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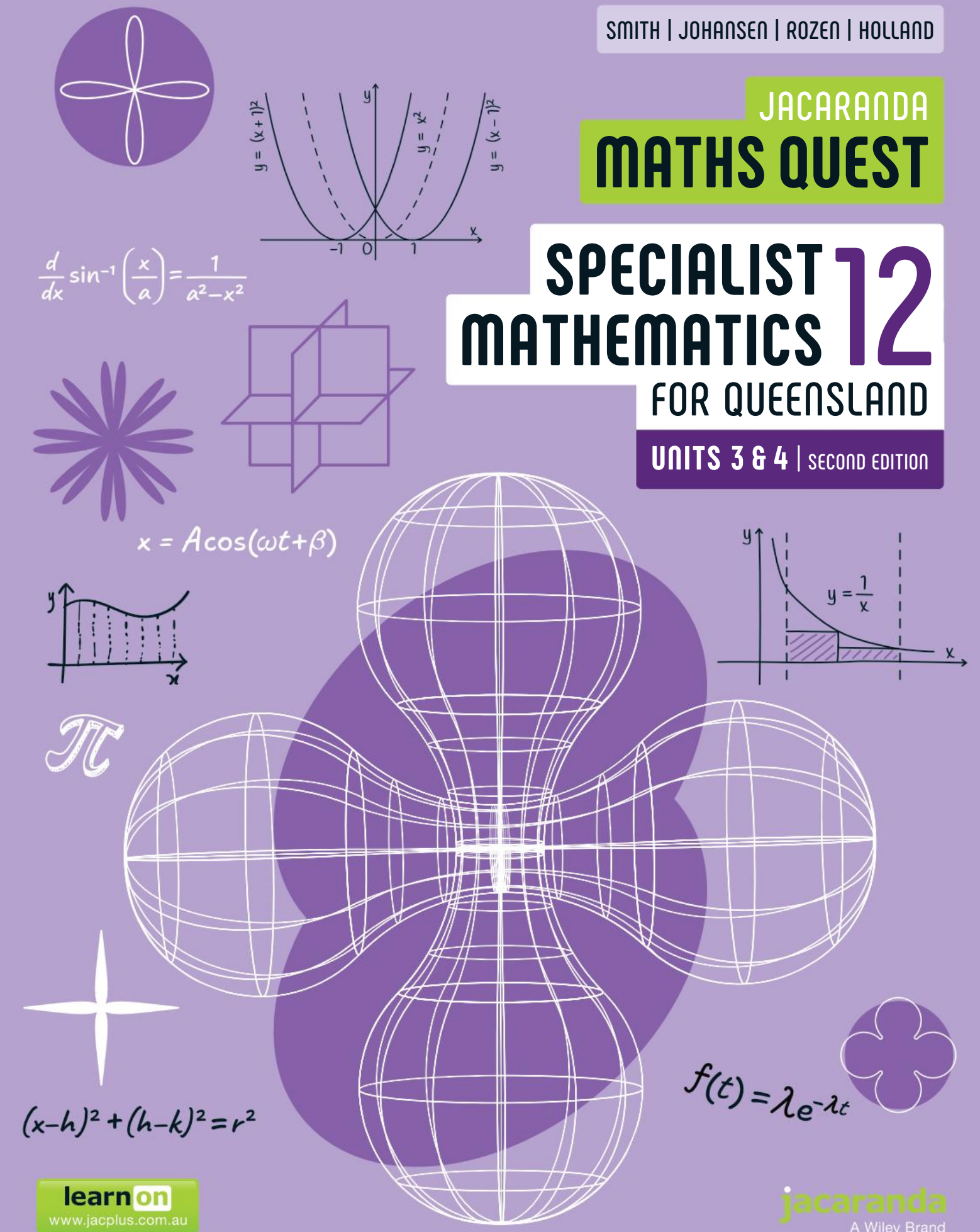
JACARANDA

MATHS QUEST

SPECIALIST MATHEMATICS 12

FOR QUEENSLAND

UNITS 3 & 4 | SECOND EDITION



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Second edition published 2025 by
John Wiley & Sons Australia, Ltd
Level 4, 600 Bourke Street, Melbourne, Vic 3000

First edition published 2020

Typeset in 10.5/13 pt TimesLTStd

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ISBN: 978-1-394-31777-6

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This suite of resources may include references to (including names, images, footage or voices of) people of Aboriginal and/or Torres Strait Islander heritage who are deceased. These images and references have been included to help Australian students from all cultural backgrounds develop a better understanding of Aboriginal and Torres Strait Islander Peoples' history, culture and lived experience.

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The publisher acknowledges ongoing discussions related to gender-based population data. At the time of publishing, there was insufficient data available to allow for the meaningful analysis of trends and patterns to broaden our discussion of demographics beyond male and female gender identification.

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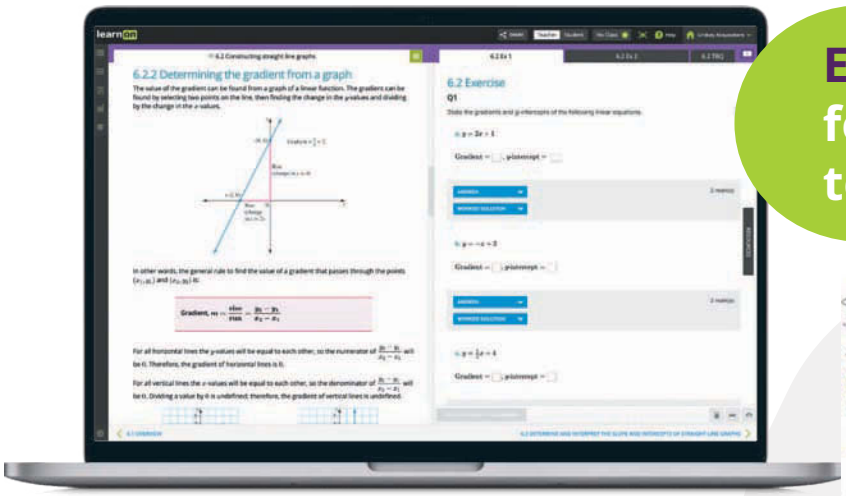
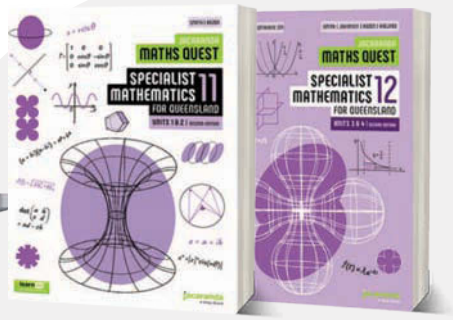
PRACTICE ASSESSMENT 4
Units 3 & 4 Examination

Glossary ..... online only

Learning with learnON



Everything you need for your students to succeed



JACARANDA MATHS QUEST

SPECIALIST MATHEMATICS 12

UNITS 3 AND 4 FOR QUEENSLAND
SECOND EDITION

Developed by expert teachers for students

Tried, tested and trusted. The completely revised and updated second edition of *Jacaranda Maths Quest 12 Specialist Mathematics Units 3 & 4 for Queensland* continues to focus on helping teachers achieve learning success for every student — ensuring no student is left behind and no student is held back.

Because both what and how students learn matter

Learning is personal

Whether students need a challenge or a helping hand, you'll find what you need to create engaging lessons.

Whether in class or at home, students can get unstuck and progress! Scaffolded lessons include detailed worked examples with both TI and Casio calculator support. Differentiated question sets are all supported by detailed worked solutions. And brand-new exam-style questions and Past QCAA exam questions to support in-depth skill acquisition.

Learning is effortful

Learning happens when students push themselves. With learnON, Australia's most powerful online learning platform, students can challenge themselves, build confidence and ultimately achieve success.

Learning is rewarding

Through real-time results data, students can track and monitor their own progress and easily identify areas of strength and weakness.

And for teachers, Learning Analytics provide valuable insights to support student growth and drive informed intervention strategies.

Learn online with Australia's most

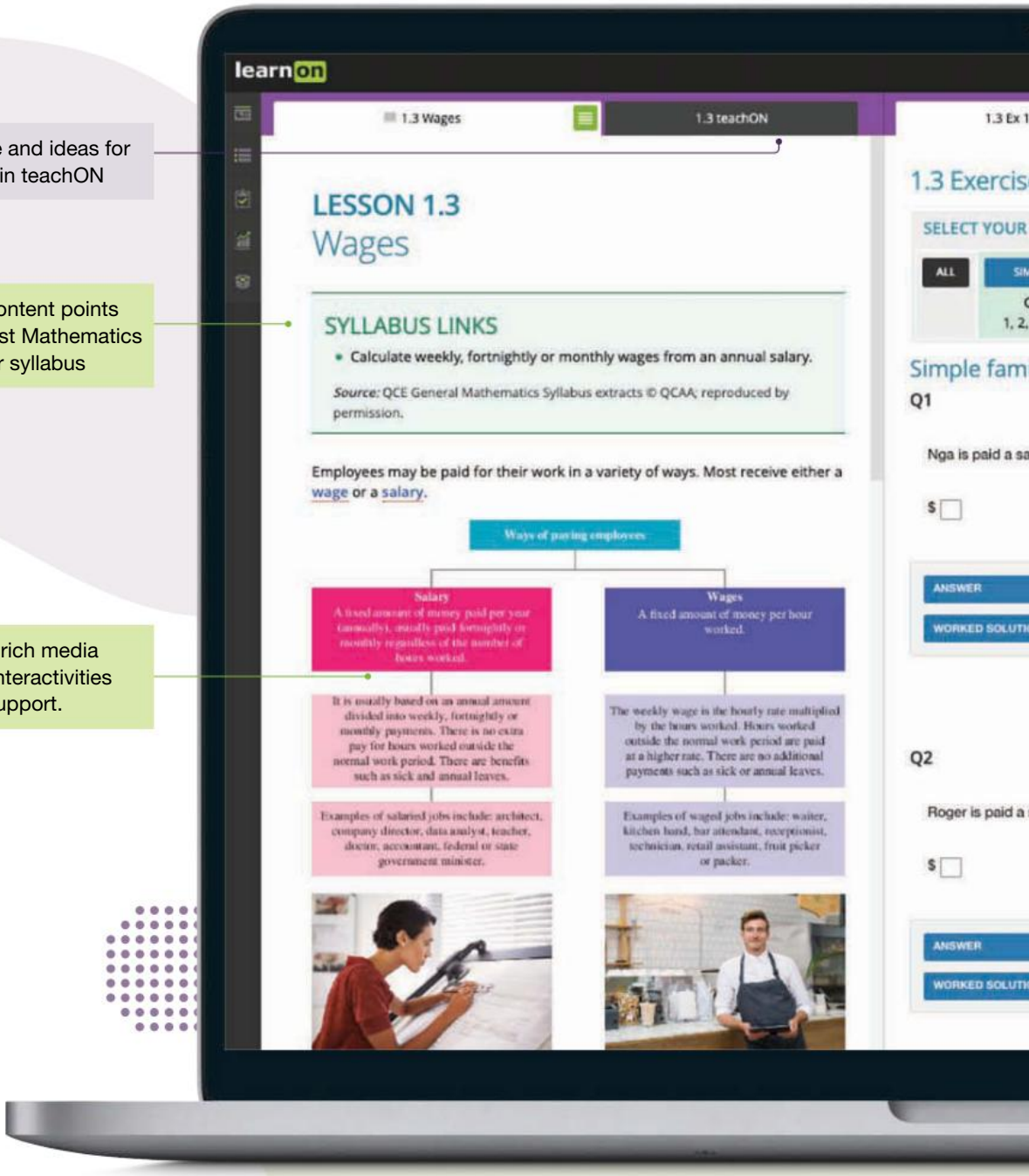
Everything you need for each of your lessons in one simple view

- **Trusted, syllabus-aligned theory**
- **Engaging, rich multimedia**
- **All the teacher support resources you need**
- **Deep insights into progress**
- **Immediate feedback for students**
- **Create custom assignments in just a few clicks.**

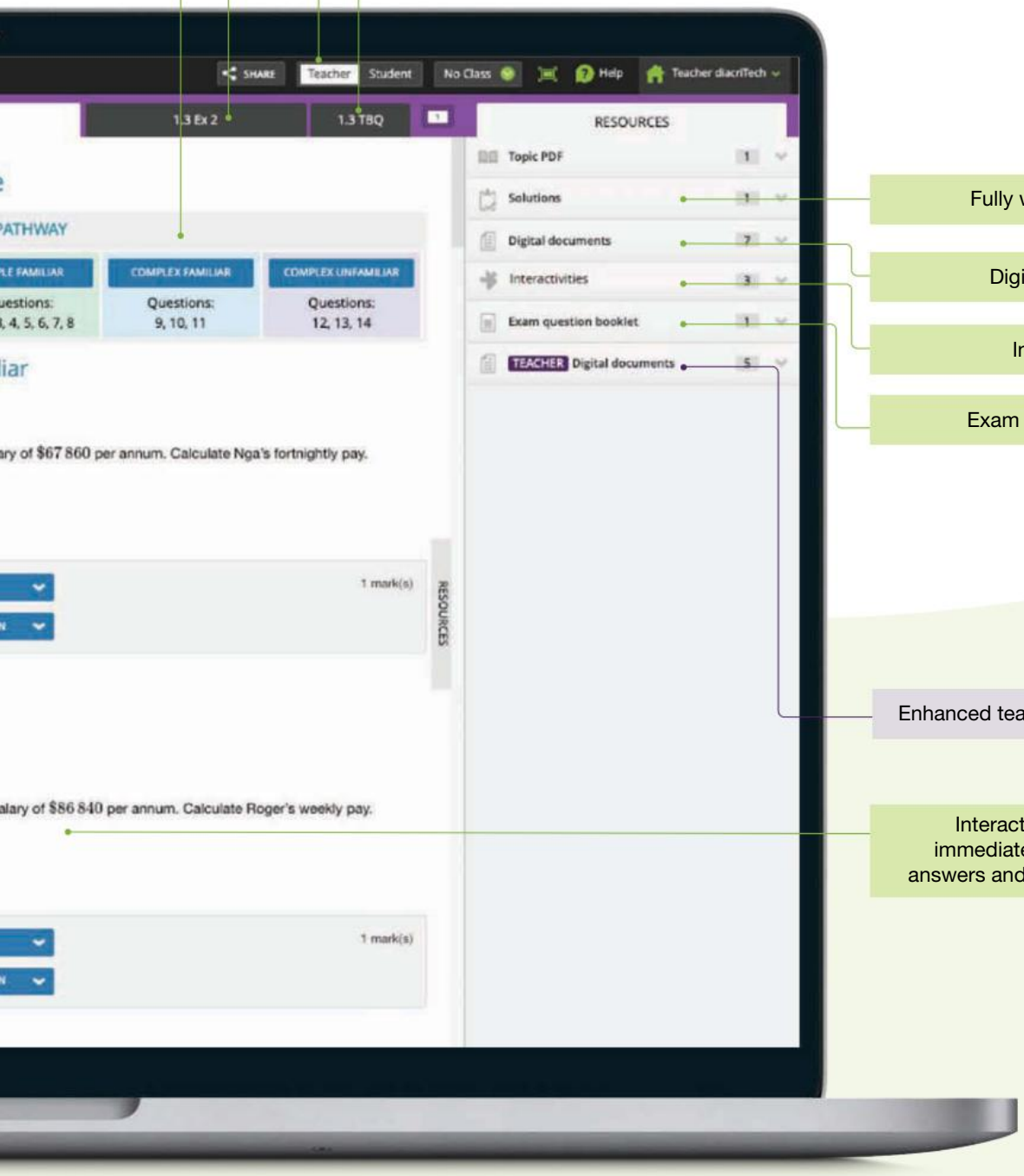
Practical teaching advice and ideas for each lesson provided in teachON

Each lesson linked to content points from the QCAA Specialist Mathematics 2025 General senior syllabus

Reading content and rich media including embedded interactivities and calculator support.



powerful learning tool, learnON



Differentiated question sets

Practice exam questions

Teacher and student views

Textbook questions

Fully worked solutions

Digital documents

Interactivities

Exam question booklet

Enhanced teaching support resources

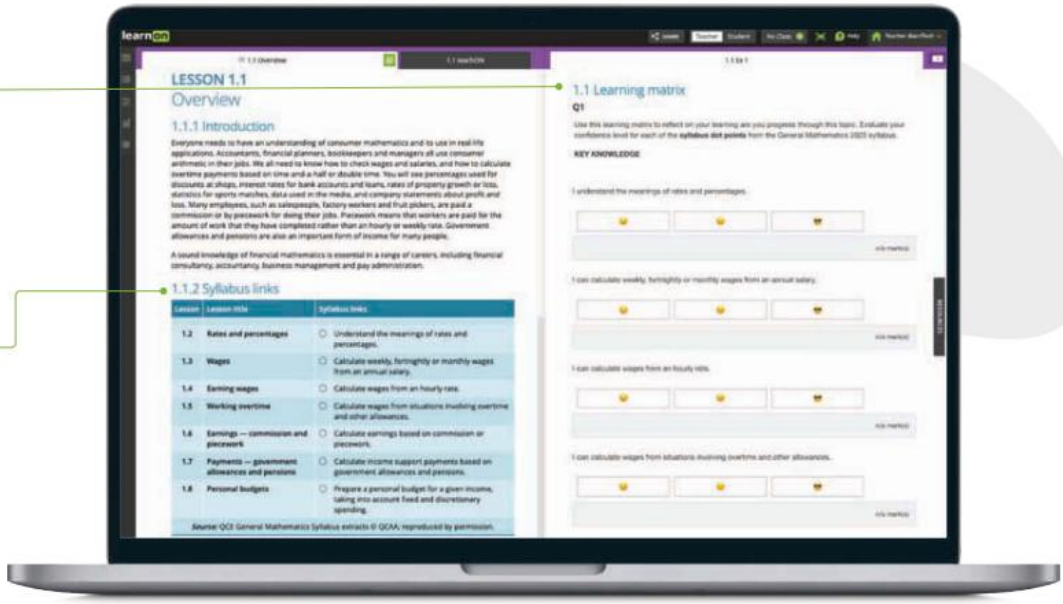
Interactive questions with immediate feedback including answers and fully worked solutions

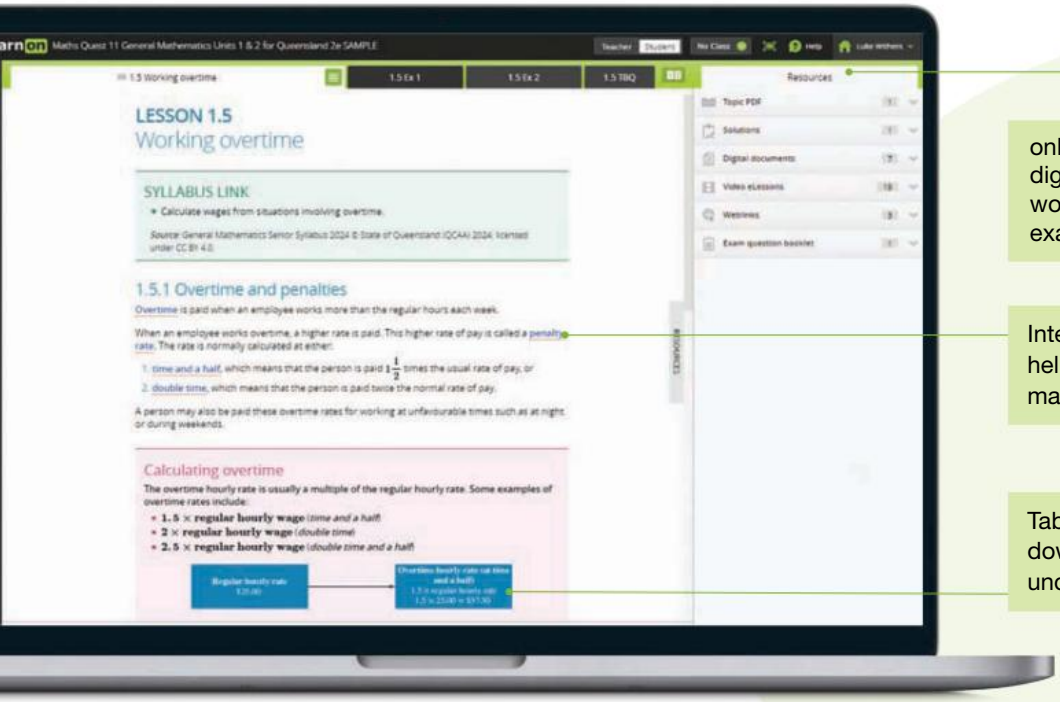
Online, these new editions are the complete package

Trusted Jacaranda theory, plus tools to support teaching and make learning more engaging, personalised and visible.

Learning matrix to monitor student's confidence level throughout topics.

Each topic is linked to content points from the QCAA Specialist Mathematics 2025 General senior syllabus.

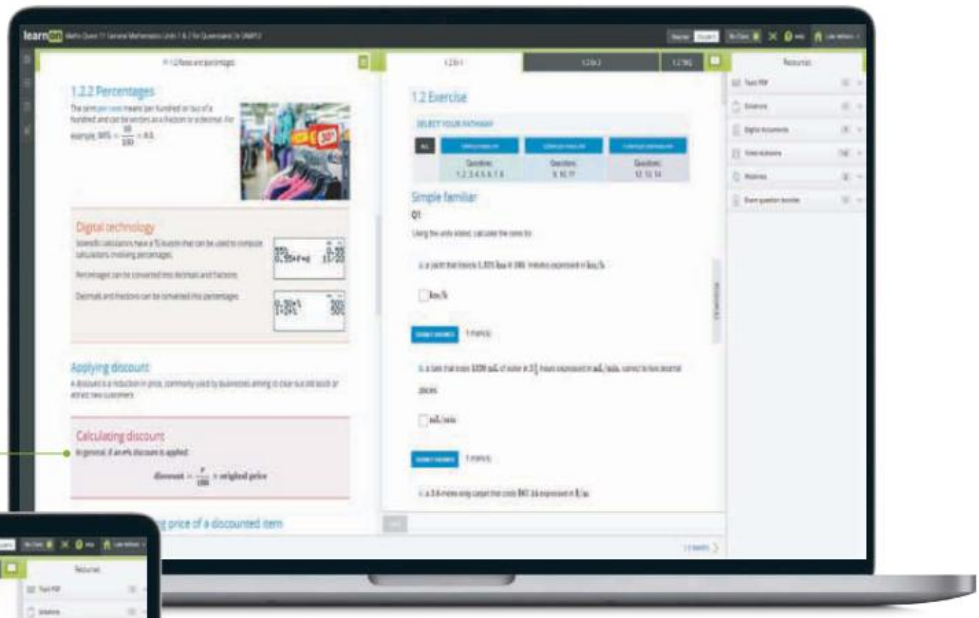




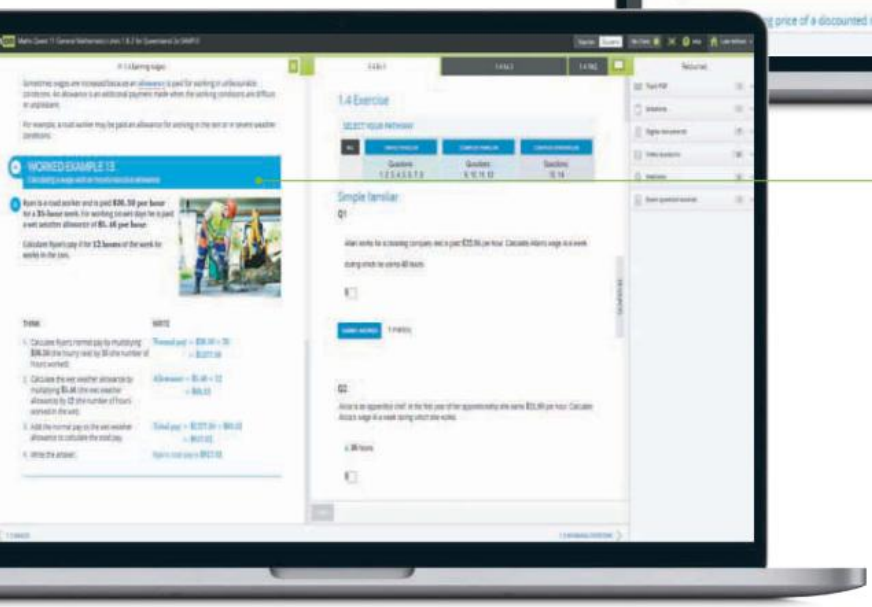
onResources link to targeted digital resources including fully worked solutions, interactivites and exam question booklets.

Interactive glossary terms help develop and support mathematical literacy.

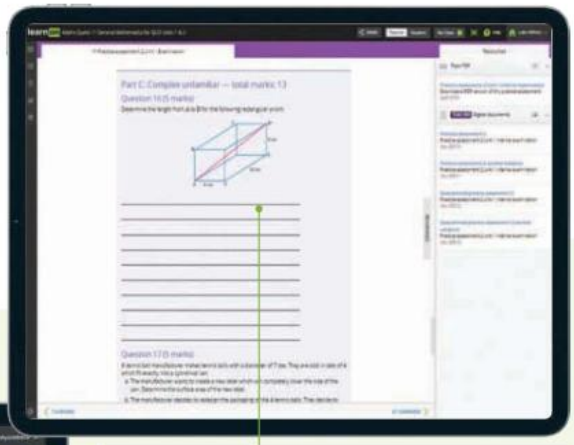
Tables, flow charts and images break down content, allowing students to understand complex concepts.



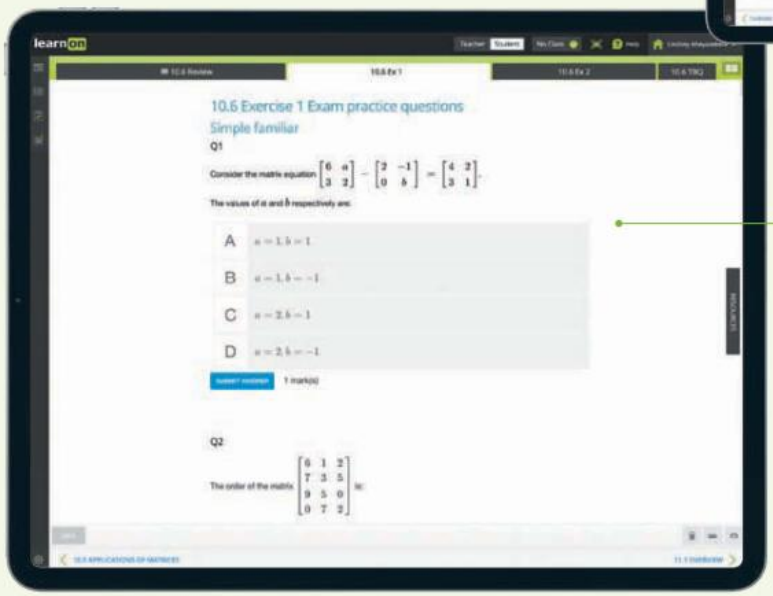
Pink highlight boxes summarise key information.



Worked examples break down the process of answering questions using a think/write format, including calculator support.



Each unit has practice assessments (Problem-solving and modelling task or examination) with sample responses to help build skills.



- Online and offline question sets contain practice questions and exam-style questions with exemplary responses and marking guides.
- Every question has immediate, corrective feedback to help students to overcome misconceptions as they occur and to study independently — in class and at home.

The new jacTUTOR

An AI tutor to build confidence and get unstuck

With jacTUTOR, every student can have the help of a personal tutor when they get stuck, in class or at home. This purpose-built tool sits safely within your favourite Jacaranda resource, so guidance will always be specific to that lesson, question and curriculum.

A personal tutor for every student

Now every student can get the help they need, when they need it. jacTUTOR uses AI to create a fair and level playing field for all students.

Get guidance, not the answer

jacTUTOR doesn't just give the answer away. Students are given prompts to help understand what they should be asking to get to a correct answer.

Combat anxiety

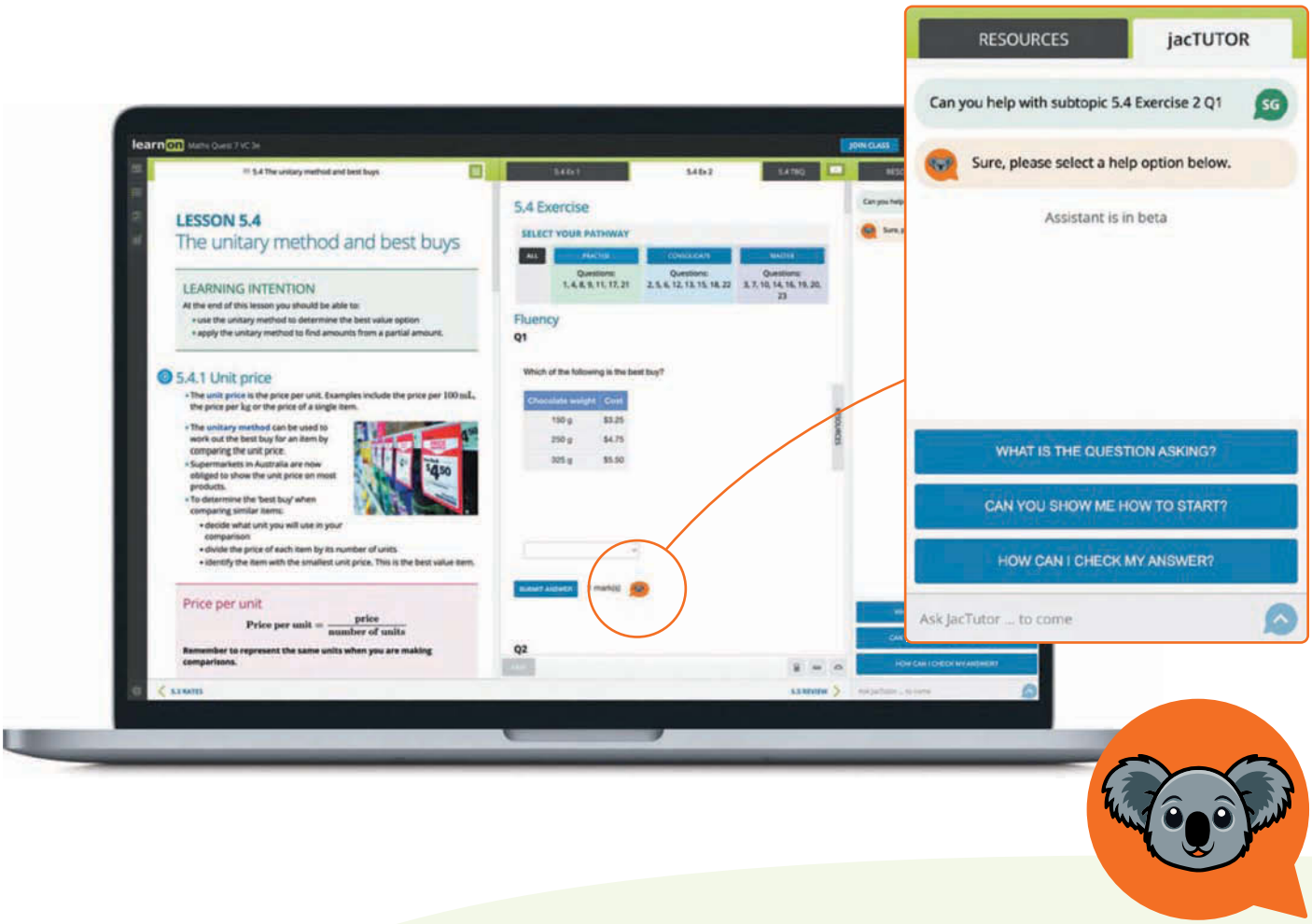
Afraid to raise a hand or ask more questions? With jacTUTOR, students will find a safe space to ask questions, get clarification and try again.

Consistency over confusion

There are so many ways to learn. To avoid confusion, jacTUTOR is built on the same trusted method used in the Jacaranda text.

A safe space

To keep students safe, any concerning or inappropriate comments are automatically flagged and sent to their teacher.



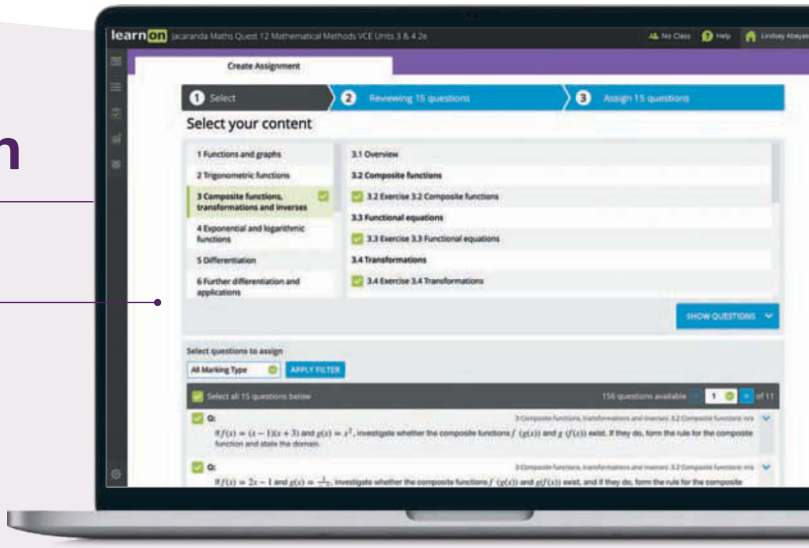
Teaching with learnON

Enhanced teacher support resources, including:

- work programs and syllabus grids
- teaching advice and additional activities
- quarantined topic tests (with solutions)
- Quarantined PSMTs and examinations Custom exam-builder with question differentiation (SF/CF/CU) question filters

Customise and assign

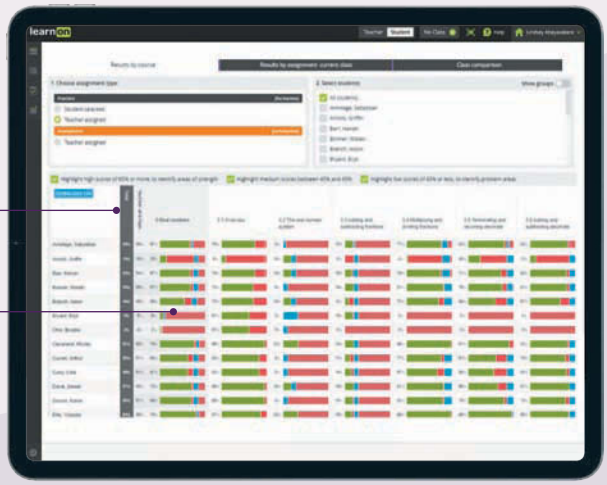
A testmaker enables you to create custom tests from the complete bank of thousands of questions (including past QCAA exam questions in year 12).



Reports and results

Data analytics and instant reports provide data-driven insights into performance across the entire course.

Show students (and their parents or carers) their own assessment data in fine detail. You can filter their results to identify areas of strength and weakness.



Acknowledgements

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UNIT 3

Further complex numbers, proof, vectors and matrices

TOPIC 1 **Further complex numbers**

1 Further complex numbers 3

TOPIC 2 **Mathematical induction and trigonometric proofs**

2 Mathematical induction and trigonometric proofs 69

TOPIC 3 **Vectors in two and three dimensions**

3 Vectors in two and three dimensions 101

TOPIC 4 **Vector calculus**

4 Vector calculus 203

TOPIC 5 **Further matrices**

5 Further matrices 291

PRACTICE ASSESSMENT 1 **Problem-solving and Modelling Task** ..... online only

PRACTICE ASSESSMENT 2 **Unit 3 Examination** ..... online only

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1 Further complex numbers

LESSON SEQUENCE

1.1 Overview 4
1.2 Review of complex numbers 5
1.3 Complex arithmetic using polar form 23
1.4 Roots of complex numbers 31
1.5 Factorisation of polynomials 43
1.6 Review 59
Answers 63

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 1
Exam questions	Exam question booklet — Chapter 1
Digital documents	Learning matrix — Chapter 1 Chapter summary — Chapter 1

LESSON
1.1 Overview

Hey students! Bring these pages to life online

Engage with interactivities | **Answer questions and check results** | **Track your progress**

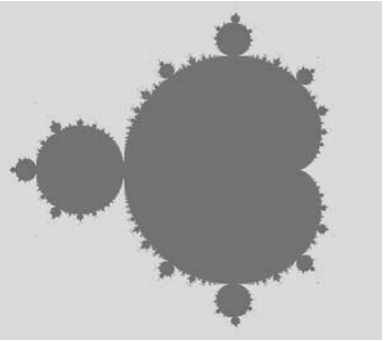
Find all this and MORE in jacPLUS

1.1.1 Introduction

Complex numbers have many real-world applications and are commonly used in engineering and physics. In mathematics, complex numbers are used to solve equations that have no real solutions. The introduction of the imaginary number i extends the set of real numbers, $\mathbb{R}$, to include another infinite set: the complex numbers, $\mathbb{C}$, producing a new field of mathematics.

This chapter begins with an optional review of the complex numbers subject matter covered in Unit 2, before extending complex arithmetic using polar form and investigating further the roots of complex numbers and the factorisation of higher order polynomials.

The beautiful Mandelbrot set shown in the image is generated using complex numbers.



1.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
1.2	**Review of complex numbers**	[Review of Unit 2 Topic 1 Complex numbers and Topic 2 Complex arithmetic and algebra]
1.3	**Complex arithmetic using polar form**	◯ Prove complex number identities involving modulus and argument, e.g. $z\bar{z} = \lvert z\rvert^2$, $\lvert z_1\rvert\lvert z_2\rvert = \lvert z_1 z_2\rvert$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. ◯ Use De Moivre's theorem for integral powers. • $z^n = r^n \operatorname{cis}(n\theta)$
1.4	**Roots of complex numbers**	◯ Determine and examine the nth roots of unity and their location on the unit circle. ◯ Determine and examine the nth roots of complex numbers and their location in the complex plane.
1.5	**Factorisation of polynomials**	◯ Apply the factor theorem and the remainder theorem for polynomials. ◯ Understand and use the complex conjugate root theorem for polynomials with real coefficients, e.g. factorise a cubic polynomial with real coefficients given one factor. ◯ Solve polynomial equations over $\mathbb{C}$ to order 4 including those with real and imaginary coefficients, e.g. solve $z^4 + z^3 - z^2 + z - 2 = 0$ and $z^3 - 2iz^2 + z - 2i = 0$.

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LESSON
1.2 Review of complex numbers

SYLLABUS LINKS

[Review of Unit 2 Topic 1 Complex numbers and Topic 2 Complex arithmetic and algebra]

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1.2.1 Definition of a complex number

The **complex number system** is an extension of the real number system that includes the **imaginary unit** i.

The imaginary unit

The imaginary unit i is defined by its property $i^2 = -1$.

A **complex number** (generally denoted by the letter z) is defined as a quantity consisting of a real number added to a multiple of the imaginary unit i. For real numbers a and b, $a + bi$ is a complex number. This is referred to as the standard or **Cartesian form**. $\mathbb{C} = \{z = a + bi, a, b \in \mathbb{R}\}$ defines the set of complex numbers.

Definition of a complex number

A complex number is of the form:

$$z = a + bi$$

where $a, b \in \mathbb{R}$.

$a = \text{Re}(z)$ is the real part of z and $b = \text{Im}(z)$ is the imaginary part of z.

A number is purely real:

$$z = a + 0i$$

A number is purely imaginary:

$$z = 0 + bi$$

TIP

When performing calculations without technology, it is useful to remember these patterns in the powers of i. $\forall n \in \mathbb{N}$:

$$\begin{array}{llll}
i^{4n} = 1 & (-i)^{4n} = 1 & \dfrac{1}{i^{4n}} = 1 & \dfrac{1}{(-i)^{4n}} = 1 \\
i^{4n+1} = i & (-i)^{4n+1} = -i & \dfrac{1}{i^{4n+1}} = -i & \dfrac{1}{(-i)^{4n+1}} = i \\
i^{4n+2} = -1 & (-i)^{4n+2} = -1 & \dfrac{1}{i^{4n+2}} = -1 & \dfrac{1}{(-i)^{4n+2}} = -1 \\
i^{4n+3} = -i & (-i)^{4n+3} = i & \dfrac{1}{i^{4n+3}} = i & \dfrac{1}{(-i)^{4n+3}} = -i
\end{array}$$

Remember the geometric interpretation of multiplying by i (rotation of $\frac{\pi}{2}$ anticlockwise about the origin) and dividing by i (rotation of $\frac{\pi}{2}$ clockwise about the origin) to help recall these patterns.

WORKED EXAMPLE 1 Determining the real and imaginary parts

Evaluate each of the following.

a. $\text{Re}(6-12i)$ **b.** $\text{Im}(3-i+i^3)$

THINK	WRITE
a. The real part of the complex number $6-12i$ is 6.	**a.** $\text{Re}(6-12i)=6$
b. 1. Simplify $3-i+i^3$ first by recalling that $i^2=-1$.	**b.** $\text{Im}(3-i+i^3)=\text{Im}(3-i+i^2\times i)$ $=\text{Im}(3-i-i)$ $=\text{Im}(3-2i)$
2. Look for the coefficient of i to determine the imaginary part of any complex number.	$\text{Im}(3-2i)=-2$

1.2.2 Basic operations on complex numbers

Complex numbers can be added, subtracted, multiplied and divided. In general, the solutions obtained when performing these operations are presented in the **standard form** $z=a+bi$.

Basic operations in Cartesian form

If $z=a+bi$, $w=c+di$, and $k\in\mathbb{R}$, then:

$$\begin{aligned} z+w &= (a+c)+(b+d)i \\ z-w &= (a-c)+(b-d)i \\ kz &= ka+kbi \\ zw &= (ac-bd)+(ad+bc)i \end{aligned}$$

WORKED EXAMPLE 2 Performing basic operations with and without technology

If $p=5-3i$ and $q=-2+7i$, evaluate each of the following without using a calculator.

a. $2p-q$ **b.** $p\times q$ **c.** $(p+q)^2$

THINK	WRITE
a. 1. Substitute for p and q.	**a.** $2p-q=2(5-3i)-(-2+7i)$
2. Multiply each part of p by 2.	$=10-6i+2-7i$
3. Simplify the real and imaginary parts.	$=12-13i$
b. Substitute for p and q.	**b.** $p\times q=(5-3i)\times(-2+7i)$
Use the Distributive Law to expand the brackets. Simplify the real and imaginary parts by recalling that $i^2=-1$.	$=-10+35i+6i-21i^2$ $=-10+41i-21\times-1$ $=11+41i$

c. 1. Substitute for p and q.

2. Simplify the real and imaginary parts.

3. Expand by recalling $(a+b)^2 = a^2 + 2ab + b^2$.

4. Simplify $9 + 24i + 16i^2$ by recalling that $i^2 = -1$.

c. $(p+q)^2 = ((5-3i)+(-2+7i))^2$

$= (3+4i)^2$

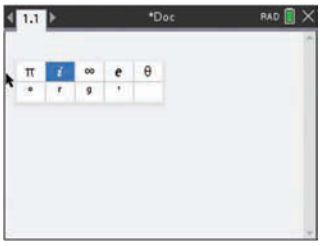
$= 9 + 24i + 16i^2$

$= 9 + 24i - 16$

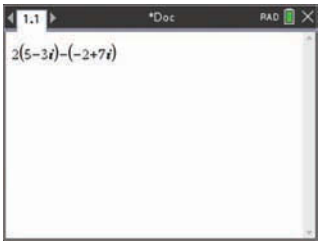
$= -7 + 24i$

TI | THINK **WRITE**

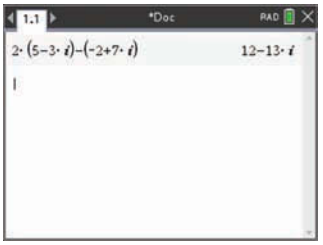
a. 1. *Note:* i can be entered by pressing the $\pi \triangleright$ button to open a menu.



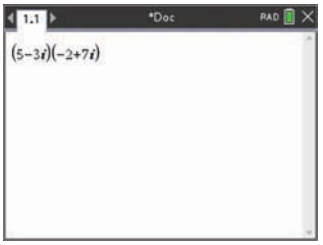
2. On a Calculator page, complete the entry line as: $2(5-3i) - (-2+7i)$



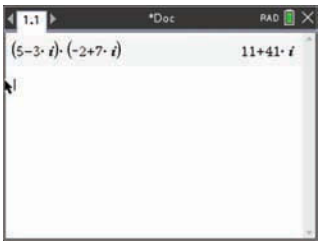
3. Press the ENTER button.
The answer appears on the screen.



b. 1. On a Calculator page, complete the entry line as: $(5-3i)(-2+7i)$

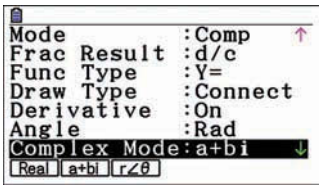


2. Press the ENTER button.
The answer appears on the screen.

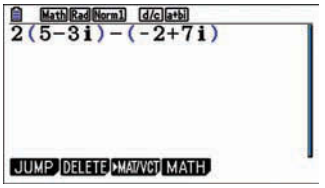


CASIO | THINK **WRITE**

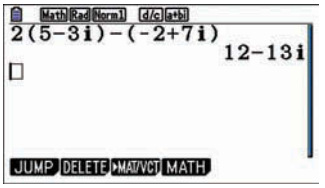
a. 1. On a Run-Matrix screen, select SHIFT MENU Complex Mode $a + bi$ Once this has been set, press the EXIT button.



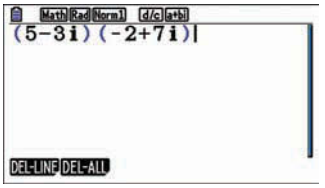
2. On a Run-Matrix screen, complete the entry line as: $2(5-3i) - (-2+7i)$
Note: i can be entered by selecting SHIFT followed by 0.



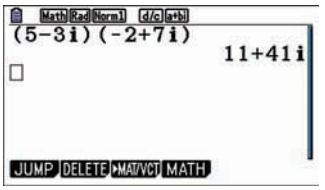
3. Press the EXE button.
The answer appears on the screen.



b. 1. On a Run-Matrix screen, complete the entry line as: $(5-3i)(-2+7i)$
Note: i can be entered by selecting SHIFT followed by 0.



2. Press the EXE button.
The answer appears on the screen.



Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Equality of complex numbers

If $z = a + bi$ and $w = \alpha + \beta i$, then:

$$z = w \Leftrightarrow \begin{cases} a = \alpha \\ b = \beta \end{cases}$$

1.2.3 Complex conjugates

The **conjugate of a complex number** is obtained by changing the sign of *only* the imaginary component.

The conjugate of a complex number

For a complex number $z = a + bi$, the conjugate is:

$$\bar{z} = a - bi$$

Multiplication of a complex number and its conjugate results in a real number:

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2 \end{aligned}$$

WORKED EXAMPLE 3 Determining the conjugate of a complex number

Determine the complex conjugate of each of the following complex numbers.

a. $v = 12 - 6i$ **b.** $w = -3 + \sqrt{6}i$

THINK	WRITE
a. Change the sign of the imaginary component.	$\bar{v} = 12 + 6i$
b. Change the sign of the imaginary component.	$\bar{w} = -3 - \sqrt{6}i$

The conjugates of complex numbers are useful for divisions of complex numbers.

Dividing complex numbers

To express the quotient of two complex numbers $\frac{z}{w}$ in the form $a + bi$, multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{z}{w} = \frac{z}{w} \times \frac{\bar{w}}{\bar{w}}$$

For any non-zero complex number z, its **multiplicative inverse** $z^{-1} = \frac{1}{z}$ is such that $zz^{-1} = 1$.

Recall that z^{-1} can be expressed in Cartesian form by multiplying the numerator and denominator by $\bar{z}$:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{a+bi}\\ &= \frac{1}{a+bi}\times\frac{\bar{z}}{\bar{z}}\\ &= \frac{1}{a+bi}\times\frac{a-bi}{a+bi}\\ &= \frac{a-bi}{a^2+b^2}\end{aligned}$$

Multiplicative inverse of a complex number

If $z = a + bi$, then:

$$z^{-1} = \frac{a-bi}{a^2+b^2}$$

WORKED EXAMPLE 4 Calculating the inverse of a complex number, with and without technology

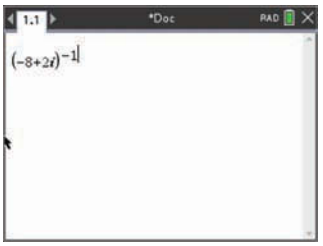
Determine z^{-1} given $z = -8 + 2i$.

THINK	WRITE
1. Write z^{-1} as a rational expression: $z^{-1} = \frac{1}{z}$.	$z^{-1} = \frac{1}{z} = \frac{1}{-8+2i}$
2. Multiply the numerator and denominator by the complex conjugate of $-8 + 2i$.	$= \frac{1}{-8+2i}\times\frac{-8-2i}{-8-2i}$ $= \frac{-8-2i}{64-4i^2}$ $= \frac{-8-2i}{68}$
3. Express in standard form, $a + bi$.	$= -\frac{2}{17} - \frac{1}{34}i$

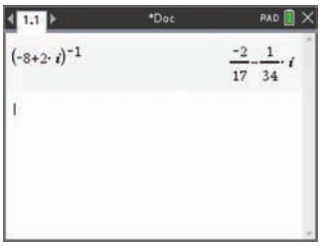
TI | THINK

1. On a Calculator page, complete the entry line as: $(-8 + 2i)^{-1}$

WRITE



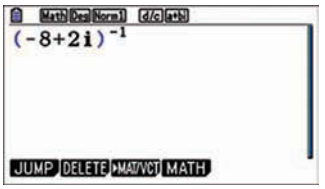
2. Press the ENTER button. The answer appears on the screen.



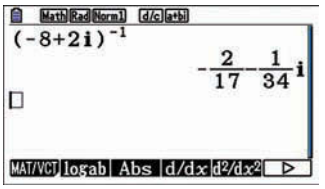
CASIO | THINK

1. On a Run-Matrix screen, complete the entry line as: $(-8 + 2i)^{-1}$
Note: i can be entered by selecting SHIFT followed by 0.

WRITE



2. Press the EXE button. The answer appears on the screen.



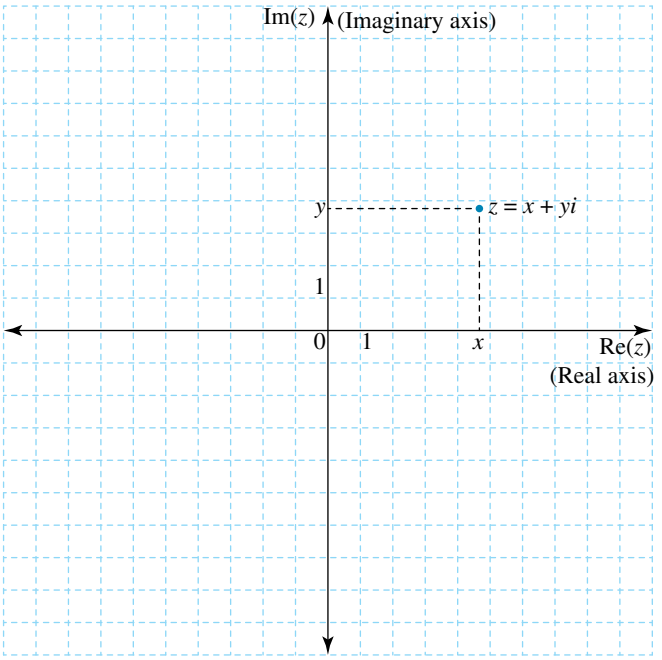
WORKED EXAMPLE 5 Expressing the division of complex numbers in Cartesian form

Express $\dfrac{1-3i}{4+i}$ in the form $a+bi$.

THINK	WRITE
1. Multiply both the numerator and denominator by the complex conjugate of $4+i$.	$\dfrac{1-3i}{4+i}\times\dfrac{4-i}{4-i}$
2. Expand and simplify the expressions obtained in the numerator and denominator.	$=\dfrac{4-i-12i+3i^2}{4^2-4i+4i-i^2}$ $=\dfrac{1-13i}{17}$
3. Express in standard form, $a+bi$.	$=\dfrac{1}{17}-\dfrac{13}{17}i$

1.2.4 The Argand plane and geometric interpretation of complex numbers

The **Argand plane** allows geometric interpretation of complex numbers. The horizontal axis of the Argand plane is referred to as the real axis and the vertical axis is referred to as the imaginary axis. A complex number $z=x+yi$ can be plotted on the Argand plane as the point (x, y).



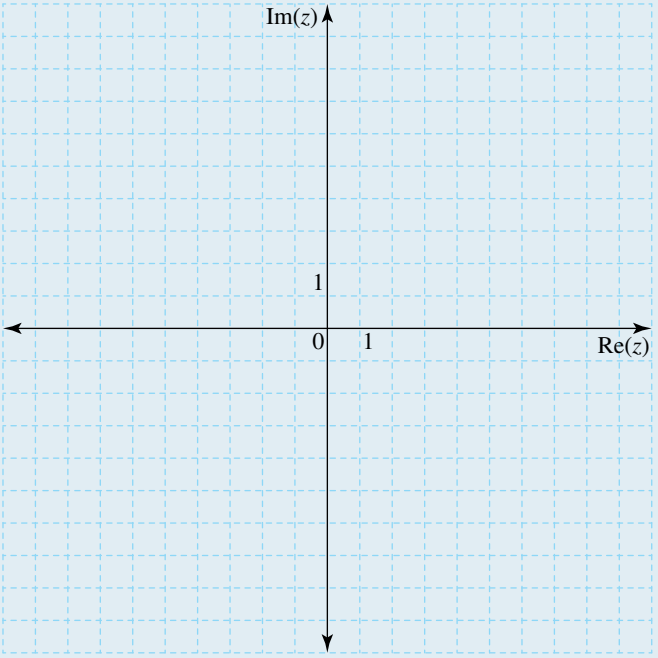
WORKED EXAMPLE 6 Representing complex numbers on the Argand plane

Represent the complex numbers $z_1 = 5 + 7i$, $z_2 = -6 + 9i$, $z_3 = -4 - 5i$ and $z_4 = 8 - 3i$ on the same Argand plane.

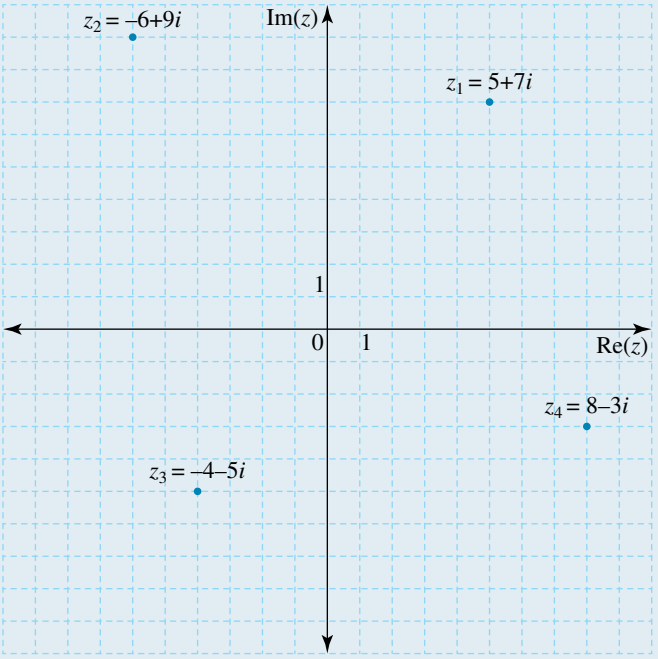
THINK

1. Draw an Argand plane.

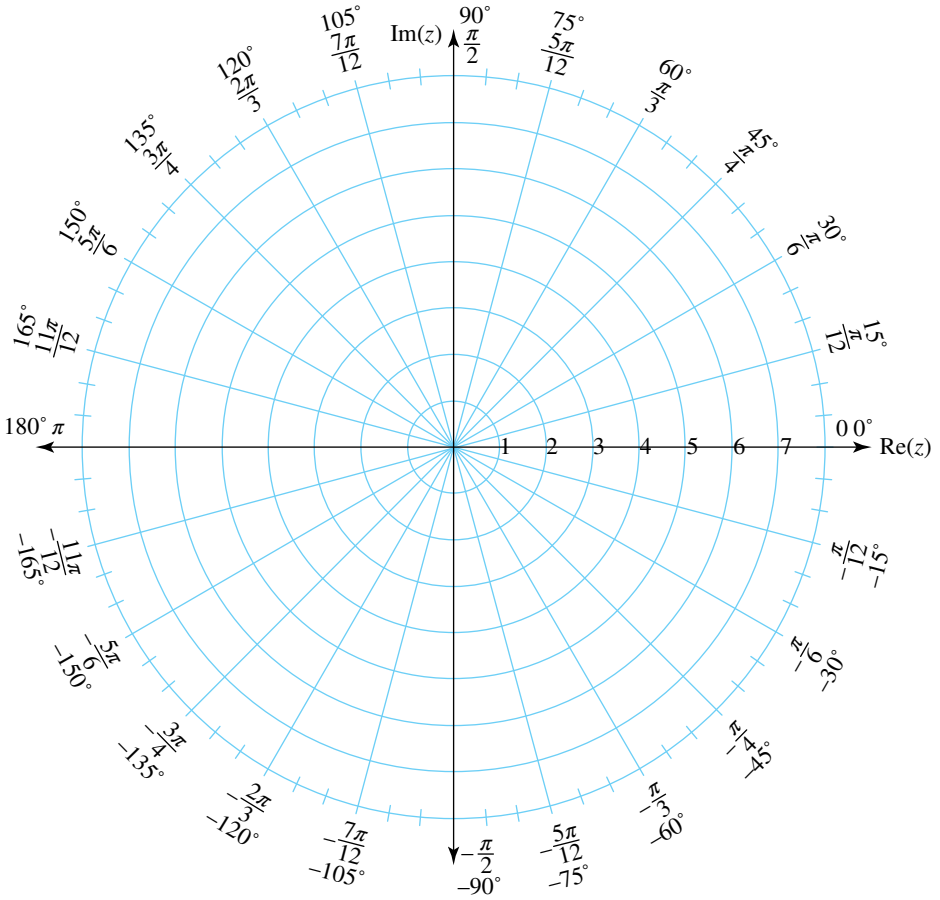
WRITE



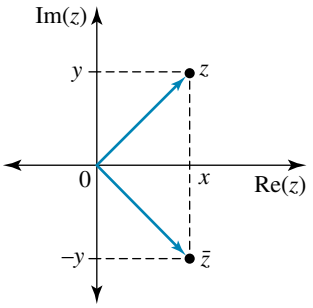
2. Recall that a complex number in the form $z = x + yi$ be plotted on the Argand plane as the point (x, y). Thus, $z_1 = 5 + 7i$ can be plotted as $(5, 7)$, $z_2 = -6 + 9i$ can be plotted as $(-6, 9)$, $z_3 = -4 - 5i$ can be plotted as $(-4, -5)$, and $z_4 = 8 - 3i$ can be plotted as $(8, -3)$.



Note that when representing complex numbers on the Argand plane, it can be convenient to use their polar form and a polar grid instead, in which the smallest circle has a radius of 1.



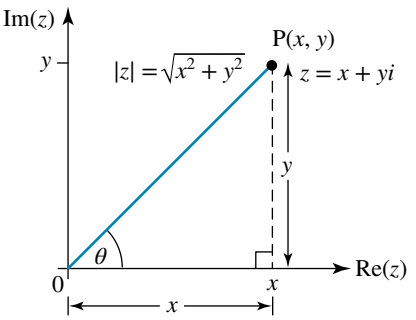
The conjugate, $\bar{z}$, of a complex number z is the reflection of z in the real axis.



1.2.5 Complex arithmetic using polar form

The magnitude or **modulus** of the complex number $z = x + yi$ is the length of the line segment joining the origin to the point z.

The **argument** of the complex number $z = x + yi$ is the angle measurement anticlockwise from the positive real axis.



Modulus and argument of a complex number

The modulus of a complex number $z = x + yi$ is:

$$|z| = \sqrt{x^2 + y^2}$$

The argument of a complex number $z = x + yi$ is:

$$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right), x \neq 0$$

The principal argument of a complex number $z = x + yi$ is:

$$\text{Arg}(z) = \theta, \tan\theta = \left(\frac{y}{x}\right), -\pi < \theta \leq \pi, x \neq 0$$

The modulus and argument of a complex number are used to express it in **polar form**.

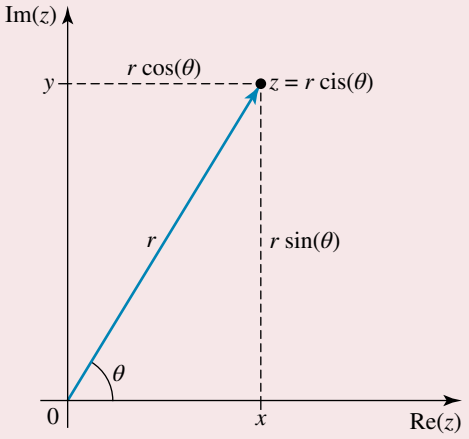
Polar form of a complex number

The polar form of a complex number $z = x + yi$ is:

$$z = r\,\text{cis}(\theta) = r(\cos(\theta) + i\sin(\theta))$$

where $r = |z| = \sqrt{x^2 + y^2}$ and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), -\pi < \theta \leq \pi, x \neq 0.$$



WORKED EXAMPLE 7 Converting between Cartesian and polar forms, with and without technology

Use your knowledge of the Cartesian and polar forms of a complex number to answer the following.

a. **Express $2 - 2\sqrt{3}\,i$ in polar form.**

b. **Express $6\,\text{cis}\left(\frac{\pi}{4}\right)$ in Cartesian form.**

THINK

a. 1. To express a complex number in polar form, r and θ are required for $r\,\text{cis}(\theta)$. Recall that $r = \sqrt{a^2 + b^2}$.

WRITE

a. $r = \sqrt{(2)^2 + \left(-2\sqrt{3}\right)^2}$
$= \sqrt{4 + 12}$
$= 4$

2. Calculate θ in radians. Recall $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ for $a + bi$.

$$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right)$$
$$= -\frac{\pi}{3}$$

3. Express $2 - 2\sqrt{3}i$ in polar form, that is, in the form $r\,\text{cis}(\theta)$.

$$2 - 2\sqrt{3}i = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$$

b. 1. Identify r and θ.

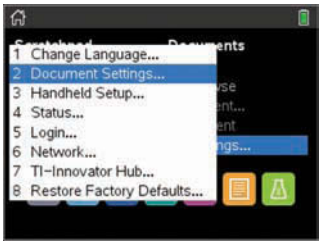
b. For $6\,\text{cis}\left(\frac{\pi}{4}\right)$: $r = 6$ and $\theta = \frac{\pi}{4}$

2. To convert into Cartesian form, use the relationship $z = r\cos(\theta) + r\sin(\theta)i$.

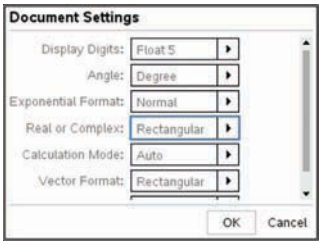
$$6\,\text{cis}\left(\frac{\pi}{4}\right) = 6\cos\left(\frac{\pi}{4}\right) + 6\sin\left(\frac{\pi}{4}\right)i$$
$$= 6 \times \frac{\sqrt{2}}{2} + 6i \times \frac{\sqrt{2}}{2}$$
$$= 3\sqrt{2} + 3\sqrt{2}i \text{ or } 3\sqrt{2}(1 + i)$$

TI | THINK **WRITE**

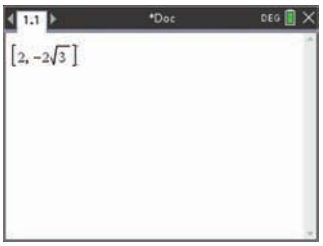
a. 1. On a Home page, select:
5 Settings
2: Document Settings



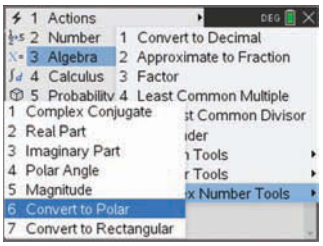
2. Set the calculator to the appropriate format by completing the entry lines as:
Angle: Degree
Real or Complex: Rectangular



3. On a Calculator page, complete the entry line as: $\left[2, -2\sqrt{3}\right]$

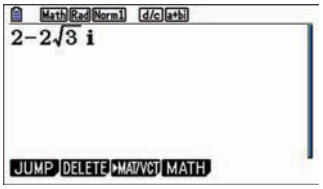


4. To convert the complex number to polar form, select:
MENU
2: Number
9: Complex Number Tools
6: Convert to Polar

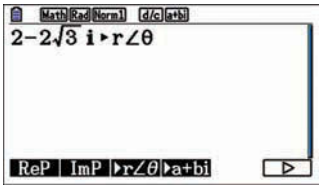


CASIO | THINK **WRITE**

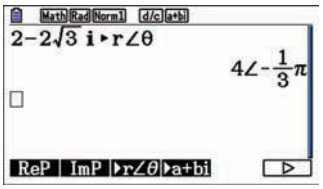
a. 1. On a Run-Matrix screen, complete the entry line as:
$2 - 2\sqrt{3}i$



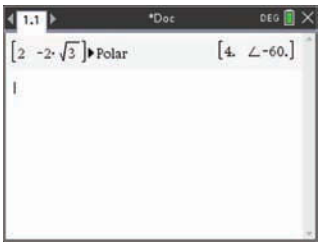
2. To convert the complex number into polar form, select:
OPTN
COMPLEX
$\triangleright r\angle\theta$



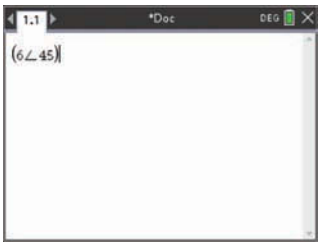
3. Press the EXE button. The answer appears on the screen.



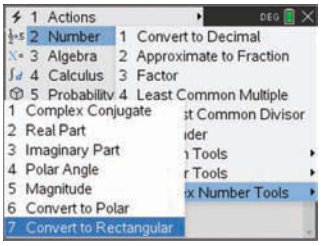
5. Press the ENTER button.
The answer appears on the screen.
Note: The answer states the angle in degrees $\left(-60° = -\frac{\pi}{3}\right)$.



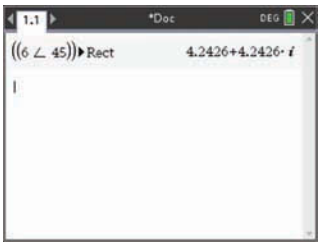
b. 1. On a Calculator page, complete the entry line as: (6 ∠ 45)
Note: The ∠ symbol can be entered by pressing CTRL .



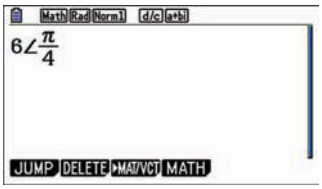
2. To convert a complex number from polar to Cartesian form, select:
MENU
2: Number
9: Complex Number Tools
7: Convert to Rectangular



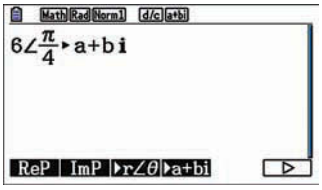
3. Press ENTER.
The answer appears on the screen.



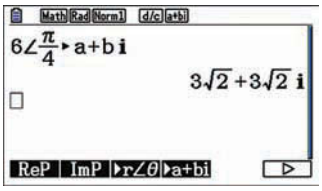
b. 1. On a Run-Matrix screen, complete the entry line as:
$6 \angle \frac{\pi}{4}$



2. To convert the complex number from polar to Cartesian form, select:
OPTN
COMPLEX
$\triangleright a + bi$



3. Press the EXE button.
The answer appears on the screen.



The polar form of complex numbers is convenient for multiplying and dividing complex numbers.

Multiplication and division of complex numbers in polar form

If $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$, then

$$z_1 \times z_2 = r_1 \times r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 + \theta_2)$$

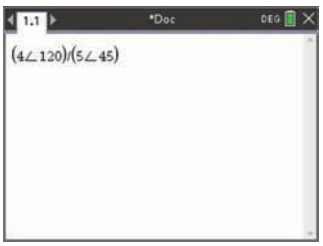
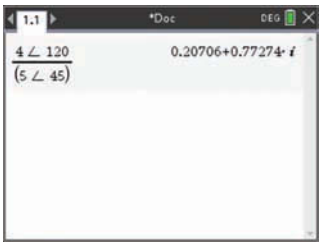
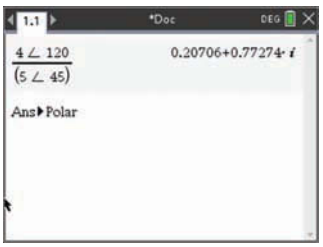
WORKED EXAMPLE 8 Multiplying and dividing complex numbers in polar form, with and without technology

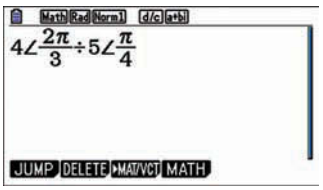
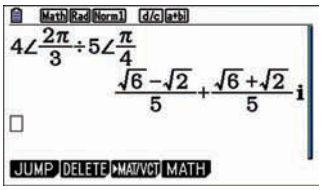
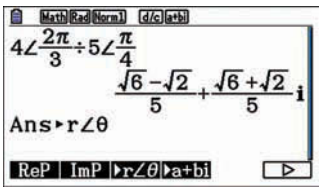
If $z_1 = 4 \text{ cis}\left(\frac{2\pi}{3}\right)$ and $z_2 = 5 \text{ cis}\left(\frac{\pi}{4}\right)$, determine:

a. $z_1 \times z_2$

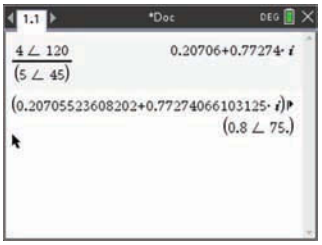
b. $\frac{z_1}{z_2}$

THINK	WRITE
a. 1. Simplify using the multiplication rule $z_1 \times z_2 = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$.	**a.** $z_1 \times z_2 = 4 \text{ cis}\left(\frac{2\pi}{3}\right) \times 5 \text{ cis}\left(\frac{\pi}{4}\right)$ $= 4 \times 5 \text{ cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$
2. The argument lies within $-\pi < \theta \leq \pi$, so it does not need to be simplified. Express the answer in the required format.	$= 20 \text{ cis}\left(\frac{11\pi}{12}\right)$
b. 1. Simplify using the rule $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis}(\theta_1 - \theta_2)$.	**b.** $\frac{z_1}{z_2} = \frac{4 \text{ cis}\left(\frac{2\pi}{3}\right)}{5 \text{ cis}\left(\frac{\pi}{4}\right)}$
2. The argument lies within $-\pi\ \theta \leq \pi$, so it does not need to be simplified. Express the answer in the required format.	$= \frac{4}{5} \text{ cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$ $= \frac{4}{5} \text{ cis}\left(\frac{5\pi}{12}\right)$

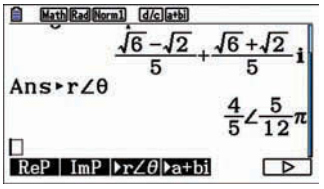
TI | THINK	WRITE
b. 1. On a Calculator page, complete the entry line as: (4∠120) / (5∠45)	
2. Press the ENTER button. The answer appears on the screen in Cartesian form.	
3. To convert the answer to polar form, select: CTRL (−) MENU 2: Number 9: Complex Number Tools 6: Convert to Polar	

CASIO | THINK	WRITE
b. 1. On a Run-Matrix screen, complete the entry line as: $4\angle\frac{2\pi}{3} \div 5\angle\frac{\pi}{4}$	
2. Press the EXE button. The answer appears on the screen in Cartesian form.	
3. To convert the answer to polar form, select: SHIFT (−) OPTN COMPLEX $\triangleright r\angle\theta$	

4. Press the ENTER button.
The answer appears on the screen.



4. Press the EXE button.
The answer appears on the screen in Polar form.



When multiplying or dividing two complex numbers together, the resulting complex number is dilated and rotated.

Geometric interpretation of complex multiplication and division

If z_1 and z_2 are two complex numbers with $z_2 \neq 0$:

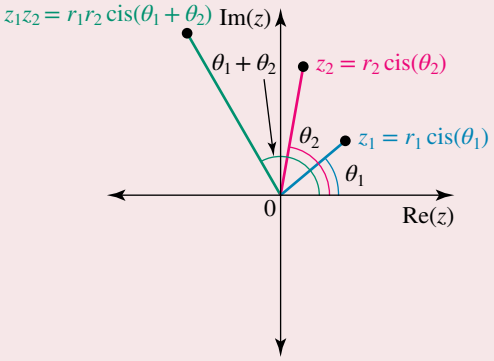
$$|z_1 \times z_2| = |z_1| \times |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

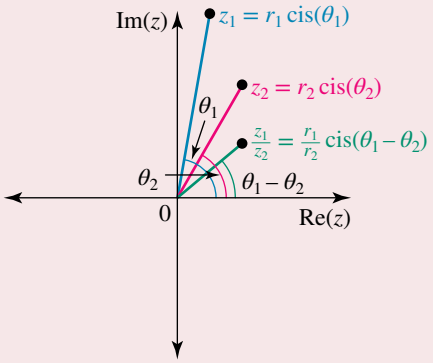
$$\arg(z_1 \times z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

When multiplying z_1 by z_2, z_1 is dilated by a factor $|z_2|$ and rotated by θ_2 anticlockwise about the origin.



When dividing z_1 by z_2, z_1 is dilated by a factor $\dfrac{1}{|z_2|}$ and rotated by θ_2 clockwise about the origin.



WORKED EXAMPLE 9 Interpreting geometrically the multiplication and division of complex numbers

Let $z_1 = -3 + 3i$ and $z_2 = 1 - \sqrt{3}i$.

a. Express z_1z_2 and $\frac{z_1}{z_2}$ in polar form.

b. Provide a geometric interpretation of z_1z_2 and $\frac{z_1}{z_2}$, and represent z_1, z_2, z_1z_2 and $\frac{z_1}{z_2}$ in the Argand plane.

THINK	WRITE
a. 1. Convert z_1 and z_2 from Cartesian form $z = x + yi$ to polar form $z = r\,\text{cis}(\theta)$ using $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $-\pi < \theta \le \pi, x \ne 0$.	**a.** $\lvert z_1\rvert = 3\sqrt{2}$ $\tan(\theta_1) = -1$ $\theta_1 = \frac{3\pi}{4}$ (2nd quadrant) $z_1 = 3\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$ $\lvert z_2\rvert = 2$ $\tan(\theta_2) = -\sqrt{3}$ $\theta_1 = -\frac{\pi}{3}$ (4th quadrant) $z_1 = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$
2. Calculate z_1z_2.	$z_1z_2 = 3\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right) \times 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ $= 6\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{3}\right)\right)$ $= 6\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$ $\text{LHS} = 6\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$
3. Calculate $\frac{z_1}{z_2}$.	$\frac{z_1}{z_2} = \frac{3\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)}{2\,\text{cis}\left(-\frac{\pi}{3}\right)}$ $= \frac{3\sqrt{2}}{2}\,\text{cis}\left(\frac{3\pi}{4} - \left(-\frac{\pi}{3}\right)\right)$ $= \frac{3\sqrt{2}}{2}\,\text{cis}\left(\frac{13\pi}{12}\right)$ $= \frac{3\sqrt{2}}{2}\,\text{cis}\left(-\frac{11\pi}{12}\right)$ $\text{LHS} = \frac{3\sqrt{2}}{2}\,\text{cis}\left(-\frac{11\pi}{12}\right)$
4. State the answer.	$z_1z_2 = 6\sqrt{2}\text{cis}\left(\frac{5\pi}{12}\right)$ and $\frac{z_1}{z_2} = \frac{3\sqrt{2}}{2}\text{cis}\left(-\frac{11\pi}{12}\right)$

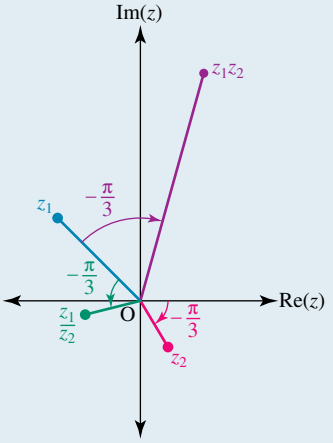
b. 1. Recall that multiplying or dividing two complex numbers together corresponds to a dilation and a rotation.

2. Represent z_1 and z_2 in the Argand plane, and use the geometric interpretation of z_1z_2 and $\frac{z_1}{z_2}$ to represent them in the Argand plane.

b. When multiplying z_1 by z_2, z_1 is dilated by a factor $|z_2| = 2$ and rotated by $\theta_2 = -\frac{\pi}{3}$ **anticlockwise** about the origin.

When dividing z_1 by z_2, z_1 is dilated by a factor $\frac{1}{|z_2|} = \frac{1}{2}$ and rotated by $\theta_2 = -\frac{\pi}{3}$ **clockwise** about the origin.

(Alternatively, when multiplying z_2 by z_1, z_2 is dilated by a factor $|z_1| = 3\sqrt{2}$ and rotated by $\theta_1 = \frac{3\pi}{4}$ anticlockwise about the origin.)



WORKED EXAMPLE 10 Representing dilation and rotation about the origin

Let $z = 4\,\text{cis}\left(\frac{\pi}{6}\right)$. The complex number z is rotated by $\frac{\pi}{4}$ clockwise about the origin and dilated by $\frac{1}{2}$, giving z'.

a. Represent z and z' on the Argand plane.

b. Express z' in polar form.

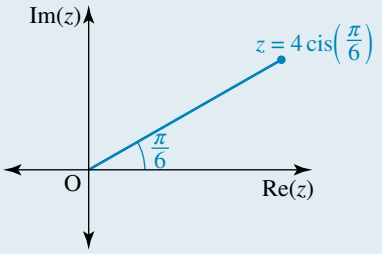
THINK

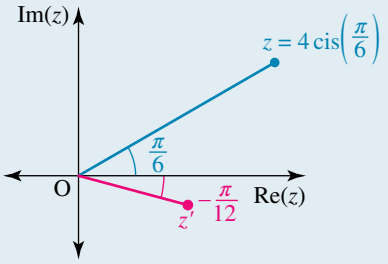
a. 1. The modulus of z is 4 and its principal argument is $\frac{\pi}{6}$.

Represent z on a circle of radius 4 centred on the origin, and at an angle of $\frac{\pi}{6}$ anticlockwise from the real axis.

2. The number z is rotated by $\frac{\pi}{4}$ clockwise about the origin and dilated by $\frac{1}{2}$. Thus, z' is on a circle of radius $\frac{1}{2} \times 4$ centred on the origin, at an angle $\frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$ to the real axis.

WRITE





b. 1. Recall that a rotation about the origin by θ clockwise and a dilation by $\frac{1}{r}$ are given by $\frac{z}{r\,\text{cis}(\theta)}$.

$$z' = \frac{z}{2\,\text{cis}\left(\frac{\pi}{4}\right)}$$

2. Express z' in polar form.

$$z' = \frac{4\,\text{cis}\left(\frac{\pi}{6}\right)}{2\,\text{cis}\left(\frac{\pi}{4}\right)}$$

$$= 2\,\text{cis}\left(\frac{\pi}{6}, -, \frac{\pi}{4}\right)$$

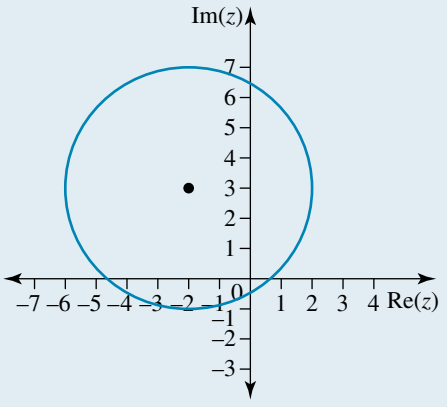
$$= 2\,\text{cis}\left(-, \frac{\pi}{12}\right)$$

Complex numbers can also be used to identify subsets of the Argand plane, such as straight lines and circles.

WORKED EXAMPLE 11 Sketching circles in the complex plane

Determine the Cartesian equation and sketch the graph of $\{z: |z+2-3i| = 4\}$.

THINK	WRITE
1. Consider the equation.	$\lvert z+2-3i\rvert = 4$ Substitute $z = x + yi$: $\lvert x+yi+2-3i\rvert = 4$
2. Group the real and imaginary parts.	$\lvert (x+2)+i(y-3)\rvert = 4$
3. Use the definition of the modulus.	$\sqrt{(x+2)^2+(y-3)^2} = 4$
4. Square both sides.	$(x+2)^2+(y-3)^2 = 16$
5. Sketch and identify the graph of the Argand plane.	The equation represents a circle with centre at $(-2, 3)$ and radius 4.



1.2.6 Roots of real quadratic equations

The roots of quadratic equations are the focus of Lesson 1.5. You have already studied the roots of real quadratic equations in Unit 2.

The quadratic equation $az^2 + bz + c = 0$ where the coefficients a, b and c are real has real roots if its discriminant $\Delta = b^2 - 4ac$ is positive.

If $\Delta < 0$, the equation has one pair of **complex conjugate roots**.

Roots of real quadratic equations

If $a, b, c \in \mathbb{R}, a \neq 0$ and $\Delta = b^2 - 4ac < 0$, then

$$az^2 + bz + c = 0 \Leftrightarrow z = \frac{-b \pm i\sqrt{-\Delta}}{2a}$$

Note that the sum of the conjugate roots is equal to $-\frac{b}{a}$ and their product is equal to $\frac{c}{a}$.

WORKED EXAMPLE 12 Solving quadratic equations

Solve for z, given $z^2 - 6z + 25 = 0$.

THINK	WRITE
Method 1: Using completing the square	$z^2 - 6z + 25 = 0$
1. Complete the square.	$z^2 - 6z + 9 + (25 - 9) = 0$ $(z-3)^2 + 16 = 0$ $(z-3)^2 = -16$
2. Substitute i^2 for -1.	$(z-3)^2 = 16i^2$
3. Solve for z.	$z - 3 = \pm 4i$
4. State the two solutions for z.	$z = 3 \pm 4i$
Method 2: Using the general quadratic formula	$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
1. Write the general quadratic formula.	
2. State the values for a, b and c.	$z^2 - 6z + 25 = 0$ $a = 1, b = -6, c = 25$
3. Substitute the values into the formula and simplify.	$z = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(25)}}{2}$ $= \frac{6 \pm \sqrt{-64}}{2}$ $= \frac{6 \pm \sqrt{64i^2}}{2}$ $= \frac{6 \pm 8i}{2}$ $= 3 \pm 4i$
4. State the two solutions.	$z = 3 \pm 4i$

Exercise 1.2 Review of complex numbers

learnon

1.2 Exercise | 1.2 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. WE1 Evaluate each of the following.

a. $\text{Re}(8-9i)$ **b.** $\text{Im}(12-i^2+2i^5)$ **c.** $\text{Im}(17-i)$ **d.** $\text{Im}(7-3i^2+i^4-2i)$

2. WE2 If $p=6-2i$ and $q=-3+3i$, evaluate each of the following without using a calculator.

a. $2p-q$ **b.** $p\times q$ **c.** $(p+q)^2$

3. WE3 Determine the complex conjugate of each of the following complex numbers.

a. $e=14+9i$ **b.** $f=-2+\sqrt{7}i$ **c.** $g=9-2\sqrt{7}i$ **d.** $h=-1-i$

4. WE4 Determine z^{-1} given z in each of the following cases.

a. $z=1+i$ **b.** $z=2-6i$ **c.** $z=-3+2i$ **d.** $z=-4-4i$

5. WE5 Express $\dfrac{3-i}{2+5i}$ in the form $a+bi$.

6. WE6 Represent the complex numbers $z_1=-3+2i$, $z_2=9-6i$, $z_3=5+4i$ and $z_4=-3-8i$ on the same Argand plane.

7. WE7 Use your knowledge of the Cartesian and polar forms of a complex number to answer the following.

a. Express $2\sqrt{3}+2i$ in polar form.

b. Express $15\text{ cis}\left(\dfrac{2\pi}{3}\right)$ in Cartesian form.

c. Express $-\dfrac{1}{\sqrt{3}}-\dfrac{1}{3}i$ in polar form.

d. Express $4\text{ cis}\left(-\dfrac{\pi}{4}\right)$ in Cartesian form.

8. WE8 If $z_1=2\text{ cis}\left(\dfrac{\pi}{6}\right)$ and $z_2=6\text{ cis}\left(\dfrac{3\pi}{4}\right)$, determine:

a. $z_1\times z_2$

b. $\dfrac{z_2}{z_1}$

9. WE9 Let $z_1=-1+\sqrt{3}i$ and $z_2=2-2i$.

a. Express z_1z_2 and $\dfrac{z_2}{z_1}$ in polar form.

b. Provide a geometric interpretation of z_1z_2 and $\dfrac{z_2}{z_1}$, and represent z_1, z_2, z_1z_2 and $\dfrac{z_2}{z_1}$ in the Argand plane.

10. WE10 Let $z=2\text{ cis}\left(\dfrac{5\pi}{6}\right)$. The complex number z is rotated by $\dfrac{3\pi}{4}$ anticlockwise about the origin and dilated by 3, giving z'.

a. Represent z and z' on the Argand plane.

b. Express z' in polar form.

11. WE11 Sketch and describe the region of the complex plane defined by $\{z:|z-3+2i|=4\}$.

12. WE12 Solve for z, given $z^2-4z+29=0$.

Complex familiar

13. Determine the values of a and b that satisfy each of the following.

a. $(3i-8)(a+bi)=-23-37i$

b. $(7+6i)(a+bi)=4-33i$

14. Write $\dfrac{2+i}{1+i}+\dfrac{9-2i}{2-i}+\dfrac{7+i}{1-i}$ in the form $a+bi$.

15. Solve for z given $z^4+z^2+1=0$. Give your answers in Cartesian form.

16. Determine $(2+2i)^2(1-\sqrt{3}i)^4$ in standard form.

Complex unfamiliar

17. Determine values for a and b so that $z=a+bi$ satisfies $\dfrac{z+i}{z+2}=i$.

18. Let $z=-\dfrac{3}{4}+2i$. The complex number z is rotated by 30° clockwise, dilated by 2 and then reflected in the real axis, giving z'. Express z'.

19. Let T be the clockwise rotation by $\dfrac{\pi}{3}$ about the origin O and a dilation of factor 2.

Let A $\left(a=2\operatorname{cis}\left(\dfrac{2\pi}{3}\right)\right)$, B $(b=T(a)$, M(m) on the middle of AB, and C(k), where $k\in\mathbb{R}$, be four points on the complex plane.

Determine k so that OMC is a right-angled triangle where the right angle is at angle M. Give your answer to 3 decimal places.

20. Describe and sketch the region defined by $\{z\colon 1\le|z-2|\le 2\}$.

Fully worked solutions for this chapter are available online.

LESSON
1.3 Complex arithmetic using polar form

SYLLABUS LINKS

- Prove complex number identities involving modulus and argument, e.g. $z\bar{z}=|z|^2$, $|z_1||z_2|=|z_1z_2|$ and $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$.
- Use De Moivre's theorem for integral powers.
  - $z^n=r^n\operatorname{cis}(n\theta)$

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1.3.1 Proving identities involving the modulus and argument

You have been told in Unit 2 that when you are performing operations in polar form, there is no convenient method for adding and subtracting complex numbers. For addition or subtraction, the complex numbers need to be converted to Cartesian form first.

TIP

To add or subtract complex numbers, use their Cartesian form.

To multiply or divide complex numbers, use their polar form.

Multiplying and dividing complex numbers is much easier when they are expressed in polar form, as the product of complex numbers can be determined by multiplying the moduli and adding the arguments, while the quotient of complex numbers can be determined by dividing the moduli and subtracting the arguments. This can easily be proven.

Consider the two complex numbers z_1 and z_2 given by$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

To determine the product $z_1 \times z_2$:

$$\begin{aligned}
z_1 \times z_2 &= r_1(\cos(\theta_1) + i\sin(\theta_1)) \times r_2(\cos(\theta_2) + i\sin(\theta_2)) \\
&= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\
&= r_1 r_2\left(\cos(\theta_1)\cos(\theta_2) + \cos(\theta_1)i\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i^2\sin(\theta_1)\sin(\theta_2)\right) \\
&= r_1 r_2(\cos(\theta_1)\cos(\theta_2) + \cos(\theta_1)i\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) \\
&= r_1 r_2\left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))\right]
\end{aligned}$$

Using the following trigonometric identities,

$$\sin(\theta_1 \pm \theta_2) = \sin(\theta_1)\cos(\theta_2) \pm \cos(\theta_1)\sin(\theta_2) \qquad [1]$$

$$\cos(\theta_1 \pm \theta_2) = \cos(\theta_1)\cos(\theta_2) \mp \sin(\theta_1)\sin(\theta_2) \qquad [2]$$

$$\cos^2(\theta) + \sin^2(\theta) = 1 \qquad [3]$$

we have

$$\begin{aligned}
z_1 \times z_2 &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\
&= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)
\end{aligned}$$

We can follow a similar process for division of complex numbers in polar form, shown below.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))} \\
&= \frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))} \times \frac{(\cos(\theta_2) - i\sin(\theta_2))}{(\cos(\theta_2) - i\sin(\theta_2))} \\
&= \frac{r_1(\cos(\theta_1)\times\cos(\theta_2) + \cos(\theta_1)\times -i\sin(\theta_2) + i\sin(\theta_1)\times\cos(\theta_2) + i\sin(\theta_1)\times -i\sin(\theta_2))}{r_2(\cos(\theta_2)\times\cos(\theta_2) + \cos(\theta_2)\times -i\sin(\theta_2) + i\sin(\theta_2)\times\cos(\theta_2) + i\sin(\theta_2)\times -i\sin(\theta_2))} \\
&= \frac{r_1(\cos(\theta_1)\cos(\theta_2) - i\cos(\theta_1)\sin(\theta_2) + i\cos(\theta_2)\sin(\theta_1) + \sin(\theta_1)\sin(\theta_2))}{r_2\left(\cos^2(\theta_2) - i\cos(\theta_2)\sin(\theta_2) + i\cos(\theta_2)\sin(\theta_2) + \sin^2(\theta_2)\right)} \\
&= \frac{r_1(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_2)\sin(\theta_1) - \cos(\theta_1)\sin(\theta_2)))}{r_2\left(\cos^2(\theta_2) + \sin^2(\theta_2)\right)}
\end{aligned}$$

Using [1], [2] and [3]

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))\\ &= \frac{r_1}{r_2}\text{cis}(\theta_1-\theta_2)\end{aligned}$$

There are other identities involving the modulus and the argument, already seen in Unit 2, that can also be proved. For example, let's use a Cartesian approach to prove the identity $z\bar{z}=||z^2|$. Consider the following where $a, b \in \mathbb{R}$.

$$\begin{aligned}z\bar{z} &= (a+bi)\times(a-bi)\\ &= a^2-abi+abi-b^2i^2\\ &= a^2+b^2\\ &= |z|^2\end{aligned}$$

Below is a list of some common complex number identities that can be proved using the polar form of a complex number.

- $z\bar{z}=|z|^2$
- $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2),\ z_2\neq 0$
- $|z^n|=|z|^n$ and $\arg(z^n)=n\ \arg(z)$
- $\left|\frac{1}{z^n}\right|=\frac{1}{|z|^n}$ and $\arg\left(\frac{1}{z^n}\right)=-n\ \arg(z),\ z\neq 0$
- $|z_1z_2|=|z_1||z_2|$ and $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$
- $\overline{z_1}+\overline{z_2}=\overline{z_1+z_2}$
- $\overline{z_1}\times\overline{z_2}=\overline{z_1\times z_2}$
- $z+\bar{z}=2\text{Re}(z)$
- $z-\bar{z}=2i\text{Im}(z)$

WORKED EXAMPLE 13 Proving identities involving the modulus and the argument

a. Prove the identity $z\bar{z}=|z|^2$ using polar arithmetic.

b. Verify the identity $z-\bar{z}=2i\ \text{Im}(z)$ using $z=2\ \text{cis}\left(\frac{\pi}{6}\right)$.

THINK	WRITE
a. 1. Define the complex number z and its conjugate $\bar{z}$ in polar form.	**a.** Let $z=\|z\|\ \text{cis}(\theta)$ and $\bar{z}=\|z\|\ \text{cis}(-\theta)$.
2. Begin with the LHS. Substitute the expressions for z and $\bar{z}$.	$\text{LHS}=z\bar{z}$ $=\left(\|z\|\ \text{cis}(\theta)\right)\times\left(\|z\|\ \text{cis}(-\theta)\right)$
3. Recall and apply the rule $z_1\times z_2=r_1r_2\ \text{cis}(\theta_1+\theta_2)$ to expand $z\bar{z}$.	$=\|z\|\times\|z\|\ \text{cis}(\theta+(-\theta))$
4. Simplify the modulus and argument components.	$=\|z\|^2\ \text{cis}(0)$

5. Recall that $\text{cis}(\theta)$ is short for $\cos(\theta) + i \times \sin(\theta)$. Use exact values to evaluate $\cos(0)$ and $\sin(0)$.

$$= |z|^2 (\cos(0) + i\sin(0))$$
$$= |z|^2 (1 + i \times 0)$$

6. The LHS has been simplified to $|z|^2$.

$$= |z|^2$$
$$= \text{RHS}$$

7. Finalise the proof with a suitable statement.

Since the LHS = RHS, then $z\bar{z} = |z|^2$.

b. 1. Define the complex number z and its conjugate $\bar{z}$ in polar form.

b. Let $z = 2\text{ cis}\left(\frac{\pi}{6}\right)$ and $\bar{z} = 2\text{ cis}\left(-\frac{\pi}{6}\right)$.

2. Begin with the LHS. Substitute the expressions for z and $\bar{z}$.

$$\text{LHS} = z - \bar{z}$$
$$= 2\text{ cis}\left(\frac{\pi}{6}\right) - 2\text{ cis}\left(-\frac{\pi}{6}\right)$$

Since there is no rule to follow for subtracting complex numbers in polar form, we must convert them into Cartesian form first. Recall that $\text{cis}(\theta)$ is short for $\cos(\theta) + i \times \sin(\theta)$.

$$= 2\cos\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{6}\right)i - \left(2\cos\left(-\frac{\pi}{6}\right) + 2\sin\left(-\frac{\pi}{6}\right)i\right)$$

3. Use exact values and simplify where possible.

$$= 2 \times \frac{\sqrt{3}}{2} + 2 \times \frac{1}{2} \times i - 2 \times \frac{\sqrt{3}}{2} - 2 \times \frac{-1}{2}i$$
$$= \cancel{\sqrt{3}} + i - \cancel{\sqrt{3}} + i$$
$$= 2i$$

4. In this case, polar arithmetic needs to be applied to simplify the RHS. Make a substitution for $z = 2\text{ cis}\left(\frac{\pi}{6}\right)$.

$$\text{RHS} = 2i\,\text{Im}(z)$$
$$= 2i\,\text{Im}\left(2\text{ cis}\left(\frac{\pi}{6}\right)\right)$$

5. Recall that $\text{cis}(\theta)$ is short for $\cos(\theta) + i \times \sin(\theta)$.

$$= 2i\,\text{Im}\left(2\cos\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{6}\right)i\right)$$

6. Extract the imaginary component of z.

$$= 2i \times 2\left(\sin\left(\frac{\pi}{6}\right)\right)$$

7. Simplify the RHS.

$$= 2i \times 2 \times \frac{1}{2}$$
$$= 2i$$
$$= \text{LHS}$$

8. Finalise the proof with a suitable statement.

Since the LHS = RHS, then $z - \bar{z} = 2i\,\text{Im}(z)$ for $z = 2\text{ cis}\left(\frac{\pi}{6}\right)$.

1.3.2 De Moivre's theorem

You have already observed that multiplying and dividing complex numbers is a simpler process when the complex numbers are in polar form that it is when they are in Cartesian form.

Multiplication and division of complex numbers in polar form

If $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$, then

$$z_1 \times z_2 = r_1 \times r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 + \theta_2)$$

To calculate the power of z, it is necessary to complete repeated multiplications.

Consider $z^n = \underbrace{z \times z \times z \times \cdots \times z}_{n \text{ times}}$.

Using the polar form $z = r\operatorname{cis}(\theta)$:

$$\begin{aligned}
z^1 &= r^1 \operatorname{cis}(\theta) \\
z^2 &= r\operatorname{cis}(\theta) \times r\operatorname{cis}(\theta) = r \times r \times \operatorname{cis}(\theta + \theta) = r^2 \operatorname{cis}(2\theta) \\
z^3 &= z^2 z = r^2 \operatorname{cis}(2\theta) \times r\operatorname{cis}(\theta) = r^3 \operatorname{cis}(3\theta) \\
z^3 &= z^3 z = r^3 \operatorname{cis}(3\theta) \times r\operatorname{cis}(\theta) = r^4 \operatorname{cis}(4\theta) \\
&\ldots \\
z^n &= r^n \operatorname{cis}(n\theta)
\end{aligned}$$

The generalisation is known as **De Moivre's theorem**.

De Moivre's theorem

If $z = r\operatorname{cis}(\theta)$, then

$$z^n = r^n \operatorname{cis}(n\theta).$$

or

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This theorem can be proven using mathematical induction. You will study mathematical induction in Chapter 2 and will see a formal demonstration of De Moivre's theorem there.

De Moivre's theorem is also applicable for negative integer values of n. A proof will not be conducted here. However, an opportunity to do so exists in Chapter 2, Exercise 2.4.

WORKED EXAMPLE 14 Using De Moivre's theorem with positive and negative integers

Simplify the following using De Moivre's theorem.

a. $\left(3\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^5$

b. $\left(8\operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^{-3}$

THINK	WRITE
a. 1. Use De Moivre's theorem for calculating the result of raising a complex number to a power.	**a.** $\left(3\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^5 = 3^5 \operatorname{cis}\left(5 \times \frac{\pi}{4}\right)$

2. Evaluate 3^5 and convert the angle into a principal value.
$\frac{5\pi}{4} = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$.

$$= 243 \text{ cis}\left(-\frac{3\pi}{4}\right)$$

b. 1. Use De Moivre's theorem for calculating the result of raising a complex number to a power.

b. $$\left(8 \text{ cis}\left(\frac{2\pi}{3}\right)\right)^{-3} = 8^{-3} \text{ cis}\left(-3 \times \frac{2\pi}{3}\right)$$

2. Use index laws to help simplify 8^{-3} and simplify the argument.

$$= \frac{1}{8^3} \text{ cis}\left(-\cancel{3} \times \frac{2\pi}{\cancel{3}}\right)$$

3. Express the angle as a principal value.

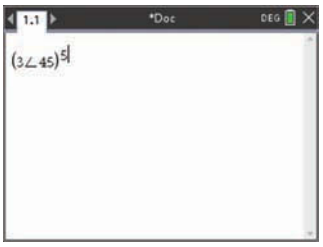
$$= \frac{1}{512} \text{ cis}(-2\pi)$$
$$= \frac{1}{512} \text{ cis}(0)$$

4. Write $r\,\text{cis}(\theta)$ in expanded form to simplify further.

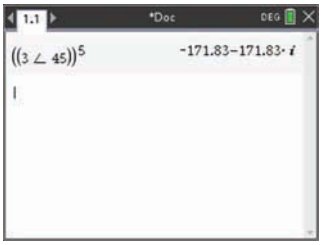
$$= \frac{1}{512}\cos(0) + i \times \frac{1}{512}\sin(0)$$
$$= \frac{1}{512}$$

TI | THINK **WRITE**

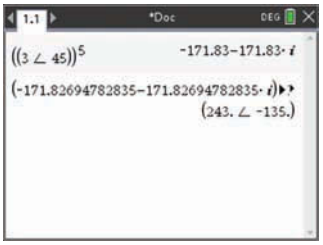
a. 1. On a Calculator page, complete the entry line as:
$(3\angle 45)^5$



2. Press the ENTER button.
The answer appears on the screen in Cartesian form.

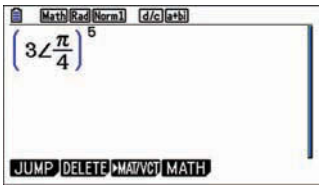


3. Convert the answer to polar form.
The answer appears on the screen.

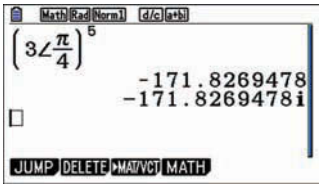


CASIO | THINK **WRITE**

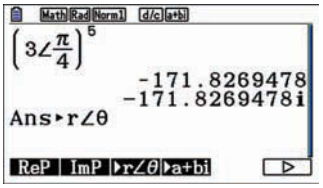
a. 1. On a Run-Matrix screen, complete the entry line as:
$\left(3\angle\frac{\pi}{4}\right)^5$



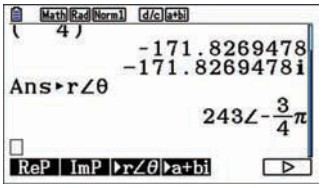
2. Press the EXE button.
The answer appears on the screen in Cartesian form.



3. To convert the answer to polar form, select:
SHIFT (−)
OPTN
COMPLEX
▷ $r\angle\theta$



4. Press the EXE button.
The answer appears on the screen.



Exercise 1.3 Complex arithmetic using polar form

learnon

1.3 Exercise	1.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	16, 17	18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. WE13 Verify the identity $z - \bar{z} = 2i\,\text{Im}(z)$ using $z = 3\text{ cis}\left(\frac{\pi}{3}\right)$.

2. Use polar arithmetic to prove the following identities.

a. $|z_1 z_2| = |z_1||z_2|$

b. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

3. Prove the following identities using a polar approach.

a. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

b. $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

4. Prove the following identities using polar arithmetic.

a. $|z^n| = |z|^n$

b. $\arg(z^n) = n\arg(z)$

5. Given $z = r\text{ cis}(\theta)$, prove the following identities.

a. $\left|\frac{1}{z^n}\right| = \frac{1}{|z|^n}$, $z \neq 0$

b. $\arg\left(\frac{1}{z^n}\right) = -n\arg(z)$, $z \neq 0$

6. Verify the following identities if $z_1 = 4\text{ cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 7\text{ cis}\left(\frac{\pi}{6}\right)$.

a. $|z_1 z_2| = |z_1||z_2|$

b. $z_1 + \bar{z}_1 = 2\text{Re}(z_1)$

c. $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

d. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

7. Prove the identity $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$ using polar arithmetic.

8. Prove the identity $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$ by considering all complex numbers in polar form.

9. Use polar arithmetic to prove the identity $z + \bar{z} = 2\text{Re}(z)$.

10. Given $z = |z|\text{cis}(\theta)$, prove the identity $z - \bar{z} = 2i\text{Im}(z)$.

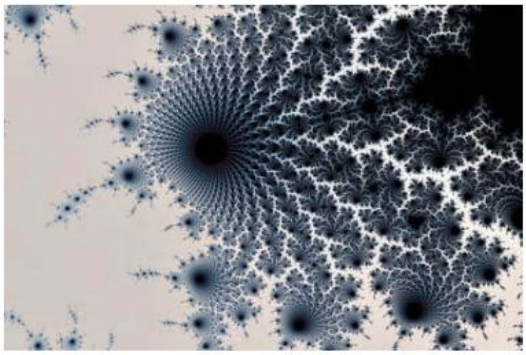
11. WE14 Simplify the following using De Moivre's theorem.

a. $\left(2\text{ cis}\left(\frac{\pi}{4}\right)\right)^5$

b. $\left(3\text{ cis}\left(-\frac{3\pi}{4}\right)\right)^6$

c. $\left(3\text{ cis}\left(\frac{\pi}{6}\right)\right)^{-4}$

d. $\left(\text{cis}\left(-\frac{\pi}{3}\right)\right)^{-6}$



12. Use De Moivre's theorem to simplify each of the following.

a. $\left(2\operatorname{cis}\left(\frac{3\pi}{4}\right)\right)^5 \times \left(3\operatorname{cis}\left(\frac{\pi}{8}\right)\right)^6$

b. $\left(4\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^5 \times \left(2\operatorname{cis}\left(\frac{5\pi}{6}\right)\right)^{-3}$

13. Use De Moivre's theorem to simplify each of the following.

a. $\dfrac{\left(3\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^4}{\left(\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^3}$

b. $\dfrac{\left(4\operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^7}{\left(2\operatorname{cis}\left(-\frac{3\pi}{5}\right)\right)^{-6}}$

14. Use De Moivre's theorem to express the following in the form $a + bi$.

a. $(2-2i)^4$

b. $\left(\sqrt{2}+\sqrt{6}i\right)^{-6}$

15. **a.** If $z = 2 + 2i$, determine each of the following.

i. z^8

ii. $\operatorname{Arg}(z^8)$

b. If $z = -3\sqrt{3} + 3i$, calculate each of the following.

i. z^6

ii. $\operatorname{Arg}(z^6)$

Complex familiar

16. If $4\left(\left(\frac{4+4i}{1-i}\right)^4 - \left(\frac{1}{1+i}\right)^4\right) = a + bi$, determine the value of a and b.

17. If $z = \operatorname{cis}(\theta)$, show that:

a. $|z+1| = 2\cos\left(\frac{\theta}{2}\right)$

b. $\operatorname{Arg}(1+z) = \frac{\theta}{2}$

Complex unfamiliar

18. Determine all values of n such that $\left(-\sqrt{3}+1\right)^n + \left(-\sqrt{3}-i\right)^n = 0$.

19. By considering the real and imaginary parts of the equation $(\operatorname{cis}(\theta))^2 = \operatorname{cis}(2\theta)$, show that:

i. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

ii. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

20. By considering the real and imaginary parts of the equation $(\operatorname{cis}(\theta))^3 = \operatorname{cis}(3\theta)$, show that:

i. $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$

ii. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

Fully worked solutions for this chapter are available online.

LESSON
1.4 Roots of complex numbers

SYLLABUS LINKS

- Determine and examine the nth roots of unity and their location on the unit circle.
- Determine and examine the nth roots of complex numbers and their location in the complex plane.

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1.4.1 Determine and examine the *n*th roots of unity

Since $1 = 1 + 0i = 1 \times \text{cis}(0)$, De Moivre's theorem can be used to solve equations in the form $z^n = 1$.

Consider a complex number $z = r\,\text{cis}(\theta)$.

If $z^n = 1$, then $r^n\text{cis}(n\theta) = 1 \times \text{cis}(0)$. Two complex numbers are equal if their real parts are equal and their imaginary parts are equal, that is, if they have equal moduli and their arguments are the same or differ by a multiple of 2π.

Therefore,

$$r^n\text{cis}(n\theta) = 1 \times \text{cis}(0) \Rightarrow \begin{cases} r = \sqrt[n]{1} \\ n\theta = 0 + 2k\pi, k = 0, 1, \cdots \end{cases} \Rightarrow \begin{cases} r = 1 \\ \theta = \dfrac{2k\pi}{n}, k = 0, 1, \cdots \end{cases}$$

Given that the trigonometric functions cosine and sine have a period of 2π, when $k \geq n$, the roots are repeated:

$$\begin{aligned} \frac{2n\pi}{n} &= 2\pi = 0 = \frac{2k\pi}{n}, k = 0 \\ \frac{2(n+1)\pi}{n} &= \frac{2\pi}{n} + 2\pi = \frac{2\pi}{n} = \frac{2k\pi}{n}, k = 1 \\ \frac{2(n+2)\pi}{n} &= \frac{4\pi}{n} + 2\pi = \frac{4\pi}{n} = \frac{2k\pi}{n}, k = 2 \\ &\ldots \\ \frac{2(n+n-1)\pi}{n} &= \frac{2(n-1)\pi}{n} + 2\pi = \frac{2(n-1)\pi}{n} = \frac{2k\pi}{n}, k = n-1 \end{aligned}$$

Because the solutions start repeating for $k = n$, we only keep the values of $k < n$.

Therefore,

$$z^n = 1 \Rightarrow \begin{cases} r = 1 \\ \theta = \dfrac{2k\pi}{n}, k = 0, 1, 2, \ldots, n-1 \end{cases}$$

If we interpret the solutions geometrically, $r = \sqrt[n]{1}$ means that the roots sit on the unit circle, which is the circle of radius 1 centred at the origin.

$\theta = \dfrac{2k\pi}{n}, k = 0, 1, 2, \ldots, n-1$ means that the n roots are evenly spaced around the circle by $\dfrac{2\pi}{n}$. Thus, n roots represent the n vertices of a regular n-sided polygon inscribed in the unit circle.

The nth roots of unity

If $z^n = 1$, where $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then $z = \text{cis}\left(\frac{2k\pi}{n}\right), k = 0, 1, \cdots, n-1.$

The roots are generally denoted $z_1, z_2, z_3, \cdots, z_n$ where z_1 represents the first root (for $k = 0$). The first root is commonly known as the principal nth root of the complex number.

Let's apply this to solve $z^8 = 1$.

$$z^8 = 1 \Rightarrow \begin{cases} r = 1 \\ \theta = \dfrac{2k\pi}{n}, k = 0, 1, 2, \ldots, 7 \end{cases} \Rightarrow z = \text{cis}\left(\frac{k\pi}{4}\right), k = 0, 1, \cdots 7$$

The solution set, in polar form and in Cartesian form is displayed below.

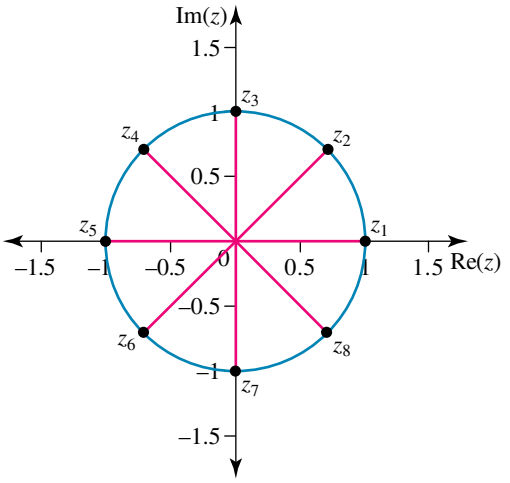
$$z = \left\{\text{cis}\left(\frac{0\pi}{4}\right), \text{cis}\left(\frac{\pi}{4}\right), \text{cis}\left(\frac{\pi}{2}\right), \text{cis}\left(\frac{3\pi}{4}\right), \text{cis}(\pi), \text{cis}\left(\frac{5\pi}{4}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}\left(\frac{7\pi}{4}\right)\right\}$$

$$z = \left\{1, -1, i, -i, \sqrt{2}\left(\frac{1}{2} - \frac{1}{2}i\right), \sqrt{2}\left(\frac{1}{2} + \frac{1}{2}i\right), \sqrt{2}\left(-\frac{1}{2} - \frac{1}{2}i\right), \sqrt{2}\left(-\frac{1}{2} + \frac{1}{2}i\right)\right\}$$

If you examine the non-real roots, you will observe that they occur as conjugate pairs. In this case, the real roots are $\{1, -1\}$, and the conjugate pairs are

$$\{i, -i\}, \left\{\sqrt{2}\left(\frac{1}{2} - \frac{1}{2}i\right), \sqrt{2}\left(\frac{1}{2} + \frac{1}{2}i\right)\right\} \text{ and}$$

$$\left\{\sqrt{2}\left(-\frac{1}{2} - \frac{1}{2}i\right), \sqrt{2}\left(-\frac{1}{2} + \frac{1}{2}i\right)\right\}$$



The eight 8th roots of 1 can be interpreted geometrically using an Argand diagram. We can observe that the 8 roots are indeed evenly spaced by $\frac{2\pi}{8} = \frac{\pi}{4}$, as expected from the discussion at the start of this section.

WORKED EXAMPLE 15 Determining the cubic roots of unity

Answer the following.

a. Determine all solutions to the equation $z^3 = 1$ over $\mathbb{C}$. Express the solutions in polar form.

b. Represent these solutions using a polar grid.

THINK	WRITE
a. 1. Express 1 in polar form.	a. $1 = \text{cis}(0)$
2. Use De Moivre's theorem.	$z^3 = \text{cis}(0)$ $z = \sqrt[3]{1}\,\text{cis}\left(\frac{0 + 2k\pi}{3}\right)$
3. Simplify the modulus and argument.	$z = 1\,\text{cis}\left(\frac{2k\pi}{3}\right)$

4. Let $k = 0$ to calculate the first root.

Let $k = 0$.

$$z_1 = \text{cis}\left(\frac{2 \times 0 \times \pi}{3}\right)$$
$$= \text{cis}(0)$$
$$= 1$$

5. Let $k = 1$ to calculate the second root.

Let $k = 1$.

$$z_2 = \text{cis}\left(\frac{2 \times 1 \times \pi}{3}\right)$$
$$= \text{cis}\left(\frac{2\pi}{3}\right)$$

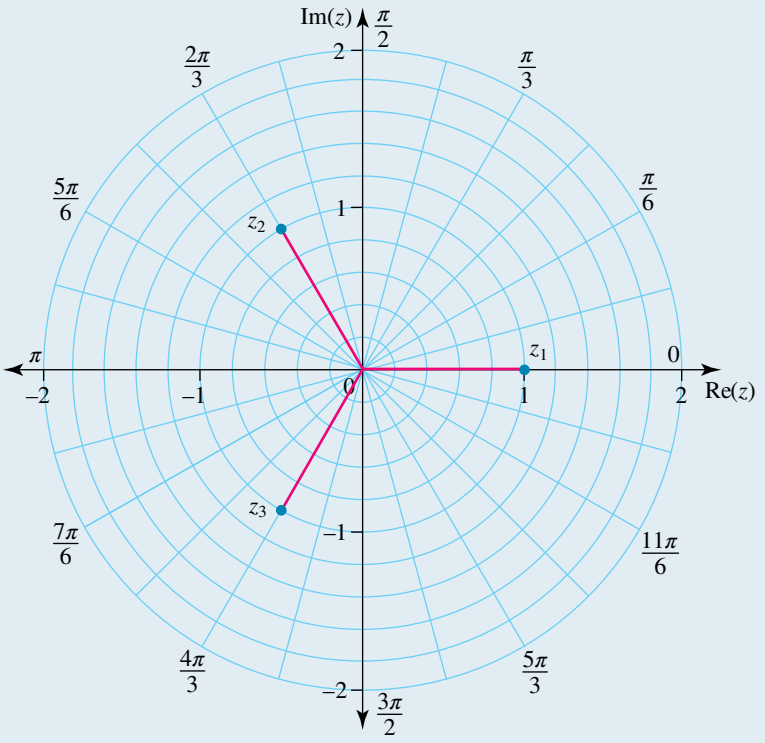
6. Let $k = 2$ to calculate the third root.

Let $k = 2$.

$$z_3 = \text{cis}\left(\frac{2 \times 2 \times \pi}{3}\right)$$
$$= 2\ \text{cis}\left(\frac{-2\pi}{3}\right)$$

b. Plot the three roots on a polar grid. The circle has a centre $(0, 0)$ and radius $r = 1$.
The three solutions are spaced by $\frac{2\pi}{3}$.

b.



1.4.2 Solving $z^n = a, a \in \mathbb{R}$

Equations in the form $z^n = a, a \in \mathbb{R}$ can be solved using a similar method, as any real number a can be written as $a = |a|\ \text{cis}(0)$ if $a > 0$ and as $a = |a|\ \text{cis}(\pi)$ if $a < 0$.

Consider a complex number $z = r\,\text{cis}(\theta)$.

If $z^n = a, a \in \mathbb{R}^+$, then $r^n \text{cis}(n\theta) = a \times \text{cis}(0)$.

Therefore,

$$z^n = a \times \text{cis}(0) \Rightarrow \begin{cases} r = \sqrt[n]{a} \\ \theta = 0 + \dfrac{2k\pi}{n}, k = 0, 1, \cdots, n-1 \end{cases}$$

If $z^n = a, a \in \mathbb{R}^-$, then $r^n \text{cis}(n\theta) = |a| \times \text{cis}(\pi)$.

Therefore,

$$z^n = |a| \times \text{cis}(\pi) \Rightarrow \begin{cases} r = \sqrt[n]{|a|} \\ \theta = \dfrac{\pi}{n} + \dfrac{2k\pi}{n}, k = 0, 1, \cdots, n-1 \end{cases}$$

WORKED EXAMPLE 16 Solving $z^n = a, a \in \mathbb{R}$

Determine the solutions to $\{z : z^3 = -27\}$, stating the solution set in Cartesian form.

THINK	WRITE
1. Express -27 in polar form.	$-27 = \lvert -27 \rvert \text{ cis}(\pi)$ $= 27 \text{ cis}(\pi)$
2. Use De Moivre's theorem.	$z^3 = 27 \text{ cis}(\pi)$ $z = \sqrt[3]{27} \text{ cis}\left(\dfrac{\pi + 2k\pi}{3}\right)$ $z = 3 \text{ cis}\left(\dfrac{\pi + 2k\pi}{3}\right)$
3. Let $k = 0$ to calculate the first root.	Let $k = 0$. $z_1 = 3 \text{ cis}\left(\dfrac{\pi + 2 \times 0 \times \pi}{3}\right)$ $= 3 \text{ cis}\left(\dfrac{\pi}{3}\right)$
4. Let $k = 1$ to calculate the second root.	Let $k = 1$. $z_2 = 3 \text{ cis}\left(\dfrac{\pi + 2 \times 1 \times \pi}{3}\right)$ $= 3 \text{ cis}(\pi)$
5. Let $k = 2$ to calculate the third root.	Let $k = 2$. $z_1 = 3 \text{ cis}\left(\dfrac{\pi + 2 \times 2 \times \pi}{3}\right)$ $= 3 \text{ cis}\left(\dfrac{5\pi}{3}\right)$ $= 3 \text{ cis}\left(-\dfrac{\pi}{3}\right)$
6. State the solution set in Cartesian form. Note that the imaginary roots occur as a conjugate pair.	$z_1 = \dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i,\ z_2 = -3,\ z_3 = \dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i$

Note that when solving $z^n = a, a \in \mathbb{R}$, the first root always lies on the real axis.

1.4.3 Determine and examine the *n*th roots of complex numbers

We can now generalise the method used to determine the nth roots of unity and of any real number to determine the nth roots of complex numbers, that is, to solve $z^n = w, w \in \mathbb{C}$.

Consider the complex numbers $z = r\,\text{cis}(\theta)$ and $w = \rho\,\text{cis}(\phi)$.

Using De Moivre's theorem and the fact that two complex numbers are equal if they have equal moduli and their arguments differ by a multiple of 2π to solve $z^n = w$:

$$r^n \text{cis}(n\theta) = \rho\,\text{cis}(\phi) \Rightarrow \begin{cases} r = \sqrt[n]{\rho} \\ \theta = \dfrac{\phi + 2k\pi}{n}, k = 0, 1, \cdots, n-1 \end{cases}$$

Remember that because the solutions repeat when $k \geq n$, we only keep the values of $k < n$.

In general, equations in the form of $z^n = w$, where $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$, have the following properties.

Solutions of $z^n = w$ where $w = r\,\text{cis}(\theta)$

The n solutions to $z^n = w$, where $z \in \mathbb{C}$, are in the form

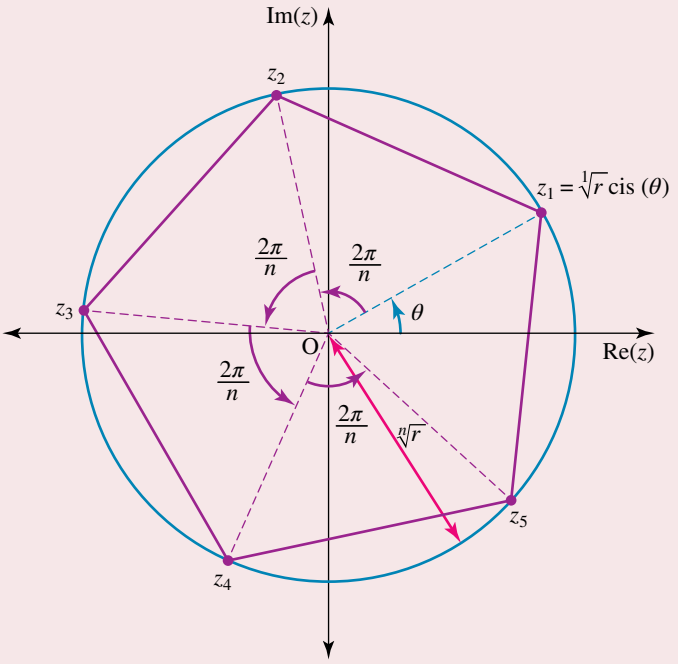
$$z = \sqrt[n]{r}\,\text{cis}\left(\frac{\theta + 2k\pi}{n}\right), k = 0, 1, \cdots, n-1$$

The solutions are equidistant from the origin and lie on a circle of radius $\sqrt[n]{r}$ centred on the origin.

The solutions are evenly spaced around the circle, with their arguments differing by $\frac{2\pi}{n}$.

The solutions lie on the circle of radius $\sqrt[n]{r}$ centred on the origin.

Therefore, for $n \geq 3$, the solutions are the vertices of a regular n-sided polygon inscribed in the circle of radius $\sqrt[n]{r}$ centred on the origin.



The first root will be at an angle with the positive direction of the real axis equal to θ, the argument of w.

WORKED EXAMPLE 17 Determining the square roots of complex numbers

Use De Moivre's theorem to solve $z^2 = 4\,\text{cis}\left(\frac{\pi}{3}\right)$ for z.

THINK	WRITE
1. Use De Moivre's theorem.	$z = \sqrt[2]{4}\,\text{cis}\left(\frac{\frac{\pi}{3} + 2k\pi}{2}\right)$

2. Simplify the modulus and argument.

$$z = 2\,\text{cis}\left(\frac{\pi}{6} + k\pi\right)$$

3. Let $k = 0$ to calculate the first root.

Let $k = 0$.

$$z_1 = 2\,\text{cis}\left(\frac{\pi}{6} + 0 \times \pi\right)$$
$$= 2\,\text{cis}\left(\frac{\pi}{6}\right)$$

4. Let $k = 1$ to calculate the second root.

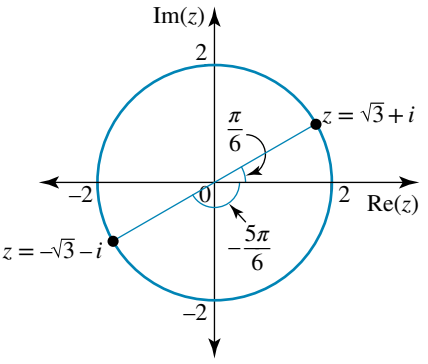
Let $k = 1$.

$$z_2 = 2\,\text{cis}\left(\frac{\pi}{6} + 1 \times \pi\right)$$
$$= 2\,\text{cis}\left(\frac{7\pi}{6}\right)$$
$$= 2\,\text{cis}\left(-\frac{5\pi}{6}\right)$$

5. State both answers.

$$z_1 = 2\,\text{cis}\left(\frac{\pi}{6}\right),\ z_2 = 2\,\text{cis}\left(-\frac{5\pi}{6}\right)$$

In Cartesian form, the roots are given by $z_1 = \sqrt{3} + i$ and $z_2 = -\sqrt{3} - i$. Plotting the solutions on the Argand plane, it can be seen that the roots of the equation lie on a circle with a radius of 2. The two solutions are separated around the circle by 180°. Although there are two roots, they are not complex conjugates of one another.



WORKED EXAMPLE 18 Determining the cubic roots of complex numbers

Answer the following.

a. If $z^3 = 4 - 4\sqrt{3}\,i$, solve the equation for z using a polar method.

b. Plot the solutions on an Argand diagram and comment on their location.

THINK

a. 1. Express z in polar form ($z = r\,\text{cis}(\theta)$).

WRITE

a.

$$|z| = \sqrt{(4)^2 + \left(-4\sqrt{3}\right)^2}$$
$$|z| = 8$$
$$\theta = \tan^{-1}\left(\frac{-4\sqrt{3}}{4}\right)$$
$$\theta = \frac{-\pi}{3}$$
$$\therefore z = 4 - 4\sqrt{3}i = 8\,\text{cis}\left(\frac{-\pi}{3}\right)$$
$$z^3 = 8\,\text{cis}\left(-\frac{\pi}{3}\right)$$

2. Use De Moivre's theorem.

$$z = \sqrt[3]{8}\,\text{cis}\left(\frac{-\frac{\pi}{3} + 2k\pi}{3}\right)$$

3. Simplify the modulus and argument.

$$z = 2\,\text{cis}\left(\frac{-\pi}{9} + \frac{2k\pi}{3}\right)$$

4. Let $k = 0$ to calculate the first root. Convert z_1 to standard form.

Let $k = 0$.

$$\begin{aligned} z_1 &= 2\,\text{cis}\left(\frac{-\pi}{9} + \frac{2 \times 0 \times \pi}{3}\right) \\ &= 2\,\text{cis}\left(\frac{-\pi}{9}\right) \\ &\approx 1.879 - 0.684i \end{aligned}$$

5. Let $k = 1$ to calculate the second root. Convert z_2 to standard form.

Let $k = 1$.

$$\begin{aligned} z_2 &= 2\,\text{cis}\left(\frac{-\pi}{9} + \frac{2 \times 1 \times \pi}{3}\right) \\ &= 2\,\text{cis}\left(\frac{5\pi}{9}\right) \\ &\approx -0.347 + 1.970i \end{aligned}$$

6. Let $k = 2$ to calculate the third root. Convert z_3 to standard form.

Let $k = 2$.

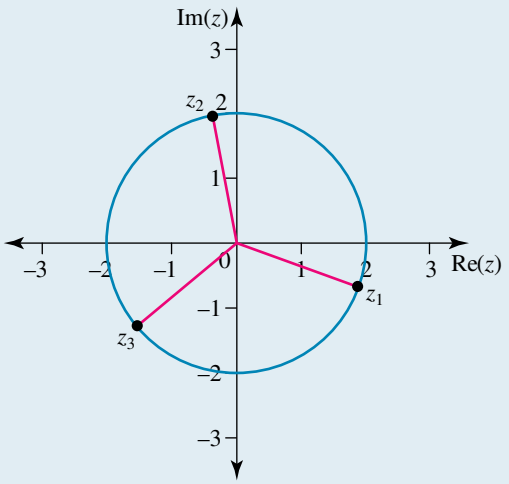
$$\begin{aligned} z_3 &= 2\,\text{cis}\left(\frac{-\pi}{9} + \frac{2 \times 2 \times \pi}{3}\right) \\ &= 2\,\text{cis}\left(\frac{-7\pi}{9}\right) \\ &\approx -1.532 - 1.286i \end{aligned}$$

State all answers.

$$z = 2\,\text{cis}\left(-\frac{\pi}{9}\right),\ 2\,\text{cis}\left(\frac{5\pi}{9}\right) \text{ and } 2\,\text{cis}\left(-\frac{7\pi}{9}\right)$$

b. Use the Cartesian form to plot the solutions on an Argand diagram.

b.



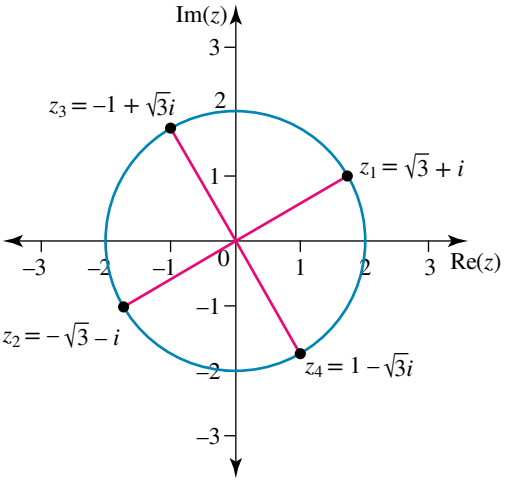
The three roots are equally spaced around a circle of radius 2. The angle between each solution is $\frac{2\pi}{3}$ (120°).

In Worked example 18 there were 3 solutions that were evenly spaced by $\frac{2\pi}{n} = \frac{2\pi}{3}$.

Consider the problem below, in which the fourth roots of a complex number are determined.

$$z^4 = -8 + 8\sqrt{3}i$$

$$\text{Solutions} \begin{cases} z_1 = \sqrt{3} + i \\ z_2 = -\sqrt{3} - i \\ z_3 = -1 + \sqrt{3}i \\ z_4 = 1 - \sqrt{3}i \end{cases}$$



In this case there are 4 solutions that are evenly spaced by $\frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$.

Worked example 19 demonstrates an alternative method for solving equations of the form $z^n = w$ where $w \in \mathbb{C}$.

WORKED EXAMPLE 19 Determining the square roots of a complex number using a Cartesian approach

Solve $z^2 = 3 - 4i$ using a Cartesian approach.

THINK	WRITE
1. Express z in Cartesian form and substitute this into the equation.	$z = a + bi$ $(a + bi)^2 = 3 - 4i$
2. Expand the brackets using $(a + b)^2 = a^2 + 2ab + b^2$ and use $i^2 = -1$ to simplify the equation.	$a^2 + 2abi + b^2i^2 = 3 - 4i$ $a^2 - b^2 + 2abi = 3 - 4i$
3. Equate the coefficients of the real parts and the imaginary parts.	$a^2 - b^2 = 3$ and $2ab = -4$
4. Rearrange $2ab = -4$ for b.	$2ab = -4 \Rightarrow b = -\frac{2}{a}$
5. Substitute this new equation into $a^2 - b^2 = 3$ and simplify.	$a^2 - \left(-\frac{2}{a}\right)^2 = 3$ $a^2 - \frac{4}{a^2} = 3$
6. Multiply through by a^2.	$a^4 - 4 = 3a^2$ $a^4 - 3a^2 - 4 = 0$
7. Factorise this quartic by identifying the hidden quadratic equation.	$(a^2)^2 - 3(a^2) - 4 = 0$ $(a^2 - 4)(a^2 + 1) = 0$

8. Use the Null Factor Law to calculate the answers for a. Recall that for any complex number $z = a + bi$ that $a, b \in R$. This means we can ignore any solutions to $(a^2 + 1) = 0$.

$a^2 - 4 = 0$ or $a^2 + 1 = 0$

$(a-2)(a+2) = 0$ since $a, b \in R$

$\therefore\ a = 2$ or $a = -2$

9. Determine the corresponding values of b for each value of a.

When $a = 2,\ b = -1$, and when $a = -2,\ b = 1$.

10. State the solutions to the equation.

The solutions to the equation $z^2 = 3 - 4i$ are $z = 2 - i$ and $z = -2 + i$.

Exercise 1.4 Roots of complex numbers

learn on

1.4 Exercise	1.4 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	16, 17	18, 19, 20

These questions are even better in jacPLUS!

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Simple familiar

1. WE15 Answer the following.
   a. Determine all solutions to the equation $z^4 = 1$ over $\mathbb{C}$. Express the solutions in polar form.
   b. Represent these solutions using a polar grid.

2. Determine the product of the three cubic roots of 1.

3. Determine the sum of the four quartic (4th) roots of 1.

4. a. Determine all solutions to the equation $z^5 = 1$ over $\mathbb{C}$. Express the solutions in polar form.
   b. Represent these solutions using a polar grid.

5. Plot the solutions to $z^{10} = 1$ on a polar grid without determining the solutions using De Moivre's theorem.

6. WE16 Determine the solutions to the following equations, stating the solution set in Cartesian form.
   a. $z^3 = 8$
   b. $z^4 = 256$

7. Let $\omega_0, \omega_1, \omega_2, \omega_3$ and ω_4 be the five 5th roots of 1. Describe how these solutions would compare to the solutions to the equation $z^5 = 10$ over $\mathbb{C}$.

8. WE17 Use De Moivre's theorem to solve $z^2 = 5 \text{ cis}\left(\frac{\pi}{6}\right)$ for z.

9. Use De Moivre's theorem to solve $z^3 = 27 \text{ cis}\left(\frac{2\pi}{3}\right)$ for z.

10. Use De Moivre's theorem to solve $z^4 = 16 \text{ cis}\left(\frac{-3\pi}{4}\right)$ for z.

11. **WE18** Answer the following.

a. If $z^3 = -1 - \sqrt{3}i$, solve the equation for z using a polar method.
b. Plot the solutions on an Argand diagram and comment on their location.

12. **a.** If $z^4 = -5 - 5i$, solve the equation for z using a polar method.
b. Plot the solutions on an Argand diagram and comment on their location.

13. **a.** If $z^6 = 3 + \sqrt{3}i$, solve the equation for z using a polar method.
b. Plot the solutions on an Argand diagram and comment on their location.

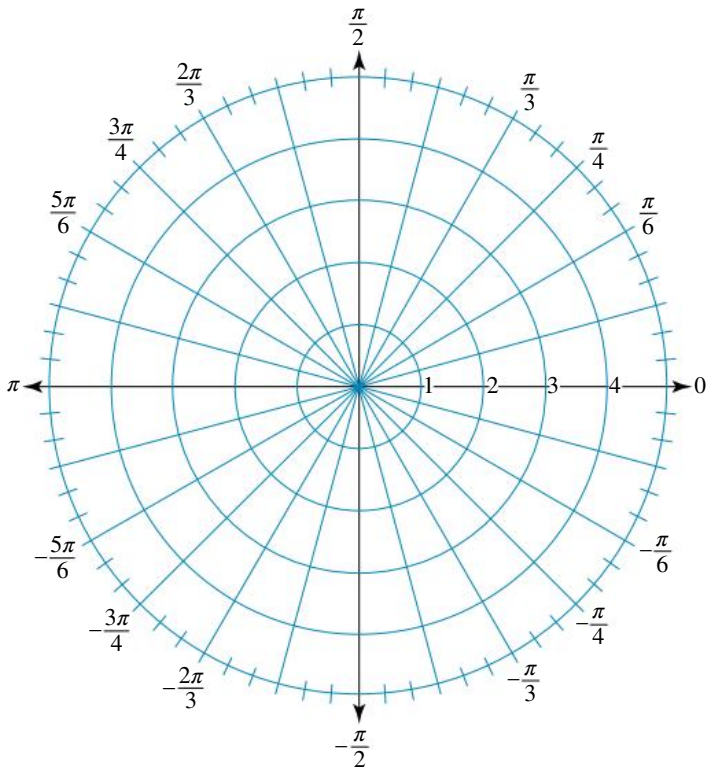
14. Use De Moivre's theorem to solve for z in the following cases.

a. $z^3 = i$
b. $z^4 = 64i$

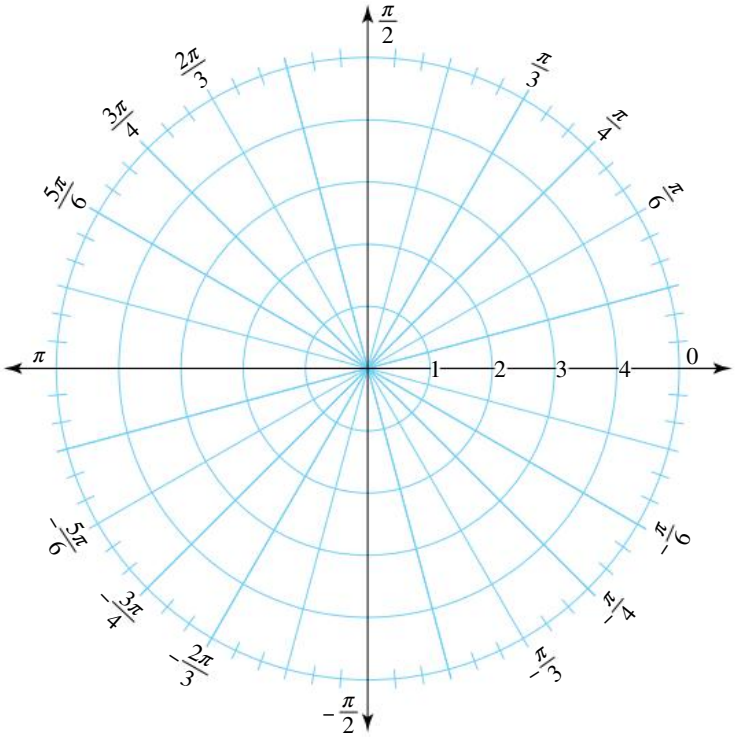
15. **WE19** Solve $z^2 = 2 + 2\sqrt{3}i$ using a Cartesian approach.

Complex familiar

16. Plot the solutions to $\left(z^2 - 4\sqrt{2}(1+i)\right)\left(z^4 - 8\left(1+\sqrt{3}i\right)\right) = 0$ on the polar grid.



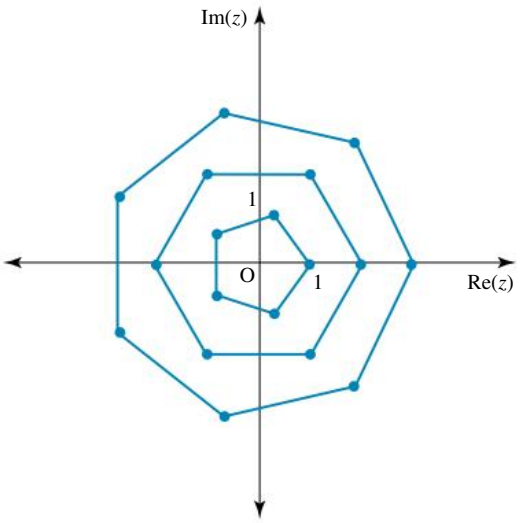
17. A student has plotted the four roots to the equation $z^3 = w$, where $w = r\,\text{cis}(\theta)$, on the polar grid below. Plot the roots to $z^4 = w^2$ on the same polar grid.



Justify your answer.

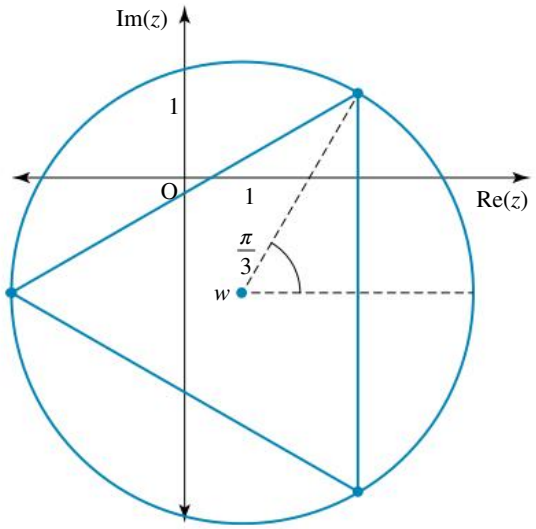
Complex unfamiliar

18. A student has created the following diagram on the Argand plane.

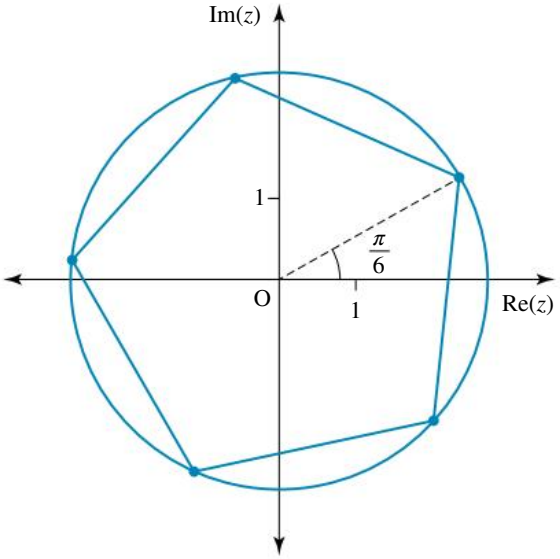


Determine the coordinates of the vertices of these three regular polygons.

19. Determine the coordinates of the vertices of the equilateral triangle represented in the diagram. Give your answers in Cartesian form.



20. Determine the coordinates of the vertices of the regular pentagon represented in the diagram.



Fully worked solutions for this chapter are available online.

