITE**
1. Calculate the effective interest rate for each option. Write the effective interest rate formula.	$i_{effective} = \left(1 + \frac{i}{n}\right)^n - 1$
2. Write the values for *i* and *n* for option 1	$i = 0.039, n = 4$
3. Substitute the values into the formula and calculate.	$i_{effective} = \left(1 + \frac{0.039}{4}\right)^{4} - 1$ $= 0.039574$ $= 3.96\%$ p.a.
4. The simple interest rate is the effective interest rate.	The effective interest rate for option 2 is 3.95% p.a.
5. Write the values for *i* and *n* for option 3	$i = 0.0385, n = 365$
6. Substitute the values into the formula and calculate.	$i_{effective} = \left(1 + \frac{0.0385}{365}\right)^{365} - 1$ $= 0.039249$ $= 3.92\%$ p.a.

7. Compare the interest rates and write the answer.	The effective interest rate for option 1 is 3.96% p.a. The effective interest rate for option 2 is 3.95% p.a. The effective interest rate for option 3 is 3.92% p.a. The option that would provide Jack with the greatest return is 3.9% p.a. compounding quarterly.

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Units 3 & 4 > Area 5 > Sequence 1 > Concept 2

Effective annual rate of interest Summary screen and practice questions

Exercise 7.3 Effective annual rate of interest

1. Tamara's bank offered her an interest rate of 4% p.a. compounding quarterly. What is the equivalent effective interest rate?

2. If the bank offered Tamara the same interest rate but compounded daily, what effective interest rate would this represent?

3. What simple interest rate would be equivalent to a rate of 3.75% p.a. compounding 6-monthly?

4. Which would provide the better return as an investment?
  a. 4.2% p.a. compounding quarterly
  b. 4.175% p.a. compounding monthly

5. Patrick was offered an investment rate of 4.97% p.a. compounding daily or 5% p.a. compounding monthly. Which should he choose?

6. An interest rate of 4.5% p.a. compounding quarterly is equivalent to what effective interest rate?

7. Marilyn can invest in two different funds:
  a. 6% p.a. simple interest
  b. 5.75% p.a. compounding monthly.

She is inclined to choose the 6% p.a. simple interest because this is a higher value. Is this a wise choice? Explain your answer.

The following information relates to questions 8–10.

William takes a personal loan of $14 000 to buy a car. The interest rate is 9% p.a. compounding monthly. The loan is to be paid back in 2 years.

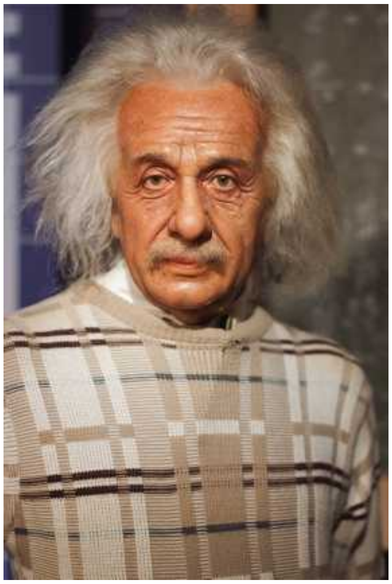
8. **MC** The total interest that William will pay on this loan is closest to
A. $2520.00. **B.** $2749.80. **C.** $3244.51. **D.** $3978.03.

9. The monthly instalment is closest to
A. $700. **B.** $688. **C.** $514. **D.** $499.

10. The effective annual interest rate is
A. 8.11%. **B.** 8.67%. **C.** 9%. **D.** 9.38%.

7.4 Compound interest problems: future values and present values

7.4.1 The effect of compound interest



The effect of compounding interest (which oil billionaire J. P. Getty called the 'eighth wonder in the world' and theoretical physicist Albert Einstein below described as 'the driving force of the Universe') is a secret of financial wealth creation.

7.4.2 The compound interest rule

We have seen in previous sections that we can write the value of a compounding investment as a recurrence relation. We can also use a compound interest rule to solve problems involving compound interest loans and investments.

Calculating compound interest

$$A = P(1+i)^n$$

where

A = future value of the investment ($)

P = present value (or principal) of the investment ($)

$$i = \frac{r}{100}, r = \frac{\text{annual interest rate}}{\text{number of compounding periods per year}}$$

n = number of compounding periods.

To find the amount of interest earned, the principal value is subtracted from the future value.

$$\mathrm{I} = \mathrm{A} - \mathrm{P}$$

WORKED EXAMPLE 5

Use the compound interest formula to calculate the future value of $5000 invested at 4% p.a. compounded annually for 5 years. Write your answer correct to the nearest cent.

THINK	WRITE
1. Determine the number of compounding periods, n.	$n = 1 \times 5 = 5$
2. Determine r and hence i.	$r = 4, \ i = 0.04$
3. Substitute the values P, n and i into the compound interest formula.	$A = P(1+i)^n$ $= 5000\,(1+0.04)^5$ $= 5000\,(1.04)^5$ $= 6083.26$
4. Answer the question.	The future value of the investment is $6083.26.

on Resources

Interactivity Non-annual compounding (int-6462)

WORKED EXAMPLE 6

\$4500 is invested for 3 years at 6% p.a. compounded monthly. Using the compound interest formula, determine the future value of the investment. Write your answer correct to the nearest cent.

THINK	WRITE
1. Determine the number of compounding periods, n.	$n = 12 \times 3 = 36$
2. Determine r and hence i.	$r = \frac{6}{12} = 0.5,\ i = \frac{0.5}{100} = 0.005$
3. Substitute the values P, n and i into the compound interest formula.	$A = P\,(1 + i)^n$ $= 4500\,(1 + 0.005)^{36}$ $= 4500\,(1.005)^{36}$ $= 5385.06$
4. Answer the question.	The future value of the investment is \$5385.06.

WORKED EXAMPLE 7

Use the compound interest formula to find the present value required to yield a future value of \$6842.85 after compounding every 6 months at a rate of 8% per annum for 4 years. Write your answer correct to the nearest dollar.

THINK	WRITE
1. Determine the number of compounding periods, n.	$n = 2 \times 4 = 8$
2. Determine r and hence i.	$r = \frac{8}{2} = 4$ $i = \frac{4}{100} = 0.04$
3. Substitute the values A, n and i into the compound interest formula and solve for P.	$A = P\,(1 + 0.04)^n$ $6842.85 = P\,(1 + 0.04)^8$ $6842.85 = P \times 1.04^8$ $6842.85 = P \times 1.368...$ $\frac{6842.85}{1.368...} = P$ $5000.00 = P$
4. Answer the question.	The present value of the investment is \$5000.00.

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Units 3 & 4 > Area 5 > Sequence 1 > Concept 3

Applications of compound interest Summary screen and practice questions

Exercise 7.4 Compound interest problems: future values and present values

1. **WE5** Use the compound interest formula to calculate the future value of each of the following investments with interest compounded annually. Write your answers correct to the nearest cent.
   a. \$4000 at 5% p.a. for 3 years
   b. \$8000 at 3% p.a. for 5 years
   c. \$18 000 at 8% p.a. for 4 years
   d. \$11 500 at 5.5% p.a. for 3 years
2. Use the compound interest formula to calculate the amount of compound interest earned on the following investments with interest rates compounded annually for the times shown. Write your answers correct to the nearest cent.
   a. \$4655 at 4.55% p.a. for 3 years
   b. \$12 344 at 6.35% p.a. for 6 years
   c. \$3465 at 2.015% p.a. for 8 years
   d. \$365 000 at 7.65% p.a. for 20 years
3. **WE6** A principal of \$3800 is compounded monthly at 3% per annum for 4 years. Use the compound interest formula to determine the future value of the investment. Write your answer correct to the nearest cent.
4. Calculate the future value of each of the following investments.
   a. \$960 for 1 year at 4.50% p.a. with interest compounded six monthly
   b. \$7500 for $3\frac{1}{2}$ years at 5.6% p.a. with interest compounded quarterly
   c. \$152 000 for $2\frac{1}{2}$ years at 7.2% p.a. with interest compounded six monthly
   d. \$14 000 for 4 years at 9% p.a. with interest compounded monthly
5. **MC** A sum of \$5000 is invested for 2 years at the rate of 4.75% p.a., compounded quarterly. The interest paid on this investment, to the nearest dollar, is
   A. \$475.
   B. \$495.
   C. \$1900.
   D. \$5475.
6. **WE7** Using the compound interest formula calculate the present value or principal required to yield a future value of the following. Write your answers correct to 2 decimal places with interest compounding annually.
   a. \$15 000 after compounding at a rate of 5.25% p.a. for 8 years
   b. \$22 500 after compounding at a rate of 7.15% p.a. for 10 years
   c. \$1000 after compounding at a rate of 1.25% p.a. for 2 years
7. Carla is to invest \$45 000 at 9.2% p.a. for 5 years with interest compounded six-monthly. Calculate the future value of the investment. Write your answer correct to the nearest cent.
8. Vicky invests \$30 000 in a one-year fixed deposit at an interest rate of 6% p.a., with interest compounding monthly. Calculate the future value of the investment upon maturity. Write your answer correct to the nearest cent.
9. **MC** If \$14 000 is invested for $3\frac{1}{2}$ years at 5.75%, compounding fortnightly, the amount of interest that will accrue will be closest to
   A. \$8435.
   B. \$2113.
   C. \$3117.
   D. \$17 117.
10. \$1500 is invested for 2 years into an account paying 8% p.a. Calculate the balance in the account if
    a. interest is compounded yearly
    b. interest is compounded quarterly
    c. interest is compounded monthly
    d. interest is compounded weekly.
    e. Compare your answers to parts a–d.
11. Calculate the interest generated from an account that pays compound interest at a nominal rate of
    a. 7% p.a. if \$2600 is invested for 3 years (compounded monthly)
    b. 8% p.a. if \$3500 is invested for 4 years (compounded monthly)

   - **c.** 11% p.a. if $960 is invested for $5\frac{1}{2}$ years (compounded fortnightly)
   - **d.** 7.3% p.a. if $2370 is invested for 5 years (compounded weekly)
   - **e.** 15.25% p.a. if $4605 is invested for 2 years (compounded daily).

12. Determine the present value or principal that will grow to $15 600 in 4 years, if interest is added quarterly at 8.5%.

13. Calculate the present value or principal that will grow to:
   - **a.** $3000 in 4 years, if interest is compounded 6 monthly at 9.5% p.a.
   - **b.** $2000 in 3 years, if interest is compounded quarterly at 9% p.a.
   - **c.** $5600 in $5\frac{1}{4}$ years, if interest is compounded quarterly at 8.7% p.a.
   - **d.** $10 000 in $4\frac{1}{4}$ years, if interest is compounded monthly at 15% p.a.

7.5 Compound interest problems: interest rates and number of compounding periods

7.5.1 Using technology to compare the effect on investments of changing of interest rates

A graph of an investment for different compound interest rates can assist in comparing the effect changing interest rates can have on a loan or investment.

For compound interest graphs, the interest rate affects the slope of the curve. An increase in interest rates increases the value of the investment, which causes the curve to move away from the horizontal axis more quickly. This is an example of exponential growth.

WORKED EXAMPLE 8

Using technology, show on one set of axes how each of the following compound interest rates affect the future value of $4000 compounding yearly for 10 years. Hence, describe the effect increasing the compound interest rate has on the investment, using calculations.

a. 4% **b. 6%** **c. 8%** **d. 10%**

THINK

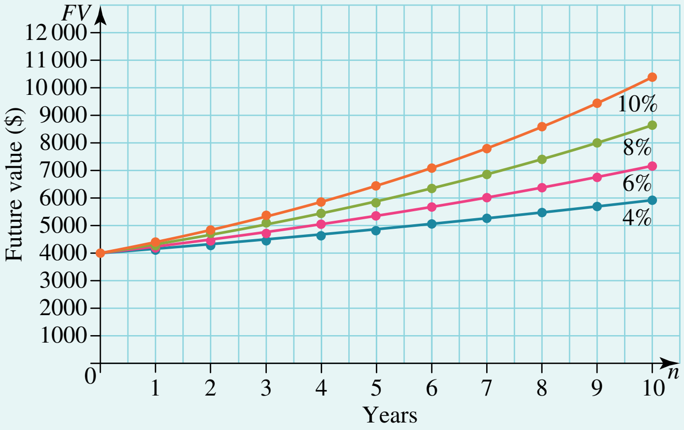
WRITE

1. Using the technology of your choice, insert a formula to calculate the future value of $4000 each year for 10 years for each of the interest rates.

	A	B	C	D	E
1	Year	4%	6%	8%	10%
2	0	4000	4000	4000	4000
3					
4	Formula	= 1.04*b2	= 1.06*c2	= 1.08*d2	= 1.1*e2

2. Determine the future value of $4000 each year for 10 years for each of the compound interest rates.

	A	B	C	D	E
1	Year	4%	6%	8%	10%
2	0	4000.00	4000.00	4000.00	4000.00
3	1	4160.00	4240.00	4320.00	4400.00
4	2	4326.40	4494.40	4665.60	4840.00
5	3	4499.46	4764.06	5038.85	5324.00
6	4	4679.43	5049.91	5441.96	5856.40
7	5	4866.61	5352.90	5877.31	6442.04
8	6	5061.28	5674.08	6347.50	7086.24
9	9	5263.73	6014.52	6855.30	7794.87
10	8	5474.28	6375.39	7403.72	8574.36
11	9	5693.25	6757.92	7996.02	9431.79
12	10	5920.98	7163.39	8635.70	10374.97

THINK	WRITE
3. Using the values from step **2**, construct a graph.	Effect of compound interest on \$4 000 
4. Answer the question.	Increasing the compound interest rate has the effect of increasing the future value of the investment each year. This is shown on the graph as the curves becoming steeper.

7.5.2 Using the compound interest formula to determine the interest rate

The compound interest formula $A = P(1 + i)^n$ can be used to determine the interest rate required to reach a certain future value in a set number of years.

WORKED EXAMPLE 9

Calculate the interest rate per annum required that would enable an investment of \$4000 to grow to \$4200 over 3 years if the interest is compounded every 6 months. Give your answer correct to two decimal places.

THINK	WRITE
1. Write the values of A, P and n.	$A = 4200$ $P = 4000$ $n = 3 \times 2 = 6$
2. Write the compound interest rule and substitute the known values.	$A = P(1 + i)^n$ $4200 = 4000(1 + i)^6$
3. Rearrange the equation.	$4200 = 4000(1 + i)^6$ $\frac{42}{40} = (1 + i)^6$ $(1 + i)^6 = 1.05$
4. Use the $\sqrt[y]{x}$ button on your calculator to determine the value of i.	$(1 + i) = \sqrt[6]{1.05}$ $i = \sqrt[6]{1.05} - 1$
5. Multiply i by 100 to obtain r.	$r = 100 \times i = 100 \times \left(\sqrt[6]{1.05} - 1\right) = 0.8165$

6. Multiply r by the number of compounding periods per year to obtain annual interest rate.	Annual interest rate $= 0.8165 \times 2 = 1.633$
7. Write the answer.	1.63% p.a.

When investing or borrowing money, it is important to know how long it will take the amount to reach a certain value.

WORKED EXAMPLE 10

Calculate the length of time required for $3000 to grow to $4000 at 5% p.a. compounded quarterly.

THINK	**WRITE**
1. Write the values of A, P and i.	$A = 4000$ $P = 3000$ $i = \dfrac{0.05}{4} = 0.0125$
2. Write the rule for compound interest and substitute in the values.	$A = P(1 + i)^n$ $4000 = 3000(1.0125)^n$
3. Rearrange the equation.	$(1.0125)^n = \dfrac{4000}{3000}$ $(1.0125)^n = \dfrac{4}{3}$
4. Try some different values of n.	Let $n = 5$ $\quad 1.0125^5 = 1.0641$ Let $n = 10$ $\quad 1.0125^{10} = 1.1323$ Let $n = 20$ $\quad 1.0125^{20} = 1.2820$ Let $n = 21$ $\quad 1.0125^{21} = 1.2981$ Let $n = 22$ $\quad 1.0125^{22} = 1.3143$ Let $n = 23$ $\quad 1.0125^{23} = 1.3307$
5. Write the answer	It will take 24 periods where a period is 3 months. So, it will take 72 months or 6 years.

Exercise 7.5 Compound interest problems: interest rates and number of compounding periods

1. James has $8000 to invest at either 4% p.a., 6% p.a. or 8% p.a. compounding annually.
   a. Complete the following table to show the interest that he would earn over 5 years.

No. of years	1	2	3	4	5
Interest (5%)					
Interest (6%)					
Interest (8%)					

   b. Show this information in graph form.
2. **WE8** Using technology, show on one set of axes how the following compound interest rates affect the future value of $5000 compounding yearly for 8 years. Hence, describe the effect increasing the compound interest rate has on the investment, using calculations.
   a. 5%
   b. 10%
   c. 15%
3. Petra has $4000 to invest at 6% p.a.
   a. Complete the following table to show the future value of the investment at the end of each year, if interest is compounded
      i. annually.
      ii. six-monthly.

No. of years	1	2	3	4	5
Annually					
Six-monthly					

   b. Show this information in graphical form.
4. **WE9** Calculate the interest rate per annum required that would enable an investment of $1000 to grow to $1200 over 2 years if the interest is compounded every quarter. Give your answer correct to two decimal places.
5. Determine the interest rate per annum (correct to 2 decimal places) that would enable an investment of $10 000 to grow to $15 500 over 3 years if interest is compounded six monthly.
6. **WE10** Calculate the length of time required for $2500 to grow to $3000 at 6.5% p.a. compounded every 6 months.
7. Calculate the number of interest-bearing periods, *n*, required and hence the time it will take $2400 to amount to $5300 at a rate of 16% p.a., with interest compounded quarterly.
8. **MC** The least number of interest periods, *n*, required for $6175 to grow to at least $8500 in an account with interest paid at 6.5% p.a. and compounded half-yearly is closest to
   A. 10.
   B. 11.
   C. 12.
   D. 20.
9. **MC** The minimum interest rate per annum required for an investment of $5400 to grow to at least $8000 in 3 years if compounded quarterly would be
   A. 3.33%
   B. 3.4%
   C. 8.67%
   D. 13.32%

10. Approximately how long would it take Peter to save up for a $5000 car if he has $3500 now and elects to invest his money at 12% p.a., interest credited monthly?

11. Tai has $20 000 to invest at 4.75% compound interest for 2 years. She can choose from the following compounding periods:
    - Option A: six-monthly
    - Option B: monthly
    - Option C: yearly.

    Using technology, calculate the future value of the investment after each compounding period. Hence, state which option Tai should choose.
12. Patrick was offered an investment rate of 4.97% p.a. compounding daily or 5% p.a. compounding monthly. Which should he choose? Justify your answer using calculations. Assume 365 days = 1 year.
13. **MC** After selling their house, Mr and Mrs Dengate have $61 800. They plan to invest it at 6% p.a., with interest compounded annually. The value of their investment will first exceed $100 000 after

    A. 7 years. **B.** 8 years. **C.** 9 years. **D.** 10 years.

7.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar SF (Questions 1–12)

The following information relates to questions 1–4.

Genevieve sold her car for \$4300 and decided to invest it in a savings account. The interest accrued was compounded annually. The amount in the account after n years, A_n, can be modelled by the recurrence relation

$$A_{n+1} = 1.04A_n, \ A_0 = 4300.$$

1. **MC** The annual percentage compound interest rate for this account is
   A. 4%. **B.** 1.04%. **C.** 104%. **D.** 4.3%.
2. **MC** The interest earned in the first year is
   A. \$4472. **B.** \$430. **C.** \$172. **D.** \$104.
3. **MC** The balance at the end of the first year is
   A. \$104. **B.** \$172. **C.** \$4300. **D.** \$4472.
4. **MC** The balance at the end of the second year is closest to
   A. \$4651. **B.** \$4650. **C.** \$4644. **D.** \$4472.
5. **MC** An investment of \$6000 was placed in an account for 3 years at 4.25% p.a. compounded annually. How much more money would be collected if the investment was compounded quarterly?
   A. \$13.36 **B.** \$32.97 **C.** \$46.33 **D.** \$52.29
6. **MC** An interest rate of 4.5% p.a. compounding monthly is equivalent to an effective interest rate of
   A. 4.50% p.a. **B.** 4.55% p.a. **C.** 4.57% p.a. **D.** 4.59% p.a.

Use the following information to answer questions 7–9.

An amount of \$12 000 is invested for a period of 9 months at 3% p.a. compounded monthly. The compound interest formula to calculate the future value of an investment over a period of time is

$$A = P(1 + i)^n$$

7. **MC** The value of n in the formula would be
   A. $\frac{9}{12}$. **B.** 1. **C.** 3. **D.** 9.
8. **MC** The value of i in the formula would be
   A. $\frac{9}{12}$. **B.** 0.025. **C.** 0.0025. **D.** 0.03.
9. **MC** The value of A at the end of the time period would be closest to
   A. \$12 030. **B.** \$12 273. **C.** \$12 070. **D.** \$15 657.
10. **MC** An investment of \$10 000 at the rate of 7% per annum, compounded quarterly, will reach \$14 800 in close to
    A. 3 years. **B.** 6 years. **C.** 9 years. **D.** 12 years.
11. **MC** Warren wishes to invest \$10 000 for a period of 5 years (assume 365 days = 1 year). The following investment alternatives are suggested to him. The best investment would be
    A. simple interest at 9% p.a.
    B. compound interest at 8% p.a. with interest compounded annually
    C. compound interest at 7.8% p.a. with interest compounded six-monthly
    D. compound interest at 7.7% p.a. with interest compounded daily.
12. **MC** Which of the following compounding rates is equivalent to an effective interest rate of 2.75% p.a.?
    A. 2.7% p.a. compounding six-monthly
    B. 2.75% p.a. compounding yearly

C. 2.6% p.a. compounding quarterly
D. 2.6% p.a. compounding monthly

Complex familiar

13. Using the compound interest formula, calculate the amount of compound interest, correct to the nearest cent, on each of the following investments.

- **a.** \$4280 at 3.75% p.a. compounded yearly for 4 years
- **b.** \$10 495 at 5.50% p.a. compounded six monthly for 6 years
- **c.** \$2050 at 4.80% p.a. compounded monthly for 5 years
- **d.** \$15 250 at 6.50% p.a. compounded weekly for 2 years (assume 52 weeks = 1 year)

14. Using the compound interest formula, calculate the present values required to yield the following future values. Write your answers correct to 2 decimal places.

- **a.** \$22 000 after compounding yearly at a rate of 6.50% p.a. for 6 years
- **b.** \$17 580 after compounding monthly at a rate of 4.80% p.a. for 4 years

15. The formula $FV = 8000\,(1.025)^{10}$ calculates the future value of an investment compounded half-yearly. From the formula, write

- **a.** the sum of money invested
- **b.** the annual interest rate in % p.a.
- **c.** the number of years the sum was invested.

16. Mai has \$10 000 to invest at 3.25% compound interest for 3 years. She can choose the following compounding periods:

- Option A: six-monthly
- Option B: monthly
- Option C: yearly.

Using technology, calculate the future value of the investment after each compounding period. Hence, state which option Mai should choose.

Complex unfamiliar

17. Ben invests \$5000 at 4.75% p.a. compounding yearly for 5 years. Each year, immediately after the yearly interest is added to the investment, Ben deposits an additional \$150. The table shows the future value of his investment for the first 2 years.

Year	Future value (\$A)
0	5000
1	5387.50
2	5793.41
3	
4	
5	

- **a.** Explain how the future value is calculated each year. Support your answer by showing that the future value at the end of the 3rd year is \$6218.60.
- **b.** Complete the table for the last 3 years.
- **c.** If Ben did not deposit the additional \$150 at the end of each year, calculate the future value of his investment at the end of the 5 years. Write your answer correct to the nearest cent.
- **d.** Would it make any difference if Ben deposits the \$150 before the interest is calculated? Explain your answer using calculations.

18. Shivani has saved \$5000 from her part-time job and decides to invest it for at least 3 years. Her parents help her to find the best investment options and come up with the following choices.

- **i.** A local business promising a return of 3.5% compounded annually, with an additional 2% bonus on the total sum paid at the end of the 3-year period
- **ii.** A building society paying a fixed interest rate of 4.3% compounded monthly
- **iii.** A venture capitalist company guaranteeing a return of 3.9% compounded daily (assume 365 days = 1 year)
  - **a.** Calculate the expected return after 3 years for each of the options. Write your answers correct to the nearest cent.
  - **b.** Assuming each option is equally secure, where should Shivani invest her money?

19. Maisie wants to buy a paddleboard. Its retail price is \$3995. She needs to save up until she has enough cash to pay for the paddleboard. Maisie's first option for financing the purchase is to place the balance of her savings account, \$1983.50, into a term deposit offering 5.6% per annum for a 2 year term.

- **a.** Calculate the total value of her investment at the end of 2 years.
- **b.** Maisie uses the term deposit investment towards the purchase of the paddleboard. What extra fortnightly savings will be needed over the next 2 years to make up the balance of \$3995?

  Another option for Maisie is to place her \$1983.50 into a building society, which offers 5.4% interest, compounded monthly.
- **c.** How long will it take Maisie to accumulate enough funds for the paddleboard?

  Maisie decided to invest with the building society. Eight months later she received a \$1000 bonus from her employer.
- **d.** How much money did Maisie have in her account at the time that she received her bonus?
- **e.** If Maisie immediately deposited the entire bonus into her building society account, how much longer would she need to wait to get enough funds for the paddleboard?

20. Hugo has inherited some money from his grandparents and decides to invest it in an account that compounds monthly. The amount of money in his account after n months, A_n, can be modelled using the recurrence relation $A_{n+1} = 1.006A_n, A_0 = 15\,000$.

- **a.** What is the annual interest rate?
- **b.** What was the value of the inheritance Hugo received from his grandparents?
- **c.** What was the value of Hugo's account after 2 years? Give your answer to the nearest dollar.

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Units 3 & 4 Sit exam

Answers

7 Compound interest loans and investments

Exercise 7.2 Modelling a compound interest loan or investment using a recurrence relation

1. a. $A_{n+1} = 1.12A_n, A_0 = 8500$

   b.

A_n (\$)	Calculation	A_{n+1} (\$)
8500	$1.12 \times 8500 = 9520$	9520
9520	$1.12 \times 9520 = 10662.40$	10662.40
10662.40	$1.12 \times 10662.40 = 11941.89$	11941.89

   c. $11941.89

   d. $3441.89

2. a. $A_{n+1} = 1.01A_n, A_0 = 8500$

   b. $9578.01

   c. $1078.01

   d. $58.01

3. a. $A_{n+1} = 1.04125A_n, A_0 = 2575$

   b. $3281.76

4. a. $A_{n+1} = 1.00667A_n, A_0 = 34000$

   b. $1149.13

5. B
6. B
7. A
8. B
9. D
10. C
11. A
12. A = 6
    B = 10 600
    C = 10 600
    D = 11 236
    E = 12 624.77
    F = 12 624.77
13. A = 540.70
    B = 10 553.70
    C = 10 553.70
    D = 11 724.27
    E = 11 724.27
    F = 12 357.38
14.

A_n (\$)	A_{n+1} (\$)
2700	$1.00375 \times 2700 = 2710.13$
2710.13	$1.00375 \times 2710.13 = 2720.29$
2720.29	$1.00375 \times 2720.29 = 2730.49$
2730.49	$1.00375 \times 2730.49 = 2740.73$
2740.73	$1.00375 \times 2740.73 = 2751.01$

15. a. $A_{n+1} = 1.10A_n, A_0 = 300$

    b. $139.23

16. $2187.25

Exercise 7.3 Effective annual rate of interest

1. 4.06% p.a.
2. 4.10% p.a.
3. 3.79% p.a.
4. a. 4.27% p.a. effective rate
   b. 4.26% p.a. effective rate
   So 4.27% p.a. compounding quarterly is better.
5. 5.09% effective rate, 5.12% effective rate, so choose 5% p.a. compounding monthly.
6. 4.58% p.a.
7. 5.75% p.a. compounding monthly is equivalent to a 5.90% p.a. effective interest rate. In this case Marilyn's choice is better, but it is important to compare effective interest rates.
8. B
9. A
10. D

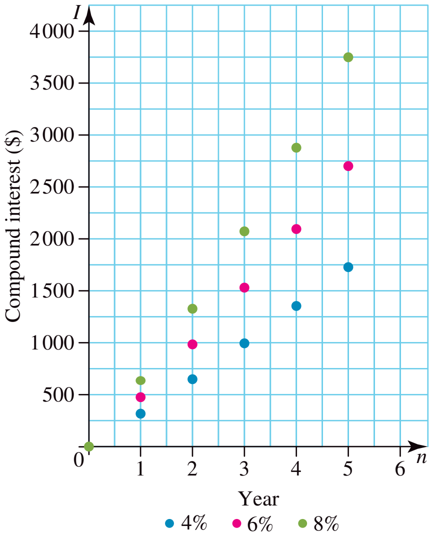
Exercise 7.4 Compound interest problems: future values and present values

1. a. $4630.50
   b. $9274.19
   c. $24 488.80
   d. $13 503.78
2. a. $664.76
   b. $5515.98
   c. $599.58
   d. $1 229 312.85
3. $4283.85
4. a. $1003.69
   b. $9111.56
   c. $181 402.12
   d. $20 039.67
5. B
6. a. $9961.26
   b. $11 278.74
   c. $975.46
7. $70 555.25
8. $31 850.33
9. C
10. a. $1749.60
    b. $1757.49
    c. $1759.47
    d. $1759.97
    e. More frequent compounding periods increases the final balance.
11. a. $605.57
    b. $1314.91
    c. $795.82
    d. $1043.28
    e. $1642.73
12. $11 143.24
13. a. $2069.61
    b. $1531.33
    c. $3564.10
    d. $5307.05

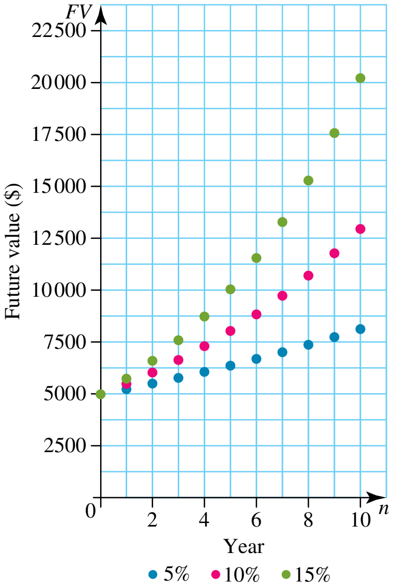
Exercise 7.5 Compound interest problems: interest rates and number of compounding periods

1. a. * See table below

b. Compound interest earned on $8000



2. Effect of compound interest on $5000



Increasing the compound interest rate means that the investment grows more quickly over time.

3. a. * See table below

b. * See image below

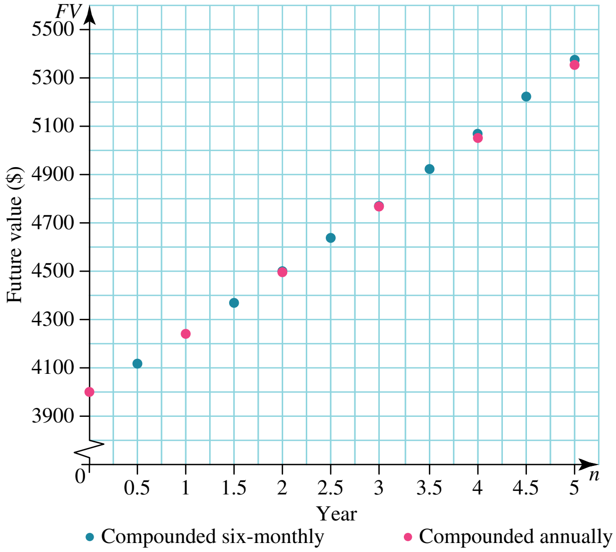
***1. a.**

No. of years	1	2	3	4	5
Interest (4%)	$320.00	$652.80	$998.91	$1358.87	$1733.22
Interest (6%)	$480.00	$988.80	$1528.13	$2099.82	$2705.80
Interest (8%)	$640.00	$1331.20	$2077.70	$2883.91	$3754.62

***3. a.**

No. of years	1	2	3	4	5
Annually	4240.00	4494.40	4764.06	5049.91	5352.90
Six-monthly	4243.60	4502.04	4776.21	5067.08	5375.67

***3. b.** Petra's investment



4. 9.22% p.a.
5. 15.16% p.a.
6. 3 years
7. $n = 21$ quarters, it will take 5 years and 3 months
8. A
9. D
10. 36 months
11. Monthly: $21 989.05; six-monthly: $21 968.77; yearly: $21 945.13. Tai should choose option B.
12. 5% p.a. compounded monthly
13. C

Exercise 7.6 Review: exam practice

1. A
2. C
3. D
4. A
5. A
6. D
7. D
8. C
9. B
10. B
11. D
12. B
13. a. $679.02
    b. $4038.31
    c. $554.81
    d. $2115.72
14. a. $15 077.35
    b. $14 514.45
15. a. $8000
    b. 5%
    c. 5 years
16. Option A: $11 015.48, option B: $11 022.66, option C: $11 007.03.
    - Option B
17. a. Sample responses can be found in the worked solutions in the eBookPLUS.
    b.

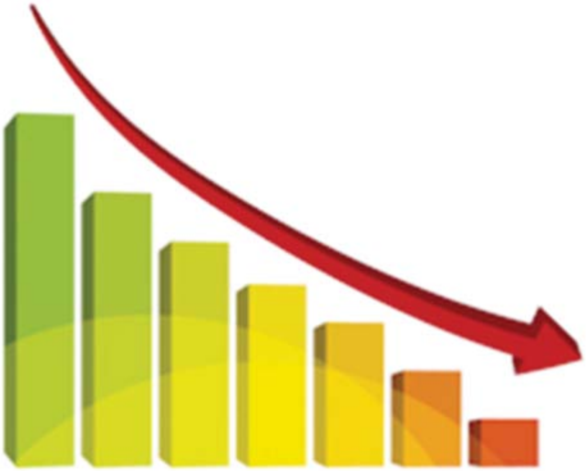
Year	Future value ($)
0	5000.00
1	5387.50
2	5793.41
3	6218.60
4	6663.98
5	7130.52

    c. $6305.80
    d. Yes, because adding $150 before the interest is calculated increases the present value; hence, the overall amount of interest added would be greater.
18. a. Option i: $5654.46, option ii: $5687.14, option iii: $5620.56
    b. Option ii
19. a. $2205.65   b. $34.41   c. 13 years
    d. $2056.04   e. 5 years
20. a. 7.2%   b. $15 000   c. $17 316

8 Reducing balance loans

8.1 Overview

8.1.1 Introduction



The biggest loan that most people will ever take out will be a loan to buy a home. These loans are usually for large amounts of money, often over $800 000 and are taken over long periods of time. Most commonly they are taken over 10, 15, 20 or 25 years but they can be taken over even longer periods of up to 35 years.

Home loans are not charged at a flat rate of interest. The interest on these loans is reducible, which means that the interest is calculated on the amount of money owing on the loan at the time, rather than on the amount initially borrowed. This is known as a **reducing balance loan**.

In reducing balance loans, interest is usually charged every month by the financial institution and repayments are made by the borrower on a regular basis. These repayments nearly always amount to more than the interest for the same period of time and so the amount still owing is reduced. Since the amount still owing is continually decreasing and interest is calculated on the current balance but debited monthly, the amount of interest charged also decreases throughout the life of the loan.

This means that less of the amount borrowed is paid off in the early stages of the loan compared to the end. That is, the rate at which the loan is paid off increases as the loan progresses.

It is possible to have an 'interest only' loan account whereby the repayments equal the interest added and so the balance doesn't reduce. This option is available to a borrower who wants to make the smallest repayment possible.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.2 Modelling a reducing balance loan using a recurrence relation

8.2.1 Recurrence relations

A recurrence relation can be used to model a reducing balance loan.

Reducing balance loan modelled by recurrence relation

$$A_{n+1} = rA_n - R, A_0 = a,$$

where,

A_0 is the initial amount borrowed

A_n is the balance of the loan after n payments

$$r = 1 + \frac{\text{interest rate per compounding period}}{100}$$

R is the monthly repayment.

WORKED EXAMPLE 1

Jeffrey borrows \$3500 at a rate of 6.5% p.a. (interest compounded monthly) which is to be paid back with monthly instalments of \$711.42.

a. Write a recurrence relation to describe this situation.

b. Use your calculator to determine the number of monthly repayments needed to pay off this loan.

THINK	WRITE
a. 1. State the initial values for A_0, r and R.	a. $A_0 = 3500$ $r = 1 + \frac{\frac{6.5}{12}}{100} = 1.00542$ $R = 711.42$
2. Write the general form of the recurrence relation.	$A_{n+1} = rA_n - R, A_0 = a,$
3. Substitute in the values for A_0, r and R.	$A_{n+1} = 1.00542A_n - 711.42, A_0 = 3500$
b. 1. On a calculator screen, type 3500, press ENTER/EXE and then type × 1.00542 − 711.42 Continue to press ENTER/EXE until the output is less than 0. Count the number of iterations of the recurrence relation until the balance is less than 0.	b. 3500 3500 3500 × 1.00542 − 711.42 2807.55 2807.55 × 1.00542 − 711.42 2111.35 2111.35 × 1.00542 − 711.42 1411.37 1411.37 × 1.00542 − 711.42 707.60 707.60 × 1.00542 − 711.42 0.02
2. Write the answer.	After 5 months the account balance reads 0.02, so it would take 5 months to pay off the loan.

Note: As the above calculations have been rounded to two decimal places, this can impact on the final balance. If there was an underpayment of 2 cents, the final payment would be adjusted by 2 cents.

8.2.2 Paying off a reducing balance loan using a table

The process of paying off a loan by regular payments over a period of time is known as **amortisation**. The amortisation of a loan can be tracked on a step-by-step basis by following the payments made, the interest and reduction in the principal. For each payment period, an amortisation table provides your loan balance, interest charges on your loan, and the amount of principal that you pay off.

Consider an amortisation table for Worked example 1.

- The *scheduled payment* of \$711.42 remains fixed
- The *interest rate* of 0.542% per month remains fixed.
- The *interest* is calculated on the previous month's balance.
- The *payment off the principal* is the *Scheduled payment — Interest for that month*
- The *balance of the loan* is the *Previous month's balance — Payment off the principal*

Payment number (n)	**Payment (\$)**	**Interest (\$)**	**Principal reduction (\$)**	**Balance of loan (\$)**
0	0.00	0.00	0.00	3500.00
1	711.42	$0.00542 \times 3500 = 18.97$	$711.42 - 18.97 = 692.45$	$3500 - 692.45 = 2807.55$
2	711.42	$0.00542 \times 2807.55 = 15.22$	$711.42 - 15.22 = 696.20$	$2807.55 - 696.20 = 2111.35$
3	711.42	$0.00542 \times 2111.35 = 11.44$	$711.42 - 11.44 = 699.98$	$2111.35 - 699.98 = 1411.37$
4	711.42	$0.00542 \times 1411.37 = 7.65$	$711.42 - 7.65 = 703.77$	$1411.37 - 703.77 = 707.60$
5	711.42	$0.00542 \times 707.60 = 3.84$	$711.42 - 3.84 = 707.58$	$707.60 - 707.58 = 0.02$

Note: The table above can be arranged differently. Another way to present the information is shown below.

Payment number (n)	**Principal (\$) P**	**Interest (\$) I**	**Repayment (\$) R**	**Balance (\$) P + I − R**
0	3500	0.00	0.00	3500.00
1	3500	$0.00542 \times 3500 = 18.97$	$711.42 - 18.97 = 692.45$	$3500 - 692.45 = 2807.55$
2	2807.55	$0.00542 \times 2807.55 = 15.22$	$711.42 - 15.22 = 696.20$	$2807.55 - 696.20 = 2111.35$
3	2111.35	$0.00542 \times 2111.35 = 11.44$	$711.42 - 11.44 = 699.98$	$2111.35 - 699.98 = 1411.37$

(Continued)

(Continued)

Payment number (n)	Principal (\$) P	Interest (\$) I	Repayment (\$) R	Balance (\$) P + I − R
4	1411.37	$0.00542 \times 1411.37 = 7.65$	$711.42 - 7.65 = 703.77$	$1411.37 - 703.77 = 707.60$
5	707.60	$0.00542 \times 707.60 = 3.84$	$711.42 - 3.84 = 707.58$	$707.60 - 707.58 = 0.02$

WORKED EXAMPLE 2

Mimi takes out a loan of \$1500 at 9% p.a. to buy a new computer. The loan is to be paid back monthly in equal instalments of \$382.06. The incomplete amortisation table for this loan is below.

a. **What is the interest rate per payment period?**
b. **How much of the second payment of \$382.06 is interest?**
c. **What is the balance of the loan after two payments have been made?**
d. **What is the principal reduction after 3 payments have been made?**
e. **What is the total interest paid on this loan?**
f. **What should be the amount of the final payment to ensure a balance of zero at the end of the loan period?**

Payment number (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0.00	0.00	1500.00
1	382.06	11.25	370.81	1129.19
2	382.06		373.59	
3	382.06	5.67		379.21
4	382.06	2.84	379.22	−0.01

THINK	WRITE
a. **1.** Write the value of the annual interest rate.	**a.** Annual interest rate = 9%
2. Calculate the monthly interest rate.	Monthly interest rate $= \frac{9}{12} = 0.75\%$
3. Write the answer	The interest rate per time period is 0.75%
b. **1.** Calculate 0.75% of 1129.19 as this is the balance of the loan at the end of the first payment period.	**b.** $0.0075 \times 1129.19 = 8.47$
2. Write the answer.	The interest component of the second payment is \$8.47.
c. **1.** The balance of the loan after two payments have been made is the previous balance – balance reduction	**c.** Previous balance − balance reduction = $1129.19 - 373.59 = 755.60$
2. Write the answer.	The balance of the loan after two payments have been made is \$755.66
d. **1.** Principal reduction = payment − interest	**d.** $382.06 - 5.67 = 376.39$
2. Write the answer.	The principal reduction after 3 payments is \$376.39.

e. 1. The interest paid is the total of the interest.	**e.** 11.25 + 8.47 + 5.67 + 2.84 = 28.23
2. Write the answer.	The total interest paid is $28.23.
f. The last payment results in an overpayment of 1 cent	**f.** The last payment should be $379.21.

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Units 3 & 4 > Area 5 > Sequence 2 > Concept 1

Modelling a reducing balance loan using recursion Summary screen and practice questions

Exercise 8.2 Modelling a reducing balance loan with a recurrence relation

1. **WE1** Susie borrows $20 540 at a rate of 6% p.a. (interest compounded monthly) and pays the loan back in monthly instalments of $4169.82.

- **a.** Write a recurrence relation to describe this situation.
- **b.** Use your calculator to determine the number of monthly repayments needed to pay off this loan. Give your answer to the nearest whole number.

2. Frankie invests $800 with interest paid at a rate of 7.5% p.a. (interest compounded monthly) and each month he adds another $150 from his part-time job to the investment.

- **a.** Write a recurrence relation to describe this situation.
- **b.** Use your calculator to determine the balance of his investment after 4 months.

3. **WE2** Ahmet borrows $2400 at 9% p.a. to buy a new laptop. The loan is to be paid back monthly in instalments of $489.64. Following is the incomplete amortisation table for this loan.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	2400.00
1	489.64		471.64	1928.36
2	489.64	14.46	475.18	1453.18
3	489.64	10.90	478.74	
4	489.64	7.31		492.11
5	489.64	3.69	485.95	

- **a.** What is the interest rate per payment period?
- **b.** How much of the first payment of $489.64 is interest?
- **c.** What is the balance of the loan after three payments have been made?
- **d.** What is the principal reduction after 4 payments have been made?
- **e.** What is the total interest paid on this loan?
- **f.** What is the balance left at the end of 5 months to pay out the loan?

4. **MC** Whitney took out a loan for $15 000 for a new car at 17% p.a. The contract required that she repay the loan over 4 years with monthly instalments of $432.83. After 6 months Whitney still owes

A. $13 866.71. **B.** $14 7661.22. **C.** $14 552.61. **D.** $13 633.30.

5. A loan of $34 000 is taken out over 6 years at a rate of 6.5% p.a. (interest debited monthly) and is to be paid back monthly with $571.54 instalments. Complete the following table for the third payment.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	34 000.00
1	571.54	184.17	387.37	33 612.63
2	571.54	182.07	389.47	33 223.16
3	571.54			

6. Nicole wanted to buy a new road bike so she took out an $8000 loan over 4 years at a rate of 9% p.a. (interest debited quarterly) and is to be paid back quarterly with instalments of $600.93.
   a. Write a recurrence relation for this situation.
   b. Complete the following table for the first three payments and state how much interest she has paid and how much she owes at the end of this time.

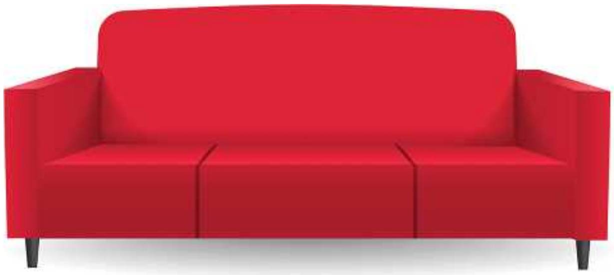
Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0	0.00	0.00	0.00	8000
1	600.93			
2	600.93			
3	600.93			

7. Su Fen borrows $550 000 at 4.5% p.a. to buy a house, compounded monthly. The loan is to be paid back monthly in instalments of $3800. Complete the following amortisation table for the first 3 repayments.

Payment number (n)	Payment ($)	Interest ($)	Principal reduction ($)	Balance of loan ($)
0				
1				
2				
3				

8. Ivona took out a loan for $25 000 over 6 years to set up her hydroponic nursery. The interest rate was set at 8.5% p.a. (interest debited monthly), which is to be paid back monthly with instalments of $444.46.
   a. Write a recurrence relation to model this loan.
   b. Use your recurrence relation to find how much Ivona owes after her third payment was made.

9. Marcus took out a loan of \$1800 to buy a new sofa, with the loan having an interest rate of 14% p.a. The loan will be repaid with 10 equal monthly payments of \$191.75.



- **a.** Write a recurrence relation to model this loan.
- **b.** Use the recurrence relation to determine how much is still owed after 5 months (5 Payments).
- **c.** Is the loan paid out exactly after 10 months? If not, how much will Marcus need to add to the last payment to pay out the loan?

10. Oscar borrows \$13 000, taken out over 2 years and to be repaid in monthly instalments. Calculate the amount still owing after 4 months if interest is debited monthly at a rate of

- **a.** 5% p.a. and the repayment is \$570.33
- **b.** 8% p.a. and the repayment is \$587.95
- **c.** 10% p.a. and the repayment is \$599.88.
  (*Note:* As the interest rate increases, the monthly repayment increases if the loan period is to remain the same.)

11. Jenny decides to borrow \$6500 for a trip to London and takes a personal loan at an interest rate of 11.5% debited fortnightly, with fortnightly repayments of \$98.72 over 3 years.

- **a.** Write a recurrence relation for this situation where A_n is the amount owing after n repayments.
- **b.** Using the answer to part **a**, how much does Jenny still owe after 6 fortnights.

12. A reducing balance loan can be modelled by the recurrence relation

$$A_{n+1} = 1.0075A_n - 600.35, \qquad A_0 = 45\,000.$$

Interest is accrued monthly and is to be paid off in monthly instalments.

- **a.** What is the monthly instalment?
- **b.** How much is the initial loan?
- **c.** What is the annual interest rate?
- **d.** What is the total interest paid after the 3rd instalment?

8.3 The effect of the interest rate and repayment amount on the time taken to repay the loan

8.3.1 Frequency of repayments

The amount of interest paid, and the time taken to repay a loan can be altered by the frequency of the repayments. For example, if payments on money borrowed are made each fortnight rather than each month, less interest will be paid and the loan will be paid out faster.

WORKED EXAMPLE 3

Oscar borrows $1500 from a bank at an interest rate of 16% p.a. He can pay it off in 6 months with fortnightly payments of $120.42 or monthly payments of $261.80.

a. **Calculate how much interest is paid with fortnightly payments.**
b. **Calculate how much interest is paid with monthly payments.**
c. **Which is the better option?**

THINK	WRITE
a. **1.** There are 13 fortnights in 6 months, so multiply $120.42 by 13 and subtract $1500 from the total.	**a.** Total fortnightly repayments = 120.42 × 13 = 1565.46 1565.46 − 1500 = 65.46
2. Write the answer.	$65.46 is paid in interest with fortnightly payments.
b. **1.** Multiply 6 months by $261.80 and subtract $1500 from the total.	**b.** Total fortnightly repayments = 261.80 × 6 = 1570.80 1570.80 − 1500 = 70.80
2. Write the answer.	$70.80 is paid in interest with monthly payments.
c. **1.** Subtract $65.46 from $70.80	**c.** Option 2 − Option 1 = 70.80 − 65.46 = 5.34
2. Write the answer.	Oscar would save $5.34 by making payments each fortnight instead of each month.

8.3.2 The effect of changing the interest rate

Most home loans are not fixed interest rate loans, which means the interest rate will fluctuate for the life of the loan. The interest rate has a significant effect on the amount of interest paid. An increase in the interest rate will increase the time required to pay off the loan.

WORKED EXAMPLE 4

Briony wants to borrow $2400 to redecorate her living room and checks out her two local banks for their terms. Bank with Me charges 15.4% p.a. compounded monthly with monthly repayments of $217.07. Better Bank charges 14.2% p.a. compounded monthly with monthly repayments of $282.69.

a. **Calculate the number of months it would take to pay off the Bank with Me loan.**
b. **Calculate the number of months it would take to pay off the Better Bank loan.**
c. **How much interest would be paid with each loan?**

THINK	WRITE
a. **1.** Write the rule for the recurrence relation for a reducing balance loan.	**a.** $A_{n+1} = rA_n - R, A_0 = 2400$
2. Write the values for r and R.	$R = 217.07, r = 1 + \frac{\frac{15.4}{12}}{100} = 1.01283$
3. Substitute the values into the rule.	$A_{n+1} = 1.01283\,A_n - 217.07$

4. Use the iterative function on your calculator to determine how many months until the loan is amortised.

```
2400                 2400
                     2213.72
                     2025.05
                     1833.97
                     1640.43
                     1444.40
                     1245.86
                     1044.78
                      841.11
                      634.83
                      425.91
                      214.30
                      −0.02
```

5. Count the number of iterations.

It will take 12 months to pay off the loan with Bank with Me.

b. Write the rule for the recurrence relation for a reducing balance loan.

b. $A_{n+1} = rA_n - R,\ A_0 = 2400$

Write the values for r and R.

$R = 282.69,\ r = 1 + \frac{\frac{14.2}{12}}{100} = 1.01183$

Substitute the values into the rule.

$A_{n+1} = 1.01183\,A_n - 282.69$

Use the iterative function on your calculator to determine how many months until the loan is amortised.

```
2400                 2400
                     2145.70
                     1888.40
                     1628.05
                     1364.62
                     1098.07
                      828.37
                      555.48
                      279.36
                      −0.03
```

Count the number of iterations.

It will take 9 months to pay off the loan with Better Bank.

c. For Bank with Me, multiply the monthly repayment of \$217.07 by 12 and subtract \$2400.

Bank with Me

$217.07 \times 12 = 2604.84$

$2604.84 - 2400 = 204.84$

For Better Bank, multiply \$282.69 by 9 and subtract \$2400.

Better Bank

$282.69 \times 9 = 2544.21$

$2544.21 - 2400 = 144.21$

Write the answer.	The interest paid for the Bank with Me loan is $204.84. The interest paid for the Better Bank loan is $144.21.

8.3.3 The effect of changing the repayment amount

Most home loans are taken for 25 to 30 years, due to the size of the loans and to give borrowers some scope for changing life circumstances. The monthly repayments are calculated based on the interest rate at the time and repayments, usually monthly, are locked in. If the interest rate increases or decreases, the repayments are varied accordingly. The repayments generally stay as first determined. If the borrower finds that they have some extra money each payment period, adding extra to the repayments will make a difference to the time needed to pay off the loan and the interest paid. Even a small amount, for example $20, will make a significant difference.

Exercise 8.3 The effect of the interest rate and repayment amount on the time taken to repay the loan

1. **WE3** Stephanie borrows $3600 from a bank at an interest rate of 12.75% p.a. She can pay it off in 6 months with fortnightly payments of $286.52 or monthly payments of $622.51.
   a. Calculate how much interest is paid with fortnightly payments.
   b. Calculate how much interest is paid with monthly payments.
   c. Which is the better option?
2. Harry borrows $2700 from a bank at an interest rate of 10.5% p.a. He can pay it off in 12 months with quarterly payments of $719.87 or monthly payments of $238.00. By paying off the loan with more frequent payments, how much did Harry save in interest?
3. **WE4** Carlos wants to borrow $1300 for a new motorcycle and looks at the online deals provided by two lending institutions. Finance Me, charges 18.4% p.a. compounded monthly with monthly repayments of $228.44 and Have it Now charges 16.6% p.a. compounded monthly with monthly repayments of $336.32.
   a. Calculate the number of months it would take to pay off the Finance Me loan.
   b. Calculate the number of months it would take to pay off the Have it Now loan.
   c. How much interest would be paid with each loan?
4. Cassandra needs to borrow $3000 to furnish her apartment before she starts her new job and makes an appointment with her two local banks to discuss what they can offer. The first bank charges 10.2% p.a. compounded monthly with monthly repayments of $389.49 and the second bank charges 8% p.a. compounded monthly with monthly repayments of $612.05. Cassandra needs to budget very carefully each month and is inclined to go with the smaller monthly repayment of $389.49. Calculate how much Cassandra would save if she could pay the larger amount each month.

5. A reducing balance loan of $4000 has been taken out over 1 year at 6% p.a. (adjusted monthly) with monthly repayments of $344.27
   a. What is the total interest paid?
   b. If the rate was 9% p.a. (adjusted monthly) and the repayments increased to $349.81, what would be the total amount of interest paid?

6. Cindy and Sean took out a loan for home renovations. The loan of \$36 000 was for 5 years and attracted interest at 8% p.a., debited quarterly on the outstanding balance. Repayments of \$2201.64 were made each quarter. After 1 year the rate changed to 4.5% p.a. (debited quarterly). The repayment value didn't change.
   a. Calculate the amount outstanding when the rate changed.
   b. Calculate the actual term of the loan.
   c. Compare the total interest paid to what it would have been if the rate had remained at 8% p.a. for the 5 years

7. Mahmoud wants to buy a shoe shop. He borrows \$18 000 at 8.3% p.a. (debited prior to each repayment) of the reducing balance. He can afford quarterly repayments of \$1709.92.

   The quarterly repayment gives the equivalent monthly repayment of \$566.55.

   The equivalent fortnightly repayment is \$261.06.

   Calculate
   a. the term of the loan using the quarterly repayment
   b. the amount still owing after the third payment if Mahmoud made repayments
      i. monthly
      ii. fortnightly.

8. Steven wants to invest money in shares. He borrows \$25 000 at 7.2% p.a. (debited prior to each repayment) of the reducing balance. He can afford quarterly repayments of \$1292.02.

   One third of the quarterly repayment gives the equivalent monthly repayment of \$430.67.

   Calculate
   a. the term of the loan and
   b. the amount still owing after 9 months if Steven made repayments
      i. quarterly
      ii. monthly.
   c. Which is the better alternative: monthly or quarterly payments?

8.4 Solving problems involving reducing balance loans with and without the use of technology

8.4.1 Solving reducing balance loan problems without spreadsheets

Problems involving reducing balance loans can be solved by hand with a scientific calculator to assist with simple calculations.

WORKED EXAMPLE 5

A couple borrow \$80 000 on their existing mortgage to renovate their home. The interest rate is 5.5% p.a. and their monthly payment is \$750 per month. Calculate for the first month of the loan, correct to the nearest cent

a. the interest paid

b. the amount that the principal is reduced by

c. the balance owing (A).

THINK	WRITE
a. 1. Calculate the monthly interest rate.	a. $i = \frac{5.5}{100} \times \frac{1}{12}$

▶

2. Calculate the interest on the amount borrowed, P, rounded to the nearest cent.	$I = P \times i$ $= 80\,000 \times \frac{5.5}{100} \times \frac{1}{12}$ $= 366.67$
3. Answer the question.	The interest paid on the first month is $366.67.
b. 1. Subtract the interest from the monthly payment.	**b.** $750 - 366.67 = \$383.33$
2. Answer the question.	The principal is reduced by $383.33.
c. 1. Add the interest to the principal and subtract the repayment.	**c.** $A = 80\,000 + 366.67 - 750$ $= 79\,616.67$
2. Answer the question.	The balance owing after the first month is $79 616.67.

8.4.2 Finding the total amount repaid

A loan of a large sum of money, such as to purchase a home, usually takes many years to repay. It is not uncommon for home loans to take 20 or 30 years to repay.

When interest is calculated monthly over such a long period of time, the amount of money required to pay off such a loan can be much more than the initial loan.

WORKED EXAMPLE 6

A loan of $120 000 is paid off at 9% p.a. reducible interest over a period of 25 years. The monthly repayment is $1007.04. Calculate the total repayments on this loan and the total amount of interest paid.

THINK	WRITE
1. Calculate the number of repayments by multiplying the number of years by 12.	$\text{Number of repayments} = 25 \times 12$ $= 300$
2. Multiply the monthly repayments by the number of repayments.	$\text{Total repayments} = 1007.04 \times 300$ $= 302\,112.00$
3. Calculate the total amount of interest paid.	$\text{Total interest paid} = \text{total repayments} - \text{loan amount}$ $= 302\,112 - 120\,000$ $= 182\,112$
4. Answer the question.	The total repayments on this loan are $302 112 and the total amount of interest paid is $182 112.

8.4.3 Solving reducing balance loan problems using technology

To calculate the balance owing and the amount of interest each month, we need the previous month's balance. Therefore, to determine the balance owing over many months and years, we would have to perform individual calculations for each month.

Using technology helps us to calculate the interest and balance owing over many months and years without having to perform multiple repeated calculations.

WORKED EXAMPLE 7

A couple borrow $250 000 from a financial institution on a reducing balance loan of 4.75% p.a. for 20 years with monthly repayments of $1620. Using technology, determine the balance after 2 years. Write your answer correct to the nearest cent.

THINK

WRITE

1. Set up a spreadsheet with the headings as shown.

	A	B	C	D	E	F
1	Month	Balance at start of month ($)	Interest ($)	Total amount ($)	Payment ($)	Balance owing ($)
2	1					
3	2					

2. Insert formula to calculate the interest and balance owing. Remember to round to the nearest cent.

	A	B	C	D	E	F
1	Month	Balance at start of month ($)	Interest ($)	Total amount ($)	Payment ($)	Balance owing ($)
2	1	250 000	= 4.75/1200*b2	= b2 + c2	1620	= d2 – e2
3	2	= f2				

3. Use fill down to copy the formula and complete the spreadsheet for 2 years.

	A	B	C	D	E	F
1	Month	Balance at start of month ($)	Interest ($)	Total amount ($)	Payment ($)	Balance owing ($)
2	1	250 000	989.58	250 989.58	1620	249 369.58
3	2	249 369.58	987.09	250 356.67	1620	248 736.67
4	3	248 736.67	984.58	249 721.25	1620	248 101.25
5	4	248 101.25	982.07	249 083.32	1620	247 463.32
6	5	247 463.32	979.54	248 442.86	1620	246 822.86
7	6	246 822.86	977.01	247 799.87	1620	246 179.87
8	7	246 179.87	974.46	247 154.33	1620	245 534.33
9	8	245 534.33	971.91	246 506.24	1620	244 886.24
10	9	244 886.24	969.34	245 855.58	1620	244 235.58
11	10	244 235.58	966.77	245 202.35	1620	243 582.35
12	11	243 582.35	964.18	244 546.53	1620	242 926.53
13	12	242 926.53	961.58	243 888.11	1620	242 268.11
14	13	242 268.11	958.98	243 227.09	1620	241 607.09
15	14	241 607.09	956.36	242 563.45	1620	240 943.45
16	15	240 943.45	953.73	241 897.18	1620	240 277.18
17	16	240 277.18	951.10	241 228.28	1620	239 608.28
18	17	239 608.28	948.45	240 556.73	1620	238 936.73
19	18	238 936.73	945.79	239 882.52	1620	238 262.52
20	19	238 262.52	943.12	239 205.65	1620	237 585.65
21	20	237 585.65	940.44	238 526.09	1620	236 906.09
22	21	236 906.09	937.75	237 843.84	1620	236 223.84
23	22	236 223.84	935.05	237 158.89	1620	235 538.89
24	23	235 538.89	932.34	236 471.24	1620	234 851.24
25	24	234 851.24	929.62	235 780.86	1620	234 160.86

4. Answer the question. The balance owing after 2 years is $234 160.86.

Using technology allows us to explore how changing interest rates, payment amounts and time of the loan can affect the total amount repaid.

WORKED EXAMPLE 8

From Worked example 7, determine the balance after 2 years if the payments were increased to $1700. Hence, explain the effect increasing the regular payment has on the loan. Write your answer correct to the nearest cent.

THINK | **WRITE**

1. Change the formula from Worked example 7.

	A	B	C	D	E	F
1	Month	Balance at start of month ($)	Interest ($)	Total amount ($)	Payment ($)	Balance owing ($)
2	1	250 000	= 4.75/1200*b2	= b2 + c2	1700	= d2 – e2
3	2	= f2				

2. Use fill down to copy the formulas and complete the spreadsheet for 2 years.

	A	B	C	D	E	F
1	Month	Balance at start of month ($)	Interest ($)	Total amount ($)	Payment ($)	Balance owing ($)
2	1	250 000	989.58	250 989.58	1700	249 289.58
3	2	249 289.58	986.77	250 276.35	1700	248 576.35
4	3	248 576.35	983.95	249 560.30	1700	247 860.30
5	4	247 860.30	981.11	248 841.42	1700	247 141.42
6	5	247 141.42	978.27	248 119.68	1700	246 419.68
7	6	246 419.68	975.41	247 395.10	1700	245 695.10
8	7	245 695.10	972.54	246 667.64	1700	244 967.64
9	8	244 967.64	969.66	245 937.30	1700	244 237.30
10	9	244 237.30	966.77	245 204.08	1700	243 504.08
11	10	243 504.08	963.87	244 467.95	1700	242 767.95
12	11	242 767.95	960.96	243 728.90	1700	242 028.90
13	12	242 028.90	958.03	242 986.93	1700	241 286.93
14	13	241 286.93	955.09	242 242.03	1700	240 542.03
15	14	240 542.03	952.15	241 494.17	1700	239 794.17
16	15	239 794.17	949.19	240 743.36	1700	239 043.36
17	16	239 043.36	946.21	239 989.57	1700	238 289.57
18	17	238 289.57	943.23	239 232.80	1700	237 532.80
19	18	237 532.80	940.23	238 473.03	1700	236 773.03
20	19	236 773.03	937.23	237 710.26	1700	236 010.26
21	20	236 010.26	934.21	236 944.47	1700	235 244.47
22	21	235 244.47	931.18	236 175.64	1700	234 475.64
23	22	234 475.64	928.13	235 403.78	1700	233 703.78
24	23	233 703.78	925.08	234 628.85	1700	232 928.85
25	24	232 928.85	922.01	233 850.86	1700	232 150.86

3. Answer the question.

The balance owing after 2 years is $232 150.86, which has reduced the balance owing by $234 160.86 – $232 150.86 = $2010.

Reducing the principal by increasing the payment will reduce the number of years required to pay off the loan, saving many dollars in interest.

Using technology allows us to determine the monthly repayments required to pay off a housing loan.

WORKED EXAMPLE 9

A couple decide to borrow \$10 000 to buy a car. The interest rate is 12.5% p.a. compounded monthly for a personal loan. They want to pay off the loan in 18 months and need to know what their repayments per month will be. Use a spreadsheet to determine their monthly repayments. Write your answer correct to the nearest cent.

THINK

WRITE

1. Open a spreadsheet and enter the headings in the cells as shown.

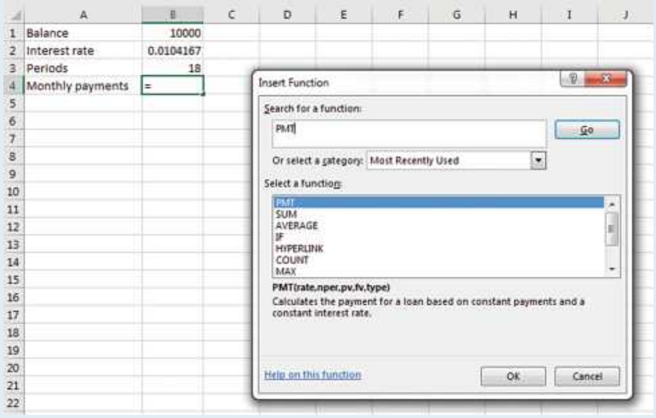
	A	B	C	D
1	Balance			
2	Interest rate			
3	Periods			
4	Monthly payments			

2. The initial balance is \$10 000. The interest rate is $\frac{\frac{12.5}{12}}{100}$ The number of periods is 18. Enter these values into B1, B2 and B3.

B2 ▼ ⋮ ✕ ✓ *fx* =12.5/12/100

	A	B	C	D	E	F
1	Balance	10000				
2	Interest rate	0.0104167				
3	Periods	18				
4	Monthly payments					

3. To determine the monthly payment, place the cursor in Cell B4 and then click on 'Formulas' in ribbon and then insert function to open the function menu.