LESSON
1.5 Factorisation of polynomials

SYLLABUS LINKS

- Apply the factor theorem and the remainder theorem for polynomials.
- Understand and use the complex conjugate root theorem for polynomials with real coefficients, e.g. factorise a cubic polynomial with real coefficients given one factor.
- Solve polynomial equations over $\mathbb{C}$ to order 4 including those with real and imaginary coefficients, e.g. solve $z^4 + z^3 - z^2 + z - 2 = 0$ and $z^3 - 2iz^2 + z - 2i = 0$.

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1.5.1 The fundamental theorem of algebra

A polynomial in z is an expression in the form $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, a_n \neq 0$, where $n \in \mathbb{N}$ is the order of $P(z)$, that is, its highest power, and a_n are coefficients. If $a_n \in \mathbb{R}$, that is, if all the coefficients are real, then $P(z)$ is said to be a polynomial over $\mathbb{R}$. If at least one of the coefficients is complex, then $P(z)$ is said to be a polynomial over $\mathbb{C}$.

For example, $P(z) = 3z^4 - 5z^2 + 6$ is a polynomial of order 4 over $\mathbb{R}$, and $P(z) = 2iz^3 + 3z^2 - 8i$ is a polynomial of order 3 over $\mathbb{C}$.

Let's consider a polynomial $P(x)$ of order 1 or greater and a real number a. When dividing $P(x)$ by $(x - a)$, there exists a polynomial $Q(x)$, the quotient polynomial, and a real number r, the remainder, such that $P(x) = (x - a) \times Q(x) + r$. Now, $P(a) = (a - a) \times Q(x) + r = r$; that is, $P(a) = r$. This is the **remainder theorem**.

The remainder theorem

When a polynomial $P(x)$ of order 1 or greater is divided by a linear polynomial $(x - a)$, the remainder is equal to $P(a)$.

If a polynomial $P(x)$ of order 1 or greater has $(x - a)$ as a factor, then $P(x)$ can be written as $P(x) = (x - a)Q(x)$ and $P(a) = (a - a)Q(a) = 0$.

That is, if $(x - a)$ is a factor of polynomial $P(x)$, then $P(a) = 0$. We can show, using the remainder theorem, that the converse is true; that is, if $P(a) = 0$, then $(x - a)$ is a factor of polynomial $P(x)$.

This result is known as the **factor theorem**.

The factor theorem

If $P(x)$ is a polynomial of order 1 or greater and a is a real number, then $(x - a)$ is a factor of $P(x)$ if $P(a) = 0$, and if $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

The factor theorem holds true for complex numbers as well.

In 1799 the German mathematician Carl Friedrich Gauss proved that every polynomial over $\mathbb{C}$ (i.e. with at least one complex coefficient) of order 1 or greater has at least one solution that is a complex number. This is the **fundamental theorem of algebra**.

The fundamental theorem of algebra

Every polynomial $P(z)$ over $\mathbb{C}$ of order 1 or greater has at least one complex root.

That is, if $P_n(z)$ is a polynomial of order n over $\mathbb{C}$, then there exists $z_1 \in \mathbb{C}$ such that $P_n(z_1) = 0$.

This important result can be used to show that a polynomial $P_n(z)$ of order n, where $n \in \mathbb{N}$, has n solutions. The proof relies on repeated application of the fundamental theorem of algebra and the factor theorem.

Firstly, the fundamental theorem of algebra guarantees that $\exists z_1 \in \mathbb{C}$ such that $P_n(z_1) = 0$.

The factor theorem states that if $P_n(z_1) = 0$ for some z_1, then $(z - z1)$ is a factor of $P_n(z)$, so that $P_n(z) = (z - z_1) \times P_{n-1}(z)$ where $P_{n-1}(z)$ is a polynomial of order $n - 1$.

Now by applying the fundamental theorem of algebra to $P_{n-1}(z)$, $\exists z_2 \in \mathbb{C}$ such that $P_{n-1}(z_2) = 0$, and the factor theorem ensures that $P_{n-1}(z) = (z - z_2) \times P_{n-2}(z)$. Hence, $P_n(z) = (z - z_1)(z - z_2) \times P_{n-2}(z)$.

By applying this method to each successive polynomial, the fundamental theorem of algebra can also be stated as follows.

The fundamental theorem of algebra — alternative statement

A polynomial of order n will always have exactly n linear factors.

$P_n(z) = (z - z_1)(z - z_2)\ldots(z - z_n)P_0(z)$ where $z_1, z_2, \cdots, z_n \in \mathbb{C}$ and $P_0(z)$ is a constant.

Note: Although n solutions are obtained, the fundamental theorem of algebra does not prescribe that they are necessarily distinct.

1.5.2 Solving quadratic equations

You have seen in Unit 2 that, given a quadratic equation with real coefficients, if the discriminant is negative, then the roots occur in complex conjugate pairs, and that a relationship can be formed between the roots and the coefficients.

Quadratic equations with real coefficients

The quadratic equation $az^2 + bz + c = 0$, where a, b and $c \in \mathbb{R}$ and $a \neq 0$, has a discriminant $\Delta = b^2 - 4ac$. Its number of roots depends on the sign of Δ.

- **If $\Delta > 0$, there are two distinct real roots, $\dfrac{-b \pm \sqrt{\Delta}}{2a}$.**
- **If $\Delta = 0$, there is one real root, $-\dfrac{b}{2a}$.**
- **If $\Delta < 0$, there is a pair of complex roots, $\dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-b \pm i\sqrt{-\Delta}}{2a}$.**

The sum of the roots is equal to $-\dfrac{b}{a}$ and the product of the roots is $\dfrac{c}{a}$.

A polynomial equation $P(z) = 0$ can be solved by first factorising $P(z)$ and then using the Null Factor Law, which states that if two or more factors multiplied together equal zero, then the solutions can be found be equating each factor separately to zero.

Solving factorised polynomial equations

If $P(z) = (z - z_1)(z - z_2)...(z - z_n) = 0$, then $z = \{z_1, z_2, \cdots, z_n\}$.

WORKED EXAMPLE 20 Solving a quadratic equation with real coefficients

Solve $z^2 + 6z + 13 = 0$ over $\mathbb{C}$ using the completing the square method.

THINK	WRITE
1. Use the 'complete the square' method.	$0 = z^2 + 6z + 13$ $0 = z^2 + 6z + 9 - 9 + 13$ $0 = (z+3)^2 + 4$
2. Express the RHS as the difference of two squares by converting 4 into complex form $(i^2 = -1)$.	$0 = (z+3)^2 - 4i^2$ $0 = (z+3)^2 - (2i)^2$
Write the expression in brackets as a quadratic.	$0 = (z+3+2i)(z+3-2i)$
3. Apply the Null Factor Law to state the solutions.	$(z+3+2i) = 0$ or $(z+3-2i) = 0$ The roots occur as a pair of complex conjugates: $z = -3-2i, \ -3+2i.$

WORKED EXAMPLE 21 Forming a quadratic equation with real coefficients, given one of the roots

Determine the equation of the quadratic $P(z)$ with real coefficients given that $P(-11 + 2i) = 0$.

THINK	WRITE
1. State the given root.	Let $\alpha = -11 + 2i$.
2. The conjugate is also a root.	Let $\beta = -11 - 2i$
3. State the linear factors.	$P(z) = (z - \alpha)(z - \beta)$ $= (z + 11 - 2i)(z + 11 + 2i)$
4. Expand the linear factors and state the quadratic equation.	$P(z) = (z+11)^2 - (2i)^2$ $= z^2 + 22z + 121 - 4i^2$ $= z^2 + 22z + 125$

1.5.3 Solving cubic equations

A cubic equation of the form $az^3 + bz^2 + cz + d$, where a, b, c and $d \in \mathbb{R}$ and $a \neq 0$ and $z \in \mathbb{C}$, will have three linear factors. The three roots can either be three real roots (not necessarily distinct) or they can be one real root and a pair of complex conjugate roots. That is, a cubic with real coefficients will have at least one real root.

We will discuss soon why there must be a conjugate pair.

TIP

Similar to the relationships between the coefficients and the roots of a quadratic equation, the following relationships exist between the three roots α, β and γ of a cubic equation and its coefficients a, b, c and d.

If $az^3 + bz^2 + cz + d = a(z-\alpha)(z-\beta)(z-\gamma), a \neq 0$, then:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

WORKED EXAMPLE 22 Solving a cubic equation with real coefficients

Determine the roots of $z^3 + 6z^2 + 61z + 106 = 0$.

THINK	WRITE
1. Determine the real root using trial and error.	$P(-1) = (-1)^3 + 6(-1)^2 + 61(-1) + 106$ $= 50$ $P(-2) = (-2)^3 + 6(-2)^2 + 61(-2) + 106$ $= 0$
2. Apply the factor theorem.	Therefore, $(z+2)$ is a factor.
3. We can find the quadratic factor using algebraic methods.	$P(z) = z^3 + 6z^2 + 61z + 106$ $= (z+2)\left(z^2 + bz + c\right)$ $= z^3 + bz^2 + cz + 2z^2 + 2bz + 2c$
4. Compare the two expressions for $P(z)$ by equating the constant term (the coefficient of the term z^0) to calculate c.	$106 = 2c$ [1] $c = 53$
5. Compare the two expressions for $P(z)$ by equating the coefficients of the other powers of z, and use equation [1] to calculate b.	$61z = 2bz + cz$ [2] $61z = 2bz + 53z$ $b = \dfrac{61z - 53z}{2z}$ $b = 4$
6. State $P(z)$ as the product of a linear and quadratic factor.	$P(z) = z^3 + 6z^2 + 61z + 106$ $= (z+2)\left(z^2 + 4z + 53\right)$

7. Factorise the quadratic equation by completing the square. Apply the Null Factor Law.	$0 = (z+2)\left(z^2+4z+4-4+53\right)$ $0 = (z+2)\left((z+2)^2+49\right)$ $0 = (z+2)\left((z+2)^2-(7i)^2\right)$ $0 = (z+2)(z+2+7i)(z+2-7i)$
8. State the three roots.	The roots are one real root and one pair of complex conjugates: $z=-2,\ -2+7i,\ -2-7i$.

The preceding results are true not only for quadratic and cubic equations but for any nth order polynomial. This is the **conjugate root theorem**.

The conjugate root theorem

If a polynomial with real coefficients has a complex root, then its conjugate is also a root of the polynomial.

Note that this does not hold true for polynomials over $\mathbb{C}$. That is, if one of the coefficients is a complex number, then the roots do not occur in conjugate pairs.

WORKED EXAMPLE 23 Solving a cubic equation with real coefficients, given a complex root

If $P(z) = z^3 + bz^2 + cz - 87 = 0$, where b and c are real, and $P(-2+5i) = 0$, determine the values of b and c, and state all the roots of $P(z) = 0$.

THINK	WRITE
1. State the given root.	Let $\alpha = -2+5i$.
2. The conjugate is also a root.	Let $\beta = -2-5i$
3. Express $P(z)$ as the product of three linear factors.	$P(z) = (z-\alpha)(z-\beta)(z-\gamma)$ $= (z+2+5i)(z+2-5i)(z-\gamma)$
4. Expand the complex conjugate factors to form a quadratic factor.	$P(z) = \left((z+2)^2-(5i)^2\right)(z-\gamma)$ $= \left(z^2+4z+4-25i^2\right)(z-\gamma)$ $= \left(z^2+4z+29\right)(z-\gamma)$
5. Compare the two expressions for $P(z)$. Expand the linear and quadratic factors.	$z^3+bz^2+cz-87 = (z-\gamma)\left(z^2+4z+29\right)$ $= z^3+4z^2+29z-\gamma z^2-4\gamma z-29\gamma$ $= z^3+(4-\gamma)z^2+(29-4\gamma)z-29\gamma$
6. Compare the two expressions for $P(z)$ and equate the constant terms to determine γ.	$-87 = -\gamma \times 29 \quad [1]$ $\gamma = 3$
7. Compare the two expressions for $P(z)$ and equate the coefficients to determine b and c.	$bz^2 = (4-\gamma)z^2$ and $cz = (29-4\gamma)z$ $b\cancel{z^2} = (4-3)\cancel{z^2}$ and $c\cancel{z} = (29-4\times 3)\cancel{z}$ $b = 1$ and $c = 17$

8. State the values for b and c along with the three zeros of $P(z)$ over $\mathbb{C}$.	$P(z) = (z-3)(z+2+5i)(z+2-5i) = 0$ $\therefore b = 1$ and $c = 17$ The roots are one real root and one pair of complex conjugates: $z = 3,\ -2-5i,\ -2+5i$.

1.5.4 Solving quartic equations

A quartic equation of the form $az^4 + bz^3 + cz^2 + dz + e$, where a, b, c, d and $e \in \mathbb{R}$ and $a \neq 0$ and $z \in \mathbb{C}$, will have four linear factors. The four roots can be four real roots (not necessarily distinct), or they can be two real roots and a pair of complex conjugate roots, or they can be two pairs of complex conjugate roots.

WORKED EXAMPLE 24 Solving a quartic equation with real coefficients

Solve for z if $z^4 + 10z^2 - 11 = 0$.

THINK	WRITE
1. Use appropriate substitution to reduce the quartic equation to a quadratic equation.	Let $a = z^2$, then $a^2 = z^4$. $z^4 + 10z^2 - 11 = 0$ $a^2 + 10a - 11 = 0$
2. Factorise the new expression.	$(a+11)(a-1) = 0$
3. Substitute z^2 for a.	$\left(z^2+11\right)\left(z^2-1\right) = 0$
4. Express each factor as the difference of two squares by recalling $i^2 = -1$.	$\left(z^2 - \left(\sqrt{11}i\right)^2\right)\left(z^2 - (1)^2\right) = 0$
5. Express $P(z)$ as the product of four linear factors.	$(z+\sqrt{11}i)(z-\sqrt{11}i)(z+1)(z-1) = 0$
6. Apply the Null Factor law and state the roots.	$z = 1,\ -1,\ \sqrt{11}i,\ -\sqrt{11}i$

In the previous worked examples, algebraic methods were used to determine unknown coefficients for quadratic factors. Polynomial long division is an alternative method and is demonstrated in the following worked example.

WORKED EXAMPLE 25 Using polynomial long division to determine the roots of a polynomial with real coefficients

Given that $P(4) = 0$ where $P(z) = z^4 - 2z^3 - 6z^2 + 32z - 160$, use polynomial long division to calculate all the roots for $P(z)$.

THINK	WRITE
1. State the given real root.	Let $z_1 = 4$.
2. Since one root is real, determine another real root by testing $P(-4)$.	$P(-4) = (-4)^4 - 2(-4)^3 - 6(-4)^2 + 32(-4) - 160$ $= 0$ Thus, $z_2 = -4$ is another root.
3. Express $P(z)$ as the product of two quadratic factors.	$P(z) = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$ $= (z-4)(z+4)(z-z_3)(z-z_4)$ $= \left(z^2-16\right)\left(az^2+bz+c\right)$

4. Complete polynomial long division to calculate the unknown quadratic factor.

$$
\begin{array}{r|l}
 & z^2 - 2z + 10 \\
\hline
z^2 - 16 & z^4 - 2z^3 - 6z^2 + 32z - 160 \\
 & -(z^4 \quad \downarrow \quad - 16z^2 \quad \downarrow \quad \downarrow) \\
\hline
 & -2z^3 + 10z^2 + 32z - 160 \\
 & -(\ -2z^3 \quad \downarrow \quad + 32z \quad \downarrow) \\
\hline
 & 10z^2 \quad - 160 \\
 & -(\quad 10z^2 \quad - 160) \\
\hline
 & 0
\end{array}
$$

5. Express $P(z)$ as the product of two quadratic factors by using the result from the long division operation.

$$P(z) = \left(z^2 - 16\right)\left(z^2 - 2z + 10\right)$$

6. Factorise the quadratic equation by completing the square.

$$
\begin{aligned}
P(z) &= \left(z^2 - 16\right)\left(z^2 - 2z + 1 - 1 + 10\right) \\
&= \left(z^2 - 16\right)\left((z-1)^2 + 9\right) \\
&= \left(z^2 - 16\right)\left((z-1)^2 - (3i)^2\right)
\end{aligned}
$$

7. Express $P(z) = 0$ as the product of four linear factors.

$$0 = (z+4)(z-4)(z-1+3i)(z-1-3i)$$

8. Apply the Null Factor Law to state the four solutions for $P(z) = 0$.

The solutions occur as a pair of real roots and a pair of complex conjugate roots:
$z = -4,\ 4,\ 1-3i,\ 1+3i.$

1.5.5 Solving polynomial equations with complex coefficients

Polynomials always have linear factors, and the number of linear factors (not necessarily distinct) is equal to the order of the polynomial (i.e. the highest power of n). When polynomials have real coefficients, the roots can be real or complex, and if they are complex, they occur in conjugate pairs. However, for polynomials over $\mathbb{C}$, that is, with at least one complex coefficient, determining the factors and solving equations require other techniques. For polynomials of order 2, the quadratic formula is convenient to use when $\sqrt{\Delta}$ is easy to determine, such as when Δ is real.

Remember that the conjugate root theorem **cannot** be used for a polynomial equation with complex coefficients.

WORKED EXAMPLE 26 Solving a quadratic equation with complex coefficients using the quadratic formula when $\Delta \in \mathbb{R}$

Solve $z^2 + 4iz + 5 = 0$ over $\mathbb{C}$ and verify your answer.

THINK	WRITE
1. Calculate $\Delta = b^2 - 4ac$.	$\Delta = b^2 - 4ac$ $= (4i)^2 - 4 \times 5$ $= -16 - 30$ $= -36$ $= (6i)^2$

2. Apply the quadratic formula $z = \dfrac{-b \pm \sqrt{\Delta}}{2a}$.

$$\begin{aligned} z &= \frac{-b \pm \sqrt{\Delta}}{2a} \\ &= \frac{-4i \pm \sqrt{-36}}{2} \\ &= \frac{-4i \pm 6i}{2} \\ &= 2i \pm 3i \end{aligned}$$

3. Write your answer.

$z = \{-5i, i\}$

4. Verify the answer by substituting $z = -5i$ and $z = i$ in the equation $z^2 + 4i + 5i = 0$ and checking LHS = RHS.

If $z = -5i$:

$$\begin{aligned} \text{LHS} &= (-5i)^2 + 4i(-5i) + 5 \\ &= -25 + 20 + 5 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

If $z = i$:

$$\begin{aligned} \text{LHS} &= (i)^2 + 4i(i) + 5 \\ &= -1 - 4 + 5 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

When Δ is not real, using the quadratic formula requires the calculation of $\sqrt{\Delta}$. This can be done using technology, but it can also be done using the following method.

Consider the complex number $z = a + bi$, where $a, b \in \mathbb{R}$ and the complex number w.

$$\text{If } z^2 = w \text{ then } (a + bi)^2 = w$$

Thus

$$(a^2 - b^2) + 2abi = \text{Re}(w) + i\text{Im}(w)$$

$$\Rightarrow \begin{cases} (a^2 - b^2) = \text{Re}(w) \\ ab = \dfrac{\text{Im}(w)}{2} \end{cases}$$

Therefore, to determine a complex number z such that $z^2 = w$, we have to determine two real numbers such that their product is equal to $\dfrac{\text{Im}(w)}{2}$ and the difference of their square is equal to $\text{Re}(w)$. This can be done by inspection, considering the real factors of $\dfrac{\text{Im}(w)}{2}$.

WORKED EXAMPLE 27 Solving a quadratic equation with complex coefficients using the quadratic formula when $\Delta \in \mathbb{C}$

Solve $z^2 + (3 - i)z + (14 - 5i) = 0$ over $\mathbb{C}$.

THINK

1. Calculate $\Delta = b^2 - 4ac$.

WRITE

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (3 - i)^2 - 4 \times (14 - 5i) \\ &= 8 - 6i - 56 + 20i \\ &= -48 + 14i \end{aligned}$$

2. Determine $a, b \in \mathbb{R}$ such that $(a+bi)^2 = \Delta$.

$$\begin{aligned}(a+bi)^2 &= \left(a^2-b^2\right)+2abi\\ &= -48+14i\\ &\Rightarrow \begin{cases} a^2-b^2=-48\\ 2ab=14\end{cases}\\ &\Rightarrow \begin{cases} a^2-b^2=-48\\ ab=7\end{cases}\end{aligned}$$

a and b are real, and the only real factors of 7 are ± 1 and ± 7.
We can observe that $1^2-7^2=-48$ and $1\times 7=7$, and that $(-1)^2-(-7)^2=-48$ and $(-1)\times(-7)=7$.
Therefore, by inspection, $(1+7i)^2=\Delta$ and $(-1-7i)^2=\Delta$.
Alternatively, if a and b cannot be determined easily by inspection:

$$\begin{cases} a^2-b^2=-48\\ 2ab=14\end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{49}{b^2}-b^2=-48\\ a=\dfrac{7}{b}\end{cases}$$

$$\Rightarrow \begin{cases} 49-b^4+48b^2=0\\ a=\dfrac{7}{b}\end{cases}$$

$$\Rightarrow \begin{cases} \left(b^2-49\right)\left(b^2+1\right)=0\\ a=\dfrac{7}{b}\end{cases}$$

$$\Rightarrow \begin{cases} (b-7)(b+7)=0\\ a=\dfrac{7}{b}\end{cases}$$

$$\Rightarrow \begin{cases} b=7 \text{ and } a=1\\ b=-7 \text{ and } a=-1\end{cases}$$

3. Apply the quadratic formula $z=\dfrac{-b\pm\sqrt{\Delta}}{2a}$.

$$\begin{aligned}(\pm\,(1+7i))^2 &= \Delta\\ \sqrt{\Delta} &= \pm(1+7i)\end{aligned}$$

$$\begin{aligned}z &= \frac{-b\pm\sqrt{\Delta}}{2a}\\ &= \frac{-(3-i)\pm(1+7i)}{2}\\ &= \{-1+4i, -2-3i\}\end{aligned}$$

4. Write your answer.

$z=\{-1+4i, -2-3i\}$

For polynomials of order 2, we can also use the fact that the sum of the roots is equal to $-\dfrac{b}{a}$ and the product of the roots is $\dfrac{c}{a}$.

WORKED EXAMPLE 28 Solving a quadratic equation with complex coefficients using the sum and product of roots

Factorise $P(z) = z^2 + 9iz - 8$ and solve $P(z) = 0$ over $\mathbb{C}$.

THINK	WRITE
1. To factorise $2z^2 + 9iz - 8$, determine two complex numbers, α and β, such that their sum is equal to $-\frac{b}{a}$ and their product is equal to $\frac{c}{a}$.	Let's determine α and $\beta \in \mathbb{C}$ such that $\alpha + \beta = -9i$ and $\alpha\beta = -8$. We can observe that $-9i = (-8i) + (-i)$ and $-8 = (-8i) \times (-i)$. Therefore, $-8i$ and $-i$ are two such numbers.
2. Factorise $P(z)$ using the two numbers determined, which are roots.	$P(z) = (z - (-8i))(z - (-i))$ $= (z + 8i)(z + i)$
3. Verify the factorisation by expanding $(z + 8i)(z + i)$.	$(z + 8i)(z + i) = z^2 + iz + 8iz + 8i^2$ $= z^2 + 9iz - 8$ $= P(z)$
4. Solve $P(z) = 0$ using the Null Factor Law.	$P(z) = 0$ $(z + 8i)(z + i) = 0$ $z = \{-8i, -i\}$

The factor theorem states that if $P(a) = 0$, then $(z - a)$ is a factor. This can be used to identify some factors first, especially for cubic and quartic equations.

TIP

When factorising polynomials, it is useful to only consider factors of the constant term when testing for values of a such that $P(a) = 0$. For instance, if the constant term is $-10i$, first test $-10i, 10i, -10, 10, -5i, 5i, -5, 5, -2i, 2i, -2, 2, -i, i, -1, 1$ as potential roots.

WORKED EXAMPLE 29 Solving a cubic equation with complex coefficients using the factor theorem

Solve $z^3 - 2iz^2 + z - 2i = 0$ over $\mathbb{C}$.

THINK	WRITE
1. Test factors of the constant term $-2i$, that is, $-2i, 2i, -i, i, -1, 1$, as potential roots.	Let $P(z) = z^3 - 2iz^2 + z - 2i$. $P(-2i) = (-2i)^3 - 2i(-2i)^2 + (-2i) - 2i$ $= 8i - 2i \times -4 - 4i$ $= 12i$ $\neq 0$ $P(2i) = (2i)^3 - 2i(2i)^2 + (2i) - 2i$ $= -8i - 2i \times -4$ $= 8i - 8i$ $= 0$

	$P(-i) = (-i)^3 - 2i(-i)^2 + (-i) - 2i$ $= i + 2i - i + 2i$ $= 0$ $P(i) = (i)^3 - 2i(i)^2 + i - 2i$ $= -i + 2i + i - 2i$ $= 0$ We could test 1 and -1, but we already have three out of three factors, so it is unnecessary.
2. Apply the factor theorem.	$P(2i) = 0$; therefore, $(z - 2i)$ is a factor of $P(z)$. $P(-i) = 0$; therefore, $(z + i)$ is a factor of $P(z)$. $P(i) = 0$; therefore, $(z - i)$ is a factor of $P(z)$.
3. Factorise the polynomial $P(z)$.	$P(z)$ is of order 3; therefore, it will have three linear factors. $(z - 2i)$, $(z + i)$ and $(z - i)$ are factors of $P(z)$; therefore, $P(z) = (z - 2i)(z - i)(z + i)$.
4. Solve $P(z) = 0$ over $\mathbb{C}$ using the Null Factor Law.	$P(z) = 0$ $(z - 2i)(z - i)(z + i) = 0$ $z = \{-i, i, 2i\}$

WORKED EXAMPLE 30 Solving a quartic equation with complex coefficients using the factor theorem

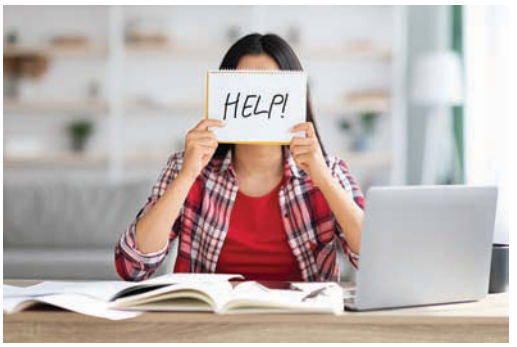
Solve $z^4 + z^3 - z^2 + z - 2 = 0$ over $\mathbb{C}$.

THINK	WRITE
1. Consider the factors of the constant term: $-2i, 2i, -2, 2, -i, i, -1, 1$.	Let $P(z) = z^4 + z^3 - z^2 + z - 2$. $P(-2) = 0$ $P(2) \neq 0$ $P(-2i) \neq 0$ $P(2i) \neq 0$ $P(-1) \neq 0$ $P(1) = 0$ $P(-i) = 0$ $P(i) = 0$
2. Apply the factor theorem.	$P(-2) = 0$; therefore, $(z + 2)$ is a factor of $P(z)$. $P(1) = 0$; therefore, $(z - 1)$ is a factor of $P(z)$. $P(-i) = 0$; therefore, $(z + i)$ is a factor of $P(z)$. $P(i) = 0$; therefore, $(z - i)$ is a factor of $P(z)$.
3. Factorise the polynomial $P(z)$.	The order of the polynomial $P(z) = z^4 + z^3 - z^2 + z - 2$ is 4; therefore, it will have four linear factors. $(z + 2)$, $(z - 1)$, $(z + i)$ and $(z + i)$ are linear factors; therefore, $P(z) = (z + 2)(x - 1)(z - i)(z + i)$.
4. Solve $P(z) = 0$ over $\mathbb{C}$ using the Null Factor Law.	$P(z) = 0$ $(z + 2)(x - 1)(z - i)(z + i) = 0$ $z = \{-2, -i, i, 1\}$

When the constant term has a large number of factors, it is not necessarily convenient to test them all.

For example, if the constant term is 60, there are 52 possible roots:

$\pm 1, \pm i$	$\pm 12, \pm 12i$
$\pm 2, \pm 2i$	$\pm 15, \pm 15i$
$\pm 3, \pm 3i$	$\pm 20, \pm 20i$
$\pm 4, \pm 4i$	$\pm 20, \pm 20i$
$\pm 5, \pm 5i$	$\pm 30, \pm 30i$
$\pm 6, \pm 6i$	$\pm 60, \pm 60i$
$\pm 10, \pm 10i$	



Testing all 52 possible roots until you have n factors would be time-consuming, unless the use of calculators is permitted.

In cases such as this, it might be quicker to determine just one factor, express the polynomial of order n as the product of that linear factor and a polynomial of order $n-1$ (using polynomial long division for example), and repeat the process until the n linear factors are determined.

WORKED EXAMPLE 31 Factorising a quartic equation with complex coefficients given two roots

Let $P(z) = z^4 - (2+i)z^3 + (17+2i)z^2 - (40-3i)z - 60$.

a. Show that $P(5i) = 0$ and $P(-4i) = 0$.

b. Factorise $P(z)$.

THINK	WRITE
a. 1. Substitute $z = 5i$ and show that LHS = RHS.	**a.** $\text{LHS} = P(5i)$ $= (5i)^4 - (2+i)(5i)^3 + (17+2i)(5i)^2 - (40-3i)(5i) - 60$ $= 625 + 125i(2+i) - 25(17+2i) - 200i - 15 - 60$ $= 625 + 250i - 125 - 425 - 50i - 200i - 15 - 60$ $= 0$ $= \text{RHS}$
2. Substitute $z = -4i$ and show that LHS = RHS.	$\text{LHS} = P(-4i)$ $= (-4i)^4 - (2+i)(-4i)^3 + (17+2i)(-4i)^2 - (40-3i)(-4i) - 60$ $= 256 - 64i(2+i) - 16(17+2i) + 160i + 12 - 60$ $= 256 - 128i + 64 - 272 - 32i - 160i + 12 - 60$ $= 0$ $= \text{RHS}$
b. 1. Apply the factor theorem.	**b.** $P(-4i) = 0$ and $P(5i) = 0$. Therefore, $(z+4i)$ and $(z-5i)$ are linear factors of $P(z)$. $P(z) = (z-5i)(az^3 + bz^2 + cz + d)$ $= az^4 + (b-5ia)z^3 + (c-5ib)z^2 + (d-5ic)z - 5id$

2. The order of $P(z)$ is 4, so two additional factors are needed. Determine $b, c \in \mathbb{C}$ such that $P(z) = (z-5i)(z+4i)(z^2+bz+c)$ by first expanding $(z-5i)(z+4i)(az^2+bz+c)$ and then equating it to $P(z)$.

$$\begin{aligned} P(z) &= \left(z^2 - iz + 20\right)(z^2 + bz + c) \\ &= z^4 - iz^3 + 20z^2 + bz^3 - biz^2 + 20bz + cz^2 - icz + 20c \\ &= z^4 + (b-i)z^3 + (20 - bi + c)z^2 + (20b - ic)z + 20c \end{aligned}$$

3. Recall that two polynomials are equal if they have the same order, and the corresponding terms have equal coefficients.

$$\begin{aligned} P(z) &= z^4 - (2+i)\,z^3 + (17+2i)\,z^2 - (40-3i)\,z - 60 \\ &= z^4 + (b-i)\,z^3 + (20 - bi + c)\,z^2 + (20b - ic)\,z + 20c \end{aligned}$$

$$\Rightarrow \begin{cases} (b-i) = -(2+i) \\ (20 - bi + c) = (17+2i) \\ (20b - ic) = -(40-3i) \\ 20c = -60 \end{cases}$$

4. Solve the system. Note that we will have 4 equations for only 2 unknowns, so we can use the remaining equations to verify our answer.

$$\begin{cases} (b-i) = -(2+i) \\ (20 - bi + c) = (17+2i) \\ (20b - ic) = -(40-3i) \\ 20c = -60 \end{cases}$$

$$\Rightarrow \begin{cases} (b-i) = -2 \\ (20 - bi + c) = (17+2i) \\ (20b - ic) = -(40-3i) \\ c = -3 \end{cases}$$

Verification:

$$\begin{aligned} (20 - bi + c) &= (20 + 2i - 3) \\ &= (17 + 2i) \end{aligned}$$

and

$$\begin{aligned} (20b - ic) &= -40i + 3i \\ &= -(40 - 3i) \end{aligned}$$

5. Write $P(z)$.

$$P(z) = (z-5i)(z+4i)(z^2 - 2z - 3)$$

6. Use the quadratic formula to factorise $(z^2 - 2z - 3)$.

$$\begin{aligned} \Delta &= 16 \\ z &= \frac{2 \pm \sqrt{16}}{2} \\ &= \{-1, 3\} \\ \therefore z^2 - 2z - 3 &= (z+1)\,(z-3) \end{aligned}$$

7. Factorise $P(z)$.

$$P(z) = (z-5i)(z+4i)(z+1)(z-3)$$

Quartic polynomials with only even powers of z can also be factorised using $Z = z^2$.

WORKED EXAMPLE 32 Solving a quartic equation with only even powers

Let $P(z) = z^4 - 8iz^2 - 25$.

Solve $P(z) = 0$ over $\mathbb{C}$.

THINK	WRITE
1. Substitute $Z = z^2$.	$P(z) = z^4 - 8iz^2 - 25$ Let $Z = z^2$. $P(Z) = Z^2 - 8iZ - 25$
2. Apply the quadratic formula.	$\Delta = (-8i)^2 - 4 \times 1 \times -25$ $= 36$ $Z = \dfrac{8i \pm 6}{2}$ $Z = -3 + 4i$ or $Z = 3 + 4i$.
3. Substitute $z^2 = Z$.	$z^2 = -3 + 4i$ or $z^2 = 3 + 4i$.
4. Determine $a, b \in \mathbb{R}$ such that $(a + bi)^2 = -3 + 4i$ $z^2 = -3 + 4i$.	Let $a, b \in \mathbb{R}$. $(a + bi)^2 = \left(a^2 - b^2\right) + 2abi$ $= -3 + 4i$ $\Rightarrow \begin{cases} a^2 - b^2 = -3 \\ 2ab = 4 \end{cases}$ $\Rightarrow \begin{cases} a^2 - b^2 = -3 \\ ab = 2 \end{cases}$ a and b are real, and the only real factors of 2 are ± 2 and ± 1. By inspection, we can observe that $1 \times 2 = 2$ and $1^2 - 2^2 = -3$, and that $(-1) \times (-2) = 2$ and $(-1)^2 - (-2)^2 = -3$. Therefore, $z^2 = -3 + 4i$ and as $Z^2 = Z$, $z = \pm(-1 + 2i)$. Alternatively, if a and b cannot be determined easily by inspection: $\begin{cases} a^2 - b^2 = -3 \\ 2ab = 4 \end{cases}$ $\Rightarrow \begin{cases} \dfrac{4}{b^2} - b^2 = -3 \\ a = \dfrac{2}{b} \end{cases}$ $\Rightarrow \begin{cases} 4 - b^4 + 3b^2 = 0 \\ a = -\dfrac{2}{b} \end{cases}$ $\Rightarrow \begin{cases} \left(b^2 - 4\right)\left(b^2 + 1\right) = 0 \\ a = -\dfrac{2}{b} \end{cases}$ $\Rightarrow \begin{cases} (b - 2)(b + 2) = 0 \\ a = -\dfrac{2}{b} \end{cases}$ $\Rightarrow \begin{cases} b = 2 \text{ and } a = -1 \\ b = -2 \text{ and } a = 1 \end{cases}$

Therefore,
$z^2 = -3 + 4i \Rightarrow z = \pm\ (-1 + 2i)$.

5. Determine $a, b \in \mathbb{R}$ such that $(a + bi)^2 = 3 + 4i$ to solve $z^2 = 3 + 4i$.

$$\begin{aligned}(a+bi)^2 &= \left(a^2 - b^2\right) + 2abi \\ &= 3 + 4i \\ &\Rightarrow \begin{cases} a^2 - b^2 = 3 \\ 2ab = 4 \end{cases} \\ &\Rightarrow \begin{cases} a^2 - b^2 = 3 \\ ab = 2 \end{cases}\end{aligned}$$

a and b are real, and the only real factors of 2 are ± 2 and ± 1. By inspection, we can observe that $2 \times 1 = 2$ and $2^2 - 1^2 = 3$, and that $(-2) \times (-1) = 2$ and $(-2)^2 - (-1)^2 = 3$.
Therefore,
$z^2 = 3 + 4i \Rightarrow z = \pm\ (-2 + i)$.

6. Factorise $P(z)$.

$P(z) = (z - 1 - 2i)(z - 1 + 2i)(z - 2 - i)(z + 2 + i)$

7. Solve $P(z) = 0$ over $\mathbb{C}$.

$P(z) = 0$
$z = \{\pm\ (-2 + i), \pm\ (-1 + 2i)\}$

Exercise 1.5 Factorisation of polynomials

learnon

1.5 Exercise | **1.5 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16, 17, 18, 19	20

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. WE20 Solve $z^2 - 14z + 74 = 0$ over $\mathbb{C}$ using the completing the square method.
2. Solve $z(z + 4) = -29$ for z using the quadratic formula.
3. WE21 Determine the equation of the quadratic $P(z)$ with real coefficients given that $P(-1 + 13i) = 0$.
4. WE22 Determine the roots of $z^3 + 9z^2 + 24z - 34 = 0$.
5. Solve the equation $z^3 + 13z^2 + 97z + 85 = 0$ over $\mathbb{C}$.
6. WE23 If $P(z) = z^3 + bz^2 + cz + 29 = 0$, where b and c are real, and $P(5 + 2i) = 0$, determine the values of b and c, and state all the roots of $P(z) = 0$.
7. Develop an equation with integer coefficients that has the following roots.
   a. $z_1 = -9,\ z_2 = 7 - 8i,\ z_3 = 7 + 8i$
   b. $z_1 = 11,\ z_2 = \alpha,\ z_3 = 4 - 7i,\ z_4 = \beta$

8. WE24 Solve for z if $z^4 + 5z^2 - 36 = 0$.

9. WE25 Given that $P(3) = 0$ where $P(z) = z^4 + 10z^3 + 25z^2 - 90z - 306$, use polynomial long division to calculate all the roots for $P(z)$.

10. If $z = 4$ is a zero of $P(z) = z^3 + 8z^2 + 69z - 468$, use polynomial long division to calculate all the roots for $P(z)$.

11. Given $P(z) = z^4 + bz^3 + 18z^2 + 32z + 32$ and $P(4i) = 0$, calculate the value of the real constant b and determine all the roots.

12. A quartic is given by the equation, $P(z) = \left(az^2 + bz + c\right)\left(z^2 - 10z + 41\right) + d$, where $d \in \mathbb{R}$. If $P(z) = z^4 - 8z^3 + 23z^2 + 62z + 81$ in expanded form, determine the value of d.

Complex familiar

13. WE26 Solve $z^2 + iz + 6 = 0$ over $\mathbb{C}$ and verify your answer.

14. WE27 Solve $z^2 - (1 - i)z + (4 + 7i) = 0$ over $\mathbb{C}$.

15. WE28 Factorise $P(z) = z^2 - 2iz + 15$ and solve $P(z) = 0$ over $\mathbb{C}$.

16. WE29 Solve $z^3 + (5 + i)z^2 + (6 + 5i)z + 6i = 0$ over $\mathbb{C}$.

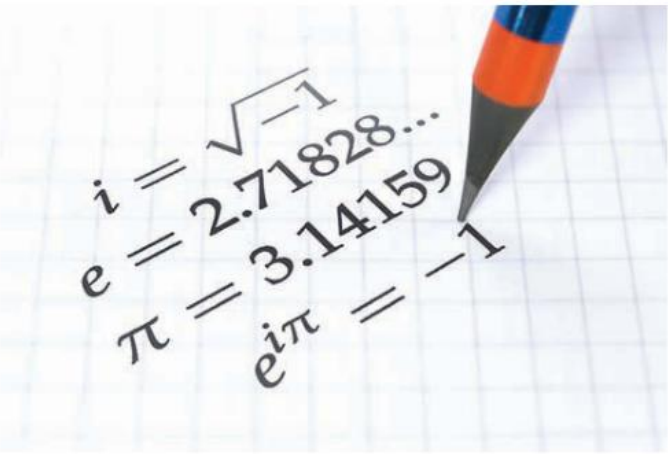
17. WE30 Solve $z^4 - (1 + 2i)z^3 + (1 + 2i)z^2 - (3 - 4i)z - 6 = 0$ over $\mathbb{C}$.

18. WE31 Let $P(z) = z^4 - (1 - 2i)z^3 + (15 + 9i)z^2 + (14 + 22i)z - (56 - 24i)$.
    a. Show that $P(-2) = 0$ and $P(4i) = 0$
    b. Factorise $P(z)$.

19. WE32 Let $P(z) = z^4 + 12iz^2 - 100$.
    Solve $P(z) = 0$ over $\mathbb{C}$.

Complex unfamiliar

20. Solve for z given $iz^8 + iz^4 + 1 + i = 0$. Give your answers in polar form.



Fully worked solutions for this chapter are available online.

LESSON
1.6 Review

1.6.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

- Access the chapter summary
- Review your results
- Practise exam questions

Find all this and MORE in jacPLUS

1.6 Exercise

learnon

1.6 Exercise	1.6 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. Determine z^{-1} given $z = 4 - 8i$.

2. Use De Moivre's theorem to simplify $\left(4 \operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^5 \div \left(2 \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^4$.

3. Use De Moivre's theorem to solve $z^3 = 125 \operatorname{cis}\left(\frac{-\pi}{4}\right)$ for z.

4. If $z = a + bi$, determine the solutions to the equation $\frac{z-1}{z+1} = z + 3$.

5. a. If $z^4 = \sqrt{3} + i$, determine the complex number z using a polar method.
   b. Plot the solutions on an Argand diagram and comment on their location.

6. a. Determine all solutions to the equation $z^6 = 1$ over $\mathbb{C}$. Express the solutions in polar form.
   b. Represent these solutions using a polar grid.

7. Solve the equation $z^4 + 55z^2 - 576 = 0$ over $\mathbb{C}$.

8. Solve $4z^2 + 4z + 82 = 0$ over $\mathbb{C}$ using the completing the square method.

9. Determine the equation of the quadratic $P(z)$ with real coefficients given that $P(6 - 5i) = 0$.

10. Determine the roots of $z^3 + 4z^2 - 2z - 20 = 0$.

11. Solve for z if $z^4 - 24z^2 - 1225 = 0$.

12. Use De Moivre's theorem to determine z^9, given $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Verify the solution by evaluating $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^9$ using technology.

Complex familiar

13. Prove the following identities using polar arithmetic.

a. $z\bar{z} = |z|^2$

b. $|z^n| = |z|^n$

14. Solve $z^4 - (2-2i)z^3 + (11-4i)z^2 + (22-12i)z - 24i = 0$ over $\mathbb{C}$.

15. If $P(z) = z^3 + bz^2 + cz + 29 = 0$, where b and c are real, and $P(5+2i) = 0$, determine the values of b and c, and state all the roots of $P(z) = 0$.

16. Given $P(z) = z^4 + 8z^3 + 30z^2 + 56z + 65 = (z+3+2i)(z-z_2)(z-z_3)(z-z_4)$, use polynomial long division to calculate all the roots for $P(z)$.

Complex unfamiliar

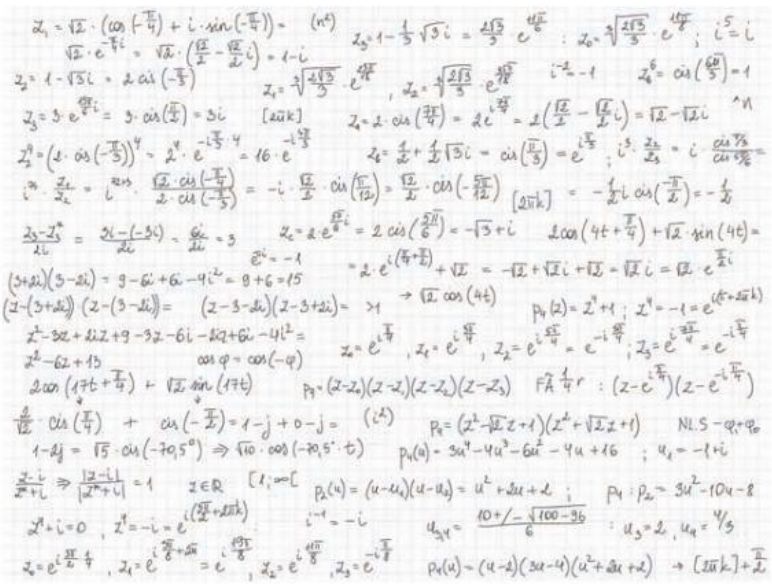
17. Determine a polynomial of degree 5 with integer coefficients that has the following roots.

$$z_1 = 6i,\ z_2 = 2-4i,\ z_3 = 3$$

18. Prove the triangle inequality given by $|z+v| \le |z| + |v|$.

19. If $z = \text{cis}(\theta)$, prove that $\frac{1+z}{1-z} = i\cot\left(\frac{\theta}{2}\right)$.

20. Let $P(z) = z^3 - 8z^2 + 23z + d$, where $d \in \mathbb{R}$.
Solve $P(z) = 0$ given that the sum of two of the roots of $P(z)$ is equal to its third root.



1.6 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2020 Specialist Mathematics, Paper 1, Section 1, Q6; © State of Queensland (QCAA) 2024.

Given $z = 2 - 2i$ and $w = -3 + i$, calculate $z^2 - \overline{w}$

A. $3 - 9i$
B. $3 - 7i$
C. $11 - 9i$
D. $11 - 7i$

Question 2 (1 mark)

Source: QCAA 2021 Specialist Mathematics, Paper 1, Section 1, Q2; © State of Queensland (QCAA) 2024.

When the polynomial $P(z) = z^3 - iz^2 - z - i$ is divided by $z - i$, the remainder is

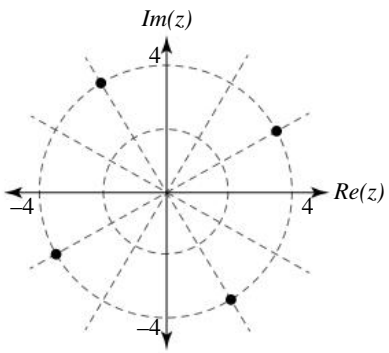
A. $-2i$
B. 0
C. $2i$
D. $4i$

Question 3 (1 mark)

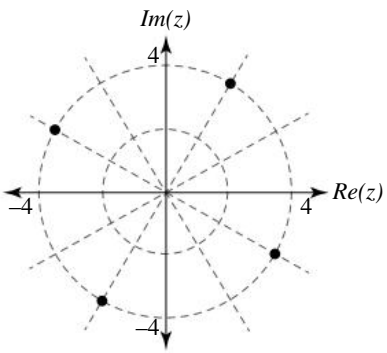
Source: QCAA 2023 Specialist Maths, Paper 2, Section 1, Q10; © State of Queensland (QCAA) 2024.

The Argand diagram that represents the solutions to $z^4 = 16\,\text{cis}\left(\frac{2\pi}{3}\right)$, $z \in C$ is

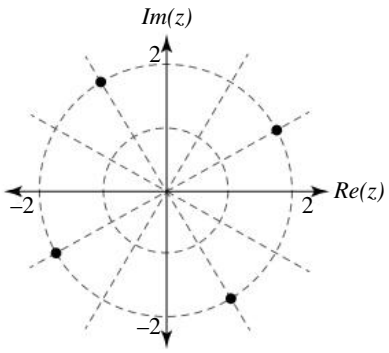
A.



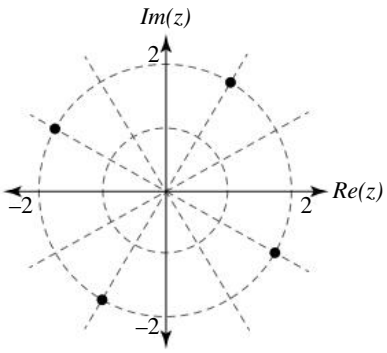
B.



C.



D.



Question 4 (6 marks)

Source: QCAA 2021 Specialist Mathematics, Paper 1, Section 2, Q13; © State of Queensland (QCAA) 2024.

Use $z = a + bi$ and $w = c + di$, where $a, b, c, d \in R$, to prove $|z - w|^2 = |z|^2 + |w|^2 - 2\,Re\left(z\overline{w}\right)$

Question 5 (6 marks)

Source: QCAA 2021 Specialist Mathematics, Paper 2, Section 2, Q18; © State of Queensland (QCAA) 2024.

Consider the polynomial $P(z) = z^3 + az^2 + bz + c$, where $a, b, c \in R$ and $z \in C$.

Two of the roots of $P(z)$ are also roots of $z^4 + z^3 + z^2 + z + 1$. The remaining root of $P(z)$ is $z = 2$.

Given $z^5 - 1 = (z - 1)\left(z^4 + z^3 + z^2 + z + 1\right)$, determine a possible expression for $P(z)$.

Leave your answer in expanded form.

Fully worked solutions for this chapter are available online.

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Answers

Chapter 1 Further complex numbers

1.2 Review of complex numbers

1.2 Exercise

1. a. 8 **b.** 2
c. -1 **d.** -2

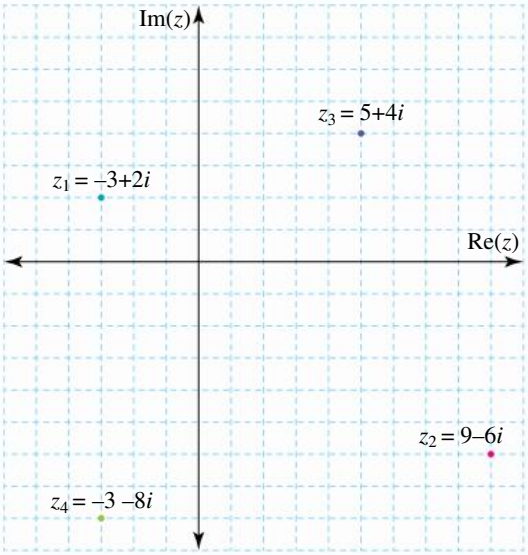
2. a. $15-7i$ **b.** $-12+24i$ **c.** $8+6i$

3. a. $\bar{e}=14-9i$ **b.** $\bar{f}=-2-\sqrt{7}i$
c. $\bar{g}=9+2\sqrt{7}i$ **d.** $h=-1+i$

4. a. $\frac{1}{2}-\frac{1}{2}i$ **b.** $\frac{1}{20}+\frac{3}{20}i$
c. $-\frac{3}{13}-\frac{2}{13}i$ **d.** $-\frac{1}{8}+\frac{1}{8}i$

5. $\frac{1}{29}-\frac{17}{29}i$

6.



7. a. $z=4\operatorname{cis}\left(\frac{\pi}{6}\right)$

b. $\frac{15}{2}\left(-1+\sqrt{3}i\right)$

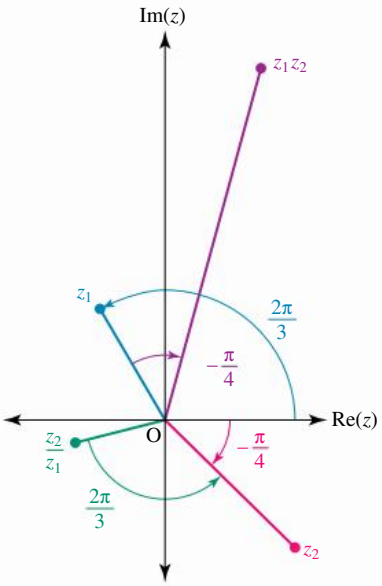
c. $z=\frac{2}{3}\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

d. $2\sqrt{2}(1-i)$

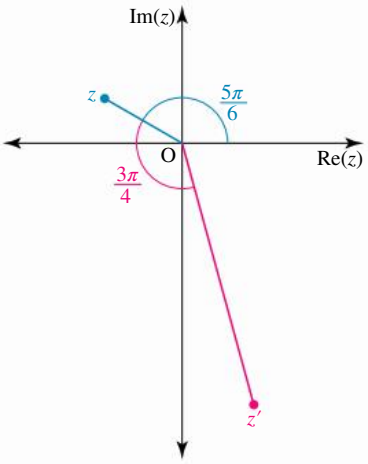
8. a. $12\operatorname{cis}\left(\frac{11\pi}{12}\right)$ **b.** $3\operatorname{cis}\left(\frac{7\pi}{12}\right)$

9. a. $z_1z_2=4\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$ and $\frac{z_2}{z_1}=\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{12}\right)$

b. z_1 is dilated by a factor $2\sqrt{2}$ and rotated by $-\frac{\pi}{4}$ anticlockwise about the origin.
z_2 is dilated by a factor $\frac{1}{2}$ and rotated by $\frac{2\pi}{3}$ clockwise about the origin.

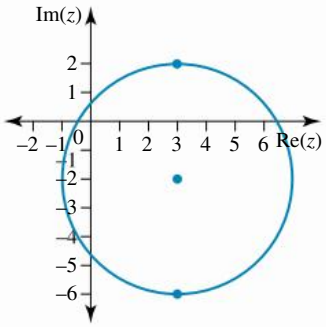


10. a.



b. $z'=6\operatorname{cis}\left(-\frac{5\pi}{12}\right)$

11. $(x-3)^2+(y+2)^2=16$; circle with centre $(3,-2)$ and radius 4.



12. $2\pm5i$

13. a. $a=1, b=5$ **b.** $a=-2,\ b=-3$

14. $\frac{17}{2}+\frac{9}{2}i$

15. $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

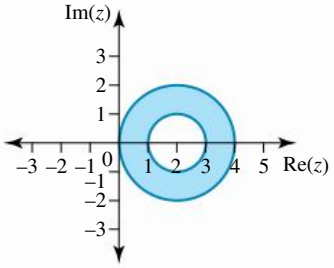
16. $-64\sqrt{3}-64i$

17. $a = -\frac{1}{2},\ b = \frac{1}{2}$

18. $z' = 2 - \frac{3\sqrt{3}}{4} - \left(2\sqrt{3} + \frac{3}{4}\right)i$

19. $k = 14$

20. The region lies on and between the two circles centred at $(2, 0)$ with radii of 1 and 2 units.



1.3 Complex arithmetic using polar form

1.3 Exercise

1–10. Sample responses can be found in the worked solutions in the online resources.

11. a. $32 \text{ cis}\left(-\frac{3\pi}{4}\right)$

b. $729 \text{ cis}\left(-\frac{\pi}{2}\right)$

c. $\frac{1}{81} \text{ cis}\left(\frac{-2\pi}{3}\right)$

d. 1

12. a. $23\,328 \text{ cis}\left(\frac{\pi}{2}\right)$

b. $128 \text{ cis}\left(\frac{3\pi}{4}\right)$

13. a. $81 \text{ cis}\left(\frac{\pi}{3}\right)$

b. $1\,048\,576 \text{ cis}\left(-\frac{14\pi}{15}\right)$

14. a. -64 **b.** $\frac{1}{512}$

15. a. i. 4096 **ii.** 0

b. i. $-46\,656$ **ii.** π

16. $a = 1025,\ b = 0$

17. Sample responses can be found in the worked solutions in the online resources.

18. $n = \frac{3(2k+1)}{5}$

19–20. Sample responses can be found in the worked solutions in the online resources.

1.4 Roots of complex numbers

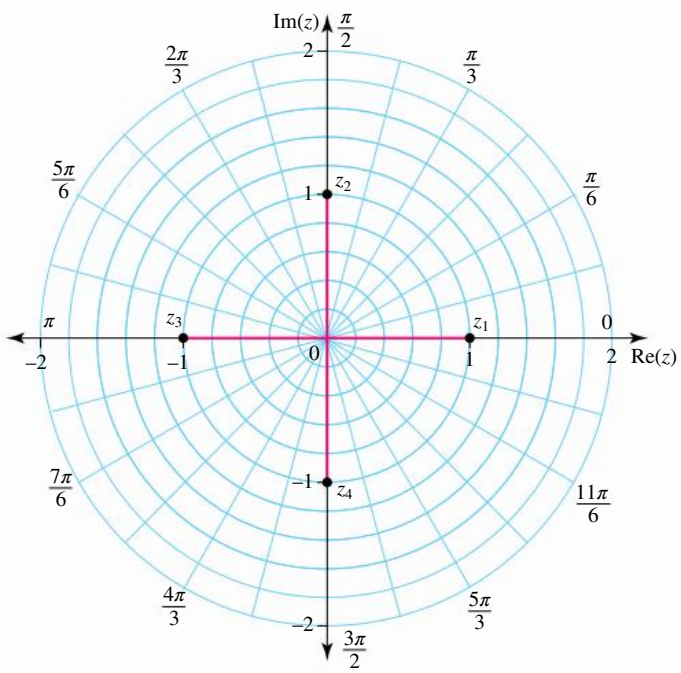
1.4 Exercise

1. a. $z = 1,\ \text{cis}\left(\frac{\pi}{2}\right), \text{cis}(\pi), \text{cis}\left(\frac{3\pi}{2}\right)$

b. See the figure at the bottom of the page*

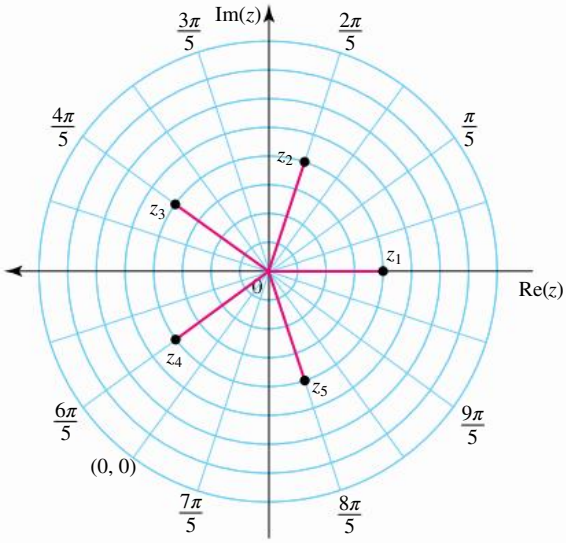
2–3. Sample responses can be found in the worked solutions in the online resources.

*1. b.



4. a. $z = 1,\ \text{cis}\left(\frac{\pi}{5}\right),\ \text{cis}\left(\frac{2\pi}{5}\right),\ \text{cis}\left(\frac{-4\pi}{5}\right),\ \text{cis}\left(\frac{-2\pi}{5}\right)$

b.



5. See the figure at the bottom of the page*

6. a. $z = -1 - \sqrt{3}i,\ z = -1 + \sqrt{3}i,\ z = 2$

b. $z = 4i,\ z = -4i,\ z = -4,\ z = 4$

7. The solutions would be equally spaced by $\frac{2\pi}{5}$ radians and lie on a circle of radius $10^{\frac{1}{5}}$. The modulus of each solution would change from $|z| = 1$ to $|z| = 10^{\frac{1}{5}}$.

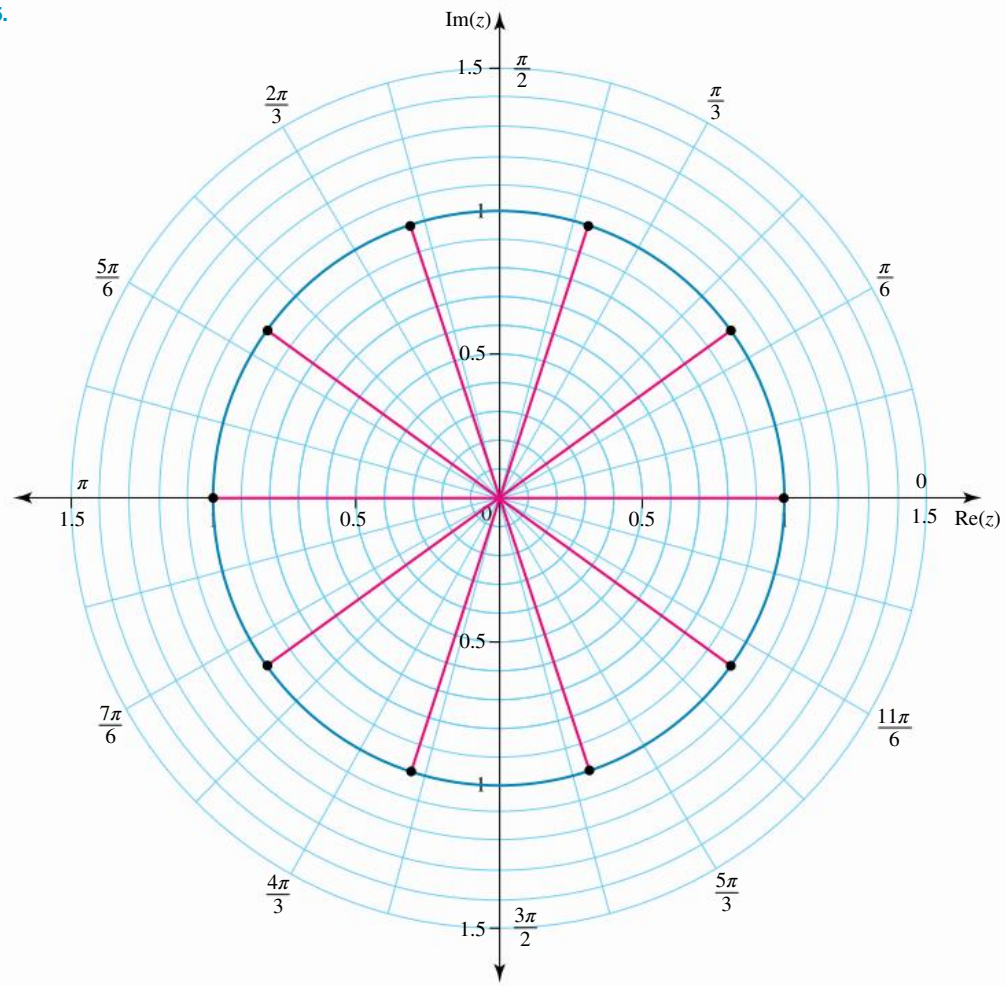
8. $z = \sqrt{5}\ \text{cis}\left(\frac{\pi}{12}\right),\ \sqrt{5}\ \text{cis}\left(-\frac{11\pi}{12}\right)$

9. $z = 3\ \text{cis}\left(\frac{2\pi}{9}\right),\ 3\ \text{cis}\left(\frac{8\pi}{9}\right),\ 3\ \text{cis}\left(-\frac{4\pi}{9}\right)$

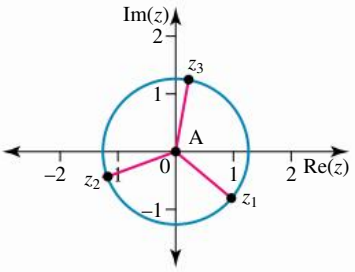
10. $z = 2\ \text{cis}\left(-\frac{3\pi}{16}\right),\ 2\ \text{cis}\left(\frac{5\pi}{16}\right),\ 2\ \text{cis}\left(\frac{13\pi}{16}\right),$
$2\ \text{cis}\left(-\frac{11\pi}{16}\right)$

11. a. $z = \sqrt[3]{2}\ \text{cis}\left(-\frac{2\pi}{9}\right),\ \sqrt[3]{2}\ \text{cis}\left(\frac{4\pi}{9}\right),\ \sqrt[3]{2}\ \text{cis}\left(-\frac{8\pi}{9}\right)$

***5.**



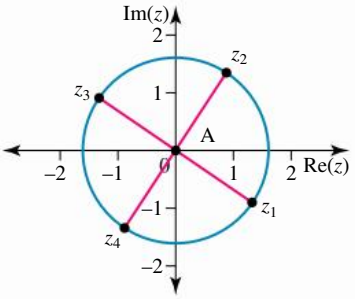
b.



The three roots are equally spaced around a circle of radius $\sqrt[3]{2}$. The angle between each solution is $\frac{2\pi}{3}$.

12. a. $z = \sqrt[4]{5\sqrt{2}}\,\text{cis}\left(-\frac{3\pi}{16}\right), \sqrt[4]{5\sqrt{2}}\,\text{cis}\left(\frac{5\pi}{16}\right),$
$\sqrt[4]{5\sqrt{2}}\,\text{cis}\left(\frac{13\pi}{16}\right), \sqrt[4]{5\sqrt{2}}\,\text{cis}\left(-\frac{11\pi}{16}\right)$

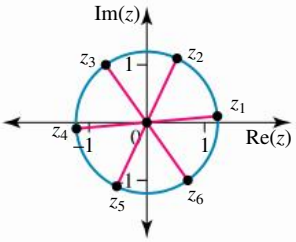
b.



The three roots are equally spaced around a circle of radius $\sqrt[4]{5\sqrt{2}} \approx 1.631$. The angle between each solution is $\frac{\pi}{2}$

13. a. $z = \sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{\pi}{36}\right), \sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{13\pi}{36}\right),$
$\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(\frac{25\pi}{36}\right), \sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{35\pi}{36}\right),$
$\sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{23\pi}{36}\right), \sqrt[6]{2\sqrt{3}}\,\text{cis}\left(-\frac{11\pi}{36}\right)$

b.



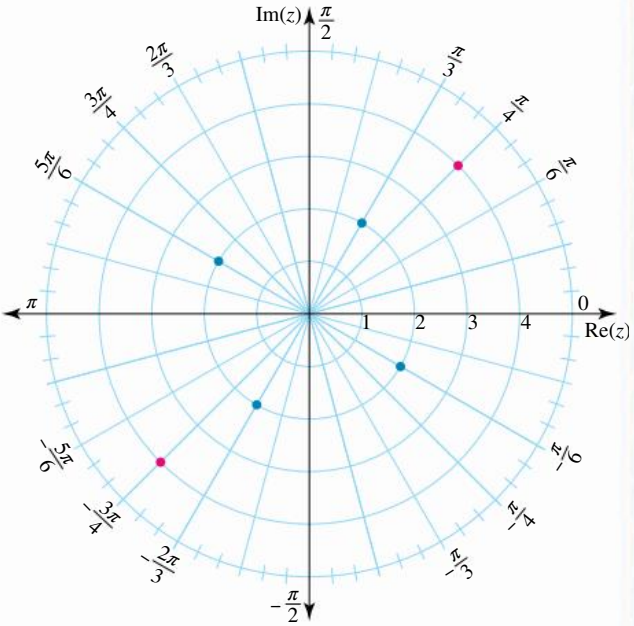
The three roots are equally spaced around a circle of radius $\sqrt[6]{2\sqrt{3}} \approx 1.230$. The angle between each solution is $\frac{\pi}{3}$.

14. a. $z = \text{cis}\left(\frac{\pi}{6}\right), \text{cis}\left(\frac{5\pi}{6}\right), \text{cis}\left(-\frac{\pi}{2}\right)$

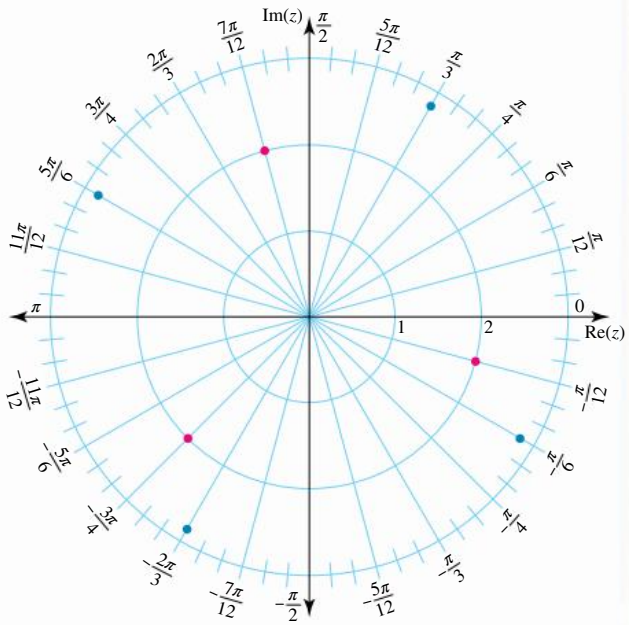
b. $z = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}\right), 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{8}\right),$
$2\sqrt{2}\,\text{cis}\left(-\frac{7\pi}{8}\right), 2\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{8}\right)$

15. $\sqrt{3} + i,\ -\sqrt{3} - i$

16.



17.



18. For the pentagon: $z = \left\{1, \text{cis}\left(\frac{2\pi}{5}\right), \text{cis}\left(\frac{4\pi}{5}\right),\right.$
$\left.\text{cis}\left(\frac{6\pi}{5}\right) \text{cis}\left(\frac{8\pi}{5}\right)\right\}$

For the hexagon: $z = \left\{2, 2\,\text{cis}\left(\frac{\pi}{3}\right), 2\,\text{cis}\left(\frac{2\pi}{3}\right),\right.$
$\left.-2, 2\,\text{cis}\left(\frac{4\pi}{3}\right), 2\,\text{cis}\left(\frac{5\pi}{3}\right)\right\}$

For the heptagon: $z = \left\{3, 3\,\text{cis}\left(\frac{2\pi}{7}\right), 3\,\text{cis}\left(\frac{4\pi}{7}\right),\right.$
$3\,\text{cis}\left(\frac{6\pi}{7}\right), 3\,\text{cis}\left(\frac{8\pi}{7}\right),\}$
$\left.3\,\text{cis}\left(\frac{10\pi}{7}\right), 3\,\text{cis}\left(\frac{12\pi}{7}\right)\right\}$

19. $z = \left\{3 + 2\left(\sqrt{3} - 1\right)i, -3 - 2i, 3 - 2\left(\sqrt{3} + 1\right)i\right\}$

20. $z = 3\,\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$

1.5 Factorisation of polynomials

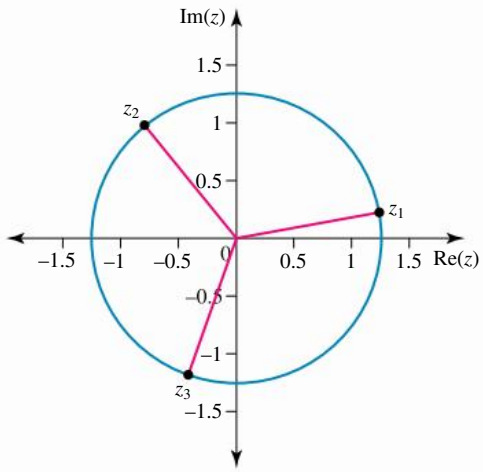
1.5 Exercise

1. $z = \{7 + 5i,\ 7 - 5i\}$
2. $z = \{-2 + 5i,\ -2 - 5i\}$
3. $P(z) = z^2 + 2z + 170$
4. $z = \{1,\ -5 - 3i,\ -5 + 3i\}$
5. $z = \{-1, -6 - 7i, -6 + 7i\}$
6. $b = -9,\ c = 19$
   $z = \{-1,\ 5 + 2i,\ 5 - 2i\}$
7. a. $P(z) = z^3 - 5z^2 - 13z + 1017$
   b. $P(z) = z^4 - 8z^3 - 56z^2 + 968z + 7865$
8. $z = \{2,\ -2,\ 3i,\ -3i\}$
9. $z = \{-3,\ 3,\ -5 - 3i,\ -5 + 3i\}$
10. $z = \{4,\ -6 - 9i,\ -6 + 9i\}$
11. $b = 2,\ z = \pm 4i,\ z = -1 \pm i$
12. $d = -1$
13. $z = \{-3i, 2i\}$
14. $z = \{2 - 3i, -1 + 2i\}$
15. $P(z) = (z - 5i)(z + 3i)$
    $P(z) = 0 \Rightarrow z = \{-3i, 5i\}$
16. $z = \{-i, -3, -2\}$
17. $z = \{-i, -1, 2, 3i\}$
18. a. Sample responses can be found in the worked solutions in the online resources.
    b. $P(z) = (z + 2)(z - 4i)(z - 2 + 5i)(z - 1 + i)$
19. $z = \{\pm(3 - i), \pm(1 - 3i)\}$
20. $z = \text{cis}\left(\frac{\pi}{8} + \frac{k\pi}{2}\right), k = 0, 1, 2, 3$ and
    $z = 2^{\frac{1}{8}}\,\text{cis}\left(\frac{5\pi}{16} + \frac{k\pi}{2}\right), k = 0, 1, 2, 3$

1.6 Review

1.6 Exercise

1. $z^{-1} = \frac{1}{20} + \frac{i}{10}$
2. $64\,\text{cis}\left(\frac{-2\pi}{3}\right)$
3. $z = \left\{5\,\text{cis}\left(\frac{-\pi}{12}\right), 5\,\text{cis}\left(\frac{7\pi}{12}\right), 5\,\text{cis}\left(\frac{-9\pi}{12}\right)\right\}$
4. $z = \left\{-\frac{3}{2} - \frac{\sqrt{7}}{2}i,\ -\frac{3}{2} + \frac{\sqrt{7}}{2}i\right\}$
5. a. $z = \sqrt[3]{2}\,\text{cis}\left(\frac{\pi}{18}\right), \sqrt[3]{2}\,\text{cis}\left(\frac{13\pi}{18}\right), \sqrt[3]{2}\,\text{cis}\left(-\frac{11\pi}{18}\right)$
   b.



The three roots are evenly spaced by $\frac{2\pi}{3}$ radians. The modulus of each root is $2^{\frac{1}{3}}$.

6. a. $z = \left\{\text{cis}(0), \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}(\pi), \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\left(-\frac{\pi}{3}\right)\right\}$

b. See the figure at the bottom of the page*

7. $z = \{8i, -8i, 3, -3\}$

8. $z = \left\{-\frac{1}{2} + \frac{9}{2}i,\ -\frac{1}{2} - \frac{9}{2}i\right\}$

9. $P(z) = z^2 - 12z + 61$

10. $z = \{-1,\ -3 - i,\ -3 + i\}$

11. $z = \{7,\ -7,\ 5i,\ -5i\}$

12. $z^9 = 1$

13. Sample responses can be found in the worked solutions in the online resources.

14. $z = \{-4i, -i, 2, 3i\}$

15. $z = \{5 + 2i,\ 5 - 2i,\ -1\}$

16. $z = \{-3 + 2i,\ -3 - 2i,\ -1 + 2i,\ -1 - 2i\}$

17. $P(z) = z^5 - 7z^4 + 68z^3 - 312z^2 + 1152z - 2160$

18-19. Sample responses can be found in the worked solutions in the online resources.

20. $z = \left\{2 - i\sqrt{3}, 2 + i\sqrt{3}, 4\right\}$

1.6 Past QCAA exam questions

1. B

2. A

3. D

4. RTP

$|z - w|^2 = |z|^2 + |w|^2 - 2\text{Re}(z\overline{w})$

$\text{LHS} = |z - w|^2$

$= |(a + bi) - (c + di)|^2$

$= |(a - c) + (b - d)i|^2$

$= (a - c)^2 + (b - d)^2$

$= a^2 - 2ac + c^2 + b^2 - 2bd + d^2$

$\text{RHS} = |z|^2 + |w|^2 - 2\text{Re}(z\overline{w})$

$= |a + bi|^2 + |c + di|^2 \ldots$

$\ldots - 2\text{Re}((a + bi)(c - di))$

$= a^2 + b^2 + c^2 + d^2 \ldots$

$\ldots - 2\text{Re}((ac + bd) + (bc - ad)i)$

$= a^2 + b^2 + c^2 + d^2 - 2(ac + bd)$

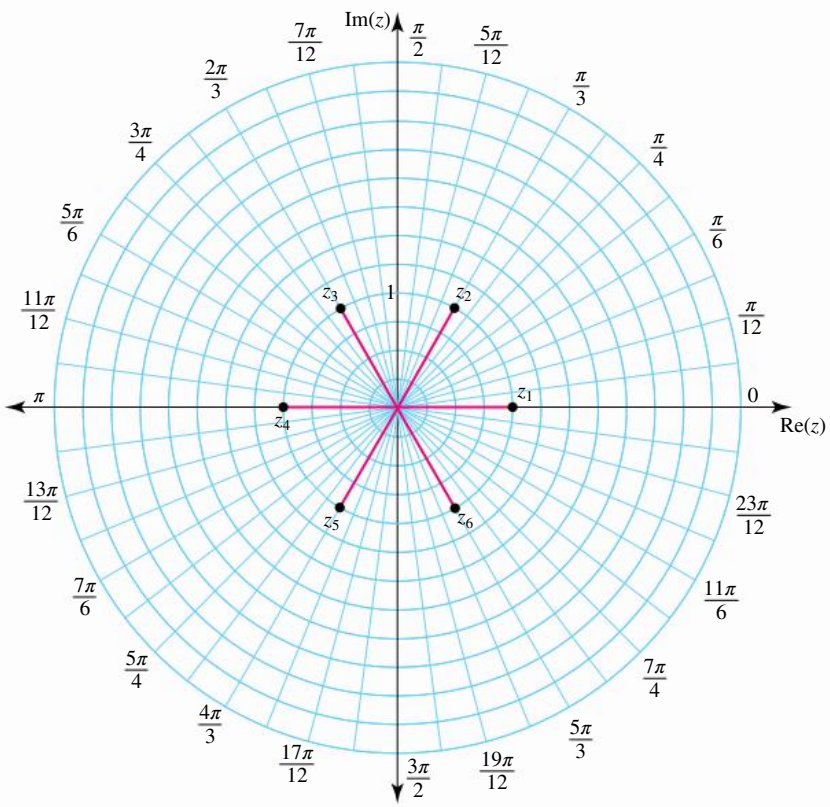
$= a^2 + b^2 + c^2 + d^2 - 2ac - 2bd$

$= a^2 - 2ac + c^2 + b^2 - 2bd + d^2$

$= \text{LHS}$

5. $P(z) = z^3 + az^2 + bz + c$ where $a \simeq -2.62$, $b \simeq 2.236$ and $c = -2$

*6. b.



2 Mathematical induction and trigonometric proofs

LESSON SEQUENCE

2.1 Overview 70
2.2 Proof by induction 71
2.3 Proofs of divisibility 85
2.4 Trigonometric proofs using De Moivre's theorem 88
2.5 Review 96
Answers 99

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 2
Exam questions	Exam question booklet — Chapter 2
Digital documents	Learning matrix — Chapter 2 Chapter summary — Chapter 2

LESSON
2.1 Overview

Hey students! Bring these pages to life online

Engage with interactivities | **Answer questions and check results** | **Track your progress**

Find all this and MORE in jacPLUS

2.1.1 Introduction

You were introduced to the concept of proofs in Unit 1 Chapter 2. In that chapter, we explored how to construct a logical argument based on definitions and axioms to prove that a statement or proposition is true.

These concepts are extended here, through proof by mathematical induction, which is a direct method of constructing a proof.

Euclid's proof that the number of primes is infinite contains suggestive use of proof by mathematical induction, but as a technique it was not until the tenth century when the Persian mathematician al-Karaji (953–1029) used a more overt style of induction-type proof to prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$. In 1838 the term mathematical induction was introduced, and a common understanding of the process was developed through the work of Augustus De Morgan (1806–1871).

2.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
2.2	**Proof by induction**	◯ Understand the nature of inductive proof including the use of initial statement, assumption statement, inductive step and conclusion. ◯ Use sigma notation $(\sum)$ to represent a sum, e.g. $\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3$ and $\sum_{j=0}^{2n+1} 2^j = 2^0 + 2^1 + 2^2 + \cdots + 2^{2n+1}$. ◯ Prove results for sums for any positive integer n.
2.3	**Proofs of divisibility**	◯ Prove divisibility results for any positive integer n.
2.4	**Trigonometric proofs using De Moivre's theorem**	◯ Prove De Moivre's theorem for powers of positive integers. ◯ Prove multi-angle trigonometric identities up to angles of $4x$ by equating parts using the binomial expansion and De Moivre's theorem, e.g. $\cos(3x) = 4\cos^3(x) - 3\cos(x)$ and $\sin(3x) = 3\sin(x) - 4\sin^3(x)$.

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LESSON
2.2 Proof by induction

SYLLABUS LINKS

- Understand the nature of inductive proof including the use of initial statement, assumption statement, inductive step and conclusion.
- Use sigma notation $\left(\sum\right)$ to represent a sum, e.g. $\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3$ and $\sum_{j=0}^{2n+1} 2^j = 2^0 + 2^1 + 2^2 + \cdots + 2^{2n+1}$.
- Prove results for sums for any positive integer n.

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2.2.1 Review of set theory and logic symbols

Set notation	**Meaning**
$\mathbb{R}$	The set of real numbers; numbers that can be represented on the number line, the set of which can be divided into irrational numbers and rational numbers ($\mathbb{Q}$).
$\mathbb{N}$	The set of natural numbers; $\mathbb{N} = \{1, 2, 3, 4...\}$
$\mathbb{Z}$ $\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$ $\mathbb{Z}^+$ $\mathbb{Z}^-$	The set of integers; $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3...\}$ The set of integers consists of zero, the positive natural numbers and the negative integers. The set of positive integers, the same as $\mathbb{N}$ The set of negative integers
$\mathbb{Q}$ $\mathbb{Q} = \left\{\frac{a}{b}, a, b \in \mathbb{R}, b \neq 0\right\}$	The set of rational numbers; numbers that can be written as the ratio of two integers, a, b where $b \neq 0$.
Logic symbols	**Meaning**
Proposition	A mathematical statement that is either true or false.
Proposition A or B	One or both of the propositions are true. Equivalent to $A \cup B$.
Proposition A and B	Both of the propositions are true. Equivalent to $A \cap B$.
$\neg A$	The negation of proposition A. The negation of a true statement is false, and the negation of a false statement is true. Equivalent to A'.
$\neg(A \text{ or } B) = \neg A \text{ and } \neg B$	Equivalent to $(A \cup B)'$
$\neg(A \text{ and } B) = \neg A \text{ or } \neg B$	Equivalent to $(A \cap B)'$
$P \Rightarrow \mathbb{Q}$ $P \Leftrightarrow \mathbb{Q}$	Implication: if P, then $\mathbb{Q}$ Equivalence: if P, then $\mathbb{Q}$ and if $\mathbb{Q}$, then P
Quantifiers $\forall$ $\exists$	 For all; all possible values for the variable are considered. There exists; there is a value for the variable that would make the propositional function true.
R.T.P iff	Required to prove If and only if

2.2.2 Sequences and series

A **sequence** is a set of numbers or terms arranged in a specific pattern or following a defined rule. For example:

$$1, 2, 3, 4, 5, \ldots \quad \text{or} \quad 5, 8, 11, 14, 17, \ldots$$

In the first sequence, the difference between consecutive terms is 1, while in the second sequence, the difference between consecutive terms is 3. The pattern might be less obvious. For instance, $1, 8, 27, 64, \ldots$ is the sequence of the cube of sequential integers.

A **series** is the result of adding up all the terms of a sequence. The addition of the terms, as well as the resulting value, are both referred to as the 'sum' or the 'summation'.

For instance, in the first sequence, the corresponding series is the sum $1 + 2 + 3 + 4 + 5 + \ldots$.

To illustrate this with **summation notation**, the sum of the first five natural numbers is:

$$\sum_{i=1}^{5} j = 1 + 2 + 3 + 4 + 5 = 15$$

This representation is useful when dealing with larger sums, as we will see in the following section.

2.2.3 Summation notation

Proof by induction often involves statements that depend on the natural numbers. Summation notation, which is frequently used in such proofs, will be briefly reviewed here before we move on to the induction itself.

We write the sum of the first n natural numbers as:

$$1 + 2 + 3 + 4 + \ldots + (n-1) + n = \sum_{j=1}^{n} j$$

The symbol $\sum$, which is the Greek letter $\sum$, pronounced sigma, denotes a **sum** of its argument over a sequence of values. In this case, the argument is j and $\sum_{j=1}^{n} j$ represents the sum of j, a natural number, starting from the smallest value $j = 1$ and going up to the maximum value $j = n$. The variable that it is summed over is called a **dummy variable** because it acts as a placeholder, taking on each integer value from 1 to n during the summation.

In other contexts, your work might use different symbols such as r, i or s for the dummy variable, but they serve the same purpose. For example, $\sum_{r=1}^{4} 3r$ can be explicitly expressed as $\sum_{r=1}^{4} 3r = 3 + 6 + 9 + 12$.

WORKED EXAMPLE 1 Understanding the summation notation

Write out the following sums explicitly.

a. $\sum_{j=1}^{4} 2j$

b. $\sum_{r=1}^{n} r^3$

THINK	WRITE
a. 1. Interpret the summation $\sum_{j=1}^{4} 2j$. The argument is $2j$. Calculate the first term t_1, for $j = 1$.	**a.** $t_1 = 2 \times 1$ $= 2$

2. Calculate the next three terms for $j = 2, 3, 4$.

$t_2 = 2 \times 2, \quad t_3 = 2 \times 3, \quad t_4 = 2 \times 4$

$t_2 = 4 \quad t_3 = 6 \quad t_4 = 8$

3. Write the sum explicitly.

$$\sum_{j=1}^{4} 2j = 2 + 4 + 6 + 8$$

b. 1. Interpret the summation $\sum_{r=1}^{n} r^3$. The argument is r^3. State the first three terms for $r = 1, 2, 3$ in index form.

b. $t_1 = 1^3, \quad t_2 = 2^3, \quad t_3 = 3^3$

2. Write the sum explicitly, noting that this series continues to the maximum value, n.

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3$$

WORKED EXAMPLE 2 Using the summation notation to represent a sum

Use the summation notation to represent the following sums.

a. $4 + 8 + 12 + 16 + \cdots + 40$ **b.** $8 + 11 + 14 + 17 + \cdots + 50$

THINK	WRITE
a. 1. Recognise the pattern of this sequence, the difference between consecutive terms is 4.	a. $4 + 8 + 12 + 16 + \cdots + 40 = 1 \times 4 + 2 \times 4 + 3 \times 4 + 4 \times 4 + \cdots + 10 \times 4$
2. Identify the argument of this sequence using a dummy variable, and the smallest and highest values for the dummy variable.	The argument is $4j$, from $j = 1$ to $j = 10$.
3. Write the sum using the summation notation	$4 + 8 + 12 + 16 + \cdots + 40 = \sum_{j=1}^{10} 4j$
b. 1. Recognise the pattern of this sequence, the difference between consecutive terms is 3.	b. $8 + 11 + 14 + 17 + \cdots + 50 = 5 + 3 + 5 + 6 + 5 + 9 + 5 + 12 + \cdots + 5 + 45$ $= 5 + 1 \times 3 + 5 + 2 \times 3 + 5 + 3 \times 3 + 5 + 4 \times 3 + \cdots + 5 + 15 \times 3$ OR $8 + 11 + 14 + 17 + \cdots + 50 = 8 + 0 + 8 + 3 + 8 + 6 + 8 + 9 + \cdots + 8 + 42$ $= 8 + 0 \times 3 + 8 + 1 \times 3 + 8 + 2 \times 3 + 8 + 3 \times 3 + \cdots + 8 + 14 \times 3$
2. Identify the argument of this sequence using a dummy variable, and the smallest and highest values for the dummy variable.	The argument is $5 + 3j$, from $j = 1$ to $j = 15$. OR The argument is $8 + 3j$, from $j = 0$ to $j = 14$.

3. Write the sum using the summation notation.

$$8+11+14+17+\cdots+50=\sum_{j=1}^{15} 5+3j$$

OR

$$8+11+14+17+\cdots+50=\sum_{j=0}^{14} 8+3j$$

Note that there isn't a unique way to write a sum using the summation notation.

2.2.4 Formulae for the sum of the first n terms of a sequence

Let's go back to the sum of the first n natural numbers, $\sum_{j=1}^{n} j = 1+2+\cdots+(n-1)+n$. If we were to calculate the sum of the first n integers, we would have to perform n additions, which could be time consuming. Thankfully, there is a formula for the sum of the first n integers, that we can deduce by using the fact that addition is commutative, writing the sum from 1 to n, and from n to 1 and adding both sums.

$$\sum_{j=1}^{n} j = 1+2+\cdots+(n-1)+n$$

$$\sum_{j=1}^{n} j = n+(n-1)+\cdots+2+1$$

$$2\sum_{j=1}^{n} j = (n+1)+(n-1+2)+\cdots+(2+(n-1))+(1+n)$$

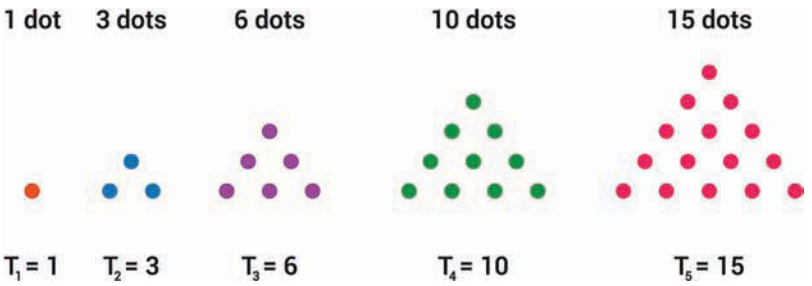
$$2\sum_{j=1}^{n} j = \underbrace{(n+1)+(n+1)+\cdots+(n+1)+(n+1)}_{n \text{ times}}$$

$$2\sum_{j=1}^{n} j = n(n+1)$$

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

The sum of the first n natural numbers is indeed given by the formula $\frac{n(n+1)}{2}$, which we will prove later. This formula is convenient, as calculating the sum of the first n integers now requires only three operations (1 addition, 1 multiplication, 1 division), instead of n additions.

Now let's consider triangular numbers, which count objects arranged in an equilateral triangle.



We can observe that, in this pattern, the jth triangular number T_j is given by $T_j = \frac{j(j+1)}{2}$. Now, if we want to sum the first n triangular numbers, we would use the summation

$$\sum_{j=1}^{n} T_j = \sum_{j=1}^{n} \frac{j(j+1)}{2} = 1 + 3 + 6 + 10 + 15 + \cdots + \frac{(n-1)n}{2} + \frac{n(n+1)}{2}$$

And, similarly to what we have done with the sum of the first n integers, having a general formula for the sum of the first n terms of this series would be more convenient. For this series, such a formula is

$$\sum_{j=1}^{n} \frac{j(j+1)}{2} = \frac{n(n+1)(n+2)}{3!}$$

We can verify this formula for $n = 5$ for instance.

$$\begin{aligned}\sum_{j=1}^{5} T_j &= T_1 + T_2 + T_3 + T_4 + T_5 \\ &= 1 + 3 + 6 + 10 + 15 \\ &= 35\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{5} \frac{j(j+1)}{2} &= \frac{5(5+1)(5+2)}{3!} \\ &= \frac{210}{6} \\ &= 35\end{aligned}$$

However, to prove this formula, or the formula for the sum of integers, is true for all values of n, the process of **proof by mathematical induction** is required.

2.2.5 Principle of mathematical induction

Imagine a series of dominoes placed close enough that if the first topples, the rest will fall in sequence. This is how a proof by mathematical induction works. For the dominoes to topple, the first must fall, and they must be spaced in a way such that if a domino falls, the one next in line will fall too.

In mathematical induction, to prove that a statement or proposition is true for all values of n, the first step is to prove that the statement holds for $n = 1$.

This is known as the **initial statement** (or base case). This step is like knocking over the first domino — establishing that the statement is true at least once.

Next, we assume that the statement holds for some arbitrary $k \in \mathbb{N}$. This assumption is called the **inductive hypothesis**.

The final step is to use this assumption to prove that the statement also holds for $n = k + 1$. This is the **inductive step**, ensuring that if one domino falls, the next one does too.

Since we have shown that the statement is true for $n = 1$ and that its truth for $n = k$ guarantees its truth for $n = k + 1$, it follows by the principle of mathematical induction that the statement holds for all natural numbers n.

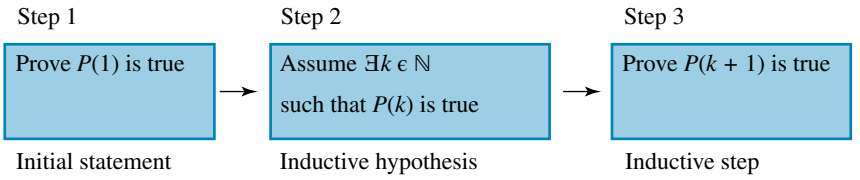
Proof by mathematical induction procedure

Let $P(n)$ represent the proposition.

1. **Initial statement: show the proposition is true for $n = 1$.**
2. **Inductive hypothesis: assume the proposition is true for $n = k$.**
   That is $\exists k \in \mathbb{N}$ (there exists a natural number k) such that $P(k)$ is true.
3. **Inductive step: using the inductive hypothesis, show the statement is true for $n = k + 1$.**
   That is : $P(k) \Rightarrow P(k+1)$
   State your conclusion:

$$P(1) \text{ is true and } P(k) \Rightarrow P(k+1)$$
$$\therefore P(n) \text{ is true } \forall n \geq 1.$$

The three steps of mathematical induction



2.2.6 Propositions

Before proving propositions using mathematical induction, let's first practise its individual steps.

WORKED EXAMPLE 3 Understanding the steps of mathematical induction

Consider the proposition $P(n) : \sum_{j=1}^{n} \frac{1}{2^j} = \left(1 - \frac{1}{2^n}\right), \forall n \in \mathbb{N}$.

a. Show that $P(1), P(2)$ and $P(3)$ are true.
b. Identify $P(k)$.
c. Identify $P(k+1)$.
d. Prove $P(k) \Rightarrow P(k+1)$.

THINK	WRITE
a. 1. Identify $P(1)$, $P(2)$ and $P(3)$.	a. $P(1) : \sum_{j=1}^{1} \frac{1}{2^j} = \left(1 - \frac{1}{2^1}\right)$ $P(2) : \sum_{j=1}^{2} \frac{1}{2^j} = \left(1 - \frac{1}{2^2}\right)$ $P(3) : \sum_{j=1}^{3} \frac{1}{2^j} = \left(1 - \frac{1}{2^3}\right)$
2. Write each sum explicitly.	$\sum_{j=1}^{1} \frac{1}{2^j} = \frac{1}{2}$ $\sum_{j=1}^{2} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4}$ $\sum_{j=1}^{3} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

3. Show that $P(1)$, $P(2)$ and $P(3)$ are true by showing LHS = RHS for each sum.

For $P(1)$

$$\text{LHS} = \sum_{j=1}^{1} \frac{1}{2^j} = \frac{1}{2} \text{ and RHS} = \left(1 - \frac{1}{2^1}\right) = \frac{1}{2}$$

$\therefore$ LHS = RHS

$\therefore P(1)$ is true

For $P(2)$

$$\text{LHS} = \sum_{j=1}^{2} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \text{ and RHS} = \left(1 - \frac{1}{2^2}\right) = \frac{3}{4}$$

$\therefore$ LHS = RHS

$\therefore P(2)$ is true

For $P(3)$

$$\text{LHS} = \sum_{j=1}^{3} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \text{ and RHS} = \left(1 - \frac{1}{2^3}\right) = \frac{7}{8}$$

$\therefore$ LHS = RHS

$\therefore P(3)$ is true

4. Conclude.

$P(1)$, $P(2)$ and $P(3)$ are true.

b. Identify $P(k)$.

b. $P(k) : \sum_{j=1}^{k} \frac{1}{2^j} = \left(1 - \frac{1}{2^k}\right)$

c. Identify $P(k+1)$.

c. $P(k+1) : \sum_{j=1}^{k+1} \frac{1}{2^j} = \left(1 - \frac{1}{2^{k+1}}\right)$

d. 1. Recall the meaning of $P \Rightarrow Q$ and apply it to $P(k) \Rightarrow P(k+1)$.

d. $P \Rightarrow Q$ is an implication: if P, then Q. Here, this means if $P(k)$ is true, then $P(k+1)$ is also true (or $P(k+1)$ is true whenever $P(k)$ is true).

2. Assume $P(k)$ is true for some positive integer k.

$\exists k \in \mathbb{N}$ such that $\sum_{j=1}^{k} \frac{1}{2^j} = \left(1 - \frac{1}{2^k}\right)$

3. Using this assumption, show that $P(k+1)$ is then true by substituting $\left(1 - \frac{1}{2^k}\right)$ for $\sum_{j=1}^{k} \frac{1}{2^j}$.

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{2^j} &= \sum_{j=1}^{k} \frac{1}{2^j} + \frac{1}{2^{k+1}} \\
&= \left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} \\
&= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\
&= \frac{2^{k+1} - 1}{2^{k+1}} \\
&= 1 - \frac{1}{2^{k+1}}
\end{aligned}$$

4. Conclude.

$P(k) \Rightarrow P(k+1)$

Note that, when using the summation notation, it is not strictly necessary to write the dummy variable under the sigma symbol if the variable is clear from the context.

For example, $\sum_{j=1}^{n} \frac{1}{2^j}$ can be written $\sum_{1}^{n} \frac{1}{2^j}$ and $\sum_{r=1}^{n} r^3$ can be written $\sum_{1}^{n} r^3$.

2.2.7 Proof by induction

Let's now apply the proof by induction method to prove that the sum of the first n integers is given by $\frac{n(n+1)}{2}$ and other propositions.

WORKED EXAMPLE 4 Using proof by mathematical induction for the formula for the sum of the first *n* integers

Use mathematical induction to prove that $1+2+3+4+\ldots+n=\frac{n(n+1)}{2}, \forall n \in \mathbb{N}$.

THINK	WRITE
1. State the proposition, $P(n)$.	Let $P(n)$ represent the proposition $1+2+3+4+\ldots+n=\frac{n(n+1)}{2}, \forall n \in \mathbb{N}$.
2. Make the initial statement by showing the proposition is true for, $n=1$.	Verify $P(1)$ is true. $\text{LHS}=1, \quad \text{RHS}=\frac{1(1+1)}{2}$ $=1$ $=\text{LHS}$ $\therefore P(1)$ is true.
3. Make the inductive hypothesis by assuming that $P(k)$ is true for some positive integer k.	Assume $P(k)$ holds, that is, $\exists k \in \mathbb{N}$ such that $1+2+3+4+\ldots+k=\frac{k(k+1)}{2}$.
4. Clarify the inductive step, which is to show that if $P(k)$ holds, then $P(k+1)$ also holds. The goal is to prove, **using the inductive hypothesis**, that the LHS equals the RHS for $P(k+1)$.	Show that, if $P(k)$ holds, then $P(k+1)$ holds, that is $1+2+3+4+\ldots+k+(k+1)=\frac{(k+1)(k+2)}{2}$.
5. Define the LHS.	$\text{LHS}=1+2+3+4+\ldots+k+(k+1)$
6. Use the inductive hypothesis to substitute $\frac{k(k+1)}{2}$ for $1+2+3+4+\ldots+k$.	$\text{LHS}=1+2+3+4+\ldots+k+(k+1)$ $=\frac{k(k+1)}{2}+(k+1)$

7. Simplify the expression by creating like denominators then factoring.	$\text{LHS} = \dfrac{k(k+1)}{2} + \dfrac{2(k+1)}{2}$ $= \dfrac{k(k+1)+2(k+1)}{2}$ $= \dfrac{(k+1)(k+2)}{2}$ $= \text{RHS}$
8. Communicate on the inductive step being proven.	$\therefore P(k+1)$ is true whenever $P(k)$ is true (or $P(k) \Rightarrow P(k+1)$).
9. Conclude the proof with a formal statement.	$P(1)$ is true and $P(k) \Rightarrow P(k+1)$. $\therefore P(n)$ is true $\forall n \in \mathbb{N}$.

WORKED EXAMPLE 5 Using proof by mathematical induction for the formula for the sum of the first *n* terms of a sequence 1

Use mathematical induction to prove that $5 + 15 + 45 + \ldots + 5(3)^{n-1} = 5\left(\dfrac{3^n - 1}{2}\right), \forall n \in \mathbb{N}$.

THINK	WRITE
1. State the proposition, $P(n)$.	Let $P(n)$ represent the proposition $5 + 15 + 45 + \ldots + 5(3)^{n-1} = 5\left(\dfrac{3^n - 1}{2}\right), \forall n \in \mathbb{N}$.
2. Make the initial statement by showing the proposition is true for, $n = 1$.	Verify $P(1)$ is true. $\text{LHS} = 5, \quad \text{RHS} = \dfrac{5(3^1 - 1)}{2}$ $= 5$ $= \text{LHS}$ $\therefore P(1)$ is true.
3. Make the inductive hypothesis by assuming that $P(k)$ is true for some positive integer k.	Assume $P(k)$ is true, that is, $\exists k \in \mathbb{N}$ such that $5 + 15 + 45 + \ldots + 5(3)^{k-1} = 5\left(\dfrac{3^k - 1}{2}\right)$.
4. Clarify the inductive step, which is to show that if $P(k)$ holds, then $P(k+1)$ also holds. The goal is to prove, **using the inductive hypothesis**, that the LHS equals the RHS for $P(k+1)$.	Show that, if $P(k)$ holds, then $P(k+1)$ holds, that is $5 + 15 + 45 + \ldots + 5(3)^{k-1} + 5(3)^k = 5\left(\dfrac{3^{k+1} - 1}{2}\right)$.
5. Define the LHS.	$\text{LHS} = 5 + 15 + 45 + \ldots + 5(3)^{k-1} + 5(3)^k$
6. Use the inductive hypothesis to substitute $5\left(\dfrac{3^k - 1}{2}\right)$ for $5 + 15 + 45 + \ldots + 5(3)^{k-1}$.	$\text{LHS} = 5 + 15 + 45 + \ldots + 5(3)^{k-1} + 5(3)^k$ $= 5\left(\dfrac{3^k - 1}{2}\right) + 5(3)^k$

7. To simplify the expression, begin by factorising. Then create like denominators to perform addition.	$= 5\left[\left(\frac{3^k - 1}{2}\right) + (3)^k\right]$ $= 5\left(\frac{3^k - 1}{2} + \frac{2(3)^k}{2}\right)$ $= 5\left(\frac{3 \times 3^k - 1}{2}\right)$ $= 5\left(\frac{3^{k+1} - 1}{2}\right)$ $= \text{RHS}$
8. Communicate on the inductive step being proven.	$\therefore P(k+1)$ is true whenever $P(k)$ is true. $\therefore P(k) \Rightarrow P(k+1)$.
9. Conclude the proof with a formal statement.	$P(1)$ is true and $P(k) \Rightarrow P(k+1)$. $\therefore P(n)$ is true $\forall n \in \mathbb{N}$.

WORKED EXAMPLE 6 Using proof by mathematical induction for the formula for the sum of the first *n* terms of a sequence 2

Use mathematical induction to prove that $\sum_{r=1}^{n} 3r = \frac{3n(1+n)}{2}, \forall n \in \mathbb{N}$.

THINK	WRITE
1. State the proposition, $P(n)$.	Let $P(n)$ represent the proposition $\sum_{r=1}^{n} 3r = \frac{3n(1+n)}{2}, \forall n \in \mathbb{N}$.
2. Make the initial statement by showing the proposition is true for, $n = 1$.	Verify $P(1)$ is true. $\text{LHS} = 3, \quad \text{RHS} = \frac{3 \times 1(1+1)}{2}$ $= 3$ $= \text{LHS}$ $\therefore P(1)$ is true.
3. Make the inductive hypothesis by assuming that $P(k)$ is true for some positive integer k.	Assume $P(k)$ is true. $\exists k \in \mathbb{N}$ such that $\sum_{r=1}^{k} 3r = \frac{3k(1+k)}{2}$
4. Clarify the inductive step, which is to show that if $P(k)$ holds, then $P(k+1)$ also holds. The goal is to prove, **using the inductive hypothesis**, that the LHS equals the RHS for $P(k+1)$.	Show that, if $P(k)$ holds, then $P(k+1)$ holds, that is $\sum_{r=1}^{k+1} 3r = \frac{3(k+1)(k+2)}{2}$.
5. Define the LHS.	$\text{LHS} = \sum_{r=1}^{k+1} 3r$

6. Express $\sum_{r=1}^{k+1} 3r$ as $\sum_{r=1}^{k} 3r + 3(k+1)$.

$$\text{LHS} = \sum_{r=1}^{k} 3r + 3(k+1)$$

7. Use the inductive hypothesis to substitute $\frac{3k(1+k)}{2}$ for $\sum_{r=1}^{k} 3r$.

$$\text{LHS} = \frac{3k(1+k)}{2} + 3(k+1)$$

8. Simplify the expression by creating like denominators to add the two terms.

$$= \frac{3k(1+k)}{2} + \frac{6(k+1)}{2}$$
$$= \frac{3k^2 + 3k + 6k + 6}{2}$$

9. Collect like terms to simplify the numerator. Express the numerator in factorised form to complete the inductive step.

$$= \frac{3k^2 + 9k + 6}{2}$$
$$= \frac{3\left(k^2 + 3k + 2\right)}{2}$$
$$= \frac{3(k+1)(k+2)}{2}$$
$$= \text{RHS}$$

10. Communicate on the inductive step being proven.

$P(k) \Rightarrow P(k+1)$.

11. Conclude the proof with a formal statement.

$P(1)$ is true and $P(k) \Rightarrow P(k+1)$.
$\therefore P(n)$ is true $\forall n \in \mathbb{N}$.

Exercise 2.2 Proof by induction

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2.2 Exercise | **2.2 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26	27, 28, 29, 30

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Simple familiar

1. WE1 Write out the following sums explicitly.

a. $\sum_{j=1}^{4} 5j$

b. $\sum_{r=1}^{n} r^4$

2. Write out the following sums explicitly.

a. $\sum_{r=1}^{n} 6r - 5$

b. $\sum_{j=1}^{4} \frac{1}{j}$

c. $\sum_{s=1}^{6} 2 - 3s$

d. $\sum_{j=0}^{n} \frac{(-1)^j}{j+2}$

3. **WE2** Use the summation notation to represent the following sums.

a. $3+6+9+12+\cdots+57$

b. $7+11+15+19+\cdots+43$

4. Use the summation notation to represent the following sums.

a. $1^6+2^6+3^6+\ldots+(n-1)^6+n^6$

b. $5+7+9+11+13$

5. **WE3** Consider the proposition $P(n): \sum_{j=1}^{n} \frac{1}{3^j} = \frac{1}{2}\left(1-\frac{1}{3^n}\right), \forall n \in \mathbb{N}$

a. Show that $P(1)$, $P(2)$ and $P(3)$ are true.

b. Identify $P(k)$.

c. Identify $P(k+1)$.

d. Prove $P(k) \Rightarrow P(k+1)$.

6. A student wants to use mathematical induction to prove the proposition made for the sum of triangular numbers:

$$P(n): \sum_{j=1}^{n} \frac{j(j+1)}{2} = \frac{n(n+1)(n+2)}{3!}, \forall n \in \mathbb{N}$$

The student has completed the following:

Step 1 — Base case, let's show that the statement is true for $n=1$:

$$\text{LHS} = \frac{1(1+1)}{2} = 1$$

$$\text{RHS} = \frac{1(1+1)(1+2)}{3!} = \frac{1\times2\times3}{3\times2\times1} = 1$$

LHS = RHS for $n=1$

$\therefore P(1)$ is true

Step 2 — Inductive hypothesis, let's assume the statement is true for $n=k$:

$\exists k \in \mathbb{N}$ such that $P(k)$ is true

$$\sum_{j=1}^{k} \frac{j(j+1)}{2} = \frac{k(k+1)(k+2)}{3!}.$$

Step 3 — Inductive step, let's show that if $P(k)$ is true, then $P(k+1)$ is true.

$P(k+1)$ is true if $\sum_{j=1}^{k+1} \frac{j(j+1)}{2} = \frac{(k+1)(k+2)(k+3)}{3!}$

$$\begin{aligned}
\text{LHS} &= \sum_{j=1}^{k+1} \frac{j(j+1)}{2} \\
&= \sum_{j=1}^{k} \frac{j(j+1)}{2} + \frac{(k+1)(k+2)}{2} \\
&= \frac{k(k+1)(k+2)}{3!} + \frac{(k+1)(k+2)}{2} \\
&= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3!} \\
&= \frac{(k+1)(k+2)(k+3)}{3!} \\
&= \text{RHS}
\end{aligned}$$

$\therefore P(k+1)$ is true.

We have shown that if $P(k)$ is true, then $P(k+1)$ is true.

Conclusion:

$P(1)$ is true and $P(k) \Rightarrow P(k+1)$.

$\therefore P(n)$ is true $\forall n \geq 1$.

Complete the student's proof.

7. Consider the statement $2+4+6+...+2n=n(n+1)$, $n \geq 1$.

a. Demonstrate that the statement is true for the first three possible values of n.

b. If $2+4+6+...+2k=k(k+1)$, demonstrate that $2+4+6+...+2k+2(k+1)=(k+1)(k+2)$.

8. Consider the statement $\sum_{r=1}^{n} 2^{r-1}=2^n-1$.

a. Demonstrate that the statement is true for the first three possible values of n.

b. If $\sum_{r=1}^{k} 2^{r-1}=2^k-1$, demonstrate that $\sum_{r=1}^{k+1} 2^{r-1}=2^{k+1}-1$.

9. Consider the statement $3+5+7+...+(1+2n)=n(n+2)$, $n \geq 1$.

a. Demonstrate that the statement is true for the first possible value of n.

b. Assume that the statement is true for $n=k$. Document this assumption by writing a suitable mathematical equation.

c. Use the assumption from part **b** to demonstrate that if the statement is true for $n=k$, it will also be true for $n=k+1$.

d. Explain how you have demonstrated that $3+5+7+...+(1+2n)=n(n+2)$.

10. Consider the statement $\sum_{r=1}^{n}(2r-1)=n^2$.

a. Demonstrate that the statement is true for the first possible value of n.

b. Assume that the statement is true for $n=k$. Document this assumption by writing a suitable mathematical equation.

c. Use the assumption from part **b** to demonstrate that if the statement is true for $n=k$, it will also be true for $n=k+1$.

d. Explain how you have demonstrated that $\sum_{r=1}^{n}(2r-1)=n^2$.

11. Consider proposition $P(n): \sum_{i=1}^{n} 2^i=2^{n+1}-2$.

a. Identify $P(1)$.

b. Identify $P(k)$.

c. Identify term $k+1$ in $\sum_{i=1}^{n} 2^i$.

d. If $P(k)$ is true, demonstrate that $P(k+1)$ is true.

12. Consider proposition $P(n): \sum_{i=1}^{n} 2i=n(n+1)$.

a. Identify $P(1)$.

b. Identify $P(k)$.

c. Identify term $k+1$ in $\sum_{i=1}^{n} 2i$.

d. If $P(k)$ is true, demonstrate that $P(k+1)$ is true.

13. **WE4** Use mathematical induction to prove that $5+6+7+...+(n+4)=\frac{n(n+9)}{2}$, $n \geq 1$.

14. **WE5** Use mathematical induction to prove that $3+6+12+...+3\times 2^{n-1}=3(2^n-1)$, $n \geq 1$.

15. **WE6** Use mathematical induction to prove that $\sum_{r=1}^{n} 2^r=2^{n+1}-2$.

Complex familiar

16. Use mathematical induction to prove that $1^2+2^2+3^2...+(n-1)^2+n^2=\frac{n(n+1)(2n+1)}{6}$.

17. Use mathematical induction to prove that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}$.

18. Use mathematical induction to prove that $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$ for $n \geq 1$.

19. Use a proof by induction to demonstrate that $P(n): \sum_{i=1}^{n} 3^i = \frac{3^{n+1}-3}{2}$ is true.

20. Use a proof by induction to demonstrate that $P(n): \sum_{i=1}^{n} 2 \times 3^{i-1} = 3^n - 1$ is true.

21. Use a proof by induction to demonstrate that $P(n): \sum_{i=1}^{n} (3i-1) = \frac{n(3n+1)}{2}$.

22. Consider the proposition $P(n): \sum_{r=2}^{n} (r-1)r = \frac{(n-1)n(n+1)}{3}$.

a. Demonstrate that $P(2)$ is true.

b. Identify $P(k)$.

c. Identify $P(k+1)$.

d. Use a proof by induction to prove $P(n)$.

23. Use mathematical induction to prove that $a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n-1)}{r-1}, \quad n \geq 1$.

24. Use mathematical induction to prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

25. Use mathematical induction to prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

26. Use mathematical induction to prove that $a + (a+d) + (a+2d) + \ldots (a + [n-1]d) = \frac{n}{2}[2a + (n-1)d]$, for $n \geq 1$.

Complex unfamiliar

27. Use mathematical induction to demonstrate that the sum of the entries in the nth horizontal row of Pascal's Triangle is 2^n. You may use Pascal's identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ in your proof.

28. Use mathematical induction to prove the binomial expansion $(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}a^0 b^n$. You may use Pascal's identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ in your proof.

29. Demonstrate al-Karaji's proof of $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. Use this proof for the sum of n numbers to prove by induction that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$.

30. Use mathematical induction to prove that $\sum_{r=1}^{n} r^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$.

Fully worked solutions for this chapter are available online.

LESSON
2.3 Proofs of divisibility

SYLLABUS LINKS

- Prove divisibility results for any positive integer n.

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2.3.1 Divisibility results

Proof by induction can also be used to prove that an expression is divisible by a certain number. An equation can be found by writing the expression as a multiple of that number. As an example, if $n^3 - n$ is divisible by 3, then we let $n^3 - n = 3m$ (where m is an integer).

WORKED EXAMPLE 7 Using mathematical induction to prove divisibility by 3

Use a proof by induction to demonstrate that $n^3 - n$ is divisible by 3, $n > 1$.

THINK	WRITE
1. Since $n > 1$, the first value of n to check is $n = 2$. Verify that the formula is true for $n = 2$.	If $n = 2$, $n^3 - n = 8 - 2$ $= 6$ As 6 is divisible by 3, that claim is true for $n = 2$.
2. There are some values of n that make the claim true. If a number is divisible by 3, it can be written as $3m$, write this as a mathematical equation where $n = k$.	Assume that it is true for some $n = k$. $k^3 - k = 3m, \quad m \in \mathbb{Z}$
3. Consider the statement when $n = k + 1$.	$(k+1)^3 - (k+1) = \left(k^3 + 3k^2 + 3k + 1\right) - (k+1)$
4. Substitute $k^3 - k = 3m$ to demonstrate that the statement can be written as a multiple of 3.	$= k^3 + 3k^2 + 3k + 1 - k - 1$ $= k^3 - k + 3k^2 + 3k$ $= 3m + 3k^2 + 3k$ $= 3(m + k^2 + k)$
5. Write a statement to explain that the statement is true for $n = k + 1$ if it is true for $n = k$.	The statement $k^3 - k = 3m$, $m \in \mathbb{Z}$ is divisible by 3. Therefore if $k^3 - k$ is divisible by 3, then $(k+1)^3 - (k+1)$ is divisible by 3.
6. Write the concluding statement.	$n^3 - n$ is divisible by 3 if $n = 2$. It is true for $n = k + 1$ if it is true for $n = k$. Therefore, by mathematical induction, the statement is true $\forall \; n > 1$.

2.3.2 Using divisibility notation

When proving an expression is divisible by a certain number, an alternative way to write the equation to be proved involves using **| notation**. In Worked example 7, it was required to prove that $n^3 - n$ is divisible by 3. An alternative way to write this is $3|(n^3 - n)$.

In the following worked example, the notation $5|(2^{4n+1} + 3)$ is used which is equivalent to $2^{4n+1} + 3$ is divisible by 5. In general terms, $a|b$ means b is divisible by a.

WORKED EXAMPLE 8 Using mathematical induction to prove divisibility by 5

Use a proof by induction to demonstrate that $5|\left(2^{4n+1} + 3\right)$, $n \geq 1$.

THINK	WRITE
1. Verify that the statement is true for $n = 1$.	If $n = 1$, $2^{4n+1} + 3 = 2^5 + 3$ $= 32 + 3$ $= 35$ As 35 is divisible by 5, the statement is true for $n = 1$.
2. There are some values of n that make the claim true. If a number is divisible by 5, it can be written as $5m$, write this as a mathematical equation where $n = k$.	Assume that it is true for some $n = k$. $2^{4k+1} + 3 = 5m$
3. Consider the statement when $n = k + 1$.	$2^{4(k+1)+1} + 3 = 2^{4k+4+1} + 3$
4. Rewrite using $2^{4k+4+1} = 2^{4k+1} \times 2^4$ and expand 2^4. As $16\left(2^{4k+1} + 3\right) = 16 \times 2^{4k+1} + 16 \times 3$ it is necessary also subtract 16×3 to maintain the equality. Substitute $2^{4k+1} + 3 = 5m$ to show that the claim is also true for $n = k + 1$.	$2^{4k+4+1} + 3 = 2^{4k+1} \times 2^4 + 3$ $= 16 \times 2^{4k+1} + 3$ $= 16\left(2^{4k+1} + 3\right) - 3 \times 16 + 3$ $= 16\left(2^{4k+1} + 3\right) - 48 + 3$ $= 16 \times 5m - 45$ $= 5(16m - 9)$
Alternatively: If $2^{4k+1} + 3 = 5m$ then we can substitute $2^{4k+1} = 5m - 3$ to show that the claim is also true for $n = k + 1$.	**Alternatively:** $2^{4(k+1)+1} + 3 = 2^{4k+4+1} + 3$ $= 2^{4k+1} \times 2^4 + 3$ $= (5m - 3) \times 16 + 3$ $= 16 \times 5m - 3 \times 16 + 3$ $= 16 \times 5m - 3 \times 15$ $= 5(16m - 9)$
5. Write a statement to explain that the statement is true for $n = k + 1$ if it is true for $n = k$.	This is divisible by 5, therefore $2^{4(k+1)+1} + 3$ is divisible by 5 if $2^{4k+1} + 3$ is divisible by 5.
6. Write the concluding statement.	$2^{4n+1} + 3$ is divisible by 5 if $n = 1$. It is true for $n = k + 1$ if it is true for $n = k$. Therefore, the statement is true $\forall\ n \geq 1$.

Exercise 2.3 Proofs of divisibility

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2.3 Exercise | 2.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8	9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19	20

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Simple familiar

1. Consider the statement that $6^n + 4$ is divisible by 5, $n \in \mathbb{Z}^+ \cup \{0\}$.
   a. Demonstrate that the statement is true for the first three possible values of n.
   b. If $6^k + 4 = 5m,\ m \in \mathbb{Z}$, demonstrate that $6^{k+1} + 4$ is divisible by 5.
2. Consider the statement that $8^n - 1$ is divisible by 7, $n \in \mathbb{Z}^+$.
   a. Demonstrate that the statement is true for the first three possible values of n.
   b. If $8^k - 1 = 7m,\ m \in \mathbb{Z}$, demonstrate that $8^{k+1} - 1$ is divisible by 7.
3. Consider the statement: $3^{2n-1} + 1$ is divisible by 4, $n \in \mathbb{Z}^+$.
   a. Demonstrate that the statement is true for the first possible value of n.
   b. Assume that the statement is true for $n = k$. Document this assumption by writing a suitable mathematical equation.
   c. Use the assumption from part **b** to demonstrate that if the statement is true for $n = k$, it will also be true for $n = k + 1$.
   d. Explain how you have demonstrated that $3^{2n-1} + 1$ is divisible by 4, $n \in \mathbb{Z}^+$.
4. Consider the statement: $10^n - 4$ is divisible by 12, $n \geq 2$.
   a. Demonstrate that the statement is true for the first possible value of n.
   b. Assume that the statement is true for $n = k$. Document this assumption by writing a suitable mathematical equation.
   c. Use the assumption from part **b** to demonstrate that if the statement is true for $n = k$, it will also be true for $n = k + 1$.
   d. Explain how you have demonstrated that $10^n - 4$ is divisible by 12, $n \geq 2$.
5. WE7 Use a proof by induction to demonstrate that $n^3 + 2n$ is divisible by 3, $n \geq 1$.
6. Use a proof by induction to demonstrate that $3^{4n} - 1$ is divisible by 80, $n \geq 1$.
7. Use a proof by induction to demonstrate that $5^n + 2 \times 11^n$ is divisible by 3, $n \geq 0$.
8. WE8 Use a proof by induction to demonstrate that $9 \mid (4^n + 15n - 1),\ n \geq 1$.

Complex familiar

9. Prove, using a proof by induction, that $9 \mid (4^n + 5^n)$, n is odd.
10. Prove, using a proof by induction, that $5 \mid (2^{3n} - 3^n),\ n > 0$.
11. Prove, using a proof by induction, that $21 \mid (4^{n+1} + 5^{2n-1}),\ n > 0$.
12. Prove, using a proof by induction, that $11 \mid (10^n - (-1)^n),\ n \geq 1$.
13. Prove, using a proof by induction, that $19 \mid \left(\frac{5}{4} \times 8^n + 3^{3n-1}\right),\ n \in \mathbb{Z}^+$.

14. Prove by induction that $9|[(3n+1)7^n - 1]$, $n \in \mathbb{Z}^+$.

15. Prove, using a proof by induction, that $7|9^n - 2^n$, $n \in \mathbb{Z}^+$.

16. Prove, using a proof by induction, that $6|8^n - 2^n$, $n \in \mathbb{Z}^+$.

17. Prove, using a proof by induction, that $16|5(5^n) - 4^n - 5$, $n \in \mathbb{Z}^+$.

18. Prove, using a proof by induction, that $5|2^{2k-1} + 3^{2k-1}, n \in \mathbb{Z}^+$.

19. Prove, using a proof by induction, that $4|5^n + 8n + 3$, $n \in \mathbb{Z}^+$.

Complex unfamiliar

20. Make a conjecture about a number which divides $2^{n+2} + 3^{2n+1}$ and prove your conjecture by induction.

Fully worked solutions for this chapter are available online.

LESSON
2.4 Trigonometric proofs using De Moivre's theorem

SYLLABUS LINKS

- Prove De Moivre's theorem for powers of positive integers.
- Prove multi-angle trigonometric identities up to angles of $4x$ by equating parts using the binomial expansion and De Moivre's theorem, e.g. $\cos(3x) = 4\cos^3(x) - 3\cos(x)$ and $\sin(3x) = 3\sin(x) - 4\sin^3(x)$.

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2.4.1 De Moivre's theorem and the binomial expansion

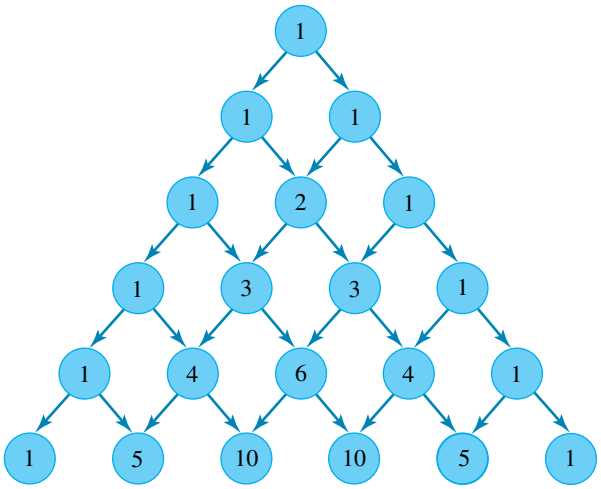
In Unit 1 of the Mathematical Methods course, you studied the binomial theorem.

Binomial theorem

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{r} x^{n-r}y^r + \cdots + y^n.$$

In Unit 1 you were also introduced to Pascal's triangle, which links the mathematical fields of combinations, probability and the binomial theorem. Recall that each row of Pascal's triangle begins and ends with a 1 and every other element is generated by adding the pair of elements directly above it. And recall that the elements in Pascal's triangle can be represented using combinatorial notation.

$$\text{Combinatorial notation: } \binom{n}{r} \text{ or } {}^nC_r.$$



$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

In Unit 3 of the Specialist mathematics course, you will revisit De Moivre's theorem, first introduced in Unit 2 Chapter 7 of this series.

De Moivre's theorem

If $z = r\text{cis}(\theta)$ then, $z^n = r^n\text{cis}(n\theta)$.

De Moivre's theorem can be proved using mathematical induction.

Let $P(n)$ be the proposition that $(r\text{cis}(\theta))^n = r^n\text{cis}(n\theta)$, $n \in \mathbb{N}$.

Initial statement:

For $n = 1$,

$$\text{LHS} = (r\text{cis}(\theta))^1 = r\text{cis}(\theta) \quad \text{and} \quad \text{RHS} = \left(r^1\text{cis}(1 \times \theta)\right) = r\text{cis}(\theta)$$

LHS = RHS; therefore, $P(1)$ is true.

Inductive hypothesis:

$\exists k \in \mathbb{N}$ such that $P(k)$ is true; that is, $(r\text{cis}(\theta))^k = r^k\text{cis}(k\theta)$, or

$$(r(\cos(\theta) + i\sin(\theta)))^k = r^k(\cos(k\theta) + i\sin(k\theta))$$

Inductive step:

Let's show that $P(k+1)$ is then true; that is, $(r\cos(\theta))^{k+1} = r^{k+1}\cos((k+1)\theta)$, or

$$(r(\cos(\theta) + i\sin(\theta)))^{k+1} = r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta))$$

$$\begin{aligned}
\text{LHS} &= (r(\cos(\theta) + i\sin(\theta)))^{k+1} \\
&= (r(\cos(\theta) + i\sin(\theta)))^k \times (r(\cos(\theta) + i\sin(\theta))) \\
&= r^k(\cos(k\theta) + i\sin(k\theta)) \times (r(\cos(\theta) + i\sin(\theta))) \\
&= r^{k+1}(\cos(k\theta) + i\sin(k\theta)) \times (\cos(\theta) + i\sin(\theta)) \\
&= r^{k+1}(\cos(k\theta) \times \cos(\theta) - \sin(k\theta) \times \sin\theta + i(\cos(k\theta) \times \sin(\theta) + \sin(k\theta) \times \cos(\theta)))
\end{aligned}$$

Using the following trigonometric identities:

$$\cos(A)\cos(B) - \sin(A)\sin(B) = \cos(A+B)$$

and

$$\cos(A)\sin(B) + \sin(A)\cos(B) = \sin(A+B)$$

$$\begin{aligned}\text{LHS} &= r^{k+1}(\cos(k\theta+\theta) + i\sin(k\theta+\theta))\\ &= r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta))\\ &= \text{RHS}\end{aligned}$$

Therefore, if $P(k)$ is true, then $P(k+1)$ is true.

Conclusion:

$P(1)$ is true and $P(k) \leftrightarrow P(k+1)$.

$\therefore P(n)$ is true $\forall n \in \mathbb{N}$.

In Unit 2 Chapter 9 of this series, you also used De Moivre's theorem to prove multiple angle identities.

De Moivre's theorem and the binomial theorem can be applied to prove various trigonometric identities involving $\cos(n\theta)$ and $\sin(n\theta)$.

The intention of this section is proving multiple-angle identities such as:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2(\theta) - 1\\ \sin(3\theta) &= 3\sin(\theta) - 4\sin^3(\theta)\end{aligned}$$

In Chapter 1, De Moivre's theorem was used with any complex number $z = r\cos(n\theta)$ for integral powers of n.

It states that for any complex number:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) \qquad [1]$$

We can also express z^n as:

$$z^n = r^n(\cos(\theta) + i\sin(\theta))^n \qquad [2]$$

Equating [1] and [2] leads to the following important result:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

This means that by using binomial expansion and equating the real and imaginary parts, we obtain the following identities.

Identities for $\sin(n\theta)$ and $\cos(n\theta)$

$$\begin{aligned}\sin(n\theta) &= \text{Im}\left[(\cos\theta + i\sin\theta)^n\right]\\ \cos(n\theta) &= \text{Re}\left[(\cos\theta + i\sin\theta)^n\right]\end{aligned}$$

Before we apply this definition to prove multi-angle trigonometric identities, it is important to review a few key points on powers of i and the binomial expansion theorem.

Powers of i	Binomial expansion
$i^2 = -1$	$(a+b)^2 = a^2 + 2ab + b^2$
$i^3 = -i$	$(a+b^3) = a^3 + 3a^2b + 3ab^2 + b^3$
$i^4 = 1$	$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$i^5 = i$	$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
$i^6 = -1$	

WORKED EXAMPLE 9 Applying De Moivre's theorem 1

Apply De Moivre's theorem and the binomial expansion to prove $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

THINK	WRITE
1. Write the general formula for $\sin(n\theta)$ where $n = 3$.	$\sin(n\theta) = \text{Im}(\cos(\theta) + i\sin(\theta))^n$ $\sin(2\theta) = \text{Im}(\cos(\theta) + i\sin(\theta))^2$
2. Apply the binomial expansion.	$(\cos(\theta) + i\sin(\theta))^2 = \cos^2(\theta) + 2\cos(\theta)\sin(\theta)\ i - \sin^2(\theta)$
3. Using the identity for $\sin(n\theta)$, extract the imaginary component to write an expression for $\sin(2\theta)$.	$\sin 2\theta = \text{Im}\left(\cos^2\theta + 2\cos\theta\sin\theta\, i - \sin^2\theta\right)$
4. State your conclusion.	$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$

In the following proof, the trigonometric identities are expressed in terms of x instead of θ.

WORKED EXAMPLE 10 Applying De Moivre's theorem 2

Use De Moivre's theorem and the binomial expansion to prove $\cos(3x) = 4\cos^3(x) - 3\cos(x)$.

THINK	WRITE
1. Write the general formula for $\cos(nx)$ where $n = 3$.	$\cos(nx) = \text{Re}(\cos(x) + i\sin(x))^n$ $\cos(3x) = \text{Re}(\cos(x) + i\sin(x))^3$
2. Apply the binomial expansion.	$(\cos(x) + i\sin(x))^3 = \cos^3(x) + 3\cos^2(x)(i\sin(x))$ $+ 3\cos(x)(i\sin(x))^2 + (i\sin(x))^3$ $= \cos^3(x) + 3i\cos^2(x)\sin(x)$ $+ 3\cos(x)\left(i^2 \times \sin^2(x)\right) + \left(i^3 \times \sin^3(x)\right)$
3. Simplify powers of i.	$= \cos^3(x) + 3i\cos^2(x)\sin(x) -$ $3\cos(x)\sin^2(x) - i\sin^3(x)$

4. Extract the real component to write an expression for $\cos(3x)$.	$\cos 3(x) = \text{Re}\left(\cos^3(x) + 3i\cos^2(x)\sin(x) - 3\cos(x)\sin^2(x) - i\sin^3(x)\right)$ $= \cos^3(x) - 3\cos(x)\sin^2(x)$
5. Use the Pythagorean identity $\sin^2(x) = 1 - \cos^2(x)$.	$= \cos^3(x) - 3\cos(x)\left(1 - \cos^2(x)\right)$ $= \cos^3(x) - 3\cos(x) + 3\cos^3(x)$ $= 4\cos^3(x) - 3\cos(x)$
6. State your conclusion.	Therefore, $\cos 3(x) = 4\cos^3(x) - 3\cos(x)$.

2.4.2 Alternative forms of multiple-angle formulas

It is possible to prove multiple-angle identities using an alternative approach. Consider the complex number $z = r\,\text{cis}(\theta)$ where $r = 1$. This means that $z = \cos(\theta) + i\sin(\theta)$.

Applying De Moivre's theorem for n and $-n$ it follows that:

$$z^n = \cos(n\theta) + i\sin(n\theta) \text{ and } z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$

Using the negative angle identities $\sin(\theta) = -\sin(\theta)$ and $\cos(\theta) = \cos(-\theta)$ it implies that:

$$z^{-n} = \cos(n\theta) - i\sin(n\theta)$$

Consider the sum of z^n and z^{-n}:

$$z^n + z^{-n} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$$

Hence,

$$z^n + z^{-n} = 2\cos(n\theta) \qquad [1]$$

Consider the difference of z^n and z^{-n}:

$$z^n - z^{-n} = \cos(n\theta) + i\sin(n\theta) - (\cos(n\theta) - i\sin(n\theta))$$

Hence,

$$z^n - z^{-n} = 2i\sin(n\theta) \qquad [2]$$

Equations [1] and [2] can be rearranged to produce multiple-angle formulas that can be used more easily when proving trigonometric identities.

Multiple-angle identities

$$\cos(n\theta) = \frac{z^n + z^{-n}}{2}, \quad \sin(n\theta) = \frac{z^n - z^{-n}}{2i}$$

WORKED EXAMPLE 11 Applying multiple-angle formulas 1

Use the multiple-angle formulas to prove that $2\sin(3\theta)\cos(3\theta) = \sin(6\theta)$.

THINK	WRITE
1. Consider the left-hand side of the equation. Use the multiple-angle formula where $n = 3$.	$\text{LHS} = 2\sin(3\theta)\cos(3\theta)$ $= 2 \times \frac{z^3 - z^{-3}}{2i} \times \frac{z^3 + z^{-3}}{2}$
2. Simplify to make the expansion process simpler.	$= \frac{2}{4i}\left(z^3 - z^{-3}\right)\left(z^3 + z^{-3}\right)$
3. Cancel a factor of 2 from the numerator and denominator and then expand the brackets.	$= \frac{1}{2i}\left(z^6 + z^0 - z^0 - z^{-6}\right)$
4. Simplify and rewrite in the form $\sin(n\theta) = \frac{z^n - z^{-n}}{2i}$.	$= \frac{z^6 - z^{-6}}{2i}$ $= \sin(6\theta)$ $= \text{RHS}$
5. State your conclusion.	$\therefore 2\sin(3\theta)\cos(3\theta) = \sin(6\theta)$

WORKED EXAMPLE 12 Applying multiple-angle formulas 2

Use the multiple-angle formulas to prove that $2\cos(3\theta)\sin(2\theta) = \sin(5\theta) - \sin(\theta)$.

THINK	WRITE
1. Consider the left-hand side of the equation. Use the multiple-angle formulae with appropriate values of n.	$\text{LHS} = 2\cos(3\theta)\sin(2\theta)$ $= 2 \times \frac{z^3 + z^{-3}}{2} \times \frac{z^2 - z^{-2}}{2i}$
2. Simplify to make the expansion process easier.	$= \frac{2}{4i}\left(z^3 + z^{-3}\right)\left(z^2 - z^{-2}\right)$
3. Cancel a factor of 2 from the numerator and denominator and then expand the brackets.	$= \frac{1}{2i}\left(z^5 - z^1 + z^{-1} - z^{-5}\right)$
4. Simplify and rewrite by combining appropriate pairings of z.	$= \frac{1}{2i}\left[\left(z^5 - z^{-5}\right) - \left(z^{-1} - z^{-1}\right)\right]$
5. Expand the brackets by multiplying by $\frac{1}{2i}$.	$= \frac{z^5 - z^{-5}}{2i} - \frac{z^{-1} - z^{-1}}{2i}$ $= \sin(5\theta) - \sin(\theta)$ $= \text{RHS}$
6. State your conclusion.	$\therefore 2\cos(3\theta)\sin(2\theta) = \sin(5\theta) - \sin(\theta)$

It is possible to prove more complex identities such as $\cos^4(x) = \frac{1}{8}(\cos(4x) + 4\cos(2x) + 3)$ by applying the multiple-angle formulas and the binomial theorem. This may be considered an extension concept, and the proofs will be limited to integer powers of $n \leq 5$.

WORKED EXAMPLE 13 Applying multiple-angle formulas 3

Use the multiple-angle formulas and the binomial theorem to prove that $\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$.

THINK	WRITE
1. Consider the left-hand side of the equation. Use the multiple-angle formulas and set $n = 1$.	$\sin(nx) = \frac{z^n - z^{-n}}{2i}$ $\sin(1x) = \frac{z^1 - z^{-1}}{2i}$ [1]
2. Multiply both sides of [1] by $2i$.	$2i\sin(x) = z^1 - z^{-1}$
3. Write an equation involving $\sin^3 x$.	$(2i\sin(x))^3 = \left(z^1 - z^{-1}\right)^3$
4. Apply the binomial expansion for $\left(z^1 - z^{-1}\right)^3$.	$8i^3\sin^3(x) = 1\left(z^1\right)^3\left(-z^{-1}\right)^0 + 3\left(z^1\right)^2\left(-z^{-1}\right)^1$ $+3\left(z^1\right)^1\left(-z^{-1}\right)^2 + 1\left(z^1\right)^0\left(-z^{-1}\right)^3$ $= z^3 - 3z^1 + 3z^{-1} - z^{-3}$
5. Rewrite the LHS using $i^3 = -i$. Group suitable pairings of z on the RHS.	$-8i\sin^3(x) = z^3 - z^{-3} + 3z^{-1} - 3z^1$ $= z^3 - z^{-3} - 3\left(z^1 - z^{-1}\right)$
6. Divide both sides by $-2i$ so that a multiple-angle formula can be used.	$4\sin^3(x) = \frac{-\left(z^3 - z^{-3}\right)}{2i} + \frac{3\left(z^1 - z^{-1}\right)}{2i}$ $= -\sin(3x) + 3\sin(x)$ $= 3\sin(x) - \sin(3x)$
7. State your conclusion.	$\therefore\ \sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$

Exercise 2.4 Trigonometric proofs using De Moivre's theorem

learnon

2.4 Exercise | 2.4 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9	10, 11, 12	13, 14, 15

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

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Simple familiar

1. WE9 Apply De Moivre's theorem and the binomial expansion to prove $\cos(2\theta) = 2\cos^2(\theta) - 1$.
2. WE10 Use De Moivre's theorem and the binomial expansion to prove $\sin(3x) = 3\sin(x) - 4\sin^3(x)$.

3. Use De Moivre's theorem and the binomial expansion to prove the following multi-angle identities.
   a. $\sin(4x) = 4\sin(x)\cos^3(x) - 4\cos(x)\sin^3(x)$
   b. $\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$

4. Use De Moivre's theorem and the binomial expansion to prove the following multi-angle identities.
   a. $\cos(5x) = 16\cos^5(x) - 20\cos^3(x) + 5\cos(x)$
   b. $\sin(5x) = 16\sin^5(x) - 20\sin^3(x) + 5\sin(x)$

5. WE11 Use the multiple-angle formulas to prove the following identities.
   a. $2\sin(2x)\cos(x) = \sin(3x) + \sin(x)$
   b. $2\cos(2x)\sin(x) = \sin(3x) - \sin(x)$

6. WE12 Use the multiple-angle formulas to prove $2\sin(4\theta)\cos(2\theta) = \sin(6\theta) + \sin(2\theta)$.

7. Use the multiple-angle formulas to prove $2\cos(3x)\cos(2x) = \cos(5x) + \cos(x)$.

8. Use the multiple-angle formulas to prove $\sin(4\theta)(1 + 2\cos(2\theta)) = \sin(2\theta) + 4\sin(4\theta) + \sin(6\theta)$.

9. Express $\cos(5\theta)$ in terms of powers of $\cos(\theta)$ and $\sin(5\theta)$ in terms of powers of $\sin(\theta)$.

Complex familiar

10. WE13 Use the multiple-angle formulas and the binomial theorem to prove $\cos^3(x) = \frac{1}{4}(\cos(3x) + 3\cos(x))$.

11. Use the multiple-angle formulas and the binomial theorem to prove $\sin^4(x) = \frac{1}{8}(\cos(4x) - 4\cos(2x) + 3)$.

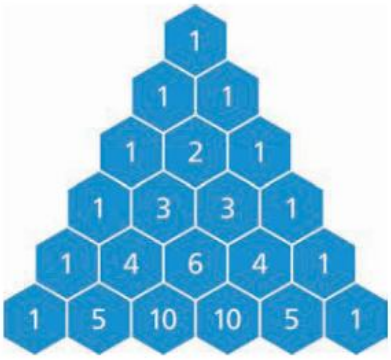
12. Use the multiple-angle formulas and the binomial theorem to prove $\sin^5(x) = \frac{1}{16}(\sin(5x) - 5\sin(3x) + 10\sin(x)$.

Complex unfamiliar

13. If $z = \cos(\theta) + i\sin(\theta)$, then show that $\tan(\theta) = \dfrac{z - z^{-1}}{i\left(z + z^{-1}\right)}$.

14. If $z = \cos(\theta) + i\sin(\theta)$, then show that $\tan^2(\theta) = \dfrac{2 - z^{-2} - z^2}{2 + z^{-2} + z^2}$.

15. If $z = \cos(\theta) + i\sin(\theta)$, then show that $\cos^4(\theta) + \sin^4(\theta) \le 6\cos^2(\theta)\sin^2(\theta)$ if $\frac{\pi}{8} \le \theta \le \frac{3\pi}{8}$.



Fully worked solutions for this chapter are available online.

LESSON
2.5 Review

2.5.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

- Access the chapter summary
- Review your results
- Practise exam questions

Find all this and MORE in jacPLUS

2.5 Exercise

learnon

2.5 Exercise	2.5 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. Consider the proposition $P(n): \sum_{r=1}^{n}(r+1)=\frac{n(n+3)}{2}, \forall n \in \mathbb{N}$.
   a. Show that $P(1)$, $P(2)$ and $P(3)$ are true.
   b. Identify $P(k)$.
   c. Identify $P(k+1)$.
   d. Prove $P(k) \Rightarrow P(k+1)$.
2. Prove by induction that $(\cos(\theta)+i\sin(\theta))^n=(\cos(n\theta)+i\sin(n\theta)), \forall n \in \mathbb{N}$.
3. Prove by induction that $\sum_{r=1}^{n}(3r-2)=\frac{n(3n-1)}{2}$.
4. Use a proof by induction to prove that $3+9+27+\ldots+3^n=\frac{3^{n+1}-3}{2}$.
5. Use a proof by induction to prove that $P(n): \sum_{1}^{n}\frac{1}{(2r-1)(2r+1)}=\frac{n}{2n+1}$.
6. Prove by induction $P(n): (1+x)^n \geq 1+nx$.
7. Use proof by induction to demonstrate that $3^{2n}-1$ is divisible by 8 for $n \geq 1$.
8. Prove by induction that $7 \mid \left(12^n+2\left(5^{n-1}\right)\right)$, $n \geq 1$.
9. Use a proof by induction to demonstrate that $9 \mid (4^n+6n-1)$, $n \geq 1$.
10. Prove by induction that $9 \mid (10^n-1)$, $n \geq 1$.
11. Prove by induction that $2^{n+2}3^n+5n-4$ is divisible by 25 for $n \in \mathbb{N}$.
12. Use a proof by induction to prove that 7^n-2^n is divisible by 5 for $n \in \mathbb{Z}^+$.

Complex familiar

13. Use proof by induction to demonstrate that $19 \mid \left(5^{2n+1} + 3^{n+2} \times 2^{n-1}\right)$, $n \geq 1$.

14. Use proof by induction to demonstrate that $\sum_{r=1}^{n} \left(r^2 + r\right) = \frac{n^3}{3} + n^2 + \frac{2n}{3}$.

15. Use proof by induction to demonstrate that $P(n) : \sum_{1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ is true.

16. Use De Moivre's theorem and the binomial expansion to prove that
$\sin^7(x) = \frac{1}{64}\left(-\sin(7x) + 7\sin(5x) - 21\sin(3x) + 35\sin(x)\right) \forall n \in \mathbb{N}$.

Complex unfamiliar

17. Apply mathematical induction to prove $\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$ for all $n \geq 1$ where $r \in \mathbb{R}$ and $r \neq 1$.

18. Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Proof by induction that $\forall n \in \mathbb{N}$, $A^n = \begin{bmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{bmatrix}$.

19. Make a conjecture about a number which divides $5^{5n+1} + 4^{5n+2} + 3^{5n}$ and prove your conjecture by induction.

20. Let $\theta \in \mathbb{R}$ such that $\sin(\theta) \neq 0$.

Prove that $\forall n \in \mathbb{N}$, $\cos(\theta) \times \cos(2\theta)\cos(4\theta)\cos(8\theta) \times \cdots \times \cos\left(2^{n-1}\theta\right) = \frac{\sin(2^n\theta)}{2^n \sin(\theta)}$.

2.5 Past QCAA exam questions

Question 1 (1 mark)

Source: *QCAA 2021 Specialist Mathematics, Paper 2, Section 1, Q3; © State of Queensland (QCAA) 2024*

Given $n \in Z^+$, for which proposition can the initial statement for mathematical induction be proven?

A. $x^{2n} - y^{2n}$ is divisible by $(x + y) \forall (x + y) \neq 0$

B. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n\left(2n^2 + 3n - 1\right)$

C. $(n+1)^3 + (n+2)^3$ is divisible by 3

D. $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{n+1}$

Question 2 (1 mark)

Source: *QCAA 2023 Specialist Maths, Paper 1, Section 1, Q2; © State of Queensland (QCAA) 2024*

Consider the proof of the following proposition using mathematical induction.

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)\ \forall\, n \in Z^{+}$$

An appropriate assumption statement within the proof is

A. $\sum_{r=1}^{k} k(k+1) = \frac{1}{3}k(k+1)(k+2)$

B. $\sum_{r=1}^{k} k(k+1) = \frac{1}{3}n(n+1)(n+2)$

C. $\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$

D. $\sum_{r=1}^{k} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

Question 3 (1 mark)

Source: *QCAA 2021 Specialist Mathematics, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024*

Use mathematical induction to prove that $2^{2n} + 3n - 1$ is divisible by $3\ \forall\, n \in Z^{+}$.

Question 4 (6 marks)

Source: *QCAA 2020 Specialist Mathematics, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024*

Given $\cos(\theta) \neq 0\ \forall\, n \in Z^{+}$, use mathematical induction to prove

$$\cos(\theta) - \cos(3\theta) + \cos(5\theta) - \cdots + (-1)^{n+1}\cos((2n-1)\theta) = \frac{1-(-1)^{n}\cos(2n\theta)}{2\cos(\theta)}$$

Question 5 (5 marks)

Source: *QCAA 2023 Specialist Maths, Paper 1, Section 2, Q15; © State of Queensland (QCAA) 2024*

The sum of a geometric progression with n terms, where the first term is 1 and the common ratio is r, is given by

$$1 + r + r^{2} + r^{3} + \cdots + r^{n-1} = \frac{r^{n}-1}{r-1}\ (\text{for } r \neq 1).$$

Prove that this rule is true $\forall\, n \in Z^{+}$ using mathematical induction by completing the steps of the proof as indicated.

A. Initial statement: **[1 mark]**

Assuming the rule is true for $n = k$,

$$1 + r + r^{2} + r^{3} + \cdots + r^{k-1} = \frac{r^{k}-1}{r-1}\ (r \neq 1).$$

B. Inductive step: **[3 marks]**

C. Conclusion: **[1 mark]**

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Answers

Chapter 2 Mathematical induction and trigonometric proofs

2.2 Proof by induction

2.2 Exercise

1. a. $\sum_{j=1}^{4} 5j = 5 + 10 + 15 + 20$

b. $\sum_{r=1}^{n} r^4 = 1^4 + 2^4 + 3^4 + \ldots + (n-1)^4 + n^4$

2. a. $\sum_{r=1}^{n} 6r - 5 = 1 + 7 + 13 + \ldots + (6(n-1) - 5) + (6n - 5)$

b. $\sum_{j=1}^{4} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

c. $\sum_{s=1}^{6} 2 - 3s = -1 - 4 - 7 - 10 - 13 - 15$

d. $\sum_{j=0}^{n} \frac{(-1)^j}{j+2} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \ldots + \frac{(-1)^n}{n+2}$

3. a. $3 + 6 + 9 + 12 + \cdots + 57 = \sum_{j=1}^{19} 3j$

b. $7 + 11 + 15 + 19 + \cdots + 43 = \sum_{j=0}^{9} 7 + 4j \left(\text{or} \sum_{j=1}^{10} 3 + 4j \right)$

4. a. $1^6 + 2^6 + 3^6 + \ldots + (n-1)^6 + n^6 = \sum_{r=1}^{n} r^6$

b. $5 + 7 + 9 + 11 + 13 = \sum_{r=0}^{4} 5 + 2r \left(\text{or} \sum_{r=1}^{5} 3 + 2r \right)$

5. a. Sample responses can be found in the worked solutions in the online resources.

b. $P(k) : \sum_{j=1}^{k} \frac{1}{3^j} = \frac{1}{2}\left(1 - \frac{1}{3^k}\right)$

c. $P(k+1) : \sum_{j=1}^{k+1} \frac{1}{3^j} = \frac{1}{2}\left(1 - \frac{1}{3^{k+1}}\right)$

d. Sample responses can be found in the worked solutions in the online resources.

6. Sample responses can be found in the worked solutions in the online resources.

7. a. $n = 1$. LHS $= 2$. RHS $= 1(1+1) = 2$. LHS = RHS
$n = 2$. LHS $= 2 + 4 = 6$. RHS $= 2(2+1) = 6$. LHS = RHS
$n = 3$. LHS $= 2 + 4 + 6 = 12$. RHS $= 3(3+1) = 12$. LHS = RHS

b. Sample responses can be found in the worked solutions in the online resources.

8. a. $n = 1$. $\sum_{r=1}^{n} 2^{r-1} = 2^0 = 1$, $2^n - 1 = 2^1 - 1 = 1$. True
$n = 2$. $\sum_{r=1}^{n} 2^{r-1} = 2^0 + 2^1 = 3$, $2^n - 1 = 2^2 - 1 = 3$. True
$n = 3$. $\sum_{r=1}^{n} 2^{r-1} = 2^0 + 2^1 + 2^2 = 7$, $2^n - 1 = 2^3 - 1 = 7$. True

b. Sample responses can be found in the worked solutions in the online resources.

9. a. $n = 1$. LHS $= 3$. RHS $= n(n+2) = 1(3) = 3$. True

b. $3 + 5 + 7 + \ldots (1 + 2k) = k(k+2)$

c. Sample responses can be found in the worked solutions in the online resources.

d. If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, the statement is true for $n \geq 1$.

10. a. $n = 1$. $\sum_{r=1}^{n} (2r - 1) = 2 - 1 = 1$. $n^2 = 1$. True

b. $\sum_{r=1}^{k} (2r - 1) = k^2$

c. Sample responses can be found in the worked solutions in the online resources.

d. If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, the statement is true for $n \geq 1$.

11. a. $P(1) : \sum_{i=1}^{1} 2^i = 2$

b. $P(k) : \sum_{i=1}^{k} 2^i = 2^{k+1} - 2$

c. 2^{k+1}

d. Sample responses can be found in the worked solutions in the online resources.

12. a. $P(1) : \sum_{i=1}^{1} 2i = 2$

b. $P(k) : \sum_{i=1}^{k} 2i = k(k+1)$

c. $2(k+1)$

d. Sample responses can be found in the worked solutions in the online resources.

13–21. Sample responses can be found in the worked solutions in the online resources.

22. a. LHS : $\sum_{2}^{2} (r-1)r = (2-1) \times 2$ RHS : $= 2$
$\frac{(2-1)2(2+1)}{3} = 2$. $P(2)$ is true.

b. $P(k) : \sum_{r=2}^{k} (r-1)r = \frac{(k-1)k(k+1)}{3}$

c. $P(k+1) : \sum_{r=2}^{k+1} (r-1)r = \frac{k(k+1)(k+2)}{3}$

d. Sample responses can be found in the worked solutions in the online resources.

23–30. Sample responses can be found in the worked solutions in the online resources.

2.3 Proofs of divisibility

2.3 Exercise

1. a. $n = 0$, $6^n + 4 = 1 + 4 = 5$. Divisible by 5.
$n = 1$, $6^n + 4 = 6 + 4 = 10$. Divisible by 5.
$n = 2$, $6^n + 4 = 36 + 4 = 40$. Divisible by 5.

b. Sample responses can be found in the worked solutions in the online resources.

2. a. $n = 1$, $8^n - 1 = 8 - 1 = 7$. Divisible by 7.
$n = 2$, $8^n - 1 = 64 - 1 = 63$. Divisible by 7.
$n = 3$, $8^n - 1 = 512 - 1 = 511 = 73 \times 7$. Divisible by 7.

b. Sample responses can be found in the worked solutions in the online resources.

3. a. $n = 1$. $3^{2n-1} + 1 = 3 + 1 = 4$. Divisible by 4.

b. $3^{2k-1} + 1 = 4m,\ m \in \mathbb{Z}$.

c. Sample responses can be found in the worked solutions in the online resources.

d. $3^{2n-1} + 1$ is divisible by 4 when $n = 1$. It is divisible by 4 for $n = k + 1$ if it is divisible by 4 for $n = k$. Therefore, it is divisible by 4 for $n \in \mathbb{Z}^+$.

4. a. $n = 2$. $10^n - 4 = 100 - 4 = 96 = 12 \times 8$. Divisible by 12.

b. $10^n - 4 = 12m,\ m \in \mathbb{Z}$.

c. Sample responses can be found in the worked solutions in the online resources.

d. $10^n - 4$ is divisible by 12 when $n = 2$. It is divisible by 12 for $n = k + 1$ if it is divisible by 12 for $n = k$. Therefore, it is divisible by 12 for $n \geq 2$.

5–20. Sample responses can be found in the worked solutions in the online resources.

2.4 Trigonometric proofs using De Moivre's theorem

2.4 Exercise

1–8. Sample responses can be found in the worked solutions in the online resources.

9. $\cos(5\theta) = 16\cos^5(\theta) - 20\cos^3(\theta) + 5\cos(\theta)$
$\sin(5\theta) = 16\sin^5(\theta) - 20\sin^3(\theta) + 5\sin(\theta)$

10–15. Sample responses can be found in the worked solutions in the online resources.

2.5 Review

2.5 Exercise

1. a. Sample responses can be found in the worked solutions in the online resources.

b. $P(k): \sum_{r=1}^{k}(r+1) = \frac{k(k+3)}{2}$

c. $P(k+1): \sum_{r=1}^{k+1}(r+1) = \frac{(k+1)(k+4)}{2}$

d. Sample responses can be found in the worked solutions in the online resources.

2–20. Sample responses can be found in the worked solutions in the online resources.

2.5 Past QCAA exam questions

1. A

2. C

3. RTP
$2^{2n} + 3n - 1$ is always divisible by 3 for $n \in Z^+$,
i.e. $2^{2n} + 3n - 1 = 3m$, for some $m \in Z^+$.
Initial statement: Let $n = 1$.
$2^2 + 3 - 1 = 6$
$= 3 \times 2$
The proposition is true for $n = 1$.
Assume proposition is true for $n = k\ \forall\ k \in Z^+$
$2^{2k} + 3k - 1 = 3m$ for some $m \in Z^+$
Inductive step: Let $n = k + 1$.
$2^{2(k+1)} + 3(k+1) - 1 = 2^{2k+2} + 3k + 3 - 1$
$= 2^2 \times 2^{2k} + 3k - 1 + 3$
$= 2^{2k} + 3k - 1 + 3 \times 2^{2k} + 3$
$= 3m + 3 \times 2^{2k} + 3$
$= 3\left(m + 2^{2k} + 1\right)$
$= 3p$ for some $p \in Z^+$
The proposition is true for $n = k + 1$. By mathematical induction, the proposition is true for $n = 1, 2, \ldots$

4. It is true for $n = k + 1$, so by mathematical induction it is true $\forall\ n \in Z^+$.

5. Sample responses can be found in the worked solutions in the online resources.

3 Vectors in two and three dimensions

LESSON SEQUENCE

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3.2 Vectors in three dimensions 104
3.3 Algebra of three dimensional vectors 120
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3.5 Vector, parametric and Cartesian equations 142
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LESSON
3.1 Overview

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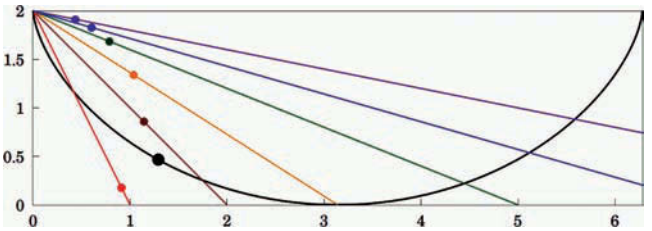
3.1.1 Introduction

Until this point, the study of vectors has been restrained to only two dimensions. By extending our study of vectors to three dimensions, it allows us to strengthen our understanding of Cartesian equations and expand our knowledge through the introduction to parametric equations.



Parametric equations link our conventional understanding of equations of two variable with a third, known as a parameter. The famous Brachistochrone problem incorporates three variables and was posed by the Mathematician James Bernoulli (1654–1705) to identify the path (x, y) in which a particle could slide from two points (P and Q) in the least possible time (t). If the particle followed a path in the shape of an inverted cycloid, it would reach its destination in the shortest possible time. Applications of equations involving the parameter t are inherent in modelling problems such as the vector equation of a straight line in three dimensions and eventually determining points of collision between the paths of two or more particles.

3.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
3.2	**Vectors in three dimensions**	◯ Use Cartesian coordinates for three dimensional space, including plotting points. ◯ Use ordered pair notation (x, y, z) and column vector notation $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to represent a position vector in three dimensions. ◯ Calculate the magnitude of a vector. • $\lvert \boldsymbol{a} \rvert = \left\lvert \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$ ◯ Calculate and use a unit vector $\hat{\boldsymbol{n}}$, in three dimensional space. • $\hat{\boldsymbol{n}} = \dfrac{\boldsymbol{n}}{\lvert \boldsymbol{n} \rvert}$ ◯ Define and use unit vectors and the perpendicular unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. ◯ Express a vector in Cartesian (component) form using the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. ◯ Define and use the altitude angle φ.

3.3	**Algebra of three dimensional vectors**	◯ Examine and use addition and subtraction of vectors in Cartesian form. ◯ Use multiplication by a scalar of a vector in Cartesian form. ◯ Determine a vector between two points. ◯ Use the scalar (dot) product. • $\boldsymbol{a} \cdot \boldsymbol{b} = \lvert\boldsymbol{a}\rvert \lvert\boldsymbol{b}\rvert \cos(\theta)$ • $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$ ◯ Use scalar and vector projections of vectors. • scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$: $\lvert\boldsymbol{a}\rvert \cos(\theta) = \boldsymbol{a} \cdot \hat{\boldsymbol{b}}$ • vector projection of $\boldsymbol{a}$ on $\boldsymbol{b}$: $\lvert\boldsymbol{a}\rvert \cos(\theta)\hat{\boldsymbol{b}} = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}} = \left(\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right)\boldsymbol{b}$ ◯ Apply the scalar product to vectors expressed in Cartesian form.
3.4	**Geometric proofs using vectors**	◯ Use a vector representing a section of a line segment, including the midpoint of a line segment. ◯ Examine properties of parallel and perpendicular vectors and determine if two vectors are parallel or perpendicular. ◯ Use vectors to prove geometric results in two dimensions (other than those listed in Unit 2 Topic 3) and in three dimensions.
3.5	**Vector, parametric and Cartesian equations**	◯ Understand and use equations of spheres. • equation of sphere: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ ◯ Use vector equations of curves in two or three dimensions involving a parameter, and determine a 'corresponding' Cartesian equation in the two dimensional case. ◯ Determine vector, parametric and Cartesian equations of straight lines and straight-line segments given the position of two points, or equivalent information, in both two and three dimensions. • vector equation of line: $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}$ • parametric equations of line: $x = a_1 + td_1$ $y = a_2 + td_2$ $z = a_3 + td_3$ • Cartesian equation of line: $\dfrac{x-a_1}{d_1} = \dfrac{y=a_2}{d_2} = \dfrac{z-a_3}{d_3}$
3.6	**The vector (cross) product**	◯ Define and use the vector (cross) product to determine a vector normal to a given plane, with and without technology. • $\boldsymbol{a} \times \boldsymbol{b} = \lvert\boldsymbol{a}\rvert \lvert\boldsymbol{b}\rvert \sin(\theta)\hat{\boldsymbol{n}}$ • $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$
3.7	**Applications of vectors**	◯ Model and solve problems that involve displacement, force, velocity and relative velocity using the above concepts. ◯ Use vectors to prove geometric results in two dimensions (other than those listed in Unit 2 Topic 3) and in three dimensions. ◯ Use vector methods in applications, including areas of shapes and determining vector and Cartesian equations of a plane and of regions in a plane. • vector equation of plane: $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ • Cartesian equation of plane: $ax + by + cz + d = 0$.

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LESSON
3.2 Vectors in three dimensions

SYLLABUS LINKS

- Use Cartesian coordinates for three dimensional space, including plotting points.
- Use ordered pair notation (x, y, z) and column vector notation $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to represent a position vector in three dimensions.
- Calculate the magnitude of a vector.
  - $|\boldsymbol{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- Calculate and use a unit vector $\hat{\boldsymbol{n}}$, in three dimensional space.
  - $\hat{\boldsymbol{n}} = \dfrac{\boldsymbol{n}}{|\boldsymbol{n}|}$
- Define and use unit vectors and the perpendicular unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$.
- Express a vector in Cartesian (component) form using the unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$.
- Define and use the altitude angle φ.

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Introduced in Unit 1 (see Unit 1, Topic 3 Vectors in the plane and Topic 4 Algebra of vectors in two dimensions), vectors have thus far only been considered in two dimensional form. In this chapter, following a review of two dimensional vectors, our study of vectors will be extended to three dimensions.

3.2.1 Review of vectors in two dimensions

Scalar and vector quantities

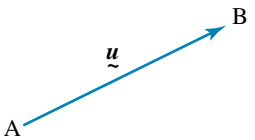
A **scalar** quantity is a real number only, it has magnitude, but no direction. Scalar quantities include temperature, mass and time.

A **vector** is a quantity that has **magnitude** and **direction**. Vectors include weight in a particular direction and displacement with a specific direction.

Vectors are directed line segments with a tail (start) and head (end). In the figure below, the head of the vector is at point B (indicated with an arrow), while the tail is at point A. In this instance, the vector could be expressed in a variety of forms.

Various forms of vector notation

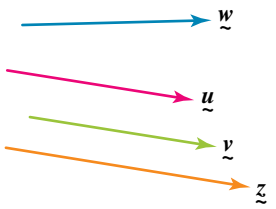
$\overrightarrow{\mathrm{AB}} = \underset{\sim}{u} = \boldsymbol{u}$



The *tilde* symbol (~) is commonly used because hand writing bold letters is difficult.

Equality of vectors

Two vectors are equal if *both* their magnitude and direction are equal.



In this figure, the following statements can be made:

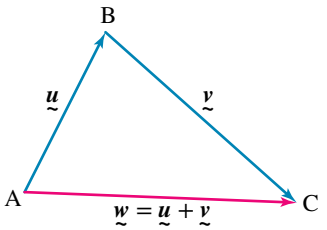
$\underset{\sim}{u} = \underset{\sim}{v}$ (direction and magnitude equal)

$\underset{\sim}{u} \neq \underset{\sim}{w}$ (directions are not equal)

$\underset{\sim}{u} \neq \underset{\sim}{z}$ (magnitudes are not equal)

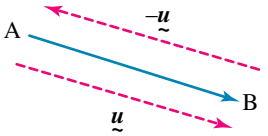
Addition and subtraction of vectors

The triangle rule of vectors can be applied in both the addition and subtraction of vectors.



To add two vectors, take the tail of one vector and join it to the head of another. The result of this addition is the vector from the tail of the first vector to the head of the second vector.

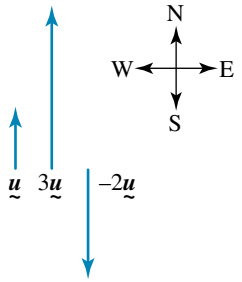
If $\underset{\sim}{u}$ is the vector from A to B, then $-\underset{\sim}{u}$ is the vector from B to A.



To subtract two vectors, add the negative of the second vector to the first vector. For example, $A - B = A + (-B)$.

Multiplying a vector by a scalar

Multiplying a vector by a *positive* number (scalar) affects only the *magnitude* (size) of the vector, not the direction.



If the scalar is negative, then the direction is reversed.

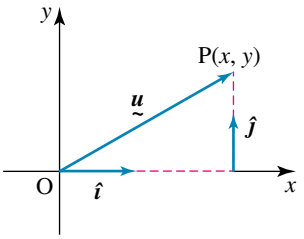
Cartesian form of a vector in two dimensions

Thus far, points have been considered in two dimensions. A point, $P(x, y)$, is determined by its distances from two mutually perpendicular axes.

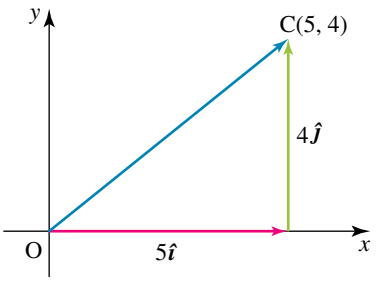
On the Cartesian plane, a **position vector** is the directed line segment joining the origin with the point P. It defines a point, P, by magnitude and direction relative to the origin, O, and can be written as $\overrightarrow{OP}$ or $\underset{\sim}{u}$.

The position vector $\underset{\sim}{u}$ is the sum of the two directional unit vectors, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ where:

- $\hat{\boldsymbol{\imath}}$ is a unit vector along the x-axis, with magnitude 1.
- $\hat{\boldsymbol{\jmath}}$ is a unit vector along the y-axis, with magnitude 1.



For example, if the point C has coordinates (5, 4), then $\underset{\sim}{u} = 5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$.



This is the **Cartesian form** of a vector, and is given by,

$$\underset{\sim}{u} = \overrightarrow{OP} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$$

The position vector, $\underset{\sim}{u} = 5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ can also be expressed as an ordered pair $(5, 4)$ or in column vector notation as, $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

The magnitude of a vector

By using Pythagoras' theorem on a position vector, we can determine its length, or magnitude.

The magnitude of the vector $\underset{\sim}{u} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ is also called the **norm** and is denoted as $|\underset{\sim}{u}|$ or r.

$$|\underset{\sim}{u}| = \sqrt{x^2 + y^2}$$

The magnitude of $\underset{\sim}{u} = 5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ can be calculated using the above formula

$$\begin{aligned} |\underset{\sim}{u}| &= \sqrt{5^2 + 4^2} \\ &= \sqrt{41} \end{aligned}$$

The unit vector

The magnitude of a vector is also used to determine the **unit vector**. If $\underset{\sim}{u}$ is the vector, then the unit vector, denoted by $\hat{\underset{\sim}{u}}$, is given by,

$$\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$$

For example, the unit vector for $\underset{\sim}{u} = 5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ is:

$$\begin{aligned} \hat{\underset{\sim}{u}} &= \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|} \\ &= \frac{5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}}{\sqrt{41}} \\ &= \frac{5}{\sqrt{41}}\hat{\boldsymbol{\imath}} + \frac{4}{\sqrt{41}}\hat{\boldsymbol{\jmath}} \quad \text{or} \quad \frac{1}{\sqrt{41}}(5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}) \end{aligned}$$

The unit vector $\hat{\underset{\sim}{u}}$ is in the same direction as vector $\underset{\sim}{u}$, but it is scaled up or down to be of length 1.

Polar form

In **polar form**, the vector $\underset{\sim}{u} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, can be expressed in terms of its magnitude $|\underset{\sim}{u}|$ and direction θ. The polar form of a vector $\underset{\sim}{u}$ is (r, θ), where:

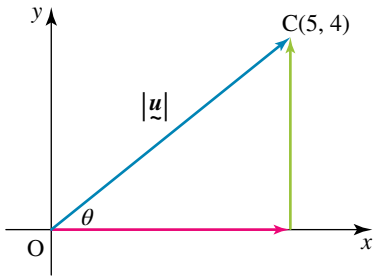
- r is the magnitude of $\underset{\sim}{u}$
- θ is the anti-clockwise angle formed from the positive x-axis (polar axis)
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

  $\theta = \tan^{-1}\left(\frac{4}{5}\right)$

  $\theta \approx 0.6747$ radians

  $\theta \approx 38.66°$

  $\therefore \underset{\sim}{u} = \left(\sqrt{41}, 38.66°\right)$



Recall that there are two possible values for θ in a 2π domain, thus you will need to check which value to keep and which value to reject, as seen in Unit 1, Topic 3 Vectors in the plane.

The position vector $\underset{\sim}{u} = 5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ can be expressed in polar form as shown.

$$\underset{\sim}{u} = 5\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$$

$|\underset{\sim}{u}| = \sqrt{41}$ (previously calculated)

Resolving vectors into components

If the two dimensional vector $\underset{\sim}{u}$ makes an angle of θ with the positive x-axis and it has a magnitude of r, then we can determine its x- and y-components using the formulas:

$$x = |r|\cos(\theta)$$
$$y = |r|\sin(\theta)$$

For example, the vector in polar form $\underset{\sim}{v} = (5, 233.13°)$, can be converted to component (Cartesian) form:

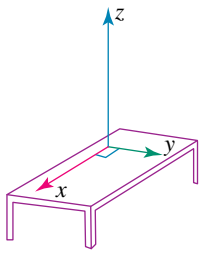
$$\begin{aligned}\underset{\sim}{v} &= 5\cos(233.13°)\,\hat{\boldsymbol{\imath}} + 5\sin(233.13°)\,\hat{\boldsymbol{\jmath}}\\ &= -3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}\end{aligned}$$

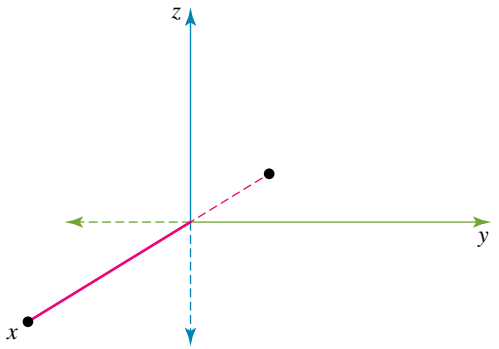
3.2.2 Three dimensional vectors in component form

All the concepts learnt in Unit 1 for vectors in the plane are applicable in three dimensions.

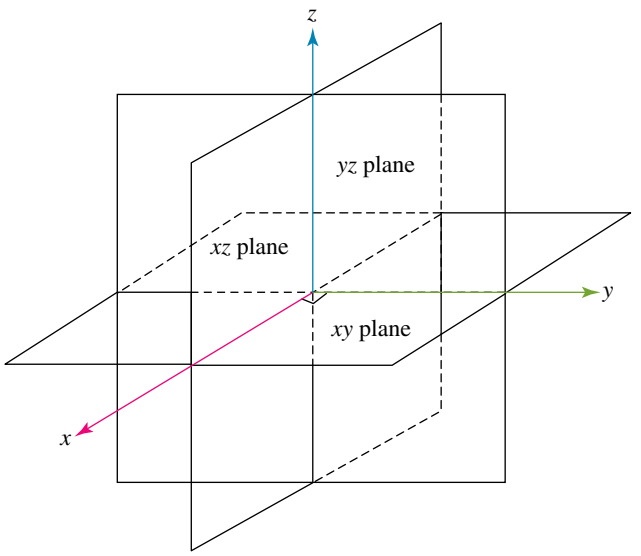
Cartesian coordinates for three dimensional planes

In the three dimensional coordinate system, the z-axis exists perpendicular to both the x- and y-axes at the origin. The three dimensions are shown here as if the x and y axes are lying flat on a desk top, at right angles to each other, and the z-axis is pointing straight up.



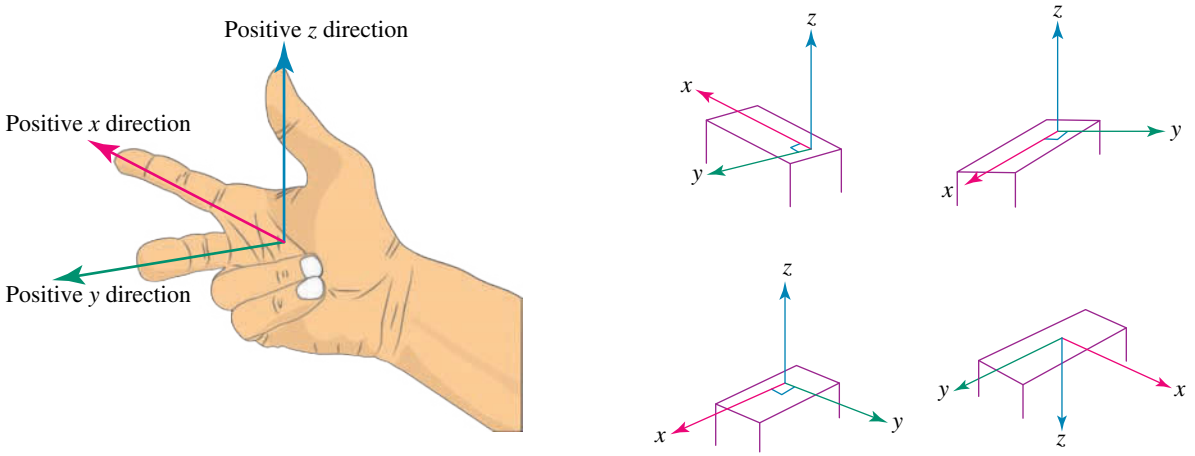


The space consists of 8 regions called octants that are divided by three perpendicular planes. Points in the first octant have positive values for x, y and z. Diagrams are usually oriented so that this octant is clearly visible.

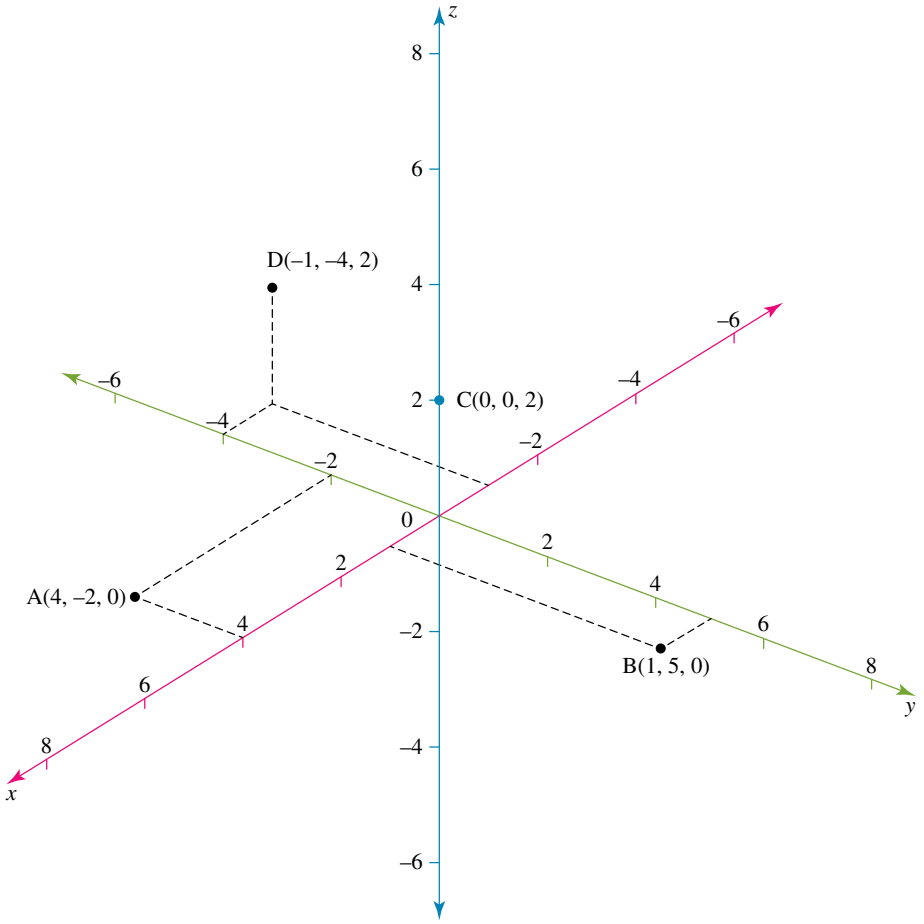


The orientation of the axes in the diagram indicates that it is a **right-handed coordinate system**. Systems can be either right- or left-handed in orientation, however the right-handed is the preferred option.

Whichever way the axes are orientated, their position relative to each other stays the same. The centre of the right hand represents the origin, If the index finger points along the direction of the positive x-axis, and the middle finger points along the positive direction of the y-axis, the thumb will always point along the positive direction of the z-axis. The following arrangements are all examples of right-handed systems.



In three dimensional space the coordinates (x, y, z) define a point $P(x, y, z)$, with each point made up of three components. Points $A(4, -2, 0)$, $B(1, 5, 0)$, $C(0, 0, 2)$ and $D(-1, -4, 2)$ are shown in the diagram.



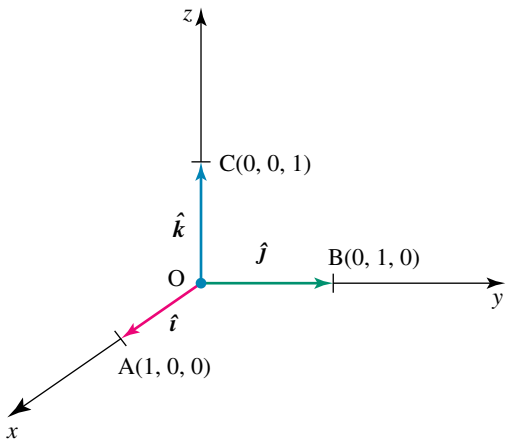
The location of a point relative to the origin can be expressed as the position vector $\overrightarrow{OP}$, in terms of the unit vectors, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. These are known as the **standard unit vectors**:

Let $\hat{\boldsymbol{\imath}}$ be a unit vector in the positive direction, parallel to the x-axis, with magnitude 1.

Let $\hat{\boldsymbol{\jmath}}$ be a unit vector in the positive direction, parallel to the y-axis, with magnitude 1.

Let $\hat{\boldsymbol{k}}$ be a unit vector in the positive direction, parallel to the z-axis, with magnitude 1.

Points A, B and C, can be expressed in **component form** in terms of the standard unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$.



$$\begin{aligned} \hat{\boldsymbol{\imath}} &= (1, 0, 0) & \therefore \overrightarrow{OA} &= \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} &= (0, 1, 0) & \therefore \overrightarrow{OB} &= \hat{\boldsymbol{\jmath}} \\ \hat{k} &= (0, 0, 1) & \therefore \overrightarrow{OC} &= \hat{k} \end{aligned}$$

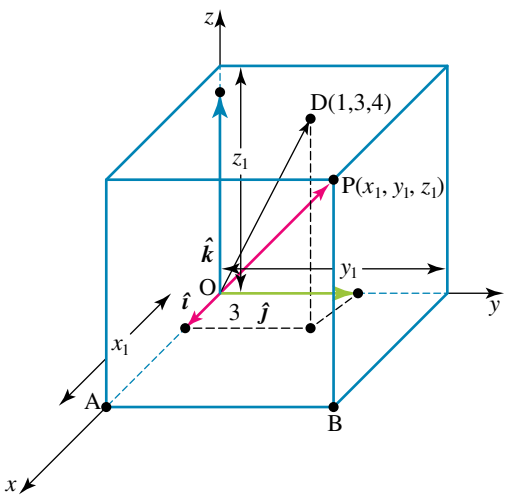
In three dimensions, a position vector beginning at the origin O(0, 0, 0) and terminating at the point P(x, y, z) is defined as,

$$\begin{aligned} \overrightarrow{OP} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \\ \underset{\sim}{u} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \end{aligned}$$

or

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

A vector expressed in terms of $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ is said to be in component form (or Cartesian form), and the process is called the **resolution of vectors**.



The position vector $\overrightarrow{OP}$ can be determined using a vector sum.

$$\begin{aligned}\therefore \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP} \\ &= x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}} + z_1\hat{\boldsymbol{k}}\end{aligned}$$

For example, if the point D has coordinates (1, 3, 4), and $\underset{\sim}{d} = \overrightarrow{OD}$, then the Cartesian form of the vector $\underset{\sim}{d}$ can be given by $\underset{\sim}{d} = \hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}}$.

Magnitude and unit vectors in three dimensions.

Pythagoras' theorem works in three dimensions, therefore, the magnitude of the three dimensional vector $\underset{\sim}{u} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ is defined by the formula,

$$|\underset{\sim}{u}| = \sqrt{x^2 + y^2 + z^2}$$

If $\underset{\sim}{u}$ is the vector, then the unit vector denoted by $\hat{\underset{\sim}{u}}$, is given by,

$$\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$$

WORKED EXAMPLE 1 Determining vector position, magnitude and unit vectors in three dimensions

If the point P has the coordinates P(−7, 3, −4), determine:

a. The vector $\overrightarrow{OP}$

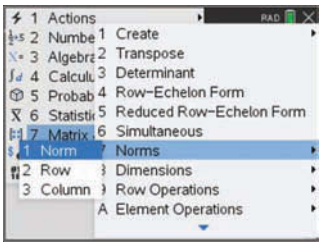
b. The magnitude of $\overrightarrow{OP}$

c. The unit vector parallel to $\overrightarrow{OP}$

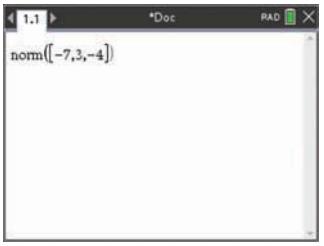
THINK	WRITE
a. The $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ components of the vector $\overrightarrow{OP}$ correspond to the x-, y- and z-coordinates respectively.	**a.** P(−7, 3, −4) has coordinates $x = -7, y = 3, z = -4$ $\overrightarrow{OP} = -7\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$
b. Recall that the magnitude of the vector $\underset{\sim}{u} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ is given by $\lvert\underset{\sim}{u}\rvert = \sqrt{x^2 + y^2 + z^2}$.	**b.** $\lvert\overrightarrow{OP}\rvert = \sqrt{(-7)^2 + (3)^2 + (-4)^2}$ $= \sqrt{74}$
c. 1. Let $\overrightarrow{OP} = \underset{\sim}{p}$, the unit vector parallel to $\overrightarrow{OP}$ is notated as $\hat{\underset{\sim}{p}}$. Recall the unit vector formula $\hat{\underset{\sim}{p}} = \frac{\underset{\sim}{p}}{\lvert\underset{\sim}{p}\rvert}$.	**c.** $\hat{\underset{\sim}{p}} = \frac{\underset{\sim}{p}}{\lvert\underset{\sim}{p}\rvert}$ $= \frac{-7\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}}{\sqrt{74}}$
2. The unit vector is often expressed as a scalar multiple of $\underset{\sim}{p}$ as shown here.	$= \frac{1}{\sqrt{74}}\left(-7\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}\right)$
3. Rationalise the denominator.	$\hat{\underset{\sim}{p}} = \frac{\sqrt{74}}{74}\left(7\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}\right)$

TI | THINK | **WRITE/DISPLAY**

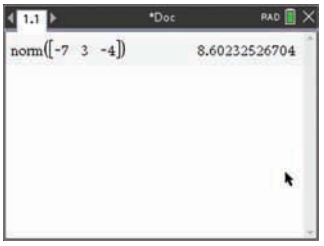
b. 1. On a Calculator page, select MENU 7: Matrix Vector 7: Norms 1: Norm



2. Complete the entry line ([−7, 3, −4])

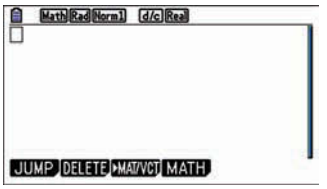


3. Press the ENTER button. The answer appears on the screen.



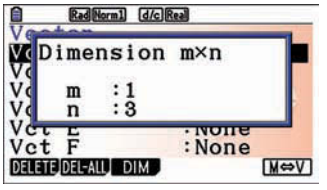
CASIO | THINK | **WRITE/DISPLAY**

b. 1. On a Run-Matrix screen, select MAT/VCT



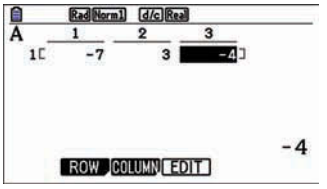
2. To switch between the vector and matrix screens, press the M ⇔ V button. Press the EXE button to define the Vct A.

3. Complete the entry line as
1
3
Press the EXE button.

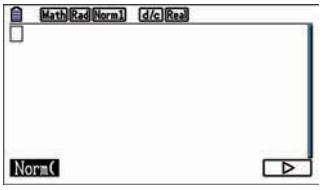


4. Complete the entry line as
–7 3 –4

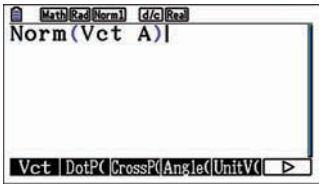
Press the EXIT button.



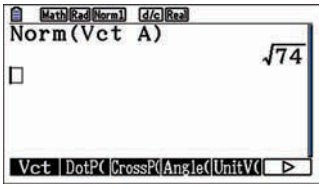
5. Press the EXIT button to return to the Run-Matrix screen. Select OPTN MAT/VCT ▷ (x3 times) The screen should appear as shown.



6. Select, Norm (▷(x3 times) Vct ALPHA X, θ, T)



7. Press the EXE button to complete the calculation. The answer appears on the screen.

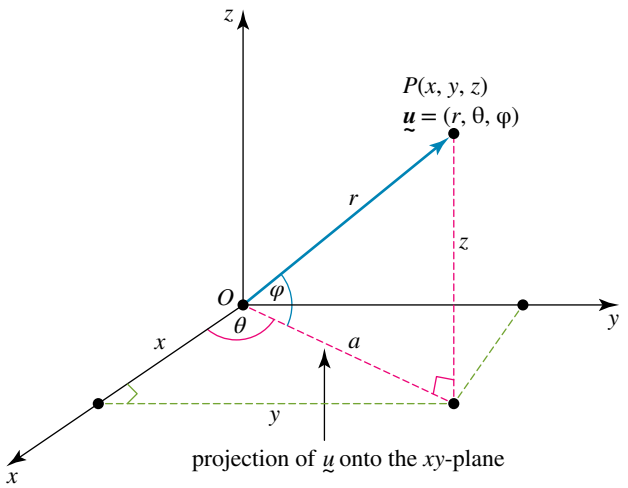


Note that the TI calculator can only display the magnitude in decimal form, so square the given value to determine the approximative surd form.

3.2.3 Three dimensional vectors in polar form

In two dimensions, a vector in polar form was defined by $\underset{\sim}{u} = (r,\ \theta)$, where r is the magnitude of $\underset{\sim}{u}$ and θ is the anti-clockwise angle formed from the positive x-axis.

In three dimensions, the point P(x, y, z) can be defined by the position vector, $\overrightarrow{\text{OP}} = \underset{\sim}{u}$.

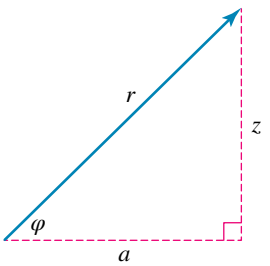


The vector, $\underset{\sim}{u}$, can then be represented in polar form $\underset{\sim}{u} = (r, \theta, \varphi)$ where,

- r is the magnitude of $\underset{\sim}{u}$, where $r \geq 0$ and is given by $r = \sqrt{x^2 + y^2 + z^2}$.
- θ (**azimuth angle**) is the anti-clockwise angle formed from the positive x-axis to the projection of $\underset{\sim}{u}$ onto the xy-plane, where $0° \leq \theta \leq 360°$. $\boldsymbol{\theta}$ is pronounced 'theta'.
- φ (**altitude angle**) is the angle between the line segment $\overrightarrow{\text{OP}}$ and the xy-plane, where $-90° \leq \varphi \leq 90°$. φ is pronounced 'phi'.
  The relationship between Cartesian (x, y, z) and polar (r, θ, φ) coordinates can be examined by referring to the diagram to the right. The findings will be useful to convert between one form to the other.
  Two trigonometric ratios can be formed by considering the triangle defining the altitude angle, φ.

$$\sin(\varphi) = \frac{z}{r} \qquad \therefore z = r\sin(\varphi) \quad [1]$$

$$\cos(\varphi) = \frac{a}{r} \qquad \therefore a = r\cos(\varphi) \quad [2]$$

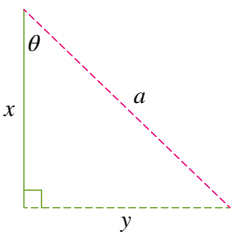


The second triangle forms a further three trigonometric ratios in terms of θ, a, x and y:

$$\sin(\theta) = \frac{y}{a} \qquad \therefore y = a\sin(\theta) \quad [3]$$

$$\cos(\theta) = \frac{x}{a} \qquad \therefore x = a\cos(\theta) \quad [4]$$

$$\tan(\theta) = \frac{y}{x} \quad [5]$$



x and y can be expressed in terms r and the two angles, through the process of substitution.

Substituting [2] into [4] and substituting [2] into [3] gives

$x = r\ \cos(\varphi)\ \cos(\theta)$ $\qquad$ $y = r\ \cos(\varphi)\ \sin(\theta)$

These formulae are useful when working with vectors in three dimensions. A summary of the necessary rules to remember are detailed below.

Vectors in three dimensions

When considering the three dimensional vector, $\underset{\sim}{u} = (r, \theta, \varphi) = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \begin{aligned} x &= r\cos(\varphi)\cos(\theta) \\ y &= r\cos(\varphi)\sin(\theta) \\ z &= r\sin(\varphi) \end{aligned} \qquad \begin{aligned} \sin(\varphi) &= \frac{z}{r} \\ \tan(\theta) &= \frac{y}{x} \end{aligned}$$

Vectors expressed in the polar form, $\underset{\sim}{u} = (r, \theta, \varphi)$, are often related to the spherical coordinate system which may be used to identify points on the Earth's surface.

WORKED EXAMPLE 2 Using the polar form of a vector in three dimensions

a. Express $\underset{\sim}{u} = (10°, 30°, 45°)$ in component form, $(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$. Give your answer as an exact value.
b. Determine the altitude angle, to 3 decimal places, between the vector, $\underset{\sim}{v} = -12\hat{\imath} + 9\hat{\jmath} - 8\hat{k}$ and the xy-plane.
c. Convert $\underset{\sim}{v} = -12\hat{\imath} + 9\hat{\jmath} - 8\hat{k}$ to polar form, giving your answer to 3 decimal places.

THINK	WRITE
a. 1. State the magnitude, r, and the two angles (θ, φ).	a. $\underset{\sim}{u} = (10, 30°, 45°)$ $r = 10, \theta = 30°, \varphi = 45°$
2. Recall the formula $x = r\cos(\varphi)\cos(\theta)$ to resolve the x-component. Substitute values (r, φ and θ) and evaluate.	$x = r\cos(\varphi)\cos(\theta)$ $= 10\cos(45°)\cos(30°)$ $= 10\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$ $= \frac{5\sqrt{6}}{2}$
3. Recall the formula $y = r\cos(\varphi)\sin(\theta)$ to resolve the y-component. Substitute values (r, φ and θ) and evaluate.	$y = r\cos(\varphi)\sin(\theta)$ $= 10\cos(45°)\sin(30°)$ $= 10\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$ $= \frac{5\sqrt{2}}{2}$
4. Recall the formula $z = r\sin(\varphi)$ to resolve the z-component. Substitute values (r and φ) and evaluate.	$z = r\sin(\varphi)$ $= 10\sin(45°)$ $= 10\left(\frac{\sqrt{2}}{2}\right)$ $= 5\sqrt{2}$

5. Conclude by representing the vector in component form.

$$\underset{\sim}{u} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
$$= \frac{5\sqrt{6}}{2}\hat{\imath} + \frac{5\sqrt{2}}{2}\hat{\jmath} + 5\sqrt{2}\hat{k}$$

b. 1. State the coefficient (x, y, z) of each component of the vector.

b. $\underset{\sim}{v} = -12\hat{\imath} + 9\hat{\jmath} - 8\hat{k} \therefore x = -12, y = 9, z = -8$

2. Calculate the magnitude of $\underset{\sim}{v}$. In polar form, $|\underset{\sim}{v}| = r$.

$$r = \sqrt{(-12)^2 + (9)^2 + (-8)^2}$$
$$= 17$$

3. Calculate the altitude angle by recalling the formula $\sin(\varphi) = \frac{z}{r}$. Substitute values for $\underset{\sim}{r}$ and φ. Solve for φ remembering $-90° \le \varphi \le 90°$.

$$\sin(\varphi) = \frac{z}{r}$$
$$\sin(\varphi) = \frac{-8}{17}$$
$$\varphi = \sin^{-1}\left(\frac{-8}{17}\right)$$
$$\varphi \approx -28.072°$$

The angle between $\underset{\sim}{v}$ and the xy-plane is $-28.072°$

c. 1. Recall the formula to calculate the azimuth angle, $\tan(\theta) = \frac{y}{x}$.
Substitute values for x and y.

c. $\tan(\theta) = \frac{y}{x} = \frac{9}{-12}$

$$\theta = \tan^{-1}\left(\frac{9}{-12}\right)$$
$$\theta \approx -36.870°$$

2. Determine in which quadrant θ lies. As the x-value is negative and the y-value is positive, the angle lies in the second quadrant. Calculate the second quadrant solution by subtracting the magnitude of the angle from 180°.

The angle is in the second quadrant as $x < 0$ and $y > 0$
$\theta = 180° - 36.870°$
$\theta = 143.130°$

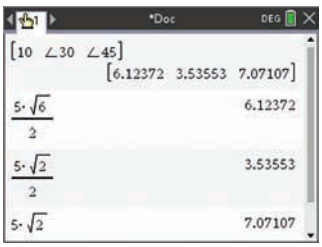
3. State the polar equivalent for $\underset{\sim}{v}$.

$\therefore \underset{\sim}{v} = -12\hat{\imath} + 9\hat{\jmath} - 8\hat{k}$ in polar form is
$\underset{\sim}{v} = (17, 143.130°, -28.072°)$

TI | THINK

1. On a Calculator page, with the document settings in degrees instead of radians for the angles, complete the line [10, ∠30, ∠45] and press the ENTER button to obtain the Cartesian form of vector (10, 30°, 45°) and verify your results. (Use [Ctrl] [catalog] to access the ∠ symbol). This provides

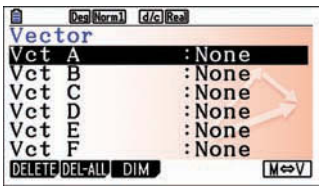
WRITE/DISPLAY



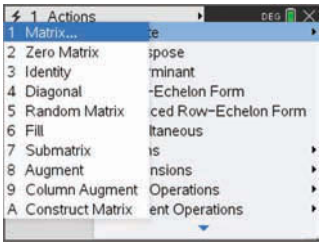
CASIO | THINK

1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.

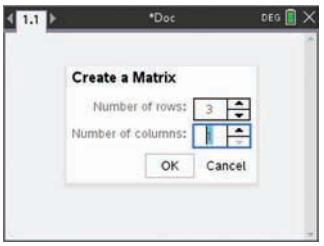
WRITE/DISPLAY



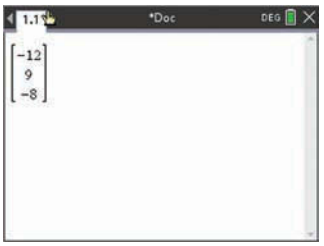
2. On a Calculator page, select MENU 7: Matrix & Vector 1: Create 1: Matrix Press the ENTER button.



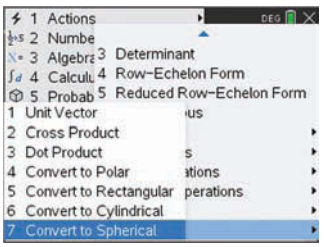
3. Complete the entry line as 3
1 Press the OK button.



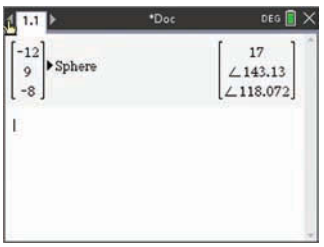
4. Complete the entry line as −12 9 − 8.



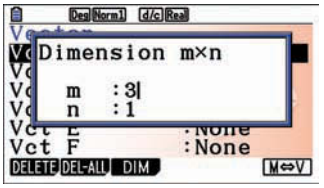
5. Select MENU 7: Matrix & Vector, C: Vector, 7 Convert to Spherical.



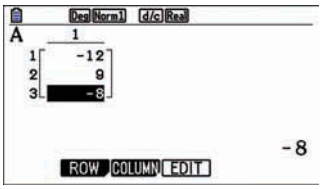
6. Press the ENTER button. The calculator displays the magnitude, the azimuth angle and the altitude angle. Note that the altitude angle φ is such that $-90° \leq \varphi \leq 90°$, always consider the quadrants in which these angles are when using the results obtained this way.



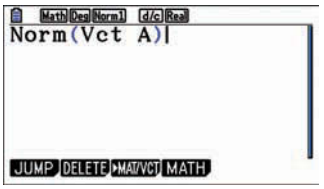
2. Press the EXE button. Complete the entry line as 1
3 Press the EXE button.



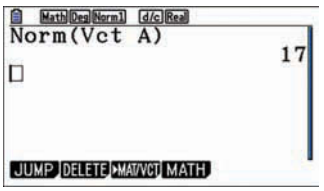
3. Complete the entry line as −12 9 − 8. Press the EXIT button when finished.



4. Press OPTN, MAT/VCT, ▷, ▷, ▷, Norm(, and complete the entry line as Norm(Vct A).



5. Press the EXE button.



6. Note that with a Casio fx-CG50 for instance, the Cartesian/Polar conversion is available for vectors in the plane only (two dimensions).

Exercise 3.2 Vectors in three dimensions

learnon

3.2 Exercise	3.2 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16, 17	18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

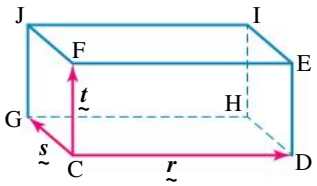
Simple familiar

1. State if the following are either vector or scalar quantities.

a. Velocity
b. Volume
c. Force applied in a downward direction
d. Time
e. Speed

2. A rectangular prism (box) CDEFGHIJ can be defined by three vectors, $\underset{\sim}{r}$, $\underset{\sim}{s}$ and $\underset{\sim}{t}$, as shown.

Express in terms of $\underset{\sim}{r}$, $\underset{\sim}{s}$ and $\underset{\sim}{t}$



a. the vector joining C to E
b. the vector joining C to I
c. the vector joining D to J.

3. State the coordinates of a point whose position vector is defined as:

a. $\overrightarrow{OA} = 5\hat{\imath} - 4\hat{\jmath}$

b. $\overrightarrow{OB} = -2\hat{\imath} + 6\hat{k}$

c. $\overrightarrow{OC} = \frac{5}{2}\hat{\imath} + 11\hat{\jmath} - 12\hat{k}$

d. $\overrightarrow{OD} = -4\hat{\imath} - \hat{\jmath} + 8\hat{k}$

4. Determine the position vector of each of the following points relative to the origin, O:

a. A(5, 3, 2)

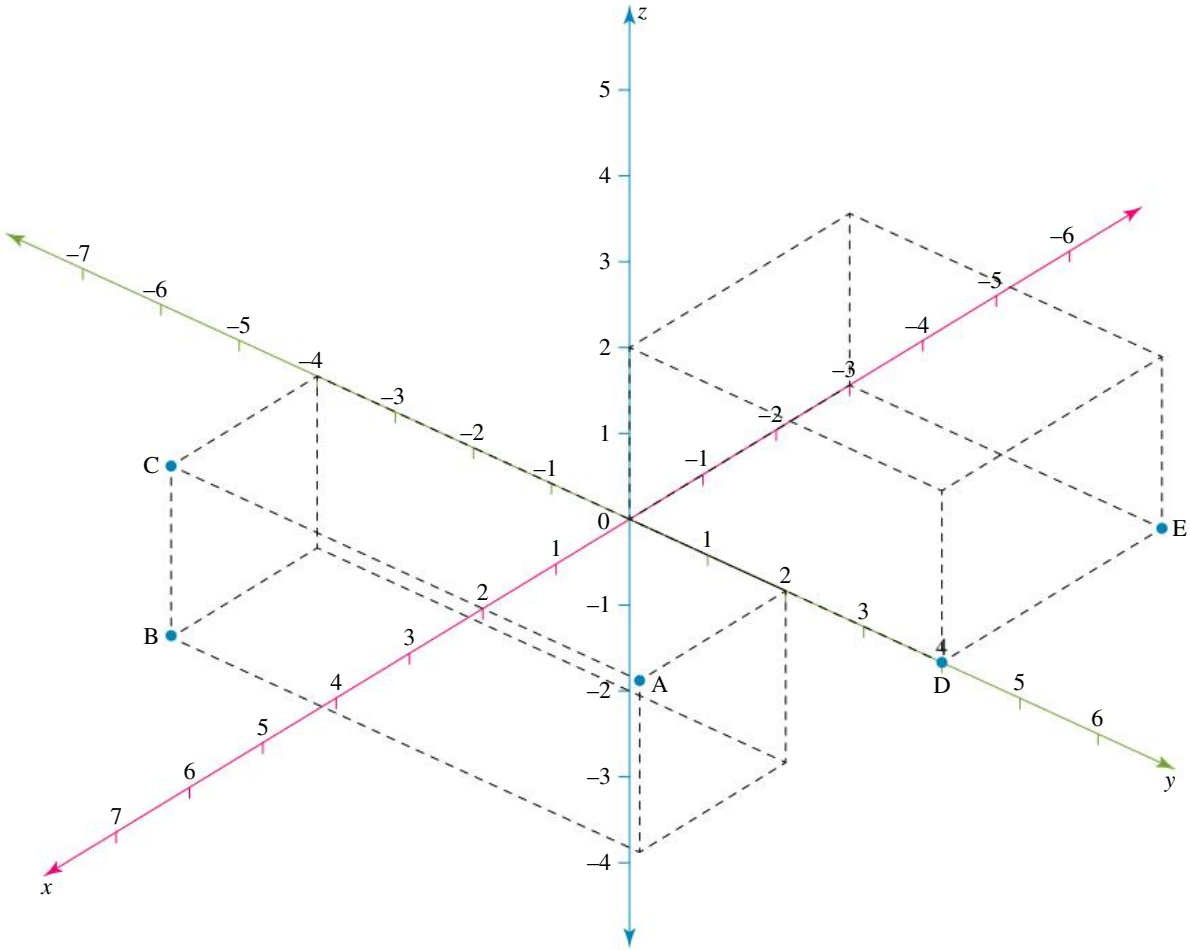
b. B(2, 0, −6)

c. $C\left(-\frac{2}{5}, \frac{2}{3}, -\frac{22}{17}\right)$

d. D(−11, 1, −4)

5. Refer to the three dimensional space below to state the coordinates of the following points.

a. Point A b. Point B c. Point C d. Point D e. Point E



6. An aeroplane travels 600 km east, then 900 km south. Calculate how far the aeroplane is from its starting point and determine the true bearing of the resultant displacement.

7. **WE1** If the point P has the coordinate (8, 4, 1), determine:

a. the vector $\overrightarrow{OP}$
b. the magnitude of $\overrightarrow{OP}$
c. the unit vector parallel to $\overrightarrow{OP}$

8. If the point P has coordinates (6, 2, −8), determine:

a. the vector $\overrightarrow{OP} = \underset{\sim}{p}$
b. the magnitude of the vector, $|\underset{\sim}{p}|$
c. the unit vector parallel to $\underset{\sim}{p}$
d. a vector 7 units long in the direction of $\underset{\sim}{p}$.

Evaluate the reasonableness of your result.

9. Three points, A, B and C have the coordinates (3, 4, 2), (8, 5, 10) and (9, 7, 11) respectively. Calculate:

a. the vector $\overrightarrow{AB}$
b. the vector $\overrightarrow{BC}$
c. the vector sum $\underset{\sim}{u} = 3\,\overrightarrow{BC} - 4\,\overrightarrow{AB}$.

10. **MC** Determine which of the following vectors has a norm equal to twice to the norm of vector $\underset{\sim}{u} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

A. $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$

B. $\underset{\sim}{b} = 8\hat{\boldsymbol{\imath}} + 12\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}$

C. $\underset{\sim}{c} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$

D. $\underset{\sim}{d} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} = -3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$

11. **MC** Select which of the following is the polar form of $\underset{\sim}{v} = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix}$.

A. $\underset{\sim}{v} = (7.07, 30.96°, 124.45°)$

B. $\underset{\sim}{v} = (7.07, 149.04°, 34.45°)$

C. $\underset{\sim}{v} = (7.07, -30.96°, -55.55°)$

D. $\underset{\sim}{v} = (7.07, -59.04°, 55.55°)$

12. **MC** Select which of the following is the Cartesian form of $\underset{\sim}{r} = \left(4, -\frac{2\pi}{3}, \frac{\pi}{6}\right)$.

A. $\underset{\sim}{r} = \begin{pmatrix} 3 \\ \sqrt{3} \\ -2 \end{pmatrix}$

B. $\underset{\sim}{r} = -\sqrt{3}\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$

C. $\underset{\sim}{r} = \begin{pmatrix} \sqrt{3} \\ 3 \\ -2 \end{pmatrix}$

D. $\underset{\sim}{r} = -\sqrt{3}\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$

Complex familiar

13. **WE2a** For each of the following, express $\underset{\sim}{u} = (r, \theta, \varphi)$ in component form, $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$. Give your answers to 2 decimal places.

a. $\underset{\sim}{u} = (6, 61°, 32°)$

b. $\underset{\sim}{u} = (20, 153°, -64°)$

14. Express the following vectors in component form. Give your answers to 2 decimal places.

a. $\underset{\sim}{u} = (3, 120°, -40°)$

b. $\underset{\sim}{u} = (15, 325°, 36°)$

15. **a.** **WE2b** Determine the altitude angle, to 2 decimal places, between the vector $\underset{\sim}{u} = 2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$ and the xy-plane.

b. Convert $\underset{\sim}{u} = 2\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$ to polar form, giving your answer to 2 decimal places.

16. **WE2c** Convert $\underset{\sim}{u} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ to polar form in each case below. Give your answers to 2 decimal places.

a. $\underset{\sim}{u} = 3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$

b. $\underset{\sim}{u} = 5\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}$

c. $\underset{\sim}{u} = -4\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} - 10\hat{\boldsymbol{k}}$

17. An injured mountain climber is located at a position relative to a camp given by the vector $\underset{\sim}{b} = 150\hat{\boldsymbol{\imath}} + 200\hat{\boldsymbol{\jmath}} + 100\hat{\boldsymbol{k}}$.
A rescue helicopter heads off from the camp in a direction parallel to the vector $\underset{\sim}{a} = 200\hat{\boldsymbol{\imath}} + 250\hat{\boldsymbol{\jmath}} + 125\hat{\boldsymbol{k}}$.
All measurements are in metres.

a. Determine how far, in metres correct to 2 decimal places, the rescue helicopter is from the camp when closest to the mountain climber.

b. Calculate the minimum distance, in metres correct to 2 decimal places, between the helicopter and the mountain climber.

Complex unfamiliar

18. The vector $\underset{\sim}{u} = \begin{pmatrix} k \\ k+1 \\ -k \end{pmatrix}$, where $k \in \mathbb{R}$, has magnitude of $\sqrt{66}$ and an azimuth angle $\theta = 38.66°$. Determine the altitude angle φ of vector $\underset{\sim}{u}$. Give your answer to 2 decimal places.

19. The vector $\underset{\sim}{u} = \begin{pmatrix} k+1 \\ 2k \\ k-1 \end{pmatrix}$, where $k \in \mathbb{R}$, has an altitude angle $\varphi = -30°$ and an azimuth angle θ. Determine the azimuth angle θ of vector $\underset{\sim}{u}$ to 2 decimal places.

20. A drone is launched from sea level, on the seashore near Brisbane, and travels 1.5 km in a north-west direction and then 2.5 km in a north-east direction, while increasing its altitude until it is h km above sea level, when the drone takes a photo.
Determine altitude h of the drone, given that the altitude angle of its vector position $\underset{\sim}{d}$ is $\varphi = 6.981°$ when the photo is taken. Give your answer to 3 decimal places.

Fully worked solutions for this chapter are available online.

LESSON
3.3 Algebra of three dimensional vectors

SYLLABUS LINKS

- Examine and use addition and subtraction of vectors in Cartesian form.
- Use multiplication by a scalar of a vector in Cartesian form.
- Determine a vector between two points.
- Use the scalar (dot) product.
  - $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos(\theta)$
  - $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Use scalar and vector projections of vectors.
  - scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$: $|\boldsymbol{a}| \cos(\theta) = \boldsymbol{a} \cdot \hat{\boldsymbol{b}}$
  - vector projection of $\boldsymbol{a}$ on $\boldsymbol{b}$: $|\boldsymbol{a}| \cos(\theta) \hat{\boldsymbol{b}} = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\, \hat{\boldsymbol{b}} = \left(\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}} \right) \boldsymbol{b}$
- Apply the scalar product to vectors expressed in Cartesian form.

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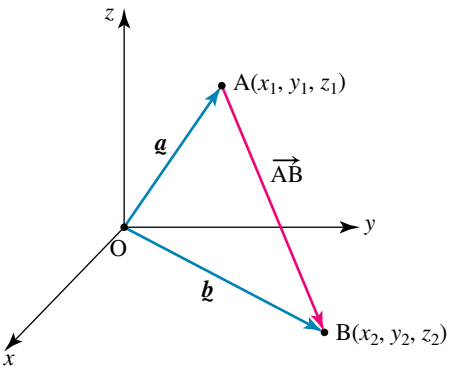
3.3.1 Operations on vectors

The vector between two points in three dimensions

In the diagram opposite, $\underset{\sim}{a} = \overrightarrow{OA}$ is the position vector of point A and $\underset{\sim}{b} = \overrightarrow{OB}$ is the position vector of point B. The vector describing the location of A relative to B denoted by $\overrightarrow{BA}$, is found using vector addition as $-\underset{\sim}{b} + \underset{\sim}{a}$ or $\underset{\sim}{a} - \underset{\sim}{b}$.



The vector describing the location of B relative to A denoted by $\overrightarrow{AB}$, shown, is $\underset{\sim}{b} - \underset{\sim}{a}$.

If A and B are points defined by the position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, then

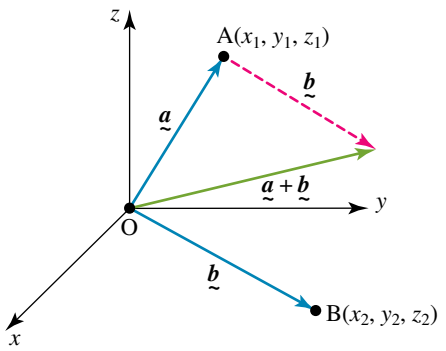
$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \underset{\sim}{b} - \underset{\sim}{a} \end{aligned}$$

Addition and subtraction of three dimensional vectors in component form

Given, $\underset{\sim}{a} = (x_1, y_1, z_1)$ and $\underset{\sim}{b} = (x_2, y_2, z_2)$

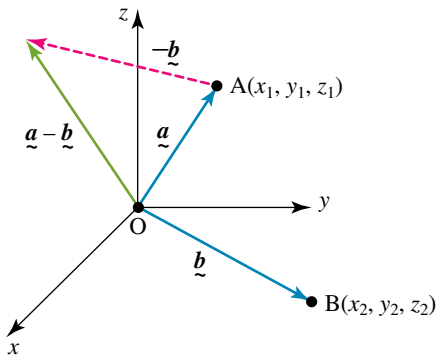
Addition of vectors in component form is:

$$\begin{aligned} \underset{\sim}{a} + \underset{\sim}{b} &= \left(x_1 \hat{\boldsymbol{\imath}} + y_1 \hat{\boldsymbol{\jmath}} + z_1 \hat{\boldsymbol{k}}\right) + \left(x_2 \hat{\boldsymbol{\imath}} + y_2 \hat{\boldsymbol{\jmath}} + z_2 \hat{\boldsymbol{k}}\right) \\ &= (x_1 + x_2)\hat{\boldsymbol{\imath}} + (y_1 + y_2)\hat{\boldsymbol{\jmath}} + (z_1 + z_2)\hat{\boldsymbol{k}} \end{aligned}$$



Subtraction of vectors in component form is:

$$\underset{\sim}{a} - \underset{\sim}{b} = \left(x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}\right) - \left(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}\right)$$
$$= (x_1 - x_2)\hat{\imath} + (y_1 - y_2)\hat{\jmath} + (z_1 - z_2)\hat{k}$$



WORKED EXAMPLE 3 Adding and subtracting vectors

Three points, A, B and C have the coordinates (−3, 7, −1), (2, −2, 11) and (8, −7, 12) respectively. Calculate:

a. The vector $\overrightarrow{AB}$

b. The vector $\overrightarrow{BC}$

c. The vector sum $\underset{\sim}{u} = 3\,\overrightarrow{BC} - 4\,\overrightarrow{AB}$

THINK	WRITE
a. 1. Define the vector $\overrightarrow{OA}$.	a. $\overrightarrow{OA} = -3\hat{\imath} + 7\hat{\jmath} - \hat{k}$
2. Define the vector $\overrightarrow{OB}$.	$\overrightarrow{OB} = 2\hat{\imath} - 2\hat{\jmath} + 11\hat{k}$
3. $\overrightarrow{AB}$ is determined by subtracting $\overrightarrow{OA}$ from $\overrightarrow{OB}$.	$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $= \left(2\hat{\imath} - 2\hat{\jmath} + 11\hat{k}\right) - \left(-3\hat{\imath} + 7\hat{\jmath} - \hat{k}\right)$ $= (2+3)\hat{\imath} + (-2-7)\hat{\jmath} + (11+1)\hat{k}$ $= 5\hat{\imath} - 9\hat{\jmath} + 12\hat{k}$
b. 1. Define the vector $\overrightarrow{OC}$.	b. $\overrightarrow{OC} = 8\hat{\imath} - 7\hat{\jmath} + 12\hat{k}$
2. $\overrightarrow{BC}$ is determined by subtracting $\overrightarrow{OB}$ from $\overrightarrow{OC}$.	$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$ $= \left(8\hat{\imath} - 7\hat{\jmath} + 12\hat{k}\right) - \left(2\hat{\imath} - 2\hat{\jmath} + 11\hat{k}\right)$ $= (8-2)\hat{\imath} + (-7+2)\hat{\jmath} + (12-11)\hat{k}$ $= 6\hat{\imath} - 5\hat{\jmath} + \hat{k}$
c. 1. The vectors $\overrightarrow{BC}$ and $\overrightarrow{AB}$ have been defined. Substitute these into the vector sum $\underset{\sim}{u} = 3\overrightarrow{BC} - 4\overrightarrow{AB}$.	c. $\underset{\sim}{u} = 3\overrightarrow{BC} - 4\overrightarrow{AB}$ $= 3\left(6\hat{\imath} - 5\hat{\jmath} + \hat{k}\right) - 4\left(5\hat{\imath} - 9\hat{\jmath} + 12\hat{k}\right)$
2. Simplify the vector sum by multiplying the respective vectors by the scalar values. Define $\underset{\sim}{u}$ by applying the rules for adding and subtracting vectors as components.	$= 18\hat{\imath} - 15\hat{\jmath} + 3\hat{k} - 20\hat{\imath} + 36\hat{\jmath} - 48\hat{k}$ $= -2\hat{\imath} + 21\hat{\jmath} - 45\hat{k}$

Scalar multiples and parallel vectors

When a vector is multiplied by a scalar, λ, each coefficient is multiplied by the scalar.

$$\lambda\underset{\sim}{a} = \lambda\left(x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}\right)$$
$$= \lambda x_1\hat{\imath} + \lambda y_1\hat{\jmath} + \lambda z_1\hat{k}$$

Two vectors can be defined as parallel if one is a scalar multiple of the other. That is, $\underset{\sim}{b}$ is parallel to $\underset{\sim}{a}$ if $\underset{\sim}{b} = \lambda\underset{\sim}{a}$.

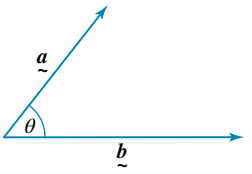
Equality of vectors.

Consider two vectors, $\underset{\sim}{a} = x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}} + z_1\hat{\boldsymbol{k}}$ and $\underset{\sim}{b} = x_2\hat{\boldsymbol{\imath}} + y_2\hat{\boldsymbol{\jmath}} + z_2\hat{\boldsymbol{k}}$. The vectors are equal if and only if, $x_1 = x_2, y_1 = y_2$ and $z_1 = z_2$.

3.3.2 The scalar (dot) product

Review of the dot product

When a vector is added or subtracted to another vector, or multiplied by a scalar quantity, the result is a vector. However, when vectors are multiplied this can either be in the form of the **dot (scalar) product** or **vector (cross) product**. The dot product results in a scalar quantity, whereas the vector product results in a vector.



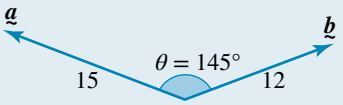
The scalar product, or dot product is also known as the **inner product** and can be calculated using two different methods. The first method is used if the magnitude of two vectors and the angle, θ, between them are known.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos(\theta)$$

where θ is the angle between the tail of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

WORKED EXAMPLE 4 Calculating the scalar (dot) product using polar form

Examine the diagram and determine $\underset{\sim}{a} \cdot \underset{\sim}{b}$.



THINK	WRITE
1. The magnitude of the two vectors and the angle between is given.	$\lvert\underset{\sim}{a}\rvert = 15, \lvert\underset{\sim}{b}\rvert = 12, \theta = 145°$
2. Recall the dot product formula and substitute in relevant values.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert\,\lvert\underset{\sim}{b}\rvert\cos(\theta)$ $= 15 \times 12 \times \cos(145°)$ $= -147.45$

The scalar (dot) product of vectors expressed in component form

The second method of calculating the dot product is used with the vectors in component form. Consider the standard unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. The magnitude of each vector is 1 and can be expressed as $|\hat{\boldsymbol{\imath}}| = 1, |\hat{\boldsymbol{\jmath}}| = 1$ and $|\hat{\boldsymbol{k}}| = 1$. Begin by examining $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}$ in detail.

- The angle between $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\imath}}$ is 0°.
- $\cos(0°) = 1$

By definition $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}$ is

$$\begin{aligned}\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} &= |\hat{\boldsymbol{\imath}}|\,|\hat{\boldsymbol{\imath}}|\cos(0°)\\ &= |\hat{\boldsymbol{\imath}}|^2 \quad (\text{where } |\hat{\boldsymbol{\imath}}| = 1)\\ &= 1\end{aligned}$$

Extrapolating from $\hat{\boldsymbol{\imath}}$ to $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$:

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = 1, \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = 1 \text{ and } \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$$

The unit vectors are mutually perpendicular to each other, that is the angle between $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are all 90°. Given that $\cos(90°) = 0$ and the dot product is commutative, it follows that $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}} = 0$, $\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\jmath}} = 0$ and $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}} = 0$.

This allows us to calculate the dot product for any two vectors, $\underset{\sim}{a} = x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}} + z_1\hat{\boldsymbol{k}}$ and $\underset{\sim}{b} = x_2\hat{\boldsymbol{\imath}} + y_2\hat{\boldsymbol{\jmath}} + z_2\hat{\boldsymbol{k}}$, where x, y and z are constants.

$$\begin{aligned}\underset{\sim}{a} \cdot \underset{\sim}{b} &= \left(x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}} + z_1\hat{\boldsymbol{k}}\right) \cdot \left(x_2\hat{\boldsymbol{\imath}} + y_2\hat{\boldsymbol{\jmath}} + z_2\hat{\boldsymbol{k}}\right) \\ &= x_1x_2\left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}\right) + y_1y_2\left(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}\right) + z_1z_2\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}\right) \\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

Alternatively, using column vector notation

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

We only need to multiply the corresponding x-, y- and z-components of two vectors to calculate their dot product.

WORKED EXAMPLE 5 Calculating the scalar (dot) product using component form

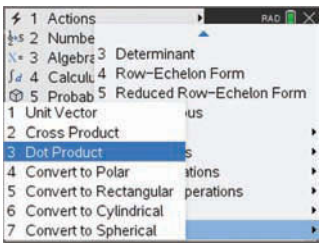
Given the vectors, $\underset{\sim}{a} = 5\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$ and $\underset{\sim}{b} = -2\hat{\imath} + \hat{\jmath} - 3\hat{k}$, determine $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

THINK	WRITE
1. Substitute for the given vectors.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left(5\hat{\imath} + 2\hat{\jmath} - 8\hat{k}\right) \cdot \left(-2\hat{\imath} + \hat{\jmath} - 3\hat{k}\right)$
2. Add the products of the corresponding x-, y-, z-components.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (5 \times -2) + (2 \times 1) + (-8 \times -3)$ $= -10 + 2 + 24$ $= 16$
3. State the final result.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = 16$

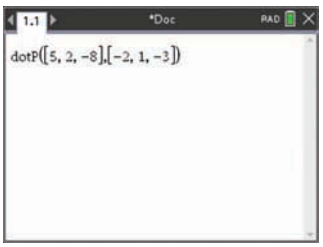
TI | THINK

1. On a Calculator page, select MENU
   7: Matrix Vector
   C: Vector
   3: Dot Product
   Press the ENTER button.

WRITE/DISPLAY



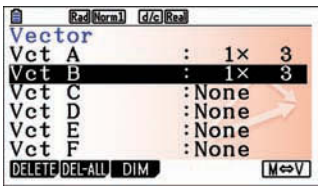
2. Complete the entry line $\left(\left[5, 2, -8\right], \left[-2, 1, -3\right]\right)$



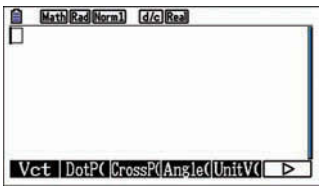
CASIO | THINK

1. On a Run-Matrix screen, select MAT/VCT
   Define Vectors A and B as shown in Worked example 1b.

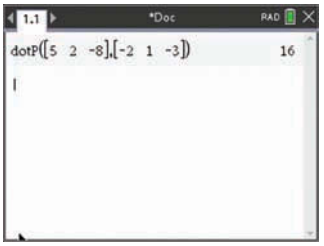
WRITE/DISPLAY



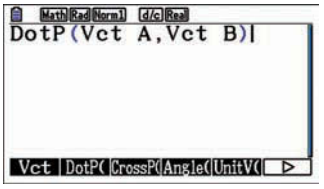
2. On a Run-Matrix screen, select OPTN MAT/VCT ▷ (x3 times) DotP(



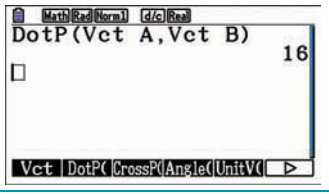
3. Press the ENTER button.
The answer appears on the screen.



3. Complete the entry line as
Vct ALPHA X, θ, T,
Vct ALPHA Log)



4. Press the EXE button to perform the calculation.
The answer appears on the screen.



WORKED EXAMPLE 6 Calculating the scalar (dot) product using column vector notation

Given $\underset{\sim}{a} = \begin{pmatrix} m \\ 2 \\ 1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 10$, determine m.

THINK	WRITE
1. Recall the dot product formula and substitute in the relevant values.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{pmatrix} m \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ $= m \times 3 + 2 \times (-2) + 1 \times 5$ $= 3m + 1$
2. Equate to 10, as $\underset{\sim}{a} \cdot \underset{\sim}{b} = 10$.	$3m + 1 = 10$
3. Solve for m.	$3m = 9$ $m = 3$
4. State the answer.	$m = 3$

Determining the angle between two vectors

Given two formulas for calculating the dot product:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos(\theta)$$
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2 + z_1z_2$$

they can be combined to give:

$$x_1x_2 + y_1y_2 + z_1z_2 = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos(\theta)$$

Rearranging this formula gives the angle between two vectors.

The angle between two vectors

$$\cos(\theta) = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} \text{ or } \cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$$

The scalar (dot) product of perpendicular vectors

It was shown that the dot product of perpendicular unit vectors is equal to zero.

$\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{\imath}=0,\ \ \hat{\jmath}\cdot\hat{k}=\hat{k}\cdot\hat{\jmath}=0$ and $\hat{\imath}\cdot\hat{k}=\hat{k}\cdot\hat{\imath}=0$

This is true for any two, non-zero vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$. If the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular (**orthogonal**) the angle between them is 90°, and the equation $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos(\theta)$ becomes,

$$\begin{aligned}\underset{\sim}{a}\cdot\underset{\sim}{b} &= |\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos(90°)\\ &= |\underset{\sim}{a}|\,|\underset{\sim}{b}|\times 0 \quad (\text{Recall } \cos(90°)=0)\\ &= 0\end{aligned}$$

If $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$, then the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

Properties of the scalar (dot) product

- **The dot product is a scalar. It is the result of multiplying scalar quantities.**
- **If the magnitude of two vectors and the angle, θ, between them are known it is calculated by: $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos(\theta)$. If the vectors are in component form, it is calculated by: $\underset{\sim}{a}\cdot\underset{\sim}{b}=x_1x_2+y_1y_2+z_1z_2$**
- **The order of multiplication is unimportant (*commutative* property): $\underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{a}$**
- **The dot product is *distributive*: $\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b})=\underset{\sim}{c}\cdot\underset{\sim}{a}+\underset{\sim}{c}\cdot\underset{\sim}{b}$**
- **Since the angle between the same vector and itself is 0°: $\underset{\sim}{a}\cdot\underset{\sim}{a}=|\underset{\sim}{a}|^2$**
- **If $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$, then the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.**

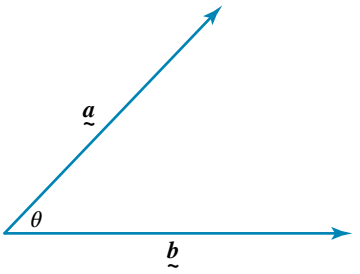
WORKED EXAMPLE 7 Using the properties of the scalar (dot) product

The vectors given by, $\underset{\sim}{a}=6\hat{\imath}+y\hat{\jmath}-\hat{k}$ and $\underset{\sim}{b}=-3\hat{\imath}+4\hat{\jmath}+2\hat{k}$, are perpendicular. Calculate the value of y.

THINK	WRITE
1. Write $\underset{\sim}{a}\cdot\underset{\sim}{b}$ as the dot product of the two vectors in component form.	$\underset{\sim}{a}\cdot\underset{\sim}{b}=\left(6\hat{\imath}+y\hat{\jmath}-\hat{k}\right)\cdot\left(-3\hat{\imath}+4\hat{\jmath}+2\hat{k}\right)$
2. Add the products of the corresponding x-, y-, z-components.	$\underset{\sim}{a}\cdot\underset{\sim}{b}=(6\times-3)+(y\times4)+(-1\times2)$ $=-18+4y-2$ $=4y-20$ [1]
3. State the dot product relationship of perpendicular vectors.	$\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$ $\therefore\ \underset{\sim}{a}\cdot\underset{\sim}{b}=0$ [2]
4. Substitute the initial solution for $\underset{\sim}{a}\cdot\underset{\sim}{b}$ [1] into [2] to solve for y.	$\underset{\sim}{a}\cdot\underset{\sim}{b}=0$ $4y-20=0$ $y=\dfrac{20}{4}$
5. The solution is stated.	$\therefore\ y=5$

3.3.3 Review of vector projections (vector resolutes)

We have resolved vectors parallel and perpendicular to the x- and y-axes. Now we will examine the more generalised process of resolving one vector parallel and perpendicular to another vector.

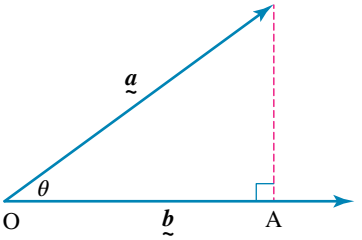


Consider the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, shown. The angle between them, as for a dot product, is given by θ. It can be shown that $\underset{\sim}{a}$ is made up of a **projection** acting in the direction of $\underset{\sim}{b}$ and another projection acting perpendicular to $\underset{\sim}{b}$.

The first step is to determine the projection in the direction of $\underset{\sim}{a}$.

The scalar projection (scalar resolute)

- The length OA is the **scalar projection** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$.
- It effectively indicates 'how much' of $\underset{\sim}{a}$ is in the direction of $\underset{\sim}{b}$. Using trigonometry $\text{OA} = |\underset{\sim}{b}|\cos(\theta)$, and the dot product $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos(\theta)$:



$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= |\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos(\theta) \\ \underset{\sim}{a} \cdot \underset{\sim}{b} &= |\underset{\sim}{b}|\,\text{OA} \\ \therefore \text{OA} &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}, \text{ where } \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} \\ &= \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} \end{aligned}$$

Scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$

The scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is given by $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$ or $|\underset{\sim}{a}|\cos(\theta)$.

The scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$ is not equal to the scalar resolute of $\underset{\sim}{a}$ on $\underset{\sim}{b}$.

Vector projection (vector resolutes)

Consider now the vector joining O to A. Its magnitude is just the scalar resolute $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$, while its direction is the same as $\underset{\sim}{b}$. This quantity is called the **vector resolute** of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$ and is denoted by the symbol $\underset{\sim}{a}_{\parallel}$.

Vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$

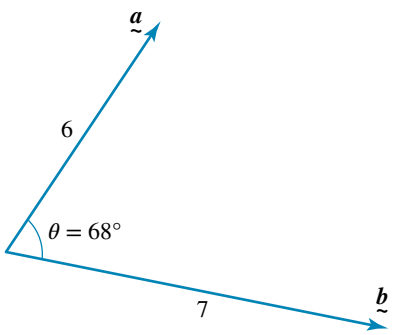
The vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is given by

$$\begin{aligned} \underset{\sim}{a}_{\parallel} &= \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} \\ &= \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b} \\ &= |\underset{\sim}{a}|\cos(\theta)\hat{\underset{\sim}{b}} \end{aligned}$$

Consider the **geometry** of the figure. The original vector $\underset{\sim}{a}$ is the sum of $\underset{\sim}{a}_{\perp}$ and $\underset{\sim}{a}_{\parallel}$. The vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ can be calculated simply as shown below.

$\underset{\sim}{a} = \underset{\sim}{a}_{\parallel} + \underset{\sim}{a}_{\perp}$ (by addition of vectors)

$\underset{\sim}{a}_{\perp} = \underset{\sim}{a} - \underset{\sim}{a}_{\parallel}$ (by rearranging the vector equation)



Vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$

The vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is given by

$$\underset{\sim}{a}_{\perp} = \underset{\sim}{a} - \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$$

WORKED EXAMPLE 8 Using vector projections

Let $\underset{\sim}{b} = 8\hat{\imath} - 6\hat{\jmath} + 5\hat{k}$ and $\underset{\sim}{a} = -6\hat{\imath} + 2\hat{\jmath} - \hat{k}$. Determine:

a. the scalar resolute of $\underset{\sim}{a}$ on $\underset{\sim}{b}$

b. the vector resolute of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ (on $\underset{\sim}{a}_{\parallel}$)

c. the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$, namely $\underset{\sim}{a}_{\perp}$

THINK	WRITE
a. 1. Calculate the magnitude of $\underset{\sim}{b}$.	a. $\lvert\underset{\sim}{b}\rvert = \sqrt{(8)^2 + (-6)^2 + (5)^2}$ $= \sqrt{125}$ $= 5\sqrt{5}$
2. Calculate the unit vector in the direction of $\underset{\sim}{b}$, $\hat{\underset{\sim}{b}}$, by dividing $\underset{\sim}{b}$ by $\lvert\underset{\sim}{b}\rvert$.	$\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $= \dfrac{(8\hat{\imath} - 6\hat{\jmath} + 5\hat{k})}{5\sqrt{5}}$
3. Determine the scalar resolute of $\underset{\sim}{a}$ on $\underset{\sim}{b}$, by substituting into the formula $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$.	$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = (-6\hat{\imath} + 2\hat{\jmath} - \hat{k}) \cdot \left(\dfrac{8\hat{\imath}}{5\sqrt{5}} - \dfrac{6\hat{\jmath}}{5\sqrt{5}} + \dfrac{5\hat{k}}{5\sqrt{5}}\right)$
4. Simplify $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$ using dot product laws for unit vectors.	$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \dfrac{-48}{5\sqrt{5}} + \dfrac{-12}{5\sqrt{5}} + \dfrac{-5}{5\sqrt{5}}$ $= -\dfrac{65}{5\sqrt{5}}$ $= -\dfrac{13}{\sqrt{5}}$

b. 1. Determine $\underset{\sim}{a}_{\parallel}$ by recalling the formula $\underset{\sim}{a}_{\parallel} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\,\hat{\underset{\sim}{b}}$

b. $\underset{\sim}{a}_{\parallel} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\,\hat{\underset{\sim}{b}}$

$$= \left(-\frac{13}{\sqrt{5}}\right)\left(\frac{8\hat{\imath}}{5\sqrt{5}} - \frac{6\hat{\jmath}}{5\sqrt{5}} + \frac{\hat{k}}{\sqrt{5}}\right)$$

2. Simplify.

$$= -\frac{104}{25}\hat{\imath} + \frac{78}{25}\hat{\jmath} - \frac{13}{5}\hat{k}$$

c. 1. Determine $\underset{\sim}{a}_{\perp}$ by recalling the formula $\underset{\sim}{a}_{\perp} = \underset{\sim}{a} - \underset{\sim}{a}_{\parallel}$

c. $\underset{\sim}{a}_{\perp} = \underset{\sim}{a} - \underset{\sim}{a}_{\parallel}$

2. Substitute in the vectors, $\underset{\sim}{a}$ and $\underset{\sim}{a}_{\parallel}$.

$$\underset{\sim}{a}_{\perp} = (-6\hat{\imath} + 2\hat{\jmath} - \hat{k}) - \left(-\frac{104}{25}\hat{\imath} + \frac{78}{25}\hat{\jmath} - \frac{13}{5}\hat{k}\right)$$

3. Simplify.

$$\underset{\sim}{a}_{\perp} = -\frac{46}{25}\hat{\imath} - \frac{28}{25}\hat{\jmath} + \frac{8}{5}\hat{k}$$

Exercise 3.3 Algebra of three dimensional vectors

learnon

3.3 Exercise | **3.3 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!
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- Access sample responses
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Simple familiar

1. Refer to the vectors, $\underset{\sim}{a} = 4\hat{\imath} - 2\hat{\jmath} + 7\hat{k}$ and $\underset{\sim}{b} = -3\hat{\imath} + 5\hat{\jmath} - \hat{k}$, to calculate:

a. $-\underset{\sim}{a}$
b. $2\underset{\sim}{b}$
c. $\underset{\sim}{a} + \underset{\sim}{b}$
d. $4\underset{\sim}{a} - 2\underset{\sim}{b}$

2. WE3 Three points A, B and C have coordinates $(1, -2, 4)$, $(3, -4, 5)$ and $(0, 4, -1)$ respectively. Calculate the following and give your answers using column vector notation:

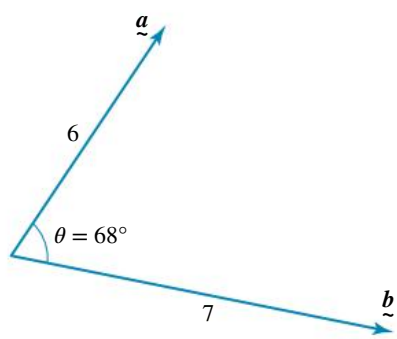
a. The vector $\overrightarrow{AB}$.
b. The vector $\overrightarrow{BC}$.
c. The vector sum $\underset{\sim}{u} = -3\overrightarrow{AB} + 2\overrightarrow{BC}$.

3. Three points A, B and C have coordinates $(4, 3, z)$, $(x, -5, 2)$ and $(5, y, -3)$ respectively. Determine x, y and z such that $2\overrightarrow{AB} - 3\overrightarrow{AC} = 4\overrightarrow{BC}$.

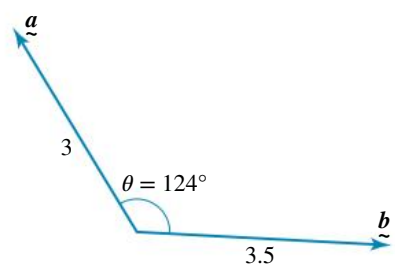
4. Three points A, B and C have coordinates $(-5, 2, -4)$, $(-1, 2, 5)$ and $(-2, 3, -6)$ respectively. Give a relationship between x, y and z such that $x\overrightarrow{AB} + y\overrightarrow{AC} = z\overrightarrow{BC}$.

5. **WE4** For each of the following, examine the diagram and determine $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

a.



b.



6. **WE5** Given the vectors $\underset{\sim}{a} = 2\hat{\imath} - 5\hat{\jmath} + \hat{k}$ and $\underset{\sim}{b} = -3\hat{\imath} + 7\hat{\jmath} - 4\hat{k}$, determine $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

7. **WE6** Given $\underset{\sim}{a} = \begin{pmatrix} 4 \\ m \\ -1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} -m \\ 3 \\ 1 \end{pmatrix}$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 5$, determine m.

8. **WE7** The vectors given by, $\underset{\sim}{a} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\underset{\sim}{b} = 4\hat{\imath} + y\hat{\jmath} + 3\hat{k}$, are perpendicular. Determine the value of y.

9. Let $\underset{\sim}{a} = \hat{\imath} - 2\hat{\jmath} + 8\hat{k}$, $\underset{\sim}{b} = 3\hat{\imath} + 6\hat{\jmath} - 9\hat{k}$ and $\underset{\sim}{c} = 5\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$. Demonstrate, using these vectors, the property:

$$\underset{\sim}{c} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{b}$$

10. Calculate the angle between the two vectors, $\underset{\sim}{a} = -5\hat{\imath} + 6\hat{\jmath} + 2\hat{k}$ and $\underset{\sim}{b} = 7\hat{\imath} + 11\hat{\jmath} - 2\hat{k}$.

11. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}$. Determine the angle θ between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Give your answer to 2 decimal places.

12. Given that the vectors $\underset{\sim}{a} = \begin{pmatrix} -5 \\ 3 \\ k \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} k \\ -8 \\ 3 \end{pmatrix}$ are perpendicular, calculate the value of k.

Complex familiar

13. **WE8** Let $\underset{\sim}{a} = 20\hat{\imath} - 18\hat{\jmath} + 8\hat{k}$ and $\underset{\sim}{b} = -5\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$. Determine:
    - **a.** the scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$
    - **b.** the vector resolute of $\underset{\sim}{b}$ parallel to $\underset{\sim}{a}$, namely $\underset{\sim}{b}_{\parallel}$
    - **c.** the vector resolute of $\underset{\sim}{b}$ perpendicular to $\underset{\sim}{a}$, namely $\underset{\sim}{b}_{\perp}$.

14. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 6 \\ 3 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} -4 \\ k \\ -1 \end{pmatrix}$. Determine k, to 2 decimal places, such that the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is twice the angle between the vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Show your working.

15. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} -6 \\ 4 \\ 2\sqrt{3} \end{pmatrix}$ and $\underset{\sim}{b}$.

Given that the scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is equal to 6, calculate, to 2 decimal places, the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

16. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -k \\ 4 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix}$.

Determine k, to 2 decimal places, such that the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ is the same as the angle between the vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Show your working.

Complex unfamiliar

17. Let $\underset{\sim}{a} = 6\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$. Determine a vector parallel to $\underset{\sim}{a}$ such that their dot product is -82.

18. Consider the points $A(3, k, -k)$, $B(6, 1, 5)$ and $C(6, k, -1)$ where $k \in \mathbb{R}$. Determine the values of k such that ΔABC has a right angle at point A and verify the reasonableness of your solution.

19. ABCD is a regular tetrahedron with side length a. Show that opposite edges such as AB and CD are perpendicular.

20. Consider the points $A(2, 2, -k)$, $B(1, -2, -1)$, $C(k, -1, -1)$ and $D(-1, 0, k)$ where $k \in \mathbb{R}$. Determine the values of k such that ABCD is a rhombus.

Fully worked solutions for this chapter are available online.

LESSON
3.4 Geometric proofs using vectors

SYLLABUS LINKS

- Use a vector representing a section of a line segment, including the midpoint of a line segment.
- Examine properties of parallel and perpendicular vectors and determine if two vectors are parallel or perpendicular.
- Use vectors to prove geometric results in two dimensions (other than those listed in Unit 2 Topic 3) and in three dimensions.

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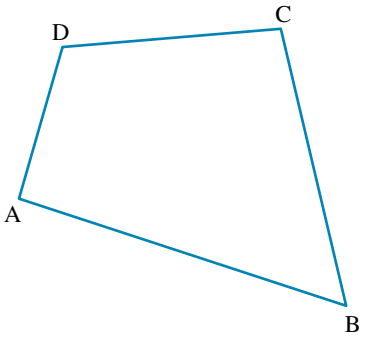
3.4.1 Geometrical shapes

Geometry is a branch of mathematics that explores the relationship between lines, shapes, solids and angles. It is often referred to as deductive geometry because problems are solved by referring to mathematical statements considered to be true, known as axioms or postulates. The most famous of which are Euclid's postulates, explored in Unit 1, Topic 2 Introduction to proof Euclid's first postulate stated that a straight line can be drawn between any two points.

Straight lines define the edges of many shapes and solids and can be expressed as directed line segments. When considered in this way, vectors can be used to prove and solve many geometric problems. Consider the common geometric shapes and their vector properties below.

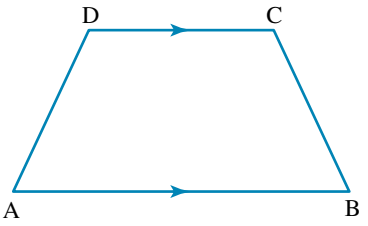
Quadrilateral – General

A plane four-sided figure with no two sides necessarily parallel nor equal in length.



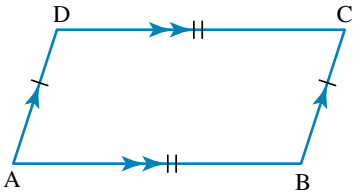
Trapezium

A plane four-sided figure with one pair of sides parallel, but not equal. In trapezium ABCD, since AB is parallel to DC, $\overrightarrow{AB} = \lambda\,\overrightarrow{DC}$; where λ is a scalar.



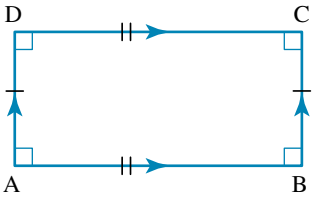
Parallelogram

A plane four-sided figure with two sets of sides parallel and equal in length. In parallelogram ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$. Also, $|\overrightarrow{AB}| = |\overrightarrow{DC}|$ and $|\overrightarrow{AD}| = |\overrightarrow{BC}|$.



Rectangle

A parallelogram with all angles being 90°.
In rectangle ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$ and thus $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{BC} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{CD} \cdot \overrightarrow{DA} = 0$ and $\overrightarrow{DA} \cdot \overrightarrow{AB} = 0$, since all these sides are perpendicular.

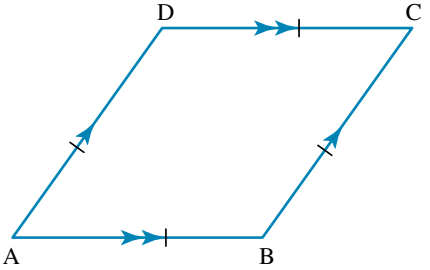


Rhombus

A parallelogram with all sides equal in length.

In rhombus ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

$|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{DC}| = |\overrightarrow{AD}|$, since all sides are equal in length.

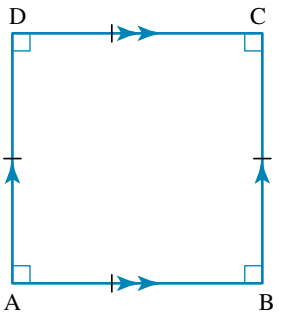


Square

A rhombus with all angles 90°.

In square ABCD, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$ and $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{BC} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{CD} \cdot \overrightarrow{DA} = 0$, $\overrightarrow{DA} \cdot \overrightarrow{AB} = 0$, since all these sides are perpendicular.

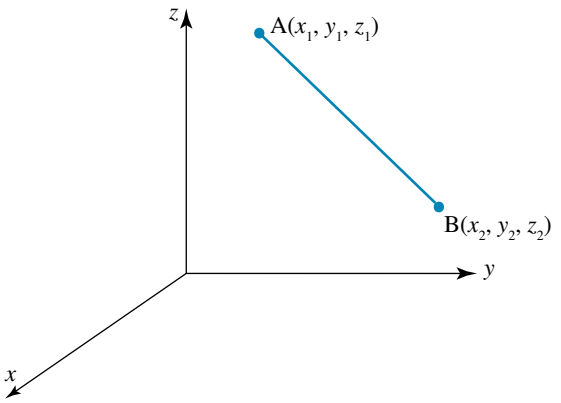
Also, $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{DC}| = |\overrightarrow{AD}|$



3.4.2 Distances between points in three dimensions

Many of the formulae previously studied can be applied to three dimensional Cartesian systems.

The distance between two points in three dimensional space, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, can be calculated using Pythagoras's theorem twice to produce the formula for the distance between two points.



Distance between two points in three dimensional space

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Like midpoint calculations in two dimensional space, the midpoint M_{AB} of a line segment defined by the Cartesian coordinates $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ can be found by taking the average of each of the coordinates.

Midpoint of a segment in three dimensional space

The midpoint of the line segment joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

$$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

WORKED EXAMPLE 9 Determining midpoint using formula

A line segment is formed between two points, A(−2, 5, 8) and B(−10, 11, −4).

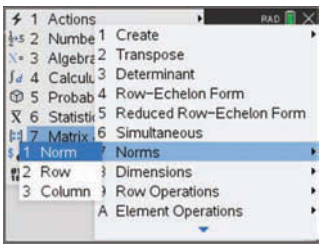
a. Calculate the exact distance between A and B.

b. Determine the midpoint of the line segment AB.

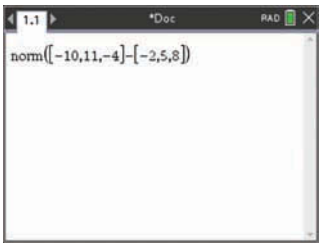
THINK	WRITE
a. 1. Recall the distance formula for two points in three dimensional space.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
2. Substitute the x, y, z coordinates into the formula for points A and B.	$d = \sqrt{(-10 - -2)^2 + (11 - 5)^2 + (-4 - 8)^2}$ $d = \sqrt{64 + 36 + 144}$
3. Express the distance in simplified surd form.	$d = \sqrt{244}$ $d = 2\sqrt{61}$ units $= 15.62$ to 2 decimal places
b. 1. Recall the midpoint formula for three dimensions.	$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
2. Substitute the x, y, z coordinates into the formula for points A and B.	$= \left(\frac{-2 + -10}{2}, \frac{5 + 11}{2}, \frac{8 + -4}{2}\right)$
3. Express the midpoint, M_{AB} as coordinates (x, y, z).	$M_{AB} = (-6, 8, 2)$

TI | THINK — **WRITE/DISPLAY**

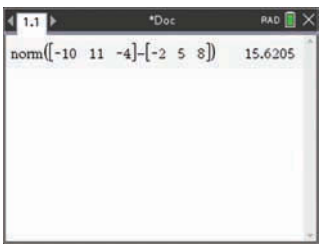
a. 1. On a Calculator page, select MENU 7: Matrix Vector 7: Norms 1: Norm Press the ENTER button.



2. Complete the entry line ([−10, 11, −4]−[−2, 5, 8])

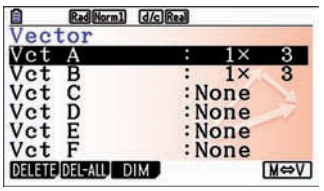


3. Press the ENTER button. The answer appears on the screen.

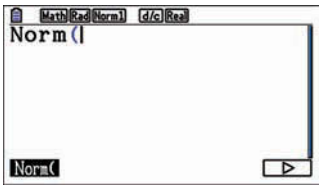


CASIO | THINK — **WRITE/DISPLAY**

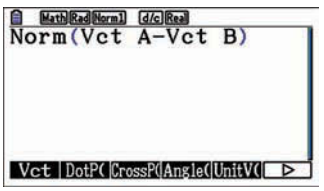
a. 1. On a Run-Matrix screen, select MAT/VCT Define Vectors A and B as shown in Worked example 1b.



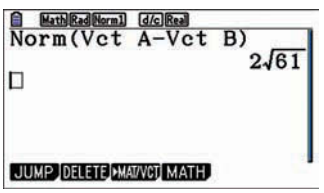
2. On a Run-Matrix screen, select OPTN MAT/VCT ▷ (x3 times) Norm(



3. Complete the entry line as Vct ALPHA X, θ, T-Vct ALPHA Log)



4. Press the EXE button. The answer appears on the screen



Consider that given one end of a segment and a midpoint of a line segment, you can determine the position vector of the other end.

The midpoint of a vector in three dimensions

Consider the two vectors,

$\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$

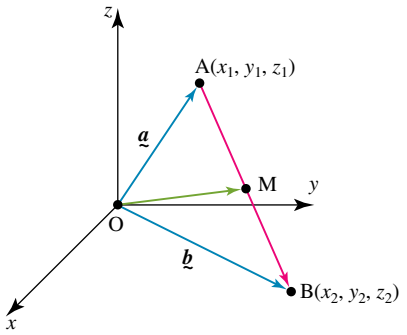
The midpoint, M, of the line segment AB, can be represented by the vector,

$\overrightarrow{OM} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$. Alternatively, it can be defined as,

$$= \frac{\underset{\sim}{a} + \underset{\sim}{b}}{2}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$$

$$= \underset{\sim}{a} + \frac{\underset{\sim}{b} - \underset{\sim}{a}}{2}$$



WORKED EXAMPLE 10 Determining the midpoint of a line segment using vectors

The location of two points are defined by P(4, 8, −2) and Q(12, −3, 6).
Calculate the position vector $\overrightarrow{OM}$, where M is the midpoint of the line segment PQ.

THINK	WRITE
1. Define the vector $\overrightarrow{OP}$.	$\overrightarrow{OP} = 4\hat{\imath} + 8\hat{\jmath} - 2\hat{k}$
2. Define the vector $\overrightarrow{OQ}$.	$\overrightarrow{OQ} = 12\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$
3. Recall the rule that defines a midpoint as a vector.	$\overrightarrow{OM} = \dfrac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$
4. Calculate $\overrightarrow{OM}$ by substituting $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ into the formula.	$= \dfrac{(4\hat{\imath} + 8\hat{\jmath} - 2\hat{k}) + (12\hat{\imath} - 3\hat{\jmath} + 6\hat{k})}{2}$ $= \dfrac{16\hat{\imath} + 5\hat{\jmath} + 4\hat{k}}{2}$ $= 8\hat{\imath} + \dfrac{5}{2}\hat{\jmath} + 2\hat{k}$
5. Conclude with a statement.	The midpoint between the points P and Q is represented by the position vector $\overrightarrow{OM} = 8\hat{\imath} + \dfrac{5}{2}\hat{\jmath} + 2\hat{k}$.

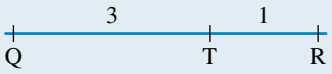
Similar logic can be applied to use a vector to represent any section of a line segment. Considering $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$. The point C, one third along the line segment $\overrightarrow{AB}$ can be represented by the vector $\underset{\sim}{c} = \overrightarrow{OC}$.

$$\overrightarrow{OC} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{3} \quad \text{or} \quad \overrightarrow{OC} = \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB}$$

$$\underset{\sim}{c} = \frac{\underset{\sim}{a} + \underset{\sim}{b}}{3} \qquad \underset{\sim}{c} = \underset{\sim}{a} + \frac{\underset{\sim}{b} - \underset{\sim}{a}}{3}$$

WORKED EXAMPLE 11 Determining the coordinates of a point using sections of a line segment

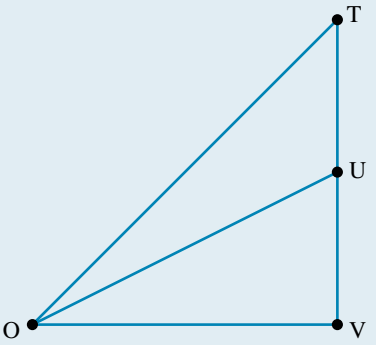
Consider the points Q(2, −24, −6) and R(−2, 12, 10). T is the point on the line segment QR such that $\overrightarrow{QT} = 3\overrightarrow{TR}$. Determine the location of point T.

THINK	WRITE
1. Define vectors.	$\overrightarrow{OQ} = 2\hat{\imath} - 24\hat{\jmath} - 6\hat{k}$ $= \underset{\sim}{q}$ $\overrightarrow{OR} = -2\hat{\imath} + 12\hat{\jmath} + 10\hat{k}$ $= \underset{\sim}{r}$ $\overrightarrow{OT} = \underset{\sim}{t}$
2. Draw the relationship.	
3. Write the equation.	$\overrightarrow{QT} = 3\overrightarrow{TR}$
4. Calculate $\overrightarrow{OT}$ by substituting $\overrightarrow{OQ}$ and $\overrightarrow{QR}$ in the equation.	$\underset{\sim}{t} - \underset{\sim}{q} = 3(\underset{\sim}{r} - \underset{\sim}{t})$ $\underset{\sim}{t} - \underset{\sim}{q} = 3\underset{\sim}{r} - 3\underset{\sim}{t}$ $4\underset{\sim}{t} = 3\underset{\sim}{r} + \underset{\sim}{q}$ $= +3\left(-2\hat{\imath} + 12\hat{\jmath} + 10\hat{k}\right) + \left(2\hat{\imath} - 24\hat{\jmath} - 6\hat{k}\right)$ $= -4\hat{\imath} + 12\hat{\jmath} + 24\hat{k}$ $\underset{\sim}{t} = -\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$
5. State the answer.	The location of point T is (−1, 3, 6).

3.4.3 Geometric proofs

WORKED EXAMPLE 12 Geometric proof using midpoints

If O is the origin and U is the midpoint of the line segment TV, prove that $\overrightarrow{OU} = \frac{1}{2}\left(\overrightarrow{OT} + \overrightarrow{OV}\right)$.



THINK	WRITE
1. Define the vector $\overrightarrow{OV}$ as the sum of the other two vectors.	Let, $\overrightarrow{OV} = \overrightarrow{OT} + \overrightarrow{TV}$ [1]
2. Rearrange [1] to make $\overrightarrow{TV}$ as the subject.	$\overrightarrow{TV} = \overrightarrow{OV} - \overrightarrow{OT}$ [2]
3. Define the vector $\overrightarrow{OU}$ as the sum of $\overrightarrow{OT}$ and $\overrightarrow{TU}$. As U is the midpoint of $\overrightarrow{TV}$, $\overrightarrow{TU} = \frac{1}{2}\overrightarrow{TV}$.	$\overrightarrow{OU} = \overrightarrow{OT} + \frac{1}{2}\overrightarrow{TV}$ [3]

4. Substitute [2] into [3] and simplify the right-hand side.
Expand $\frac{1}{2}\left(\overrightarrow{OV}-\overrightarrow{OT}\right)$.
Collect like vectors $\overrightarrow{OT}$.
Factorise to demonstrate the LHS = RHS.

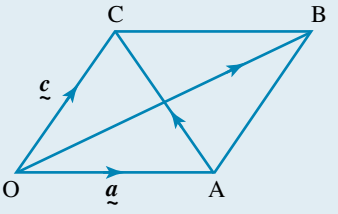
$$\overrightarrow{OU}=\overrightarrow{OT}+\frac{1}{2}\left(\overrightarrow{OV}-\overrightarrow{OT}\right)$$
$$\overrightarrow{OU}=\overrightarrow{OT}+\frac{1}{2}\overrightarrow{OV}-\frac{1}{2}\overrightarrow{OT}$$
$$=\frac{1}{2}\overrightarrow{OT}+\frac{1}{2}\overrightarrow{OV}$$
$$=\frac{1}{2}\left(\overrightarrow{OT}+\overrightarrow{OV}\right)$$

5. Write a concluding statement.

$$\therefore \overrightarrow{OU}=\frac{1}{2}\left(\overrightarrow{OT}+\overrightarrow{OV}\right)$$

WORKED EXAMPLE 13 Geometric proofs in parallelogram

Prove that if the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle.

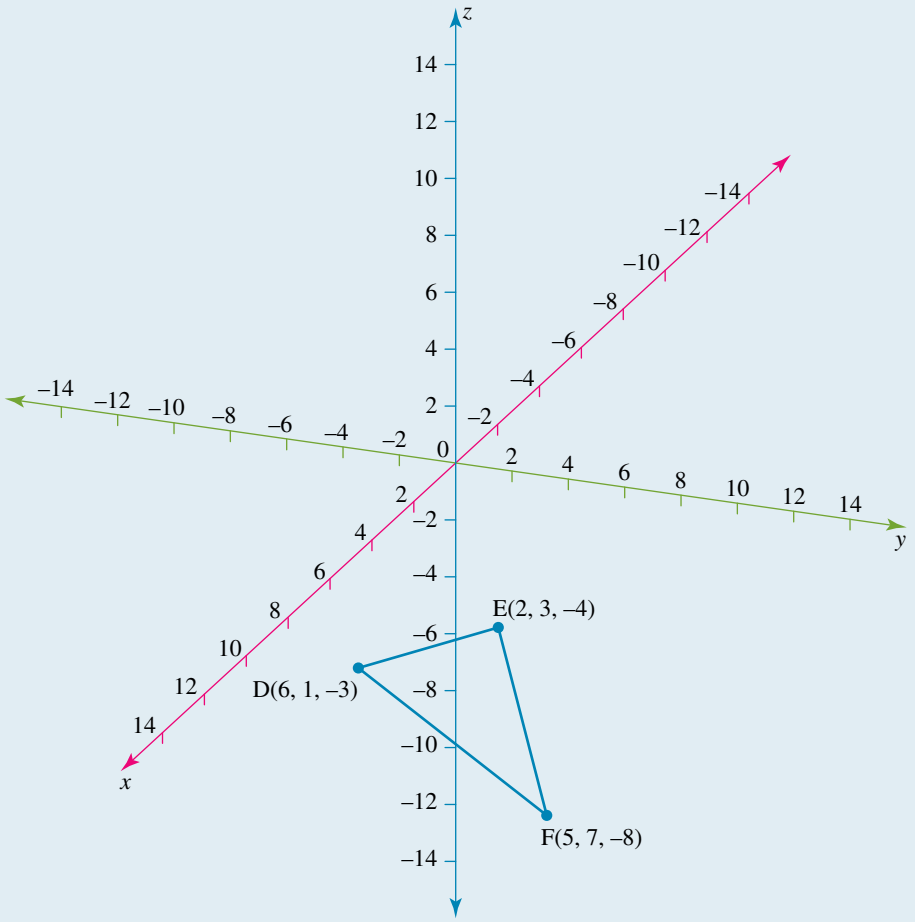


THINK	WRITE
1. Define the vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ in terms of vectors joining the points A, B, C, D. State equivalent vectors using the properties of a parallelogram.	Let O, A, B, C be a parallelogram. $\underset{\sim}{a}=\overrightarrow{OA}=\overrightarrow{CB}$ $\underset{\sim}{c}=\overrightarrow{OC}=\overrightarrow{AB}$
2. Express the diagonals in terms of the vectors, $\underset{\sim}{a}$ and $\underset{\sim}{c}$.	$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\underset{\sim}{c}-\underset{\sim}{a}$ $\quad\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\underset{\sim}{a}+\underset{\sim}{c}$
3. Assume the diagonals are equal in length.	$\lvert\overrightarrow{OB}\rvert=\lvert\overrightarrow{AC}\rvert$
4. Express the diagonals in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ and square both sides to apply the dot product property, $\lvert\underset{\sim}{a}\rvert^2=\underset{\sim}{a}\cdot\underset{\sim}{a}$.	$\lvert\overrightarrow{OB}\rvert^2=\lvert\overrightarrow{AC}\rvert^2$ $(\underset{\sim}{a}+\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{c})=(\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{c}-\underset{\sim}{a})$
5. Expand both sides. Remember that $\underset{\sim}{a}\cdot\underset{\sim}{a}=\lvert\underset{\sim}{a}\rvert^2$ and $\underset{\sim}{a}\cdot\underset{\sim}{c}=\underset{\sim}{c}\cdot\underset{\sim}{a}$.	LHS $=\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{a}+\underset{\sim}{c}\cdot\underset{\sim}{c}$ $=\lvert\underset{\sim}{a}\rvert^2+2\underset{\sim}{a}\cdot\underset{\sim}{c}+\lvert\underset{\sim}{c}\rvert^2$ RHS $=\underset{\sim}{c}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot-\underset{\sim}{a}+-\underset{\sim}{a}\cdot\underset{\sim}{c}+-\underset{\sim}{a}\cdot-\underset{\sim}{a}$ $=\lvert\underset{\sim}{c}\rvert^2-2\underset{\sim}{a}\cdot\underset{\sim}{c}+\lvert\underset{\sim}{a}\rvert^2$
6. Rewrite $\lvert\overrightarrow{OB}\rvert^2=\lvert\overrightarrow{AC}\rvert^2$ in terms of the dot product calculations and collect like terms.	$\lvert\underset{\sim}{a}\rvert^2+2\underset{\sim}{a}\cdot\underset{\sim}{c}+\lvert\underset{\sim}{c}\rvert^2=\lvert\underset{\sim}{c}\rvert^2-2\underset{\sim}{a}\cdot\underset{\sim}{c}+\lvert\underset{\sim}{a}\rvert^2$ $2\underset{\sim}{a}\cdot\underset{\sim}{c}+2\underset{\sim}{a}\cdot\underset{\sim}{c}=\lvert\underset{\sim}{c}\rvert^2+\lvert\underset{\sim}{a}\rvert^2-\lvert\underset{\sim}{a}\rvert^2-\lvert\underset{\sim}{c}\rvert^2$ $4\underset{\sim}{a}\cdot\underset{\sim}{c}=0$ $\underset{\sim}{a}\cdot\underset{\sim}{c}=0$
7. State the conclusion.	Since $\underset{\sim}{a}\cdot\underset{\sim}{c}=0$ this implies that $\angle COA=90°$. $\therefore \overrightarrow{OA}$ is $\perp$ to $\overrightarrow{OC}$, OABC is a rectangle.

This can extend to simple applications of plane shapes in three dimensions.

WORKED EXAMPLE 14 Simple three dimensional proof

Given the points D(6, 1, −3), E(2, 3, −4) and F(5, 7, −8), prove that DEF forms a right-angled triangle at E, and hence calculate the area of the triangle as an exact value.



THINK

1. Define each of the vertices as position vectors.

2. The line segments that form the triangle can be written as vectors in terms of $\overrightarrow{OD}, \overrightarrow{OE}, \overrightarrow{OF}$.

WRITE

$D(6, 1, -3) \quad \therefore \overrightarrow{OD} = 6\hat{\imath} + \hat{\jmath} - 3\hat{k}$

$E(2, 3, -4) \quad \therefore \overrightarrow{OE} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$

$F(5, 7, -8) \quad \therefore \overrightarrow{OF} = 5\hat{\imath} + 7\hat{\jmath} - 8\hat{k}$

$$\begin{aligned}\overrightarrow{DE} &= \overrightarrow{OE} - \overrightarrow{OD}\\ &= \left(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}\right) - \left(6\hat{\imath} + \hat{\jmath} - 3\hat{k}\right)\\ &= -4\hat{\imath} + 2\hat{\jmath} - \hat{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{OF} - \overrightarrow{OE}\\ &= \left(5\hat{\imath} + 7\hat{\jmath} - 8\hat{k}\right) - \left(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}\right)\\ &= 3\hat{\imath} + 4\hat{\jmath} - 4\hat{k}\end{aligned}$$

3. If $\overrightarrow{DE}$ is perpendicular to $\overrightarrow{EF}$, then $\overrightarrow{DE} \cdot \overrightarrow{EF} = 0$.

$$\begin{aligned}\overrightarrow{DE} \cdot \overrightarrow{EF} &= \left(-4\hat{\imath} + 2\hat{\jmath} - \hat{k}\right) \cdot \left(3\hat{\imath} + 4\hat{\jmath} - 4\hat{k}\right) \\ &= -12 + 8 + 4 \\ &= 0\end{aligned}$$

4. As the dot product is 0, the angle between the vectors is 90°.

Therefore $\overrightarrow{DE}$ is $\perp$ to $\overrightarrow{EF}$, hence the triangle is right-angled.

5. Recall the area of a triangle can be calculated by the rule, $A = \frac{1}{2}bh$. In this case $b = \left|\overrightarrow{DE}\right|$ and $h = \left|\overrightarrow{EF}\right|$

$$\begin{aligned}\left|\overrightarrow{DE}\right| &= \sqrt{(-4)^2 + (2)^2 + (-1)^2} \\ &= \sqrt{21} \\ \left|\overrightarrow{EF}\right| &= \sqrt{(3)^2 + (4)^2 + (-4)^2} \\ &= \sqrt{41}\end{aligned}$$

6. Substitute the magnitude of the two vectors into the area rule and state the result.

$$\begin{aligned}A &= \frac{1}{2}\left|\overrightarrow{DE}\right|\left|\overrightarrow{EF}\right| \\ &= \frac{1}{2}\sqrt{21} \times \sqrt{41} \\ &= \frac{1}{2}\sqrt{861} \text{ units}^2\end{aligned}$$

WORKED EXAMPLE 15 Three dimensional proof

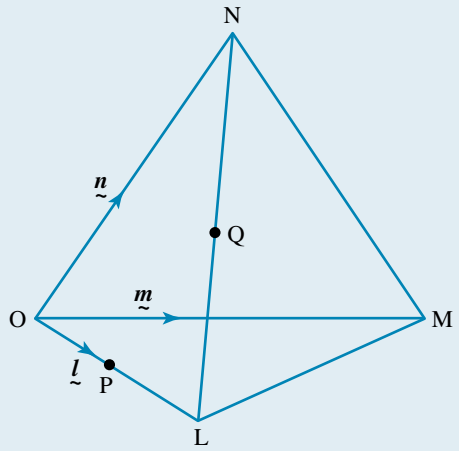
A tetrahedron is defined by the vertices, O, L, M and N. Given that, $\overrightarrow{OL} = \underset{\sim}{l}$, $\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{ON} = \underset{\sim}{n}$.

a. Express $\overrightarrow{LM}$ in terms of $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$
b. Express $\overrightarrow{MN}$ in terms of $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$
c. Express $\overrightarrow{LN}$ in terms of $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$
d. Develop an expression for the line joining the midpoints of OL to LN.

THINK	WRITE
a. 1. Choose a path from L to M that travels along edges defined as vectors $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$.	a. $\overrightarrow{LM} = \overrightarrow{LO} + \overrightarrow{OM}$
2. Recall that $\overrightarrow{LO} = -\overrightarrow{OL}$ (negative of a vector).	$\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$
3. Express the RHS in terms of $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$.	$\overrightarrow{LM} = \underset{\sim}{m} - \underset{\sim}{l}$
b. 1. Choose a path from M to N that travels along edges defined as vectors $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$.	$\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$
2. Recall that $\overrightarrow{MO} = -\overrightarrow{OM}$ (negative of a vector).	$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$
3. Express the RHS in terms of $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$.	$\overrightarrow{MN} = \underset{\sim}{n} - \underset{\sim}{m}$
c. 1. Choose a path from L to N that travels along edges defined as vectors $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$.	$\overrightarrow{LN} = \overrightarrow{LO} + \overrightarrow{ON}$
2. Recall that $\overrightarrow{LO} = -\overrightarrow{OL}$ (negative of a vector).	$\overrightarrow{LN} = \overrightarrow{ON} - \overrightarrow{OL}$
3. Express the RHS in terms of $\underset{\sim}{l}, \underset{\sim}{m}$ or $\underset{\sim}{n}$.	$\overrightarrow{LN} = \underset{\sim}{n} - \underset{\sim}{l}$

d. 1. Define the midpoints of $\overrightarrow{OL}$ and $\overrightarrow{LN}$ as P and Q respectively. Draw a sketch to assist visualising the problem.

Let P be the midpoint of OL.
Let Q be the midpoint of LN.



2. $\overrightarrow{OP}$ is parallel to $\underset{\sim}{l}\left(\overrightarrow{OL}\right)$ and is half its magnitude.

$$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OL} = \frac{1}{2}\underset{\sim}{l}$$

3. $\overrightarrow{OQ}$ is the vector sum of $\overrightarrow{OL}$ and one half of $\overrightarrow{LN}$.
$\overrightarrow{OL}$ can be expressed as $\underset{\sim}{l}$.
$\overrightarrow{LN}$ was defined as $\underset{\sim}{n} - \underset{\sim}{l}$.
Expand and simplify to then collect like vectors.

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OL} + \frac{1}{2}\overrightarrow{LN}\\ &= \underset{\sim}{l} + \frac{1}{2}(\underset{\sim}{n} - \underset{\sim}{l})\\ &= \underset{\sim}{l} + \frac{1}{2}\underset{\sim}{n} - \frac{1}{2}\underset{\sim}{l}\\ &= \frac{1}{2}\underset{\sim}{l} + \frac{1}{2}\underset{\sim}{n}\end{aligned}$$

4. $\overrightarrow{PQ}$ can be determined by subtracting $\overrightarrow{OP}$ from $\overrightarrow{OQ}$.

The vector joining P to Q is

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\ &= \left(\frac{1}{2}\underset{\sim}{l} + \frac{1}{2}\underset{\sim}{n}\right) - \left(\frac{1}{2}\underset{\sim}{l}\right)\\ &= \frac{1}{2}\underset{\sim}{n}\end{aligned}$$

Exercise 3.4 Geometric proofs using vectors

learnon

3.4 Exercise | **3.4 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. Determine the coordinates of point M, middle of segment AB, where A(2, −3, 4), and B(−6, 8, −1).

2. Determine the exact distance d between the points A(2, −1, −6), and B(−3, 5, −4).

3. Given the points A$(-1, -1, 17)$, B$(4, -2, 4)$ and C$(1, -3, 7)$, prove that A, B and C are not aligned.

4. Four points are given by P$(-2, 5, 0)$, Q$(0, 8, 4)$, R$(3, -6, 12)$ and S$(6, 17, 16)$. Refer to these points to determine:
   a. $\overrightarrow{OP}$
   b. $\overrightarrow{PQ}$
   c. $\underset{\sim}{c} = 2\overrightarrow{PQ} - 3\overrightarrow{QR}$
   d. the position vector $\overrightarrow{OM}$, where M is the midpoint of the line segment PS
   e. if $\overrightarrow{PQ}$ and $\overrightarrow{QS}$ are parallel.

5. WE9a A line segment is formed between two points, A(x_1, y_1, z_1) and B(x_2, y_2, z_2). Calculate the exact distance between A and B.
   a. A$(-5, 8, 12)$ and B$(7, 15, 21)$
   b. A$(7, -2, 9)$ and B$(3, 1, -2)$
   c. A$(5, 2, 1)$ and B$(-3, 7, -11)$
   d. A$(1, -2, 7)$ and B$(-9, 3, -2)$

6. WE9b A line segment is formed between two points, A(x_1, y_1, z_1) and B(x_2, y_2, z_2). Determine the midpoint of the line segment AB.
   a. A$(5, 2, -7)$ and B$(9, 12, 5)$
   b. A$(-10, 2, -2)$ and B$(4, 6, -12)$
   c. A$(7, -2, 10)$ and B$(1, -10, -2)$
   d. A$(2, 2, 0)$ and B$(9, -6, -9)$

7. WE10 The location of two points are defined by, P$(1, 5, -6)$ and Q$(9, 7, 4)$, respectively. Calculate the position vector $\overrightarrow{OM}$, where M is the midpoint of the line segment PQ.

8. WE11 Consider the points Q$(-5, -3, 12)$ and R$(6, -9, 7)$. T is the point on the line segment QR such that $\overrightarrow{QT} = 5\overrightarrow{TR}$. Determine the location of point T.

9. WE12 OABC is a rhombus. The points P, Q, R and S are the midpoints of the sides OA, AB, BC and OC respectively. Prove that PQRS is a rectangle.

10. WE13 Prove that if the diagonals of a parallelogram are perpendicular, the parallelogram is a rhombus.

11. WE14 Given the points A$(-3, 5, 4)$, B$(2, 3, 5)$ and C$(4, 6, 1)$, prove that ABC forms a right-angled triangle at B, and hence calculate the area of the triangle.

12. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Complex familiar

13. A median is formed inside a triangle when the midpoint of a side is joined to its opposite vertex. Prove that medians inside a triangle sum to zero.

14. Four points, ABCD, form a quadrilateral. Prove that the midpoints of the sides of the quadrilateral form a parallelogram.

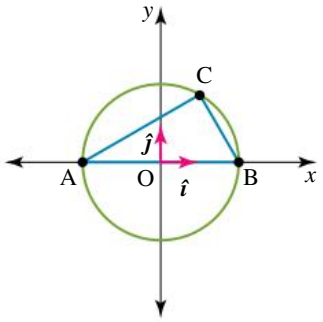
15. Given the points A$(4, 7, 3)$, B$(8, 7, 1)$ and C$(6, 5, 2)$:
   a. Show that ABC forms an isosceles triangle.
   b. Let M be the midpoint of AB, and show that MC is perpendicular to AB.

16. Prove Pythagoras' theorem.

Complex unfamiliar

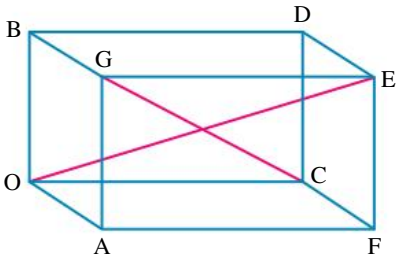
17. **a.** Prove that the angle inscribed in a semicircle is a right angle.

b. The diagram shows a circle of radius r with centre at the origin O on the x- and y-axes. The points A and B lie on the diameter of the circle and are on the x-axis; their coordinates are $(-r, 0)$ and $(r, 0)$ respectively. The point C has coordinates (a, b) and lies on the circle, where a, b and r are all positive real constants. Show that CA is perpendicular to CB.

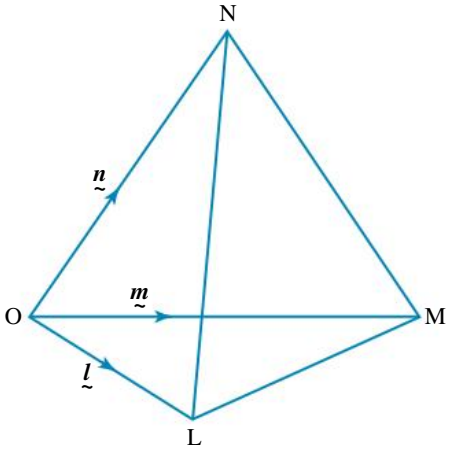


18. Prove that the vector $\hat{\underset{\sim}{a}} + \hat{\underset{\sim}{b}}$ bisects the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

19. A storage container is formed in the shape of a rectangular prism. Two support beams are being installed on the inside of the container that would join O to E and G to C. Prove that the diagonals bisect one another, if and only if, the dimensions of the container are a Pythagorean triad.



20. WE15 A tetrahedron is defined by the vertices, O, L, M and N. Given that, $\overrightarrow{OL} = \underset{\sim}{l}$, $\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{ON} = \underset{\sim}{n}$, show that the line segment formed between the midpoints of $\overrightarrow{LN}$ to $\overrightarrow{MN}$ is parallel to the $\overrightarrow{LM}$.



Fully worked solutions for this chapter are available online.

LESSON
3.5 Vector, parametric and Cartesian equations

SYLLABUS LINKS

- Understand and use equations of spheres.
  - equation of sphere: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$
- Use vector equations of curves in two or three dimensions involving a parameter, and determine a 'corresponding' Cartesian equation in the two dimensional case.
- Determine vector, parametric and Cartesian equations of straight lines and straight-line segments given the position of two points, or equivalent information, in both two and three dimensions.
  - vector equation of line: $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{d}$
  - parametric equations of line: $x=a_1+td_1$

    $y=a_2+td_2$

    $z=a_3+td_3$
  - Cartesian equation of line: $\dfrac{x-a_1}{d_1}=\dfrac{y=a_2}{d_2}=\dfrac{z-a_3}{d_3}$

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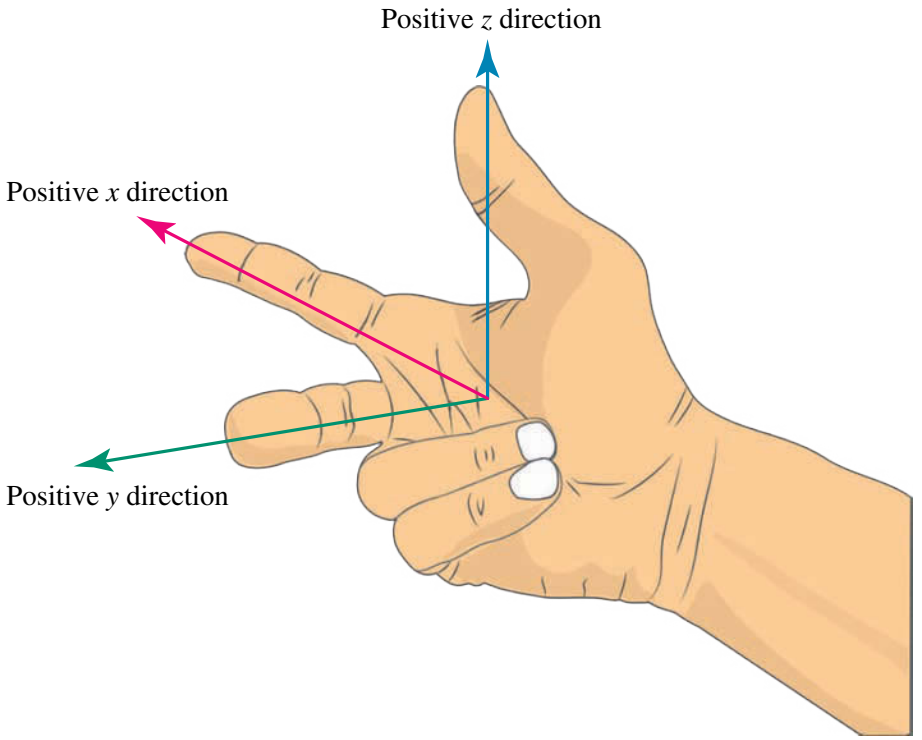
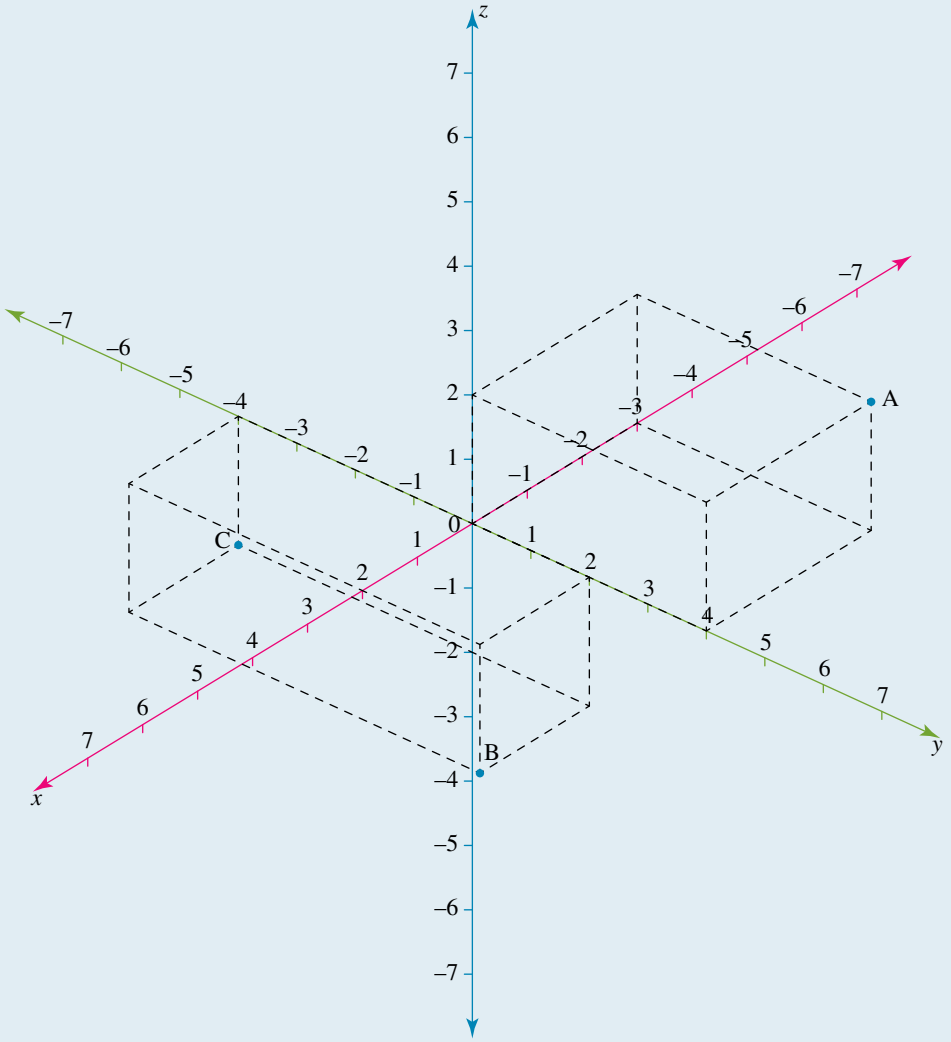
3.5.1 Cartesian coordinates for three dimensional space

The three dimensional coordinate system was introduced at the beginning of this chapter. Recall that the position of a point in three dimensional space can be defined in terms of the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ and the point P can be written in the form of an ordered triple, P(x, y, z). Three dimensional axes are commonly drawn as a right-handed system — index finger for positive x direction, middle finger for positive y direction and thumb for positive z direction. Go back to Lesson 3.2 or use the following Worked example to review the Cartesian coordinates in three dimensions.