	A	B
1	Balance	10000
2	Interest rate	0.0104167
3	Periods	18
4	Monthly payments	=

Search for a function: PMT

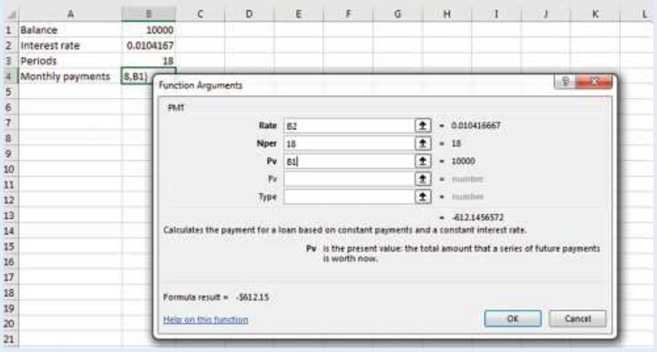
Or select a category: Most Recently Used

Select a function: PMT, SUM, AVERAGE, IF, HYPERLINK, COUNT, MAX

PMT(rate,nper,pv,fv,type)

Calculates the payment for a loan based on constant payments and a constant interest rate.

4. Search for the payment function PMT and click OK. Complete the fields as shown and then click OK.



	A	B
1	Balance	10000
2	Interest rate	0.0104167
3	Periods	18
4	Monthly payments	8,B1)

PMT

Rate B2 = 0.010416667

Nper 18 = 18

Pv B1 = 10000

Fv = number

Type = number

= -612.1456572

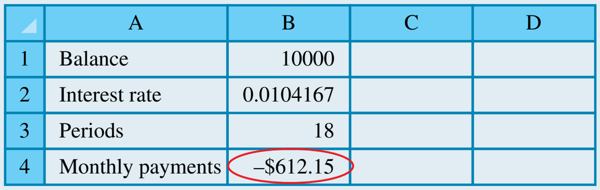
Calculates the payment for a loan based on constant payments and a constant interest rate.

Pv is the present value: the total amount that a series of future payments is worth now.

Formula result = -\$612.15

▶

5. The monthly repayment will be displayed in Cell B4.



	A	B	C	D
1	Balance	10000		
2	Interest rate	0.0104167		
3	Periods	18		
4	Monthly payments	–$612.15		

6. Answer the question.

The monthly repayment is $612.15.

studyon

Units 3 & 4 > Area 5 > Sequence 2 > Concept 2

Applications of reducing balance loans Summary screen and practice questions

Exercise 8.4 Solving problems involving reducing balance loans with and without the use of technology

1. **WE5** A couple borrow $70 000 on a reducing balance loan for 10 years. The interest rate is 7.5% p.a. and their monthly payment is $840 per month. Calculate for the first month, correct to the nearest cent
   - a. the interest paid
   - b. the amount that the principal is reduced by
   - c. the balance owing.
2. Mr and Mrs Devcich borrow $80 000 to top up their home loan. The interest rate is 6.5% p.a. and their monthly payment is $850. Write your answers to the following correct to the nearest cent.
   - a. Calculate the interest for the second month of the loan.
   - b. Calculate the balance owing at the end of the second month.
3. **MC** The first month's interest on a $60 000 reducing balance loan at 12% p.a. is
   A. $600. **B.** $7200. **C.** $59 400. **D.** $60 600.
4. **WE6** The repayments on a loan of $105 000 at 8% p.a. reducible interest over 25 years are $810.41 per month. Calculate the total repayments on this loan and the total amount of interest paid.
5. **MC** A $95 000 reducing balance loan at 8% p.a. over a 15-year term has monthly payments of $907.87. The total amount of interest paid on this loan is:
   A. $7600. **B.** $68 416.60. **C.** $102 600. **D.** $114 000.
6. Mara borrows $8000 from a financial institution on a reducing balance loan of 6.5% p.a. for 5 years with monthly repayments of $157.
   - a. **WE6** Using technology, determine the balance after 2 years. Write your answer correct to the nearest cent.
   - b. **WE7** If the payment is increased to $200, determine the balance after 2 years. Write your answer correct to the nearest cent.
   - c. Explain the effect increasing the regular payment has on Mara's loan by finding how long it takes Mara to pay back the loan in full.
7. The repayment on a loan of $50 000 at 7.5% p.a. over a 15-year term is $463.51 per month. Write your answers to the following correct to the nearest cent.
   - a. Calculate the interest for the first month of the loan and the balance owing at the end of the first month.
   - b. Calculate the amount by which the balance has reduced in the first month.
   - c. Calculate the interest for the second month of the loan and the balance at the end of the second month.
   - d. By how much has the balance of the loan reduced during the second month?

8. The repayments on a loan of \$150 000 over a 20-year term at 9.6% p.a. are \$1408.01 per month. Use technology to complete the table shown.

Month	Principal (\$)	Interest (\$)	Balance owing (\$)
1	150 000.00	1200.00	149 791.99
2	149 791.99		
3			
4			
5			
6			
7			
8			
9			
10			

9. **WE8** Nerida wants to buy a car. She needs to borrow \$22 000. The interest rate is 10.5% p.a. compounded monthly for a personal loan. She wants to pay off the loan in 3 years and needs to know what her repayments per month will be. Use a spreadsheet to determine the monthly repayments. Write your answer correct to the nearest cent.

10. A couple decide to borrow \$440 000 to buy an apartment. The interest rate is 5.75% p.a. compounded monthly for a home mortgage. They want to pay off the loan in 15 years and need to know what their repayments per month will be. Use a spreadsheet to determine their monthly repayments. Write your answer correct to the nearest cent.

11. The Taylors borrow \$140 000 over 20 years at 9% p.a.
   a. The monthly repayment on this loan is \$1259.62. Calculate the total made in repayments.
   b. The Taylors attempt to pay the loan off quickly by increasing their monthly payment to \$1500. The loan is then paid off in 161 months. Calculate the total repayments made under this plan.
   c. How much will the Taylors save by increasing each monthly payment?

12. **MC** The Nguyen family borrows \$250 000 on a reducing balance loan for 20 years at 5.25% p.a. with monthly payments of \$1684.61. Which one of the following calculations determines the balance owing, A, at the start of the 2nd month?

A. $A = 250\,000 \times \left(1 + \frac{5.25}{100}\right) + 1684.61$

B. $A = 249\,409.14 \times \left(1 + \frac{5.25}{100}\right) + 1684.61$

C. $A = 250\,000 \times \left(1 + \frac{5.25}{1200}\right) - 1684.61$

D. $A = 249\,409.14 \times \left(1 + \frac{5.25}{1200}\right) - 1684.61$

13. Mr and Mrs Roebuck borrow $85 000 as a home loan. The interest rate is 9% p.a. and over a 25-year term the monthly repayment is $764.77.

a. Use technology to complete the table shown.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000.00	637.50	84 872.73
2	84 872.73		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

b. Mr and Mrs Roebuck decide to increase their monthly payment to $800. Use technology to complete the table shown.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000.00	637.50	84 837.50
2	84 837.50		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

c. Mr and Mrs Roebuck can repay the loan in 214 months by increasing their monthly payments to $800. Show that that they save $58 231 by increasing their monthly payments to $800.

14. Mr and Mrs Chakraborty are borrowing \$100 000 on a reducing balance loan to purchase a motor home. The interest rate charged by the bank is 7% p.a. They have four payment options:

- Option 1: \$706.78 per month over a 25-year term
- Option 2: \$775.30 per month over a 20-year term
- Option 3: \$898.83 per month over a 15-year term
- Option 4: \$1161.08 per month over a 10-year term.

a. By determining the total amount paid over the term of the loan for each option, which option should Mr and Mrs Chakraborty choose? Write your answers correct to the nearest cent.

b. Mr and Mrs Chakraborty have another option to borrow the money against their existing home loan. The balance owing on their home is \$45 850. They have 3.5 years left to pay back the loan in full, with an interest rate of 5.85% p.a. and monthly repayments of \$1175. Adding \$100 000 on to their home loan will extend the loan to 16 years based on the existing monthly repayments.

Would it be a better option for Mr and Mrs Chakraborty to borrow the money against their existing home loan? Support your answer by finding the total payments for the motor home and their existing home for the best option from part **a**.

15. The Smith and Jones families each take out a \$50 000 reducing balance loan at 9.5% p.a. The Smith family repays the loan at \$500 per month and the Jones family repays the loan at \$750 per month.

a. How much does each family make in repayments in the first year?

b. Use technology to calculate the balance owing for the first 12 months for each family.

c. After one year, how much less does the Jones family owe than the Smith family?

d. The Smith family receives a \$15 000 inheritance. On their 6th payment, they put the entire amount onto their loan as an additional payment (i.e. they pay \$15 000 plus \$500). Adjust your calculations for the Smith family from part **b** to show the additional payment of \$15 000.

16. Ghan applies to borrow \$8000 from a financial institution at 4.5% p.a. for 5 years. Ghan is offered a simple interest loan. Interest is calculated for the entire term of the loan and then the amount owing (principal + interest) is repaid in equal monthly instalments. The following calculation shows the amount owing (A):

$$A = 8000 + \frac{8000 \times 4.5 \times 5}{100}.$$

a. Show that Ghan's monthly repayment, to the nearest cent, would be \$163.33.

b. The table shows the loan payments and principal owing for the first 3 months. Using technology, show that after 20 months Ghan would owe \$6533.40.

Month	Balance ($)	Interest ($)	Principal reduction ($)	Payment ($)
1	9800.00	30	133.33	163.33
2	9636.67	30	133.33	163.33
3	9473.34	30	133.33	163.33

   c. Another lender offers Ghan a reducing balance loan at the same interest rate (4.5% p.a.) and term (5 years) with monthly repayments of $149.14 (to the nearest cent). Using technology, show that if Ghan took the reducing balance loan, after 20 months he would owe $5530.37.

   d. Explain why the reducing balance loan is the most cost-effective option for Ghan. Support your explanation with calculations.

17. Helda is borrowing $15 000 on a reducing balance loan at 4.75% p.a. After budgeting she has $450 left for the monthly payments. If she takes out the loan for 10 years, her monthly payments are $157.27.

   a. Calculate the total amount she pays over the 10 years.

   b. Using technology, determine how long it would take Helda to pay off the loan if she increases her monthly payments to $450.

   c. What would you recommend Helda pays back each month? Justify your answer using calculations.

8.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** The monthly repayment on a reducing balance loan of $175 850 at 4% p.a. over 20 years is $1064.62. Over the life of the loan, the total amount paid is closest to

A. $21 290. **B.** $175 850. **C.** $197 140. **D.** $255 510.

2. **MC** The balance owing on a $215 000 reducing balance loan at 5.25% p.a. with monthly repayments of $1448.76 after 3 repayments is closest to

A. $210 650. **B.** $213 470. **C.** $213 980. **D.** $214 500.

3. **MC** An amount of money is borrowed on a reducing balance at 4.75% p.a. with monthly payments of $157.27 for 10 years. After the first payment, the balance owing is $14 902.10. The amount of money borrowed, correct to the nearest $100 dollars, is

A. $15 000. **B.** $15 100. **C.** $15 500. **D.** $15 600.

4. **MC** Increasing the frequency of repayments of a loan without changing the overall outlay of money paid, will

A. reduce the term of the loan only.
B. lower the interest rate.
C. reduce the amount originally owed and lower the interest rate.
D. reduce the term of the loan and reduce the amount of interest paid.

5. **MC** Elena needed to borrow $450 to purchase a dining table. The yearly interest rate was 3.75%, compounded monthly. The loan was to be paid back in 5 equal monthly instalments. Her monthly repayments were

A. $90.85. **B.** $37.50.
C. $139.27. **D.** $38.91.

6. **MC** If the interest rate on a loan is 7.5% p.a., compounded and debited fortnightly, then the growth rate (r) would be closest to

A. 0.002. **B.** 0.003.
C. 1.002. **D.** 1.003.

7. **MC** Jane borrows \$24 000 to purchase some farm machinery. She agrees to pay the loan over a period of 5 years at 6.8% p.a. compounded monthly. She wants to know how much of the principal has been repaid after the fifth payment. If A_n = the amount left after n payments, to do this she needs to calculate
   A. A_6.
   B. $A_5 - A_6$.
   C. $A_4 - A_5$.
   D. A_5.

Use the following information to answer questions 8 and 9.
A reducing balance loan with interest compounded monthly can be modelled by the recurrence relation

$$A_{n+1} = 1.005A_n - 600.35, \quad A_0 = 45\,000.$$

8. **MC** The interest rate per annum is:
   A. 5%. **B.** 0.5%. **C.** 6%. **D.** 0.6%.
9. **MC** The monthly repayment is:
   A. \$400.23. **B.** \$600.35. **C.** \$1005. **D.** \$45000.
10. **MC** The first month's interest on a \$55 000 reducing balance loan compounded monthly at 7% p.a. is:
   A. \$3850. **B.** \$672. **C.** \$51150. **D.** \$320.83.
11. **MC** Boris borrows \$2500 from his bank at an interest rate of 4.5% p.a. compounded quarterly. If his repayments are \$433.23, he will repay his loan in
   A. 6 years. **B.** 1.5 years. **C.** 7 years. **D.** 2 years.
12. **MC** Mary Johnson borrows \$305 000 for 25 years to buy an apartment. Her bank charges 6.4% interest compounded monthly. Using a spreadsheet, the monthly repayment is closest to:
   A. \$1604.45.
   B. \$11028.71.
   C. \$1556.23.
   D. \$2040.36.

Complex familiar

13. Jamal borrows \$1300 with interest paid at a rate of 4.5% p.a. (interest compounded monthly) and each month he makes a repayment of \$188.51.
   a. Write a recurrence relation to describe this situation.
   b. Use your calculator to determine how much he owes after 4 months.
14. A couple borrow \$100 000 on a reducing balance loan of 6.75% p.a. with monthly payments of \$1148.24. Using technology, complete the table shown and hence state the balance owing after 5 months of payments.

Month	Principal (\$)	Interest (\$)	Balance owing (\$)
1	100 000	562.50	99 414.26
2	99 414.26		
3			
4			
5			

15. David borrows \$25 000 from a financial institution on a reducing balance loan of 7.50% p.a. for 8 years with monthly repayments of \$500.95.

- **a.** Using technology, determine the balance after 2 years. Write your answer correct to the nearest cent.
- **b.** If the payment is increased to \$650, determine the balance after 2 years. Write your answer correct to the nearest cent.

16. A couple borrow \$25 000 on a reducing balance loan for 8 years. The interest rate is 6.35% p.a. and their monthly payment is \$332.81 per month.

Calculate for the first month, correct to the nearest cent:

- **a.** the interest paid
- **b.** the amount that the principal is reduced by
- **c.** the balance owing.

Complex unfamiliar

17. A couple borrow \$250 000 on a reducing balance loan at 4.85% p.a. with \$1629.24 monthly payments for 20 years.

- **a.** Calculate the total amount they will pay over the life of the loan. Write your answer correct to the nearest cent.
- **b.** After the 10th payment, they increase their payments to \$2000. It takes them another 14 years to pay off their loan. Show that they will save \$38 725, correct to the nearest dollar, by increasing their monthly payment to \$2000.

18. Gretin borrows \$12 000 on a reducing balance loan at 4.75% p.a. with monthly payments of \$225 for 5 years.

- **a.** Determine the total interest, in dollars, Gretin pays for the loan.
- **b.** Show that over the 5 years Gretin repays 12.5% more than the original loan.

The following table shows the balance of Gretin's loan after 23 payments.

Payment	Balance owing at start of month (\$)	Interest (\$)	Balance owing (\$)
23	7928.33	31.38	7734.72

- **c.** Show that the interest Gretin pays in the 24th month is \$30.62.
- **d.** Gretin increases her payment to \$250 after the 24th payment. Write a calculation that determines the balance owing at the start of the 26th month.

19. Suki borrows \$4000 on a reducing balance loan to pay for new kitchen appliances. The interest rate is 5.50% p.a. with monthly payments of \$176.38 for 2 years.

- **a.** Show that the total amount in interest Suki pays is \$233.12, correct to the nearest cent.
- **b.** To calculate the amount owing (A) after her second payment, Suki performs the following calculation:

$$A = 4000 + 4000 \times \frac{5.50}{100} - 2 \times 176.38.$$

Explain why Suki's calculation is incorrect by finding the correct amount owing after her second payment.

- **c.** After her second payment, Suki is offered a lower interest rate of 5.20% p.a. with monthly payments of \$175.89.
How much, correct to the nearest cent, does Suki save?

20. A sound system in a nightclub depreciates at 17% p.a. After one year its salvage value is $4565.

a. Show that the original value of the sound system was $5500.

b. Determine the salvage value of the system after 2 years. Write your answer correct to the nearest dollar.

c. The DJ has taken out a personal loan on a reducing balance with an interest rate of 8.75% p.a. to pay for the sound system. His monthly payments are $250.64 for 2 years. The table shows the balance owing after 12 payments. Complete the table by finding the interest and principal reduction for the 9th month.

Month	Balance owing at start of month ($)	Interest ($)	Principal reduction ($)	Balance owing ($)
1	5500.00	40.10	210.54	5289.46
2	5289.46	38.57	212.07	5077.39
3	5077.39	37.02	213.62	4863.78
4	4863.78	35.47	215.17	4648.60
5	4648.60	33.90	216.74	4431.86
6	4431.86	32.32	218.32	4213.53
7	4213.53	30.72	219.92	3993.62
8	3993.62	29.12	221.52	3772.10
9	3772.10			3548.96
10	3548.96	25.88	224.76	3324.20
11	3324.20	24.24	226.40	3097.80
12	3097.80	22.59	228.05	2869.75

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Units 3 & 4 Sit exam

Answers

8 Reducing balance loans

Exercise 8.2 Modelling a reducing balance loan using a recurrence relation

1. a. $A_{n+1} = 1.005A_n - 4169.82, A_0 = 20\,540$
b. 5 repayments

2. a. $A_{n+1} = 1.00625A_n + 150, A_0 = 800$
b. \$1425.84

3. a. 0.75% **b.** \$18 **c.** \$974.44
d. \$482.33 **e.** \$54.36 **f.** \$6.16

4. D

5.

Payment number (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0.00	0.00	34 000.00
1	571.54	184.17	387.37	33 612.63
2	571.54	182.07	389.47	33 223.16
3	571.54	179.96	391.58	32 831.58

6. a. $A_{n+1} = 1.0225A_n - 600.93, A_0 = 8000$
b.

Payment number (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0.00	0.00	8000
1	600.93	180	420.93	7579.07
2	600.93	170.53	430.40	7148.67
3	600.93	160.85	440.08	6708.58

After 3 payments, Nicole has paid \$511.38 in interest and still owes \$6708.58.

7.

Payment number (n)	Payment (\$)	Interest (\$)	Principal reduction (\$)	Balance of loan (\$)
0	0.00	0	0	5 50 000
1	3800	2062.50	1737.50	54 8262.50
2	3800	2055.98	1744.02	54 6518.48
3	3800	2049.44	1750.56	54 4767.93

8. a. $A_{n+1} = 1.0071A_n - 444.46, A_0 = 25\,000$
b. \$24 193.42

9. a. $A_{n+1} = 1.0117A_n - 191.75, A_0 = 1800$
b. \$926.34
c. No, Marcus will need to add 37 cents to his last payment.

10. a. \$10 922.42 **b.** \$10 974.73 **c.** \$11 009.10

11. a. $A_{n+1} = 1.0044A_n - 98.72, A_0 = 6500$
b. \$6074.62

12. a. \$600.35 **b.** \$45 000
c. 9% **d.** \$1006.57

Exercise 8.3 The effect of the interest rate and repayment amount on the time taken to repay the loan

1. a. \$124.76
b. \$135.06
c. More frequent payments reduce the amount of interest, so paying each fortnight is the better option.

2. \$23.48

3. a. 6 months
b. 4 months
c. Finance Me interest = \$70.64
Have it Now interest = \$45.28

4. \$55.67

5. a. \$131.24 **b.** \$197.72

6. a. \$29 893.26
b. 19 quarters (to the nearest quarter). The last repayment will be 2201.64 − 401.42 = 1800.22
c. \$2603.06

7. a. 3 years
b. i. \$16 664.67
ii. \$17 387.24

8. a. 6 years
b. i. \$22 428.20 **ii.** \$22 412.50
c. Steven would be better off by \$15.70 if he made monthly repayments.

Exercise 8.4 Solving problems involving reducing balance loans with and without the use of technology

1. a. \$437.50 **b.** \$402.50 **c.** \$69 597.50

2. a. \$431.08 **b.** \$79 164.41

3. A

4. Total amount paid: \$243 123, total interest paid: \$138 123

5. B

6. a. \$5095.12
b. \$3996.21
c. Increasing the monthly repayments reduces the balance. Hence, the overall interest paid is reduced, in addition to the time taken to pay back the loan.

7. a. Interest = \$312.50, balance owing = \$49 848.99
b. \$151.01
c. Interest = \$311.56, balance owing = \$49 697.04
d. \$151.95

8.

Month	Principal (\$)	Interest (\$)	Balance owing (\$)
1	150 000.00	1200.00	149 791.99
2	149 791.99	1198.34	149 582.32
3	149 582.32	1196.66	149 370.96
4	149 370.96	1194.97	149 157.92
5	149 157.92	1193.26	148 943.18
6	148 943.18	1191.55	148 726.71
7	148 726.71	1189.81	148 508.51
8	148 508.51	1188.07	148 288.57
9	148 288.57	1186.31	148 066.87
10	148 066.87	1184.53	147 843.40

9. $715.05

10. $3653.80

11. **a.** $302 308.80 **b.** $241 500 **c.** $60 808.80

12. C

13. **a.**

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000	637.50	84 872.73
2	84 872.73	636.54	84 744.50
3	84 744.51	635.58	84 615.31
4	84 615.32	634.61	84 485.16
5	84 485.16	633.63	84 354.03
6	84 354.03	632.65	84 221.91
7	84 221.92	631.66	84 088.81
8	84 088.81	630.66	84 954.70
9	83 954.71	629.66	84 819.59
10	83 819.06	628.64	84 683.47
11	83 683.48	627.62	83 546.33
12	83 546.33	626.59	83 408.15

b.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000	637.50	84 837.50
2	84 837.50	636.28	84 673.78
3	84 673.78	635.05	84 508.83
4	84 508.83	633.81	84 342.65
5	84 342.65	632.56	84 175.22
6	84 175.22	631.31	84 006.53
7	84006.53	630.04	83 836.58
8	83836.58	628.77	83 665.35
9	83 665.36	627.49	83 492.84
10	83 492.85	626.19	83 319.04
11	83 319.04	624.89	83143.93
12	83 143.94	623.57	82 967.51

c. Sample responses can be found in the worked solutions in eBookPLUS.

14. **a.** Option 1 : $212 034
Option 2 : $186 072
Option 3 : $161 789.40
Option 4 : $139 329.60
Mr and Mrs Chakraborty should choose option 4.

b. If they can afford the monthly payments of $2336.08, Mr and Mrs Chakraborty should borrow the money for the motor home on a reducing balance loan at 7% p.a.

15. **a.** Smith family: $6000, Jones family: $9000

b. **Jones family**

Month	Principal ($)	Interest ($)	Balance owing ($)
1	50 000	395.83	49 645.83
2	49 645.83	393.02	49 288.86
3	49 288.86	390.20	48 929.06
4	48 929.07	387.35	48 566.42
5	48 566.42	384.48	48 200.90
6	48 200.91	381.59	47 832.49
7	47 832.05	378.67	47 461.17
8	47 461.17	375.73	47 086.90
9	47 086.90	372.77	46 709.67
10	46 709.68	369.78	46 329.46
11	46 329.46	366.77	45 946.23
12	45 946.24	363.74	45 559.97

Smith family

Month	Principal ($)	Interest ($)	Balance owing ($)
1	50 000	395.83	49 895.83
2	49 895.83	395.00	49 790.84
3	49 790.84	394.17	49 685.01
4	49 685.02	393.33	49 578.35
5	49 578.36	392.49	49 470.85
6	49 470.85	391.64	49 362.49
7	49 362.49	390.78	49 253.28
8	49 253.29	389.92	49 143.20
9	49 143.21	389.05	49 032.25
10	49 032.25	388.17	48 920.42
11	48 920.43	387.28	48 807.71
12	48 807.72	386.39	48 694.11

c. $3134.13

d.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	50 000	395.83	49 895.83
2	49 895.83	395.00	49 790.84
3	49 790.84	394.17	49 685.01
4	49 685.02	393.33	49 578.35
5	49 578.36	392.49	49 470.85
6	49 470.85	391.64	34 362.49
7	34 362.5	272.03	34 134.53
8	34 134.54	270.23	33 904.76
9	33 904.77	268.41	33 673.17
10	33 673.18	266.57	33 439.75
11	33 439.76	264.73	33 204.49
12	33 204.49	262.86	32 967.35

16. a.–c. Sample responses can be found in the worked solutions in eBookPLUS.

d. Ghan will pay $851.60 less with the reducing balance loan. This is because the amount that he owes (the balance) is decreasing each month, so the amount of interest owing also decreases.

17. a. $18 872.40

b. 36 months

c. Helda should increase her monthly payments to $450, saving $2755.64 and paying back the loan in full in 3 years.

Exercise 8.5 Review: exam practice

1. D
2. B
3. A
4. D
5. A
6. D
7. C
8. C
9. B
10. D
11. B
12. D

13. a. $A_{n+1} = 1.00375A_n - 188.51, A_0 = 1300$

b. $561.32

14.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	100 000	562.50	99 414.26
2	99 414.26	559.21	98 825.23
3	98 825.23	555.89	98 232.88
4	98 232.88	552.56	97 637.20
5	97 637.20	549.21	97 038.17

15. a. $16 104.42 **b.** $12 257.93

16. a. $132.29 **b.** $200.52 **c.** $24 799.48

17. a. $391 017.60

b. Sample responses can be found in the worked solutions in eBookPLUS.

18. a. $1500

b. $\frac{1500}{12\,000} \times 100 = 12.5\%$

c. $7734.72 \times \left(\frac{4.75}{1200}\right) \approx 30.62$

d. $7540.34 + 7540.34 \times \left(\frac{4.75}{1200}\right) - 250$

19. a. See the worked solutions in your eBookPLUS.

b. There are two errors in Suki's calculation.

- The interest rate should be the monthly rate, not the annual interest rate.
- Her loan is a reducing balance loan, so interest is calculated on the amount owing at the end of each month. Suki's method does not take into account that the balance is reducing and the interest will therefore be decreasing with each payment.

c. $10.78

20. a. $4565 = x \times (1 - 0.17)$

$$x = \frac{4565}{(1 - 0.17)}$$

$$x = \$5500$$

b. $3789

c. Interest = $27.50, principal reduction = $223.14

9 Annuities and perpetuities

9.1 Overview

The retirement age for working Australians is now 70 years. Entering the workforce at 20 and retiring at 70 means that many people will work for a significant proportion of their lives. But what happens when work is no longer part of a person's life and earning money is no longer an option?

There is much discussion from financial advisers about how much a person will need in a superannuation fund to retire comfortably. Figures of up to $1 million and more have been suggested, depending on a person's individual circumstances and lifestyles. How can we save enough for the future?

In this topic, you will learn about investing regularly and how much needs to be saved on a regular basis to result in a desired amount in the future. Remember that even the smallest amount of money, if invested regularly in an interest-bearing account, will grow in value without you having to do any more physical work for it.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

9.2 Introduction to annuities

9.2.1 Identifying annuities

An **annuity** is a form of compound interest investment into which regular and equal payments are made periodically for a fixed period of time, or a single sum investment from which regular and equal payments are received. The interest in an annuity is compounded at the end of each period before the payment is made or received.

Superannuation is a common example of an annuity. In many employment situations, the employer makes contributions (currently set at 9.5% of an employee's wage) into an account on a regular basis. The interest on the investment compounds. The annuity is usually set aside for the employee's working life and is used to fund their retirement. This type of annuity is also referred to as a superannuation fund, or 'super fund'.

House mortgages (home loans) and car loans are other examples of annuities. In these cases, the regular and equal monthly payments are subtracted from the loan amount, which is compounded each month along with the interest.

WORKED EXAMPLE 1

Which of the following are examples of annuities?

a. $5000 is invested into a savings account with simple interest of 5.85% p.a.

b. $2000 is invested into an account compounded monthly. At the end of each month an additional contribution of $150 is made.

c. Monthly mortgage payments of $1750 are made on a home loan of $250 000.

d. Withdrawals are made from an investment sum of $500 000 when needed.

THINK	WRITE
a. **1.** An annuity is a form of a compounded investment. What type of interest is earned?	**a.** $5000 is invested at simple interest.
2. An annuity consists of a series of regular payments or withdrawals. Check if regular payments are made.	There are no regular and equal payments made to the investment.
3. Answer the question.	This is not an example of an annuity.
b. **1.** An annuity is a form of compounded investment.	**b.** $2000 is invested and compounded monthly.
2. An annuity consists of a series of regular payments or withdrawals. Check if regular payments are made.	Payments of $150 are made monthly. Therefore, equal and regular payments are received.
3. Answer the question.	Yes, this is an example of an annuity.
c. **1.** An annuity is a single sum investment.	**c.** $250 000 is the invested single sum (loan).
2. An annuity consists of a series of regular and equal payments that are received. Check if regular payments are made.	Payments of $1750 are made monthly. Therefore, equal and regular payments are received.
3. Answer the question.	Yes, this is an example of an annuity.
d. **1.** An annuity is a single sum investment.	**d.** $500 000 is the invested single sum.
2. An annuity consists of a series of regular and equal payments that are received. Check if regular payments are made.	Withdrawals are made as needed (payments are neither regular nor equal).
3. Answer the question.	This is not an example of an annuity.

9.2.2 Calculating annuities using the compound interest formula

An annuity is the sum of compound interest on an investment. Therefore, the compound interest formula can be used to calculate the value of an annuity.

The amount to which an annuity grows by the end of the investment period is known as the **future value**, A, of the annuity:

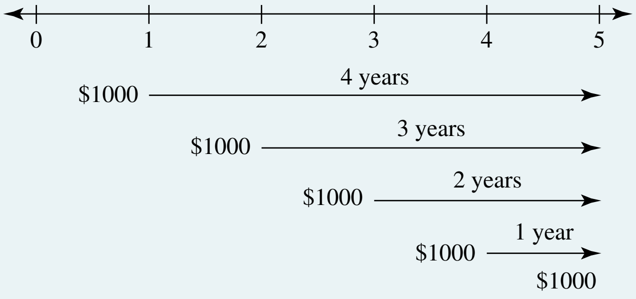
Future value

$$A = P(1 + i)^n$$

where A is the future value, P is the present value (or principal), i is the interest rate per compounding period expressed as a decimal and n is the number of compounding periods.

WORKED EXAMPLE 2

Calculate the future value of an annuity in which \$1000 is invested each year at 10% p.a. for 5 years.

THINK	**WRITE**
1. The compounding periods can be illustrated on a diagram.	
2. Use the compound interest formula to calculate the amount to which the first \$1000 will grow. The first \$1000 will have 4 compounding periods, so $n = 4$.	$n = 4$ $A = P(1 + i)^n$ $= 1000(1 + 0.1)^4$ $= 1000 \times 1.1^4$ $= \$1464.10$
3. Use the compound interest formula to calculate the amount to which the second \$1000 will grow. The second \$1000 will have 3 compounding periods, so $n = 3$.	$n = 3$ $A = P(1 + i)^n$ $= 1000(1 + 0.1)^3$ $= 1000 \times 1.1^3$ $= \$1331.00$
4. Use the compound interest formula to calculate the amount to which the third \$1000 will grow. The third \$1000 will have 2 compounding periods, so $n = 2$.	$n = 2$ $A = P(1 + i)^n$ $= 1000(1 + 0.1)^2$ $= 1000 \times 1.1^2$ $= \$1210.00$
5. Use the compound interest formula to calculate the amount to which the fourth \$1000 will grow. The fourth \$1000 will have 1 compounding period, so $n = 1$.	$n = 1$ $A = P(1 + i)^n$ $= 1000(1 + 0.1)^1$ $= 1000 \times 1.1$ $= \$1100.00$

6. Find the total of the separate \$1000 investments, remembering to add the final \$1000. The fifth \$1000 will have 0 compounding periods.	$\text{Total value} = \$1464.10 + \$1331.00 + \$1210.00$ $+\$1100.00 + \$1000$ $= \$6105.10$

WORKED EXAMPLE 3

Calculate the future annuity value for the first 3 months of a car loan of \$15 880 compounded monthly at 6% p.a. with monthly repayments of \$210 each month. Write your answers correct to the nearest cent.

THINK	WRITE
1. Determine the monthly interest rate.	$i = \frac{6}{100} \times \frac{1}{12} = 0.005$
2. Use the compound interest formula to calculate the future annuity value after interest is added for the first month.	$A = P(1+i)^n$ $= 15\,880 \times 1.005^1$ $= 15\,959.40$
3. Find the future value at the end of the first month by subtracting the payment.	$A = 15\,959.40 - 210$ $= 15\,749.40$
4. Use the compound interest formula to calculate the future annuity value after interest is added for the second month.	$A = P(1+i)^n$ $= 15\,749.40 \times 1.005^1$ $= 15\,828.15$
5. Find the future value at the end of the second month by subtracting the payment.	$A = 15\,828.15 - 210$ $= 15\,618.15$
6. Use the compound interest formula to calculate the future annuity value after interest is added for the third month.	$A = P(1+i)^n$ $= 15\,618.15 \times 1.005^1$ $= 15\,696.24$
7. Find the future value at the end of the third month by subtracting the payment.	$A = 15\,696.24 - 210$ $= 15\,486.24$
8. Answer the question.	1st month: \$15 749.40 2nd month: \$15 618.15 3rd month: \$15 486.24

Exercise 9.2 Introduction to annuities

1. **WE1** Which of the following are examples of annuities?
   - **a.** \$150 is received monthly from a superannuation fund of \$145 000.
   - **b.** \$150 is paid at the end of each month on a car loan with compound interest at 4.75% p.a.
   - **c.** \$2500 is invested at compound interest of 5.45% p.a. with additional payments of \$150, \$350 and \$1000 made in the first year.
   - **d.** \$10 000 is invested in a term deposit at simple interest of 7.5% p.a.

2. Using the compound interest formula, determine the future values of the following. Write your answers correct to the nearest cent.
   a. \$2000 at 5% p.a. for 6 years with interest calculated annually
   b. \$1500 at 4% p.a. for 4 years with interest compounded six monthly
   c. \$10 000 at 7% p.a. for 2 years with interest compounded quarterly
   d. \$3750 at 6% p.a. for 3 years with interest compounded monthly
3. **MC** Which one of the following is *not* an example of an annuity?
   A. A pension received fortnightly from a pension fund
   B. Monthly payments to pay off a car loan
   C. Receiving a scholarship from a fund
   D. Weekly payments from an investment fund
4. **WE2** Calculate the future value of an annuity in which \$500 is being invested each year at 6.75% p.a. for 4 years. Write your answer to the correct nearest cent.
5. **MC** The future value of an annuity in which \$2000 is invested every 6 months at 5.25% p.a. for 2 years, correct to the nearest whole dollar, is
   A. \$2454. B. \$4105. C. \$8321. D. \$8652.
6. At the end of each year for four years Rodney invests \$1000 into an investment fund that pays 7.5% p.a. interest, compounded annually. By calculating each investment of \$1000 separately, use the compound interest formula to calculate the future value of Rodney's investment after four years. Write your answer correct to the nearest cent.
7. **WE3** Calculate the future annuity value for the first 4 months of a car loan of \$18 950 compounded monthly at 5% p.a. with monthly repayments of \$200 at the end of each month.
8. **MC** Monthly repayments of \$1200 are made to pay off a \$150 000 home loan that is compounding monthly at 4.25% p.a. Which one of the following calculates the future value of the annuity after the first payment?

   A. $150\,000 \times \left(1 + \frac{4.25}{100}\right) - 1200$
   B. $150\,000 \times \left(1 + \frac{4.25}{1200}\right) - 1200$
   C. $(150\,000 - 1200) \times \frac{4.25}{1200}$
   D. $(150\,000 - 1200) \times \left(1 + \frac{4.25}{1200}\right)$
9. \$5000 is invested at 3.75% p.a. compounded monthly. Each month, after interest is added, \$250 is deposited. The table shows the value of the annuity each month for 5 months. Complete the table for the next 2 months.

Payment no.	Payment (\$)	Future value (\$)
1	5000	5000.00
2	250	5265.63
3	250	5532.08
4	250	5799.37
5	250	6067.49
6		
7		

10. **MC** Alexi makes monthly repayments of \$250 to pay off a \$19 500 car loan that is compounding monthly at 5.5% p.a. Interest is added before the payment is made at the end of the month. The balance of the loan after the 3rd payment is made is closest to

- **A.** \$18 750.
- **B.** \$18 839.
- **C.** \$19 015.
- **D.** \$19 016.

11. Annie invests \$5000 at 3.5% p.a. compounded monthly.
   - **a.** Explain why this is not an example of an annuity.
   - **b.** How could this type of investment become an example of an annuity? Write a question that would make this an example of an annuity to support your explanation.

12. Bertrand invests \$1500 each year for 9 years at 6% p.a. compounded annually.
   - **a.** Write down the calculations that determine the future value of the annuity after the second and third payments.
   - **b.** A spreadsheet is constructed as follows to calculate the future value of the annuity each year for the 10 payments.

	A	B	C
1	Payment no.	Payment	*FV*
2	1	1500	1500.0
3	2	1500	=
4	3	1500	
5	4	1500	

The formula '$= 1.06 * \text{c}2$' is typed into cell c3.

   - **i.** Explain why this formula will not determine the future value of the annuity.
   - **ii.** Write a formula that will calculate the future value of the annuity each year for the 10 payments.

   - **c.** Using your formula from part **b ii**, determine the future value of the annuity after the 10 payments.

13. The following calculation finds the future value of an annuity after the third payment when interest has been added and the payment made.

$$A = 2004.50 \times 1.0045 + 1000$$

   - **a.** Show that the interest rate on each payment is 0.45%.
   - **b.** What additional information is needed to determine the annual interest rate?
   - **c.** Show that the first payment was \$1000.

14. A company contributes 9.5% of its employees' salaries into a superannuation fund each week.
   - **a.** Explain why this is an example of an annuity.
   - **b.** Max is an employee with this company with a weekly wage of \$895 before tax. How much, in dollars correct to the nearest cent, does the company contribute to his superannuation fund each week?

c. Max's superannuation fund currently has \$205 850 and is invested at 6.5% p.a. compounded weekly. The following calculation shows the amount in dollars, A, in his fund after his next weekly contribution. Assume 52 weeks = 1 year.

$$A = 205\,850\,(1 + i)^1 + Q$$

Write down the values of i and Q, and hence determine Max's superannuation balance after his next weekly contribution, correct to the nearest cent.

15. Monthly mortgage payments of \$1485 are paid towards a loan of \$250 000 which is compounded monthly at 3.75%. The following calculation shows the amount in dollars, A, of the balance of the loan after the first payment.

$$A = 250\,000\,(1 \ + i)^1 \ - 1485$$

a. Show that the value of r is 0.003 125, and hence find the balance of the loan after the first payment, correct to the nearest cent.
b. Using technology, determine the balance of the loan after the first 12 payments.
c. In the context of this problem, explain why mortgage payments are an example of an annuity.

16. At the end of each year for 5 years, Bec invests \$1200 in an investment fund that pays 6% p.a. interest compounded annually. By calculating each individual \$1200 separately, use the compound interest formula to calculate the future value of Bec's investment after 5 years. Write your answer correct to the nearest cent.

9.3 Using recurrence relations to model annuities

9.3.1 Recurrence relations

Annuities can be modelled using **recurrence relations**. Recurrence relations have been studied in previous sections.

A recurrence relation is a relation in which each term of the sequence depends only on the previous term of the sequence. This means that only the initial value is needed to generate the remaining terms of the sequence.

Annuities are compound investments. Therefore, the value of the annuity at the end of each payment period forms a geometric sequence (exponential growth).

For compound investments, each new term is found by multiplying the previous term by a percentage (the interest rate).

For example, \$100 compounded monthly by 1% (that is, multiplied by 1.01) would generate the following terms in the sequence:

$$\begin{aligned}&100\\&100 \times 1.01 = 101\\&101 \times 1.01 = 102.01\\&102.01 \times 1.01 = 103.03\\&\text{Geometric sequence: } 100, 101, 102.01, 103.03\end{aligned}$$

A recurrence relation representing the value of an annuity at the end of each payment period will be of the following form.

Value of an annuity at the end of each payment period

$$A_{n+1} = rA_n + d, \;\; A_0 = a$$

where $r = \left(1 + \frac{x}{100}\right)$, x is the compounded interest rate per payment period, a is the value of the annuity at the end of the first payment or the initial investment and d is the payment amount.

on Resources

Interactivity Initial values and first-order recurrence relations (int-6262)

WORKED EXAMPLE 4

At the end of each year $1000 is invested at 5.75% p.a. compounded annually.

a. Set up a recurrence relation that represents the value of the annuity after each payment is made.

b. Using the recurrence relation from part a, find the value of the annuity for the first 4 years. Write your answers correct to the nearest cent.

THINK	**WRITE**
a. 1. Identify the components of the recurrence relation.	**a.** $a = 1000, x = 5.75$ $r = \left(1 + \frac{5.75}{100}\right) = 1.0575$ $d = 1000$
2. Substitute the components into the recurrence relation. *Note*: The initial term must be included in the recurrence relation so that the next term can be found.	$A_{n+1} = rA_n + d, \qquad A_0 = a$ $A_{n+1} = 1.0575 \times A_n, A_0 = 1000$
b. 1. Substitute $n = 1$ and $A_0 = 1000$ into the recurrence relation to find the value of the annuity after the second payment is made.	**b.** $n = 1, A_0 = 1000$ $A_1 = 1.0575 \times 1000 + 1000$
2. Calculate term A_1.	$A_1 = 2057.50$
3. Substitute $n = 2$ and $A_1 = 2057.50$ into the recurrence relation to find the value of the annuity after the third payment is made.	$n = 2, A_1 = 2057.50$ $A_{1+1} = A_2 = 1.0575 \times 2057.50 + 1000$
4. Calculate term A_2.	$A_2 = 3175.81$
5. Substitute $n = 3$ and $A_2 = 3175.81$ into the recurrence relation to find the value of the annuity after the fourth payment is made.	$n = 3, t_3 = 3175.81$ $A_{2+1} = A_3 = 1.0575 \times 3175.81 + 1000$
6. Calculate term A_3	$A_3 = 4358.42$
7. Answer the question.	The values of the annuity for the first four years are \$1000, \$2057.50, \$3175.81 and \$4358.42.

9.3.2 Using technology to determine the future value of an annuity

Technology can be used to determine the future value of an annuity by using formulas to perform the repetitive calculations required.

WORKED EXAMPLE 5

Using technology, determine the future value of an annuity of $500 that is invested every six months at 5.75% p.a. for 6 years.

THINK

1. Determine the value of r. (Note that the investment is compounded twice a year.)

WRITE

$$r = \left(1 + \frac{5.75}{200}\right) = 1.02875$$

2. Set up a spreadsheet as shown.

	A	B	C
1	Payment period, *n*	Payment ($)	Future value of annuity ($)
2	1	500.00	500.00

3. Use a spreadsheet function to complete the table.

	A	B	C
	Formulas		= c2*1.02875 + b3 (starting from cell c3)
1	Payment period, *n*	Payment ($)	Future value of annuity ($)
2	1	500.00	500.00
3	2	500.00	1014.38
4	3	500.00	1543.54
5	4	500.00	2087.92
6	5	500.00	2647.94
7	6	500.00	3224.07
8	7	500.00	3816.76
9	8	500.00	4426.49
10	9	500.00	5053.76
11	10	500.00	5699.05
12	11	500.00	6362.90
13	12	500.00	7045.83

4. Answer the question.

After 6 years (on maturity) the future value of the annuity is $7045.83.

9.3.3 Other factors affecting the future value of an annuity

The growth of an annuity is affected by the number of payments each year and the compounding interest rate.

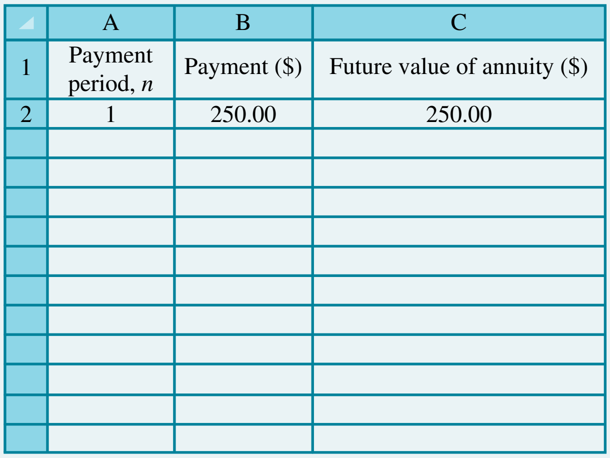
Using technology, we can investigate the effects that varying the amount and frequency of contributions made, and the interest rate, have on the future value of annuities.

WORKED EXAMPLE 6

Using technology, determine the future value of an annuity of \$250 invested every 3 months (quarterly) at 5.75% p.a. compounded quarterly.

- a. **Determine the future value of the annuity after 6 years.**
- b. **Construct a graph that compares the future value of the annuity at an interest rate of 5.75% p.a. for the following investments.**
  - i. **\$1000 being invested every year and compounded annually**
  - ii. **\$500 being invested every 6 months, compounded half-yearly**
  - iii. **\$250 invested every 3 months, compounded quarterly**

  (For parts ii and iii, use your values from Worked examples 4 and 5.)
- c. **Comment on the effect that changing the number of payments has on the future value of the annuity.**

THINK	WRITE
a. 1. Determine the value of r. (Note that the investment is compounded four times a year.)	a. $r = \left(1 + \dfrac{5.75}{400}\right) = 1.014375$
2. Set up a spreadsheet as shown.	

	A	B	C
1	Payment period, n	Payment (\$)	Future value of annuity (\$)
2	1	250.00	250.00

3. Use a spreadsheet function to complete the table.

	A	B	C
		Formula	= c2*1.014375 + b3 (starting from cell c3)
1	Payment period, *n*	Payment ($)	Future value of annuity ($)
2	1	250.00	250.00
3	2	250.00	503.59
4	3	250.00	760.83
5	4	250.00	1021.77
6	5	250.00	1286.46
7	6	250.00	1554.95
8	7	250.00	1827.30
9	8	250.00	2103.57
10	9	250.00	2383.81
11	10	250.00	2668.08
12	11	250.00	2956.43
13	12	250.00	3248.93
14	13	250.00	3545.63
15	14	250.00	3846.60
16	15	250.00	4151.90
17	16	250.00	4461.58
18	17	250.00	4775.71
19	18	250.00	5094.37
20	19	250.00	5417.60
21	20	250.00	5745.47
22	21	250.00	6078.07
23	22	250.00	6415.44
24	23	250.00	6757.66
25	24	250.00	7104.80

4. Answer the question.

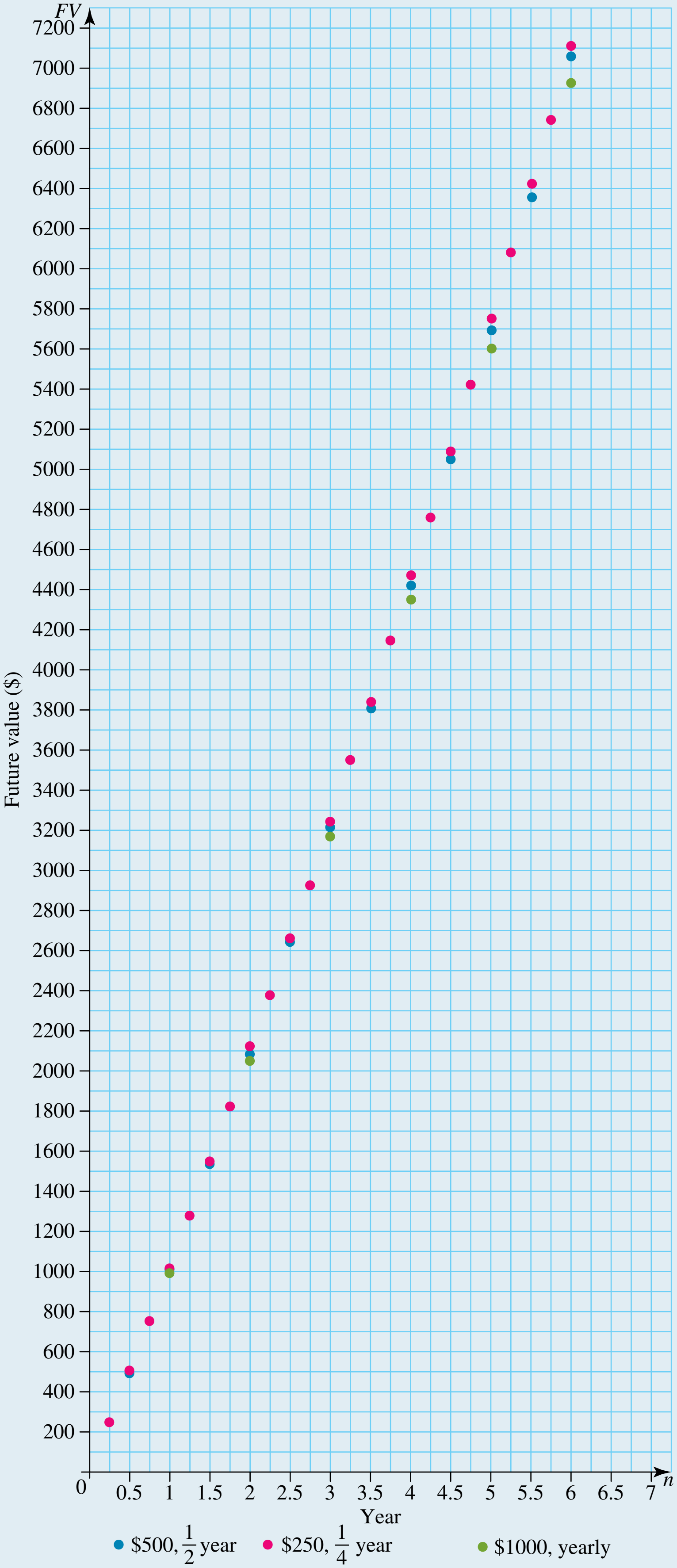
After 6 years (on maturity), the future value of the annuity is $7104.80.

b. 1. Determine the value of the annuity of $1000 invested each year at 5.75% p.a. compounded annually. (Note that this is also done in **Worked example 4**).

b.

	A	B	C
1		Formula	= c2 * 1.0575+ b3 (starting from cell c3)
2	Payment period, *n*	Payment ($)	Future value of annuity ($)
3	1	1000.00	2057.50
4	2	1000.00	3175.81
5	3	1000.00	4358.42
6	4	1000.00	5609.02
7	5	1000.00	6931.54

2. Using the values obtained from part **a** and **Worked examples 4** and **5**, construct a graph.



c. **1.** Compare the future values of the annuity for each of the 3 situations.

c. **i.** For $1000 invested each year, the future value after 6 years is $6931.54.
ii. For $500 invested half-yearly, the future value after 6 years is $7045.83.
iii. For $250 invested quarterly, the future value after 6 years is $7104.80.

2. Comment on the findings.

If payments are more frequent, the future value of the annuity increases.

In this example the yearly payment amount did not change, but the payments were made with different frequencies. Making 4 payments per year resulted in a future value of $7104.80, which is $173.26 more than the result from a $1000 payment once a year and $58.97 more than the result from a $500 payment twice a year.

WORKED EXAMPLE 7

Use technology to find the future value of $1000 invested each year for 6 years at the following interest rates, compounded annually. Hence, explain the effect increasing interest rates has on the future value of an annuity.

a. **4%** **b.** **8%** **c.** **12%**

THINK

WRITE

1. Determine the value of r for each investment.

$$4\%: r = \left(1 + \frac{4}{100}\right) = 1.04$$

$$8\%: r = \left(1 + \frac{8}{100}\right) = 1.08$$

$$12\%: r = \left(1 + \frac{12}{100}\right) = 1.12$$

2. Set up a spreadsheet as shown.

	A	B	C	D	E
		Formulas	= 1.04*c2 + b3	= 1.08*d2 + b3	= 1.12*e2 + b3
1	Payment number	Payment ($)	*FV* ($), 4%	*FV* ($), 8%	*FV* ($), 12%
2	1	1000	1000.00	1000.00	1000.00
3	2	1000	2040.00	2080.00	2120.00
4	3	1000	3121.60	3246.40	3374.40
5	4	1000	4246.46	4506.11	4779.33
6	5	1000	5416.32	5866.60	6352.85
7	6	1000	6632.98	7335.93	8115.19

3. Answer the question.

i. Future value at 4% is $6632.98.
ii. Future value at 8% is $7335.93.
iii. Future value at 12% is $8115.19.

$$\frac{7335.93 - 6632.98}{6632.98} \times 100 = 10.6\%$$

$$\frac{8115.19 - 7335.93}{7335.93} \times 100 = 10.62\%$$

Increasing the interest rate increases the future value of the annuity, with an approximate 10.6% increase for each 4% increase in the interest rate.

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Units 3 & 4 > Area 5 > Sequence 3 > Concept 1

Modelling an annuity using recursion Summary screen and practice questions

Exercise 9.3 Using recurrence relations to model annuities

Note: Assume the investment is at the end of the year for the following questions.

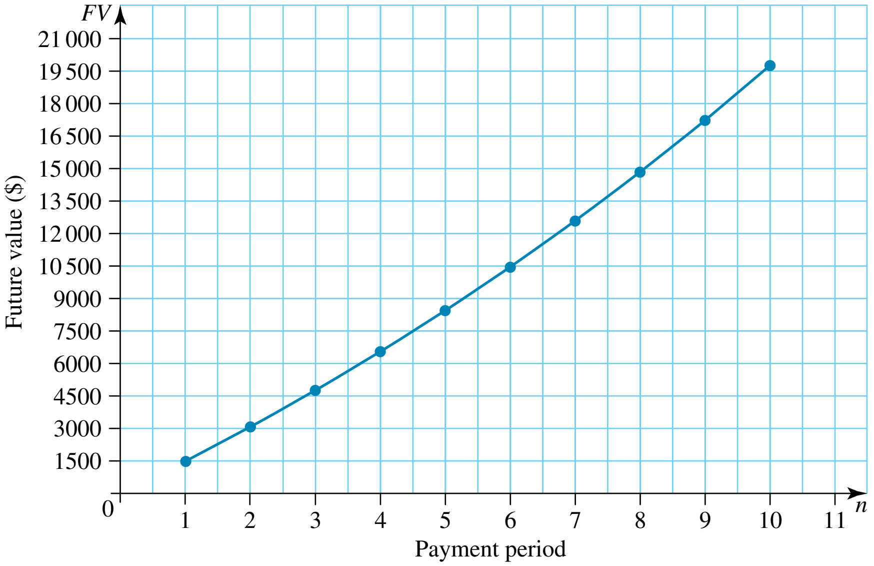
1. **WE4** At the end of each year, \$200 is invested at 6% p.a. compounded annually.
   - a. Set up a recurrence relation to represent the value of the annuity at the end of each payment.
   - b. Using the recurrence relation from part **a**, find the value, in dollars, of the annuity after the first 3 payments. Write your answers correct to the nearest cent.
2. The following recurrence relation represents the future value of an annuity.

$$A_{n+1} = 1.025A_n + 750, \; A_0 = 750$$

   - a. What is the interest rate per payment period?
   - b. Write down the regular payment amount, in dollars.
   - c. Using the recurrence relation, find the future value of the annuity after the first 3 payments, correct to the nearest cent.
3. Set up recurrence relations for each of the following annuities.
   - a. \$1500 invested each year at 8% p.a.
   - b. \$600 invested every 6 months at 4% p.a.
   - c. \$450 invested every month at 7.5% p.a.
   - d. \$600 invested quarterly at 9.5% p.a.

4. **MC** Which one of the following recurrence relations represents monthly \$500 payments on a \$25 000 car loan at 6% p.a. compounded monthly?
   - **A.** $A_{n+1} = 1.06A_n - 500, \; A_0 = 25\,000$
   - **B.** $A_{n+1} = 1.005A_n - 500, \; A_0 = 25\,000$
   - **C.** $A_{n+1} = 1.005A_n + 500, \; A_0 = 25\,000$
   - **D.** $A_{n+1} = 1.06A_n, \; A_0 = 25\,000$
5. **MC** The recurrence relation $A_{n+1} = 1.02A_n + 500, \; A_0 = 500$ represents the future value of an annuity with regular payments made 4 times a year. Which one of the following statements is correct?
   - **A.** The annual interest rate is 2%.
   - **B.** The interest rate per payment is 1.02%.
   - **C.** The future value of the annuity after the 3rd payment is \$1530.20.
   - **D.** The future value of the annuity after the 2nd payment is \$510.00.

6. **WE5** Using technology, determine the future values of the following annuities.
   a. \$1000 invested each year for 5 years at 4% p.a. compounded annually
   b. \$500 invested every 6 months for 4 years at 5% p.a. compounded half-yearly
   c. \$700 invested every 4 months for 3 years at 4.5% p.a. compounded every 4 months
7. **MC** The future value of an annuity of \$250 invested each month for 2 years at 5.5% p.a. compounded monthly has a future value, correct to the nearest dollar, of
   A. \$6347. **B.** \$6328. **C.** \$7052. **D.** \$8342.
8. Each of the following recurrence relations determines the future value of an annuity. For each relation, find the future value after the first 3 payments, correct to the nearest cent.
   a. $A_{n+1} = 1.005A_n + 100,\ A_0 = 100$
   b. $A_{n+1} = 1.0004A_n - 1650,\ A_0 = 250\,000$
   c. $A_{n+1} = 1.015A_n + 2000,\ A_0 = 3000$
9. The same sum of money is invested each year. The following graph shows the future value of an annuity for the first 10 years.



   a. What is the sum of money invested each period?
   b. After how many payments will the future value of the annuity first be greater than \$10 000?
   c. After the second payment, the value of the annuity is \$3090. Calculate the annual interest rate.
10. **MC** Instead of \$1000 invested at 6% p.a. compounded annually each year for 4 years, \$500 is invested every half year at 6% p.a. compounded every six months for 4 years. The effect that increasing the frequency of payments has on the future value of the annuity is
    A. \$72 (to the nearest dollar) more in interest.
    B. \$72 (to the nearest dollar) less in interest.
    C. \$574 (to the nearest dollar) less in interest.
    D. \$574 (to the nearest dollar) more in interest.
11. **WE6** A payment of \$2000 is invested at the end of each year at 5.5% p.a. compounded annually.
    a. Using technology, determine the future value of the annuity after 4 years.
    b. Construct a graph that compares the future value of the annuity after 4 years at an interest rate of 5.5% p.a. for the following investments.
       i. \$2000 invested each year, compounded annually
       ii. \$1000 invested every 6 months, compounded half-yearly
       iii. \$500 invested every 3 months, compounded quarterly
    c. Comment on the effect that changing the number of payments has on the future value of the annuity.

12. **WE7** Use technology to find the future value of \$100 invested each year for 10 years at the following interest rates, compounded annually. Hence, explain the effect that increasing interest rates has on the future value of an annuity.
    a. 5%
    b. 7.5%
    c. 10%
13. The future values of an annuity after the first 3 payments are as follows.

$$\$1000, \$2003, \$3009$$

    a. Show that $r = 1.003$ and hence state the interest rate, as a percentage, per payment period.
    b. Set up a recurrence relation that finds the future value of the annuity after each payment.
    c. Using the recurrence relation from part b, find the future value of the annuity after the fourth and fifth payments.
14. Mortgage payments are made monthly on a home loan. The amount owing on the home loan can be represented by the following recurrence relation.

$$A_{n+1} = 1.003125A_n - 1500,\ A_0 = 400\,000$$

    a. What is the amount of the loan?
    b. Show that the annual interest on the loan is 3.75%.
    c. Explain why mortgage payments are a type of annuity.
    d. Using the recurrence relation, determine the amount owing on the loan after 3 payments.

15. The following recurrence relation finds the future value of an annuity.

$$A_{n+1} = 1.02A_n + x,\ A_0 = x$$

    If $A_3 = \$1530.20$, show that $x = \$500$
16. Using technology, determine the future value of each of the following annuities after the final payment has been made. (For part c, assume 26 fortnights = 1 year, 26 weeks = $\frac{1}{2}$ year.)
    a. \$900 invested every 6 months for 4 years at 6% p.a. compounded every six months
    b. \$5000 invested every year for 10 years at 8% p.a. compounded annually
    c. \$100 invested every fortnight for 1year at 5% p.a. compounded fortnightly.
17. A payment of \$2000 is invested each year for 12 years at 7% p.a. compounded annually.
    a. Write down the calculations that determine the future value of the annuity after the first and second payments.
    b. Using technology, determine the future value of the annuity each year until maturity. Write your answers correct to the nearest cent.
18. The following calculation finds the future value of an annuity after the third payment when interest has been added and the payment made.

$$FV = 1.012 \times 503 + 250$$

    a. Write down the interest rate per payment period, as a decimal.
    b. What additional information is needed to determine the annual interest rate?
    c. Show that the first payment was \$250.
19. Monthly mortgage payments of \$1358 are paid towards a loan of \$200 000 that is compounded monthly at 3.25%. The following calculation shows the amount in dollars, A, of the balance of the loan after each payment.

$$A = 200\,000\,(1 + r)^1 - 1358$$

a. Show that the value of r is 0.002 708 and hence find the balance of the loan after the first payment.
b. Using technology, determine the balance of the loan for the first 12 months.
c. In the context of this problem, explain why mortgage payments are an example of an annuity.

20. Use technology to find the future value of \$500 invested each year for 10 years at the following interest rates, compounded annually. Hence, explain the effect that increasing interest rates has on the future value of an annuity.

a. 4% **b.** 5% **c.** 6%

21. The future values of an annuity after the first 3 payments are as follows.

$$\$650, \$1326, \$2029.04$$

a. Show that $r = 1.04$ and hence state the interest rate, as a percentage, per payment period.
b. Set up a recurrence relation that finds the future value of the annuity after each payment.
c. Using the recurrence relation from part **b**, find the future value of the annuity after the fourth and fifth payments.

22. Anna has a super fund and decides to increase her fortnightly contribution to \$800. The amount in her super fund after interest is added and the next payment is made is represented by the following recurrence relation:

$$t_{n+1} = 1.002\,692 t_n + 800, t_1 = 125\,700$$

a. How much was in Anna's super fund before she increased her fortnightly contribution to \$800?
b. Show that the annual interest on the loan, to the nearest whole percentage, is 7%. (Assume 26 weeks = 1 year.)
c. Explain why Anna's super fund contributions are an example of an annuity.
d. Using the recurrence relation, determine the balance in Anna's super fund after she makes the next 3 payments. Write your answer correct to the nearest cent.

9.4 Annuity and perpetuity calculations

9.4.1 Calculating the future value of an annuity using a formula

In this section, we will calculate the future value of an annuity using a formula. However, you will be asked to calculate the future value of an annuity using a table of future values in the exam. This is addressed in section 9.4.2.

The future value of an annuity can be calculated using the following formula.

Future value of an annuity

$$A = M\left(\frac{(1+i)^n - 1}{i}\right)$$

where M is the amount of each regular payment, i is the interest rate per payment period expressed as a decimal, and n is the number of payments.

WORKED EXAMPLE 8

Using the annuity future value formula, calculate the future value of $500 invested every six months for 10 years at 6% p.a. compounded six-monthly. Write your answer correct to the nearest cent.

THINK	WRITE
1. Identify the components of the annuity future value formula.	$M = 500$ $n = 2 \times 10 = 20$ $i = \frac{6}{200} = 0.03$
2. Substitute the values into the formula.	$A = M\left(\frac{(1+i)^n - 1}{i}\right)$ $= 500\left(\frac{(1+0.03)^{20} - 1}{0.03}\right)$ $= 13\,435.19$
3. Answer the question	In 10 years the future value of the annuity is $13 435.19.

Rearranging the annuity formula enables us to calculate the payment amount required to obtain a specific annuity future value:

Payment amount of annuity future value

$$M = A\left(\frac{i}{(1+i)^n - 1}\right)$$

where A is the desired future value of the annuity, i is the interest rate per payment period expressed as a decimal, n is number of payments, and M is the regular payment amount.

WORKED EXAMPLE 9

Calculate the annual contribution needed to obtain $50 000 in 10 years at 4.5% p.a. with interest compounded annually.

THINK	WRITE
1. Identify the key components of the formula.	$A = 50\,000, n = 10$ $i = \frac{4.5}{100} = 0.045$
2. Substitute the values into the formula.	$M = A\left(\frac{i}{(1+i)^n - 1}\right)$ $= 50\,000\left(\frac{0.045}{(1+0.045)^{10} - 1}\right)$ $= 4068.94$
3. Answer the question.	Annual regular payments of $4068.94 are needed to obtain $50 000 in 10 years at 4.5% interest compounded annually.

9.4.2 Table of future values

Future annuity values can also be found by creating a table that shows the future value of a $1 annuity invested per interest period.

To create a table of future values, set up a spreadsheet and insert the formula as shown.

	A	B	C	D	E	F	G	H	I	J	K
1	Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
2	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600	2.0700	2.0800	2.0900	2.1000
4	3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836	3.2149	3.2464	3.2781	3.3100
5	4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746	4.4399	4.5061	4.5731	4.6410
6	5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371	5.7507	5.8666	5.9847	6.1051
7	6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753	7.1533	7.3359	7.5233	7.7156
8	7	7.2135	7.4343	7.6625	7.8983	8.1420	8.3938	8.6540	8.9228	9.2004	9.4872
9	8	8.2857	8.5830	8.8923	9.2142	9.5491	9.8975	10.2598	10.6366	11.0285	11.4359
10	9	9.3685	9.7546	10.1591	10.5828	11.0266	11.4913	11.9780	12.4876	13.0210	13.5795
11	10	10.4622	10.9497	11.4639	12.0061	12.5779	13.1808	13.8164	14.4866	15.1929	15.9374
	Formula (type in cell B2 '= ((1 + B$1)^$A2 – 1)/B$1' and use the fill function to complete the table)										

WORKED EXAMPLE 10

Use the previous table to find the future value (A) of an annuity where $1500 is deposited at the end of each year into an account that pays 7% p.a. interest, compounded annually for 9 years.

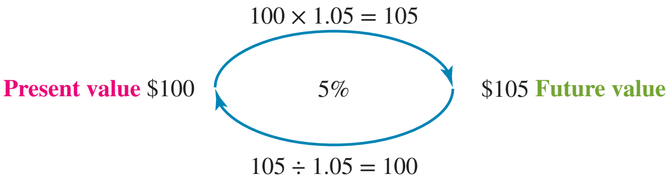
THINK	WRITE
1. Using the previous table, look up the future value of $1 at 7% p.a. for 9 years.	$A = \$1500 \times 11.9780$
2. Multiply this value by 1500.	$= \$17\,967$

9.4.3 Calculating the present value of an annuity

In this section, we will calculate the present value of an annuity using a formula. However, you may be asked to calculate the present value of an annuity using a table of present values in the exam. This is addressed in section 9.4.4

The **present value** of an annuity is the single sum of money that, invested on the same terms as the annuity, will produce the same future value.

For example, consider $100 invested at 5% for one year.



The future value of $105 in one year's time is $100 now. This means that for the payment to be worth $105 next year, $100 needs to be invested this year.

Consider 3 annual payments of \$1000 at 5% p.a.

The 1st payment of \$1000 in one year is currently worth $\frac{1000}{1.05} = \$952.38$.

The 2nd payment of \$1000 in 2 years is currently worth $\frac{1000}{1.05 \times 1.05} = \$907.03$.

The 3rd payment of \$1000 in 3 years is currently worth $\frac{1000}{1.05 \times 1.05 \times 1.05} = \$863.84$.

Therefore, 3 payments of \$1000 at 5% p.a. in 3 years' are worth 952.38 + 907.03 + 863.84 = \$2723.25.

In the context of present value annuities, this means that if \$2723.25 is invested as a single sum investment now at 5% p.a. compounded annually, it will have the same future value as the annuity of \$1000 invested each year for 3 years at 5% p.a.

Check:

If \$2723.25 is invested at compound interest of 5% p.a. compounded annually:

$$\begin{aligned} A &= 2723.25 \times 1.05^3 \\ &= 3152.50 \end{aligned}$$

If \$1000 is invested each year for 3 years at 5% p.a. compounded annually:

$$\begin{aligned} A &= M\left(\frac{(1+i)^n - 1}{i}\right) \\ &= 1000\left(\frac{(1+0.05)^3 - 1}{0.05}\right) \\ &= 3152.50 \end{aligned}$$

The following formula can be used to determine the present value of an annuity, PV:

Present value of an annuity

$$PV = M\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$

where M is the regular payment amount, i is the interest rate period per payment period expressed as a decimal, and n is the number of payments.

WORKED EXAMPLE 11

Using the formula for the present value of an annuity, determine the present value of an annuity of \$1500 per year for 10 years invested at 5% p.a. Hence, in the context of this problem, explain the present value.

THINK	WRITE
1. Identify the key components of the formula.	$M = 1500, \quad n = 10$ $i = \frac{5}{100} = 0.05$
2. Substitute the values into the formula.	$PV = M\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$ $= 1500\left(\frac{(1+0.05)^{10} - 1}{0.05(1+0.05)^{10}}\right)$ $= 11\,582.60$

3. Answer the question.	The present value of the annuity is \$11 582.60. This means that \$11 582.60 invested now as a single sum investment would have the same future value as an annuity of \$1500 invested each year for 10 years at 5% p.a.

9.4.4 Table of present values

As with a table of future values, a table of present values can be created to determine the present value of a \$1 annuity.

	A	B	C	D	E	F	G	H	I	J	K
1	Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
2	1	0.9901	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346	0.9259	0.9174	0.9091
3	2	1.9704	1.9416	1.9135	1.8861	1.8594	1.8334	1.8080	1.7833	1.7591	1.7355
4	3	2.9410	2.8839	2.8286	2.7751	2.7232	2.6730	2.6243	2.5771	2.5313	2.4869
5	4	3.9020	3.8077	3.7171	3.6299	3.5460	3.4651	3.3872	3.3121	3.2397	3.1699
6	5	4.8534	4.7135	4.5797	4.4518	4.3295	4.2124	4.1002	3.9927	3.8897	3.7908
7	6	5.7955	5.6014	5.4172	5.2421	5.0757	4.9173	4.7665	4.6229	4.4859	4.3553
8	7	6.7282	6.4720	6.2303	6.0021	5.7864	5.5824	5.3893	5.2064	5.0330	4.8684
9	8	7.6517	7.3255	7.0197	6.7327	6.4632	6.2098	5.9713	5.7466	5.5348	5.3349
10	9	8.5660	8.1622	7.7861	7.4353	7.1078	6.8017	6.5152	6.2469	5.9952	5.7590
11	10	9.4713	8.9826	8.5302	8.1109	7.7217	7.3601	7.0236	6.7101	6.4177	6.1446
	Formula (type in cell b2 '= ((1 + B\$1)^\$A2 – 1)/(B\$1*(1 + B\$1)^\$A2)' and use the fill function to complete the table)										

WORKED EXAMPLE 12

Liam invests \$750 per year into an annuity at 6% per annum for 8 years, with interest compounded annually. Use the previous table to calculate the present value of Liam's annuity.

THINK	**WRITE**
1. Use the table to find the present value of a \$1 annuity at 6% for 8 interest periods.	The present value of \$1 for 8 years is 6.2098.
2. Multiply this value by 750.	$\text{Present value} = \$750 \times 6.2098$ $= \$4657.35$

9.4.5 Perpetuities

A **perpetuity** is a special type of annuity where an amount of money is invested to provide regular payments that continue indefinitely. Scholarships and bursaries are forms of perpetuities.

The investment amount remains the same for a perpetuity as the funds paid out must never exceed the interest earned on the invested money.

The formula for the present value of an annuity is;

$$PV = M\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$

This can be expressed as:

$$PV = M\left(\frac{(1+i)^n}{i(1+i)^n} - \frac{1}{i(1+i)^n}\right)$$
$$= M\left(\frac{1}{i} - \frac{1}{i(1+i)^n}\right)$$

But $i(1+i)^n$ is a very large number when n is a very large number, so $\frac{1}{i(1+i)^n}$ is very close to zero when n is very large.

So for large values of n, $PV = \frac{M}{i}$.

Perpetuity formula

$$A = \frac{M}{i}$$

where
A is the amount invested for an indefinite period
M is the amount of the regular payment per period
i is the interest rate per period.

WORKED EXAMPLE 13

A wealthy businesswoman decides to set up a scholarship fund to assist a disadvantaged young person in her community. She invests in a perpetuity that is guaranteed an interest rate of 6% p.a. Determine the amount to be invested in this perpetuity to provide a regular monthly income of $500.

THINK	WRITE
1. Write the perpetuity formula	$A = \frac{M}{i}$
2. Write the values of M and i.	$M = 500,\ i = \frac{\frac{6}{12}}{100} = 0.005$
3. Substitute the values of M and I into the perpetuity formula.	$A = \frac{500}{0.005} = 100\,000$
4. Write the answer.	An investment of $100 000 is required to provide a monthly income of $500.

The perpetuities formula can be rearranged to make the interest rate or the payment per period the subject.

$$M = A \times i \qquad i = \frac{M}{A}$$

WORKED EXAMPLE 14

Genevieve's grandparents set up a perpetuity to help pay for her tertiary education. They invest \$125 000 at an interest rate of 4.5% p.a., with the interest rate for this perpetuity compounded quarterly.

a. **Calculate the amount Genevieve can pay off her university fees each quarter.**

b. **Her grandparents want Genevieve to receive \$2000 each quarter and so decide to shop around for a better interest rate. Calculate the interest rate that is needed to ensure a quarterly payment of \$2000.**

THINK	WRITE
a. 1. Write the perpetuity formula with M as the subject.	**a.** $M = A \times i$
2. Write the values of A and i.	$A = 125\,000,\ i = \dfrac{\frac{4.5}{4}}{100} = 0.01125$
3. Substitute the values of A and i into the perpetuity formula.	$M = 125\,000 \times 0.01125 = 1406.25$
4. Write the answer.	Genevieve will be able to pay \$1406.25 off her university fees each quarter.
b. 1. Write the perpetuity formula with i as the subject.	**b.** $i = \dfrac{M}{A}$
Write the values of A and M.	$A = 125\,000,\ M = 2000$
Substitute the values of A and M into the perpetuity formula.	$i = \dfrac{2000}{125\,000} = 0.016$
$i = \dfrac{\frac{\text{interest rate}}{n}}{100}$	Interest rate $= 0.016 \times 100 \times 4 = 6.4\%$
Write the answer.	The interest rate required is 6.4% p.a.

studyon

Units 3 & 4 > Area 5 > Sequence 3 > Concepts 2, 3 & 4

Perpetuities Summary screen and practice questions
Annuity calculations Summary screen and practice questions
Applications of annuities Summary screen and practice questions

Exercise 9.4 Annuity and perpetuity calculations

1. **WE8** Using the future value annuity formula, calculate the future value of each of the following annuities on maturity. Write your answers correct to the nearest cent.

- **a.** \$400 invested at the end of every six months for 12 years at 12% p.a. with interest compounded six monthly
- **b.** \$1000 invested at the end of every quarter for 5 years at 8% p.a. with interest compounded every quarter
- **c.** \$2500 invested at the end of each quarter at 7.2% p.a. for 4 years with interest compounded quarterly
- **d.** \$1000 invested at the end of every month for 5 years at 6% p.a. with interest compounded monthly

2. **MC** Tracey invests \$500 into a fund at the end of each year for 20 years. The fund pays 12% p.a. interest, compounded annually. The total amount of interest that Tracey earns on this fund investment is

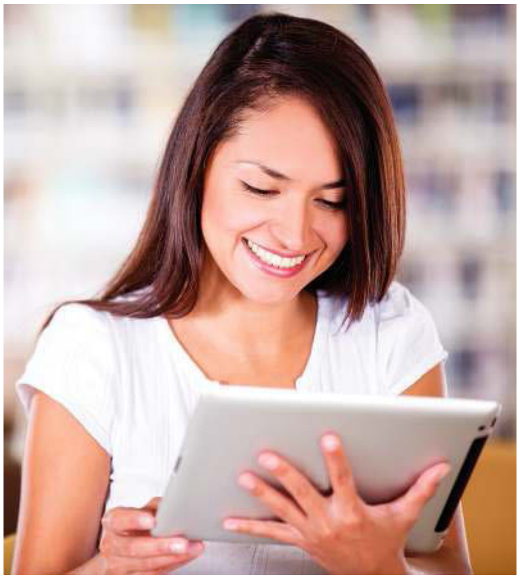
A. \$1200.
B. \$4323.15.
C. \$4823.23.
D. \$26 026.22.

3. **WE9** Calculate the amount of each annual contribution needed to obtain each of the following amounts, correct to the nearest cent.
   a. \$25 000 in 5 years at 5% p.a., with interest compounded annually
   b. \$100 000 in 10 years at 7.5% p.a., with interest compounded annually
   c. \$500 000 in 40 years at 8% p.a., with interest compounded annually

4. Thomas has the goal of saving \$400 000 for his retirement in 25 years. If the best interest rate that Thomas can obtain is 10% p.a., with interest compounded annually, calculate the amount of each annual contribution that Thomas will need to make.
5. **WE10** Use the table of future values in section 9.4.2 to determine the future value of an annuity of \$800 invested per year for 5 years at 9% p.a., with interest compounded annually.
6. Use the table of future values to determine the future value of each of the following annuities.
   a. \$400 invested per year for 3 years at 10% p.a., with interest compounded annually
   b. \$2250 invested per year for 8 years at 8% p.a., with interest compounded annually
   c. \$625 invested per year for 10 years at 4% p.a., with interest compounded annually
   d. \$7500 invested per year for 7 years at 6% p.a., with interest compounded annually

7. **MC** Use the table of future values to determine which of the following annuities will have the greatest financial outcome.
   A. 5% p.a. for 9 years, with interest compounded annually
   B. 6% p.a. for 8 years, with interest compounded annually
   C. 8% p.a. for 6 years, with interest compounded annually
   D. 7% p.a. for 7 years, with interest compounded annually
8. **WE11** Using the formula $PV = M\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$, determine the present value of an annuity of \$1250 per year for 8 years invested at 9% p.a.
9. **WE12** Use the table of present values in section 9.4.4 to determine the present value of each of the following annuities.
   a. \$450 per year for 5 years at 7% p.a., with interest compounded annually
   b. \$2000 per year for 10 years at 10% p.a., with interest compounded annually
   c. \$850 per year for 6 years at 4% p.a., with interest compounded annually
   d. \$3000 per year for 8 years at 9% p.a., with interest compounded annually
10. **MC** Which of the following present values gives the same value as an annuity of \$2000 invested every six months for 4 years at 6% p.a. compounded half-yearly?
    A. \$6930.20 B. \$7434.20 C. \$12 419.60 D. \$14 039.40
11. **WE13** A college alumni decides to set up a scholarship fund for a gifted musician at his old college, He invests in a perpetuity that is guaranteed an interest rate of 7.5% p.a., compounded every six months. Determine the amount to be invested in this perpetuity to provide a regular six-monthly income of \$3000.
12. The director of a theatre uses some of an inheritance to create a drama scholarship for talented young actors. She invests in bonds that pay 8% p.a. and secures a payment each year of \$22 000. How much did she invest in the bonds?

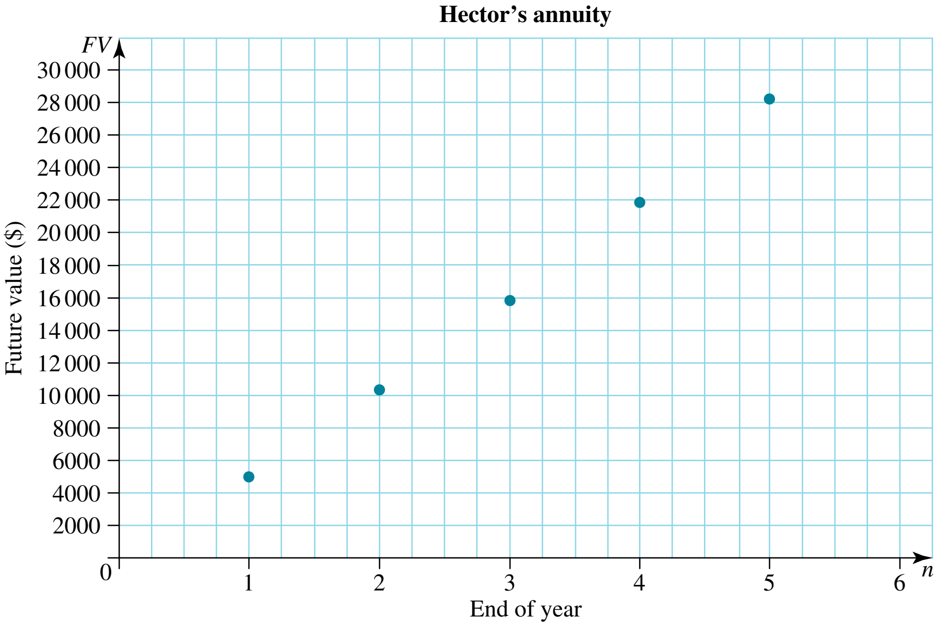
13. **WE14** A local Lions Club donates $100 000 to the football and netball club to be invested and hence provide an ongoing source of money to provide uniforms and sports shoes to junior players. The club invests in a perpetuity paying an interest rate of 5.5% p.a., with the interest rate for this perpetuity compounded quarterly.
    a. Calculate the amount the club receives each quarter.
    b. The club need $3000 each quarter and so decide to shop around for a better interest rate. Calculate the interest rate that is needed to ensure a quarterly payment of $3000.
14. Use the perpetuity formula to calculate the payment as specified in each of the following.
    a. $150 000 invested at 7.2%, compounded monthly.
    b. $350 000 invested at 12% p.a. compounded six monthly and paid out annually.
15. Samantha invests $500 every 6 months for 5 years into an annuity at 8% p.a. with interest compounded every 6 months.



    a. What is the interest rate, as a percentage, per interest period?
    b. How many interest periods are there in Samantha's annuity?
    c. Use the table of future values in section 9.4.2 to determine the future value of Samantha's annuity.
    d. Samantha's financial institution offers to increase her rate to 9% p.a. To calculate the future value of the annuity, Samantha finds the future value of \$1 at 4% and at 5% and then finds the average. Determine the future value of Samantha's annuity:
        i. using her method
        ii. using the formula $A = M\left(\frac{(1+i)^n - 1}{i}\right)$.
    e. Comment on your results from part d
16. Use the table of future values to determine whether an annuity at 5% p.a. for 6 years or an annuity at 6% p.a. for 5 years will produce the greater financial outcome. Explain your answer by calculating $(1 + r)^n$, correct to 4 decimal places.
17. Jacinta's financial adviser has told her that if she plans on retiring in 25 years, she will need $500 000 more in her retirement fund to live comfortably. The interest rate on her retirement fund is 6% p.a. and she intends to make monthly payments.
    a. Show that if Jacinta's monthly contribution is $721.51, correct to the nearest cent, the future value of the annuity in 25 years will be $500 000.
    b. Calculate the present value of Jacinta's annuity. In the context of this problem, explain what the present value means.
18. Emerton makes monthly contributions of $950 into a compounding investment at 5% p.a. compounded monthly. His investment will mature in 10 years.

    a. Calculate the future value of Emerton's annuity using the formula $A = M\left(\frac{(1+i)^n - 1}{i}\right)$. Write your answer correct to the nearest cent.
    b. Emerton's friend Gilroy decides to invest a sum of money at the same interest rate and conditions as Emerton. If the future value of Gilroy's investment is the same as Emerton's, how much does Gilroy invest? Explain your method.

19. After speaking to his financial adviser, Hector wants \$275 000 more in his superannuation upon retirement in 25 years. His super fund is expected to grow at 6% per annum. Hector makes regular and equal annual contributions to his fund. The following graph shows the future value of Hector's annuity for the first 5 years, after speaking to his financial adviser.



a. Write down Hector's annual contribution.

b. The value of Hector's annuity after the second payment is \$10 300, correct to the nearest dollar. Show that Hector's annuity is growing at 6% p.a. compounded annually.

c. If Hector's annuity continues to grow at 6% p.a. compounded annually, the value of the annuity after 24 years will be \$254 078, correct to the nearest dollar. Calculate the future value of the annuity on maturity, correct to the nearest dollar.

d. After 10 years, a change to interest rates increases the growth of Hector's super fund to 7% p.a. compounded annually. After 24 years, Hector's super fund is \$282 688. How much will be in Hector's super fund upon maturity? Write your answer correct to the nearest dollar.

20. Use the perpetuity formula to calculate the interest rate p.a. required for each of the following.

a. \$200 000 provides \$3500 per annum with interest compounded annually

b. \$600 000 provides \$25 000 every six months with interest compounded biannually.

c. \$350 000 provides \$500 per fortnight with interest calculated fortnightly.

21. Use a table of future values to determine the future values of the following annuities.

a. \$1250 invested each year for 4 years at 3% p.a. with interest compounded annually

b. \$2000 invested every year for 10 years at 7% p.a. with interest compounded annually

22. A sum of money was invested each year at 4% p.a. for 5 years. Upon maturity the value of the annuity was \$3249.78. Calculate the sum of money invested.

9.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** An annuity is
   - **A.** a form of compound investment.
   - **B.** investing for retirement.
   - **C.** a single sum investment.
   - **D.** regular payments made to an investment which is compounding.
2. **MC** The future value of \$500 invested every 6 months compounding biannually at 4% p.a. after 2 years is:
   A. \$1506.38. **B.** \$2060.80. **C.** \$3415.60. **D.** 4001.38.
3. **MC** The future value of \$300 invested every 3 months for 1 year at 6% p.a. compounded quarterly is
   A. \$1227.27. **B.** \$1010.49. **C.** \$655.32. **D.** \$2114.36.
4. **MC** If \$1000 is invested every year for 5 years at 3% p.a. compounded annually, the future value of the annuity after the third payment, correct to the nearest dollar, is:
   A. \$1093. **B.** \$2060. **C.** \$3091. **D.** \$3184.

Use the following information to answers questions 5 & 6.
The following recurrence relation represents the future annuity value of a car loan.

$$A_{n+1} = 1.005A_n - 375,\ A_0 = 25\,450$$

5. **MC** The regular payment amount in dollars is
   A. \$1005. **B.** \$375. **C.** \$25 450. **D.** \$50.
6. **MC** The future value of the annuity after the first 3 payments is closest to:
   A. \$30 499. **B.** \$26 100. **C.** \$25 450. **D.** \$24 703.
7. **MC** Which one of the following recurrence relations represents \$1000 invested every 3 months at 7.5% p.a. compounded quarterly?
   - **A.** $A_{n+1} = 1.075A_n + 1000,\ A_0 = 1000$
   - **B.** $A_{n+1} = 1.018\,75A_n + 1000, A_0 = 1000$
   - **C.** $A_{n+1} = 1.075A_n + 250,\ A_0 = 250$
   - **D.** $A_{n+1} = 1.018\,75A_n + 250,\ A_0 = 250$
8. **MC** The future annuity value of a \$19 500 car loan compounded monthly at 5.75% p.a. with monthly repayments of \$380 after the second payment, correct to the nearest dollar, is
   A. \$18 740. **B.** \$18 926. **C.** \$19 213. **D.** \$19 861.
9. **MC** The following recurrence relation determines the future value of an annuity.

$$A_{n+1} = 1.0125A_n + 450,\ A_0 = 450$$

   Which one of the following shows the values of the annuity, correct to the nearest dollar, after each of the first 4 payments?
   - **A.** 450, 456, 461, 467
   - **B.** 450, 900, 1350, 1800
   - **C.** 456, 911, 1373, 1840
   - **D.** 450, 906, 1367, 1834

10. **MC** Grei invests \$200 into a fund at the end of each fortnight for 15 years. The fund pays 8% p.a. interest compounded fortnightly (assume 26 fortnights = 1 year). The total amount of interest, correct to the nearest dollar, that Grei earns on this investment is
   - **A.** \$72 410.
   - **B.** \$78 000.
   - **C.** \$145 217.
   - **D.** \$150 410.

11. **MC** The amount of each annual contribution, correct to the nearest dollar, needed to obtain \$150 000 in 12 years at 6% p.a. compounded annually is
   A. \$8565. **B.** \$8714. **C.** \$8892. **D.** \$11 792.

12. **MC** Using a table of future values, the future value of \$1500 invested every six months for 4 years at 8% p.a. compounded every six months is
   A. \$6369.75. **B.** \$6759.15. **C.** \$10 099.05. **D.** \$13 821.30.

Complex familiar

13. A payment of \$1500 is being invested at the end of every year at 6% p.a. for 4 years. Complete the table that shows the future value of the annuity every year for 4 years. Write your answers correct to the nearest cent.

Payment, *n*	**Payment amount (\$)**	**Future value (\$)**
1	1500	1500
2	1500	
3	1500	
4	1500	

14. For each of the following investments, set up a recurrence relation that represents the value of the annuity at the end of each payment. (For part **c**, assume 26 fortnights = 1 year.)
   - **a.** \$450 invested each month at 5.75% p.a. compounded monthly
   - **b.** \$2000 invested at the end of each year at 4% p.a. compounded annually
   - **c.** \$150 invested every fortnight at 6% p.a. compounded fortnightly

15. Using the future value annuity formula, $A = M\left(\frac{(1+i)^n - 1}{i}\right)$ or a table, calculate the future value, correct to the nearest cent, of each of the following annuities on maturity. (For part **d**, assume 52 weeks = 1 year.)
   - **a.** \$1600 invested at the end of each year for 10 years at 6% p.a. compounded annually
   - **b.** \$2500 invested at the end of every quarter for 6 years at 8% p.a. compounded quarterly
   - **c.** \$700 invested at the end of every month for 12 years at 4% p.a. compounded monthly
   - **d.** \$100 invested at the end of each week for 5 years at 3% p.a. compounded weekly

16. Determine the amount of contribution, correct to the nearest cent, needed to obtain each of the following.
   - **a.** \$10 000 in 4 years at 5% compounded annually
   - **b.** \$250 000 in 20 years at 7% compounded monthly

17. Using the formula $PV = M\left(\frac{(1+i)^n - 1}{r(1+i)^n}\right)$ or a table, determine the present value, correct to the nearest cent, of an annuity of \$3500 per year for 12 years invested at 8% p.a.

Complex unfamiliar

18. Use a table of present values to determine the present value of each of the following annuities
  a. \$2000 per year for 6 years at 4% p.a. with interest compounded annually
  b. \$500 per year for 8 years at 7% p.a. with interest compounded annually

19. A payment of \$1000 is invested at the end of each year at 5% p.a. compounded annually.
  a. Using technology, determine the future value of the annuity after 4 years
  b. Construct a graph that compares the future value of the annuity after 4 years, at an interest rate of 5% p.a. for the following investments.
    i. \$1000 being invested each year, compounded annually
    ii. \$500 being invested every 6 months, compounded half-yearly
    iii. \$250 being invested every 3 months, compounded quarterly
  c. Comment on the effect that changing the number of payments has on the future value of the annuity.

20. According to Jett's financial adviser, over the next 15 years Jett will need to save \$275 000 in his super fund, which grows at 6% p.a. compounded monthly.
  a. Calculate Jett's monthly contributions, correct to the nearest cent.
  b. Calculate the present value of the \$275 000 matured value at 6% p.a. compounded monthly. Write your answer correct to the nearest dollar.
  c. At the start of the 15 years, Jett decides to sell his rental investment property and invest this money. How much, in dollars, does he need to sell his property for so that he will have the same amount of money as his super fund upon maturity? Write your answer correct to the nearest dollar.

21. In 10 years, Billie would like to go on the trip of a lifetime and spend 12 months visiting every continent. Her travel agent suggests that she will need at least \$85 000. Billie makes regular monthly contributions at the end of each month towards her trip at 5% p.a. compounded monthly.

  a. Calculate the monthly contribution, correct to the nearest cent, that Billie needs to make so that she will have the suggested amount by the end of the 10 years.
  b. Billie's budget allows her to save \$800 each month. She invests this money at 5% p.a. compounded monthly. Determine how much in dollars, correct to the nearest cent, Billie will have for her trip at the end of the 10 years.
  c. Billie makes her monthly payments for 5 years, but then a change in her circumstances means that she can no longer save \$800 each month. Determine the value of her annuity after 5 years. Write your answer correct to the nearest cent.

d. After her change in circumstances, Billie can make monthly payments of \$200 at the same interest rate of 5% p.a.

  i. Write a recurrence relation that find the future value of Billie's annuity with the adjusted monthly payment of \$200.

  ii. The value of Billie's annuity, correct to the nearest cent, after 58 payments is \$82 897.11. Show that Billie will have to postpone her trip by 2 months.

studyon

Units 3 & 4 Sit exam

Answers

9 Annuities and perpetuities

Exercise 9.2 Introduction to annuities

1. a. Yes
   b. Yes
   c. No
   d. No
2. a. $2680.19
   b. $1757.49
   c. $11 488.82
   d. $4487.55
3. C
4. $2211.77
5. C
6. $4472.93
7. 1st month: $18 828.96, 2nd month: $18 707.42, 3rd month: $18 585.37, 4th month: $18 462.82
8. B
9.

Payment no.	Payment ($)	Future value ($)
1	5000	5000.00
2	250	5265.63
3	250	5532.08
4	250	5799.37
5	250	6067.49
6	**250**	**6336.45**
7	**250**	**6606.25**

10. D
11. a. No regular and equal payments are being made to the compounded investment.
    b. This could be an annuity if Annie made regular and equal payments to the compounded investment.
    An example to make this an annuity is:
    Annie invests $5000 each month at 3.5% p.a. compounded monthly.
12. a. Second payment: $FV = 1500 \times 1.06 + 1500$
    Third payment: $FV = 3090 \times 1.06 + 1500$
    b. i. The formula does not include the payment of $1500 being made after interest is calculated.
       ii. ‘$= 1.06 * c2 + b3$’
    c. Payment 1: $1500, payment 2: $3090.00, payment 3: 4775.40, payment 4: $6561.92, payment 5: $8455.64, payment 6: $10 462.98, payment 7: $12 590.76, payment 8: $14 846.20, payment 9: $17 236.97, payment 10: $19 771.19
13. a. Sample responses can be found in the worked solutions in the eBookPLUS.
    b. How many payments are made each year (annually)
    c. Sample responses can be found in the worked solutions in the eBookPLUS.
14. a. Regular weekly payments are made of 9.5% of the employees’ wage.
    b. $85.03
    c. $i = 0.001\,25, Q = \$85.03$
    Super fund balance after next weekly contribution: $206 192.34
15. a. $249 296.25
    b. 1st payment: $249 296.25, 2nd payment: $248 590.30, 3rd payment: $247 882.15, 4th payment: $247 171.78, 5th payment: $246 459.19, 6th payment: $245 744.37, 7th payment: $245 027.33, 8th payment: $244 308.04, 9th payment: $243 586.50, 10th payment: $242 862.71, 11th payment: $242 136.65, 12th payment: $241 408.33
    c. Regular and equal monthly payments are made.
16. $6764.51

Exercise 9.3 Using recurrence relations to model annuities

1. a. $A_{n+1} = 1.06A_n + 200, A_0 = 200$
   b. 1st payment: $200, 2nd payment: $412, 3rd payment: $636.72
2. a. 2.5%
   b. $750
   c. 1st payment: $750, 2nd payment: $1518.75, 3rd payment: $2306.72
3. a. $A_{n+1} = 1.08A_n + 1500, A_0 = 1500$
   b. $A_{n+1} = 1.02A_n + 600, A_0 = 600$
   c. $A_{n+1} = 1.006\,25A_n + 450, A_0 = 450$
   d. $A_{n+1} = 1.023\,75A_n + 600, A_0 = 600$
4. B
5. C
6. a. $5416.32
   b. $4368.06
   c. $6591.07
7. A
8. a. $301.50
   b. $246 899.38
   c. $7120.68
9. a. $1500
   b. 6 payments
   c. 6%
10. A

11. a. \$8684.53
  b. **i, ii, iii.** *See the diagram at the bottom of the page.
  c. If payments are more frequent, the future value of the annuity increases. Making 4 payments per year results in a future value of \$8880.38, which is \$195.85 more than the result from \$2000 once a year and \$66.54 more than the result from \$1000 twice a year.

12. a. \$1257.79
  b. \$1414.71
  c. \$1593.74
  Increasing the interest rate increases the future value of the annuity, with an approximate 5% increase with each 1% increase in interest rate.

13. a. Interest rate = 0.3% per payment period
  b. $A_{n+1} = 1.003A_n + 1000, A_0 = 1000$
  c. 4th payment: \$4018.03, 5th payment: \$5030.08

14. a. \$400 000
  b. Sample responses can be found in the worked solutions in the eBookPLUS.
  c. This is an example of an annuity because there are regular and equal monthly payments.
  d. \$399 247.65

15. Sample responses can be found in the worked solutions in the eBookPLUS.

16. a. \$8003.10
  b. \$72 432.81
  c. \$2663.47

17. a. 1st payment: $FV = \$2000 \times 1.07 + 2000$
  2nd payment: $FV = \$4140 \times 1.07 + 2000$
  b. 1st payment: \$2000.00, 2nd payment: \$4140.00, 3rd payment: \$6429.80, 4th payment: \$8879.89, 5th payment: \$11 501.48, 6th payment: \$14 306.58, 7th payment: \$17 308.04, 8th payment: \$20 519.61, 9th payment: \$23 955.98, 10th payment: \$27 632.90, 11th payment: \$31 567.20, 12th payment: \$35 776.90

18. a. 0.012
  b. The number of payment periods per year
  c. Sample responses can be found in the worked solutions in the eBookPLUS.

19. a. $i = \dfrac{3.25}{1200}$
  $= 0.002708$
  After 1st payment: \$199 183.60
  b. 1st month: \$199 183.60, 2nd month: \$198 364.99, 3rd month: \$197 544.16, 4th month: \$196 721.11, 5th month: \$195 895.83, 6th month: \$195 068.32, 7th month: \$194 238.56, 8th month: \$193 406.56, 9th month: \$192 572.31, 10th month: \$191 735.79, 11th month: \$190 897.01, 12th month: \$190 055.96
  c. There are equal and regular monthly payments of \$1358.

20. a. \$6003.05
  b. \$6288.95
  c. \$6590.40
  Increasing the interest rate increases the future value of the annuity, with an approximate 4.8% increase for each 1% increase in interest rate.

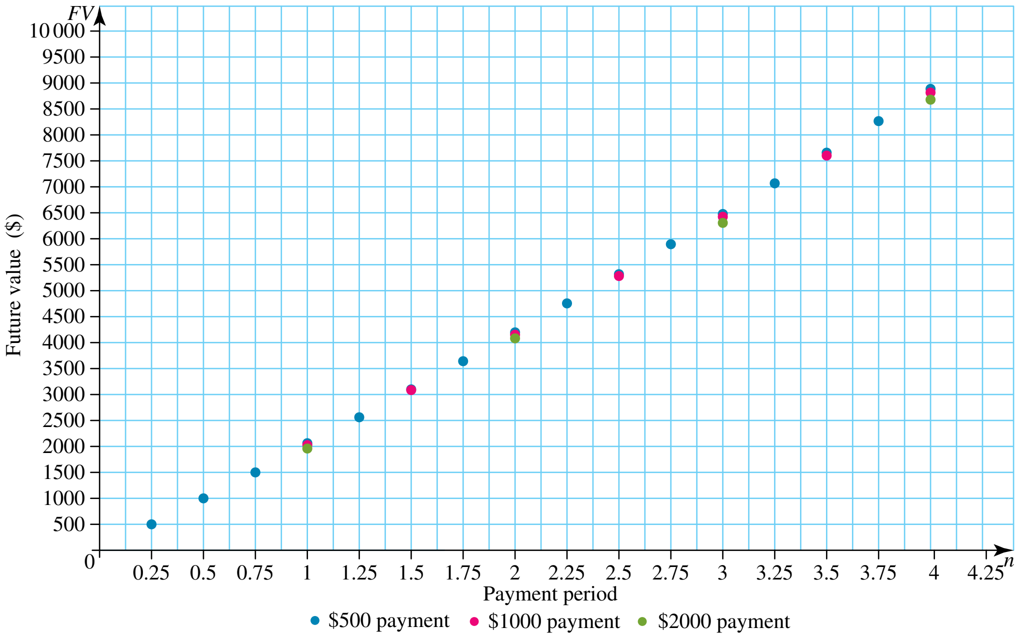
21. a. 4%
  b. $A_{n+1} = 1.04A_n + 650, A_0 = 650$
  c. 4th payment: \$2760.20, 5th payment: \$3520.61

22. a. \$125 700
  b. Sample responses can be found in the worked solutions in the eBookPLUS.
  c. Anna is making regular and equal fortnightly payments of \$800.
  d. \$129 124.36

Exercise 9.4 Annuity and perpetuity calculations

1. a. \$20 326.23 **b.** \$24 297.37
  c. \$45 881.32 **d.** \$69 770.03
2. D
3. a. \$4524.37 **b.** \$7068.59 **c.** \$1930.08
4. \$4067.23
5. \$4787.76

***11. b. i, ii, iii**



6. a. $1324 b. $23 932.35
   c. $7503.81 d. $62 953.50

7. A

8. $6918.52

9. a. $1845.09 b. $12 289.20
   c. $4455.79 d. $16 604.40

10. D

11. $80 000

12. $275 000

13. a. $1375 b. 12%

14. a. $900 b. $42 000

15. a. 4% b. 10
   c. $6003.05
   d. i. $6146
      ii. $6144.10
   e. The results from the two methods are similar but not the same ($6146 compared to $6144.10).

16. 5% for 6 years will have a greater value than 6% for 5 years, as $1.05^6 > 1.06^5$.

17. a. Sample responses can be found in the worked solutions in the eBookPLUS.
   b. $111 983.30; this means that $111 983.20 invested now as a single sum investment would have the same future value as an annuity of $721.51 invested each month for 25 years at 6% p.a. compounded monthly.

18. a. $147 549.74 b. $89 550.77
   Method shown in the worked solutions in your eBookPLUS.

19. a. $5000
   b. Sample responses can be found in the worked solutions in the eBookPLUS.
   c. $274 323
   d. $307 476.16

20. a. 1.75% b. 8.33% c. 3.7%

21. a. $5229.50 b. $27 632.80

22. $600

Exercise 9.5 Review: exam practice

1. D
2. B
3. A
4. C
5. B
6. D
7. B
8. B
9. D
10. A
11. C
12. D
13.

Payment, n	Payment amount ($)	Future value ($)
1	1500	1500.00
2	1500	**3090.00**
3	1500	**4775.40**
4	1500	**6561.92**

14. a. $A_{n+1} = 1.004\,79A_n + 450, A_0 = 450$
   b. $A_{n+1} = 1.04A_n + 2000, A_0 = 2000$
   c. $A_{n+1} = 1.002\,308A_n + 150, A_0 = 150$

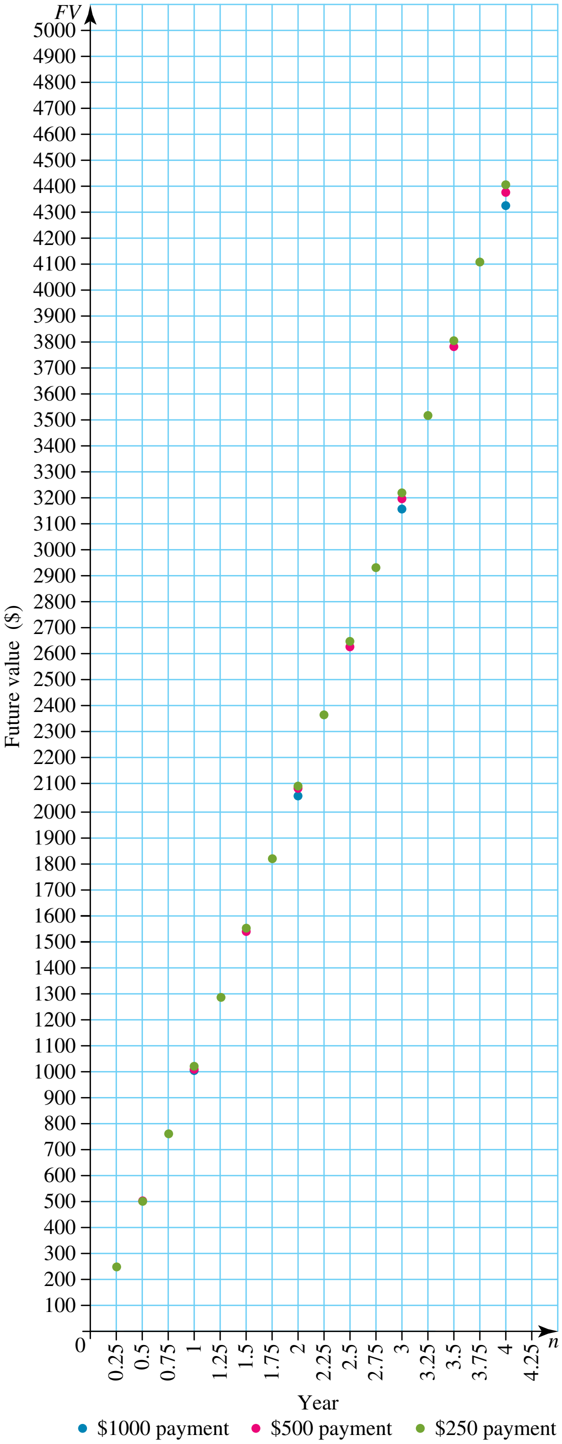
15. a. $21 089.27 b. $76 054.66
   c. $129 104.83 d. $28 042.56

16. a. $2320.12 b. $479.91

17. $26 376.27

18. a. $10 484.20 b. $2985.65

19. a. $4310.13
   b.



c. If payments are more frequent, the future value of the annuity increases. Making 4 payments per year results in a future value of \$4397.79, which is \$29.73 more than the result of \$500 payments twice a year and \$87.66 more than the result of a \$1000 payment once a year.

20. **a.** \$945.61

b. \$112 058

c. Jett will need to sell his rental for \$112 058, which is the present value from part **b**.

21. **a.** \$547.38

b. \$124 228.48

c. \$54 405.42

d. **i.** $A_{n+1} = 1.004\,167A_n + 200, A_0 = 54\,405.42$

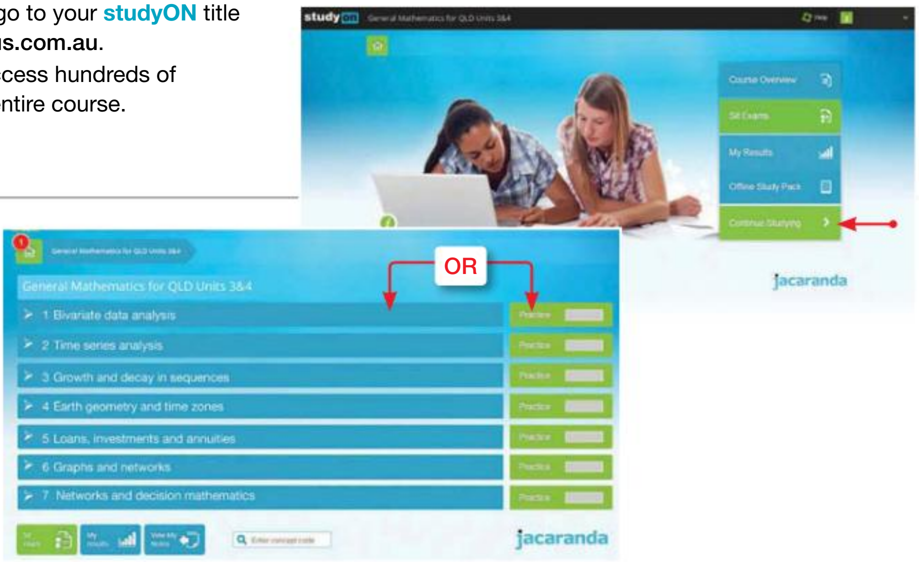
ii. Sample responses can be found in the worked solutions in the eBookPLUS.

REVISION UNIT 4 Investing and networking

TOPIC 1 Loans, investments and annuities

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

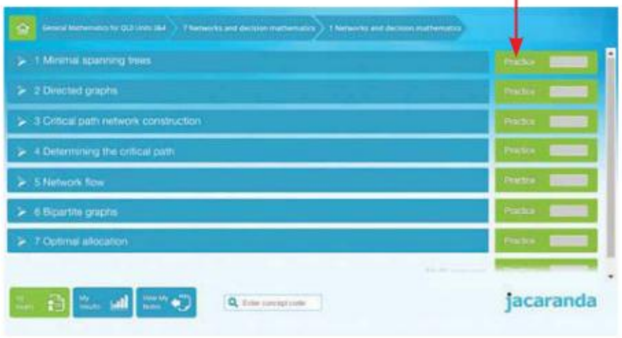
- Select your **course** *General Mathematics for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.



General Mathematics for QLD Units 3&4

- 1 Bivariate data analysis
- 2 Time series analysis
- 3 Growth and decay in sequences
- 4 Earth geometry and time zones
- 5 Loans, investments and annuities
- 6 Graphs and networks
- 7 Networks and decision mathematics

- Select **Practice** at the sequence level to access all questions in the sequence.

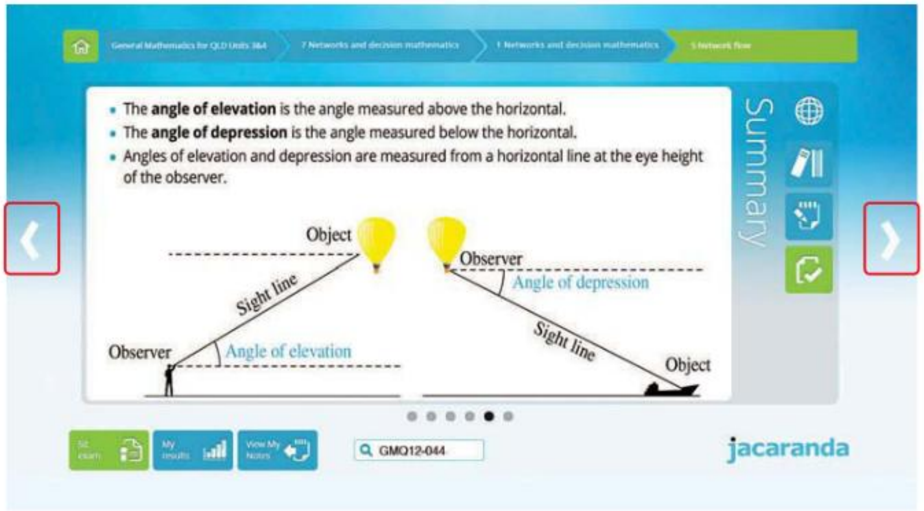


- 1 Minimal spanning trees
- 2 Directed graphs
- 3 Critical path network construction
- 4 Determining the critical path
- 5 Network flow
- 6 Bipartite graphs
- 7 Optimal allocation

- At **sequence level**, drill down to concept level.

- 7 Networks and decision mathematics
  - 1 Networks and decision mathematics

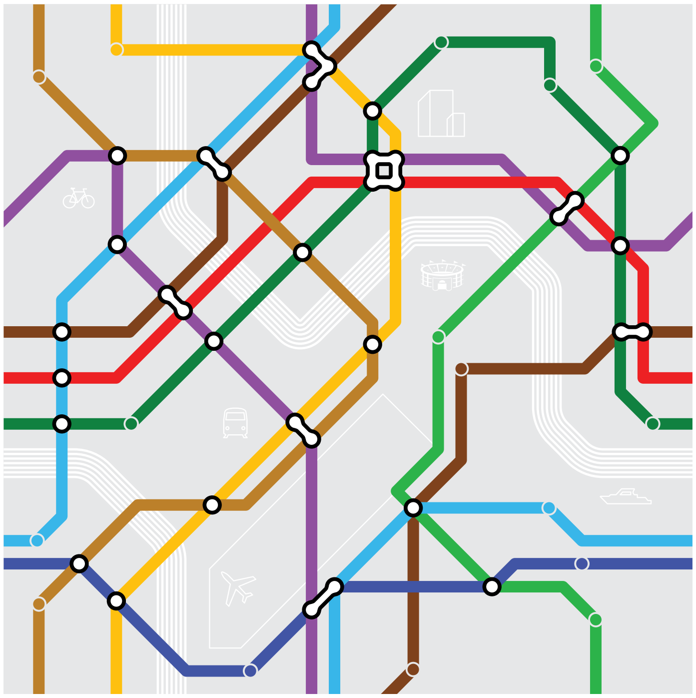
- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



- The **angle of elevation** is the angle measured above the horizontal.
- The **angle of depression** is the angle measured below the horizontal.
- Angles of elevation and depression are measured from a horizontal line at the eye height of the observer.

10 Graphs and networks

10.1 Overview



Just like matrices, networks are used to show how things are connected. The idea of networks and graph theory is usually credited to Leonhard Euler's 1736 work, *Seven Bridges of Königsberg*. This work carried on with the analysis situs initiated by Leibniz.

One of the most famous problems in graph theory is the 'four colour problem', which poses the question: 'Is it true that any map drawn in the plane may have its regions coloured with four colours, in such a way that any two regions having a common border have different colours?' This question was first posed by Francis Guthrie in 1852. There have been many failed attempts to prove this. The 'four colour problem' remained unsolved for more than a century, until in 1969 Heinrich Heesch published a method for solving the problem using a computer. Computers allowed networks to be used to solve problems that previously took too long due to the multitude of combinations.

Procedures called algorithms are applied to networks to find maximum and minimum values. This further study of constructed networks belongs to a field of mathematics called operational research. This developed rapidly during and after World War II, when mathematicians, industrial technicians and members of the armed services worked together to improve military operations.

In more recent times, graph theory and networks have been used to deliver mail around the neighbourhood, land people on the moon, organise train timetables and improve the flow of traffic. Graph theory and networks have also been applied to a wide range of disciplines, from social networks, used to examine the structure of relationships and social identities, to biological networks, which analyse molecular networks.

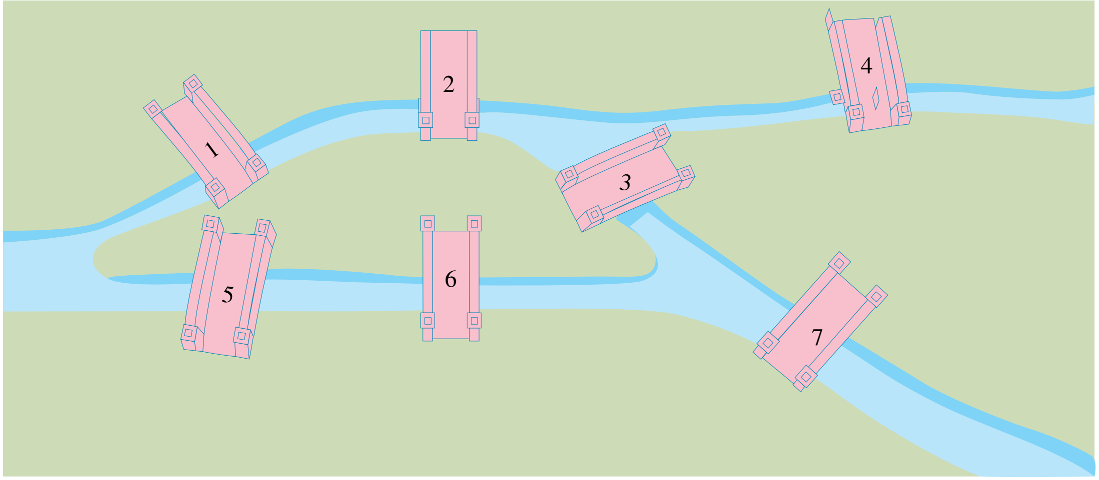
LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.2 Graphs, associated terminology and the adjacency matrix

As you will have noticed in previous years, it is a common practice to draw diagrams and other visual and graphic representations when solving many mathematical problems. In the branch of mathematics known as graph theory, diagrams involving points and lines are used as a planning and analysis tool for systems and connections. Applications of graph theory include business efficiency, transportation systems, design projects, building and construction, food chains and communications networks. The graphs referred to in graph theory are different from the graph of a function and are called networks.

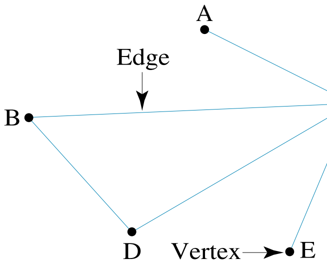
The mathematician Leonhard Euler (1707–83) is usually credited with being the founder of graph theory. He famously used it to solve a problem known as the 'Bridges of Königsberg'. For a long time it had been pondered whether it was possible to travel around the European city of Königsberg (now called Kaliningrad) in such a way that the seven bridges would only have to be crossed once each.



Bridges of Königsberg

10.2.1 Graphs and networks

A **graph** or **network** is a series of points and lines that can be used to represent the connections that exist in various settings.



In a graph or network, the lines are called **edges** (sometimes referred to as 'arcs') and the points are called **vertices** (or 'nodes'), with each edge joining a pair of vertices.

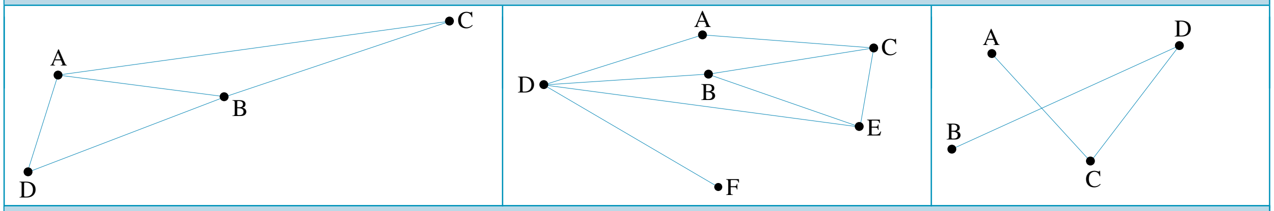
Although edges are often drawn as straight lines, they don't have to be.

When vertices are joined by an edge, they are known as 'adjacent' vertices. Note that the edges of a graph can intersect without there being a vertex. For example, the graph above has five edges and five vertices.

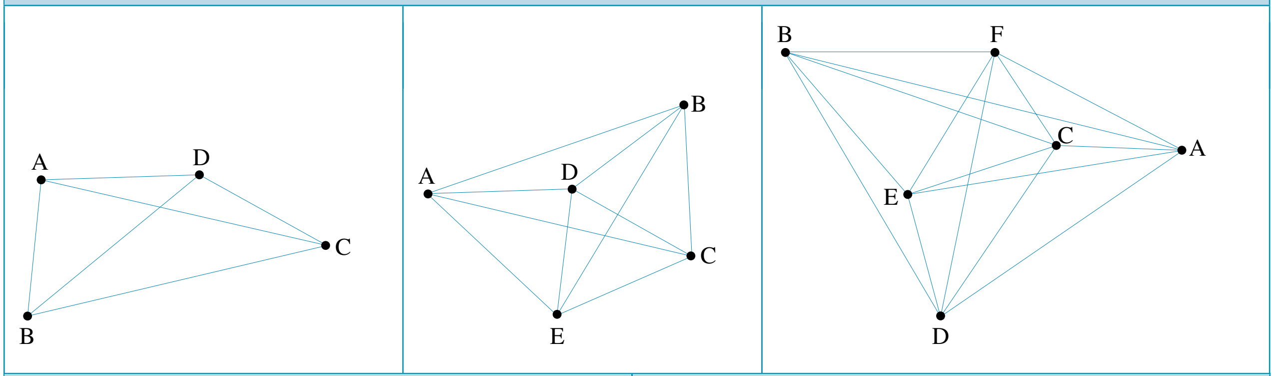
A **simple graph or network** is one in which pairs of vertices are connected by one edge at most.
If there is an edge connecting each vertex to all other vertices in the graph, it is called a **complete graph or network**.
If it is possible to reach every vertex of a graph by moving along the edges, it is called a **connected graph or network**; otherwise, it is a **disconnected graph or network**.

A sub-graph of a graph is one whose vertices and edges are contained in that graph or network.

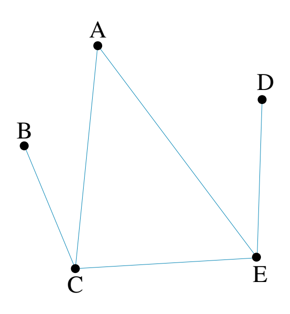
Simple graphs or networks



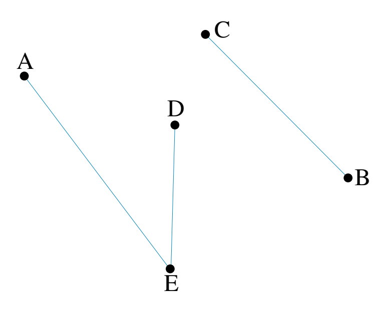
Complete graphs or networks



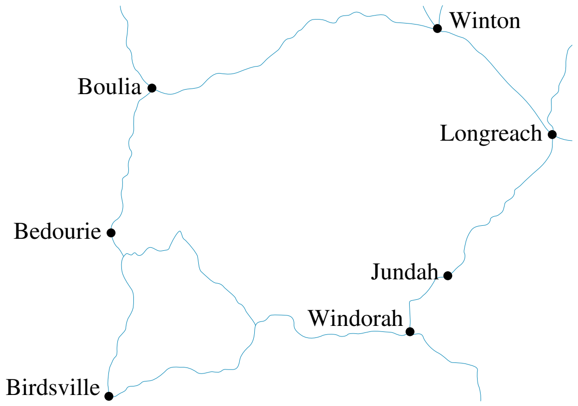
Connected graph



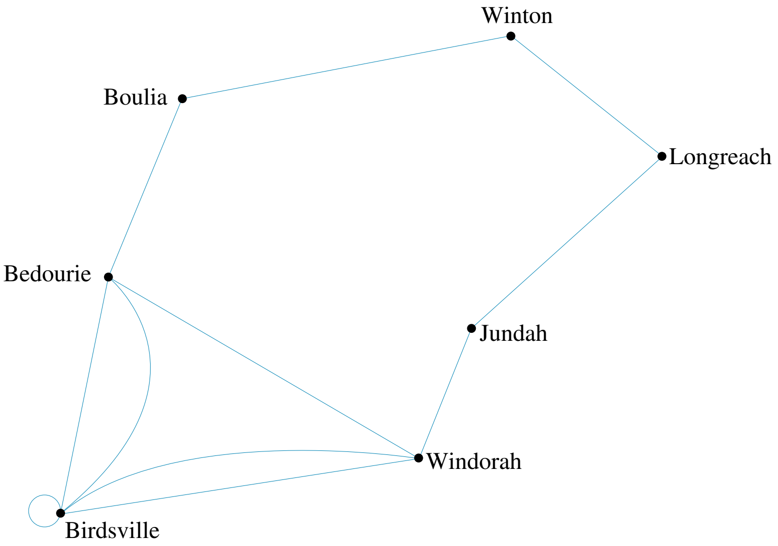
Disconnected graph



Consider the road map shown.



This map can be represented by the following graph or network.

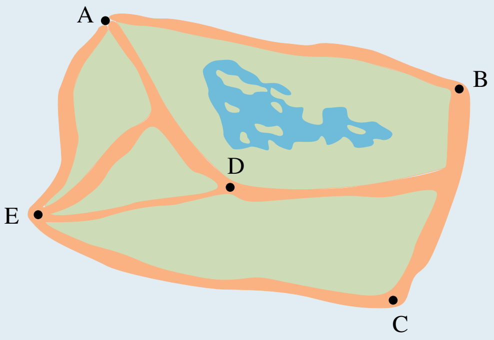


As there is more than one route connecting Birdsville to Windorah and Birdsville to Bedourie, they are each represented by an edge in the graph. In this case we say there are multiple edges. Also, as it is possible to travel along a road from Birdsville that returns without passing through another town, this is represented by an edge. When this happens, the edge is called a **loop**.

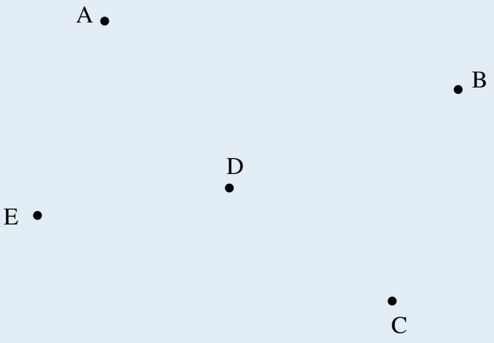
> If it is only possible to move along the edges of a graph in one direction, the graph is called a **directed graph** (digraph) and the edges are represented by arrows. Otherwise it is an **undirected graph**.

WORKED EXAMPLE 1

The diagram represents a system of paths and gates in a large park. Draw a graph to represent the possible ways of travelling to each gate in the park.

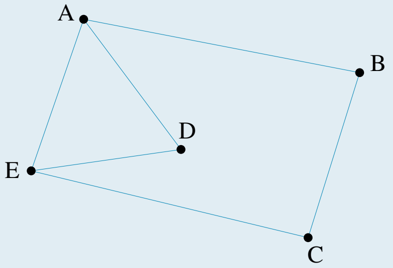


THINK	WRITE/DRAW
1. Identify, draw and label all possible vertices	Represent each of the gates as vertices.



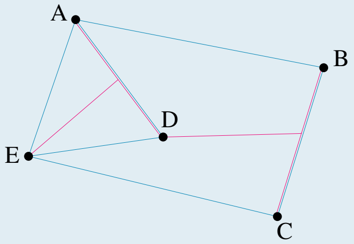
2. Draw edges to represent all the direct connections between the identified vertices.

Direct pathways exist for A – B, A – D, A – E, B – C, C – E and D – E.

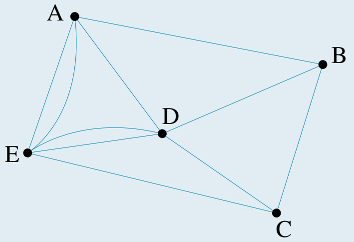


3. Identify all the other unique ways of connecting vertices.

Other unique pathways exist for A – E, D – E, B – D and C – D.

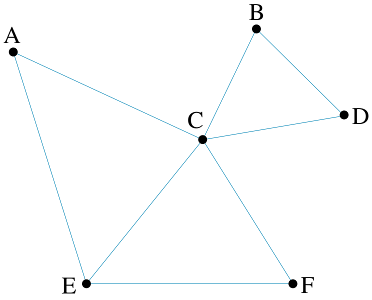


4. Draw the final graph.



10.2.2 The degree of a vertex

When analysing the situation that a graph is representing, it can often be useful to consider the number of edges that are directly connected to a particular vertex. This is referred to as the **degree** of the vertex and is given the notation deg(V), where V represents the vertex.



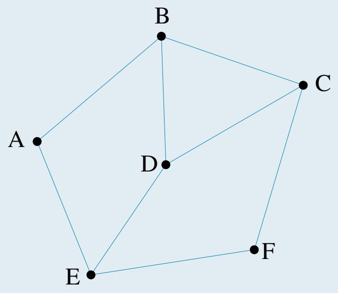
The degree of a vertex =
the number of edges directly connected to that vertex.

In the diagram, $\deg(\text{A}) = 2, \deg(\text{B}) = 2, \deg(\text{C}) = 5, \deg(\text{D}) = 2, \deg(\text{E}) = 3$ and $\deg(\text{F}) = 2$.

Notice that the sum of the degrees in this graph is 16. The total number of edges in the graph should always be half of the sum of the degrees. In an undirected graph, a vertex with a loop counts as having a degree of 2.

WORKED EXAMPLE 2

For the graph in the following diagram, show that the number of edges is equal to half the sum of the degree of the vertices.

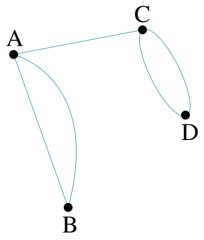


THINK	WRITE/DRAW
1. Identify the degree of each vertex.	$\deg(A) = 2, \deg(B) = 3, \deg(C) = 3,$ $\deg(D) = 3, \deg(E) = 3 \text{ and } \deg(F) = 2$
2. Calculate the sum of the degrees for the graph.	The sum of the degrees for the graph $= 2 + 3 + 3 + 3 + 3 + 2$ $= 16$
3. Count the number of edges for the graph.	The graph has the following edges: A – B, A – E, B – C, B – D, C – D, C – F, D – E, E – F. The graph has 8 edges.
4. State the final answer.	The total number of edges in the graph is therefore half the sum of the degrees.

10.2.3 Adjacency matrices

Matrices are often used when working with graphs. A matrix that represents the number of edges that connect the vertices of a graph is known as an adjacency matrix.

Each column and row of an adjacency matrix corresponds to a vertex of the graph, and the numbers indicate how many edges are connecting them.



Graph

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Adjacency Matrix

In the adjacency matrix, column 3 corresponds to vertex C and row 4 to vertex D. The ‘2’ indicates the number of edges joining these two vertices.

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

10.2.4 Characteristics of adjacency matrices

Adjacency matrices are square matrices with n rows and columns, where ‘n’ is equal to the number of vertices in the graph.

$$\begin{array}{c c c} \text{Column:} & \begin{array}{ccccc} 1 & 2 \ldots & & n-1 & n \end{array} & \text{Row} \\ & \begin{bmatrix} 0 & 2 \ldots & & 1 & 0 \\ 2 & 0 \ldots & & 0 & 0 \\ \vdots & \vdots \ldots & & \vdots & \vdots \\ 1 & 0 \ldots & & 0 & 2 \\ 0 & 0 \ldots & & 2 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array} \end{array}$$

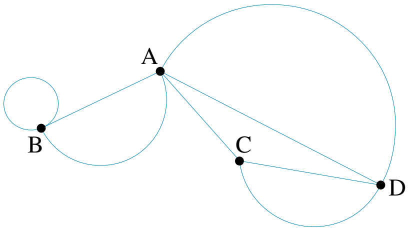
Adjacency matrices are symmetrical around the leading diagonal.

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Any non-zero value in the leading diagonal will indicate the existence of a loop.

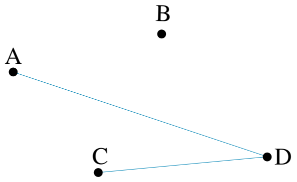
The ‘1’ indicates that a loop exists at vertex B:

$$\begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & \textcircled{1} & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$



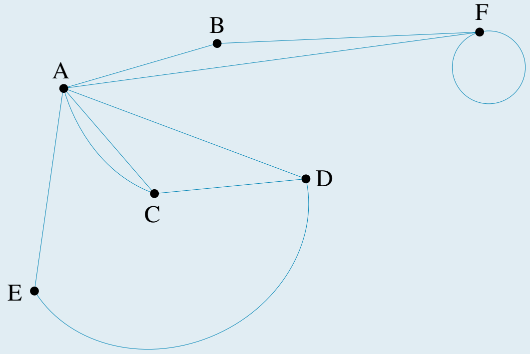
A row consisting of all zeros indicates an isolated vertex (a vertex that is not connected to any other vertex).

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



WORKED EXAMPLE 3

Construct the adjacency matrix for the given graph.



THINK

WRITE

1. Draw up a table with rows and columns for each vertex of the graph.

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

2. Count the number of edges that connect vertex A to the other vertices and record these values in the corresponding space for the first row of the table.