WORKED EXAMPLE 16 Identifying coordinates in three dimensions

Refer to the three dimensional space below to state the coordinates of A, B and C.

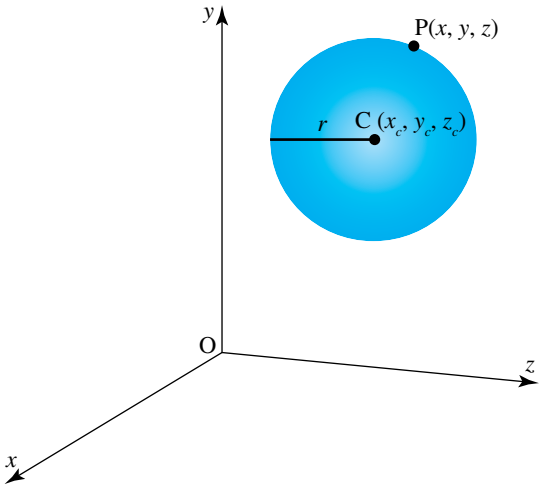


THINK	WRITE
1. Point A is 3 units in the negative direction of the x-axis, 4 units in the positive direction of the y-axis and 2 units in the positive direction of the z-axis.	Point A is defined as, A$(-3, 4, 2)$
2. Point B is 2 units in the positive direction of the x-axis, 3 units in the positive direction of the y-axis and 2 units in the negative direction of the z-axis.	Point B is defined as, B$(2, 3, -2)$
3. Point C does not have an x-axis component, but is 4 units in the negative direction of the y-axis and 1 unit in the negative direction of the z-axis.	Point C is defined as, C$(0, -4, -1)$

3.5.2 The Cartesian equation of a sphere

The **sphere** can be defined, in formal terms, as the locus of points P(x, y, z) that is equidistant from a given point C, where C is, C(x_c, y_c, z_c). More simply, a sphere is simply the set of all points P(x, y, z) that are an equal distance from the centre C(x_c, y_c, z_c).

The equation of a sphere, with radius r and centre C(x_c, y_c, z_c), can be determined by applying the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ where the distance d is the radius of the sphere, r.



Equation of a sphere in three dimensional space

If P(x, y, z) is a point on the sphere of centre C(x_c, y_c, z_c) and radius r, the equation for a sphere is

$$r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$$

WORKED EXAMPLE 17 Using Cartesian equation of spheres

a. Determine the Cartesian equation for the sphere with centre C$(-3, 7, 2)$ and radius $r = 9$.
b. Determine the Cartesian equation for the sphere that has A$(6, 12, 11)$ and B$(-2, 4, -1)$ as the endpoints of its diameter.
c. Determine the centre and radius of the sphere with the Cartesian equation $x^2 + y^2 + z^2 - 12x + 4y + 8z + 20 = 0$.

THINK	WRITE
a. 1. Identify the coordinates of the centre of the circle. Express these values in terms of the variables x_c, y_c, z_c.	**a.** $C(x_c, y_c, z_c) = C(-3, 7, 2) \therefore x_c = -3,\ y_c = 7,\ z_c = 2$
2. Square the given radius so that it can be expressed as r^2.	Radius, $r = 9$ $\therefore r^2 = 9^2 = 81$
3. Recall the general form for the equation of a sphere and substitute in the values.	$r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$ $81 = (x + 3)^2 + (y - 7)^2 + (z - 2)^2$

b. 1. State the given points, which define the endpoints of a diameter.

b. A(6, 12, 11) and B(−2, 4, −1)

2. The midpoint, M_{AB}, defines the centre of the circle. Recall the rule for the midpoint and substitute the coordinates for points A and B.

$$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$
$$= \left(\frac{6 + -2}{2}, \frac{12 + 4}{2}, \frac{11 + -1}{2}\right)$$
$$= (2, 8, 5)$$

∴ The centre is given by C(2, 8, 5)

3. The centre of the circle defines the value for the associated variables x_c, y_c, z_c.

$C(x_c, y_c, z_c = C(2, 8, 5)$
$x_c = 2, y_c = 8, z_c = 5$

4. The radius of a sphere, r, is the distance from the centre to any point on its surface. The radius can then be defined by calculating the distance from the centre C(2, 8, 5) to the point A(6, 12, 11).

Calculate the distance from the centre to one of the endpoints of the diameter.

$$r = \sqrt{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2}$$
$$r = \sqrt{(6 - 2)^2 + (12 - 8)^2 + (11 - 5)^2}$$
$$r = \sqrt{16 + 16 + 36}$$
$$r = \sqrt{68}$$
$$r^2 = 68 \text{ units}$$

5. Recall the general form for the equation of a sphere and substitute in the values.

The Cartesian equation of the sphere is
$$r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$$
$$68 = (x - 2)^2 + (y - 8)^2 + (z - 5)^2$$

c. 1. Complete the square on each of the x-, y- and z-components (group the x parts together, the y parts together and the z part together) to rearrange the equation into the general form of a sphere $r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$.
Remember to add any values required to complete the square to both side of the equation.

c. $x^2 + y^2 + z^2 - 12x + 4y + 8z + 20 = 0$
$$x^2 - 12x + y^2 + 4y + z^2 + 8z + 20 = 0$$
$$x^2 - 12x + y^2 + 4y + z^2 + 8z = -20$$
$$x^2 - 12x + \left(\frac{-12}{2}\right)^2 + y^2 + 4y + \left(\frac{4}{2}\right)^2 + z^2 + 8z + \left(\frac{8}{2}\right)^2$$
$$= -20 + \left(\frac{-12}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \left(\frac{8}{2}\right)^2$$
$$x^2 - 12x + 36 + y^2 + 4y + 4 + z^2 + 8z + 16$$
$$= -20 + 36 + 4 + 16$$

2. The centre of the circle can be recognised from the equation generated.

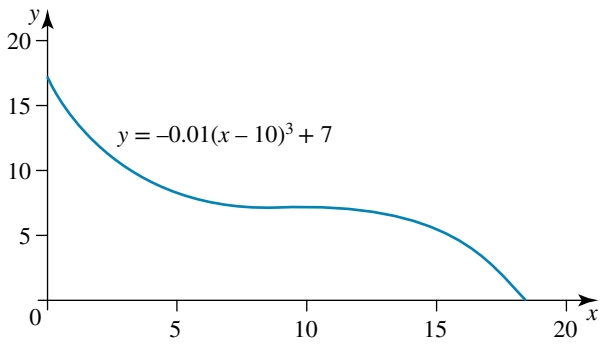
$$(x - 6)^2 + (y + 2)^2 + (z + 4)^2 = 36$$
$$(x - 6)^2 + (y - (-2))^2 + (z - (-4))^2 = 36$$
∴ C(6, −2, −4)

3. To determine the radius, recall the general form of the sphere $r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$.

The centre is (6, −2, −4) with a radius of 6.
$$r^2 = 36$$
$$r = \sqrt{36}$$
$$= 6$$

3.5.3 Equation of curves

A function defines a dependent variable, y, in terms of an independent variable x. These types of equations are in the form $y = f(x)$ and have graphs that are intersected only once by a vertical line. That is, for every x-value there is a unique y-value.



Consider the Cartesian equation, $y = -0.01\,(x - 10)^3 + 7$ shown graphically in the diagram. In this case, the function could model the position of a particle travelling along a water slide. The function can then define the point (x, y), where the position of the particle is as a function of x.

The table exhibits this relationship as ordered pairs.

x	0	5	10	15
$y = f(x)$	17	8.25	7	5.75

While the path of the particle can be described by the Cartesian equation, $y = -0.01\,(x - 10)^3 + 7$, it would also be useful to know *when* the particle reaches a point location. This is achieved by introducing *time* as a third variable. Here, time is referred to as the **parameter**.

Written as a **parametric equation**, the location of the particle on the slide can be defined in terms of the parameter, that is:

$$x = t,\ y = -0.01(t - 10)^3 + 7$$

For each number t, there are corresponding values of x and y (and hence corresponding positions), shown in the table below.

t	0	1	5	10	18
$x = t$	0	1	5	10	18
$y = -0.01\,(t - 10)^3 + 7$	17	14.29	8.25	7	1.88
(x, y)	(0, 17)	(1, 14.29)	(5, 8.25)	(10, 7)	(18, 1.88)

Any letter can be used to represent the parameter. While the letter t is commonly used, it does not have to be defined as time. Where equations of curves are being examined, the angle θ is often the parameter.

Plane curve

If $f(t)$ and $g(t)$ are two continuous functions defined by the parameter t, then the equations $x = f(t)$ and $y = g(t)$ are referred to as parametric equations. The collection of points (x, y) obtained as t varies is called a plane curve.

Simple algebraic techniques can be used to convert a set of parametric equations into Cartesian form (and vice versa). The following examples will focus on parametric equations in two dimensions. Recall that a Cartesian equation is any equation where y is expressed in terms of x.

WORKED EXAMPLE 18 Determining equivalent Cartesian equation for parametric equations 1

Determine the equivalent Cartesian equation (y in terms of x) for the following parametric equations.

a. $x = 2t, y = t + 3$ **b.** $x = t - 3, y = 2t^2$ **c.** $x = \dfrac{2t}{1+t^2}, y = \dfrac{1-t^2}{1+t^2}$

THINK	WRITE
a. 1. Define each parametric equation as equation [1] and [2].	a. $x = 2t$ [1] $y = t + 3$ [2]
2. To convert to a Cartesian equation, the parameter t needs to be eliminated. Thus isolate t in the x equation and then substitute in the y equation. Solve for t in equation [1].	$x = 2t$ $t = \dfrac{x}{2}$
3. Substitute the solution for t into [2]. That is, t is being replaced with x. The Cartesian equation represents the graph of a straight line.	$y = t + 3$ [2] $y = \dfrac{x}{2} + 3$
b. 1. Define each parametric equation as equation [1] and [2].	b. $x = t - 3$ [1] $y = 2t^2$ [2]
2. The parameter t needs to be eliminated. Solve for t in [1].	$x = t - 3$ $t = x + 3$
3. Substitute the solution for t into [2]. That is, t is being replaced with x. The Cartesian equation represents the graph of a parabola.	$y = 2t^2$ [2] $y = 2(x+3)^2$ $y = 2(x^2 + 6x + 9)$ $y = 2x^2 + 12x + 18$
c. 1. Define each parametric equation as equation [1] and [2].	c. $x = \dfrac{2t}{1+t^2}$ [1] $y = \dfrac{1-t^2}{1+t^2}$ [2]

2. As the parameter t is not easily eliminated, a different approach is needed. Try squaring both sides of equation [1] and [2]. We do this because in many of the equations we know (ellipses, circles, hyperbolas), the x and y variables are squared then added (or subtracted in the case of hyperbolas).

Squaring [1] gives, $x^2 = \left(\frac{2t}{1+t^2}\right)^2$

Squaring [2] gives, $y^2 = \left(\frac{1-t^2}{1+t^2}\right)^2$

3. Add the equations, [1] + [2].

$$x^2 + y^2 = \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2$$

4. Rewrite both fractions as the square of the numerators and denominators. Add the fractions as they have like denominators.

$$= \frac{(2t)^2}{\left(1+t^2\right)^2} + \frac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2}$$

$$= \frac{4t^2 + 1 - 2t^2 + t^4}{\left(1+t^2\right)^2}$$

$$= \frac{t^4 + 2t^2 + 1}{\left(1+t^2\right)^2}$$

5. The numerator can be factorised as $t^4 + 2t^2 + 1 = \left(t^2+1\right)^2$ using the substitution $\propto = t^2$.

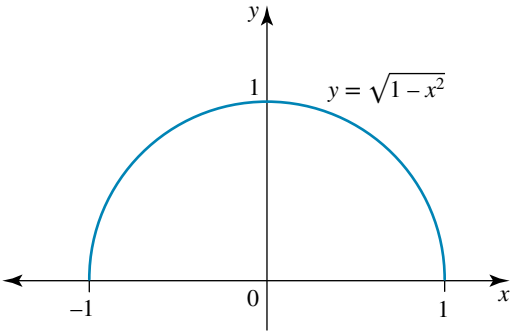
$$= \frac{\left(t^2+1\right)^2}{\left(t^2+1\right)^2}$$

$$= 1$$

6. Cancel like terms and state the relation. In this case, the Cartesian equation represents a circle of radius 1 and centre (0, 0).

$$\therefore x^2 + y^2 = 1$$

Parametric equations are useful when describing the curve of complex graphs and relations. In Worked example 18c, the parametric equation described a circle. If the equation of a circle was written in the form $y = f(x)$, it would be $y = \sqrt{1-x^2}$. When graphed, it produces the top half of a circle. The bottom half of the circle could be expressed as $y = -\sqrt{1-x^2}$.



Curves such as circles and ellipses can be expressed in parametric form when the angle θ or A is the parameter. When parametric equations involve trigonometric functions, we often need to make use of standard trigonometric identities to eliminate the parameter. Recalling the Pythagorean identities, addition identities, double-angle identities and factorisation identities will help calculate and solve Cartesian equations of parametric equations.

In parametric equations involving $\cos(x)$ and $\sin(x)$, it is often a good idea to isolate these trigonometric functions, then square them and add them together to form a Pythagorean identity.

Similar steps are used with other trigonometric identities. You will practice recognising and selecting an appropriate trigonometric identity in Worked example 19.

Standard trigonometric identities

1. The Pythagorean identities

$$\sin^2(A) + \cos^2(A) = 1$$
$$\sec^2(A) = 1 + \tan^2(A)$$
$$\text{cosec}^2(A) = 1 + \cot^2(A)$$

2. Angle sum and angle difference identities

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\cos(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\cos(B)$$

3. The double-angle identities

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

4. Product identities

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$
$$\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$
$$\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

Additional identities may be useful

5. Multiple-angle identities

$$\sin(3A) = 2\sin(A) - 4\sin^2(A)$$
$$\cos(3A) = 4\cos^2(A) - 3\cos(A)$$
$$\sin(4A) = \cos(A)(4\sin(A) - 8\sin^2(A))$$
$$\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$$

6. Half-angle identities

$$\sin(A) = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$
$$\cos(A) = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$$
$$= 2\cos^2\left(\frac{A}{2}\right) - 1$$
$$= 1 - 2\sin^2\left(\frac{A}{2}\right)$$

WORKED EXAMPLE 19 Determining equivalent Cartesian equation for parametric equations 2

Calculate the Cartesian equation for the following parametric equations:

a. $x = \cos(\theta), y = \sin(\theta)$
b. $x = 3\cos(\theta), y = 5\sin(\theta)$
c. $x = 2\sec(\theta), y = 3\tan(\theta)$
d. $x = \sin(\theta), y = \cos 2(\theta)$

THINK	WRITE
a. 1. Select the trigonometric identity to use.	a. $\cos^2(A) + \sin^2(A) = 1.$
2. Define each parametric equation as equation [1] and [2].	$x = \cos(\theta)$ [1] $y = \sin(\theta)$ [2]
3. Square equations [1] and [2] to allow the Pythagorean identities to be used, and define new equations.	$x^2 = \cos^2(\theta)$ [3] $y^2 = \sin^2(\theta)$ [4]
4. Add equations [3] and [4].	$x^2 + y^2 = \cos^2(\theta) + \sin^2(\theta)$ [3] + [4]

5. Substitute the Pythagorean identity $\sin^2(A) + \cos^2(A) = 1$. The result is the Cartesian equation of a circle, with radius 1 and centre $(0, 0)$.

$$x^2 + y^2 = 1$$

b. 1. Select the trigonometric identity to use.

b. $\cos^2(A) + \sin^2(A) = 1.$

2. Define each parametric equation as equation [1] and [2].

$$x = 3\cos(\theta) \quad [1]$$
$$y = 5\sin(\theta) \quad [2]$$

3. Begin by squaring equations [1] and [2]. Rewrite the resulting equation in terms of either $\cos^2\theta$ or $\sin^2\theta$.

$$x^2 = 9\cos^2(\theta)$$
$$\frac{x^2}{9} = \cos^2(\theta) \quad [3]$$
$$y^2 = 25\sin^2(\theta)$$
$$\frac{y^2}{25} = \sin^2(\theta) \quad [4]$$

4. Add equations [3] and [4].

$$\frac{x^2}{9} + \frac{y^2}{25} = \cos^2(\theta) + \sin^2(\theta) \quad [3] + [4]$$

5. Substitute the Pythagorean identity $\sin^2(A) + \cos^2(A) = 1$. The result is the Cartesian equation of an ellipse, with minor axis 3, major axis 5 and centre $(0, 0)$.

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

c. 1. Select the trigonometric identity to use.

c. $\sec^2(A) - \tan^2(A) = 1.$

2. Define each parametric equation as equation [1] and [2].

$$x = 2\sec(\theta) \quad [1]$$
$$y = 3\tan(\theta) \quad [2]$$

3. Square equation [1] and rearrange the result making $\sec^2\theta$ the subject.

$$x^2 = 4\sec^2(\theta) \therefore \sec^2(\theta) = \frac{x^2}{4} \quad [3]$$

4. Square equation [2] and rearrange the result making $\tan^2\theta$ the subject.

$$y^2 = 9\tan^2(\theta) \therefore \tan^2(\theta) = \frac{y^2}{9} \quad [4]$$

5. Subtract equation [4] from [3].

$$\sec^2(\theta) - \tan^2(\theta) = \frac{x^2}{4} - \frac{y^2}{9} \quad [3] - [4]$$

6. Substitute the Pythagorean identity $\sec^2(A) - \tan^2(A) = 1$. The result is the Cartesian equation of a hyperbola.

$$\sec^2(\theta) - \tan^2(\theta) = 1$$
$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

d. 1. Select the trigonometric identity to use.

d. $\cos(2A) = 1 - 2\sin^2(A).$

2. Define each parametric equation as equation [1] and [2].

$$x = \sin(\theta) \quad [1]$$
$$y = \cos(2\theta) \quad [2]$$

3. Begin by squaring equation [1].

$$x^2 = \sin^2(\theta) \quad [3]$$

4. Substitute the double angle identity into equation [2] then [3] to form the Cartesian equation.

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$
$$y = 1 - 2x^2$$

3.5.4 Parametric equations as vector functions

Recall the vector location, by definition, of a point P(x, y), is given by the position vector $\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath}$. Now consider a vector where the x- and y-components are not constants but vary dependent on a parameter, t. If we write $\underset{\sim}{r}(t)$ to represent $\overrightarrow{OP}$, according to the parameter t, then the position of any particle can be defined by a parametric equation.

Vector function

The position of any particle in two dimensional space whose location varies according to a parameter, t can be defined by

$$\underset{\sim}{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

where $x(t)$ and $y(t)$ are the respective horizontal and vertical components of the position vector at any real value of the parameter t.

Equations written in this form are defined as a **vector function**. That is, the vector location of a particle is a function of t. In many instances, the parameter t, is time.

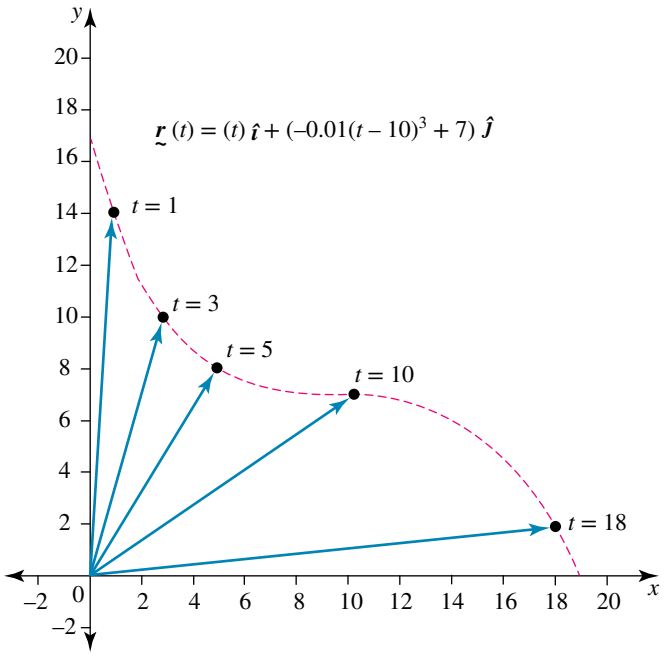
Returning to the example of the path of a particle on a water slide defined by the parametric equations

$$x = t, y = -0.01(t - 10)^3 + 7.$$

The parametric equation can then be used to define the path of the same particle as a vector function.

$$\underset{\sim}{r}(t) = (t)\hat{\imath} + \left(-0.01\,(t - 10)^3 + 7\right)\hat{\jmath}$$

A sketch of the particle's path can be depicted using vectors. For each new value of t, a new position vector, $\underset{\sim}{r}(t)$, is formed.



It is common to define the domain for a vector function. In the case above, the curve was graphed for the interval, $0 \le t \le 18.88$.

WORKED EXAMPLE 20 Using the vector function

Let a particle's position (metres), as a function of time (seconds), be given by $\underset{\sim}{r}(t) = (t)\hat{\imath} + \left(-t^2 + 6t\right)\hat{\jmath}$, for $0 \le t \le 4$.

a. Determine the particles location at $t = 1$ second.

b. Generate a table of values to model the position, $\underset{\sim}{r}(t) = x\hat{\imath} + y\hat{\jmath}$, of the particle for integer values of t.

THINK

a. Substitute $t = 1$ into the equation $\underset{\sim}{r}(t) = (t)\hat{\imath} + \left(-t^2 + 6t\right)\hat{\jmath}$ to determine the particle's position at $t = 1$.

b. The process used to determine $\underset{\sim}{r}(1)$ is repeated for all valid integer values of t for the stated domain $0 \le t \le 4$. Summarise the results in table form.

WRITE

a. For $\underset{\sim}{r}(t)$, let $t = 1$.

$$\underset{\sim}{r}(t) = (t)\hat{\imath} + \left(-t^2 + 6t\right)\hat{\jmath}$$

$$\underset{\sim}{r}(1) = (1)\hat{\imath} + \left(-(1)^2 + 6 \times 1\right)\hat{\jmath}$$

$$= \hat{\imath} + 5\hat{\jmath}$$

b.

t	$x(t)$	$y(t)$	$\underset{\sim}{r}(t)$
0	0	0	$0\hat{\imath} + 0\hat{\jmath}$
1	1	5	$\hat{\imath} + 5\hat{\jmath}$
2	2	8	$2\hat{\imath} + 8\hat{\jmath}$
3	3	9	$3\hat{\imath} + 9\hat{\jmath}$
4	4	8	$4\hat{\imath} + 8\hat{\jmath}$

WORKED EXAMPLE 21 Determining the path of a particle

The position of a particle is given by the vector function $\underset{\sim}{r}(t) = (2\sqrt{t})\hat{\imath} + (5t - 2)\hat{\jmath}$, for $t \ge 0$.

a. Express the position of the particle as a Cartesian equation.

b. Calculate the domain and range of the equation, then graph the path of the particle on a Cartesian plane.

c. At a certain point on the particles path, its location is $\underset{\sim}{r}(t) = x\hat{\imath} + 8\hat{\jmath}$. Use the vector function to calculate the value for x.

THINK

a. 1. Define each parametric equation as equation [1] and [2].

2. The parameter t needs to be eliminated. Solve for t in the equation [1].

WRITE

$$x = 2\sqrt{t} \quad [1]$$
$$y = 5t - 2 \quad [2]$$

$$x = 2\sqrt{t}$$
$$\frac{x}{2} = \sqrt{t}$$
$$t = \left(\frac{x}{2}\right)^2$$
$$t = \frac{x^2}{4} \quad [3]$$

3. Substitute [3] into [2]. Simplify and state the Cartesian equation.

Substitute [3] into [2]

$$t = \frac{x^2}{4} \qquad [3]$$

$$y = 5t - 2 \qquad [2]$$

$$y = 5\left(\frac{x^2}{4}\right) - 2$$

$$y = \frac{5x^2}{4} - 2$$

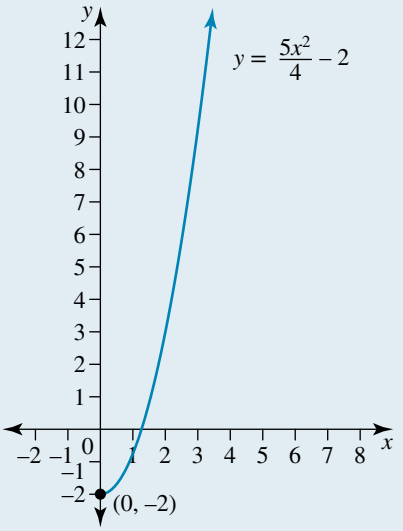
b. 1. As $t \geq 0$ calculate the domain and range by substituting $t = 0$ into $x = 2\sqrt{t}$ and $y = 5t - 2$.

b. When $t = 0$

$$x = 2\sqrt{t}$$
$$= 2\sqrt{0}$$
$$= 0$$
$$y = 5t - 2$$
$$= 5 \times 0 - 2$$
$$= -2$$

As $t \geq 0$, it follows from the parametric equations that $x \geq 0$ and $y \geq -2$.

2. Construct the graph of the Cartesian equation using suitable technology.



c. 1. Equate the two position vectors.

c. $2\sqrt{t}\,\hat{\imath} + (5t - 2)\hat{\jmath} = x\hat{\imath} + 8\hat{\jmath}$

2. Equate the y-component of $r(t) = x\hat{\imath} + 8\hat{\jmath}$ with the known parametric equation $y = (5t - 2)\hat{\jmath}$

$(5t - 2)\hat{\jmath} = 8\hat{\jmath}$

3. Solve for t.

$$5t - 2 = 8$$
$$t = \frac{8 + 2}{5}$$
$$t = 2$$

4. Determine the value of x by evaluating the parametric equation for when $t = 2$

$x\hat{\imath}$ can be calculated by substituting $t = 2$ in $x = 2\sqrt{t}$

$$x = 2\sqrt{2}$$

5. State the answer.

$$x = 2\sqrt{2}$$

Parametric equations as vector functions in three dimensions

The use of parameters can be extended to three dimensional spaces. The vector function, $\underset{\sim}{r}(t)$, would contain the parametric equations,

$$x = f(t),\ y = g(t),\ z = h(t)$$

Vector functions in three dimensional space

The position of any particle in three dimensional space whose location varies according to a parameter, t, would consist of three components

$$\underset{\sim}{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

where $x(t)$, $y(t)$, $z(t)$ are the respective components of the position vector at any real value of the parameter t.

The procedures studied in the two dimensional examples, are applicable in three dimensions, as shown in the following worked example.

WORKED EXAMPLE 22 Using parametric equation as vector functions in three dimensions

The position (metres) of a bird is given by $\underset{\sim}{r}(t) = (2t)\hat{\imath} + \left(0.01t^3\right)\hat{\jmath} + \left(0.5t^2\right)\hat{k}$ for $t \geq 0$ seconds.

a. Determine the position of the bird at $t = 0$ and $t = 15$ seconds.

b. Calculate the straight-line distance of the bird from the origin at $t = 15$ seconds, to 2 decimal places.

THINK	WRITE
a. 1. Substitute $t = 0$ into the equation $\underset{\sim}{r}(t) = (2t)\hat{\imath} + \left(0.01t^3\right)\hat{\jmath} + \left(0.5t^2\right)\hat{k}$ to determine the bird's position at $t = 0$.	**a.** Let $t = 0$. $\underset{\sim}{r}(0) = (2 \times 0)\hat{\imath} + \left(0.01\,(0)^3\right)\hat{\jmath} + \left(0.5\,(0)^2\right)\hat{k}$ $= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$ Hence, at $t = 0$ the bird's position is the origin.
2. Substitute $t = 15$ into the equation $\underset{\sim}{r}(t) = (2t)\hat{\imath} + \left(0.01t^3\right)\hat{\jmath} + \left(0.5t^2\right)\hat{k}$ to determine the bird's position at $t = 15$.	Let $t = 15$. $\underset{\sim}{r}(15) = (2 \times 15)\hat{\imath} + \left(0.01\,(15)^3\right)\hat{\jmath} + \left(0.5\,(15)^2\right)\hat{k}$ $= 30\hat{\imath} + 33.75\hat{\jmath} + 112.5\hat{k}$ Hence at $t = 15$, the bird's position is $30\hat{\imath} + 33.75\hat{\jmath} + 112.5\hat{k}$.
b. 1. The distance of the bird from the origin at $t = 15$ is the magnitude of the position vector at $t = 15$. Recall the magnitude formula for three dimensional vectors $\lvert\underset{\sim}{u}\rvert = \sqrt{x^2 + y^2 + z^2}$.	**b.** $\lvert\underset{\sim}{r}(15)\rvert = \sqrt{x^2 + y^2 + z^2}$ $= \sqrt{(30)^2 + (33.75)^2 + (112.5)^2}$ $= 121.22$
2. Conclude with a statement.	After 15 seconds of flight the bird is 121.22 metres from the origin.

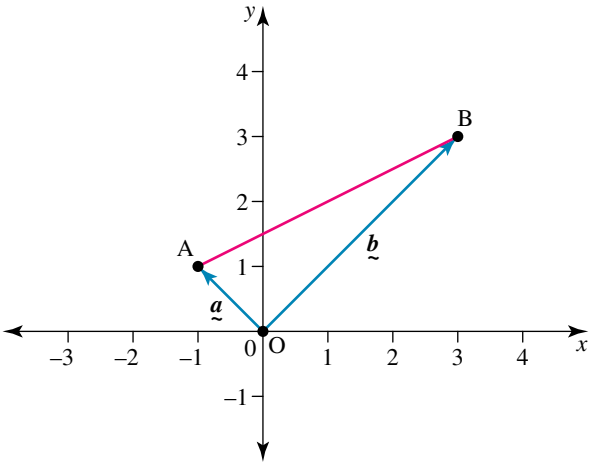
3.5.5 The vector equation of a straight line

In your studies thus far, the equation of a line has been considered only in terms of a scalar equation in the form $y = mx + c$. Equations in this form describe the straight-line relationship existing between two variables (x, y). We will now explore how position vectors can be used to define a straight line in both two and three dimensions.

Lines in two dimensions

Consider the two dimensional plane containing the points A$(-1, 1)$ and B$(3, 3)$. The location of these points can be expressed as position vectors:

- $\overrightarrow{OA} = \underset{\sim}{a} = -1\hat{\imath} + 1\hat{\jmath}$
- $\overrightarrow{OB} = \underset{\sim}{b} = 3\hat{\imath} + 3\hat{\jmath}$

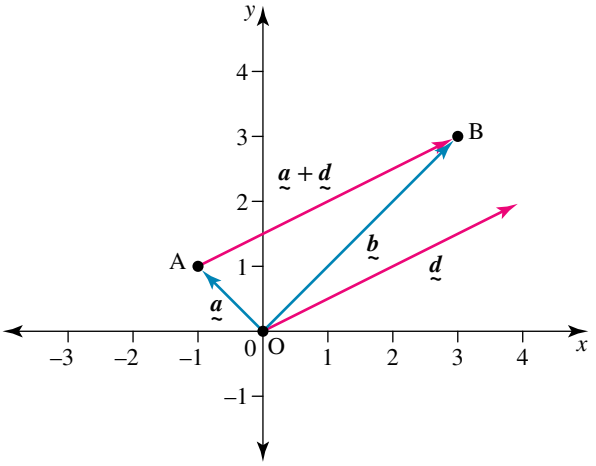


Similarly, the location of the vector connecting the two points is defined as

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{d} = 4\hat{\imath} + 2\hat{\jmath}$$

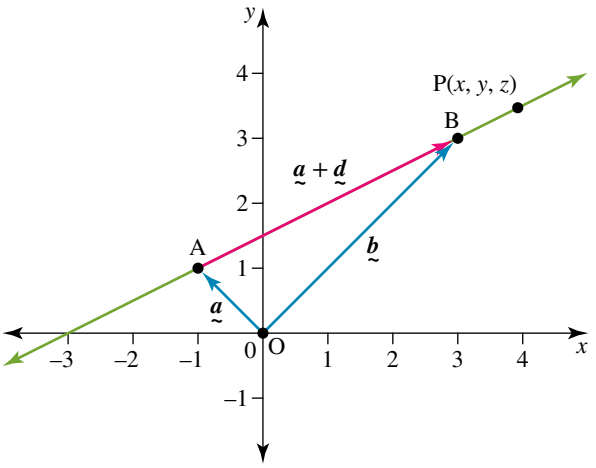
The **line segment** joining points A and B can now be described with the two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{d}$. Note that $\underset{\sim}{a}$ gets us onto the line AB, while $\underset{\sim}{d}$ let us slide along the line.

- The 'starting point' is the head of position vector $\underset{\sim}{a} = -1\hat{\imath} + 1\hat{\jmath}$
- The line segment is formed by travelling (adding) in the direction of $\underset{\sim}{d} = 4\hat{\imath} + 2\hat{\jmath}$.



Vectors can also be used to define a line in both two and three dimensions. In this example a line can be formed by extending the line segment connecting A and B.

The position of any point P(x, y, z) on this line can be calculated by adding scalar multiples of $\underset{\sim}{d}$ to the initial position vector $\underset{\sim}{a}$. Once again, $\underset{\sim}{a}$ gets us onto the line AB, while $\underset{\sim}{d}$ let us slide along the line.



The vector equation of a straight line in two dimensions

The **vector equation** of a line is given by $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$.

Where:

- $\underset{\sim}{r}$ is the position vector of any point on the line
- $\underset{\sim}{a}$ is the position vector of a known point on the line
- $\underset{\sim}{d}$ is the vector travelling in the same direction (parallel) to the line
- the varying (scalar) parameter is t, where $t \in \mathbb{R}$.

In changing the parameter t, the vector $\underset{\sim}{r}$ can have its head at any point on the line. Consider again the two dimensional plane containing the points A(−1, 1) and B(3, 3).

- The line between points A and B is defined by the vector equation, $\underset{\sim}{r} = (-1\hat{\imath} + 1\hat{\jmath}) + t(4\hat{\imath} + 2\hat{\jmath})$.
- Two points on the line will be located by varying the parameter t.

Scenario 1:
$t = 0$

$$\begin{aligned}\underset{\sim}{r} &= (-1\hat{\imath} + 1\hat{\jmath}) + 0(4\hat{\imath} + 2\hat{\jmath}) \\ &= -\hat{\imath} + 1\hat{\jmath}\end{aligned}$$

$$\begin{aligned}P_1(x, y) &= (-1, 1) \\ &= A\end{aligned}$$

Scenario 2:
$t = 1$

$$\begin{aligned}\underset{\sim}{r} &= (-1\hat{\imath} + 1\hat{\jmath}) + 1(4\hat{\imath} + 2\hat{\jmath}) \\ &= 3\hat{\imath} + 3\hat{\jmath}\end{aligned}$$

$$\begin{aligned}P_2(x, y) &= (3, 3) \\ &= B\end{aligned}$$

Scenario 3:
$t = 1.3$

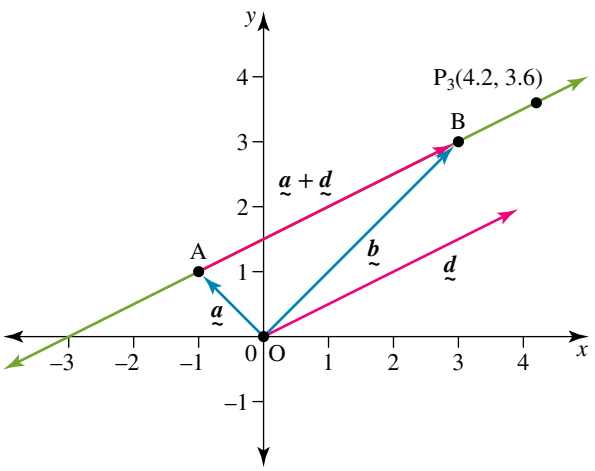
$$\begin{aligned}\underset{\sim}{r} &= (-1\hat{\imath} + 1\hat{\jmath}) + 1.3(4\hat{\imath} + 2\hat{\jmath}) \\ &= -1\hat{\imath} + 1\hat{\jmath} + 5.2\hat{\imath} + 2.6\hat{\jmath} \\ &= 4.2\hat{\imath} + 3.6\hat{\jmath}\end{aligned}$$

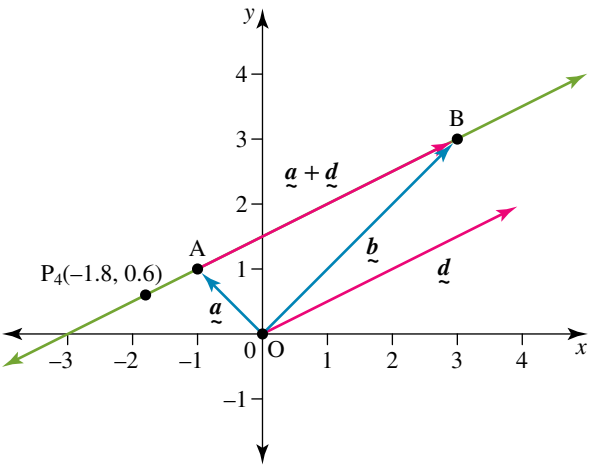
$$\therefore\ P_3(x, y) = (4.2, 3.6)$$

Scenario 4:
$t = -0.2$

$$\begin{aligned}\underset{\sim}{r} &= (-1\hat{\imath} + 1\hat{\jmath}) - 0.2(4\hat{\imath} + 2\hat{\jmath}) \\ &= -1\hat{\imath} + 1\hat{\jmath} - 0.8\hat{\imath} - 0.4\hat{\jmath} \\ &= -1.8\hat{\imath} + 0.6\hat{\jmath}\end{aligned}$$

$$\therefore\ P_4(x, y) = (-1.8, 0.6)$$





Notice that when $t = 0$, the point lands on A, when $t = 1$ it lands on B, when $t > 1$, it lands on the line to the right of B, and when $t < 0$, it lands on the line, to the left of A. For $0 < t < 1$, the point would land on the line, between A and B.

When $t \neq 0$, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AP}$ are collinear. (Recall that two vectors u and v are collinear if $\exists k \in R \setminus \{0\}$ such that $u = kv$.)

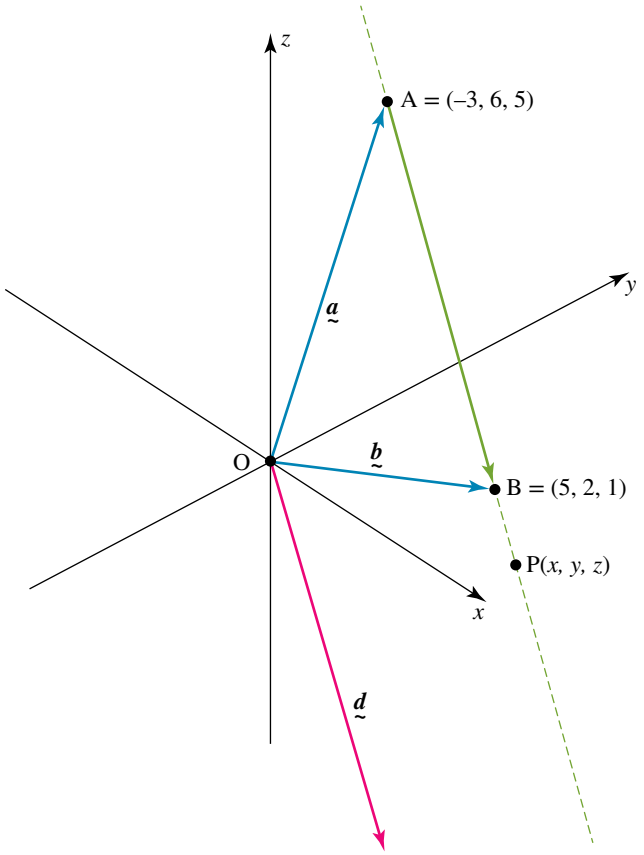
WORKED EXAMPLE 23 Determining a vector equation in two dimensions

a. Determine a vector equation, in the two dimensional space (x, y), for the line passing through the points A$(-1, 7)$ and B$(4, -2)$.

b. Calculate the vector location of the point $\underset{\sim}{r}$ on the line if the parameter $t = 2$.

THINK	WRITE
a. 1. Define the position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	**a.** $\overrightarrow{OA} = \underset{\sim}{a} = -1\hat{\imath} + 7\hat{\jmath}$ $\overrightarrow{OB} = \underset{\sim}{b} = 4\hat{\imath} - 2\hat{\jmath}$
2. Calculate the direction vector, $\underset{\sim}{d}$, between points A and B. This vector is parallel to the line.	$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a}$ $\underset{\sim}{d} = (4\hat{\imath} - 2\hat{\jmath}) - (-1\hat{\imath} + 7\hat{\jmath})$ $\underset{\sim}{d} = 5\hat{\imath} - 9\hat{\jmath}$
3. Recall the formula for the vector equation of a line and substitute in the expressions for $\underset{\sim}{d}$ and $\underset{\sim}{a}$. The vector equation of the line has been determined.	$\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$ $\underset{\sim}{r} = (-\hat{\imath} + 7\hat{\jmath}) + t(5\hat{\imath} - 9\hat{\jmath})$
b. The location of $\underset{\sim}{r}$ can be calculated by substituting $t = 2$ into the vector equation. The solution is expressed as a position vector.	**b.** Let $t = 2$ $\underset{\sim}{r} = (-\hat{\imath} + 7\hat{\jmath}) + 2(5\hat{\imath} - 9\hat{\jmath})$ $= (-\hat{\imath} + 7\hat{\jmath}) + (10\hat{\imath} - 18\hat{\jmath})$ $= 9\hat{\imath} - 11\hat{\jmath}$

Lines in three dimensions



Vector equations can be developed in three dimensional space providing we have:

- the position vector $\underset{\sim}{a} = \left(x_1\hat{\imath}, y_1\ \hat{\jmath}, z_1\hat{k}\right)$ that corresponds to a point on the line and,
- a second vector $\underset{\sim}{u} = \left(x_2\hat{\imath}, y_2\ \hat{\jmath}, z_2\hat{k}\right)$ that is a vector in the direction of the line.

  Alternatively,
- the equation of a line can be defined if two points (A and B) on the same line are known, as shown in the diagram.

As for two dimensional space, the vector equation $\underset{\sim}{r} = \underset{\sim}{a} + t\,\underset{\sim}{d}$ will define the location of any point P (x, y, z) in a three dimensional space, in most instances.

Consider any point P(x, y, z) which lies along a line and can be located in terms of $\underset{\sim}{a}$ and $\underset{\sim}{d}$, where:

- $\underset{\sim}{d}$ is the position vector between the points A and B.
- The line containing points A and B and runs in the same direction as $\underset{\sim}{d}$.
- The position of any point P(x, y, z) can be determined by starting at $\underset{\sim}{a}$ and then travelling along the line in scalar (t) products of $\underset{\sim}{d}$.
- Hence, the formula is given by $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$.

WORKED EXAMPLE 24 Using lines in three dimensions

a. Determine a vector equation, in the three dimensional space (x, y, z), for the line passing through the points A$(-3, 6, 5)$ and B$(5, 2, 1)$.

b. b. Verify if the point C$(65, -28, -29)$ exists on the line.

c. The position vector $\underset{\sim}{e} = -27\hat{\imath} + 18\hat{\jmath} + z\hat{k}$ terminates on the line formed between points A and B. Calculate the value of z.

THINK	WRITE
a. 1. Define the position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	**a.** $\overrightarrow{OA} = \underset{\sim}{a} = -3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}$ $\overrightarrow{OB} = \underset{\sim}{b} = 5\hat{\imath} + 2\hat{\jmath} + 1\hat{k}$
2. Calculate the direction vector, $\underset{\sim}{d}$, between points A and B. This vector is parallel to the line.	$\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{a}$ $\underset{\sim}{d} = \left(5\hat{\imath} + 2\hat{\jmath} + 1\hat{k}\right) - \left(-3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}\right)$ $\underset{\sim}{d} = 8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$
3. Recall the formula for the vector equation of a line and substitute in the expressions for $\underset{\sim}{d}$ and $\underset{\sim}{a}$. The vector equation of the line has been determined.	$\underset{\sim}{r} = \underset{\sim}{a} + t\,\underset{\sim}{d}$ $\underset{\sim}{r} = \left(-3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}\right) + t\left(8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}\right)$
b. 1. In most instances it is possible to verify a solution by substituting values into an equation. As t is unknown this is not possible. The first step is to equate one of the components $\underset{\sim}{r}\left(\hat{\imath}, \hat{\jmath}, \hat{k}\right)$ with the point C$(65, -28, -29)$.	**b.** $\underset{\sim}{r} = \left(-3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}\right) + t\left(8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}\right)$ C$(65, -28, -29)$ Equate the $\hat{\imath}$ components to calculate t. $65 = -3 + 8t$
2. Solve for t.	$65 = -3 + 8t$ $\dfrac{65 + 3}{8} = t$ $t = 8.5$
3. Substitute $t = 8.5$ into the vector equation $\underset{\sim}{r}$ to determine if the point C$(65, -28, -29)$ lies on the line.	$\underset{\sim}{r}(8.5) = \left(-3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}\right) + 8.5\left(8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}\right)$ $= \left(-3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}\right) + \left(68\hat{\imath} - 34\hat{\jmath} - 34\hat{k}\right)$ $= 65\hat{\imath} - 28\hat{\jmath} - 29\hat{k}$ $\therefore$ C$(65, -28, -29)$ does lie on the line.

c. 1. The vector equation of the line was developed in part **a**. Determine the value of t by equating either the $\hat{\boldsymbol{\imath}}$ or $\hat{\boldsymbol{\jmath}}$ components from the vectors $\underset{\sim}{e}$ and $\underset{\sim}{r}$.	**c.** $\underset{\sim}{r} = \left(-3\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) + t\left(8\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}\right)$ $\underset{\sim}{e} = -27\hat{\boldsymbol{\imath}} + 18\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ Equate the $\hat{\boldsymbol{\imath}}$ components to calculate t. $-27 = -3 + 8t$ $\dfrac{-27+3}{-8} = t$ $t = -3$
2. Substitute $t = -3$ into the vector equation $\underset{\sim}{r} = \left(-3\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) + t\left(8\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}\right)$ to determine the position vector $\underset{\sim}{e}$.	Let $t = -3$ $\underset{\sim}{r} = \left(-3\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) + -3\left(8\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}\right)$ $= -27\hat{\boldsymbol{\imath}} + 18\hat{\boldsymbol{\jmath}} + 17\hat{\boldsymbol{k}}$ Hence the position vector $\underset{\sim}{e}$ is $\underset{\sim}{e} = -27\hat{\boldsymbol{\imath}} + 18\hat{\boldsymbol{\jmath}} + 17\hat{\boldsymbol{k}}$
3. Equate the vectors $\underset{\sim}{r}$ and $\underset{\sim}{e}$.	$-27\hat{\boldsymbol{\imath}} + 18\hat{\boldsymbol{\jmath}} + 17\hat{\boldsymbol{k}} = -27\hat{\boldsymbol{\imath}} + 18\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$
4. Solve for z by equating the $\hat{\boldsymbol{k}}$ components.	$z\hat{\boldsymbol{k}} = 17\hat{\boldsymbol{k}}$ $z = 17$

Vector equations in parametric and Cartesian form

We have seen how lines can be expressed as vector equations in the form $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$. These equations may also be expressed in either parametric or Cartesian form. To demonstrate this, the vector equation from Worked example 24 will be used.

Parametric form

- The vector equation was $\underset{\sim}{r} = \left(-3\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) + t\left(8\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}\right)$
- Points on the line are in the form $P(x, y, z)$
- Therefore, any point on the line is given by, $(x, y, z) = (-3 + 8t,\ 6 - 4t,\ 5 - 4t)$
  The vector equation can be rewritten in parametric form by equating the components:

$$x = -3 + 8t,\ y = 6 - 4t,\ z = 5 - 4t$$

Parametric equation of a line in three dimensional space

The parametric equation of a line in three dimensional space is given by the general formula

$$x = a_1 + td_1, \qquad y = a_2 + td_2, \qquad z = a_3 + td_3$$

Cartesian form

The position of the point $P(x, y, z)$ is defined by the parameter t. Consequently, each of the equations that exist in the parametric form of a line, can be rewritten in terms of t.

$x = -3 + 8t$	$y = 6 - 4t$	$z = 5 - 4t$
$\dfrac{x+3}{8} = t$	$\dfrac{y-6}{-4} = t$	$\dfrac{z-5}{-4} = t$
$t = \dfrac{x+3}{8}$	$t = \dfrac{y-6}{-4}$	$t = \dfrac{z-5}{-4}$

The line connecting points A and B can now be written in Cartesian form as the value for k is equivalent in the three equations above. The Cartesian equation is given by,

$$\frac{x+3}{8} = \frac{y-6}{-4} = \frac{z-5}{-4} = t$$
$$\frac{x+3}{8} = \frac{y-6}{-4} = \frac{z-5}{-4}$$

The Cartesian equation of a line is likewise referred to as the **symmetric equation of a line**.

Cartesian equation of a line in three dimensional space

The Cartesian equation of line in three dimensional space is given by the general formula

$$\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$$

In this form, the line passes through the point $(x, y, z) = (a_1, a_2, a_3)$, and is parallel to the vector $\underset{\sim}{d} = d_1\hat{\imath} + d_2\hat{\jmath} + d_3\hat{k}$.

The relationship between the three forms of a line in vector space is summarised below using the vector equation $\underset{\sim}{r} = (-3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}) + t(8\hat{\imath} - 4\hat{\jmath} - 4\hat{k})$ as an example.

Vector equation	Parametric form	Cartesian equation
$\underset{\sim}{r} = \left(-3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}\right) + t\left(8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}\right)$ $\underset{\sim}{a} = -3\hat{\imath} + 6\hat{\jmath} + 5\hat{k}$ $\underset{\sim}{d} = 8\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$	$x = -3 + 8t$ $y = 6 - 4t$ $z = 5 - 4t$ $(a_1, a_2, a_3) = (-3, 6, 5)$ $(d_1, d_2, d_3) = (8, -4, -4)$	$\frac{x+3}{8} = \frac{y-6}{-4} = \frac{z-5}{-4}$ $(a_1, a_2, a_3) = (-3, 6, 5)$ $(d_1, d_2, d_3) = (8, -4, -4)$

WORKED EXAMPLE 25 Using different forms for lines in three dimensions

The vector equation of a line is given by $\underset{\sim}{r} = (8\hat{\imath} + 2\hat{\jmath} - 3\hat{k}) + t(3\hat{\imath} - 4\hat{\jmath} + 2\hat{k})$.

a. Convert the vector equation into parametric form.

b. Express the parametric form of the line as a Cartesian equation.

THINK	WRITE
a. 1. Write an equation linking the general location of the point P(x, y, z) to the components of the vector equation.	a. $(x, y, z) = (8 + 3t,\ 2 - 4t, -3 + 2t)$
2. Equate the components to write three equations representing each of the three dimensions.	$x = 8 + 3t$ $y = 2 - 4t$ $z = -3 + 2t$
b. 1. Rewrite the equation for the x-component in terms of t.	b. $x = 8 + 3t$ $t = \frac{x-8}{3}$
2. Rewrite the equation for the y-component in terms of t.	$y = 2 - 4t$ $t = \frac{y-2}{-4}$

3. Rewrite the equation for the z-component in terms of t.

$$z = -3 + 2t$$
$$t = \frac{z+3}{2}$$

4. Equate the three equations as they are equivalent values of t.

$$\frac{x-8}{3} = \frac{y-2}{-4} = \frac{z+3}{2} = t$$

5. Express the final answer as a Cartesian equation in terms of only x, y and z.

$$\frac{x-8}{3} = \frac{y-2}{-4} = \frac{z+3}{2}$$

Exercise 3.5 Vector, parametric and Cartesian equations

learnon

3.5 Exercise | **3.5 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	16, 17, 18, 19	20, 21, 22, 23

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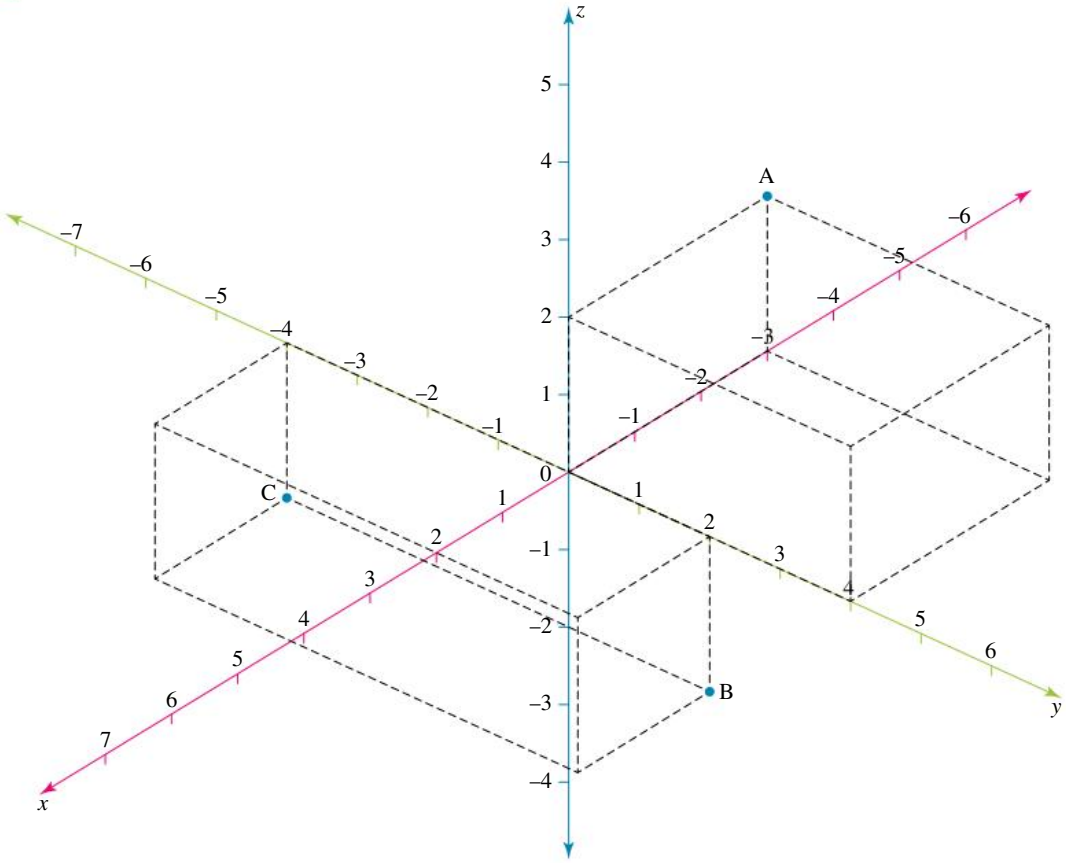
Simple familiar

1. WE16 Refer to the three dimensional space below to state the coordinates of the points A, B and C.



2. Determine the equivalent Cartesian equation (y in terms of x) for the following vector functions.

a. $\underset{\sim}{r}(t) = 5t\ \hat{\imath} + \frac{t}{3}\hat{\jmath}$

b. $\underset{\sim}{r}(t) = 4t\ \hat{\imath} + \left(t^2 - t\right)\hat{\jmath}$

c. $\underset{\sim}{r}(t) = 3t\ \hat{\imath} + \left(t^3 + 2\right)\hat{\jmath}$

d. $\underset{\sim}{r}(t) = (2\cos(t))\hat{\imath} + (2\sin(t))\hat{\jmath}$

3. For each of the following vector equations, use technology to sketch the equation of the path.

a. The Spiral: $\underset{\sim}{r}(t) = t\cos(t)\hat{\imath} + t\sin(t)\hat{\jmath}$, for $0 \le t \le 6\pi$

b. The Cycloid: $\underset{\sim}{r}(t) = 4(t - \sin(t))\hat{\imath} + 4(1 - \cos(t))\hat{\jmath}$, for $-2\pi \le t \le 4\pi$

c. The Witch of Agnesi: $\underset{\sim}{r}(t) = 2\cot(t)\hat{\imath} + 2\sin^2(t)\hat{\jmath}$, for $-\pi \le t \le \pi$

d. Lissajous curve: $\underset{\sim}{r}(t) = 3\cos(t)\hat{\imath} + 2\sin(2t)\hat{\jmath}$, for $-\pi \le t \le \pi$

4. **WE17a** Determine the Cartesian equation for the sphere with centre $C(x_c, y_c, z_c)$ and radius r.

a. $r = 6$ and $C(2, 8, -3)$

b. $r = 3$ and $C(-3, 1, -9)$

c. $r = 7$ and $C(1, 0, 6)$

d. $r = \sqrt{11}$ and $C(-9, -2, 7)$

5. **WE17b** Determine the Cartesian equation for the sphere that has $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as the endpoints of its diameter.

a. $A(7, 13, 8)$ and $B(-1, 3, 2)$

b. $A(-5, -3, -1)$ and $B(3, 1, 11)$

c. $A(2, -4, 5)$ and $B(-8, 0, -25)$

d. $A(1, -4, 8)$ and $B(9, -8, -2)$

6. Express the following equations in the expanded form of a sphere, $x^2 + y^2 + z^2 + ax + by + cz + d = 0$.

a. $(x-5)^2 + (y-7)^2 + (z-1)^2 = 144$

b. $(x+1)^2 + (y-3)^2 + (z+2)^2 = 81$

c. $(x+3)^2 + (y-1)^2 + z^2 = 15$

d. $(x-1)^2 + (y-4)^2 + (z+3)^2 = 56$

7. **WE17c** Each of the following equations have been expressed in the expanded form of a sphere. Determine the centre and radius of the following spheres.

a. $x^2 - 10x + y^2 + 4y + z^2 + 12z - 16 = 0$

b. $x^2 + y^2 + z^2 + 2x + 8y - 16z = 0$

c. $x^2 + y^2 + z^2 + 20x - 12y + 24z + 180 = 0$

d. $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$

8. **WE18** Determine the equivalent Cartesian equation for the following parametric equations.

a. $x = 3t + 2, y = 2t + 5$

b. $x = t - 5, y = -t^2 + t$

c. $x = 2t - 1, y = 4t^2 + 7$

d. $x = t^3, y = \frac{t^2}{4}$

e. $x = 4 + \frac{1}{t}, y = 4 - t$

f. $x = \frac{2}{\sqrt{t+1}}, y = \frac{t}{t+1}$

9. **WE19** Calculate the Cartesian equation for the following parametric equations.

a. $x = \sin(\theta), y = \cos(\theta)$

b. $x = \cos(3\theta), y = \sin(3\theta)$

c. $x = 4\cos(\theta), y = 7\sin(\theta)$

d. $x = \operatorname{cosec}(\theta), y = \cot(\theta)$

10. **WE20** A particle's position (metres), as a function of time (seconds), is given by $\underset{\sim}{r}(t) = (t)\ \hat{\imath} + \left(-0.5t^2 + 4t\right)\hat{\jmath}$, for $0 \le t \le 4$.

a. Determine the particle's location at $t = 1$ second.

b. Generate a table of integer values to model the position, $\underset{\sim}{r}(t) = x\hat{\imath} + y\hat{\jmath}$, of the particle for integer values of t.

11. Calculate the three dimensional vector equation for the line passing through each of the following pairs of points (x, y, z).

a. $A(-2, 2, 1)$ and $B(0, 6, 7)$

b. $A(4, -1, 11)$ and $B(-13, 9, -12)$

c. $A(5, 4, 11)$ and $B(-1, 1, 8)$

d. $A(6, -1, 2)$ and $B(5, 3, -9)$

e. $A(-3, -2, 1)$ and $B(5, 7, -13)$

f. $A(1, 5, 8)$ and $B(7, 0, -1)$

12. WE24 a. Determine a vector equation, in the three dimensional space (x, y, z), for the line passing through the points A$(-4, 7, 6)$ and B$(6, 3, 2)$.

b. Verify if the point C$(96, -33, -34)$ exists on the line.

c. The position vector $\underset{\sim}{e} = -34\hat{\imath} + 19\hat{\jmath} + z\hat{k}$ terminates on the line formed between points A and B. Calculate the value of z.

13. WE23 a. Determine a vector equation, in the two dimensional space (x, y), for the line passing through the points A$(-2, 8)$ and B$(5, -3)$.

b. Calculate the vector location of the point $\underset{\sim}{r}$ on the line if the parameter $t = 3$.

14. Determine the vector equation of the line through the point P$(6, 5, 1)$ that is parallel to the vector $\underset{\sim}{u} = 6\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

15. WE25a The vector equation of a line is given by $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$. Convert the following vector equations into parametric form.

a. $\underset{\sim}{r} = (-\hat{\imath} + 3\hat{\jmath} + 7\hat{k}) + t(2\hat{\imath} + 5\hat{\jmath} - 3\hat{k})$

b. $\underset{\sim}{r} = (-7\hat{\imath} + 8\hat{\jmath} + 3\hat{k}) + t(19\hat{\imath} + 11\hat{\jmath} + k)$

c. $\underset{\sim}{r} = (5\hat{\imath} + \hat{\jmath} - 3\hat{k}) + t(2\hat{\imath} + 2\hat{\jmath} - 6\hat{k})$

Complex familiar

16. WE21 The position of a particle is given by the vector function $\underset{\sim}{r}(t) = \left(\sqrt[3]{\frac{t}{4}}\right)\hat{\imath} + (2t + 7)\hat{\jmath}$, for $t \geq 0$.

a. Express the position of the particle as a Cartesian equation.

b. Calculate the domain and range of the equation, then graph the path of the particle on a Cartesian plane.

c. At a certain point on the particle's path, its location is $\underset{\sim}{r}(t) = x\hat{\imath} + 71\hat{\jmath}$. Use the vector function to calculate the value for x.

17. WE22 The position (metres) of a bird is given by $\underset{\sim}{r}(t) = (3t)\hat{\imath} + \left(0.02t^3\right)\hat{\jmath} + \left(0.4t^2\right)\hat{k}$ for $t \geq 0$ seconds.

a. Determine the position of the bird at $t = 0$ and $t = 12$ seconds.

b. Calculate the straight-line distance of the bird from the origin at $t = 12$ seconds.

18. WE25b Express the following parametric equations of lines as Cartesian equations.

a. $x = 1 + 5t, y = 2 + 5t, z = 3 + 5t$

b. $x = -2 + 6t, y = 4 - 9t, z = 8 - 2t$

c. $x = 11 + 12t, y = 3 + 21t, z = 12 + 13t$

19. Convert the following Cartesian equations into the form $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$.

a. $\dfrac{x-2}{3} = \dfrac{y-5}{2} = \dfrac{z+8}{5}$

b. $\dfrac{x+7}{-2} = \dfrac{y+1}{6} = \dfrac{z-2}{3}$

Complex unfamiliar

20. The path of particle is defined by the vector function, $\underset{\sim}{r}(t) = \dfrac{3t}{1+t^3}\hat{\imath} + \dfrac{3t^2}{1+t^3}\hat{\jmath}$, for $-10 \leq t \leq 10$.

Show that the Cartesian equation is given by $x^3 + y^3 - 3xy = 0$.

21. A paragliding enthusiast begins their flight from a cliff face. The path of the paraglider is given by $\underset{\sim}{r}(t) = (850 - 15t)\hat{\boldsymbol{\imath}} + \left(950 - 0.4t^2\right)\hat{\boldsymbol{\jmath}} + \left(350 - 0.01t^3\right)\hat{\boldsymbol{k}}$ for $0 \le t \le b$. The pilot aims to finish the flight somewhere near the airport, defined as the point O(0, 0, 0).

a. Calculate the initial position of the paraglider.

b. Determine the domain by calculating the largest possible value for b. State any assumptions made in developing a solution.

c. Determine the ground level distance the pilot is from the airport when the paraglider lands.

d. Analyse the flight path of the paraglider and determine if it at any point it ascended in altitude.

22. Two lines are formed between the points D, E, F and G. The first line joins D(−1, 2, 1) and E(3, −3, 5). The second line passes through F(2, −1 , 4) and G(12, −2, 9). Calculate the acute angle between the two lines.

23. Determine the point at which the following lines intersect the specified plane.

	Cartesian equation	Plane
a.	$\dfrac{x-2}{3} = \dfrac{y-5}{2} = \dfrac{z+8}{5}$	xy-plane
b.	$\dfrac{x+7}{-2} = \dfrac{y+1}{6} = \dfrac{z-2}{3}$	xz-plane

Fully worked solutions for this chapter are available online.

LESSON
3.6 The vector (cross) product

SYLLABUS LINKS

- Define and use the vector (cross) product to determine a vector normal to a given plane, with and without technology.
  - $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin(\theta)\,\hat{\boldsymbol{n}}$
  - $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$

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3.6.1 The vector product in polar form

Up until this point, the product of two vectors has only been calculated as a scalar quantity. The dot product (or scalar or inner product) for two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined by:

- $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\ \ |\underset{\sim}{b}| \cos(\theta)$: when the magnitudes of, and angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are known.
- $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$: when $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are expressed in component form.

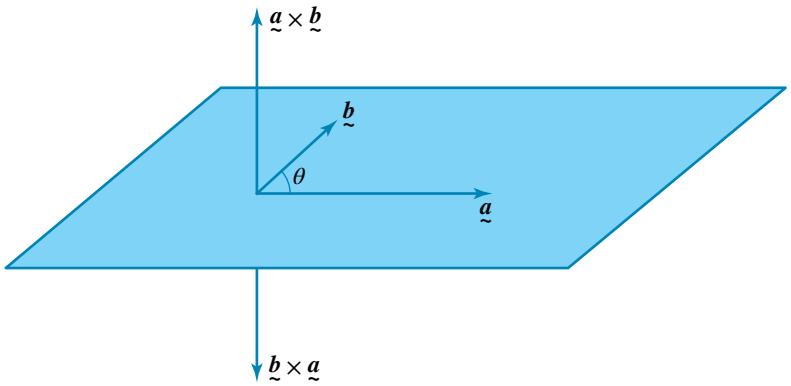
Applications of the dot product include:

- determining if vectors are perpendicular
- calculating the angle between two vectors
- calculating projections and components of vectors.

Is it possible to multiply two vectors so that their product isn't restricted to a scalar quantity?

The vector product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, unlike the scalar product, is a vector. For this reason, it is called the **vector product**, and has applications in fields as diverse as geology, physics, mechanics and engineering. Furthermore, it is extremely useful in describing relationships between lines and planes in three dimensions.

We have seen how two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, can be used to calculate and define the vector equation of a line between two points. The same two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, provided that they are not parallel, define a **plane**. A plane is a flat surface extending indefinitely in all directions. When two vectors exist on the same plane, we say that they are **coplanar**.



The orientation of the plane can be described by a vector perpendicular (orthogonal) to the plane defined by vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. This perpendicular vector is called a normal vector, (or simply the **normal**) and is the cross product of the two vectors which define the plane the direction of $\underset{\sim}{a} \times \underset{\sim}{b}$ is 'up' (above) the plane, while the direction of $\underset{\sim}{b} \times \underset{\sim}{a}$ is 'down' (below) the plane.

Vector (cross) product

The vector product is defined as

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\hat{\underset{\sim}{n}}$$

where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\hat{\underset{\sim}{n}}$ is the unit vector perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the right-hand sense.

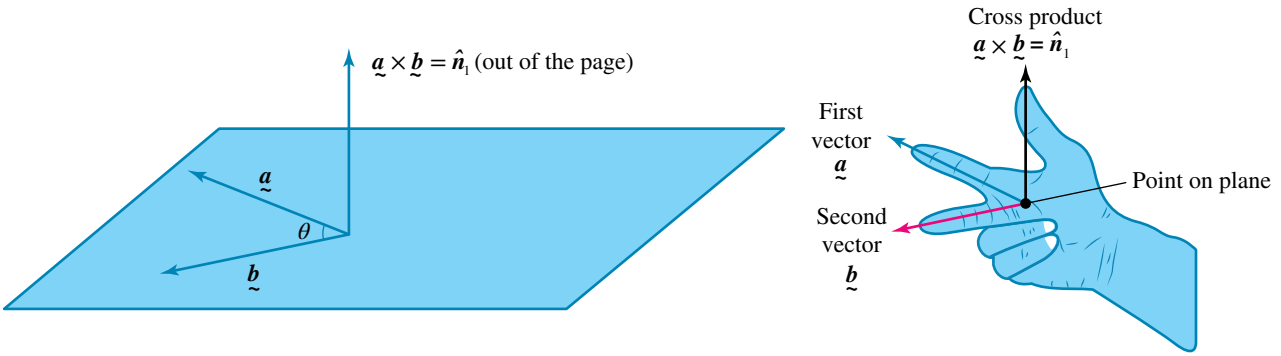
Note: Vector product is commonly called **cross product**.

Recall the right-handed coordinates system seen in Lesson 3.2. The vector product produces a vector perpendicular to the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, in a direction that is either out of the page or into the page. The order of the vectors being cross multiplied matter: the scalar (dot) product is commutative, the vector(cross) product is not. This is explained below. You will have to orientate your right hand correctly. This is how we distinguish between a normal vector pointing up (above) the plane or pointing down (below) the plane.

Normal vector $\hat{\underset{\sim}{n}}_1$ — out of the page

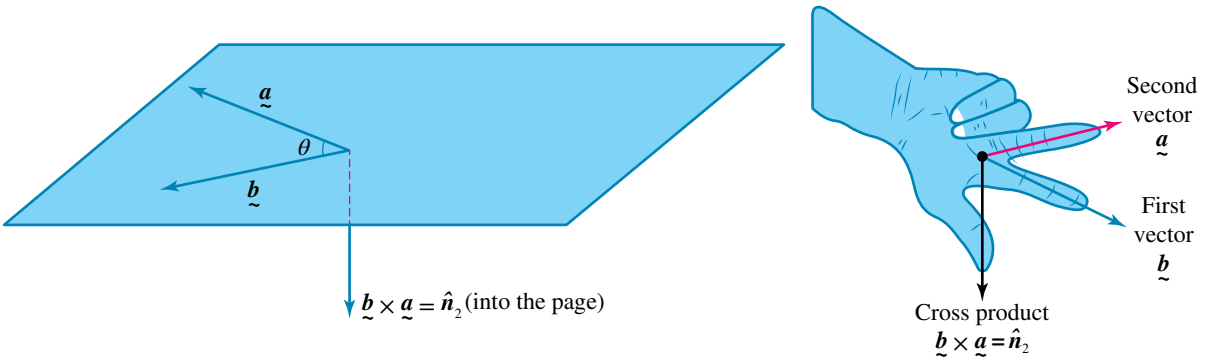
The centre of the right hand represents a point on the plane, represented by a page on your desk top. If the index finger points along the direction of $\underset{\sim}{a}$, the first vector in the cross product, and the middle finger points along the

direction of $\underset{\sim}{b}$, the second vector in the cross product, then the thumb will always point along the direction of the normal to the plan ($\hat{\underset{\sim}{n}}_1$), in this case, up, or out of the page.



Normal vector $\hat{\underset{\sim}{n}}_2$ — into the page

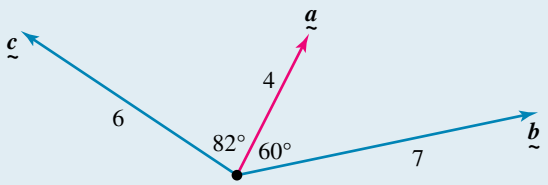
If you were calculating $\underset{\sim}{b} \times \underset{\sim}{a}$, then the first vector (your index finger) is $\underset{\sim}{b}$ and the second vector (your middle finger) is $\underset{\sim}{a}$ and your thumb will point in the direction of the normal to the plane ($\hat{\underset{\sim}{n}}_2$), in this case, down or into the page.



WORKED EXAMPLE 26 Calculating the cross product using the right-hand rule

The vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ exist in a horizontal plane, a page on your desk top. Calculate the following vector products and in each case state the direction of $\hat{\underset{\sim}{n}}$.

a. $\underset{\sim}{a} \times \underset{\sim}{b}$ **b.** $\underset{\sim}{b} \times \underset{\sim}{c}$



THINK

a. 1. Recall the vector product formula.

2. Substitute the values for $|\underset{\sim}{a}| = 4$, $|\underset{\sim}{b}| = 7$ and $\theta = 60°$.

WRITE

a. $\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\hat{\underset{\sim}{n}}$

$= 4 \times 7 \times \sin(60°)\hat{\underset{\sim}{n}}$

$= 28 \times \dfrac{\sqrt{3}}{2}\hat{\underset{\sim}{n}}$

3. Calculate the vector product, including the direction using the right-hand rule. $= 14\sqrt{3}\hat{\underset{\sim}{n}}$ into the page (down)

b. 1. Recall the vector product formula. **b.** $\underset{\sim}{b} \times \underset{\sim}{c} = |\underset{\sim}{b}|\,|\underset{\sim}{c}| \sin(\theta)\hat{\underset{\sim}{n}}$

2. Substitute the values for $|\underset{\sim}{b}| = 7$, $|\underset{\sim}{c}| = 6$ and $\theta = 142°$. $= 7 \times 6 \times \sin(142°)\,\hat{\underset{\sim}{n}}$

3. Calculate the vector product using the right-hand rule. $= 25.86\hat{\underset{\sim}{n}}$ out of the page (up)

WORKED EXAMPLE 27 Calculating the cross product using the polar form of vectors in a plane

The vectors $\underset{\sim}{c} = \left(2\sqrt{2}, \frac{\pi}{4}\right)$ and $\underset{\sim}{d} = \left(6, \frac{5\pi}{6}\right)$ exist in a horizontal plane. Calculate the exact value of the vector product, $\underset{\sim}{c} \times \underset{\sim}{d}$.

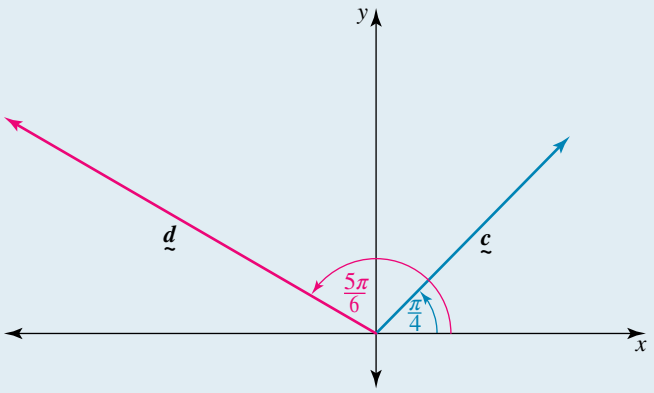
THINK

1. Recognise that the vectors are in polar form, $\underset{\sim}{v} = (|\underset{\sim}{v}|, \theta)$ and state their magnitude.

WRITE

The magnitudes of the vectors are, $|\underset{\sim}{c}| = 2\sqrt{2}$ and $|\underset{\sim}{d}| = 6$.

2. Sketch the vectors to get a clearer perspective of their orientation.



Both vectors are in a positive direction from the x-axis. The angle, θ, between the vectors is, $\theta = \frac{5\pi}{6} - \frac{\pi}{4} = \frac{7\pi}{12}$

3. Recall the rule for vector product, $\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin\theta\hat{\underset{\sim}{n}}$. The direction of the normal vector can be determined by applying the right hand rule. In this case, $\underset{\sim}{c} \times \underset{\sim}{d}$ is turning in an anticlockwise direction, which is out of the page.

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\hat{\underset{\sim}{n}}$$
$$\underset{\sim}{c} \times \underset{\sim}{d} = 2\sqrt{2} \times 6 \times \sin\left(\frac{7\pi}{12}\right)\hat{\underset{\sim}{n}}$$
$$= 12\sqrt{2}\sin\left(\frac{7\pi}{12}\right)\hat{\underset{\sim}{n}}$$

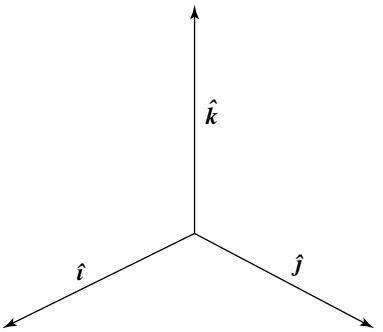
out of the page.

3.6.2 Calculating the vector cross product

We are now going to see two methods to calculate vector cross products in three dimension: in component form and using determinants.

Method 1: Calculating the vector product in component form

Recall that the vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the mutually perpendicular unit vectors that can be used to describe three dimensional space. The x-, y- and z-directions cannot be arranged in any order, even though they may be mutually perpendicular. The form is a right-hand triad. This means that the direction of z must be that obtained by anticlockwise rotation of x to y. Use the right-hand rule, recognising that x is the first axis, and y is the second axis.



Consider some special vector products. Remember that $\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\,\hat{\underset{\sim}{n}}$ where $\hat{\underset{\sim}{n}}$ is a unit vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the right-hand sense.

$\hat{\imath} \times \hat{\imath}$

$\underset{\sim}{a} = \hat{\imath}$ and $\underset{\sim}{b} = \hat{\imath}$

$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\hat{\underset{\sim}{n}}$

$\hat{\imath} \times \hat{\imath} = 1 \times 1 \sin(0°)\hat{\underset{\sim}{n}}$

$\hat{\imath} \times \hat{\imath} = \underset{\sim}{0}$

$\boxed{\therefore\ \hat{\imath} \times \hat{\imath} = \underset{\sim}{0}}$

$\hat{\jmath} \times \hat{k}$

$\underset{\sim}{a} = \hat{\jmath}$ and $\underset{\sim}{b} = \hat{k}$

$\hat{\jmath} \times \hat{k} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\hat{\underset{\sim}{n}}$

$\hat{\jmath} \times \hat{k} = 1 \times 1 \sin(90°)\hat{\underset{\sim}{n}}$

$\hat{\jmath} \times \hat{k} = 1\hat{\underset{\sim}{n}}$ in the direction of $+\hat{\imath}$

$\boxed{\therefore\ \hat{\jmath} \times \hat{k} = \hat{\imath}}$

$\hat{\imath} \times \hat{k}$

$\underset{\sim}{a} = \hat{\imath}$ and $\underset{\sim}{b} = \hat{k}$

$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\hat{\underset{\sim}{n}}$

$\hat{\imath} \times \hat{k} = 1 \times 1 \sin(90°)\hat{\underset{\sim}{n}}$

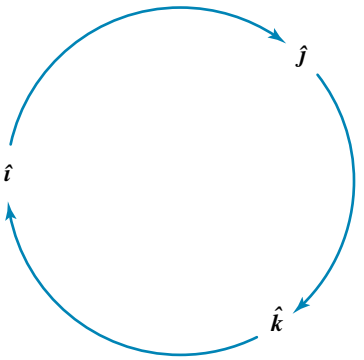
$\hat{\imath} \times \hat{k} = 1\hat{\underset{\sim}{n}}$ in the direction of $-\hat{\jmath}$

$\boxed{\therefore\ \hat{\imath} \times \hat{k} = -\hat{\jmath}}$

Continuing in this way, we can show that the cross product for the basic unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are:

$\hat{\imath} \times \hat{\imath} = \underset{\sim}{0}$	$\hat{\imath} \times \hat{\jmath} = \hat{k}$	$\hat{\imath} \times \hat{k} = -\hat{\jmath}$
$\hat{\jmath} \times \hat{\imath} = -\hat{k}$	$\hat{\jmath} \times \hat{\jmath} = \underset{\sim}{0}$	$\hat{\jmath} \times \hat{k} = \hat{\imath}$
$\hat{k} \times \hat{\imath} = \hat{\jmath}$	$\hat{k} \times \hat{\jmath} = -\hat{\imath}$	$\hat{k} \times \hat{k} = \underset{\sim}{0}$

A useful way of summarising these results is shown in the figure. A product following the direction indicated (clockwise) produces a positive value.



For example, $\hat{\imath} \times \hat{\jmath} = \hat{k}$

A product in the anticlockwise direction produces a negative value.

For example, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$

To evaluate the vector product of two vectors that are expressed in component form we can use the Distributive Law. This is illustrated in the following worked example.

WORKED EXAMPLE 28 Using the distributive law with vector (cross) product

If $\underset{\sim}{a} = 3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$, $\underset{\sim}{b} = -2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\underset{\sim}{c} = -3\hat{\imath} + 4\hat{\jmath} - 7\hat{k}$, calculate the following vector products using the distributive law of vectors.

a. $\underset{\sim}{a} \times \underset{\sim}{b}$

b. $\underset{\sim}{c} \times \underset{\sim}{b}$

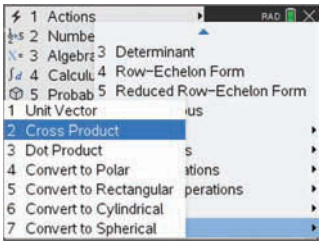
c. Justify that the vector product from b is orthogonal to vectors $\underset{\sim}{c}$ and $\underset{\sim}{b}$ respectively.

THINK	WRITE
a. 1. Express $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as a product in component form.	**a.** $\underset{\sim}{a} \times \underset{\sim}{b} = \left(3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}\right) \times \left(-2\hat{\imath} + 3\hat{\jmath} - \hat{k}\right)$
2. Calculate the product $\underset{\sim}{a} \times \underset{\sim}{b}$ by applying the Distributive Law for vectors.	$= 3\hat{\imath} \times -2\hat{\imath} + 3\hat{\imath} \times 3\hat{\jmath} + 3\hat{\imath} \times -\hat{k}$ $+ 2\hat{\jmath} \times -2\hat{\imath} + 2\hat{\jmath} \times 3\hat{\jmath} + 2\hat{\jmath} \times -\hat{k}$ $+ -4\hat{k} \times -2\hat{\imath} + -4\hat{k} \times 3\hat{\jmath} + -4\hat{k} \times -\hat{k}$
3. Simplify the calculation by collecting like vector components.	$= 9\hat{k} + 3\hat{\jmath} + 4\hat{k} - 2\hat{\imath} + 8\hat{\jmath} + 12\hat{\imath}$ $= 10\hat{\imath} + 11\hat{\jmath} + 13\hat{k}$
b. 1. Express $\underset{\sim}{c}$ and $\underset{\sim}{b}$ as a product in component form.	**b.** $\underset{\sim}{c} \times \underset{\sim}{b} = \left(-3\hat{\imath} + 4\hat{\jmath} - 7\hat{k}\right) \times \left(-2\hat{\imath} + 3\hat{\jmath} - \hat{k}\right)$
2. Calculate the product $\underset{\sim}{a} \times \underset{\sim}{b}$ applying the Distributive Law for vectors.	$= -3\hat{\imath} \times -2\hat{\imath} + -3\hat{\imath} \times 3\hat{\jmath} + -3\hat{\imath} \times -\hat{k}$ $+ 4\hat{\jmath} \times -2\hat{\imath} + 4\hat{\jmath} \times 3\hat{\jmath} + 4\hat{\jmath} \times -\hat{k}$ $+ -7\hat{k} \times -2\hat{\imath} + -7\hat{k} \times 3\hat{\jmath} + -7\hat{k} \times -\hat{k}$
3. Simplify the calculation by collecting like vector components.	$= -9\hat{k} - 3\hat{\jmath} + 8\hat{k} - 4\hat{\imath} + 14\hat{\jmath} + 21\hat{\imath}$ $= 17\hat{\imath} + 11\hat{\jmath} - \hat{k}$
c. 1. Let $\underset{\sim}{c} \times \underset{\sim}{b} = \underset{\sim}{n}$. Vectors, $\underset{\sim}{c}$ and $\underset{\sim}{n}$, are perpendicular if $\underset{\sim}{c} \cdot \underset{\sim}{n} = 0$.	**c.** Let $\underset{\sim}{c} \times \underset{\sim}{b} = \underset{\sim}{n}$ $\underset{\sim}{c}$ and $\underset{\sim}{n}$ are orthogonal if $\underset{\sim}{c} \cdot \underset{\sim}{n} = 0$
2. Calculate $\underset{\sim}{c} \cdot \underset{\sim}{n}$ by recalling the formula $\underset{\sim}{c} \cdot \underset{\sim}{n} = x_1x_2 + y_1y_2 + z_1z_2$. Multiply the corresponding components.	$\underset{\sim}{c} \cdot \underset{\sim}{n} = \left(-3\hat{\imath} + 4\hat{\jmath} - 7\hat{k}\right) \cdot \left(17\hat{\imath} + 11\hat{\jmath} - \hat{k}\right)$ $= -51 + 44 + 7$ $= 0$
3. Vectors, $\underset{\sim}{b}$ and $\underset{\sim}{n}$, are perpendicular if $\underset{\sim}{b} \cdot \underset{\sim}{n} = 0$. Calculate $\underset{\sim}{c} \cdot \underset{\sim}{n}$ by recalling the formula $\underset{\sim}{c} \cdot \underset{\sim}{n} = x_1x_2 + y_1y_2 + z_1z_2$. Multiply the corresponding components.	$\underset{\sim}{b}$ and $\underset{\sim}{n}$ are orthogonal if $\underset{\sim}{b} \cdot \underset{\sim}{n} = 0$ $\underset{\sim}{b} \cdot \underset{\sim}{n} = \left(-2\hat{\imath} + 3\hat{\jmath} - \hat{k}\right) \cdot \left(17\hat{\imath} + 11\hat{\jmath} - \hat{k}\right)$ $= -34 + 33 + 1$ $= 0$
4. Write a justifying statement.	$\therefore$ The vector product $\underset{\sim}{c} \times \underset{\sim}{b} = \underset{\sim}{n}$ is orthogonal to both vectors $\underset{\sim}{c}$ and $\underset{\sim}{b}$.

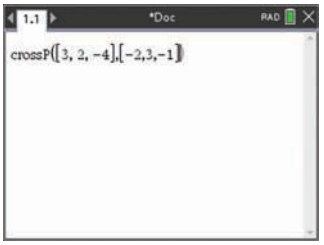
TI| THINK

a. 1. On a Calculator page, select MENU
7: Matrix Vector
C: Vector
2: Cross Product
Press the ENTER button.

WRITE/DISPLAY



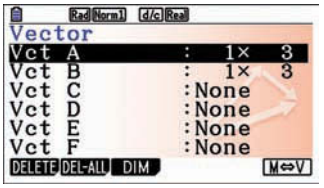
2. Complete the entry line $\left(\left[3, 2, -4\right], \left[-2, 3, -1\right]\right)$



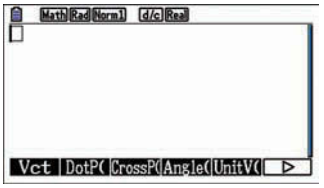
CASIO| THINK

a. 1. On a Run-Matrix screen, select MAT/VCT Define Vectors A and B as shown in Worked example 1b.

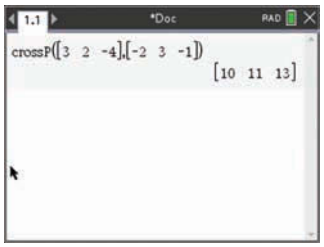
WRITE/DISPLAY



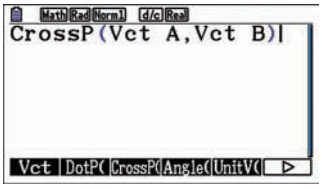
2. On a Run-Matrix screen, select OPTN MAT/VCT ▷ (x2 times) CrossP(



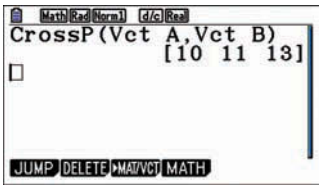
3. Press the ENTER button. The answer appears on the screen.



3. Complete the entry line as Vct ALPHA X, θ, T, Vct ALPHA) Log)



4. Press the EXE button. The answer appears on the screen



Method 2: calculating the vector product using determinants

The Distributive Law for vectors is a valid method for calculating the vector product when the vectors are in component form. The process is rather arduous, and the same result can be calculated more efficiently using determinants. First, we shall review the definitions of a determinant in both two and three dimensions. Note that we are using the notation |A| for det(A) below, for convenience.

For a 2×2 matrix:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

For a 3×3 matrix:

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then, $\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

The elements of a determinant are generally numerical figures. It is possible to replace the first row of the matrix with the vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. When calculating the $\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ d & e & f \\ g & h & i \end{vmatrix}$ determinant in this case, the sub-determinants (minors) become scalar multiples of the vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$.

$$\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ d & e & f \\ g & h & i \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} d & f \\ g & i \end{vmatrix} + \hat{\boldsymbol{k}}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

If the vectors exist in the form, $\underset{\sim}{a} = (a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}})$ and $\underset{\sim}{b} = (b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}})$, then the vector product can be defined in terms of determinants as,

$$\begin{aligned} \underset{\sim}{a} \times \underset{\sim}{b} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\boldsymbol{k}}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \hat{\boldsymbol{\imath}}(a_2b_3 - a_3b_2) - \hat{\boldsymbol{\jmath}}(a_1b_3 - a_3b_1) + \hat{\boldsymbol{k}}(a_1b_2 - a_2b_1) \end{aligned}$$

or, using the column vector notation, $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Hopefully, it is now clearer as to why the vector product can also be called the cross product.

WORKED EXAMPLE 29 Calculating the vector (cross) product using determinants and the vector column notation

a. Calculate $\underset{\sim}{a} \times \underset{\sim}{b}$ using determinants, given $\underset{\sim}{a} = -\hat{\imath} + 4\hat{\jmath} + 6\hat{k}$ and $\underset{\sim}{b} = 4\hat{\imath} + 2\hat{\jmath} - 5\hat{k}$.
b. Calculate $\underset{\sim}{b} \times \underset{\sim}{a}$ to investigate if the vector product is commutative.
c. Give a geometric interpretation of your answers to parts a and b.

THINK	WRITE
a. 1. Express the vector product, $\underset{\sim}{a} \times \underset{\sim}{b}$ as the determinant of a 3×3 matrix.	a. $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 4 & 6 \\ 4 & 2 & -5 \end{vmatrix}$
2. Calculate the determinant for the matrix. Remember to write out the expansion with the signed minors (cofactors).	$= \hat{\imath}\begin{vmatrix} 4 & 6 \\ 2 & -5 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -1 & 6 \\ 4 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 4 \\ 4 & 2 \end{vmatrix}$ $= \hat{\imath}(4 \times -5 - 6 \times 2) - \hat{\jmath}(-1 \times -5 - 6 \times 4)$ $+ \hat{k}(-1 \times 2 - 4 \times 4)$ $= \hat{\imath}(-20 - 12) - \hat{\jmath}(5 - 24) + \hat{k}(-2 - 16)$
3. Complete the expansion being careful with + and − signs.	$= -32\hat{\imath} + 19\hat{\jmath} - 18\hat{k}$
b. 1. Express the vector product $\underset{\sim}{b} \times \underset{\sim}{a}$ using the column vector notation.	b. $\underset{\sim}{b} \times \underset{\sim}{a} = \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix}$
2. Calculate the vector product.	$= \begin{pmatrix} 2 \times 6 - (-5) \times 4 \\ (-5) \times (-1) - 4 \times 6 \\ 4 \times 4 - 2 \times (-1) \end{pmatrix}$ $= \begin{pmatrix} 32 \\ -19 \\ 18 \end{pmatrix}$
3. State the result in component form.	$\underset{\sim}{b} \times \underset{\sim}{a} = 32\hat{\imath} - 19\hat{\jmath} + 18\hat{k}$
4. Recall that an operation is commutative if $a \circ b = b \circ a$. Conclude with a statement in response to the investigation.	The cross product is not commutative because $\underset{\sim}{a} \times \underset{\sim}{b} \neq \underset{\sim}{b} \times \underset{\sim}{a}$

c. 1. Compare $\underset{\sim}{a} \times \underset{\sim}{b}$ and $\underset{\sim}{b} \times \underset{\sim}{a}$.

2. Give a geometric interpretation.

c. $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} -32 \\ 19 \\ -18 \end{pmatrix}$ and $\underset{\sim}{b} \times \underset{\sim}{a} = \begin{pmatrix} 32 \\ -19 \\ 18 \end{pmatrix}$

Thus $\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{b} \times \underset{\sim}{a}$

The normal vector $\underset{\sim}{b} \times \underset{\sim}{a}$ points in the opposite direction to the normal vector $\underset{\sim}{a} \times \underset{\sim}{b}$ (if one is out of the page, the other one is into the page, and vice-versa).

3.6.3 Properties of the vector product

Although the vector product does not form a commutative operation with vectors, it is interesting to note that $\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{b} \times \underset{\sim}{a}$. This property is known as being anti-commutative. The following six theorems summarise some of the properties of the vector product.

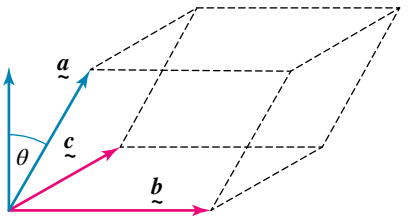
Six properties of the cross product

1. $\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{b} \times \underset{\sim}{a}$
2. $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$
3. $(\underset{\sim}{a} + \underset{\sim}{b}) \times \underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{c} + \underset{\sim}{b} \times \underset{\sim}{c}$
4. $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}$
5. $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})\underset{\sim}{b} - (\underset{\sim}{a} \times \underset{\sim}{b})\underset{\sim}{c}$
6. $(k\underset{\sim}{a}) \times \underset{\sim}{b} = k(\underset{\sim}{a} \times \underset{\sim}{b}) = \underset{\sim}{a} \times (k\underset{\sim}{b})$

where $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are vectors, and k is a scalar.

3.6.4 The scalar triple product

Let us now investigate a common and useful application of the vector and dot products. The fourth theorem, $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$, is referred to as the **scalar triple product**. As the name suggests, the product is a scalar quantity, and is sometimes referred to as the **triple product.**



The triple product has numerous applications, one of which is related to the volume of a parallelepiped. The volume of a parallelepiped whose adjacent sides are the vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ is,

$$\text{Volume} = |\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$$

In three steps, this formula can be proved by combining the theories of both the dot and vector products.

1. The area of the base is given by the vector product $A = |\underset{\sim}{b} \times \underset{\sim}{c}|$.
2. If θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b} \times \underset{\sim}{c}$, the height of the parallelepiped will be $h = |\underset{\sim}{a}|\,|\cos(\theta)|$.
3. The vector formula for the volume of a parallelepiped is then developed by combining the equations, $A = |\underset{\sim}{b} \times \underset{\sim}{c}|$ and $h = |\underset{\sim}{a}|\,|\cos(\theta)|$,

$$\begin{aligned} \text{Volume} &= A \times h \\ &= |\underset{\sim}{a}|\,|\cos(\theta)| \times |\underset{\sim}{b} \times \underset{\sim}{c}| \\ &= |\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})| \end{aligned}$$

Determinants can be used to simplify the triple product calculation, $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$.

If $\underset{\sim}{a} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k})$, $\underset{\sim}{b} = (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$ and $\underset{\sim}{c} = (c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k})$, the scalar triple product can be calculated by

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Worked examples 30 and 31 demonstrate how to calculate by hand the scalar triple product using two different methods, which, of course, provide the same answer.

WORKED EXAMPLE 30 Using the scalar triple product

Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$.

Calculate the scalar triple product $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$.

THINK	WRITE
1. State the formula for the scalar triple product.	$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$
2. Substitute the correct values.	$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} \right)$
3. Calculate the cross product first.	$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \times (-1) - (-4) \times (-4) \\ (-4) \times 3 - 5 \times (-1) \\ 5 \times (-4) - 2 \times 3 \end{pmatrix}$ $= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -18 \\ -7 \\ -26 \end{pmatrix}$
4. Calculate the dot product second.	$= 1 \times (-18) + 2 \times (-7) - 1 \times (-26)$ $= -58$
5. State the answer, ensuring your answer is a scalar.	$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = -58$

WORKED EXAMPLE 31 Determining the volume of a parallelepiped

Determine the volume of the parallelepiped containing the adjacent vectors, $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ by using a matrix determinant.

$\underset{\sim}{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\underset{\sim}{b} = 5\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$ and $\underset{\sim}{c} = 3\hat{\imath} - 4\hat{\jmath} - \hat{k}$.

THINK	WRITE
1. State the formula for the volume, which is the scalar triple product.	Volume $= \lvert \underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) \rvert$
2. Express the scalar triple product as the determinate of a 3×3 matrix. Calculate the determinant of the matrix.	$V = \begin{vmatrix} 1 & 2 & 1 \\ 5 & 2 & -4 \\ 3 & -4 & -1 \end{vmatrix}$

3. Remember to write out the expansion with the signed minors (cofactors).

$$V = 1\begin{vmatrix} 2 & -4 \\ -4 & -1 \end{vmatrix} - 2\begin{vmatrix} 5 & -4 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 5 & 2 \\ 3 & -4 \end{vmatrix}$$

4. Complete the expansion being careful with + and − signs.

$$\begin{aligned} V &= |-18 - 14 - 26| \\ &= |-58| \\ &= 58 \end{aligned}$$

Alternatively, using the column vector notation.

1. State the formula for the volume, which is the absolute value of the scalar triple product.

$$V = |\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$$

2. Calculate the cross product first.

$$\begin{aligned} V &= \left|\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}\right)\right| \\ &= \left|\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \times (-1) - (-4) \times (-4) \\ 3 \times (-4) - 5 \times (-1) \\ 5 \times (-4) - 3 \times 2 \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -18 \\ -7 \\ -26 \end{pmatrix}\right| \end{aligned}$$

3. Calculate the dot product second.

$$\begin{aligned} V &= |1 \times (-18) + 2 \times (-7) + 1 \times (-26)| \\ &= |-58| \\ &= 58 \text{ units}^3 \end{aligned}$$

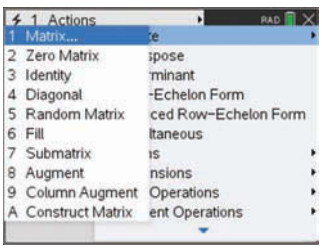
4. State the answer, with units, and remember that the volume is a positive scalar.

$$V = 58 \text{ units}^3$$

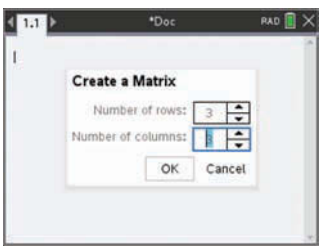
TI| THINK

1. On a Calculator page, select MENU
   7: Matrix & Vector
   1: Create
   1: Matrix
   Press the ENTER button.

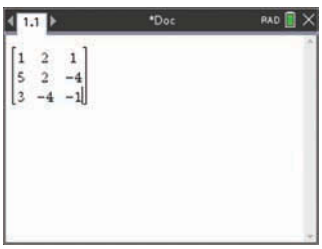
WRITE/DISPLAY



2. Complete the entry line as 3 3 Press the OK button.



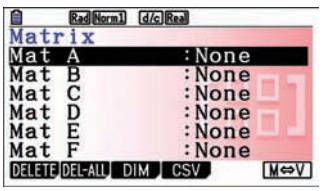
3. Complete the entry line as
   1 2 1
   5 2 −4
   3 −4 −1
   Press the ENTER button when finished.



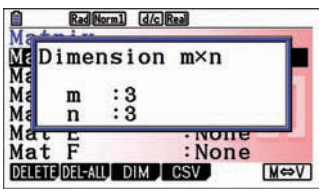
CASIO| THINK

1. On a Run-Matrix screen, select MAT/VCT
   To switch between the vector and matrix screens, press the M ⇔ V button.

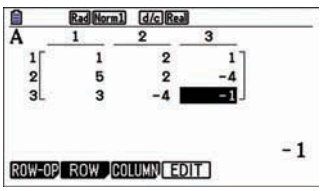
WRITE/DISPLAY



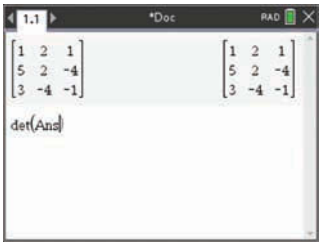
2. Press the EXE button. Complete the entry line as
   3 3 Press the EXE button.



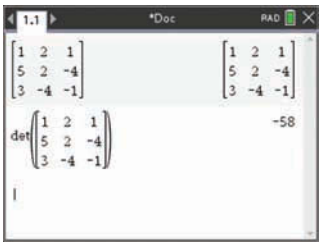
3. Complete the entry line as
   1 2 1
   5 2 −4
   3 −4 −1
   Press the EXIT button when finished.



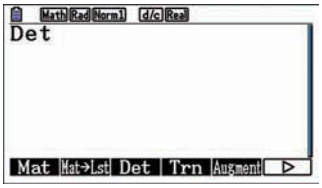
4. On a Calculator page, select MENU
7: Matrix & Vector
3: Determinant
CTRL (-)



5. Press the ENTER button. The answer appears on the screen.
Note: The absolute value of determinant is the volume of the parallelepiped.

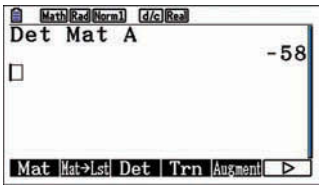


4. On a Run-Matrix screen, select OPTN
MAT/VCT
DET



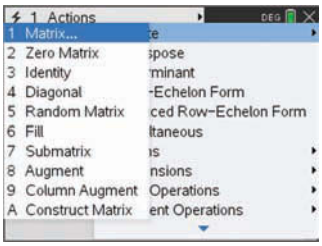
5. Complete the entry line as
SHIFT
2
ALPHA
X, θ, T

6. Press the EXE button. The answer appears on the screen.
Note: The absolute value of determinant is the volume of the parallelepiped.

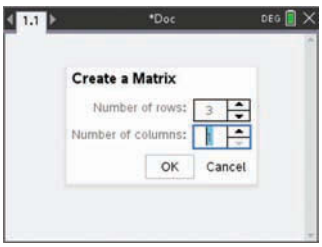


Alternatively, using the column vector notation.

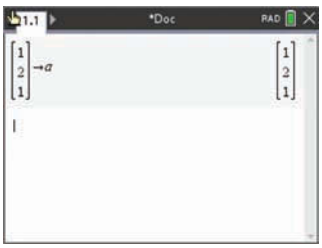
1. On a Calculator page, select MENU
7: Matrix & Vector
1: Create
1: Matrix
Press the ENTER button.



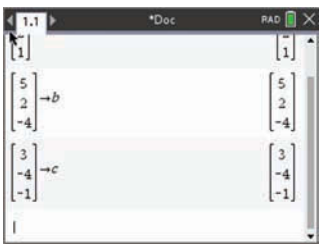
2. Complete the entry line as 3 1 Press the OK button.



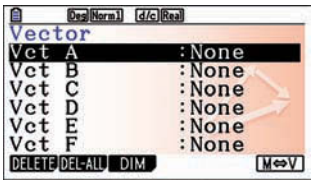
3. Complete the entry line as 1 2 1, then press ctrl var A and ENTER to store this vector.



4. Repeat steps 1–3 with 5 2 −4 stored in b and 3 −4 −1 stored in c.

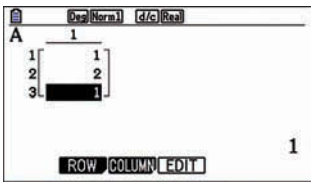


1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.



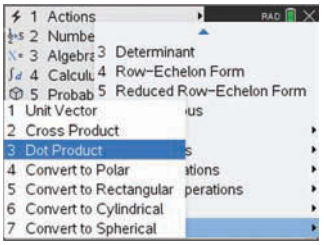
2. Press the EXE button. Complete the entry line as
1 3 Press the EXE button.

3. Complete the entry line as1 2 1 . Press the EXIT button when finished.

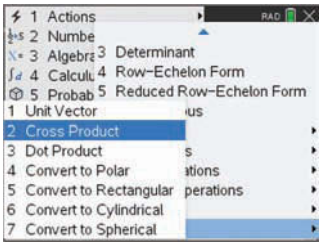


4. Repeat steps 1–3 with 5 2 −4 stored in Vct B and 3 −4 −1 stored in Vct C.

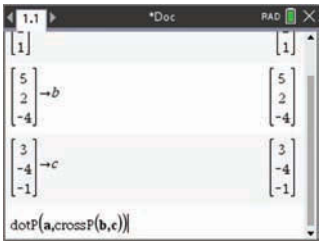
5. Select MENU 7: Matrix & Vector, C: Vector, 3 Dot Product



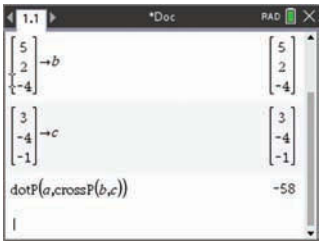
6. Key in a, then select MENU 7: Matrix & Vector, C: Vector, 2 Cross Product.



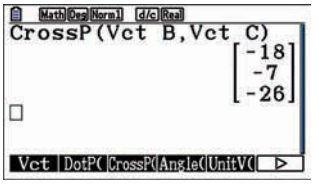
7. Finish completing the line as dotP(a, crossP(b,c)).



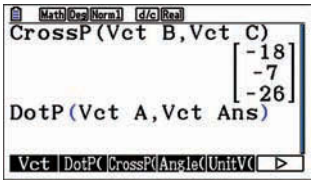
8. Press the ENTER button. Do not forget to take the absolute value of the scalar triple product.



5. Press OPTN, MAT/VCT, ▷,▷, CrossP(and complete the entry line as CrossP(Vct B, Vct C) to calculate $\underset{\sim}{b} \times \underset{\sim}{c}$.



6. Press OPTN, MAT/VCT, ▷,▷, DotP(and complete the entry line as DotP(VctA, Vct Ans) to calculate $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$



7. Press the EXE button.

8. Do not forget to take the absolute value of the scalar triple product.

WORKED EXAMPLE 32 Investigating the order of the scalar triple product (extension)

Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}$.

Use technology to calculate $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$, $\underset{\sim}{b} \cdot (\underset{\sim}{a} \times \underset{\sim}{c})$, $\underset{\sim}{c} \cdot (\underset{\sim}{b} \times \underset{\sim}{a})$ and $(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}$ and write a statement regarding the volume V of the parallelepiped whose adjacent sides are the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

THINK	WRITE
1. Use technology to calculate $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$.	$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = -135$
2. Use technology to calculate $\underset{\sim}{b} \cdot (\underset{\sim}{a} \times \underset{\sim}{c})$.	$\underset{\sim}{b} \cdot (\underset{\sim}{a} \times \underset{\sim}{c}) = 135$
3. Use technology to calculate $\underset{\sim}{c} \cdot (\underset{\sim}{b} \times \underset{\sim}{a})$.	$\underset{\sim}{c} \cdot (\underset{\sim}{b} \times \underset{\sim}{a}) = -135$
4. Use technology to calculate $(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}$.	$(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c} = -135$
5. Observe that all these have the same absolute value.	$\lvert \underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) \rvert = \lvert \underset{\sim}{b} \cdot (\underset{\sim}{a} \times \underset{\sim}{c}) \rvert = \lvert \underset{\sim}{c} \cdot (a \times b) \rvert = \lvert (a \times b) \cdot \underset{\sim}{c} \rvert$
6. Write a statement.	When calculating the volume V of the parallelepiped whose adjacent sides are the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, the order of the vectors in the scalar triple product does not matter.

Exercise 3.6 The vector (cross) product

learnon

3.6 Exercise	3.6 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
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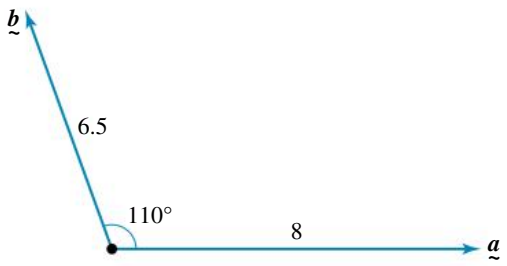
Simple familiar

1. WE26 The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ exist in a horizontal plane. Calculate $\underset{\sim}{a} \times \underset{\sim}{b}$, and state the direction of $\hat{\underset{\sim}{n}}$.

a.



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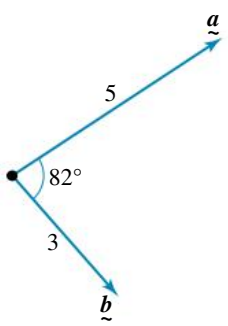
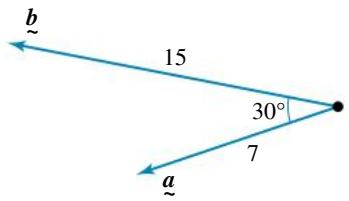
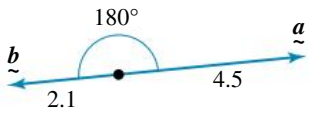


c.



d.



2. WE27 The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ exist in a horizontal plane. Calculate the vector product, $\underset{\sim}{a} \times \underset{\sim}{b}$, for the following pairs of vectors.

a. $\underset{\sim}{a} = (12, 25°)$ and $\underset{\sim}{b} = (10, 75°)$

b. $\underset{\sim}{a} = (21, 182°)$ and $\underset{\sim}{b} = (16, 267°)$

c. $\underset{\sim}{a} = (4, 173°)$ and $\underset{\sim}{b} = (7, 22°)$

d. $\underset{\sim}{a} = (5, 25°)$ and $\underset{\sim}{b} = (7, 345°)$

3. The table below summaries the nine possibilities when the vector product is calculated using the unit vectors $\hat{\imath}$, $\hat{\jmath}$ or $\hat{k}$. Complete the table by using your knowledge of the Distributive Law for vectors.

		Second vector $\underset{\sim}{b}$		
	$\times$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
	$\hat{\imath}$			$-\hat{\jmath}$
First Vector $\underset{\sim}{a}$	$\hat{\jmath}$		$\underset{\sim}{0}$	
	$\hat{k}$			

4. Use the vector product of vectors to calculate a vector perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

a. $\underset{\sim}{a} = 2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$ and $\underset{\sim}{b} = -\hat{\imath} + 7\hat{\jmath} - 2\hat{k}$

b. $\underset{\sim}{a} = -5\hat{\imath} - \hat{\jmath} - 3\hat{k}$ and $\underset{\sim}{b} = 11\hat{\imath} - 4\hat{\jmath} - 2\hat{k}$

c. $\underset{\sim}{a} = \hat{\imath} + 12\hat{\jmath} + 8\hat{k}$ and $\underset{\sim}{b} = -9\hat{\imath} + 3\hat{\jmath} + 7\hat{k}$

d. $\underset{\sim}{a} = 3.5\hat{\imath} + 1.5\hat{\jmath} - 2\hat{k}$ and $\underset{\sim}{b} = 0.5\hat{\imath} - 2\hat{\jmath}$

5. WE28 If $\underset{\sim}{a} = 4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$, $\underset{\sim}{b} = -3\hat{\imath} + 5\hat{\jmath} - \hat{k}$ and $\underset{\sim}{c} = -7\hat{\imath} + \hat{\jmath} - 2\hat{k}$, calculate the following vector products using the Distributive Law for vectors

a. $\underset{\sim}{a} \times \underset{\sim}{b}$
b. $\underset{\sim}{c} \times \underset{\sim}{b}$
c. Justify that the vector product from **b**, is orthogonal to vectors $\underset{\sim}{c}$ and $\underset{\sim}{b}$ respectively.

6. WE29 a. Calculate $\underset{\sim}{a} \times \underset{\sim}{b}$ using determinants, given $\underset{\sim}{a} = -3\hat{\imath} + 7\hat{\jmath} - 5\hat{k}$, $\underset{\sim}{b} = 8\hat{\imath} + \hat{\jmath} - 2\hat{k}$.

b. Calculate $\underset{\sim}{b} \times \underset{\sim}{a}$ to investigate if the vector product is commutative.

7. Use determinants to calculate the following vector products, $\underset{\sim}{a} \times \underset{\sim}{b}$.

a. $\underset{\sim}{a} = 5\hat{\imath} - 2\hat{\jmath} - 9\hat{k}$ and $\underset{\sim}{b} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$
b. $\underset{\sim}{a} = 3\hat{\jmath} - 8\hat{k}$ and $\underset{\sim}{b} = -4\hat{\imath} + 6\hat{\jmath} + 7\hat{k}$
c. $\underset{\sim}{a} = 3\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$ and $\underset{\sim}{b} = 7\hat{\imath} + 2\hat{\jmath} - \hat{k}$
d. $\underset{\sim}{a} = -7\hat{\imath} + \hat{\jmath} - 6\hat{k}$ and $\underset{\sim}{b} = 6\hat{\imath} - 12\hat{\jmath} + 13\hat{k}$

8. WE29 Determine the volume of the parallelepiped containing the adjacent vectors, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

a. $\underset{\sim}{a} = 5\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$, $\underset{\sim}{b} = 7\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\underset{\sim}{c} = 8\hat{\imath} - 7\hat{\jmath} - 3\hat{k}$
b. $\underset{\sim}{a} = \hat{\imath} - 2\hat{\jmath} + 8\hat{k}$, $\underset{\sim}{b} = -3\hat{\imath} - 2\hat{\jmath} + 9\hat{k}$ and $\underset{\sim}{c} = -5\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$
c. $\underset{\sim}{a} = 8\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$, $\underset{\sim}{b} = -4\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\underset{\sim}{c} = 7\hat{\imath} - 11\hat{\jmath} + 6\hat{k}$
d. $\underset{\sim}{a} = 7\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$, $\underset{\sim}{b} = -2\hat{\imath} - 4\hat{\jmath} - 6\hat{k}$ and $\underset{\sim}{c} = 2\hat{\imath} - 4\hat{\jmath} + 8\hat{k}$

9. WE30 Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 4 \\ -6 \\ 5 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} -1 \\ 0 \\ 6 \end{pmatrix}$. Calculate the scalar triple product $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})$.

10. WE32 Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$. Use technology to calculate $(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}$, $(\underset{\sim}{a} \times \underset{\sim}{c}) \cdot \underset{\sim}{b}$, $(\underset{\sim}{c} \times \underset{\sim}{a}) \cdot \underset{\sim}{b}$, and $\underset{\sim}{b} \cdot (\underset{\sim}{c} \times \underset{\sim}{a})$ and write a statement regarding the volume V of the parallelepiped whose adjacent sides are the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

11. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$ and $\underset{\sim}{c} = 3\underset{\sim}{a} - 2\underset{\sim}{b} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Using the scalar triple product, determine the volume V of the parallelepiped whose adjacent sides are the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

12. Use the vector product to prove the property $\underset{\sim}{a} \times \underset{\sim}{a} = \underset{\sim}{0}$, given that $\underset{\sim}{a} = (x, y, z)$.

Complex familiar

13. Calculate the scalar triple product for the vectors, $\underset{\sim}{a} = 3\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$, $\underset{\sim}{b} = 2\hat{\imath} + 4\hat{\jmath}$ and $\underset{\sim}{c} = 3\hat{\imath} + 9\hat{\jmath} + 6\hat{k}$, to determine if they are coplanar (exist on the same plane).

14. Calculate each of the following scalar triple products.

a. $\hat{\jmath} \cdot \hat{k} \times \hat{k}$
b. $\hat{\imath} \cdot \hat{\jmath} \times \hat{k}$
c. $\hat{\imath} \cdot \hat{k} \times \hat{\jmath}$
d. $\hat{k} \cdot \hat{\jmath} \times \hat{\imath}$

15. The vector product of two vectors $\underset{\sim}{a} = -2\hat{\imath} + 11\hat{\jmath} - \hat{k}$ and $\underset{\sim}{b} = -3\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$ is $\underset{\sim}{a} \times \underset{\sim}{b} = 12\hat{\imath} + 5\hat{\jmath} + 31\hat{k}$. Calculate the values of b_y and b_z.

16. The vector product of two vectors $\underset{\sim}{a} = 3\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$ and $\underset{\sim}{b} = 4\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$ is $\underset{\sim}{a} \times \underset{\sim}{b} = -7\hat{\imath} - 15\hat{\jmath} - 32\hat{k}$. Calculate the values of b_y and b_z.

Complex unfamiliar

17. A tetrahedron is a pyramid consisting of four triangular faces. The volume of a tetrahedron is given by the rule, $V = \frac{1}{6}|\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$. Calculate the volume of a tetrahedron with vertices A(6, 2, 0), B(7, 5, 0), C(3, 8, 1) and D(13, 7, 2).

18. Use the vector product to calculate the angle between the vectors $\underset{\sim}{a} = 3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$.

19. The vectors $\underset{\sim}{u} = \left(3\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{2}\right)$ and $\underset{\sim}{v} = \left(4, -\frac{\pi}{4}, \frac{\pi}{2}\right)$ exists in a horizontal plane. Using $\underset{\sim}{u} \times \underset{\sim}{v}$, show that $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$.

20. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -3 \\ 6 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}$. Determine the unit vector $\hat{\underset{\sim}{n}}$ perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Give your answer using the column vector notation.

Fully worked solutions for this chapter are available online.

LESSON
3.7 Applications of vectors

SYLLABUS LINKS

- Model and solve problems that involve displacement, force, velocity and relative velocity using the above concepts.
- Use vectors to prove geometric results in two dimensions (other than those listed in Unit 2 Topic 3) and in three dimensions.
- Use vector methods in applications, including areas of shapes and determining vector and Cartesian equations of a plane and of regions in a plane.
  - vector equation of plane: $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
  - Cartesian equation of plane: $ax + by + cz + d = 0$.

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3.7.1 Velocity and relative velocity

Velocity is a vector quantity. It has a magnitude (speed, with units, e.g. m/s) and a direction (e.g. north, up, 45° clockwise from the vertical, in the direction of travel, towards the Sun etc.).

In a three dimensional space, a velocity vector can be defined like a displacement vector can be defined.

For instance:

$$\underset{\sim}{v} = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}} \text{ m/s}$$

$$= \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \text{ m/s}$$

$$= \left(10, \frac{\pi}{2}, \frac{\pi}{3}\right) \text{ m/s}$$

WORKED EXAMPLE 33 Calculating velocity and speed of a particle in three dimensional space

A particle travels 2 cm in the positive x-direction, 6 cm in the negative y-direction and 4 cm in the positive z-direction in 2 seconds.

a. Define the velocity vector.

b. Calculate the speed of the particle.

THINK	WRITE
a. 1. Define the velocity vector by considering that velocity $= \dfrac{\text{displacement}}{\text{time}}$ and include the units.	a. $\underset{\sim}{v} = \dfrac{2\hat{\imath} - 6\hat{\jmath} + 4\hat{k}}{2}$ $= \hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ cm/s
2. State the answer.	The velocity of the particle is $\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ cm/s.
b. 1. Recall that speed is the magnitude of the velocity vector.	b. Speed $= \lvert \underset{\sim}{v} \rvert$
2. Use Pythagoras' theorem in three dimensions.	$= \sqrt{1^2 + (-3)^2 + 2^2}$
3. Include units.	$= \sqrt{14}$ cm/s
4. State the answer.	The speed of the particle is $= \sqrt{14}$ cm/s.

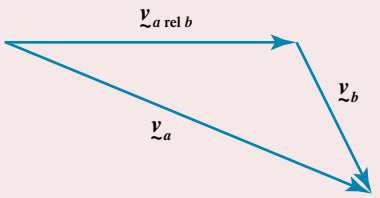
In Unit 1, Topic 4 (Algebra of vectors in two dimensions), you investigated the concept of relative velocities by recognising the difference between vectors. In three dimensions, the concept is the same.

Relative velocity

$$\underset{\sim}{v}_a = \underset{\sim}{v}_{a \text{ rel } b} + \underset{\sim}{v}_b$$

where $\underset{\sim}{v}_{a \text{ rel } b}$ is the velocity of $\underset{\sim}{a}$ ($\underset{\sim}{v}_a$) relative to the velocity of $\underset{\sim}{b}$ ($\underset{\sim}{v}_b$)

$$\underset{\sim}{v}_{a \text{ rel } b} = \underset{\sim}{v}_a - \underset{\sim}{v}_b$$



WORKED EXAMPLE 34 Determining relative velocity in three dimensional space.

A first particle is travelling on a velocity vector $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix}$ m/s while a second particle is travelling on a velocity vector $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ m/s.

a. Determine the velocity of the first particle relative to the second one.

b. Determine the velocity of the second particle related to the first one.

THINK	WRITE
a. 1. Write the relative velocity formula for $\underset{\sim}{a}$ relative to $\underset{\sim}{b}$.	a. $\underset{\sim}{v}_{a \text{ rel } b} = \underset{\sim}{v}_a - \underset{\sim}{v}_b$
2. Substitute the appropriate vectors.	$= \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

3. Calculate the relative velocity, and include units.

$$= \begin{pmatrix} 2 \\ 10 \\ -3 \end{pmatrix} \text{m/s}$$

b. 1. Write the relative velocity formula for $\underset{\sim}{b}$ relative to $\underset{\sim}{a}$.

2. Recognise that $\underset{\sim}{v}_{a \text{ rel } b} = -\underset{\sim}{v}_{b \text{ rel } a}$ and state the relative velocity, including units.

b. $\underset{\sim}{v}_{b \text{ rel } a} = \underset{\sim}{v}_b - \underset{\sim}{v}_a$

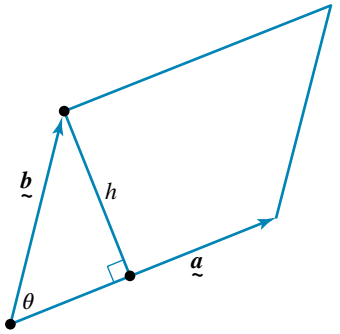
$$= -\underset{\sim}{v}_{a \text{ rel } b}$$

$$= \begin{pmatrix} 2 \\ 10 \\ -3 \end{pmatrix} \text{m/s}$$

3.7.2 The area of shapes

The vector product can be used to calculate the area of particular shapes. The area of a parallelogram is given by the rule, Area = base × height. Consider the relationship between the area of a parallelogram and the vector product.

The height of the parallelogram can be expressed using basic trigonometric ratios.



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin(\theta) = \frac{h}{|\underset{\sim}{b}|}$$

$$h = |\underset{\sim}{b}| \sin(\theta)$$

The area of the parallelogram is then given by.

$$\begin{aligned} \text{Area} &= \text{base} \times \text{height} \\ &= |\underset{\sim}{a}| \times |\underset{\sim}{b}| \sin(\theta) \\ &= |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta) \end{aligned}$$

Recalling the formula for the vector product, $\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin(\theta)\hat{\underset{\sim}{n}}$, we can recognise that the area of a parallelogram, defined by the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, is equal to the magnitude of the vector product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. That is, the area is given by:

$$\text{Area}_{\text{parallelogram}} = |\underset{\sim}{a} \times \underset{\sim}{b}|$$

The vector product can be used to calculate the area of a triangle via a similar procedure.

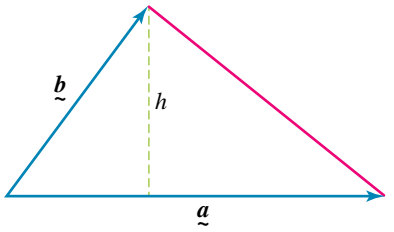
Recall that the area of any triangle is given by, Area $= \frac{1}{2} \times$ base $\times$ height.

Given that $h = |\underset{\sim}{b}| \sin(\theta)$, the area can be calculated

$$\text{Area} = \frac{1}{2} \times |\underset{\sim}{a}| \times |\underset{\sim}{b}| \ \sin(\theta)$$

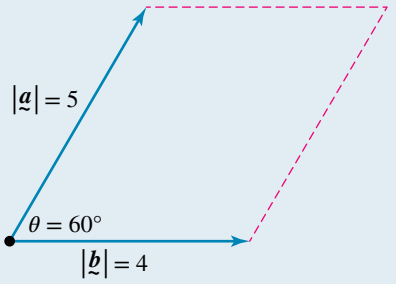
Thus, the area of a triangle, defined by the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, is equal to,

$$\begin{aligned}\text{Area}_{\text{triangle}} &= \frac{1}{2}|\underset{\sim}{a}\times\underset{\sim}{b}|\\ &= \frac{1}{2}|\underset{\sim}{a}|\times|\underset{\sim}{b}|\sin(\theta)\\ &= \frac{1}{2}|\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)\\ &= \frac{1}{2}\underset{\sim}{a}\times\underset{\sim}{b}\end{aligned}$$

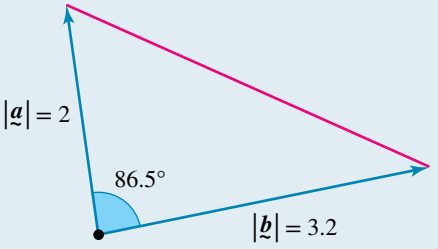


WORKED EXAMPLE 35 Calculating the area of shapes using the cross product

a. Use the vector formula, Area $= |\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)$ to determine the area of the parallelogram formed by the vectors with adjacent sides, $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Give your answer to 2 decimal places.



b. Use the formula Area $= \frac{1}{2}|\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)$ to calculate the area of the triangle. Give your answer to 2 decimal places.



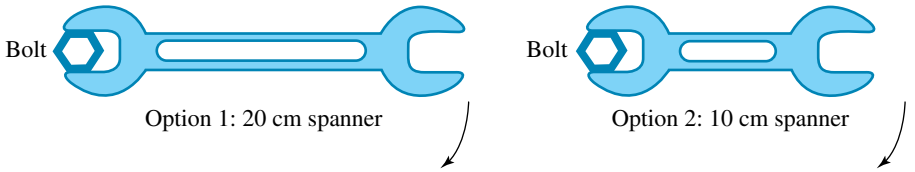
THINK	WRITE
a. 1. Recall and state the formula for the area of a parallelogram. Area $= |\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)$	**a.** Area $= |\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)$
2. Substitute the values for $|\underset{\sim}{a}| = 5$, $|\underset{\sim}{b}| = 4$ and $\theta = 60°$.	$= 5\times4\times\sin(60°)$ $= 20\times\frac{\sqrt{3}}{2}$
3. Calculate the area and simplify the answer to 2 decimal places.	$= 10\sqrt{3}$ ≈ 17.32 square units
b. 1. Recall and state the formula for the area of a triangle. Area $= \frac{1}{2}|\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin\theta$	**b.** Area $= \frac{1}{2}|\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)$
2. Substitute the values for $|\underset{\sim}{a}| = 2$, $|\underset{\sim}{b}| = 3.2$ and $\theta = 86.5°$.	$= \frac{1}{2}\times2\times3.2\times\sin(86.5°)$
3. Calculate the area and simplify the solution to 2 decimal places.	≈ 3.19 square units

3.7.3 Force and displacement applications

The vector product has further applications in both engineering and physics. Consider tightening a bolt on an item that you hope will never loosen. In order to turn the bolt inward, the spanner needs to be rotated clockwise.

The length of the spanner, and the rotation of the spanner are in the same plane, but the bolt will move up or down, in a direction perpendicular to that plane. If you visualise this as a vector $\underset{\sim}{a}$ rotated by a certain angle giving a vector $\underset{\sim}{b}$, then the movement of the bolt, perpendicular to the plane, is represented by the cross product $\underset{\sim}{a} \times \underset{\sim}{b}$.

Now let's consider the two options below.



Which spanner do you think would be more effective if both were applied with the same force? The turning force depends on how far the force is applied from the point of application.

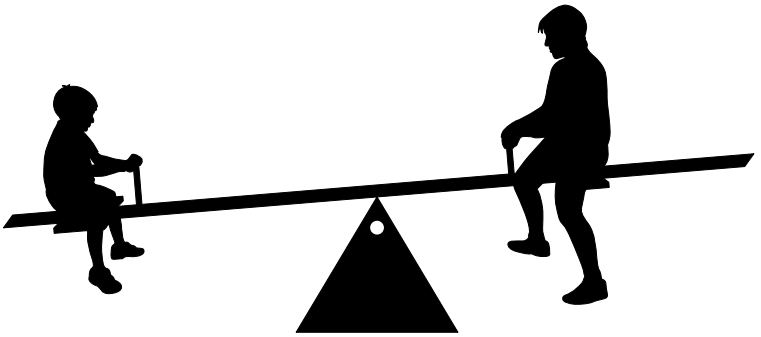
This is similar to how $|2\underset{\sim}{a} \times \underset{\sim}{b}| > |\underset{\sim}{a} \times \underset{\sim}{b}|$ for instance, if you increase the magnitude of one of the vectors, then the magnitude of the cross product will increase.

Thus, this explains why the longer spanner is the best choice here.

This principle is observed in many other applications such as:

1. The location of the handle on a hinged door. If the handle was placed too close to the hinges, a person would find it difficult to open the door. In most instances the door handle is on the opposite side to the hinges thereby increasing the distance from the point of application.
2. If an adult and a child are trying to balance a seesaw, the adult must sit much closer to the centre of balance than the child sits.

This is because the product of the weight force and the distance from the centre along the seesaw must be the same on both side, or the seesaw will tip.



Torque

The vector product of the applied force vector ($\underset{\sim}{F}$) and the displacement vector ($\underset{\sim}{r}$) to the point of application of the applied force is called the torque, generally noted τ (pronounced tau) in physics.

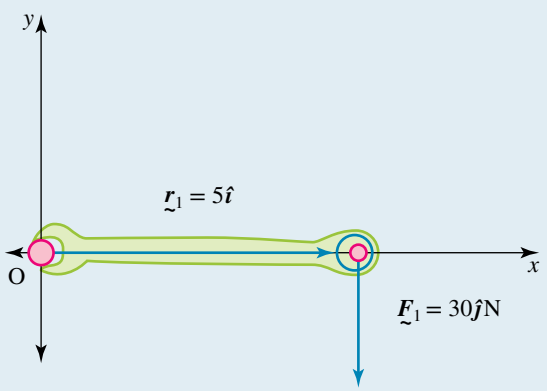
$$\textbf{Torque} = |\underset{\sim}{r} \times \underset{\sim}{F}|$$

As this equation suggests, the size of the torque depends not only on the force and the distance the force is from the point of application, but also on the angle at which the force is applied. The torque is a maximum when the angle between $\underset{\sim}{F}$ and $\underset{\sim}{r}$ is 90°. If $\underset{\sim}{F}$ is measured in Newtons (N) and $\underset{\sim}{r}$ is measured in metres (m) then torque is measured in Newton metres (Nm).

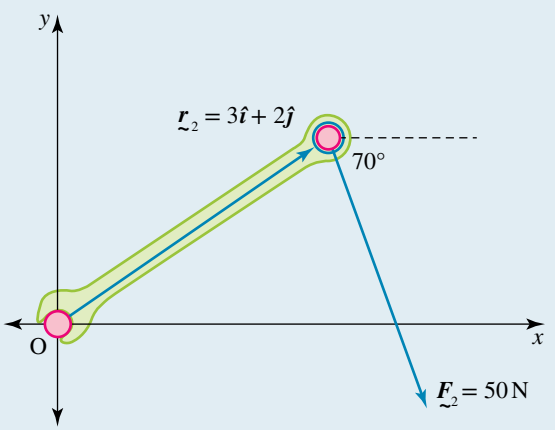
WORKED EXAMPLE 36 Calculating torque

The diagrams depict two different spanners being acted on by a force, $\underset{\sim}{F}_n$. For each scenario, calculate the torque about the point O ($\underset{\sim}{r}$ is in metres).

a.



b.



THINK	WRITE
a. 1. Recall the torque formula. Substitute in the vectors, $\underset{\sim}{r}$ and $\underset{\sim}{F}$.	**a.** $\underset{\sim}{r} = \overrightarrow{OP} = 5\hat{\imath}, \underset{\sim}{F} = -30\hat{\jmath}$ Torque $= \lvert\underset{\sim}{r} \times \underset{\sim}{F}\rvert$ $= \lvert(5\hat{\imath}) \times (-30\hat{\jmath})\rvert$
2. Calculate the vector product using the determinant method.	$\underset{\sim}{r} \times \underset{\sim}{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 0 & 0 \\ 0 & -30 & 0 \end{vmatrix}$ $= \hat{\imath}\begin{vmatrix} 0 & 0 \\ -30 & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 5 & 0 \\ 0 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 5 & 0 \\ 0 & -30 \end{vmatrix}$ $= -150\hat{k}$
3. The vector product is in component form. Determine the magnitude of $\underset{\sim}{r} \times \underset{\sim}{F}$.	$\lvert\underset{\sim}{r} \times \underset{\sim}{F}\rvert = \lvert -150\rvert$ $= 150$ Nm The torque is 150 Nm.
b. 1. Force needs to be expressed as a vector in component form. $\underset{\sim}{F} = \lvert\underset{\sim}{F}\rvert(\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath})$	**b.** $\underset{\sim}{r} = \overrightarrow{OP} = 3\hat{\imath} + 2\hat{\jmath}$ $\underset{\sim}{F} = 50\cos(70°)\hat{\imath} - 50\sin(70°)\hat{\jmath}$ $= 17.10\hat{\imath} - 46.98\hat{\jmath}$

2. State the torque formula. Substitute in the vectors, $\underset{\sim}{r}$ and $\underset{\sim}{F}$.

$$\text{Torque} = |\underset{\sim}{r} \times \underset{\sim}{F}|$$
$$= |(3\hat{\imath} + 2\hat{\jmath}) \times (17.10\hat{\imath} - 46.98\hat{\jmath})|$$

3. Calculate the vector product using the determinant method.

$$|\underset{\sim}{r} \times \underset{\sim}{F}| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 2 & 0 \\ 17.10 & -46.98 & 0 \end{vmatrix}$$
$$= \hat{\imath}\begin{vmatrix} 2 & 0 \\ -46.98 & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 3 & 0 \\ 17.10 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 2 \\ 17.10 & -46.98 \end{vmatrix}$$
$$= -175.2\hat{k}$$

4. The vector product is in component form. Determine the magnitude of $\underset{\sim}{r} \times \underset{\sim}{F}$.

$$|\underset{\sim}{r} \times \underset{\sim}{F}| = |-175.2|$$
$$= 175.2 \text{ Nm}$$

The torque is 175.2 Nm.

Alternatively, using the column vector notation.

a. 1. Write the vectors $\underset{\sim}{r}$ and $\underset{\sim}{F}$ using the column vector notation.

a. $$(\underset{\sim}{r}) = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, (\underset{\sim}{F}) = \begin{pmatrix} 0 \\ -30 \\ 0 \end{pmatrix}$$

2. Recall the torque formula and substitute in the vectors $\underset{\sim}{r}$ and $\underset{\sim}{F}$.

$$\text{Torque} = \left|\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -30 \\ 0 \end{pmatrix}\right|$$

3. Calculate the cross product.

$$\text{Torque} = \left|\begin{pmatrix} 0 \times 0 - 0 \times (-30) \\ 0 \times 0 - 5 \times 0 \\ 5 \times (-30) - 0 \times 0 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} 0 \\ 0 \\ -150 \end{pmatrix}\right|$$

4. Calculate the norm of the cross product and include units.

$$= 150 \text{ Nm}$$

State the answer.

The torque is 150 Nm

b. 1. Write the vectors $\underset{\sim}{r}$ and $\underset{\sim}{F}$ using the column vector notation.

b. $$(\underset{\sim}{r}) = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, (\underset{\sim}{F}) = \begin{pmatrix} 50\cos(70°) \\ -50\sin(70°) \\ 0 \end{pmatrix}$$

2. Recall the torque formula and substitute in the vectors $\underset{\sim}{r}$ and $\underset{\sim}{F}$.

$$\text{Torque} = \left|\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 50\cos(70°) \\ -50\sin(70°) \\ 0 \end{pmatrix}\right|$$

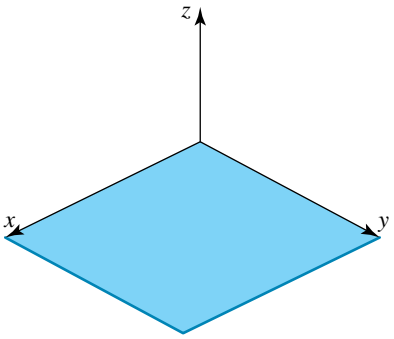
3. Calculate the cross product.

$$\text{Torque} = \left|\begin{pmatrix} 2 \times 0 - 0 \times (-50\sin(70°)) \\ 0 \times 50\cos(70°) - 3 \times 0 \\ 3 \times (-50\sin(70°)) - 2 \times 50\cos(70°) \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} 0 \\ 0 \\ -50(3\sin(70°) + 2\cos(70°)) \end{pmatrix}\right|$$

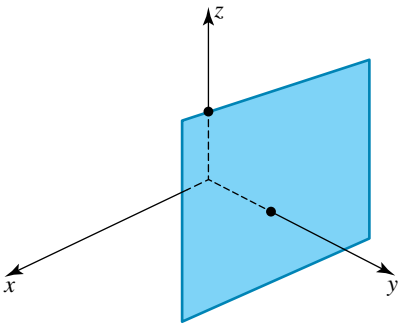
4. Calculate the norm of the cross product and include units.	$= 175.2$ Nm
5. State the answer.	The torque is 175.2 Nm

3.7.4 The vector and Cartesian equation of a plane in three dimensions

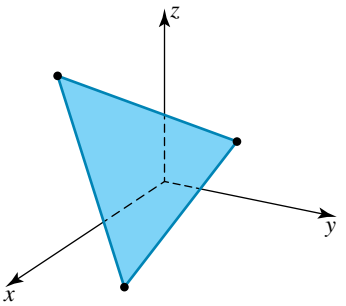
A two dimensional line can be represented by the scalar equation $y = mx + c$. In three dimensional space, the line between two points is modelled by the vector equation of a line, $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{d}$. In this section we will model a plane in three dimensions through the application of vectors. A plane can be described as a 'flat' 2 dimensional surface extending indefinitely. To help you visualise a plane, three graphical examples are shown below.



The xy plane, where the z-coordinate is always zero.



A plane parallel to the xz plane, where the y-coordinate is a constant.



A plane that cuts through all three axes.

You've already seen in Unit 1, Topic 5 Matrices (Lesson 5.7 Matrices equations of the U1 & 2 title of this series) how to use 2×2 matrices to solve systems of linear equations as well as the geometrical interpretation of the number of solutions (whether two lines are parallel, concurrent or intersect).

Note that this mathematical knowledge from Unit 1 is extended beyond 2×2 matrices in Unit 3, Topic 5 Further matrices (Chapter 5 of this title).

You will find the mathematical knowledge and skills in that chapter useful for this section of Chapter 3, particularly the three cases for solutions of systems of linear equations, and Gaussian techniques of elimination.

The section in Chapter 5 on the use of technology for matrix calculation is a useful resource for this section.

Determining the equation of a plane using a point and the normal

The equation of a plane written in Cartesian form, is expressed as $ax + by + cz + d = 0$. To calculate the equation of a plane, two important features are required:

1. A point $P(x, y, z)$ that exists on the plane.
2. A vector that is perpendicular to the plane. Remember this vector is called the normal.

Consider the diagram shown. The locations of points P and P_0 can be defined by position vectors.

That is, $P(x, y, z) = \underset{\sim}{r}$ and $P_0(a, b, c) = \underset{\sim}{a}$. The vector connecting the two points can then be defined as $\overrightarrow{P_0P} = \underset{\sim}{r} - \underset{\sim}{a}$.

As we are asserting that $\underset{\sim}{r}-\underset{\sim}{a}$ lies on the plane it will always be perpendicular to the normal. That is,

$$(\underset{\sim}{r}-\underset{\sim}{a})\cdot\underset{\sim}{n}=0$$

This is defined as the **vector equation of a plane**.

Vector equation of a plane

The vector equation of a plane is given by

$$(\underset{\sim}{r}-\underset{\sim}{a})\cdot\underset{\sim}{n}=0$$

This can be written as

$$\underset{\sim}{r}\cdot\underset{\sim}{n}=\underset{\sim}{a}\cdot\underset{\sim}{n}$$

To translate this into Cartesian form, the vector, $\underset{\sim}{r}-\underset{\sim}{a}$ is to be expressed in terms of its Cartesian coordinates.

$$\begin{aligned}\overrightarrow{\mathrm{P_0P}} &= \underset{\sim}{r}-\underset{\sim}{a}\\ \underset{\sim}{r}-\underset{\sim}{a} &= \left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)-\left(a\hat{\imath}+b\hat{\jmath}+c\hat{k}\right)\\ &= (x-a)\hat{\imath}+(y-b)\hat{\jmath}+(z-c)\hat{k}\end{aligned}$$

In component form, the normal vector is $\underset{\sim}{n}=n_1\hat{\imath}+n_2\hat{\jmath}+n_3\hat{k}$. The dot product calculation results in:

$$\begin{aligned}(\underset{\sim}{r}-\underset{\sim}{a})\cdot\underset{\sim}{n} &= 0\\ \left[(x-a)\hat{\imath}+(y-b)\hat{\jmath}+(z-c)\hat{k}\right]\cdot\left[n_1\hat{\imath}+n_2\hat{\jmath}+n_3\hat{k}\right] &= 0\\ n_1(x-a)+n_2(y-b)+n_3(z-c) &= 0\end{aligned}$$

This is the equation of a plane containing the point, $\mathrm{P}_0\,(a,b,c)$, that is perpendicular to

$$\underset{\sim}{n}=n_1\hat{\imath}+n_2\hat{\jmath}+n_3\hat{k}$$

The equation can then be transformed into Cartesian form, $ax+by+cz+d=0$, by expanding out and collecting like terms. It is interesting to note that the coefficients in this form are the components of the normal vector.