	A	B	C	D	E	F
A	0	1	2	1	1	1

3. Repeat step 2 for all the other vertices.

	A	B	C	D	E	F
A	0	1	2	1	1	1
B	1	0	0	0	0	1
C	2	0	0	1	0	0
D	1	0	1	0	1	0
E	1	0	0	1	0	0
F	1	1	0	0	0	1

4. Display the numbers as a matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Resources

Interactivity The adjacency matrix (int-6466)

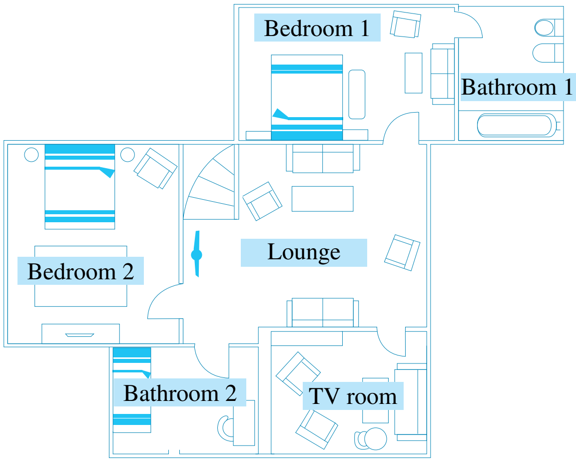
studyon

Units 3 & 4 > Area 6 > Sequence 1 > Concepts 1, 2 & 3

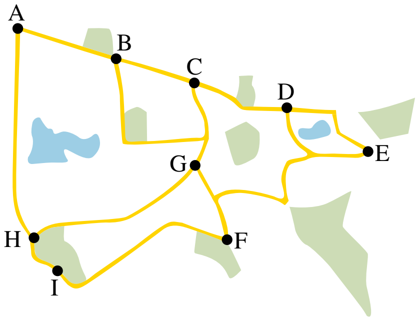
Introduction to graphs (networks) Summary screen and practice questions
Adjacency matrix Summary screen and practice questions
Types of graphs (networks) Summary screen and practice questions

Exercise 10.2 Graphs, associated terminology and the adjacency matrix

1. **WE1** The diagram shows the plan of a floor of a house. Draw a graph to represent the possible ways of travelling between each room of the floor.

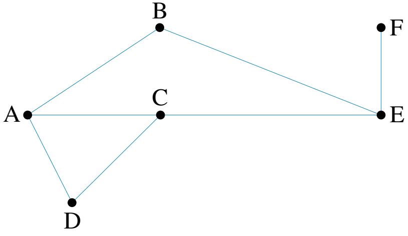


2. Draw a graph to represent the following tourist map.

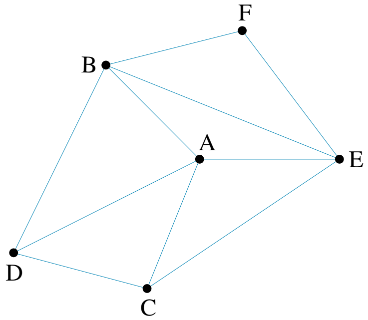


3. **WE2** For each of the following graphs, verify that the number of edges is equal to half the sum of the degree of the vertices.

a.

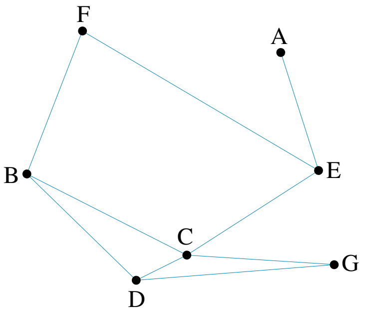


b.

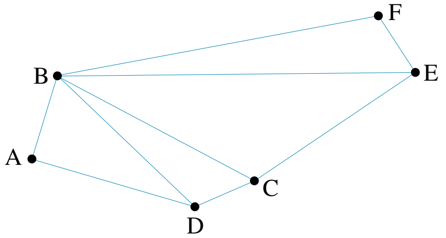


4. For each of the following graphs, verify that the number of edges is equal to half the sum of the degree of the vertices.

a.

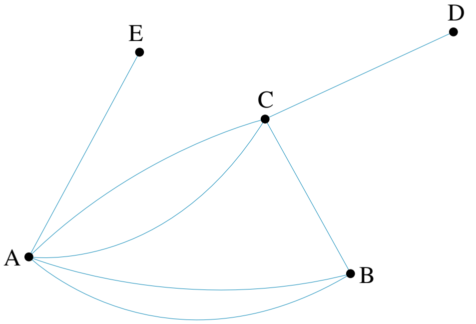


b.

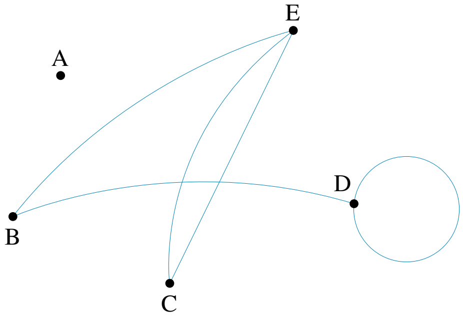


5. Identify the degree of each vertex in the following graphs.

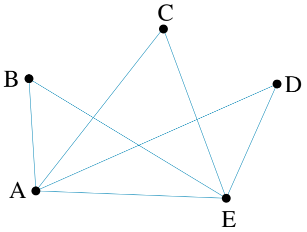
a.



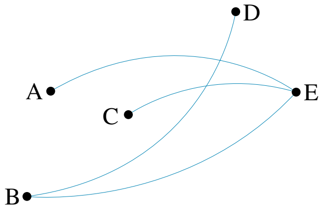
b.



c.

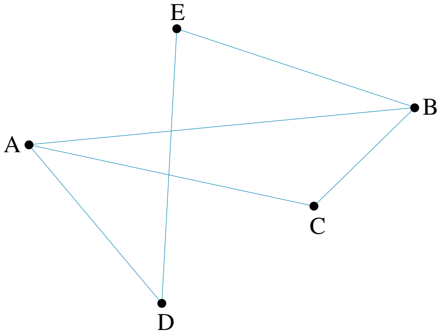


d.

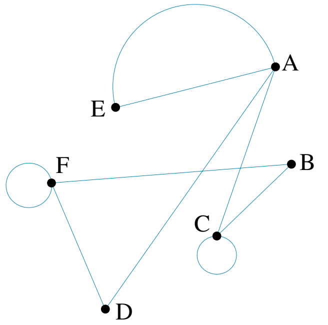


6. **WE3** Construct adjacency matrices for the following graphs.

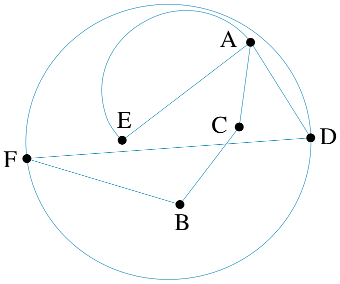
a.



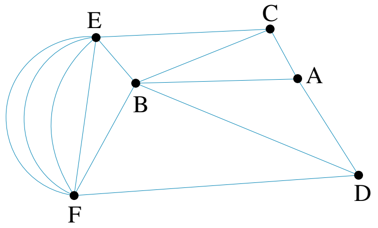
b.



c.



d.



7. Draw graphs to represent the following adjacency matrices.

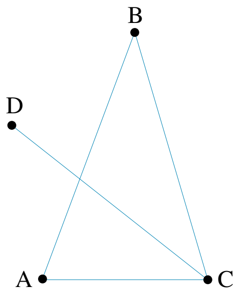
a. $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$

8. Complete the following adjacency matrices.

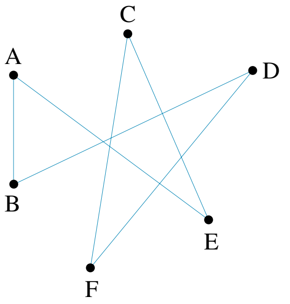
a. $\begin{bmatrix} 0 & 0 & \\ 0 & 2 & 2 \\ 1 & & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 1 & & 0 \\ & 0 & & \\ 0 & 1 & 0 & 1 \\ & 2 & & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & & 1 & & 0 \\ 0 & 0 & & & 0 \\ & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ & & & 0 & 0 \\ 0 & & & 0 & 1 \end{bmatrix}$

9. Complete the following table for the graphs shown.

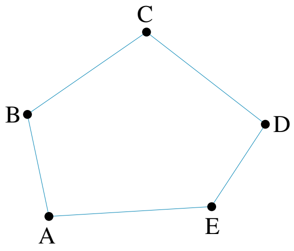
	Simple	**Complete**	**Connected**
Graph 1	Yes	No	Yes
Graph 2			
Graph 3			
Graph 4			
Graph 5			



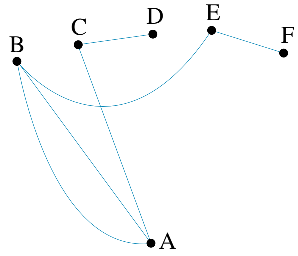
Graph 1



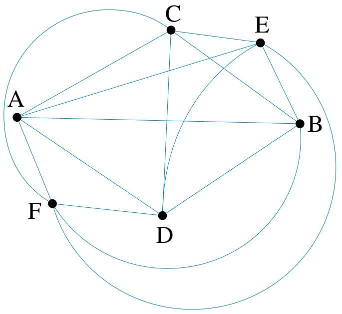
Graph 2



Graph 3



Graph 4



Graph 5

10. Construct the adjacency matrices for each of the graphs shown in question **9**.

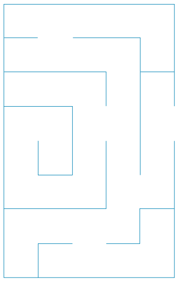
11. Enter details for complete graphs in the following table.

Vertices	Edges
2	
3	
4	
5	
6	
n	

12. Draw a graph of:

- **a.** a simple, connected graph with 6 vertices and 7 edges
- **b.** a simple, connected graph with 7 vertices and 7 edges, where one vertex has degree 3 and five vertices have degree 2
- **c.** a simple, connected graph with 9 vertices and 8 edges, where one vertex has degree 8.

13. By indicating the passages with edges and the intersections and passage endings with vertices, draw a graph to represent the maze shown in the diagram.



Maze

14. Five teams play a round robin competition.

- **a.** Draw a graph to represent the games played.
- **b.** What type of graph is this?
- **c.** What does the total number of edges in the graph indicate?

15. The diagram shows the map of some of the main suburbs of Beijing.

- **a.** Draw a graph to represent the shared boundaries between the suburbs.
- **b.** Which suburb has the highest degree?
- **c.** What type of graph is this?

16. The map shows some of the main highways connecting some of the states on the west coast of the USA.

a. Draw a graph to represent the highways connecting the states shown.
b. Use your graph to construct an adjacency matrix.
c. Which state has the highest degree?
d. Which state has a degree of 5?

17. Jetways Airlines operates flights in South East Asia.

The table indicates the number of direct flights per day between key cities.

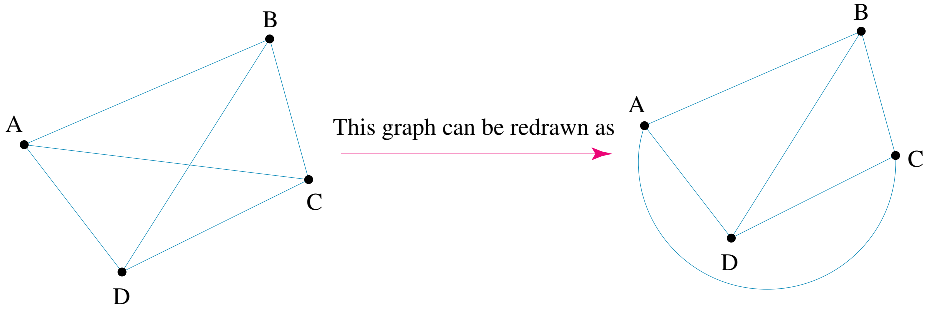
From: **To:**	**Bangkok**	**Manila**	**Singapore**	**Kuala Lumpur**	**Jakarta**	**Hanoi**	**Phnom Penh**
Bangkok	0	2	5	3	1	1	1
Manila	2	0	4	1	1	0	0
Singapore	5	4	0	3	4	2	3
Kuala Lumpur	3	1	3	0	0	3	3
Jakarta	1	1	4	0	0	0	0
Hanoi	1	0	2	3	0	0	0
Phnom Penh	1	0	3	3	0	0	0

a. Draw a graph to represent the number of direct flights.
b. Would this graph be considered to be directed or undirected? Why?
c. How many routes can you use to travel from:
  i. Phnom Penh to Manila through Bangkok
  ii. Hanoi to Bangkok?

10.3 Planar graphs

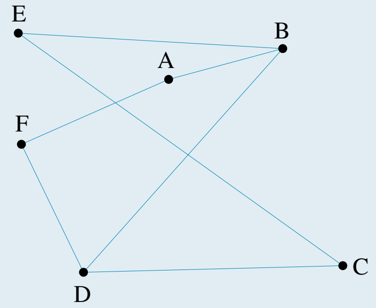
As indicated in Section 10.2, graphs can be drawn with intersecting edges. However, in many applications intersections may be undesirable. Consider a graph of an underground railway network. In this case intersecting edges would indicate the need for one rail line to be in a much deeper tunnel, which could add significantly to construction costs.

In some cases it is possible to redraw graphs so that they have no intersecting edges. When a graph can be redrawn in this way, it is known as a **planar graph**. For example, in the graph shown below, it is possible to redraw one of the intersecting edges so that it still represents the same information.



WORKED EXAMPLE 4

Redraw the graph so that it has no intersecting edges.



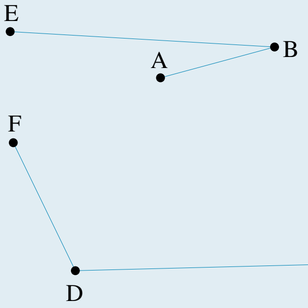
THINK

WRITE/DRAW

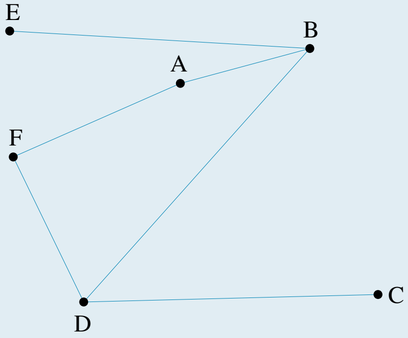
1. List all connections in the original graph.

Connections:
AB; AF; BD; BE; CD; CE; DF

2. Draw all vertices and any section(s) of the graph that have no intersecting edges.

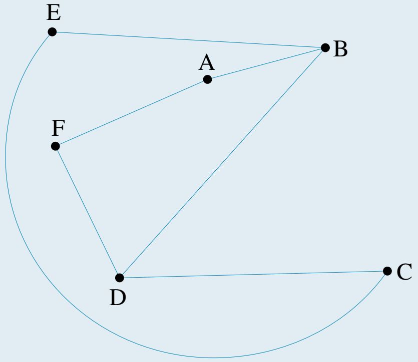


3. Draw any further edges that don't create intersections. Start with edges that have the fewest intersections in the original drawing.

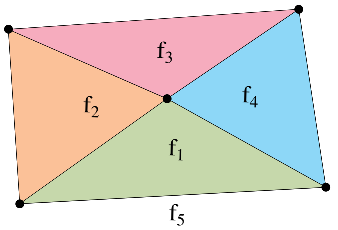


4. Identify any edges yet to be drawn and redraw so that they do not intersect with the other edges.

Connections:
~~AB~~; ~~AF~~; ~~BD~~; ~~BE~~; ~~CD~~; CE; ~~DF~~



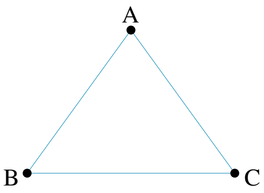
10.3.1 Euler's formula



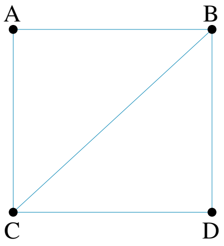
In all planar graphs, the edges and vertices create distinct areas referred to as **faces**.

The planar graph shown in the diagram at right has five faces including the area around the outside.

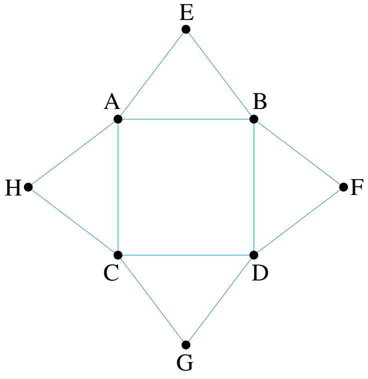
Consider the following group of planar graphs.



Graph 1



Graph 2



Graph 3

The number of vertices, edges and faces for each graph is summarised in the following table.

Graph	Vertices	Edges	Faces
Graph 1	3	3	2
Graph 2	4	5	3
Graph 3	8	12	6

For each of these graphs, we can obtain a result that is well known for any planar graph: the difference between the vertices and edges added to the number of faces will always equal 2.

Graph 1: $3 - 3 + 2 = 2$
Graph 2: $4 - 5 + 3 = 2$
Graph 3: $8 - 12 + 6 = 2$
This is known as Euler's formula for connected planar graphs.

Euler's formula

$v - e + f = 2$, where v is the number of vertices, e is the number of edges and f is the number of faces.

WORKED EXAMPLE 5

How many faces will there be for a connected planar graph of 7 vertices and 10 edges?

THINK	WRITE
1. Substitute the given values into Euler's formula.	$v - e + f = 2$ $7 - 10 + f = 2$
2. Solve the equation for the unknown value.	$7 - 10 + f = 2$ $f = 2 - 7 + 10$ $f = 5$
3. State the final answer.	There will be 5 faces in a connected planar graph with 7 vertices and 10 edges.

on Resources

Interactivities Planar graphs (int-6467)

Euler's formula (int-6468)

studyon

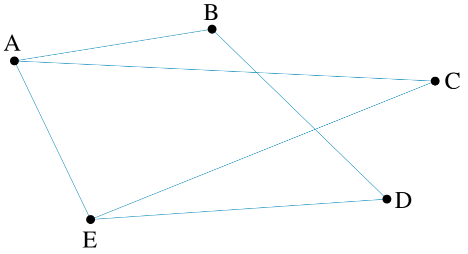
Units 3 & 4 › Area 6 › Sequence 1 › Concept 4

Planar graphs and Euler's formula Summary screen and practice questions

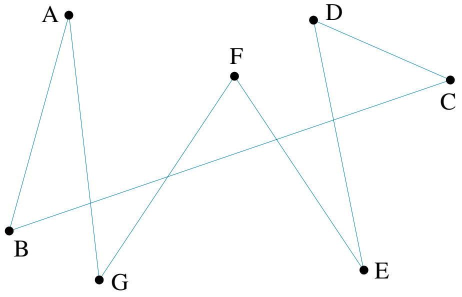
Exercise 10.3 Planar graphs

1. **WE4** Redraw the following graphs so that they have no intersecting edges.

a.

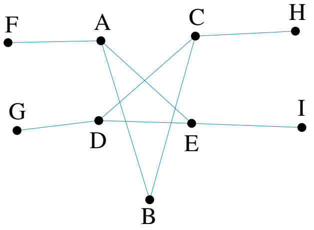


b.

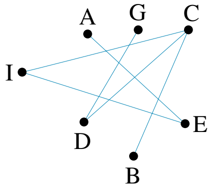


2. **WC** Which of the following are planar graphs?

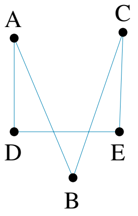
a. **A.**



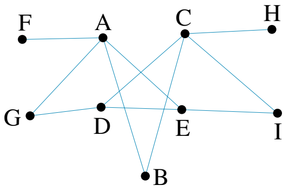
B.



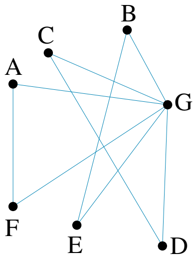
C.



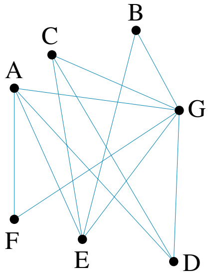
D.



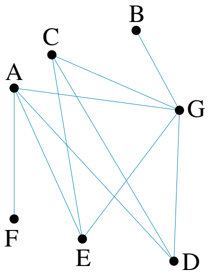
b. **A.**



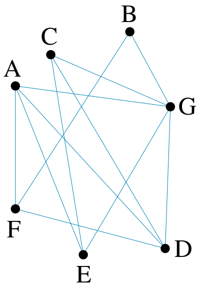
B.



C.

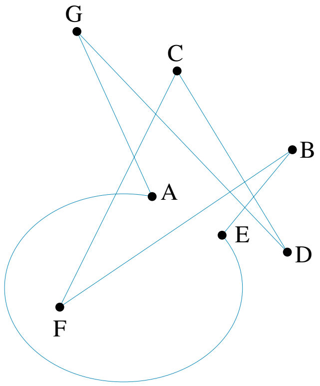


D.

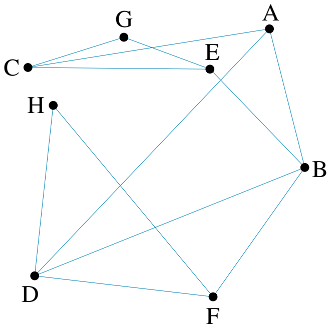


3. Redraw the following graphs to show that they are planar.

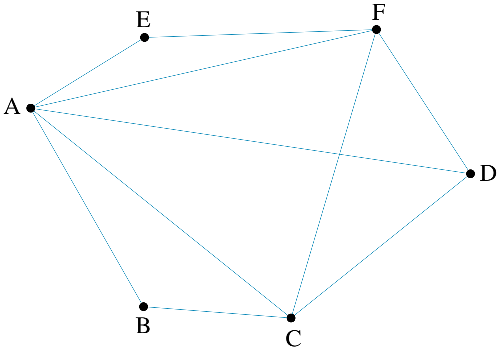
a.



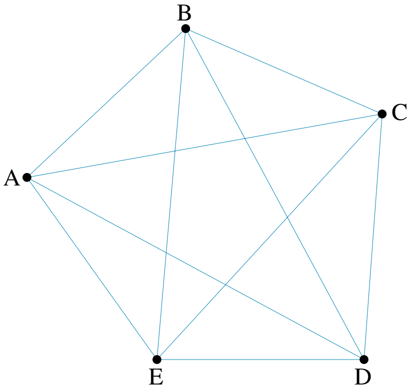
b.



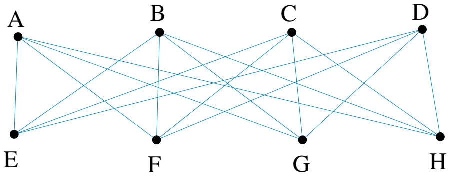
4. Which of the following graphs are not planar?



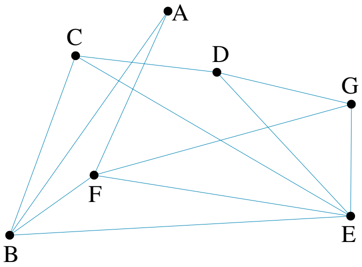
Graph 1



Graph 2



Graph 3



Graph 4

5. **WE5** How many faces will there be for a connected planar graph of

a. 8 vertices and 10 edges

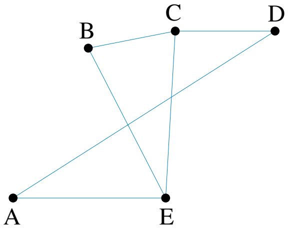
b. 11 vertices and 14 edges?

6. **a.** For a connected planar graph of 5 vertices and 3 faces, how many edges will there be?

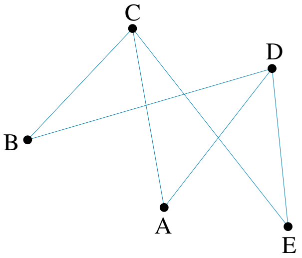
b. For a connected planar graph of 8 edges and 5 faces, how many vertices will there be?

7. For each of the following planar graphs, identify the number of faces.

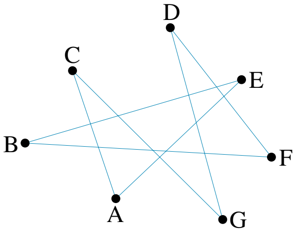
a.



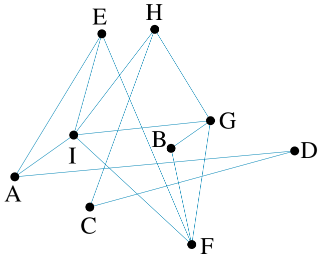
b.



c.



d.



8. Construct a connected planar graph with

a. 6 vertices and 5 faces

b. 11 edges and 9 faces.

9. Use the following adjacency matrices to draw graphs that have no intersecting edges.

a.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

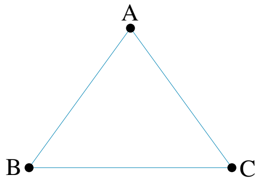
10. For the graphs in question **9**:

i. identify the number of enclosed faces

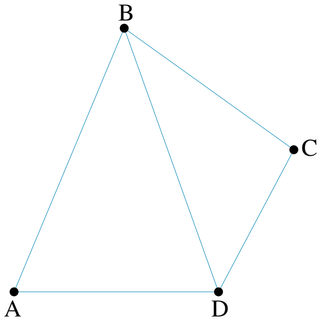
ii. identify the maximum number of additional edges that can be added to maintain a simple planar graph.

11. a. Use the planar graphs shown to complete the table.

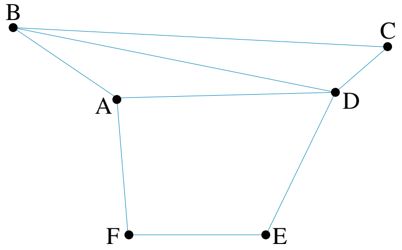
Graph	Total edges	Total degrees
Graph 1		
Graph 2		
Graph 3		
Graph 4		



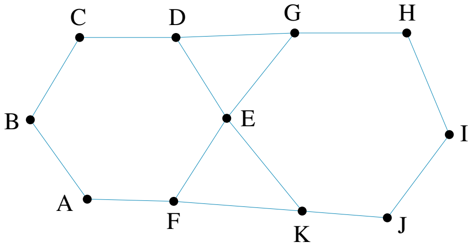
Graph 1



Graph 2



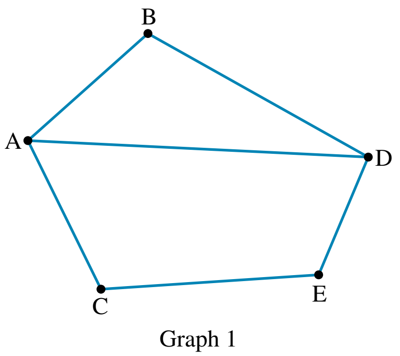
Graph 3



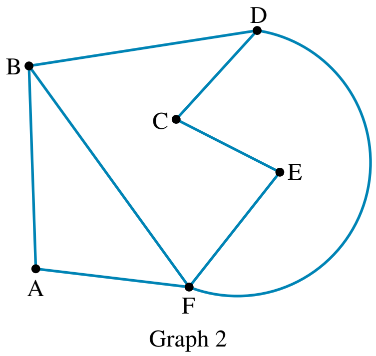
Graph 4

b. What pattern is evident from the table?

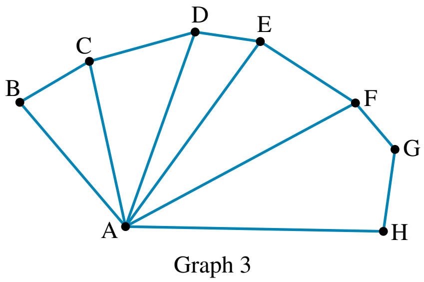
12. **a.** Use the planar graphs shown to complete the table.



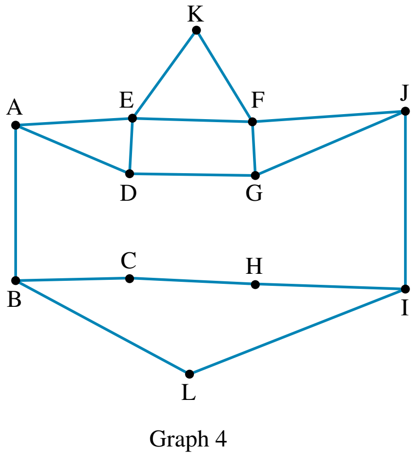
Graph 1



Graph 2



Graph 3

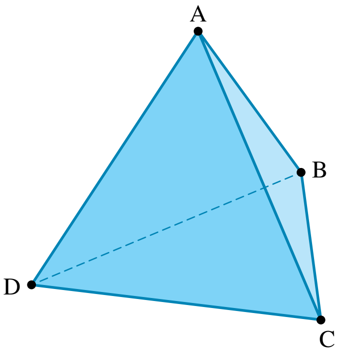


Graph 4

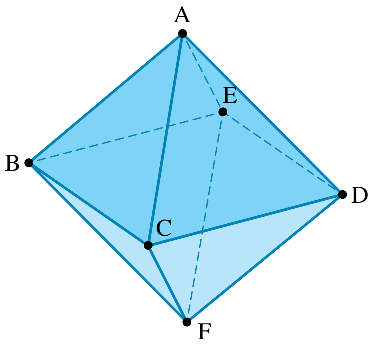
Graph	Total vertices of even degree	Total vertices of odd degree
Graph 1		
Graph 2		
Graph 3		
Graph 4		

b. Is there any pattern evident from this table?

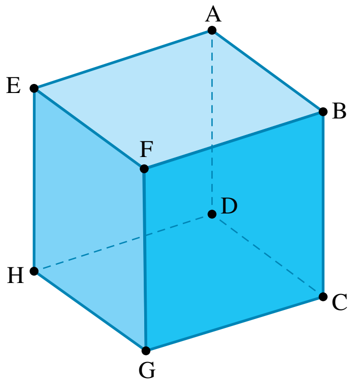
13. Represent the following 3-dimensional shapes as planar graphs.



Tetrahedron



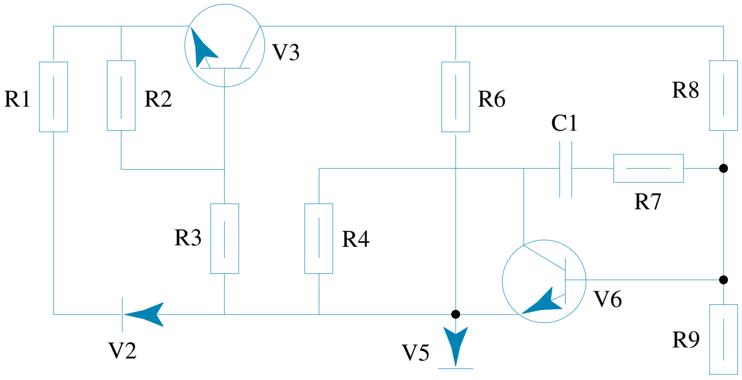
Octahedron



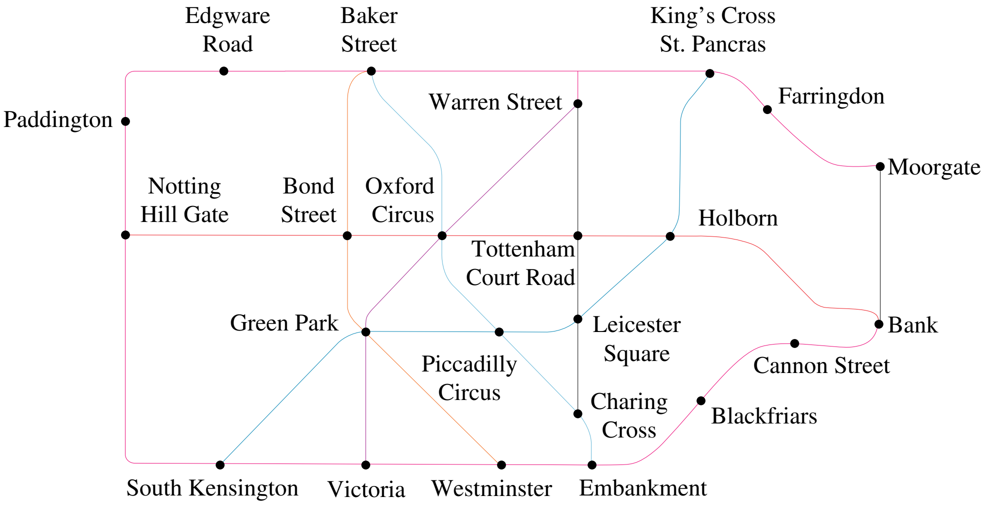
Cube

14. A section of an electric circuit board is shown in the diagram.

- **a.** Draw a graph to represent the circuit board, using vertices to represent the labelled parts of the diagram.
- **b.** Is it possible to represent the circuit board as a planar graph?



15. The diagram shows a section of the London railway system.



- **a.** Display this information using an adjacency matrix.
- **b.** What does the sum of the rows of this adjacency matrix indicate?

16. The table displays the most common methods of communication for a group of people.

	Email	**Facebook**	**SMS**
Adam	Ethan, Liam	Ethan, Liam	Ethan
Michelle		Sophie, Emma, Ethan	Sophie, Emma
Liam	Adam		
Sophie		Michelle, Chloe	Michelle, Chloe
Emma	Chloe	Chloe, Ethan, Michelle	Chloe, Ethan
Ethan		Emma, Adam, Michelle	Emma
Chloe	Emma, Sophie	Emma, Sophie	Emma, Sophie

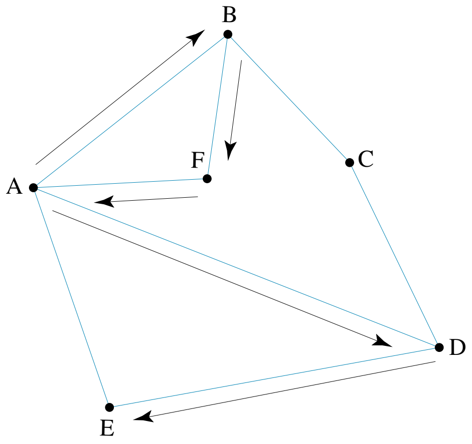
- **a.** Display the information for the entire table in a graph.
- **b.** Who would be the best person to introduce Chloe and Michelle?
- **c.** Display the Facebook information in a separate graph.
- **d.** If Liam and Sophie began communicating through Facebook, how many faces would the graph from part **c** then have?

10.4 Connected graphs

10.4.1 Traversing connected graphs

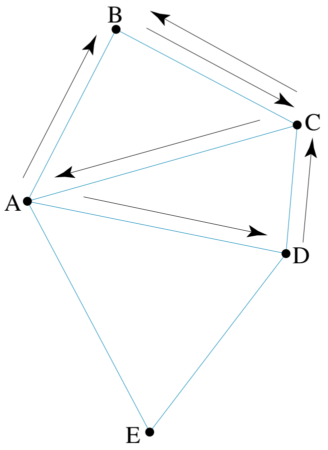
Many applications of graphs involve an analysis of movement around a network. These could include fields such as transport, communications or utilities, to name a few. Movement through a simple connected graph is described in terms of starting and finishing at specified vertices by travelling along the edges. This is usually done by listing the labels of the vertices visited in the correct order. In more complex graphs, edges may also have to be indicated, as there may be more than one connection between vertices.

The definitions of the main terms used when describing movement across a network are as follows.

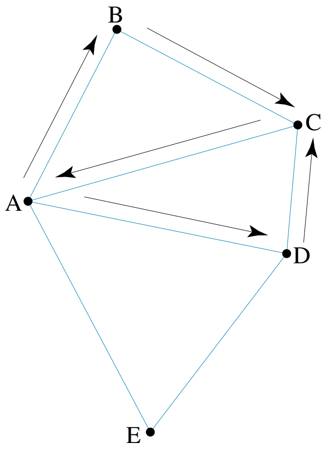


Route: ABFADE

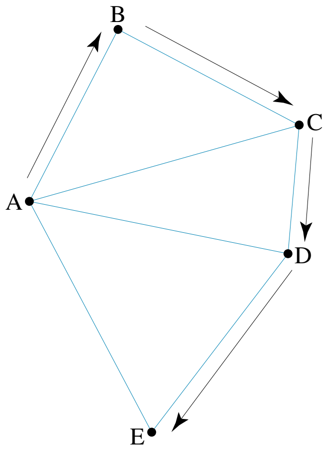
Walk: Any route taken through a network, including routes that repeat edges and vertices
Closed walk: A route taken through a network that starts and ends at the same vertex.
Trail: A walk in which no edges are repeated
Path: A walk in which no vertices are repeated, except possibly the start and finish
Cycle: A path beginning and ending at the same vertex
Closed trail: A trail beginning and ending at the same vertex



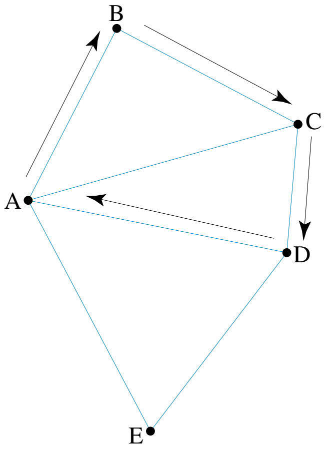
Walk: ABCADCB



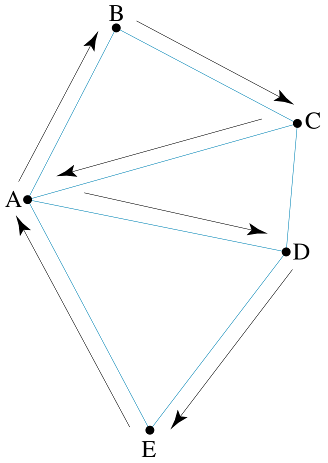
Trail: ABCADC



Path: ABCDE



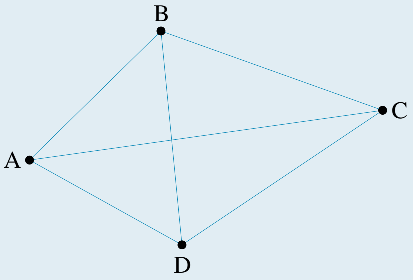
Cycle: ABCDA



Closed trail: ABCADEA

WORKED EXAMPLE 6

In the following network, identify two different routes: one cycle and one closed trail.



THINK	WRITE
1. For a cycle, identify a route that doesn't repeat a vertex apart from the start/finish.	Cycle: ABDCA
2. For a closed trail, identify a route that doesn't repeat an edge and ends at the starting vertex.	Closed trail: ADBCA

10.4.2 Eulerian graphs and semi-Eulerian graphs

In some practical situations, it is most efficient if a route travels along each edge only once. Examples include parcel deliveries and council rubbish collections. In each of these cases, it is desirable to return to the start of the route.

An **Eulerian graph** is a connected graph which allows you to start at a vertex, traverse each edge only once and return to the vertex from which you started.

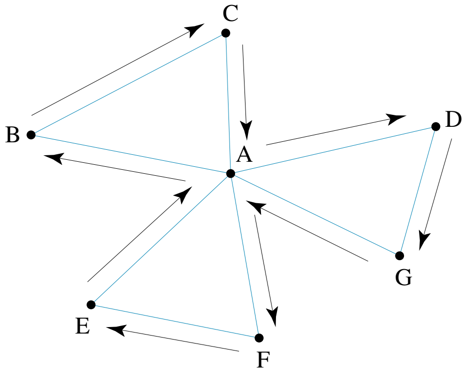
An **Eulerian trail** is the trail by which you travel around an Eulerian graph.

A graph is Eulerian if each vertex has an even degree.

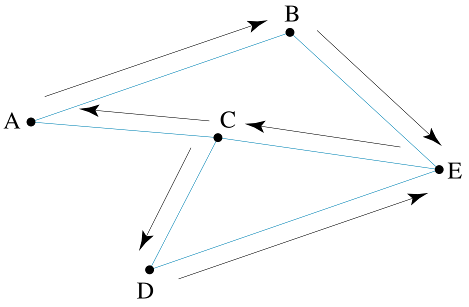
A **semi-Eulerian graph** is a connected graph which allows you to start at a vertex and traverse each edge only once without returning to the start.

A **semi-Eulerian trail** is the trail by which you travel around a semi-Eulerian graph.

A graph is semi-Eulerian if it has only one pair of odd vertices.



Eulerian trail ABCADGAFEA

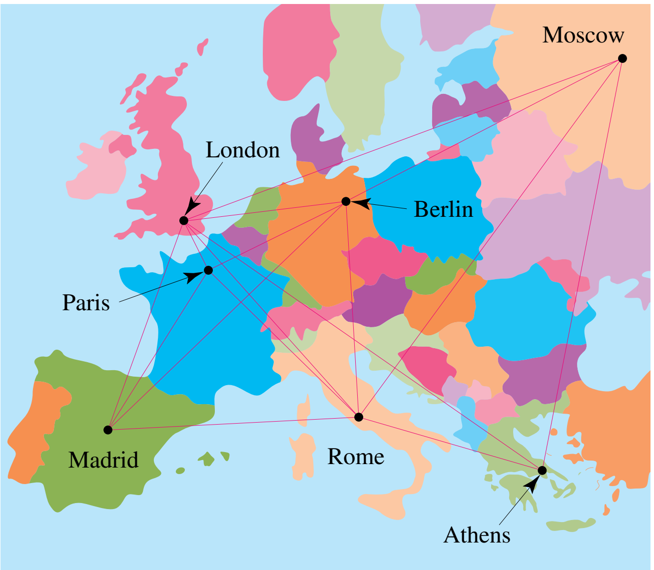


Semi-Eulerian trail CDECABE

If all of the vertices of a connected graph are even, then an Eulerian trail exists. The diagram is a Eulerian graph.
If exactly 2 vertices of a connected graph are odd, then a semi-Eulerian trail exists. The diagram is a semi-Eulerian graph.

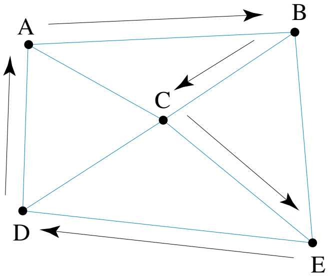
10.4.3 Hamiltonian graphs and semi-Hamiltonian graphs

In other situations, it may be more practical if all vertices can be reached without using all of the edges of the graph. For example, if you wanted to visit a selection of the capital cities of Europe, you wouldn't need to use all the available flight routes shown in the diagram.

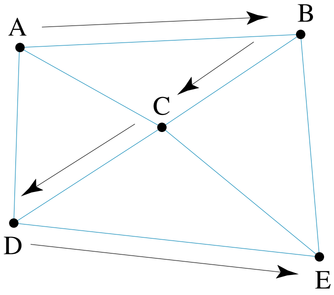


If a graph has a path that includes every vertex exactly once, while ending at the initial vertex, the graph is **a Hamiltonian graph**. The path is called **a Hamiltonian cycle**.
If a graph has a path that includes every vertex exactly once but ends at a vertex other than the starting one, then the graph is **a semi-Hamiltonian graph**. The path is called **a Semi-Hamiltonian cycle**.

Hamiltonian cycles and semi-Hamiltonian cycles reach all vertices of a network once without necessarily using all of the available edges.



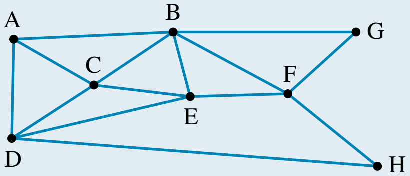
Hamiltonian cycle: ABCEDA

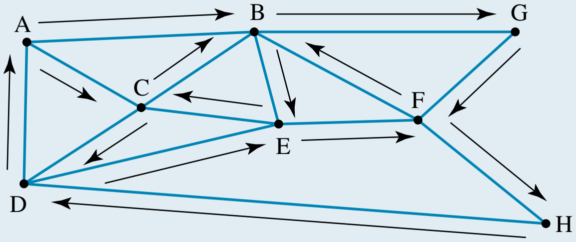
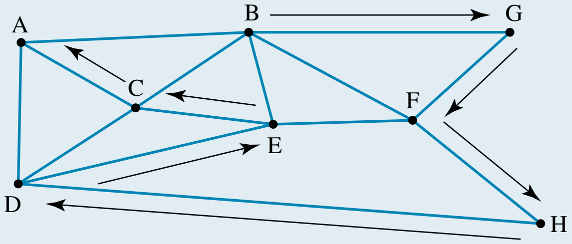


Semi-Hamiltonian cycle: ABCDE

WORKED EXAMPLE 7

Identify a semi-Eulerian trail and a semi-Hamiltonian cycle in the following graph.



THINK	**WRITE/DRAW**
1. For a semi-Eulerian trail to exist, there must be exactly 2 vertices with an odd-numbered degree.	$\text{Deg}(A) = 3,\ \deg(B) = 5,\ \deg(C) = 4,\ \deg(D) = 4,$ $\deg(E) = 4,\ \deg(F) = 4,\ \deg(G) = 2,\ \deg(H) = 2$ As there are only two odd-degree vertices, a semi-Eulerian trail must exist.
2. Identify a route that uses each edge once.	 semi-Eulerian trail: ABGFHDEFBECDACB
3. Identify a route that reaches each vertex once.	 semi-Hamiltonian cycle: BGFHDECA
4. State the answer.	semi-Eulerian trail: ABGFHDEFBECDACB semi-Hamiltonian cycle: BGFHDECA

on Resources

Interactivity Traversing connected graphs (int-6469)

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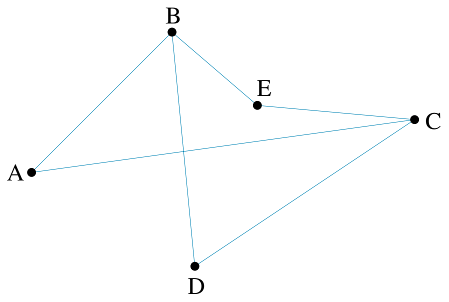
Units 3 & 4 › Area 6 › Sequence 1 › Concepts 5, 6 & 7

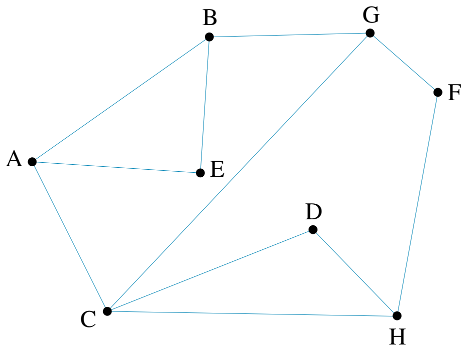
Walks, paths, trails, cycles and bridges Summary screen and practice questions
Hamiltonian graphs Summary screen and practice questions
Eulerian graphs Summary screen and practice questions

Exercise 10.4 Connected graphs

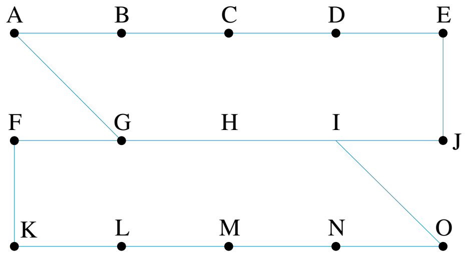
1. **WE6** In the following network, identify two different routes: one cycle and one closed trail.



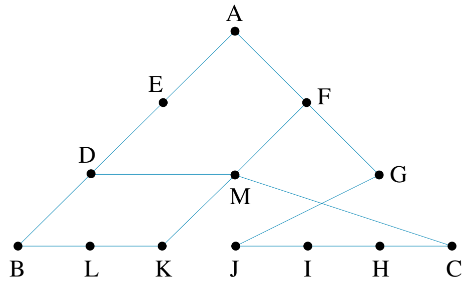
2. In the following network, identify three different routes: one path, one cycle and one closed trail.



3. Which of the terms walk, trail, path, cycle and closed trail could be used to describe the following routes on the graph shown?
   - a. AGHIONMLKFGA
   - b. IHGFKLMNO
   - c. HIJEDCBAGH
   - d. FGHIJEDCBAG

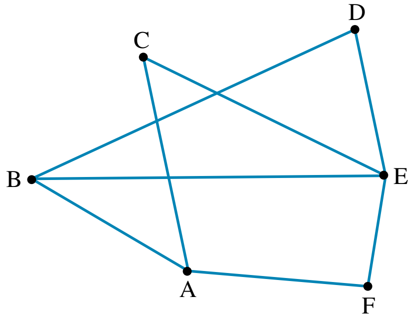


4. Use the following graph to identify the indicated routes.
   - a. A path commencing at M, including at least 10 vertices and finishing at D
   - b. A trail from A to C that includes exactly 7 edges
   - c. A cycle commencing at M that includes 10 edges
   - d. A closed trail commencing at F that includes 7 vertices

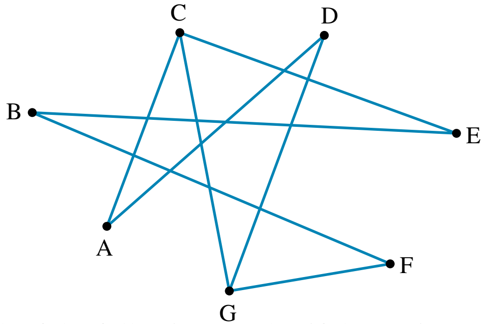


5. **WE7** Identify a semi-Eulerian trail and a semi-Hamiltonian cycle in each of the following graphs.

a.

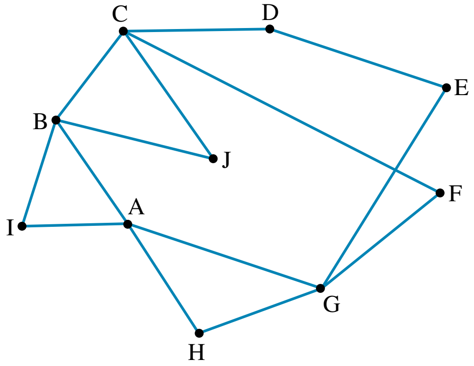


b.

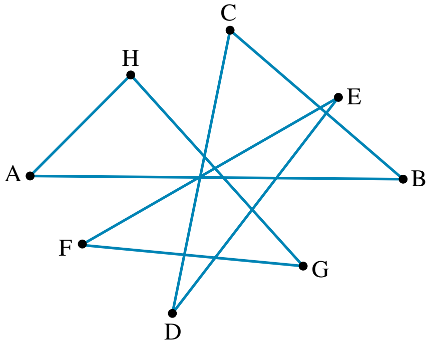


6. Identify an Eulerian trail and a Hamiltonian cycle in each of the following graphs, if they exist.

a.

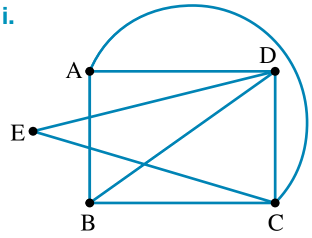


b.

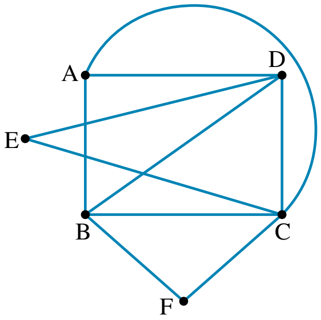


7. **a.** Identify which of the following graphs have a semi-Eulerian trail.

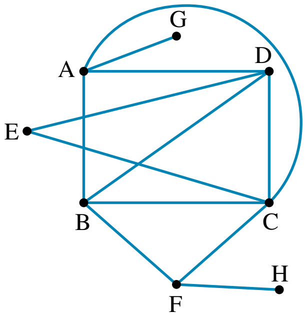
i.



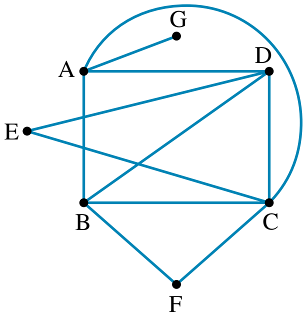
ii.



iii.



iv.



b. Identify the semi-Eulerian trails found.

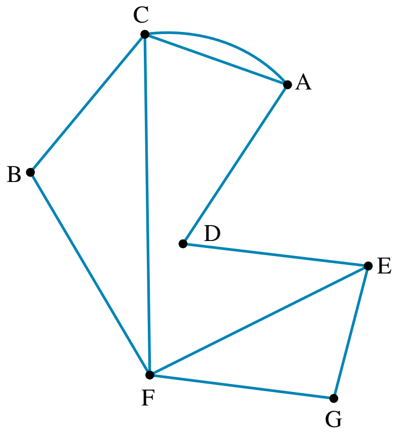
8. **a.** Identify which of the graphs from question 7 have a Hamiltonian cycle.

b. Identify the Hamiltonian cycles found.

9. **a.** Construct adjacency matrices for each of the graphs in question 7.

b. How might these assist with making decisions about the existence of Eulerian trails and semi-Eulerian trails, and Hamiltonian cycles and semi-Hamiltonian cycles?

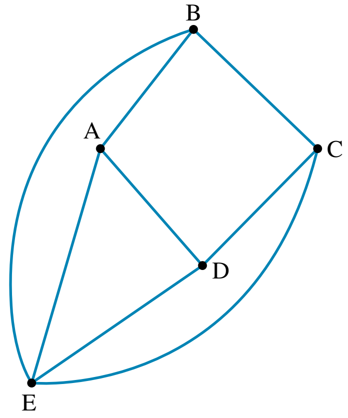
10. In the following graph, if a semi-Eulerian trail commences at vertex A, at which vertices could it finish?



11. In the following graph, at which vertices could a semi-Hamiltonian cycle finish if it commences by travelling from

a. B to E

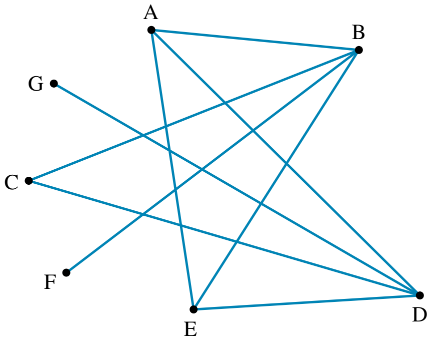
b. E to A?



12. In the following graph, other than from G to F, between which 2 vertices must you add an edge in order to create a semi-Hamiltonian cycle that commences from vertex

a. G

b. F?

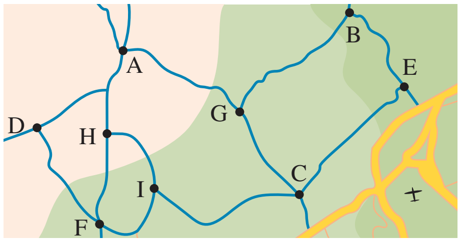


13. On the map shown, a school bus route is indicated in yellow. The bus route starts and ends at the school indicated.

- **a.** Draw a graph to represent the bus route.
- **b.** Students can catch the bus at stops that are located at the intersections of the roads marked in yellow. Is it possible for the bus to collect students by driving down each section of the route only once? Explain your answer.
- **c.** If road works prevent the bus from travelling along the two sections indicated by the Xs, will it be possible for the bus to still collect students on the remainder of the route by travelling each section only once? Explain your answer.

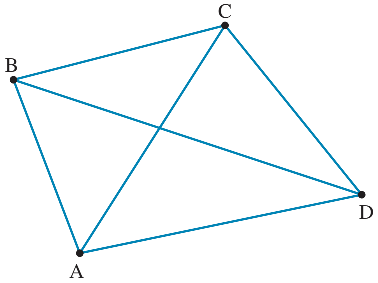
14. The map of an orienteering course is shown. Participants must travel to each of the nine checkpoints along any of the marked paths.

- **a.** Draw a graph to represent the possible ways of travelling to each checkpoint.
- **b.** What is the degree of checkpoint H?
- **c.** If participants must start and finish at A and visit every other checkpoint only once, identify two possible routes they could take.
- **d.** **i.** If participants can decide to start and finish at any checkpoint, and the paths connecting D and F, H and I, and A and G are no longer accessible, it is possible to travel the course by moving along each remaining path only once. Explain why.
  - **ii.** Identify the two possible starting points.



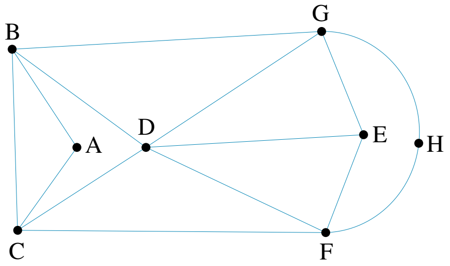
15. **a.** Use the following graph to complete the table to identify all of the Hamiltonian cycles commencing at vertex A.

	Hamiltonian cycle
1.	ABCDA
2.	
3.	
4.	
5.	
6.	



b. Are any other Hamiltonian cycles possible?

16. The graph shown outlines the possible ways a tourist bus can travel between eight locations.



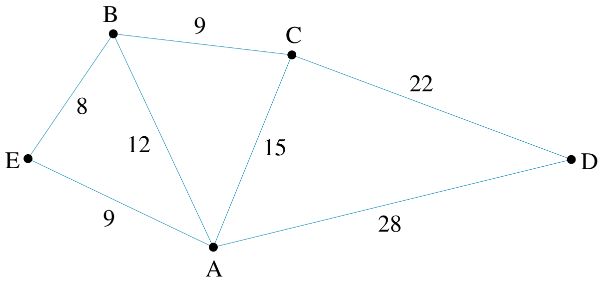
   a. If vertex A represents the second location visited, list the possible starting points.
   b. If the bus also visited each location only once, which of the starting points listed in part a could not be correct?
   c. If the bus also needed to finish at vertex D, list the possible paths that could be taken.
   d. If instead the bus company decides to operate a route that travelled to each connection only once, what are the possible starting and finishing points?
   e. If instead the company wanted to travel to each connection only once and finish at the starting point, which edge of the graph would need to be removed?

10.5 Weighted graphs and trees

10.5.1 Weighted graphs

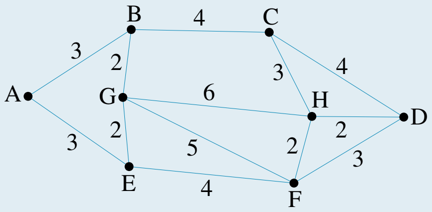
In many applications using graphs, it is useful to attach a value to the edges. These values could represent the length of the edge in terms of time or distance, or the costs involved with moving along that section of the path. Such graphs are known as **weighted graphs**.

Weighted graphs can be particularly useful as analysis tools. For example, they can help determine how to travel through a network in the shortest possible time.



WORKED EXAMPLE 8

The graph represents the distances in kilometres between eight locations.



Identify the shortest distance to travel from A to D that goes to all vertices.

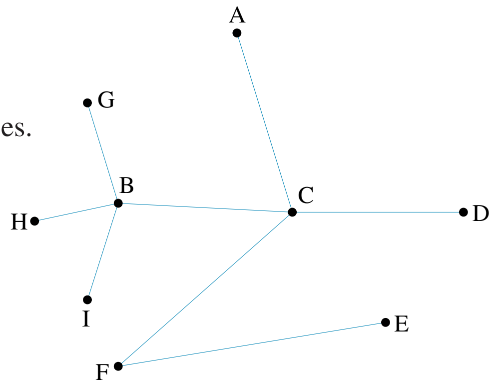
THINK	WRITE
1. Identify the Hamiltonian paths that connect the two vertices.	Possible paths: a. ABGEFHCD b. ABCHGEFD c. AEGBCHFD d. AEFGBCHD e. AEFHGBCD

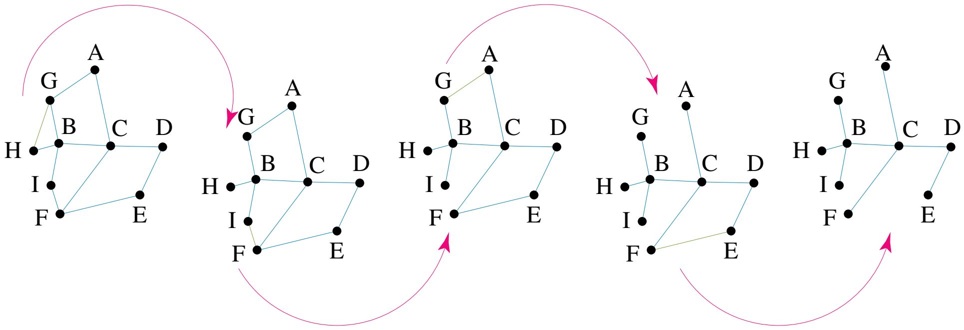
2. Calculate the total distances for each path to find the shortest.	**a.** $3 + 2 + 2 + 4 + 2 + 3 + 4 = 20$ **b.** $3 + 4 + 3 + 6 + 2 + 4 + 3 = 25$ **c.** $3 + 2 + 2 + 4 + 3 + 2 + 3 = 19$ **d.** $3 + 4 + 5 + 2 + 4 + 3 + 2 = 23$ **e.** $3 + 4 + 2 + 6 + 2 + 4 + 4 = 25$
3. State the final answer.	The shortest distance from A to D that travels to all vertices is 19 km.

10.5.2 Trees

A **tree** is a simple connected graph with no circuits. As such, any pairs of vertices in a tree are connected by a unique path, and the number of edges is always 1 less than the number of vertices.

Spanning trees are sub-graphs (graphs that are formed from part of a larger graph) that include all of the vertices of the original graph. In practical settings, they can be very useful in analysing network connections. For example, a *minimum* spanning tree for a weighted graph can identify the lowest-cost connections. Spanning trees can be obtained by systematically removing any edges that form a circuit, one at a time.





10.5.3 Prim’s algorithm

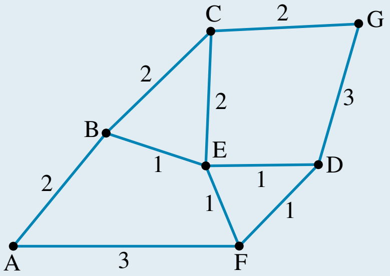
Prim’s algorithm is a set of logical steps that can be used to identify the minimum spanning tree for a weighted connected graph.

The steps for prim’s algorithm are:

Step 1: Begin at a vertex with low weighted edges.
Step 2: Progressively select edges with the lowest weighting (unless they form a circuit).
Step 3: Continue until all vertices are selected.

WORKED EXAMPLE 9

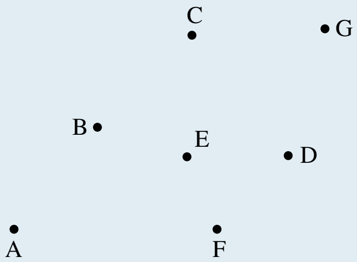
Use Prim's algorithm to identify the minimum spanning tree of the graph shown.



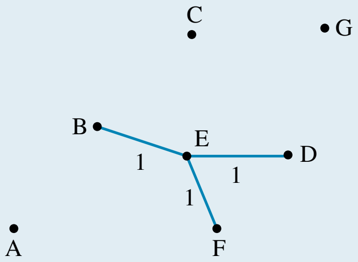
THINK

DRAW

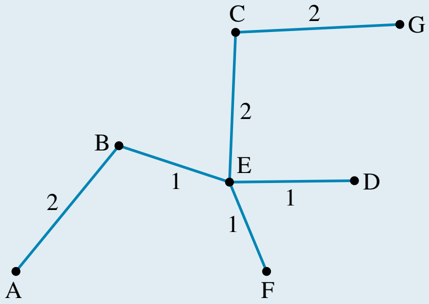
1. Draw the vertices of the graph.



2. Draw in any edges with the lowest weighting that do not complete a circuit.



3. Draw in any edges with the next lowest weighting that do not complete a circuit. Continue until all vertices are connected.



on Resources

Interactivity Minimum spanning trees and Prim's algorithm (int-6285)

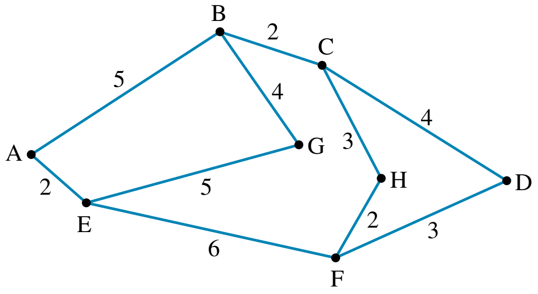
studyon

Units 3 & 4 | Area 6 | Sequence 1 | Concept 8

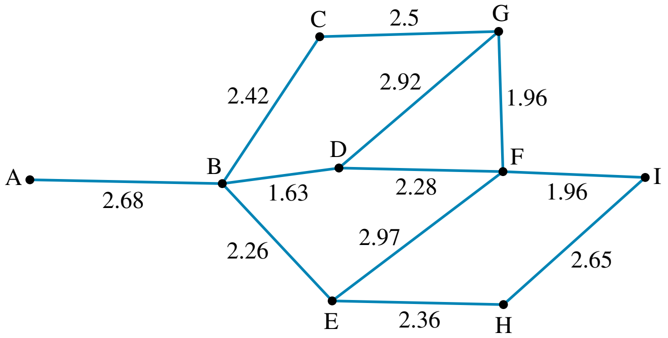
Weighted graphs and shortest paths Summary screen and practice questions

Exercise 10.5 Weighted graphs and trees

1. **WE8** Use the graph to identify the shortest distance to travel from A to D that goes to all vertices.

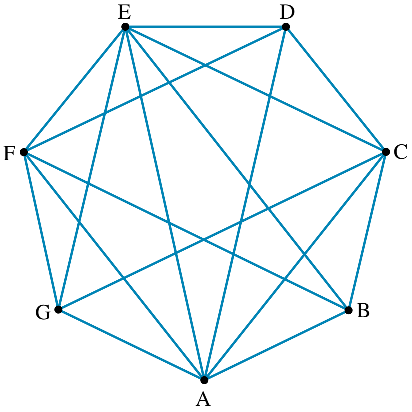


2. Use the graph to identify the shortest distance to travel from A to I that goes to all vertices.

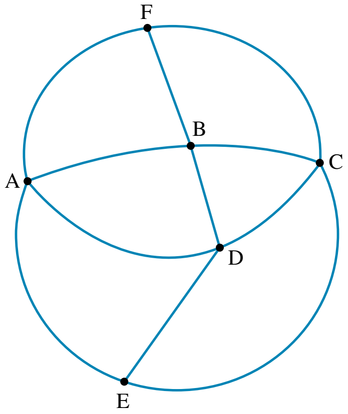


3. Draw three spanning trees for each of the following graphs.

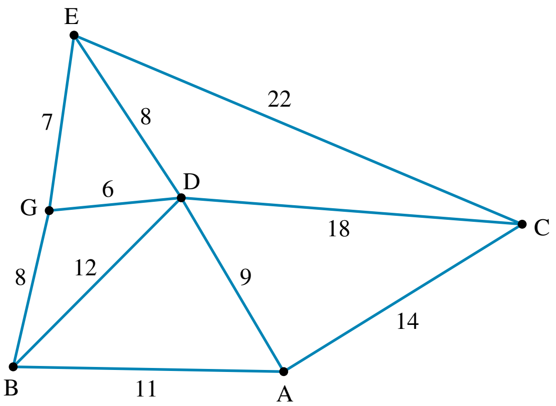
a.



b.



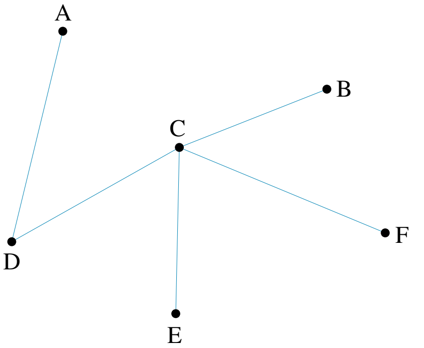
4. A truck starts from the main distribution point at vertex A and makes deliveries at each of the other vertices before returning to A. What is the shortest route the truck can take?



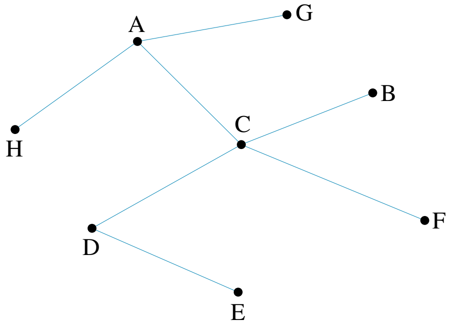
5. For the following trees

- **i.** add the minimum number of edges to create a semi-Eulerian trail.
- **ii.** identify the semi-Eulerian trail created.

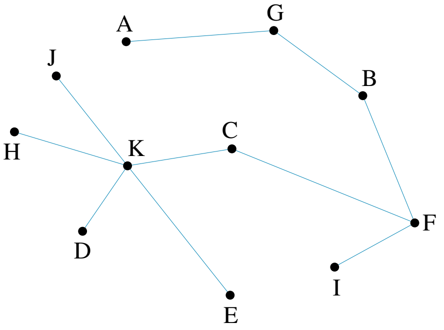
a.



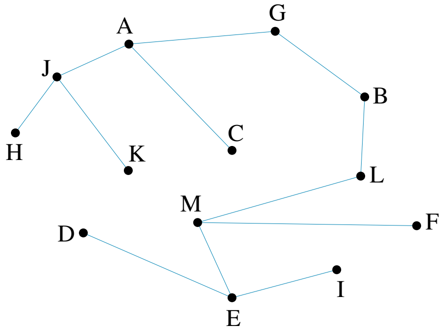
b.



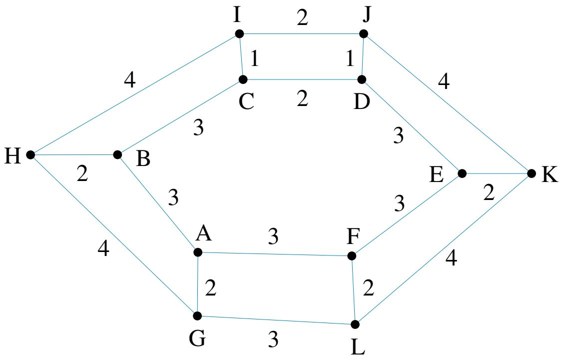
c.



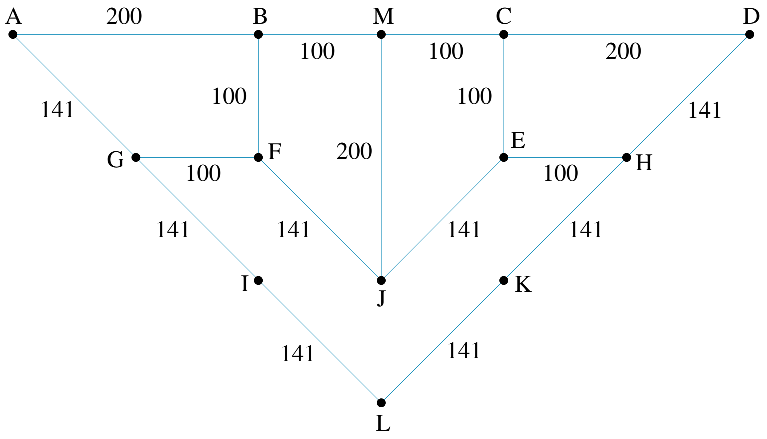
d.



6. **WE9** Use Prim’s algorithm to identify the minimum spanning tree of the graph shown.

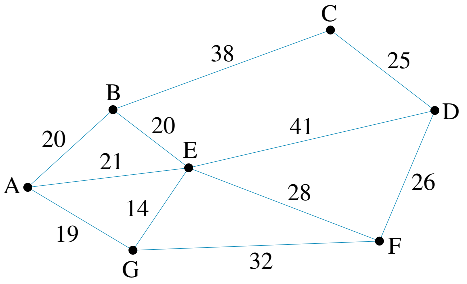


7. Use Prim’s algorithm to identify the minimum spanning tree of the graph shown.

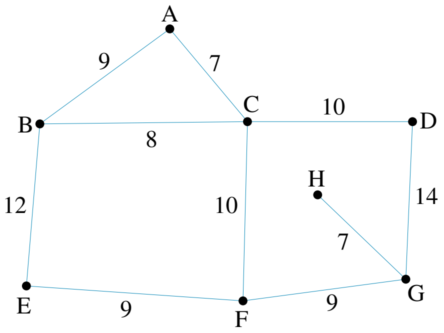


8. Identify the minimum spanning tree for each of the following graphs.

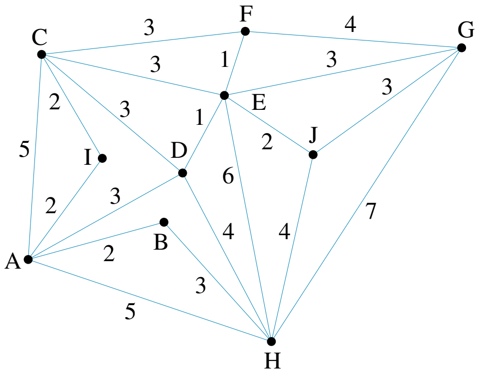
a.



b.



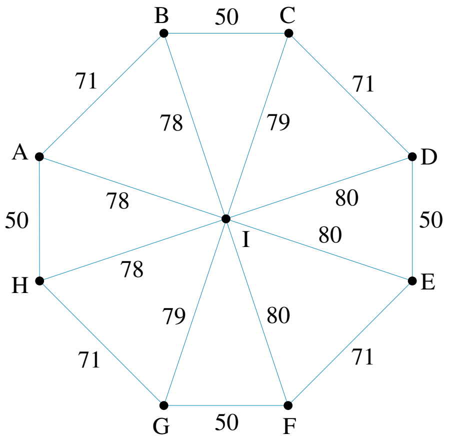
9. Draw diagrams to show the steps you would follow when using Prim's algorithm to identify the minimum spanning tree for the following graph.



10. Part of the timetable and description for a bus route is shown in the table. Draw a weighted graph to represent the bus route.

Bus stop	Description	Time
Bus depot	The northernmost point on the route	7.00 am
Northsea Shopping Town	Reached by travelling south-east along a highway from the bus depot	7.15 am
Highview Railway Station	Travel directly south along the road from Northsea Shopping Town.	7.35 am
Highview Primary School	Directly east along a road from the railway station	7.40 am
Eastend Medical Centre	Continue east along the road from the railway station.	7.55 am
Eastend Village	South-west along a road from the medical centre	8.05 am
Southpoint Hotel	Directly south along a road from Eastend Village	8.20 am
South Beach	Travel south-west along a road from the hotel.	8.30 am

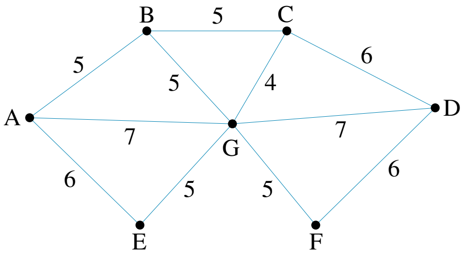
11. Consider the graph shown.



a. Identify the longest and shortest semi-Hamiltonian cycle.

b. What is the minimum spanning tree for this graph?

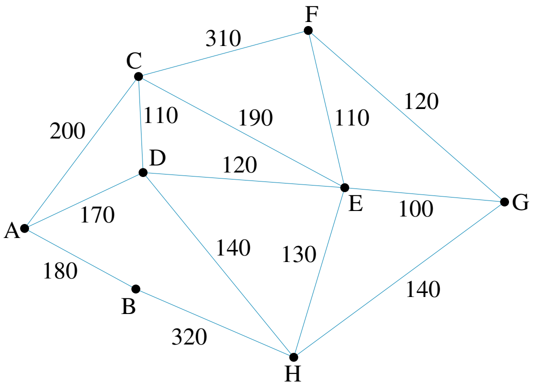
12. Consider the graph shown.



a. If an edge with the highest weighting is removed, identify the shortest semi-Hamiltonian cycle.

b. If the edge with the lowest weighting is removed, identify the shortest semi-Hamiltonian cycle.

13. The weighted graph represents the costs incurred by a salesman when moving between the locations of various businesses.



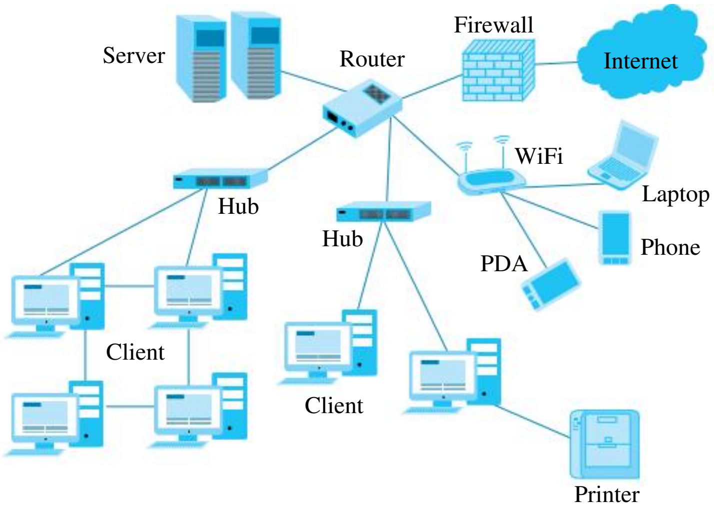
a. What is the cheapest way of travelling from A to G?

b. What is the cheapest way of travelling from B to G?

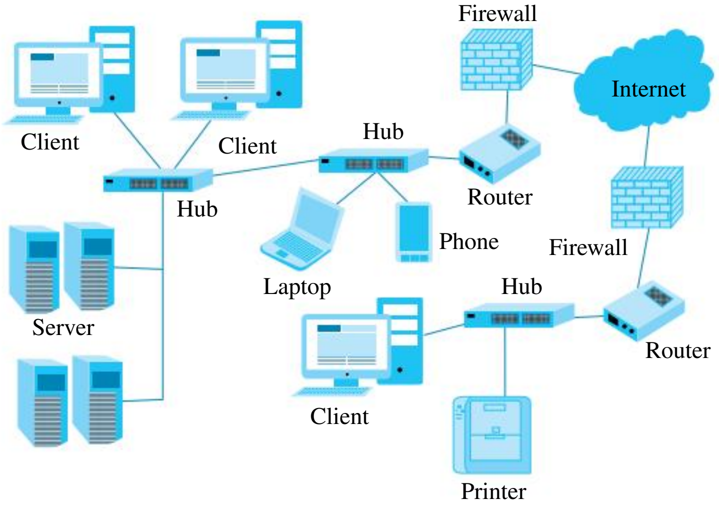
c. If the salesman starts and finishes at E, what is the cheapest way to travel to all vertices?

14. The diagrams show two options for the design of a computer network for a small business.

Option 1



Option 2



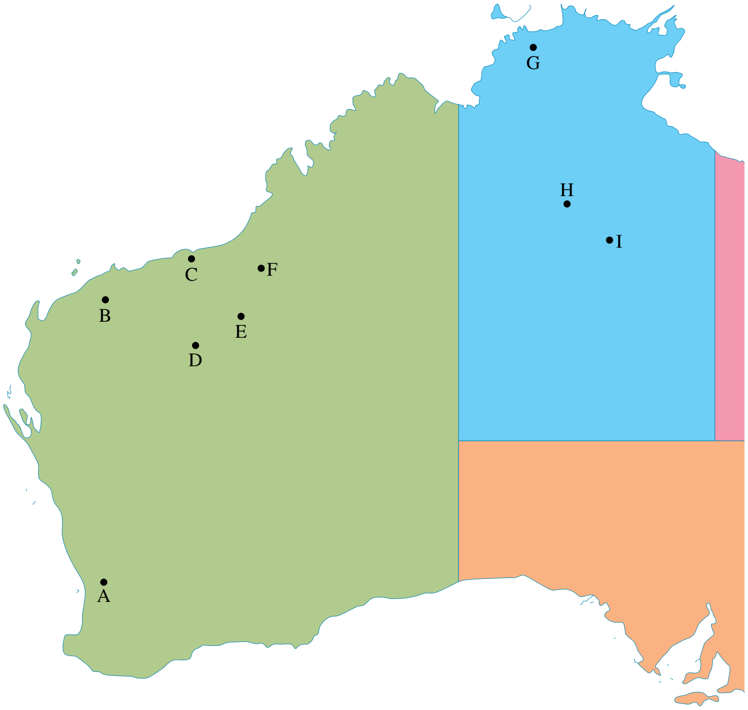
Information relating to the total costs of setting up the network is shown in the following table.

Connected to:	Server	Client	Hub	Router	Firewall	Wifi	Printer
Server			$995	$1050			
Client		$845	$355				$325
Hub			$365	$395			$395
Router	$1050		$395		$395	$395	
Laptop			$295			$325	
Phone			$295			$325	
PDA						$325	
Internet					$855		

a. Use this information to draw a weighted graph for each option.

b. Which is the cheaper option?

15. A mining company operates in several locations in Western Australia and the Northern Territory, as shown on the map.

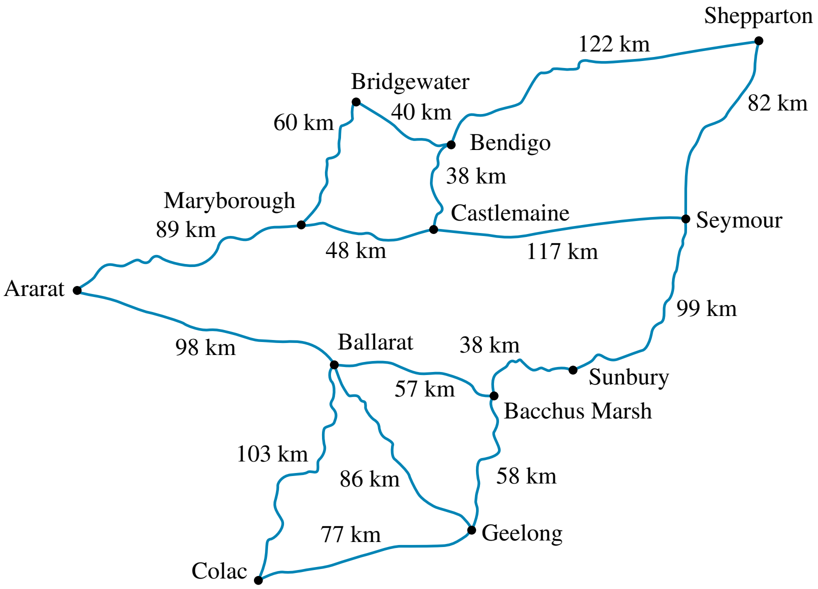


Flights operate between selected locations, and the flight distances (in km) are shown in the following table.

	A	B	C	D	E	F	G	H	I
A		1090		960			2600		2200
B	1090		360	375	435				
C		360							
D	960	375							
E		435							
F							1590	1400	
G	2600					1590		730	
H						1400	730		220
I	2200							220	

a. Show this information as a weighted graph.
b. Does a semi-hamiltonian cycle exist? Explain your answer.
c. Identify the shortest distance possible for travelling to all sites the minimum number of times if you start and finish at
   i. A
   ii. G.
d. Draw the minimum spanning tree for the graph.

16. The organisers of the 'Tour de Vic' bicycle race are using the following map to plan the event.



a. Draw a weighted graph to represent the map.

b. If they wish to start and finish in Geelong, what is the shortest route that can be taken that includes a total of nine other locations exactly once, two of which must be Ballarat and Bendigo?

c. Draw the minimum spanning tree for the graph.

d. If the organisers decide to use the minimum spanning tree as the course, what would the shortest possible distance be if each location had to be reached at least once, starting anywhere and finishing anywhere?

10.6 Review: exam practice

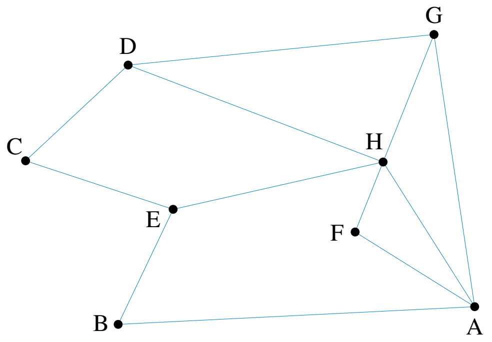
A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

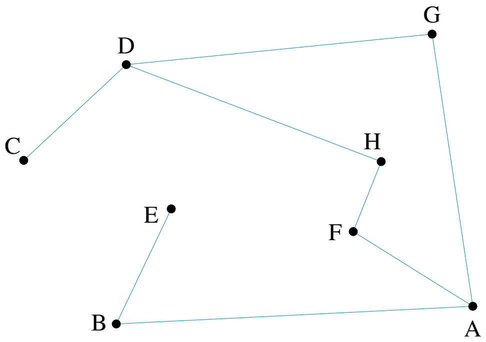
1. **MC** The minimum number of edges in a connected graph with eight vertices is

A. 5. **B.** 6. **C.** 7. **D.** 8.

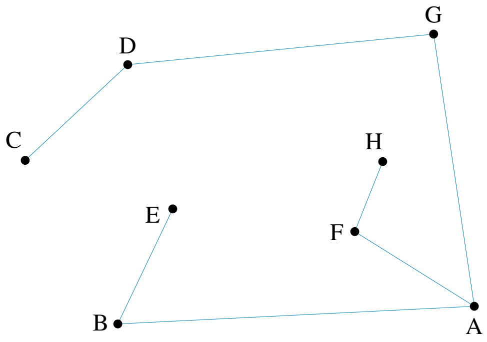
2. **MC** Which graph is a spanning tree for the following graph?



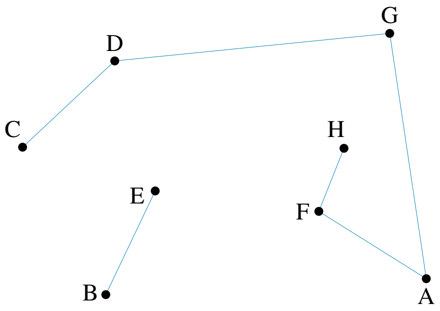
A.



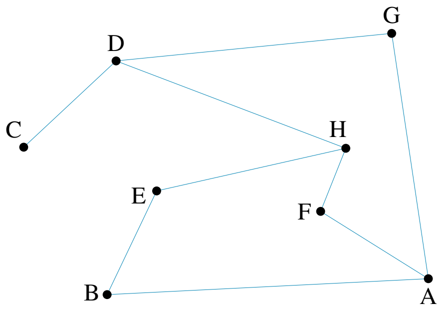
B.



C.



D.

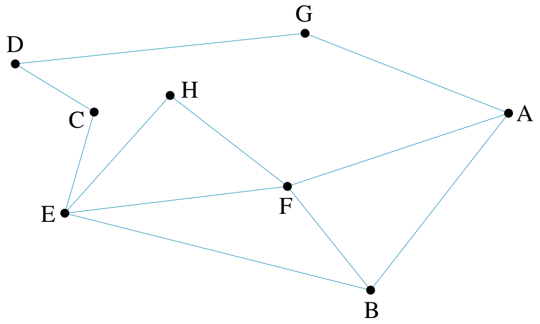


3. **MC** A connected graph with 9 vertices has 10 faces. The number of edges in the graph is

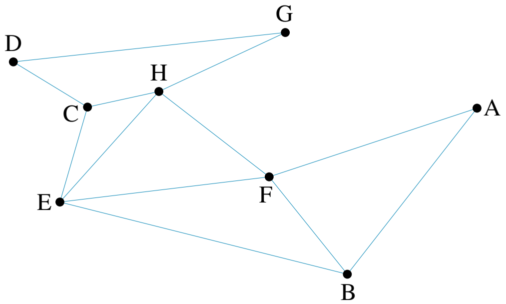
A. 15. **B.** 16. **C.** 17. **D.** 18.

4. **MC** Which of the following graphs will not have a semi-Eulerian trail?

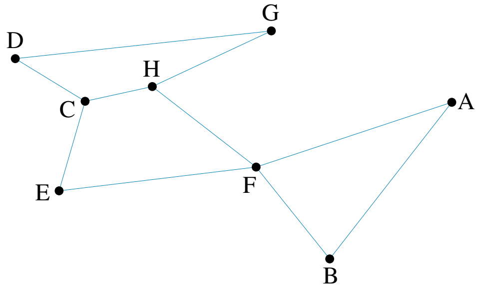
A.



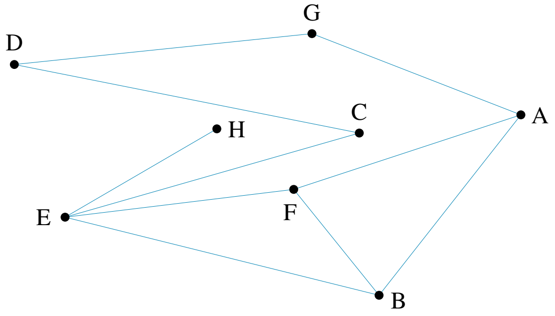
B.



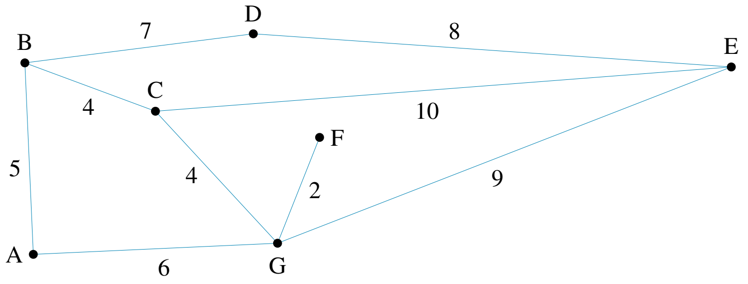
C.



D.

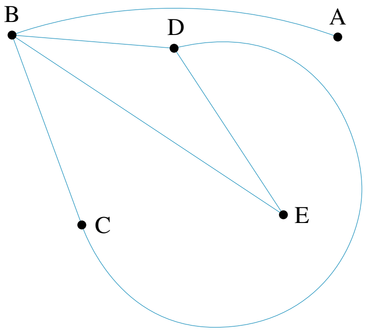


5. **MC** What is the length of the minimum spanning tree of the following graph?



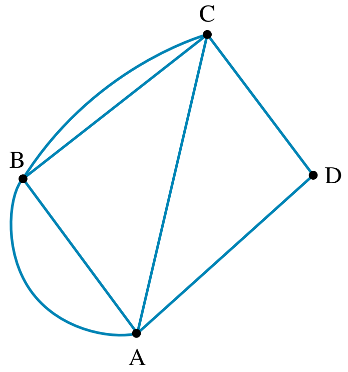
A. 33 **B.** 26 **C.** 34 **D.** 30

6. **MC** An Eulerian trail can be created in the following graph by adding an edge between the vertices:



A. A and D. **B.** A and B. **C.** A and C. **D.** B and C.

7. **MC** The adjacency matrix that represents the following graph is



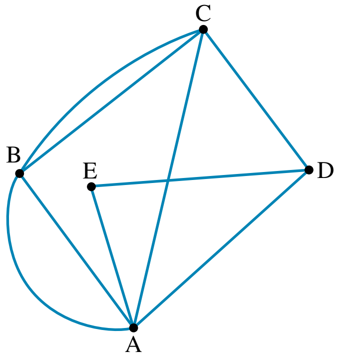
A. $\begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

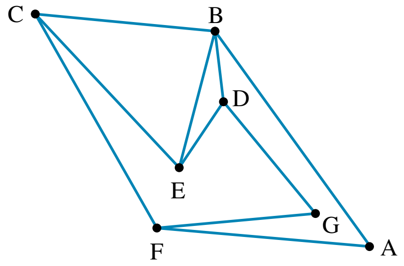
D. $\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

8. **MC** The number of faces in the following planar graph is



A. 6. **B.** 7. **C.** 8. **D.** 9.

9. **MC** A Hamiltonian cycle for the following graph is



A. ABCEDGFA. **B.** ABDGFCEA. **C.** ABDGFCEDEBCFA.
D. ABDGFCECFA.

10. **MC** A complete graph with 7 vertices will have a total number of edges of
A. 7. **B.** 8. **C.** 14. **D.** 21.

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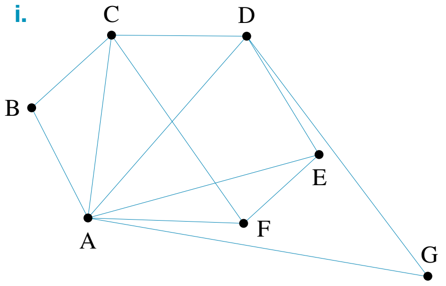
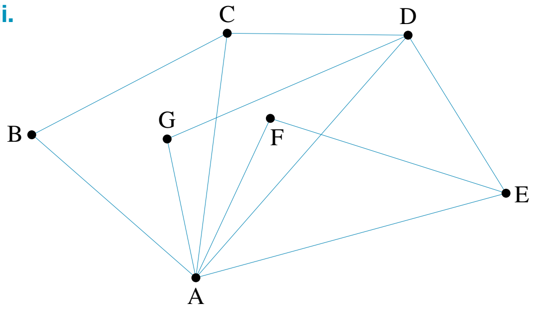
11. **a.** Identify whether the following graphs are planar or not planar.

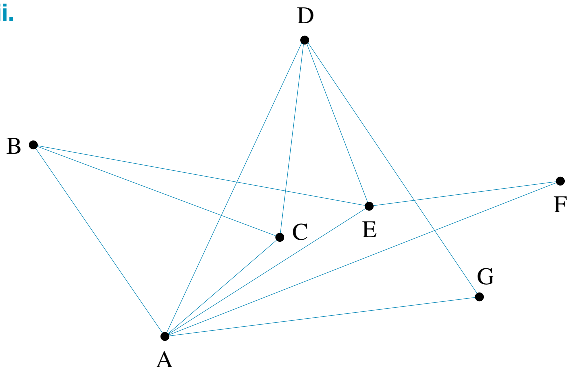
i.



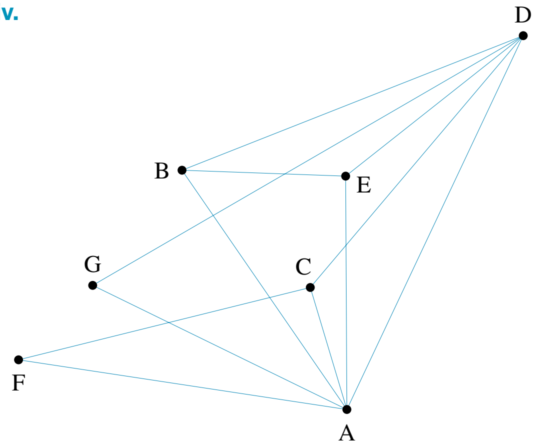
ii.



iii.



iv.



b. Redraw the graphs that are planar without any intersecting edges.

12. Complete the following adjacency matrices.

a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ & 0 & & 0 \\ & 3 & 1 & \\ & & 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 2 & 1 & \\ & 0 & & 0 & 1 \\ & 2 & 0 & 2 & \\ & & & 2 & 2 \\ 1 & & 3 & & 0 \end{bmatrix}$

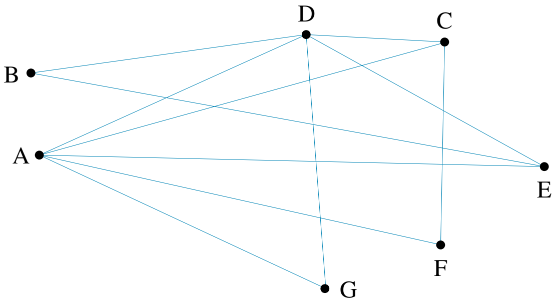
c. $\begin{bmatrix} 0 & & 1 & 3 & 1 & \\ 2 & 0 & & & 1 & \\ & 3 & 0 & 2 & & \\ & 1 & & 2 & 2 & 1 \\ & & 3 & & 3 & 1 \\ 2 & 0 & 1 & & & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & & & & 0 & & \\ 2 & 0 & & 1 & & & \\ 1 & 2 & 0 & 1 & 1 & 0 & \\ 3 & & & 0 & & & 1 \\ & 2 & & 0 & 0 & & 3 \\ 1 & 1 & & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & & & & 0 \end{bmatrix}$

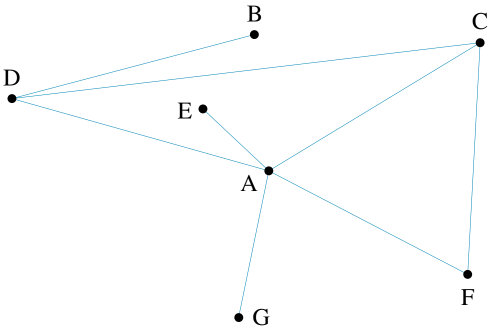
13. Identify which of the following graphs are:

i. simple
ii. complete
iii. planar.

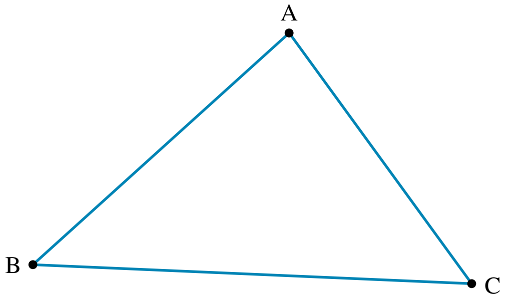
a.



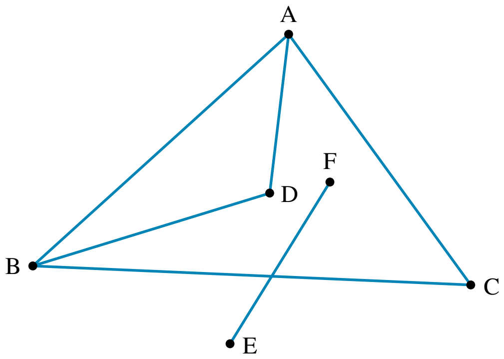
b.



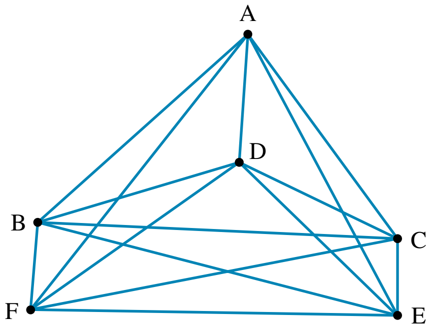
c.



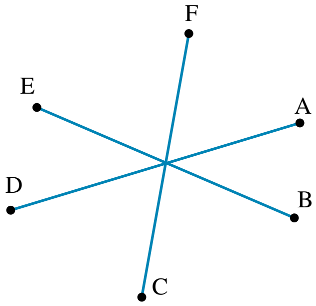
d.



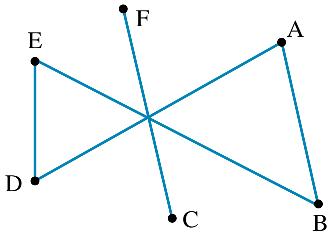
e.



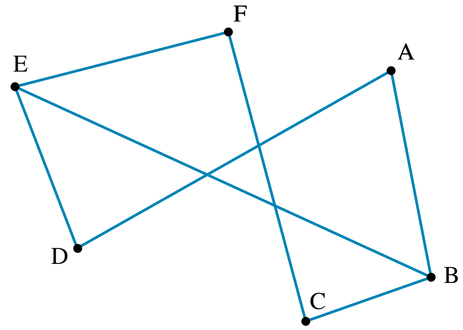
f.



g.

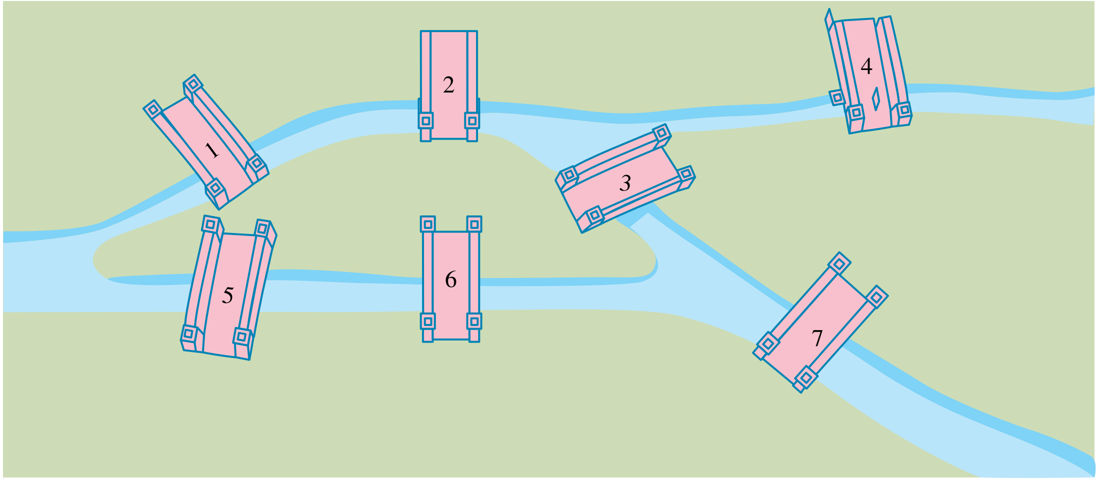


h.



14. The Bridges of Königesberg problem

The European city of Königesberg (now called Kaliningrad) is set on the banks of the River Pregel. Seven bridges were arranged as shown to connect the mainland of the city to two large islands.



Bridges of Königsberg

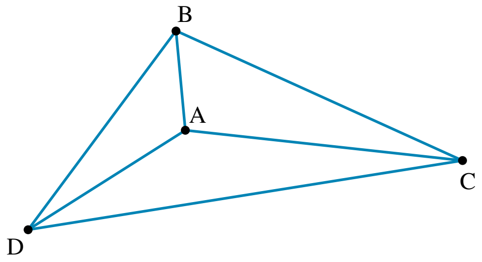
For a long time the townspeople wondered it was possible to travel around the city in such a way that all seven bridges would only have to be crossed once each.

- **a.** Is there a way to cross all 7 bridges without crossing any bridge more than once?
- **b.** Explain why or why not.

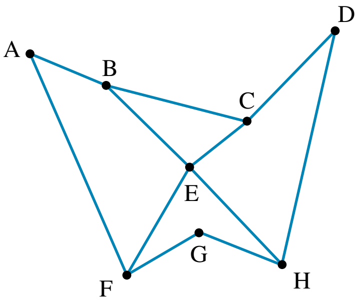
15. For each of the following graphs

- **i.** add the minimum number of edges to the following graphs in order to create a semi-Eulerian trail.
- **ii.** state the semi-Eulerian trail created.

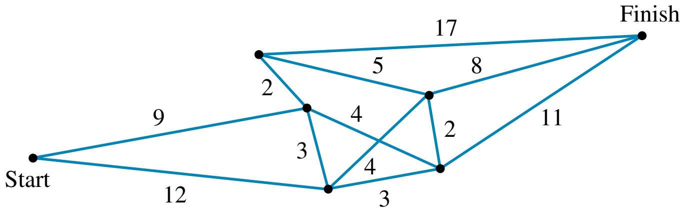
a.



b.



16. **a.** What is the shortest distance from start to finish in the following graph?



- **b.** What is the total length of the shortest semi-Hamiltonian cycle from start to finish?
- **c.** Draw the minimum spanning tree for this graph.

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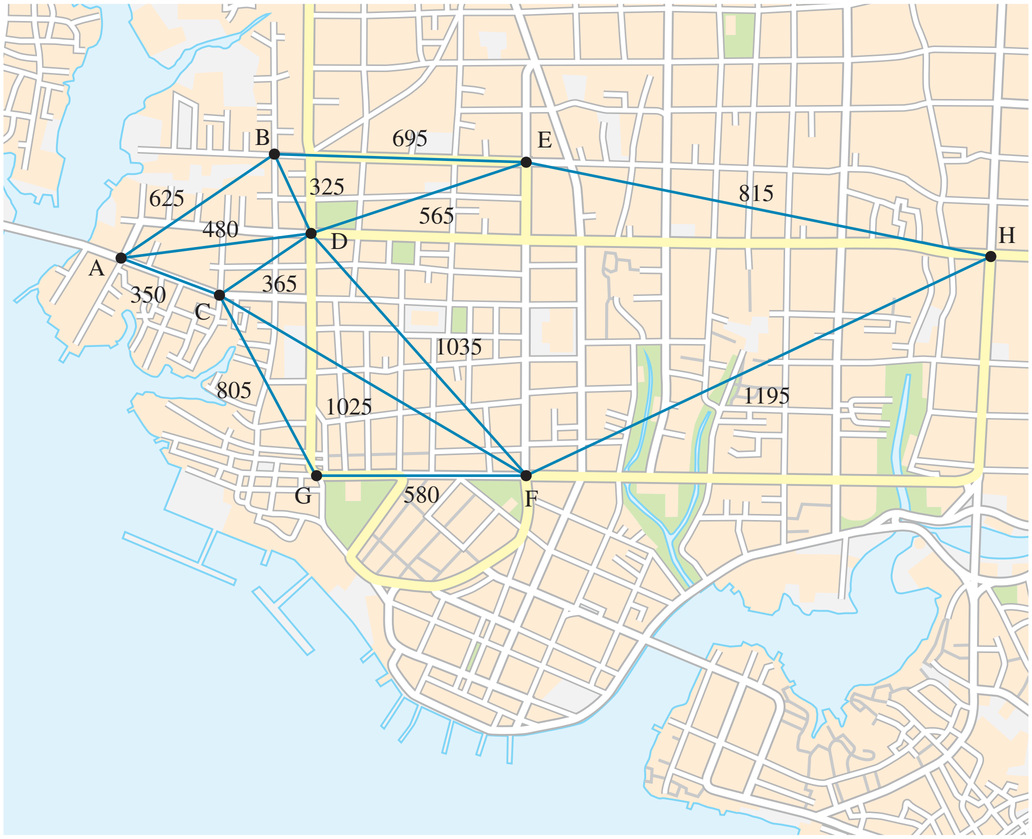
17. The flying distances between the capital cities of Australian mainland states and territories are listed in the following table.

	Adelaide	**Brisbane**	**Canberra**	**Darwin**	**Melbourne**	**Perth**	**Sydney**
Adelaide		2055	1198	3051	732	2716	1415
Brisbane	2055		1246	3429	1671	4289	982
Canberra	1198	1246		4003	658	3741	309
Darwin	3051	3429	4003		3789	4049	4301
Melbourne	732	1671	658	3789		3456	873
Perth	2716	4289	3741	4049	3456		3972
Sydney	1415	982	309	4301	873	3972	

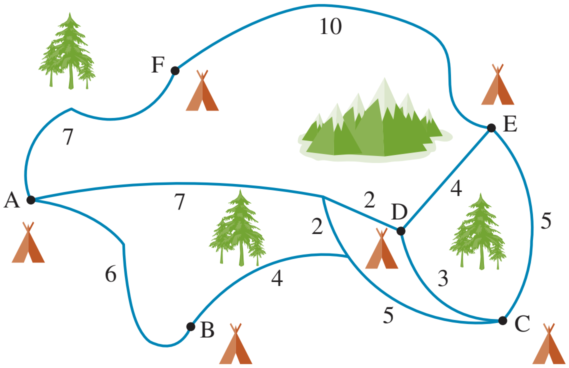
- **a.** Draw a weighted graph to show this information.
- **b.** If technical problems are preventing direct flights from Melbourne to Darwin and from Melbourne to Adelaide, what is the shortest way of flying from Melbourne to Darwin?
- **c.** If no direct flights are available from Brisbane to Perth or from Brisbane to Adelaide, what is the shortest way of getting from Brisbane to Perth?
- **d.** Draw the minimum spanning tree for the graph and state its total distance.

18. The following diagram shows the streets in a suburb of a city with a section of underground tunnels shown in black. Weightings indicate distances in metres. The tunnels are used for utilities such as electricity, gas, water and drainage.

- **a.** **i.** If the gas company wishes to run a pipeline that minimises its total length but reaches each vertex, what will be the total length required?
  - **ii.** Draw a graph to show the gas lines.
- **b.** If drainage pipes need to run from H to A, what is the shortest path they can follow? How long will this path be in total?
- **c.** A single line of cable for a computerised monitoring system needs to be placed so that it starts at D and reaches every vertex once. What is the minimum length possible, and what is the path it must follow?
- **d.** If a power line has to run from D so that it reaches every vertex at least once and finishes back at the start, what path must it take to be a minimum?



19. A brochure for a national park includes a map showing the walking trails and available camping sites at the park.



a. Draw a weighted graph to represent all the possible ways of travelling to the camp sites.
b. Draw the adjacency matrix for the graph.
c. Is it possible to walk a route that travels along each edge exactly once? Explain your answer, and indicate the path if it is possible.
d. If the main entrance to the park is situated at A, what is the shortest way to travel to each campsite and return to A?

20. A cruise ship takes passengers around Tasmania between the seven locations marked on the map.

The sailing distances between locations are indicated in the table.

	Hobart	Bruny I.	Maria I.	Flinders I.	Devonport	Robbins I.	King I.
Hobart	–	65 km	145 km	595 km	625 km	–	–
Bruny I.	65 km	–	130 km	–	–	715 km	–
Maria I.	145 km	130 km	–	450 km	–	–	–
Flinders I.	595 km	–	450 km	–	330 km	405 km	465 km
Devonport	625 km	–	–	330 km	–	265 km	395 km
Robbins I.	–	715 km	–	405 km	265 km	–	120 km
King I.	–	–	–	465 km	395 km	120 km	–

a. Draw a weighted graph to represent all possible ways of travelling to the locations.
b. What is the shortest route from Hobart to Robbins Island?

c. What is the shortest way of travelling from Hobart to visit each location only once?

d. What is the shortest way of sailing from King Island, visiting each location once and returning to King Island?

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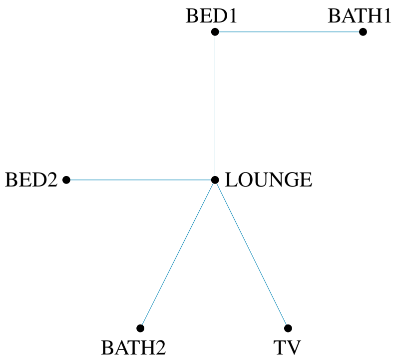
Answers

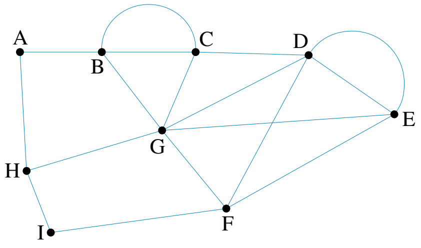
10 Graphs and networks

Exercise 10.2 Graphs, associated terminology and the adjacency matrix.

1.



2.



3. a. Edges = 7; Degree sum = 14

b. Edges = 10; Degree sum = 20

4. a. Edges = 9; Degree sum = 18

b. Edges = 9; Degree sum = 18

5. a. deg (A) = 5; deg (B) = 3; deg (C) = 4; deg (D) = 1; deg (E) = 1

b. deg (A) = 0; deg (B) = 2; deg (C) = 2; deg (D) = 3; deg (E) = 3

c. deg (A) = 4; deg (B) = 2; deg (C) = 2; deg (D) = 2; deg (E) = 4

d. deg (A) = 1; deg (B) = 2; deg (C) = 1; deg (D) = 1; deg (E) = 3

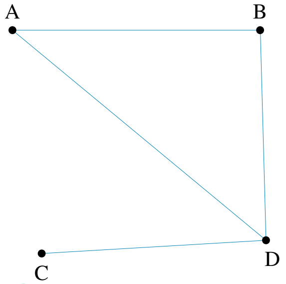
6. a.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

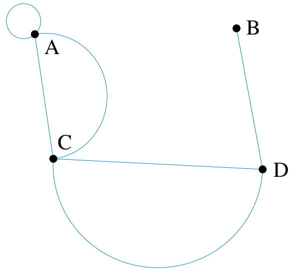
c.
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \end{bmatrix}$$

d.
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 4 & 0 \end{bmatrix}$$

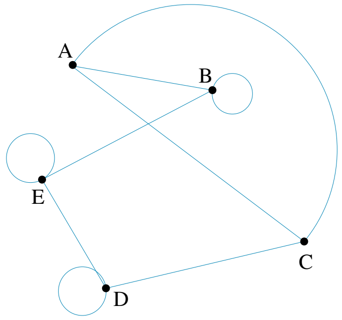
7. a.



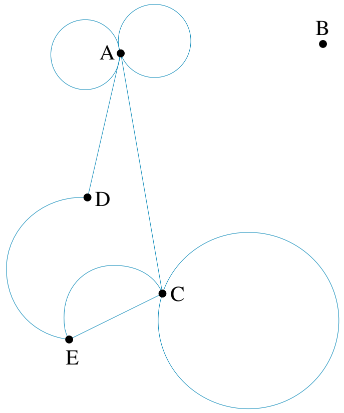
b.



c.



d.



8. a.
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

b.
$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

c. $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

9.

Graph	Simple	Complete	Connected
Graph 1	Yes	No	Yes
Graph 2	Yes	No	Yes
Graph 3	Yes	No	Yes
Graph 4	No	No	Yes
Graph 5	Yes	Yes	Yes

10. **Graph 1** $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Graph 2 $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

Graph 3 $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

Graph 4 $\begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

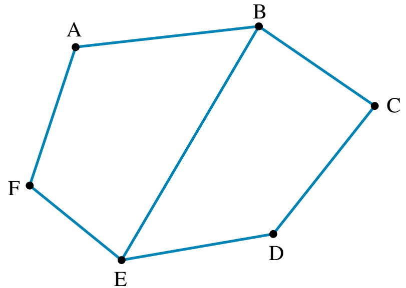
Graph 5 $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

11.

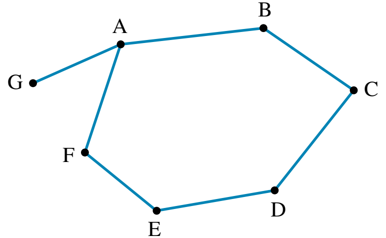
Vertices	Edges
2	1
3	3
4	6
5	10
6	15
n	$\frac{n(n-1)}{2}$

12. Answers will vary. Possible answers are shown.

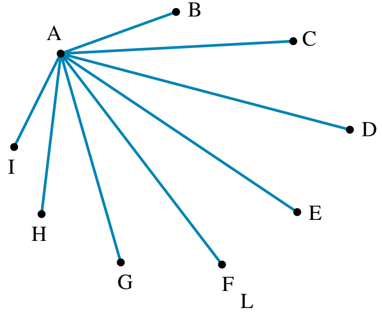
a.



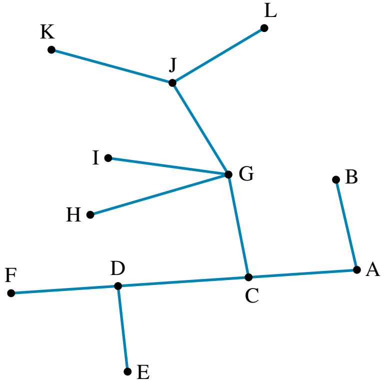
b.



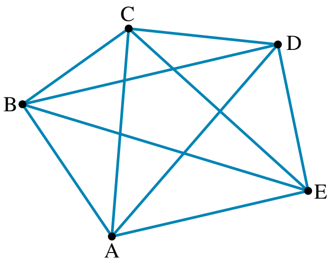
c.



13.



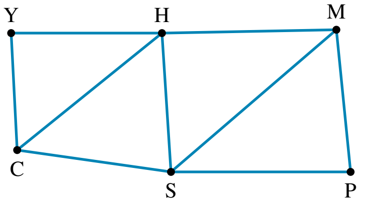
14. a.



b. Complete graph

c. Total number of games played

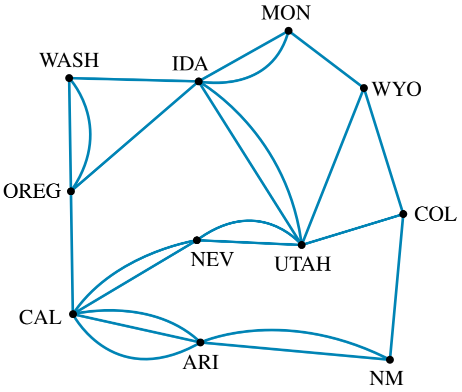
15. a.



b. Huairou and Shunyi

c. Simple connected graph

16. a.



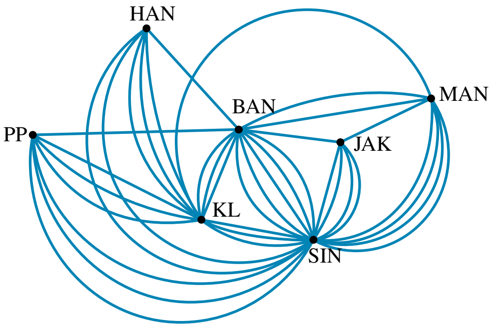
b.

$$\begin{array}{c} \\ \text{Wash} \\ \text{OREG} \\ \text{CAL} \\ \text{IDA} \\ \text{NEV} \\ \text{ARI} \\ \text{MON} \\ \text{UTAH} \\ \text{WYO} \\ \text{Col} \\ \text{NM} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} \text{Wash} & \text{OREG} & \text{CAL} & \text{IDA} & \text{NEV} & \text{ARI} & \text{MON} & \text{UTAH} & \text{WYO} & \text{Col} & \text{NM} \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

c. California, Idaho and Utah

d. Arizona

17. a.



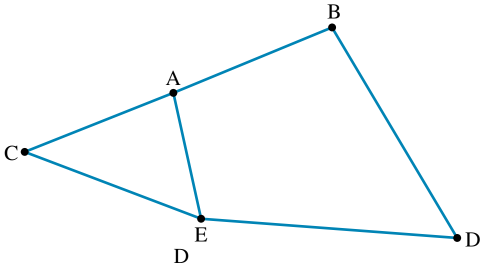
b. Directed, as it would be important to know the direction of the flight

c. **i.** 10

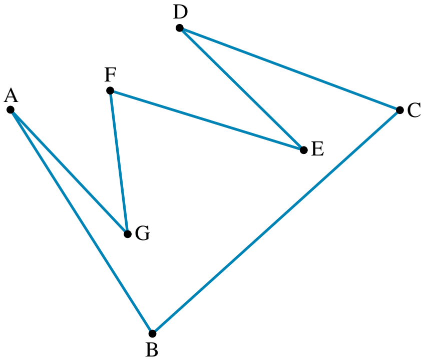
ii. 7

Exercise 10.3 Planer graphs

1. a.



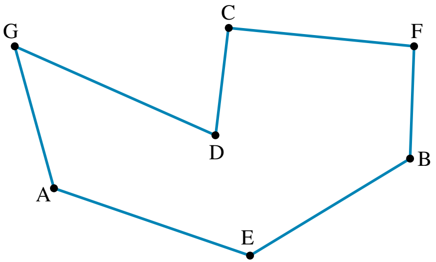
b.



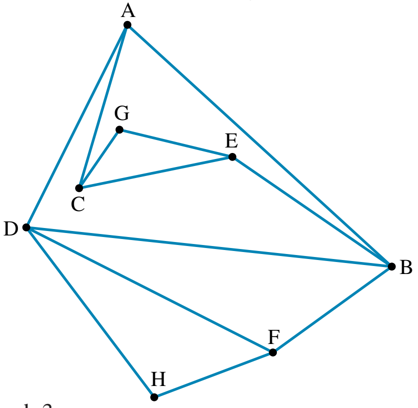
2. a. All of them

b. All of them

3. a.



b.



4. Graph 3

5. a. 4

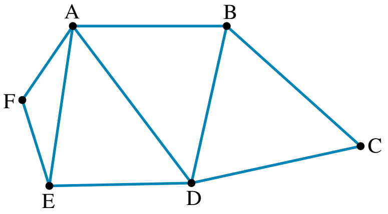
b. 5

6. a. 6

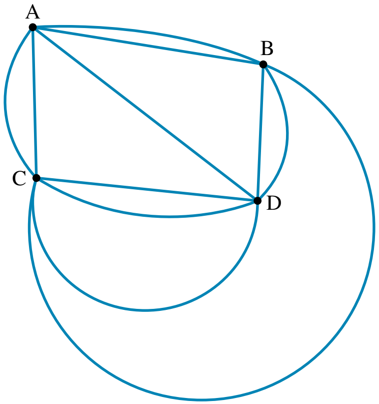
b. 5

7. a. 3 **b.** 3 **c.** 2 **d.** 7

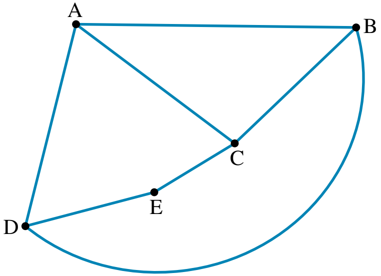
8. a.



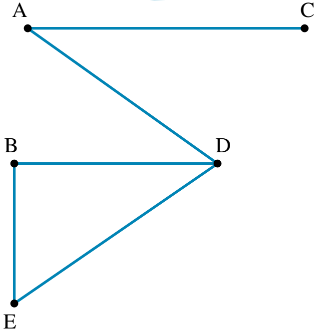
b.



9. a.



b.



10. a. i. 3

ii. 2

b. i. 1

ii. 4

11. a.

Graph	Total edges	Total degrees
Graph 1	3	6
Graph 2	5	10
Graph 3	8	16
Graph 4	14	28

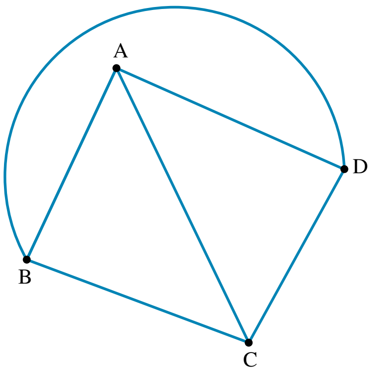
b. Total degrees = 2 × total edges

12. a.

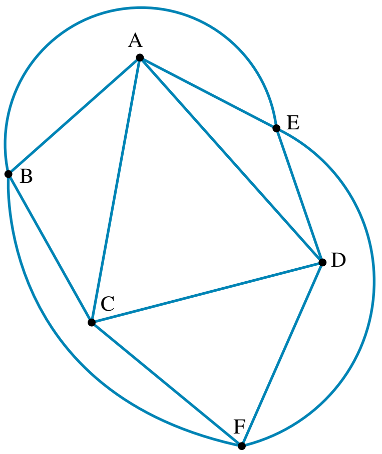
Graph	Total vertices of even degree	Total vertices of odd degree
Graph 1	3	2
Graph 2	4	2
Graph 3	4	4
Graph 4	6	6

b. No clear pattern evident.

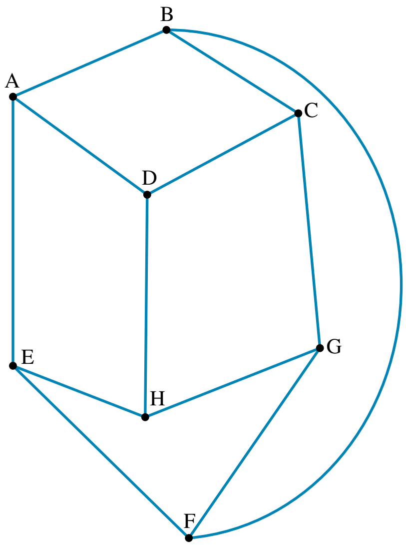
13.



Tetrahedron

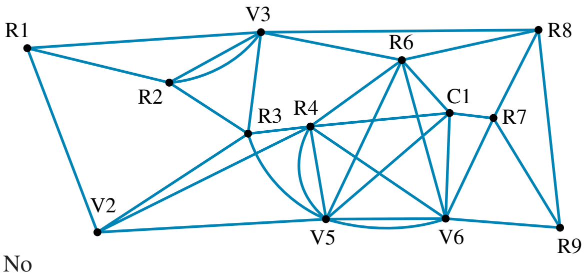


Octahedron



Cube

14. a.



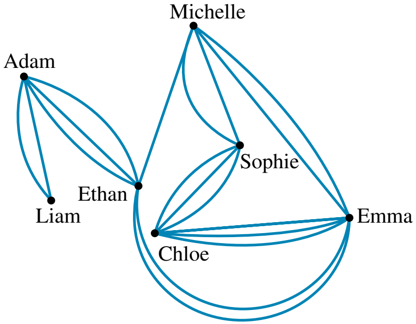
b. No

15. a.

$$\begin{array}{l|ccccccccccccccccccccccc|}
 & Pa & Ed & Bak & Wa & Ki & Fa & Mo & No & Bo & Ox & To & Ho & Ba & So & Vi & Gr & Pi & We & Em & Bl & Ca & Le & Ch \\
Pa & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Ed & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Bak & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Wa & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Ki & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Fa & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Mo & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
No & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Bo & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Ox & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
To & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Ho & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Ba & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
So & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Vi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
Gr & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
Pi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
We & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
Em & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
Bl & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
Ca & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
Le & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
Ch & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}$$

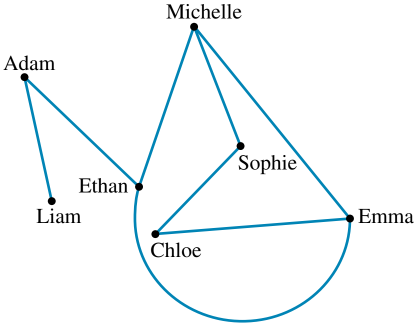
b. The sum of the rows represents the sum of the degree of the vertices, or twice the number of edges (connections).

16. a.



b. Sophie or Emma

c.



d. 4

Exercise 10.4 Connected graphs

1. Cycle: ABECA (others exist)
Closed trail: BECDB (others exist)

2. Path: ABGFHDC (others exist)
Cycle: DCGFHD (others exist)
Closed trail: AEBGFHDCA (others exist)

3. a. Walk
b. Walk, trail and path
c. Walk, trail, path, cycle and closed trail
d. Walk and trail

4. a. MCHIJGFAED
b. AEDBLKMC
c. MDEAFGJIHCM
d. FMCHIJGF

5. a. Semi-Eulerian trail: AFEDBECAB; semi-Hamiltonian cycle: BDECAF
b. Semi-Eulerian trail: GFBECGDAC; semi-Hamiltonian cycle: BECADGF

6. a. Eulerian trail: AIBAHGFCJBCDEGA; Hamiltonian cycle: none exist
b. Eulerian trail: ABCDEFGHA (others exist); Hamiltonian cycle: HABCDEFGH (others exist)

7. a. Graphs **i**, **ii** and **iv**
b. Graph **i**: ACDABDECB (others exist)
Graph **ii**: CFBCEDBADCA (others exist)
Graph **iv**: CFBCEDCADBAG (others exist)

8. a. Graphs **i** and **ii**
b. Graph **i**: CEDABC
Graph **ii**: CEDABFC

9. a. i.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

ii.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

iii.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

iv.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

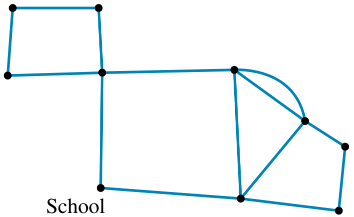
b. The presence of semi-Eulerian trails and Eulerian trails can be identified by using the adjacency matrix to check the degree of the vertices. The presence of Hamiltonian and semi-Hamiltonian cycles can be identified by using the adjacency matrix to check the connections between vertices.

10. E

11. a. A or C
b. B or D

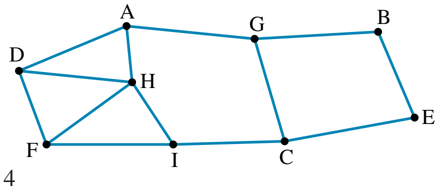
12. a. G to C
b. F to E

13. a.



b. Yes, because the degree of each intersection or corner point is an even number.
c. Yes, because the degree of each remaining intersection or corner point is still an even number.

14. a.



b. 4
c. i. ADHFICEBGA
ii. AHDFICEBGA
d. i. Two of the checkpoints have odd degrees.
ii. H and C

15. a.

	Hamiltonian cycle
1.	ABCDA
2.	ABDCA
3.	ACBDA
4.	ACDBA
5.	ADBCA
6.	ADCBA

b. Yes, commencing on vertices other than A

16. a. B, C, D, F or G

b. B or C

c. None possible

d. D or E

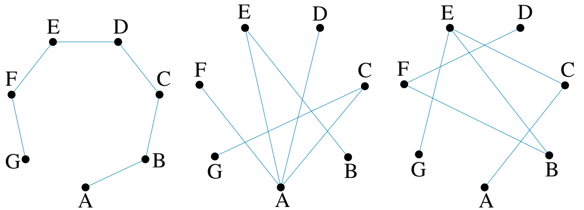
e. D to E

Exercise 10.5 Weighted graphs and trees

1. 21

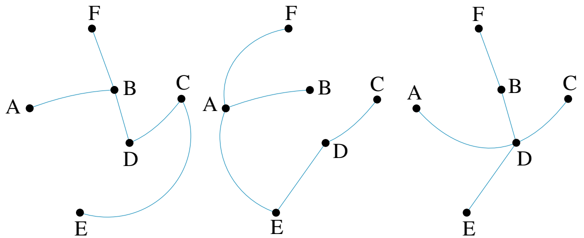
2. 20.78

3. a.



Other possibilities exist.

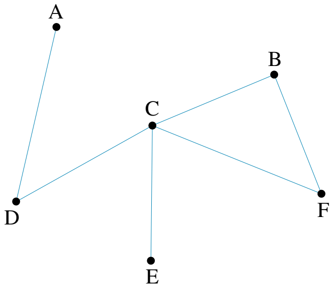
b.



Other possibilities exist.

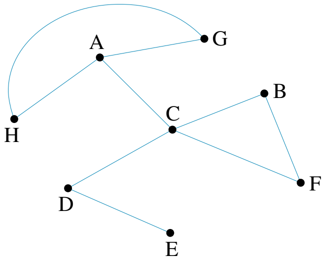
4. ABGEDCA or ACDEGBA (length 66)

5. a. i.



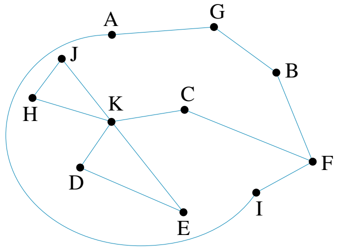
ii. ADCBFCE or ADCFBCE

b. i.



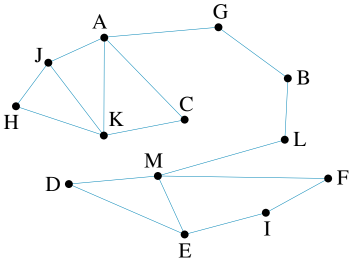
ii. AHGACBFCDE or similar

c. i.



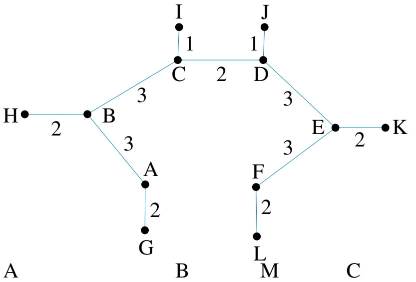
ii. KDEKHJKCFIAGBF or similar

d. i.

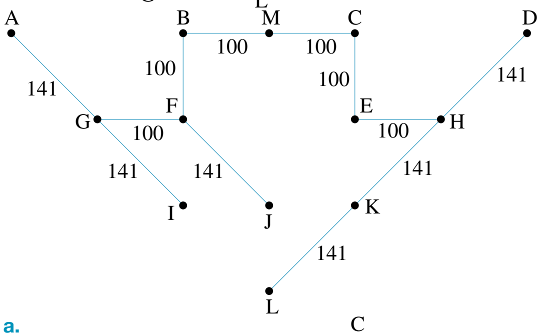


ii. EDMEIFMLBGACKHJKA or similar

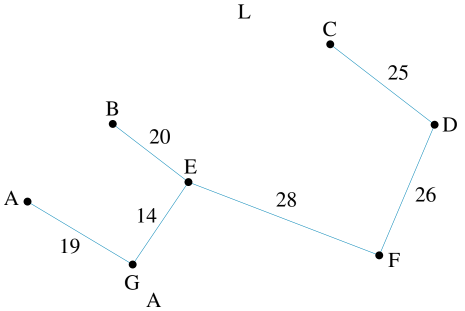
6.