Cartesian equation of a plane

The equation of a plane containing the points $\mathrm{P}(x,y,z)$ and $\mathrm{P_0}(a,b,c)$ that is perpendicular to the normal $\underset{\sim}{n}=n_1\hat{\imath}+n_2\hat{\jmath}+n_3\hat{k}$ is given by

$$n_1(x-a)+n_2(y-b)+n_3(z-c)=0$$

WORKED EXAMPLE 37 Calculating the Cartesian equation of a plane 1

Calculate the equation of the plane perpendicular to $\underset{\sim}{n} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$, containing the point $P_0\,(2, 4, -1)$ using a dot product calculation.

THINK	WRITE
1. Define the vectors $\underset{\sim}{r}$, $\underset{\sim}{a}$ and $\underset{\sim}{n}$.	$\underset{\sim}{r} = (x, y, z)$, $\underset{\sim}{a} = (2, 4, -1)$ and $\underset{\sim}{n} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$
2. Determine the vector $\underset{\sim}{r} - \underset{\sim}{a}$ by applying the formula $\overrightarrow{P_0P} = \underset{\sim}{r} - \underset{\sim}{a}$ $= (x-a)\hat{\imath} + (y-b)\hat{\jmath} + (z-c)\hat{k}$	$(\underset{\sim}{r} - \underset{\sim}{a}) = (x-2)\hat{\imath} + (y-4)\hat{\jmath} + (z+1)\hat{k}$
3. Recall the formal dot product definition for a plane and substitute in the vectors.	$(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$ $\left((x-2)\hat{\imath} + (y-4)\hat{\jmath} + (z+1)\hat{k}\right) \cdot \left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right) = 0$
4. Complete the dot product calculations.	$2\,(x-2) + 3\,(y-4) + 4\,(z+1) = 0$
5. Expand and simplify.	$2x - 4 + 3y - 12 + 4z + 4 = 0$ $2x + 3y + 4z - 12 = 0$

The procedure completed in Worked example 37 was very detailed. The same result can be achieved in fewer steps by recalling the formula, $n_1(x-a) + n_2(y-b) + n_3(z-c) = 0$. This is demonstrated in the next example and is the preferred option unless specified otherwise.

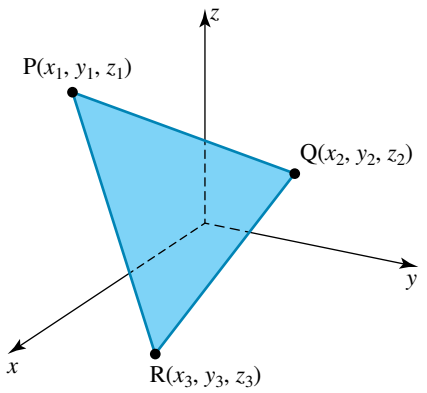
WORKED EXAMPLE 38 Calculating the Cartesian equation of a plane 2

Determine the equation of the plane containing the point $(-5, 3, 6)$ if the vector $\underset{\sim}{n} = 2\hat{\imath} + 4\hat{\jmath} - 7\hat{k}$ is normal to it.

THINK	WRITE
1. Recall the general formula for the equation of a plane.	$n_1(x-a) + n_2(y-b) + n_3(z-c) = 0$
2. Substitute the components of point (a, b, c) and $\underset{\sim}{n}$.	$2(x+5) + 4(y-3) - 7(z-6) = 0$
Expand and collect like terms.	$2x + 10 + 4y - 12 - 7z + 42 = 0$ $2x + 4y - 7z + 40 = 0$
3. State your answer.	The equation of the plane is $2x + 4y - 7z + 40 = 0$

Determining the equation of a plane using three points

In both worked examples above, the equation was calculated based on a known point on the plane and a vector normal to it. Consider the plane shown, defined by the three non-collinear points. A normal vector can be calculated by applying our knowledge of the vector product. The following example demonstrates this procedure.



WORKED EXAMPLE 39 Calculating the Cartesian equation of a plane 3

a. Calculate the Cartesian equation of the plane determined by the points P(−2, 1, 3), Q(3, 5, 6) and R(−5, 3, 5).

b. Determine the axis intercepts for the plane.

THINK	WRITE
a. 1. Calculate and define two vectors, starting from the same point, relating to the points P, Q and R.	**a.** $\overrightarrow{PQ} = (3 - -2)\hat{\imath} + (5-1)\hat{\jmath} + (6-3)\hat{k}$ $= 5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ $= \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$ $\overrightarrow{PR} = (-5 - -2)\hat{\imath} + (3-1)\hat{\jmath} + (5-3)\hat{k}$ $= -3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ $= \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$
2. Determine a vector perpendicular to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. This can be found by calculating $\overrightarrow{PQ} \times \overrightarrow{PR}$. This is calculated as the determinant or alternatively using the column vector notation.	$\underset{\sim}{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ $\underset{\sim}{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 4 & 3 \\ -3 & 2 & 2 \end{vmatrix}$ $\underset{\sim}{n} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 4 \times 2 - 3 \times 2 \\ 3 \times (-3) - 5 \times 2 \\ 5 \times 2 - 4 \times (-3) \end{pmatrix}$ $= \begin{pmatrix} 2 \\ -19 \\ 22 \end{pmatrix}$
3. State the normal vector.	$\underset{\sim}{n} = 2\hat{\imath} - 19\hat{\jmath} + 22\hat{k}$

4. Recall the general formula for the equation of a plane.

$$n_1(x-a)+n_2(y-b)+n_3(z-c)=0$$

5. Substitute the components of any of the points P, Q or R, for instance point P(−2, 1, 3) and the normal vector into the formula.

$$2(x+2)-19(y-1)+22(z-3)=0$$
$$2x+4-19y+19+22z-66=0$$
$$2x-19y+22z-43=0$$

6. State your answer.

$$\therefore 2x-19y+22z-43=0$$

b. 1. Calculate the x-intercept $(x, 0, 0)$ when $y=0, z=0$.

b. Let $y=0$ and $z=0$,

$$2x-19(0)+22(0)=43$$
$$x=\frac{43}{2}$$

2. Calculate the y-intercept $(0, y, 0)$ when $x=0, z=0$.

Let $x=0$ and $z=0$,

$$2(0)-19y+22(0)=43$$
$$y=-\frac{43}{19}$$

3. Calculate the z-intercept $(0, 0, z)$ when $x=0, y=0$.

Let $x=0$ and $y=0$,

$$2(0)-19(0)+22z=43$$
$$z=\frac{43}{22}$$

4. State the three intercepts.

The plane intersects the axes at

$$\left(\frac{43}{2}, 0, 0\right), \left(0, -\frac{43}{19}, 0\right) \text{ and } \left(0, 0, \frac{43}{22}\right)$$

TIP

If the equation of a plane is written in the form $ax+by+cz=d$, the intercepts can be calculated by the following formulae, $x=\frac{d}{a}, y=\frac{d}{b}, z=\frac{d}{c}$.

Exercise 3.7 Applications of vectors

learnon

3.7 Exercise	3.7 Exam practice on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16, 17	18, 19, 20, 21, 22

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Simple familiar

1. WE33 A particle travels 16 cm in the negative x-direction, 10 cm in the positive y-direction and 9 cm in the negative z-direction in 5 seconds.
   a. Define the velocity vector.
   b. Calculate the speed of the particle. Give your answer to 2 decimal places.

2. **WE34** A first particle is travelling on a velocity vector $\underset{\sim}{a} = \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix}$ cm/s while a second particle is travelling on a velocity vector $\underset{\sim}{b} = \begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$ cm/s.

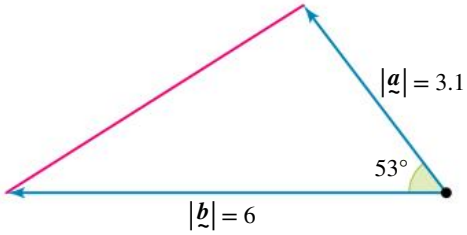
   a. Determine the velocity of the first particle relative to the second one. Give your answer in column vector form.
   b. Determine the velocity of the second particle related to the first one. Give your answer in column vector form.

3. Triangular shaped shade sails are often used in parks and schools. The vertices of a shade sail can be located by the points L(3, −5, 3), M(−4, −1, 4) and N(1, 2, 5), where all units are in metres.

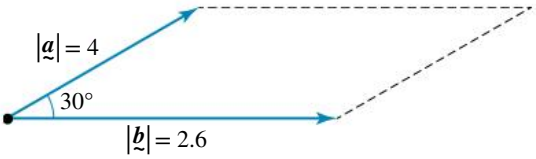
   a. Develop an equation to model the plane created by the erected triangular shade cloth. State the assumptions considered true when developing models and solutions.
   b. Calculate the area of the shade cloth to 2 decimal places.

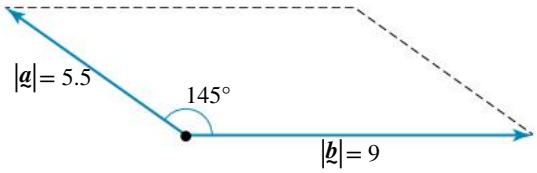
4. **WE35b** Use the formula Area $= \frac{1}{2}|\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)$ to calculate the area of the following triangles.

   a.

   b.


5. **WE35a** Use the vector formula, Area $= |\underset{\sim}{a}|\,|\underset{\sim}{b}|\sin(\theta)$ to determine the area of the parallelograms formed by the vectors with adjacent sides, $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

   a.


   b.


6. **WE36** A force, $\underset{\sim}{F}$, acts on an object at a point, P. In each of the following cases, calculate the torque about O. The coordinate location of points is measured in metres.

   a. P(10, 0, 0) and $\underset{\sim}{F} = -20\hat{k}$ N
   b. P(6, 1, −2) and $\underset{\sim}{F} = (4\hat{\imath} - 7\hat{\jmath} + \hat{k})$N
   c. P(2, −5, 6) and $\underset{\sim}{F} = (-5\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$N

7. **WE37&38** Calculate the equation of the plane perpendicular to $\underset{\sim}{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$, containing the point $P_0 = (x_0, y_0, z_0)$.

   a. $P_0 = (3, 6, -2)$ and $\underset{\sim}{n} = 5\hat{\imath} - 2\hat{\jmath} + \hat{k}$
   b. $P_0 = (-2, 1, 4)$ and $\underset{\sim}{n} = -3\hat{\imath} + 5\hat{\jmath} + 4\hat{k}$
   c. $P_0 = (-1, 0, 3)$ and $\underset{\sim}{n} = -7\hat{\imath} + 4\hat{\jmath} - 6\hat{k}$
   d. $P_0 = (4, -1, -4)$ and $\underset{\sim}{n} = -4\hat{\imath} - \hat{\jmath} + 3\hat{k}$

8. Determine a unit vector $\hat{\underset{\sim}{n}}$, which is normal to each of the following planes.

a. $4x - 2y - z = 12$ b. $-x + 2y - 12 = 2z$ c. $4 = -6x + 7y - 3z$

9. WE39 Calculate the Cartesian equation of the plane determined by the points below and determine the axis intercepts for each plane.

a. P(1, 4, 2), Q(2, 3, 5) and R(1, 9, 8)
b. P(−2, 5, 3), Q(3, −1, 2) and R(−7, 1, 0)

10. For each of the following, determine if the point P (x, y, z), lies on the given plane.

a. P(2, 9, −1), plane: $4x - 2y + z = -11$
b. P(−1, −6, 4), plane: $3x - z + 1 = 10$
c. P(1, 4, −7), plane: $-x + 3y + 5z = -24$

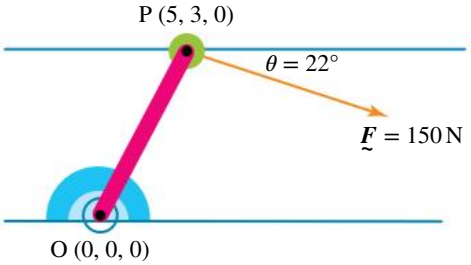
11. Explain why the equation of a plane cannot be formed by the points P(−2, 6, −18), Q(−12, 8, −68) and R(13, 3, 57).

12. The vector equation of the line, $\underset{\sim}{r} = \left(-3\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}\right) + k\left(2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}\right)$, defines the path of particle. Determine the point of intersection between the particle and the plane, $7x + 4y + 8z = 1$.

Complex familiar

13. The vertices of a triangle are defined by the vectors, $\underset{\sim}{a} = 7\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}}$, $\underset{\sim}{b} = \hat{\boldsymbol{\imath}} + 14\hat{\boldsymbol{\jmath}} - 8\hat{\boldsymbol{k}}$ and $\underset{\sim}{c} = 8\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$. Calculate the area using the rule $A = \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$.

14. To rotate the level arm shown on the right, the turning force must be greater than 400 Nm. Calculate the torque about the point O, given the force, $\underset{\sim}{F} = \left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right)$, acting on the object at P.



15. Two planes are parallel if both normal vectors are parallel. Similarly, if two planes are perpendicular, then so too are their normal vectors.

a. Examine the two planes and determine if they are parallel. P_1: $4x - 2y + 4z = 15$ and P_2: $2x - y + 2z = -23$.
b. Determine the equation of the plane that is parallel to $11x - 2y + 6z = 12$ and contains the point P(1, −8, 2).
c. Calculate the value of c if the two planes, P_1: $8x - 6y - 2z = 1$ and P_2: $2x - y + cz = -6$, are perpendicular.

16. a. Calculate the equation of the plane that is perpendicular to the plane $8x + 2y + z = 0$, passing through the points P(2, 3, 4) and Q(5, 7, 8).
b. Justify that the intersection of the two planes form a right-angle.

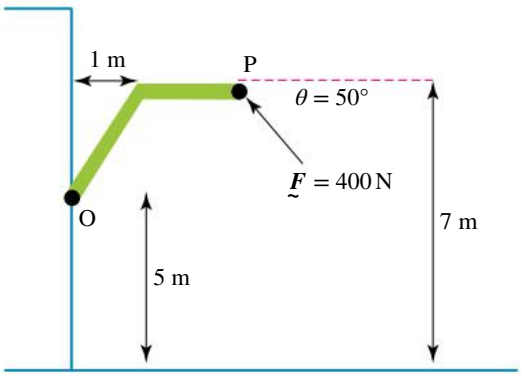
17. The parametric equation, $x = 2 + 4t$, $y = -4 - 2t$, $z = 3 + 3t$, defines the path of particle. Determine the point of intersection between the particle and the plane, $-4x + 3y + 7z = 21$.

Complex unfamiliar

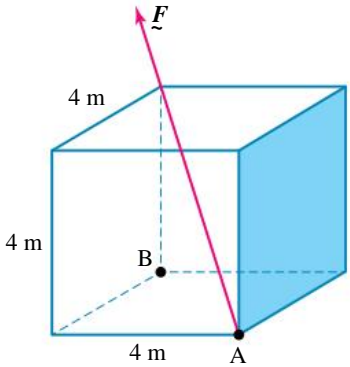
18. Verify through suitable calculations, that the quadrilateral with vertices defined by $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ is a parallelogram. Hence, calculate its area using determinants.

$$\begin{aligned}
\underset{\sim}{a} &= 6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} \\
\underset{\sim}{b} &= 3\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}} \\
\underset{\sim}{c} &= 3\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} + 9\hat{\boldsymbol{k}} \\
\underset{\sim}{d} &= 6\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}
\end{aligned}$$

19. A metal bracket is fixed to the wall at a rotatable point O. The force, $\underset{\sim}{F}$, is applied at P. The magnitude of the torque at point O is 1586.7 Nm.
Calculate the vector location of point P.



20. A force of 400 N is applied on the cube at point A, as shown. Calculate the torque of the force about B to 2 decimal places. Evaluate the reasonableness of your result.



21. a. A plane is perpendicular to the vector $\underset{\sim}{n}$ and contains the point P. Prove that the shortest distance, d, from the plane to a point Q is given by $d = \dfrac{|\underset{\sim}{n} \cdot \overrightarrow{PQ}|}{|\underset{\sim}{n}|}$.

b. Apply the formula, $d = \dfrac{|\underset{\sim}{n} \cdot \overrightarrow{PQ}|}{|\underset{\sim}{n}|}$, to calculate the shortest distance from Q(6, 5, 4) to the plane, $x + 5y - 2z = 8$. (Hint, P (x, y, z) is any point on the plane)

c. If a plane is in the form, $ax + by + cz = d$, the shortest distance to a point $Q = (x_1, y_1, z_1)$ can be calculated by the equation, $d = \dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$. Evaluate the reasonableness of your result.

22. A line is formed when two non-parallel planes intersect. A vector in the direction of the line of intersection can be calculated by the vector product, $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$. This implies that the vector product of the two normals are parallel to the line of intersection. Calculate the parametric equation for the line of intersection of the planes, $8x - 9y - 2z = 4$ and $-6x + 2y - z = 13$.

Fully worked solutions for this chapter are available online.

LESSON
3.8 Review

3.8.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

- Access the chapter summary
- Review your results
- Practise exam questions

Find all this and MORE in jacPLUS

3.8 Exercise

learnon

3.8 Exercise	3.8 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. Express the vector $\underset{\sim}{u} = (15, 325°, 36°)$ in component form.
2. Consider the points P(−2, 5, 1), Q(5, 8, 4) and R(9, −6, 12). Determine the resultant vector of the vector sum, $2\overrightarrow{OP} + \overrightarrow{QP} - 4\overrightarrow{QR}$.
3. Determine the angle between the given vectors, $\underset{\sim}{a} = 2\hat{\imath} - 5\hat{\jmath} + 8\hat{k}$ and $\underset{\sim}{b} = 9\hat{\imath} + 7\hat{\jmath} - 8\hat{k}$.
4. A sphere contains the points, A(7, 13, 8) and B(−1, 3, 2), that are the endpoints of its diameter. Calculate the Cartesian equation of the sphere.
5. Determine the equivalent Cartesian equation for $x = 2t - 7, y = \frac{t^2}{7} + t$.
6. Calculate $\underset{\sim}{a} \times \underset{\sim}{b}$ for the given vectors, $\underset{\sim}{a} = (19, 7°)$ and $\underset{\sim}{b} = (24, 145°)$.
7. OABC is a square. The points P, Q, R and S are the midpoints of the sides OA, AB, BC and OC respectively. Prove that PQRS is a square.
8. Calculate a vector perpendicular to both $\underset{\sim}{a} = 5\hat{\imath} + 6\hat{\jmath} - 4\hat{k}$ and $\underset{\sim}{b} = 4\hat{\imath} - 9\hat{\jmath} + 7\hat{k}$.
9. The two vectors $\underset{\sim}{a} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\underset{\sim}{b} = a\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$ exist in the same three dimensional space, and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 28$. Calculate $\underset{\sim}{a} \times \underset{\sim}{b}$ to determine if it is possible to find a unit vector orthogonal to $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
10. Determine the equation of the plane containing the point (−2, 9, 1) if the vector $\underset{\sim}{n} = 3\hat{\imath} + 5\hat{\jmath} + 8\hat{k}$ is normal to it.
11. Calculate the equation of the plane determined by the following points: P(3, 8, 9), Q(4, −8, −5) and R(−5, 6, 2).

12. Calculate the Cartesian equation of the plane determined by the points below and determine the axis intercepts for each plane.

a. P$(-1, 4, 8)$, Q$(1, 8, -3)$ and R$(12, 9, -5)$

b. P$(-6, 7, 1)$, $(-9, 2, 4)$ and R$(3, -3, 8)$

Complex familiar

13. From base camp, a group of explorers can observe a famous ruin that can be located by the position vector, $\underset{\sim}{v} = 150\hat{\boldsymbol{\imath}} + 160\hat{\boldsymbol{\jmath}} + 650\hat{\boldsymbol{k}}$, where all units are in metres.
Justify whether the path from camp to the ruin is reasonable.

14. **a.** Calculate the vector equation for the line passing through the given points, A$(5, -2, 13)$ and B$(6, 4, 7)$.

b. Determine the value for $y \times z$ if the head of vector, $\underset{\sim}{c} = 12\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$, touches the line from part **a.** at P(x, y, z).

15. Consider the vectors $\underset{\sim}{a} = \left(5\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} - \lambda\hat{\boldsymbol{k}}\right)$ and $\underset{\sim}{b} = \left(\lambda\hat{\boldsymbol{\imath}} - 9\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}\right)$ where $\lambda \in \mathbb{R}$.

Determine the vector that is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and is also collinear with $\begin{pmatrix} 1 \\ 20 \\ 5 \end{pmatrix}$

16. Given $\underset{\sim}{a} = 3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 9\hat{\boldsymbol{k}}$, $\underset{\sim}{b} = 8\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{k}}$ and $\underset{\sim}{c} = 6\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}$, prove that the vector product is not associative. That is, $\underset{\sim}{a} \times (\underset{\sim}{b} \times c) \neq (\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c}$.

Complex unfamiliar

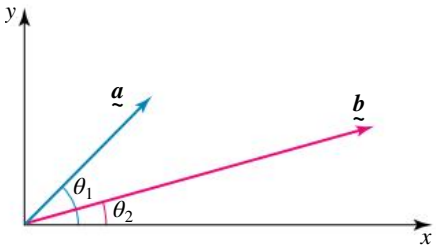
17. AB and CD are two diameters of a circle with centre O. Letting $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$, prove that ABCD is a rectangle.

18. A golfer is trying to hit the golf ball as far as possible from ground level. The path of the golf ball, according to time t (seconds), can be modelled by the parametric equations,

$$x = (v_0\cos(\theta))\,t,\ \ y = (v_0\sin(\theta))\,t - \frac{1}{2}gt^2 \qquad \text{(where } g = 9.8\text{)}$$

When the golfer strikes the ball, $\theta = 30°$ and $v_0 = 66\ \text{ms}^{-1}$.
Calculate the maximum height the ball reaches and when it strikes the ground. State any assumptions considered to be true when developing models and/or solutions. Justify the reasonableness of the results by considering the location of the golf ball when it returns to ground level.

19. Consider the two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, in the diagram below.
Verify the identity $\sin(\theta_a - \theta_b) = \sin(\theta)_a\cos(\theta)_b - \cos(\theta)_a\sin(\theta)_b$.
The proof should be constructed by calculating the vector product, $\underset{\sim}{a} \times \underset{\sim}{b}$.



20. Calculate the equation of the plane containing the point P$(-3, 2, 5)$, and containing the line of intersection of the planes $x - 2y + 4z = 3$ and $-3x + 5y - 4z = -5$. Technology can be used to determine a particular point on the line formed by the intersection of the planes.

3.8 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 1, Q1; © State of Queensland (QCAA) 2024.

The position of a particle is given by $r = (t+2)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$ for $t \geq 0$.

Determine the corresponding Cartesian equation.

A. $y = x^2 - 4$

B. $y = x^2 + 4$

C. $y = x^2 - 4x + 4$

D. $y = x^2 + 4x + 4$

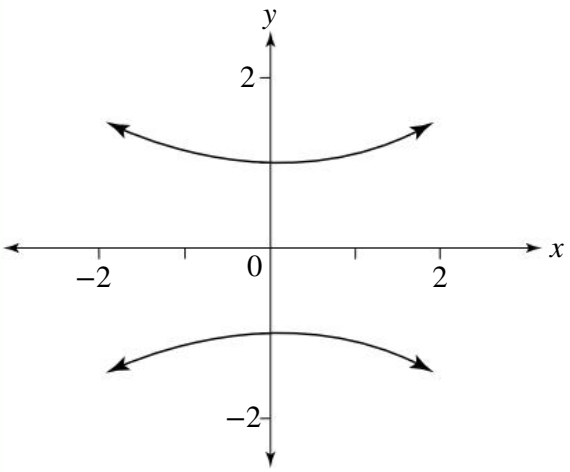
Question 2 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 1, Q4; © State of Queensland (QCAA) 2024.

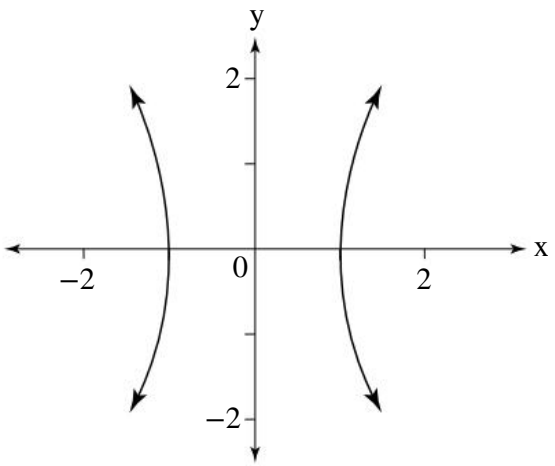
The position of a particle can be modelled using $r = \cos(t)\hat{\boldsymbol{i}} - 2\sin(t)\hat{\boldsymbol{j}},\ t \geq 0$.

Which curve best represents the path of the particle?

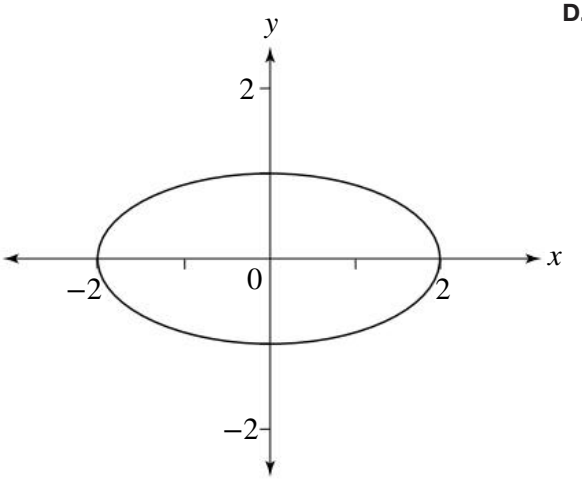
A.



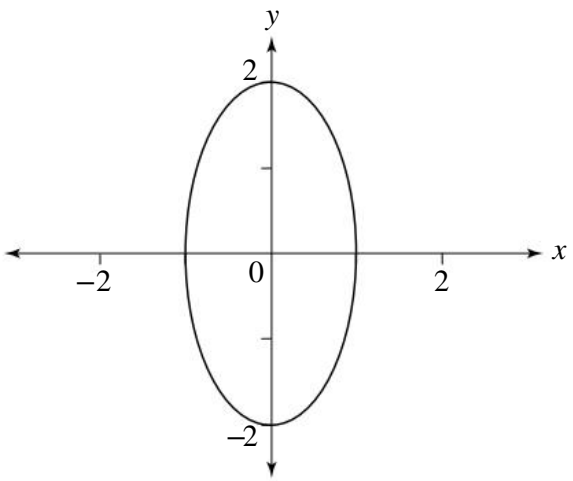
B.



C.



D.



Question 3 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 1, Q5; © State of Queensland (QCAA) 2024.

A plane contains the origin and the points $(1, 2, 3)$ and $(3, 2, 1)$.

A vector normal to the plane is

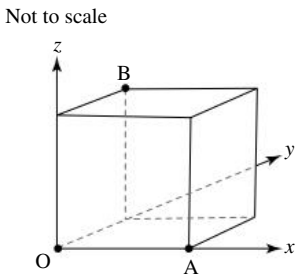
A. $\begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}$ **B.** $\begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}$ **C.** $\begin{pmatrix} -4 \\ -8 \\ -4 \end{pmatrix}$ **D.** $\begin{pmatrix} -4 \\ -8 \\ 4 \end{pmatrix}$

Question 4 (6 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 2, Q14; © State of Queensland (QCAA) 2024.

Consider a cube with three edges positioned along the x-, y- and z-axes on the Cartesian plane as shown.

Points O, A and B are vertices of the cube.



a. Given $\overrightarrow{OA} = 2\hat{\boldsymbol{i}}$, determine $\overrightarrow{OB}$. Express your answer in terms of $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$. **[1 mark]**

b. Calculate $\overrightarrow{OA} \times \overrightarrow{OB}$. **[1 mark]**

Consider the triangle formed by joining points O, A and B.

c. Use the result from Question 14b to determine the area of the triangle. **[2 marks]**

Let points M and N be the midpoints of the triangle's sides OA and OB respectively.

d. Determine $\overrightarrow{MN}$. **[1 mark]**

e. Use the result from Question 14d to show that the length of AB is twice the length of MN. **[1 mark]**

Question 5 (5 marks)

Source: QCAA 2022 Specialist Mathematics, Paper 2, Section 2, Q15; © State of Queensland (QCAA) 2024.

Consider points $A(3, -1, 3)$ and $B(1, 1, 6)$.

a. Determine $\overrightarrow{AB}$. **[1 mark]**

b. Determine the Cartesian equation of the line that passes through points A and B. **[2 marks]**

Point A lies on the plane, φ, and $\overrightarrow{AB}$ is perpendicular to this plane.

c. Determine the Cartesian equation of the plane. **[2 marks]**

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Answers

Chapter 3 Vectors in two and three dimensions

3.2 Vectors in three dimensions

3.2 Exercise

1. a. Vector b. Scalar c. Vector
   d. Scalar e. Scalar
2. a. $\overrightarrow{CE} = \underset{\sim}{r} + \underset{\sim}{t}$
   b. $\overrightarrow{CI} = \underset{\sim}{r} + \underset{\sim}{t} + \underset{\sim}{s}$
   c. $\overrightarrow{DJ} = -\underset{\sim}{r} + \underset{\sim}{s} + \underset{\sim}{t}$
3. a. A$(5, -4)$ b. B$(-2, 6)$
   c. C$\left(\frac{5}{2}, 11, -12\right)$ d. D$(-4, -1, 8)$
4. a. $5\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ b. $2\hat{\imath} - 6\hat{k}$
   c. $-\frac{2}{5}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{22}{17}\hat{k}$ d. $-11\hat{\imath} + \hat{\jmath} - 4\hat{k}$
5. a. A$(2, 2, 0)$ b. B$(2, -4, -2)$ c. C$(2, -4, 0)$
   d. D$(0, 4, 0)$ e. E$(-3, 4, 0)$
6. The aeroplane is 1081.67 km at a bearing 146.31°T from its starting location.
7. a. $8\hat{\imath} + 4\hat{\jmath} + \hat{k}$ b. 9 c. $\frac{1}{9}\left(8\hat{\imath} + 4\hat{\jmath} + \hat{k}\right)$
8. a. $6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$ b. $2\sqrt{26}$
   c. $\frac{1}{\sqrt{26}}\left(3\hat{\imath} + \hat{\jmath} - 4\hat{k}\right)$ d. $\frac{7}{\sqrt{26}}\left(3\hat{\imath} + \hat{\jmath} - 4\hat{k}\right)$
9. a. $5\hat{\imath} + \hat{\jmath} + 8\hat{k}$ b. $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ c. $-17\hat{\imath} + 2\hat{\jmath} - 29\hat{k}$
10. C
11. B
12. D
13. a. $2.47\hat{\imath} + 4.45\hat{\jmath} + 3.18\hat{k}$
    b. $-7.81\hat{\imath} + 3.98\hat{\jmath} - 17.98\hat{k}$
14. a. $-1.15\hat{\imath} + 1.99\hat{\jmath} - 1.93\hat{k}$
    b. $9.94\hat{\imath} - 6.96\hat{\jmath} + 8.82\hat{k}$
15. a. $\varphi = -29.12°$
    b. $\left(\sqrt{38}, 68.20°, -29.12°\right)$
16. a. $\left(\sqrt{29}, 306.87°, 21.80°\right)$
    b. $\left(\sqrt{110}, 50.19°, 41.87°\right)$
    c. $\left(\sqrt{141}, 128.66°, -57.37°\right)$
17. The helicopter has travelled 269.14 metres when it reaches the closest point. The distance to the mountain climber at that point is 8.13 metres.
18. $\varphi = 37.99°$
19. $\theta = 69.09°$
20. The altitude of the drone when the photo is taken is 0.357 km.

3.3 Algebra of three dimensional vectors

3.3 Exercise

1. a. $-4\hat{\imath} + 2\hat{\jmath} - 7\hat{k}$ b. $-6\hat{\imath} + 10\hat{\jmath} - 2\hat{k}$
   c. $\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ d. $22\hat{\imath} - 18\hat{\jmath} + 30\hat{k}$
2. a. $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ b. $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 8 \\ -6 \end{pmatrix}$ c. $\underset{\sim}{u} = \begin{pmatrix} -12 \\ 22 \\ -15 \end{pmatrix}$
3. $x = \frac{31}{6}$
   $y = -\frac{27}{7}$
   $z = -39$
4. $x = -y = -z$
5. a. 15.73 b. -5.87
6. -45
7. $m = -6$
8. $y = -18$
9. $-76 = -76$
   $\therefore \underset{\sim}{c} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{b}$
10. $\theta = 75.29°$
11. $\theta = 49.45°$
12. $k = -12$
13. a. $\frac{-94}{\sqrt{197}}$
    b. $-\frac{940}{197}\hat{\imath} + \frac{846}{197}\hat{\jmath} - \frac{376}{197}\hat{k}$
    c. $-\frac{45}{197}\hat{\imath} - \frac{58}{197}\hat{\jmath} - \frac{18}{197}\hat{k}$
14. $k = 0.46$
15. $\theta = 41.41°$
16. $k = 19.20$
17. $\lambda = -2$; $\lambda\underset{\sim}{a} = -12\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$
18. $k = 2$
19. Sample response can be found in the worked solutions in the online resources.
20. $k = -1$ or $k = 3$.

3.4 Geometric proofs using vectors

3.4 Exercise

1. $\left(-2, \frac{5}{2}, \frac{3}{2}\right)$
2. $d = \sqrt{65}$
3. Sample responses can be found in the worked solutions in the online resources.
4. a. $-2\hat{\imath} + 5\hat{\jmath}$
   b. $2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$
   c. $-5\hat{\imath} + 48\hat{\jmath} - 16\hat{k}$
   d. $2\hat{\imath} + 11\hat{\jmath} + 8\hat{k}, \therefore$ M$(2, 11, 8)$
   e. $3\overrightarrow{PQ} = \overrightarrow{QS}$ Therefore, the vectors are parallel.
5. a. $\sqrt{274}$ b. $\sqrt{146}$ c. $\sqrt{233}$ d. $\sqrt{206}$

6. a. $(7, 7, -1)$ **b.** $(-3, 4, -7)$

c. $(4, -6, 4)$ **d.** $\left(\frac{11}{2}, -2, \frac{-9}{2}\right)$

7. $5\hat{\imath} + 6\hat{\jmath} - \hat{k}$

8. $\mathrm{T}\left(\frac{25}{6}, -8, \frac{47}{6}\right)$

9–10. Sample responses can be found in the worked solutions in the online resources.

11. $\frac{1}{2}\sqrt{870}$ units squared

12–20. Sample responses can be found in the worked solutions in the online resources.

3.5 Vector, parametric and Cartesian equations

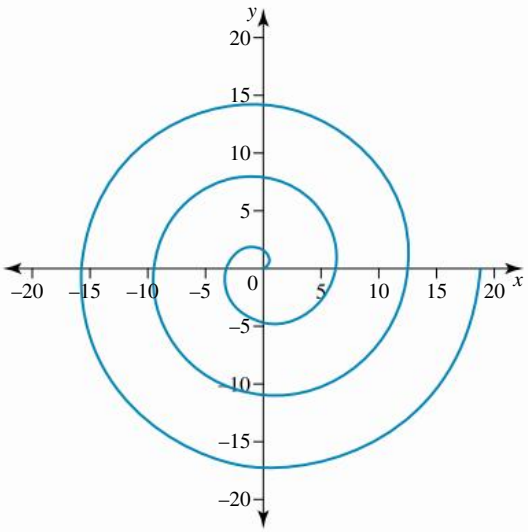
3.5 Exercise

1. A $(-5, 0, 2)$, B $(2, 2, -4)$ and C $(0, -4, -2)$

2. a. $y = \frac{x}{15}$ **b.** $y = \frac{x^2}{16} - \frac{x}{4}$

c. $y = \frac{x^3}{27} + 2$ **d.** $x^2 + y^2 = 4$

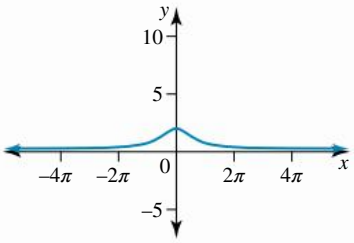
3. a. $\underset{\sim}{r}(t) = t\cos(t)\,\hat{\imath} + t\sin(t)\,\hat{\jmath}$, for $0 \le t \le 6\pi$



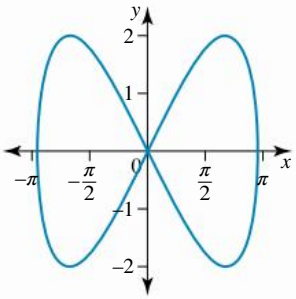
b. $\underset{\sim}{r}(t) = 4(t - \sin(t))\,\hat{\imath} + 4(1 - \cos(t))\,\hat{\jmath}$, for $-2\pi \le t \le 4\pi$

See the figure at the bottom of the page*

c. $\underset{\sim}{r}(t) = 2\cot(t)\,\hat{\imath} + 2\sin^2(t)\,\hat{\jmath}$, for $-\pi \le t \le \pi$



d. $\underset{\sim}{r}(t) = 3\cos(t)\,\hat{\imath} + 2\sin(2t)\,\hat{\jmath}$, for $-\pi \le t \le \pi$



4. a. $36 = (x-2)^2 + (y-8)^2 + (z+3)^2$

b. $9 = (x+3)^2 + (y-1)^2 + (z+9)^2$

c. $49 = (x-1)^2 + (y)^2 + (z-6)^2$

d. $11 = (x+9)^2 + (y+2)^2 + (z-7)^2$

5. a. $50 = (x-3)^2 + (y-8)^2 + (z-5)^2$

b. $56 = (x+1)^2 + (y+1)^2 + (z-5)^2$

c. $254 = (x+3)^2 + (y+2)^2 + (z+10)^2$

d. $45 = (x-5)^2 + (y+6)^2 + (z-3)^2$

6. a. $x^2 + y^2 + z^2 - 10x - 14y - 2z - 69 = 0$

b. $x^2 + y^2 + z^2 + 2x - 6y + 4z - 67 = 0$

c. $x^2 + y^2 + z^2 + 6x - 2y - 5 = 0$

d. $x^2 + y^2 + z^2 - 2x - 8y + 6z - 30 = 0$

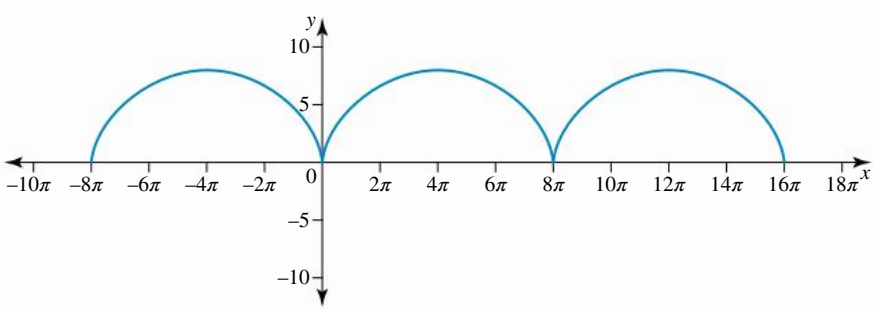
7. a. C $(5, -2, -6)$, $r = 9$

b. C $(-1, -4, 8)$, $r = 9$

c. C $(-10, 6, -12)$, $r = 10$

d. C $(-1, -2, -3)$, $r = 5$

*3b



8. a. $y = \frac{2}{3}x + \frac{11}{3}$ b. $y = -x^2 - 9x - 20$

c. $y = x^2 + 2x + 8$ d. $y = \frac{\left(\sqrt[3]{x}\right)^2}{4}$ or $y = \frac{1}{4}x^{\frac{2}{3}}$

e. $y = \frac{4x - 17}{x - 4}$

f. $y = \frac{4 - x^2}{4}$ or $y = -0.25x^2 + 1$

9. a. $x^2 + y^2 = 1$ b. $x^2 + y^2 = 1$

c. $\frac{x^2}{16} + \frac{y^2}{49} = 1$ d. $x^2 - y^2 = 1$

10. a. $\hat{\imath} + 3.5\hat{\jmath}$

b.

t	$x(t)$	$y(t)$	$\underset{\sim}{r}(t)$
0	0	0	$0\hat{\imath} + 0\hat{\jmath}$
1	1	3.5	$\hat{\imath} + 3.5\hat{\jmath}$
2	2	6	$2\hat{\imath} + 6\hat{\jmath}$
3	3	7.5	$3\hat{\imath} + 7.5\hat{\jmath}$
4	4	8	$4\hat{\imath} + 8\hat{\jmath}$

11. a. $\underset{\sim}{r} = (-2\hat{\imath} + 2\hat{\jmath} + \hat{k}) + t(2\hat{\imath} + 4\hat{\jmath} + 6\hat{k})$

b. $\underset{\sim}{r} = (4\hat{\imath} - 1\hat{\jmath} + 11\hat{k}) + t(-17\hat{\imath} + 10\hat{\jmath} - 23\hat{k})$

c. $\underset{\sim}{r} = (5\hat{\imath} + 4\hat{\jmath} + 11\hat{k}) + t(-6\hat{\imath} - 3\hat{\jmath} - 3\hat{k})$

d. $\underset{\sim}{r} = (6\hat{\imath} - 1\hat{\jmath} + 2\hat{k}) + t(-\hat{\imath} + 4\hat{\jmath} - 11\hat{k})$

e. $\underset{\sim}{r} = (-3\hat{\imath} - 2\hat{\jmath} + \hat{k}) + t(8\hat{\imath} + 9\hat{\jmath} - 14\hat{k})$

f. $\underset{\sim}{r} = (\hat{\imath} + 5\hat{\jmath} + 8\hat{k}) + t(6\hat{\imath} - 5\hat{\jmath} - 9\hat{k})$

12. a. $\underset{\sim}{r} = (-4\hat{\imath} + 7\hat{\jmath} + 6\hat{k}) + t(10\hat{\imath} - 4\hat{\jmath} - 4\hat{k})$

b. C (96, −33, −34) does lie on the line.

c. $z = 18$

13. a. $\underset{\sim}{r} = (-2\hat{\imath} + 8\hat{\jmath}) + t(7\hat{\imath} - 11\hat{\jmath})$

b. $19\hat{\imath} - 25\hat{\jmath}$

14. $\underset{\sim}{r} = (6\hat{\imath} + 5\hat{\jmath} + \hat{k}) + t(6\hat{\imath} - 2\hat{\jmath} + 3\hat{k})$

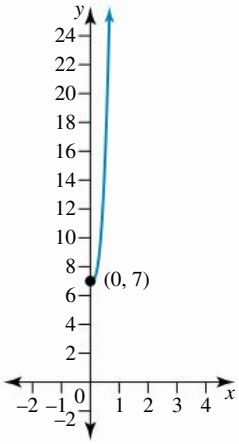
15. a. $x = -1 + 2t, y = 3 + 5t, z = 7 - 3t$

b. $x = 5 + 2t, y = 1 + 2t, z = -3 - 6t$

c. $x = 5 + 2k, y = 1 + 2t, z = -3 - 6t$

16. a. $y = 8x^3 + 7$

b. $0 \le x \le +\infty$ and $7 \le y \le +\infty$



c. $x = 2$

17. a. $\underset{\sim}{r}(0) = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

$\underset{\sim}{r}(12) = 36\hat{\imath} + 34.56\hat{\jmath} + 57.6\hat{k}$

b. 76.21 m

18. a. $x - 1 = y - 2 = z - 3$

b. $\frac{x + 2}{6} = \frac{4 - y}{9} = \frac{8 - z}{2}$

c. $\frac{x - 11}{12} = \frac{y - 3}{21} = \frac{z - 12}{13}$

19. a. $\underset{\sim}{r} = (2\hat{\imath} + 5\hat{\jmath} - 8\hat{k}) + t(3\hat{\imath} + 2\hat{\jmath} + 5\hat{k})$

b. $\underset{\sim}{r} = (-7\hat{\imath} - \hat{\jmath} + 2\hat{k}) + t(-2\hat{\imath} + 6\hat{\jmath} + 3\hat{k})$

20. Sample responses can be found in the worked solutions in the online resources.

21. a. $850\hat{\imath} + 950\hat{\imath} + 350\hat{k}$

b. The largest value for $b = 32.71$

c. 633.75 m

d. No, it never ascends in flight path, it is always descending.

22. $\theta = 39.92°$

23. a. $\left(\frac{34}{5}, \frac{41}{5}, 0\right)$

b. $\left(-\frac{44}{6}, 0, \frac{15}{6}\right)$

3.6 The vector (cross) product

3.6 Exercise

1. a. 48.86 $\hat{\underset{\sim}{n}}$ out of the page b. 14.854 $\hat{\underset{\sim}{n}}$ into the page

c. 52.5$\hat{\underset{\sim}{n}}$ into the page d. 0

2. a. 91.93$\hat{\underset{\sim}{n}}$ b. 334.72$\hat{\underset{\sim}{n}}$

c. 13.57$\hat{\underset{\sim}{n}}$ d. 22.50$\hat{\underset{\sim}{n}}$

3.

$\times$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	$\underset{\sim}{0}$	$\hat{k}$	$-\hat{\jmath}$
$\hat{\jmath}$	$-\hat{k}$	$\underset{\sim}{0}$	$\hat{\imath}$
$\hat{k}$	$\hat{\jmath}$	$-\hat{\imath}$	$\underset{\sim}{0}$

4. a. $43\hat{\imath} + 11\hat{\jmath} + 17\hat{k}$ b. $-10\hat{\imath} - 43\hat{\jmath} + 31\hat{k}$

c. $60\hat{\imath} - 79\hat{\jmath} + 111\hat{k}$ d. $-4\hat{\imath} - \hat{\jmath} - \frac{31}{4}\hat{k}$

5. a. $\underset{\sim}{a} \times \underset{\sim}{b} = 7\hat{\imath} + 10\hat{\jmath} + 29\hat{k}$

b. $\underset{\sim}{b} \times \underset{\sim}{a} = 9\hat{\imath} - \hat{\jmath} - 32\hat{k}$

c. Sample responses can be found in the worked solutions in the online resources.

6. a. $-9\hat{\imath} - 46\hat{\jmath} - 59\hat{k}$

b. $\underset{\sim}{a} \times \underset{\sim}{b} \neq \underset{\sim}{b} \times \underset{\sim}{a}$. Not commutative.

7. a. $19\hat{\imath} - 38\hat{\jmath} + 19\hat{k}$ b. $69\hat{\imath} + 32\hat{\jmath} + 12\hat{k}$

c. $10\hat{\imath} - 25\hat{\jmath} + 20\hat{k}$ d. $-59\hat{\imath} + 55\hat{\jmath} + 78\hat{k}$

8. a. 246 units^3 b. 172 units^3

c. 164 units^3 d. 276 units^3

9. $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = 49$

10. Sample responses can be found in the worked solutions in the online resources.

11. V = 17 units^3

12. Sample responses can be found in the worked solutions in the online resources.

13. The vectors are coplanar as the scalar triple product equals zero.

14. a. 0 b. 1 c. -1 d. -1

15. $b_z = 1$ and $b_y = 1$

16. $b_z = 1$ and $b_y = -4$

17. 7.67 units^3

18. 68.199°

19. Sample response can be found in the worked solutions in the online resources.

20. $\hat{\underset{\sim}{n}} = \pm \dfrac{\sqrt{20}}{20}\begin{pmatrix}0\\4\\2\end{pmatrix}$

3.7 Applications of vectors

3.7 Exercise

1. a. $\underset{\sim}{v} = -3.2\hat{\imath} + 2\hat{\jmath} - 1.8\hat{k}$ cm/s
   b. $|\underset{\sim}{v}| = 4.18$ cm/s

2. a. $\begin{pmatrix}-4\\8\\-3\end{pmatrix}$ cm/s
   b. $\begin{pmatrix}4\\-8\\3\end{pmatrix}$ cm/s

3. a. $x + 12y - 41z = -180$
   b. 21.37m^2

4. a. 12.04 units^2 b. 7.43 units^2

5. a. 5.2 units^2 b. 28.39 units^2

6. a. 200 Nm b. 49.81 Nm c. 32.03 Nm

7. a. $5x - 2y + z = 1$
   b. $-3x + 5y + 4z = 27$
   c. $-7x + 4y - 6z = -11$
   d. $-4x - y + 3z = -27$

8. a. $\dfrac{1}{\sqrt{21}}\left(4\hat{\imath} - 2\hat{\jmath} - \hat{k}\right)$
   b. $\dfrac{1}{3}\left(-\hat{\imath} + 2\hat{\jmath} - 2\hat{k}\right)$
   c. $\dfrac{1}{\sqrt{94}}\left(6\hat{\imath} - 7\hat{\jmath} + 3\hat{k}\right)$

9. a. Cartesian equation of the plane: $21x + 6y - 5z = 35$
   Axes intercepts: $x = \dfrac{5}{3}, y = \dfrac{35}{6}, z = -7$
   b. Cartesian equation of the plane: $7x + 10y - 25z = -39$
   Axes intercepts: $x = \dfrac{-39}{7}, y = \dfrac{-39}{10}, z = \dfrac{39}{25}$

10. a. P$(2, 9, -1)$ exists on the plane.
    b. P$(-1, -6, 4)$ doesn't exist on plane.
    c. P$(1, 4, -7)$ exists on the plane.

11. Points are collinear. A plane cannot be formed.

12. $(-4.2, 4.4, 1.6)$

13. $6\sqrt{59}$ units^2

14. 698.19 Nm > 400 Nm, hence sufficient force.

15. a. $\underset{\sim}{n}_1 = 2\underset{\sim}{n}_2 \quad \therefore$ The planes are parallel
    b. $11x - 2y + 6z = 39$
    c. 11

16. a. $4x - 29y + 26z = 25$
    b. $P_1 \perp P_2$

17. $(-78, 36, -57)$

18. 30.61 units^2

19. $\overrightarrow{OP} = 3.5\hat{\imath} + 2\hat{\jmath}$

20. 1306.39 Nm

21. a. Sample responses can be found in the Worked Solutions in the online resources.
    b. 2.74 units
    c. Sample responses can be found in the Worked Solutions in the online resources.

22. $x = -\dfrac{11}{10} + 13k, y = 20k, z = -\dfrac{32}{5} - 38k$

3.8 Review

3.8 Exercise

1. $\underset{\sim}{u} = 9.94\hat{\imath} - 6.96\hat{\jmath} + 8.82\hat{k}$

2. $-27\hat{\imath} + 63\hat{\jmath} - 33\hat{k}$

3. $\theta = 127.09°$

4. $50 = (x-3)^2 + (y-8)^2 + (z-5)^2$

5. $y = \dfrac{1}{28}x^2 + x + \dfrac{21}{4}$

6. $\underset{\sim}{a} \times \underset{\sim}{b} = 305.12\,\hat{\underset{\sim}{n}}$

7. Sample responses can be found in the worked solutions in the online resources.

8. $6\hat{\imath} - 51\hat{\jmath} - 69\hat{k}$

9. A unit vector normal to vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is not possible. The two vectors do not define a plane as they are scalar multiples of each other.

10. $3x + 5y + 8z = 47$

11. $84x + 119y - 130z = 34$

12. a. $x - 39y - 14z = -269$
    $$x = -269, y = \frac{269}{39}, z = \frac{269}{14}$$
    b. $-5x + 48y + 75z = 441$
    $$x = -\frac{441}{5},\ y = \frac{147}{16},\ z = \frac{147}{25}$$

13. The straight-line distance is 686 metres. The altitude angle is 71.36°. This may be considered very steep and hence unreasonable to climb.

14. a. $\underset{\sim}{r} = \left(5\hat{\imath} - 2\hat{\jmath} + 13\hat{k}\right) + t\left(\hat{\imath} + 6\hat{\jmath} - 6\hat{k}\right)$
    b. $y \times z = -1160$

15. $\begin{pmatrix}-3\\-60\\-15\end{pmatrix}$

16-17. Sample responses can be found in the worked solutions in the online resources.

18. Horizontal distance is 384.94 metres. The maximum height is 55.56 metres which occurred at $t = 3.367$ seconds. The result is unreasonable in terms of general driving distances. It is assumed that no other forces, aside from gravity, act on the ball during its flight, that the ground is level and no spin/sideways movement occurs.

19. Sample responses can be found in the worked solutions in the online resources.

20. $34x - 58y + 56z - 62 = 0$

3.8 Past QCAA exam questions

1. C
2. D
3. A
4. a. $\overrightarrow{OA} = 2\hat{\imath}$, $\overrightarrow{OB} = 2\hat{\jmath} + 2\hat{k}$
   b. $-4\hat{\jmath} + 4\hat{k}$
   c. $2\sqrt{2}$
   d. $\overrightarrow{MN} = -\hat{\imath} + \hat{\jmath} + \hat{k}$
   e. Sample responses can be found in the worked solutions in the online resources.
5. a $\overrightarrow{AB} = -2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ or $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$
   b $\dfrac{x-3}{-2} = \dfrac{y+1}{2} = \dfrac{z-3}{3}$
   c $-2x + 2y + 3z = 1$

4 Vector calculus

LESSON SEQUENCE

4.1 Overview 204
4.2 Position vectors as functions of time — circles, ellipses and hyperbolas 205
4.3 Collision problems 221
4.4 Differentiation and integration of vectors 231
4.5 Straight line motion with constant and variable acceleration 247
4.6 Applications of vector calculus 258
4.7 Review 278
Answers 283

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 4
Exam questions	Exam question booklet — Chapter 4
Digital documents	Learning matrix — Chapter 4 Chapter summary — Chapter 4

LESSON
4.1 Overview

Hey students! Bring these pages to life online

Engage with interactivities

Answer questions and check results

Track your progress

Find all this and MORE in jacPLUS

4.1.1 Introduction

The motion of a particle has been studied since ancient times and has intrigued both mathematicians and physicists. The motion of particles can take many forms, for example, a piece of gum stuck to a rotating bicycle wheel or a ball travelling toward the ground in a straight line. In this chapter we will examine the motion of particles in both two and three dimensions and how differentiation and integration techniques can be used to analyse the motion. Consider the image the young athlete throwing a javelin. The initial speed and the angle at which the javelin is released will both have an effect on the distance the javelin will travel. These variables will be investigated in the study of projectile motion in this chapter.

4.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
4.2	**Position vectors as functions of time — circles, ellipses and hyperbolas**	Understand and use position of vectors as a function of time. Understand and use the Cartesian equation of a path given as a vector equation in two dimensions, including circles, ellipses and hyperbolas. • equation of circle: $(x-h)^2+(y-k)^2=r^2$ • equation of ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ • equation of hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$
4.3	**Collision problems**	Understand and use the position of two particles, each described as a vector function of time, and determine if their paths cross or if the particles meet.
4.4	**Differentiation and integration of vectors**	Differentiate and integrate a vector function with respect to time.
4.5	**Straight line motion with constant and variable acceleration**	Use vector calculus to determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration.
4.6	**Applications of vector calculus**	Apply vector calculus to model and solve problems that involve motion in a plane, including projectile and circular motion, with and without technology.

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LESSON
4.2 Position vectors as functions of time — circles, ellipses and hyperbolas

SYLLABUS LINKS

- Understand and use position of vectors as a function of time.
- Understand and use the Cartesian equation of a path given as a vector equation in two dimensions, including circles, ellipses and hyperbolas.
  - equation of circle: $(x-h)^2+(y-k)^2=r^2$
  - equation of ellipse: $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$
  - equation of hyperbola: $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$ or $\dfrac{(y-k)^2}{a^2}+\dfrac{(x-h)^2}{b^2}=1$

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4.2.1 Review of position vectors as functions of time

Parametric equations

The **parametric equations** of a **particle** as it moves in two dimensions were examined in Chapter 3, Lesson 3.5. The path traced out by the particle is defined in terms of a third variable, t, which is called a parameter. In the two-dimensional case the path is described by two parametric equations, as both the x- and y-coordinates depend upon the parameter, t: $x=x(t)$ and $y=y(t)$.

The position vector of the particle is given by $\underset{\sim}{r}(t)=x(t)\hat{\boldsymbol{\imath}}+y(t)\hat{\boldsymbol{\jmath}}$, where $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are unit vectors in the x and y directions. This is also called a vector function of the scalar real variable t, where t is called the parameter, and often represents time.

If we can eliminate the parameter from the two parametric equations and obtain an equation of the form $y=f(x)$, this is called an **explicit relationship** or the Cartesian equation of the path. It may not be possible to obtain an explicit relationship, but often an **implicit relationship** of the form $f(x, y)=0$ can be formed. The relationship between x and y is called the **Cartesian equation of the path**.

Closest approach to origin

Given the position vector of a particle, $\underset{\sim}{r}(t)=x(t)\hat{\boldsymbol{\imath}}+y(t)\hat{\boldsymbol{\jmath}}$, where $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are unit vectors in the x and y directions, it is possible to determine the position or coordinates of the particle at a given value of t. It is also possible to calculate the value of t and the coordinates when the particle is closest to the origin, using calculus (Pythagoras, theorem in three dimensions) for the magnitude of the distance to the origin.

WORKED EXAMPLE 1 Using vector equations and closest approach

A particle moves so that its vector equation is given by $\underset{\sim}{r}(t)=(3t-4)\hat{\boldsymbol{\imath}}+(4t-3)\hat{\boldsymbol{\jmath}}+1\hat{\boldsymbol{k}}$ for $t\geq 0$.
Determine:

a. the distance of the particle from the origin when $t=2$
b. the distance of the particle from the origin at any time t
c. the closest distance of the particle from the origin.

THINK	WRITE
a. 1. To determine the position of the particle at $t=2$, substitute the value of t.	**a.** $\underset{\sim}{r}(2)=(6-4)\hat{\boldsymbol{\imath}}+(8-3)\hat{\boldsymbol{\jmath}}+1\hat{\boldsymbol{k}}$ $=2\hat{\boldsymbol{\imath}}+5\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}$ At $t=2$ the particle is at $(2, 5, 1)$.

2. To calculate the distance of the particle from the origin, determine the magnitude of the vector by applying the formula $|\underset{\sim}{u}| = \sqrt{x^2 + y^2 + z^2}$.

$$|\underset{\sim}{r}(2)| = \sqrt{2^2 + 5^2 + 1^2}$$

3. State the distance from the origin at this time.

$$|\underset{\sim}{r}(2)| = \sqrt{30} \text{ units}$$

b. 1. Determine the magnitude of the vector at time t by applying the formula $|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2}$.

b. $\underset{\sim}{r}(t) = (3t - 4)\hat{\boldsymbol{\imath}} + (4t - 3)\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$

$$|\underset{\sim}{r}(t)| = \sqrt{(3t-4)^2 + (4t-3)^2 + 1^2}$$

2. Expand and simplify to state the distance in terms of t.

$$\begin{aligned} |\underset{\sim}{r}(t)| &= \sqrt{9t^2 - 24t + 16 + 16t^2 - 24t + 9 + 1} \\ &= \sqrt{25t^2 - 48t + 26} \end{aligned}$$

c. 1. For the closest (or minimum distance), we need to use calculus to optimise.

c. The minimum distance occurs when $\frac{d}{dt}(|\underset{\sim}{r}(t)|) = 0$.

2. Use the chain rule.

$$|\underset{\sim}{r}(t)| = (25t^2 - 48t + 26)^{\frac{1}{2}}$$

$$\frac{d}{dt}(|\underset{\sim}{r}(t)|) = \frac{\frac{1}{2} \times (50t - 48)}{\sqrt{25t^2 - 48t + 26}}$$

3. Set the expression equal to zero. Recall that a rational fraction is equal to zero if its numerator is equal to zero.

$$\frac{1}{2}\frac{50t - 48}{\sqrt{25t^2 - 48t + 26}} = 0$$

4. Solve for the value of t.

$$\begin{aligned} 50t - 48 &= 0 \\ t &= \frac{24}{25} \end{aligned}$$

5. Determine the position vector at this value.

Substitute $t = \frac{24}{25}$:

$$\begin{aligned} \underset{\sim}{r}\left(\frac{24}{25}\right) &= \left(3 \times \frac{24}{25} - 4\right)\hat{\boldsymbol{\imath}} + \left(4 \times \frac{24}{25} - 3\right)\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}} \\ &= -\frac{28}{25}\hat{\boldsymbol{\imath}} + \frac{21}{25}\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} \end{aligned}$$

6. Determine the magnitude of the vector at this time, which represents the closest distance of the particle from the origin.

$$|\underset{\sim}{r}(t)|_{\min} = \sqrt{\left(-\frac{28}{25}\right)^2 + \left(\frac{21}{25}\right)^2 + 1^2}$$

7. State the final result.

$$|\underset{\sim}{r}(t)|_{\min} = \frac{\sqrt{74}}{5}$$

4.2.2 Cartesian equations of circles and ellipses

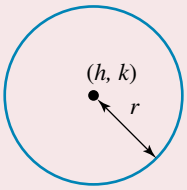
When eliminating the parameter a number of different methods may be required. These could include direct substitution or trigonometric identities.

When the path traced out by a particle is given as a vector equation in two dimensions, converting to parametric equations and then eliminating the parameter may produce recognisable Cartesian equations.

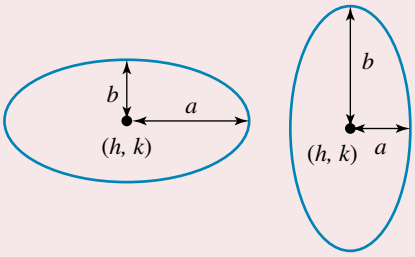
The Cartesian equations for paths of different curves are set out here.

Cartesian equations of circles and ellipses

Equation of a circle with centre (h, k) and radius r: $(x-h)^2+(y-k)^2=r^2$



Equation of an ellipse with centre (h, k), horizontal radius a and vertical radius b: $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$



Note that the circle and ellipse equations take the form of the ***sum*** of two terms.

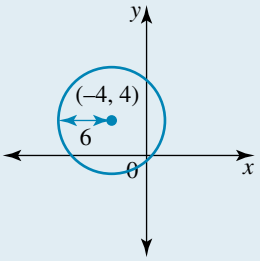
Eliminating the parameters to produce the Cartesian equations from vector equations may require several different methods. These could include direct substitution or using known trigonometric identities to manipulate the form of the equations to appear as expected.

Trigonometric identities that could be useful include the Pythagorean identity $\cos^2(A)+\sin^2(A)=1$.

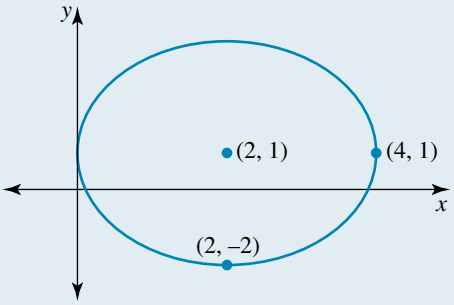
WORKED EXAMPLE 2 Understanding Cartesian equations of circles and ellipses

Write Cartesian equations for the following circle and ellipse.

a.



b.



THINK

a. 1. The circle has a centre C and radius r defined.

WRITE

a. $C(-4, 4)$, $r=6$

2. Write the Cartesian equation of the circle.	$(x+4)^2+(y-4)^2=36$
b. 1. The ellipse has a centre C and horizontal and vertical radii a and b defined.	**b.** $C(2,1)$, $a=4-2$ $=2$ and $b=1-(-2)$ $=3$
2. Write the Cartesian equation of the ellipse.	$\frac{(x-2)^2}{4}+\frac{(y-1)^2}{9}=1$

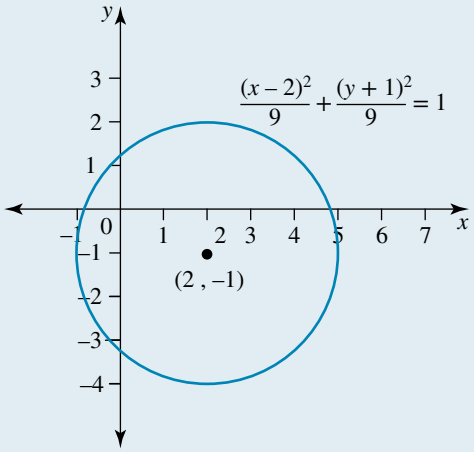
Careful consideration must be given to the possible values of the parameter t, which then specify the x-values (domain) and y-values (range) of the equation of the path.

WORKED EXAMPLE 3 Determining the Cartesian equation of a circle from its vector equation

Given the vector equation, $\underset{\sim}{r}(t)=(2+3\cos(t))\hat{\imath}+(3\sin(t)-1)\hat{\jmath}$, for $t\geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

THINK	WRITE
1. Define the two parametric equations as [1] and [2].	$x=2+3\cos(t)$ [1] $y=3\sin(t)-1$ [2]
2. Express the trigonometric ratios $\cos(t)$ and $\sin(t)$ in terms of x and y respectively.	$\frac{x-2}{3}=\cos(t)$ [3] $\frac{y+1}{3}=\sin(t)$ [4]
3. Square equations [3] and [4] to utilise a Pythagorean identity and define results as [5] and [6].	$\frac{(x-2)^2}{9}=\cos^2(t)$ [5] $\frac{(y+1)^2}{9}=\sin^2(t)$ [6]
4. Add equations [5] and [6].	$\frac{(x-2)^2}{9}+\frac{(y+1)^2}{9}=\cos^2(t)+\sin^2(t)$
5. Apply the Pythagorean identity, $\cos^2(t)+\sin^2(t)=1$. The result is the Cartesian equation of a circle.	$\frac{(x-2)^2}{9}+\frac{(y+1)^2}{9}=1$ $\therefore (x-2)^2+(y+1)^2=9$ This is a circle of radius 3, with centre at $(2,-1)$
6. Determine the domain.	Since $-1\leq\cos(t)\leq 1$, it follows from the parametric equation $x(t)=2+3\cos(t)$ that the domain is $-1\leq x\leq 5$, that is $[-1,5]$.
7. Determine the range.	Since $-1\leq\sin(t)\leq 1$, it follows from the parametric equation $y(t)=3\sin(t)-1$ that the range is $-4\leq y\leq 2$, that is $[-4,2]$.

8. The graph is the whole circle. The exact coordinates of the x- and y-intercepts are not required in this case.



WORKED EXAMPLE 4 Understanding and using the equation of a path (circle)

A particle is found to be travelling in the path of a circle with Cartesian equation $(x-1)^2 + (y+2)^2 = 25$.

a. Determine the corresponding parametric equations.

b. Determine the corresponding vector equation of the path.

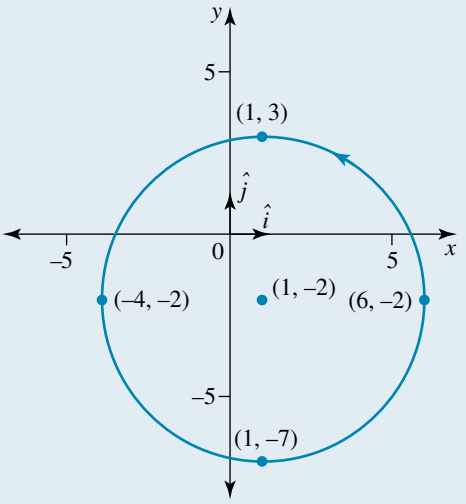
c. Locate the position of the particle at $t = \left(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right)$, sketch the circle, showing these positions and define the direction of the path.

THINK	WRITE
a. 1. Rearrange the equation so it equates to 1.	a. $\frac{(x-1)^2}{25} + \frac{(y+2)^2}{25} = 1$
2. Define terms to use a Pythagorean identity using $\cos(t)$ for the x term.	$\cos^2(t) + \sin^2(t) = 1$ $\cos^2(t) = \frac{(x-1)^2}{25}$ $\sin^2(t) = \frac{(y+2)^2}{25}$ $\cos(t) = \frac{x-1}{5}$ $\sin(t) = \frac{y+2}{5}$
3. Express each term as a function of t to define the parametric equations.	$x = 5\cos(t) + 1$ $y = 5\sin(t) - 2$
b. Write the vector equation by combining the parametric equations.	b. $\underset{\sim}{r}(t) = (5\cos(t) + 1)\hat{\imath} + (5\sin(t) - 2)\hat{\jmath}$

c. 1. Substitute $t = \left(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right)$.

c.
$$\begin{aligned}
\underset{\sim}{r}(0) &= (5\times 1+1)\hat{\imath} + (5\times 0-2)\hat{\jmath} \\
&= 6\hat{\imath} - 2\hat{\jmath} \\
\underset{\sim}{r}\left(\frac{\pi}{2}\right) &= (5\times 0+1)\hat{\imath} + (5\times 1-2)\hat{\jmath} \\
&= \hat{\imath} + 3\hat{\jmath} \\
\underset{\sim}{r}(\pi) &= (5\times(-1)+1)\hat{\imath} + (5\times 0-2)\hat{\jmath} \\
&= -4\hat{\imath} - 2\hat{\jmath} \\
\underset{\sim}{r}\left(\frac{3\pi}{2}\right) &= (5\times 0+1)\hat{\imath} + (5\times(-1)-2)\hat{\jmath} \\
&= \hat{\imath} - 7\hat{\jmath} \\
\underset{\sim}{r}(2\pi) &= (5\times 1+1)\hat{\imath} + (5\times 0-2)\hat{\jmath} \\
&= 6\hat{\imath} - 2\hat{\jmath}
\end{aligned}$$

2. Sketch the circle with centre $(1, -2)$ and radius 5, mark the position of the points and determine the direction of the path.



The particle is travelling in an anticlockwise direction around the circular path.

WORKED EXAMPLE 5 Determining the Cartesian equation of an ellipse from its vector equation

Given the vector equation, $\underset{\sim}{r}(t) = (6\cos(t))\hat{\imath} + (5\sin(t))\hat{\jmath}$, for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

THINK	WRITE	
1. Define the two parametric equations as [1] and [2].	$x = 6\cos(t)$ $y = 5\sin(t)$	[1] [2]
2. Square equations [1] and [2] and define results as [3] and [4].	$x^2 = 36\cos^2(t)$ $\frac{x^2}{36} = \cos^2(t)$ $y^2 = 25\sin^2(t)$ $\frac{y^2}{25} = \sin^2(t)$	 [3] [4]
3. Add equations [3] and [4].	$\frac{x^2}{36} + \frac{y^2}{25} = \cos^2(t) + \sin^2(t)$	[3] + [4]

4. Apply the Pythagorean identity, $\cos^2(t) + \sin^2(t) = 1$. The result is the Cartesian equation of an ellipse.

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

This is an ellipse, with centre at (0, 0) and semi-major and semi-minor axes 6 and 5.

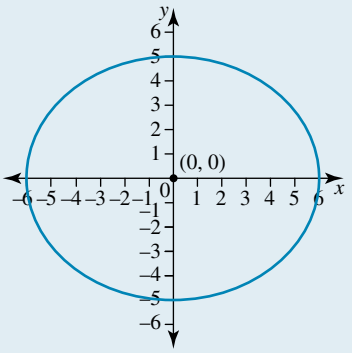
5. Determine the domain.

Since $-1 \le \cos(t) \le 1$, it follows from the parametric equation $x(t) = 6\cos(t)$ that the domain is $-6 \le x \le 6$, that is, $[-6, 6]$.

6. Determine the range.

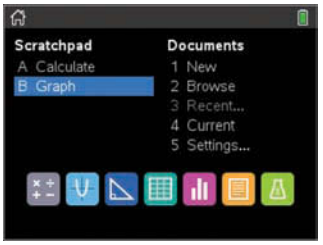
Since $-1 \le \sin(t) \le 1$, it follows from the parametric equation $y(t) = 5\sin(t)$ that the range is $-5 \le y \le 5$, that is, $[-5, 5]$.

7. The graph is the whole ellipse. The exact ordinates of the x- and y-intercepts are not required in this case.

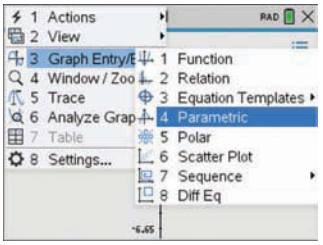


TI | THINK **WRITE**

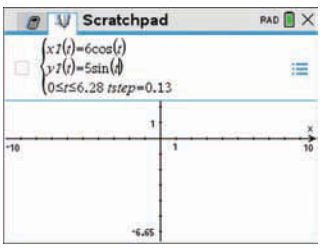
1. On a Home page, select Graph.



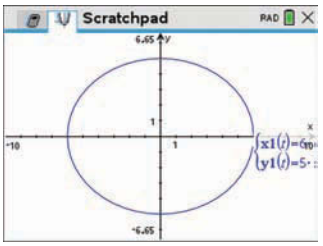
2. Parametric equations can be graphed by selecting,
MENU
3: Graph Entry/Edit
4: Parametric.



3. Complete the entry line as
$6\cos(t)$
$5\sin(t)$
Note: t is entered by pressing the T button on the alpha keyboard.

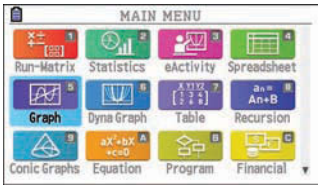


4. Press the ENTER button. The answer appears on the screen.

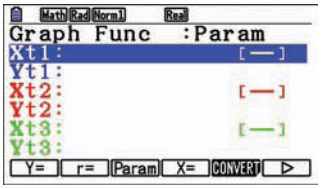


CASIO | THINK **WRITE**

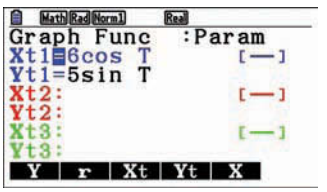
1. On a Menu screen, select GRAPH.



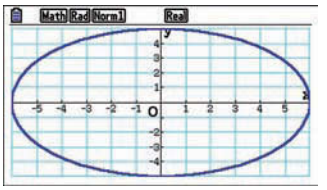
2. Parametric equations can be graphed by selecting, TYPE Param.



3. Complete the entry line as
6 cos T
5 sin T
Note: T is entered by pressing the X, θ, T button.



4. Press the EXE button.The answer appears on the screen.

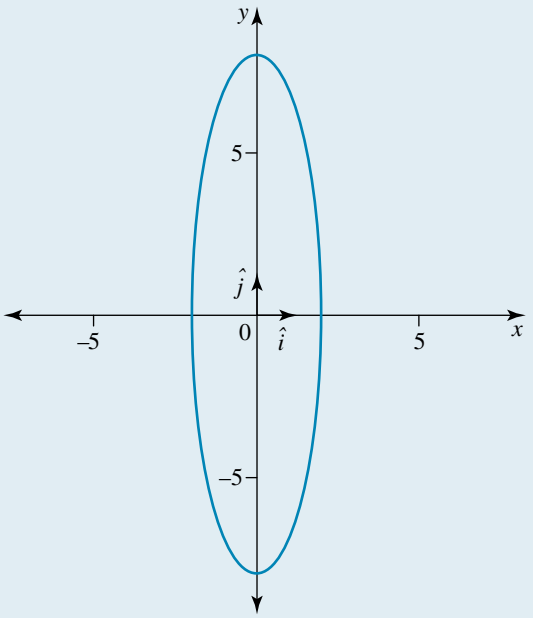


WORKED EXAMPLE 6 Understanding and using the equation of a path (ellipse)

Consider the sketch of the path of vector equation $\underset{\sim}{r}(t) = 2\cos(t)\hat{\imath} + 8\sin(t)\hat{\jmath}$ for $t \geq 0$.

a. Determine the Cartesian equation of the path.

b. Determine whether the particle is moving faster at $t = 0$ or at $t = \frac{\pi}{2}$ and justify your decision.



THINK

a. 1. Define the parameters.

2. Square both sides of the equations and rearrange.

3. Add the equations and apply the Pythagorean identity.

4. Write the Cartesian equation.

b. 1. Substitute the values $t = \left(0, \frac{\pi}{4}, \frac{\pi}{2}\right)$ into the vector equation.

WRITE

a. $x = 2\cos(t)$
$y = 8\sin(t)$

$$x^2 = 4\cos^2(t)$$
$$y^2 = 64\sin^2(t)$$
$$\frac{x^2}{4} = \cos^2(t)$$
$$\frac{y^2}{64} = \sin^2(t)$$

$$\frac{x^2}{4} + \frac{y^2}{64} = \cos^2(t) + \sin^2(t)$$
$$= 1$$

The Cartesian equation of the path is $\frac{x^2}{4} + \frac{y^2}{64} = 1$.

b.
$$\underset{\sim}{r}(0) = 2 \times 1\hat{\imath} + 8 \times 0\hat{\jmath}$$
$$= 2\hat{\imath}$$
$$\underset{\sim}{r}\left(\frac{\pi}{4}\right) = 2 \times \frac{\sqrt{2}}{2}\hat{\imath} + 8 \times \frac{\sqrt{2}}{2}\hat{\jmath}$$
$$= \sqrt{2}\hat{\imath} + 4\sqrt{2}\hat{\jmath}$$
$$\underset{\sim}{r}\left(\frac{\pi}{2}\right) = 2 \times 0\hat{\imath} + 8 \times 1\hat{\jmath}$$
$$= 8\hat{\jmath}$$

2. Consider the position at $t = \frac{\pi}{4}$.

The change of position from $t = 0$ to $t = \frac{\pi}{4}$ is greater than the change of position from $t = \frac{\pi}{4}$ to $t = \frac{\pi}{2}$. The particle moves more quickly in the vertical direction than in the horizontal direction.
The particle is moving faster at $t = 0$ that at $t = \frac{\pi}{2}$.

Alternatively, determine the velocity vector, substitute $t = \left(0, \frac{\pi}{2}\right)$ and determine the speed of the particle at $t = 0$ and $t = \frac{\pi}{2}$.

$$\dot{\underset{\sim}{r}}(t) = -2\sin(t)\hat{\boldsymbol{\imath}} + 8\cos(t)\hat{\boldsymbol{\jmath}}$$

$$\dot{\underset{\sim}{r}}(0) = 8\hat{\boldsymbol{\jmath}}$$

$$\dot{\underset{\sim}{r}}\left(\frac{\pi}{2}\right) = -2\hat{\boldsymbol{\imath}}$$

$$|\dot{\underset{\sim}{r}}(0)| = 8$$

$$\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{2}\right)\right| = 2$$

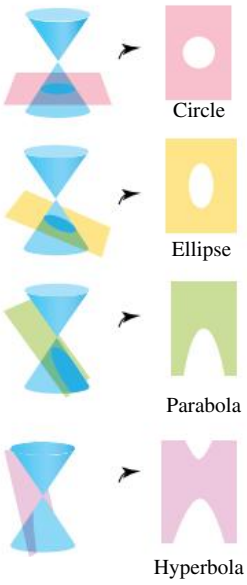
$|\dot{\underset{\sim}{r}}(0)| \geq \left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{2}\right)\right|$, the particle is moving faster at $t = 0$ that at $t = \frac{\pi}{2}$.

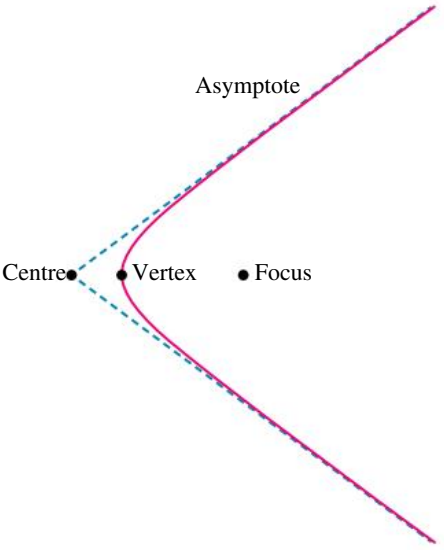
4.2.3 Cartesian equations of hyperbolas

A conic section is a curve obtained from the intersection of a plane with a cone's surface. A hyperbola, like circles, ellipses and parabolas, is another type of curve that can be described in the Cartesian plane.

The curves can be described by a set of points (x, y) such that the difference in the distance from (x, y) to each of the **foci** is always the same.

The result is two curves that are mirrored and approach slant asymptotes.



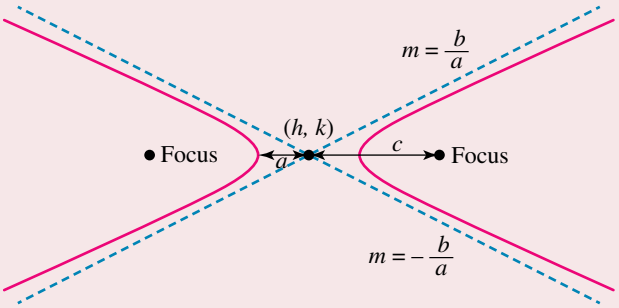


Like an ellipse, a hyperbola can have two different orientations. The traverse axis, which is the line containing the centre, both vertices and both foci, can be horizontal or vertical.

Cartesian equations of hyperbolas

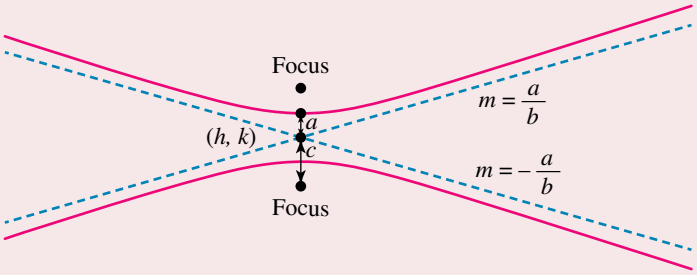
Equation of a horizontal hyperbola with centre (h, k):

$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ **where $\pm a$ is the horizontal distance from the centre to the vertex, $\pm c$ is the horizontal distance from the centre to the focus, $\pm \dfrac{b}{a}$ is the gradient of the asymptote and $c^2 = a^2 + b^2$.**



Equation of a vertical hyperbola with centre (h, k):

$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ **where $\pm a$ is the vertical distance from the centre to the vertex, $\pm c$ is the vertical distance from the centre to the focus, $\pm \dfrac{a}{b}$ is the gradient of the asymptote and $c^2 = a^2 + b^2$.**



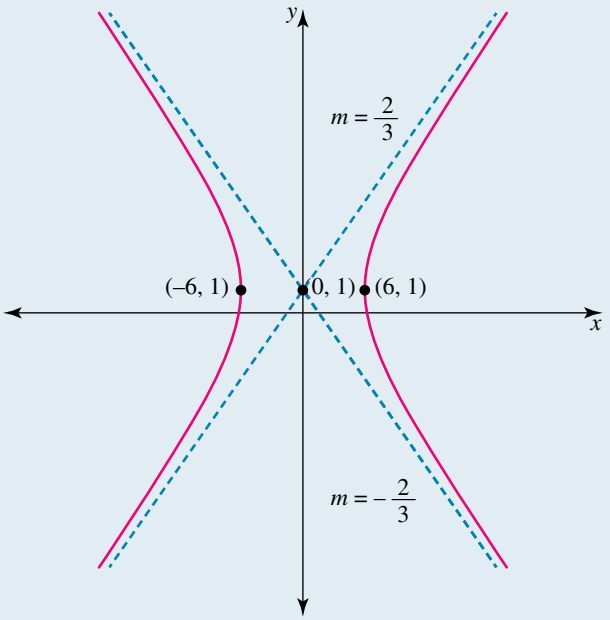
Note that the hyperbola equations take the form of the ***difference*** of two terms.

WORKED EXAMPLE 7 Understanding the Cartesian equation of a horizontal hyperbola

Given the Cartesian equation $\dfrac{x^2}{36} - \dfrac{(y-1)^2}{16} = 1$, sketch the hyperbola.

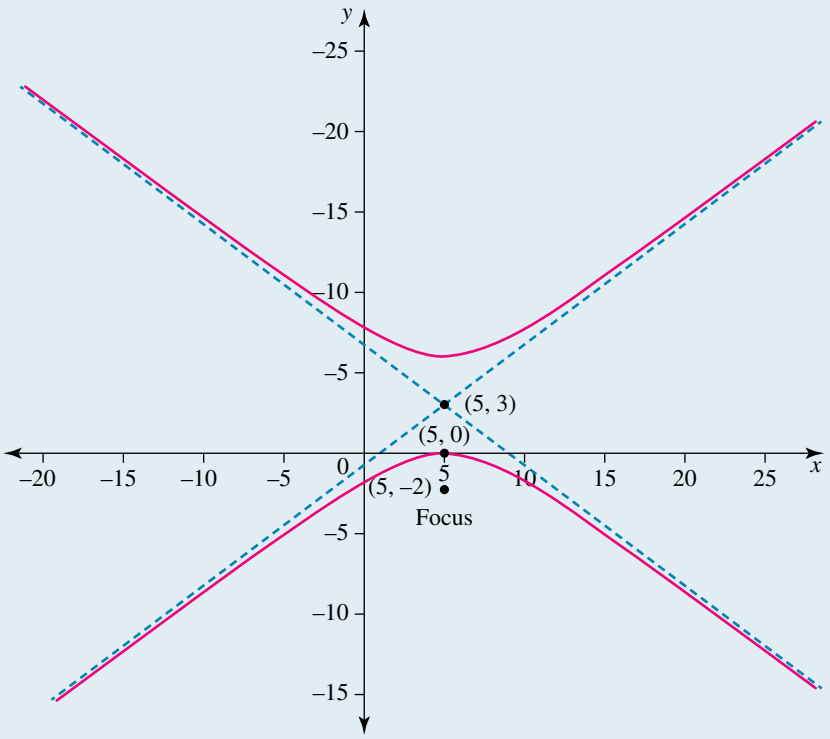
THINK	WRITE
1. Consider the form of the equation.	This is a horizontal hyperbola of the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$
2. Identify the values of h, k, a and b.	$h = 0, \quad k = 1$ $a = 6, \quad b = 4$
3. Determine the centre C of the hyperbola.	$C(0, 1)$
4. Determine the gradients of the asymptotes.	$m = \pm \dfrac{4}{6}$ $= \pm \dfrac{2}{3}$
5. Determine the location of the vertices.	$(h-a, k)$ that is $(-6, 1)$ and $(h+a, k)$ that is $(6, 1)$

6. Sketch the hyperbola.



WORKED EXAMPLE 8 Understanding the Cartesian equation of a vertical hyperbola

Determine the Cartesian equation of the given hyperbola.



THINK

1. Consider the form of the equation.

WRITE

This is a vertical hyperbola of the form

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

2. Identify the centre C of the hyperbola and thus the values of h and k.	$C(5, 3)$ $h = 5$ and $k = 3$
3. Identify the vertical distance centre-vertex a and the vertical distance centre-focus c.	$a = 3 - 0 = 3$ and $c = 3 - (-2) = 5$
4. Calculate the value of b using $a^2 + b^2 = c^2$.	$3^2 + b^2 = 5^2$ $b = \sqrt{25 - 9}$ $= 4$
5. Write the Cartesian equation of the hyperbola.	$\dfrac{(y-3)^2}{9} - \dfrac{(x-5)^2}{16} = 1$

To determine the Cartesian equation of a path of a hyperbola given as a vector equation in two dimensions, a rearrangement of trigonometric equations to the difference form may be required.

Trigonometric identities that could be useful include:

$$\sec^2(A) - \tan^2(A) = 1$$
$$\cot^2(A) - \operatorname{cosec}^2(A) = 1$$

WORKED EXAMPLE 9 Determining the Cartesian equation of a hyperbola from its vector equation

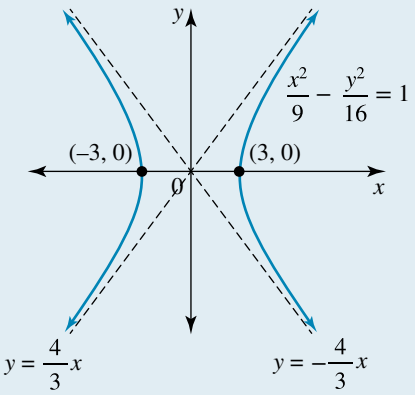
Given the vector equation $\underset{\sim}{r}(t) = 3\sec(2t)\hat{\imath} + 4\tan(2t)\hat{\jmath}$ for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

THINK	WRITE
1. Define two parametric equations as [1] and [2].	$x = 3\sec(2t)$ [1] $y = 4\tan(2t)$ [2]
2. Express the trigonometric ratios $\sec(2t)$ and $\tan(2t)$ in terms of x and y respectively.	$[1] \Rightarrow \sec(2t) = \dfrac{x}{3}$ $[2] \Rightarrow \tan(2t) = \dfrac{y}{4}$
3. Eliminate the parameter using an appropriate trigonometric identity (rearranged as a difference) to determine the Cartesian equation of the path.	Since $\sec^2(2t) - \tan^2(2t) = 1$, it follows that $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ This is a hyperbola with centre at the origin. A hyperbola of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has asymptotes when $y = \pm\dfrac{b}{a}x$. Therefore the asymptotes of $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ are $y = \pm\dfrac{4x}{3}$.
4. Determine the domain.	It follows from the parametric equation $x(t) = 3\sec(2t)$ that the domain is $(-\infty, -3] \cup [3, \infty)$.

5. Determine the range.

It follows from the parametric equation $y(t) = 4\tan(2t)$ that the range is $\mathbb{R}$.

6. The graph is the whole hyperbola.



The parametric representation of a curve is not necessarily unique. This can be seen by comparing Worked examples 9 and 10.

WORKED EXAMPLE 10 Using alternative forms

Show that the parametric equations $x(t) = \frac{3}{2}\left(t + \frac{1}{t}\right)$ and $y(t) = 2\left(t - \frac{1}{t}\right)$ where $t \in \mathbb{R}\backslash\{0\}$ represent the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

THINK	WRITE
1. Define the two parametric equations as [1] and [2].	$x = \frac{3}{2}\left(t + \frac{1}{t}\right)$ [1] $y = 2\left(t - \frac{1}{t}\right)$ [2]
2. Express the equations in a form to eliminate the parameter.	$\frac{2x}{3} = t + \frac{1}{t}$ [3] $\frac{y}{2} = t - \frac{1}{t}$ [4]
3. Square both equations [3] and [4].	$\frac{4x^2}{9} = t^2 + 2 + \frac{1}{t^2}$ [5] $\frac{y^2}{4} = t^2 - 2 + \frac{1}{t^2}$ [6]
4. Subtract the equations [5] and [6] to eliminate the parameter.	[5] – [6] : $\frac{4x^2}{9} - \frac{y^2}{4} = t^2 + 2 + \frac{1}{t^2} - \left(t^2 - 2 + \frac{1}{t^2}\right)$
5. Divide by 4.	$\frac{4x^2}{9} - \frac{y^2}{4} = 4$ $\frac{x^2}{9} - \frac{y^2}{16} = 1$ This gives the hyperbola as required.

Exercise 4.2 Position vectors as functions of time — circles, ellipses and hyperbolas

learnon

4.2 Exercise | 4.2 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	15, 16, 17, 18, 19, 20, 21, 22	23, 24, 25

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Simple familiar

1. WE1 A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (t-2)\hat{\boldsymbol{\imath}} + (3t-1)\hat{\boldsymbol{\jmath}} - 2t\hat{\boldsymbol{k}}$ for $t \geq 0$.
Determine:

a. the distance of the particle from the origin when $t = 5$
b. the distance of the particle from the origin at any time t
c. the closest distance of the particle from the origin.

2. A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = \sqrt{t}\,\hat{\boldsymbol{\imath}} + (2t+3)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$.
Determine:

a. the distance of the particle from the origin when $t = 4$
b. the value t when the distance of the particle from the origin is $15\sqrt{2}$.

3. a. A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2t-1)\hat{\boldsymbol{\imath}} + (t-3)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$.
Determine:

i. the distance of the particle from the origin when $t = 4$
ii. the distance of the particle from the origin at any time t
iii. the closest distance of the particle from the origin.

b. A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (4t-3)\hat{\boldsymbol{\imath}} + (3t+4)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$.
Determine:

i. the distance of the particle from the origin when $t = 2$
ii. the distance of the particle from the origin at any time t
iii. the closest distance of the particle from the origin.

4. a. A boat moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2t-3)\hat{\boldsymbol{\imath}} + 2\sqrt{t}\,\hat{\boldsymbol{\jmath}}$ for $t \geq 0$, where distance is measured in kilometres.
Determine:

i. the distance of the boat from the origin when $t = 4$
ii. the closest distance of the boat from the origin
iii. the times when the boat is 3 kilometres from the origin.

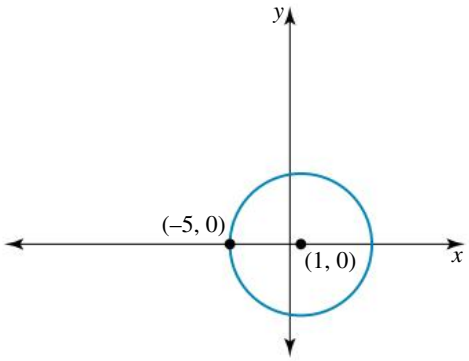
b. A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (2t-7)\hat{\boldsymbol{\imath}} + (2t+2)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$.
Determine:

i. the closest distance of the particle from the origin
ii. the time when the particle is $9\sqrt{5}$ units from the origin.

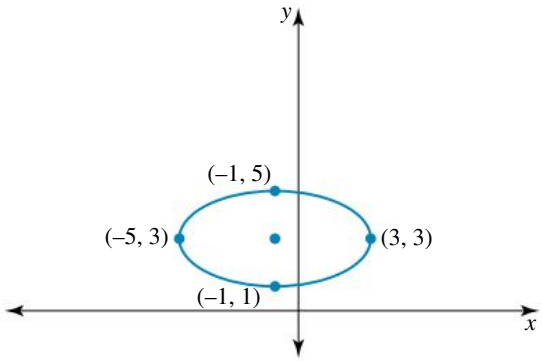
5. a. A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (at+b)\hat{\boldsymbol{\imath}} + (ct^2+d)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$. If $\underset{\sim}{r}(2) = 5\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(4) = 13\hat{\boldsymbol{\imath}} + 19\hat{\boldsymbol{\jmath}}$, calculate the values of a, b, c and d.
b. A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = (at+b)\hat{\boldsymbol{\imath}} + (ct^2+dt)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$. If $\underset{\sim}{r}(4) = 13\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(6) = 17\hat{\boldsymbol{\imath}} + 18\hat{\boldsymbol{\jmath}}$, calculate the values of a, b, c and d.

6. **WE2** Write Cartesian equations for the following circle and ellipse.

a.



b.



7. **WE3** Given the vector equation $\underset{\sim}{r}(t) = 3\cos(t)\hat{\boldsymbol{\imath}} + 4\sin(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

8. **WE4** A particle is found to be travelling in the path of a circle with Cartesian equation $(y-3)^2 = 16 - (x+2)^2$.

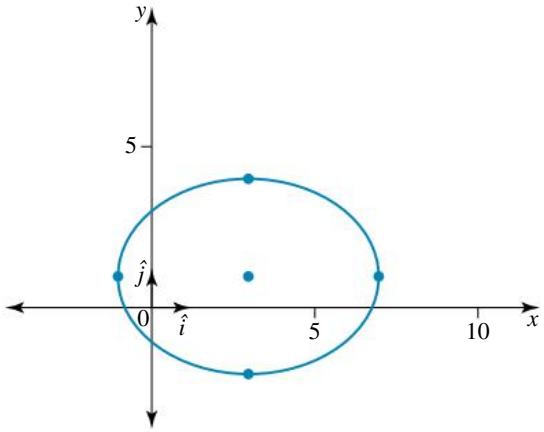
   a. Determine the corresponding parametric equations.

   b. Determine the corresponding vector equation of the path.

   c. Locate the position of the particle at $t = \left(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right)$, sketch the circle, showing these positions and define the direction of the path.

9. **WE5** Given the vector equation $\underset{\sim}{r}(t) = 5\sec(2t)\hat{\boldsymbol{\imath}} + 3\tan(2t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

10. **WE6** Consider the sketch of the path of the vector equation $\underset{\sim}{r}(t) = (3 - 4\cos(t))\hat{\boldsymbol{\imath}} + (1 + 3\sin(t))\hat{\boldsymbol{\jmath}}$ for $t \geq 0$.

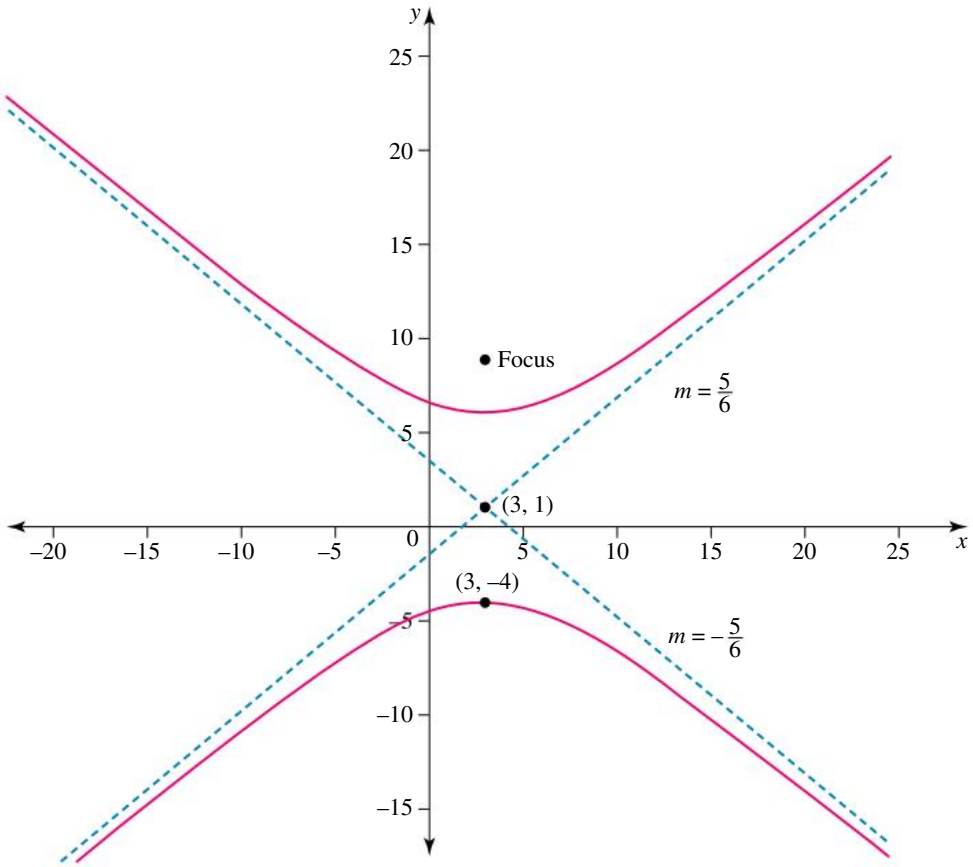


   a. Determine the Cartesian equation of the path.

   b. Determine whether the particle is moving faster at $t = 0$ or at $t = \frac{\pi}{2}$ and justify your decision.

11. **WE7** Given the Cartesian equation $\frac{x^2}{9} - \frac{(y+2)^2}{16} = 1$, sketch the hyperbola.

12. **WE8** Determine the Cartesian equation of the given hyperbola.



13. **WE9** Given the vector equation $\underset{\sim}{r}(t) = 4\cot\left(\frac{t}{2}\right)\hat{\boldsymbol{\imath}} + 3\operatorname{cosec}\left(\frac{t}{2}\right)\hat{\boldsymbol{\jmath}}$ for $t > 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

14. **WE10** Show that the parametric equations $x = \frac{5}{2}\left(t + \frac{1}{t}\right)$ and $y = \frac{3}{2}\left(t - \frac{1}{t}\right)$ where $t \in \mathbb{R} \setminus \{0\}$ represent the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

Complex familiar

15. **a.** A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = t\,\hat{\boldsymbol{\imath}} + \frac{1}{t}\hat{\boldsymbol{\jmath}}$ for $t > 0$. Determine the closest distance of the particle from the origin.

b. A particle moves so that its vector equation is given by $\underset{\sim}{r}(t) = e^{-t}\hat{\boldsymbol{\imath}} + e^{t}\hat{\boldsymbol{\jmath}}$ for $t \in \mathbb{R}$. Determine the closest distance of the particle from the origin.

For questions 16–21, determine and sketch the Cartesian equation of the path for each of the given vector equations, and state the domain and range.

16. **a.** $\underset{\sim}{r}(t) = 2t\hat{\boldsymbol{\imath}} + 4t^2\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

b. $\underset{\sim}{r}(t) = (t-1)\hat{\boldsymbol{\imath}} + 3t\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

17. **a.** $\underset{\sim}{r}(t) = 2t\hat{\boldsymbol{\imath}} + \frac{1}{t}\hat{\boldsymbol{\jmath}}$ for $t > 0$

b. $\underset{\sim}{r}(t) = 2t\hat{\boldsymbol{\imath}} + (t^2 - 4t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

c. $\underset{\sim}{r}(t) = \left(t + \frac{1}{t}\right)\hat{\boldsymbol{\imath}} + \left(t - \frac{1}{t}\right)\hat{\boldsymbol{\jmath}}$ for $t > 0$

18. **a.** $\underset{\sim}{r}(t) = e^{-2t}\hat{\boldsymbol{\imath}} + e^{2t}\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
b. $\underset{\sim}{r}(t) = e^{-t}\hat{\boldsymbol{\imath}} + \left(2 + e^{2t}\right)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
c. $\underset{\sim}{r}(t) = e^{t}\hat{\boldsymbol{\imath}} + \left(2 + e^{2t}\right)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

19. **a.** $\underset{\sim}{r}(t) = 3\cos(t)\hat{\boldsymbol{\imath}} + 3\sin(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
b. $\underset{\sim}{r}(t) = 4\cos(t)\hat{\boldsymbol{\imath}} + 3\sin(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
c. $\underset{\sim}{r}(t) = 4\sec(t)\hat{\boldsymbol{\imath}} + 3\tan(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

20. **a.** $\underset{\sim}{r}(t) = (1 + 3\cos(t))\hat{\boldsymbol{\imath}} + (3\sin(t) - 2)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
b. $\underset{\sim}{r}(t) = (4 + 3\cos(t))\hat{\boldsymbol{\imath}} + (2\sin(t) - 3)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
c. $\underset{\sim}{r}(t) = (2 - 3\sec(t))\hat{\boldsymbol{\imath}} + (5\tan(t) - 4)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

21. **a.** $\underset{\sim}{r}(t) = \cos^2(t)\hat{\boldsymbol{\imath}} + \sin^2(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
b. $\underset{\sim}{r}(t) = \cos^3(t)\hat{\boldsymbol{\imath}} + \sin^3(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
c. $\underset{\sim}{r}(t) = \cos^4(t)\hat{\boldsymbol{\imath}} + \sin^4(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

22. For each of the following vector equations, sketch the equation of the path, using technology.
a. $\underset{\sim}{r}(t) = \cos(2t)\hat{\boldsymbol{\imath}} + \sin(4t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
b. $\underset{\sim}{r}(t) = \cos(2t)\hat{\boldsymbol{\imath}} + \sin(6t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

Complex unfamiliar

23. Show that the parametric equations $x = \dfrac{6t}{1+t^2}$ and $y = \dfrac{3(1-t^2)}{1+t^2}$ represent the circle $x^2 + y^2 = 9$.

24. If a and b are positive real numbers, show that the following vector equations give the same Cartesian equation.
a. $\underset{\sim}{r}(t) = a\cos(t)\hat{\boldsymbol{\imath}} + a\sin(t)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(t) = \left(\dfrac{2at}{1+t^2}\right)\hat{\boldsymbol{\imath}} + \left(\dfrac{a(1-t^2)}{1+t^2}\right)\hat{\boldsymbol{\jmath}}$
b. $\underset{\sim}{r}(t) = a\cos(t)\hat{\boldsymbol{\imath}} + b\sin(t)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(t) = \left(\dfrac{2at}{1+t^2}\right)\hat{\boldsymbol{\imath}} + \left(\dfrac{b(1-t^2)}{1+t^2}\right)\hat{\boldsymbol{\jmath}}$

25. For each of the following vector equations, sketch the equation of the path, using technology.
a. The cycloid $\underset{\sim}{r}(t) = 2(t - \sin(t))\hat{\boldsymbol{\imath}} + 2(1 - \cos(t))\hat{\boldsymbol{\jmath}}$ for $t \geq 0$
b. The cardioid $\underset{\sim}{r}(t) = 2\cos(t)(1 + \cos(t))\hat{\boldsymbol{\imath}} + 2\sin(t)(1 + \cos(t))\hat{\boldsymbol{\jmath}}$ for $t \geq 0$

Fully worked solutions for this chapter are available online.

LESSON
4.3 Collision problems

SYLLABUS LINKS

- Understand and use the position of two particles, each described as a vector function of time, and determine if their paths cross or if the particles meet.

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There are a number of problems that can be formulated around the motion of two moving particles on different curves in two dimensions.

1. Do the particles collide?
2. Do the paths of the particles cross without colliding?
3. What is the distance between the particles at a particular time?

Collision problems

For two particles to collide, they must be at exactly the same coordinates at exactly the same time.

Two particles cross paths without colliding when they are at the same coordinates but at different times.

The distance between two particles at any given time is determined by the magnitude of the difference between their respective position vectors.

4.3.1 Collisions

Vector equations of lines, vector functions and parametric equations are convenient methods to describe the position of a particle in two or three dimensions according to a given parameter, t. As shown in many of the previous sections of work, time is a common parameter. The following examples explore various applications of vector functions of time in both abstract and real life instances.

In Worked example 11, we will see two methods to determine whether two particles collide. The first method consists of equating the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components of both paths, and comparing the values of the time t.

If both values of t are the same, this means that the paths cross at the same time, that is, the objects collide. Otherwise, the objects cross paths without colliding.

The second method consists of equating only the $\hat{\boldsymbol{\imath}}$ (or the $\hat{\boldsymbol{\jmath}}$) components of both paths, calculating the value(s) of t and determining the position vectors of both object at those times t to check whether they are equal. If they are, the objects collide (as they are at the same position at the same time). Otherwise, they cross paths without colliding.

WORKED EXAMPLE 11 Determining collision in two dimensions

The paths of two model sail boats are defined by the following vector functions of time, where time is measured in minutes and distance in metres.

$$\text{Boat A: } \underset{\sim}{r}_A(t) = (t+13)\hat{\boldsymbol{\imath}} + 6t\hat{\boldsymbol{\jmath}}$$

$$\text{Boat B: } \underset{\sim}{r}_B(t) = (3t-5)\hat{\boldsymbol{\imath}} + (5+t)\hat{\boldsymbol{\jmath}}$$

a. Calculate the distance between the two boats at $t = 8$ minutes.

b. Calculate if, at any given time t (seconds), they are in the same location. That is, determine if there is any possibility of a collision.

THINK	WRITE
a. 1. Evaluate the position of boat A by letting $t = 8$, expressed as $\underset{\sim}{r}_A(8)$.	a. $\underset{\sim}{r}_A(8) = (8+13)\hat{\boldsymbol{\imath}} + 6\times 8\hat{\boldsymbol{\jmath}}$ $= 21\hat{\boldsymbol{\imath}} + 48\hat{\boldsymbol{\jmath}}$
2. Evaluate the position of boat B by letting $t = 8$, expressed as $\underset{\sim}{r}_B(8)$.	$\underset{\sim}{r}_B(8) = (3\times 8-5)\hat{\boldsymbol{\imath}} + (5+8)\hat{\boldsymbol{\jmath}}$ $= 19\hat{\boldsymbol{\imath}} + 13\hat{\boldsymbol{\jmath}}$

3. The vector joining the two boats at $t = 8$ min, $\underset{\sim}{r}_B(8) - \underset{\sim}{r}_A(8)$, defines the directed line segment between the two boats. The magnitude of the resultant vector is then calculated. This represents the straight line distance between boat A and B.

$$\begin{aligned}|\underset{\sim}{r}_B(8) - \underset{\sim}{r}_A(8)| &= |(19\hat{\imath} + 13\hat{\jmath}) - (21\hat{\imath} + 48\hat{\jmath})| \\ &= |-2\hat{\imath} - 35\hat{\jmath}| \\ &= \sqrt{(-2)^2 + (-35)^2} \\ &= \sqrt{1229}\end{aligned}$$

The boats are approximately 35.06 metres apart.

Method 1:

b. 1. The boats will collide if they reach the same point at the same time. Equate the two vector functions.

b. $\underset{\sim}{r}_A = \underset{\sim}{r}_B$

$(t + 13)\hat{\imath} + 6t\hat{\jmath} = (3t - 5)\hat{\imath} + (5 + t)\hat{\jmath}$

2. Equate the $\hat{\imath}$ and $\hat{\jmath}$ components and solve for t in both cases.

$\hat{\imath}$ component:	$\hat{\jmath}$ component:
$t + 13 = 3t - 5$	$6t = 5 + t$
$18 = 2t$	$5t = 5$
$2t = 18$	$t = 1$
$t = 9$	

3. State your observations.

There is no common time for when the two functions are in the same location.

4. State the answer.

The boats do not collide.

Method 2:

1. The boats will collide if they reach the same point at the same time. Equate the two vector functions.

b. $\underset{\sim}{r}_A = \underset{\sim}{r}B$

$(t + 13)\hat{\imath} + 6t\hat{\jmath} = (3t - 5)\hat{\imath} + (5 + t)\hat{\jmath}$

2. Equate the $\hat{\imath}$ components and solve for t.

$$\begin{aligned} t + 13 &= 3t - 5 \\ 2t &= 18 \\ t &= 9 \end{aligned}$$

3. Substitute $t = 9$ into the $\hat{\jmath}$ components.

For boat A:

$6t = 54$

For boat B:

$5 + t = 14$

4. State your observations.

The boats have the same $\hat{\imath}$ components only when $t = 9$, but when $t = 9$, they do not have the same the $\hat{\jmath}$ components. There is no common time when the two functions are in the same location.

5. State the answer.

The boats do not collide.

Similarly, for collisions in three dimensions, two methods can be used.

- Method 1: equate the $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ components of both paths, and compare the values of the time t. If the three values of t are the same, this means that the paths cross at the same time, that is, the objects collide. Otherwise, the objects cross paths without colliding.
- Method 2: equate only one of the components of both paths (e.g. the $\hat{\imath}$ component), calculate the value(s) of t and determine the position vectors at those time to check whether they are equal. If they are, the objects collide (as they are at the same position at the same time). Otherwise, they cross paths without colliding.

WORKED EXAMPLE 12 Determining collision in three dimensions

The location of two particles can be defined by the following vector functions of time.

$$\underset{\sim}{r}_A(t) = \left(144 - t^2\right)\hat{\imath} + \left(2t^2\right)\hat{\jmath} + \left(0.25\,(t-2)^3\right)\hat{k}$$
$$\underset{\sim}{r}_B(t) = \left(t^2 - 5t - 6\right)\hat{\imath} + (240 - 4t)\hat{\jmath} + \left((t-4)^2 + 92\right)\hat{k}$$

Calculate if the particles collide at any given time t, $t \geq 0$ seconds.

THINK	WRITE
Method 1:	
1. Equate the $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ components and collect like terms.	$\hat{\imath}$ component $144 - t^2 = t^2 - 5t - 6$ $2t^2 - 5t - 150 = 0$ $\hat{\jmath}$ component $2t^2 = 240 - 4t$ $2t^2 + 4t - 240 = 0$ $t^2 + 2t - 120 = 0$ $\hat{k}$ component $0.25\,(t-2)^3 = (t-4)^2 + 92$
2. Factorise both quadratic equations and apply the Null Factor Law to solve for t and use technology to solve the cubic equation for t.	$\hat{\imath}$ component $2t^2 - 5t - 150 = 0$ $(2t + 15)(t - 10) = 0$ $t = -\frac{15}{2}$ or $t = 10$ $\hat{\jmath}$ component $t^2 + 2t - 120 = 0$ $(t - 10)(t + 12) = 0$ $t = 10$ or $t = -12$ $\hat{k}$ component $(t-2)^3 = (t-4)^2 + 92$ $t = 10$
3. State your observations.	There is a common time in which both particles have the same $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ components, that is, the same location. The particles collide at $t = 10$ s.
4. Determine the position vector of both particles at $t = 10$. You could substitute $t = 10$ in only one of the position vectors but substituting $t = 10$ in both position vectors enables you to check your previous working, as both position vectors should be equal.	$\underset{\sim}{r}_A(10) = \left(144 - 10^2\right)\hat{\imath} + 2 \times 10^2\hat{\jmath} + 0.25 \times (10-2)^3\,\hat{k}$ $= 44\hat{\imath} + 200\hat{\jmath} + 128\hat{k}$ $\underset{\sim}{r}_B(10) = \left(10^2 - 5 \times 10 - 6\right)\hat{\imath} + (240 - 4 \times 10)\hat{\jmath} + \left((10-4)^2 + 92\right)\hat{k}$ $= 44\hat{\imath} + 200\hat{\jmath} + 128\hat{k}$ We have verified $\underset{\sim}{r}_A(10) = \underset{\sim}{r}_B(10) = 44\hat{\imath} + 200\hat{\jmath} + 128\hat{k}$
Summarise your findings.	The two particles will be in the same location when $t = 10$ s. The position vector at this location is $44\hat{\imath} + 200\hat{\jmath} + 128\hat{k}$
Method 2:	
1. The particles will be in the same location if they reach the same point at the same time. Equate the two vector functions. Note that here we are using the column vector notation for practice with all notations.	$\underset{\sim}{r}_A(t) = \underset{\sim}{r}_B(t)$ $\begin{pmatrix} 144 - t^2 \\ 2t^2 \\ 0.25\,(t-2)^3 \end{pmatrix} = \begin{pmatrix} t^2 - 5t - 6 \\ 240 - 4t \\ (t-4)^2 + 92 \end{pmatrix}$

2. Equate the $\hat{\boldsymbol{\imath}}$ components, factorise and use the Null factor law to solve for t.	$\hat{\boldsymbol{\imath}}$ component $144 - t^2 = t^2 - 5t - 6$ $2t^2 - 5t - 150 = 0$ $(2t + 15)(t - 10) = 0$ $t = -\frac{15}{2}$ or $t = 10$
3. Eliminate any negative solution.	The position vectors are defined for $t \geq 0$ thus we can eliminate $t = -\frac{15}{2}$
4. Determine the position vectors of each particles at $t = 10$.	$\underset{\sim}{r}_A(10) = \begin{pmatrix} 144 - 10^2 \\ 2 \times 10^2 \\ 0.25(t-2)^3 \end{pmatrix} = \begin{pmatrix} 44 \\ 200 \\ 128 \end{pmatrix}$ $= \underset{\sim}{r}_B(10) = \begin{pmatrix} t^2 - 5 \times 10 - 6 \\ 240 - 4 \times 10 \\ (10-4)^2 + 92 \end{pmatrix} = \begin{pmatrix} 44 \\ 200 \\ 128 \end{pmatrix}$
5. Summarise your findings.	The two particles will be in the same location when $t = 10$ s. The position vector at this location is $\begin{pmatrix} 44 \\ 200 \\ 128 \end{pmatrix}$.

Note that the second method is generally quicker, as you are solving only one equation for t, and then determining the position vector for these values of t. It is recommended that you select $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ or $\hat{\boldsymbol{k}}$ components to equate for both paths depending on which one is the easiest equation to solve.

WORKED EXAMPLE 13 Determining intersection of path of particle with plane

The parametric equation, $x = 4 - t$, $y = 1 + 3t$, $z = 4 + 2t$, defines the path of particle. Determine the point of intersection between the particle and the plane, $2x + 4y - 2z = 1$.

THINK	WRITE
1. Define the three parametric equations as [1] to [3].	$x = 4 - t$ [1] $y = 1 + 3t$ [2] $z = 4 + 2t$ [3]
2. The point of intersection must satisfy both the equation of the plane and line. Substitute [1], [2] and [3] into the equation of the plane. Continue by solving for t.	$2(4 - t) + 4(1 + 3t) - 2(4 + 2t) = 1$ $8 - 2t + 4 + 12t - 8 - 4t = 1$ $6t + 4 = 1$ $t = \frac{1 - 4}{6}$ $t = -0.5$
3. Evaluate the parametric equation, for $t = -0.5$ to calculate the point of intersection.	$x = 4 - (-0.5)$ $= 4.5$ $y = 1 + 3(-0.5)$ $= -0.5$ $z = 4 + 2(-0.5)$ $= 3$
4. State the solution as the point of intersection.	The particle intersects the plane at $(4.5, -0.5, 3)$

4.3.2 Crossing paths

If two particles move so that they are at the same position at different times, they will not collide, but their paths will cross.

To solve this kind of problem, use a different pronumeral than t for the time parameter for one of the vector equations of motions, equate the components and solve for one of the parameters.

Investigate the position of each of the particles to determine whether this corresponds to a collision or to a crossing of paths.

WORKED EXAMPLE 14 Determining where particles collide or cross path

Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (3t - 8)\hat{\imath} + (t^2 - 18t + 87)\hat{\jmath}$ and $\underset{\sim}{r}_B(t) = (20 - t)\hat{\imath} + (2t - 4)\hat{\jmath}$ for $t \geq 0$. Determine:

a. when and where the particles collide

b. where their paths cross.

THINK	WRITE
a. 1. Equate the $\hat{\imath}$ components for each particle.	**a.** $\underset{\sim}{r}_A(t) = (3t - 8)\hat{\imath} + (t^2 - 18t + 87)\hat{\jmath}$ $\underset{\sim}{r}_B(t) = (20 - t)\hat{\imath} + (2t - 4)\hat{\jmath}$ $3t - 8 = 20 - t$
2. Solve this equation for t.	$4t = 28$ $t = 7$
3. Equate the $\hat{\jmath}$ components for each particle.	$t^2 - 18t + 87 = 2t - 4$
4. Solve this equation for t.	$t^2 - 20t + 91 = 0$ $(t - 7)(t - 13) = 0$ $t - 7 = 0$ or $t - 13 = 0$ $\therefore t = 7, 13$
5. Evaluate the position vectors at the common time, $t = 7$.	The common solution is when $t = 7$. $\underset{\sim}{r}_A(7) = 13\hat{\imath} + 10\hat{\jmath}$ $\underset{\sim}{r}_B(7) = 13\hat{\imath} + 10\hat{\jmath}$
6. State the result for when the particles collide.	The particles collide when $t = 7$ at the point $(13, 10)$.
b. 1. The particles, paths cross at different values of t. Introduce a different parameter, s.	**b.** Let $\underset{\sim}{r}_A(s) = (3s - 8)\hat{\imath} + (s^2 - 18s + 87)\hat{\jmath}$ and $\underset{\sim}{r}_B(t) = (20 - t)\hat{\imath} + (2t - 4)\hat{\jmath}$.
2. Equate the $\hat{\imath}$ components for each particle.	$3s - 8 = 20 - t$
3. Solve this equation for t.	$t = 28 - 3s$
4. Equate the $\hat{\jmath}$ components for each particle.	$s^2 - 18s + 87 = 2t - 4$
5. Solve this equation for s by substituting $t = 28 - 3s$.	Substitute $t = 28 - 3s$: $s^2 - 18s + 87 = 2(28 - 3s) - 4$ $s^2 - 18s + 87 = 56 - 6s - 4$ $s^2 - 12s + 35 = 0$ $(s - 5)(s - 7) = 0$ Using the Null Factor Law $s = 5, 7$
6. Note that when $s = 7$, the particles collide, so use $s = 5$.	$\underset{\sim}{r}_A(5) = (15 - 8)\hat{\imath} + (25 - 90 + 87)\hat{\jmath}$ $= 7\hat{\imath} + 22\hat{\jmath}$

7. Determine the time when the paths cross by solving for t when $s = 5$.	When $s = 5$, $t = 28 - 15$ $= 13$
8. Evaluate and check the position vectors at the required time, $t = 13$ or $s = 5$.	$\underset{\sim}{r}_B(13) = (20 - 13)\hat{\imath} + (26 - 4)\hat{\jmath}$ $= 7\hat{\imath} + 22\hat{\jmath}$ $\underset{\sim}{r}_A(5) = (15 - 8)\hat{\imath} + (25 - 18 \times 5 + 87)\hat{\jmath}$ $= 7\hat{\imath} + 22\hat{\jmath}$
9. State the required result.	The paths cross at the point (7, 22).

4.3.3 Closest approach

Knowing whether particles collide or cross paths will be important in many contexts, but it may also be useful to know how close they come to each other.

Calculating the distance between particles at a particular time, or at any time, is a matter of using the magnitude of the difference between their position vectors, as seen in Lesson 4.2.

To calculate the closest approach, calculus must be used to minimise the distance between the position vectors of the particles. Technology may be used to solve equations.

WORKED EXAMPLE 15 Determining the distance between particles and their closest approach

Two particles move so that their positions vectors are given by $\underset{\sim}{r}_A(t) = (t + 3)\hat{\imath} + (5t - 2)\hat{\jmath}$ and $\underset{\sim}{r}_B(t) = (5t + 1)\hat{\imath} + (2t)\hat{\jmath}$.

a. Calculate the distance between the particles when $t = 2$.

b. Determine an expression for the distance between the particles at any time.

c. Calculate the minimum distance between the paths of the particles.

THINK	WRITE
a. 1. Evaluate both position vectors when $t = 2$.	**a.** $\underset{\sim}{r}_A(2) = 5\hat{\imath} + 8\hat{\jmath}$ and $\underset{\sim}{r}_B(2) = 11\hat{\imath} + 4\hat{\jmath}$.
2. Calculate the difference between the two vectors.	$\underset{\sim}{r}_A(2) - \underset{\sim}{r}_B(2) = -6\hat{\imath} + 4\hat{\jmath}$ $\lvert\underset{\sim}{r}_A(2) - \underset{\sim}{r}_B(2)\rvert = \sqrt{6^2 + 4^2}$ $= 2\sqrt{13}$ units
3. State the distance between the particles when $t = 2$.	The distance between the particles when $t = 2$ is $2\sqrt{13}$ units.
b. 1. Determine the vector displacement at any time.	**b.** $\underset{\sim}{r}_A(t) - \underset{\sim}{r}_B(t) = ((t + 3) - (5t + 1))\hat{\imath} + ((5t - 2) - (2t))\hat{\jmath}$ $= (2 - 4t)\hat{\imath} + (3t - 2)\hat{\jmath}$
2. Determine the distance between the particles by calculating the magnitude of the difference of the vectors.	$\lvert\underset{\sim}{r}_A(t) - \underset{\sim}{r}_B(t)\rvert = \sqrt{(2 - 4t)^2 + (3t - 2)^2}$ $= \sqrt{25t^2 - 28t + 8}$ units
3. State the distance between the particles at any time.	The distance between the particles at time t is $\sqrt{25t^2 - 28t + 8}$ units.

c. 1. Use calculus to determine the minimum distance between the particles.

c. The minimum distance occurs when $\frac{d}{dt}|\underset{\sim}{r}_A(t) - \underset{\sim}{r}_B(t)| = 0$.

$$\begin{aligned}\frac{d}{dt}\sqrt{25t^2 - 28t + 8} &= \frac{1}{2}\frac{50t - 28}{\sqrt{25t^2 - 28t + 8}}\\ &= 0\\ 50t - 28 &= 0\\ t &= \frac{14}{25}\end{aligned}$$

The minimum distance between the particles occurs when $t = \frac{14}{25}$.

2. Substitute $t = \frac{14}{25}$ into the expression for the distance between the particles.

$$\begin{aligned}\left|\underset{\sim}{r}_A\left(\frac{14}{25}\right) - \underset{\sim}{r}_B\left(\frac{14}{25}\right)\right| &= \sqrt{25\left(\frac{14}{25}\right)^2 - 28\left(\frac{14}{25}\right) + 8}\\ &= \sqrt{0.16}\\ &= 0.4 \text{ units}\end{aligned}$$

3. State the minimum distance between the paths of the particles.

The minimum distance between the paths of the particles is 0.4 units.

Exercise 4.3 Collision problems

learn on

4.3 Exercise | **4.3 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	12, 13	14, 15

These questions are even better in jacPLUS!
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Simple familiar

1. WE11 The paths of two model sail boats are defined by the following vector functions of time, where time is measured in minutes and distance in metres.

$$\text{Boat A: } \underset{\sim}{r}_A(t) = (9t - 7)\hat{\boldsymbol{\imath}} + 11t\hat{\boldsymbol{\jmath}}$$
$$\text{Boat B: } \underset{\sim}{r}_A(t) = (2t + 14)\hat{\boldsymbol{\imath}} + (9 + t)\hat{\boldsymbol{\jmath}}$$

a. Calculate the distance between the two boats at $t = 5$ minutes.

b. Calculate if, at any given time t (minutes), they are in the same location. That is, determine if there is any possibility of a collision.

2. **WE12** The flight path, measured in metres, of two helicopters is given by the vector functions of time (minutes) below.

$$\underset{\sim}{r}_A(t) = \left(169 + 0.5t^2\right)\hat{\boldsymbol{\imath}} + (10t + 120)\hat{\boldsymbol{\jmath}} + \left(0.45t^3 + 2\right)\hat{\boldsymbol{k}}$$
$$\underset{\sim}{r}_B(t) = (4t + 179)\hat{\boldsymbol{\imath}} + \left(240 - 0.2t^2\right)\hat{\boldsymbol{\jmath}} + \left(4t^2 + 20\right)\hat{\boldsymbol{k}}$$

Determine if the helicopters will ever cross paths. State any assumptions used in developing models and solutions.

3. **WE13** Two particles are moving in a circular motion in a three-dimensional space. The position of the first particle is defined by

$$x_a = 6\sin(t), \quad y_a = 4\cos(t), \quad z_a = 5 \qquad 0 \le t \le 10\pi$$

The position of the second particle is defined by

$$x_b = 2\cos(t) - 6, \quad y_b = 2 + 2\sin(t), \quad z_b = 5 \qquad 0 \le t \le 10\pi$$

The position of both particles are functions of time (t) given in seconds. Calculate the time and position of when the two paths first intersect.

4. **WE14** Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (2t + 6)\hat{\boldsymbol{\imath}} + (t^2 - 6t + 45)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (t + 11)\hat{\boldsymbol{\imath}} + (7t + 5)\hat{\boldsymbol{\jmath}}$ for $t \ge 0$. Determine:
   - **a.** when and where the particles collide
   - **b.** where their paths cross
   - **c.** the distance between the particles when $t = 10$.

5. **WE15** Two particles move so that their positions vectors are given by $\underset{\sim}{r}_A(t) = (2t - 1)\hat{\boldsymbol{\imath}} + (5 + t)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (4t + 3)\hat{\boldsymbol{\imath}} + (3t - 2)\hat{\boldsymbol{\jmath}}$.
   - **a.** Calculate the distance between the particles when $t = 1$.
   - **b.** Determine an expression for the distance between the particles at any time.
   - **c.** Calculate the minimum distance between the paths of the particles. Give your answer to 2 decimal places.

6. Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (-t^2 + 12t - 22)\hat{\boldsymbol{\imath}} + (19 - 3t)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (18 - 2t)\hat{\boldsymbol{\imath}} + (t + 3)\hat{\boldsymbol{\jmath}}$ for $t \ge 0$. Determine:
   - **a.** when and where the particles collide
   - **b.** where their paths cross
   - **c.** the distance between the particles when $t = 5$.

7. Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (3t - 43)\hat{\boldsymbol{\imath}} + (-t^2 + 26t - 160)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (17 - t)\hat{\boldsymbol{\imath}} + (2t - 25)\hat{\boldsymbol{\jmath}}$ for $t \ge 0$. Determine:
   - **a.** when and where the particles collide
   - **b.** where their paths cross
   - **c.** the distance between the particles when $t = 12$.

8. Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (t^2 - 6)\hat{\boldsymbol{\imath}} + (2t + 2)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (7t - 16)\hat{\boldsymbol{\imath}} + \frac{1}{5}(17t - t^2)\hat{\boldsymbol{\jmath}}$ for $t \ge 0$. Determine:
   - **a.** when and where the particles collide
   - **b.** the distance between the particles when $t = 10$.

9. Two particles move so that their position vectors are given by $\underset{\sim}{r}_A(t) = (-t^2 + 12t + 53)\hat{\boldsymbol{\imath}} + (3t + 38)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (2t + 29)\hat{\boldsymbol{\imath}} + (86 - t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$. Determine:
   a. when and where the particles collide
   b. where their paths cross
   c. the distance between the particles when $t = 20$.

10. The path (measured in metres) of two objects in flight is given by the following vector functions of time (minutes):

$$\underset{\sim}{r}_a(t) = \left(t^2 + 4\right)\hat{\boldsymbol{\imath}} + (4t + 9)\hat{\boldsymbol{\jmath}} + \left(0.5t^2 + 1\right)\hat{\boldsymbol{k}}, \quad 0 \leq t \leq 20$$
$$\underset{\sim}{r}_b(t) = (8t + 184)\hat{\boldsymbol{\imath}} + \left(0.25t^2\right)\hat{\boldsymbol{\jmath}} + (3t + 109)\hat{\boldsymbol{k}}, \quad 0 \leq t \leq 20$$

Determine if the particles will cross paths at the same time, stating the time it occurs and the location at that point in time.

11. Two particles move so that their positions vectors are given in metres by $\underset{\sim}{r}_A(t) = \begin{pmatrix} 2t + 4 \\ t^2 - 1 \\ 6t + 4 \end{pmatrix}$ and $\underset{\sim}{r}_B(t) = \begin{pmatrix} t^2 + 2t + 5 \\ t^2 - 4t + 7 \\ 3t^2 + 4 \end{pmatrix}$ for $t \geq 0$ in seconds.
Calculate the minimum distance between the paths of the particles. Give your answer to two decimal places.

Complex familiar

12. Two particles move so that their positions vectors are given by $\underset{\sim}{r}_A(t) = \left(4 - t^2\right)\hat{\boldsymbol{\imath}} + (1 + t)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (2 - t)\hat{\boldsymbol{\imath}} + \left(1 - kt^2\right)\hat{\boldsymbol{\jmath}}$, $t \geq 0$, $\forall t \geq 0$, and where $k \in \mathbb{R}$, the distance are in centimetres and the time is in seconds. Determine the speed of particle B when the particles collide.

13. The position of a particle A is defined by the vector equation $\underset{\sim}{r}_A(t) = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} k \\ -1 \\ -2 \end{pmatrix}$, $t \geq 0$, where $k \in \mathbb{R}$ and the position of a particle B is defined by the vector equation $\underset{\sim}{r}_B(t) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$, $t \geq 0$. The distance is in metres and the time is in seconds. Determine the closest distance between the particles.

Complex unfamiliar

14. Two particles move so that their positions vectors are given by $\underset{\sim}{r}_A(t) = (1 + t)\hat{\boldsymbol{\imath}} + (1 - kt)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_B(t) = (2 + t)\hat{\boldsymbol{\imath}} + (1 - 6t)\hat{\boldsymbol{\jmath}}$, $t \geq 0$, $\forall t \geq 0$, and where $k \in \mathbb{R}$, the distance are in metres and the time is in seconds.
Tanya affirms that the particles are always at least 1 m apart. Verify the reasonableness of this claim.

15. The position of a particle A is defined by the vector equation $\underset{\sim}{r}_A(t) = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -k \end{pmatrix}$, $t \geq 0$, where $k \in \mathbb{R}$.
Particle B is moving on a straight-line path, and its location was recorded on two occasions, defined by the vectors $\underset{\sim}{a} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$. Distances are in metres, and time is in seconds.
Determine, to two decimal places, the maximum value of k such that the distance between the particles is always greater than 5 m and verify the reasonableness of your solution.

Fully worked solutions for this chapter are available online.

LESSON
4.4 Differentiation and integration of vectors

SYLLABUS LINKS

- Differentiate and integrate a vector function with respect to time.

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4.4.1 Differentiating and integrating vectors with respect to time

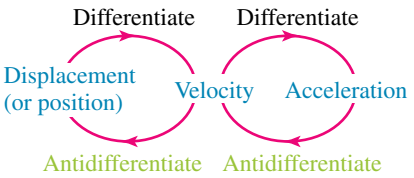
Consider $\underset{\sim}{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$ the position vector of a particle moving in three dimensions.

The function $\underset{\sim}{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$ is called a vector function. As t changes, the x-, y- and z-coordinates changes, the particle is moving along a curve. The equation of the curve is called the Cartesian equation of the path.

Differentiating or integrating a vector function with respect to time enables the motion of a particle on a vector path to be investigated.

The vectors describing motion are the position vector, the velocity vector and the acceleration vector.

The relationships between the equations describing motion are shown in the diagram.



These connections work in two and three dimensions, and are used to model real world, as well as abstract situations.

Differentiating and integrating a vector with respect to time

The derivative of a vector is determined by differentiating separately each component of the vector with respect to the variable time, using the rules for differentiation.

The integral of a vector is determined by integrating separately each component of the vector with respect to the variable time, using the rules for integration.

Note that you already have all the skills required to differentiate and integrate vectors from the differentiating and integrating functions skills covered in the Mathematical Methods course, as you only need to consider each component separately.

The rules for differentiation and for integration still apply, we merely differentiate or integrate each component separately.

When differentiating with respect to time, we use the following notation $\frac{d}{dt}\underset{\sim}{r}(t) = \dot{\underset{\sim}{r}}(t)$. The single dot above indicates the derivative with respect to time.

Similarly, $\frac{d^2}{dt^2}\underset{\sim}{r}(t) = \ddot{\underset{\sim}{r}}(t)$ is the second derivative with respect to time.

Differentiating a vector with respect to time

Consider a vector $\underset{\sim}{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

Its derivative with respect to time is $\frac{d}{dt}\underset{\sim}{r}(t) = \dot{\underset{\sim}{r}}(t)$.

It is determined by differentiating each component with respect to time.

$$\dot{\underset{\sim}{r}}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$$= \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}$$

Note that when working in the plane instead of in three dimensions, you just need to consider the x- and y-components.

You can, of course, use the component form $\dot{\underset{\sim}{r}}(t) = \dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\hat{\boldsymbol{\jmath}} + \dot{z}(t)\hat{\boldsymbol{k}}$ instead.

However, the column vector notation makes it easier to separate the components to treat them independently.

WORKED EXAMPLE 16 Calculating the first and second derivative of a vector

Consider the vector $\underset{\sim}{r}(t) = \begin{pmatrix} 2t+4 \\ t^2 \\ 4t^3+2t-1 \end{pmatrix}, t \geq 0$.

a. Calculate $\frac{d}{dt}\underset{\sim}{r}(t)$.

b. Calculate $\frac{d^2}{dt^2}\underset{\sim}{r}(t)$.

THINK

a. 1. Calculate the first derivative of each component.

WRITE

a. $\frac{d}{dt}\underset{\sim}{r}(t) = \frac{d}{dt}\begin{pmatrix} 2t+4 \\ t^2 \\ 4t^3+2t-1 \end{pmatrix}$

$$= \begin{pmatrix} \frac{d}{dt}(2t+4) \\ \frac{d}{dt}(t^2) \\ \frac{d}{dt}(4t^3+2t-1) \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2t \\ 12t^2+2 \end{pmatrix}$$

2. State the answer.

$$\frac{d}{dt}\underset{\sim}{r}(t) = \begin{pmatrix} 2 \\ 2t \\ 12t^2 + 2 \end{pmatrix}$$

b. 1. Calculate the second derivative of each component.

b.
$$\begin{aligned}
\frac{d^2}{dt^2}\underset{\sim}{r}(t) &= \frac{d^2}{dt^2}\begin{pmatrix} 2t+4 \\ t^2 \\ 4t^3 + 2t - 1 \end{pmatrix} \\
&= \frac{d}{dt}\begin{pmatrix} 2 \\ 2t \\ 12t^2 + 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{d}{dt}(2) \\ \frac{d}{dt}(2t) \\ \frac{d}{dt}\left(12t^2 + 2\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2 \\ 24t \end{pmatrix}
\end{aligned}$$

2. State the answer.

$$\frac{d^2}{dt^2}\underset{\sim}{r}(t) = \begin{pmatrix} 0 \\ 2 \\ 24t \end{pmatrix}$$

Recall that when integrating a function, you need to include a constant of integration, which is determined by using initial conditions.

Similarly, when integrating a vector, you need to include a constant vector, which is also determined by using initial conditions.

If $\underset{\sim}{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$, then $\int \underset{\sim}{r}(t) = \begin{pmatrix} \int x(t)\,dt + c_1 \\ \int x(t)\,dt + c_2 \\ \int x(t)\,dt + c_3 \end{pmatrix}$ where c_1, c_2 and c_3 are constant.

This can be expressed as

$$\int \underset{\sim}{r}(t) = \begin{pmatrix} \int x(t)\,dt \\ \int x(t)\,dt \\ \int x(t)\,dt \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Integrating a vector with respect to time

Consider a vector $\underset{\sim}{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

Its integral with respect to time is $\int \underset{\sim}{r}(t)\,dt$.

It is determined by integrating each component with respect to time and adding a constant vector.

$$\int \underset{\sim}{r}(t) = \begin{pmatrix} \int x(t)\,dt \\ \int x(t)\,dt \\ \int x(t)\,dt \end{pmatrix} + \underset{\sim}{c}$$

where $\underset{\sim}{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ **is a constant vector with respect to time, that is, its components do not depend on** t.

WORKED EXAMPLE 17 Integrating vectors with respect to time

Consider the vector $\ddot{\underset{\sim}{r}}(t) = \begin{pmatrix} t+5 \\ 2t \\ 3-t \end{pmatrix}, t \geq 0$.

a. Calculate $\dot{\underset{\sim}{r}}(t)$**, given that** $\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

b. Calculate $\underset{\sim}{r}(t)$**, given that** $\underset{\sim}{r}(0) = \begin{pmatrix} 5 \\ -6 \\ 3 \end{pmatrix}$.

THINK

a. 1. Integrate each component and include a constant vector of integration.

WRITE

a. $\dot{\underset{\sim}{r}}(t) = \int \ddot{\underset{\sim}{r}}(t)\,dt$

$$= \begin{pmatrix} \int (t+5)\,dt \\ \int (2t)\,dt \\ \int (3-t)\,dt \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}t^2 + 5t \\ t^2 \\ 3t - \frac{1}{2}t^2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

2. Use the initial condition to determine the constant vector of integration.

$$\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

3. State the answer.

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} \frac{1}{2}t^2 + 5t + 1 \\ t^2 + 1 \\ 3t - \frac{1}{2}t^2 - 2 \end{pmatrix}$$

b. 1. Integrate each component again and include a constant vector of integration.

$$\underset{\sim}{r}(t) = \int \dot{\underset{\sim}{r}}(t)\, dt$$

$$= \begin{pmatrix} \int \left(\frac{1}{2}t^2 + 5t + 1\right) dt \\ \int \left(t^2 + 1\right) dt \\ \int \left(3 - \frac{1}{2}t^2 - 2\right) dt \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6}t^3 + \frac{5}{2}t^2 + t \\ \frac{1}{3}t^3 + t \\ 3t - \frac{1}{6}t^3 - 2t \end{pmatrix} + \begin{pmatrix} c_4 \\ c_5 \\ c_6 \end{pmatrix}$$

2. Use the initial condition to determine the constant vector of integration.

$$\underset{\sim}{r}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} c_4 \\ c_5 \\ c_6 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -6 \\ 3 \end{pmatrix}$$

3. State the answer.

$$\underset{\sim}{r}(t) = \begin{pmatrix} \frac{1}{6}t^3 + \frac{5}{2}t^2 + t + 5 \\ \frac{1}{3}t^3 + t - 6 \\ 3t - \frac{1}{6}t^3 - 2t + 3 \end{pmatrix}$$

As you can see from Worked examples 16 and 17, differentiating and integrating vectors with respect to time is merely applying the rules for differentiation and the rules for integration to each component independently.

4.4.2 Vectors describing motion

Velocity vector

Because $\underset{\sim}{r}(t)$ represents the position vector, $\underset{\sim}{v}(t) = \frac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}}(t)$ represents the velocity vector.

Speed and momentum

The speed of a moving particle is the magnitude of the velocity vector. The speed at time t is given by $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{k}^2}$.

Momentum is the quantity of motion that an object has, and it is dependent upon both the mass of the object and the speed of the object. As all objects have mass, all moving objects have momentum. If the particle has a mass of m, then the momentum, $\underset{\sim}{p}$, acting on the particle is given by $\underset{\sim}{p} = m\dot{\underset{\sim}{r}}(t)$.

Acceleration vector

Since $\underset{\sim}{v}(t) = \dfrac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}}(t)$ represents the velocity vector, differentiating again with respect to t gives the acceleration vector. The acceleration vector is given by $\underset{\sim}{a}(t) = \dfrac{d}{dt}(\dot{r}(t)) = \ddot{r}(t) = \ddot{x}(t)\hat{\boldsymbol{\imath}} + \ddot{y}(t)\hat{\boldsymbol{\jmath}}$. Note that the two dots above the variables indicate the second derivative with respect to time.

Vectors describing motion in three dimensions

Position vector: $\underset{\sim}{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Velocity vector:
$$\begin{aligned}\underset{\sim}{v}(t) &= \frac{d\underset{\sim}{r}}{dt}\\ &= \dot{\underset{\sim}{r}}(t)\\ &= \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} + \dot{z}(t)\hat{k}\end{aligned}$$

Speed: $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

Momentum: $\underset{\sim}{p} = m\dot{\underset{\sim}{r}}(t)$

Acceleration:
$$\begin{aligned}\underset{\sim}{a}(t) &= \frac{d^2\underset{\sim}{r}}{dt^2}\\ &= \frac{d\underset{\sim}{v}}{dt}\\ &= \frac{d}{dt}(\dot{\underset{\sim}{r}}(t))\\ &= \ddot{\underset{\sim}{r}}(t)\\ &= \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath} + \ddot{z}(t)\hat{k}\end{aligned}$$

WORKED EXAMPLE 18 Deriving vectors in two dimensions

A particle spirals outwards so that its position vector is given by $\underset{\sim}{r}(t) = t\cos(t)\hat{\imath} + t\sin(t)\hat{\jmath}$ for $t \geq 0$. Calculate:

a. the velocity vector

b. the speed of the particle at time t and hence determine the speed when $t = \dfrac{3\pi}{4}$

c. the acceleration vector.

THINK	WRITE
a. 1. State the parametric equations.	a. $\underset{\sim}{r}(t) = t\cos(t)\hat{\imath} + t\sin(t)\hat{\jmath}$ Then $x(t) = t\cos(t)$ and $y(t) = t\sin(t)$.

2. Differentiate x with respect to t. The dot notation is used for the derivative with respect to t. Recall and apply the product rule.

$$\begin{aligned}\frac{dx}{dt} &= \dot{x} = \frac{d}{dt}(t\cos(t)) \\ &= \cos(t)\frac{d}{dt}(t) + t\frac{d}{dt}(\cos(t)) \\ &= \cos(t) - t\sin(t)\end{aligned}$$

3. Differentiate y with respect to t. Recall and apply the product rule.

$$\begin{aligned}\frac{dy}{dt} &= \dot{y} = \frac{d}{dt}(t\sin(t)) \\ &= \sin(t)\frac{d}{dt}(t) + t\frac{d}{dt}(\sin(t)) \\ &= \sin(t) + t\cos(t)\end{aligned}$$

4. State the velocity vector.

$$\dot{\underset{\sim}{r}}(t) = (\cos(t) - t\sin(t))\hat{\boldsymbol{\imath}} + (\sin(t) + t\cos(t))\hat{\boldsymbol{\jmath}}$$

b. 1. Calculate the speed of the particle at time t, by applying the speed formula, $|\dot{\underset{\sim}{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$. Substitute for the derivatives and expand.

b.
$$\begin{aligned}|\dot{\underset{\sim}{r}}(t)| &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ \dot{x}^2 &= (\cos(t) - t\sin(t))^2 \\ &= \cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t) \\ \dot{y}^2 &= (\sin(t) + t\cos(t))^2 \\ &= \sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t)\end{aligned}$$

2. Simplify using trigonometry, by recalling the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$.

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 &= \cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t) + \sin^2(t) \\ &\quad + 2t\sin(t)\cos(t) + t^2\cos^2(t) \\ &= \cos^2(t) + \sin^2(t) + t^2\sin^2(t) + t^2\cos^2(t) \\ &= 1 + t^2\end{aligned}$$

3. State the speed at time t.

$$|\dot{\underset{\sim}{r}}(t)| = \sqrt{1 + t^2}$$

4. Determine the speed at the required time, $t = \dfrac{3\pi}{4}$.

$$\begin{aligned}\left|\dot{\underset{\sim}{r}}\left(\frac{3\pi}{4}\right)\right| &= \sqrt{1 + \left(\frac{3\pi}{4}\right)^2} \\ &= \sqrt{\frac{16 + 9\pi^2}{16}}\end{aligned}$$

5. State the speed in simplified form.

$$\left|\dot{\underset{\sim}{r}}\left(\frac{3\pi}{4}\right)\right| = \frac{1}{4}\sqrt{16 + 9\pi^2}$$

c. 1. Recall the acceleration vector formula $\underset{\sim}{a}(t) = \dfrac{d}{dt}(\dot{\underset{\sim}{r}}(t)) = \ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\hat{\boldsymbol{\imath}} + \ddot{y}(t)\hat{\boldsymbol{\jmath}}$ and apply first to the $\hat{\boldsymbol{\imath}}$ component.

c.
$$\begin{aligned}\frac{d^2x}{dt^2} &= \ddot{x} = \frac{d}{dt}(\cos(t) - t\sin(t)) \\ &= -\sin(t) - \frac{d}{dt}(t\sin(t)) \\ &= -\sin(t) - \sin(t) - t\cos(t) \\ &= -2\sin(t) - t\cos(t)\end{aligned}$$

2. Determine the $\hat{\boldsymbol{\jmath}}$ component of the acceleration vector.

$$\begin{aligned}\frac{d^2y}{dt^2} &= \ddot{y} = \frac{d}{dt}(\sin(t) + t\cos(t)) \\ &= \cos(t) + \frac{d}{dt}(t\cos(t)) \\ &= \cos(t) + \cos(t) - t\sin(t) \\ &= 2\cos(t) - t\sin(t)\end{aligned}$$

3. State the acceleration vector in terms of t.

$$\begin{aligned}\underset{\sim}{r}(t) &= x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} \\ \ddot{\underset{\sim}{r}}(t) &= -(2\sin(t) + t\cos(t))\hat{\boldsymbol{\imath}} + (2\cos(t) - t\sin(t))\hat{\boldsymbol{\jmath}}\end{aligned}$$

WORKED EXAMPLE 19 Deriving vectors in three dimensions

A particle has a position vector given by $\underset{\sim}{r}(t) = t^3\hat{\imath} + 6\sin(3t)\hat{\jmath} + 12e^{-\frac{t}{2}}\hat{k}$ for $t \geq 0$. Determine:

a. the velocity vector

b. the acceleration vector.

THINK	WRITE
a. 1. Differentiate the position vector to determine the velocity vector.	a. $\underset{\sim}{r}(t) = t^3\hat{\imath} + 6\sin(3t)\hat{\jmath} + 12e^{-\frac{t}{2}}\hat{k}$ $\dot{\underset{\sim}{r}}(t) = \frac{d}{dt}(t^3)\hat{\imath} + \frac{d}{dt}(6\sin(3t))\hat{\jmath} + \frac{d}{dt}\left(12e^{-\frac{t}{2}}\right)\hat{k}$
2. State the derivative or velocity vector.	$\dot{\underset{\sim}{r}}(t) = 3t^2\hat{\imath} + 18\cos(3t)\hat{\jmath} - 6e^{-\frac{t}{2}}\hat{k}$
b. 1. Differentiate the velocity vector to determine the acceleration vector.	b. $\ddot{\underset{\sim}{r}}(t) = \frac{d}{dt}(3t^2)\hat{\imath} + \frac{d}{dt}(18\cos(3t)\hat{\jmath} - \frac{d}{dt}\left(6e^{-\frac{t}{2}}\right)\hat{k}$
2. State the acceleration vector.	$\ddot{\underset{\sim}{r}}(t) = 6t\hat{\imath} - 54\sin(3t)\hat{\jmath} + 3e^{-\frac{t}{2}}\hat{k}$

Acceleration, velocity and position vectors in the plane

Given the acceleration vector $\underset{\sim}{a}(t) = \frac{d\underset{\sim}{v}(t)}{dt} = \ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath}$ we can obtain the:

Velocity vector: $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \int \ddot{x}(t)dt\hat{\imath} + \int \ddot{y}(t)dt\hat{\jmath} = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} + \underset{\sim}{c}_1$

Position vector: $\underset{\sim}{r}(t) = \int \dot{x}(t)dt\hat{\imath} + \int \dot{y}(t)dt\hat{\jmath} = x(t)\hat{\imath} + y(t)\hat{\jmath} + \underset{\sim}{c}_2$

WORKED EXAMPLE 20 Determining the position vector from the velocity vector

The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = 2\hat{\imath} + 6t\hat{\jmath}$ for $t \geq 0$. If $\underset{\sim}{r}(1) = 3\hat{\imath} + \hat{\jmath}$, determine the position vector at any time t.

THINK	WRITE
1. Integrate the velocity vector to obtain the position vector using the given rules.	$\dot{\underset{\sim}{r}}(t) = 2\hat{\imath} + 6t\hat{\jmath}$ $\underset{\sim}{r}(t) = \int 2dt\,\hat{\imath} + \int 6tdt\,\hat{\jmath}$
2. Perform the integration. Do not forget to add in a constant vector.	$\underset{\sim}{r}(t) = 2t\hat{\imath} + 3t^2\hat{\jmath} + \underset{\sim}{c}$
3. Substitute to determine the value of the constant vector.	Substitute $t = 1$ and use the given condition. $\underset{\sim}{r}(1) = 2\hat{\imath} + 3\hat{\jmath} + \underset{\sim}{c} = 3\hat{\imath} + \hat{\jmath}$
4. Solve for the constant vector.	$2\hat{\imath} + 3\hat{\jmath} + \underset{\sim}{c} = 3\hat{\imath} + \hat{\jmath}$ $\underset{\sim}{c} = 3\hat{\imath} + \hat{\jmath} - (2\hat{\imath} + 3\hat{\jmath})$ $\underset{\sim}{c} = \hat{\imath} - 2\hat{\jmath}$

5. Substitute this value of the constant vector into the position vector.	$\underset{\sim}{r}(t) = 2t\hat{\imath} + 3t^2\hat{\jmath} + \underset{\sim}{c}$ $= (2t\hat{\imath} + 3t^2\hat{\jmath}) + (\hat{\imath} - 2\hat{\jmath})$
6. Simplify the position vector to give the final result.	$\underset{\sim}{r}(t) = (2t+1)\hat{\imath} + (3t^2 - 2)\hat{\jmath}$

WORKED EXAMPLE 21 Integrating vectors, and using constants of integration

The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = 6t\hat{\imath}$, where $t \geq 0$. Given that $\dot{\underset{\sim}{r}}(2) = 6\hat{\imath} - 3\hat{\jmath}$ and $\underset{\sim}{r}(2) = 4\hat{\imath} - 2\hat{\jmath}$, determine the position vector at time t.

THINK	WRITE
1. Integrate the acceleration vector to obtain the velocity vector using the given rules.	$\ddot{\underset{\sim}{r}}(t) = 6t\hat{\imath}$ $\dot{\underset{\sim}{r}}(t) = \int 6t\,dt\,\hat{\imath}$
2. Perform the integration. Do not forget to add in the first constant vector.	$\dot{\underset{\sim}{r}}(t) = 3t^2\hat{\imath} + \underset{\sim}{c}_1$
3. To determine the first constant vector substitute $t = 2$ into the integration and equate with the first given condition, $\dot{\underset{\sim}{r}}(2) = 6\hat{\imath} - 3\hat{\jmath}$.	$\dot{\underset{\sim}{r}}(2) = 3t^2\hat{\imath} + \underset{\sim}{c}_1$ $= 3(2)^2\hat{\imath} + \underset{\sim}{c}_1$ $= 12\hat{\imath} + \underset{\sim}{c}_1$ $12\hat{\imath} + \underset{\sim}{c}_1 = 6\hat{\imath} - 3\hat{\jmath}$
4. Solve for the first constant vector.	$\underset{\sim}{c}_1 = (6\hat{\imath} - 3\hat{\jmath}) - 12\hat{\imath}$ $= -6\hat{\imath} - 3\hat{\jmath}$
5. Substitute back for the first constant vector, $\dot{\underset{\sim}{r}}(t) = 3t^2\hat{\imath} + \underset{\sim}{c}_1$.	$\dot{\underset{\sim}{r}}(t) = 3t^2\hat{\imath} + (-6\hat{\imath} - 3\hat{\jmath})$
6. Simplify to give the velocity vector.	$\dot{\underset{\sim}{r}}(t) = (3t^2 - 6)\hat{\imath} - 3\hat{\jmath}$
7. To obtain the position vector, integrate the velocity vector using the given rules.	$\underset{\sim}{r}(t) = \int (3t^2 - 6)\,dt\,\hat{\imath} - \int 3\,dt\,\hat{\jmath}$
8. Perform the integration. Do not forget to add in a second constant vector.	$\underset{\sim}{r}(t) = (t^3 - 6t)\hat{\imath} - 3t\hat{\jmath} + c_2$
9. To determine the second constant vector, substitute $t = 2$ into the integration and equate with the second given condition, $\underset{\sim}{r}(2) = 4\hat{\jmath} - 2\hat{\jmath}$.	Substitute $t = 2$ and use the second given condition. $\underset{\sim}{r}(2) = (8 - 12)\hat{\imath} - 6\hat{\jmath} + \underset{\sim}{c}_2 = 4\hat{\imath} - 2\hat{\jmath}$ $-4\hat{\imath} - 6\hat{\jmath} + \underset{\sim}{c}_2 = 4\hat{\imath} - 2\hat{\jmath}$
10. Solve for the second constant vector.	$\underset{\sim}{c}_2 = (4\hat{\imath} - 2\hat{\jmath}) - (-4\hat{\imath} - 6\hat{\jmath})$ $= 8\hat{\imath} + 4\hat{\jmath}$
11. Substitute back for the second constant vector, $\underset{\sim}{r}(t) = (t^3 - 6t)\hat{\imath} - 3t\hat{\jmath} + c_2$.	$\underset{\sim}{r}(t) = (t^3 - 6t)\hat{\imath} - 3t\hat{\jmath} + (8\hat{\imath} + 4\hat{\jmath})$
12. Simplify the position vector to give the final result.	$\underset{\sim}{r}(t) = (t^3 - 6t + 8)\hat{\imath} + (4 - 3t)\hat{\jmath}$

4.4.3 Determining the Cartesian equation

If the position vector is known then the parametric equations of the path of the particle can be determined. Eliminating the parameter, we can determine the Cartesian equation of the curve along which the particle moves.

WORKED EXAMPLE 22 Determining the Cartesian equation

The acceleration vector of a moving particle is given by $\cos\left(\frac{t}{2}\right)\hat{\imath} - \sin\left(\frac{t}{2}\right)\hat{\jmath}$ for $0 \le t \le 4\pi$, where t is the time. The initial velocity is $2\hat{\jmath}$ and the initial position is $\hat{\imath} - 3\hat{\jmath}$. Determine the Cartesian equation of the path.

THINK	WRITE
1. To obtain the velocity vector, state and integrate the acceleration vector using the given rules.	$\ddot{\underset{\sim}{r}}(t) = \cos\left(\frac{t}{2}\right)\hat{\imath} - \sin\left(\frac{t}{2}\right)\hat{\jmath}$ $\dot{\underset{\sim}{r}}(t) = \int \cos\left(\frac{t}{2}\right)\, dt\hat{\imath} + \int \sin\left(\frac{t}{2}\right)\, dt\hat{\jmath}$
2. Perform the integration. Do not forget to add in the first constant vector. Recall that $\int \cos(kx)dx = \frac{1}{k}\sin(kx)$ and $\int \sin(kx)dx = -\frac{1}{k}\cos(kx)$.	Since $\int \cos(kx)dx = \frac{1}{k}\sin(kx)$ and $\int \sin(kx)dx = -\frac{1}{k}\cos(kx)$ with $k = \frac{1}{2}$, $\dot{\underset{\sim}{r}}(t) = 2\sin\left(\frac{t}{2}\right)\hat{\imath} + 2\cos\left(\frac{t}{2}\right)\hat{\jmath} + \underset{\sim}{c}_1$.
3. Initially means when $t = 0$. Substitute $t = 0$ and equate with the first given condition to solve for the first constant vector.	$\dot{\underset{\sim}{r}}(0) = 2\sin(0)\hat{\imath} + 2\cos(0)\hat{\jmath} + \underset{\sim}{c}_1$ $= 2 \times 0\hat{\imath} + 2\hat{\jmath} + \underset{\sim}{c}_1$ $2\hat{\jmath} + \underset{\sim}{c}_1 = 2\hat{\jmath}$ $\underset{\sim}{c}_1 = \underset{\sim}{0}$
4. Substitute back for the first constant vector and integrate the velocity vector to obtain the position vector, using the given rules.	$\dot{\underset{\sim}{r}}(t) = 2\sin\left(\frac{t}{2}\right)\hat{\imath} + 2\cos\left(\frac{t}{2}\right)\hat{\jmath}$ $\underset{\sim}{r}(t) = \int 2\sin\left(\frac{t}{2}\right)\, dt\hat{\imath} + \int 2\cos\left(\frac{t}{2}\right)\, dt\hat{\jmath}$
5. Perform the integration. Do not forget to add in a second constant vector.	$\underset{\sim}{r}(t) = -4\cos\left(\frac{t}{2}\right)\hat{\imath} + 4\sin\left(\frac{t}{2}\right)\hat{\jmath} + \underset{\sim}{c}_2$
6. Initially means when $t = 0$. Substitute $t = 0$ and equate with the second given condition to solve for the second constant vector.	$\underset{\sim}{r}(0) = -4\cos(0)\hat{\imath} + 4\sin(0)\hat{\jmath} + \underset{\sim}{c}_2$ $= -4 \times 1\hat{\imath} + 4 \times 0\hat{\jmath} + \underset{\sim}{c}_2$ $-4\hat{\imath} + \underset{\sim}{c}_2 = \hat{\imath} - 3\hat{\jmath}$ $\underset{\sim}{c}_2 = 5\hat{\imath} - 3\hat{\jmath}$
7. Substitute back for the second constant vector and state the position vector, $\underset{\sim}{r}(t) = -4\cos\left(\frac{t}{2}\right)\hat{\imath} + 4\sin\left(\frac{t}{2}\right)\hat{\jmath} + \underset{\sim}{c}_2$.	$\underset{\sim}{r}(t) = -4\cos\left(\frac{t}{2}\right)\hat{\imath} + 4\sin\left(\frac{t}{2}\right)\hat{\jmath} + (5\hat{\imath} - 3\hat{\jmath})$ $\underset{\sim}{r}(t) = \left(5 - 4\cos\left(\frac{t}{2}\right)\right)\hat{\imath} + \left(4\sin\left(\frac{t}{2}\right) - 3\right)\hat{\jmath}$
8. State the parametric equations.	$x = 5 - 4\cos\left(\frac{t}{2}\right)$, $y = 4\sin\left(\frac{t}{2}\right) - 3$

9. Eliminate the parameter and label the equations.	$\cos\left(\frac{t}{2}\right) = \frac{5-x}{4}$, $\sin\left(\frac{t}{2}\right) = \frac{y+3}{4}$
10. Square equations and utilise the Pythagorean identity $\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$.	$\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$ $\left(\frac{5-x}{4}\right)^2 + \left(\frac{y+3}{4}\right)^2 = 1$
11. State the Cartesian equation in the implicit form.	$(x-5)^2 + (y+3)^2 = 16$ This is a circle with centre at $(5, -3)$ and radius 4.

4.4.4 The gradient of the curve

The gradient of the curve can be determined using either explicit or implicit differentiation. Alternatively, the gradient of the curve can be found using parametric differentiation, since $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$. Techniques such as these will be studied in later chapters in Unit 4. It may be necessary to solve equations to calculate the maximum or minimum speeds of a particle, the maximum or minimum values of the acceleration, or the force acting on a particle, given its mass.

WORKED EXAMPLE 23 Determining the gradient of a curve

A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (3 - 2\cos(2t))\hat{\imath} + (2 + 3\sin(2t))\hat{\jmath}$ for $0 \le t \le \pi$.

a. Determine the coordinates where the tangent of the curve is parallel to the x-axis.

b. Determine and sketch the Cartesian equation of the path, stating its domain and range.

c. Determine the maximum and minimum values of the speed.

THINK	WRITE
a. 1. State the parametric equations.	**a.** $\underset{\sim}{r}(t) = (3 - 2\cos(2t))\hat{\imath} + (2 + 3\sin(2t))\hat{\jmath}$ $x(t) = 3 - 2\cos(2t)$ and $y(t) = 2 + 3\sin(2t)$.
2. Differentiate x with respect to t. The dot notation can be used for the derivative with respect to t.	$x(t) = 3 - 2\cos(2t)$ $\frac{dx}{dt} = \dot{x} = 4\sin(2t)$
3. Differentiate y with respect to t.	$y(t) = 2 + 3\sin(2t)$ $\frac{dy}{dt} = \dot{y} = 6\cos(2t)$
4. Apply the chain rule to determine $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$
5. Substitute for the derivatives.	$\frac{dy}{dx} = \frac{6\cos(2t)}{4\sin(2t)}$ $= \frac{3\cos(2t)}{2\sin(2t)}$

6. The tangent is parallel to the x-axis when the gradient is zero.

$\frac{dy}{dx} = 0 \Rightarrow \cos(2t) = 0$, but $\sin(2t) \neq 0$.

7. Solve for the values of t.

Since $\cos(2t) = 0$, $0 \leq t \leq \pi$

$$2t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$t = \frac{\pi}{4}, \frac{3\pi}{4}$$

8. Calculate the coordinates when the tangent is parallel to the x-axis by substituting the values of t into the parametric equations.

When $t = \frac{\pi}{4}$,

$x\left(\frac{\pi}{4}\right) = 3 - 2\cos\left(\frac{\pi}{2}\right) = 3$ and

$y\left(\frac{\pi}{4}\right) = 2 + 3\sin\left(\frac{\pi}{2}\right) = 5$.

At (3, 5), the gradient is zero.

9. Calculate the other coordinate.

When $t = \frac{3\pi}{4}$,

$x\left(\frac{3\pi}{4}\right) = 3 - 2\cos\left(\frac{3\pi}{2}\right) = 3$ and

$y\left(\frac{3\pi}{4}\right) = 2 + 3\sin\left(\frac{3\pi}{2}\right) = -1$.

At (3, −1), the gradient is zero.

b. 1. To determine the Cartesian equation of the path, eliminate the parameter from the parametric equations.

b. $x = 3 - 2\cos(2t) \Rightarrow \cos(2t) = \frac{x-3}{-2}$

$y = 2 + 3\sin(2t) \Rightarrow \sin(2t) = \frac{y-2}{3}$

2. State the Cartesian equation of the path by substituting the x- and y-terms into the Pythagorean identity $\cos^2(2t) + \sin^2(2t) = 1$.

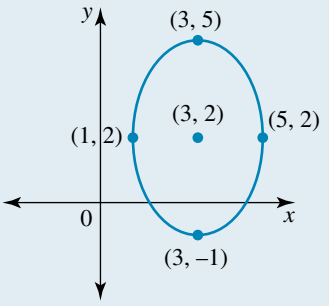
$$\frac{(x-3)^2}{4} + \frac{(y-2)^2}{9} = 1$$

3. State the curve and its domain and range.

The curve is an ellipse with centre at (3, 2) and semi-major and minor axes of 2 and 3. The domain is 3 ± 2 or [1, 5] and the range is 2 ± 3 or [−1, 5].

4. Sketch the curve.

Note that at the points (3, 5) and (3, −1) the gradient is zero.



c. 1. Determine the speed at time t by applying the formula $\lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{\dot{x}^2 + \dot{y}^2}$ and substituting for the derivatives.	**c.** $\lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{\dot{x}^2 + \dot{y}^2}$ $= \sqrt{(4\sin(2t))^2 + (6\cos(2t))^2}$ $= \sqrt{16\sin^2(2t) + 36\cos^2(2t)}$
2. Substitute the Pythagorean identity $\sin^2(2t) = 1 - \cos^2(2t)$ to express the speed in terms of $\cos(2t)$ only.	$\lvert\dot{\underset{\sim}{r}}(t)\rvert = \sqrt{16\,(1 - \cos^2(2t)) + 36\cos^2(2t)}$ $= \sqrt{16 + 20\cos^2(2t)}$ $= \sqrt{4(4 + 5\cos^2(2t))}$
3. State the speed at time t.	$\lvert\dot{\underset{\sim}{r}}(t)\rvert = 2\sqrt{4 + 5\cos^2(2t)}$
4. Determine the maximum value of the speed.	The maximum value of the speed occurs when $\cos^2(2t) = 1$. $\lvert\dot{\underset{\sim}{r}}(t)\rvert_{\max} = \sqrt{4(4 + 5\cos^2(2t))}$ $= 2\sqrt{4 + 5}$
5. State the maximum value of the speed.	$\lvert\dot{\underset{\sim}{r}}(t)\rvert_{\max} = 6$
6. Determine the minimum value of the speed.	The minimum value of the speed occurs when $\cos^2(2t) = 0$. $\lvert\dot{\underset{\sim}{r}}(t)\rvert_{\min} = 2\sqrt{4}$
7. State the minimum value of the speed.	$\lvert\dot{\underset{\sim}{r}}(t)\rvert_{\min} = 4$

4.4.5 The unit tangent vector

The vector $\frac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}}$, where the dot indicates the derivative with respect to t, is a vector parallel to the tangent T to the curve at the point P. It is the instantaneous slope of the function.

Unit tangent vector

The unit tangent vector at $t = a$ is denoted by $\hat{\underset{\sim}{s}} = \frac{\dot{\underset{\sim}{r}}(a)}{\lvert\dot{\underset{\sim}{r}}(a)\rvert}$ and is defined as the unit vector in the direction of $\underset{\sim}{r}$ at $t = a$.

WORKED EXAMPLE 24 Determining the unit tangent vector

Calculate a unit tangent vector to $\underset{\sim}{r}(t) = e^{3t}\hat{\imath} + \sin(2t)\hat{\jmath}$ at the point where $t = 0$.

THINK	WRITE
1. Differentiate the vector by differentiating the individual components of the vector.	$\underset{\sim}{r}(t) = e^{3t}\hat{\imath} + \sin(2t)\hat{\jmath}$ $\frac{d\underset{\sim}{r}}{dt} = \frac{d}{dt}\left(e^{3t}\right)\hat{\imath} + \frac{d}{dt}(\sin(2t))\hat{\jmath}$
2. State the derivative vector.	$\frac{d\underset{\sim}{r}}{dt} = 3e^{3t}\hat{\imath} + 2\cos(2t)\hat{\jmath}$

3. Evaluate the derivative vector at the required value, $t = 0$.

$$\left.\frac{d\underset{\sim}{r}}{dt}\right|_{t=0} = 3\hat{\imath} + 2\hat{\jmath}$$

4. Determine the magnitude of the derivative vector by recalling the formula.

$$\left|\frac{d\underset{\sim}{r}(0)}{dt}\right| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

5. State the required result, which is a unit vector by substituting into the formula $\hat{\underset{\sim}{s}} = \dfrac{\dot{\underset{\sim}{r}}(a)}{\left|\dot{\underset{\sim}{r}}(a)\right|}$.

$$\hat{\underset{\sim}{r}} = \frac{1}{\sqrt{13}}(3\hat{\imath} + 2\hat{\jmath})$$

Exercise 4.4 Differentiation and integration of vectors

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4.4 Exercise | **4.4 Exam questions** on

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Simple familiar

1. WE16 Consider the vector $\underset{\sim}{r}(t) = \begin{pmatrix} \cos(2t) \\ \sin(t) \\ 2t^2 + t + 3 \end{pmatrix}$, $t \geq 0$.

   a. Calculate $\dfrac{d}{dt}\underset{\sim}{r}(t)$.

   b. Calculate $\dfrac{d^2}{dt^2}\underset{\sim}{r}(t)$.

2. WE17 Consider the vector $\ddot{\underset{\sim}{r}}(t) = \begin{pmatrix} \sin(t) \\ t - 2 \\ \cos(3t) \end{pmatrix}$, $t \geq 0$.

   a. Calculate $\dot{\underset{\sim}{r}}(t)$, given that $\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$.

   b. Calculate $\underset{\sim}{r}(t)$, given that $\underset{\sim}{r}(0) = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$.

3. WE18 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = te^{-2t}\hat{\imath} + te^{2t}\hat{\jmath}$ for $t \geq 0$. Calculate:

   a. the velocity vector

   b. the speed of the particle at time t and hence determine its speed when $t = \dfrac{1}{2}$

   c. the acceleration vector.

4. A particle of mass 4 kg moves so that its position vector is given by $\underset{\sim}{r}(t) = \cos^2(t)\hat{\boldsymbol{\imath}} + \sin^2(t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$, where t is measured in seconds and the distance is in metres. Calculate the magnitude of the momentum acting on the particle at time $t = \dfrac{3\pi}{8}$.

5. **WE19** A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 2t^4\hat{\boldsymbol{\imath}} + 4\cos(2t)\hat{\boldsymbol{\jmath}} + 6e^{-2t}\hat{\boldsymbol{k}}$ for $t \geq 0$. Determine:
   a. the velocity vector
   b. the acceleration vector.

6. A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 8\cos\left(\dfrac{\pi t}{4}\right)\hat{\boldsymbol{\imath}} + 8\sin\left(\dfrac{\pi t}{4}\right)\hat{\boldsymbol{\jmath}} + 4e^{-2t}\hat{\boldsymbol{k}}$ for $t \geq 0$. Calculate the magnitude of the acceleration vector when $t = 1$.

7. **WE24** Calculate a unit tangent vector to $\underset{\sim}{r}(t) = (e^{2t} + e^{-2t})\hat{\boldsymbol{\imath}} + (e^{2t} - e^{-2t})\hat{\boldsymbol{\jmath}}$ at the point where $t = 0$.

8. Calculate a unit tangent vector to $\underset{\sim}{r}(t) = \cos(2t)\hat{\boldsymbol{\imath}} + \sin(2t)\hat{\boldsymbol{\jmath}}$ at the point where $t = \dfrac{\pi}{6}$.

9. Determine a unit tangent vector to each of the following at the point indicated.
   a. $\underset{\sim}{r}(t) = 2t\hat{\boldsymbol{\imath}} + 4t^2\hat{\boldsymbol{\jmath}}$, $t \geq 0$ at $t = 1$
   b. $\underset{\sim}{r}(t) = 2t\hat{\boldsymbol{\imath}} + 8t^3\hat{\boldsymbol{\jmath}}$, $t \geq 0$ at $t = 1$
   c. $\underset{\sim}{r}(t) = 3t^2\hat{\boldsymbol{\imath}} + (t^2 - 4t)\hat{\boldsymbol{\jmath}}$, $t \geq 0$ at $t = 3$
   d. $\underset{\sim}{r}(t) = \left(t + \dfrac{1}{t}\right)\hat{\boldsymbol{\imath}} + \left(t - \dfrac{1}{t}\right)\hat{\boldsymbol{\jmath}}$, $t \geq 0$ at $t = 2$

10. **WE20** The velocity vector of a particle is given by $\dot{\underset{\sim}{r}}(t) = (4t - 4)\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$ for $t \geq 0$. If $\underset{\sim}{r}(1) = 3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$, determine the position vector at time t.

11. The initial position of a particle is given by $3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$. If the velocity vector of the particle is given by $\underset{\sim}{v}(t) = 6\sin(3t)\hat{\boldsymbol{\imath}} + 4e^{-2t}\hat{\boldsymbol{\jmath}}$, determine the position vector.

12. **WE21** The acceleration vector of a particle is given by $\ddot{\underset{\sim}{r}}(t) = -12t^2\hat{\boldsymbol{\jmath}}$, where $t \geq 0$ is the time. Given that $\dot{\underset{\sim}{r}}(2) = -2\hat{\boldsymbol{\imath}} - 16\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(2) = \hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}}$, determine the position vector at time t.

13. A particle is moving such that $\underset{\sim}{r}(1) = -2\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}}$ and $\dot{\underset{\sim}{r}}(1) = 6\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}}$. The acceleration vector of the particle is given by $\ddot{\underset{\sim}{r}}(t) = 6\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$, where $t \geq 0$ is the time. Determine the position vector at time t.

14. **WE22** The acceleration vector of a moving particle is given by $-45\cos(3t)\hat{\boldsymbol{\imath}} + 45\sin(3t)\hat{\boldsymbol{\jmath}}$, where $t \geq 0$ is the time. The initial velocity is $-15\hat{\boldsymbol{\jmath}}$ and the initial position is $3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$. Determine the Cartesian equation of the path.

Complex familiar

15. a. The acceleration vector of a moving particle is given by $\ddot{\underset{\sim}{r}}(t) = 4e^{-2t}\hat{\boldsymbol{\imath}} + 4e^{2t}\hat{\boldsymbol{\jmath}}$, where $t \in \mathbb{R}$. If $\dot{\underset{\sim}{r}}(0) = -2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(0) = 5\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$, determine the Cartesian equation of the path.
    b. The acceleration vector of a moving particle is given by $\ddot{\underset{\sim}{r}}(t) = -e^{-\frac{t}{2}}\hat{\boldsymbol{\imath}} + 2e^{\frac{t}{2}}\hat{\boldsymbol{\jmath}}$, where $t \in \mathbb{R}$. If $\dot{\underset{\sim}{r}}(0) = 2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(0) = -2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$, determine the Cartesian equation of the path.

16. a. A particle is moving such that its initial position is $3\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}}$ and its initial velocity $-6\hat{\boldsymbol{\jmath}}$. The acceleration vector of the particle is given by $\underset{\sim}{a}(t) = 9\cos(3t)\hat{\boldsymbol{\imath}} + 18\sin(3t)\hat{\boldsymbol{\jmath}}$, where t is the time and $0 \leq t \leq 2\pi$. Determine the Cartesian equation of the path.
    b. The acceleration vector of a moving particle is given by $-3\cos\left(\dfrac{t}{2}\right)\hat{\boldsymbol{\imath}} + \sin\left(\dfrac{t}{2}\right)\hat{\boldsymbol{\jmath}}$, where $0 \leq t \leq 4\pi$. If $\underset{\sim}{v}(0) = -2\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(0) = 5\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$, determine the Cartesian equation of the path.

17. **a.** The acceleration vector of a moving particle is given by $-2\cos(t)\hat{\boldsymbol{\imath}} - 8\cos(2t)\hat{\boldsymbol{\jmath}}$, where $t \geq 0$. If initially the velocity vector is zero and the initial position is $2\hat{\boldsymbol{\imath}}$, determine the Cartesian equation of the path, stating the domain and range.

b. A moving particle is such that its initial position is $\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$ and its initial velocity $4\hat{\boldsymbol{\imath}}$. The acceleration vector of the particle is given by $-8\sin(2t)\hat{\boldsymbol{\imath}} - 96\cos(4t)\hat{\boldsymbol{\jmath}}$, where t is the time and $0 \leq t \leq \pi$. Determine the Cartesian equation of the path, stating the domain and range.

18. **a.** Particle A has an acceleration of $2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$, an initial velocity of $-6\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}}$ and an initial position of $13\hat{\boldsymbol{\imath}} - 17\hat{\boldsymbol{\jmath}}$. Particle B has an acceleration of $6\hat{\boldsymbol{\imath}} + 8\hat{\boldsymbol{\jmath}}$, an initial velocity of $-8\hat{\boldsymbol{\imath}} - 20\hat{\boldsymbol{\jmath}}$ and an initial position of $\hat{\boldsymbol{\imath}} + 40\hat{\boldsymbol{\jmath}}$. Show that the two particles collide and calculate the time and point of collision.

b. Car A has an acceleration of $2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ at time t, and after 1 second its velocity is $-5\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}}$ and its position is $-3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$. Car B has an acceleration of $2\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}}$ at time t, and after 1 second its velocity is $-2\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}}$ and its position is $-15\hat{\boldsymbol{\imath}} + 34\hat{\boldsymbol{\jmath}}$. Show that the two cars collide and calculate the time and point of collision.

Complex unfamiliar

19. **a.** The acceleration vector of a moving particle is given by $(4\cos(2t) - 2\cos(t))\hat{\boldsymbol{\imath}} + (4\sin(2t) - 2\sin(t))\hat{\boldsymbol{\jmath}}$, where $0 \leq t \leq 2\pi$, and $\dot{\underset{\sim}{r}}(\pi) = -4\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}(0) = -3\hat{\boldsymbol{\imath}}$. Determine the position vector and sketch the equation of the path.

b. A particle is moving such that $\underset{\sim}{r}\left(\frac{\pi}{2}\right) = \hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ and $\dot{\underset{\sim}{r}}\left(\frac{\pi}{2}\right) = -4\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$. The acceleration vector of the particle is given by $-(16\cos(4t) + 4\cos(t))\hat{\boldsymbol{\imath}} + (16\sin(4t) - 4\sin(t))\hat{\boldsymbol{\jmath}}$. Determine the position vector and sketch the equation of the path.

c. The acceleration vector of a moving particle is given by $-8\cos(2t)\hat{\boldsymbol{\imath}} - 108\sin(6t)\hat{\boldsymbol{\jmath}}$, where $0 \leq t \leq 2\pi$ and $\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right) = -4\hat{\boldsymbol{\imath}}$ and $\underset{\sim}{r}\left(\frac{\pi}{4}\right) = -3\hat{\boldsymbol{\jmath}}$. Determine the position vector and sketch the equation of the path.

20. WE23 A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = (4 + 3\cos(2t))\hat{\boldsymbol{\imath}} + (3 - 2\sin(2t))\hat{\boldsymbol{\jmath}}$ for $0 \leq t \leq \pi$.

a. Determine the coordinates where the gradient of the curve is parallel to the x-axis.

b. Determine and sketch the Cartesian equation of the path, stating its domain and range.

c. Determine the maximum and minimum values of the speed.

21. A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 3\cos(2t)\hat{\boldsymbol{\imath}} + 3\sin(2t)\hat{\boldsymbol{\jmath}}$. A student affirms that the particle has a circular path, that the speed of this particle is constant, its acceleration vector is a scalar multiple of the position vector and is perpendicular to the direction of the velocity vector. Verify the reasonableness of these claims.

22. A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 3\operatorname{cosec}(t)\hat{\boldsymbol{\imath}} + 4\cot(t)\hat{\boldsymbol{\jmath}}$ for $0 \leq t \leq 2\pi$.

a. Determine and sketch the Cartesian equation of the path, stating its domain and range.

b. Calculate the gradient when $t = \frac{\pi}{3}$.

c. Calculate the minimum value of the speed.

Fully worked solutions for this chapter are available online.

LESSON
4.5 Straight line motion with constant and variable acceleration

SYLLABUS LINKS

- Use vector calculus to determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration.

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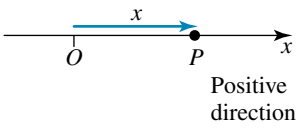
4.5.1 Rectilinear motion

Kinematics is the name given to the study of the motion of bodies, objects or particles. This section will deal with motion in a straight line only — **rectilinear motion** — and all objects considered will be treated as points for mathematical convenience; that is, the objects do not rotate or change shape. First, some basic concepts are reviewed.

The vector nature of calculations of rectilinear motion needs only assume a positive or negative direction at the start of the solution. The direction of the motion will be relative to the initial assumption.

Position

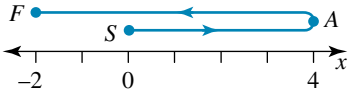
The position of a particle moving in a straight line measures its location from a fixed point of reference, usually the origin O on the line. Positions to the right of O are usually taken as positive x-direction.



For ease of explanation, the point P in this direction has a position coordinate x, but it could be y or z, or any direction in a straight line.

Displacement

The displacement of a moving particle is its change in position relative to a fixed point. Displacement gives both the distance and direction that a particle is from a fixed point.



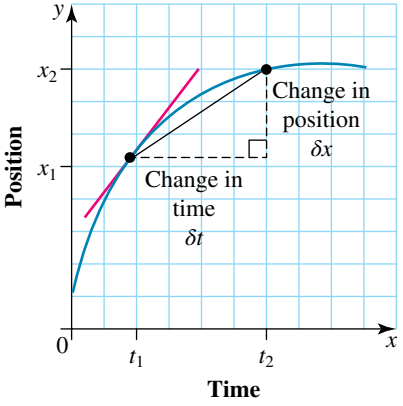
For example, a particle which moves from S to F via A is shown here on a displacement–time line (or position–time line).

The distance travelled from S to F is 10 units. The displacement of F from S is -2.

Velocity

The average velocity of a particle is the rate of change of its position with respect to time. This can be illustrated on a position–time graph. For this curve the position x at time t is shown. The curve $x(t)$ is shown.

$$\begin{aligned}\text{Average velocity} &= \frac{\text{change in position}}{\text{change in time}}\\ &= \frac{x_2 - x_1}{t_2 - t_1}\\ &= \frac{\delta x}{\delta t}\end{aligned}$$



The instantaneous velocity, $v(t)$, at time t is defined as the limiting value of the average velocity as δt approaches 0.

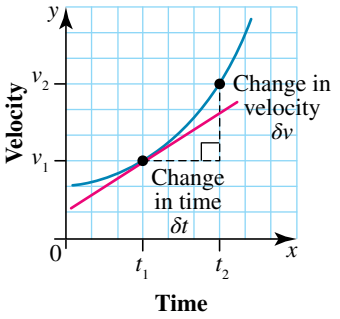
That is, velocity is the gradient of a position–time graph

or $v(t) = \lim_{\delta x \to 0} \frac{\delta x}{\delta t}$ or $v(t) = \frac{dx}{dt}$.

Acceleration

The average acceleration of a particle a during a time interval $t_2 - t_1$ is the rate of change of its velocity with respect to time.

$$\text{Average acceleration} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\delta v}{\delta t}$$



The instantaneous acceleration, a, at time t is the limiting value of the average acceleration as t approaches 0.

That is, acceleration is the gradient of a velocity–time graph at a particular time; or

$$a = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$$

4.5.2 Motion with variable acceleration

We can apply these definitions of velocity and acceleration to equations defining the path of a particle as a function of time. This allows the calculation of a particle's velocity and acceleration at any given time.

Recall from Lesson 4.4 that if the acceleration, a, of a particle is known in terms of time, t, then the velocity, $v(t)$, can be found by anti-differentiation. Thus, $v(t) = \int a(t)dt + c_1$

where c_1 is a constant due to integration; generally it is the initial velocity, $v(0)$.

Similarly, the position, $x(t)$, can be found by anti-differentiating velocity with respect to time:

$$x(t) = \int v(t)dt + c_2$$

where c_2 is a constant due to integration; generally it is the initial position, $x(0)$.

Determining the constant

The constant of anti-differentiation can be determined if initial conditions are given.

Velocity: $v(t) = \int a(t)\, dt + c_1$

Displacement: $x(t) = \int v(t)\, dt + c_2$

WORKED EXAMPLE 25 Determining motion, given velocity, in the case of variable acceleration

The velocity of an object which is initially 3 metres to the left of O is given by:

$$v(t) = 3t^2 - 2t - 5 \text{ m/s } t \geqslant 0$$

Determine:

a. **the displacement from O at any time t**
b. **the acceleration at any time t**
c. **when the object is at rest**
d. **the distance travelled in the first second**
e. **the acceleration when the velocity is 0.**

THINK	WRITE
a. 1. To determine the displacement, x, anti-differentiate with respect to t.	a. $v(t) = 3t^2 - 2t - 5$ $x(t) = \int (3t^2 - 2t - 5)dt$ $= t^3 - t^2 - 5t + c$
2. To calculate c, substitute the initial condition $t = 0$ and $x = -3$ into the equation.	When $t = 0$, $x = -3$. $-3 = 0^3 - 0^2 - 5(0) + c$
3. Solve the equation for c.	$-3 = c$
4. State the rule for x.	$x(t) = t^3 - t^2 - 5t - 3$
b. To determine the acceleration, $\dot{v}(t)$, differentiate the velocity, that is, $\dot{v}(t) = \frac{d}{dt}[v(t)]$.	b. $a(t) = \dot{v}(t) = \frac{d}{dt}(3t^2 - 2t - 5)$ $a(t) = \dot{v}(t) = 6t - 2$
c. 1. The object is at rest when $v(t) = 0$.	c. $v(t) = 3t^2 - 2t - 5 = 0$
2. Factorise the LHS.	$(3t - 5)(t + 1) = 0$
3. Solve for t by applying the Null Factor Law.	$t = \frac{5}{3}$ or $t = -1$,
4. State the answer (t cannot be negative). (The answer can be verified by graphing the velocity function on a graphics calculator.)	The domain is $t \geq 0$, so only the first solution is valid. The object is at rest after $1\frac{2}{3}$ seconds.
d. 1. The velocity is always negative during the first second and so the distance travelled will equal the magnitude of the displacement during the first second. The object is always moving to the left during the first second.	d. At $t = 0$ $v(0) = 3t^2 - 2t - 5$ $= 3 \times 0^2 - 2 \times 0 - 5$ $= -5$ m/s $t = 1$ $v(1) = 3t^2 - 2t - 5$ $= 3 \times 1^2 - 2 \times 1 - 5$ $= -4$ m/s Also, $v \neq 0$ during the first second. Thus, the distance travelled will equal the magnitude of the displacement during the first second.

2. The displacement is given by $x(t) = t^3 - t^2 - 5t - 3$ as per step **a. 1.** The distance is thus $\lvert x(1) - x(0)\rvert$.	$x(t) = t^3 - t^2 - 5t - 3$ Distance travelled $= \lvert x(1) - x(0)\rvert$ $= \lvert(1^3 - 1^2 - 5 - 3) - (-3)\rvert$ $= \lvert -8 + 3\rvert$ $= \lvert -5\rvert$ $= 5$ The object moves 5 m to the left during the first second. Therefore the distance travelled is 5 m.
e. 1. Determine when the velocity is 0 by solving $v(t) = 0$ as in part **c**.	**e.** $v(t) = 3t^2 - 2t - 5 = 0$ $(3t - 5)(t + 1) = 0$
2. Solve by applying the Null Factor Law. Accept only positive solutions. Justify your decision.	$t = \frac{5}{3}$ or $t = -1$, but the domain was given as $t \geq 0$, so only $t = \frac{5}{3}$ is valid.
3. Substitute the solution into the equation for acceleration.	$a = \frac{dv}{dt}$ $v(t) = 3t^2 - 2t - 5$ $a(t) = 6t - 2$ When $t = \frac{5}{3}$, $a = 6 \times \frac{5}{3} - 2$
4. Evaluate for a.	$a = 10 - 2$ $= 8$
5. State the answer.	When the velocity is 0, the acceleration is 8 m/s^2 in the direction of the original motion.

WORKED EXAMPLE 26 Calculating velocity and acceleration in the case of variable acceleration

The position of a particle moving in a straight line is given by:

$$x(t) = 2t^3 + t\ln(t) - 4,\ t > 0$$

Determine:

a. the velocity at any time t

b. the acceleration at any time t.

THINK	WRITE
a. 1. Recall velocity is the gradient of a position–time graph, that is $v(t) = \frac{dx}{dt}$. Differentiate x with respect to t, using the product rule on $t\ln(t)$.	**a.** $x = 2t^3 + t\ln(t) - 4$ $v = \frac{dx}{dt}$ $= 6t^2 + \left(t \times \frac{1}{t}\right) + \ln(t) \times 1$
2. Simplify.	$= 6t^2 + \ln(t) + 1$

b. Recall acceleration is the gradient of a velocity–time graph, that is $a = \frac{dv}{dt}$. Differentiate v with respect to t.	**b.** $a = \frac{dv}{dt}$ $= 12t + \frac{1}{t}$

WORKED EXAMPLE 27 Determining motion, given acceleration, in the case of variable acceleration

The acceleration of a particle moving in a straight line is given by: $\frac{dv}{dt} = 5e^t - 6t + 4$ cm/s^2, where v is the velocity at any time t.
If the particle starts at the origin with a velocity of -1 cm/s, calculate:

a. the velocity at any time t
b. the displacement $x(t)$ from the origin at any time t
c. the displacement from the origin after 1 second.

THINK	WRITE
a. 1. To calculate v, anti-differentiate the acceleration, $\frac{dv}{dt}$.	**a.** $\frac{dv}{dt} = 5e^t - 6t + 4$ $v(t) = \int (5e^t - 6t + 4)dt + c$ $= 5e^t - 3t^2 + 4t + c$
2. To determine the constant substitute the initial condition $v = -1$ when $t = 0$.	When $t = 0$, $v = -1$. $-1 = 5e^0 - 3(0)^2 + 4(0) + c$
3. Solve for the constant of anti-differentiation.	$-1 = 5 + c$ $c = -6$
4. State the velocity.	Therefore, $v(t) = 5e^t - 3t^2 + 4t - 6$.
b. 1. The displacement is determined by anti-differentiating v with respect to t.	**b.** $x(t) = \int (5e^t - 3t^2 + 4t - 6)\, dt$ $= 5e^t - t^3 + 2t^2 - 6t + c$
2. Substitute the initial condition $x = 0$ when $t = 0$.	When $t = 0$, $x = 0$. $0 = 5e^0 - (0)^3 + 2(0)^2 - 6(0) + c$
3. Solve for the constant of anti-differentiation.	$0 = 5 + c$ $c = -5$
4. State the displacement.	Therefore, $x(t) = 5e^t - t^3 + 2t^2 - 6t - 5$.
c. 1. Substitute $t = 1$ into the rule for x.	**c.** $x(1) = 5e^1 - (1)^3 + 2(1)^2 - 6(1) - 5$
2. Evaluate.	$= 5e - 1 + 2 - 6 - 5$ $= 5e - 10$
3. Give an approximate answer correct to a useful number of decimal places. (The answer can be verified by graphing the displacement function on a graphics calculator.)	Therefore, the displacement from O after 1 second is approximately 3.59 cm in the direction of the original acceleration.

4.5.3 Motion under constant acceleration

Having investigated the link between position, velocity and acceleration, we shall now review the situation where acceleration is constant. These quantities are vectors, but as motion is in a straight line, we are going to use the sign of the quantity to indicate direction. Later in this chapter, we will specify vector directions that apply to specific situations.

Note that this lesson, does not use the same type of displacement, velocity and acceleration relationships that would be familiar to Physics students, vector calculus should be used.

WORKED EXAMPLE 28 Determining motion under constant acceleration

A particle moves from rest with a constant acceleration, reaching a speed of 20 m/s in 8 seconds. Calculate:

a. the acceleration

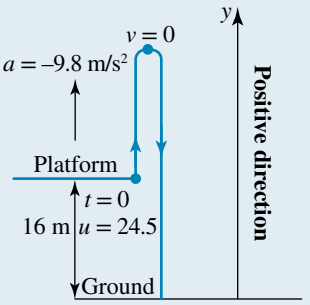
b. the distance travelled by the particle.

THINK	WRITE
a. Use the definition of acceleration formula. When the acceleration is constant, the average acceleration is the instantaneous acceleration.	$\text{Average acceleration} = \dfrac{\text{change in velocity}}{\text{change in time}} = \dfrac{20-0}{8-0} = \dfrac{20}{8} = 2.5\ \text{m/s}^2$
b. 1. To determine the velocity, v, integrate with respect to t.	$a(t) = 2.5$ $v(t) = \int 2.5\ dt = 2.5t + c_1$
2. The particle is initially at rest. To calculate c, substitute $t = 0$ and $v = 0$ into the equation and solve for c_1.	When $t = 0$ and $v = 0$: $0 = 2.5 \times 0 + c_1$ $0 = c_1$
3. State the rule for v.	$v(t) = 2.5t$
4. The determine the position, x, integrate with respect to t.	$x(t) = \int v(t)\ dt = \int 2.5t\ dt = 1.25t^2 + c_2$
5. The particle is initially at rest. To calculate c, substitute $t = 0$ and $x = 0$ into the equation and solve for c_2.	When $t = 0$ and $x = 0$: $0 = 1.25 \times 0^2 + c_2$ $0 = c_2$
6. State the rule for x.	$x(t) = 1.25t^2$
7. Calculate the distance travelled by the particle after 8 seconds by making the substitution $t = 8$.	$x(8) = 1.25(8)^2 = 80$
8. State the answer.	The distance covered is 80 m in the direction of the initial acceleration.

WORKED EXAMPLE 29 Determining vertical motion under gravity (constant acceleration)

A ball is thrown directly upward from a platform 16 metres above the ground with an initial velocity of 24.5 m/s. Determine:

a. the time taken to reach the maximum height
b. the maximum height above the ground that the ball reaches
c. the total time the ball is in the air
d. the time taken for the ball to fall to the ground from the maximum height
e. the speed of the ball when it hits the ground
f. the speed of the ball when it returns to the level of the platform.

THINK	WRITE
a. 1. Sketch a diagram of the motion of the ball. Consider *up* to be positive and *down* to be negative.	**a.** $v_0 = 24.5$ and $v_t = 0$ 
2. The path of the ball is considered as a vertical straight line with the origin O at ground level. Vertically up is taken as the positive direction.	$\underset{\sim}{a} = \dfrac{dv}{dt} = -9.8\hat{\boldsymbol{j}}$
3. To determine the velocity, v, integrate with respect to t.	$v(t) = \int \dfrac{dv}{dt}\,dt$ $= \int -9.8\,dt$ $= -9.8t + c_1$
4. The particle has an initially velocity of 24.5 m/s. To calculate c, substitute $t = 0$ and $v = 24.5$ into the equation and solve for c_1.	When $t = 0$ and $v = 24.5$: $24.5 = -9.8 \times 0 + c_1$ $24.5 = c$
5. State the rule for v.	$\underset{\sim}{v}(t) = (-9.8t + 24.5)\hat{\boldsymbol{j}}$
6. At maximum height, the velocity of the ball is 0 m/s.	$0 = -9.8t + 24.5$ $t = \dfrac{24.5}{9.8}$ $t = 2.5$ s
7. State the solution.	It takes 2.5 s for the ball to reach its maximum height.
b. 1. To determine the position, y, integrate $v(t)$ with respect to t.	$y(t) = \int (-9.8t + 24.5)\,dt$ $= -4.9t^2 + 24.5t + c_2$

2. The ball is initially released from a height of 16 m above the ground. To calculate c, substitute $t = 0$ and $x = 16$ into the equation and solve for c_2.

When $t = 0$ and $x = 16$:
$16 = -4.9 \times 0^2 + 24.5 \times 0 + c_2$
$16 = c_2$

3. State the rule for y.

$\underset{\sim}{y}(t) = \left(-4.9t^2 + 24.5t + 16\right)\hat{\boldsymbol{j}}$

4. The maximum height is reached when the ball is instantaneously at rest, that is, when $v = -9.8t + 24.5 = 0$, which has previously been calculated to occur when $t = 2.5$.

Determine y when $t = 2.5$.
$y(2.5) = -4.9(2.5)^2 + 24.5(2.5) + 16$
$= 46.625$

5. State the solution.

The maximum height above the ground is 46.625 m.

c. 1. When the ball lands, $y = 0$. Solve for t. Use the quadratic formula or graphics calculator to solve. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

c. When the ball lands, $y = 0$
$0 = -4.9t^2 + 24.5t + 16$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-24.5 \pm \sqrt{(24.5)^2 - 4 \times -4.9 \times 16}}{-9.8}$$
$t = 5.585,\ -0.585$
$t = 5.585$ or $t = -0.585$

2. Justify the decision to ignore the negative solution.

Ignore the negative solution because the motion starts at $t = -0$.

3. State the solution.

The ball takes 5.585 s to reach the ground.

d. 1. The ball took 2.5 s to reach its maximum height, and 5.585 s to land.

d. Time to fall from the maximum height
$= 5.585 - 2.5$
$= 3.085$

2. State the solution.

It takes 3.085 s to fall from the maximum height.

e. 1. It takes 5.585 s for the ball to reach the ground. Determine the velocity at $t = 5.585$ s.

e. $v = -9.8 \times 5.585 + 24.5$
$= -30.23$

2. State the solution (speed doesn't have direction).

The speed when it hits the ground is 30.23 m/s.

f. 1. Calculate the time when the position is 16 m by substituting $y = 16$ into position equation.

f. $y = 16$
$16 = -4.9t^2 + 24.5t + 16$
$0 = -4.9t^2 + 24.5t$
$= -4.9t(t - 5)$
$t = 0$ or $t = 5$
The ball will return to the platform level after 5 s.

2. Calculate the velocity after 5 s by using the velocity equation.
State the solution (speed doesn't have direction).

$t = 5$,
$v = -9.8 \times 5 + 24.5$
$= -24.5$
The speed when it passes the platform will be 24.5 m/s.

Exercise 4.5 Straight line motion with constant and variable acceleration

learnon

4.5 Exercise	4.5 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	15, 16, 17	18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

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Simple familiar

1. The displacement from O of a particle travelling in a straight line is: $x = t^3 - 6t^2 + 11t - 6$, where x is in cm and t is in seconds.

a. At what time does it pass through the origin?
b. Determine the velocity at any time.
c. When is the velocity equal to 0?
d. Calculate the minimum velocity.
e. Calculate the distance travelled in the first second.

2. WE26 For each of the following, determine

i. the velocity and
ii. the acceleration at any time t.

a. $x = t^3 + 2t^2 - 5t$
b. $x = t(5 - t)(t + 2)$
c. $x = e^t + 2e^{-t}$
d. $x = t^2 - \ln(t + 3)$
e. $x = 4t - \sqrt{t + 2}$
f. $x = \sin^{-1}\dfrac{t}{3}$

3. An object travelling in a straight line has its displacement given by:

$$x = 2\cos(3t - 1) + 3$$

Determine:

a. the minimum and maximum displacement
b. when the velocity is first equal to 0
c. how long after it is first at rest is it next at rest
d. the acceleration in terms of the displacement.

4. WE25 The velocity of an object which is initially 2 m right of O is given by:

$$v(t) = 6t^2 - 2t - 8 \text{ m/s}$$

Determine:

a. the displacement $x(t)$ from the origin at any time t
b. the acceleration $\ddot{x}(t)$ at any time t
c. when the object is at rest
d. the distance travelled in the first second
e. the acceleration when the velocity is 0.

5. The acceleration of a body moving in a straight line is given by:

$$a(t) = 6 - 12t \text{ m/s}^2$$

Initially, the body is at rest at $x = 4$.
Determine:

a. the velocity at any time
b. the position at any time
c. the time and position when the velocity is 0 m/s
d. the distance travelled in the first 3 seconds.

6. WE27 The acceleration of a particle travelling in a straight line is given by:

$$\frac{dv}{dt} = 4e^{2t} - 2t + 3 \text{ cm/s}^2$$

If the particle starts from the origin with a velocity of 2 cm/s, then determine:

a. the velocity at any time t
b. the displacement from the origin at any time t
c. the displacement from the origin after 3 seconds (correct to 2 decimal places).

7. A particle travelling in a straight line has an initial velocity of -1 m/s and initial displacement of -2 m. The acceleration of the particle is given by:

$$\ddot{x} = 6t - 2 \text{ m/s}^2$$

a. Determine the time t when the particle is stationary.
b. Determine the time t when the particle is at the origin.

8. If the velocity of a body moving in a straight line is: $\dot{x} = t^2 + 2t - 3$ cm/s then calculate the distance, in cm, travelled in the first 2 seconds.

9. A body is travelling in a straight line with a velocity given by:

$$\frac{dx}{dt} = \frac{t}{t^2 + 1} \text{ m/s}$$

If the body starts at the origin, determine:

a. the initial velocity
b. the displacement at any time
c. the acceleration at any time
d. the maximum velocity
e. the minimum acceleration.

10. The acceleration of a stone falling from rest from a height of 100 m is given by:

$$\ddot{x} = -10e^{-t} \text{ m/s}^2$$

a. Determine the velocity at any time t.
b. Sketch a velocity versus time graph.
c. Estimate the time taken, to the nearest second, to reach the ground.

11. The velocity of a particle is given by:

$$\frac{dx}{dt} = \frac{4}{1+t^2} \text{ m/s}$$

where the initial displacement is 1 m.
Determine:

a. the displacement at any time t
b. the maximum velocity and when it occurs
c. the acceleration at any time t
d. the acceleration when $t = 1$.

12. The acceleration of a particle is given by:

$$a(t) = 4\sin(2t)$$

where $t = 0$, $v = -2$ and $x = 0$.
Determine:

a. the velocity at any time t
b. the displacement at any time t
c. the acceleration in terms of the displacement x
d. the acceleration in terms of the velocity v.

13. A particle moves from rest with constant acceleration of 2 m/s^2. Answer the following questions on the particle, which has travelled 25 metres.

a. Calculate the velocity at this time.
b. Calculate the time taken for the particle to reach a speed of 20 m/s.
c. Calculate the distance travelled when it reaches 20 m/s.

14. WE28 A particle moving from rest with constant acceleration reaches a speed of 20 m/s in 5 seconds.
Calculate:

a. the acceleration
b. the distance travelled.

Complex familiar

15. A train decelerates uniformly for 20 seconds over a distance of 320 metres before coming to a stop.
Calculate:

a. the acceleration
b. the initial speed.

16. A car accelerates from rest at 12 m/s^2 over a distance of 400 metres.
Calculate:

a. the time taken to travel the 400 metres
b. the final velocity of the car.

17. An object accelerates at 9 m/s^2 over a distance of 40 metres until it reaches a speed of 60 m/s.
Calculate:

a. the initial velocity
b. the time taken.

Complex unfamiliar

18. An object is thrown from the ground at 39.2 m/s.
Calculate:

a. the maximum height reached by the object
b. the total time that the object is in flight.

19. WE29 A stone is dropped from a tower that is 50 metres above ground level.
Determine:
   a. how long it takes the stone to reach the ground
   b. the speed of the stone on impact.

20. A stone is dropped down a deep well and it takes 4 seconds for it to hit the water at the bottom.
   a. Determine the length of the fall.
   b. Determine the speed of the stone on impact with the water.

Fully worked solutions for this chapter are available online.

LESSON
4.6 Applications of vector calculus

SYLLABUS LINKS

- Apply vector calculus to model and solve problems that involve motion in a plane, including projectile and circular motion, with and without technology.

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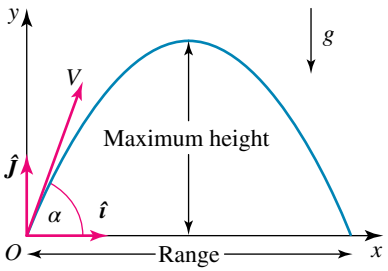
4.6.1 Projectile motion

The motion of a particle when acted upon by gravity and air resistance is called projectile motion.

In the modelling approach to projectile motion in this course, certain assumptions are made. The first is to assume that the projectile, no matter how big, is treated as a point particle. Further assumptions are to ignore air resistance, assume that Earth is flat at the scale concerned, ignore the rotation of Earth and ignore the variations in gravity due to height. For projectiles moving close to Earth's surface at heights of no more than approximately two hundred metres, these assumptions are generally valid.

General theory of a projectile

Consider a projectile fired in a vertical two-dimensional plane from the origin, O, with an initial speed of V m/s at an angle of α degrees to the horizontal.



As the motion is in two dimensions, $x = x(t)$ is the horizontal displacement and $y = y(t)$ is the vertical displacement at time t seconds, where $0 \leq t$.

Taking $\hat{\boldsymbol{\imath}}$ as a unit vector of 1 metre in the positive x direction and $\hat{\boldsymbol{\jmath}}$ as a unit vector of 1 metre in the positive upwards y direction,

$\underset{\sim}{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ is the position vector,

$\dot{\underset{\sim}{r}}(t) = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}$ is the velocity vector and

$\ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath}$ is the acceleration vector.

The only force acting on the projectile is the weight force, which acts downwards, so that $\underset{\sim}{g} = -g\hat{\jmath}$, where $g = 9.8\ \text{m/s}^2$.

Particular language and phrases are sometimes used in a projectile motion situation. They are listed here, with information on how to determine them.

Time of flight	The total time a projectile spends in the air. To determine the time of flight, determine t when the y component of the position vector has hit the ground (or stopped the flight).
Range	The range is the horizontal distance travelled from the time the projectile is launched until it hits the ground. The range can be determined by calculating the x component of the position vector when t is equal to the time of flight.
Maximum height	The maximum height is the highest point reached by the projectile during its flight (when its vertical velocity becomes zero). The maximum height can be determined by calculating the time t when the y component of the velocity vector is equal to zero, and substituting this value in the y component of the position vector.
Launch angle	The launch angle is the angle at which the projectile is launched, relative to the horizontal plane. The launch angle can be determined using trigonometry to resolve a given velocity at a given angle into x and y components of the velocity vector when the time t equals zero.

WORKED EXAMPLE 30 Determining time of flight, range and maximum height

A golf ball is hit off the ground at an angle of 53.13° to the horizontal with an initial speed of 25 m/s.

a. Determine the acceleration, velocity and position vectors.
b. Calculate the time of flight.
c. Calculate the range.
d. Calculate the maximum height reached.

THINK	WRITE
a. 1. Set up the acceleration vector.	a. $\ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath}$ $= 0\hat{\imath} - 9.8\hat{\jmath}$
2. Determine the initial velocity.	$\dot{\underset{\sim}{r}}(0) = 25\cos(53.13°)\hat{\imath} + 25\sin(53.13°)\hat{\jmath}$
3. Antidifferentiate and use initial condition to determine the velocity vector.	$\dot{\underset{\sim}{r}}(t) = \int \ddot{r}(t)\,dt$ $= 0\hat{\imath} - 9.8t\hat{\jmath} + c_1$ $c_1 = 25\cos(53.13°)\hat{\imath} + 25\sin(53.13°)\hat{\jmath}$ $\dot{\underset{\sim}{r}}(t) = 25\cos(53.13°)\hat{\imath} + (25\sin(53.13°) - 9.8t)\hat{\jmath}$
4. Antidifferentiate and use initial condition to determine the position vector.	$\underset{\sim}{r}(t) = \int \dot{r}(t)\,dt$ $= (25\cos(53.13°)\,t)\hat{\imath} + \left(25\sin(53.13°)\,t - 4.9t^2\right)\hat{\jmath} + c_2$ $\underset{\sim}{r}(0) = 0\hat{\imath} + 0\hat{\jmath}$ $c_2 = 0$ $\underset{\sim}{r}(t) = (25\cos(53.13°)\,t)\hat{\imath} + \left(25\sin(53.13°)\,t - 4.9t^2\right)\hat{\jmath}$

5. State the answers.	$\ddot{\underset{\sim}{r}}(t) = 0\hat{\boldsymbol{\imath}} - 9.8\hat{\boldsymbol{\jmath}}$ $\dot{\underset{\sim}{r}}(t) = 25\cos(53.13°)\,\hat{\boldsymbol{\imath}} + (25\sin(53.13°) - 9.8t)\hat{\boldsymbol{\jmath}}$ $\underset{\sim}{r}(t) = (25\cos(53.13°)\,t)\,\hat{\boldsymbol{\imath}} + \left(25\sin(53.13°)\,t - 4.9t^2\right)\hat{\boldsymbol{\jmath}}$
b. 1. Time of flight is calculated by determining the time when the y component of the position vector is zero.	**b.** $y(t) = 25\sin(53.13°)\,t - 4.9t^2$ $= 0$ $t = 0\text{ s}$ or $t = \dfrac{25\sin(53.13°)}{4.9}$ $= 4.08\text{ s}$
2. State the time of flight.	The time of flight is 4.08 s.
c. 1. Range is calculated by determining the x component of the position vector is zero.	**c.** $x\left(\dfrac{25\sin(53.13°)}{4.9}\right) = \left(25\cos(53.13°) \times \dfrac{25\sin(53.13°)}{4.9}\right)$ $= 61.22\text{ m}$
2. State the range.	The range is 61.22 m.
d. 1. Maximum height occurs when the y component of the velocity vector is zero.	**d.** $\dot{y}(t) = (25\sin(53.13°) - 9.8t)$ $= 0$ $t = \dfrac{25\sin(53.13°)}{9.8}$ $= 2.04\text{ s}$ $y(2.04\cdots) = \left(25\sin(53.13°) \times 2.04\cdots - 4.9\,(2.04\cdots)^2\right)$ $= 20.41\text{ m}$
2. State the maximum height.	The maximum height is 20.41 m.

Note that column vector notation can be used instead of the component form. Some students might find this notation cleaner visually. For instance, for an object at the origin initially, with an initial speed V and a launch angle α:

$$\ddot{\underset{\sim}{r}}(t) = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} c_1 \\ -gt + c_2 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} V\cos(\alpha) \\ V\sin(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} V\cos(\alpha) \\ -gt + V\sin(\alpha) \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} V\cos(\alpha)t + c_3 \\ -\frac{1}{2}gt^2 + V\sin(\alpha)t + c_4 \end{pmatrix}$$

$$\underset{\sim}{r}(0) = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} V\cos(\alpha)t \\ -\frac{1}{2}gt^2 + V\sin(\alpha)t \end{pmatrix}$$

Also note that when an object is in free fall (only force acting is the force due to gravity), its movement is an inverted parabola.

The Cartesian equation of the parabola can easily be determined from the position vector $\underset{\sim}{r}(t) = (V\cos(\alpha)t)\hat{\boldsymbol{\imath}} + \left(V\sin(\alpha)t - \frac{1}{2}gt^2\right)\hat{\boldsymbol{\jmath}}$, where V is the initial speed, and α is the launch angle, by eliminating the parameter t from the equations

$$\begin{aligned} x(t) &= (V\cos(\alpha)t) \\ y(t) &= \left(V\sin(\alpha)t - \frac{1}{2}gt^2\right) \\ t &= \frac{x}{V\cos(\alpha)} \\ y(x) &= V\sin(\alpha) \times \frac{x}{V\cos(\alpha)} - \frac{1}{2}g \times \left(\frac{x}{V\cos(\alpha)}\right)^2 \\ &= -\frac{g}{2V^2}\sec^2(\alpha)x^2 + \tan(\alpha)x \end{aligned}$$

which is the Cartesian equation of an inverted parabola.

Thus, the time when the maximum height is reached is half the time of flight, as observed in Worked example 28, as it is the vertex of an inverted parabola with roots $t = 0$ and $t =$ time of flight, and the x coordinate of the vertex of a parabola is half the sum of the roots.

TIP

When a projectile is initially at the origin, the maximum height occurs at $t = \frac{\text{Time of flight}}{2}$ (in seconds), which can be useful to remember to check your results.

WORKED EXAMPLE 31 Determining time of flight, range, landing speed and final velocity angle

A boy throws a tennis ball from the top of a hill 2 metres above ground level with an initial speed of 10 m/s at an angle 40° to the horizontal.

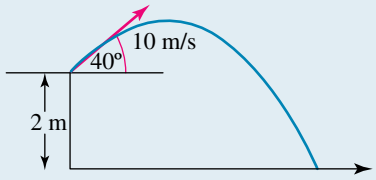
a. Calculate the time of flight and the horizontal distance travelled.
b. Calculate the maximum height reached.
c. Determine the speed and angle at which the ball lands.
d. Show that the ball travels in a parabolic path.

THINK

a. 1. State the value of the parameters.

WRITE

a. $V = 10$ m/s and $\alpha = 40°$



2. Set up the acceleration vector, antidifferentiate and use the initial condition to determine the velocity vector and then antidifferentiate and use the initial condition to determine the position vector.

$$\ddot{\underset{\sim}{r}}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(t) = \int \ddot{r}(t)\,dt$$

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} c_1 \\ -9.8t + c_2 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} 10\cos(40°) \\ 10\sin(40°) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} 10\cos(40°) \\ -9.8t + 10\sin(40°) \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \int \dot{\underset{\sim}{r}}(t)\,dt$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 10\cos(40°)\,t + c_3 \\ -4.9t^2 + 10\sin(40°)\,t + c_4 \end{pmatrix}$$

$$\underset{\sim}{r}(0) = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 10\cos(40°)\,t \\ -4.9t^2 + 10\sin(40°)\,t + 2 \end{pmatrix}$$

3. State the position vector at time t.

$$\underset{\sim}{r}(t) = \begin{pmatrix} 10\cos(40°)\,t \\ -4.9t^2 + 10\sin(40°)\,t + 2 \end{pmatrix}$$

4. The ball hits the ground at t = time of flight when the y component of the position vector is equal to zero. Use technology to solve the quadratic equation.

$$y(t) = -4.9t^2 + 10\sin(40°)\,t + 2$$
$$= 0$$
$$t = -0.259\cdots\text{s or } t = 1.571\cdots\text{s}$$

5. Use the positive root as the time of flight and state when the ball hits the ground.

Since $t \geq 0$, the ball hits the ground at $t = 1.57$ s.

6. The horizontal distance travelled (the range) is determined by substituting t = time of flight into the x component of the position vector.

$$x(t) = 10\cos(40°)\,t$$
$$x(1.571\cdots) = 12.038\cdots\text{ m}$$

7. State the horizontal distance travelled.

The horizontal distance travelled is 12.04 m.

b. 1. The maximum height is reached when the y component of the velocity vector is zero.

b. $\dot{y}(t) = -9.8t + 10\sin(40°)$
$= 0$
$t = \dfrac{10\sin(40°)}{9.8}$
$= 0.655\cdots \text{ s}$

2. Substitute this time into the y component of the position vector and state the maximum height reached.

$y(t) = -4.9t^2 + 10\sin(40°)\,t + 2$
$y(0.655\cdots) = 4.763\cdots \text{ m}$
The maximum height reached is 4.76 m.

c. 1. The speed when the ball lands is determined by substituting the time of flight into the velocity vector and calculating its magnitude at that time.

c. $\dot{\underset{\sim}{r}}(1.571\cdots) = \begin{pmatrix} 10\cos(40°) \\ -9.8 \times 1.571\cdots + 10\sin(40°) \end{pmatrix}$
$= \begin{pmatrix} 7.660\cdots \\ -8.972\cdots \end{pmatrix}$
$|\dot{\underset{\sim}{r}}(1.571\cdots)| = \sqrt{7.660\cdots^2 + -8.972\cdots^2}$
$= 11.798\cdots \text{ m/s}$

2. State the speed when the ball lands.

The balls lands with a speed of 11.8 m/s.

3. Use the ratio of the y and x components of the velocity vector at the time when the ball lands to determine the tangent of the angle when the ball lands.

$\dot{\underset{\sim}{r}}(1.571\cdots) = \begin{pmatrix} 7.660\cdots \\ -8.972\cdots \end{pmatrix}$
Let β be the angle when the ball lands.
$\tan(\beta) = \dfrac{y(1.571\cdots)}{x(1.571\cdots)}$
$= -12.22\cdots$

4. Determine the angle, given that it is in the fourth quadrant (y component is negative, x component is positive) and state the velocity angle when the ball hits the ground.

$\beta = |-49.511\cdots°|$
$= 49.51°$
The angle at which the ball lands is 49.51°.

d. 1. State the parametric equations and eliminate the parameter t.

d. $x(t) = 10\cos(40°)\,t$
$y(t) = -4.9t^2 + 10\sin(40°)\,t + 2$
$t = \dfrac{x}{10\cos(40°)}$
$y(x) = \dfrac{-4.9}{100\cos^2(40°)}x^2 + \tan(40°)\,x + 2$

2. The Cartesian equation is parabolic, as it is in the form $y = ax^2 + bx + c$. Note that it is an inverted parabola as $a < 0$.

$y(x) = \dfrac{-4.9\sec^2(40°)}{100}x^2 + \tan(40°)\,x + 2$

Note that in this case, when the object is not at the origin initially, the time at which the maximum height is reached is not equal to half the time of flight, but it is equal to half the sum of the roots:
$\dfrac{-0.259\ldots + 1.571\ldots}{2} = 0.655\ldots.$

Given the vector equation of a moving particle, its motion can be analysed.

WORKED EXAMPLE 32 Analysing motion from vector position

The position vector, $\underset{\sim}{r}(t)$, of a golf ball at a time t seconds is given by $\underset{\sim}{r}(t) = 15t\hat{\imath} + (20t - 4.9t^2)\hat{\jmath}$ for $t \geq 0$, where the distance is in metres, $\hat{\imath}$ is a unit vector horizontally forward and $\hat{\jmath}$ is a unit vector vertically upwards above ground level.

a. Determine when the golf ball hits the ground.
b. Determine where the golf ball hits the ground.
c. Determine the initial speed and angle of projection.
d. Calculate the maximum height reached.
e. Show that the golf ball travels in a parabolic path.

THINK	WRITE
a. 1. The time when the golf ball is at ground level is when the $\hat{\jmath}$ component is zero.	**a.** $y(t) = 20t - 4.9t^2 = 0$ $t(20 - 4.9t) = 0$ $t = 0$ or $20 - 4.9t = 0$ $t = \dfrac{20}{4.9}$ $t \approx 4.08$ At $t = 0$ and at $t = 4.08$ seconds the ball is at the ground level.
2. We can ignore the initial time, $t = 0$, when the ball is at ground level. State when the golf ball hits the ground.	The golf ball hits the ground after 4.08 seconds.
b. 1. The distance, or range, $\mathbb{R}$ travelled when the golf ball hits the ground is the value of the $\hat{\imath}$ component at this time. Substitute $t = 4.08$ into the $\hat{\imath}$ component.	**b.** $x(4.08) = 15 \times 4.08$ $= 61.2$
2. State where the golf ball hits the ground.	The golf ball hits the ground at a distance of 61.2 metres from the initial point.
c. 1. To determine the velocity vector, differentiate the position vector.	**c.** $\underset{\sim}{r}(t) = 15t\hat{\imath} + (20t - 4.9t^2)\hat{\jmath}$ $\dot{\underset{\sim}{r}}(t) = 15\hat{\imath} + (20 - 9.8t)\hat{\jmath}$
2. Calculate the initial velocity vector, that is, when $t = 0$.	$\dot{\underset{\sim}{r}}(0) = 15\hat{\imath} + 20\hat{\jmath}$
3. Determine the initial speed by calculating the magnitude of the initial velocity vector.	The initial speed is the magnitude of the initial velocity vector. $\lvert\dot{\underset{\sim}{r}}(0)\rvert = \sqrt{15^2 + 20^2} = 25$ m/s.
4. The initial angle of projection is the angle the initial velocity vector makes with the $\hat{\imath}$ axis.	The angle the initial velocity vector makes with the $\hat{\imath}$ axis is $\tan^{-1}\left(\dfrac{20}{15}\right) = 53.13°$.
5. State the required results.	The golf ball is struck with an initial speed of 25 m/s at an angle of 53.13°.

d. 1. The golf ball will rise until the vertical component of its velocity is zero. Set the j component of the velocity vector to 0 to calculate the time this occurs.

d. $\dot{y}(t) = 20 - 9.8t = 0$

$$t = \frac{20}{9.8}$$
$$= 2.04$$

2. The maximum height reached, is the value of the $\hat{\boldsymbol{j}}$ component of the position vector at this time, $t = 2.04$.

$$y(2.04) = 20 \times 2.04 - 4.9 \times 2.04^2$$
$$= 20.41$$

3. State the maximum height reached.

The golf ball reaches a maximum height of 20.41 metres.

e. 1. Write the parametric equations.

e. $x = 15t \Rightarrow t = \dfrac{x}{15}$

$y = 20t - 4.9t^2$

2. Substitute the value of t into the equation for y.

$$y = 20\left(\frac{x}{15}\right) - 4.9\left(\frac{x}{15}\right)^2$$

3. Simplify and form common denominators.

$$y = -\frac{x}{2250}(49x - 3000)$$

4. State the result.

The parametric equation is of the form of a parabola, $y = ax(x - b)$ with $a < 0$. Therefore, the golf ball travels in a parabolic path.

WORKED EXAMPLE 33 Calculating the launch angle

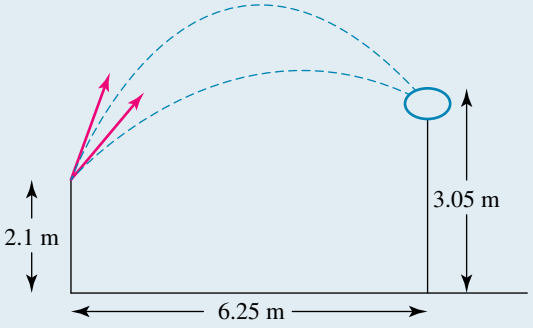
A basketballer shoots for goal from the three-point line. He throws the ball with an initial speed of 15 m/s and the ball leaves his hands at a height of 2.1 metres above the ground. Calculate the possible angles of projection if he is to score a goal.
Data: Distance from goal to three-point line: 6.25 m
Height of ring: 3.05 m

THINK

1. State the value of the parameters. Drawing a diagram will help.

WRITE

$V = 15$ but α is unknown and required to be found.



2. Set up the acceleration vector, antidifferentiate and use the initial condition to determine the velocity vector and then antidifferentiate and use the initial condition to determine the position vector.

$$\ddot{\underset{\sim}{r}}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(t) = \int \ddot{r}(t)\,dt$$

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} c_1 \\ -9.8t + c_2 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} 15\cos(\alpha) \\ 15\sin(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} 15\cos(\alpha) \\ -9.8t + 15\sin(\alpha) \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \int \dot{\underset{\sim}{r}}(t)\,dt$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 15\cos(\alpha)t + c_3 \\ -4.9t^2 + 15\sin(\alpha)t + c_4 \end{pmatrix}$$

$$\underset{\sim}{r}(0) = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2.1 \end{pmatrix}$$

$$\underset{\sim}{r}(t) = \begin{pmatrix} 15\cos(\alpha)t \\ -4.9t^2 + 15\sin(\alpha)t + 2.1 \end{pmatrix}$$

3. Use the information to score a goal to determine conditions on the x and y components of the position vector.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 15\cos(\alpha)t \\ -4.9t^2 + 15\sin(\alpha)t + 2.1 \end{pmatrix}$$

$$= \begin{pmatrix} 6.25 \\ 3.05 \end{pmatrix}$$

4. Use the x component of the position vector to express t in relation to α and substitute this value into the y component of the position vector.

$$t = \frac{6.25}{15\cos(\alpha)}$$

$$-4.9\left(\frac{6.25}{15\cos(\alpha)}\right)^2 + 15\sin(\alpha) \times \frac{6.25}{15\cos(\alpha)} + 2.1 = 3.05$$

5. Use technology to solve for α.

$$-4.9\left(\frac{6.25}{15\cos(\alpha)}\right)^2 + 6.25\tan(\alpha) = 0.95$$

$$-0.13611\cdots\sec^2(\alpha) + \tan(\alpha) = 0.152$$

$$\alpha = 16.72° \text{ or } \alpha = 81.92°$$

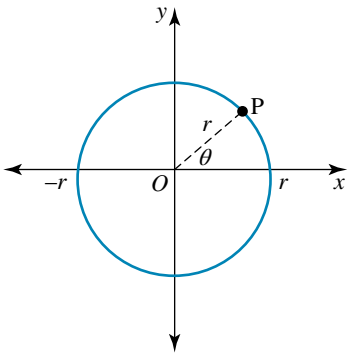
6. State the answer.

To score, the initial angle can be either 16.72° or 81.92°.

4.6.2 Circular motion

Uniform circular motion

Parametric equations in the form $x = (r\cos(t))\hat{\boldsymbol{\imath}}$ and $y = (r\sin(t))\hat{\boldsymbol{\jmath}}$ were studied in the first section of this chapter when describing the position of particle. The Cartesian form of this equation is given by $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$, which is the general form of a circle.



A particle P travelling at a constant speed along a circular path of radius r is defined as being in **uniform circular motion**. Consider the diagram on the right, which exhibits the Cartesian equation of the path defined as $x^2 + y^2 = r^2$, where $r > 0$.

Even though the speed is constant, the direction of motion is constantly changing. The particle has a changing velocity, so it is accelerating.

We have previously defined that the anticlockwise direction is considered to be positive.

The rate at which the position vector $\overrightarrow{OP}$ of a rotating particle changes with respect to time is given by the derivative $\frac{d\theta}{dt}$. This is referred to as the **angular velocity** of the particle and is often denoted by the lowercase Greek letter omega, ω, so that $\frac{d\theta}{dt} = \omega$. Assuming the particle moves at a constant speed around the circle, then the angular velocity will be constant. By integrating the equation, $\int \frac{d\theta}{dt} = \omega$, with respect to t, we obtain $\theta = \omega t$ (as ω is constant and $\theta = 0$ when $t = 0$).

The position of a particle, P, moving with angular velocity ω can be described by the vector,

$$\underset{\sim}{r}(t) = r\cos(\omega t)\hat{\boldsymbol{\imath}} + r\sin(\omega t)\hat{\boldsymbol{\jmath}}$$

where the unit of angular velocity is radians per second.

Period of circular motion

The period of the motion is the time taken, T, to complete one revolution and is given by $T = \frac{2\pi}{\omega}$, where ω is the angular velocity.

Position of a particle on a circle

The position vector of particle, P, on a circle of radius, r, can be defined as

$$\begin{aligned}\underset{\sim}{r} &= \overrightarrow{OP}\\ &= x\hat{\imath} + y\hat{\jmath}\\ &= (r\cos(\theta))\,\hat{\imath} + (r\sin(\theta))\,\hat{\jmath}\\ &= r\cos(\omega t)\,\hat{\imath} + r\sin(\omega t)\,\hat{\jmath}\end{aligned}$$

where $\theta = r\omega$ is the angle r makes with the positive x-axis and ω represents the angular velocity. The position of the particle varies according to time, t, hence $\theta = f(t)$. Note that when $\theta = 0$, $t = 0$.

Velocity vector

If particle P is travelling in a circular path of radius r at a constant speed, the following statements can be made:

1. **The velocity vector, $v(t)$, can be defined as: $\underset{\sim}{v}(t) = -r\omega\sin(\omega t)\hat{\imath} + r\omega\cos(\omega t)\hat{\jmath}$**
2. **$v(t)$ is perpendicular to $\underset{\sim}{r}(t)$**
3. **The magnitude of velocity (speed) of the particle is defined as $|v(t)| = v = r\omega$**

The proofs to these three statements exist below.

Keep in mind that not all of these are on the Specialist mathematics formula sheet.

1. The velocity vector

$$\begin{aligned}\underset{\sim}{v}(t) &= \frac{d}{dt}\underset{\sim}{r}(t)\\ &= \frac{d}{dt}((r\cos(\omega r))\hat{\imath} + (r\sin(\omega r))\hat{\jmath})\\ &= (-r\omega\sin(\omega t))\hat{\imath} + (r\omega\cos(\omega r))\hat{\jmath}\end{aligned}$$

2. $v(t)$ is perpendicular to $r(t)$.
   If the velocity and position vectors are perpendicular, the dot product will equal zero.

$$\begin{aligned}\underset{\sim}{r}\cdot v &= ((r\cos(\omega r))\hat{\imath} + (r\sin(\omega r))\hat{\jmath})\cdot((-r\omega\sin(\omega r))\hat{\imath} + (r\omega\cos(\omega r))\hat{\jmath})\\ &= -r^2\omega\cos(\omega r)\sin(\omega r) + r^2\omega\cos(\omega r)\sin(\omega \mathrm{r})\\ &= 0\\ \therefore r(t) &\perp y(t)\end{aligned}$$

3. The magnitude of velocity (speed) of the particle is defined as $|v(t)| = \underset{\sim}{v} = r\omega$.

$$\begin{aligned}v &= \sqrt{(-r\omega\sin(\omega r))^2 + (r\omega\cos(\omega r))^2}\\ &= r\omega\end{aligned}$$

WORKED EXAMPLE 34 Solving problems involving circular motion

A particle follows a circular path about the origin, with a radius of 8 cm. It takes 6 s to complete one revolution.

a. Write the Cartesian equation of the path.
b. Calculate the angular velocity ω of the particle.
c. Determine the position vector of the particle.

THINK	WRITE
a. Write the Cartesian equation of the path	a. $x^2 + y^2 = r^2$, where $r = 8$ cm.
b. Calculate the angular velocity of the particle.	b. $T = \dfrac{2\pi}{\omega}$ $\omega = \dfrac{2\pi}{T}$ $= \dfrac{2\pi}{6}$ $= \dfrac{\pi}{3}$ rad/s

c. Determine the position vector of the particle.

c. $\underset{\sim}{r} = x\hat{\imath} + y\hat{\jmath}$
$= r\cos(\omega t)\hat{\imath} + r\sin(\omega t)\hat{\jmath}$
$= 8\cos\left(\frac{\pi}{3}t\right)\hat{\imath} + 8\sin\left(\frac{\pi}{3}t\right)\hat{\jmath}$

WORKED EXAMPLE 35 Determining angular velocity and position and velocity vectors

A particle of mass 6 kg rotates with a constant speed of 10 m/s in a circular path of radius 5 metres. Determine:

a. the angular velocity of the particle
b. the position vector of the particle after 7 seconds if $\theta = 0$ when $t = 0$
c. the velocity vectors after 7 seconds.

THINK	WRITE
a. Recall the formula for speed. Rearrange the equation to make ω the subject. Substitute in values for the speed (v) and radius (r) to calculate the angular velocity.	**a.** $v = r\omega$ $\omega = \frac{v}{r}$ $= \frac{10}{5}$ $= 2$ rad/s
b. 1. Substitute in values for ω and r to develop the position vector $r(t) = (r\cos(\omega r))\hat{\imath} + (r\sin(\omega r))\hat{\jmath}$	**b.** At $t = 0, \theta = 0$. $r(t) = (r\cos(\omega t))\hat{\imath} + (r\sin(\omega t))\hat{\jmath}$ $= (5\cos(2t))\hat{\imath} + (5\sin(2t))\hat{\jmath}$
2. Determine the position of the particle at 7 seconds by letting $t = 7$.	$r(7) = (5\cos(2\times 7))\hat{\imath} + (5\sin(2\times 7))\hat{\jmath}$ $\approx 0.684\hat{\imath} + 4.953\hat{\jmath}$

WORKED EXAMPLE 36 Determining angular velocity and speed

A particle moves in a circular path of radius 6 metres, completing 120 revolutions per minute. Calculate:

a. the angular velocity in radians per second
b. the speed of the particle in metres per second.

THINK	WRITE
a. Convert 1 minute to seconds. Calculate the angular velocity given one revolution is 2π radians.	**a.** $\omega = 120$ revolutions per minute (rpm) $= \frac{120\times 2\pi}{60}$ $= 4\pi$ rad/s
b. Recall the formula for speed (v) and substitute values for r and ω.	**b.** $v = r\omega$ $= 6\times 4\pi$ $= 24\pi$ m/s ($v \approx 75.398$ m/s)

Acceleration of circular motion

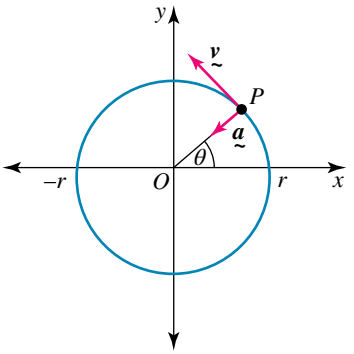
If P is travelling in a circular path of radius r at a constant speed, then the following statements can be made about the acceleration vector, $\underset{\sim}{a}(t)$:

1. **The acceleration vector, $\underset{\sim}{a}(t)$, can be defined as: $\underset{\sim}{a}(t) = -\omega^2 \underset{\sim}{r}(t)$**
2. **The magnitude of acceleration is given by, $|\underset{\sim}{a}(t)| = \dfrac{v^2}{r}$**

The proofs to these statements exist below.

1. $\underset{\sim}{a}(t) = -\omega^2 \underset{\sim}{r}(t)$

$$\begin{aligned} a(t) &= \frac{dy}{dt} \\ &= \frac{d}{dt}((-r\omega\sin(\omega t))\hat{\boldsymbol{i}} + (r\omega\cos(\omega t))\hat{\boldsymbol{j}}) \\ &= -r\omega^2\cos(\omega r)\hat{\boldsymbol{i}} - r\omega^2\sin(\omega t)\hat{\boldsymbol{j}} \\ &= -\omega^2(r\cos(\omega t)\hat{\boldsymbol{i}} + r\sin(\omega x)\hat{\boldsymbol{j}}) \\ &= -\omega^2 r(t) \end{aligned}$$



Consider the diagram. The acceleration vector is heading to the centre of its circular path (referred to as centripetal acceleration).

2. The magnitude of acceleration is given by, $|\underset{\sim}{a}(t)| = \dfrac{v^2}{r}$

$$\begin{aligned} a &= |\underset{\sim}{a}(t)| \\ &= \sqrt{(-\omega^2 r\cos(\omega t))^2 + (-\omega^2 r\sin(\omega t))^2} \\ &= \sqrt{\omega^4 r^2\cos^2(\omega t) + \omega^4 r^2\sin^2(\omega t)} \\ &= \sqrt{\omega^4 r^2\left(\cos^2(\omega t) + \sin^2(\omega t)\right)} \\ &= \sqrt{\omega^4 r^2} \\ &= \omega^2 r \\ &= \frac{v^2}{r^2} r \quad \left(\text{given } \omega = \frac{v}{r}\right) \\ &= \frac{v^2}{r} \end{aligned}$$

WORKED EXAMPLE 37 Determining acceleration in circular motion

A particle is travelling in a circular path with a radius of 2 metres at a constant speed. The period of the motion is 4 seconds. Calculate:

a. **the angular velocity**
b. **the speed of the particle**
c. **the magnitude of the acceleration after 6 seconds.**

THINK	WRITE
a. The period of motion is given as 4 seconds. Determine the angular velocity, ω, by recalling the formula for the time T for a revolution.	a. $T = \dfrac{2\pi}{\omega}$ $\omega = \dfrac{2\pi}{T}$ $\omega = \dfrac{2\pi}{4}$ $\omega = \dfrac{\pi}{2}\text{rad/s}$

b. Recall the equation for the magnitude of velocity (speed). Substitute in values for r and ω to calculate v.

b. $v = r\omega$
$= 2 \times \frac{\pi}{2}$
$= \pi$ m/s

c. Recall either formula for the magnitude of acceleration $|a| = \omega^2 r$ or $a = \frac{v^2}{r}$. Calculate the magnitude by substituting in values for ω and r.

c. $a = \omega^2 r \quad \left(a = \frac{v^2}{r} \text{ is also applicable}\right)$
$a = \left(\frac{\pi}{2}\right)^2 \times 2$
$a = \frac{\pi^2}{2}$ m/s^2

4.6.3 Collisions applications

Collision problems were investigated earlier in Lesson 4.3. Generally, collision problems start by defining the position vector of two different particles in two or three dimensions, and then investigating whether particles collide or cross paths, as well as how close they come to each other.

Applications of these questions may require the translation of information to achieve the position vectors, using Cartesian or parametric equations, or by using vector calculus to define the position vectors.

WORKED EXAMPLE 38 Solving a collision problems given velocity vector

Two paths of two model sail boats are defined by the following vector functions of time, where time is measured in minutes and distance is measured in metres.
Boat A: $\underset{\sim}{r}_A(t) = (9t - 7)\hat{\imath} + 11t\hat{\jmath}$
Boat B: $\dot{\underset{\sim}{r}}_B(t) = 2\hat{\imath} + \hat{\jmath}$, $\underset{\sim}{r}_B(0) = 14\hat{\imath} + 9\hat{\jmath}$.
Calculate if, at any time, they are in the same location.

THINK	WRITE
1. Antidifferentiate the velocity of boat B.	$\underset{\sim}{r}_B(t) = \int \dot{\underset{\sim}{r}}_B(t)\,dt$ $= 2t\hat{\imath} + t\hat{\jmath} + c$
2. Use the initial condition to determine the position vector for boat B.	$\underset{\sim}{r}_B(0) = c$ $= 14\hat{\imath} + 9\hat{\jmath}$ $\underset{\sim}{r}_B(t) = (2t + 14)\hat{\imath} + (t + 9)\hat{\jmath}$
3. Equate the $\hat{\imath}$ components for boats A and B.	$(9t - 7) = (2t + 14)$ $7t = 21$ $t = 3$ minutes
4. Determine the position of both boats at $t = 3$ minutes.	$\underset{\sim}{r}_A(3) = 20\hat{\imath} + 33\hat{\jmath}$ $\underset{\sim}{r}_B(3) = 20\hat{\imath} + 12\hat{\jmath}$
5. Answer the question.	The two boats are never in the same location.

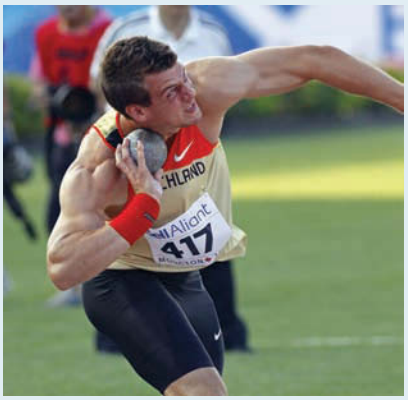
In reality, projectiles will move in a three-dimensional framework rather than a two-dimensional plane. Also, with the effects of air resistance being included, the path is not necessarily a parabola.

WORKED EXAMPLE 39 Solving complex collision problems involving motion in three dimensions (extension)

A shot is thrown by a shot-put competitor on level ground. At a time t in seconds measured from the point of release of the shot, the position vector $\underset{\sim}{r}(t)$ of the shot is given by

$$\underset{\sim}{r}(t) = 7t\hat{\imath} + \left(5t + 6\left(e^{-\frac{t}{2}} - 1\right)\right)\hat{\jmath} + (2 + 12t - 5t^2)\hat{k}$$

where $\hat{\imath}$ is a unit vector in the east direction, $\hat{\jmath}$ is a unit vector in the north direction and $\hat{k}$ is a unit vector vertically up. The origin, O, of the coordinate system is at ground level and displacements are measured in metres. Let P be the point where the shot hits the ground.



a. Calculate the time taken for the shot to hit the ground.
b. Determine how far from O the shot hits the ground.
c. Calculate the initial speed of projection.
d. Calculate the speed and angle at which the shot hits the ground.

THINK	WRITE
a. 1. Determine when the shot hits the ground.	**a.** The shot hits the ground when the $\hat{k}$ component is zero, that is, when $2 + 12t - 5t^2 = 0$.
2. Solve for the values of t using the quadratic formula.	$\Delta = b^2 - 4ac$ $= (12)^2 - 4 \times -5 \times 2$ $= 184$ $\sqrt{\Delta} = 2\sqrt{46}$ $t = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ $= \dfrac{-12 \pm 2\sqrt{46}}{2 \times -5}$ $= -0.1565,\ 2.5565$ $t = -0.1565$ or $t = 2.5565$. Since $t \geq 0$, the shot hits the ground at 2.56 seconds.
b. 1. Determine the position vector where the shot hits the ground at $t = 2.56$ seconds.	**b.** $\underset{\sim}{r}(2.56) = 7 \times 2.56\hat{\imath} + \left(5 \times 2.56 + 6\left(e^{-\frac{2.56}{2}} - 1\right)\right)\hat{\jmath}$ $+ (2 + 12 \times 2.56 - 5 \times 2.56^2)\hat{k}$ $\underset{\sim}{r}(2.56) = 17.896\hat{\imath} + 8.453\hat{\jmath}$
2. Determine the distance where the shot hits the ground by calculating the magnitude of the position vector at $t = 2.56$ seconds.	$\lvert\underset{\sim}{r}(2.56)\rvert = \sqrt{17.895^2 + 8.453^2}$ $= 19.79$ metres

c. 1. Determine the velocity vector.

c. $\dot{\underset{\sim}{r}}(t) = 7\hat{\boldsymbol{\imath}} + \left(5 - 3e^{-\frac{t}{2}}\right)\hat{\boldsymbol{\jmath}} + (12 - 10t)\hat{\boldsymbol{k}}$

2. Determine the initial velocity vector, at $t = 0$.

$$\begin{aligned}\dot{\underset{\sim}{r}}(0) &= 7\hat{\boldsymbol{\imath}} + (5 - 3e^{0})\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}} \\ &= 7\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}\end{aligned}$$

3. Determine the initial speed of projection by calculating the magnitude of the velocity vector.

$$\begin{aligned}|\dot{\underset{\sim}{r}}(0)| &= \sqrt{7^2 + 2^2 + 12^2} \\ &= \sqrt{197}\end{aligned}$$

The initial speed of projection is 14.04 m/s.

d. 1. Determine the velocity vector when the shot hits the ground, that is, at $t = 2.56$ seconds.

d. $$\begin{aligned}\dot{\underset{\sim}{r}}(2.56) &= 7\hat{\boldsymbol{\imath}} + \left(5 - 3e^{-\frac{2.56}{2}}\right)\hat{\boldsymbol{\jmath}} + (12 - 10 \times 2.56)\hat{\boldsymbol{k}} \\ &= 7\hat{\boldsymbol{\imath}} + 4.16\hat{\boldsymbol{\jmath}} - 13.56\hat{\boldsymbol{k}}\end{aligned}$$

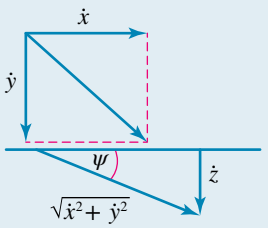
2. Calculate the magnitude of the velocity vector.

$$\begin{aligned}|\dot{\underset{\sim}{r}}(2.56)| &= \sqrt{7^2 + 4.16^2 + (-13.56)} \\ &= \sqrt{250.34} \\ &\approx 15.82\end{aligned}$$

3. Determine the angle at which the shot hits the ground.

The required angle, ψ, is the angle between the downwards component and the combined east and north components.

$$\tan(\psi) = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$



4. Determine the angle.

$$\begin{aligned}\tan(\psi) &= \frac{13.56}{\sqrt{7^2 + 4.16^2}} \\ \psi &= \tan^{-1}(1.665)\end{aligned}$$

5. State the final result.

The shot hits the ground with a speed of 15.82 m/s at an angle of 59.015°.

Exercise 4.6 Applications of vector calculus

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4.6 Exercise | 4.6 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

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Simple familiar

1. WE30 A soccer ball is kicked off the ground at an angle of 67.38° with an initial speed of 13 m/s. Calculate:

a. the time of flight
b. the range
c. the maximum height reached.

2. WE31 A cricketer smashes a cricket ball at an angle of 35° to the horizontal from a point 0.5 m above the ground with an initial speed of 30 m/s.

a. Calculate the time of flight and the horizontal distance travelled.
b. Calculate the maximum height reached.
c. Determine the speed and angle at which the ball lands.
d. Show that the ball travels in a parabolic path.

3. A cricket outfielder attempts to throw the cricket ball back towards the stumps. He throws the ball from a height of 1.8 m above the ground with an initial speed of 20 m/s at an angle of 20° to the horizontal.

a. Calculate the time of flight and the horizontal distance travelled.
b. Calculate the maximum height reached.
c. Determine the speed and angle at which the ball lands.
d. Show that the ball travels in a parabolic path.

4. WE34 A particle follows a circular path about the origin, with a radius of $\frac{5}{2}$ cm. It takes $\frac{3}{2}$ s to complete one revolution.

a. Write the Cartesian equation of the path.
b. Calculate the angular velocity ω of the particle.
c. Determine the position vector of the particle.

5. Given the vector equation, $\underset{\sim}{r}(t) = (2\cos(3t))\hat{\boldsymbol{\imath}} + (2\sin(3t))\hat{\boldsymbol{\jmath}}$ for $t \geq 0$, determine:

a. the velocity vector, $\underset{\sim}{v}(t)$
b. the acceleration vector, $\underset{\sim}{a}(t)$.

6. A particle rotates with constant speed in a circular path according to the position vector, $\underset{\sim}{r}(t) = (5\cos(2t))\hat{\boldsymbol{\imath}} + (5\sin(2t))\hat{\boldsymbol{\jmath}}$, for $t \geq 0$, where distances are in metres and time is in seconds.

a. Determine the radius of the circular path.
b. Calculate the speed, v, at which the particle moves.

7. WE35 A particle of mass 8 kg rotates with a constant speed of 12 m/s in a circular path of radius 8 metres. Determine the following:

a. the angular velocity of the particle
b. the position vector of the particle after 4 seconds if $\theta = 0$ when $t = 0$
c. the velocity and acceleration vectors after 4 seconds.

8. A toy train travels in a circular of radius 3 metres at a constant speed. The train completes 2 revolutions each second. Calculate:

a. the speed of the object
b. the magnitude of the acceleration of the train.

9. WE36 A particle moves in a circular path of radius 24 metres, completing 360 revolutions per minute. Calculate:

a. the angular velocity in radians per second
b. the speed of the particle in metres per second.

10. A particle connected to a string 4 metres long is moving in a circular path at a speed of 35 metres per second. Calculate the angular velocity of the particle.

11. WE37 A particle is travelling in a circular path with a radius of 7 metres at a constant speed. The period of the motion is 124. Calculate:

a. the angular velocity
b. the speed of the particle
c. the magnitude of the acceleration after 3 seconds.

12. WE32 The position vector $\underset{\sim}{r}(t)$ of a soccer ball at a time $t \geq 0$ seconds is given by $\underset{\sim}{r}(t) = 5t\hat{\boldsymbol{\imath}} + (12t - 4.9t^2)\hat{\boldsymbol{\jmath}}$, where the distance is in metres, $\hat{\boldsymbol{\imath}}$ is a unit vector horizontally forward and $\hat{\boldsymbol{\jmath}}$ is a unit vector vertically upwards above ground level.

a. Determine when the soccer ball hits the ground.
b. Determine where the soccer ball hits the ground.
c. Determine the initial speed and angle of projection.
d. Calculate the maximum height reached.
e. Show that the soccer ball travels in a parabolic path.

Complex familiar

13. WE39 A girl throws a tennis ball. The position vector, $\underset{\sim}{r}(t)$, of the tennis ball at a time $t \geq 0$ seconds is given by $\underset{\sim}{r}(t) = 24t\hat{\boldsymbol{\imath}} + (2 + 7t - 4.9t^2)\hat{\boldsymbol{\jmath}}$, where the distance is in metres, $\hat{\boldsymbol{\imath}}$ is a unit vector horizontally forward and $\hat{\boldsymbol{\jmath}}$ is a unit vector vertically upwards above ground level.

a. How long before the tennis ball hits the ground?
b. Determine where the tennis ball hits the ground.
c. Determine the initial speed and angle of projection.
d. Calculate the maximum height reached.
e. Show that the tennis ball travels in a parabolic path.

14. A girl throws a tennis ball for her dog. The position vector $\underset{\sim}{r}(t)$ of the tennis ball at a time $t \geq 0$ seconds is given by $\underset{\sim}{r}(t) = 10t\hat{\boldsymbol{\imath}} + (10t - 4.9t^2)\hat{\boldsymbol{\jmath}}$ where the distance is in metres, $\hat{\boldsymbol{\imath}}$ is a unit vector horizontally forward and $\hat{\boldsymbol{\jmath}}$ is a unit vector vertically upwards above ground level.

a. Determine the time taken to reach the ground.
b. Determine the horizontal distance covered.
c. Determine the initial speed and angle of projection.
d. Calculate the maximum height reached.
e. Show that the tennis ball travels in a parabolic path.

15. WE33 A tennis player hits the ball 4 feet above the baseline of a tennis court with a speed of 100 feet per second. If the ball travels in a vertical plane towards the centre of the net and just grazes the top of the net, calculate the angles at which it could have been hit.
Data: Tennis court dimensions 78×27 feet
Net height at the centre 3 feet
Use $g = 32\ \text{ft/s}^2$.

16. WE38 The velocity of a model car moving along the ground is $\dot{\underset{\sim}{r}}_{\text{Car}}(t) = e^{-\frac{t}{3}}\hat{\boldsymbol{\imath}} + 4t^3\hat{\boldsymbol{\jmath}}$ where $t \geq 0$, and it starts at the origin.
Determine whether it will collide with a toy truck with a position vector $\underset{\sim}{r}_{\text{Truck}}(t) = 2\hat{\boldsymbol{\imath}} + 16\hat{\boldsymbol{\jmath}}$.

Complex unfamiliar

17. A ball is thrown so that its time of flight is $\dfrac{10\sqrt{3}}{g}$ and the ratio of the range to the maximum height reached is $\dfrac{4\sqrt{3}}{3}$. Calculate the initial speed and angle of projection.

18. A projectile is fired at an angle of α with an initial speed of V. It reaches a maximum height of H and has a horizontal range of R. Show that:

a. $R\tan(\alpha) = \dfrac{1}{2}gT^2$

b. $\tan(\alpha) = \dfrac{4H}{R}$

c. $T^2 = \dfrac{8H}{g}$

d. $V = \sqrt{2g\left(H + \dfrac{R^2}{16H}\right)}$

e. the equation of the path is $y = \dfrac{4Hx}{R^2}(R - x)$ for $0 \leq x \leq \mathbb{R}$.

19. a. An object is thrown horizontally with a speed of V m/s from the top of a cliff face h metres high and strikes the ground at a distance of R metres from the base of the cliff after a time of T seconds. Show that:

i. $T = \sqrt{\dfrac{2h}{g}}$

ii. $R = VT$

iii. the speed at which the object hits the ground is $\sqrt{V^2 + 2gh}$

iv. the angle at which the object strikes the ground is $\tan^{-1}\left(\dfrac{2h}{R}\right)$

v. the object travels in a parabolic path, $y = h\left(1 - \dfrac{x^2}{R}\right)$.

b. A stone is projected horizontally from the top of a cliff of height H metres with a speed of U m/s. At the same instant another object is fired from the base of the cliff with a speed of V m/s at an angle of α. Given that the two objects collide after a time of T seconds, show that

$$\alpha = \cos^{-1}\left(\frac{U}{V}\right) \text{ and } V^2 = U^2 + \frac{H^2}{T^2}$$

20. Kristen is spinning her 1.2-metre diameter umbrella horizontally at a rate of 2 revolutions in 4 seconds. If she holds the rim 1.1 m above the ground, calculate the diameter of the horizontal circle where the droplets meet the ground. (The drops leave the horizontal rim of the umbrella tangentially.)

Fully worked solutions for this chapter are available online.

LESSON
4.7 Review

4.7.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

- Access the chapter summary
- Review your results
- Practise exam questions

Find all this and MORE in jacPLUS

4.7 Exercise

learnon

4.7 Exercise	4.7 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. Consider the vector equation, $\underset{\sim}{r}(t) = (1 + 6\cos(t))\hat{\boldsymbol{\imath}} + (6\sin(t) - 5)\hat{\boldsymbol{\jmath}}$, for $t \geq 0$. Determine the Cartesian equation.

2. A particle moves in such a way that its velocity vector at a time t is given by $4e^{\frac{t}{2}}\hat{\boldsymbol{\imath}} - 2\sin\left(\frac{t}{2}\right)\hat{\boldsymbol{\jmath}}$. Initially the position vector of the particle is $3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$. Determine the position vector of the particle at time t.

3. A particle moves so that its position vector at a time t, is given by $\underset{\sim}{r}(t) = 2t\hat{\boldsymbol{\imath}} + 2\sqrt{t}\hat{\boldsymbol{\jmath}}$ for $t \geq 0$. Calculate the distance of the particle from the origin after 4 seconds.

4. A golf ball is hit off the ground, its position vector, at a time t seconds after being hit, is given by the following: $\underset{\sim}{r}(t) = 15\sqrt{2}t\hat{\boldsymbol{\imath}} + \left(15\sqrt{2}t - \frac{1}{2}gt^2\right)\hat{\boldsymbol{k}}$ for $t \geq 0$, where $\hat{\boldsymbol{\imath}}$ is a unit vector in metres horizontally forward, and $\hat{\boldsymbol{k}}$ is a unit vector in metres vertically upwards.

   Students, when analysing the motion of the golf ball, stated some propositions:
   - Alicia stated that the golf ball is hit with an initial velocity of 30 m/s for maximum range.
   - Betty stated that the golf ball hits the ground again after a time of 4.33 seconds.
   - Colin stated that the golf ball reaches a maximum height of 22.96 metres.
   - David stated that the golf ball first hits the ground at a distance of 91.84 metres from where it was hit.
   - Edward stated that the golf ball travels in a parabolic path.

5. A stone is attached to string and is swung around in a circle at a constant speed of 4π m/s. The radius of the circular path is 16π metres. Calculate the magnitude of the acceleration of the stone.

6. Two particles, P and Q, have position vectors $\underset{\sim}{p} = 3\cos(2t)\hat{\boldsymbol{\imath}} + 4\sin(2t)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{q} = 4\sin(n\pi)\hat{\boldsymbol{\imath}} + 3\cos(n\pi)\hat{\boldsymbol{\jmath}}$ respectively at a time, t seconds, $t \geq 0$.
   a. Determine the Cartesian equation of the paths for both P and Q.
   b. Determine the coordinates where the paths of P and Q cross.

7. A particle moves so that at time t its position vector is given by $\underset{\sim}{r}(t) = 8\cos(t)\hat{\boldsymbol{\imath}} + 6\sin(t)\hat{\boldsymbol{\jmath}}$, for $0 \leq t \leq 2\pi$. The particle moves along a curve C. Determine:
   a. the coordinates of the point A when $t = \dfrac{\pi}{4}$
   b. the coordinates of the point B when $t = \dfrac{5\pi}{6}$
   c. the Cartesian equation of the curve C.

8. Two particles move so that their position vectors are given by $\underset{\sim}{r}_i(t) = (2t+5)\hat{\boldsymbol{\imath}} + \left(t^2 - 6t + 48\right)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{r}_\beta(t) = (t+11)\hat{\boldsymbol{\imath}} + (7t+6)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$. Calculate:
   a. when and where the particles collide
   b. the distance between the particles when $t = 9$.

9. A particle moves along a curve defined by $\underset{\sim}{r}(t) = (2 + 3\sec(t))\hat{\boldsymbol{\imath}} + (4\tan(t) - 1)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$. Determine the Cartesian equation of the path, and sketch the graph, stating its domain and range.

10. The acceleration of a body moving in a straight line is given by: $a(t) = 8 - 14t\ \text{m/s}^2$. Initially, the body is at rest at $x = 5$. Calculate:
   a. the velocity at any time
   b. the position at any time
   c. the time and position when the velocity is 0 m/s.

11. A particle moves so that its position vector is given by $\underset{\sim}{r}(t) = 3t^4\hat{\boldsymbol{\imath}} + 2t^2\hat{\boldsymbol{\jmath}} + e^2\hat{\boldsymbol{k}}$. Determine:
   a. the velocity vector
   b. the acceleration vector.

12. The acceleration of a particle is given by $\ddot{\underset{\sim}{r}}(t) = (6t - 4)\hat{\boldsymbol{\imath}}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underset{\sim}{r}}(t) = 5\hat{\boldsymbol{\jmath}}$ and the initial position is $\underset{\sim}{r}(0) = 4\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$, calculate the position vector at time t.

Complex familiar

13. A particle of mass 13 kg rotates with a constant speed of 36 m/s in a circular path of radius 12 metres. Determine the following:
   a. the angular velocity of the particle
   b. the position vector of the particle after 4 seconds if $\theta = 0$ when $t = 0$
   c. the acceleration vector after 4 seconds.

14. A ball is dropped from a tower that is 60 metres above the ground level.
   a. Calculate how long it takes the stone to reach the ground.
   b. Determine the speed of the stone on impact.

15. A girl throws a ball from the top of a hill 3 metres above ground level with an initial speed of 12 m/s at an angle of 42° to the horizontal.
   a. Calculate the time of flight and the horizontal distance.
   b. Calculate the maximum height reached.
   c. Determine the speed and angle at which the ball lands.

16. A particle rotates with constant speed in a circular path according to the position vector, $\underset{\sim}{r}(t) = (10\cos(5t))\hat{\boldsymbol{\imath}} + (10\sin(5t))\hat{\boldsymbol{\jmath}}$, for $t \geq 0$.
   a. Determine the radius of the circular path.
   b. Calculate the speed, v, at which the particle moves.

Complex unfamiliar

17. **a.** When a projectile is fired at an angle of α with an initial speed of V from a cliff of height h for maximum range R, provided that $V < \sqrt{gR}$, show that:

i. $\tan(\alpha) = \dfrac{V^2}{gR}$

ii. $h = \dfrac{g^2R^2 - V^4}{2V^2g}$

iii. $R = \dfrac{V}{g}\sqrt{V^2 + 2gH}$

iv. $T = \dfrac{\sqrt{2\left(V^2 + gh\right)}}{g}$.

b. A shot-putter can throw the shot with a release speed of 20 m/s. The shot leaves her hand 2 metres above the ground. Calculate the maximum range, the angle of projection and the time of flight.

18. A curve given by the Cartesian equation $y = \dfrac{abx}{x^2 + a^2}$, where a and b are positive constants, is called the serpentine curve.

a. Show that $\dfrac{dy}{dx} = \dfrac{ab(a-x)(a+x)}{\left(x^2 + a^2\right)^2}$ and that the graph of the serpentine curve has turning points at $\left(a, \dfrac{b}{2}\right)$ and $\left(-a, -\dfrac{b}{2}\right)$.

b. Show that $\dfrac{d^2y}{dx^2} = \dfrac{2abx\left(x^2 - 3a^2\right)}{\left(x^2 + a^2\right)^3}$ and hence show that the points of inflection on the serpentine curve are given by $\left(\pm\sqrt{3}a, \pm\dfrac{\sqrt{3}b}{4}\right)$.

c. Show that the vector equation of the serpentine curve is given by $\underset{\sim}{r}(t) = a\cot(t)\hat{\boldsymbol{\imath}} + \dfrac{b}{2}\sin(2t)\hat{\boldsymbol{\jmath}}$ for $t \geq 0$.

d. Sketch the graph of the serpentine curve.

19. **a.** The position vector of a moving particle is given by $\underset{\sim}{r}(t) = 2\sin(t)\hat{\boldsymbol{\imath}} + (2\sin(t)\tan(t))\hat{\boldsymbol{\jmath}}$, for $t \geq 0$. Show that the particle moves along the curve $y = \dfrac{x^2}{\sqrt{4 - x^2}}$.

b. A curve called the Witch of Agnesi is defined by the parametric equations $x = at$ and $y = \dfrac{a}{1 + t^2}$. Show that Cartesian equation is given by $y = \dfrac{a^3}{a^2 + x^2}$.

20. The position of drone A is defined by the vector equation $\underset{\sim}{r}_{\mathrm{A}}(t) = \begin{pmatrix} 9 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

A second drone, drone B, is flying on a straight-line path, and its location was recorded on two occasions, defined by the vectors $\underset{\sim}{a} = \begin{pmatrix} 5 \\ 8 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 9 \\ -17 \\ -15 \end{pmatrix}$.

The distance are in metres, the time is in seconds and the drones have a diameter of 40 cm.
In the interest of public safety, determine whether this is safe or whether there will be a collision, and consider the reasonableness of your solution.

4.7 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2020 Specialist Mathematics, Paper 2, Section 1, Q9; © State of Queensland (QCAA) 2024.

Two objects, P and Q, move in three-dimensional space such that their positions, r, over time, t, are described by the following vectors until they collide.

$$\boldsymbol{r}_P = \left(t^2 - 4t\right)\hat{\boldsymbol{\imath}} + \left(2t^2 - t + 3\right)\hat{\boldsymbol{\jmath}} - (6 - 5t)\hat{\boldsymbol{k}}$$
$$\boldsymbol{r}_Q = \left(-t^2 + 2t\right)\hat{\boldsymbol{\imath}} + \left(3t + t^2\right)\hat{\boldsymbol{\jmath}} + t^2\hat{\boldsymbol{k}}$$

The objects will collide at

A. $t = 0$ **B.** $t = 1$ **C.** $t = 2$ **D.** $t = 3$

Question 2 (1 mark)

Source: QCAA 2021 Specialist Mathematics, Paper 1, Section 1, Q3; © State of Queensland (QCAA) 2024.

An object has a velocity $\boldsymbol{v}(t) = e^{-2t}\hat{\boldsymbol{\imath}} + \left(\frac{1}{t}\right)\hat{\boldsymbol{k}}$, where t represents time $(t > 0)$.

The displacement $\boldsymbol{r}(t)$ of the object could be

A. $-2e^{-2t}\hat{\boldsymbol{\imath}} + \ln(t)\hat{\boldsymbol{k}}$ **B.** $-2e^{-2t}\hat{\boldsymbol{\imath}} - \frac{1}{t^2}\hat{\boldsymbol{k}}$

C. $-\frac{1}{2}e^{-2t}\hat{\boldsymbol{\imath}} + \ln(t)\hat{\boldsymbol{k}}$ **D.** $-\frac{1}{2}e^{-2t}\hat{\boldsymbol{\imath}} - \frac{1}{t^2}\hat{\boldsymbol{k}}$

Question 3 (1 mark)

Source: QCAA 2022 Specialist Mathematics, Paper 1, Section 1, Q3; © State of Queensland (QCAA) 2024.

A particle travels in a straight line over time, t, with a constant acceleration, $a(t)$.

Which function could represent the particle's displacement, $x(t)$?

A. $x(t) = t^3$ **B.** $x(t) = t^2$ **C.** $x(t) = \frac{1}{t}$ **D.** $x(t) = \sqrt{t}$

Question 4 (6 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 2, Q19; © State of Queensland (QCAA) 2024.

Object A is released from the origin with constant velocity, $\boldsymbol{v}_A$, such that its position after t seconds is given by

$\boldsymbol{r}_A = 2\sqrt{3}t\hat{\boldsymbol{\imath}} + 3t\hat{\boldsymbol{\jmath}} + 2t\hat{\boldsymbol{k}}, t \geq 0.$

At a later time, object B is released from point $P(3\sqrt{3}, 6, 0)$ and travels towards point $Q(5\sqrt{3}, 8, 4)$ with constant velocity, $\boldsymbol{v}_B$, such that $|\boldsymbol{v}_B| = \sqrt{2}\,|\boldsymbol{v}_A|$.

Given that objects A and B collide, determine the time between the release of the two objects. Assume all positions are given in metres and all velocities are given in metres per second.

Question 5 (6 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 2, Q17; © State of Queensland (QCAA) 2024.

An object is projected upwards from ground level with an initial velocity of $15\,\mathrm{m\,s^{-1}}$ at an angle of $54°$ to the horizontal.

The object just passes over a drone hovering in the air. An observer is positioned directly below the drone and at a horizontal distance of $20\,\mathrm{m}$ from where the object is projected.

The observer commented that:

- it took the object around 2 to 2.5 seconds after its projection to reach the drone
- the object was still moving in an upwards direction as it passed the drone.

Assuming that air resistance is negligible, use a vector calculus approach to evaluate the reasonableness of the observer's comments.

Fully worked solutions for this chapter are available online.

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Answers

Chapter 4 Vector Calculus

4.2 Position vectors as functions of time — circles, ellipses and hyperbolas

4.2 Exercise

1. a. $\sqrt{305}$

b. $\sqrt{14t^2 - 10t + 5}$

c. $\dfrac{3\sqrt{70}}{14}$

2. a. $5\sqrt{2}$ **b.** 9

3. a. i. $5\sqrt{2}$

ii. $\sqrt{5t^2 - 10t + 10}$

iii. $\sqrt{5}$

b. i. $5\sqrt{5}$

ii. $5\sqrt{t^2 + 1}$

iii. 5

4. a. i. $\sqrt{41}$

ii. $\sqrt{5}$

iii. 0, 2

b. i. $\dfrac{9\sqrt{2}}{2}$

ii. 8

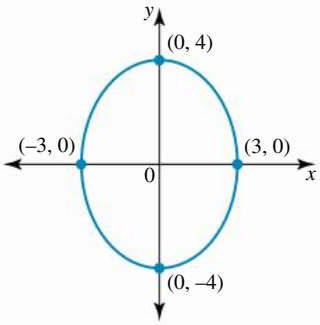
5. a. $a = 4,\ b = -3,\ c = 1,\ d = 3$

b. $a = 2,\ b = 5,\ c = 1,\ d = -3$

6. a. $(x-1)^2 + y^2 = 36$

b. $\dfrac{(x+1)^2}{16} + \dfrac{(y-3)^2}{4} = 1$

7. Ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$; domain $[-3, 3]$, range $[-4, 4]$

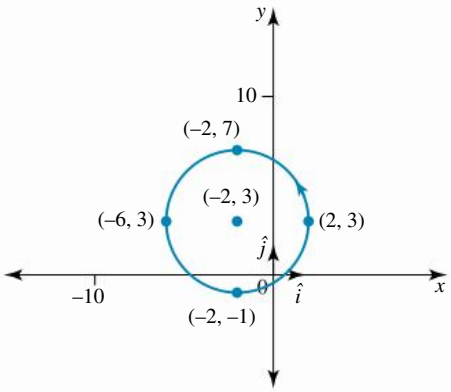


8. a. $x = 4\cos(t) - 2$

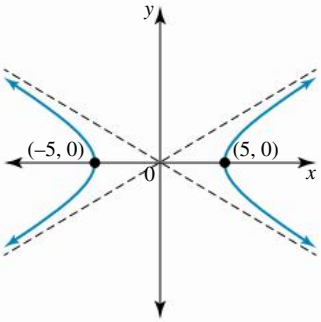
$y = 4\sin(t) + 3$

b. $\underset{\sim}{r}(t) = (4\cos(t) - 2)\hat{\boldsymbol{\imath}} + (4\sin(t) + 3)\hat{\boldsymbol{\jmath}}$

c.



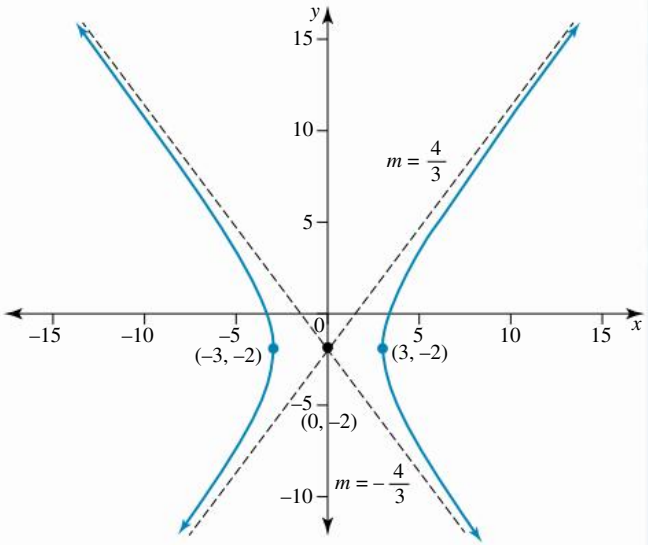
9. Hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$, asymptotes $y = \pm\dfrac{3x}{5}$; domain $(-\infty, -5] \cup [5, \infty)$, range $\mathbb{R}$



10. a. $\dfrac{(x-3)^2}{16} + \dfrac{(y-1)^2}{9} = 1$

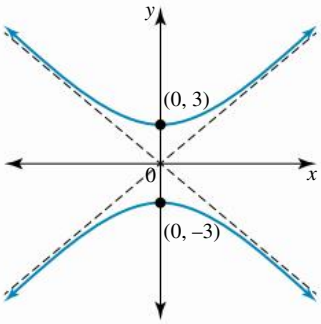
b. $|\dot{\underset{\sim}{r}}(0)| \le \left|\dot{\underset{\sim}{r}}\left(\dfrac{\pi}{2}\right)\right|$, the particle is moving faster at $t = \dfrac{\pi}{2}$ than at $t = 0$.

11.



12. $\dfrac{(y-1)^2}{25} - \dfrac{(x-3)^2}{36} = 1$

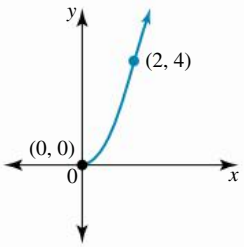
13. Hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$, asymptotes $y = \pm \frac{3x}{4}$; domain $\mathbb{R}$, range $(-\infty, -3] \cup [3, \infty)$



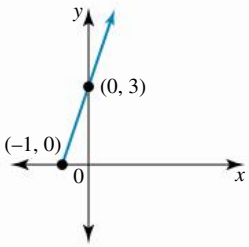
14. Sample responses can be found in the worked solutions in the online resources.

15. a. $\sqrt{2}$ **b.** $\sqrt{2}$

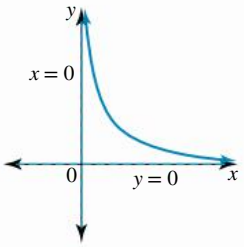
16. a. Part of a parabola, $y = x^2$; domain $[0, \infty)$, range $[0, \infty)$,



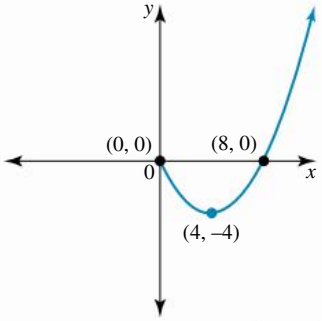
b. Part of a straight line, $y = 3x + 3$; domain $[-1, \infty)$, range $[0, \infty)$



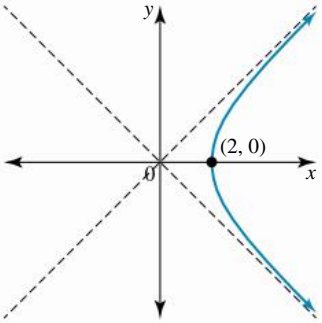
17. a. $y = \frac{2}{x}$; domain $[0, \infty)$, range $[0, \infty)$



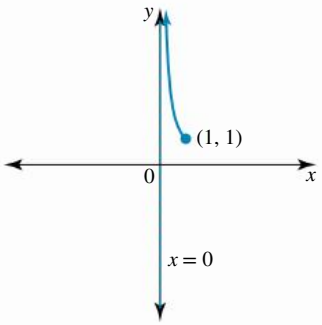
b. Part of a parabola, $y = \frac{1}{4}\left(x^2 - 8x\right)$; domain $[0, \infty)$, range $[-4, \infty)$



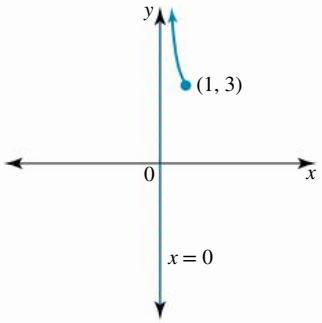
c. Part of a hyperbola, $y = \sqrt{x^2 - 4}$; domain $[2, \infty)$, range $\mathbb{R}$



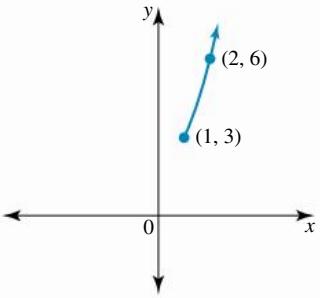
18. a. Part of a hyperbola, $y = \frac{1}{x}$; domain $(0, 1]$, range $[1, \infty)$



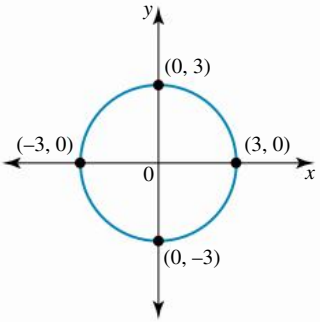
b. Part of a truncus, $y = 2 + \frac{1}{x^2}$; domain $(0, 1]$, range $[3, \infty)$



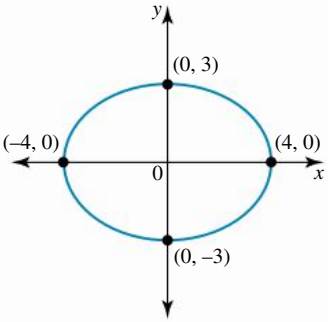
c. Part of a parabola, $y = 2 + x^2$; domain $[1, \infty)$, range $[3, \infty)$



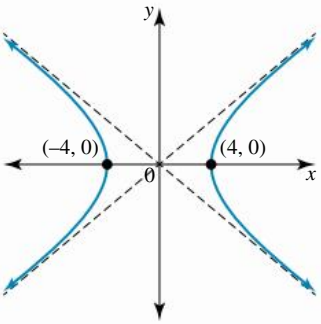
19. a. Circle with centre at the origin, radius 3, $x^2 + y^2 = 9$; domain $[-3, 3]$, range $[-3, 3]$



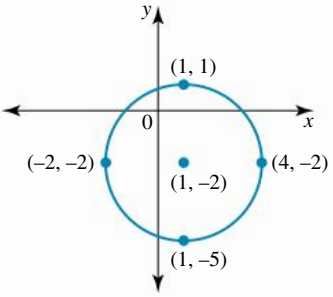
b. Ellipse with centre at the origin, $\frac{x^2}{16} + \frac{y^2}{9} = 1$; domain $[-4, 4]$, range $[-3, 3]$



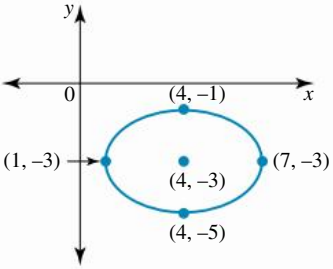
c. Hyperbola with centre at the origin, $\frac{x^2}{16} + \frac{y^2}{9} = 1$, asymptotes $y = \pm \frac{3x}{4}$; domain $(-\infty, -4] \cup [4, \infty)$, range $\mathbb{R}$



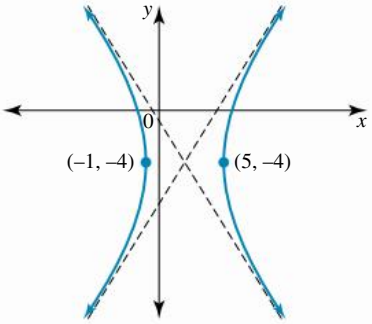
20. a. Circle with centre at $(1, -2)$, radius 3, $(x - 1)^2 + (y + 2)^2 = 9$; domain $[-2, 4]$, range $[-5, 1]$



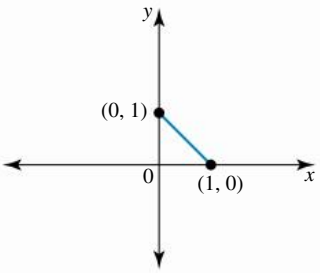
b. Ellipse with centre at $(4, -3)$, $\frac{(x-4)^2}{9} + \frac{(y+3)^2}{4} = 1$; domain $[1, 7]$, range $[-5, -1]$



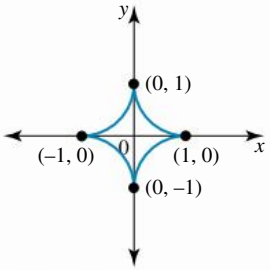
c. Hyperbola with centre at $(2, -4)$, $\frac{(x-2)^2}{9} - \frac{(y+4)^2}{25} = 1$, asymptotes $y = \frac{5x}{3} - \frac{22}{3}$, $y = -\frac{5x}{3} - \frac{2}{3}$; domain $(-\infty, -1] \cup [5, \infty)$, range $\mathbb{R}$



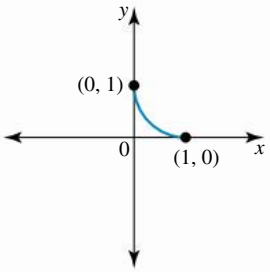
21. a. Part of a straight line, $y = 1 - x$; domain $[0, 1]$, range $[0, 1]$



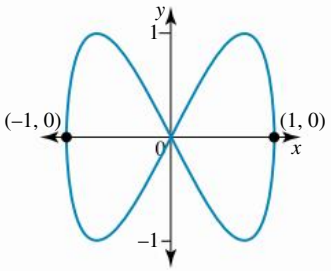
b. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$; domain $[-1, 1]$, range $[-1, 1]$



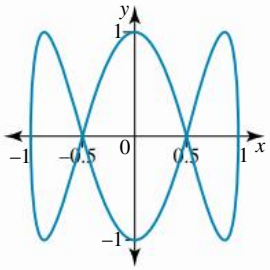
c. $\sqrt{y} + \sqrt{x} = 1$; domain $[0, 1]$, range $[0, 1]$



22. a.



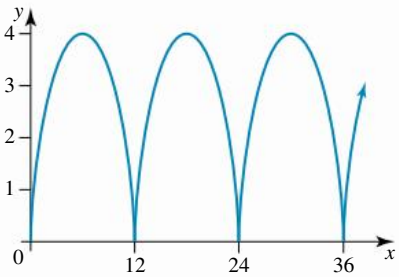
b.



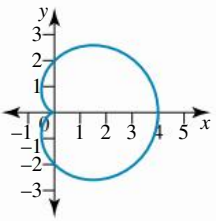
23. Sample responses can be found in the worked solutions in the online resources.

24. Sample responses can be found in the worked solutions in the online resources.

25. a.



b.



4.3 Collision problems

4.3 Exercise

1. a. The boats are approximately 43.32 metres apart.

b. $\underset{\sim}{r}_A(3) \neq \underset{\sim}{r}_B(3)$. No collision.