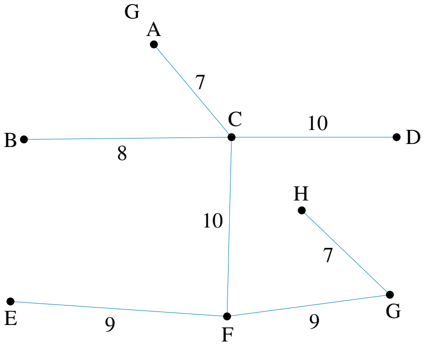
7.



8. a.

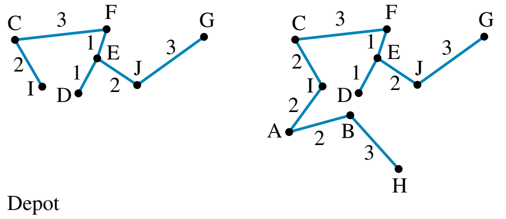


b.

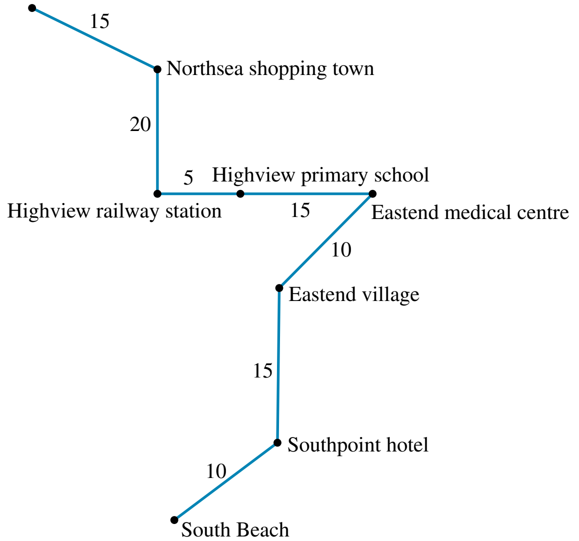


9. Step 1 Step 2

Step 3 Step 4



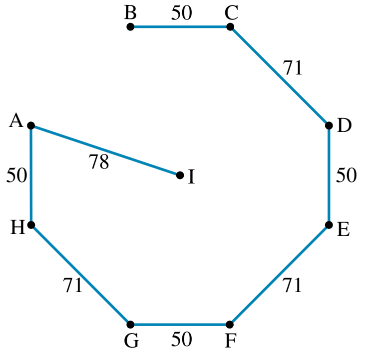
10. Depot



11. a. Longest: IFEDCBAHG (or similar variation of the same values)

Shortest: IAHGFEDCB (or similar variation of the same values)

b.



12. a. FDCGBAE (other solutions exist)

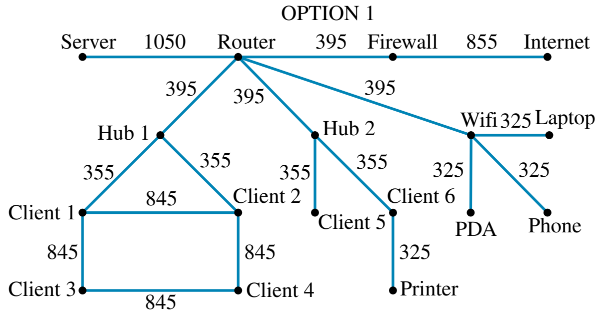
b. FDCBAEG (other solutions exist)

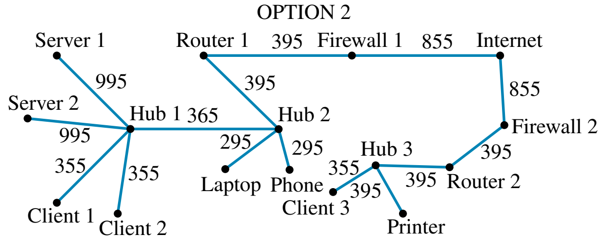
13. a. ADEG

b. BHG

c. EGFCDABHE

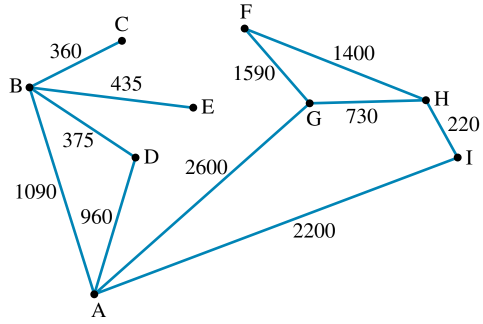
14. a.





b. Option 2

15. a.

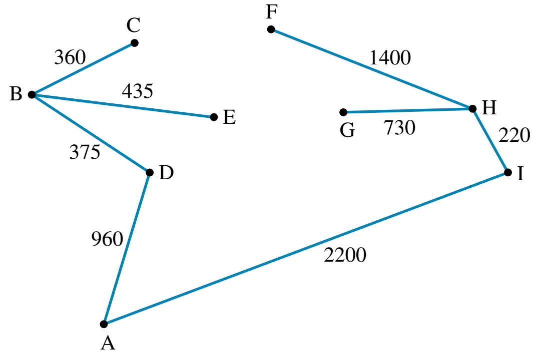


b. No; C and E are both only reachable from B.

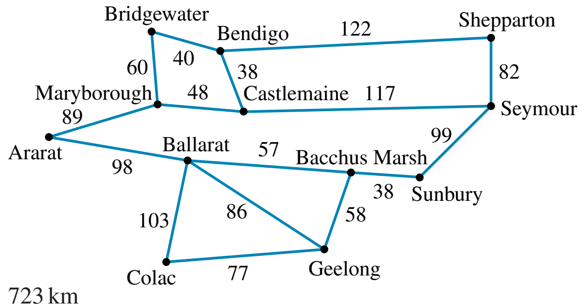
c. i. 12 025

ii. 12 025

d.

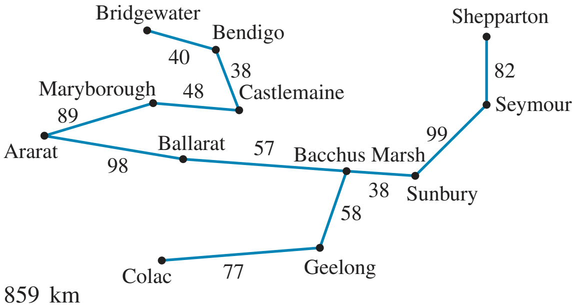


16. a.



b. 723 km

c.



d. 859 km

10.6 Review: exam practice

1. C

2. B

3. C

4. D

5. D

6. A

7. D

8. B

9. A

10. D

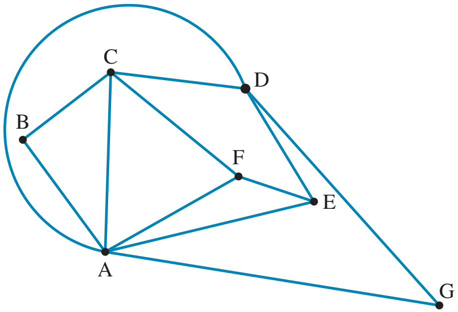
11. **a.** **i.** Planar

ii. Planar

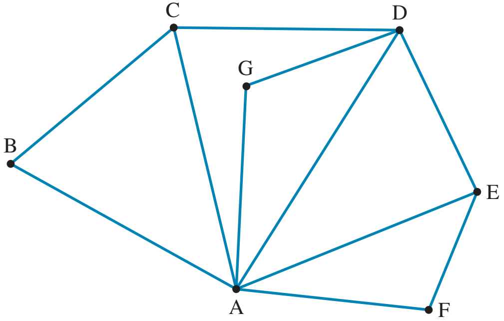
iii. Planar

iv. Planar

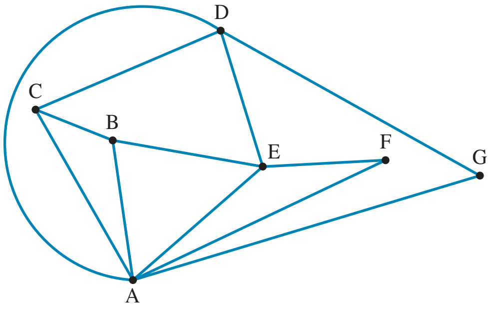
b. **i.**



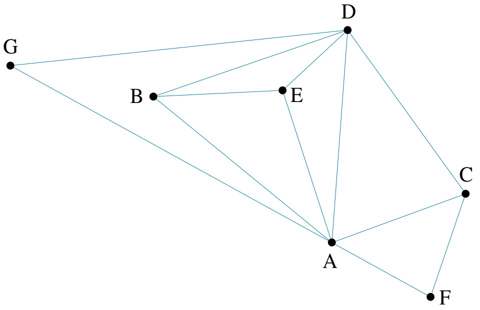
ii.



iii.



iv.



12. a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 2 & 0 & 2 & 3 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$

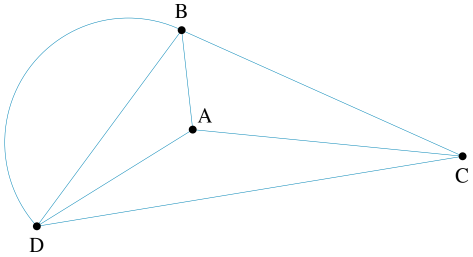
c. $\begin{bmatrix} 0 & 2 & 1 & 3 & 1 & 2 \\ 2 & 0 & 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 3 & 1 \\ 3 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 2 & 1 & 3 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 & 2 & 0 \end{bmatrix}$

13. a. Simple, planar
b. Simple, planar
c. Simple, complete, planar
d. Simple, planar
e. Simple, complete
f. Simple, planar
g. Simple, planar
h. Simple, planar

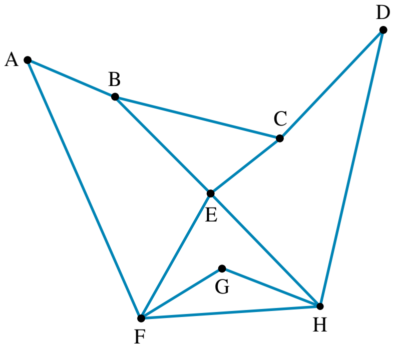
14. a. No
b. In this network, all 4 vertices have an odd degree. For an Eulerian or semi-Eulerian trail to exist 0 or 2 vertices of a connected network must have an odd numbered degree.

15. a. i. 3



ii. ABDBCADC

b. i.

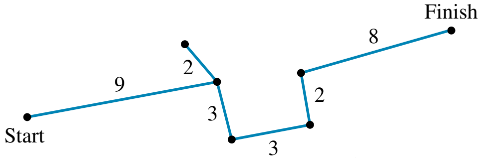


ii. BAFEHGFHDCEBC

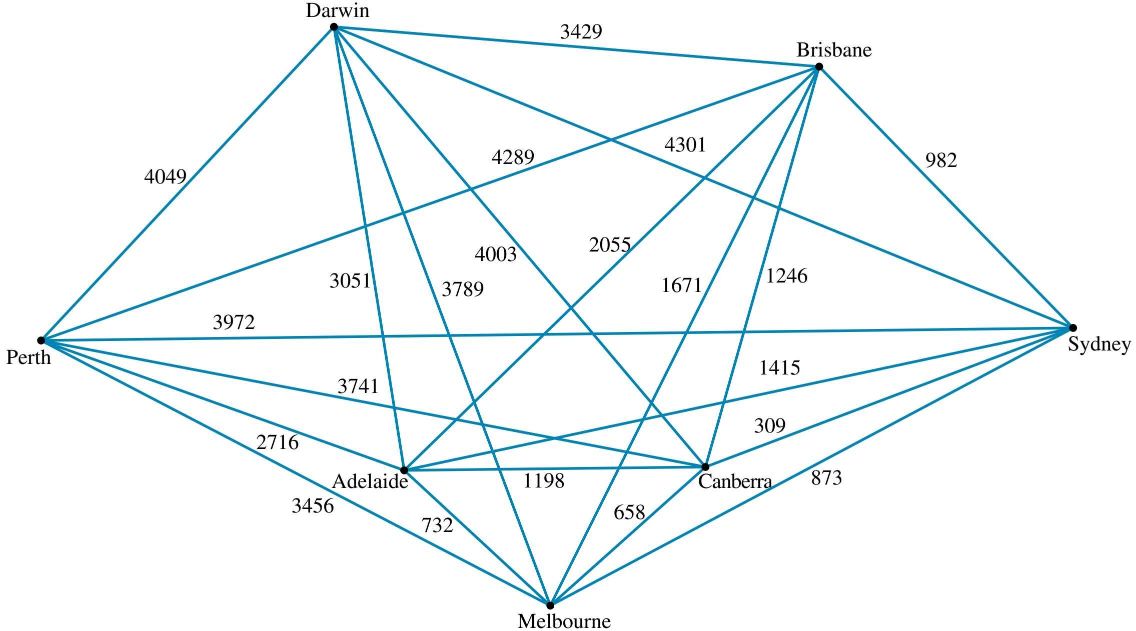
16. a. 23

b. 34

c.



17. a.



b. Via Canberra (4661 km)

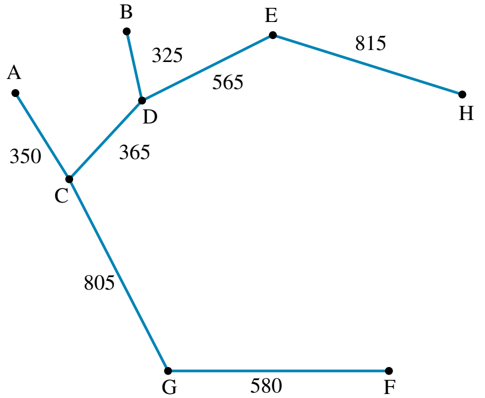
c. Via Sydney (4954 km)

d.

The total distance is 8448 km.

18. a. i. 3805 m

ii.

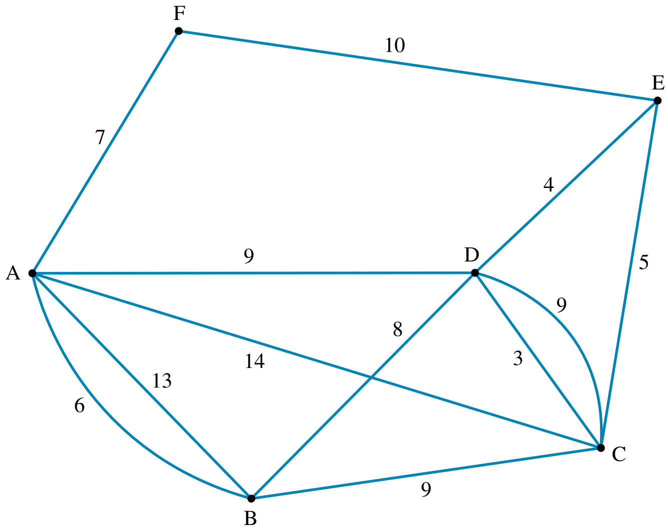


b. HEDA, 1860 m

c. 4905 m, DFGCABEH

d. DEHFGCABD, 5260 m

19. a.

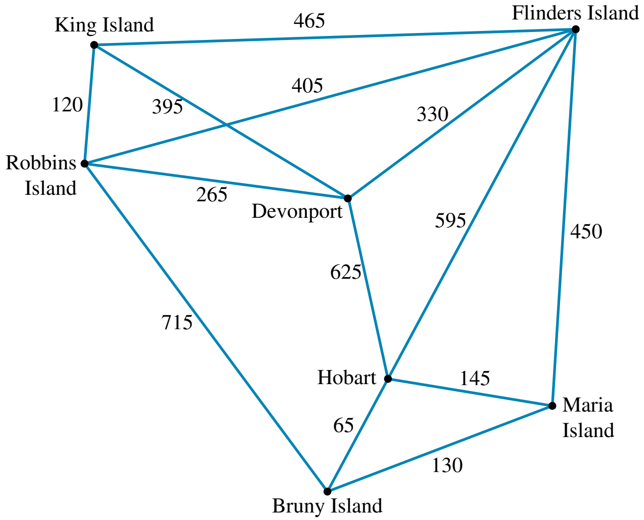


b. $\begin{bmatrix} 0 & 2 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

c. No, as there are more than two vertices of odd degree.

d. AFEDCBA(39)

20. a.



b. Hobart–Bruny–Robbins (780 km)

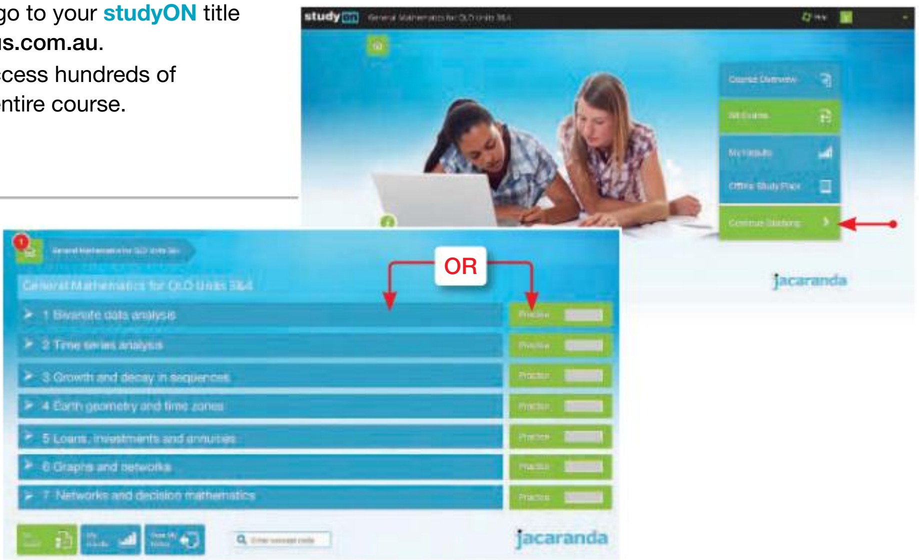
c. Hobart–Bruny–Maria–Flinders–Devonport–Robbins–King (1360 km)

d. King–Robbins–Devonport–Hobart–Bruny–Maria–Flinders–King (2120 km)

REVISION UNIT 4 Investing and networking

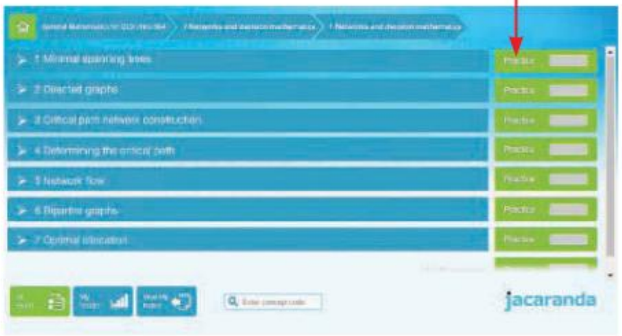
TOPIC 2 Graphs and networks

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



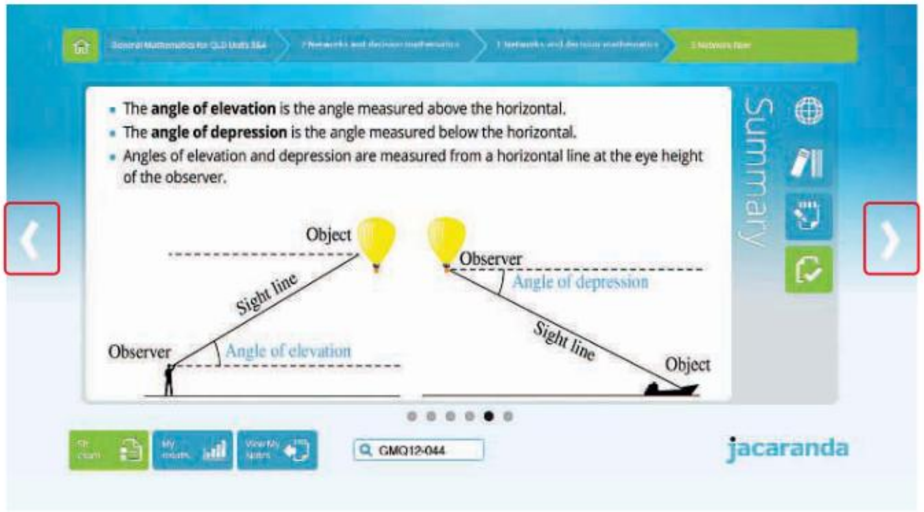
- Select your **course** *General Mathematics for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level *OR* select **Practice** to answer all practice questions available for each area.

- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



- The **angle of elevation** is the angle measured above the horizontal.
- The **angle of depression** is the angle measured below the horizontal.
- Angles of elevation and depression are measured from a horizontal line at the eye height of the observer.

11 Networks and decision mathematics

11.1 Overview

Whether you are building a skyscraper or working on various projects in your small business, critical path analysis is essential for staying on budget, meeting deadlines, and managing personnel efficiently. Critical path techniques can be done with a pen and paper, but as a project becomes larger, it becomes more necessary to implement modern software. Although there are many types of software to assist you in finding critical paths, you will still need to understand the basic mathematical concepts in order to correctly reach your project goals.

Critical path analysis handles human resources and materials; the mathematics of network flow deal with the flow of resources such as water, oil, electricity or information through a given network. Imagine trying to maximise the amount of water that flows through a pipe system, determine the quickest way to shut down an oil leak in a gas field, or increase the amount of data flowing through a communications network. Problems such as these can be solved using the algorithms of network flow.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

11.2 Critical paths

11.2.1 Activity charts and networks

No matter what we do in our lives, there are many tasks that we must fit into our daily schedule. If the daily tasks are not organised, we tend to run out of time or double-book ourselves. Similarly, operations such as major construction tasks must be efficiently planned so that the right people and materials are at the right place, at the right time. If one of these components is wrong, then time, and therefore money, is wasted. To demonstrate the advantages of planning, we will use a simple example.

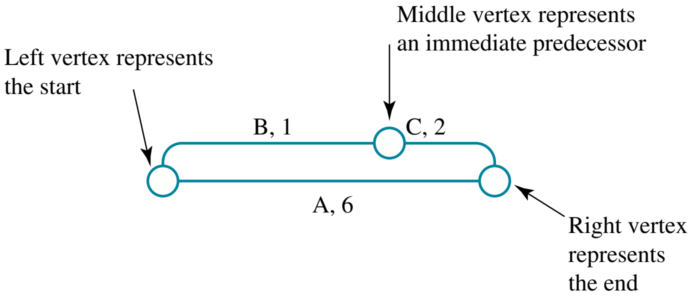
Michelle organises the things she *has* to do, to allow time for the activities she *wants* to do. (These activities include getting up early for half an hour of tai chi, and taking her dog for a run.)

While getting ready for school in the morning, Michelle is faced with the following problem. She has three tasks to do: downloading her emails from the computer, reading the emails and eating her breakfast. The first two tasks take 1 minute and 2 minutes respectively, while the last takes 6 minutes in all. Michelle needs to complete all these tasks in 7 minutes. How might she accomplish this?

Clearly, Michelle needs to be able to do some tasks simultaneously. The following table demonstrates what might happen each minute.

Time	Activity	Activity
1st minute		Download emails
2nd minute	Eat breakfast	
3rd minute	Eat breakfast	
4th minute	Eat breakfast	
5th minute	Eat breakfast	Read emails
6th minute	Eat breakfast	Read emails
7th minute	Eat breakfast	

More complex activities require a much greater amount of planning and analysis. Another way of representing this information is in a network diagram. A network diagram can be used to represent the 'flow' of activities.



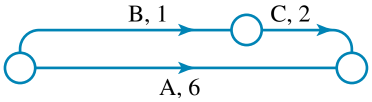
In the network diagram, the *edges* of our network represent the three activities of eating (A), downloading (B) and reading (C). The left vertex represents the start of all activity, the right vertex represents the end of all activity, and the middle vertex indicates that the downloading must occur before the reading starts. In other words, downloading (B) is the **immediate predecessor** of reading (C).

Another way of representing this information is in an **activity chart**, which shows the durations (times) of interdependencies (predecessors).

Activity letter	Activity	Predecessor	Time (min)
A	Eat breakfast	—	6
B	Download emails	—	1
C	Read emails	B	2

This activity chart also shows that activity B (downloading) is the immediate predecessor of activity C (reading), and that the other two activities (B and A) have no predecessors.

Following is an alternative network diagram.



This network diagram also indicates a direction. The activities can only be undertaken in a certain sequence (C must follow B), so arrowheads are shown on the edges. Because of the implied direction, networks of this type are called **directed networks**. (The edges in a directed network represent a one-way path between the vertices.)

WORKED EXAMPLE 1

Prepare a network diagram from the activity chart below.

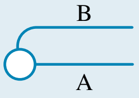
Activity letter	Activity	Predecessor	Time (min)
A	Eat breakfast	—	6
B	Download emails	—	1
C	Read emails	B	2

THINK

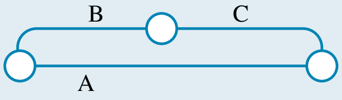
WRITE

1. Begin the network diagram by drawing the starting vertex.

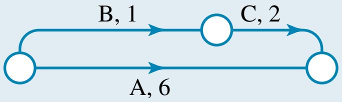
2. The table shows that two activities have no predecessor, so they are the starting activities in the network diagram. To show this, draw two edges out from the first vertex in step 1.



3. As activity A is not the predecessor of any other activities, its path will end at the last vertex. Activity B is a predecessor for activity C, so a new vertex needs to be drawn to link the two paths.



4. Add arrows to the network diagram to indicate the direction in which the activities flow. Lastly, write the times next to each activity.

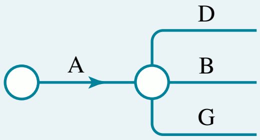


Let us now extend the activity chart to a more complex set of activities for Michelle's morning routine.

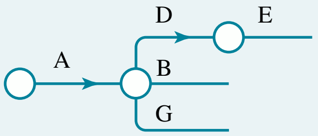
WORKED EXAMPLE 2

From the following activity chart, prepare a network diagram of Michelle's morning schedule.

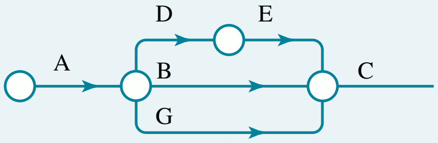
Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Download email	A	1
H	Read email	B, E, G	2
		Total time	25

THINK	WRITE
1. Begin the network diagram by drawing the starting vertex.	
2. a. Examine the table looking for activities that have no predecessors. There must be at least one of these.	Activity A has no predecessors.
b. This activity becomes the first edge and is labelled with its activity letter.	
3. a. List all activities for which A is the immediate predecessor. This is provided in the table.	Activity B has A as an immediate predecessor. Activity D has A as an immediate predecessor. Activity G has A as an immediate predecessor.
b. Add a vertex to the end of the edge for activity A, and an arrow indicating direction of flow.	
c. Create one edge from this vertex for each of the listed activities. Label these edges with their activity letters. *Note*: The end vertex for each of these activities is not drawn until either you are certain that it is not the immediate predecessor of any later activities, or all activities have been completed.	

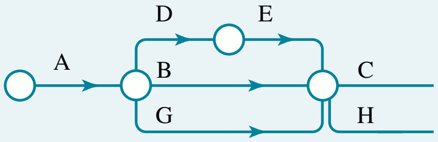
4. Repeat step 3 for activity D. Since it is the *only* immediate predecessor of activity E, this can be added to the diagram. Otherwise, activity E could not be added yet.



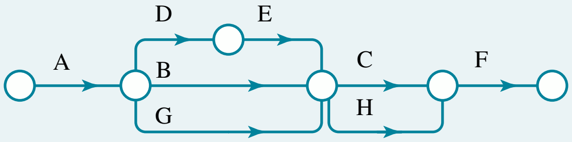
5. a. Repeat step 3 for activities B and G. They have no activities for which they are the only predecessors. Since activity C is preceded by all of B, G and E, join all the edges at a single vertex.
   b. Add activity C after this joining vertex. Note that activity H is also preceded by all of B, G and H but *not* by activity C.



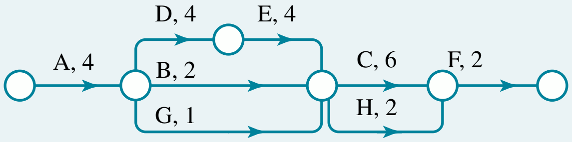
6. Determine whether activity C and H are independent of each other. Neither C nor H are predecessors of each other. Since they are independent, activity H starts from the same vertex as activity C.



7. The last activity is F, which has C and H as its immediate predecessors. Therefore join C and H with a vertex, then add an edge for F.
   Since F is the final activity, also add the end vertex.



8. Add the time required for each activity next to its letter.



Now that the tasks have been reduced to a network diagram, we can use the diagram to help Michelle reduce the total time spent (duration) on all these tasks. If all the tasks were spread out in a straight line so that no tasks were completed at the same time, then her morning routine would take 25 minutes (see the activity chart). The diagram shows that some of Michelle's tasks can take place at the same time. Let us investigate the time savings available.

11.2.2 Forward scanning and earliest start time

By forward scanning through a network we can calculate the earliest start times for each activity and the earliest completion time for the whole project.

The **earliest start time** (EST) is the earliest that any activity can be started after all prior activities have been completed.

The EST is determined by looking at all the previous activities, starting with the immediate predecessors, and working back to the start of the project. An activity can start no earlier than the *completion* of such predecessors. Obviously, the EST for the first activity is 0.

To determine the time saving, first determine the earliest start time for each activity. As an example, we will return to Michelle's initial three tasks.

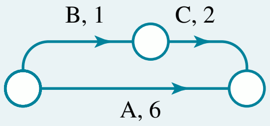
WORKED EXAMPLE 3

Use forward scanning to determine the earliest completion time for Michelle's initial three tasks from Worked example 1.

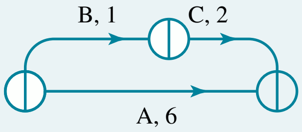
THINK

WRITE

1. Begin with the network diagram.

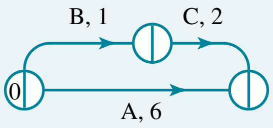


2. Separate all vertices into two halves.

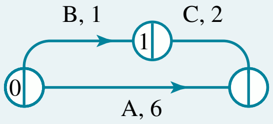


3. The earliest start time for each vertex is entered in the left-hand side of the vertex. Vertices with no immediate predecessors are given the value of zero.

As activities B and A have no immediate predecessor, their earliest start time is zero.



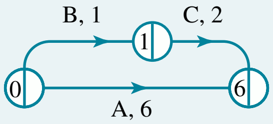
4. Move to another vertex and enter the earliest start time in the left-hand side. In the case of activity C it must wait one minute while its immediate predecessor, B, is completed.



5. The last vertex's earliest start time is entered. When more than one edge joins at a vertex then the earliest start time is the largest value of the paths to this vertex. This is because all tasks along these paths must be completed before the job is finished.

   There are two paths converging at the final vertex. The top path takes 3 minutes to complete and the bottom path, 6 minutes. The larger value, 6, is entered in the vertex.

Path B − C = 1 + 2 = 3 minutes
Path A = 6 minutes



6. The earliest completion time is the value in the final vertex.

All tasks can be completed in 6 minutes.

It is important for anybody planning many tasks to know which tasks can be delayed and which tasks must be completed immediately. In **Worked example 3**, the eating must be commenced immediately if the six-minute time is to be attained, whereas downloading the emails could be delayed three minutes and still allow enough time for them to be read while Michelle is eating.

on Resources

Interactivity Critical path analysis (int-6290)

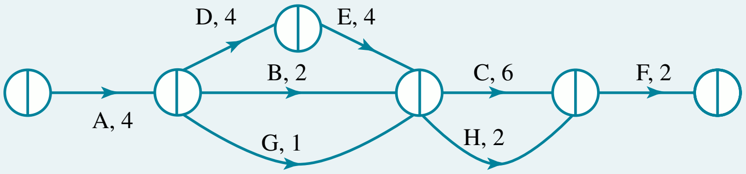
WORKED EXAMPLE 4

Using all the activities listed in Michelle's morning routine in Worked example 2, find the earliest completion time and hence identify those tasks that may be delayed without extending the completion time.

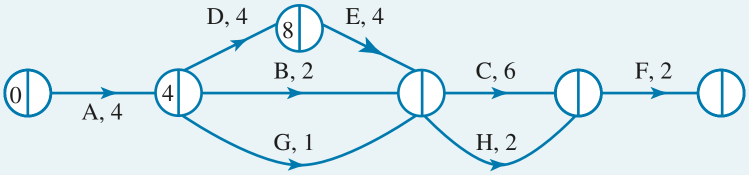
THINK

WRITE

1. Draw a directed network with split circles at each vertex. Include arrows to show the direction of Michelle's morning routine.

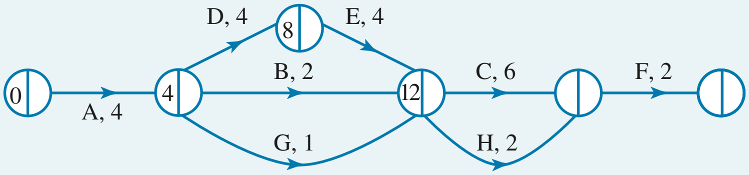


2. Begin forward scanning. The earliest start time for the first three vertices in the path can be entered immediately.



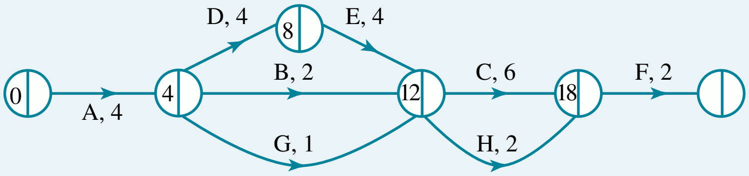
3. Calculate the time values for the paths to the fourth vertex. As there are multiple time values that could be entered, enter the largest value into the left-hand side of the vertex. This ensures all tasks are completed along these paths.

$$\begin{aligned} A–D–E &= 4 + 4 + 4 \\ &= 12 \text{ minutes} \\ A–B &= 4 + 2 \\ &= 6 \text{ minutes} \\ A–G &= 4 + 1 \\ &= 5 \text{ minutes} \end{aligned}$$



4. Repeat step 3 for the next vertex. Note that calculations begin by using the time from the previous vertex (12 minutes). Again, select the higher time value and enter it into the vertex.

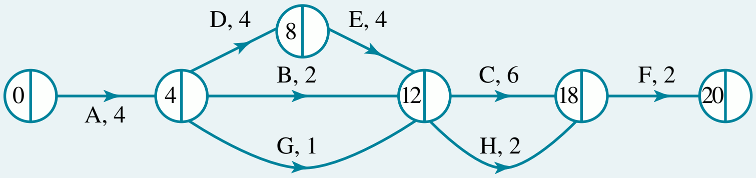
$$\begin{aligned} A–E–C &= 12 + 6 \\ &= 18 \text{ minutes} \\ A–E–H &= 12 + 2 \\ &= 14 \text{ minutes} \end{aligned}$$

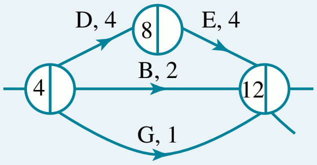
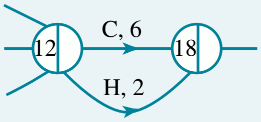


5. There is only one path to the last activity (F). Add its time requirement to that of the previous vertex (18 minutes).

$$\begin{aligned} A–C–F &= 18 + 2 \\ &= 20 \text{ minutes} \end{aligned}$$

Earliest completion time is 20 minutes.



6. The time in the last vertex indicates the earliest completion time of the morning routine.	Earliest completion time is 20 minutes.
7. Now find out which tasks can be delayed without extending the completion time of Michelle's routine. Identify the vertices that have more than one path to reach them. There are two verticies in this network, the 4th and 5th vertices. Examine the first one (the 4th vertex).	
8. List and total the time along each path to the 4th vertex of the network. The largest value indicates the path that cannot be delayed. Write down the paths that can be delayed.	D–E = 4 + 4 = 8 minutes B = 2 minutes G = 1 minute Paths B and G can be delayed.
9. Repeat step 8 for the 5th vertex.	 C = 6 minutes H = 2 minutes H can be delayed.

11.2.3 Critical paths and latest start time

The path through the network that follows those activities that cannot be delayed without causing the entire project to be delayed is called the **critical path**.

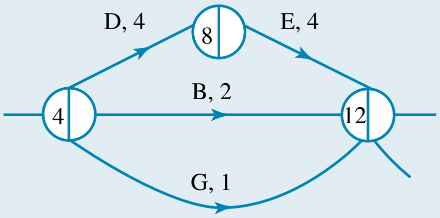
Therefore, the critical path for the activities in the previous section would be A–D–E–C–F, which is the path with the longest time (20 minutes).

Float time is the maximum time an activity can be suspended without delaying the entire project; that is, non-critical activities.

The **latest start time** (LST) for such activities is defined as the latest time they may be started without delaying the project.

WORKED EXAMPLE 5

Work out the float time for activities B and G in Worked example 4, and hence identify the latest starting time for these activities.



THINK

1. List all the paths to the 4th vertex and the times for these alternatives.

2. To calculate the float times for B and G, subtract the smaller times separately from the maximum time.

3. To identify the latest starting time for B and G, look up the **earliest completion time** for the activity on the critical path and subtract the activities' times.

WRITE

1.
$$\begin{aligned} D\text{–}E &= 4 + 4 \\ &= 8 \text{ minutes} \\ B &= 2 \text{ minutes} \\ G &= 1 \text{ minute} \end{aligned}$$

2.
$$\begin{aligned} \text{Float time for activity B} &= 8 - 2 \\ &= 6 \text{ minutes} \\ \text{Float time for activity G} &= 8 - 1 \\ &= 7 \text{ minutes} \end{aligned}$$

3. D–E is on the critical path. Earliest completion time = 12 minutes

$$\begin{aligned} \text{Latest start time for activity B} &= 12 - 2 \\ &= 10 \text{ minutes} \\ \text{Latest start time for activity G} &= 12 - 1 \\ &= 11 \text{ minutes} \end{aligned}$$

The float times indicate the amount of time for which these activities can be suspended without delaying the completion of all tasks. Furthermore, activity B could begin up to 6 minutes (10 − 4) after the start of the critical activity (D), while G could begin up to 7 minutes (11 − 4) after the same critical activity (D). There will be a more formal treatment of float time in the next section.

studyon

Units 3 & 4 > Area 7 > Sequence 1 > Concepts 1, 2 & 3

Directed graphs Summary screen and practice questions
Critical path network construction Summary screen and practice questions

Exercise 11.2 Critical paths

1. **WE 1** From each of the following activity charts, prepare a network diagram.

a.

Activity	Immediate predecessor
A	—
B	—
C	A

b.

Activity	Immediate predecessor
D	—
E	D
F	D
G	E, F

2. From each of the following activity charts, prepare a network diagram.

a.

Activity	Immediate predecessor
A	—
B	A
C	A
D	C
E	B
F	B
G	F
H	D, E, G
I	J, H
J	D, E, G

b.

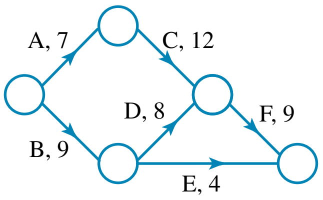
Activity	Immediate predecessor
N	—
O	N
P	O, T
Q	P
R	—
S	N
T	S, Y
U	O, T
V	O, T
W	V
X	Y
Y	R
Z	U, X

3. **WE 2, 3** When a laptop computer is being assembled the following processes must be performed.

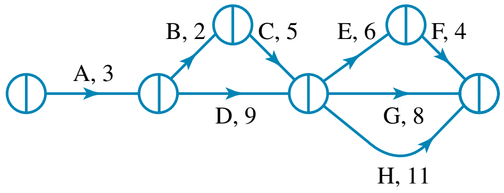
Activity letter	Activity	Predecessor	Time (min)
A	Install memory board	—	2
B	Test hard drive	A	20
C	Install hard drive	B, E	4
D	Install I/O ports	A	5
E	Install CD-ROM	D	3
F	Test CD-ROM	E	5
G	Install operating system	C, F	10
H	Test assembled computer	G	12
		Total time	61

a. From the activity chart, prepare a network diagram of the laptop assembly process.
b. Determine the earliest completion time for all tasks to be completed.

Refer to the network diagram shown to answer questions 4 and 5. Times shown are in minutes.

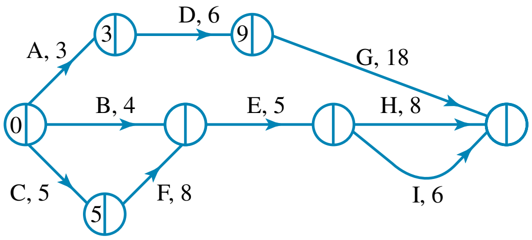


4. **MC** Which of the following statements is true?
   A. Activity A is an immediate predecessor of F.
   B. Activity D is an immediate predecessor of F.
   C. Activity F must be done before activity D.
   D. Activity F must be done before activity E.
5. **MC** The minimum time taken to complete all activities is
   A. 13 minutes. **B.** 21 minutes. **C.** 26 minutes. **D.** 28 minutes.
6. **WE 3, 4** Refer to the network diagram shown.



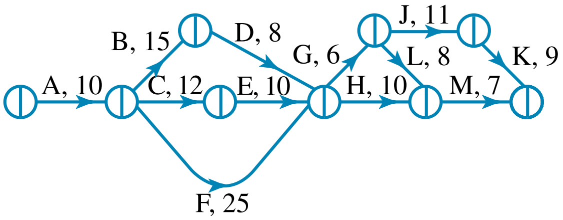
   a. Produce an activity chart for the network diagram.
   b. Use forward scanning to determine the earliest completion time.
   c. Identify tasks that may be delayed without increasing the earliest completion time.
   d. Determine the critical path for the network.

Refer to the network diagram shown to answer questions 7 to 10.



7. **a.** Produce an activity chart for the network diagram.
   b. Find the critical path.
   c. Determine which activities have float time and hence calculate their float times.
   d. Determine the latest start time for all non-critical activities.
8. **MC** The number required in the left-hand side of the vertex after activities B and F is
   A. 4. **B.** 13. **C.** 5. **D.** 8.
9. **MC** The number required in the left-hand side of the vertex after activity E is
   A. 5. **B.** 9. **C.** 10. **D.** 18.
10. **MC** The earliest completion time for all tasks is
   A. 27. **B.** 24. **C.** 21. **D.** 17.

11. **a.** Refer to the network diagram. Find the earliest start time for each vertex shown.
b. Hence, find the earliest completion time for the project.
c. Find the critical path.
d. Determine which activities have float time.



12. Following is a recipe for cheese and broccoli muffins.

Cheese and broccoli muffins

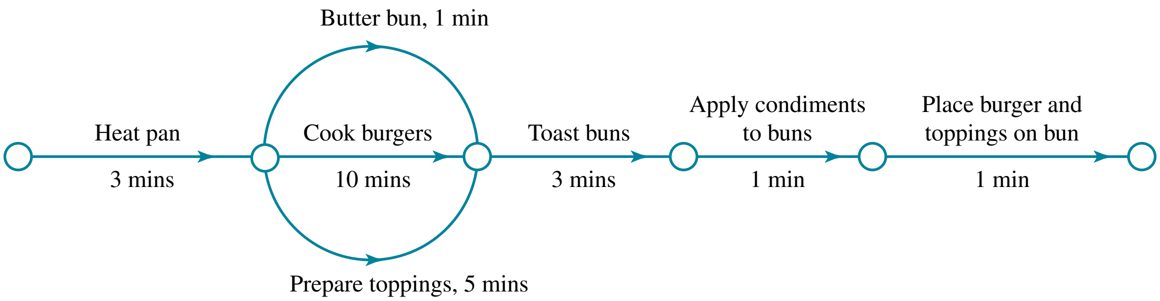
Ingredients
1.5 cups of raw broccoli
1.25 cups of cheese
0.75 cup of milk
2 large eggs
0.25 cup of vegetable oil
1.5 cups of self-raising flour
Salt and pepper
Cooking at 180 °C for 25 min

Directions
A — Chop broccoli (4 min).
B — Steam/boil broccoli. While steaming, measure out your ingredients in separate bowls (8 min).
Note: Steps C, D and E are being performed **simultaneously** as the broccoli is steaming.
C — Measure cheese (2 min).
D — Measure wet ingredients (2 min).
E — Measure dry ingredients (2 min).
F — Whisk wet ingredients (2 min).
G — Remove broccoli from steamer (1 min).
H — Combine broccoli, cheese and wet ingredients (3 min).
I — Combine these with the dry ingredients (3 min).
J — Preheat oven to 180 °C (10 min).
K — Prepare muffin tins (2 min).
L — Spoon mixture into prepared muffin tins (5 min).
M — Place in hot oven for 25 minutes (25 min)
N — Check with skewer in centre after 25 minutes to make sure the muffins are cooked through (1 min).
O — Once cooked, remove from oven and allow to cool for 5 minutes (5 min).

a. Create an activity chart for making cheese and broccoli muffins.
b. Use the activity chart to draw a network diagram.
c. Determine the critical path and minimum completion time for the entire recipe.
d. Determine which activities have float times and calculate these float times.
e. Would there be a change to the minimum completion time if C, D and E each took 1 minute longer?
f. Explain your answer to part **e**.
g. Would there be a change to the minimum completion time if C, D and E each took 5 minutes longer?
h. Explain your answer to part **g**.
i. If step L took 9 minutes, what would happen to the project time?

13. **a.** Give an example of a familiar project where simultaneous activities can be implemented in order to reduce the minimum completion time for the entire project.
b. Describe a familiar project where it is impossible to create simultaneous activities.
c. For part **a**, draw a network diagram of the project.
d. For part **b**, draw a network diagram of the project.

14. The following network diagram describes the process of preparing a burger. John made an incorrect assumption when drawing the network diagram.



a. Describe John's incorrect assumption.
b. Redraw the network to simulate the correct cooking process, including changes to the earliest start times.
c. How does John's incorrect assumption affect the earliest completion time?
d. What is the new earliest completion time?

15. a. Can a project have no critical path? Explain your answer.
b. Can a project have multiple critical paths? Explain your answer.

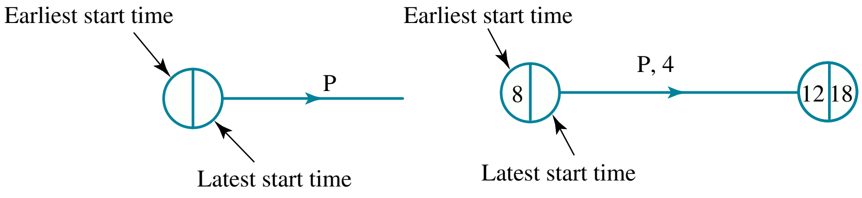
11.3 Critical path analysis with backward scanning

11.3.1 Backward scanning

With more complex projects requiring the coordination of many activities, it is necessary to record more information on the network diagrams and to display the information using charts.

In the previous section the float times and the critical path were worked out using somewhat informal methods. In this section a more formal method will be shown to enable float times to be calculated and the critical path to be determined. This method involves **backward scanning**.

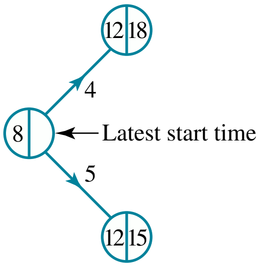
To complete critical path analysis, a procedure called *backward scanning* must be performed. In forward scanning, we record the *earliest start time* for an activity in the left-hand side of each vertex; in backward scanning, we record the *latest start time* in the *right-hand side* of each vertex — that is, the latest time that this activity can start without delaying the project.



Latest start time = 18 – 4
= 14

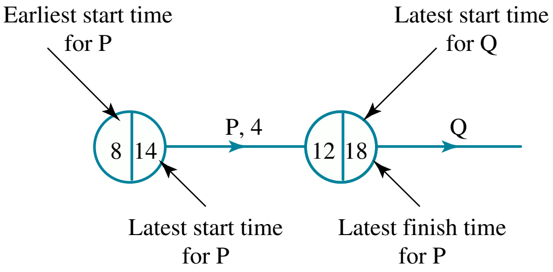
Backward scanning starts at the end vertex and moves backward through the network, subtracting the time of each edge from the earliest start time of each succeeding vertex.

When two or more paths are followed back to the same vertex the smallest such difference is recorded. The results of each backward scanning step yield the latest start time for each activity. Latest start time is the latest time an activity can start without delaying the project.



$$\begin{aligned}\text{Latest start time} &= 15 - 5 \text{ (because it is smaller than } 18 - 4)\\ &= 10\end{aligned}$$

Latest finish time for an activity is equal to the latest start time of the following activity. *Float time* is the maximum time that an activity can be delayed without delaying a subsequent activity on the critical path and thus affecting the earliest completion time.



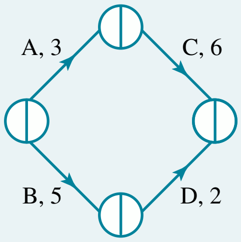
From the above it can be seen that there is a relationship between float time and the other quantities, namely:

Float time = latest finish time − earliest start time − activity time

The technique of backward scanning is best explained with an example.

WORKED EXAMPLE 6

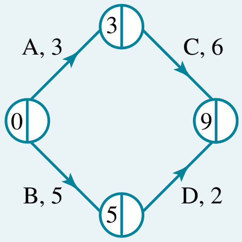
The network diagram at right has been constructed for a project manager. Use forward and backward scanning to clearly display the critical path and to list any float times.



THINK

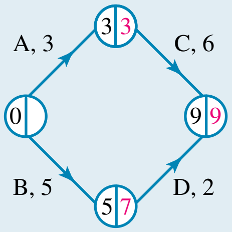
1. Forward scan through the network and record the earliest start time for each activity in the left-hand side of the appropriate vertex. The beginning vertex is between paths A and B.

WRITE



Along path C: $9 - 6 = 3$
Along path D: $9 - 2 = 7$

2. a. Begin backward scanning.
Start at the end vertex and trace backwards along all paths from this vertex.

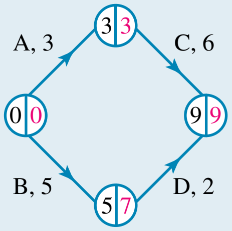


b. Subtract the times of the activities along each path from the earliest completion time (9) and record the value in the right-hand side of the previous vertex. These values are the latest start times for the activities along the path.

Latest start time for activity C = 3
Latest start time for activity D = 7

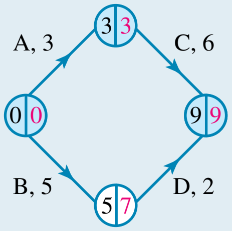
3. Repeat the process backwards through the diagram. Where two (or more) paths come together at a vertex (activities A and B), record the *smaller* value in the right-hand side of the vertex.

Along path A: 3 − 3 = 0
Along path B: 7 − 5 = 2
Smaller value = 0



4. The critical path can now be clearly identified. It is the path that has the same numbers in both the left and right sides of any vertex. Remember to include *all* such vertices in the critical path.

Critical path shown in blue



5. a. Float times can now be calculated.

b. Construct a table with the headings shown. Record the activity times (T), then record the times from the left-hand side of the vertices in the earliest start times (EST) column and the times in the right-hand side of the vertices in the latest finish times (LFT) column. Calculate float times using the equation:

Float = LFT − EST − T

In this example the float times are also the differences between the corresponding times in the vertices. This is not the rule in the general case.

Activity	Activity time	Earliest start time	Latest finish time	Float time
A	3	0	3	0
B	5	0	7	2
C	6	3	9	0
D	2	5	9	2

For activity D: Float = 9 − 5 − 2 = 2
For activity C: Float = 9 − 3 − 6 = 0
For activity B: Float = 7 − 0 − 5 = 2
For activity A: Float = 3 − 0 − 3 = 0

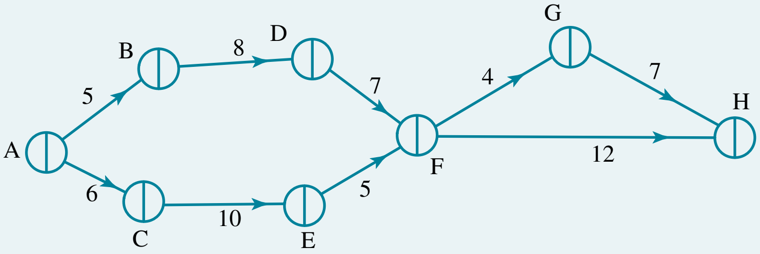
11.3.2 Crashing

The original completion time of some projects may need to be reduced to meet a given deadline. This can be done by a process known as **crashing**, which involves shortening the time of activities on the critical path. Once the time of activities on the critical path has been shortened, we can determine the new critical path as well as the new earliest completion time for the project.

WORKED EXAMPLE 7

Reggie is a project manager for a construction project. John, his boss, told him that he would receive a bonus if he could reduce the original completion times for the phases according to the following chart. Find the new critical path and new completion time for the project.

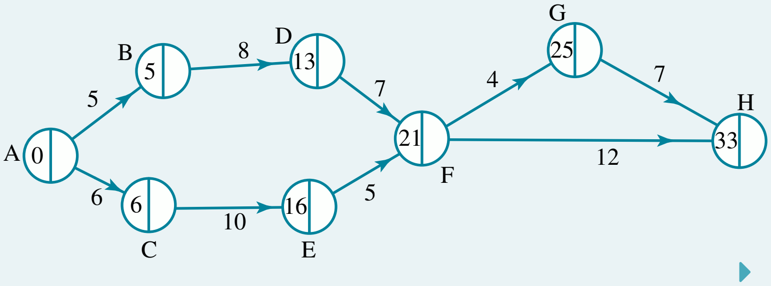
Phases	Original completion times (weeks)	Time reduction crash (weeks)
A–B	5	0
A–C	6	2
B–D	8	1
C–E	10	2
D–F	7	1
E–F	5	0
F–G	4	1
F–H	12	1
G–H	7	1



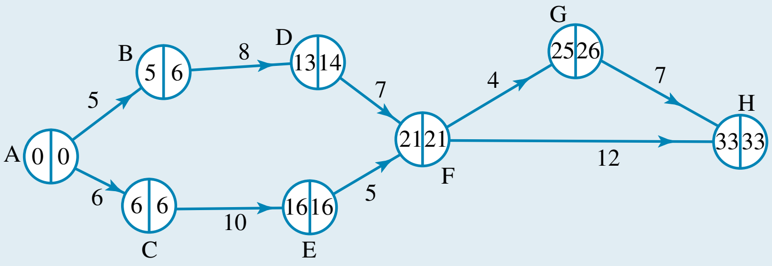
THINK

1. First, determine the critical path for the original project. To start, forward scan through the network and record the earliest start time for each phase in the left-hand side of the appropriate vertex.

WRITE



2. Backward scan through the network and record the latest finish time for each phase in the right-hand side of the appropriate vertex.



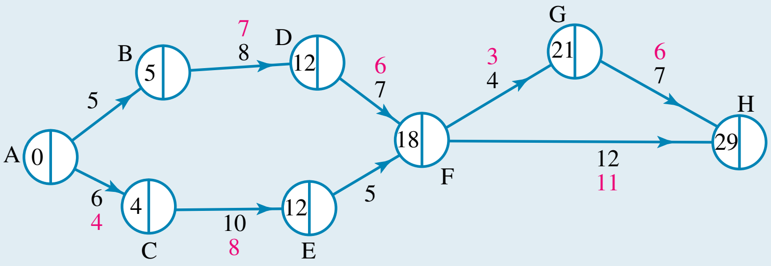
3. Determine the critical path.

A–C–E–F–H

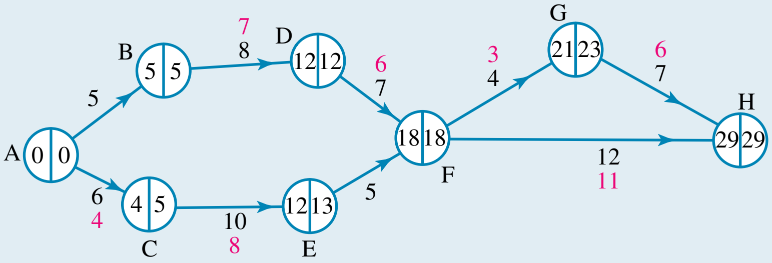
4. Determine the earliest completion time for the original project. Look at the last vertex (H). This number indicates the earliest completion time for this project: 33 hours.

The earliest completion time for the original project was 33 hours.

5. After subtracting the time reduction crashes for each phase, determine the new critical paths. Use the new phase times to forward scan and backward scan.
Forward scan through the network and record the earliest start time for each phase in the left-hand side of the appropriate vertex.



6. Backward scan through the network and record the latest finish time for each phase in the right-hand side of the appropriate vertex.



7. Determine the new critical path after crashing.

A–B–D–F–H

8. Determine the new minimum completion time after crashing. Look at the last vertex (H). This number indicates the earliest completion time for this project after crashing: 29 hours.

The earliest completion time for the project after crashing is 29 hours.

on Resources

Interactivity Crashing (int-6286)

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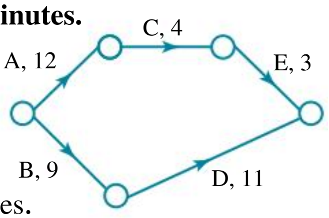
Units 3 & 4 › Area 7 › Sequence 1 › Concept 4

Determining the critical path Summary screen and practice questions

Exercise 11.3 Critical path analysis with backward scanning

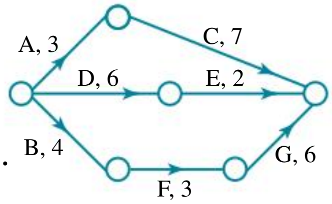
For questions 1 and 2, refer to the network diagram shown. Times are in minutes.



1. **WE 6** For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times. Times are in minutes.

2. **MC** The earliest completion time for all tasks is
   A. 13 minutes. **B.** 17 minutes. **C.** 19 minutes. **D.** 20 minutes.

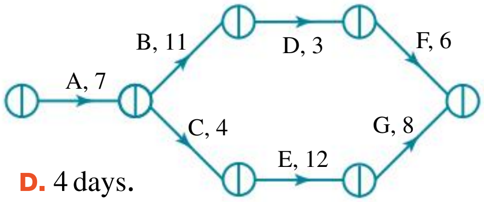
For questions 3 and 4, refer to the network diagram at right. Times are in hours.



3. For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times for non-critical activities. Times are in hours.

4. **MC** The earliest completion time for all tasks is
   A. 19 hours. **B.** 10 hours. **C.** 12 hours. **D.** 13 hours.

5. Complete the figure at right by forward and backward scanning and hence
   a. determine the earliest completion time
   b. indicate the critical path.
   Note: Times are in days.



6. **MC** The float time for activity D in question 5 is
   A. 1 day. **B.** 2 days. **C.** 3 days. **D.** 4 days.

7. **MC** The latest start time for activity D in question 5 is
   A. 18 days. **B.** 21 days. **C.** 22 days. **D.** 25 days.

8. The manufacturing of bicycles can be considered as a 7-step process

A — Collect all the parts — 12 minutes	
B— Paint frame — 35 minutes	(requires A to be completed first)
C — Assemble brakes — 16 minutes	(requires A to be completed first)
D — Assemble gears — 20 minutes	(requires B to be completed first)
E — Install brakes — 12 minutes	(requires C to be completed first)
F — Install seat — 5 minutes	(requires C to be completed first)
G — Final assembly — 18 minutes	(requires D and E to be completed first)

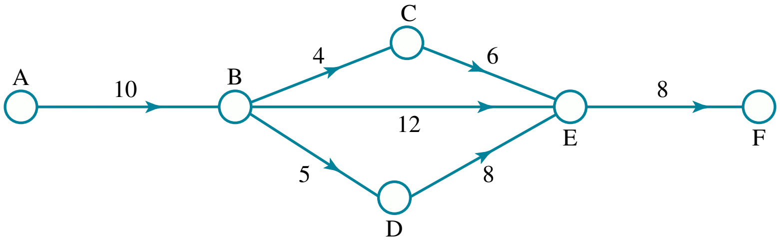
   a. Construct an activity chart.
   b. Construct a network diagram.
   c. Determine the earliest completion time using forward and backward scanning.
   d. Determine the critical path.
   e. Determine the amount of time saved, as a percentage, using the critical path approach versus completing each task sequentially.

9. In the bicycle manufacturing system described in question 8, activities with float time are
   A. A, B, C, D, E, F, G. **B.** A, B, C, D. **C.** C, E, F. **D.** C only.

10. **WE 7** Byron was assigned to create a computer network in his company. The table and network diagram below detail the process and times that Byron configured for each phase of the project.

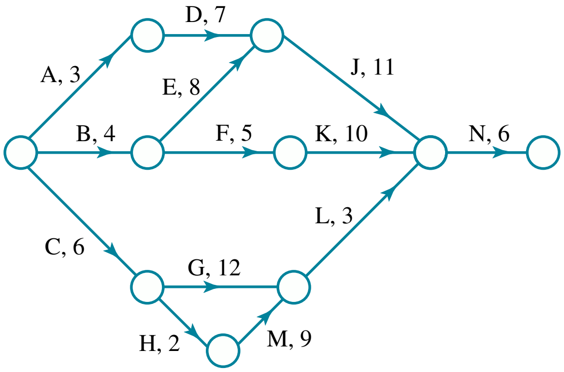
Byron's manager looked at the proposed plan and created amended reductions to the timetable.

Phases	Original completion times (weeks)	Time reduction crash (weeks)
A–B	10	0
B–C	4	2
B–D	5	1
B–E	12	2
C–E	6	2
D–E	8	1
E–F	8	2



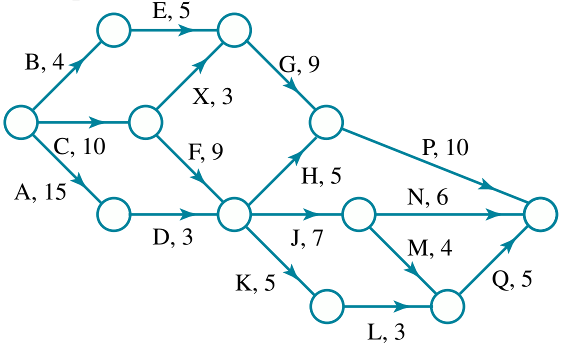
a. Determine the original critical path for this project.
b. Determine the original completion time for this project.
c. Determine the new critical path after crashing.
d. Determine the new completion time for this project after crashing.

11. From the network diagram below

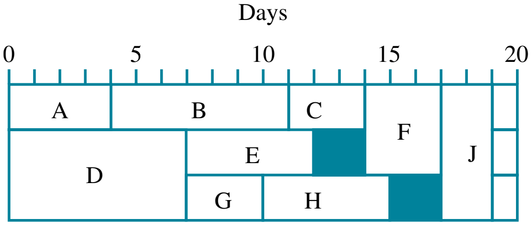


a. complete a forward scan and hence determine the earliest completion time
b. complete a backward scan and hence determine the critical path.

12. From the network diagram at right
a. forward scan to determine the earliest completion time
b. backward scan to determine the critical path
c. determine the float time for activity X.



13. A method often used in business to display the critical path is a critical path chart, as shown.



The chart indicates that the activities A–B–C–F–J are the critical path. The chart works as follows. Activities immediately to the left are immediate predecessors. For example, A is the immediate predecessor of B, while D is the immediate predecessor of E and G.

The length of activity is read off the scale (days) at the top. For example, activity C is 3 days long.

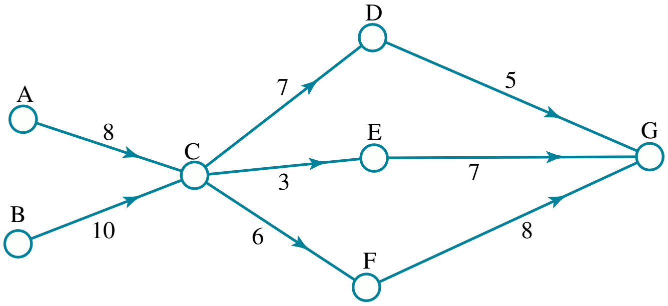
a. Construct a network diagram.

b. Determine the float times for each non-critical activity. (*Hint*: This can be determined directly from the critical path chart.)

Questions 14 to 16 refer to the following information.

The construction schedule for a pier in the seaside town of Bendalong is shown below. The network diagram corresponding to the construction table is also provided.

Phase	Normal time (weeks)	Crash time (weeks)	Normal cost ($)	Crash cost ($)	Crash cost ($)(per week)
A–C	8	0	4000	0	0
B–C	10	0	8000	0	0
C–D	7	1	6000	8000	2000
C–E	3	0	1500	0	0
C–F	6	1	6000	7000	1000
D–G	5	1	10 000	12 500	2500
E–G	7	0	5000	0	0
F–G	8	1	4000	5000	1000



14. a. What is the critical path of this project?

b. What is the original completion time for this project?

c. What is the earliest this project can be finished after crashing?

15. Calculate each of the following, assuming that the pier construction has a normal cost and crash cost added for each phase of the project.

a. The total cost of the project before crashing

b. The total cost of the project after crashing

c. The total additional cost of crashing the project

16. a. The project manager wanted no float times in this project. Create a new table showing the most efficient way to crash the project further without changing the critical path.
    b. Do you think crashing the project will reduce or increase the cost of the project? Explain.
    c. Do you have enough information to calculate the cost of crashing the project as described in part a? Explain.
17. Consider a construction project in which work hours, equipment and materials are used.
    a. What would be the purpose of crashing this project?
    b. Does crashing a project save money or cost money? Explain your answer.
    c. What costs increase as the duration of a project increases?
    d. What costs decrease as the duration of a project decreases?
    e. What costs typically stay the same regardless of the duration of a project?
    f. What problems can occur if you become too aggressive when crashing a project?

11.4 Network flow

11.4.1 Maximum flow

An application of networks used to analyse flow of traffic or water is **network flow**. These usually involve directed networks where arrows show the direction of flow. An example follows.