2. $\underset{\sim}{r}(10) = 219\hat{\boldsymbol{\imath}} + 220\hat{\boldsymbol{\jmath}} + 452\hat{\boldsymbol{k}}$

3. $t = \dfrac{3\pi}{2}$ seconds

4. a. $t = 5,\ (16, 40)$ **b.** $(36, 180)$ **c.** $5\sqrt{5}$

5. a. $\sqrt{61}$

b. $\sqrt{8t^2 - 12t + 65}$

c. 8.86

6. a. $t = 4,\ (10, 7)$ **b.** $(-2, 13)$ **c.** $\sqrt{41}$

7. a. $t = 15,\ (2, 5)$ **b.** $(8, -7)$ **c.** 15

8. a. $t = 2,\ (-2, 6)$; $t = 5,\ (19, 12)$

b. $8\sqrt{26}$

9. a. $t = 12,\ (53, 74)$

b. $(89, 56)$

c. $80\sqrt{5}$

10. $\underset{\sim}{r}_b(18) = \underset{\sim}{r}_a(18) = 328\hat{\boldsymbol{\imath}} + 81\hat{\boldsymbol{\jmath}} + 163\hat{\boldsymbol{k}}$,

11. The minimum distance is 4.34 m.

12. $\sqrt{5}$ cm/s

13. $\sqrt{2 - \dfrac{1}{1+k}}$

14. Tanya's claim is correct.

15. $k \leq 5.29$ or $k \geq 12.71$

4.4 Differentiation and integration of vectors

4.4 Exercise

1. a. $\dfrac{d}{dt}\underset{\sim}{r}(t) = \begin{pmatrix} -2\sin(2t) \\ \cos(t) \\ 4t + 1 \end{pmatrix}$

b. $\dfrac{d^2}{dt^2}\underset{\sim}{r}(t) = \begin{pmatrix} -4\cos(2t) \\ -\sin(t) \\ 4 \end{pmatrix}$

2. a. $\dot{\underset{\sim}{r}}(t) = \begin{pmatrix} -\cos(t) + 5 \\ \frac{1}{2}t^2 - 2t - 1 \\ \frac{1}{3}\sin(3t) + 2 \end{pmatrix}$

b. $\underset{\sim}{r}(t) = \begin{pmatrix} 2 - \sin(t) \\ \frac{1}{6}t^3 - t^2 - 4 \\ \frac{1}{9}(1 - \cos(3t)) \end{pmatrix}$

3. a. $e^{-2t}(1-2t)\hat{\imath} + e^{2t}(1+2t)\hat{\jmath}$

b. $2e$

c. $4(t-1)e^{-2t}\hat{\imath} + 4(t+1)e^{2t}\hat{\jmath}$

4. 4 kg m/s

5. a. $8t^3\hat{\imath} - 8\sin(2t)\hat{\jmath} - 12e^{-2t}\hat{k}$

b. $24t^2\hat{\imath} - 16\cos(2t)\hat{\jmath} + 24e^{-2t}\hat{k}$

6. $\frac{1}{2}\sqrt{\pi^4 + 1024e^{-4}}$

7. $\hat{\jmath}$

8. $\frac{1}{2}\left(-\sqrt{3}\,\hat{\imath} + \hat{\jmath}\right)$

9. a. $\frac{1}{\sqrt{17}}(\hat{\imath} + 4\hat{\jmath})$

b. $\frac{1}{\sqrt{145}}(\hat{\imath} + 12\hat{\jmath})$

c. $\frac{1}{\sqrt{82}}(9\hat{\imath} + \hat{\jmath})$

d. $\frac{1}{\sqrt{34}}(3\hat{\imath} + 5\hat{\jmath})$

10. $(2t^2 - 4t + 5)\hat{\imath} + (4 - 3t)\hat{\jmath}$

11. $(5 - 2\cos(3t))\hat{\imath} + (3 - 2e^{-2t})\hat{\jmath}$

12. $(5 - 2t)\hat{\imath} + (16t - t^4 - 10)\hat{\jmath}$

13. $(3t^2 - 5)\hat{\imath} + (t^2 + 8t - 2)\hat{\jmath}$

14. $(x+2)^2 + (y-4)^2 = 25$; circle with centre $(-2, 4)$, radius 5

15. a. $y = \frac{1}{x-4} + 3$ b. $y = \frac{32}{2-x} - 5$

16. a. $(x-4)^2 + \frac{(y-5)^2}{4} = 1$

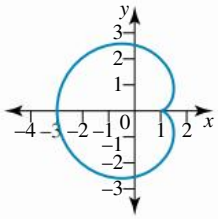
b. $\frac{(x+7)^2}{144} + \frac{(y-3)^2}{16} = 1$

17. a. $y = x^2 - 4$, $[-2, 2]$, $[-4, 0]$

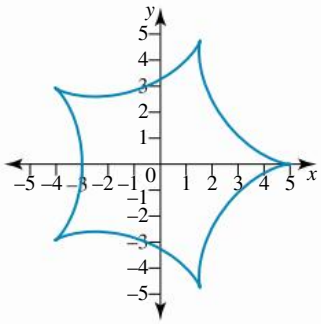
b. $y = -3x(x-2)$, $[-1, 3]$, $[-9, 3]$

18. a. $t = 3$, $(4, 16)$ b. $t = 5$, $(-7, 10)$

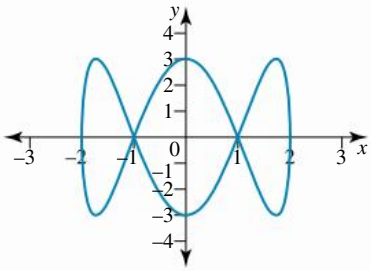
19. a. $\underset{\sim}{r}(t) = (2\cos(t) - \cos(2t))\hat{\imath} + (2\sin(t) - \sin(2t))\hat{\jmath}$



b. $\underset{\sim}{r}(t) = (4\cos(t) + \cos(4t))\hat{\imath} + (4\sin(t) - \sin(4t))\hat{\jmath}$

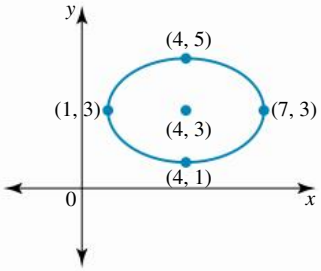


c. $\underset{\sim}{r}(t) = 2\cos(2t)\hat{\imath} + 3\sin(6t)\hat{\jmath}$



20. a. $(4, 1), (4, 5)$

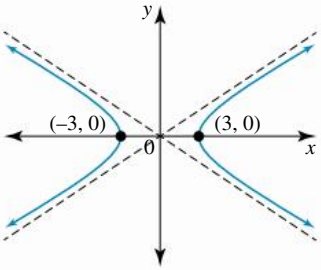
b. $\frac{(x-4)^2}{9} + \frac{(y-3)^2}{4} = 1$; domain $[1, 7]$, range $[1, 5]$



c. $|\dot{\underset{\sim}{r}}(t)|_{\max} = 6\,|\dot{\underset{\sim}{r}}(t)|_{\min} = 4$

21. All the claims are correct.

22. a. Hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$; domain $|x| \geq 3$, range $\mathbb{R}$; asymptotes $y = \pm\frac{4x}{3}$



b. $\frac{8}{3}$

c. 4

4.5 Straight line motion with constant and variable acceleration

4.5 Exercise

1. a. $t = 1, 2$ and 3

b. $v = 3t^2 - 12t + 11$

c. $t = \frac{6-\sqrt{3}}{3}$ and $\frac{6+\sqrt{3}}{3}$

d. -1 cm/s

e. 6 cm

2. a. i. $3t^2 + 4t - 5$

ii. $6t + 4$

b. i. $10 + 6t - 3t^2$

ii. $6 - 6t$

c. i. $e^t - 2e^{-t}$

ii. $e^t + 2e^{-t}$

d. i. $2t - \dfrac{1}{t+3}$

ii. $2 + \dfrac{1}{(t+3)^2}$

e. i. $4 - \dfrac{1}{2\sqrt{t+2}}$

ii. $\dfrac{1}{4(t+2)^{\frac{3}{2}}}$

f. i. $\dfrac{1}{\sqrt{9-t^2}}$

ii. $\dfrac{t}{(9-t^2)^{\frac{3}{2}}}$

3. a. Min, $x = 1$, max, $x = 5$

b. $t = \dfrac{1}{3}$

c. $t = \dfrac{\pi}{3}$

d. $a = -9x + 27$

4. a. $x = 2t^3 - t^2 - 8t + 2$

b. $\ddot{x} = 12t - 2$

c. $t = \dfrac{4}{3}$ s

d. 7 m

e. 14 m/s^2

5. a. $v = 6t - 6t^2$

b. $x = 3t^2 - 2t^3 + 4$

c. $t = 0, x = 4$ and $t = 1, x = 5$.

d. 29 m

6. a. $v = 2e^{2t} - t^2 + 3t$

b. $x = e^{2t} - \dfrac{1}{3}t^3 + \dfrac{3}{2}t^2 - 1$

c. 406.93 cm

7. a. 1 s b. 2 s

8. The distance travelled is 4 cm.

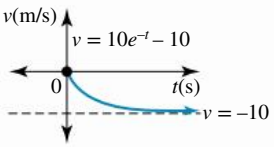
9. a. 0 m/s b. $x = \dfrac{1}{2}\ln\left(t^2 + 1\right)$

c. $a = \dfrac{1 - t^2}{(t^2+1)^2}$ d. 0.5 m/s

e. $-\dfrac{1}{8}$ m/s^2

10. a. $\dot{x} = 10e^{-t} - 10$

b.



c. 11 s

11. a. $x = 4\tan^{-1}(t) + 1$ b. $v = 4$ m/s, $t = 0$

c. $a = \dfrac{-8t}{(1+t^2)^2}$ d. -2 m/s^2

12. a. $v = -2\cos 2t$ b. $x = -\sin 2t$

c. $a = -4x$ d. $a = 2\sqrt{4 - v^2}$

13. a. 10 m/s b. 10 s c. 100 m

14. a. 4 m/s b. 50 m

15. a. -1.6 m/s^2 b. 32 m/s

16. a. 8.16 s b. 97.98 m/s

17. a. ± 53.67 m/s b. 12.6 s or 0.7 s

18. a. 78.4 m b. 8 s

19. a. 3.2 s b. 31.3 m/s

20. a. 78.4 m b. 39.2 m/s

4.6 Applications of vector calculus

4.6 Exercise

1. a. 2.45 s b. 12.24 m c. 7.35 m

2. a. 3.54 s, 87 metres

b. 15.61 metres

c. 30.16 m/s, 35.43°

d. Sample responses can be found in the worked solutions in the online resources.

3. a. 1.92 s, 34.7 metres

b. 5.45 metres

c. 20.86 m/s, 29.68°

d. Sample responses can be found in the worked solutions in the online resources.

4. a. $x^2 + y^2 = \dfrac{25}{4}$

b. $\dfrac{4\pi}{3}$ rad/s

c. $\dfrac{5}{2}\cos\left(\dfrac{4\pi}{3}t\right)\hat{\imath} + \dfrac{5}{2}\sin\left(\dfrac{4\pi}{3}t\right)\hat{\jmath}$

5. a. $\underset{\sim}{v}(t) = (-6\sin(3t))\hat{\imath} + (6\cos(3t))\hat{\jmath}$

b. $\underset{\sim}{a}(t) = -18\left(\cos(3t)\hat{\imath} + \sin(3t)\hat{\jmath}\right) = -9\,\underset{\sim}{r}(t)$

6. a. $r = 5$

b. $v = 10$ m/s

7. a. $\omega = \dfrac{3}{2}$ rad/s

b. $\underset{\sim}{r}(4) \approx 7.681\hat{\imath} - 2.235\hat{\jmath}$

c. $\dot{\underset{\sim}{r}}(4) \approx 3.353\hat{\imath} + 11.522\hat{\jmath}$

$\ddot{\underset{\sim}{r}}(4) \approx -17.283\hat{\imath} + 5.029\hat{\jmath}$

8. a. $v = 12\pi$ m/s

b. $|\ddot{\underset{\sim}{r}}(t)| = |\underset{\sim}{a}(t)| = 48\pi^2$ m/s^2

9. a. $\omega = 12\pi$ rad/s

b. $v = 288\pi$ m/s

10. $\omega = \dfrac{35}{4}$ rad/s

11. a. $\omega = \frac{\pi}{62}$ rad/s

b. $v = \frac{7\pi}{62}$ m/s ≈ 0.355 m/s

c. $|\underset{\sim}{a}| = \frac{7\pi^2}{3844}$ m/s^2 $\left(a \approx 0.018 \text{ m/s}^2\right)$

12. a. 2.45 s

b. 12.25 m

c. 13 m/s, 67.38°

d. 7.35 m

e. $y = -\frac{x}{250}(49x - 600)$

13. a. 1.67 s

b. 40.14 m

c. 25 m/s, 16.26°

d. 4.5 m

e. $y = 2 + \frac{7x}{24} - \frac{49x^2}{5760}$

14. a. 2.04 s

b. 20.4 m

c. 14.14 m/s, 45°

d. 5.1 m

e. $y = -\frac{x}{1000}(49x - 1000)$

15. 86.42°, 2.11°

16. The toy car and the toy truck will not collide.

17. 10 m/s, 60°

18. Sample responses can be found in the worked solutions in the online resources.

19. Sample responses can be found in the worked solutions in the online resources.

20. The diameter of the circle: $D = 2.986$ m

4.7 Review

4.7 Exercise

1. $\frac{(x-1)^2}{36} + \frac{(y+5)^2}{36} = 1$

2. $\left(8e^{\frac{t}{2}} - 5\right)\hat{\imath} + \left(4\cos\left(\frac{t}{2}\right) - 7\right)\hat{\jmath}$

3. $4\sqrt{5}$

4. All of Alicia, Betty, Colin, David and Edward are all correct.

5. π m s^{-2}

6. a. p: $\frac{x^2}{9} + \frac{y^2}{16} = 1,\ t \geq 0$

q: $\frac{x^2}{16} + \frac{y^2}{9} = 1,\ t \geq 0$

b. $\left(\frac{12}{5}, \frac{12}{5}\right), \left(\frac{-12}{5}, \frac{12}{5}\right), \left(\frac{-12}{5}, \frac{-12}{5}\right) \left(\frac{12}{5}, \frac{-12}{5}\right)$

7. a. $\underset{\sim}{r}\left(\frac{\pi}{4}\right) = 4\sqrt{2}\hat{\imath} + 3\sqrt{2}\hat{\jmath}$

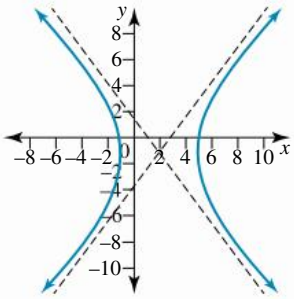
b. $\underset{\sim}{r}\left(\frac{5\pi}{6}\right) = -4\sqrt{3}\hat{\imath} + 3\hat{\jmath}$

c. $\frac{x^2}{64} + \frac{y^2}{36} = 1$

8. a. The particles collide at $17\hat{\imath} + 48\hat{\jmath}$ after 6 seconds of flight.

b. $|\underset{\sim}{r}_B(9) - \underset{\sim}{r}_A(9)| = 3\sqrt{5}$ m

9. Hyperbola, centre $(2, -1)$ domain: $(-\infty, -1] \cup [5, \infty)$, range: $\mathbb{R}$



10. a. $v(t) = 8t - 7t^2$

b. $x(t) = -\frac{7}{3}t^3 + 4t^2 + 5$

c. $x(0) = 5$ m, $x\left(\frac{8}{7}\right) = \frac{991}{147}$ m

11. a. $\underset{\sim}{v}(t) = 12t^3\hat{\imath} + 4t\hat{\jmath} + 2e^{2t}\hat{k}$

b. $\underset{\sim}{a}(t) = 36t^2\hat{\imath} + 4\hat{\jmath} + 4e^{2t}\hat{k}$

12. $\underset{\sim}{r}(t) = \left(t^3 - 2t^2 + 4\right)\hat{\imath} + (5t - 2)\hat{\jmath} + \hat{k}$

13. a. $\omega = 3$ rad/s

b. $\underset{\sim}{r}(4) = 10.126\hat{\imath} - 6.439\hat{\jmath}$

c. $\ddot{\underset{\sim}{r}}(4) = -91.136\hat{\imath} + 57.950\hat{\jmath}$

14. a. $t = 3.50$ s b. 34.30 m/s

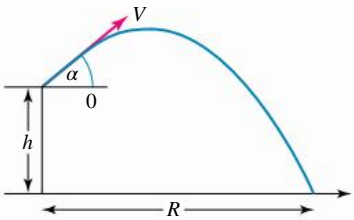
15. a. The ball travels a horizontal distance of 17.251 metres. The total time of flight is 2.033 seconds.

b. The maximum height reached is 6.673 metres.

c. The ball makes a 53.340° angle with the ground and lands with a speed of 14.242 m/s.

16. a. $r = 10$ b. $v = 50$ m/s

17.

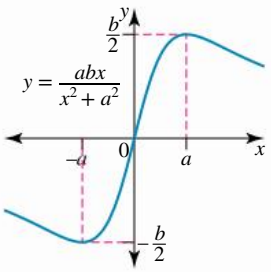


a. Sample responses can be found in the worked solutions in the online resources.

b. $\mathbb{R} = 42.77$ m
$\alpha = 43.7°$
$T = 2.96$ s

18. a–c. Sample responses can be found in the worked solutions in the online resources.

d.



19–20. Sample responses can be found in the worked solutions in the online resources.

4.7 Past QCAA exam questions

1. D

2. C

3. B

4. The time between the collision is 1.8 seconds.

5. Sample responses can be found in the worked solutions in the online resources.

5 Further matrices

LESSON SEQUENCE

5.1 Overview 292
5.2 Solving linear equations using matrix algebra 294
5.3 Solving a system of linear equations using Gaussian elimination 315
5.4 The three cases for solutions of systems of linear equations 323
5.5 Dominance and Leslie matrices 347
5.6 Applications of matrices 357
5.7 Review 377
Answers 383

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 5
Exam questions	Exam question booklet — Chapter 5
Digital documents	Learning matrix — Chapter 5 Chapter summary — Chapter 5

LESSON
5.1 Overview

Hey students! Bring these pages to life online

Engage with interactivities

Answer questions and check results

Track your progress

Find all this and MORE in jacPLUS

5.1.1 Introduction

Systems of linear equations, and their applications, are closely related to the study of algebra, geometry, vectors and matrices. Often referred to as linear algebra, this branch of mathematics simplifies higher order systems of equations by translating the information into matrices. The equation of a plane was studied in the previous chapter using both a vector and an analytical geometry approach. If you were to consider three planes simultaneously, you would be modelling three equations in a system containing three variables (x, y, z). Systems in three dimensions may have zero, one or infinitely many solutions. The diagrams represent generalised geometric interpretations of each solution type.

A unique solution

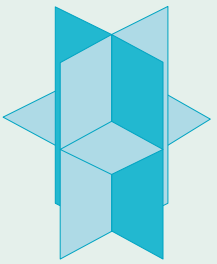
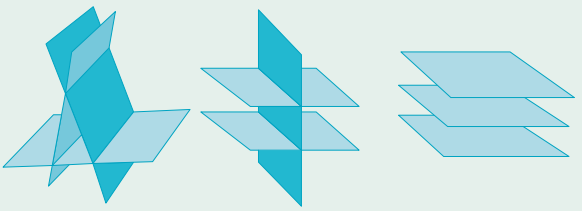
No solution

Infinitely many solutions

Matrices are used to determine solutions across a wide range of disciplines including ranking players at competitive tournaments through dominance matrices, network flow analysis, code breaking (cryptology), polynomial curve fitting, input/output models in economics using Leontief matrices and the use of Leslie matrices in ecological studies.

Leslie matrices are primarily used in ecology as a powerful predictor of population growth. Combining the principles of both matrix multiplication and powers of matrices, estimates can be made on future population numbers based on the species reproductive and survival rates.

5.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
5.2	**Solving linear equations using matrix algebra**	◯ Calculate the determinant and inverse of square matrices of any order, with technology. ◯ Use the determinant to determine whether a square matrix of any order is singular or non-singular. ◯ Use matrix algebra to solve matrix equations that involve matrices of beyond dimension 2×2, including those of the form $AX = B$, $XA = B$ and $AX + BX = C$, with technology. ◯ Model and solve problems that involve matrices of beyond dimension 2×2, including the solution of systems of linear equations, with technology.
5.3	**Solving a system of linear equations using Gaussian elimination**	◯ Recognise the general form of a system of linear equations in several variables and use Gaussian techniques of elimination on an augmented matrix to solve a system of linear equations, with and without technology.
5.4	**The three cases for solutions of systems of linear equations**	◯ Examine the three cases for solutions of systems of equations — a unique solution, no solution and infinitely many solutions — and the geometric interpretation of a solution of a system of equations with three variables including ◯ A unique solution  ◯ No solution  ◯ Infinitely many solutions
5.5	**Dominance and Leslie matrices**	◯ Model and solve problems that involve real-life situations using matrices, including Dominance and Leslie matrices.
5.6	**Applications of matrices**	◯ Investigate how matrices have been applied in other real-life situations, e.g. Leontief, Markov, area, cryptology, eigenvectors and eigenvalues. ***Note:*** The external examination may assess only Dominance and Leslie matrices.

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LESSON
5.2 Solving linear equations using matrix algebra

SYLLABUS LINKS

- Calculate the determinant and inverse of square matrices of any order, with technology.
- Use the determinant to determine whether a square matrix of any order is singular or non-singular.
- Use matrix algebra to solve matrix equations that involve matrices of beyond dimension 2×2, including those of the form $AX = B$, $XA = B$ and $AX + BX = C$, with technology.
- Model and solve problems that involve matrices of beyond dimension 2×2, including the solution of systems of linear equations, with technology.

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5.2.1 System of linear equations

Linear algebra is fundamental in the study of engineering, business, economics, network analysis, electrical circuits and vectors. This branch of mathematics builds on the concepts of **linear equations** introduced in earlier years (such as Years 9 and 10 and Year 11 Methods) and extends them to systems of linear equations and the various techniques used to solve them.

In Unit 1, you learned key matrix terminology, techniques and how to solve problems using matrix algebra. In Unit 3, the focus extends beyond 2×2 matrices to 3×3 and larger systems, applying these methods to more complex scenarios.

Recall that a linear equation in two dimensions is given by,

$ax + by = c$, where a, b and $c \in \mathbb{R}$

A linear equation in three dimensions contains three variables, x, y and z. It is commonly expressed in the form,

$ax + by + cz = d$, where a, b, c and $d \in \mathbb{R}$

Linear equations contain **variables** raised only to the first power. The variables are not multiplied together, do not appear under roots, and are not part of logarithmic, exponential or trigonometric functions.

In general, a linear equation with n variables, is of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots\ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and $b \in \mathbb{R}$ and are the coefficients of $x_1, x_2, \ldots, x_n$.

Other examples of linear and non-linear equations include:

Linear equations	Non-linear equations
$4x - 11y = 12$	$2yz + 4xy - z^2 = 12$
$2x - y + 7z = 14$	$\cos^2(x) + \ln(y) + z = 100$
$7x - \frac{121}{7}y + 3z = \pi$	$4\sqrt[5]{x} - \frac{2}{y} = -50$

A **system of linear equations** is a set of multiple linear equations that relate the same variables.

The general form of a system of m linear equations in n variables is given by

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots\cdots & + & a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots\cdots & + & a_{2n}x_n = b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots\cdots & + & a_{3n}x_n = b_3 \\ \vdots & & \vdots & & \vdots & & & & \vdots \quad \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots\cdots & + & a_{mn}x_n = b_m \end{array}$$

where $a_1, a_2, \ldots, a_n$ are the coefficients of the variables $x_1, x_2, \ldots, x_n$ and b is a constant term.

For example, the following system consists of two linear equations in two variables, x and y.

$$\begin{aligned} 55x + 38y &= 207 \\ 17x - 6y &= -7 \end{aligned}$$

As the number of equations and the number of variables is equal, it is referred to as a **square system**.

This is because the left-hand side of these equations forms a structure 2×2, which you will later see as a 2×2 matrix.

In three dimensions, an example of a square system is shown.

$$\begin{aligned} 4x + 2y - 7z &= -27 \\ 5x - 5y + z &= 0 \\ -2x + 3y - 4z &= -16 \end{aligned}$$

In a system of linear equations, the variables are called the unknowns. Consider the following system of linear equations, where x and y represent the price of two different items.

$$\begin{aligned} 120x + 10y &= 70 \\ 38x + 20y &= 39 \end{aligned}$$

The **unique solution** for this set of simultaneous equations is $x = \frac{1}{2}$, $y = 1$. These scalar values represent a sequence of numbers that satisfy every equation in the system as shown below.

$$\begin{aligned} \text{LHS} &= 120x + 10y \\ &= 120 \times \frac{1}{2} + 10 \times 1 \\ &= 70 \\ &= \text{RHS} \end{aligned} \qquad \begin{aligned} \text{LHS} &= 38x + 20y \\ &= 38 \times \frac{1}{2} + 20 \times 1 \\ &= 39 \\ &= \text{RHS} \end{aligned}$$

If there is only one possible solution, such as the example above, then it is known as a unique solution. The point of intersection of two lines is an example of a unique solution. The three possible solution types for a system of linear equations will be examined in detail later in this chapter. Until that point we will focus on systems with one unique solution.

Number of solutions of a system of linear equations

Systems of linear equations have either zero, one or an infinite number of solutions. The set of all possible solutions is called the solution set.

WORKED EXAMPLE 1 Identifying linear and non-linear equations

Identify which of the following equations are non-linear:

a. $-3x - 2y + 6z = 1$
b. $3x - 4yz = 12$
c. $7x + 9y + z = -2$
d. $12x^2 + y - z = 14$

THINK	WRITE
a. This equation contains variables raised to the first power.	**a.** $-3x - 2y + 6z = 1$ is linear.
b. This equation contains a term that is product of two variables, yz.	**b.** $3x - 4yz = 12$ is non-linear.
c. This equation contains variables raised to the first power.	**c.** $7x + 9y + z = -2$ is linear.
d. This equation contains a variable, x, that is not raised to the first power.	**d.** $12x^2 + y - z = 14$ is non-linear

WORKED EXAMPLE 2 Identifying a square system and verifying its solution

Given the following systems of linear equations:

a. $-4x + y - 6z = -7$
$x + 5y - 8z = 13$

b. $x + 3y - 4z = -21$
$9z - 12z + 2z = 41$
$-6x + y - 2z = -16$

i. **State the number of m equations and n variables in the system.**
ii. **Justify whether the system is square or not.**
iii. **Determine if $x = 1,\ y = -2,\ z = 4$ is a valid solution for all equations in the system.**

THINK	WRITE
a. i. The system contains two equations, each containing three variables.	**a.** m equations $= 2$ n variables $= 3$
ii. Square systems exist if $m = n$.	$m \neq n$, therefore it is not a square system.
iii. Assess the validity of the solution $x = 1$, $y = -2,\ z = 4$ by substituting the corresponding values for x, y and z into each equation. The solution satisfies each equation if the LHS equals the RHS.	$-4x + y - 6z = -7$ $\text{LHS} = -4 \times 1 + -2 - 6 \times 4$ $= -4 - 2 - 24$ $= -30$ $\neq \text{RHS}$ $x + 5y - 8z = 13$ $\text{LHS} = 1 + 5 \times -2 - 8 \times 4$ $= 1 - 10 - 32$ $= -41$ $\neq \text{RHS}$
iv. Conclude with a statement about the validity of the solution.	$(1, -2,\ 4)$ is not a valid solution as it doesn't satisfy all equations in the system.

b. **i.** The system contains three equations, each containing three variables.	**b.** m equations $= 3$ n variables $= 3$
ii. Square systems exist if $m = n$.	$m = n$, therefore it can be categorised as a square system.
iii. Assess the validity of the solution $x = 1$, $y = -2$, $z = 4$. Substitute the corresponding values for x, y and z into each equation. The solutions satisfies each equations if the LHS equals the RHS.	$x + 3y - 4z = -21$ $\text{LHS} = 1 + 3 \times -2 - 4 \times 4$ $= 1 - 6 - 16$ $= -21$ $= \text{RHS}$ $9x - 12y + 2z = 41$ $\text{LHS} = 9 \times 1 - 12 \times -2 + 2 \times 4$ $= 9 + 24 + 8$ $= 41$ $= \text{RHS}$ $-6x + y - 2z = -16$ $\text{LHS} = 6 \times 1 + -2 - 2 \times 4$ $= 6 - 2 - 8$ $= 16$ $= \text{RHS}$
iv. Conclude with a statement about the validity of the solution.	$(1, -2, 4)$ is a valid solution as it satisfies all equations in the system.

5.2.2 Matrix operations using technology

Most of the basic matrix operations can be completed using technology. It is important that you are competent in this area when answering technology active questions. The aim of the worked examples that follow is to build your proficiency in the use of technology.

WORKED EXAMPLE 3 Using technology to calculate matrix operations

Refer to the matrices below and use technology to calculate the following matrix operations.

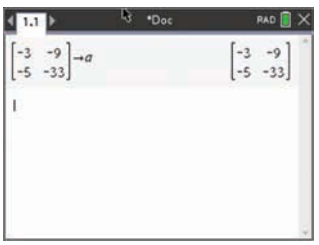
$$A = \begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}$$

a. $2A - B$ **b.** A^4 **c.** $(A \times B)^2$

TI | THINK

a. 1. On a Calculator page, select

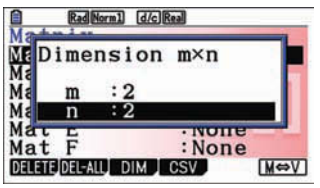
WRITE



CASIO | THINK

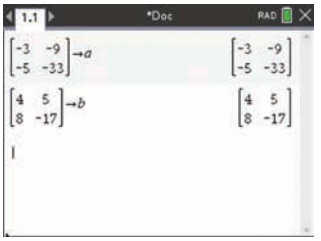
a. 1. On a RUN-MATRIX screen, press MAT/VCT by pressing F3 to bring up the Matrix menu.

WRITE

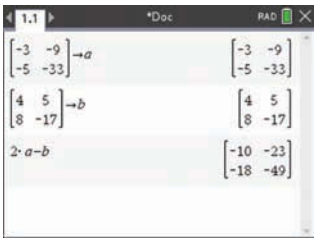


2. Complete the entry for Matrix A and then press CTRL+sto + A to store as A.

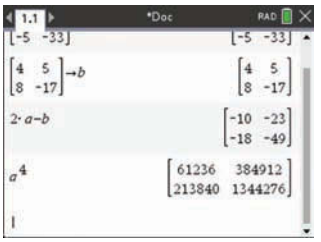
3. Repeat for Matrix B.



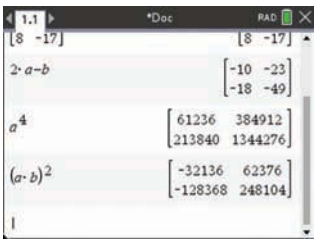
4. Complete the entry line:
$2A - B$
The answer appears on screen.



b. Complete the entry line:
A^4

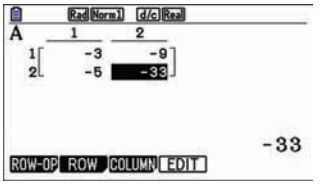


c. Complete the entry line:
$(A \times B)^2$
The answer appears on screen.

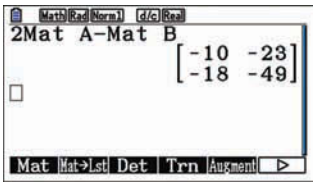


2. Enter dimensions 2×2 and enter the values in Matrix A.

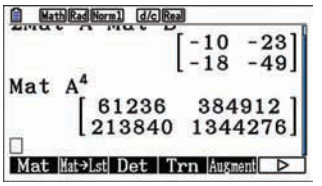
3. Press EXIT and go back and repeat for Matrix B.



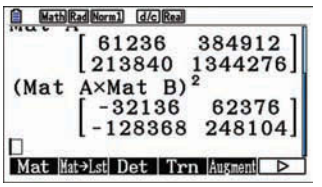
4. Press EXIT twice to return to the RUN-MATRIX screen. Complete the entry line:
2Mat A − Mat B
and press EXE.
Note: Mat can be found by pressing OPTN then F2 for MAT/VCT then 1 for MAT.
The answer appears on screen.



5. Complete the entry line:
Mat A^4
and press EXE.



b. 1. Complete the entry line:
(Mat $A \times$ Mat $B)^2$
and press EXE.
The answer appears on screen.



5.2.3 The determinant

Recall from Unit 1 that every square matrix has an associated number called the determinant. Determinants first arose as part of the process to solve systems of linear equations.

For every square matrix, the determinant, can be calculated.

For the square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant can then be used to calculate the inverse of a matrix.

Note that the notation $|A|$ is sometimes used for $\det(A)$.

The determinant of higher order matrices can be calculated using numerical approaches. For a 3×3 matrix the determinant is found by using principles involving minors and cofactors, which isn't studied in this course. Technology can be used to calculate the determinant of any order matrix in this course.

WORKED EXAMPLE 4 Calculating the determinant with and without technology.

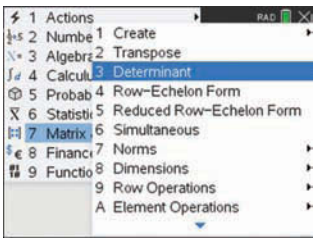
a. **Apply algebraic methods to calculate det (A) if $A = \begin{bmatrix} 6 & -2 \\ 9 & 3 \end{bmatrix}$.**

b. **Use technology to calculate the determinant of $B = \begin{bmatrix} -2 & 5 & 0 \\ 3 & -4 & 1 \\ 5 & 0 & 2 \end{bmatrix}$.**

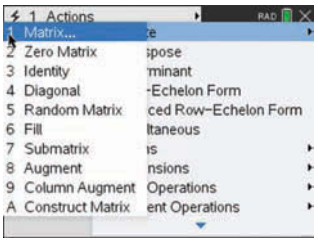
THINK	WRITE
a. State the rule for the determinant of a 2×2 matrix.	**a.** $\det(A) = ad - bc$
Substitute values for a, b, c and d to calculate the determinant.	$\det(A) = 6 \times 3 - (-2) \times 9$ $= 18 + 18$ $= 36$
b. Use the graphics calculator to define matrix B and calculate the determinant.	**b.** $\det(B) = 11$

TI | THINK — **WRITE**

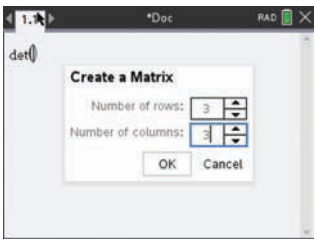
b. 1. On Calculator screen, press MENU:
7 Matrix & Vector
3 Determinant



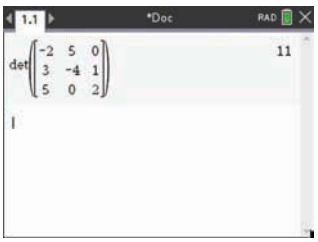
2. Press MENU:
7 Matrix & Vector
1 Create
1 Matrix…



3. Enter:
Number of rows: 3
Number of columns: 3

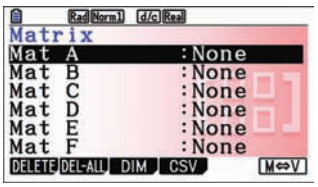


4. Enter the values into the matrix and press ENTER.

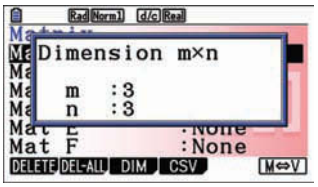


CASIO | THINK — **WRITE**

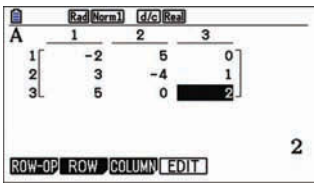
b. 1. On a RUN MATRIX screen, press F3 for MAT/VCT.



2. Press EXE on Mat A and enter the dimensions of the matrix:
m: 3
n: 3 and Press EXE.



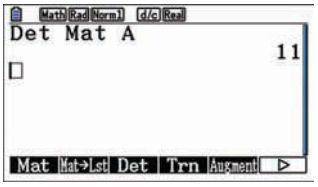
3. Enter the values into the matrix and press EXE.



4. Return the RUN MATRIX screen, Press OPTN select MAT/VCT by pressing F2, then Det by pressing F3.

5. The answer appears on screen. The determinant is 11.

5. Enter Det Mat A and then press EXE.



6. The answer appears on screen. The determinant is 11.

5.2.4 Multiplicative inverse of square matrices

For any given square matrix A, the determinant is used to calculate the multiplicative inverse, A^{-1}. When A is multiplied by its inverse, the product is the identity matrix I. That is $A \times A^{-1} = A^{-1} \times A = I$, where for a 2×2 matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Inverse of a 2×2 matrix

For the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse, A^{-1} is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For matrices of a higher order than a 2×2, the inverse will need to be calculated using technology.

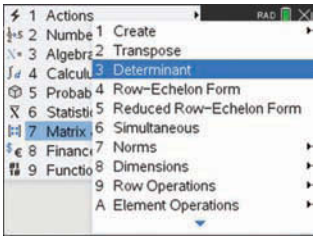
WORKED EXAMPLE 5 Calculating the inverse of a 2×2 matrix

Calculate the multiplicative inverse of matrix $A = \begin{bmatrix} -4 & 2 \\ -8 & 6 \end{bmatrix}$.

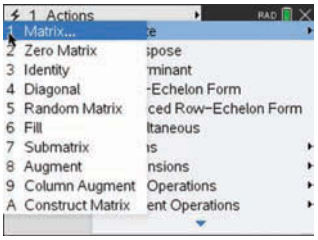
THINK	WRITE
1. Use the rule $\det(A) = ad - bc$ to calculate the determinant.	$\det(A) = ad - bc$ $= -4 \times 6 - 2 \times -8$ $= -24 + 16$ $= -8$
2. State the formula for A^{-1} and substitute values of a, b, c and d.	$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $= \frac{1}{-8} \begin{bmatrix} 6 & -2 \\ 8 & -4 \end{bmatrix}$ $= \begin{bmatrix} -\frac{6}{8} & \frac{-2}{-8} \\ -\frac{8}{8} & \frac{-4}{-8} \end{bmatrix}$ $= \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} \\ -1 & \frac{1}{2} \end{bmatrix}$
3. Alternatively, this can be detemined using technology.	

TI | THINK | **WRITE**

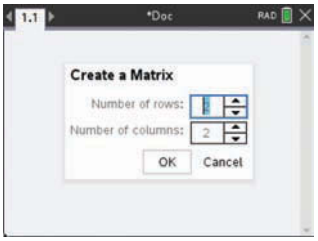
b. 1. On Calculator screen, press MENU:
7 Matrix & Vector
3 Determinant



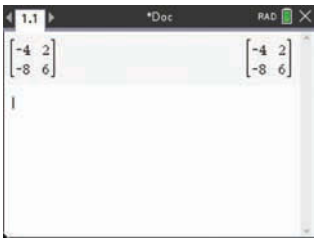
2. Press MENU:
7 Matrix & Vector
1 Create
1 Matrix…



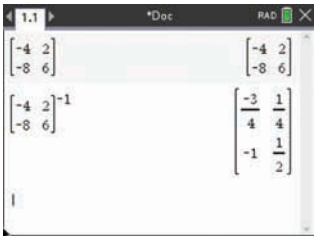
3. Enter:
Number of rows: 2
Number of columns: 2



4. Enter the values into the matrix and press ENTER.



5. Press Ans and raise to the power of −1 (inverse)

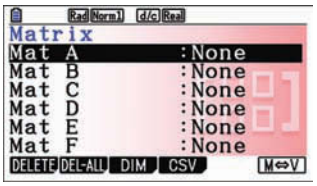


6. The answer appears on screen.

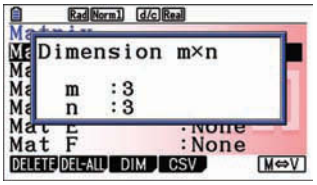
$$A^{-1} = \begin{bmatrix} \frac{-3}{4} & \frac{1}{4} \\ -1 & \frac{1}{2} \end{bmatrix}$$

CASIO | THINK | **WRITE**

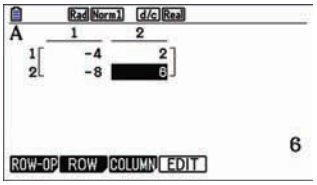
b. 1. On a RUN MATRIX screen, press F3 for MAT/VCT.



2. Press EXE on Mat A and enter the dimensions of the matrix:
m: 3
n: 3
and Press EXE.

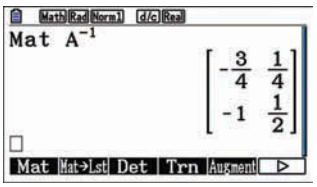


3. Enter the values into the matrix and press EXE.



4. Return the RUN MATRIX screen, Press OPTN select MAT/VCT by pressing F2, then Mat by pressing F1.

5. Enter Mat A^{-1} and then press EXE.



6. The answer appears on screen.

$$A^{-1} = \begin{bmatrix} \frac{-3}{4} & \frac{1}{4} \\ -1 & \frac{1}{2} \end{bmatrix}$$

Singular inverse matrix

Remember that an inverse does not exist for a singular matrix because the determinant equals zero. A non-singular matrix is often referred to as an invertible matrix because an inverse does exist.

WORKED EXAMPLE 6 Using the determinant to determine if square matrices are singular or non-singular.

a. Determine if $A = \begin{bmatrix} -4 & 5 \\ 3 & 7 \end{bmatrix}$ is a singular matrix.

b. Given matrix $B = \begin{bmatrix} -4 & 8 \\ x & 6 \end{bmatrix}$ is singular, calculate the matrix element x.

THINK	WRITE
a. 1. State the rule for the determinant of a 2×2 matrix.	**a.** $\det(A) = ad - bc$
2. Substitute values for a, b, c and d to calculate the determinant.	$\det(A) = -4 \times 7 - 5 \times 3$ $= -28 - 15$ $= -43$
3. The determinant of a singular matrix equals zero. Make a concluding statement.	$\det(A) \neq 0$ therefore A is a non-singular matrix.
b. 1. The determinant of a singular matrix equals zero. Let the determinant rule equal zero.	**b.** $\det(B) = ad - bc$ $= 0$
2. Substitute values for a, b, c and d and then solve for x.	$0 = -4 \times 6 - 8 \times x$ $0 = -24 - 8x$ $\frac{-24}{-8} = x$ $x = 3$
3. State the solution.	$x = 3$

5.2.5 Solving systems of linear equations using inverse matrices

In this section, the solution of a system of linear equations will be calculated by means of matrix algebra and inverses. Consider the system of equations $ax + by = e$ and $cx + dy = f$. The coefficients, variables and scalar quantities can be translated into matrix form.

Coefficient matrix is	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
Unknown/variable matrix (column vector) is	$\begin{bmatrix} x \\ y \end{bmatrix}$
Value/scalar matrix (column vector) is	$\begin{bmatrix} e \\ f \end{bmatrix}$

If we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$, the system of equations $ax + by = e$ and $cx + dy = f$ can be written in the form of a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$A \times X = B$$

To solve matrix equations for X in this form, inverse matrices must be used. Recall that matrix division is not possible. Begin by pre-multiplying both sides by A^{-1}, remembering that the order is important.

$$A^{-1}AX = A^{-1}B$$

Given that $A^{-1}A = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the matrix equation has a unique solution in the form,

$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Solution of matrix equation $AX = B$

If A is a non-singular matrix, in a system of m linear equations with n variables, the unique solution is given by $X = A^{-1}B$.

Inverse matrices are not limited to 2×2 systems. For any square system, the inverse matrix can be found using technology. Once the inverse matrix has been calculated, the general formula $X = A^{-1}B$ can then be applied to calculate the solution.

WORKED EXAMPLE 7 Solving a matrix equation in the form $AX = B$ with technology

Solve the following system of linear equations by using an inverse matrix that has been calculated using technology.

$$\begin{aligned} 3x - 5y + z &= 25 \\ -2x - 7y - 10z &= -23 \\ -6x + 2y - 9z &= -54 \end{aligned}$$

THINK	WRITE
1. Convert the system of linear equations to coefficient (A), variable (X) and value (B) matrices.	$A = \begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$
2. Express matrices A, B and X in the form of a matrix equation.	$AX = B$ $\begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$
3. Multiply both sides from the left by A^{-1}. A general matrix solution for X now exists.	Assuming A is invertible, $A^{-1}AX = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$

4. Complete the matrix operations using appropriate technology.

$$X = A^{-1}B$$

$$X = \begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}^{-1} \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

5. Apply the rules for the equality of matrices to then state the solution.

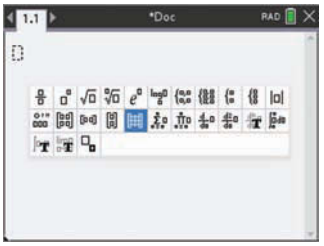
$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$\therefore$ The solution is $x = 2$, $y = -3$ and $z = 4$.

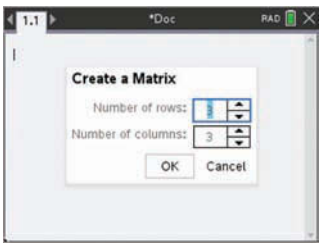
$(x, y, z) = (2, -3, 4)$

TI | THINK **WRITE**

1. On a Calculator page, select

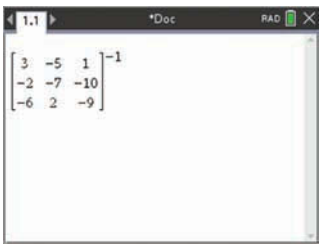


2. Complete the entry line as
3
3
Press the OK button.



3. Complete the entry line as

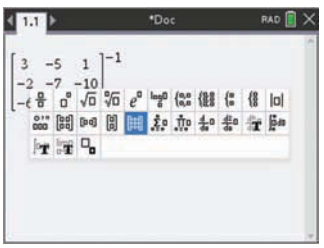
$$\begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}^{-1}$$



4. Press the

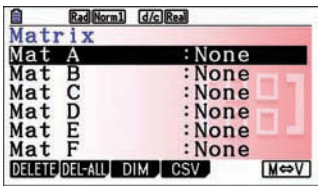
button, followed by the

button.

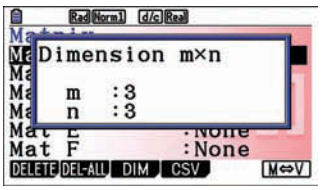


CASIO | THINK **WRITE**

1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.



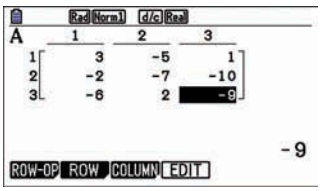
2. Press the EXE button. Complete the entry line as
3
3
Press the EXE button.



3. Complete the entry line as

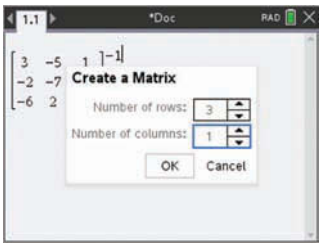
$$\begin{matrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{matrix}$$

Press the EXIT button when finished.

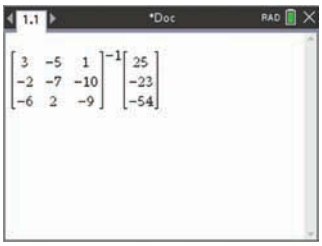


4. Highlight the Mat B row and press the EXE button.
Complete the entry line as
3
1
Press the EXE button

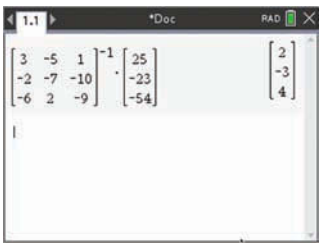
5. Complete the entry line as
3
1
Press the OK button.



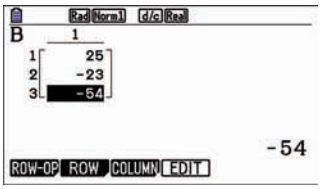
6. Complete the entry line as $\begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$



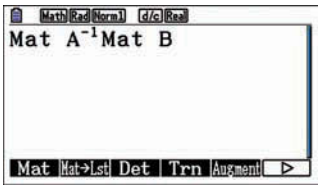
7. Press the ENTER button.
The answer appears on the screen.
Note: This can also be calculated using a row-reduction method. This is shown in a later example.



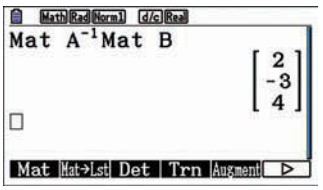
5. Complete the entry line as
25
−23
−54
Press the EXIT button twice when finished.



6. On a Run-Matrix screen, select
OPTN,
MAT/VCT
Mat
ALPHA
$X.\theta, T$
^
−1
×
Mat
ALPHA
Log



7. Press the EXE button.
The answer appears on the screen.
Note: This can also be calculated using a row-reduction method. This is shown in a later example.



In this course, solving matrix equations that involve matrices beyond a 2×2 dimension can be done using technology; however, it is possible to also solve these by hand.

WORKED EXAMPLE 8 Solving a matrix equation in the form $AX = B$ without technology (Optional)

Solve the following system of linear equations by using an inverse matrix.

$$4x + y = 12$$
$$3x + 2y = 14$$

THINK	WRITE
1. Convert the system of linear equations to coefficient (A), variable (X) and value (B) matrices.	$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$
2. Express matrices A, B and X in the form of a matrix equation.	$AX = B$ $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$
3. Multiply both sides from the left by A^{-1} to create a general matrix solution for X.	Assuming A is invertible, $A^{-1}AX = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$

4. Determine the inverse matrix, A^{-1}, using algebraic techniques.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{4 \times 2 - 1 \times 3}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

5. Apply the rules for matrix multiplication. The product is a 2×1 matrix. Apply the rules for scalar multiplication.

$$X = A^{-1}B$$

$$X = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} 12 \\ 14 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 24 + -14 \\ -36 + 56 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

6. Apply the rules for the equality of matrices to then state the solution.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$\therefore$ The solution is $x = 2,\ y = 4\ (x,\ y) = (2,\ 4)$

5.2.6 Matrix algebra

When solving matrix equations up until this point, both sides of the equation were multiplied by A^{-1} to yield the solution $X = A^{-1}B$. This process is an example of how a matrix equation can be solved through the application of matrix algebra. **Matrix algebra** shares many similarities with the algebra used to solve elementary linear equations, however not all.

For instance, the algebraic properties of a matrix equation and a simple linear equation are equivalent in the following examples.

1. The use of an additive inverse to solve a linear system.

Matrix algebra	Algebra
$X + \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix}$ $X + \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix}$ $X = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$	$x + 2 = 3$ $x + 2 - 2 = 3 - 2$ $x = 1$

2. The use of a scalar multiple to a solve a linear system.

Matrix algebra	Algebra
$3X = \begin{bmatrix} 12 & -6 \\ 9 & -3 \end{bmatrix}$ $\frac{1}{3} \times 3X = \frac{1}{3} \times \begin{bmatrix} 12 & -6 \\ 9 & -3 \end{bmatrix}$ $X = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$	$3x = 12$ $\frac{1}{3} \times 3x = \frac{1}{3} \times 12$ $x = 4$

In contrast, the method for factorising is not consistent in both systems of algebra. Consider three matrices of equal dimension C, D and X.

The matrix expression, $XC + XD$, can be factorised as it follows the left distributive law of matrix algebra.

$$XC + XD = X(C + D)$$

A common misconception occurs when attempting to factorise a matrix expression such as $XC + DX$.

$$XC + DX \neq X(C + D)$$

In this case X is common to both terms, however it is incorrect to assume that $XC = CX$ and $DX = XD$. It is also important to note that the common factor, X, does not appear on the same side of both terms.

If the left distributive law of matrices was applied to $X(C + D)$ the resultant expression would be $XC + XD$.

The application of matrix algebra is vital when solving systems of linear equations. This is especially true when attempting to solve more complex equations such as:

$AX + 2BX - C = D$ or $3A(X + I) = C - D$.

Summary of matrix algebra laws

$\boldsymbol{A + B = B + A}$	**Commutative law for addition**
$\boldsymbol{(A + B) + C = A + (B + C)}$	**Associative law for addition**
$\boldsymbol{A + 0 = A}$	**Additive identity**
$\boldsymbol{A + (-A) = 0}$	**Additive inverse**
$\boldsymbol{(AB)C = A(BC)}$	**Associative law for multiplication**
$\boldsymbol{AI = A = IA}$	**Multiplicative identity**
$\boldsymbol{AA^{-1} = A^{-1}A = I}$	**Multiplicative inverse**
$\boldsymbol{A(B + C) = AB + AC}$	**Left distributive law**
$\boldsymbol{(B + C)A = BA + CA}$	**Right distributive law**

The following worked examples will apply the above laws to solve a range of matrix equations.

WORKED EXAMPLE 9 Applying matrix algebra laws 1

Determine a general solution for the variable matrix X in the following matrix equations.

a. $XA - B = C$

b. $AX + BX = C$

THINK	WRITE
a. 1. Apply the additive inverse law to isolate XA by adding B to both sides of the equation.	**a.** $XA - B = C$ $XA - B + B = C + B$ $XA = C + B$
2. Apply the multiplicative inverse law by multiplying both sides by A^{-1} to the right.	Assuming A is invertible and $C + B$ exists, $XAA^{-1} = (C + B)A^{-1}$
3. Simplify $AA^{-1} = I$	$XI = (C + B)A^{-1}$
4. State the general solution for X.	$\therefore X = (C + B)A^{-1}$
b. 1. The right distributive law can be applied to factorise the LHS as X appears on the right of both terms.	**b.** $AX + BX = C$ $(A + B)X = C$
2. Apply the multiplicative inverse law by multiplying both sides by $(A + B)^{-1}$ to the left.	Assuming $(A + B)$ is non-singular and $A + B$ exists, $(A + B)^{-1}(A + B)X = (A + B)^{-1}C$
3. Simplify $(A + B)(A + B)^{-1} = I$	$IX = (A + B)^{-1}C$
4. State the general solution for X.	$\therefore X = (A + B)^{-1}C$

WORKED EXAMPLE 10 Applying matrix algebra laws 2

Use $A = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$ to determine a unique solution for each of the following linear systems.

a. $B + XA = C$

b. $AX - BX = C$

THINK	WRITE
a. 1. Apply the additive inverse law to isolate XA by adding $-B$ to both sides of the equation.	**a.** $B + XA = C$ $B + (-B) + XA = C + (-B)$
2. Simplify both sides.	$XA = C - B$
3. Apply the multiplicative inverse law by multiplying both sides by A^{-1} to the right.	Assuming A is non-singular. $XAA^{-1} = (C - B)A^{-1}$ $XI = (C - B)A^{-1}$
4. A general solution for X has been determined.	$X = (C - B)A^{-1}$

5. From here you can use technology to solve. Alternatively, recall the inverse matrix formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and apply it to determine $A^{-1} = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}^{-1}$.

$$A^{-1} = \frac{1}{2\times 3 - -6\times -4}\begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-18}\begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ -\frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

6. Substitute for the given matrices C, B and A^{-1} into $X = (C-B)A^{-1}$ and then simplify the RHS by applying the rule for subtraction of matrices to simplify $(C-B)$.

$$X = \left(\begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}\right) \times \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ -\frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

7. Apply the rules for matrix multiplication.

$$X = \begin{bmatrix} 4 & -10 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ -\frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

8. State the solution.

$$X = \begin{bmatrix} \frac{14}{9} & \frac{-2}{9} \\ \frac{5}{18} & \frac{8}{9} \end{bmatrix}$$

b. 1. The LHS can be factorised because X appears on the right of both terms.

b. $AX - BX = C$

$(A-B)X = C$

2. Apply the multiplicative inverse law by multiplying both sides by $(A-B)^{-1}$ to the left.

Assuming $(A-B)$ is invertible.

$(A-B)^{-1}(A-B)X = (A-B)^{-1}C$

3. Simplify $(A-B)^{-1}(A-B) = I$ by applying the multiplicative inverse law.

$IX = (A-B)^{-1}C$

$X = (A-B)^{-1}C$

4. From here you can use technology to solve. Alternatively, simplify $(A-B)$ by applying the matrix rule for subtraction.

$$A - B = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -9 \\ -9 & -1 \end{bmatrix}$$

5. Determine $(A-B)^{-1}$ by applying the rule for the inverse of a 2×2 matrix $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$(A-B)^{-1} = \frac{1}{3\times -1 - -9\times -9}\begin{bmatrix} -1 & 9 \\ 9 & 3 \end{bmatrix}$$

$$= \frac{1}{-84}\begin{bmatrix} -1 & 9 \\ 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{84} & -\frac{3}{28} \\ -\frac{3}{28} & -\frac{1}{28} \end{bmatrix}$$

6. Substitute for the given matrices into $X = (C - B)A^{-1}$.
Continue by applying the rules for matrix multiplication.

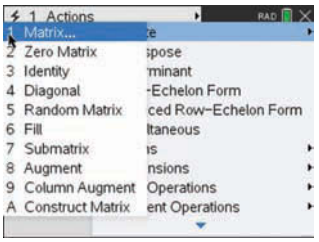
$$X = (A - B)^{-1}C$$
$$= \begin{bmatrix} \frac{1}{84} & -\frac{3}{28} \\ -\frac{3}{28} & -\frac{1}{28} \end{bmatrix} \times \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$$

7. State the solution.

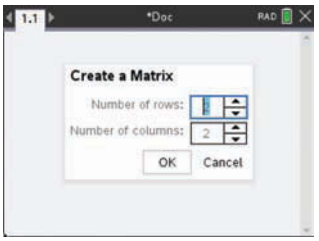
$$X = \begin{bmatrix} -\frac{5}{28} & -\frac{13}{21} \\ -\frac{11}{28} & \frac{4}{7} \end{bmatrix}$$

TI | THINK **WRITE**

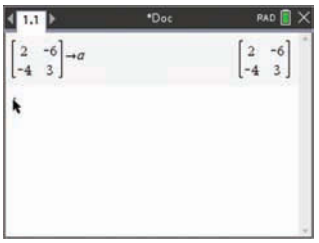
a. 1. On a Calculator screen, Press MENU:
7 Matrix & Vector
1 Create
1 Matrix…



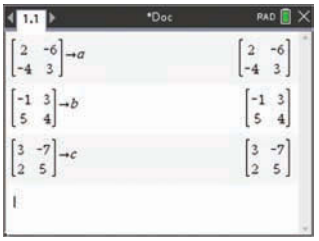
2. Enter:
Number of rows: 2
Number of columns: 2



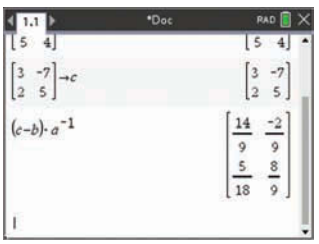
3. Enter the values into the matrix and press CTRL+VAR to store as a.



4. Repeat steps 1–3 to store
Matrices B and C.

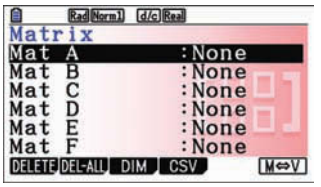


5. Enter the equation shown and press ENTER.
The answer appears on screen.

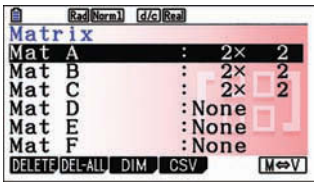


CASIO | THINK **WRITE**

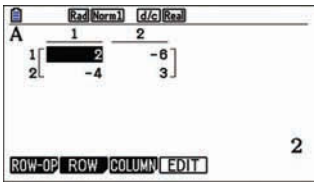
a. 1. On a RUN MATRIX screen, press F3 for MAT/VCT.



2. Press EXE on Mat A and enter the dimensions of the matrix:
m: 2
n: 2
and Press EXE.

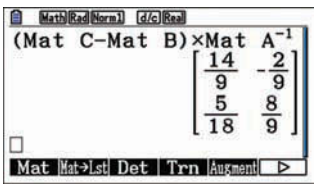


3. Enter the values into the matrix and press EXE.

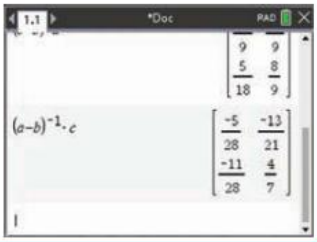


4. Repeat the process for Matrices B and C.

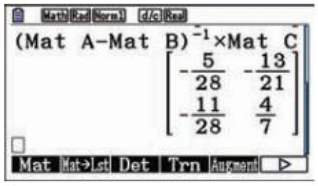
5. Return the RUN MATRIX screen, Enter the equation as shown by pressing OPTN select MAT/VCT by pressing F2, then Mat by pressing F1.
The answer appears on screen



b. 1. Enter the equation shown and press ENTER.
The answer appears on screen.



b. 1. Enter the equation as shown then press EXE.
The answer appears on screen.



Exercise 5.2 Solving linear equations using matrix algebra

learnon

5.2 Exercise | **5.2 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16	17, 18, 19, 20	21, 22, 23, 24, 25

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. WE1 Identify which of the following equations are non-linear.

a. $4x - 5\sin(y) + z = 1$
b. $8x + 9y + 11z = 98$
c. $-8x + y = 7$
d. $4x^2 - 8yz = 12$

2. WE2 Given the following systems of linear equations.

a.
$$\begin{aligned} 6x + 3y - 5z &= 23 \\ -2x + y + 3z &= -5 \end{aligned}$$

b.
$$\begin{aligned} x + 3y - 4z &= 23 \\ 9x - 12y + 2z &= -62 \\ -6x + y - 2z &= 23 \end{aligned}$$

i. State the number of m equations and n variables in the system.
ii. Justify whether the system is square or not.
iii. Determine if $x = -2,\ y = 3,\ z = -4$ is a valid solution set for all equations in the system.

3. WE3 If $A = \begin{bmatrix} 6 & -2 \\ 1 & 7 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 \\ 2 & -1 \end{bmatrix}, C = \begin{bmatrix} -10 \\ -2 \end{bmatrix}, D = \begin{bmatrix} 3 & -2 & 1 \\ 0 & -4 & 3 \\ 5 & -1 & 2 \end{bmatrix}, E = \begin{bmatrix} 10 \\ 9 \\ 24 \end{bmatrix}$, determine:

a. A^3
b. $A^2 \times \left(B^{-1}\right)^3$
c. $2D^3$
d. X if, $\begin{bmatrix} 122 & 121 & 120 \\ 119 & 118 & 117 \\ 116 & 115 & 114 \end{bmatrix} = D^2 - 5X$

4. Consider the two matrices $A = \begin{bmatrix} -8 & 2 \\ -3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 \\ 4 & -9 \end{bmatrix}$ to calculate the following.

a. $(A - B)(A + B)$
b. $(A + B)^2$

5. **WE4a** Apply algebraic methods to calculate the determinant of the following matrices.

a. $A = \begin{bmatrix} 9 & 2 \\ 4 & 6 \end{bmatrix}$ b. $B = \begin{bmatrix} -2 & 0 \\ 8 & 10 \end{bmatrix}$ c. $C = \begin{bmatrix} -4 & -5 \\ 2 & 1 \end{bmatrix}$ d. $D = \begin{bmatrix} 7 & 2 \\ -3 & 1 \end{bmatrix}$

6. **WE4b** Use technology to calculate the determinant for each matrix.

a. $A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 3 & 4 \\ 1 & 1 & -2 \end{bmatrix}$ b. $B = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -1 & 3 \\ 2 & 5 & 1 \end{bmatrix}$

c. $C = \begin{bmatrix} 1 & 0 & 4 & -3 \\ 2 & 6 & -8 & 1 \\ -5 & 7 & 1 & 0 \\ -3 & -1 & 4 & 6 \end{bmatrix}$ d. $D = \begin{bmatrix} -6 & -1 & 5 & -2 \\ 3 & 0 & 7 & -4 \\ 8 & 1 & 1 & 3 \\ -2 & -5 & 4 & 9 \end{bmatrix}$

7. **WE5** Apply algebraic methods to calculate the multiplicative inverse of the following matrices.

a. $A = \begin{bmatrix} -2 & -4 \\ 1 & 6 \end{bmatrix}$ b. $B = \begin{bmatrix} 1 & 1 \\ -4 & -12 \end{bmatrix}$ c. $C = \begin{bmatrix} 10 & -4 \\ -3 & 2 \end{bmatrix}$ d. $D = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$

8. **WE6a** Determine whether the following matrices are singular.

$A = \begin{bmatrix} -1 & -2 \\ 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -11 & 7 \end{bmatrix}$, $C = \begin{bmatrix} -4 & -16 \\ 1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 12 \\ 2 & -8 \end{bmatrix}$

9. **WE6b** Calculate the unknown value x in the following singular matrices.

a. $A = \begin{bmatrix} 10 & x \\ 8 & 4 \end{bmatrix}$ b. $B = \begin{bmatrix} x & 21 \\ -2 & 7 \end{bmatrix}$

10. **WE8** Solve the following system of linear equations by using an inverse matrix.

a. $5x + 2y = 26$
$6x + 3y = 30$

b. $-4x + 9y = 15$
$x - 3y = -6$

c. $5x - 4y = 14$
$9x + 10y = 94$

d. $-7x + 8y = 92$
$6x - 3y = -48$

11. **WE7** Solve the following system of linear equations by using an inverse matrix that has been calculated using technology.

a. $2x + 2y - 8z = 38$
$-5x - y - 7z = 40$
$8x - 7y - 5z = 17$

b. $x + 7y - 4z = 34$
$-7x - 7y - 9z = -129$
$-7x - 5z = -60$

c. $-x + 8y - 4z = -38$
$-x - 5y + 4z = 17$
$x + 3y - 5z = -13$

d. $3x + 3y + 2z = -31$
$-5x - 4y - z = 42$
$3x + 3y - 4z = -19$

12. Use technology to determine the unknown matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in the following systems of linear equations.

a. $\begin{bmatrix} 4 & -2 & 1 \\ -5 & 17 & 0 \\ 2 & 1 & 11 \end{bmatrix} X = \begin{bmatrix} 20 \\ -30 \\ -32 \end{bmatrix}$ b. $\begin{bmatrix} -1 & 13 & 5 \\ 0 & -3 & -4 \\ 7 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 51 \\ 13 \\ 55 \end{bmatrix}$ c. $\begin{bmatrix} 4 & 2 & -8 \\ 2 & 4 & -3 \\ 3 & -9 & 13 \end{bmatrix} X = \begin{bmatrix} 12 \\ -1 \\ 11 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 6 & 5 \\ -8 & -2 & 4 \\ 9 & 7 & 0 \end{bmatrix} X = \begin{bmatrix} 105 \\ -40 \\ 133 \end{bmatrix}$ e. $\begin{bmatrix} -9 & 5 & 7 \\ 12 & -3 & 0 \\ 6 & -2 & 1 \end{bmatrix} X = \begin{bmatrix} -44 \\ -12 \\ -10 \end{bmatrix}$ f. $\begin{bmatrix} 5 & 5 & 2 \\ -1 & -2 & -3 \\ 3 & -6 & -9 \end{bmatrix} X = \begin{bmatrix} -15 \\ 5 \\ 9 \end{bmatrix}$

13. Solve the following system of linear equations using the inverse matrix, computed with technology.

a. $\begin{bmatrix} 2 & 3 & 6 & 0 \\ -1 & 2 & 5 & 12 \\ 0 & -5 & 4 & 2 \\ 8 & 1 & 1 & 7 \end{bmatrix} X = \begin{bmatrix} 21 \\ 62 \\ -27 \\ 35 \end{bmatrix}$

b. $\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 8 & 1 & 5 \\ -3 & -2 & -5 & 10 \\ 1 & 2 & 2 & 0 \end{bmatrix} X = \begin{bmatrix} 27 \\ 61 \\ 80 \\ 6 \end{bmatrix}$

c. $\begin{bmatrix} 8 & 10 & -7 & 1 \\ 0 & 3 & -7 & 2 \\ -13 & 1 & 27 & 16 \\ -5 & -3 & 10 & 19 \end{bmatrix} X = \begin{bmatrix} 228 \\ 54 \\ -432 \\ -268 \end{bmatrix}$

d. $\begin{bmatrix} 5 & 4 & 21 & 9 \\ -8 & -5 & -3 & 1 \\ 0 & 6 & 14 & 22 \\ -16 & 23 & -17 & 1 \end{bmatrix} X = \begin{bmatrix} 24 \\ -11 \\ -58 \\ -117 \end{bmatrix}$

14. MC Identify which of the following matrix operations exhibit the right distributive law.

A. $A + O = A$
B. $P(Q + R) = PQ + PR$
C. $(T + U)V = TV + UV$
D. $C^{-1}C = CC^{-1} = I$

15. MC Determine which of the following statements is false. It is assumed that all the matrices are square and of the same dimension.

A. $AI = IA$
B. $CX - XD = X(C - D)$
C. $TX - XT = -XT + TX$
D. $XM + XN = X(M + N)$

16. WE9 Determine a general solution for the variable matrix X for the following matrix equations. (It is assumed that all the matrices are square matrices and non-singular).

a. $AXB = C$
b. $XA = B$
c. $AXB = I$
d. $AX + X = C$

Complex familiar

17. Determine a general solution for the variable matrix X for the following matrix equations. (It is assumed that all the matrices are square matrices and non-singular).

a. $AX + BX + C = D$
b. $2X + 3AX = B$
c. $AX + 3BX = C$
d. $7A(X - I) = B + C$

18. WE10 Use $A = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$, to determine a unique solution for each of the following linear systems.

a. $XA = B$
b. $B + XA = C$
c. $AX - BX = C$
d. $CXD = 2I$

19. Use $A = \begin{bmatrix} 3 & -1 \\ 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -8 \\ -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 7 & 5 \\ -1 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -5 \\ 3 & 4 \end{bmatrix}$, to determine a unique solution for each of the following linear systems.

a. $CX = B$
b. $ACX = 4I$
c. $CX - 5DX = B$
d. $3A(X + I) = C - D$

20. Construct an expression for X from the matrix equation $EX + X + F = G$.

Complex unfamiliar

21. Consider the matrices, $A = \begin{bmatrix} 5 & 1 & 2 \\ -3 & 0 & 8 \\ 7 & 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & -1 \\ 8 & 11 & -4 \\ 0 & 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -354 & -483 & 672 \\ -288 & -250 & 1077 \\ -136 & -126 & 476 \end{bmatrix}$ and $D = \begin{bmatrix} -18 & 41 & 22 \\ -12 & 54 & 19 \\ -19 & 32 & 21 \end{bmatrix}$.

Determine the unique solution to following systems of linear equations.

a. $A^2B = X$
b. $2A - B = X$
c. $AX + BX = D$
d. $(D - X)(AB) - C = 0$

22. The points awarded at a school athletics carnival was determined by tallying the number of team members finishing in either 1st, 2nd, 3rd or 4th place in the events.

Team	1st	2nd	3rd	4th	Total Points
Blue	5	6	2	1	105
Green	3	2	8	2	90
Yellow	8	2	2	2	110
Red	10	2	3	4	141

Determine the points awarded for each of the respective placings using an inverse matrix.

23. Every month a food stall operates at the local market. Four food items are sold, shown in the table below. Create a matrix equation and then use technology to calculate the individual cost per food item.

Number of items sold					
Month	**Lolly bag**	**Pizza**	**Meat Pie**	**Hamburger**	**Total sales**
January	12	61	12	11	\$565
February	17	36	17	13	\$432
March	23	25	23	8	\$361
April	38	51	9	9	\$494

24. Assuming that A is a square invertible matrix, use matrix algebra to justify the mathematical statement below.

$$(A+I)(A-I)=A^2-I$$

25. Solve the following system of nonlinear equations for the unknown angles α and β, where $0 \le \alpha \le \frac{\pi}{2}$ and $0 \le \beta \le \frac{\pi}{2}$.

$$2\sin(\alpha)+3\cos(\beta)=\frac{3\sqrt{2}+2}{2}$$

$$\sin(\alpha)-\cos(\beta)=\frac{1-\sqrt{2}}{2}$$

Fully worked solutions for this topic are available online.

LESSON
5.3 Solving a system of linear equations using Gaussian elimination

SYLLABUS LINKS

- Recognise the general form of a system of linear equations in several variables and use Gaussian techniques of elimination on an augmented matrix to solve a system of linear equations, with and without technology.

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5.3.1 Gaussian elimination

A variety of methods can be used to solve a system of simultaneous equations, including those covered in the previous section. In mathematics, simultaneous equations are often solved using the substitution or elimination method. Both techniques, especially elimination, are foundational for solving matrix equations using the **Gaussian elimination method**.

Consider a set of two simultaneous linear equations in two variables, x and y.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

As we saw earlier, this system can be translated into the following matrices,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ B = \begin{bmatrix} e \\ f \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \end{bmatrix}$$

where A is the coefficient matrix, B is the value matrix and X is the variable or unknown matrix.

We can combine matrices A and B as $\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$ where this is referred to as an **augmented matrix**, denoted by $[A\,|B]$.

Consider the system below.

System	Coefficient matrix	Value matrix	Augmented matrix
$5x - 2y = 8$ $3x + 4y = 10$	$\begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 10 \end{bmatrix}$	$\left[\begin{array}{cc\|c} 5 & -2 & 8 \\ 3 & 4 & 10 \end{array}\right]$

The purpose of Gaussian elimination is to solve a system of linear equations using elementary row operation procedures which correspond to the operations used when solving equations by elimination.

Row operation		Example notation
1.	Multiply a row (equation) by a constant.	If the third row was multiplied by -2 it would be denoted as $-2 \times R_3$
2.	Add and subtract one row (equation) to or from another.	If row 2 was multiplied by 4 and the result is then added to row 3 it would be denoted as $R_3 + 4R_2$
3.	Swap rows (equations).	If we interchange rows one and two it is denoted as $R_1 \leftrightarrow R_2$

The objective of these procedures is to produce a matrix with ones in the **leading diagonal** and zeros below it, known as row-echelon form. The following augmented matrices are in **row-echelon form**:

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & 10 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 3 \end{array}\right]$$

Once row-echelon form is achieved, the solution can be calculated using an algebraic method called **back-substitution.**

In the second row of the augmented matrix $\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 3 \end{array}\right]$, we immediately determine that $y = 3$. This value is then substituted into the equation represented by the first row, $x + 3y = 7$, which can be solved for x.

The following worked example demonstrates this approach.

WORKED EXAMPLE 11 Solving a system of linear equations using Gaussian elimination

Solve the following system of linear equations using Gaussian elimination and back substitution.

$$\begin{aligned} 4x + 8y &= 24 \\ -3x + y &= 17 \end{aligned}$$

THINK	WRITE
1. Translate the linear equations to matrix form.	$A = \begin{bmatrix} 4 & 8 \\ -3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 24 \\ 17 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$
2. Express A and B in augmented matrix form.	$[A\|B] = \left[\begin{array}{cc|c} 4 & 8 & 24 \\ -3 & 1 & 17 \end{array}\right]$
3. Begin the process of elementary row operations to reduce the augmented matrix to row-echelon form. Divide the first row by four.	$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ -3 & 1 & 17 \end{array}\right]_{\frac{1}{4}R_1}$
4. Continue to produce row-echelon form so $a_{21} = 0$. Add $3R_1$ to R_2.	$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 7 & 35 \end{array}\right] R_2 + 3R_1 \rightarrow R_2$
5. Multiply R_2 by $\frac{1}{7}$ so $a_{22} = 1$. The second row has now reached the desired form.	$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & 5 \end{array}\right]_{\frac{1}{7}R_2}$
6. The second row represents $0x + 1y = 5$. Therefore we can state that solution for y.	$\therefore y = 5.$
7. Rewrite the first row of the augmented matrix in terms of x and y. Substitute $y = 5$ into row one to then solve for x.	Use back-substitution to determine a solution. $x + 2y = 6$ $x + 2(5) = 6$ $x + 10 = 6$ $x + 10 - 10 = 6 - 10$ $x = -4$
8. State the unique solution for the system of linear equations.	The unique solution for the system of linear equations is $(x, y) = (-4, 5)$.

In the previous worked example the coefficient matrix was transformed into row-echelon form. To determine the solution, the process of back substitution was applied.

Another approach to determine a solution using Gaussian elimination is the **reduced row-echelon form**. In this method, the solution set $(x,\ y)$ is attained by continuing elementary row operations until all elements above the leading diagonal are zero. In this form, the augmented matrix will have I on the left-hand side and the solution will be on the right-hand side.

$$[A \mid B] \quad \rightarrow \quad [I \mid X]$$

The solution from the previous worked example was $x = -4,\ y = 5$. The reduced row echelon matrix for this system would be $\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 5 \end{array}\right]$. The following matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 11 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right]$$

$$x = -8,\ y = -3,\ z = 11 \qquad w = -2,\ x = 4, y = -5,\ z = 10$$

WORKED EXAMPLE 12 Using Gaussian elimination and back substitution

Solve the following system of linear equations using Gaussian elimination and back substitution.

$$\begin{aligned} 5x + 2y - 2z &= 2 \\ 2x + 5y + 3z &= -8 \\ x + 3y - 4z &= -11 \end{aligned}$$

THINK	WRITE
1. Translate the linear equations to matrix form.	$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & 3 \\ 1 & 3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -8 \\ -11 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
2. Express A and B in augmented matrix form.	$[A \mid B] = \left[\begin{array}{ccc|c} 5 & 2 & -2 & 2 \\ 2 & 5 & 3 & -8 \\ 1 & 3 & -4 & -11 \end{array}\right]$
3. The coefficient for x in row 3 is 1. Interchange rows 1 and 3.	$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 2 & 5 & 3 & -8 \\ 5 & 2 & -2 & 2 \end{array}\right] R_1 \leftrightarrow R_3$
4. Multiples of row 1 can be added/subtracted to row 2 and 3 so that $a_{21} = 0$ and $a_{31} = 0$.	$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & -1 & 11 & 14 \\ 0 & -13 & 18 & 57 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array}$
5. Subtract $13 \times R_2$ from R_3 so that $a_{32} = 0$.	$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & -1 & 11 & 14 \\ 0 & 0 & -125 & -125 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 13 \times R_2 \end{array}$

6. Complete two operations.
First multiply R_2 by -1 so $a_{22} = 1$.
Follow this by multiplying R_2 by $-\dfrac{1}{125}$ so $a_{33} = 1$.

$$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & 1 & -11 & -14 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ -1 \times R_2 \to R_2 \\ -\frac{1}{125} \times R_3 \to R_3 \end{array}$$

7. State the solution for the third row.

$\therefore z = 1$

8. Rewrite the second row of the augmented matrix in terms of x and y. Substitute in $z = 1$ to then solve for y.

Use back-substitution to determine a solution.

$$\begin{aligned} y - 11z &= -14 \quad \text{(From } R_2\text{)} \\ y - 11(1) &= -14 \\ y - 11 + 11 &= -14 + 11 \\ y &= -3 \end{aligned}$$

9. Rewrite the first row of the augmented matrix in terms of x, y and z. Substitute $y = -3$ and $z = 1$ to solve for x.

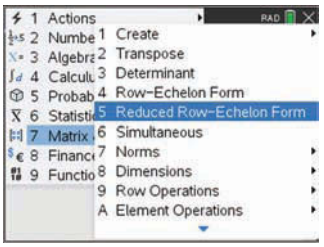
$$\begin{aligned} x + 3y - 4z &= -11 \quad \text{(From } R_1\text{)} \\ x + 3(-3) - 4(1) &= -11 \\ x - 9 - 4 &= -11 \\ x - 13 &= -11 \\ x &= 2 \end{aligned}$$

10. State the unique solution for the system of linear equations.

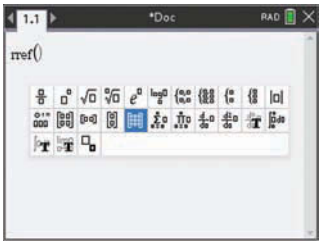
The unique solution for the system of linear equations is $x = 2$, $y = -3$ and $z = 1$.
$\therefore (x, y, z) = (2, -3, 1)$

TI | THINK **WRITE**

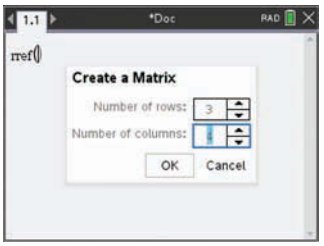
1. On a Calculator page, select
MENU
7: Matrix and Vector
5: Reduced Row-Echelon Form



2. On a calculator screen, select



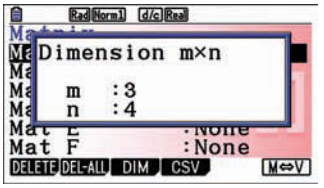
3. Complete the entry line as
3
4
Press the OK button.



CASIO | THINK **WRITE**

1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.

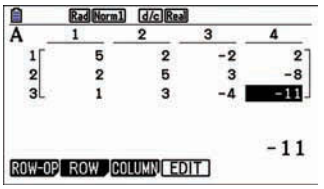
2. Press the EXE button.
Complete the entry line as
3
4
Press the EXE button.



3. Complete the entry line as

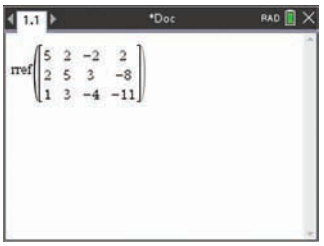
5	2	−2	2
2	5	3	−8
1	3	−4	−11

Press the EXIT button twice when finished.

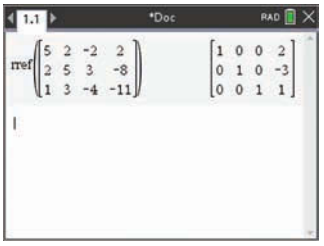


4. Complete the entry line as

5	2	−2	2
2	5	3	−8
1	3	−4	−11



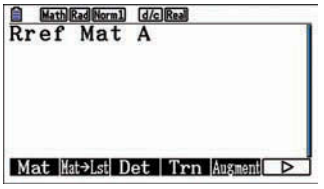
5. Press the ENTER button.



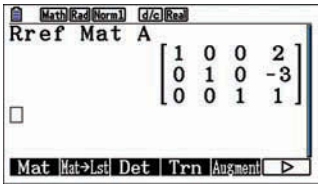
6. The answer appears on the screen, in the fourth column.

$(x, y, z) = (2, -3, 1)$

4. On a Run-Matrix screen, select OPTN, MAT/VCT ▷ Rref ▷ (x3) Mat SHIFT $X.\theta$, T



5. Press the EXE button.



6. The answer appears on the screen, in the fourth column.

$(x, y, z) = (2, -3, 1)$

The row-echelon form and reduced row-echelon form of a matrix give arise to the various types of matrices.

A **triangular matrix** is a type of square matrix in which either all the elements above or all the elements below the leading diagonal are zero. Triangular matrices come in two types:

1. An **upper triangular matrix** is a type of square matrix in which all the elements below the leading diagonal are zero.
2. A **lower triangular matrix** is a type of square matrix in which all the elements above the leading diagonal are zero.

Examples of triangular matrices are shown below:

$$\begin{bmatrix} 1 & 6 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 0 \\ 6 & 9 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$

Upper triangular matrix Lower triangular matrix

Row-echelon form is a type of an upper triangular matrix.

A **diagonal matrix** is a square matrix where all elements off the leading diagonal are zero.

For example,

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Reduced row-echelon form can be classified as a diagonal matrix where the elements on the leading diagonal are all 1s.

In the case of a square matrix, this corresponds to the **identity matrix**.

Exercise 5.3 Solving a system of linear equations using Gaussian elimination

learnon

5.3 Exercise	5.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13	14, 15, 16

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. MC Identify which description is about a matrix in row echelon form.

A. Contains ones in the leading diagonal and ones under the leading diagonal.
B. Contains ones in the leading diagonal and zeros under the leading diagonal. In addition, a row consisting of only zeros may exist as the bottom row of the matrix.
C. Contains ones in the leading diagonal and zeros in every other cell.
D. Contains ones in every cell.

2. MC Identify which of the following matrices is reduced row echelon form.

A. $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & -7 \end{array}\right]$ **B.** $\left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 0 & 5 \end{array}\right]$ **C.** $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 8 \end{array}\right]$ **D.** $\left[\begin{array}{cc|c} 1 & 9 & 4 \\ 0 & 1 & 7 \end{array}\right]$

3. MC The augmented matrix $\left[\begin{array}{ccc|c} 2 & 7 & -4 & 5 \\ 1 & 2 & 3 & 8 \\ 4 & 8 & -12 & -20 \end{array}\right]$ changes to $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 2 & 7 & -4 & 5 \\ 1 & 2 & -3 & -5 \end{array}\right]$. Identify which row operations were completed for that change to occur.

A. $R_2 - R_1,\ R_3 \leftrightarrow R_2$ **B.** $R_2 \leftrightarrow R_1,\ R_3 \times \frac{1}{4}$
C. $R_2 \leftrightarrow R_3,\ R_3 \times 4$ **D.** $R_2 \leftrightarrow R_3,\ R_3 \div 4$

4. WE11 Solve the following system of linear equations using Gaussian elimination and back substitution.

a. $\begin{aligned} 6x - 4y &= 10 \\ -2x + 7y &= 8 \end{aligned}$ **b.** $\begin{aligned} x + 3y &= -2 \\ 3x - 9y &= -24 \end{aligned}$

c. $\begin{aligned} 9x + 8y &= 34 \\ -3x + 2y &= -2 \end{aligned}$ **d.** $\begin{aligned} -5x - 2y &= 19 \\ 4x + y &= -8 \end{aligned}$

5. WE12 Solve the following system of linear equations using Gaussian elimination and back substitution.

a. $\begin{aligned} 6x + 6y + z &= -12 \\ x + 4y - z &= -8 \\ 3x + 3y - 5z &= -6 \end{aligned}$ **b.** $\begin{aligned} -2x + 6y &= -20 \\ -x - 6y - 3z &= 26 \\ -3x - 4z &= 6 \\ 5x + 3y - z &= -23 \end{aligned}$ **c.** $\begin{aligned} 6x - 6y + 2z &= -8 \\ -5x + 7y - 2z &= 16 \\ -6x + 7y - 5z &= 7 \end{aligned}$

d. $\begin{aligned} -4x + 2y + 2z &= 38 \\ 4x - y + 2z &= -17 \\ -2x + y + 5z &= 35 \end{aligned}$ **e.** $\begin{aligned} -x + 4y + 6z &= -15 \\ -2x + 4y + 6z &= -12 \end{aligned}$

6. Convert the augmented matrix, $\left[\begin{array}{ccc|c} 5 & 2 & 4 & 32 \\ -4 & 7 & 8 & -15 \\ -3 & 11 & 1 & 9 \end{array}\right]$, into a system of linear equations.

7. Solve the following set of simultaneous equations, containing four variables, using Gaussian elimination and back substitution.

$$\begin{aligned} 2w + 8x - 4y + 2z &= -46 \\ -10w + x - 8y - 6z &= -25 \\ 4w - 5x + 2y - 5z &= 56 \\ -8w - 16x + 4y + 3z &= 33 \end{aligned}$$

8. The row echelon matrix of a system of linear equations is $\left[\begin{array}{ccc|c} 1 & 4 & -2 & 30 \\ 0 & 1 & 3 & 110 \\ 0 & 0 & 1 & 30 \end{array}\right]$.

Assuming the independent variables are x, y and z, use back substitution to calculate the unique solution for the system.

9. One group of students purchase 4 regular coffees and 3 large coffees for \$33. A second group purchase 2 regular and 5 large coffees for \$34. Use Gaussian elimination to calculate the cost of both coffee sizes.

10. The cost of a movie ticket varies according to the classification of student, adult or pensioner. The table below summarises the cost for three different families to attend the cinema.

Number of students	Number of adults	Number of pensioners	Total cost (\$)
6	2	1	82
4	1	2	58
2	2	2	70

Use Gaussian elimination to determine the total cost for a family containing two students, one adult and two pensioners.

11. Solve the following system of linear equations using Gaussian elimination, continuing until reduced row-echelon form is obtained.

a. $\begin{aligned} 7x - 2y &= 8 \\ x + 5y &= 17 \end{aligned}$

b. $\begin{aligned} 6x + 5y &= 3 \\ -3x + 2y &= 12 \end{aligned}$

c. $\begin{aligned} 12x - 3y &= 78 \\ 4x + 7y &= 10 \end{aligned}$

12. Solve the following system of linear equations using Gaussian elimination, continuing until reduced row-echelon form is obtained.

a. $\begin{aligned} 4x - 2y + 7z &= 32 \\ x + 8y + z &= 5 \\ 3x - y + 5z &= 23 \end{aligned}$

b. $\begin{aligned} 5x + 3y + 4z &= 28 \\ 8x + y - 6z &= 33 \\ 9x + 4y - 2z &= 43 \end{aligned}$

c. $\begin{aligned} -7x - 4y + 2z &= 9 \\ 2x + 9y + 3z &= -19 \\ -3x + y + 5z &= 4 \end{aligned}$

Complex familiar

13. The initial stages in finding a solution to a system of linear equations was attempted below. Identify the location of the five errors in the solution below.

$$\left[\begin{array}{rrr|r} 1 & -5 & 1 & -6 \\ 3 & 1 & -2 & 4 \\ -5 & 4 & -1 & -4 \end{array}\right]$$

$$\begin{aligned} x-5y+z&=-6 \\ 3x+y-2z&=4 \\ -5x+4z-y&=-4 \end{aligned} \quad \left[\begin{array}{rrr|r} 1 & -5 & 1 & -6 \\ 0 & 16 & -5 & 22 \\ 0 & -21 & 4 & -34 \end{array}\right] \quad \begin{aligned} &R_2-3R_1 \to R_2 \\ &R_3+5R_1 \to R_3 \end{aligned}$$

$$\left[\begin{array}{rrr|r} 1 & -5 & 1 & -6 \\ 0 & 1 & -\frac{5}{16} & \frac{11}{8} \\ 0 & 0 & -1 & 44 \end{array}\right] \quad \begin{aligned} &R_2 \div 16 \to R_2 \\ &5R_3-21R_1 \to R_3 \end{aligned}$$

Complex unfamiliar

14. A quadratic in the form, $y(x)=a_0+a_1x+a_2x^2$, contains the points $A(2,1)$, $B(3,7)$ and $C(4,17)$. If n points from a curve are known it is possible to develop a polynomial to the $n-1$ degree through a process known as interpolating a polynomial. Points A, B and C can be translated into an augmented matrix given by,

$$\left[\begin{array}{rrr|r} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \end{array}\right] \quad \text{where} \quad \begin{aligned} A(2,1)&=(x_1,y_1) \\ B(3,7)&=(x_2,y_2) \\ C(4,17)&=(x_3,y_3) \end{aligned}$$

The values for a_0, a_1 and a_2 are determined when the augmented matrix has reached reduced row-echelon form.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & a_0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 1 & a_2 \end{array}\right]$$

Develop an augmented matrix and perform elementary row operations to determine the values for a_0, a_1 and a_2.

15. In the previous questions we used augmented matrices in the form $[A|B]$ to find solutions. This method can be used to determine the inverse of matrix A. To achieve this, matrix A is augmented with the identity matrix I, $[A|\,I]$. The inverse is found by completing elementary row reductions until you produce $[I|A^{-1}]$. Follow this method to determine the inverse of matrix A where:

$$A=\begin{bmatrix} 3 & 12 & -6 \\ 2 & 4 & -6 \\ 1 & 3 & -4 \end{bmatrix}$$

16. Apply gaussian elimination to solve the system of non-linear equations for x, y and z.

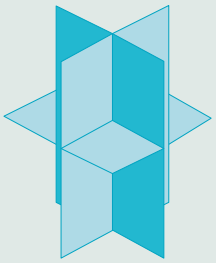
$$\left(\begin{array}{rrr|r} 2x^2 & -3y^2 & z^2 & 2 \\ 3x^2 & y^2 & z^2 & 29 \\ x^2 & -2y^2 & 2z^2 & 3 \end{array}\right)$$

Fully worked solutions for this chapter are available online.

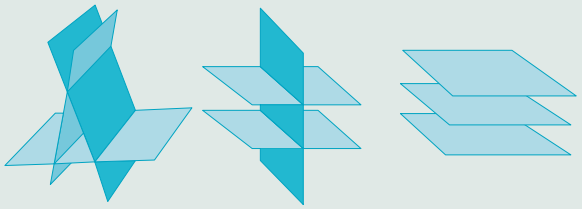
LESSON
5.4 The three cases for solutions of systems of linear equations

SYLLABUS LINKS

- Examine the three cases for solutions of systems of equations — a unique solution, no solution and infinitely many solutions — and the geometric interpretation of a solution of a system of equations with three variables including
  - a unique solution



  - no solution



  - infinitely many solutions

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The study of linear equations until this point has been focussed on the numerous methods that can be used to solve a system. You may have observed that all systems were common in that each had exactly one solution. That is, each system had a **unique solution**. However, the solution set is not limited to only one solution type.

Systems of linear equations have either:
- A unique (one) solution
- No solution
- Infinitely many solutions

A system containing at least one solution is defined as **consistent**. An **inconsistent** system has no solution.

5.4.1 Geometric interpretation of solutions of lines in two dimensions

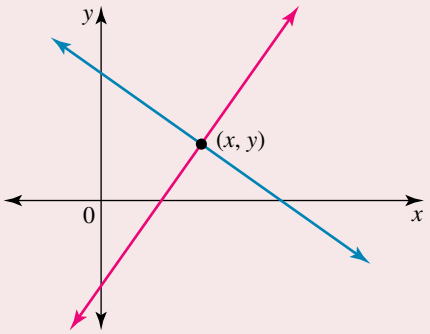
Recall the general form of a system in two dimensions. These systems are given by

$$a_1x + b_1y = c_1 \quad [1]$$
$$a_2x + b_2y = c_2 \quad [2]$$

Previously, systems in this form were represented as the matrix equation, $AX = B$. The unknown matrix X was solved using various techniques, with the most functional method being the inverse matrix approach. This was possible if A was non-singular ($\Delta \neq 0$). The coefficient matrix, A, will be the point of reference when investigating the three possible solution types below.

A unique solution (non-parallel, distinct lines)

- **A is *non-singular* (det $A \neq 0$)**
- **The system is *consistent*.**
- **The two lines are *not parallel*.**
- **The two lines intersect at a one point that represents a *unique solution*.**

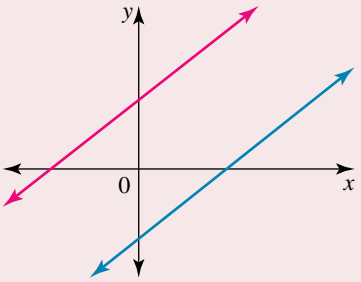


Properties of the coefficients

- $\dfrac{a_2}{a_1} \neq \dfrac{b_2}{b_1}$
- **Using technology, for the system** $\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}$**, that is for** $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$**, use** $\text{rref}\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\right)$ **to obtain** $\begin{bmatrix} 1 & 0 & s_1 \\ 0 & 1 & s_2 \end{bmatrix}$ **where $x = s_1$ and $y = s_2$ is the unique solution of the system.**

  For example, for the system $\begin{matrix} 2x + 7y = 1 \\ 4x + 8y = 5 \end{matrix}$**, using** $\text{rref}\left(\begin{bmatrix} 2 & 7 & 1 \\ 4 & 8 & 5 \end{bmatrix}\right)$ **gives** $\begin{bmatrix} 1 & 0 & \frac{9}{4} \\ 0 & 1 & \frac{-1}{2} \end{bmatrix}$**, where** $x = \dfrac{9}{4}$ **and** $y = -\dfrac{1}{2}$ **is the unique solution.**

No solution (parallel but distinct lines)



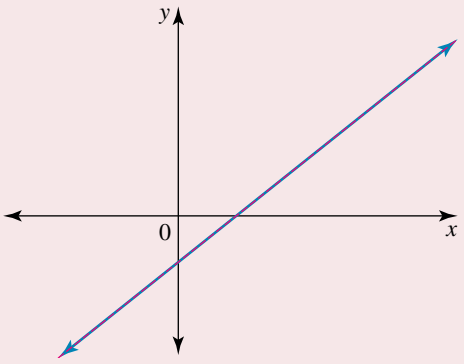
- A is *singular* (det $A = 0$)
- The system is *inconsistent.*
- The two lines are *parallel* and *distinct* (the lines will have different y-intercepts).
- The two lines do not intersect and there is *no solution*.

Properties of the coefficients

- $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} \neq \dfrac{c_2}{c_1}$
- Using rref $\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\right)$ will give $\begin{bmatrix} 1 & \dfrac{b_1}{a_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, indicating that there is no solution to the system.

  For example, for the system $\begin{matrix} 2x + 7y = 1 \\ 4x + 14y = 10 \end{matrix}$, using rref $\left(\begin{bmatrix} 2 & 7 & 1 \\ 4 & 14 & 10 \end{bmatrix}\right)$ gives $\begin{bmatrix} 1 & \dfrac{7}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, meaning the system does not have any solution.

Infinitely many solutions (parallel and coincident lines)



- A is singular (det $A = 0$)
- The system is *consistent*.
- The two lines are *parallel* and *not distinct* (the lines will have the same y-intercept).
- The two lines are *coincident* meaning they overlap and there is an *infinite number of solutions*.

Properties of the coefficients

- $\dfrac{a_2}{a_1} = \dfrac{b_2}{b_1} = \dfrac{c_2}{c_1}$
- Using rref $\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}\right)$ will give $\begin{bmatrix} 1 & \dfrac{b_1}{a_1} & \dfrac{c_1}{a_1} \\ 0 & 0 & 0 \end{bmatrix}$, indicating that $x + \dfrac{b_1}{a_1}y = \dfrac{c_1}{a_1}$ is the solution to the system (that is, there is infinitely many solutions).

  For example, for the system $\begin{matrix} 2x + 7y = 1 \\ 4x + 14y = 2 \end{matrix}$ using rref $\left(\begin{bmatrix} 2 & 7 & 1 \\ 4 & 14 & 2 \end{bmatrix}\right)$ gives $\begin{bmatrix} 1 & \dfrac{7}{2} & \dfrac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$, meaning that $x + \dfrac{7}{2}y = \dfrac{1}{2}$ is the equation of the line that is solution to the system.

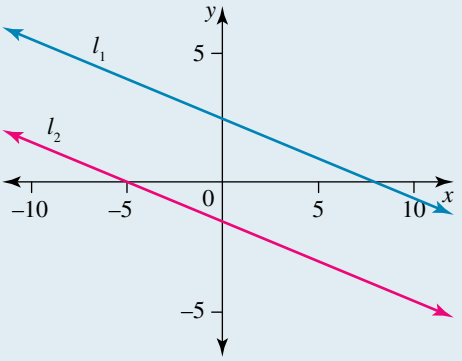
WORKED EXAMPLE 13 Solving a system of linear equations and interpreting the solution geometrically

Solve the following system of linear equations by using matrices and interpret the solution geometrically.

$$\begin{aligned} 2x + 7y &= 16 \\ -4x - 14y &= 20 \end{aligned}$$

THINK	WRITE
1. Define the matrices A, X and B.	$A = \begin{bmatrix} 2 & 7 \\ -4 & -14 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$
2. Translate into a matrix equation.	$\begin{bmatrix} 2 & 7 \\ -4 & -14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$
3. Write as a general equation and solve for X.	$AX = B$ $A^{-1}AX = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$
4. Calculate the determinant using $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ to determine if A is singular.	$\det(A) = 2 \times -14 - 7 \times -4 = 0$
5. As $\det(A) = 0$, matrix A is singular, hence A^{-1} does not exist.	$\det(A) = 0$, so A is singular. The system either has no solution or infinitely many solutions.
6. Rearrange both equations into the form $y = mx + c$.	$2x + 7y = 16$ $7y = 16 - 2x$ $\rightarrow y_1 = -\frac{2}{7}x + \frac{16}{7}$ $-4x - 14y = 20$ $14y = -4x - 20$ $\rightarrow y_2 = -\frac{2}{7}x - \frac{10}{7}$
7. The lines are parallel because the gradients are equal. The two lines are distinct because the y-intercepts are different.	Line [1]: $m_1 = \frac{-2}{7}$ y_1-intercept $= \frac{16}{7}$ Line [2]: $m_2 = -\frac{2}{7}$ y_2-intercept $= -\frac{10}{7}$ The lines are parallel, yet distinct as they have different y-intercepts.

8. Sketch the graphs. Note that the two lines are parallel with distinct y-intercepts, which means they will never intersect and hence have no solution.



9. State the answer.

There is no solution as the lines are parallel and distinct.

WORKED EXAMPLE 14 Solving a system of linear equations and describing the solution in parametric form

a. Solve the following system of linear equations by using matrices and interpret the solution geometrically.

b. Describe the solution set in parametric form.

$$3x - 4y = -2$$
$$9x - 12y = -6$$

THINK	WRITE
a. 1. Define the matrices A, X and B.	a. $A = \begin{bmatrix} 3 & -4 \\ 9 & -12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$
2. Translate the matrices into a matrix equation.	$\begin{bmatrix} 3 & -4 \\ 9 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$
3. Write as a general equation and solve for X.	$AX = B$ $A^{-1}AX = A^{-1}B$ $IX = A^{-1}B$ $X = A^{-1}B$
4. Recall the determinant rule $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and calculate $\lvert A \rvert$ to determine if A is singular.	$\det(A) = 3 \times -12 - -4 \times 9 = 0$
5. As $\det(A) = 0$, matrix A is singular, hence A^{-1} does not exist.	$\det(A) = 0$, so A is singular. The system either has no solution or infinitely many solutions.

6. Rearrange both equations into the form $y = mx + c$.

$$3x - 4y = -2$$
$$4y = 3x + 2$$
$$y_1 = \frac{3}{4}x + \frac{1}{2}$$
$$9x - 12y = -6$$
$$12y = 9x + 6$$
$$y_2 = \frac{3}{4}x + \frac{1}{2}$$

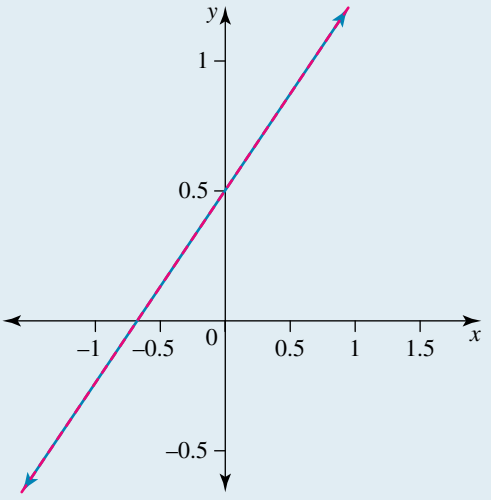
7. The lines are parallel because the gradients are equal.
The two lines are coincident because the y-intercepts are the same.

Line [1]: $m_1 = \frac{3}{4}$, y_1-int $= \frac{1}{2}$

Line [2]: $m_2 = \frac{3}{4}$, y_2-int $= \frac{1}{2}$

The two lines are coincident because they are identical.

8. Sketch the graphs. Note that the two lines overlap, meaning there are an infinite number of solutions.



b. 1. The two lines are identical. Rewrite either equation in parametric form. Begin by making x the subject of the equation.

b. $3x - 4y = -2$
$$3x = 4y - 2$$
$$x = \frac{4y - 2}{3}$$

2. Express y in terms of t.

Let $y = t$ so that $x = \frac{4t - 2}{3}$

As a coordinate: $\left(\frac{4t - 2}{3}, t\right)$

3. State the answer.

There are an infinite number of solutions of the form $(x, y) = \left(\frac{4t - 2}{3}, t\right)$.

5.4.2 Geometric interpretation of solutions of lines in three dimensions

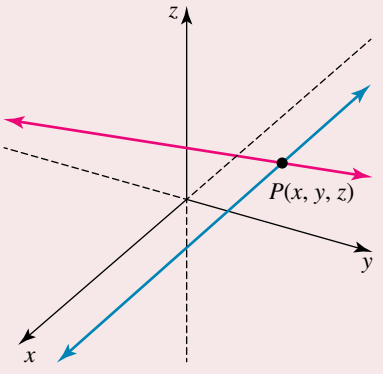
The vector equation of a line in three dimensions was studied in Chapter 2. Recall that a line can be expressed in three different forms.

Vector equation form	Parametric form	Cartesian form
$\underset{\sim}{r} = \left(-5\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}\right) + k\left(3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - 9\hat{\boldsymbol{k}}\right)$ $\underset{\sim}{a} = -5\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}$ $\underset{\sim}{d} = 3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - 9\hat{\boldsymbol{k}}$ where $k \in \mathbb{R}$	$x = -5 + 3k$ $y = 2 - 4k$ $z = 7 - 9k$ $(a_1, a_2, a_3) = (-5, 2, 7)$ $(d_1, d_2, d_3) = (3, -4, -9)$	$\frac{x+5}{3} = \frac{y-2}{-4} = \frac{z-7}{-9}$ $(a_1, a_2, a_3) = (-5, 2, 7)$ $(d_1, d_2, d_3) = (3, -4, -9)$

When examining two lines (l_1 and l_2) in three dimensions, three solution types are possible.

Consistent system — one unique solution

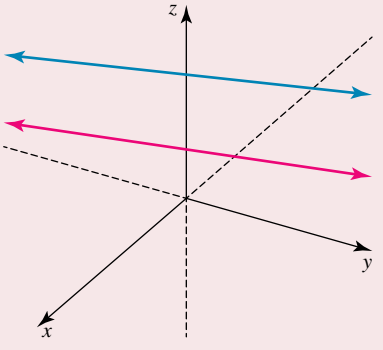
- **The system is consistent in three-dimensional space.**
- **The two lines are not parallel, meaning their direction vectors are not scalar multiples of each other.**
- **The two lines intersect at a point $P(x, y, z)$ that represents a *unique solution*.**



Inconsistent system — two cases

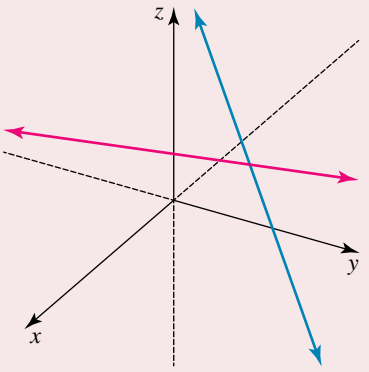
Case 1: Parallel but distinct lines

- **The system is inconsistent in a three-dimensional space.**
- **The two lines are parallel, meaning their direction vectors are scalar multiples of each other.**
- **However, they are distinct, meaning they do not share any common points, so no solution exists.**



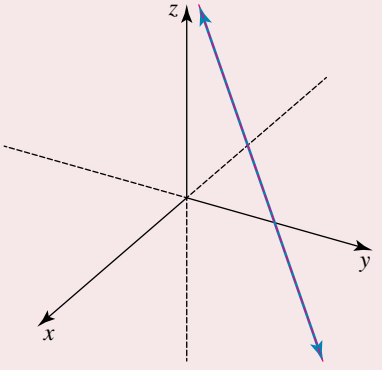
Case 2: Not parallel but distinct lines

- **The system is inconsistent in three-dimensional space.**
- **The two lines are not parallel, meaning their direction vectors are not scalar multiples of each other.**
- **However, they are distinct and do not intersect, making them skew lines, so no solution exists.**



Consistent system — infinitely many solutions (parallel and coincident lines)

- **The system is consistent in three-dimensional space.**
- **The two lines are parallel, meaning their direction vectors are scalar multiples of each other.**
- **They are not distinct but coincident, meaning they completely overlap, resulting in an *infinite number of solutions*.**



WORKED EXAMPLE 15 Describing the geometric nature of the solution of a system of linear equations

a. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2.
b. Describe the geometric nature of the solution.

$$l_1: \left(5\hat{\imath} - 3\hat{\jmath} - 7\hat{k}\right) + k\left(-4\hat{\imath} + \hat{\jmath} + 5\hat{k}\right)$$
$$l_2: \left(-5\hat{\imath} + 2\hat{\jmath} - 9\hat{k}\right) + k\left(6\hat{\imath} - 4\hat{\jmath} + 7\hat{k}\right)$$

THINK	WRITE
a. 1. Analyse the direction vectors to determine if the lines are parallel.	**a.** $\underset{\sim}{d}_1 = -4\hat{\imath} + \hat{\jmath} + 5\hat{k}$ $\underset{\sim}{d}_2 = 6\hat{\imath} - 4\hat{\jmath} + 7\hat{k}$ $\underset{\sim}{d}_1 \neq m \times \underset{\sim}{d}_2$, where $m \in \mathbb{R}$
They are not equal, nor is one a scalar multiple of the other.	The two lines are not parallel.

2. Rewrite the equations of both lines in parametric form.	l_1: $x = 5 - 4k$ $y = -3 + k$ $z = -7 + 5k$ l_2: $x = -5 + 6k$ $y = 2 - 4k$ $z = -9 + 7k$
3. Both lines are in terms of the same parameter, k, where $k \in \mathbb{R}$. Equate the x component of l_1 and l_2 to determine a possible value for k.	Solve for k by equating the x component of l_1 and l_2: $5 - 4k = -5 + 6k$ $10 = 10k$ $k = 1$
4. Determine the position on both lines when the parameter is set to $k = 1$ by substituting $k = 1$ into the parametric equations.	Let $k = 1$, for lines, l_1 and l_2: l_1: $x = 5 - 4 \times 1 = 1$ $y = -3 + 1 = -2$ $z = -7 + 5 \times 1 = -2$ $(x_1, y_1, z_1) = (1, -2, -2)$ l_2: $x = -5 + 6 \times 1 = 1$ $y = 2 - 4 \times 1 = -2$ $z = -9 + 7 \times 1 = -2$ $(x_2, y_2, z_2) = (1, -2, -2)$
5. State the solution.	$(x_1, y_1, z_1) = (x_2, y_2, z_2)$. A unique solution exists at $(x, y, z) = (1, -2, -2)$.
b. Describe the geometric nature of the solution.	**b.** The two distinct lines intersect at one unique point, $(1, -2, -2)$.

In the previous worked example, both vector equations are in terms of the parameter, k, where $k \in \mathbb{R}$. When determining a point of intersection, it was simply found by equating the components (x, y, z). In some of the more complex systems you may find that two lines are defined by different parameters, k_1 and k_2.

WORKED EXAMPLE 16 Describing the geometric nature of a system of linear equations

A system of linear equations in three variables contains the lines l_1 and l_2.

a. Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.

b. Describe the geometric nature of the system by referring to any possible solution/s.

$$l_1: \frac{x+6}{2} = \frac{y+2}{5} = \frac{z+4}{-3} = k_1,\ l_2: \frac{x+3}{4} = \frac{y+6}{10} = \frac{z}{-6} = k_2$$

THINK	WRITE
a. 1. Analyse the direction vector to determine if the lines are parallel. Both lines are in Cartesian form, $\frac{x - a_1}{d_1} = \frac{x - a_2}{d_2} = \frac{x - a_3}{d_3} = k$. The direction of the line can be expressed as a vector given by $\underset{\sim}{d}_1 = d_1\hat{\imath} + d_2\hat{\jmath} + d_3\hat{k}$.	**a.** $\underset{\sim}{d}_1 = 2\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$ $\underset{\sim}{d}_2 = 4\hat{\imath} + 10\hat{\jmath} - 6\hat{k}$ $\underset{\sim}{d}_1 = \frac{1}{2} \times \underset{\sim}{d}_2 \rightarrow \therefore \underset{\sim}{d}_1 \parallel \underset{\sim}{d}_2$ The two lines are parallel.

2. As the lines are parallel, they may overlap (coincident) or be distinct. Rewrite the equations of both lines parametric form.

$$l_1: \quad x = -6 + 2k_1$$
$$y = -2 + 5k_1$$
$$z = -4 - 3k_1$$
$$l_2: \quad x = -3 + 4k_2$$
$$y = -6 + 10k_2$$
$$z = -6k_2$$

3. Determine a point on the first line by setting the parameter to any given value. Let $k_1 = 0$.

For l_1, let $k_1 = 0$

$$l_1: \quad x = -6 + 2 \times 0 = -6$$
$$y = -2 + 5 \times 0 = -2$$
$$z = -4 - 3 \times 0 = -4$$

The following point exists on l_1: $(x_1, y_1, z_1) = (-6, -2, -4)$

4. If the lines coincident, then the same point found on l_1 will satisfy the second line, l_2. Verify whether $(-6, -2, -4)$ lies on the second line.

Substitute $(-6, -2, -4)$ into the parametric form of l_2. $x = -3 + 4k_2$

$$-6 = -3 + 4k_2$$
$$k_2 = \frac{-3}{4}$$
$$y = -6 + 10k_2$$
$$-2 = -6 + 10k_2$$
$$k_2 = \frac{2}{5}$$
$$z = -6k_2$$
$$-4 = -6k_2$$
$$k_2 = \frac{2}{3}$$

The values for k_2 are not equal which indicates the point $(-6, -2, -4)$ does not lie on l_2.

5. State the solution.

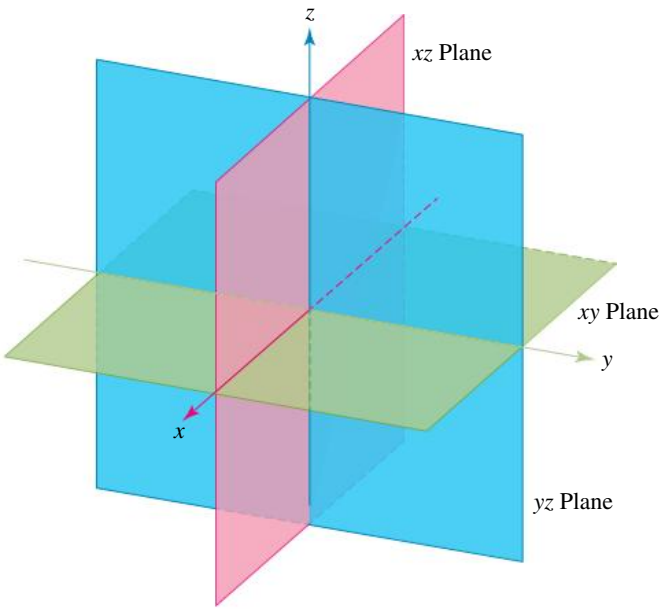
A solution does not exist for the simultaneous system.

b. Describe the geometric nature of the solution.

b. The two lines are parallel and distinct which means they will never intersect.

5.4.3 Geometric interpretation of solutions of planes in three dimensions

Coordinate geometry in three dimensions was introduced in Chapter 2. These systems are defined by three variables to specify the location of a point $P(x, y, z)$. A right-handed system, consisting of three orthogonal planes is shown. Each of these planes could be written in Cartesian form, $ax + by + cx + d = 0$. When considering the equations of these planes simultaneously you are investigating the solution for a 3×3 linear system. In the system shown in the figure, the unique solution exists as the intersection of the three planes at the origin, $P(0, 0, 0)$.



Systems comprising of three planes can have either one, zero or infinitely many solutions.

Review of properties of a plane

To begin this study, it important to refresh your memory on the properties of a plane and the associated terminology.

Equation of a plane

The equation of the plane passing through (a, b, c) and perpendicular to $\underset{\sim}{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$ is given by:

$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$

or

$$(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$$

To determine if two or more planes are parallel you need to compare their respective normal vectors.

$$\underset{\sim}{n} = n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}$$

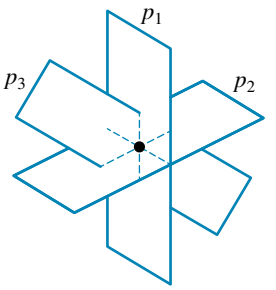
The planes are parallel if the normal vectors are equal or scalar multiples of one another. In other words, $\underset{\sim}{n}_2 = \lambda \underset{\sim}{n}_1$.

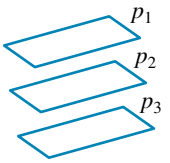
Coplanar objects are those lying in the same plane.

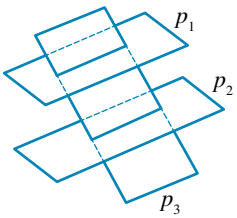
Three vectors are coplanar objects if their scalar triple product, $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3)$ is zero.

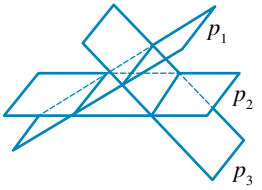
Note, the scalar triple product is simplifying the determinant of a 3×3 matrix which can be calculated using technology.

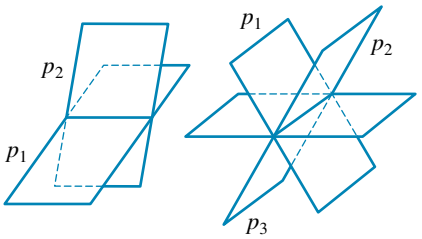
$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Consistent system — one unique solution		
Geometric interpretation $\boldsymbol{AX = B}$ **Not coplanar** **Augmented matrix in triangular form**	Planes intersect at one unique point, $P(x_1, y_1, z_1)$ $\det(A) \neq 0$, A is non-singular (invertible), therefore A^{-1} exists. $\underset{\sim}{\hat{n}}_1 \cdot (\underset{\sim}{\hat{n}}_2 \times \underset{\sim}{\hat{n}}_3) \neq 0$ $\left[\begin{array}{ccc\|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$	

Inconsistent system — zero solutions		
Geometric interpretation $\boldsymbol{AX = B}$ **Coplanar** **Augmented matrix in triangular form**	The normal vector of three planes are parallel yet distinct. $\det(A) = 0$, A is singular (not invertible), therefore A^{-1} does not exist. $\underset{\sim}{\hat{n}}_1 \cdot (\underset{\sim}{\hat{n}}_2 \times \underset{\sim}{\hat{n}}_3) = 0$ $\left[\begin{array}{ccc\|c} 1 & a_{12} & a_{13} & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \end{array}\right]$ The final row is a false statement, so this system has no solution.	

Inconsistent system — zero solutions		
Geometric interpretation $\boldsymbol{AX = B}$ **Coplanar** **Augmented matrix in triangular form**	Two planes are distinct with parallel normals. The third plane is not parallel. $\det(A) = 0$, A is singular (not invertible), therefore A^{-1} does not exist. $\underset{\sim}{\hat{n}}_1 \cdot (\underset{\sim}{\hat{n}}_2 \times \underset{\sim}{\hat{n}}_3) = 0$ $\left[\begin{array}{ccc\|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array}\right]$ The final row is a false statement, so this system has no solution (they intersect as lines in parallel pairs).	

Inconsistent system — zero solutions		
Geometric interpretation $\boldsymbol{AX = B}$ **Coplanar** **Augmented matrix in triangular form**	The normals are not parallel. $\det(A) = 0$, A is singular (not invertible), therefore A^{-1} does not exist. $\underset{\sim}{\hat{n}}_1 \cdot (\underset{\sim}{\hat{n}}_2 \times \underset{\sim}{\hat{n}}_3) = 0$ $\left[\begin{array}{ccc\|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array}\right]$ The final row is a false statement, so this system has no solution. The planes intersect as pairs in a line.	

Consistent system — infinitely many solutions		
Geometric interpretation $\boldsymbol{AX = B}$ **Coplanar** **Augmented matrix in triangular form**	The normals are coplanar and the planes intersect as a line. Infinitely many solutions can be defined parametrically. $\det(A) = 0$, A is singular (not invertible), therefore A^{-1} does not exist. $\underset{\sim}{\hat{n}}_1 \cdot (\underset{\sim}{\hat{n}}_2 \times \underset{\sim}{\hat{n}}_3) = 0$ $\left[\begin{array}{ccc\|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & 0 \end{array}\right]$ The final row is a true statement with the system having infinitely many solutions. The third equation is eliminated, hence the free variable is expressed in terms of t, $z = t$.	
Geometric interpretation $\boldsymbol{AX = B}$ **Coplanar** **Augmented matrix in triangular form**	The three planes are coincident. There are infinitely many solutions. $\det(A) = 0$, A is singular (not invertible), therefore A^{-1} does not exist. $\underset{\sim}{\hat{n}}_1 \cdot (\underset{\sim}{\hat{n}}_2 \times \underset{\sim}{\hat{n}}_3) = 0$ $\left[\begin{array}{ccc\|c} 1 & a_{12} & a_{13} & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ The final two rows are true statements with the system having infinitely many solutions.	

WORKED EXAMPLE 17 Making inference on the type of solutions of systems of linear equations

The following descriptions are the geometric and/or algebraic properties of a system of linear equations in three variables. Make an inference on the type of solution set each system would produce based on the given information.

a. The three planes are not parallel, $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$ and the final row of the triangular matrix is a false statement.

b. The three normals are not parallel. The coefficient matrix in the equation $AX = B$ is non-singular.

c. When solving the system using Gaussian elimination the augmented matrix reaches the triangular form $\left[\begin{array}{ccc|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array}\right]$. The planes intersect as pairs in the form of parallel lines.

THINK	WRITE
a. The three planes are not parallel, hence the system could be consistent or inconsistent. The normals are coplanar therefore it can't be unique. The final row of the triangular matrix is false, so the planes may intersect in pairs of lines.	a. It is an inconsistent system with zero solutions.
b. As the coefficient matrix is invertible, and the planes are not parallel the only solution type is unique.	b. It is a consistent system containing one unique solution.
c. The final row of the augmented triangular matrix is false so the system is inconsistent. As the planes intersect as pairs in the form of parallel lines there is no unique solution.	c. It is an inconsistent system containing no solution.

WORKED EXAMPLE 18 Interpreting geometrically the solution of a system of linear equations 1

Consider the following system of linear equations.

$$\begin{aligned} 3x + 2y + 5z &= 27 \\ x + 5y - 4z &= -2 \\ -4x + 3y - 6z &= -10 \end{aligned}$$

a. Determine the number of possible solutions by examining the normal vectors algebraically.

b. Convert the equations to an augmented matrix and then use technology to solve the system.

c. Describe how the solution can be interpreted geometrically.

THINK	WRITE
a. 1. Determine if the normal vectors are equivalent or a scalar multiple of another normal vector. State if any of the planes are parallel.	a. $\underset{\sim}{n}_1 = 3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$ $\underset{\sim}{n}_2 = \hat{\imath} + 5\hat{\jmath} - 4\hat{k}$ $\underset{\sim}{n}_3 = -4\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$ The normals are not parallel, hence the planes are not parallel.

2. If a unique solution exists, the normals will not be coplanar. Recall the scalar triple product formula. This can be found by calculating the determinant of the 3×3 matrix.
The determinant may be calculated using technology (see Section 5.2.2 Matrix operations using technology).

$$\det\left(\begin{bmatrix} 3 & 2 & 5 \\ 1 & 5 & -4 \\ -4 & 3 & -6 \end{bmatrix}\right)$$

$$\begin{aligned}
&\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 \times \underset{\sim}{n}_3 \\
&= \left(3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}\right) \cdot \left(\hat{\imath} + 5\hat{\jmath} - 4\hat{k}\right) \times \left(-4\hat{\imath} + 3\hat{\jmath} - 6\hat{k}\right) \\
&= \det \begin{bmatrix} 3 & 2 & 5 \\ 1 & 5 & -4 \\ -4 & 3 & -6 \end{bmatrix} \\
&= 3 \det \begin{bmatrix} 5 & -4 \\ 3 & -6 \end{bmatrix} - 2 \det \begin{bmatrix} 1 & -4 \\ -4 & -6 \end{bmatrix} + 5 \det \begin{bmatrix} 1 & 5 \\ -4 & 3 \end{bmatrix} \\
&= 105
\end{aligned}$$

As the determinant is different from zero, the planes are not coplanar and intersect at a unique point.

b. 1. Convert the linear system of equations into augmented matrix form where:
$A =$ Coefficient matrix.
$B =$ Value matrix.

b. $$[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{array}\right]$$

2. The augmented matrix can be reduced into triangular form using technology (see Worked example 12).

$$\text{rref}\left(\begin{bmatrix} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{bmatrix}\right)$$

Note: The same could be achieved using Gaussian elimination (see Worked example 11).

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The three planes intersect at $(x,\ y,\ z) = (-2,\ 4,\ 5)$.

c. State the type of solution and describe it geometrically.

c. The system is consistent and has one unique solution. Geometrically, the three planes intersect at a unique point.

WORKED EXAMPLE 19 Interpreting geometrically the solution of a system of linear equations 2

Consider the system of linear equations to answer the following.

$$\begin{aligned} -x - 5y - 8z &= -22 \\ 6x + 9y + 6z &= 69 \\ -2x - y + 2z &= -17 \end{aligned}$$

a. **Determine the number of possible solutions by examining the normal vectors algebraically.**
b. **Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.**
c. **Solve the system of linear equations, expressing the solution in parametric form.**
d. **Describe how the solution can be interpreted geometrically.**

THINK	WRITE
a. 1. Examine the normal vectors to determine if they equivalent or a scalar multiple of another normal vector. State if any of the planes are parallel.	a. $\underset{\sim}{n}_1 = -\hat{\imath} - 5\hat{\jmath} - 8\hat{k}$ $\underset{\sim}{n}_2 = 6\hat{\imath} + 9\hat{\jmath} + 6\hat{k}$ $\underset{\sim}{n}_3 = -2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ The normal vectors are not parallel, hence the planes are not parallel.
2. If a unique solution exists, the normals will not be coplanar. Recall the scalar triple product formula. This can be found by calculating the determinant of the 3×3 matrix.	$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 \times \underset{\sim}{n}_3$ $= \left(-\hat{\imath} - 5\hat{\jmath} - 8\hat{k}\right) \cdot \left(6\hat{\imath} + 9\hat{\jmath} + 6\hat{k}\right) \times \left(-2\hat{\imath} - \hat{\jmath} + 2\hat{k}\right)$ $= \det \begin{bmatrix} -1 & -5 & -8 \\ 6 & 9 & 6 \\ -2 & -1 & 2 \end{bmatrix}$ $= -1 \det \begin{bmatrix} 9 & 6 \\ -1 & 2 \end{bmatrix} - (-5) \det \begin{bmatrix} 6 & 6 \\ -2 & 2 \end{bmatrix} + -8 \det \begin{bmatrix} 6 & 9 \\ -2 & -1 \end{bmatrix}$ $= 0$
The determinant may be calculated using technology (see Section 5.2.2 Matrix operations using technology). $\det\left(\begin{bmatrix} -1 & -5 & -8 \\ 6 & 9 & 6 \\ -2 & -1 & 2 \end{bmatrix}\right)$	As the determinant is equal to zero the three planes are coplanar and it has already been noted that they are not parallel. Geometrically, this could be the planes intersecting in a line or not at all.
b. 1. Convert the linear system of equations into augmented matrix form $[A\|B]$ where, $A =$ Coefficient matrix. $B =$ Constant matrix.	$[A\|B] = \left[\begin{array}{ccc\|c} -1 & -5 & -8 & -22 \\ 6 & 9 & 6 & 69 \\ -2 & -1 & 2 & -17 \end{array}\right]$

2. The augmented matrix can be reduced into triangular form using technology (see Worked example 12). $\text{rref}\left(\begin{bmatrix} -1 & -5 & -8 & -22 \\ 6 & 9 & 6 & 69 \\ -2 & -1 & 2 & -17 \end{bmatrix}\right)$	$[A\|B] = \left[\begin{array}{ccc\|c} 1 & 0 & -2 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$
c. 1. The third row of the triangular matrix is equivalent to, $0x + 0y + 0z = 0$. This informs us that there are infinitely many solutions.	**c.** The system has infinitely many solutions because the planes intersect in a line.
2. Express the first two rows of the triangular matrix as two equations in terms of x, y and z. Let the free variable $z = t$.	$x - 2z = 7 \quad [1]$ $y + 2z = 3 \quad [2]$ Let $z = t$ as it is the free variable.
3. Rewrite equation [2] by substituting t for z and then make y the subject.	$y + 2t = 3$ $y = 3 - 2t \quad [3]$
4. Rewrite equation [1] by substituting t for z and then make x the subject.	$x - 2t = 7$ $x = 7 + 2t$
5. State the solution in terms of the parameter, t.	The intersection line is given by the parametric equations: $x = 7 + 2t$ $y = 3 - 2t$ $z = t$
d. State the type of solution and describe it geometrically.	**d.** This is a consistent system with the solution representing the intersection of three planes in the form of a line.

Note: The solution could easily be converted to two other forms.

- Vector equation: $\underset{\sim}{r} = \left(7\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}\right) + t\left(2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right)$
- Cartesian form: $\dfrac{x-7}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$

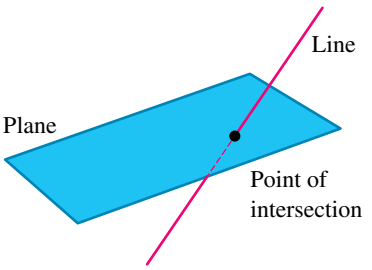
5.4.4 The intersection of lines and planes

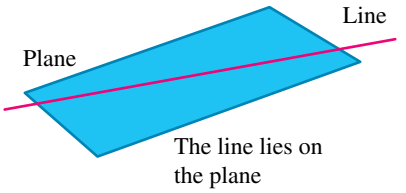
When considering the intersection of a plane and line in three dimensions, there are three possible solution types.

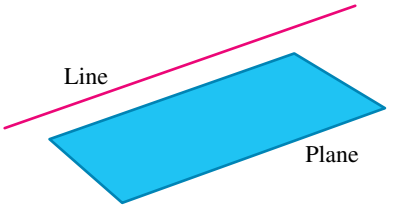
- **One (unique) solution** where the plane and line intersect at a single point $P(x, y, z)$.
- **Infinitely many solutions** in which the line lies on the plane.
- **No solution** as the line and plane are parallel yet distinct.

To examine these solutions geometrically, a vector approach will be applied in the first instance. The plane and line are defined as follows.

- **Line** (l_1): $\underset{\sim}{r} = \underset{\sim}{a} + k\underset{\sim}{d}$ where $\underset{\sim}{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ is the position vector of a known point on the line and $\underset{\sim}{d} = d_1\hat{\boldsymbol{\imath}} + d_2\hat{\boldsymbol{\jmath}} + d_3\hat{\boldsymbol{k}}$ is the direction vector *parallel* to the line.
- **Plane** (p_1): $ax + by + cz + d = 0$, where $\underset{\sim}{n} = a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{k}}$ is *perpendicular* to the plane.

Consistent system — one unique solution		
Examine the direction and normal vectors. **Geometric interpretation**	$\underset{\sim}{d} \cdot \underset{\sim}{n} \neq 0$. Therefore, the line and plane are *not parallel*. The line and plane intersect at a unique point, $P(x,\ y,\ z)$ so *one solution* exists.	

Consistent system — infinitely many solutions		
Examine the direction and normal vectors. **Infinitely many or no solutions?** **Geometric interpretation**	$\underset{\sim}{d} \cdot \underset{\sim}{n} = 0$ Therefore, the line and plane are *parallel*. Determine a point $P(x,\ y,\ z)$ that lies on the line l_1. This point *will satisfy* the equation of the plane p_1. The line lies on the plane and *infinitely many solutions* exist.	

Inconsistent system — zero solutions		
Examine the direction and normal vectors. **Infinitely many or no solutions?** **Geometric interpretation**	$\underset{\sim}{d} \cdot \underset{\sim}{n} = 0$ Therefore, the line and plane are *parallel*. (i.e. the line is above or below the plane) Determine a point $P(x,\ y,\ z)$ that lies on the line l_1. This point *will not satisfy* the equation of the plane p_1. The line is parallel yet distinct to the plane. *No solution* exists.	

WORKED EXAMPLE 20 Determining the intersection of a plane and a line

Consider a three-dimensional system containing the line l_1: $\underset{\sim}{r} = \left(2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}\right) + k\left(-\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$ and plane given by p_1: $2x + 7y - 4z = 61$.

a. Examine the line and plane to identify if they intersect.

b. If they do intersect, determine the solution, otherwise state the system has zero solutions.

THINK	WRITE
a. 1. Examine the normal $\underset{\sim}{n}$ and direction $\underset{\sim}{d}$ vectors. Determine if they are perpendicular by calculating the dot product.	**a.** $\underset{\sim}{n} = 2\hat{\imath} + 7\hat{\jmath} - 4\hat{k},\ \underset{\sim}{d} = -\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ $\underset{\sim}{n} \cdot \underset{\sim}{d} = \left(2\hat{\imath} + 7\hat{\jmath} - 4\hat{k}\right) \cdot \left(-\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$ $= 2 \times -1 + 7 \times 2 + -4 \times -3$ $= 24$ $\underset{\sim}{n} \cdot \underset{\sim}{d} \neq 0$
2. State if the line and plane intersect.	As $\underset{\sim}{n} \cdot \underset{\sim}{d} \neq 0$ the normal and direction vectors are not perpendicular. The plane and line are not parallel, hence a single point of intersection must exist.

b. 1. Convert the vector equation of the line into parametric equations.	**b.** $l_1: \underset{\sim}{r} = \left(2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}\right) + k\left(-\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right)$ $l_1: \begin{cases} x = 2 - k \\ y = 3 + 2k \\ z = -3 - 3k \end{cases}$
2. Substitute the parametric equations into the equation of the plane.	$2(2-k) + 7(3+2k) - 4(-3-3k) = 61$
3. Solve for k.	$2(2-k) + 7(3+2k) - 4(-3-3k) = 61$ $4 - 2k + 21 + 14k + 12 + 12k = 61$ $24k + 37 = 61$ $k = \dfrac{61-37}{24}$ $k = 1$
4. Determine the point of intersection by substituting $k = 1$ for the parametric equations.	$l_1,\ k = 1: \begin{cases} x = 2 - 1 = 1 \\ y = 3 + 2 \times 1 = 5 \\ z = -3 - 3 \times 1 = -6 \end{cases}$
5. State the solution.	The line and plane intersect at a single (unique) point $(1, 5, -6)$.

When determining the intersection of a line and plane the vector approach is not the only method. In Worked example 19, the equation of the line was converted to parametric equations and subsequently substituted into the equation of the plane. A single value for k was found, indicating that a unique solution existed. The following worked example will examine systems containing infinitely many or no solutions.

WORKED EXAMPLE 21 Determining whether a plane and a line intersect

For each of the following systems, calculate if the plane and line intersect. If so, determine how many solutions exist.

a. $l_1: \begin{cases} x = 2 + 4k \\ y = -3 + 2k \\ z = 3 - 2k \end{cases}, \quad p_1: 3x + y + 7z = 24$

b. $l_2: \begin{cases} x = 3 + 5k \\ y = -2 + 3k \\ z = 3 - k \end{cases}, \quad p_2: -2x + 4y + 2z = 1$

THINK	WRITE
a. 1. Substitute the parametric equations into the equation of the plane.	**a.** $3(2+4k) + 1(-3+2k) + 7(3-2k) = 18$
2. Solve for k.	$6 + 12k - 3 + 2k + 21 - 14k = 24$ $24 + 0k = 24$ $0k = 0$
3. Interpret the solution.	For any value of k the equation is true. The line lies on the plane meaning there are infinitely many solutions.

b. 1. Substitute the parametric equations into the equation of the plane.

b. $-2(3+5k)+4(-2+3k)+2(3-k)=1$

2. Solve for k.

$$\begin{aligned}-6-10k-8+12k+6-2k&=1\\0k-8&=1\\0k&=9\end{aligned}$$

3. Interpret the solution.

There are no possible values for k that make this equation true. The line is parallel yet distinct from the plane meaning there are zero solutions.

Exercise 5.4 The three cases for solutions of systems of linear equations

learnon

5.4 Exercise | **5.4 Exam questions** on

simple familiar	complex familiar	complex unfamiliar
1, 2, 3, 4	5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21	22, 23

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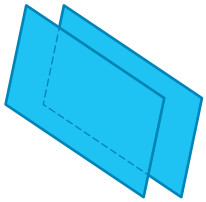
Simple familiar

1. MC Determine the geometric feature that indicates a unique solution exists in a square system of two dimensions.

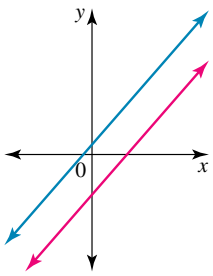
A. Parallel and distinct **B.** Coincidental **C.** Not parallel **D.** Homogenous

2. Refer to the following graphs and state how many solutions exist.

a.



b.



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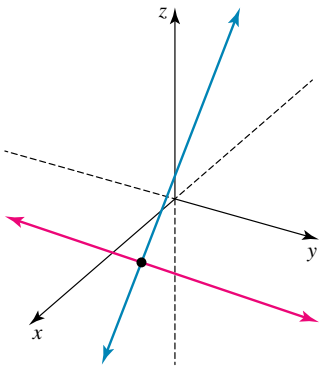
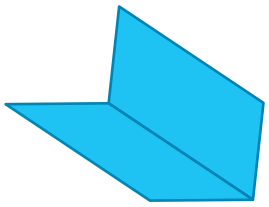


d.



3. WEB13 Solve the following system of linear equations by using matrices and interpret the solution geometrically.

a. $8x + 2y = 12$
$4x - 7y = -10$

b. $-2x + y = -2$
$6x - 3y = 7$

c. $-3x + y = -14$
$-2x - 5y = 2$

d. $4x - 8y = 4$
$6x - 12y = -3$

4. Consider the following system of linear equations to answer the following.

a. $5x + 8y + z = -5$
$-7x - 4y - 5z = -11$
$-3x - y - 6z = 4$

b. $3x + 6y - 2z = -25$
$8x - 4y + 6z = -58$
$-5x - 5y - 3z = 43$

c. $-6x - 6y - z = -64$
$x - y + 2z = 4$
$x + 7y - 7z = 24$

d. $6x + 3y - 24z = 8$
$-3x - y + 6z = -2$
$4x + 8z = 10$

e. $-4x - 5y - 4z = 5$
$4x + 5y + 4z = -7$
$7x - 6y + 7z = 4$

i. Determine the number of possible solutions by examining the normal vectors algebraically.
ii. Convert the equations to an augmented matrix and then use technology to solve the system.
iii. Describe how the solution can be interpreted geometrically.

Complex familiar

5. WE14 a. Solve the following systems of linear equations by using matrices and interpret the solution geometrically.

b. Describe the solution set in parametric form.

i. $10x - 6y = 12$
$5x - 3y = 6$

ii. $x - 2y = -4$
$-4x + 8y = 16$

6. The following systems contain two lines in the form, l_1: $a_1x + b_1y = c_1$ and l_2: $a_2x + b_2y = c_2$. Determine the nature of the solution set by interpreting the values for a_n, b_n and c_n algebraically.

a. $x + 2y = 15$
$3x - y = 10$

b. $3x + 7y = 9$
$6x + 14y = 18$

c. $-16x + 6y = -14$
$-4x + \frac{3}{2}y = -4$

d. $-x + 4y = -3$
$-5x + 20y = -12$

7. In each case below, the number of solutions has been defined for the system. Determine the restrictions on the values for a, b and c that would make the statement on the solution type true.

a. $\left.\begin{array}{r} 5x - 2y = a \\ 10x - 4y = 24 \end{array}\right\}$ No solution

b. $\left.\begin{array}{r} 8x - 2y = -12 \\ -4x + by = c \end{array}\right\}$ Infinitely many solutions

8. WE15 a. Determine the unique solution for the following systems of linear equations defined by the lines l_1 and l_2.

b. Describe the geometric nature of the solution.

i. l_1: $(\hat{\imath} - \hat{\jmath} - 4\hat{k}) + k(-5\hat{\imath} - 4\hat{\jmath} - 5\hat{k})$
l_2: $(-15\hat{\imath} + 7\hat{\jmath} - 40\hat{k}) + k(-\hat{\imath} - 6\hat{\jmath} + 4\hat{k})$

ii. l_1: $(-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) + k(-\hat{\imath} - 4\hat{\jmath} - 4\hat{k})$
l_2: $(-9\hat{\imath} - 5\hat{\jmath} - 9\hat{k}) + k(7\hat{\imath} + 3\hat{\jmath} + 7\hat{k})$

iii. l_1: $(4\hat{\imath} + 7\hat{\jmath} + 3\hat{k}) + k(6\hat{\imath} + \hat{\jmath} + 5\hat{k})$
l_2: $(60\hat{\imath} - 14\hat{\jmath} + 3\hat{k}) + k(-2\hat{\imath} + 4\hat{\jmath} + 5\hat{k})$

9. **a.** Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2.

b. Describe the geometric nature of the solution.

i. $l_1: \begin{cases} x = 3 - 3k \\ y = 3 - 4k \\ z = 4 + 4k \end{cases}$ $l_2: \begin{cases} x = -33 + k \\ y = -15 - 2k \\ z = -14 + 6k \end{cases}$

ii. $l_1: k = \dfrac{x+1}{4} = \dfrac{y-5}{-3} = \dfrac{z+8}{4}$ $l_2: \begin{cases} x = 53 - 5k \\ y = -25 + 2k \\ z = -14 + 5k \end{cases}$

10. **WE16** A system of linear equations in three variables contains the lines l_1 and l_2.

a. Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.

b. Describe the geometric nature of the system by referring to any possible solution/s.

i. $l_1: \dfrac{x+4}{-6} = \dfrac{y-5}{1} = \dfrac{z+4}{8} = k_1,$

$l_2: \dfrac{x+1}{-12} = \dfrac{y-2}{2} = \dfrac{z+7}{16} = k_2$

ii. $l_1: \dfrac{x-4}{-5} = \dfrac{y+4}{-5} = \dfrac{z-1}{6} = k_1,\ l_2: \dfrac{x+18}{-2} = \dfrac{y+17}{7} = \dfrac{z-27}{2} = k_2$

iii. $l_1: \dfrac{x-6}{-8} = \dfrac{y-3}{5} = \dfrac{z-1}{-6} = k_1,\ l_2: \dfrac{x+34}{4} = \dfrac{y-62}{6} = \dfrac{z+13}{7} = k_2$

11. A system of linear equations in three variables contains the lines l_1 and l_2.

a. Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.

b. Describe the geometric nature of the system by referring to any possible solution/s.

i. $l_1: \dfrac{x+6}{-6} = \dfrac{y+4}{-4} = \dfrac{z+7}{-2} = k_1$

$l_2: x = -1 - 30k_2, y = -7 - 20k_2, z = -10 - 10k_2$

ii. $l_1: \begin{cases} x = 2 + k_1 \\ y = 6 - 3k_1 \\ z = 2 + 5k_1 \end{cases}$ $l_2: \begin{cases} x = 1 - 4k_2 \\ y = -9 + 3k_2 \\ z = 39 + k_2 \end{cases}$

12. **WE17** The following descriptions relate to the geometric and/or algebraic properties of a system of linear equations in three variables. Make an inference on the type of solution set each system would produce based on given information.

a. The three planes are coincident.

b. The diagonal augmented matrix is $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$.

c. The three planes are not parallel, $\hat{\underset{\sim}{n}}_1 \cdot (\hat{\underset{\sim}{n}}_2 \times \hat{\underset{\sim}{n}}_3) = 0$ and the final row of the triangular matrix is a false statement. The three planes meet along a line defined by the vector equation, $\underset{\sim}{r} = \left(4\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} - 11\hat{\boldsymbol{k}}\right) + k\left(-3\hat{\boldsymbol{\imath}} + 13\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}\right)$.