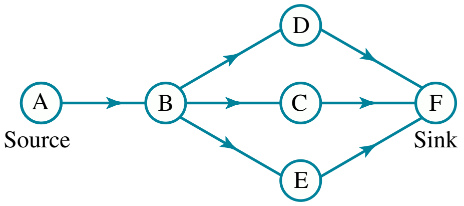
A driver starts for work in the city at 7.30 am each morning. He lives in an outer suburb and as he travels from his driveway through a few streets in his local neighbourhood, there is not much traffic on the roads. As he joins the road that connects his suburb to the next suburb, he notices an increase in the volume of the traffic. As this two-lane road joins the four-lane freeway into the city, the flow of traffic becomes immense. Cars are following bumper to bumper, with drivers changing lanes to drive in the fastest lane. The costs involved, financial and otherwise, for those who participate in the morning rush are significant.

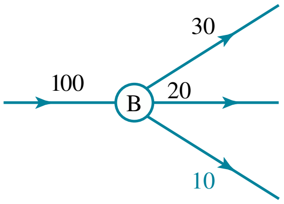
It is in everyone's best interest that the traffic flow smoothly and that traffic jams be avoided at all costs. Engineers use mathematical models of network flow to ensure smooth flow of traffic.

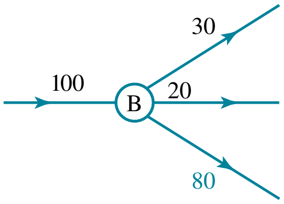
The network's starting vertex(es) is called the **source**. This is where all flows commence. The flow goes through the network to the end vertex(es) which is called the **sink**.

The **flow capacity** (capacity) of an edge is the maximum amount of flow that an edge can allow if it is not connected to any other edges. The **inflow** of a vertex is the total of the flows of all edges leading into the vertex. The **outflow** of a vertex is the *minimum* value obtained when one compares the inflow to the sum of the capacities of all the edges leaving the vertex.

Consider the following figures.







All flow commences at A. It is therefore the source. All flow converges on F indicating it is the sink.

B has an inflow of 100. The flow capacity of the edges leaving B is 30 + 20 + 10 = 60. The outflow is the minimum of 100 and 60, which is 60.

In this figure, B still has an inflow of 100 but now the capacity of the edges leaving B is 80 + 20 + 30 = 130. The outflow from B is now 100.

The *flow capacity of the network* is the total flow possible through the entire network.

WORKED EXAMPLE 8

Consider the information presented in the following flow table.

From	To	Quantity (kilolitres per minute)	Demand (E)
Rockybank Reservoir (R)	Marginal Dam (M)	1000	—
Marginal Dam (M)	Freerange (F)	200	200
Marginal Dam (M)	Waterlogged (W)	200	200
Marginal Dam (M)	Dervishville (D)	300	300

a. **Convert the information presented in the table into a network diagram, clearly indicating the direction and quantity of the flow.**

b. **Determine the flow capacity of the network.**

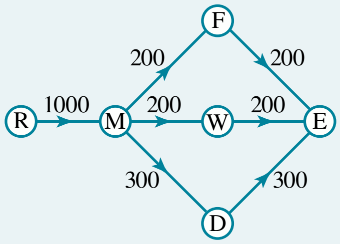
c. **Determine whether the flow through the network is sufficient to meet the demand of all the towns.**

THINK

a. Construct and label the required number of vertices. The vertices are labelled with the names of the source of the flow and the corresponding quantities are recorded on the edges. Make the final vertex the Demand (E).

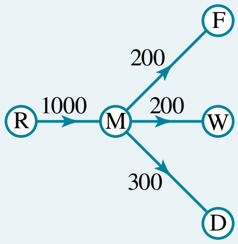
WRITE

a.



b. **1.** Examine the flow into and out of the Marginal Dam vertex. Record the smaller of the two at the vertex. This is the maximum flow through this point in the network.

b.



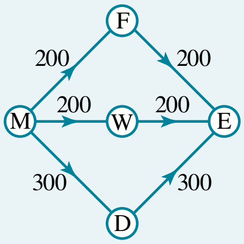
Even though it is possible for the reservoir to send 1000 kL/min (in theory), the maximum flow that the dam can pass on is 700 kL/min (the minimum of the inflow and the sum of the capacities of the edges leaving the dam).

2. In this case the maximum flow through Marginal Dam is also the maximum flow of the entire network.

Maximum flow is 700 kL/min.

c. 1. Determine that the maximum flow through Marginal Dam meets the total flow demanded by the towns. Add all the values in the demand column of the table to get the demand.

c.



Flow through Marginal Dam = 700 kL/min

Flow demanded = 200 + 300 + 200

= 700 kL/min

2. If the requirements of step 1 are able to be met, then determine that the flow into each town is equal to the flow demanded by them.

By inspection of the table, all town inflows equal town demands (capacity of edges leaving the town vertices).

Note: In **Worked example 8** there does not exist a location called Demand (E). It is preferable for a network diagram to have both a single source and a single sink, so the vertex Demand (E) has been included to simplify the diagram. **Worked example 8** is a simple case of a network in which the direction and quantity of flow are evident. Such a network diagram allows for analysis of the flow in the network; it allows us to see if various edges in the network are capable of handling the required flow.

on Resources

Interactivity Network flow (int-6287)

11.4.2 Excess flow capacity

Consider what would happen to the system if Rockybank Reservoir continually discharged 1000 kL/ min into Marginal Dam while its output remained at 700 kL / min. Such flow networks enable future planning. Future demand may change, the population may grow or a new industry that requires more water may come to one of the towns. **Worked example 9** examines such a case.

Excess flow capacity is the surplus of the capacity of an edge less the flow into the edge.

WORKED EXAMPLE 9

Use the information contained in Worked example 8 for this worked example. A new dairy factory, Creamydale (C), is to be set up on the outskirts of Dervishville. The factory will require 250 kL/min of water.

a. Determine whether the original flow to Dervishville is sufficient.

b. If the answer to part a is no, is there sufficient flow capacity into Marginal Dam to allow for a new pipeline to be constructed directly to the factory to meet their demand?

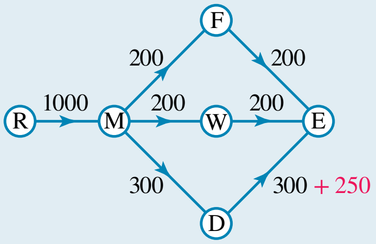
c. Determine the maximum flow through the network if the new pipeline was constructed.

THINK

WRITE

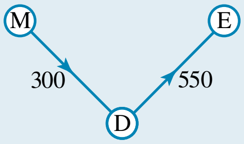
a. 1. Add the demand of the new factory to Dervishville's original flow requirements. If this value exceeds the flow into Dervishville then the new demand cannot be met.

a.



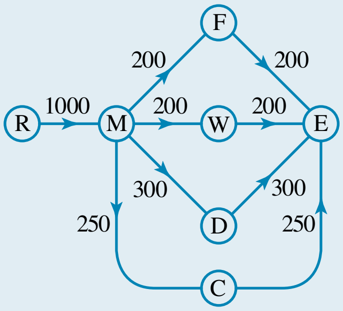
2. The new requirements exceed the flow.

The present network is not capable of meeting the new demands.



b. 1. Reconstruct the network including a new path for the factory after Marginal Dam.

b.



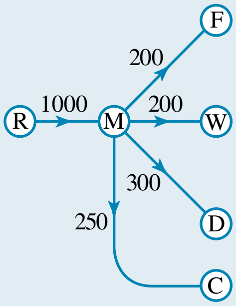
2. Add all outflow paths from M together to find the outflow of vertex M.

Marginal Dam inflow = 1000

Marginal Dam outflow

= 200 + 200 + 300 + 250

= 950



	3. Determine if the flow is sufficient for a new pipeline to be constructed.	There is excess flow capacity of 300 into Marginal Dam, which is greater than the 250 demanded by the new factory. The existing flow capacity to Marginal Dam is sufficient.
c.	This answer can be gained from part **b** step 2 on the previous page.	**c.** The maximum flow through the new network is 950 kL/ min.

The maximum flow through most simple networks can be determined using these observational methods, but more complex networks require an algorithm or method that guarantees the maximum flow can be achieved every time.

11.4.3 The maximum-flow minimum-cut theorem

Maximum-flow minimum-cut algorithms are frequently used to program computer networks. These algorithms can also be used to solve problems in electrical engineering, hydraulic design, airline timetable scheduling and many other technical areas.

When applying the maximum-flow minimum-cut algorithm, it is necessary to ensure that the network meets the following three requirements:

- The network is *weighted* and *directed;* that is, there is no inflow and there is no outflow.
- All edges connected to the source are leaving the *source*.
- All edges connected to the sink are entering the *sink*.

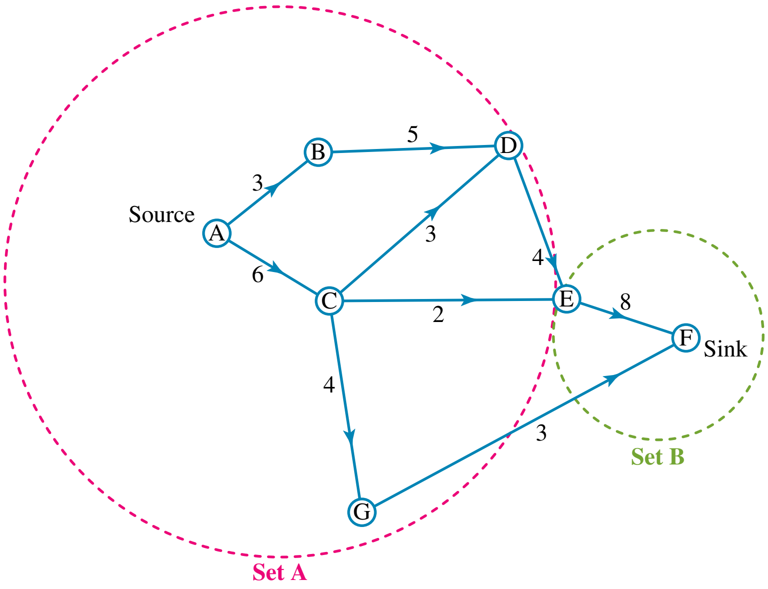
The maximum-flow minimum-cut theorem states that:

- The maximum flow through any weighted network can be calculated by finding the minimum cut set that is necessary to separate the source from the sink.
- The edges that both sets share have to be leaving the vertices of the source set in order to be included in the value of the cut set.

$$\text{Maximum flow} = \text{weight of the minimum cut set}$$

- To make a cut, break the network into two sets of vertices, where one set of vertices (A) contains the source and the other set of vertices (B) contains the sink.
  - The cut set is the set of edges that are in both set A and set B.
  - The capacity of a cut set is the sum of the weights of the edges in the cut set.
  - The minimum cut set is the sum of the weights of the cut set that has the smallest value.
- For example, in the diagram shown:

  Set A = [A, B, C, D, G]
  Set B = [E, F]
  Cut set = [DE, CE, GF].



The capacity of the cut set is 9. This cut set is the minimum cut set; therefore, the maximum flow is 9.

on Resources

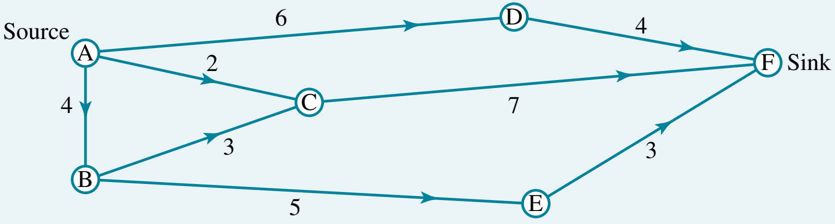
Interactivity Network flow cuts (int-6288)

WORKED EXAMPLE 10

For the weighted network below, find:

a. the capacity of each cut set

b. the capacity of the minimum cut set and maximum flow through the network.



THINK

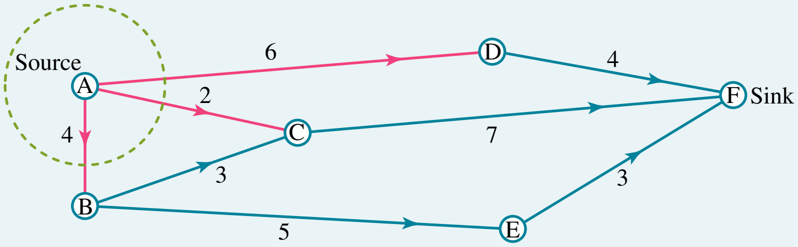
a. 1. The 9 drawings shown are just some of the cut capacities. Start with the source (vertex A). Circle the vertex to find all the edges that exit vertex A. Colour the edges blue that are either completely inside the circle or completely outside the circle. Colour the cut set edges pink. Add the capacity of the edges that are pink: this is the capacity of the cut set.

Continue this process, finding all the combinations of vertices that include the source (vertex A), and exclude any combinations that include the sink (vertex F).

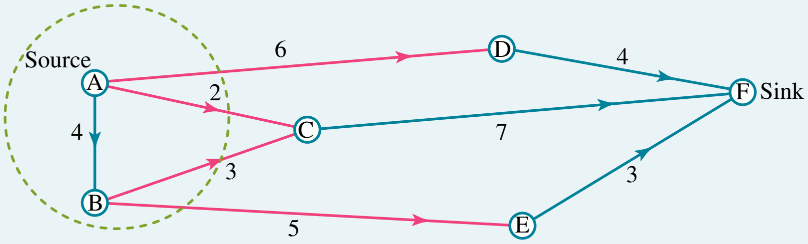
2. Circle vertices A and B. Follow the same process as in step 1.

WRITE

a.



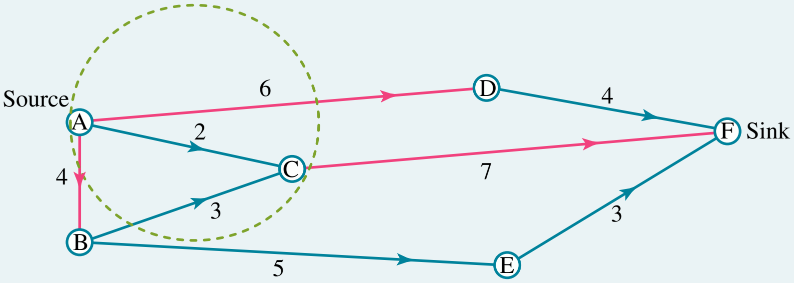
Capacity of $\{A\} = 6 + 4 + 2 = 12$



Capacity of $\{A, B\} = 6 + 2 + 3 + 5 = 16$

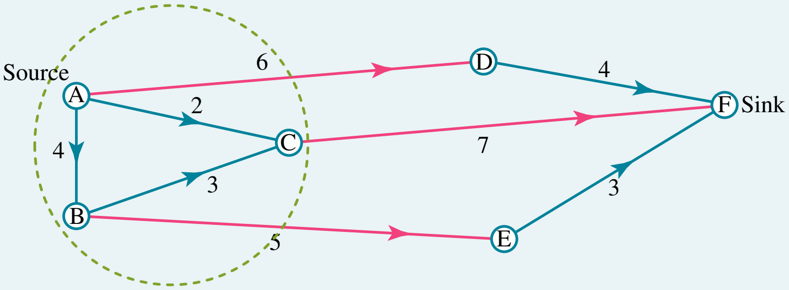
3. Circle vertices A and C. Follow the same process as in step 1.

   Notice that edge BC is not included because it is entering the cut.



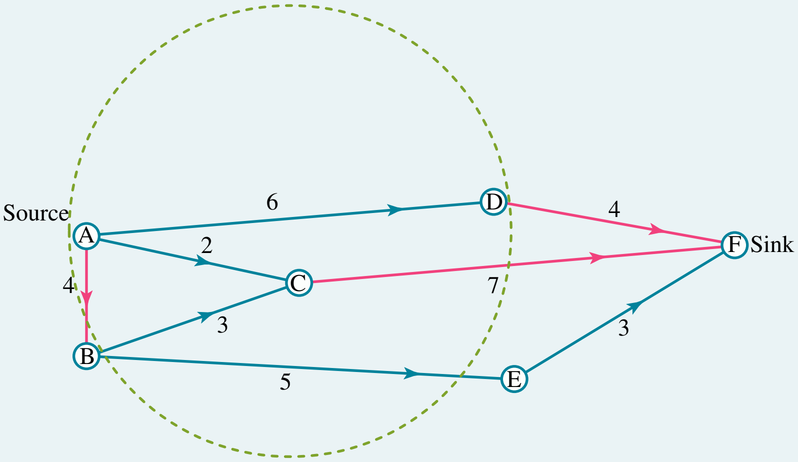
Cut set capacity of {A, C} = 6 + 4 + 7 = 17

4. Circle vertices A, B and C. Follow the same process as in step 1.



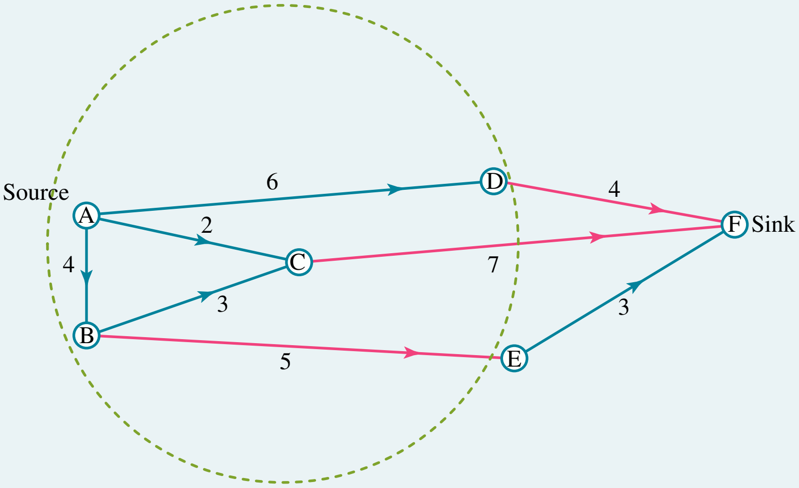
Capacity of {A, B, C} = 6 + 7 + 5 = 18

5. Circle vertices A, C and D. Follow the same process as in step 1.



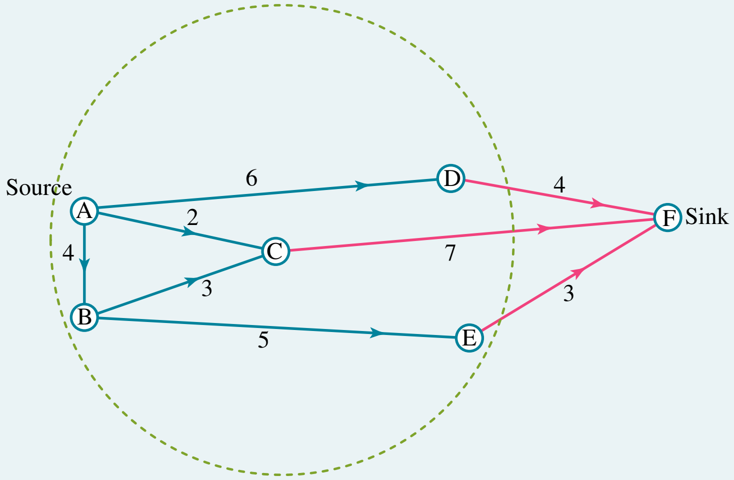
Capacity of {A, C, D} = 4 + 7 + 4 = 15

6. Circle vertices A, B, C and D. Follow the same process as in step 1.



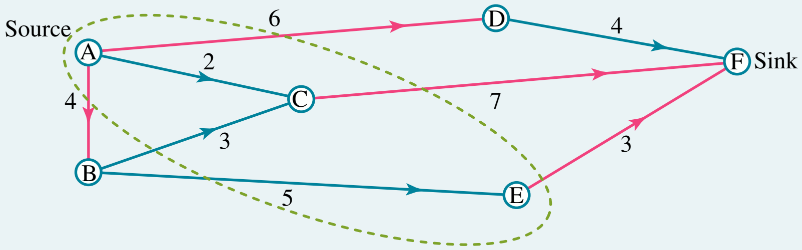
Capacity of {A, B, C, D} = 4 + 7 + 5 = 16

7. Circle vertices A, B, C, D and E. Follow the same process as in step 1.



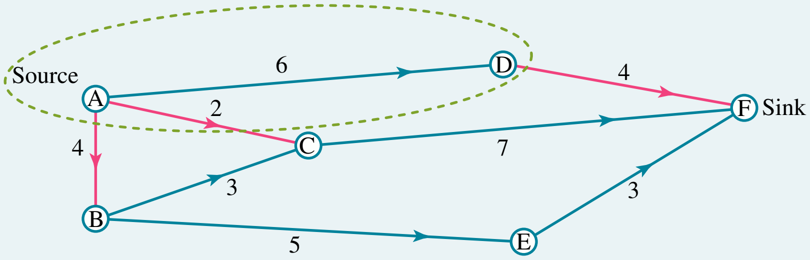
Capacity of $\{A, B, C, D, E\} = 4 + 7 + 3 = 14$

8. Circle vertices A, C and E. Follow the same process as in step 1.



Capacity of $\{A, C, E\} = 6 + 7 + 4 + 3 = 20$

9. Circle vertices A and D. Follow the same process as in step 1.



Capacity of $\{A, D\} = 4 + 2 + 4 = 10$

b. 1. There are 16 combinations of letters A, B, C, D and E. Each combination has a cut capacity.

b. $\{A\} = 12$
$\{A, B\} = 16$
$\{A, C\} = 17$
$\{A, D\} = 10$
$\{A, E\} = 14$
$\{A, B, C\} = 18$
$\{A, B, D\} = 14$
$\{A, B, E\} = 14$
$\{A, C, D\} = 15$
$\{A, C, E\} = 20$
$\{A, D, E\} = 13$
$\{A, B, C, D\} = 16$
$\{A, B, C, E\} = 16$
$\{A, B, D, E\} = 12$
$\{A, C, D, E\} = 18$
$\{A, B, C, D, E\} = 14$

2. State the answer.	The minimum cut set/maximum flow is {A, D}, with a capacity of 10.

studyon

Units 3 & 4 > Area 7 > Sequence 1 > Concept 5

Network flow Summary screen and practice questions

Exercise 11.4 Network flow

1. **WE 8** Consider the information presented in the following flow table.

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300

   a. Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.
   b. Determine the flow capacity.
   c. Determine whether the flow through the network is sufficient to meet the demand

2. Consider the information presented in the following flow table.

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	E	100
U	E	50

   a. Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.
   b. Determine the flow capacity.
   c. Determine whether the flow through the network is sufficient to meet the demand

3. Consider the information presented in the following flow table.

From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	E	12
R	E	12

a. Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.

b. Determine the flow capacity.

c. Determine whether the flow through the network is sufficient to meet the demand

4. Consider the information presented in the following flow table.

From	To	Flow capacity
D	F	8
D	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	E	8
H	E	8

a. Convert the information into a network diagram, clearly indicating the direction and quantity of the flow.

b. Determine the flow capacity.

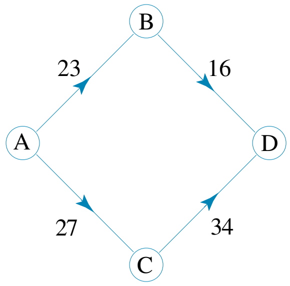
c. Determine whether the flow through the network is sufficient to meet the demand.

5. For vertex B in the network at right, state

a. the inflow at B

b. the edge capacities flowing out of B

c. the outflow from B.

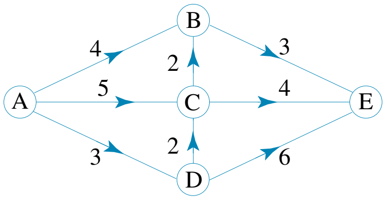


6. For vertex B in the network at right, state

a. the inflow at B

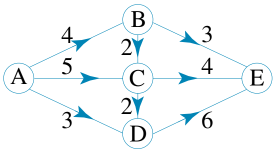
b. the edge capacities flowing out of B

c. the outflow from B.

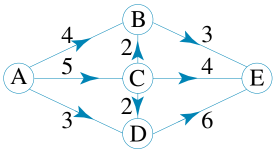


7. Consider the information presented in the following network diagrams.

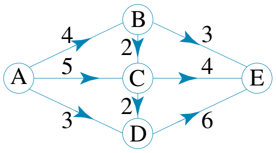
a.



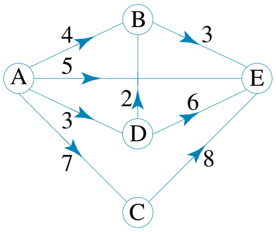
b.



c.



d.



For each network diagram:

i. convert the information into a flow table

ii. calculate the capacity of the network.

8. Consider the tables below. New flows have been added to the flow tables from questions 1– 4.

a.

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300
B	E	100

b.

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	E	100
U	E	50
S	T	100

c.

From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	E	12
R	E	12
N	E	5

d.

From	To	Flow capacity
D	F	8
D	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	E	8
H	E	8
D	E	10

For each table

i. convert this information into new network diagrams

ii. calculate the new network flow capacities.

9. **MC** Referring to question 8c, the outflow from N is

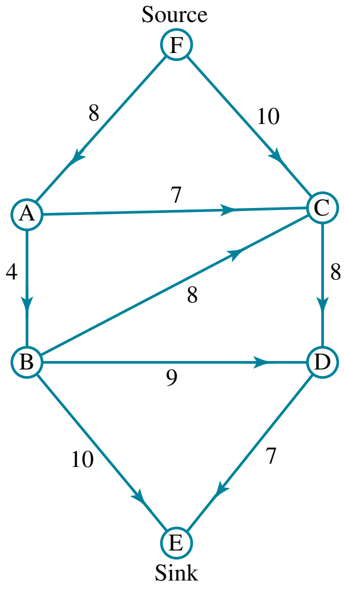
A. 5. B. 20. C. 15. D. 25.

10. **WE 9,10** Refer to the weighted network shown and answer the following questions.

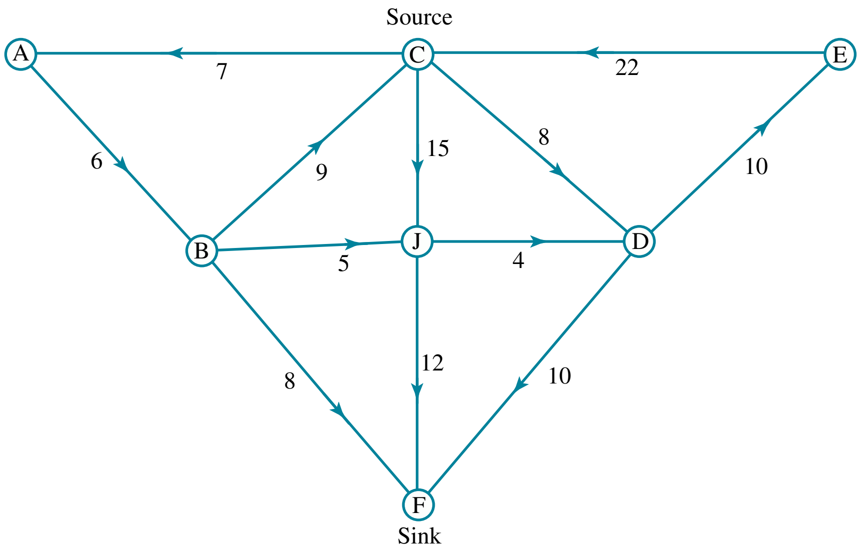
a. Calculate the capacity of each cut set.

b. Draw a diagram showing the minimum cut set.

c. Calculate the capacity of the minimum cut set and maximum flow through the network.



11. Consider the weighted network shown.



a. Draw three different cut sets for the network, including the minimum cut set.

b. Find the maximum flow for the network.

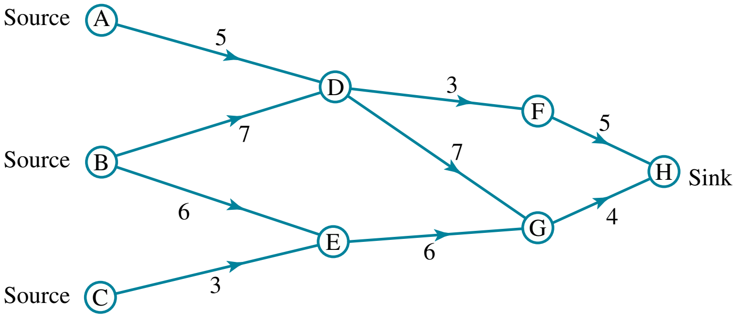
c. Does more than 1 minimum cut set have the same capacity? If so, how many are there?

12. The network shown has three sources.

a. Will the maximum-flow minimum-cut theorem operate correctly in this network?

b. Give an explanation to your response to part **a**.

c. What is the maximum flow in this network?



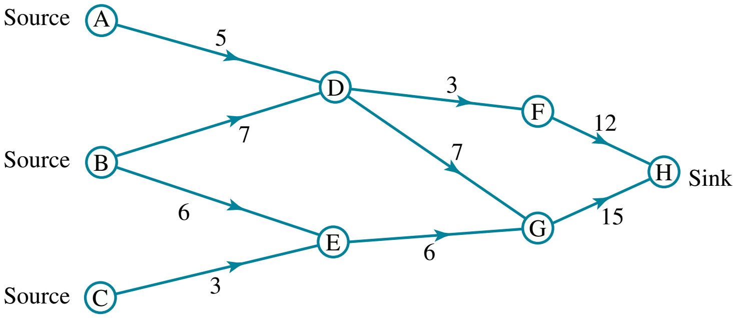
13. Consider the network shown.

a. Will the maximum-flow minimum-cut theorem operate correctly in this network?

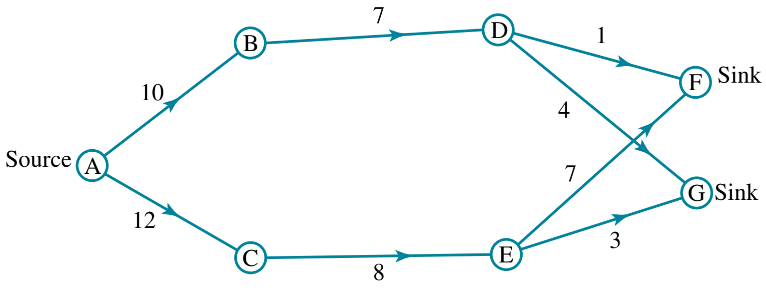
b. Give an explanation for your response to part **a**.

c. What is the maximum flow in this network?

d. If there are an infinite number of sources in a directed network, can you apply the maximum-flow minimum-cut theorem? Explain.



14. The network shown has one source and two sinks.



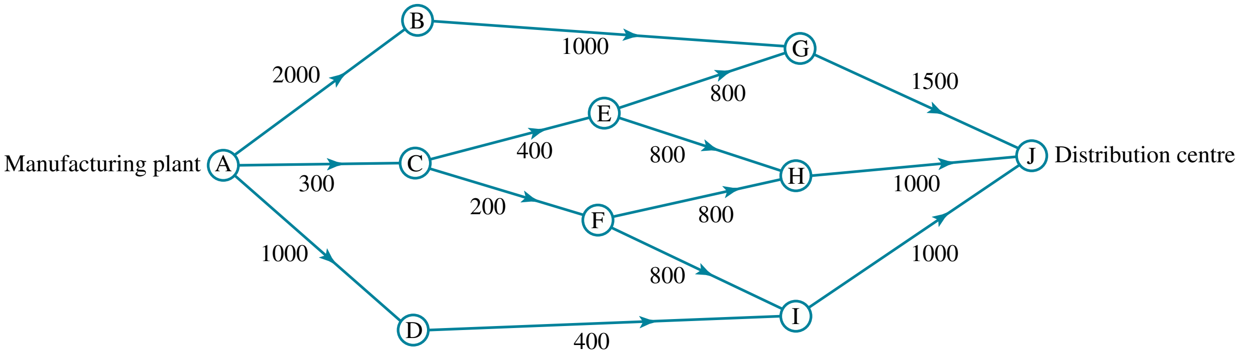
a. Will the maximum-flow minimum-cut theorem operate correctly in this network?

b. Give an explanation for your response to part **a**.

c. What is the maximum flow in this network?

d. If there are an infinite number of sinks in a directed network, can you apply the maximum-flow minimum-cut theorem? Explain.

15. A car manufacturing parts company (Rac Inc.) has to deliver various numbers of parts to their distribution centres around Australia and 1000 parts to their final distribution location in Europe (see network below).

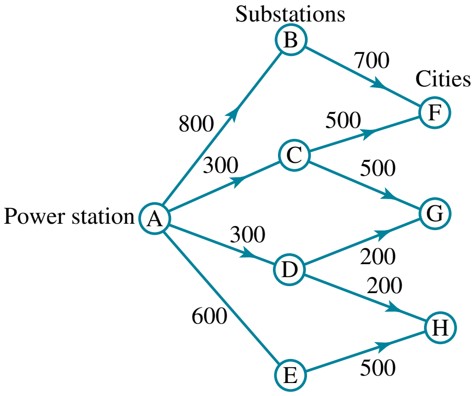


a. According to the distribution network, will it be possible to achieve this goal?

b. If you answered 'Yes' in part **a**, what is the number of parts that will be delivered to Europe? If you answered 'No' in part **a**, what is the number of parts that will be delivered to Europe?

c. If you have a surplus in part **b**, how can you reduce the number of parts to 1000? If you have a shortage in part **b**, how can you increase the number of parts to 1000? (If necessary, draw a new network to show the change(s) used to meet the 1000-part limit.)

16. The network at right represents the electricity usage of three cities in megawatt-hours (MWh). The power station that supplies electricity to these cities can produce 2000 MWh.



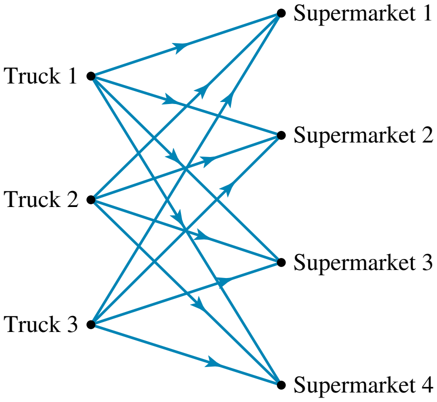
a. What is the maximum number of hours the three cities are using?

b. If you can only change the distribution of electricity to the substations, how can you ensure that the three cities will receive a total of 2000 MWh? Draw a picture of your new network. (*Hint:* You can use any numbers for AB, AC, AD, or AE to achieve your goal.)

11.5 Bipartite graphs and the Hungarian algorithm

11.5.1 Bipartite graphs

Network flow problems involve, for example, the analysis of traffic flow or water flow in a particular direction. In many situations, it is important to analyse the flow through each path in the network and ensure that each edge is used to capacity. For example, a transport company has three large trucks which deliver to four supermarkets, so that each supermarket gets the necessary supplies, regardless of the truck used. This type of problem is called an allocation problem as each truck has to be allocated a supermarket to ensure the most cost or time effective means of delivering the goods.



Bipartite graphs can be useful to solve this type of problem. The trucks can be considered to be the supply vertices and the supermarket the demand vertices.

A bipartite graph can be split into two groups of vertices; supply and demand, where no two vertices in the same group share an edge.

Consider the bipartite graph to the right which illustrates the situation mentioned above. In this case, each truck can deliver to all three supermarkets but if the graph is weighted, then there may be differences in the distance each truck may need to travel and hence the cost of each delivery may differ.

WORKED EXAMPLE 11

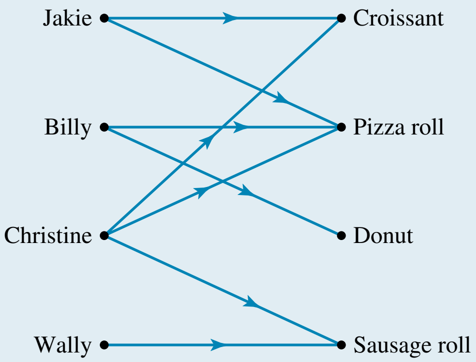
The following table lists four customers and the four menu items offered at a pop-up bakery.

Customer	Menu items
Jackie	Croissant, pizza roll
Billy	Pizza roll, donut
Christine	Croissant, pizza roll, sausage roll
Wally	Sausage roll

a. Represent this information as a bipartite graph.

b. Determine whether the following statements are true or false.

1. **Jackie purchases more items than Christine.**
2. **Between Christine and Billy, all menu items are chosen.**
3. **Between Jackie and Billy, more menu items are chosen than by Christine.**

THINK	WRITE
a. List the students down the left-hand side and the menu items down the right-hand side. Link the students with the menu items according to the table.	a. 

b. 1. Determine the truth of statement 1.	**b.** Jackie purchases only 2 items, Christine purchases 3, so the statement is false.
2. Determine the truth of statement 2.	Christine purchases a croissant, pizza roll and sausage roll. Billy purchases a pizza roll and a donut. Their choices include all 4 menu items. The statement is true.
3. Determine the truth of statement 3.	Christine chooses 3 of the menu items and between Jackie and Billy 3 menu items are chosen. The statement is false.

11.5.2 The assignment or allocation problem

In an office in the city there are four employees and four tasks that need to be completed. Each person can do the task in a different amount of time. What is the best way for their manager to allocate these tasks, one per person, so that the time can be minimised? This is known as optimal allocation.

In the general case the jobs and people can be put in an allocation matrix similar to the one below.

	Task 1	**Task 2**	**Task 3**	**Task 4**
Employee 1	10	4	9	6
Employee 2	8	11	10	7
Employee 3	6	8	7	9
Employee 4	8	5	3	9

Employee 1 would take 10 hours to complete Task 1, 4 hours to complete Task 2, 9 hours to complete Task 3 and 6 hours to complete Task 4 and so on.

Worked example 12 demonstrates the process of using row reduction to determine the optimal allocation.

WORKED EXAMPLE 12

In an office in the city there are four employees and four tasks that need to be completed. The times taken by each employee to do the four jobs are given in the following table. Determine the optimal allocation and hence state the minimum time.

	Task A	**Task B**	**Task C**	**Task D**
Employee 1	10	4	9	6
Employee 2	8	11	10	7
Employee 3	6	8	7	9
Employee 4	8	5	3	9

THINK | **WRITE**

1. Set up the matrix of employees against tasks.

	A	B	C	D
1	10	4	9	6
2	8	11	10	7
3	6	8	7	9
4	8	5	3	9

2. Perform row reduction by locating the smallest value in each row and subtracting it from all numbers in that row.
The smallest number in Row 1 is 4.
The smallest number in Row 2 is 7.
The smallest number in Row 3 is 6.
The smallest number in Row 4 is 3.

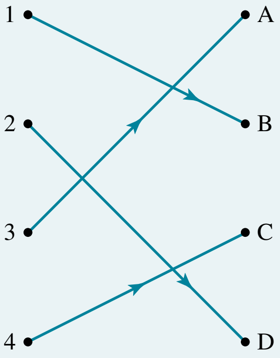
	A	B	C	D
1	6	0	5	2
2	1	4	3	0
3	0	2	1	3
4	5	2	0	6

3. Cover all the zeroes with the smallest number of straight lines; horizontal or vertical but not diagonal.
If the number of lines equals the number of tasks continue to the next step.
If the number of lines does not equal the number of tasks, another method of allocation will need to be used.

	A	B	C	D
~~**1**~~	~~6~~	~~0~~	~~5~~	~~2~~
~~**2**~~	~~1~~	~~4~~	~~3~~	~~0~~
~~**3**~~	~~0~~	~~2~~	~~1~~	~~3~~
~~**4**~~	~~5~~	~~2~~	~~0~~	~~6~~

There are four lines and four tasks.

4. Draw a bipartite graph, where the zeroes connect the employee to the tasks.



5. Write the possible allocations and determine the minimum number of hours required.

There is only one possible allocation.
$1 \rightarrow B$
$2 \rightarrow D$
$3 \rightarrow A$
$4 \rightarrow C$
Total time: $4 + 7 + 6 + 3 = 20$ hours

6. Write the answer.

The minimum time to complete the 4 tasks is 20 hours.

11.5.3 The Hungarian algorithm

When the number of zeroes does not equal the number of tasks after the first row reduction, the **Hungarian algorithm** needs to be used.

The Hungarian algorithm involves the following steps:

Step 1: Subtract the row minimum from each row.

Step 2: Subtract the column minimum from each column.

Step 3: Cover all zeroes with a minimum number of lines.

- If the number of lines equals the number of tasks, draw a bipartite graph and allocate the tasks.
- If the number of lines is less than the number of tasks, continue to step 4.

Step 4: Find the smallest uncovered number. Subtract this number from all uncovered elements and add it to all elements that are covered twice.

Step 5: Cover all zeros with a minimum number of lines. If the number of lines equals the number of tasks, draw a bipartite graph and allocate tasks.

WORKED EXAMPLE 13

Four workers need to be allocated to four tasks. The time required for each worker for each task is summarised in the table below. Use the Hungarian algorithm to minimise the time required to complete the tasks by allocating one job to each worker.

	Task 1	Task 2	Task 3	Task 4
Worker 1	80	81	67	90
Worker 2	75	35	47	90
Worker 3	9	67	3	84
Worker 4	6	7	96	21

THINK

1. Set up the matrix of employees against tasks.

WRITE

	T1	T2	T3	T4
W1	80	81	67	90
W2	75	35	47	90
W3	9	67	3	84
W3	6	7	96	21

2. Perform row reduction by locating the smallest value in each row and subtracting it from all numbers in that row.
The smallest number in Row 1 is 67.
The smallest number in Row 2 is 35.
The smallest number in Row 3 is 3.
The smallest number in Row 4 is 6.
Only 3 lines are required to cover the zeros, so continue to the Hungarian algorithm.

	T1	T2	T3	T4
W1	13	14	0	23
W2	40	0	12	55
W3	6	64	0	81
W4	0	1	90	15

3. Perform a column reduction by subtracting the smallest number in each column from all the numbers in the column.
The smallest number in Column 1 is 0.
The smallest number in Column 2 is 0.
The smallest number in Column 3 is 0.
The smallest number in Column 4 is 15.

	T1	T2	T3	T4
W1	13	14	0	8
W2	40	0	12	40
W3	6	64	0	66
W4	0	1	90	15

4. Cover all the zeroes with the smallest number of straight lines; horizontal or vertical but not diagonal.
If the number of lines does not equal the number of tasks, continue to step 5.

	T1	T2	T3	T4
W1	13	14	0	8
W2	40	0	12	40
W3	6	64	0	66
W4	0	1	90	0

There are only 3 lines and four tasks.

The smallest uncovered number is 6.

5. The smallest uncovered number is 6. Subtract 6 from all uncovered elements and add it to all elements that are covered twice.

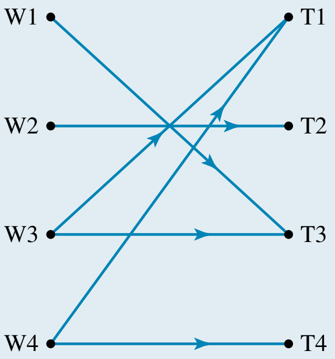
	T1	T2	T3	T4
W1	7	8	0	2
W2	40	0	18	40
W3	0	58	0	60
W4	0	1	96	15

6. Cover all zeroes with a minimum number of lines.

	T1	T2	T3	T4
W1	7	8	0	2
W2	40	0	18	40
W3	0	58	0	60
W4	0	1	96	0

There are four lines and four tasks.

7. Draw a bipartite graph, where the zeroes connect the employee to the tasks.



8. Write the possible allocations and determine the minimum number of hours required.	Worker 1 is allocated Task 3. Worker 2 is allocated Task 2 Worker 3 is allocated Task 1 (Task 3 is already allocated) Worker 4 is allocated Task 4 (Task 1 is already allocated. $67 + 35 + 9 + 21 = 132$ hours.
9. Write the answer.	The minimum time required to complete all tasks is 132 hours.

In Worked example 13, the objective was to minimise the time. Some problems require a maximum to be determined. In this process, all elements in the matrix are subtracted from the largest one first. From then on, the procedure is the same as that set out in Worked example 13.

studyon

Units 3 & 4 > Area 7 > Sequence 1 > Concepts 6, 7 & 8

Bipartite graphs Summary screen and practice questions
Optimal allocation Summary screen and practice questions
The Hungarian algorithm Summary screen and practice questions

Exercise 11.5 Bipartite graphs and the Hungarian algorithm

1. **WE11** Five customers (Will, Penny, Fan, Roya and Su Yi) go to the local fish and chip shop for dinner and place the orders as shown in the table.

Diner	Dishes
Will	Fish, chips
Penny	Chips
Fan	Fish, potato cakes, dim sims
Roya	Potato cakes, dim sims
Su Yi	Chips, dim sims

- **a.** Represent this information as a bipartite graph
- **b.** Determine whether the following statements are true or false.
  - **1.** Will and Penny between them have more different items than Fan and Roya.
  - **2.** Fan and Roya together have tried all the options.
  - **3.** Roya and Su Yi between them have more variety than Penny and Fan.
  - **4.** Penny and Fan between them have more variety than Roya and Su Yi.

2. Four visitors on a tour to Europe have a choice of five countries to visit. The countries are France, Germany, Italy, Spain and Ireland, they can visit as many of these countries as they wish. Sally decides to visit Italy and Germany, but not the others. Joe decides to spend all his time in France. Mike wants to see Germany, Spain and Ireland. Genevieve is keen to visit all of the countries on this trip.

- **a.** Why is a bipartite graph suitable to represent this information?
- **b.** Draw a bipartite graph to represent the information.
- **c.** What is the degree of the vertex representing Mike?

3. Five teachers can teach a variety of five subjects as indicated by the bipartite graph.

Are the following statements True or false?

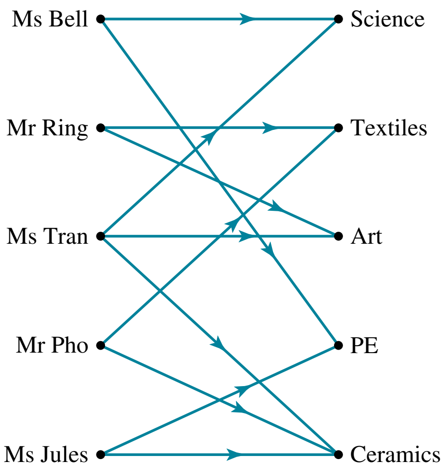
a. Ms Bell and Ms Tran can teach all five subjects between them

b. Mr Ring and Ms Tran, in total, can teach more subjects than Ms Bell and Ms Jules

c. Ms Bell and Ms Tran each teach the same number of subjects

d. Ms Bell and Mr Pho, in total teach fewer subjects than Mr Ring and Ms Tran

e. Ms Jules teaches fewer subjects than all other teachers.



4. **WE12** A shipping company has 4 ships that deliver fuel to four different oil rigs lying off shore. The times taken by each ship to do the four deliveries are given in the following table. Determine the optimal allocation and hence state the minimum time required to complete all four deliveries.

	Rig A	**Rig B**	**Rig C**	**Rig D**
Ship 1	16	14	20	13
Ship 2	15	16	17	16
Ship 3	19	13	13	18
Ship 4	22	26	20	24

For questions 5 and 6, perform row reduction on the matrices, which represent times (in hours), and attempt an optimal allocation for the minimum time. State the minimum time.

5. $$\begin{bmatrix} 5 & 2 & 6 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

6. $$\begin{bmatrix} 5 & 4 & 8 & 4 \\ 10 & 5 & 7 & 6 \\ 6 & 7 & 8 & 9 \\ 5 & 9 & 4 & 6 \end{bmatrix}$$

7. **WE13** Four workers need to be allocated to four tasks. The time required for each worker for each task is summarised in the table below. Use the Hungarian algorithm to minimise the time required to complete the tasks by allocating one job to each worker.

	Task 1	**Task 2**	**Task 3**	**Task 4**
Worker 1	7	10	10	5
Worker 2	11	10	10	8
Worker 3	5	10	7	4
Worker 4	6	9	9	7

8. Four delivery vans need to deliver to four different supermarkets. The distance of the four drivers from each of the four supermarkets are given in the table below. If the drivers take their loaded vans home in the evening before the delivery day, use the optimal allocation method to minimise the total distance travelled by the vans to reach all four supermarkets.

$$\begin{array}{c|cccc} & S1 & S2 & S3 & S4 \\ \hline \text{Driver1} & 7 & 25 & 21 & 6 \\ \text{Driver2} & 13 & 31 & 8 & 16 \\ \text{Driver3} & 23 & 19 & 16 & 15 \\ \text{Driver4} & 22 & 29 & 24 & 10 \end{array}$$

9. A florist wishes to purchase peonies, roses and lillies for three bouquets from three different flower wholesalers. The peonies cost \$40, \$55 and \$60 from the three stores, the roses cost \$65, \$60 and \$70 and the lillies cost \$45, \$50 and \$40. Determine the optimal allocation for the flower order.

10. Consider the following matrix.

$$\begin{bmatrix} 8 & 4 & 8 \\ 4 & 4 & 6 \\ 7 & 6 & 6 \end{bmatrix}$$

The total value of the optimal allocation is

A. 9. **B.** 11. **C.** 14. **D.** 16.

11. Perform an optimal allocation on the following matrices column by conducting in order each of the following until the allocation is complete. State the minimum value.

- **i.** Row reduction
- **ii.** Column reduction
- **iii.** The Hungarian algorithm

a. $\begin{bmatrix} 20 & 18 & 13 \\ 21 & 16 & 18 \\ 25 & 28 & 26 \end{bmatrix}$

b. $\begin{bmatrix} 20 & 30 & 40 & 50 \\ 60 & 20 & 30 & 60 \\ 50 & 40 & 50 & 20 \\ 10 & 70 & 40 & 60 \end{bmatrix}$

12. A large holiday park has four maintenance workers and four tasks that need one person to complete each morning.

The time it takes each of the workers to complete the four tasks is summarised in the following table.

	Task 1	Task 2	Task 3	Task 4
Worker 1	50	60	70	80
Worker 2	90	50	60	90
Worker 3	80	70	80	30
Worker 4	30	100	70	90

- **a.** Perform row and column reduction.
- **b.** Perform the Hungarian algorithm
- **c.** Determine the optimal allocations and display these allocations using a bipartite graph.
- **d.** Determine the total time required to complete all four tasks.

13. Four swimmers are to compete in a medley relay. During training each swimmer has swum each leg of the relay and their times have been recorded in the following table.

	Leg 1	Leg 2	Leg 3	Leg 4
Mary	2	4	3	5
Jenny	3	5	3	4
Pauline	2	3	4	2
Jacinta	2	4	2	3

Determine the optimal allocation and the minimum time to complete the race.

11.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

Questions 1 to 4 refer to the following table.

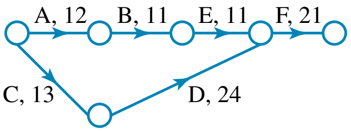
Activity	Time	Immediate predecessor
A	12	—
B	11	A
C	13	—
D	24	C
E	11	D, B
F	21	E

1. MC Using the table above, the activities that come before activity E are

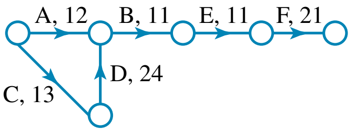
A. D and B.
B. A, D and B.
C. A, B, C and D.
D. A, C and D.

2. MC The correct diagram for the table above is

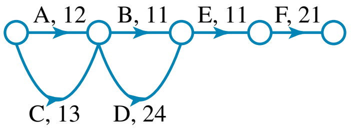
A.



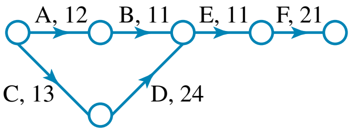
B.



C.



D.



E. None of the above.

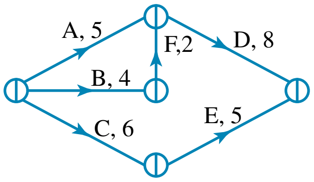
3. MC The earliest start time for activity E is

A. 23.
B. 24.
C. 30.
D. 37.

4. MC The earliest completion time for the network is

A. 56.
B. 55.
C. 68.
D. 69.

Questions 5 to 10 refer to the following diagram.



5. **MC** Which of the following is an immediate predecessor to E?

A. E **B.** C **C.** D **D.** B

6. **MC** The earliest finishing time for activity C is

A. 6. **B.** 8. **C.** 4. **D.** 5.

7. **MC** The earliest finishing time for activity D is

A. 11. **B.** 12. **C.** 13. **D.** 14.

8. **MC** The latest starting time for activity E is

A. 9. **B.** 10. **C.** 11. **D.** 12.

9. **MC** The float time for activity C is

A. 7. **B.** 5. **C.** 3. **D.** 2.

Questions 10 and 11 refer to the following matrix.

$$\begin{bmatrix} 15 & 11 & 16 & 6 \\ 13 & 5 & 2 & 12 \\ 4 & 2 & 11 & 14 \\ 12 & 7 & 12 & 10 \end{bmatrix}$$

10. **MC** which of the following is the row-reduced matrix for the previous matrix?

A. $\begin{bmatrix} 9 & 5 & 10 & 0 \\ 11 & 3 & 2 & 10 \\ 2 & 0 & 9 & 12 \\ 5 & 0 & 5 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 11 & 6 & 14 & 0 \\ 9 & 3 & 0 & 6 \\ 0 & 0 & 9 & 8 \\ 8 & 5 & 10 & 4 \end{bmatrix}$

C. $\begin{bmatrix} 9 & 5 & 10 & 0 \\ 11 & 3 & 0 & 10 \\ 2 & 0 & 9 & 12 \\ 5 & 0 & 5 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 9 & 5 & 10 & 6 \\ 11 & 3 & 2 & 10 \\ 2 & 2 & 9 & 12 \\ 7 & 5 & 5 & 5 \end{bmatrix}$

11. **MC** After column reduction is performed to the matrix obtained in question 10, the resultant matrix is

A. $\begin{bmatrix} 7 & 5 & 10 & 0 \\ 9 & 3 & 0 & 10 \\ 0 & 0 & 9 & 12 \\ 3 & 0 & 5 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 7 & 3 & 8 & 1 \\ 9 & 1 & 0 & 5 \\ 0 & 0 & 7 & 7 \\ 5 & 3 & 3 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 7 & 5 & 8 & 0 \\ 9 & 3 & 0 & 10 \\ 0 & 0 & 7 & 12 \\ 3 & 0 & 3 & 5 \end{bmatrix}$

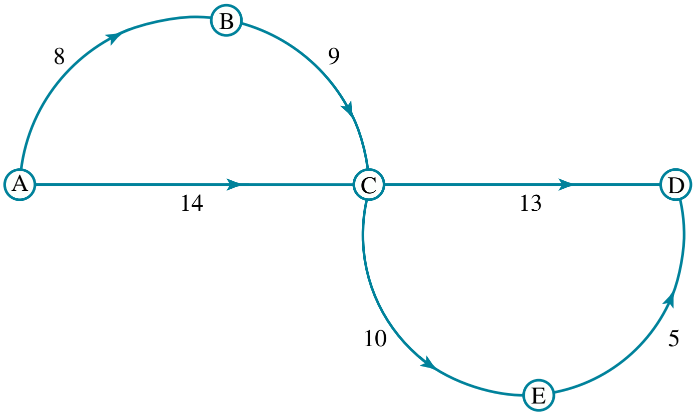
D. $\begin{bmatrix} 11 & 6 & 14 & 0 \\ 9 & 3 & 0 & 6 \\ 0 & 0 & 9 & 8 \\ 8 & 5 & 10 & 4 \end{bmatrix}$

Complex familiar

12. MC The network flow table and chart below show the project times (in weeks) for constructing a backyard swimming pool.

The charts below represent proposed crash times for the construction of the pool. Which chart represents crash times that will reduce the earliest completion date?

Phase	Time (weeks)
A– B	8
A– C	14
B– C	9
C– E	10
C– D	13
E– D	5



A.

Phase	Time (weeks)	Crash time (weeks)
A– B	8	0
A– C	14	2
B– C	9	0
C– E	10	0
C– D	13	2
E– D	5	0

B.

Phase	Time (weeks)	Crash time (weeks)
A– B	8	2
A– C	14	0
B– C	9	0
C– E	10	2
C– D	13	0
E– D	5	0

C.

Phase	Time (weeks)	Crash time (weeks)
A– B	8	0
A– C	14	3
B– C	9	0
C– E	10	0
C– D	13	2
E– D	5	0

D.

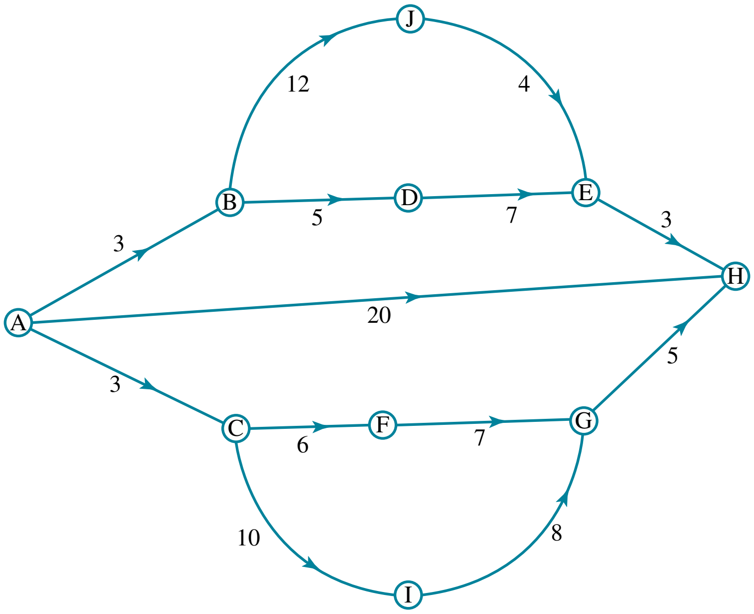
Phase	Time (weeks)	Crash time (weeks)
A– B	8	0
A– C	14	0
B– C	9	1
C– E	10	0
C– D	13	2
E– D	5	0

13. A team of four — Barnie, Ruth, Shelley and Carlos have been selected, to play four holes of golf for a social event for their workplace. Each member of the team will play one of the holes. Their scores for their previous rounds of golf for these four holes have been summarised in the following table.

	Hole 1	Hole 2	Hole 3	Hole 4
Barnie	5	7	5	9
Ruth	6	10	10	7
Shelley	7	5	3	8
Carlos	7	8	8	9

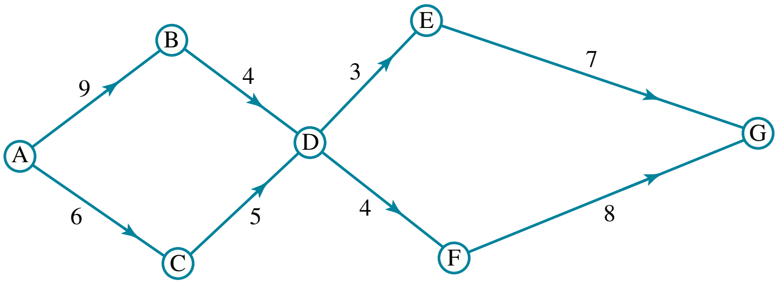
a. Perform a row reduction on the matrix formed from this table.
b. Perform a column reduction on the matrix from part **a**.
c. Apply the Hungarian algorithm if necessary.
d. State the optimal team for this event.

14. The following network diagram is a representation of the processes required to build a tennis court (times in hours).



a. Draw a table to represent the different phases of the project.
b. Find the critical path.
c. What is the earliest completion time?
d. If you crash phase A— H by 2 hours, will this change the earliest completion time?
e. What is the maximum number of hours you can crash path A— C— I— G— H without creating a new critical path?
f. What is the minimum number of hours needed to crash path A— C— I— G— H in order to create a new critical path? What is the new critical path?

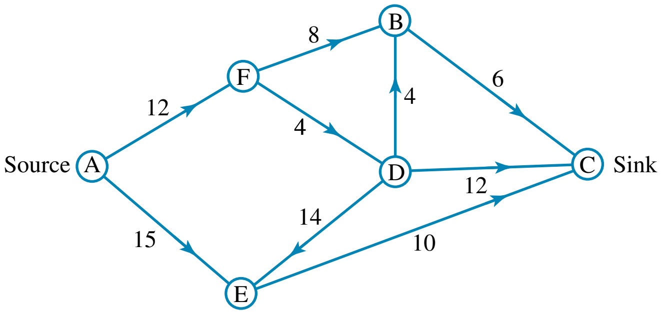
15. The earliest completion time for the project represented in the network below has to be reduced by 2 hours.



a. Create a table to represent the different phases of the project and their times.
b. Create a table showing how you can reduce the earliest completion time by 2 hours.
c. If your colleague created a crashing table for this same project, would their table be identical to your table? Explain.

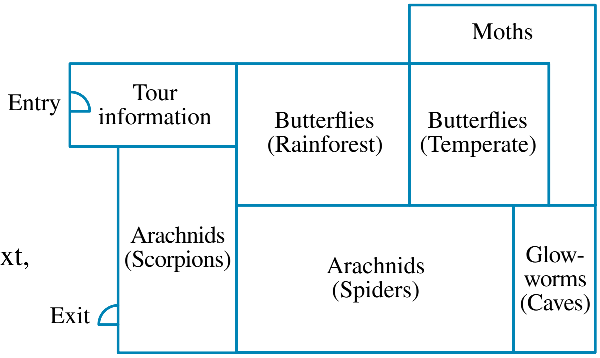
16. Answer the following questions for the network shown.

- **a.** What is the maximum flow in the network?
- **b.** Would the maximum flow change if the capacity for AF was infinite? Why or why not?
- **c.** Would the maximum flow change if the capacity for EC was infinite? Why or why not?



Complex unfamiliar

17. A Lepidoptera and Arachnid building is to be set up at the zoo. The floor plan is shown at right. The building is to be designed so that people can flow through in only one direction. Each doorway will open only one way and is designed to ensure that there is no mixing of the exhibits.



- **a.** Draw the doors leading from one section to the next, clearly indicating in which direction they open. (The entry and exit doors have been completed for you.)

An analysis of the flow of visitors through various exhibits at similar zoos in other cities has provided the following table.

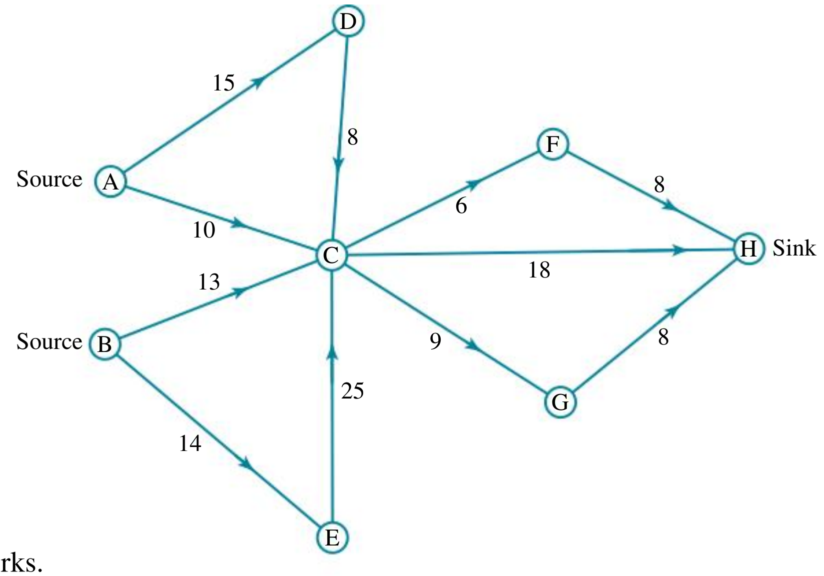
Section from	Section to	Arrival rate (number of people per minute)
Entry	Tour information	12
Tour information	Rainforest butterflies	13
Rainforest butterflies	Temperate butterflies	12
Temperate butterflies	Moths	2
Temperate butterflies	Glow-worms	4
Temperate butterflies	Spiders	2
Moths	Glow-worms	4
Glow-worms	Spiders	6
Spiders	Scorpions	5
Scorpions	Exit	12

- **b.** If the doors can be represented by an edge (A) with a capacity of 12, convert the information given in the table and plan into a network flow diagram using letters A– J.
- **c.** Analyse the inflows, capacities and outflows, and then describe what would happen to the number of people in the rainforest butterflies' room.
- **d.** At what rate should people be admitted so that they can flow smoothly through the building?

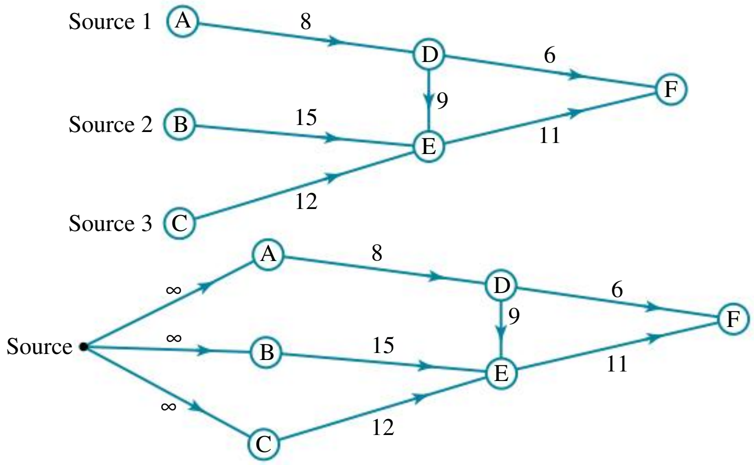
18. Answer the following questions for the weighted network shown.

a. Draw a diagram showing the minimum cut set. (*Hint:* There are two sources.)

b. Calculate the capacity of the minimum cut set/maximum flow through the network.



19. Compare the following two networks.

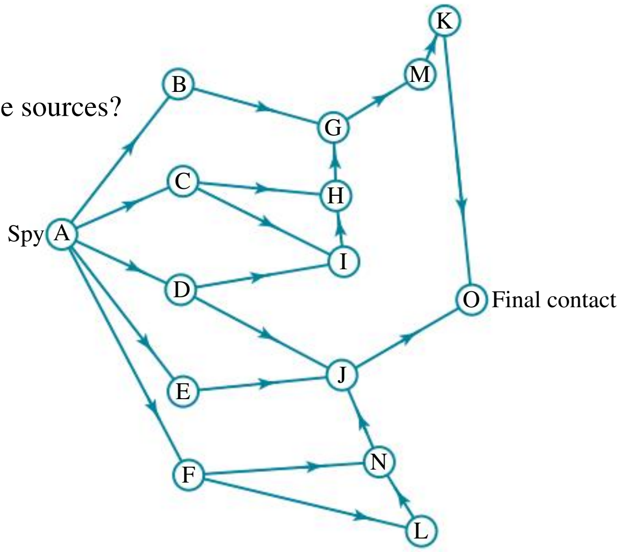


a. Do they have the same maximum flow?

b. Explain your answer to part **a**.

c. Can you make a general statement about multiple sources?

20. A spy is about to send a timely message from Australia to an important contact in Russia. The network at right represents the flow of information among his contacts. Australian security officers want to find the most efficient way to reduce the spy's dissemination of the information to his final contact. The security officers do not have the time to prevent the spy from sending the initial message. What is the minimum number of people they need to intercept to prevent the message from reaching the final contact?



studyon

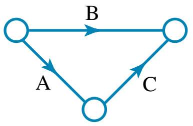
Units 3 & 4 Sit exam

Answers

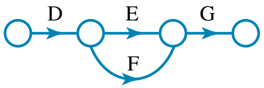
Topic 11 Networks and decision mathematics

Exercise 11.2 Critical paths

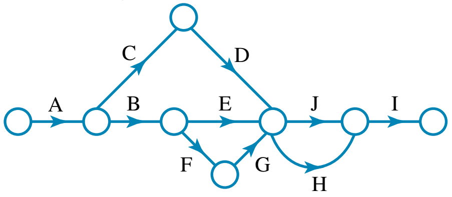
1. a.



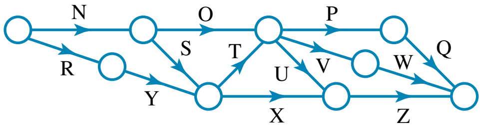
b.



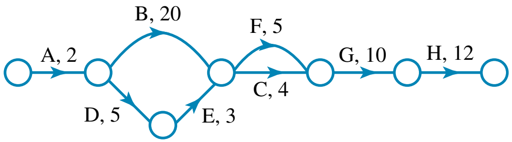
2. a.



b.



3. a.



b. 49 minutes

4. B

5. D

6. a.

Activity	Immediate predecessor	Time (min)
A	–	3
B	A	2
C	B	5
D	A	9
E	C, D	6
F	E	4
G	C, D	8
H	C, D	11

b. 23 minutes

c. B, C, E, F and G can be delayed.

d. A–D–H = 23 minutes

7. a.

Activity	Immediate predecessor	Time
A	–	3
B	–	4
C	–	5
D	A	6
E	B, F	5
F	C	8
G	D	18
H	E	8
I	E	6

b. A– D– G = 27 minutes

c. and **d.**

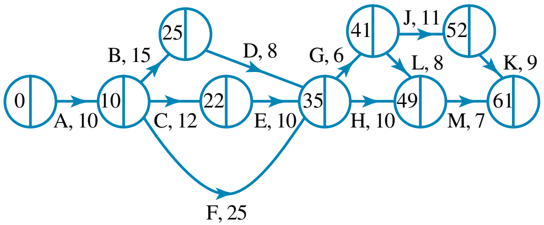
Activity B can be delayed 10 minutes, Activity C can be delayed 1 minute, Activity E can be delayed 1 minute, Activity F can be delayed 1 minute, Activity H can be delayed 1 minute and Activity I can be delayed 3 minutes.

8. B

9. D

10. A

11. a.



b. 61 minutes

c. A– F– G– J– K

d. B, D, C, E, H, L and M

12. a.

Activity	Activity	Predecessor	Time (min)
A	Chop broccoli		4
B	Steam/boil broccoli	A	8
C	Measure cheese	A	2
D	Measure wet	C	2
E	Measure dry	D	2
F	Whisk wet	E	2
G	Remove broccoli	B, F	1
H	Combine broccoli, cheese, wet	G	3
I	Combine with dry	H	3
J	Preheat oven to 180	I	10
K	Prepare muffin tins	I	2
L	Spoon mix into tins	K	5
M	Place in hot oven	J, L	25
N	Check with skewer after 25 min	M	1
O	Remove from oven and cool	M	5
	Total time		75 min

b. *See the image at the bottom of the page.

c. Minimum completion time = 60 min
**See the diagram at the bottom of the page.

d. Float times:
$C = 6 - 2 - 4 = 0$
$D = 8 - 2 - 6 = 0$
$E = 10 - 2 - 8 = 0$
$F = 12 - 2 - 10 = 0$
$K = 24 - 2 - 19 = 3$
$L = 29 - 5 - 21 = 3$

e. Yes

f. If you add an extra 1 minute to C, D and E, they would form the new critical path because the path C– F would become 11 minutes, causing the earliest completion time to be 63 minutes.

g. Yes.

h. If you add an extra 5 minutes to C, D and E, they would form the new critical path because the path C– F would become 23 minutes, causing the earliest completion time to be 75 minutes.

i. If you change L to 9 minutes, then K– L becomes the new critical path at 11 minutes, 1 more minute than J. The new earliest completion time becomes 61 minutes.

13. Answers may vary.

14. a. John should be toasting the buns 3 minutes before the burgers are cooked, so the burger does not get cold while he is waiting for the buns to toast.

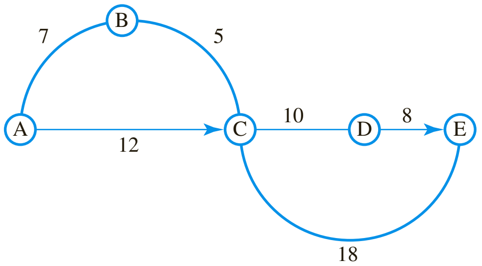
b. ***See the image at the bottom of the page.

c. It makes the earliest completion time 3 minutes longer than is needed.

d. 15 min

15. a. No. All projects have a critical path. There is a path in every project that has time associated with its completion. Therefore, there is always a critical path.

b. Yes. See the following example.



Exercise 11.3 Critical path analysis with backward scanning

1. Critical path = B – D
Float times for the non-critical activities:
A: 1minute
C: 1minute
E: 1minute

2. D

3. Critical path = B—F—G
Float time for the non-critical activities:
A: 3 hours
C: 3 hours
D: 5 hours
E: 5 hours

4. D

5. a. 31 days

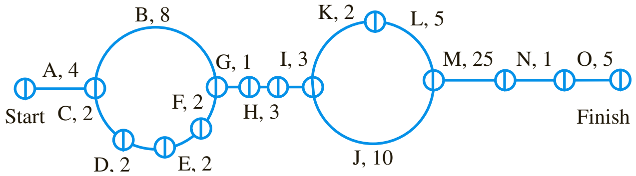
b. A– C– E– G

6. D

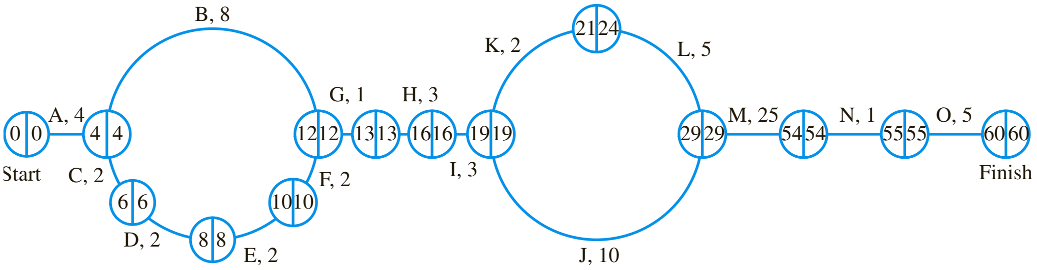
7. C

---

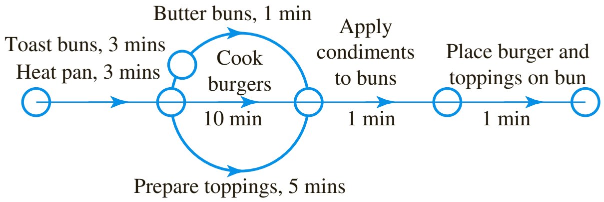
*12.b.



**12.c.



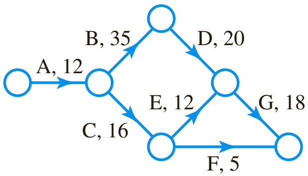
***14.b.



8. a.

Activity	Immediate predecessor	Time
A	–	12
B	A	35
C	A	16
D	B	20
E	C	12
F	C	5
G	D, E	18

b.



c. 85 minutes

d. A– B– D– G

e. 28%

9. C

10. a. ABDEF **b.** 31 weeks

c. ABDEF **d.** 27 weeks

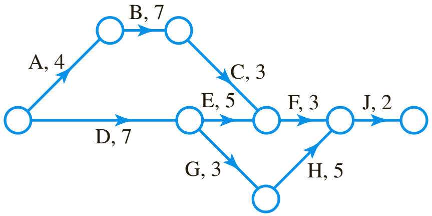
11. a. 29 minutes **b.** B– E– J– N

12. a. 35 days

b. C– F– J– M- Q

c. 3 days

13. a.



b. Float times

Activity	Times	EST	LFT	Float time
D	7	0	9	2
E	5	7	14	2
G	3	7	12	2
H	5	10	17	2

14. a. BCFG **b.** 24 weeks **c.** 22 weeks

15. a. $44 500 **b.** $51 000 **c.** $6500

16. a.

Phase	Normal time (weeks)	Crash time (weeks)
AC	8	0
BC	10	2
CD	7	1
CE	3	0
CF	6	1
DG	5	1
EG	7	0
FG	8	3

b. You can assume that crashing the project further would cost more money, and increase the risk that major problems could arise.

c. No

17. a. The main purpose of crashing a project is to save money, by being more efficient.

b. Crashing a project can save money if done correctly. However, if the project manager is too aggressive, the process of crashing can be more expensive than the original cost projection of the project.

c. The personnel hours increase, thus the salaries paid increase.

The rental or leasing equipment costs increase.

d. The personnel hours decrease, thus the salaries paid decreases.

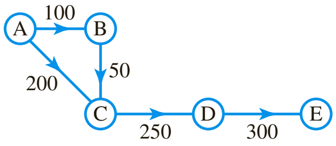
The rental or leasing equipment cost decreases.

e. The material costs stay the same.

f. If the crashing process is too aggressive, mistakes can be made in the quality of the work. These mistakes can cause the original project's expected time to increase, rather than decrease. These mistakes can also dramatically increase the cost of the project, because certain parts of the project may have to be redone, or fixed.

Exercise 11.4 Network flow

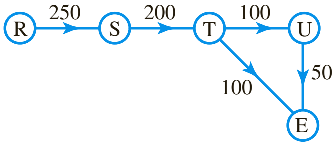
1. a.



b. 250

c. This doesn't meet the demand.

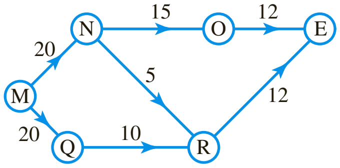
2. a.



b. 150

c. This does meet the demand.

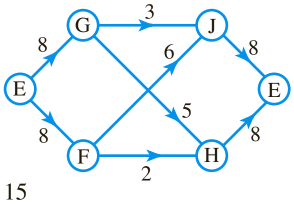
3. a.



b. 24

c. This does meet the demand.

4. a.



b. 15

c. This doesn't meet the demand.

5. a. 23 **b.** 16 **c.** 16

6. a. 6 **b.** 3 **c.** 3

7. a. i.

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
B	C	2
C	E	4
C	D	2
D	E	6

ii. 10

b. i.

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
C	B	2
C	E	4
C	D	2
D	E	6

ii. 11

c. i.

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	C	2
B	E	3
C	D	2
C	E	4
D	E	6

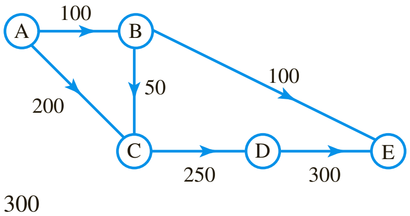
ii. 12

d. i.

From	To	Flow capacity
A	B	4
A	C	7
A	D	3
A	E	5
B	E	3
C	E	8
D	B	2
D	E	6

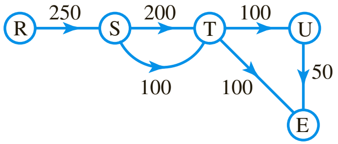
ii. 18

8. a. i.



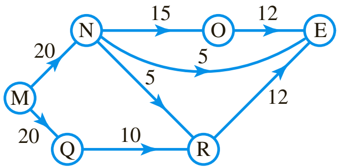
ii. 300

b. i.



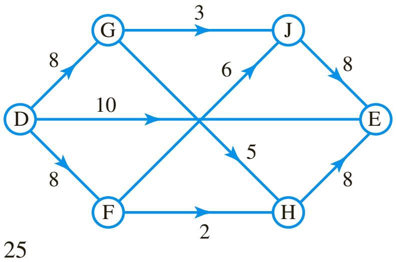
ii. 150

c. i.



ii. 29

d. i.



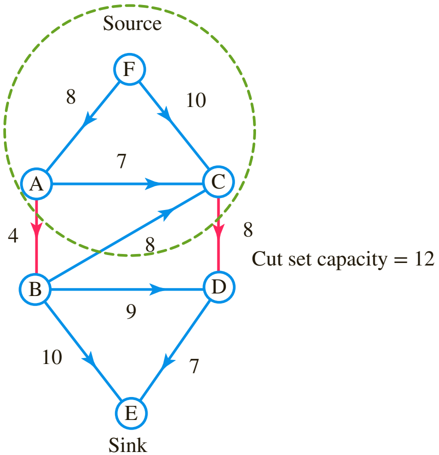
ii. 25

9. D

10. a.

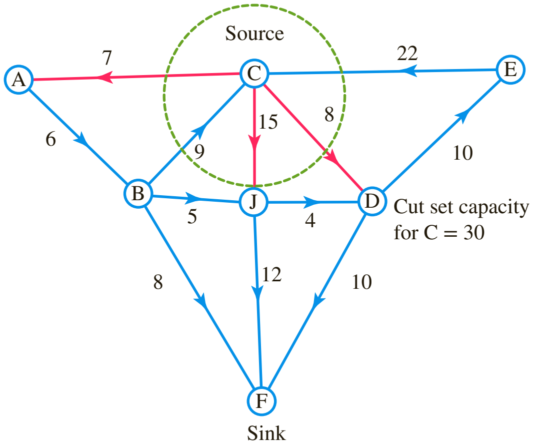
F = 18 FAC = 12
FA = 21 FCD = 15
FC = 16 FAB = 44
FD = 25 FAD = 28
FB = 45 FCB = 35

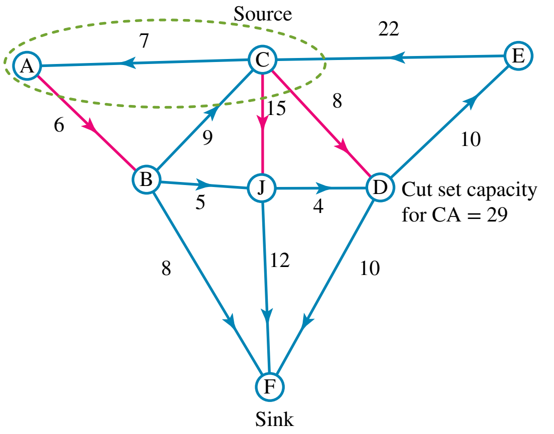
b.

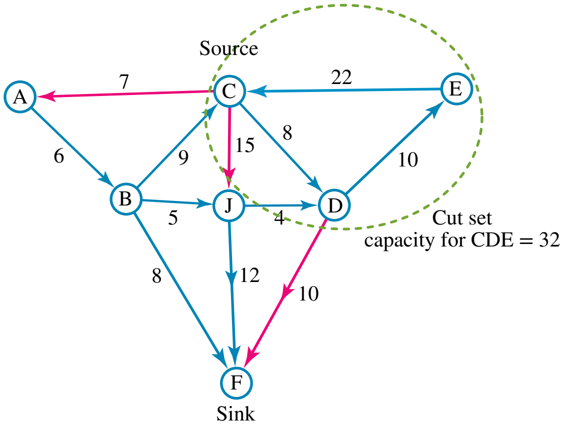


c. Minimum cut set = 12

11. a.







b. Maximum flow = 29

c. Yes, there are 3: CA, CDE, CAE

12. a. Yes

b. You have to include the three sources in each cut set.

c. 7

13. a. Yes

b. You have to include the three sources, or set up a dummy source.

c. 16

d. Yes

14. a. Yes

b. You can set up a dummy sink with the same capacity as the multiple sinks.

c. 13

d. Yes

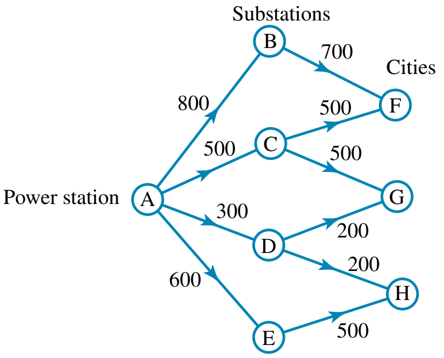
15. a. Yes

b. 1700

c. One option is to reduce the path from BG to 300 parts, though there are many other options.

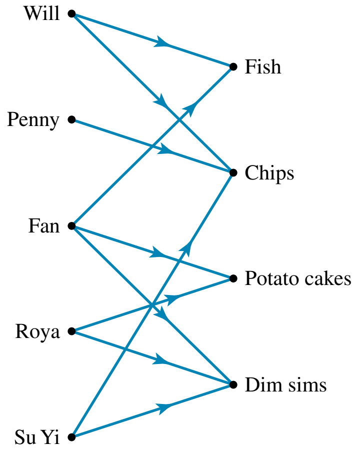
16. a. 1800 MW

b.



Exercise 11.5 Bipartite graphs and the Hungarian algorithm

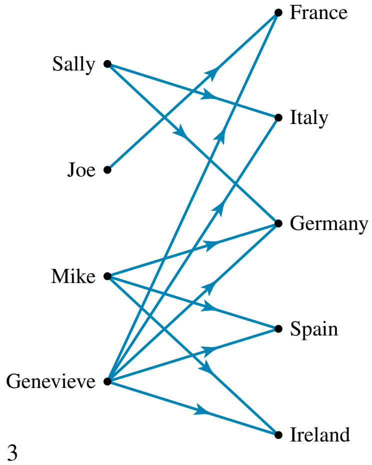
1. a.



b. False, False, False, True

2. a. The visitors can be considered the supply vertices and the countries the demand vertices. No two vertices in each of these groups share an edge, so a bipartite graph is suitable.

b.



c. 3

3. a. False

b. True

c. False

d. False

e. False

4. $S1 \rightarrow D$

$S2 \rightarrow A$

$S3 \rightarrow B$

$S4 \rightarrow C$

61 hours.

5. $R1 \rightarrow C2$

$R2 \rightarrow C1$

$R3 \rightarrow C3$

$2 + 1 + 1 = 4$ hours.

6. $R1 \rightarrow C4$

$R2 \rightarrow C2$

$R3 \rightarrow C1$

$R4 \rightarrow C3$

$4 + 5 + 6 + 4 = 19$ hours.

7. 28 hours

8. 44 km

9. $140

10. C

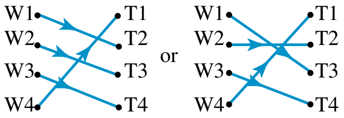
11. a. 54

b. 90

12. a. $\begin{bmatrix} 0 & 10 & 20 & 30 \\ 40 & 0 & 10 & 40 \\ 50 & 40 & 50 & 0 \\ 0 & 70 & 40 & 60 \end{bmatrix}\begin{bmatrix} 0 & 10 & 10 & 30 \\ 40 & 0 & 0 & 40 \\ 50 & 40 & 40 & 0 \\ 0 & 70 & 30 & 60 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 0 & 30 \\ 50 & 0 & 0 & 50 \\ 50 & 30 & 30 & 0 \\ 0 & 60 & 20 & 60 \end{bmatrix}$

c.



d. 180 hours

13. One possible allocation

$M \rightarrow L1$

$Je \rightarrow L3$

$P \rightarrow L2$

$Ja \rightarrow L4$

11 minutes

Exercise 11.6 Review: exam practice

1. C
2. D
3. D
4. D
5. B
6. A
7. D
8. A
9. C
10. C
11. A
12. B

13. a. $\begin{bmatrix} 0 & 2 & 0 & 4 \\ 0 & 4 & 4 & 1 \\ 4 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

b. The minimum number of lines needed to cover all zeros is 4, so all 4 tasks can be allocated.

c. Hungarian algorithm is not necessary.

d. The minimum score is $5 + 7 + 3 + 8 = 23$

14. a.

Phases	Hrs
AB	3
AC	3
AH	20
BJ	12
BD	5
CF	6
CI	10
DE	7
EH	3
FG	7
GH	5
IG	8
JE	4

b. ACIGH

c. 26 hours

d. No

e. 3 hours

f. 4 hours will create a second critical path ABJEH but 5 hours will make ABJEH the only critical path.

15. a. Earliest completion time = 25 hrs ABDFG

Phases	Hrs
AB	9
AC	6
BD	4
CD	5
DE	3
DF	4
EG	7
FG	8

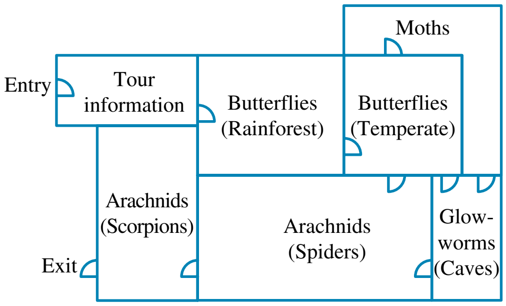
b. Earliest completion time = 25 hrs ABDFG

Phases	Hrs
AB	9
AC	6
BD	4
CD	5
DE	3
DF	4
EG	7
FG	$8 - 2 = 6$ crashed 2 hrs

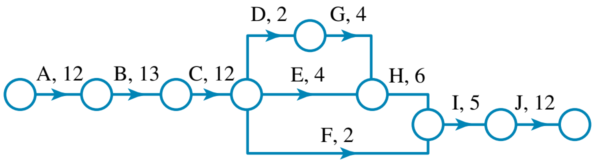
c. There is a possibility that his/her crash times could be the same, but there is also a chance that it could be different.

16. a. 20 = AFBE b. No c. Yes

17. a.



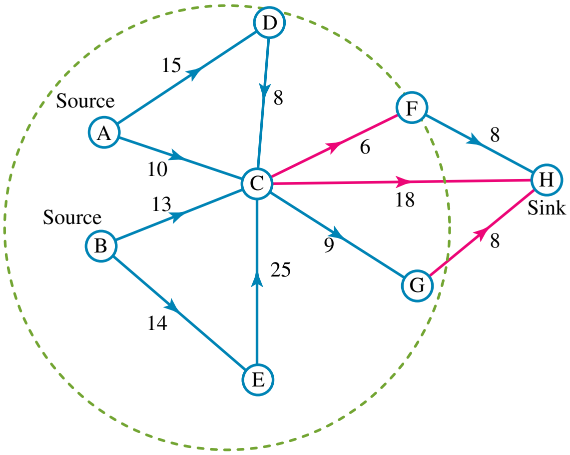
b.



c. The number of entries to the rainforest butterflies room is greater than the number who exit (and enter the temperate butterflies room), so the room will become increasingly crowded.

d. 5 people per minute

18. a.



b. 32

19. a. Yes

b. Both networks have the same flow of 17.

c. Multiple sources and maximum flow are not affected when a dummy source with infinite flows enters the multiple sources.

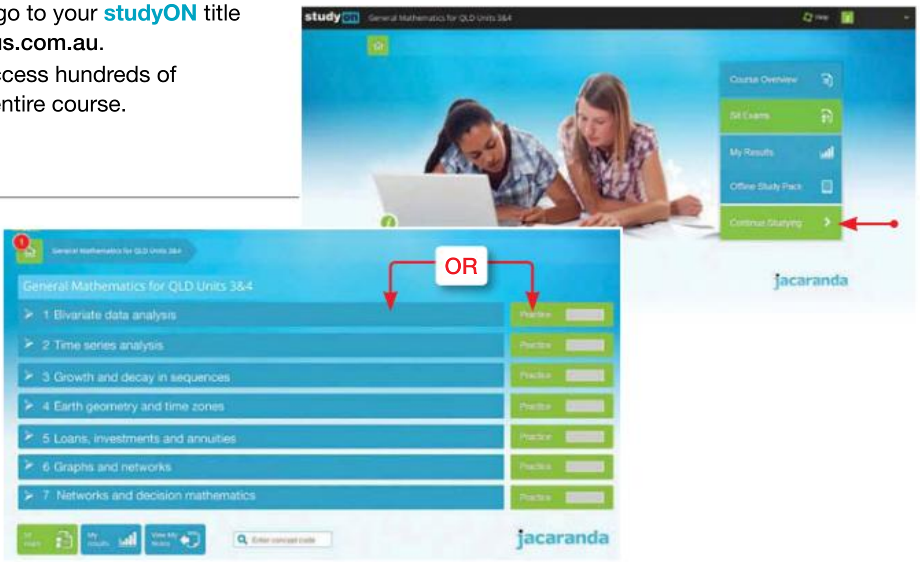
20. Only two people need to be stopped.

REVISION UNIT 4 Investing and networking

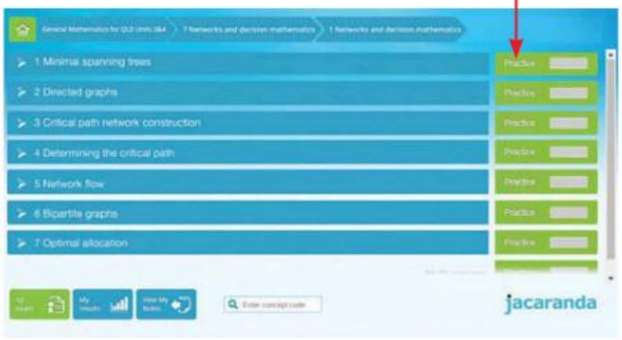
TOPIC 3 Networks and decision mathematics

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *General Mathematics for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level *OR* select **Practice** to answer all practice questions available for each area.

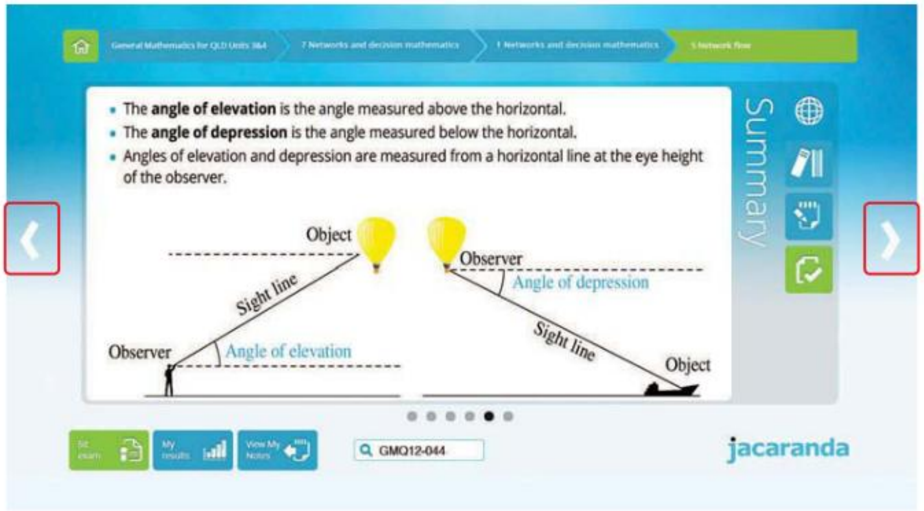


- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



- The **angle of elevation** is the angle measured above the horizontal.
- The **angle of depression** is the angle measured below the horizontal.
- Angles of elevation and depression are measured from a horizontal line at the eye height of the observer.

PRACTICE ASSESSMENT 3

General Mathematics: Unit 4 examination

Unit

Unit 4: Investing and networking

Topic

Topic 1: Loans, investments and annuities
Topic 2: Graphs and networks
Topic 3: Networks and decision mathematics

Conditions

Response type	Duration	Reading
Short response	120 minutes	5 minutes

Other	Instructions
• Only the QCAA formula sheet must be provided • Notes are not permitted • Scientific calculator permitted	• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.

Criterion	Marks allocated	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	63	

Part A: Simple familiar — total marks 40

Question 1 (6 marks)

Rylie decides to invest his savings of $\$4500$ in a savings account at 5.5% per annum compounded annually.

a. Using A_0, to represent the initial investment, write this investment as a recurrence relation.

b. Complete the following table to show the growth of the investment over 3 years.

A_n ($)	Calculation	A_{n+1} ($)
5000		

c. How much money was in the account at the end of the 3 years?

d. How much interest was made in 3 years?

Question 2 (4 marks)

A person invests $\$7600$ in a financial institution at 6.5% p.a. compounded quarterly. Calculate the value of the investment after years. Write your answer correct to the nearest cent.

Question 3 (3 marks)

An interest rate is quoted at 6.5% p.a. compounding monthly. What is the effective interest rate? Give your answer correct to two decimal places.

Question 4 (3 marks)

Calculate the length of time required for $\$1800$ to grow to $\$2200$ at 6% p.a. compounded quarterly. Give your answer to the nearest month.

Question 5 (4 marks)

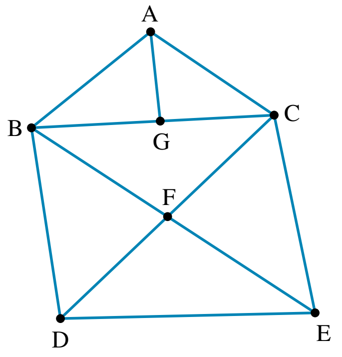
Calculate the second month's interest on a $\$50\,000$ reducing balance loan at 9.5% p.a., if the monthly repayment is $\$600$.

Question 6 (2 marks)

A wealthy businessman decides to set up a scholarship fund to give a grant to a young person volunteering in the community. He invests in a perpetuity that is guaranteed an interest rate of 5% p.a. Determine the amount to be invested in this perpetuity to provide a regular monthly income of $\$400$.

Question 7 (4 marks)

The following graph does not have an Euler trail.

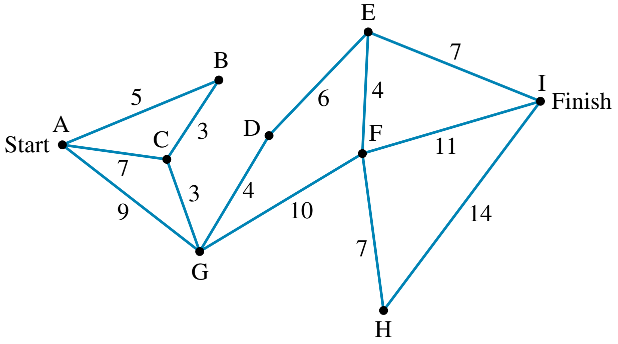


a. What is the minimum amount of edges that you will need to add to create an Euler trail? (No loops or repeats of edges are allowed.)

b. Draw a graph showing the edge(s) you added to form an Euler trail.

Question 8 (8 marks)

The following graph represents a running course. The numbers represent the distances (in km) between checkpoints.



a. What is the shortest path from start to finish in the graph?

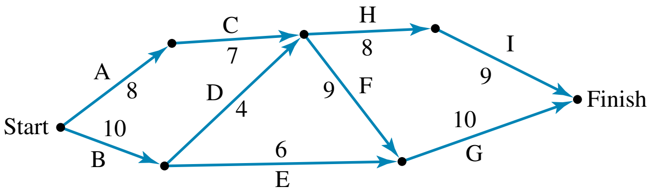
b. What is the length of the shortest path?

c. What is the route you will need to traverse if you want to begin at the start, arrive at every checkpoint only once, and end at the finish?

d. What is the distance of the path in part **c**?

Question 9 (3 marks)

Consider the network shown.

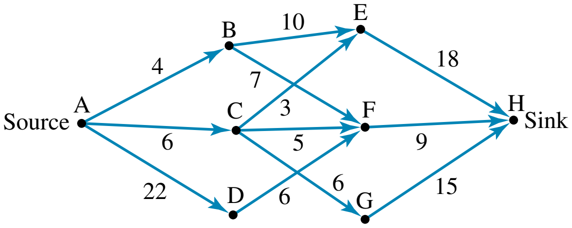


a. What is the critical path for the network?

b. What is the duration of the critical path (in hours)?

Question 10 (3 marks)

Consider the network shown.



a. Draw the minimum cut that defines that maximum flow in the previous network.

b. What is the maximum flow?

Part B: Complex familiar — total marks 10

Question 11 (7 marks)

To upgrade his office equipment, Nathan took a reducing balance loan of $\$30\,000$ from the bank. The interest rate is 8.5% per year, compounded monthly. The loan is to be repaid in full in 5 years.
The monthly repayment is $\$615.50$.

a. To monitor his repayments, Nathan keeps an amortisation table for his loan. Complete the following table for the first 4 payments.

Payment number	Principal outstanding	Interest due	Payment	Loan outstanding
1	30 000			
2				
3				
4				

b. Calculate the amount of principal and the amount of interest repaid during the fifth repayment.

Question 12 (3 marks)

Sylvie has sold a block of land for $\$75\,000$ and invests the money for 3 years in an account which pays compound interest of 7.5% per annum compounding each quarter.

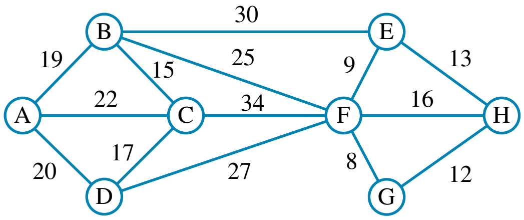
a. What will be the amount in Sylvie's account at the end of 2 years?

b. How much interest will Sylvie make during the 2-year period?

Part C: Complex unfamiliar — total marks 13

Question 13 (6 marks)

The following figure shows the time (in minutes) it takes to travel along roads connecting 8 towns.



a. Determine the path with the shortest time connecting A to H.

b. Determine the minimum spanning tree for the graph.

Question 14 (7 marks)

The following is a chart of Sam's schedule for the day before a final exam.

Activity letter	Activity	Predecessor	Time
A	Prepare breakfast	None	10 min
B	Prepare lunch	None	10 min
C	Pack backpack	A, B	10 min
D	Ride bicycle to school	C	30 min
E	Call library for study room reservation	C	5 min
F	Call study partner to confirm	E	3 min
G	Drink water from backpack	C	30 min
H	Study 3 hours	D, E, G	180 min
I	Eat lunch	H	60 min
J	Meet with professor	H	30 min
K	Study in library for 2 hours	I, J	120 min

a. Construct a network to simulate this chart. Include the times and activity letters in the graph.
b. What is the earliest completion time (ECT) for Sam?
c. What is the earliest start time (EST) for activity H?
d. What is the latest start time (LST) for activity F?
e. What is the float time for activity J?

PRACTICE ASSESSMENT 4

General Mathematics: Units 3 & 4 examination

Topic

Unit 3 **Topic 1: Bivariate data analysis**
Topic 2: Time series analysis
Topic 3: Growth and decay in sequences
Topic 4: Earth geometry and time zones
Unit 4 **Topic 1: Loans, investments and annuities**
Topic 2: Graphs and networks
Topic 3: Networks and decision mathematics

Conditions

Technique	Response Type	Duration	Reading
Paper 1: Simple familiar Paper 2: Simple familiar, Complex familiar, Complex unfamiliar	Short response	Paper 1: 90 minutes Paper 2: 90 minutes	5 minutes for each paper

Other	Instructions
• Only the QCAA formula sheet must be provided • Notes are not permitted • Scientific calculator permitted • Combined paper consisting of simple familiar, complex familiar and complex unfamiliar questions	• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.

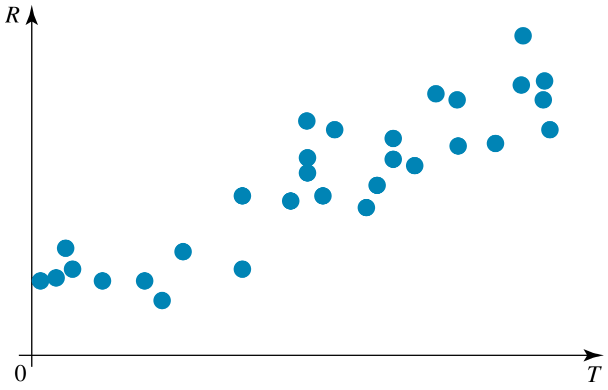
Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	40	66	

Paper 1 — Simple familiar

Part A: Multiple choice — total marks: 30

Question 1 (1 mark)

A scatterplot is shown.



Which of the following best describes the relationship between R and T?

A. Weak positive

B. Moderate positive

C. Weak negative

D. Moderate negative

Question 2 (1 mark)

For a group of 17-year-old students who regularly used social media apps on their digital devices, the Pearson moment correlation coefficient between the time spent using social media apps and fitness level was found to be $r = -0.6$. Based on this information, it can be concluded that

A. 60% of these students were not very fit.

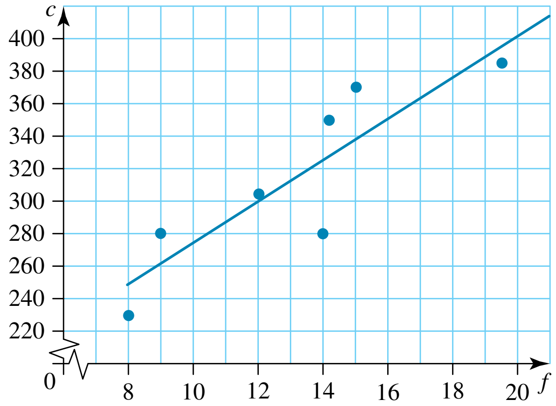
B. 60% of these students were very fit.

C. the students in the group who spent a short amount of time using social media apps tended to be fitter.

D. the students in the group who spent a large amount of time using social media apps tended to be fitter.

Question 3 (1 mark)

The following scatterplot shows the calorie content, c, and fat content, f (in grams) for the same amount of pizza from various brands.



The regression line equation is most likely to be

A. $f = 15c + 112$. **B.** $f = 112c + 15$. **C.** $c = 112f + 15$. **D.** $c = 15f + 112$.

Question 4 (1 mark)

Data showing the choice of sport offered at a holiday camp by males and females are given in this two-way frequency table.

Texture	**Males**	**Females**	**Total**
Tennis	23	14	37
Volleyball	31	41	72
Total	54	55	109

We can conclude that

A. 23% of males chose tennis.

B. 57.4% of males chose volleyball.

C. 31% of males chose volleyball.

D. 62.1% of males chose tennis.

Question 5 (1 mark)

The data set in the table compares the price of textbooks from a shopping centre bookstore to those from an online bookstore.

Online bookstore price of textbook ($)	Shopping centre bookstore price ($)
7.50	15.60
48.75	46.70
26.35	27.95
16.80	24.00
10.70	14.85
45.50	66.35
75.00	92.50

Which of the following is an example of interpolating data?

A. Finding the price at the online bookstore when the price at the shopping centre bookstore is $101.50

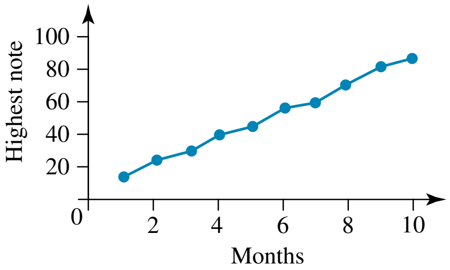
B. Finding the price at the online bookstore when the price at the shopping centre bookstore is $13.25

C. Finding the price at the shopping centre bookstore when the price at the online bookstore is $68.70

D. Finding the price at the shopping centre bookstore when the price at the online bookstore is $5.95

Question 6 (1 mark)

The following figure shows the highest note (measured on a scale out of 100) played by a saxophonist over a 10-month period.



The data are

A. seasonal.

B. random.

C. showing a trend.

D. there is no trend.

Question 7 (1 mark)

The following data represent the number of tomatoes canned in a day by Mary Farmer.

Day	**Mon**	**Tue**	**Wed**	**Thu**
Tomatoes	350	320	300	310

Day	**Fri**	**Sat**	**Sun**	**Mon**
Tomatoes	270	240	200	160

The data are smoothed using a 3-point moving average. The first two points in the smoothed trend line are

A. 320 and 300. **B.** 320 and 310. **C.** 335 and 310. **D.** 323 and 310.

Question 8 (1 mark)

How many points would be obtained from the smoothed trend line in question 7?

A. 8 **B.** 7 **C.** 6 **D.** 5

Question 9 (1 mark)

The scouts committee have $1267 in the bank for the whole of January. If the bank pays 5.3% p.a. paid at the end of each month, the amount the club has at the end of January is

A. $5.60. **B.** $1272.60. **C.** $1334.15. **D.** $67.15.

Question 10 (1 mark)

A truck is purchased for $45 000 and depreciates by 8% of its previous value. Which one of the following determines the value of the truck at the end of 10 years?

A. $S = 45\,000 \times 0.08 \times 10$ **B.** $S = 45\,000 \times 0.92 \times 10$

C. $S = 45\,000 \times 0.08^{10}$ **D.** $S = 45\,000 \times 0.92^{10}$

Question 11 (1 mark)

$450 is invested in an account paying a flat rate interest of 3.75% p.a. The recurrence relation for the value of the investment after n years is

A. $A_{n+1} = A_n + 16.88, A_0 = 450.$

B. $A_{n+1} = 450 + \frac{3.75 \times 450}{100}, A_0 = 450.$

C. $A_{n+1} = 450 + A_n, V_0 = 450.$

D. $A_{n+1} = 450 + d, V_0 = 450.$

Question 12 (1 mark)

The 3rd term of an arithmetic sequence is 8 and the 5th term is 22. The 20th term of this sequence is

A. 45. **B.** 82. **C.** 127. **D.** 133.

Question 13 (1 mark)

A geometric sequence is described with the following recurrence relation

$$t_1 = -2, t_{n+1} = -3 \times t_n$$

Which of the following sequences is described by this recurrence relation?

A. $-3, 6, -12, 24, ...$

B. $5, 15, 45, 135, 405, ...$

C. $-3, -6, -12, -24, ...$

D. $-2, 6, -18, 54, ...$

Question 14 (1 mark)

Perth $(32°\text{S}, 116°\text{ E})$ and Denpasar $(8°\text{S}, 116°\text{E})$ are on the same meridian of longitude. If two points represented these cities, the angle subtended between them at the centre of the earth would be

A. $40°$. **B.** $34°$. **C.** $30°$. **D.** $24°$.

Question 15 (1 mark)

The distance between Denpasar $(8°\text{S}, 16°\text{E})$ and Perth $(32°\text{S}, 116°\text{E})$ is closest to

A. 1033.34 km. **B.** 2680.83 km. **C.** 3445.87 km. **D.** 4132.43 km.

Question 16 (1 mark)

At a point on the Earth's surface, the coordinates are $(25°\text{N}, 150°\text{W})$. The standard time at this point would be

A. GMT – 10. **B.** GMT + 10. **C.** GMT – 15. **D.** GMT + 15.

Question 17 (1 mark)

It is 7.00 am Thursday at point X, with coordinates $(24°\text{S}, 75°\text{W})$. At point Y, with coordinates $(45°\text{N}, 135°\text{E})$, what is the time if daylight saving time applies at Y?

A. 9.00 pm Wednesday **B.** 11.00 pm Thursday

C. 10.00 pm Wednesday **D.** 10.00 pm Thursday

Question 18 (1 mark)

A wealthy alumnus of a university sets up a scholarship fund. She invests in a perpetuity that is guaranteed an interest rate of 5% p.a. The amount to be invested in this perpetuity to provide a regular monthly income of $700 would need be closest to

A. $145 618.
B. $155 634.
C. $167 980.
D. $177 609.

Question 19 (1 mark)

Mavis invests $1500 at the end of each year at 4.5% p.a. compounded annually. The recurrence relation that represents the value of the annuity after each payment is made is

A. $A_{n+1} = 1.045A_n, A_1 = 1000$.
B. $A_{n+1} = A_n + 45, A_1 = 1000$.
C. $A_{n+1} = 45A_n + 1000, A_1 = 1000$.
D. $A_{n+1} = 1.045A_n + 1000, A_1 = 1000$.

Use the following information to answer questions 20, 21 and 22.

Abdul takes out a loan of $5500 at 8% p.a. to buy a car. The loan is to be paid back in half-yearly instalments of $1515.20. The amortisation table for this loan is shown.

Instalment number	Payment	Interest	Principal reduction	Balance
0	0.00	0.00	0.00	5500
1	1515.20		1295.20	4204.80
2	1515.20	168.18	1347.01	
3	1515.20	114.30		1456.90
4	1515.20	58.28	1456.92	−0.02

Question 20 (1 mark)

For the first payment of $1515.20 the amount of interest is closest to

A. $220.
B. $250.
C. $280.
D. $310.

Question 21 (1 mark)

The balance of the loan after two payments have been made is closest to

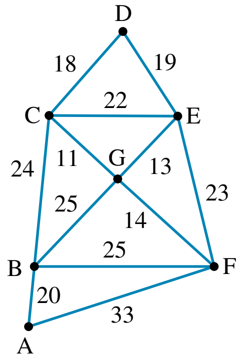
A. \$2766. **B.** \$2858. **C.** \$2999. **D.** \$3105.

Question 22 (1 mark)

After 3 payments have been made, the principal is reduced by

A. \$1401. **B.** \$1322. **C.** \$1500. **D.** \$1455.

Questions 23 and 24 refer to the following network where distances between towns are given in kilometres.



Question 23 (1 mark)

If cable is to be laid linking all towns, determine the minimum length of cable needed.

A. 103 km

B. 88 km

C. 21 km

D. 100 km

Question 24 (1 mark)

What is the shortest distance from A to D?

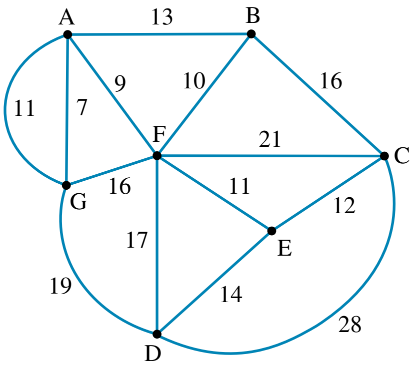
A. 62 km

B. 65 km

C. 58 km

D. 44 km

Questions 25 and 26 refer to the following network.



Question 25 (1 mark)

The length of the minimum spanning tree is

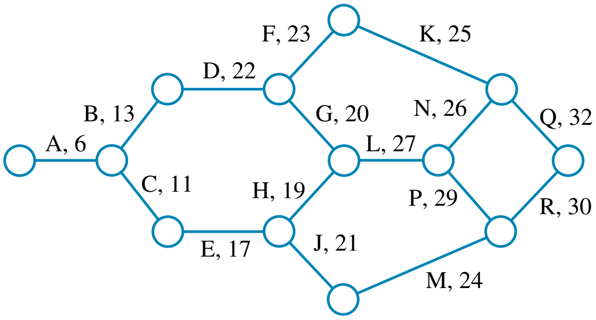
A. 58. **B.** 60. **C.** 62. **D.** 65.

Question 26 (1 mark)

The shortest path between A and D is

A. 25. **B.** 26. **C.** 27. **D.** 28.

The project plan for a new computer software program shown in the following figure relates to questions 27 to 30.



Time is measured in days.

Question 27 (1 mark)

The earliest completion time is

A. 134. **B.** 137. **C.** 144. **D.** 147.

Question 28 (1 mark)

The critical path is

A. A – C – H – L – N – Q.

B. A –B –D – G – L – N – Q.

C. A – B – D – G – L – P – R.

D. A –C – E – H – L – P – R.

Question 29 (1 mark)

What is the float time, in days, of activity Q?

A. 0 **B.** 1 **C.** 32 **D.** 114

Question 30 (1 mark)

How long can activity H be delayed, in days, before the entire project is delayed?

A. 0 **B.** 8 **C.** 4 **D.** 6

Part B: Short response (10 marks)

Question 1 (2 marks)

Describe when you would use seasonal indices and adjustments when analysing a set of data.

Question 2 (4 marks)

a. X (20°N, 50°E) and Y (40°S, 50°E) are two points on the Earth's surface. Determine the distance (in km) between these two points.

b. Determine the shortest distance between points D (35°N, 120°W) and F (35°N, 100°E).

Question 3 (4 marks)

The following contingency table shows the composition of left- and right-handed students in Years 11 and 12 at a school.

	Left handers	**Right handers**
Year 12	15	105
Year 11	10	90

a. Calculate the percentage of left-handers who are in Year 11.

b. Calculate the percentage of students who are in Year 12.

Paper 2 — Simple familiar, Complex familiar and Complex unfamiliar

Part A: Simple familiar — total marks: 16

Question 1 (3 marks)

The following set of data gives the number of cars sold by a car dealer each quarter for 3 years and the average number of cars sold each year.

Quarter	2017	2018	2019
1	26	20	28
2	31	33	25
3	18	23	20
4	22	19	19
Average	24.25	23.75	23

Calculate the seasonal index for Quarter 2, correct to 4 decimal places.

Question 2 (6 marks)

A company car purchased for $\$42\,000$ depreciates at a reducing rate of 10% per annum.

a. Calculate the value of the car after 4 years.

b. Calculate the total depreciation over the first 7 years.

c. How long will it take until the car reaches its scrap value of $\$8000$?

Question 3 (2 marks)

The AFL grand final commences on Saturday at 2.30 pm in Melbourne (UTC + 10)

a. What time will an avid AFL fan on holidays in Dublin, Ireland (UTC + 1) be watching the start of the broadcast of the game?

b. What time will an avid AFL fan on holidays in Wellington, New Zealand (UTC + 12) be watching the start of the broadcast of the game?

Question 4 (5 marks)

Kanye borrows $\$5800$ at a rate of 7.55% p.a. (interest compounded monthly) which is to be paid back with monthly instalments of $\$988.06$.

a. Write a recurrence relation to describe this situation.

b. Use your calculator to determine the number of monthly repayments needed to pay off this loan.

c. What would be the final payment to ensure that Kanye had paid out the loan in the minimum number of months?

Part B: Complex familiar — total marks 25

Explanation for marking scheme

Question 5 (6 marks)

A loan of $\$16\,000$ is taken out over 4 years at a rate of 5.5% p.a. (interest debited monthly) and is to be paid back monthly with $\$372.10$ instalments. Complete the following table for the third payment.

Payment number (*n*)	Payment	Interest	Principal reduction	Balance of loan
0	0.00	0.00	0.00	16000.00
1	372.10	73.33	298.77	15701.2
2	372.10	71.96	300.14	15401.09
3	372.10			

Question 6 (7 marks)

A reducing balance loan can be modelled by the recurrence relation

$$A_{n+1} = 1.01125A_n - 1795.69, A_1 = 32000.$$

Interest is accrued quarterly and is to be paid off in quarterly instalments.

a. What is the quarterly instalment?

b. How much is the initial loan?

c. What is the annual interest rate?

d. What is the total interest paid after the 5th instalment?

Question 7 (8 marks)

The following is a chart of Sam's schedule for the day before a final exam.

a. Construct a network to simulate this chart. Include the times and activity letters in the graph.

Activity letter	**Activity**	**Predecessor**	**Time**
A	Prepare breakfast	None	10 min
B	Prepare lunch	None	10 min
C	Pack backpack	A, B	10 min
D	Ride bicycle to school	C	30 min
E	Call library for study room reservation	C	5 min
F	Call study partner to confirm	E	3 min
G	Drink water from backpack	C	30 min
H	Study 3 hours	D, E, G	180 min
I	Eat lunch	H	60 min
J	Meet with professor	H	30 min
K	Study in library for 2 hours	I, J	120 n

b. What is the earliest completion time (ECT) for Sam?

c. What is the earliest start time (EST) for activity H?

d. What is the latest start time (LST) for activity F?

e. What is the float time for activity J?

Question 8 (4 marks)

The population of a small town was 3000 at the end of 2017. At the end of 2018 the population was 3450 and at the end of 2019 the population was 3968. If the population continued in this pattern:

a. what was the population at the end of 2021?

b. if the population had been growing in this way for the last 10 years, what was the population at the start of 2015?

Part C: Complex unfamiliar — total marks 20

Question 9 (10 marks)

An investigation is undertaken with synchronised swimmers to explore the link between years of experience and lung capacity. The data are following.

Lung capacity (x)	Years of experience
10	2
10	4
15	2
17.5	3
25	4
22.5	8
17.5	4
23.5	6
27.5	10
27.5	7
21	4.5
25	8.5
12.5	3
15	6
20	6

a. Identify the explanatory and response variables.

b. Display the data on a scatterplot.

c. Describe the association between the two variables in terms of direction, form and strength.

d. Is it appropriate to use Pearson's correlation coefficient to explain the link between years of experience and lung capacity?

e. Are the swimmers' lung capacities affected by their years of experience?

f. If Pearson's product moment correlation coefficient is 0.75, determine the value of the coefficient of determination.

g. What does the coefficient of determination say about the relationship between lung capacity and years of experience of the swimmers?

Question 10 (6 marks)

Maureen finds it difficult to manage the fluctuations in expenses involving the buying of gifts for family and friends. She decides to set up a savings account for two years with regular quarterly deposits of $\$500$. She makes an initial deposit of $\$5000$ in an account that offers a rate of 4.8% p.a. adjusted quarterly.

a. Write down the recurrence relation that represents the information given.

b. Use the recurrence relation set up in part (a) to fill in the following table.

Payment number (*n*)	Payment (*d*)	Interest	Addition to savings	Balance of savings
0				
1				
2				
3				
4				
5				

c. How much money does Maureen have in the account after one year and how much interest has she earned in that time?

Question 11 (4 marks)

In a survey, 152 boys and 157 girls were asked whether they preferred eggs or cornflakes for breakfast. The results showed that 98 boys and 23 girls preferred cornflakes, 33 boys and 93 girls preferred eggs, and the rest had no preference. Present these data in a percentaged two-way frequency table. Comment on any difference between the preferences of boys and girls.

Question 12 (5 marks)

Four workers need to be allocated four tasks. The cost for each worker to complete each task is summarised in the following table.

	Task 1	**Task 2**	**Task 3**	**Task 4**
Worker 1	472	427	502	499
Worker 2	389	411	422	478
Worker 3	512	417	508	480
Worker 4	404	412	401	460

a. Perform row and column reduction.

b. Perform the Hungarian algorithm

c. Display allocations using a bipartite graph.

d. Determine the optimal allocation and the total time required to complete all four tasks.

Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
Paper 1 — Section A				
1	1			Unit 3, Topic 1
2	1			Unit 3, Topic 1
3	1			Unit 3, Topic 1
4	1			Unit 3, Topic 1
5	1			Unit 3, Topic 1
6	1			Unit 3, Topic 2
7	1			Unit 3, Topic 2
8	1			Unit 3, Topic 2
9	1			Unit 3, Topic 3
10	1			Unit 3, Topic 3
11	1			Unit 3, Topic 3
12	1			Unit 3, Topic 3
13	1			Unit 3, Topic 3
14	1			Unit 3, Topic 4
15	1			Unit 3, Topic 4
16	1			Unit 3, Topic 4
17	1			Unit 3, Topic 4

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
18	1			Unit 4, Topic 1
19	1			Unit 4, Topic 1
20	1			Unit 4, Topic 1
21	1			Unit 4, Topic 1
22	1			Unit 4, Topic 1
23	1			Unit 4, Topic 2
24	1			Unit 4, Topic 2
25	1			Unit 4, Topic 2
26	1			Unit 4, Topic 3
27	1			Unit 4, Topic 3
28	1			Unit 4, Topic 3
29	1			Unit 4, Topic 3
30	1			Unit 4, Topic 3
Paper 1 — Section B				
1	2			Unit 3, Topic 1
2	4			Unit 3, Topic 4
3	4			Unit 3, Topic 1
Total	40			
Paper 2				
1	3			Unit 3, Topic 2
2	6			Unit 3, Topic 3
3	2			Unit 3, Topic 4
4	5			Unit 4, Topic 1
5		6		Unit 4, Topic 1
6		7		Unit 4, Topic 1
7		8		Unit 4, Topic 3
8		4		Unit 3, Topic 3
9			10	Unit 3, Topic 1
10			6	Unit 4, Topic 1
11			4	Unit 3, Topic 1
12			5	Unit 4, Topic 3
Totals	**16**	**25**	**25**	

GLOSSARY

activity chart a way of representing a network diagram in a table, that shows the durations of interdependencies.

amortisation paying off a loan through regular payments over a period time.

annuity a compound interest investment where equal payments are made on a recurring basis for a set period.

association the relationship between two variables.

backward scanning determining the critical path by calculating the latest time that each activity can begin without delaying the project.

bipartite graphs a graph whose vertices can be split into two groups so that each edge joins a vertex from one group to a vertex in the other group.

bivariate data data showing the relationship between two variables.

Cartesian plane a plane containing a horizontal x-axis and a vertical y-axis.

categorical data data that can be organised into groups or categories and is often an ‘object’, ‘thing’ or ‘idea’. Examples include brand names, colours, general sizes and opinions.

causation a change in the explanatory variable results in a change in the response variable.

centring a process used to smooth data when calculating even numbered moving averages.

circles of latitude any circle drawn around a sphere that is smaller than a great circle.

coefficient of determination the proportion of the total variation in the y-axis that can be attributed to variation in the x-axis. It is defined as Pearson’s correlation coefficient squared, r^2, and is usually expressed as a percentage.

coincidence an association between variables that has no causal connection.

common difference the difference between each term in an arithmetic sequence: $d = t_{n+1} - t_n$.

common response an underlying cause of an association between variables.

complete graph or network where an edge is connecting each vertex to all other vertices in the graph.

confounding variables variables that are not being investigated, yet still need to be considered.

connected graph or network a graph or network where it is possible to travel to each vertex through the edges.

continuous data numerical data that can take any value that lies within an interval. Continuous data values are subject to the accuracy of the measuring device being used.

crashing shortening the time of activities on the critical path to reduce the total completion time.

critical path the path through the network that follows the activities that can’t be delayed without delaying the rest of the project.

cyclical pattern fluctuations that usually take longer than a year to repeat.

degree the number of edges connected to a vertex.

depreciation the loss of value of an asset over time.

deseasonalising data the process of removing seasonal fluctuations from data to make trends more visible.

directed graph a graph where it is only possible to move along the edges in one direction, where the edges contain arrows.

directed networks a network that must be followed in a certain sequence.

disconnected graph or network a graph or network where it is not possible to travel to each vertex through the edges.

discrete data numerical data that is counted in exact values, with the values often being whole numbers.

downward trend a graph that will trend downwards over time.

earliest start time the earliest time that any activity can be started at after all previous activity has been completed.

edges lines that connect points on a graph.

effective interest rate the rate of interest after adjusting for the effect of compounding.

Eulerian graph a connected graph where you can start at a vertex and move along each edge only once and be able to return to the vertex at which you began.

Eulerian trail the trail that is created when you travel around an Eulerian graph.

excess flow capacity the surplus of the capacity of an edge minus the flow into the edge.

experimentation the testing of new methods or ideas.

explanatory variable the variable that is being adjusted or controlled. It is typically placed on the x-axis of a scatterplot.

extrapolation data being investigated which is outside the variable range. Extrapolation occurs through predictions and estimates that assumes the general trends of the known data will continue.

faces enclosed areas created by edges and vertices.

float time the maximum time that an activity can be delayed without delaying the rest of the project.

flow capacity the maximum amount of flow an edge can allow if it's connected to any other edges.

future value the full amount that an annuity has grown to at the end of the investment period.

general upward trend a graph that will trend upwards over time.

geometric sequence a sequence where the ratio between any two successive terms is the same.

GPS (global positioning system) A satellite navigation system that provides users with information on their position on the Earth, altitude, speed, direction of travel and time.

graph a series of lines that are used to show connections between points.

great circle the largest possible circle that can be drawn around a sphere.

Greenwich Mean Time (GMT) the mean solar time of the meridian through Greenwich, England. It is used to calculate local time around the world.

Greenwich Meridian half a great circle running from the North to the South Pole. All other places on the globe can be said to be either east or west of the Greenwich Meridian.

Hamiltonian cycle the trail that is created when you travel around a Hamiltonian graph.

Hamiltonian graph a graph where you can start at a vertex and travel to each vertex only once and be able to return to the vertex at which you began.

Hungarian algorithm used to achieve optimal allocation when choosing between multiple options in assignment problems.

immediate predecessor an edge that must be travelled along first, before the edge directly following it can be reached.

inflow the total value of the flows of all edges leading into the vertex.

International Date Line the meridian of longitude opposite the Greenwich Meridian. On either side of this line the day changes.

interpolated data being investigated which is within the variable range.

irregular fluctuations random fluctuations that occur in the short term.

latest start time the latest time that an activity can begin at without delaying the project.

latitude the angular distance from the centre of the Earth either north or south of the equator to a point on the Earth's surface.

line of best fit a straight line used to show the general relationship between two variables. The line is placed so that the average distance between it and the data points is at its lowest.

loop an edge that is connected to only one vertex.

meridian of longitude the half great circle passing through the poles on which a place lies.

model of exponential decay a graph of a geometrical sequence that is decreasing at an exponential rate and converging towards 0 if $0 < r < 1$.

model of exponential growth a graph of a geometrical sequence that is increasing at an exponential rate if $r > 1$.

model of linear decay a graph that has a negative gradient and the common difference is negative.

model of linear growth a graph that has a positive gradient and the common difference is positive.

moving average a way of smoothing data where the averages are taken of each observation and the observations closest to it to form a moving average table.

network an arrangement of interconnecting lines which shows the pathways between points.

network flow networks that are used to analyse the flow of things such as traffic or water.

nominal data categorical data that has no natural order or ranking.

nominal interest rate the rate of interest before adjusting for the effect of compounding.

numerical data data that can be counted or measured.

observation information or analysis needed for solving a problem or to further develop a model.

ordinal data categorical data that can be placed into a natural order or ranking.

outflow the minimum value obtained from a vertex when the inflow is compared to the sum of the capacities of all the edges that are leaving the vertex.

parallels of latitude small circles parallel to the equator

planar graph a graph that has no intersecting edges.

present value the current value of an amount of money that, if invested on the same terms, will produce the same future value as an investment.

Prim's algorithm a set of steps that can be used to identify the minimum spanning tree for a weighted connected graph.

qualitative values or statements that are non-numerical and refer to a quality or qualities.

quantitative data data that can be counted or measured, often referred to as numerical data.

recurrence method a technique used to create an arithmetic sequence if the first term (t_1) and the common difference (d) are known.

recurrence relations a sequence where each term is found using the previous term.

recursive method a technique used to create a geometric sequence if the first term (t_1) and the common ratio (r) are known.

reducing balance loan a loan where the interest is calculated based on the amount of money owing on the loan at that time, instead of the initial amount that was borrowed.

residual plot a graph where the residuals are plotted on the vertical axis.

residuals the lengths of the vertical lines joining the data points to the regression line.

response variable the variable that is being tested and responds to changes in the explanatory variable. It is typically placed on the y-axis of a scatterplot.

scrap value once an asset reaches a certain value, or after a certain amount of time, it is considered to be no longer of any worth.

seasonal index a comparison between a particular season and the average season, where the average value of a seasonal index is 1. It can be calculated using $\text{seasonal index} = \dfrac{\text{data value}}{\text{seasonal average}}$

seasonal pattern fluctuations that occur systematically and are calendar related. They will usually repeat every week, month or quarter.

semi-Eulerian graph a connected graph where you can start at a vertex and move along each edge only once without being able to return to the vertex at which you began.

semi-Eulerian trail the trail that is created when you travel around a semi-Eulerian graph.

semi-Hamiltonian cycle the trail that is created when you travel around a semi-Hamiltonian graph.

semi-Hamiltonian graph a graph where you can start at a vertex and travel to each vertex only once without being able to return to the vertex at which you began.

simple graph or network where pairs of vertices are connected by one edge at most.

simple interest interest that is calculated as percentage of the initial investment or loan, when money has been borrowed or invested.

sink the finishing vertex or vertices of a network flow.

small circles any circle drawn around a sphere that is smaller than a great circle.

smoothing process the action of smoothing data in order to see underlying trends.

source the starting vertex or vertices of a network flow.

spanning trees graphs that are formed from part of a larger graph that include all the vertices of the original graph.

time series data data that has time as the explanatory variable.

tree a connected graph with no possible circuits.

trend the general long-term direction that a graph slopes over time.

undirected graph a graph where it is possible to move along the edges in either direction.

variable quantity that can take on a range of values depending on its relationship to other values; typically represented by pronumerals.

vertices points on a graph that are joined by edges.

weighted graphs graphs that have values attached to the edges.

INDEX

Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
1	7			1
2	4			1
3	5			1
4	3			2
5	3			3
6	2			3
7	3			3
8	3			3
9	3			3
10	6			4
11	6			4
12		6		3
13		6		3
14		3		3
15			5	1
16			10	1
Totals	**45**	**15**	**15**	
Percentage	**60%**	**20%**	**20%**	

Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
1	6			1
2	4			1
3	3			1
4	3			1
5	4			1
6	2			1
7	4			2
8	8			2
9	3			3
10	3			3
11		7		1
12		3		1
13			6	2
14			7	3
Totals	**40**	**10**	**13**	
Percentage	**60%**	**20%**	**20%**	