13. In each of the following cases, determine the equation of the line formed by the intersection between the two planes. Give all possible answers in parametric form.

a. $p_1: 8x - 2y + 3z = 1$

$p_2: -8x + 6y + z = -5$

b. $p_1: -3x - 2y + z = 1$

$p_2: 7x + y + 5z = 6$

c. $p_1: \dfrac{2}{3}x - \dfrac{7}{6}y + \dfrac{1}{2}z = 2$

$p_2: -4x + 7y + -3z = -11$

14. Investigate the number of possible solutions by examining the normal vectors algebraically. Describe how the systems could be interpreted geometrically.

a. $\begin{aligned} 2x+3y-4z&=8\\ 2x+3y-4z&=-5\\ 2x+3y-4z&=-4 \end{aligned}$

b. $\begin{aligned} -6x+8y+5z&=11\\ -12x+16y+10z&=-5\\ 24x-32y-20z&=0 \end{aligned}$

c. $\begin{aligned} 4x+7y-4z&=5\\ 16x+28y-16z&=-3\\ 7x-z&=-1 \end{aligned}$

d. $\begin{aligned} -7x+4y-z&=5\\ -42x+24y-6z&=-1\\ -3x-2y+5z&=7 \end{aligned}$

15. WE18/19 Consider the following systems of linear equations to answer the following.

a. $\begin{aligned} 3x+2y-16z&=22\\ -4x-4y+20z&=-28\\ x+3y-3z&=5 \end{aligned}$

b. $\begin{aligned} 6x+5y-4z&=42\\ -3x+4y+28z&=135\\ x-4y-20z&=-109 \end{aligned}$

c. $\begin{aligned} -3x+y+\frac{11}{2}z&=40\\ 5x+4y+5z&=58\\ 3x+5y+\frac{19}{2}z&=92 \end{aligned}$

d. $\begin{aligned} 4x+2y-18z&=58\\ 8x+8y-32z&=120\\ x+y-4z&=15 \end{aligned}$

i. Determine the number of possible solutions by examining the normal vectors algebraically.
ii. Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.
iii. Solve the system of linear equations, expressing the solution in parametric form.
iv. Describe how the solution can be interpreted geometrically.

16. Determine if the solution set is valid for the following systems.

a. $\begin{aligned} -5x+7y+44z&=79\\ 9x+8y-38z&=2\\ 4x-2y-28z&=-38 \end{aligned}$ Solution: $\begin{cases} x=-6+6t\\ y=7-2t\\ z=t \end{cases}$

b. $\begin{aligned} -7x-3y+22z&=136\\ 2x+2y-4z&=-24\\ 2x-y-10z&=-63 \end{aligned}$ Solution : $\begin{cases} x=-25+4t\\ y=13-2t\\ z=t \end{cases}$

17. WE20 **a.** Consider a three-dimensional system containing the line l_1: $\underset{\sim}{r}=(2\hat{\imath}+3\hat{\jmath}+\hat{k})+t(4\hat{\imath}+6\hat{\jmath}-2\hat{k})$ and plane given by p_1: $5x-2y+4z=8$.

i. Examine the line and plane to identify if they intersect.
ii. Determine the solution if they do intersect, otherwise state the system has zero solutions.

b. Consider a three-dimensional system containing the line l_1: $\underset{\sim}{r}=(2\hat{\imath}+7\hat{\jmath}-9\hat{k})+t(3\hat{\imath}+\hat{\jmath}-\hat{k})$ and plane given by p_1: $2x-3y+3z=4$.

i. Examine the line and plane to identify if they intersect.
ii. Determine the solution if they do intersect, otherwise state the system has zero solutions.

c. Consider a three-dimensional system containing the line l_1: $\underset{\sim}{r}=(\hat{\imath}-2\hat{\jmath}+5\hat{k})+t(2\hat{\imath}+4\hat{\jmath}-6\hat{k})$ and plane given by p_1: $-3x-2y+z+34=0$.

i. Examine the line and plane to identify if they intersect.
ii. Determine the solution if they do intersect, otherwise state the system has zero solutions.

18. WE21 For each of the following systems, calculate if the plane and line intersect. If so, determine how many solutions exist.

a. $l_1: \begin{cases} x = 2 + 4k \\ y = -1 - 2k, \\ z = 7 - 3k \end{cases} \quad p_1: 5x - 3y + 2z - 47 = 0$

b. $l_1: \begin{cases} x = 5 - 2k \\ y = 4 - 9k, \\ z = 1 - 3k \end{cases} \quad p_1: 3x - 2y + 4z = 37$

c. $l_1: \begin{cases} x = 1 + 3k \\ y = -3 + 2k, \\ z = -2 - k \end{cases} \quad p_1: 5x - 7y + z - 24 = 0$

19. An object is travelling a straight-line path according to the vector equation, $l_1: (-2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} - 5\hat{\boldsymbol{k}}) + t(-6\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}})$. The sea floor is a flat plane modelled by the equation, $24x - 16y - 31z = -16$. Determine if the object will ever intersect the sea floor and calculate a solution if possible.

20. Consider this system of equations that corresponds to three planes:

$$\begin{aligned} x - 8y - 22z &= -99 \\ -3x + y &= 44 - 20z \\ -2x + 8y + 28z &= \lambda \end{aligned}$$

a. Use a Gaussian elimination technique to determine the value of λ for which this system of equations has infinitely many solutions.

b. Use the result from (**a**) to determine the infinitely many solutions. Express your answer in the form of a parametric equation.

21. Consider this system of equations that corresponds to three planes:

$$\begin{aligned} x - 3y - 19z &= -157 \\ 2x - y - 13z &= -109 \\ -x + y + \lambda &= 75 \end{aligned}$$

a. Use a Gaussian elimination technique to determine the value of λ for which this system of equations has infinitely many solutions.

b. Use the result from **a** to determine the infinitely many solutions. Express your answer as a vector equation.

Complex unfamiliar

22. The equations below are of three coincidental planes. Determine the real values for d_2 and a_3.

$$\begin{aligned} p_1&: \quad 8x + 12y - 4z = 16 \\ p_2&: \quad 4x + 6y - 2z = d_2 \\ p_3&: \quad a_3x + 18y - 6z = 24 \end{aligned}$$

23. **a.** Determine the line of intersection between the following planes. Express the solution as a vector equation.

$$p_1: -4x - 2y + 5z = -18$$
$$p_2: 5x + 3y - \frac{11}{2}z = 26$$

b. Calculate $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2$. Describe how this relates to the solution obtained in **a**.

Fully worked solutions for this chapter are available online.

LESSON
5.5 Dominance and Leslie matrices

SYLLABUS LINKS

- Model and solve problems that involve real-life situations using matrices, including Dominance and Leslie matrices.

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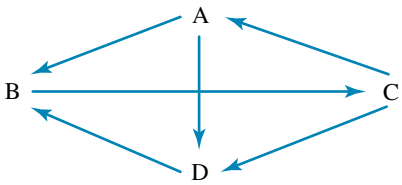
5.5.1 Dominance matrices

Have you ever wondered how tennis players are seeded or ranked? It obviously has something to do with their performance against past opponents where points are earned at every tournament based on how far a player advances in the event. In simple games such as a knock-out competition, one loss and you are out of the competition. Only the winners continue to play. **Dominance matrices** are often used to determine player rankings.

The following example will explain how matrices are used to establish the seedings or rankings of players in round-robin situations where each player plays every other player, thereby creating a more just system of ranking.

Consider 4 players Anna, Belle, Carla and Diane (A, B, C, D), who on past performances have shown that A defeats D and B, D defeats B, C defeats A and D, and B defeats C.



This situation can be represented on a **digraph** — a network diagram that has arrows on the edges, where A → B indicates that A defeats B.

The information from the digraph can be converted into a dominance matrix as shown below:

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

where 1s are used to indicate ‘defeats’ and 0s to indicate otherwise. Obviously A can’t defeat A so a ‘ 0’ is used along the leading diagonal.

Notice also that:

1. there are as many 1s as there are paths
2. corresponding elements occur on either side of the leading diagonal. That is, if A defeats B(1), then a 0 will be stored in the B defeats A element on the opposite side of the leading diagonal.

When the elements of each of the rows are added they yield a **dominance vector**, showing how many players each has defeated.

$$V = \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

This result can be readily checked from the original digraph by counting the number of arrows *out of* each node.

Note, from now on the row/column labels will be omitted.
It can be seen from this information that A and C are ranked equally, and B and D are ranked equally.

So we still need to distinguish between A and C, and B and D to establish the ranking.

We assume in most ranking situations that if A defeats B and B defeats C then A will defeat C. This relationship is described as being *transitive*, where if variable $a < b$ and $b < c$, then $a < c$.

In our example, A defeats B who defeats C, and A defeats D who defeats B. This means that A has second-order influence over C and B but not D. (A doesn’t defeat anyone who defeats D.) The matrix M^2 can be used to investigate second-order influence.

$$\begin{aligned} M^2 &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Notice that the leading diagonal is still 0. It is impossible for a player to have second-order influence over themselves. Row 1 represents the second-order influence of player A over the other players. The element 2 in row 3 occurs because C defeats 2 players (A and D) who defeat B.

We can determine the second-order dominance vector, $V_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, but how much importance should it be given?

If we assign equal importance, we calculate the sum of the first-order influence and the second-order influence.

$$\begin{aligned} M + M^2 &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

This gives a dominance vector $\begin{bmatrix} 4 \\ 3 \\ 5 \\ 2 \end{bmatrix} = V_1 + V_2$ and allows us to rank the competitors in the order C, A, B, D.

If we wanted to investigate third-order influence, we could calculate $M^3 \left(= M^2 M\right)$.

$$M^2 M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ giving } V_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

At this stage, notice that the leading diagonal is no longer 0. If there were more players in the tournament, we could continue determining powers of M, but with 4 players, we stop at M^3. In general, if there are n players, we stop at M^{n-1}.

In most scenarios, it is probably unfair to assign equal importance to first-, second- and third-order influence. We can allocate arbitrary constants to weight the influence; that is, $M + xM^2 + yM^3$. The resulting dominance vector can be found by calculating $V_1 + xV_2 + yV_3$.

If we choose $x = 0.5$ and $y = 0.25$, the dominance vector would be

$$V_1 + 0.5V_2 + 0.25V_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} + 0.25 \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3.75 \\ 2.75 \\ 4.25 \\ 2 \end{bmatrix}$$

This would rank the four players as C, A, B, D.

When you compare the final seeding with the initial information, we can see that A and C both won 2 games, but the wins by A were against the lower ranked B and D. Players B and D both won 1 game but B managed to defeat the higher placed C. This justifies the seeding as produced.

WORKED EXAMPLE 22 Using dominance matrices

We want to seed 4 badminton players, Oliver, Benjamin, Brooke and Millicent. In past matches, Millicent defeated Oliver and Brooke, both Oliver and Brooke defeated Benjamin, Benjamin defeated Millicent, and Oliver defeated Brooke.

a. Construct a digraph to represent this information.

b. By giving equal importance to first- and second-order influence, use dominance matrices to rank the players.

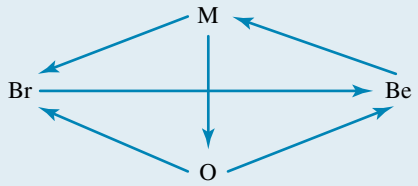
THINK

a. A diagraph represents the matches played in the past. M → O indicates that M defeats O.

b. 1. The information from the digraph is translated to a dominance matrix, M. The 1s indicate 'defeats'.

WRITE

a.



b.

$$M = \begin{array}{c} \begin{array}{cccc} \text{O} & \text{Be} & \text{Br} & \text{M} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Oliver} \\ \text{Benjamin} \\ \text{Brooke} \\ \text{Millicent} \end{array}$$

2. M^2 helps us investigate the second order influence.

$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

3. Dominance vectors are calculated by adding the elements of each row from the matrices. The resulting dominance vector is the sum of V_1 and V_2.

$$V_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

$$V_1 + V_2 = \begin{array}{c} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 5 \end{bmatrix} \end{array} \begin{array}{l} \text{O} \\ \text{Be} \\ \text{Br} \\ \text{M} \end{array}$$

4. State the ranked order of players according to the resulting dominance vector, $V_1 + V_2$.

The players would be ranked Millicent, Oliver, Benjamin, Brooke.

5.5.2 Leslie matrices

In this next section, **Leslie matrices** are studied. Unlike dominance matrices, they have no connection to hierarchy or competition. Instead, they are used to model population growth over time.

There are many factors that affect the size of populations of animal species. The first of these is the survival rate; that is, the proportion of animals that survive until the next year (or whatever time period is being used). Clearly, a more accurate picture of the survival rate can be obtained if we use survival rates specific to the different age groups. For example, a saltwater crocodile may produce many offspring in a year but not many of the babies survive their first year. However, once they reach 1 year of age it is much more likely that they will get to 2 years of age. A second factor influencing the population of a species is its reproductive rates or how many babies are produced per year. Again, reproductive rates that are specific to particular age groups will yield a more accurate analysis.

To see how matrices can be used to model animal populations, consider the following example.

A wildlife officer is interested in the number of a certain species of rat in a habitat and has compiled the following information on this species.

We have a population breakdown by age:

Age (x) (at the end of the year)	Number (N) (of females)
1	55
2	32
3	25

It is assumed that the rats do not live longer than three years and that there are the same number of males and females.

Fecundity (reproductive) rates, F_x, (the number of female offspring per female per year) have been calculated:

Age (x)	Fecundity (F_x)
1	0.6
2	2.4
3	0.5

The survival rates, S_x, of each age group have also been estimated:

Age (x)	Survival rate (S_x)
1	0.5
2	0.6

Given this information and an initial starting population, it is possible to predict the future population.

Note that the 1-year-old rats that survive the year become 2-year-old rats in the next year.

As each age group of females produce babies, the number of 1-year-old rats in the next year can be calculated as:

$$\begin{aligned}\text{Number of 1-year-old rats} = {} & \text{number of 1-year-olds} \times F_1 \\ & \text{number of 2-year-olds} \times F_2 \\ & \text{number of 3-year-olds} \times F_3\end{aligned}$$

The number of 2-year-old rats in the next year $= 0.5 \times$ number of 1-year-olds.

The number of 3-year-old rats in the next year $= 0.6 \times$ number of 2-year-olds.

It is useful to formulate the relationships within a population of rats in any year, and between the populations of rats over several years, using matrices. The population of rats at the end of year 1 can be written:

$$N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$$

A matrix called the **Leslie matrix** can be used to estimate the next year's population given the current population. The Leslie matrix is given by:

$$L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$$

The estimated rat population at the end of the second year is N_2, where $N_2 = L \times N_1$.

A Leslie matrix is a square matrix with fecundity rates in the top row, survival rates along the sub-diagonal and 0s elsewhere.

$L = \begin{bmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix}$ where F_i is the fecundity rate of the ith age group and S_i is the survival rate of the ith age group.

We can also calculate the number of females after any year using the Leslie matrix and the number of females after year 1.

Consider the number of females after years $2, 3, 4, \ldots, r$.

After year 2: $N_2 = L \times N_1$

After year 3: $N_3 = L \times N_2 = L \times L \times N_1 = L^2 \times N_1$

After year 4: $N_4 = L \times N_3 = L \times L^2 \times N_1 = L^3 \times N_1$

After year r: $N_r = L^{r-1} \times N_1$

Population over time using Leslie matrices

In general terms, $N_r = L \times N_{r-1}$ and $N_r = L^{r-1} \times N_1$.

Where $L = \begin{bmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix}$ is the Leslie matrix.

F_i is the fecundity of the ith age group and S_i is the survival rate of the ith age group.

The initial population vector is $N_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ representing the initial age-class distribution of females.

WORKED EXAMPLE 23 Using a Leslie matrix to model population over time

Using the Leslie matrix, $L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$, and the female rat population in year 1, $N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$.

a. State the meaning of the number 2.4 in the matrix.
b. Calculate the female rat population at the start of year 2.
c. Calculate the total rat population after 6 years, assuming equal numbers of male and female rats.

THINK

a. The first row of the Leslie matrix gives fecundity rates. The second column refers to 2-year-olds.

b. 1. The population in the next year is found by multiplying the Leslie matrix by the population this year.

2. Perform the matrix multiplications.

WRITE

a. The number 2.4 refers to the number of female rats born on average to a 2-year-old rat.

b. $N_2 = L \times N_1$

$$N_2 = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$$
$$= \begin{bmatrix} 122.3 \\ 27.5 \\ 19.2 \end{bmatrix}$$

3. Write the answer.

The total female population is $122.3 + 27.5 + 19.2 = 169$.

c. 1. Calculate the population after 6 years using the formula $N_r = L^{r-1} \times N_1$.

c. $N_6 = L^5 \times N_1$

$$N_6 = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}^5 \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$$

2. Perform the matrix multiplications.

$$= \begin{bmatrix} 521.9 \\ 171.9 \\ 73.3 \end{bmatrix}$$

3. Add the female populations.

The total female population is $521.9 + 171.9 + 73.3 = 767.1$.

4. Calculate the total rat population assuming that there are equal numbers of males and females.

Number of males and females $= 2 \times 767.1$
$= 1534.2$

5. Answer the question.

The total rat population after 6 years is approximately 1534.

Exercise 5.5 Dominance and Leslie matrices

learnon

5.5 Exercise	5.5 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7	8, 9, 10, 11, 12, 13	14

These questions are even better in jacPLUS!

- Receive immediate feedback
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- Track results and progress

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Simple familiar

1. The Leslie matrix L for an invasive species with a lifespan of three years is given. $L = \begin{bmatrix} 0.7 & 2.5 & 0.2 \\ 0.8 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}$

   An initial population has been detected on an Australian island at the start 1782. The initial female population is $N_0 = \begin{bmatrix} 240 \\ 120 \\ 43 \end{bmatrix}$

   Estimate the total female population at the start of 1787 if nothing is done to curtail the invasion of the island. Give your answer to the nearest integer.

2. A round-robin tennis match was arranged between Luke Callum, Grant and Kevin. Luke lost only to Grant. Grant lost to Callum and Kevin and Kevin lost to Callum. Construct a digraph to represent this information.

3. WE22 The top four chess players in a tournament are Breanna, Kayley, Teagan and Cameron. In past matches, Cameron defeated Breanna and Teagan, both Breanna and Teagan defeated Kayley, Kayley defeated Cameron, and Breanna defeated Teagan.

   a. Construct a digraph to represent this information.
   b. By giving equal importance to first- and second-order influence, use dominance matrices to rank the players.

4. Three friends have noticed that when they played chess, Mair defeated Ann and Janine, and Ann defeated Janine. Use dominance matrices to rank these players.

5. A round-robin netball match was arranged for house competitions where Barnes lost to all but Cunningham, Cunningham lost to Leslie but defeated Hamilton. No teams went undefeated.
   a. If it is decided to give equal importance to first- and second-order influence, use dominance matrices to rank the students' houses.
   b. If house points are allocated as 20, 15, 10, 5 for the overall ranking, how many points did each house receive?

6. WE23 The change in the rat population in a certain habitat is described by the following Leslie matrix:

$$L = \begin{bmatrix} 0.7 & 1.4 & 0.7 \\ 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

The initial female rat population is $N_1 = \begin{bmatrix} 33 \\ 18 \\ 8 \end{bmatrix}$.

   a. State the meaning of the number 1.4 in the matrix.
   b. State the meaning of the number 0.5 in the matrix.
   c. Calculate the female rat population for year 3.
   d. Calculate the total rat population after 9 years, assuming equal numbers of male and female rats.

7. The numbers of a particular species of bird have been monitored over three years and the following Leslie matrix has been developed to model the population. The birds are aged 1, 2 or 3 years.

$$L = \begin{bmatrix} 0 & 3.1 & 2.2 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

   a. Determine how many new birds do 1-year-old birds produce on average.
   b. Determine how many new birds do 2-year-old birds produce on average.
   c. Calculate what proportion of 1-year-old birds survive the first year.
   d. If the female population in year 1 is $N_1 = \begin{bmatrix} 245 \\ 101 \\ 32 \end{bmatrix}$, calculate the population of female birds in year 3.

Complex familiar

8. Five schools are debating in a round-robin tournament — the following table shows the results.

	Clifton	Warwick	Goondiwindi	Stanthorpe	Ipswich
Clifton	—	L	W	W	L
Warwick		—	W	W	L
Goondiwindi			—	L	W
Stanthorpe				—	W

Clifton lost to Warwick and Ipswich and so on.

a. Construct a dominance matrix of this information.

b. A total of 15 points is divided in the ratio 5 : 4 : 3 : 2 : 1 and awarded according to the ranking of the schools at the end of the tournament. If this division of points is allocated according to the figures produced by the dominance matrix sum $M + 0.8M^2 + 0.5M^3$, list the number of points each school wins.

9. Suppose that only 15% of baby saltwater crocodiles live to 1 year of age and that 60% of 1-year-olds survive to 2 years of age. In one season, 40 crocodile eggs hatch. How many of these survive to:

a. 1 year of age?

b. 2 years of age?

10. Consider the following data associated with sheep flocks.

Age (years)	Birth rate	Survival rate	Number (of females) initially
0	0	0.90	450
1	0.50	0.85	2200
2	0.45	0.50	860
3	0.40	0	55

a. Construct a Leslie matrix for the sheep flock.

b. Use the Leslie matrix to predict the sheep population in 5 years' time.

11. On a farm, the following data are recorded for the sheep flock in a particular year.

Age	Number of females at start of year	Number of females at the end of the year	Number of females born to this group
0	330	280	0
1	880	870	350
2	150	0	12

Assume that there are no female sheep aged over 2 years.

a. Calculate the survival rate of:

i. the 0-year-old group

ii. the 1-year-old group

iii. the 2-year-old group.

b. Calculate is the birth rate for:

i. the 0-year-old group

ii. the 1-year-old group

iii. the 2-year-old group.

c. Construct a Leslie matrix for the sheep flock.

d. Use the matrix to predict the sheep population in 3 years' time.

12. The Leslie matrix (L) below shows the fecundity rate of a female mouse population while N_1 shows the population in the first year.

$$L = \begin{bmatrix} 0.8 & 2.2 & 0.6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}, N_1 = \begin{bmatrix} 65 \\ 28 \\ 31 \end{bmatrix}$$

Calculate the total number of rats in the ninth year. It is assumed that there is equal number of male and female rats.

13. Anya, Beatrice, Christina, Danielle and Emma participated in a competition with the following results:

- Emma lost to Beatrice
- Beatrice, Anya and Danielle all lost to Christina
- Beatrice and Danielle lost to Anya
- Christina, Anya and Danielle lost to Emma Beatrice lost to Danielle.

  To rank the competitors from first to fifth place, the organiser decide to give twice as much importance to the first-order influence than to the second-order influence. Determine who won the competition. Show your working.

Complex unfamiliar

14. At the start of a reintroduction program, an initial population of the endangered species of spotted-tailed quolls is introduced in a reserve. An environmental scientist collects data for the females of the species. The details are shown in the table.

Age (years)	0 – 1	1 – 2	2 – 3	3 – 4
Initial population	125	95	76	39
Breeding rate	0.48	0.76	0.52	0.16
Survival rate	0.62	0.32	0.19	0

Fully worked solutions for this chapter are available online.

LESSON
5.6 Applications of matrices

SYLLABUS LINKS

- Investigate how matrices have been applied in other real-life situations, e.g. Leontief, Markov, area, cryptology, eigenvectors and eigenvalues.

Note:
The external examination may assess only Dominance and Leslie matrices. Content from section 5.6 is not included in the external assessment.

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5.6.1 Markov chains and eigenvectors

Andrei Markov was a Russian mathematician whose name is given to a technique that calculates the probability associated with the state of a system as it makes various transitions. It answers questions such as, 'What is the probability that James will be late to work today given that he was late yesterday?' or 'What is the likelihood that the next note in a piece of music will be an A, given that the previous note was a G?' or 'What can be said about the long-term behaviour of the piece of music?'.

In Palmdale only two brands of soft drink are sold — Blue or Red. Suppose, at the start of the first month, 50% of the population drank Blue and 50% of the population drank Red. The people of Palmdale do not always stick to the same drink. At the end of the month some of them change to the other drink. Suppose at the end of each month 40% of people who drink Blue change to Red and 70% of those who drink Red change to Blue.

As time goes by the proportions are as follows.

	Start	End of month 1	End of month 2
Blue	50%	$50\% \times 60\% + 70\% \times 50\% = 65\%$	$65\% \times 60\% + 70\% \times 35\% = 63.5\%$
Red	50%	$50\% \times 30\% + 40\% \times 50\% = 35\%$	$35\% \times 30\% + 40\% \times 65\% = 36.5\%$

What is the long-term behaviour of the way in which the market share is distributed? Even the calculation of the behaviour for three months is complicated. The proportion of market share each month is called the **state** of the system. The sequence of states from month to month forms a **Markov chain**.

A **transition matrix**, T, can be used to simplify the calculations. As its name suggests, the matrix assists in the calculation of the transition in drinking habits from one month to the next.

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} \text{Blue} & \text{Red} \end{array} \\ T = & \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{array}{l} \text{Blue} \\ \text{Red} \end{array} \text{ To} \end{array}$$

Note: 0.7 represents 70% of red changing to blue.

Note that the columns of a transition matrix always sum to 1. At any time, the state is represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, where x is the proportion of Blue and y is the proportion of Red. Thus, if the initial state, S_0, is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, then the state after:

1 month, S_1, is: $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$

2 months, S_2, is: $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.635 \\ 0.365 \end{bmatrix}$.

Often we are interested in the long-term behaviour of the market or the **steady state**, as it is often called. The steady state will be a value of $\begin{bmatrix} x \\ y \end{bmatrix}$ such that:

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$

This means the market share remains the same from one month to the next.

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$

In this case

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives $x = 0.6x + 0.7y$ and $y = 0.4x + 0.3y$.

Recalling that x and y are proportions that add to 1.

$$x + y = 1$$

Solving the three equations simultaneously gives:

$$x = 0.6364$$
$$y = 0.3636.$$

That is, the steady state or long-term behaviour of the market is that Blue has

63.64% of the market and Red has 36.36%

Markov chains and long-term behaviour

1. **A transition matrix, T, is used to calculate the likelihood of certain states given the states at a previous stage.**
2. **The sequence of states, S_i, is called a Markov chain.**
3. **The values of T are arranged as follows:**

$$\begin{array}{cc} & \textbf{From} \\ & \begin{array}{cc} A & B \end{array} \\ & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{array}{l} A \\ B \end{array} \textbf{To} \end{array}$$

4. **If a system can be in one of r states, the $r \times 1$ column matrix (or vector) p gives the likelihood of each of the states.**
5. **If a system is initially in a state p after n stages the system is in a state given by $T^n p$.**
6. **The sum of columns in a transition matrix is 1.**

7. **The sum of probabilities in the state matrix is 1.**
8. **The long-term behaviour of the system can be found by considering $T^{50}p$, first checking that T^{51} is the same as T^{50}.**

WORKED EXAMPLE 24 Using Markov chains

The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & x \end{bmatrix}$.

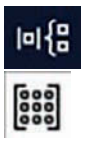
a. Calculate the value of x.

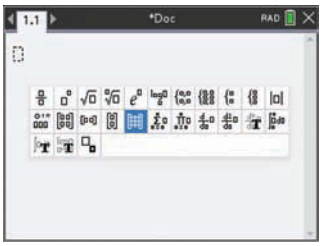
b. If the state of the system is $\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$, calculate the state of the system 2 stages later, using technology.

THINK	WRITE
a. Form an equation by adding the elements of the third column. The sum of each column must be 1.	**a.** $0.5 + 0.4 + x = 1$ $x = 0.1$
b. 1. The state of the system will be $p = T^2 p_0$ where p_0 is the initial state of the system. (Use a graphics calculator to make the calculations, see instructions below.)	**b.** $p = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}^2 \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$
2. Write the answer.	The steady state will be $p = \begin{bmatrix} 0.3125 \\ 0.3625 \\ 0.3250 \end{bmatrix}$.

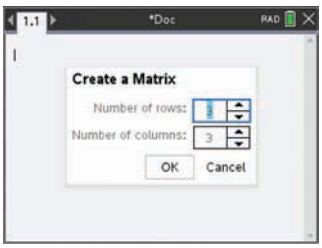
TI | THINK **WRITE**

1. On a Calculator page, select



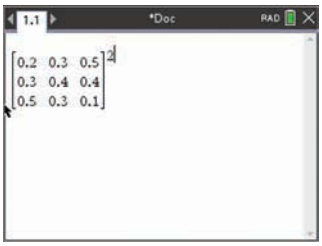


2. Complete the entry line as
3
3
Press the OK button.



3. Complete the entry line as

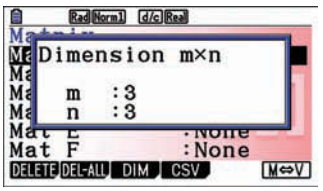
$$\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}^2$$



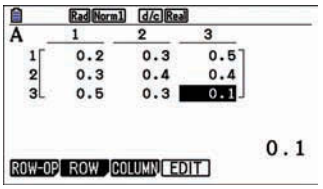
CASIO | THINK **WRITE**

1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the M ⇔ V button.

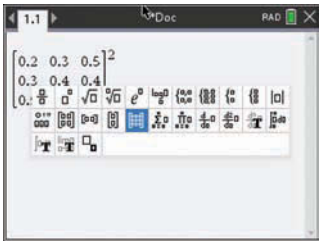
2. Press the EXE button. Complete the entry line as
3
3
Press the EXE button.



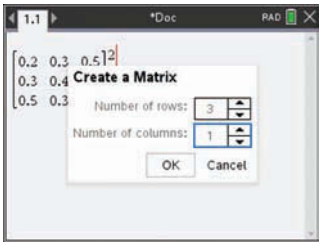
3. Complete the entry line as
0.2 0.3 0.5
0.3 0.4 0.4
0.5 0.3 0.1
Press the EXIT button when finished.



4. Select followed by .

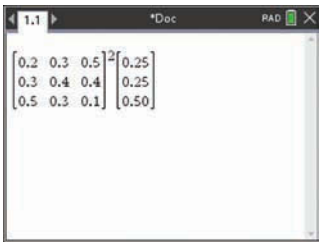


5. Complete the entry line as
3
1
Press the OK button.

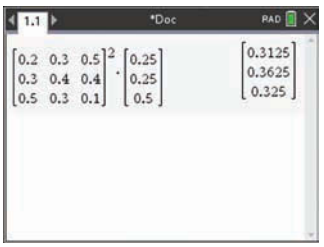


6. Complete the entry line as

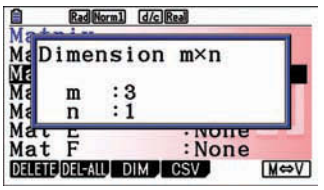
$$\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$



7. Press the ENTER button.
The answer appears on the screen.



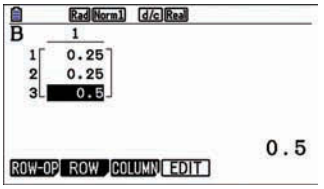
4. Highlight the Mat B row and press the EXE button.
Complete the entry line as
3
1
Press the EXE button.



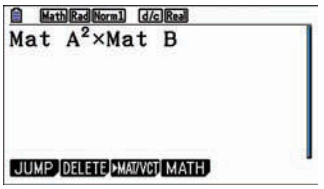
5. Complete the entry line as

$$\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$

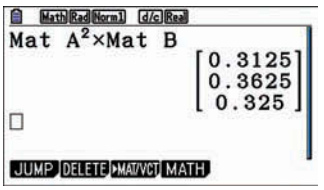
Press the EXIT button twice when finished.



6. On a Run-Matrix screen, select
OPTN,
MAT/VCT
Mat
ALPHA
$X.\theta$, T
^
2
×
Mat
ALPHA
Log



7. Press the EXE button.
The answer appears on the screen.
example.



WORKED EXAMPLE 25 Determining long-term behaviour

The Nee Islands are very wet. If it is raining on a particular day, the chance that it will rain the next day is 60%. If it is not raining on a particular day, the chance that it will rain on the following day is 50%.

a. Construct a transition matrix to represent this situation.

b. Using the transition matrix, calculate the probability that, if it is raining on Tuesday, it will also be raining on Friday of the same week.

c. In the long term, determine what proportion of days it rains in the Nee Islands.

THINK	WRITE
a. Label the columns and rows with the possible states and enter the appropriate values for the transition matrix.	**a.** From Rain Dry $\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ Rain Dry To $\therefore T = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$
b. 1. Define the initial state; that is, define the matrix for the state on Tuesday.	**b.** The state on Tuesday (rain) is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
2. Determine the state on Friday by calculating $T^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where T is the transition matrix. The matrix T is raised to the power of 3 as Friday is three days after Tuesday.	The state on Friday, $\begin{bmatrix} x \\ y \end{bmatrix}$, is given by: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.556 & 0.555 \\ 0.444 & 0.445 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.556 \\ 0.444 \end{bmatrix}$.
3. Answer the question.	The likelihood of rain on Friday is 0.556.
c. 1. Determine the long-term behaviour by considering a large (say $n = 50$) power of the transition matrix, T.	**c.** Consider $T^{50} = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^{50}$ $= \begin{bmatrix} 0.5556 & 0.5556 \\ 0.4444 & 0.4444 \end{bmatrix}$
2. Check that T^{51} is the same. They are the same (to 4 decimal places), which indicates that a steady-state solution is found.	and $T^{51} = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^{51}$ $= \begin{bmatrix} 0.5556 & 0.5556 \\ 0.4444 & 0.4444 \end{bmatrix}$
3. Choose a starting value, say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and calculate the long-term proportions. *Note*: Any starting value could have been used.	If the starting value is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the long-term behaviour is given by: $T^{50} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5556 & 0.5556 \\ 0.4444 & 0.4444 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.5556 \\ 0.4444 \end{bmatrix}$.
4. Answer the question.	So, the long-term probability of rain is 0.5556 and dry is 0.4444.

5.6.2 Eigenvalues

For any $n \times n$ matrix A, if there is a $n \times 1$ column matrix, X, such that $AX = \lambda X$, then X is called an **eigenvector** of A and λ is called an **eigenvalue** of the matrix, A. That is, for matrix A, the eigenvalue λ acts as a scalar on the eigenvector X. It is important to remember that eigenvalues are numbers whereas eigenvectors are vectors.

In the vector equation $AX = \lambda X$, both λ and X are unknowns. To solve this, we need to rewrite the equation as a linear system by expressing λX as λIX.

$$\begin{aligned} AX &= \lambda X \\ AX - \lambda X &= 0 \end{aligned} \quad [1]$$

$AX - \lambda IX = 0$ where I is the identity matrix.

$$(A - \lambda I)X = 0 \quad [2]$$

In order for the linear system to have a non-trivial (nonzero) solution, $A - \lambda I$ must be singular. That is, $\det(A - \lambda I) = 0$. It follows from equation [2] that:

$$\det(A - \lambda I) = 0 \quad [3]$$

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$.

If A is a 2×2 matrix, equation [3] is a quadratic equation in λ.

$$\begin{aligned} \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= ad - a\lambda - bc - d\lambda + \lambda^2 \\ &= \lambda^2 - \lambda(a+d) + ad - bc \end{aligned}$$

If A is a 3×3 matrix then equation [3] is an equation in λ of degree 3.

$$\begin{aligned} \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) &= \det\begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{bmatrix} \\ &= -\lambda^3 + \lambda^2(a+e+i) + \lambda(fh - ei + cg + bd - ea - ai) \\ &\quad + cdh - ecg + bfg - bdi + eai - afh \end{aligned}$$

In general, any $n \times n$ matrix will have n eigenvalues. Additionally, the sum of the diagonal entries of the square matrix is equal to the sum of the eigenvalues (check this in Worked example 24).

Eigenvectors of a matrix, A, are important because they represent objects that are in a sense unchanged by a transformation represented by the matrix, A. In physics, economics and biology, eigenvectors are important features in mathematical modelling. In the previous section on Markov chains, the values for a steady state were to be found by solving the equation $\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$. If we substitute $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and use eigenvector notation, we obtain $X = TX$.

Thus, the steady state, X, is an eigenvector corresponding to an eigenvalue, $\lambda = 1$.

Eigenvectors and eigenvalues

For a matrix, A, if there is a column matrix (vector), X, such that:

$$AX = \lambda X$$

then X is called an eigenvector of A and λ is called an eigenvalue of the matrix, A.

The steady state, $X = TX$, is an eigenvector corresponding to an eigenvalue, $\lambda = 1$.

WORKED EXAMPLE 26 Determining eigenvalues and eigenvectors

If $A = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$, determine:

a. eigenvalues associated with the matrix

b. eigenvectors associated with the matrix.

THINK	WRITE
a. 1. To determine the eigenvalues, λ, substitute matrix values into the expression $A - \lambda I$.	**a.** $A - \lambda I = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ $= \begin{bmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{bmatrix}$
2. Substitute for $A - \lambda I$ in $\det(A - \lambda I) = 0$.	So $\det \begin{bmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{bmatrix} = 0$
3. Determine the determinant and solve for λ.	$(1-\lambda)(2-\lambda) - 12 = 0$ $\lambda^2 - 3\lambda - 10 = 0$ $(\lambda - 5)(\lambda + 2) = 0$ Using the Null Factor Law $\lambda = 5$ and $\lambda = -2$
4. Write the eigenvalues.	Thus, the eigenvalues are 5 and -2.
b. 1. Recall the relationship linking the eigenvectors with the eigenvalues.	**b.** $AX = \lambda X$
2. Substitute $\lambda = 5$ and $\lambda = -2$.	$AX = 5X \cdot$ or $\cdot AX = -2X$
3. Use $\lambda = 5$ first. Substitute values for matrix A and assign unknowns to matrix X.	Let $X = \begin{bmatrix} a \\ b \end{bmatrix}$ then with $\lambda = 5$, $AX = 5X$.
4. Substitute for A and X. Consider the first row of the product. (You will obtain the same result if the second row is used.)	$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 5 \begin{bmatrix} a \\ b \end{bmatrix}$ $a + 6b = 5a$ $6b = 4a$ $b = \frac{4a}{6}$ $= \frac{2a}{3}$

5. As you cannot determine independent values for a and b, state the relationship between them.

X is any vector of the form $\begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}$.

6. Repeat this procedure for $\lambda = -2$. Again, substitute for A and X. Consider the first row of the product only.

$$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{aligned} a + 6b &= -2a \\ 6b &= -3a \\ b &= \frac{-3a}{6} \\ &= -\frac{a}{2} \end{aligned}$$

7. State the eigenvector for $\lambda = -2$.

X is any vector of the form $\begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}$.

8. Answer the question.

The eigenvectors for the given matrix are any vectors of the form $\begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}$ or $\begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}$.

5.6.3 Leontief Matrices

Wassily Leontief was born in St Petersburg in 1906. Although he trained in Russia and Germany his main contribution to economic theory took place while he was a professor of economics at Harvard University in the United States. He received the Nobel Prize for economics in 1973 and died in 1999. Leontief developed input–output analysis, a technique greatly valued by economists involved in planning and development.

Consider an industry or economy that produces energy (E), raw materials (R) and manufactured goods (M). There is an inter-relationship between these three activities. To produce energy you require raw materials and manufactured goods. Likewise, to produce raw materials requires energy and manufactured goods.

The Leontief model looks at an economy such as this and relates input and output. It answers questions such as: 'what inputs (how much of each of E, R and M) are needed to produce a certain output?'.

Matrices are used to record the relationship between the industries.

Suppose the relationship can be quantified and represented by a consumption matrix, C.

$$C = \begin{array}{c} \begin{array}{ccc} E & R & M \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ R \\ M \end{array}$$

This matrix should be interpreted as follows. The number 0.4 in the second row, first column means that every unit of E produced requires 0.4 units of R.

Output and input can be represented using 3×1 matrices, p and q. For example, $p = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ represents an output of 3 units of E, 4 units of R and units of .

The input, q, required to give an output, p, can be obtained by using the equation:

$$q = Cp$$

WORKED EXAMPLE 27 Using Leontief matrices

Consider the consumption matrix

$$C = \begin{array}{c} \begin{array}{ccc} E & R & M \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ R \\ M \end{array}$$

a. State the meaning of the value 0.4 in the third row of the matrix.
b. Calculate the inputs needed to produce 1 unit of E, 0 units of M and 0 units of R.
c. Calculate the inputs needed to produce 2 units of E, 2 units of *R* and 3 units of M.

THINK	WRITE
a. Consider the row and column for element 0.4 in the third row. The row is linked to M and the column is linked to R.	**a.** This entry means it takes 0.4 units of M to make 1 unit of R.
b. 1. 1 Write the output as a 3×4 matrix.	**b.** Let $p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ because the amount of $E = 1$ with the quantities of R and M being 0.
2. Recall the equation relating input to output, $q = Cp$.	$q = Cp$
3. Multiply the consumption matrix given, C, by the output matrix p.	$q = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix}$
4. State the answer as a sentence.	The input q required is 0.2 units of E, 0.4, units of R and 0.3 units of M.
c. 1. Write the output as a 3×1 matrix.	**c.** c $q = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$
2. Recall the equation relating input to output, $q = Cp$.	

3. Multiply the consumption matrix given, C, by the output matrix p.

$$q = Cp$$

4. State the answer as a sentence.

$$q = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$$

The input needed is 1.9 units of E, 1.9 units of R and 1.7 units of M.

In the previous worked example, you will notice that the products of the process $p = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ exceeded the inputs $q = \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$. This surplus, $p - q = \begin{bmatrix} 0.1 \\ 0.1 \\ 1.3 \end{bmatrix}$, can be used to meet external demand, d.

In a well-balanced economy, it is possible to choose a production, p, such that it matches a given demand, d, and $p - q = d$. Note that, as $q = Cp$, this equation is equivalent to $p - Cp = d$. That is, production less consumption inputs required is equal to demand.

This gives:

$$\begin{aligned} p - Cp &= d \\ (I - C)p &= d \\ p &= (I - C)^{-1}d \end{aligned}$$

where I is the identity matrix.

You may think that, provided $(I - C)^{-1}$ exists, the equation $p = (I - C)^{-1}d$ ensures that we can always determine production levels, such that the demand will always be met. This is not always the case, for although the values can be computed in theory, in practice some of the elements of p may turn out to be negative. This would not make sense and thus the process would fail for that value of d.

WORKED EXAMPLE 28 Determining production levels

Given the consumption matrix

$$C = \begin{array}{c} \begin{array}{ccc} E & R & M \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ R \\ M \end{array}$$

a. Determine the production, p, correct to 2 decimal places, that will match the demand $d = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$.

b. Demonstrate that it is always possible to determine values for production that will exactly match a given demand, d.

THINK	WRITE
a. 1. Recall the production equation $p=(I-C)^{-1}d$ to determine production, given demand.	**a.** $p=(I-C)^{-1}d$
2. Substitute values for C and d.	$p=\left[\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right]-\begin{bmatrix}0.2&0.3&0.3\\0.4&0.1&0.3\\0.3&0.4&0.1\end{bmatrix}^{-1}\begin{bmatrix}5\\2\\4\end{bmatrix}$
3. Determine using a graphics calculator.	$p=\begin{bmatrix}19.6875\\17.1875\\18.64583\end{bmatrix}$
4 Write the answer.	When the demand is: $E=5$, $R=2$ and $M=4$ the production that exactly matches it is: $E=19.69$, $R=17.19$ and $M=18.65$.
b. 1. Use the equation, $p=(I-C)^{-1}d$ to determine p.	**b.** $(I-C)^{-1}=\begin{bmatrix}2.40&1.35&1.25\\1.56&2.19&1.25\\1.49&1.42&2.08\end{bmatrix}$
2. Recognise that the values of production need to be positive.	As long as $(I-C)^{-1}$ exists and has positive entries, it will be always possible to determine values of to match the demand, d.

The next worked example uses a two-industry economy and illustrates how we can form a consumption matrix from information about the economy.

WORKED EXAMPLE 29 Using consumption matrices

Consider the following two industry input–output models. To produce one tonne of coal requires 0.2 tonnes of coal and 0.3 tonnes of steel. To produce one tonne of steel requires 0.4 tonnes of coal and 0.2 tonnes of steel.

a. Construct the consumption matrix for this economy.

b. Determine the production required to meet the external demand of 80 tonnes of coal and 20 tonnes of steel.

THINK	WRITE
a. Construct the consumption matrix using C and S across the top and down the side, where C is coal and S is steel.	**a.** $C=\begin{array}{c}\begin{array}{cc}C&S\end{array}\\\begin{bmatrix}0.2&0.4\\0.3&0.2\end{bmatrix}\end{array}\begin{array}{c}\\C\\S\end{array}$

b. 1. Recall the formula relating production, p, to demand, d and substitute the consumption matrix calculated in **a**. and the external demand given, d.

b. $p = (I - C)^{-1}d$

$$= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] - \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 138.5 \\ 76.9 \end{bmatrix}$$

2. Write the answer in words.

The production that meets the demand is 138.5 tonnes of coal and 76.9 tonnes of steel.

5.6.4 Cryptology

Cryptology is the science of rendering a communication meaningless to anyone intercepting the message. It is an area of mathematics that has grown rapidly, and is used on a daily basis through secure messaging apps and internet banking.

Enciphering is the coding of a message (plain text) into a form that is meaningless to a third party. **Deciphering** is the reverse process; taking the code and translating it back into the original plain text message.

Consider the simplest of ciphers — a mono-alphabetic substitution; that is, the substitution of one letter for another. A cipher such as this is easy to crack if one has enough text to work with. A disadvantage of substitution ciphers is that they preserve the frequencies of individual letters, making it relatively easy to break the code by statistical methods. The percentage frequency of each letter in English correct to 1 decimal place is:

A	B	C	D	E	F	G	H	I	J	K	L	M
8.2	1.5	2.8	4.3	12.7	2.2	2.0	6.1	7.0	0.2	0.8	4.0	2.4
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
6.7	7.5	1.9	0.1	6.0	6.3	9.1	2.8	1.0	2.3	0.1	2.0	0.1

Naturally, coded texts vary in content, and the frequency of letters in a code will not follow these general proportions exactly. However, they do provide a starting point for someone attempting to decipher a text. The appearance of apostrophes and frequent letter associations in common words such as ‘the’ may also provide clues.

Enciphering

In the discussion to follow, we assume that each plain-text and cipher-text letter, except Z, is assigned the numerical value that specifies its position in the standard alphabet.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
14	15	16	17	18	19	20	21	22	23	24	25	0

Modulo arithmetic

In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by a procedure that involves **modulo arithmetic**. Modulo arithmetic plays an important role in cryptology.

Let us consider this first.

Recall that we say:

$45 \equiv 3$ modulo 7, because the remainder when 45 is divided by 7 is 3.

This statement is usually abbreviated to:

$$45 \equiv 3 (\text{mod} 7)$$

When operating in mod7 there are only 7 distinct numbers 0, 1, 2, 3, 4, 5 and 6. Every other number can be linked to one of these remainders.

As there are 26 letters in the English alphabet, modulo 26 is frequently used in cryptology.

For example, $87 = 9(\text{mod}26)$. This means that the remainder when 87 is divided by 26 is 9. Another way of looking at this is $87 = 26 \times 3 + 9$.

For negative numbers, we still look for a positive remainder. For example, $-38 = 14(\text{mod}26)$. That is, $-38 = 26 \times -2 + 14$.

For modulo 26, $Z_{26} = \{0, 1, 2, ..., 25\}$.

If a is a number in Z_m, then a number, a^{-1}, in Z_m is called a *reciprocal* or *multiplicative inverse* of a modulo m if $aa^{-1} = a^{-1}a = 1(\text{mod}m)$.

Not every element of Z_m will have a multiplicative inverse.

The easiest way to search for a numbers inverse is to use the multiplication table and search along its row for the number 1.

- For instance, if we choose the number 2 in mod26, and then searched its row for the number 1, we will not find it.
- The reciprocal of 3 in mod26 is 9 because the product of the two numbers is 1.

The results for multiplication modulo 26 are listed in the table below.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	2	4	6	8	10	12	14	16	18	20	22	24	0	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	1	4	7	10	13	16	19	22	25	2	5	8	11	14	17	20	23
4	4	8	12	16	20	24	2	6	10	14	18	22	0	4	8	12	16	20	24	2	6	10	14	18	22
5	5	10	15	20	25	4	9	14	19	24	3	8	13	18	23	2	7	12	17	22	1	6	11	16	21
6	6	12	18	24	4	10	16	22	2	8	14	20	0	6	12	18	24	4	10	16	22	2	8	14	20
7	7	14	21	2	9	16	23	4	11	18	25	6	13	20	1	8	15	22	3	10	17	24	5	12	19
8	8	16	24	6	14	22	4	12	20	2	10	18	0	8	16	24	6	14	22	4	12	20	2	10	18
9	9	18	1	10	19	2	11	20	3	12	21	4	13	22	5	14	23	6	15	24	7	16	25	8	17
10	10	20	4	14	24	8	18	2	12	22	6	16	0	10	20	4	14	24	8	18	2	12	22	6	16
11	11	22	7	18	3	14	25	10	21	6	17	2	13	24	9	20	5	16	1	12	23	8	19	4	15
12	12	24	10	22	8	20	6	18	4	16	2	14	0	12	24	10	22	8	20	6	18	4	16	2	14
13	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13
14	14	2	16	4	18	6	20	8	22	10	24	12	0	14	2	16	4	18	6	20	8	22	10	24	12
15	15	4	19	8	23	12	1	16	5	20	9	24	13	2	17	6	21	10	25	14	3	18	7	22	11
16	16	6	22	12	2	18	8	24	14	4	20	10	0	16	6	22	12	2	18	8	24	14	4	20	10
17	17	8	25	16	7	24	15	6	23	14	5	22	13	4	21	12	3	20	11	2	19	10	1	18	9
18	18	10	2	20	12	4	22	14	6	24	16	8	0	18	10	2	20	12	4	22	14	6	24	16	8
19	19	12	5	24	17	10	3	22	15	8	1	20	13	6	25	18	11	4	23	16	9	2	21	14	7
20	20	14	8	2	22	16	10	4	24	18	12	6	0	20	14	8	2	22	16	10	4	24	18	12	6
21	21	16	11	6	1	22	17	12	7	2	23	18	13	8	3	24	19	14	9	4	25	20	15	10	5
22	22	18	14	10	6	2	24	20	16	12	8	4	0	22	18	14	10	6	2	24	20	16	12	8	4
23	23	20	17	14	11	8	5	2	25	22	19	16	13	10	7	4	1	24	21	18	15	12	9	6	3
24	24	22	20	18	16	14	12	10	8	6	4	2	0	24	22	20	18	16	14	12	10	8	6	4	2
25	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

From the table we can see that 9 is the multiplicative inverse of 3(mod26) as $9 \times 3 = 1(\text{mod}26)$. That is, $3^{-1}\text{mod}26 = 9$.

WORKED EXAMPLE 30 Using modular arithmetic

Simplify each of the following.

a. **200mod26** **b.** **5^{-1}mod26** **c.** **-9mod26**

THINK

a. Since mod26 refers to the remainder when dividing by 26, divide 200 by 26 and determine the remainder.

b. Use the table to determine the multiplicative inverse of 5.5^{-1} means $5^{-1} \times 5 = 1$.

c. Divide -9 by 26 and determine the positive remainder. Use $-9 = -1 \times 26 + 17$ rather than $-9 = 0 \times 26 - 9$.

WRITE

a. $200 \div 26 = 7$ remainder 18.
$200\text{mod}26 = 18$.

b. From the table, $5 \times 21 = 1(\text{mod}26)$.
Therefore $5^{-1}\text{mod}26 = 21$.

c. $-9 \div 26 = -1$ remainder 17.
Therefore -9 mod$26 = 17$.

Now we can use this to transform plain text to cipher text. In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by the following procedure

Step 1 Choose a 2×2 matrix with integer entries $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This matrix will be used to perform the encoding.

Step 2 Group successive plain-text letters into pairs, adding an arbitrary 'dummy' letter to fill out the pair if the plain text has an odd number of letters, and replace each plain-text letter by its last numerical value.

Step 3 Successively convert each plain-text pair, p_1p_2, into a column matrix: $P = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, and form the product AP. We will call P a *plain-text* matrix and AP the corresponding *cipher-text* matrix.

Step 4 Convert each cipher-text matrix into its alphabetic equivalent using modulo 26 if the numbers are greater than 26.

WORKED EXAMPLE 31 Enciphering using matrices

Use the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ to encipher the message: THE GAME IS UP.

THINK	WRITE
1. Group the letters in pairs, doubling the last if there is an odd number.	TH EG AM EI SU PP
2. Convert each letter to a number from 0 to 25. Use the table on page 171.	20 8 5 7 1 13 5 9 19 21 16 16
3. Premultiply the first pair of numbers by the given cipher matrix.	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 48 \\ 36 \end{bmatrix}$
4. Convert each number in the resulting matrix to modulo 26; that is, determine the remainder when each number is divided by 26. In this case, $48 = 22(\text{mod}26)$ and $36 = 10(\text{mod}26)$.	$= \begin{bmatrix} 22 \\ 10 \end{bmatrix} \text{mod}26$
5. Repeat steps 3 and 4 for the remaining pairs.	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \end{bmatrix} \text{mod}26$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \end{bmatrix} = \begin{bmatrix} 15 \\ 27 \end{bmatrix} = \begin{bmatrix} 15 \\ 1 \end{bmatrix} \text{mod}26$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \end{bmatrix} \text{mod}26$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 19 \\ 21 \end{bmatrix} = \begin{bmatrix} 59 \\ 61 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \text{mod}26$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 16 \\ 16 \end{bmatrix} = \begin{bmatrix} 48 \\ 48 \end{bmatrix} = \begin{bmatrix} 22 \\ 22 \end{bmatrix} \text{mod}26$ So 20 8 5 7 1 13 5 9 19 21 16 16 becomes 22 10 17 19 15 1 19 23 7 9 22 22
6. Convert each number to a letter of the alphabet using the table under Enciphering.	The enciphered message is V J Q S O A S W G I V V.

Because the plain text was grouped in pairs and enciphered by a 2×2 matrix, the cipher in the previous worked example is referred to as a 2-*cipher*. It is possible to group the plain text in triples and encipher by a 3×3 matrix with integer entries; this is called a 3-*cipher*. In general, for an *n-cipher*, plain text is grouped into sets of n letters and enciphered by an $n \times n$ matrix with integer entries.

Deciphering

To decipher a message that has been encrypted using matrix A, we use the inverse of A, A^{-1}. That is, for the enciphering 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the deciphering 2×2 matrix, A^{-1} is given by

$$A^{-1} = (ad - bc)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \pmod{26}$$

where $(ad - bc)^{-1}$ is the multiplicative inverse of $(ad - bc)$ in modulo 26.

Note that not all numbers have an inverse in mod26. The inverse of 3 is 9 because $9 \times 3 = 27 = 1 \pmod{26}$; but there is no inverse for 2 or 4 or 13. (See the table on page 174.)

Note also that the inverse of -3 is different from the inverse of 3. The inverse of 3 is 9 but $-3 \times 9 = -27 = 25 \pmod{26}$. The inverse of -3 is 17 as $-3 \times 17 = -51 = 1 \pmod{26}$.

WORKED EXAMPLE 32 Deciphering using matrices

Decode this cipher text that has been encoded using the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

DPOOOHZZVOQOVYYVQQ

THINK	WRITE
1. To determine the inverse matrix, A^{-1}, first calculate the determinant of matrix A.	$\det A = 2 \times 2 - 1 \times 1 = 3$
2. Determine the multiplicative inverse (mod26) of det A using the relationship that $(\det A)^{-1} \times (\det A) = 1$ for modulo 26. (Use the table.)	From the table, the multiplicative inverse of 3 is 9.
3. Calculate A^{-1}.	$9 \times 3 = 27 = 1 \pmod{26}$ $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = 9 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ $= \begin{bmatrix} 18 & -9 \\ -9 & 18 \end{bmatrix}$ $= \begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}$
4. Convert the elements in the resulting matrix to modulo 26.	

5. Convert the cipher text to a string of numbers. (Use the table.) Group the numbers in pairs.

D	P	O	O	O	H	Z	Z	V
4	16	15	15	15	8	0	0	22
O	Q	O	V	Y	Y	V	Q	Q
15	17	25	22	25	25	22	17	17

6. Multiply the first pair of numbers by the deciphering matrix $\begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}$.

$$\begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix} \begin{bmatrix} 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 344 \\ 356 \end{bmatrix}$$
$$= \begin{bmatrix} 6 \\ 18 \end{bmatrix} \text{mod}26$$

7. Convert the elements in the resulting matrix to modulo 26.

8. Repeat steps 6 and 7 for all pairs of numbers. List the decoded numbers.

4 16 15 15 15 8 0 0 22 15 17 25 22 25 25 22 17 17
gives 6 18 5 5 16 9 0 0 1 20 15 13 15 18 18 15 23 23

9. Convert the decoded numbers to text.

This gives FREEPIZZATOMORROWW and can be interpreted as: Free Pizza Tomorrow.

Exercise 5.6 Applications of matrices

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5.6 Exercise | 5.6 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3	4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,17, 18, 19	–

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Simple familiar

1. WE24 The transition matrix for a system is $\begin{bmatrix} 0.3 & 0.2 & 0.50 \\ 0.3 & 0.2 & 0.25 \\ 0.4 & x & 0.25 \end{bmatrix}$.

a. Calculate the value of x.

b. If the current state of the system is $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$, calculate the state of the system at each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

c. If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

d. If the current state of the system is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, calculate the state of the system at each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

2. The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.2 & c \\ 0.3 & b & 0.25 \\ a & 0.1 & 0.20 \end{bmatrix}$.

a. Calculate the values of a, b and c.

b. If the current state of the system is $\begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$, calculate the state of the system after each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

c. If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:

i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.

3. WE30 Simplify each of the following.

a. $300 \bmod 26$ b. $7^{-1} \bmod 26$ c. $-14 \bmod 26$ d. $200 \bmod 26$ e. $8^{-1} \bmod 26$

f. $-5 \bmod 26$ g. $431 \bmod 26$ h. $15^{-1} \bmod 26$ i. $-15 \bmod 26$

Complex familiar

4. WE25 The Robinson Islands are very wet. If it is raining on a particular day, the chance that it will rain the next day is 70%. If it is not raining on a particular day, the chance that it will rain on the following day is 52%.

a. Construct a transition matrix to represent this situation.

b. Using the transition matrix, calculate the probability that, if it is raining on Tuesday, it will also be raining on Friday of the same week.

c. Using technology, determine the long-term proportion of days it rains in the Nee Islands.

5. WE26 If $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$, determine the:

a. eigenvalues associated with the matrix.

b. eigenvectors associated with the matrix.

6. If $A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$, determine the:

a. eigenvalues associated with the matrix.

b. eigenvectors associated with the matrix.

7. If matrix T is a 2×2 transition matrix, then the columns of T will each add to give 1. Show that $\lambda = 1$ is always an eigenvalue for such a transition matrix.

8. a. Show that the eigenvalues of a matrix, $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ are found by solving the following equation for λ:

$$\lambda^2 - 5\lambda - 2 = 0$$

b. Show that the matrix A is a solution to the following equation.

$$A^2 - 5A - 2I = 0$$

9. **WE27** Given the consumption matrix $C = \begin{array}{c} \begin{array}{ccc} G & H & T \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ G \\ H \\ T \end{array}$,

a. state the meaning of the value 0.6 in the third row of the matrix
b. calculate the inputs needed to produce: 1 unit of H
c. calculate the inputs needed to produce 12 units of G, 15 units of H and 30 units of T.

10. Given the consumption matrix $C = \begin{array}{c} \begin{array}{ccc} G & H & T \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ G \\ H \\ T \end{array}$, calculate the inputs needed to produce:

a. 3 units of G.
b. 8 units of G, 25 units of H and 10 units of T.

11. **WE28** Consider the consumption matrix:

$$C = \begin{array}{c} \begin{array}{ccc} G & H & T \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} \end{array} \begin{array}{c} \\ G \\ H \\ T \end{array}$$

a. Calculate the production needed to match the demand $d = \begin{bmatrix} 30 \\ 45 \\ 25 \end{bmatrix}$.
b. Show that a suitable production can be found to match any demand.

12. Consider the following three-industry economy:
The external demands for A, B and C are 20, 30 and 10 respectively. The exchange matrix is: $M = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.5 & 0.6 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$

a. What is the meaning of the entry 0.5 in the matrix?
b. Calculate the required output for each of the three industries.

13. **WE29** A two-industry economy consists of coal and steel. To produce one tonne of coal requires 0.3 tonnes of coal and 0.4 tonnes of steel. To produce one tonne of steel requires 0.45 tonnes of coal and 0.2 tonnes of steel.

a. Construct the consumption matrix.
b. What production is required to meet the demand of 45 tonnes of coal and 25 tonnes of steel?

14. Consider the following hypothetical three-industry economy.

To manufacture 1 unit of *A* requires	**0.2 units of *A***	**0.3 units of *B***	**0.1 units of *C***
To manufacture 1 unit of *B* requires	0.1 units of *A*	0.2 units of *B*	0.2 units of *C*
To manufacture 1 unit of *C* requires	0.2 units of *A*	0.1 units of *B*	0.3 units of *C*

a. Construct the consumption matrix for this economy.
b. What inputs are required to produce 10 units of A, 20 units of B and 60 units of C?
c. How much of each should be produced to meet the external demand of 45 units of A, 30 units of B and 40 units of C?

15. WE31 Encode the message: LEAVE IMMEDIATELY using the enciphering matrix $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$.

16. Encode the message: THE IDES OF MARCH using the enciphering matrix $\begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$.

17. The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ has been used to encrypt the following message:
O L Y P X F R K R M E L.

a. What is A^{-1} mod26?

b. Use this matrix to decipher the message.

18. WE32 Decode this cipher text that has been encoded using the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$

I V WL A E D C O U E D V W K R Y R

19. Encode the message: ALAN TURING IS A LEGEND using a 3×3 enciphering matrix $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$.

Fully worked solutions for this topic are available online.

LESSON
5.7 Review

5.7.1 Summary

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5.7 Exercise

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5.7 Exercise	5.7 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

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Simple familiar

1. Given the following systems of linear equations,

a. $7x + 2y - 4z = 24$
$x - 8y + 2z = 22$

b. $2x - y + 3z = 25$
$3x + 5y - 4z = -3$
$8x + 2y - 5z = 26$

i. state the number of m equations and n variables in the system

ii. justify whether the system is square or not

iii. determine if $x = 6, y = -1, z = 4$ is a valid solution set all equations in the system.

2. MC Identify which of the following matrices is in row echelon form.

A. $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & -7 \end{array}\right]$ **B.** $\left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 2 & 5 \end{array}\right]$ **C.** $\left[\begin{array}{cc|c} 4 & 3 & 3 \\ 0 & 2 & 8 \end{array}\right]$ **D.** $\left[\begin{array}{cc|c} 1 & 9 & 4 \\ 0 & 1 & 7 \end{array}\right]$

3. State if the following are *true* of *false*.

a. A consistent system could have an infinite number of solutions.

b. A linear system in three unknowns can have no solution.

c. A line is formed when three planes intersect. This system is inconsistent.

d. Two parallel lines in three dimensions are distinct. This system is inconsistent.

e. A 3×3 linear system is consistent because $\underset{\sim}{\hat{\sim}} \cdot (\underset{\sim}{n} \times \underset{\sim}{n} \times \underset{\sim}{n})$.

4. MC A system consists of the following equations.

$$a_1x_1 + b_1y_1 + c_1z_1 = d_1$$
$$a_2x_2 + b_2y_2 + c_2z_2 = d_2$$
$$a_3x_3 + b_3y_3 + c_3z_3 = d_3$$

The solution for this system is $x = -2 + 4t,\ \ y = 3 + 8t,\ \ z = 4 - t$ where $t \in \mathbb{R}$. In geometric terms, the system and solution are best described by which of the following:

A. It is an inconsistent system.

B. It is a consistent system containing three planes. The planes intersecting with the solution forming a line. Infinitely many solutions exist.

C. It is a consistent system containing three planes. The planes intersecting at the unique position defined by $(x, y, z) = (-2, 3, 4)$

D. It is an inconsistent system containing three planes. Two of the planes are parallel and distinct.

5. Consider $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -4 \\ -3 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, when calculating if the following statements are true or false.

a. $A^2(A + B) = A^3 + AB$

b. $\left(A^3 - B^3\right)A^3 = A^3A^3 - A^3B^3$

6. In a round robin tennis competition, Tim recorded two losses to Nelly and Peter. Nelly's only loss was to Peter. Peter lost to Earnest. Use dominance matrices and the formula $M + 0.5M^2$ to rank the players.

7. Brooke Cove is a very monsoonal area with prevalent heavy rainfall. If it is raining on a particular day, the chance that it will rain the next day is 89%. If it is not raining on a particular day, the chance that it will rain on the following day is 71%.

a. Construct a transition matrix to represent this situation.

b. Using the transition matrix, calculate the probability that, if it is raining on Wednesday, it will also be raining on Saturday of the same week.

c. In the long term, determine on what proportion of days does it rain in the Brooke Cove.

8. Solve the following system of linear equations by using an inverse matrix.

$$\begin{aligned} 7x + 2y + 5z &= 17 \\ 3x + 8y + z &= 9 \\ 2x - 7y + z &= -15 \end{aligned}$$

9. Solve the following system of linear equations using Gaussian elimination and back substitution.

a. $\begin{aligned} 6x - 12y &= -18 \\ -2x + 7y &= 12 \end{aligned}$

b. $\begin{aligned} -2x + 6y + z &= -14 \\ -x + 6y - 3z &= -4 \\ -30y - 4z &= -30 \end{aligned}$

10. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2. Describe the geometric nature of the solution.

$$\begin{aligned} l_1&: \left(-3\hat{\imath} + 3\hat{\jmath} - 6\hat{k}\right) + t\left(6\hat{\imath} + 3\hat{\jmath} + 6\hat{k}\right) \\ l_2&: \left(6\hat{\imath} + 9\hat{\jmath} - 6\hat{\jmath}\hat{k}\right) + t\left(-3\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) \end{aligned}$$

11. Consider the following system,

$$\begin{aligned} -6x + 2y - 5z &= -22 \\ 8x - 3y - 4z &= 49 \\ -3x - 7y - 4z &= -48 \end{aligned}$$

a. Determine the number of possible solutions by examining the normal vectors algebraically.

b. Convert the equations to an augmented matrix and then use technology to solve the system.

c. Describe how the solution can be interpreted geometrically.

12. Consider the following system of linear equations,

$$\begin{aligned} 2x + 4y + 12z &= 22 \\ 8x + 3y - 30z &= 23 \\ 8x - 4y - 72z &= -12 \end{aligned}$$

a. Determine the number of possible solutions by examining the normal vectors algebraically.
b. Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.
c. Solve the system of linear equations, expressing the solution in parametric form.
d. Describe how the solution can be interpreted geometrically.

Complex familiar

13. Determine a general solution for the variable matrix X in the following matrix equations. (It is assumed that square matrices are non-singular).

a. $2X - 3B = C$ **b.** $BX - 2X + C = 0$ **c.** $A(X + I) + 2C = B$ **d.** $XA + B = XC - 2D$

14. The sum of three numbers is 10 When triple the first number is added to the sum of the second and third number the result is 26. The second number is 6 less than the third. Develop a suitable matrix equation and apply Gaussian elimination techniques to determine a unique solution.

15. Determine if the following systems are either consistent or inconsistent by completing the 'ref' or 'rref' operation on a graphics calculator.

a. $[A \mid B] = \left[\begin{array}{ccc|c} 36 & 24 & 16 & 84 \\ -9 & -6 & -4 & -21 \\ 6 & 4 & \frac{8}{3} & 14 \end{array}\right]$

b. $[A \mid B] = \left[\begin{array}{cc|c} -3 & 6 & 12 \\ 6 & -12 & -40 \end{array}\right]$

16. The numbers of a particular species of lizard, all aged 1, 2 or 3 years, have been monitored over many years and the following Leslie matrix has been developed to model the population.

$$L = \begin{bmatrix} 0 & 5.30 & 3.50 \\ 0.20 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

a. On average, how many new lizards do 1-year-old lizards produce?
b. On average, how many new lizards do 2-year-old lizards produce?
c. What proportion of 1-year-old lizards survive the first year?
d. If the initial female population is $N_1 = \begin{bmatrix} 35 \\ 46 \\ 22 \end{bmatrix}$, calculate the female population in 3 years' time.

Complex Unfamiliar

17. Determine the solution/s to the following system where $x, y, z \in \mathbb{R}$.

$$\begin{aligned} 7x^3 + 2y^2 + 3z &= -35 \\ 2x^3 + 5y^2 - 5z &= 24 \\ 3x^3 + y^2 - 4z &= -19 \end{aligned}$$

18. The linear equations below form a consistent system with infinitely many solutions.

$$\begin{aligned} p_1&: \quad -4x - 2y - 4z = -12 \\ p_2&: \quad -x + y + 5z = 9 \\ p_3&: \quad 2x + b_3 y + 4z = d_3 \end{aligned}$$

Calculate the unknowns values, b_3 and d_3.

19. A consistent linear system is given by:

$$\begin{aligned} 3x + 3y + 6z &= 3a \\ 10x + 10z &= 10b \\ 8x + 4y + 12z &= 4c \end{aligned}$$

Use Gaussian elimination to determine the relationship between a, b and c that must exist if the initial statement about the system type is true.

20. A study was carried out on a dingo population that inhabited a remote sand island. The population has the following characteristics.

- Half the dingoes survive the first year.
- Of the surviving dingoes, half survive the second year.
- In the first year of life, these dingoes produce no offspring.
- On average, 6 dingoes are produced in the second year and 8 in the third year.
- The current population is given by the initial state matrix,

$$V_1 = \begin{bmatrix} 60 \\ 40 \\ 36 \end{bmatrix} \quad \begin{matrix} 0 < t \le 1 \\ 1 < t \le 2 \\ 2 < t \le 3 \end{matrix}$$

a. Determine the dingo population in 5 and 10 years' time.

b. Compare the populations at both 5 and 10 years to show that a common ratio doesn't exist in terms of the number of dingoes per age group.

c. It is possible that a population, consisting of three age groups, will continuously abide by a fixed ratio. For example,

$$X = \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} \quad \begin{matrix} 0 < t \le 1 \\ 1 < t \le 2 \\ 2 < t \le 3 \end{matrix}$$

Determine an alternative for V_1 so that the ratio between age groups remains constant.

5.7 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 1, Q4; © State of Queensland (QCAA) 2024.

The age-specific population distribution of a particular species of animal is shown.

Age (years)	0 – 1	1 – 2	2 – 3	3 – 4
Female population	94	82	37	6
Breeding rate	0	1.3	0.9	0.2
Survival rate	0.6	0.8	0.4	0

The Leslie matrix based on this data is

A. $\begin{bmatrix} 94 & 82 & 37 & 6 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1.3 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

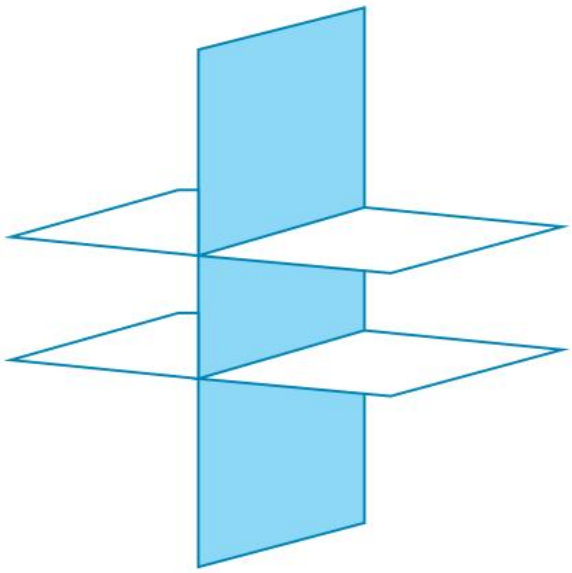
C. $\begin{bmatrix} 0.6 & 0.8 & 0.4 & 0 \\ 1.3 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1.3 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$

Question 2 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 1, Q9; © State of Queensland (QCAA) 2024.

The geometric interpretation of a certain system of three equations with no solution is shown.



Given two of the equations are $x + y - z = 0.5$ and $x - y - z = 0.5$, the third equation could be

A. $2x - 2y - 2z = 1$

B. $2x + 2y - 2z = 1$

C. $2x - 2y + 2z = 3$

D. $2x + 2y - 2z = 3$

Question 3 (5 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 2, Q12; © State of Queensland (QCAA) 2024.

Given $\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix}$ determine $\boldsymbol{X}$ in the matrix equation $\boldsymbol{XA} - \boldsymbol{XC} = \boldsymbol{B}$.

Question 4 (7 marks)

Source: QCAA 2022 Specialist Mathematics, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024.

Consider this system of equations that corresponds to three planes.

$$\begin{aligned} x + 5y &= 1 + 2z \\ x + z &= 3y + 3 \\ 8y - \lambda &= 3z \end{aligned}$$

a. Use a Gaussian technique to determine the value of λ for which this system of equations has infinitely many solutions. **[4 marks]**

b. Use the result from Question 16a to determine the infinitely many solutions.
Express your answer in the form of a vector equation of a line. **[3 marks]**

Question 5 (5 marks)

Source: QCAA 2022 Specialist Mathematics, Paper 2, Section 2, Q12; © State of Queensland (QCAA) 2024.

A scientist collects data for a species of tree frog in a protected area. Details for the female tree frog population are shown in the table.

Age (years)	0 – 1	1 – 2	2 – 3	3 – 4
Population in Year 1	150	101	84	62
Birth (breeding) rate	0.4	0.7	0.5	0.1
Survival rate	0.6	0.3	0.2	0

The scientist uses a Leslie matrix model to make predictions about the female tree frog population.

a. State the initial population matrix. **[1 mark]**

b. Determine the Leslie matrix. **[1 mark]**

A species is considered to be endangered if the female population in a restricted area is predicted to fall to less than 125 in the next 20 years.

c. Determine whether this species of tree frog is considered to be endangered. **[3 marks]**

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Answers

Chapter 5 Further matrices

5.2 Solving linear equations using matrix algebra

5.2 Exercise

1. a. and d. are non-linear.

2. a. i. $m = 2,\ n = 3$
   ii. Not a square system as $m \neq n$
   iii. Not a valid solution.
   b. i. $m = 3,\ n = 3$
   ii. It is a square system as $m = n$
   iii. The solution set satisfies all equations in the system.

3. a. $\begin{bmatrix} 178 & -250 \\ 125 & 303 \end{bmatrix}$
   b. $\begin{bmatrix} \frac{1114}{343} & \frac{586}{343} \\ -\frac{599}{343} & \frac{281}{343} \end{bmatrix}$
   c. $\begin{bmatrix} 74 & -62 & 30 \\ 30 & -152 & 84 \\ 210 & -48 & 26 \end{bmatrix}$
   d. $\begin{bmatrix} -\frac{108}{5} & -24 & -\frac{121}{5} \\ -\frac{104}{5} & -21 & -\frac{123}{5} \\ -\frac{91}{5} & -\frac{123}{5} & -\frac{108}{5} \end{bmatrix}$

4. a. $\begin{bmatrix} 71 & 4 \\ 63 & -28 \end{bmatrix}$ b. $\begin{bmatrix} 45 & 44 \\ -11 & 12 \end{bmatrix}$

5. a. $\det(A) = 46$
   b. $\det(B) = -20$
   c. $\det(C) = 6$
   d. $\det(D) = 13$

6. a. $\det(A) = 10$
   b. $\det(B) = 54$
   c. $\det(C) = 1417$
   d. $\det(D) = 1175$

7. a. $A^{-1} = \begin{bmatrix} \frac{-3}{4} & \frac{-1}{2} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$
   b. $B^{-1} = \begin{bmatrix} 10 & -4 \\ -3 & 2 \end{bmatrix}$
   c. $C^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{8} & \frac{5}{4} \end{bmatrix}$
   d. $D^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{1}{5} & \frac{-3}{5} \end{bmatrix}$

8. A is singular.
   B is non-singular.
   C is singular.
   D is singular.

9. a. $x = 5$ b. $x = -6$

10. a. $x = 6,\ y = -2$
    b. $x = 3,\ y = 3$
    c. $x = 6,\ y = 4$
    d. $x = -4,\ y = 8$

11. a. $x = -1,\ y = 0,\ z = -5$
    b. $x = 5,\ y = 7,\ z = 5$
    c. $x = 6,\ y = -3,\ z = 2$
    d. $x = -4,\ y = -5,\ z = -2$

12. a. $\mathbf{X} = \begin{bmatrix} 6 \\ 0 \\ -4 \end{bmatrix}$ b. $\mathbf{X} = \begin{bmatrix} 8 \\ 9 \\ -10 \end{bmatrix}$
    c. $\mathbf{X} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$ d. $\mathbf{X} = \begin{bmatrix} 7 \\ 10 \\ 9 \end{bmatrix}$
    e. $\mathbf{X} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$ f. $\mathbf{X} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$

13. a. $\mathbf{X} = \begin{bmatrix} 0 \\ 7 \\ 0 \\ 4 \end{bmatrix}$ b. $\mathbf{X} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 9 \end{bmatrix}$
    c. $\mathbf{X} = \begin{bmatrix} 14 \\ 8 \\ -6 \\ -6 \end{bmatrix}$ d. $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$

14. C

15. B

16. a. $X = \left(A^{-1}C\right)B^{-1}$ b. $X = BA^{-1}$
    c. $X = A^{-1}IB^{-1}$ d. $X = (A + I)^{-1}C$

17. a. $X = (A + B)^{-1}(D - C)$
    b. $X = (I2 + 3A)^{-1}B$
    c. $X = (A + 3B)^{-1}C$
    d. $X = (7A)^{-1}(B + C - 7AI)$

18. a. $X = \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{31}{18} & -\frac{19}{9} \end{bmatrix}$
    b. $X = \begin{bmatrix} -\frac{1}{18} & -\frac{7}{9} \\ -\frac{4}{9} & -\frac{2}{9} \end{bmatrix}$
    c. $X = \begin{bmatrix} -\frac{43}{84} & -\frac{53}{84} \\ -\frac{11}{28} & -\frac{9}{28} \end{bmatrix}$
    d. $X = \begin{bmatrix} \frac{64}{203} & \frac{78}{203} \\ -\frac{2}{7} & -\frac{2}{7} \end{bmatrix}$

19. a. $X = \begin{bmatrix} -\frac{5}{8} & -\frac{19}{16} \\ \frac{7}{8} & \frac{1}{16} \end{bmatrix}$

b. $X = \begin{bmatrix} -\frac{7}{20} & \frac{3}{10} \\ \frac{13}{20} & -\frac{11}{30} \end{bmatrix}$

c. $X = \begin{bmatrix} \frac{30}{217} & \frac{11}{31} \\ -\frac{2}{217} & -\frac{9}{31} \end{bmatrix}$

d. $X = \begin{bmatrix} -\frac{31}{45} & \frac{23}{45} \\ -\frac{16}{15} & -\frac{14}{5} \end{bmatrix}$

20. $X = (E + I)^{-1}(G - F)$

21. a. $X = \begin{bmatrix} 128 & 233 & -28 \\ 122 & 189 & -151 \\ 54 & 119 & 66 \end{bmatrix}$

b. $X = \begin{bmatrix} 8 & -2 & 5 \\ -14 & -11 & 20 \\ 14 & 1 & -9 \end{bmatrix}$

22. 1st place is awarded 10 points
2nd place is awarded 7 points
3rd place is awarded 5 points
5th place is awarded 3 points

23. The cost per individual item are:
Lolly bag $1
Pizza $7
Meat pie $5
Hamburger $6

24. True, LHS = RHS

25. $\alpha = \frac{\pi}{6}, \ \beta = \frac{\pi}{4}$

5.3 Solving a system of linear equations using Gaussian elimination

5.3 Exercise

1. B
2. C
3. B
4. a. $x = 3, \ y = 2$ b. $x = -5, \ y = 1$
c. $x = 2, \ y = 2$ d. $x = 1, \ y = -12$
5. a. $x = 0, \ y = -2, \ z = 0$
b. $x = -2, \ y = -4, \ z = 0$
c. $x = 3, \ y = 5, \ z = 2$
d. $x = -5, \ y = 5, \ z = 4$
e. $x = -3, \ y = -3, \ z = -1$
6. $5x + 2y + 4z = 32$
$-4x + 7y + 8z = -15$
$-3x + 11y + z = 9$
7. $w = 2, \ x = -3, y = 4, \ z = -5$
8. $x = 10, \ y = 20, \ z = 30$
9. Large = \$5, Regular = \$4.50
10. $A = 22, \ S = 5, \ P = 8$
Two students, 1 adult and 2 pensioners will cost
Cost = \$48
11. a. $(x, y) = (2, 3)$
b. $(x, y) = (-2, 3)$
c. $(x, y) = (6, -2)$
12. a. $(x, y, z) = (1, 0, 4)$
b. $(x, y, z) = (10, -2, 1)$
c. $(x, y, z) = (1, -3, 2)$
13. $\left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 3 & 1 & -2 & 4 \\ -5 & \boxed{4} & \boxed{-1} & -4 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 0 & 16 & -5 & \boxed{22} \\ 0 & \boxed{-21} & 4 & -34 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 + 5R_1 \to R3 \end{array}$

$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 0 & 1 & -5/16 & 11/8 \\ 0 & 0 & -1 & \boxed{44} \end{array}\right] \begin{array}{l} R_2 \div 16 \to R_2 \\ 5R_3 - 21R_1 \to R_3 \end{array}$

14. $y(x) = 2x^2 - 4x + 1$ or $y(x) = 1 - 4x + 2x^2$

15. $A^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{5}{3} & -\frac{8}{3} \\ \frac{1}{9} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix}$

16. $x = \pm\sqrt{7}, \quad y = \pm\sqrt{5}, \quad z = \pm\sqrt{3}$

5.4 The three cases for solutions of systems of linear equations

5.4 Exercise

1. C
2. a. No solution
b. No solution
c. One unique solution
d. Infinitely many solutions
3. a. Unique solution $(x, y) = (1, 2)$ is the intersection of two lines that are not parallel.
b. Zero solutions.
The lines are parallel, yet distinct as they have different y-intercepts.
c. Unique solution $(x, y) = (4, -2)$ is the intersection of two lines that are not parallel.
d. Zero solutions. The lines are parallel, yet distinct as they have different y-intercepts.
4. a. i. The planes are not coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
ii. The three planes intersect at $(6, -4, -3)$.
iii. The planes intersect at one unique point.
b. i. The planes are coplanar. This implies that the system has zero or infinitely many solutions.

ii. The final row is a false statement, so this system has no solution.

iii. Geometrically the solution represents an inconsistent system with distinct planes intersecting as pairs.

c. i. The planes are not parallel.

ii. The system is consistent and has one unique solution. The three planes intersect at $(-7, -1, -1)$.

iii. Geometrically the solution indicates the planes intersect at one unique point.

d. i. The planes are not coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.

ii. The system is consistent and has one unique solution. The three planes intersect at $(3, 7, 4)$.

iii. Geometrically the solution indicates the planes intersect at one unique point.

e. i. The planes are coplanar. This implies that the system has zero or infinitely many solutions.

ii. The final row is a false statement, so this system has no solution.

iii. Geometrically the solution represents an inconsistent system with distinct planes intersecting as pairs.

f. i. The planes are coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.

ii. The final row is a false statement, so this system has no solution.

iii. Geometrically the solution represents an inconsistent system with distinct planes intersecting as pairs.

5. i. a. There are an infinite number of solutions.

b. Parametric form $(x, y) = \left(\frac{3t+6}{5}, t\right)$.

ii. a. There are an infinite number of solutions.

b. Parametric of form $(x, y) = (2t - 4, t)$.

6. a. One unique solution. The lines are not parallel and intersect.

b. There are infinitely many solutions. The lines are coincident, hence the same line.

c. Zero solution exist. The two lines are parallel but distinct.

d. Zero solutions exist. The two lines are parallel but distinct.

7. a. $a \neq 12$ b. $b = 1,\ c = 6$

8. i. a. A unique solution exists at $(-19, -17, -24)$.

b. The two distinct lines intersect at one unique point.

ii. a. A unique solution exists at $(-2, -2, -2)$.

b. The two distinct lines intersect at one unique point.

iii. a. A unique solution exists at $(46, 14, 38)$.

b. The two distinct lines intersect at one unique point.

9. i. a. A unique solution exists at $(-24, -33, 40)$.

b. The two distinct lines intersect at one unique point.

ii. a. A unique solution exists at $(23, -13, 16)$.

b. The two distinct lines intersect at one unique point.

10. i. a. Zero solutions exist.

b. The two are parallel and distinct which means they will never intersect.

ii. a. The system has a unique solution at $(-16, -24, 25)$.

b. The two lines intersect at $(-16, -24, 25)$.

iii. a. The system has a unique solution at $(-50, 38, -41)$.

b. The two lines intersect at $(-50, 38, -41)$.

11. i. a. Zero solutions exist.

b. The two are parallel and distinct which means they will never intersect.

ii. a. The system has a unique solution.

b. The two lines intersect at $(9, -15, 37)$.

12. a. Coincident planes are the same plane. There are infinitely many solutions.

b. One unique solution exists.

c. The system is consistent with infinitely many solutions existing along the line,
$\underset{\sim}{r} = \left(4\hat{\imath} - 6\hat{\jmath} - 11\hat{k}\right) + t\left(-3\hat{\imath} + 13\hat{\jmath} + 2\hat{k}\right)$.

13. a. $x = \frac{-1-5t}{8},\quad y = -1 - t, z = t$

b. $x = 1 - t,\quad y = -1 + 2t,\quad z = t$

c. No solution.

14. a. The planes are parallel yet distinct. It is an inconsistent system with zero solutions.

b. The planes are parallel yet distinct. It is an inconsistent system with zero solutions.

c. Two planes are parallel yet distinct. It is an inconsistent system with zero solutions.

d. Two planes are parallel yet distinct. It is an inconsistent system with zero solutions.

15. a. $x = 8 + 6t,\ y = -1 - t,\ z = t$

b. $x = -13 + 4t,\ y = 24 - 4t,\ z = t$

c. $x = -6 + t,\ y = 22 - 2.5t,\ z = t$

d. $x = 14 + 5t,\ y = 1 - t,\ z = t$

16. a. The parametric equation satisfies all equations in the system. It is a valid solution set.

b. The parametric equation satisfies all equations in the system. It is a valid solution set.

17. a. Infinitely many solutions

b. Zero solutions

c. Unique solution at $P(5, 6, -7)$

18. a. Unique solution at $P(6, -3, 4)$

b. No solution (line is parallel yet distinct from the plane).

c. Infinitely many solutions (the line lies on the plane).

19. $(x, y, z) = \left(-\frac{1124}{49}, -\frac{831}{49}, -\frac{416}{49}\right)$

20. a. $\lambda = 110$

b. $l_1: \left(x = -11 + 6k, y = 11 - 2k, z = k\right)$

21. a. $\lambda = 9$

b. $\underset{\sim}{r} = (-34\underset{\sim}{\hat{\imath}} + 41\underset{\sim}{\hat{\jmath}}) + t\left(4\underset{\sim}{\hat{\imath}} - 5\underset{\sim}{\hat{\jmath}} + \underset{\sim}{\hat{k}}\right)$

22. $d_2 = 8,\ a_3 = 12$

23. a. $x = 1 + 2t,\ y = 7 - \frac{3}{2}t,\ z = t$

Vector equation of line:

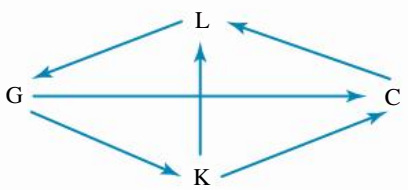
$\underset{\sim}{r} = \left(\hat{\imath} + 7\hat{\jmath}\right) + t\left(2\hat{\imath} - \frac{3}{2}\hat{\jmath} + \hat{k}\right)$

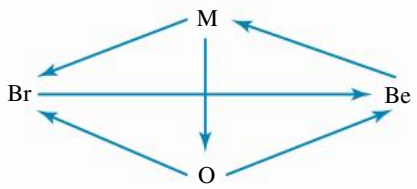
b. The vector product of the normals is parallel to the directional vector of the line of intersection formed by the two planes.

5.5 Dominance and Leslie matrices

5.5 Exercise

1. 8301

2. 

3. a. 

   b. The players would be ranked Cameron, Breanna, Kayley, Teagan.

4. Players can be ranked Mair, Ann, Janine

5. a. The houses are ranked Hamilton, Leslie, Cunningham, Barnes.

   b. Hamilton received 20 points.
   Leslie received 15 points.
   Cunningham received 10 points.
   Barnes received 5 points.

6. a. 1.4 females per year

   b. 50% live to age 3

   c. 91 female rats

   d. 674 (337 females)

7. a. 0 new birds b. 3.1 new birds
   c. 0.2 survive d. 338 females

8. a. $M = \begin{array}{c} \begin{array}{ccccc} C & W & G & S & I \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} C \\ W \\ G \\ S \\ I \end{array}$

   b. Warwick 5 points, Ipswich 4 points, Stanthorpe 3 points, Clifton 2 points, Goondiwindi 1 point

9. a. 6

   b. 3.6

10. a. $L = \begin{bmatrix} 0 & 0.50 & 0.45 & 0.4 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.85 & 0 & 0 \\ 0 & 0 & 0.50 & 0 \end{bmatrix}$

    b. 7460 sheep (3730 females)

11. a. i. 0.85 ii. 0.99 iii. 0

    b. i. 0 ii. 0.4 iii. 0.08

    c. $\begin{bmatrix} 0 & 0.40 & 0.08 \\ 0.85 & 0 & 0 \\ 0 & 0.99 & 0 \end{bmatrix}$

    d. 769 female sheep, 1538 total population.

12. $N_9 = \begin{bmatrix} 2705 \\ 870 \\ 223 \end{bmatrix}$

    7596 (3798 female).

13. Emma won the competition.

14. It will take 20 years.

5.6 Applications of matrices

5.6 Exercise

1. a. $x = 0.6$

   b. i. $P = \begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$

   iii. $P = \begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.353 \\ 0.255 \\ 0.392 \end{bmatrix}$

   c. i. $P = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.35 \\ 0.25 \\ 0.40 \end{bmatrix}$

   iii. $P = \begin{bmatrix} 0.355 \\ 0.255 \\ 0.390 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.353 \\ 0.255 \\ 0.393 \end{bmatrix}$

   d. i. $P = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$

   iii. $P = \begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$

2. a. $a = 0.5, b = 0.7, c = 0.55$

   b. i. $P = \begin{bmatrix} 0.455 \\ 0.345 \\ 0.210 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.274 \\ 0.428 \\ 0.299 \end{bmatrix}$

   iii. $P = \begin{bmatrix} 0.305 \\ 0.456 \\ 0.239 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.284 \\ 0.470 \\ 0.246 \end{bmatrix}$

   c. i. $P = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0.375 \\ 0.395 \\ 0.230 \end{bmatrix}$

   iii. $P = \begin{bmatrix} 0.281 \\ 0.447 \\ 0.273 \end{bmatrix}$ iv. $P = \begin{bmatrix} 0.296 \\ 0.465 \\ 0.240 \end{bmatrix}$

3. a. 14 b. 15 c. 12
   d. 18 e. Not possible f. 21
   g. 15 h. 7 i. 11

4. a. $\begin{bmatrix} 0.7 & 0.52 \\ 0.3 & 0.48 \end{bmatrix}$

   b. 0.636 or 63.6%

   c. The long-term probability of rain is 0.63415 and dry is 0.36585

5. a. $\lambda = 7,\ \lambda = -2$ b. $\begin{bmatrix} a \\ -\frac{4a}{5} \end{bmatrix}$ and $\begin{bmatrix} a \\ a \end{bmatrix}$

6. a. $\lambda = -0.2, \lambda = 1$ b. $\begin{bmatrix} a \\ \frac{7a}{5} \end{bmatrix}$ and $\begin{bmatrix} a \\ -a \end{bmatrix}$

7. $\lambda = 1$

8. a-b. Sample responses can be found in the worked solutions in the online resources.

9. a. This entry means it takes 0.6 units of G to make 1 unit of T.

b. The input q is 0.5 units of G, 0.4 units of H and 0.1 units of T.

c. The input q is 17.1 units of G, 17.4 units of H and 11.7 units of T.

10. a. The input needed is 0.6 units of G, 0.6 units of H and 1.2 units of T.

b. The input needed is 7.6 units of G, 14.6 units of H and 6.7 units of T.

11. a. The production that exactly matches the demand is 109 units of G, 171 units of H and 119 units of T.

b. It will always be possible to find values of P to match the demand d.

12. a. To produce 1 unit of A requires 0.5 units of C.

b. The required output is 37 units of A, 50 units of B and 59 units of C.

13. a. $c = \begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} 0.3 & 0.45 \\ 0.4 & 0.20 \end{bmatrix} \end{array} \begin{array}{c} C \\ S \end{array}$

b. The production that meets the demand is 124 tonnes of coal and 93 tonnes of steel.

14. a. $C = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array}$

b. The input q is 16 units of A, 13 units of B and 23 units of C.

c. The production that meets the demand is 90 units of A, 83 units of B and 94 units of C.

15. The enciphered message is M K P G K X M Z V Y U N C S U G

16. The enciphered message is BHCXGCIPYSNMTQ

17. a. $A^{-1} \bmod 26 = \begin{bmatrix} 8 & 11 \\ 11 & 4 \end{bmatrix}$

b. R E L A X B E H A P P Y or RELAX BE HAPPY

18. YOU SHOULD SEE ENIGMA.

19. C Z O A W J R X K K R D O D O N E Z B P P.

5.7 Review

5.7 Exercise

1. a. i. $m = 2, n = 3$

ii. The system is not square because $m \neq n$

iii. The solution is valid for all equations in the system.

b. i. $m = 3, n = 3$

ii. The system is square because $m = n$

iii. The solution is valid for all equations in the system.

2. D

3. a. True b. True c. False
d. True e. True

4. B

5. a. False: LHS $\neq$ RHS

b. False: LHS $\neq$ RHS

6. The players are ranked Peter, Nelly, Earnest and Tim.

7. a. $\begin{bmatrix} 0.89 & 0.71 \\ 0.11 & 0.29 \end{bmatrix}$

b. 0.8666

c. The long-term probability of rain is 0.86585 and dry is 0.13415.

8. The unique solution is the point $(x, y, z) = (-6, 2, 11)$.

9. a. The unique solution is the point $(x, y) = (1, 2)$.

b. The unique solution is the point $(x, y, z) = (10, 1, 0)$.

10. A unique solution exists at $(3, 6, -1)$.
The two distinct lines intersect at one unique point.

11. a. The planes are not coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.

b. The system is consistent and has one unique solution. The three planes intersect at $(7, 5, -2)$.

c. The planes intersect at one unique point, $(7, 5, -2)$.

12. a. The three planes are coplanar, and not parallel. Geometrically, this could be the planes intersecting in a line or not at all.

b. $[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & -6 & 1 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The system has infinitely many solutions because the planes intersect in a line.

c. $x = 1 + 6t$
$y = 5 - 6t$
$z = t$

d. This is a consistent system with the solution representing the intersection of three planes in the form of a line.

13. a. $X = \frac{1}{2}(C + 3B)$

b. $X = -(B - 2)^{-1}C$

c. $X = \left[A^{-1}(B - 2C)\right] - I$

d. $X = (-2D - B)(A - C)^{-1}$

14. $(x, y, z) = (8, -2, 4)$

15. a. The system is consistent (infinite number of solutions).

b. It is an inconsistent system (parallel yet distinct lines).

16. a. 0

b. 5.3

c. 0.2

d. $\begin{bmatrix} 77.4 \\ 64.2 \\ 1.8 \end{bmatrix}$; 143 female lizards

17. $(x, y, z) = (-2, \pm 3, 1)$

18. The line is defined by the parametric equations:
$x = -1 + t, y = 8 - 4t, z = t$

$$p_3: 2x + \frac{3}{2}y + 4z = 10$$

$$\therefore b_3 = \frac{3}{2}, d_3 = 10$$

19. The relationship is $0 = a + b - c$

20. a. $X_5 = \begin{bmatrix} 2076 \\ 852 \\ 85 \end{bmatrix}$; Population = 3013

$X_{10} = \begin{bmatrix} 83 & 856 \\ 20 & 246 \\ 53 & 82 \end{bmatrix}$; Population = 109 484

b. $\begin{bmatrix} 24t \\ 10t \\ t \end{bmatrix} \neq \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$

The ratios are not equivalent at 5 and 10 years

c. $X = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$

This is in a ratio of 16 : 4 : 1. If the initial dingo population conforms to $X = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$, the population each year that follows will remain in the same ratio.

5.7 Past QCAA exam questions

1. D

2. D

3. $X = \begin{pmatrix} 2 & -4 \\ 4 & -7 \end{pmatrix}$

4. a. When $\lambda = -2$ there are infinitely many solutions.

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{8}{3} \end{bmatrix} t + \begin{bmatrix} \frac{7}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$

5. a. $N_1 = \begin{bmatrix} 150 \\ 101 \\ 84 \\ 62 \end{bmatrix}$

b. $L = \begin{bmatrix} 0.4 & 0.7 & 0.5 & 0.1 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

c. $N_{20} = \begin{bmatrix} 62.7 \\ 39.7 \\ 12.6 \\ 2.7 \end{bmatrix}$

$62.7 + 39.7 + 12.6 + 2.7 < 125$

The female population has fallen below 125, thus the species is endangered.

UNIT

4 Further calculus and statistical inference

TOPIC 1 **Integration techniques**

6 Integration techniques 391

TOPIC 2 **Applications of integral calculus**

7 Applications of integral calculus 477

TOPIC 3 **Rates of change and differential equations**

8 Rates of change and differential equations 551

9 Applications of first-order differential equations 621

TOPIC 4 **Modelling motion**

10 Modelling motion 683

TOPIC 5 **Statistical inference**

11 Statistical inference 767

PRACTICE ASSESSMENT 3 **Unit 4 Examination** ..... online only

PRACTICE ASSESSMENT 4 **Units 3 & 4 Examination** ..... online only

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6 Integration techniques

LESSON SEQUENCE

6.1 Overview 392
6.2 Integration by linear substitution 394
6.3 Integration by non-linear substitutions 406
6.4 Integration using the trigonometric identities 415
6.5 Integration of inverse trigonometric functions 428
6.6 Integration by parts 450
6.7 Integration involving partial fractions 457
6.8 Review 466
Answers 470

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 6
Exam questions	Exam question booklet — Chapter 6
Digital documents	Learning matrix — Chapter 6 Chapter summary — Chapter 6

LESSON
6.1 Overview

Hey students! Bring these pages to life online

Engage with interactivities

Answer questions and check results

Track your progress

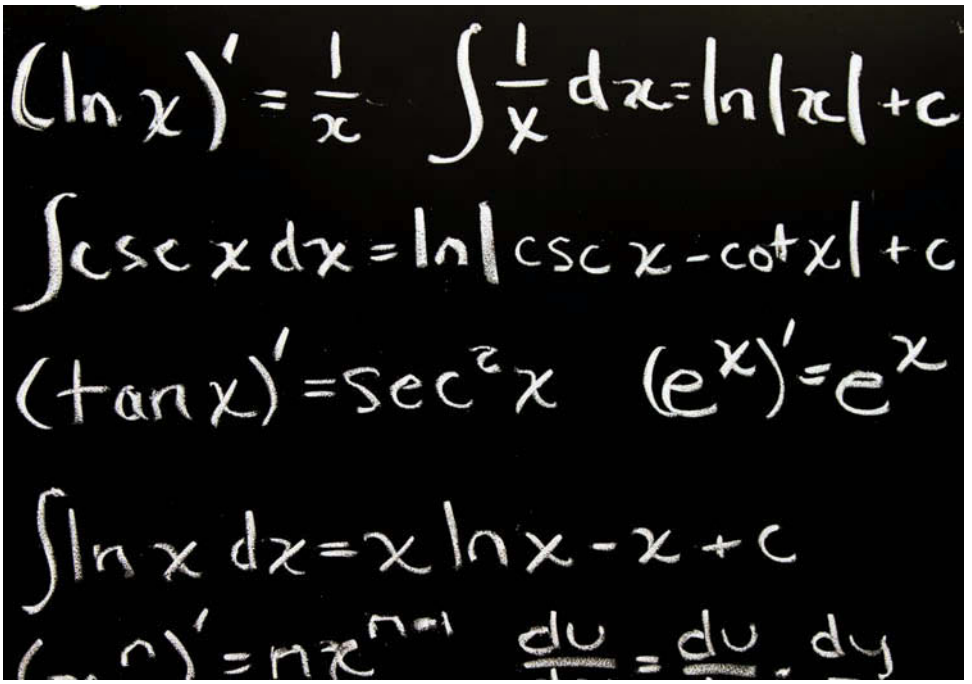
Find all this and MORE in jacPLUS

6.1.1 Introduction

Gottfried Wilhelm von Leibniz (1646–1716) and Isaac Newton (1643–1727) are credited with independently developing the foundations of calculus, identifying the inverse relationship between differentiation and integration. Their discoveries revolutionised science and mathematics, enabling us to model and solve problems involving change and accumulation.

You began your study of integration in Mathematical Methods, learning how to integrate a variety of basic functions. In Specialist Mathematics, we now take this further. This chapter introduces a powerful suite of integration techniques that allow you to handle more complex expressions — combinations of functions that cannot be integrated using basic rules alone.

Why is this important? Integration is used to calculate areas, volumes and accumulated quantities in real-world contexts. Engineers use it to design structures and systems, economists use it to model growth and optimise resources, and scientists use it to understand motion, energy and change. Whether you're analysing the trajectory of a spacecraft, the flow of electricity or the spread of a virus, integration is the tool that helps you quantify and predict.



6.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
6.2	**Integration by linear substitution**	◯ Use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))\, g'(x)$. ◯ Establish and use the formula $\int \frac{1}{x}\, dx = \ln\lvert x\rvert + c$ for $x \neq 0$ and $\int \frac{f'(x)}{f(x)}\, dx = \ln\lvert f(x)\rvert + c$ for $f(x) \neq 0$.
6.3	**Integration by non-linear substitutions**	◯ Use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))\, g'(x)$.
6.4	**Integration using the trigonometric identities**	◯ Integrate using the trigonometric identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, $1 + \tan^2(x) = \sec^2(x)$ and $\cot^2(x) + 1 = \operatorname{cosec}^2(x)$. ◯ Establish and use the formula $\int \sec^2(x)\, dx = \tan(x) + c$.
6.5	**Integration of inverse trigonometric functions**	◯ Understand and use the inverse trigonometric functions: arcsine, arccosine and arctangent. ◯ Use the derivative of the inverse trigonometric functions: arcsine, arccosine and arctangent. • $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$ • $\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$ • $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$ ◯ Integrate expressions of the form $\frac{\pm 1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$ • $\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$ • $\int \frac{-1}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + c$ • $\int \frac{a}{a^2 + x^2} = \tan^{-1}\left(\frac{x}{a}\right) + c$
6.6	**Integration by parts**	◯ Integrate by parts. • $\int u\frac{dv}{dx}\, dx = uv - \int v\frac{du}{dx}\, dx$
6.7	**Integration involving partial fractions**	◯ Use partial fractions for integration involving two distinct linear factors in the denominator, e.g. $\frac{2x - 1}{(x + 1)(x - 2)}$.

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LESSON
6.2 Integration by linear substitution

SYLLABUS LINKS

- Use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))\, g'(x)$.
- Establish and use the formula $\int \frac{1}{x}\, dx = \ln|x| + c$ for $x \neq 0$ and $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$ for $f(x) \neq 0$.

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6.2.1 Determining integrals in the form $\int (ax+b)^n\, dx$

When differentiating a function in the form $f(g(x))$, it is possible to make the substitution $u = g(x)$ and use the chain rule to determine the derivative. When determining an anti-derivative, it is also possible to make an appropriate substitution in order to rewrite the integrand as a function that has a known integral. To begin, we will explore functions when the substitution $u = ax + b$ is appropriate.

Determining integrals in the form $\int (ax+b)^n\, dx,\ n \in \mathbb{Z}^+$

Integrals in this form could be found by expanding. However, as n becomes larger, this becomes impractical.

In Mathematical Methods, you have used $\frac{du}{dx} = f(x)$, you will see in the following example that this can be manipulated to express $du = f(x)\, dx$.

WORKED EXAMPLE 1 Determining integrals in the form $\int (ax+b)^n\, dx, n \in \mathbb{Z}^+$

Determine $\int (2x-5)^4\, dx$.

THINK	WRITE
1. Although $(2x-5)^4$ could be expanded, this would result in a sum with 5 terms. Let $u = 2x - 5$ and use differentiation to determine dx.	Let $u = 2x - 5$ $\frac{du}{dx} = 2$ $du = 2\, dx$ $dx = \frac{du}{2}$
2. Rewrite the integral, substituting for $2x - 5$ and dx.	$\int (2x-5)^4\, dx = \int u^4\, \frac{du}{2}$ $= \int \frac{1}{2} u^4\, du$

3. As the integrand is now expressed as a function of u only, it can be integrated.

$$= \frac{1}{2}\left(\frac{u^5}{5}\right) + c$$

$$= \frac{1}{10}u^5 + c$$

4. As the original integral was in terms of x, substitute $u = 2x - 5$.

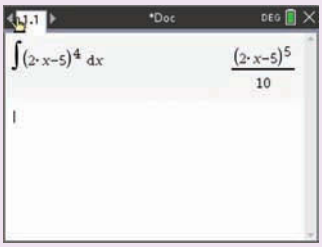
$$= \frac{1}{10}(2x-5)^5 + c$$

TIP

A calculator with CAS functionality, such as the TI-Nspire™ CX CAS for instance, can be used, with the **CAS mode ON**, to obtain an anti-derivative. This can be useful to check your work.

Note that you will not be able to use the CAS mode in examinations.

A CASIO calculator such as the fx-CG50 does not have CAS functionalities, and cannot determine an indefinite integral (without lower and upper bonds).



Determining integrals in the form $\int (ax+b)^n\,dx,\ n \in \mathbb{Q}, n \neq -1$

In this instance, it is not possible to use expansion to rewrite the integral, so substitution must be used. It may be necessary to rewrite the integrand so that it is in the form $(ax+b)^n$.

WORKED EXAMPLE 2 Determining integrals in the form $\int (ax+b)^n\,dx, n \in \mathbb{Q}, n \neq -1$

The gradient function $\frac{dy}{dx}$ of a curve is given by $\frac{1}{\sqrt{4x+9}}$.

Determine the equation of the particular curve that passes through the origin.

THINK	WRITE
1. For a curve represented by the equation $y = f(x)$, the gradient function is given by $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{1}{\sqrt{4x+9}}$
2. Integrate both sides to obtain an expression for y.	$y = \int \frac{1}{\sqrt{4x+9}}\,dx$
3. Use index laws to express the integrand as a function to a power. Note $\sqrt{u} = u^{\frac{1}{2}}$.	$y = \int (4x+9)^{-\frac{1}{2}}\,dx$
4. The integrand can be rewritten using $u = 4x+9$. Differentiate to determine du.	Let $u = 4x+9$ $\frac{du}{dx} = 4$ $du = 4\,dx$ $dx = \frac{du}{4}$

5. Rewrite the integral, substituting for $4x+9$ and dx.

$$\int (4x+9)^{-\frac{1}{2}}\,dx = \int u^{-\frac{1}{2}}\,\frac{du}{4}$$

6. As the integrand is now expressed as a function of u only, it can be integrated.

$$= \int \frac{1}{4}u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\left(2u^{\frac{1}{2}}\right) + c$$

$$= \frac{1}{2}u^{\frac{1}{2}} + c$$

7. As the original integral was in terms of x, substitute $u = 4x+9$.

$$y = \frac{1}{2}\sqrt{4x+9} + c$$

8. As the curve passes through the origin, c can be determined by substituting $x=0$ and $y=0$.

$$0 = \frac{1}{2}\sqrt{9} + c$$

$$c = -\frac{3}{2}$$

9. Substitute for c and state the equation for y.

$$y = \frac{1}{2}\sqrt{4x+9} - \frac{3}{2}$$

$$y = \frac{\sqrt{4x+9}-3}{2}$$

Determining integrals in the form $\int (ax+b)^n\,dx,\ n=-1$

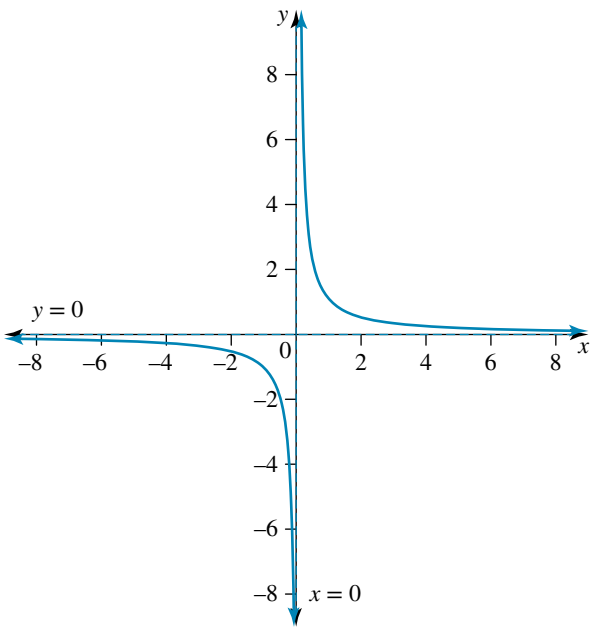
$\int \frac{1}{x}\,dx = \ln(x), x>0$. This is because if $y=\ln(x)$, then

$$x = e^y, x>0$$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$= \frac{1}{x}$$



Therefore $\int \frac{1}{x}\,dx = \ln(x) + c, x>0$.

However, as $\frac{1}{x}$ is defined for $\mathbb{R}\setminus\{0\}$, $\int \frac{1}{x}\,dx$ exists for $\mathbb{R}\setminus\{0\}$.

Consider the function $y = \ln|x|, x \neq 0$.

We can use the chain rule to differentiate this function if $u = |x|, x \neq 0$. Notice that $u > 0$ so we know that $\frac{d}{du}\ln(u) = \frac{1}{u}$.

$$y = \ln(u), u > 0$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$u = |x|, \quad x \neq 0$$

$$= \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$= \frac{du}{dx} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \begin{cases} \frac{1}{u} \times 1, & x > 0 \\ \frac{1}{u} \times 1, & x < 0 \end{cases}$$

Substituting $u = |x|$

$$= \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} \frac{1}{|x|}, & x > 0 \\ -\frac{1}{|x|}, & x < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{x}, & x > 0 \\ -\frac{1}{-x}, & x < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{x}, & x < 0 \end{cases}$$

$$= \frac{1}{x}, \quad x \neq 0$$

The anti-derivative of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + c, \; x \neq 0$$

WORKED EXAMPLE 3 Determining the anti-derivative of $\frac{1}{ax+b}$ for $x \neq -\frac{b}{a}$

Determine the anti-derivative of $\frac{1}{5x+4}$ for $x \neq -\frac{4}{5}$.

THINK	WRITE
1. Express the question using integral notation.	$\int \frac{1}{5x+4}\,dx$
2. The integrand can be rewritten using $u = 5x+4$. Differentiate to determine du.	Let $u = 5x+4$ $\frac{du}{dx} = 5$ $du = 5\,dx$ $dx = \frac{du}{5}$
3. Rewrite the integral, substituting for $5x+4$ and dx.	$\int \frac{1}{5x+4}\,dx = \int \frac{1}{u}\frac{du}{5}$ $= \frac{1}{5}\int \frac{1}{u}\,du$
4. Use $\int \frac{1}{x}\,dx = \ln\lvert x\rvert, x \neq 0$ to find the integral.	$= \frac{1}{5}\ln\lvert u\rvert + c$
5. As the original integral was in terms of x, substitute $u = 5x+4$.	$= \frac{1}{5}\ln\lvert 5x+4\rvert + c$

The last three examples can be summarised as follows

Integrals of the form $\int (ax+b)^n\,dx, n \in \mathbb{Q}$

$$\int (ax+b)^n\,dx = \begin{cases} \frac{1}{a(n+1)}(ax+b)^{n+1} + c & n \neq -1 \\ \frac{1}{a}\ln\lvert ax+b\rvert + c & n = -1 \end{cases}$$

TIP

Note that you need to check that $(ax+b)^n$ is defined before using these formulas.

For instance, $\int \frac{1}{\sqrt{4-5x}}\,dx = -\frac{2}{5}\sqrt{4-5x} + c$ when $\frac{1}{\sqrt{4-5x}}$ is defined, that is for $4-5x > 0$.

6.2.2 Evaluating definite integrals using a linear substitution

When using substitution to evaluate definite integrals, there is no need to return to an expression in terms of x provided that the terminals are expressed in terms of u.

WORKED EXAMPLE 4 Evaluating definite integrals using a linear substitution

Calculate $\displaystyle\int_0^1 \frac{4}{(3x+2)^2}\,dx$.

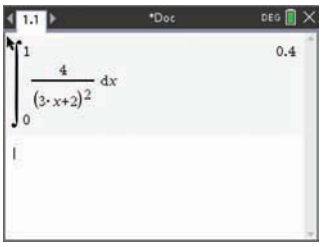
THINK	WRITE
1. Use index laws to express the integrand as a function to a power.	$\displaystyle\int_0^1 4(3x+2)^{-2}\,dx$
2. The integrand can be rewritten using $u = 3x+2$. Differentiate to determine du.	Let $u = 3x+2$ $\dfrac{du}{dx} = 3$ $du = 3\,dx$ $dx = \dfrac{du}{3}$
3. Rewrite the integral, substituting for $3x+2$ and dx. Note that the integral is occurring between $x=0$ and $x=1$.	$\displaystyle\int_{x=0}^{x=1} 4\,(3x+2)^{-2}\,dx = \int_{x=0}^{x=1} 4u^{-2}\,\frac{du}{3}$ $\displaystyle = \frac{4}{3}\int_{x=0}^{x=1} u^{-2}\,du$
4. Determine values for u for $x=0$ and $x=1$.	$x=0$: $u = 3(0)+2 = 2$ $x=1$: $u = 3(1)+2 = 5$ $\displaystyle\frac{4}{3}\int_{u=2}^{u=5} u^{-2}\,du$
5. As the integrand is now expressed as a function of u only, it can be integrated.	$\displaystyle = \frac{4}{3}\left[\frac{u^{-1}}{-1}\right]_2^5$ $\displaystyle = \frac{4}{3}\left[\frac{-1}{u}\right]_2^5$
6. Evaluate the definite integral.	$\displaystyle = \frac{4}{3}\left[\frac{-1}{5} - \frac{-1}{2}\right]$ $\displaystyle = \frac{4}{3}\left(\frac{-2}{10} + \frac{5}{10}\right)$ $\displaystyle = \frac{4}{10}$ $\displaystyle = \frac{2}{5}$
7. State the final result.	$\displaystyle\int_0^1 \frac{4}{(3x+2)^2}\,dx = \frac{2}{5}$

TI | THINK

On a calculator page, go to MENU, Calculus, Integral and complete the entry line as:

$\int_0^5 \frac{4}{(3x+2)^2}\,dx$ and press ENTER.

DISPLAY/WRITE

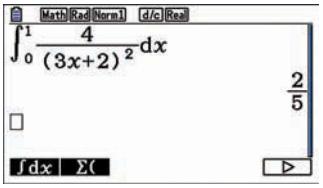


CASIO | THINK

On a Run-Matrix screen, select MATHS by pressing F4, select $\int dx$ by pressing F6 and then F1, and complete the entry line as:

$\int_0^1 \frac{4}{(3x+2)^2}\,dx$ and press EXE.

DISPLAY/WRITE



6.2.3 Other linear substitutions

Integrals of the form $\int f(x)\,[g(x)]^n\,dx$ where $g(x)$ is a linear function can be solved using the substitution $u = g(x)$. Because $g(x)$ is linear, it is possible to determine u as a function of x and therefore determine $f(x)$ as a function of u.

WORKED EXAMPLE 5 Determining integrals in the form $\int f(x)\,(g(x))^n\,dx$ where $g(x)$ is a linear function

Determine the following:

a. $\int x\,(2x-5)^4\,dx$

b. $\int \frac{6x-5}{4x^2-12x+9}\,dx$ for $x \neq \frac{3}{2}$.

THINK

a. 1. The integrand can be rewritten using $u = 2x - 5$. Differentiate to determine du.

2. Rewrite the integral, substituting for $2x - 5$ and dx.

3. The integrand is not a function of u only. Rearrange u to determine an expression for x.

4. Substitute for x in the integral.

WRITE

a. Let $u = 2x - 5$

$$\frac{du}{dx} = 2$$

$$du = 2\,dx$$

$$dx = \frac{du}{2}$$

$$\int x\,(2x-5)^4\,dx = \int x\,u^4\,\frac{du}{2}$$

$$= \frac{1}{2}\int x\,u^4\,du$$

$$u = 2x - 5$$

$$2x = u + 5$$

$$x = \frac{u+5}{2}$$

$$\frac{1}{2}\int x\,u^4\,du = \frac{1}{2}\int \frac{(u+5)}{2}\,u^4\,du$$

$$= \frac{1}{4}\int (u+5)\,u^4\,du$$

5. As the integrand is now expressed as a function of u only, it can be integrated. It will be necessary to expand the brackets before integrating.

$$= \frac{1}{4}\int \left(u^5 + 5u^4\right)\, du$$
$$= \frac{1}{4}\left(\frac{u^6}{6} + \frac{5u^5}{5}\right) + c$$
$$= \frac{1}{4}\left(\frac{u^6}{6} + u^5\right) + c$$

6. The final solution will need to be written in terms of x. If we substitute at this point, we will get $\frac{1}{4}\left[\frac{(2x-5)^6}{6} + (2x-5)^5\right] + c$. There are often two ways to show the final answer. If the question explicitly requests the answer in a factorised form, it is generally more efficient to factorise the expression before substituting.
However, if the question does not specify a particular form, we can substitute first, and the resulting expression can be simplified or factored as needed.

$$= \frac{1}{4} \times \frac{u^5}{6}(u+6) + c$$
$$= \frac{1}{24}u^5(u+6) + c$$

7. As the original integral was in terms of x, substitute $u = 2x - 5$ and simplify.

$$= \frac{1}{24}(2x-5)^5(2x-5+6) + c$$
$$= \frac{1}{24}(2x-5)^5(2x+1) + c$$

b. 1. Factorise the denominator.

b. $$\int \frac{6x-5}{4x^2-12x+9}\, dx = \int \frac{6x-5}{(2x-3)^2}\, dx$$

2. The integrand can be rewritten using $u = 2x - 3$. Differentiate to determine du.

Let $u = 2x - 3$
$$\frac{du}{dx} = 2$$
$$du = 2\, dx$$
$$dx = \frac{du}{2}$$

3. Rewrite the integral, substituting for $2x - 3$ and dx.

$$\int \frac{6x-5}{(2x-3)^2}\, dx = \int \frac{6x-5}{u^2}\frac{du}{2} \text{ for } u \neq 0$$
$$= \frac{1}{2}\int \frac{6x-5}{u^2}\, du$$

4. The integrand is not a function of u only. Rearrange u to determine an expression for $6x$.

$$u = 2x - 3$$
$$2x = u + 3$$
$$6x = 3u + 9$$

5. Substitute for x in the integral.

$$\frac{1}{2}\int \frac{6x-5}{u^2}\, du = \frac{1}{2}\int \frac{3u+9-5}{u^2}\, du$$
$$= \frac{1}{2}\int \frac{3u+4}{u^2}\, du$$

6. As the integrand is now expressed as a function of u only, it can be integrated. It will be necessary to write the integrand as the sum of two terms before integrating.

$$= \frac{1}{2}\int\left(\frac{3u}{u^2}+\frac{4}{u^2}\right)du$$

$$= \frac{1}{2}\int\left(\frac{3}{u}+4u^{-2}\right)du$$

$$= \frac{3}{2}\int\frac{1}{u}\,du+\frac{4}{2}\int u^{-2}\,du$$

7. Integrate noting that $\int\frac{1}{x}\,dx=\ln|x|, x\neq 0$.

$$= \frac{3}{2}\ln|u|+\frac{4}{2}\times\frac{u^{-1}}{-1}+c$$

$$= \frac{3}{2}\ln|u|-\frac{2}{u}+c$$

8. As the original integral was in terms of x, substitute $u=2x-3$.

$$= \frac{3}{2}\ln|2x-3|-\frac{2}{2x-3}+c$$

6.2.4 Definite integrals using other linear substitutions

WORKED EXAMPLE 6 Determining definite integrals in the form $\int_a^b f(x)\,(g(x))^n\,dx$ where $g(x)$ is a linear function

Calculate $\int_0^8\frac{x}{\sqrt{2x+9}}\,dx$.

THINK

1. Use index laws to express the integrand as a function of a power.

WRITE

$$\int_0^8\frac{x}{\sqrt{2x+9}}\,dx=\int_0^8 x\,(2x+9)^{-\frac{1}{2}}\,dx$$

2. The integrand can be rewritten using $u=2x+9$. Differentiate to determine du.

Let $u=2x+9$

$$\frac{du}{dx}=2$$

$$du=2\,dx$$

$$dx=\frac{du}{2}$$

3. Rewrite the integral, substituting for $2x+9$ and dx. Note that the integral is occurring between $x=0$ and $x=8$.

$$\int_{x=0}^{x=8}x\,(2x+9)^{-\frac{1}{2}}\,dx=\int_{x=0}^{x=8}x\,u^{-\frac{1}{2}}\,\frac{du}{2}$$

$$=\frac{1}{2}\int_{x=0}^{x=8}x\,u^{-\frac{1}{2}}\,du$$

4. The integrand is not a function of u only. Rearrange u to determine an expression for x. Determine values for u when $x=0$ and $x=8$.

$$u=2x+9$$

$$2x=u-9$$

$$x=\frac{u-9}{2}$$

$x=0$: $u=9$

$x=8$: $u=2\times 8+9=25$

5. Substitute for x in the integral and the new terminals.

$$= \frac{1}{2}\int_{u=9}^{u=25} \frac{(u-9)}{2}\, u^{-\frac{1}{2}}\, du$$

$$= \frac{1}{4}\int_{9}^{25} (u-9)\; u^{-\frac{1}{2}}\, du$$

6. As the integrand is now expressed as a function of u only, it can be integrated. It will be necessary to expand the brackets before integrating.

$$= \frac{1}{4}\int_{9}^{25} \left(u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}\right) du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}} - 9\times 2u^{\frac{1}{2}}\right]_{9}^{25}$$

$$= \frac{1}{4}\left[\frac{2}{3}\sqrt{u^3} - 18\sqrt{u}\right]_{9}^{25}$$

7. Evaluate the definite integral.

$$= \frac{1}{4}\left[\left(\frac{2}{3}\sqrt{25^3} - 18\sqrt{25}\right) - \left(\frac{2}{3}\sqrt{9^3} - 18\sqrt{9}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}\times 5^3 - 18\times 5\right) - \left(\frac{2}{3}\times 3^3 - 18\times 3\right)\right]$$

$$= \frac{22}{3}$$

8. State the final result.

$$\int_0^8 \frac{x}{\sqrt{2x+9}}\, dx = \frac{22}{3}$$

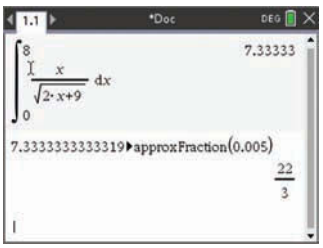
TI | THINK

On a calculator page, go to MENU, Calculus, Integral and complete the entry line as:

$\int_0^8 \frac{x}{\sqrt{2x+9}}\, dx$ and press ENTER.

Note that you can go to MENU, Numbers and Approximate to Fraction to convert from decimal to fraction (the accuracy can be modified).

DISPLAY/WRITE

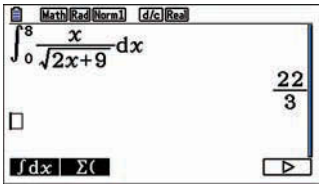


CASIO | THINK

On a Run-Matrix screen, select MATHS by pressing F4, select $\int dx$ by pressing F6 and then F1, and complete the entry line as:

$\int_0^8 \frac{x}{\sqrt{2x+9}}\, dx$ and press EXE.

DISPLAY/WRITE



When integrating functions of the form $f(g(x))$ when $g(x)$ is a linear function, the substitution $u = g(x)$ can be used. If determining an indefinite integral, it is necessary to write the final solution in terms of x.

If determining a definite integral by substitution, it is not necessary to rewrite the integral in terms of x provided that the terminals are values for u.

Exercise 6.2 Integration by linear substitution

learnon

6.2 Exercise	6.2 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. MC With a suitable substitution, $\int_{-4}^{1} (2x+1)^3\,dx$ can be expressed as

A. $\int_{-4}^{1} u^3\,du$ **B.** $\int_{-7}^{3} u^3\,du$ **C.** $\int_{-7}^{3} \frac{1}{2}u^3\,du$ **D.** $\int_{-4}^{1} 2u^3\,du$

2. MC With a suitable substitution, $\int_{0}^{2} \sqrt{4x+1}\,dx$ can be expressed as

A. $\int_{0}^{2} 4\sqrt{u}\,du$ **B.** $\int_{1}^{9} \frac{1}{4}\sqrt{u}\,du$ **C.** $\int_{1}^{9} \sqrt{u}\,du$ **D.** $\int_{0}^{2} \frac{1}{4}\sqrt{u}\,du$

3. MC With a suitable substitution, $\int_{0}^{1} \frac{2x}{\sqrt{3x+4}}\,dx$ can be expressed as

A. $\int_{4}^{7} \frac{u}{\sqrt{u}}\,du$ **B.** $\int_{4}^{7} \frac{2}{9}\frac{(u-4)}{\sqrt{u}}\,du$ **C.** $\int_{0}^{1} \frac{2}{3}\frac{(u-4)}{\sqrt{u}}\,du$ **D.** $\int_{4}^{7} 2\frac{(u-4)}{\sqrt{u}}\,du$

4. WE1 Determine:

a. $\int (5x-9)^6\,dx$ **b.** $\int (3x+4)^7\,dx$ **c.** $\int (3x-5)^6\,dx$ **d.** $\int (1-2x)^5\,dx$

5. WE2 The gradient function $\frac{dy}{dx}$ of a curve is given by $\frac{1}{\sqrt{16x+25}}$. Determine the equation of the particular curve that passes through the origin.

6. Given that $f'(x) = \frac{1}{(3x-7)^2}$ and $f(2) = 3$, determine the value of $f(1)$.

7. WE3 Determine the anti-derivative of:

a. $\frac{1}{3x-5}$ for $x \neq \frac{5}{3}$ **b.** $\frac{1}{7-2x}$ for $x \neq \frac{2}{7}$ **c.** $\frac{2}{3x-1}$ for $x \neq \frac{1}{3}$ **d.** $\frac{3}{1-2x}$ for $x \neq \frac{1}{2}$

8. Determine the anti-derivative of the following:

a. $(3x+5)^6$ **b.** $\frac{1}{(3x+5)^2}$ **c.** $\frac{1}{(3x+5)^3}$ **d.** $\frac{1}{\sqrt[3]{3x+5}}$

9. Determine each of the following:

a. $\int (6x+7)^8\,dx$ **b.** $\int \frac{1}{\sqrt{6x+7}}\,dx$

c. $\int \frac{1}{6x+7}\,dx$ for $x \neq \frac{-7}{6}$ **d.** $\int \frac{1}{(6x+7)^2}\,dx$

10. WE4 Calculate the following:

a. $\int_{-1}^{0} \frac{9}{(2x+3)^3}\,dx$ b. $\int_{0}^{4} \frac{6}{\sqrt{3x+4}}\,dx$ c. $\int_{0}^{13} (2x+1)^{\frac{1}{3}}\,dx$ d. $\int_{2}^{3} \frac{1}{(2x-3)^2}\,dx$

11. a. Given that $\frac{dx}{dt} = \frac{1}{(2-5t)^2}$ and $x(0) = 0$, determine an equation for x in terms of t.

b. A certain curve has its gradient given by $\frac{1}{\sqrt{3-2x}}$ for $x < \frac{3}{2}$. If the point $\left(-\frac{1}{2}, -2\right)$ lies on the curve, determine the equation of the curve.

c. Given that $f'(x) = \frac{3}{3-2x}$ and that $f(0) = 0$, calculate $f(1)$.

d. A certain curve has a gradient equal $\frac{x}{\sqrt{2x+9}}$, determine the equation of the particular curve which passes through the origin.

12. WE6 Calculate:

a. $\int_{1}^{2} x\sqrt{x-1}\,dx$ b. $\int_{0}^{3} \frac{x^2}{\sqrt{x+1}}\,dx$ c. $\int_{0}^{5} \frac{x}{\sqrt{3x+1}}\,dx$ d. $\int_{0}^{1} \frac{15x}{(3x+2)^2}\,dx$

Complex familiar

13. WE5 Determine:

a. $\int x(5x-9)^5\,dx$ b. $\int \frac{2x-1}{9x^2-24x+16}\,dx$ for $x \neq \frac{4}{3}$

c. $\int \frac{x}{(2x+7)^3}\,dx$ d. $\int \frac{x}{\sqrt{6x+5}}\,dx$

14. For each of the following, determine $f(x)$.

a. $f'(x) = x(3x+5)^6$ b. $f'(x) = \frac{x}{(3x+5)^2}$ for $x \neq \frac{-5}{3}$

c. $f'(x) = \frac{x}{(3x+5)^3}$ d. $f'(x) = \frac{x}{\sqrt[3]{3x+5}}$

15. Determine each of the following:

a. $\int x(6x+7)^8\,dx$ b. $\int \frac{x}{\sqrt{6x+7}}\,dx$

c. $\int \frac{x}{6x+7}\,dx$ d. $\int \frac{x}{(6x+7)^2}\,dx$ for $x \neq \frac{-7}{6}$

16. Determine each of the following.

a. $\int_{1}^{2} (3x-4)^5\,dx$ b. $\int_{1}^{2} x(3x-4)^5\,dx$

c. $\int_{0}^{13} \frac{1}{\sqrt[3]{2t+1}}\,dt$ d. $\int_{0}^{13} \frac{t}{\sqrt[3]{2t+1}}\,dt$

Complex unfamiliar

17. If a and b are non-zero real constants, determine the following:

a. $\int \sqrt{ax+b}\,dx$

b. $\int x\sqrt{ax+b}\,dx$

c. $\int \frac{1}{ax+b}\,dx$ for $x \neq \frac{-b}{a}$

d. $\int \frac{x}{ax+b}\,dx$ for $x \neq \frac{-b}{a}$

18. If a and b are non-zero real constants, determine the following:

a. $\int \frac{1}{\sqrt{ax+b}}\,dx$

b. $\int \frac{x}{\sqrt{ax+b}}\,dx$

c. $\int \frac{1}{(ax+b)^2}\,dx$

d. $\int \frac{x}{(ax+b)^2}\,dx$ for $x \neq \frac{-b}{a}$

19. If a, b, c and d are non-zero real constants, determine the following:

a. $\int \frac{cx+d}{ax+b}\,dx$ for $x \neq \frac{-b}{a}$

b. $\int \frac{cx+d}{(ax+b)^2}\,dx$ for $x \neq \frac{-b}{a}$

c. $\int \frac{cx^2+d}{(ax+b)^2}\,dx$ for $x \neq \frac{-b}{a}$

d. $\int \frac{cx^2+d}{ax+b}\,dx$ for $x \neq \frac{-b}{a}$

20. Determine $a \in \mathbb{R}^+$ such that $\int_0^a \frac{4}{(2x+1)^2}\,dx = 9\int_2^{2a} \frac{4}{(2x+1)^2}\,dx$.

Verify the reasonableness of your answer.

Fully worked solutions for this chapter are available online.

LESSON
6.3 Integration by non-linear substitutions

SYLLABUS LINKS

- Use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))\,g'(x)$.

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6.3.1 Indefinite integration using non-linear substitutions

We have looked at how it is possible to integrate functions of the form $\int f(x)\,[g(x)]^n\,dx$ where $g(x)$ is a linear function by making the substitution $u = g(x)$.

It is sometimes also possible to use a non-linear substitution. If the function is in the form $f(g(x))\ g'(x)$, then the substitution $u = g(x)$ can be used.

This method relies on the derivative, or constant multiples of the derivative, being present and recognisable. It is important that after making the substitution $u = g(x)$ that the integrand is rewritten entirely in terms of u.

WORKED EXAMPLE 7 Determining integrals in the form $\int f(g(x))\,g'(x)\,dx$

Determine $\int \frac{3x}{(x^2+9)^2}\,dx$.

THINK	WRITE
1. The function is in the form $f(g(x))\ g'(x)$ where $g(x)=x^2+9$. Also we can recognise that $g'(x)=2x$, which is the constant multiple of the numerator $3x$. Therefore, $u=x^2+9$ and differentiate to determine dx.	Let $u=x^2+9$ $\frac{du}{dx}=2x$ $du=2x\,dx$ $dx=\frac{du}{2x}$
2. Rewrite the integral substituting for x^2+9 and dx.	$\int \frac{3x}{(x^2+9)^2}\,dx=\int \frac{3x}{u^2}\,\frac{du}{2x}$
3. Simplify the integral noting that $\frac{3x}{2x}=\frac{3}{2}$.	$=\int \frac{3x}{2x}\,\frac{1}{u^2}\,du$ $=\frac{3}{2}\int \frac{1}{u^2}\,du$
4. The integrand is written in terms of u only, so it can be integrated. Rewrite the integral using power notation and integrate.	$=\frac{3}{2}\int u^{-2}\,du$ $=\frac{3}{2}\times\frac{u^{-1}}{-1}+c$ $=-\frac{3}{2u}+c$
5. Substitute for u so that the final answer is in terms of x only.	$=-\frac{3}{2(x^2+9)}+c$

Using the logarithm function

This uses the standard integral $\int \frac{1}{x}\,dx=\ln|x|+c$ where $x\neq 0$.

WORKED EXAMPLE 8 Determining integrals in the form $\int \frac{u'(x)}{u(x)}\,dx$

Determine $\int \frac{x-3}{x^2-6x+13}\,dx$.

THINK	WRITE
1. The function is in the form $f(g(x))\ g'(x)$ where $g(x)=x^2-6x+13$, and $g'(x)=2x-6=2(x-3)$, which is the multiple of the numerator. Let $u=x^2-6x+13$ and differentiate to determine dx.	Let $u=x^2-6x+13$ $\frac{du}{dx}=2x-6$ $du=(2x-6)\,dx$ $dx=\frac{du}{2(x-3)}$

2. Rewrite the integral substituting for $x^2 - 6x + 13$ and dx.

$$\int \frac{x-3}{x^2-6x+13}\,dx = \int \frac{x-3}{u}\frac{du}{2(x-3)}$$

3. Simplify the integral noting that $\frac{x-3}{2(x-3)} = \frac{1}{2}$, when $x \neq 3$

$$= \int \frac{x-3}{2(x-3)}\frac{du}{u}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

4. The integrand is written in terms of u only, so it can be integrated.

   Integrate using $\int \frac{1}{x}\,dx = \ln|x| + c$

$$= \frac{1}{2}\ln|u| + c$$

5. Substitute for u so that the final answer is in terms of x only.

$$= \frac{1}{2}\ln|x^2 - 6x + 13| + c$$

6. In the end, it will be good to always check the domain and range of $\ln(g(x))$. In here, the function $x^2 - 6x + 13 > 0$ for all x meaning the absolute value sign is not needed.

$$= \frac{1}{2}\ln(x^2 - 6x + 13) + c$$

Note that since $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$, it follows that $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$. This pattern can be used when integrating. The previous worked example could have been solved as follows:

If $f(x) = x^2 - 6x + 13$, $f'(x) = 2x - 6$

$$\begin{aligned}\int \frac{x-3}{x^2-6x+13}\,dx &= \frac{1}{2}\int \frac{2x-6}{x^2-6x+13}\,dx \\ &= \frac{1}{2}\ln|x^2 - 6x + 13| + c \\ &= \frac{1}{2}\ln(x^2 - 6x + 13) + c \text{ since } x^2 - 6x + 13 > 0.\end{aligned}$$

Using trigonometric functions

It is also possible to use the standard trigonometric integrals.

Anti-derivatives of trigonometric functions

$$\int \sin(x)\,dx = -\cos(x) + c$$

$$\int \cos(x)\,dx = \sin(x) + c$$

$$\int \sec^2(x)\,dx = \tan(x) + c$$

WORKED EXAMPLE 9 Determining integrals in the form $\int \cos(u)\,du$

Determine $\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$ when $x \neq 0$.

THINK	WRITE
1. The function is in the form $f(g(x))\ g'(x)$ where $g(x) = \frac{1}{x}$ and $g'(x) = -\frac{1}{x^2}$. Let $u = \frac{1}{x}$ and differentiate to determine dx.	Let $u = \frac{1}{x}$ $= x^{-1}$ $\frac{du}{dx} = -x^{-2}$ $= -\frac{1}{x^2}$ $x^2\,du = -dx$ $dx = -x^2\,du$
2. Rewrite the integral substituting for $\frac{1}{x}$ and dx.	$\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx = \int \frac{1}{x^2}\cos(u)\left(-x^2\,du\right)$
3. Simplify the integral noting that $\frac{-x^2}{x^2} = -1$, where $x \neq 0$.	$= \int \frac{-x^2}{x^2}\cos(u)\,du$ $= -\int \cos(u)\,du$
4. The integrand is written in terms of u only, so it can be integrated. Integrate using $\int \cos(x)\,dx = \sin(x) + c$.	$= -\sin(u) + c$
5. Substitute for u so that the final answer is in terms of x only.	$= -\sin\left(\frac{1}{x}\right) + c$

WORKED EXAMPLE 10 Determining integrals in the form $\int \sin(u)\,du$

Determine $\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$, when $x > 0$.

THINK	WRITE
1. The function is in the form $f(g(x))\ g'(x)$ where $g(x) = \sqrt{x}$, and $g'(x) = \frac{1}{2\sqrt{x}}$. Let $u = \sqrt{x}$ and differentiate to determine dx.	Let $u = \sqrt{x}$ $= x^{\frac{1}{2}}$ $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $= \frac{1}{2\sqrt{x}}$ $dx = 2\sqrt{x}\,du$

2. Rewrite the integral substituting for $\sqrt{x}$ and dx.

3. Simplify the integral noting that $\dfrac{2\sqrt{x}}{\sqrt{x}} = 2$.

$$\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = \int \frac{\sin(u)}{\sqrt{x}} 2\sqrt{x}\,du$$

$$= \int \frac{2\sqrt{x}}{\sqrt{x}} \sin(u)\,du$$

$$= 2\int \sin(u)\,du$$

4. The integrand is written in terms of u only, so it can be integrated.

   Integrate using $\int \sin(x)\,dx = -\cos(x) + c$.

$$= -2\cos(u) + c$$

5. Substitute for u so that the final answer is in terms of x only.

$$= -2\cos\left(\sqrt{x}\right) + c$$

Note: When rewriting the integral in terms of u, shown above in step 2, we notice that $\sqrt{x}$ (or u) also appears in the denominator. This could lead to trying to evaluate $\int \dfrac{\sin(u)}{u}\,dx$ or even $\int \dfrac{\sin(u)}{u}\,2\sqrt{x}\,du$, both of which cannot be determined since the variable in the integrand does not match the variable of integration. This example highlights the importance of taking care when using a substitution method to integrate.

Using other functions

Using the chain rule, $\dfrac{d}{dx}e^{g(x)} = g'(x)\,e^{g(x)}$. This means that

The anti-derivative of $g'(x)e^{g(x)}$

$$\int g'(x)e^{g(x)}\,dx = e^{g(x)} + c$$

WORKED EXAMPLE 11 Determining integrals in the form $\int e^u\,du$

Determine $\int \sin(2x)\,e^{\cos(2x)}\,dx$.

THINK

1. The function is in the form $f(g(x))\ g'(x)$ where $g(x) = \cos(2x)$ and $g'(x) = -2\sin(2x)$.
   Let $u = \cos(2x)$ and differentiate to determine dx.

WRITE

Let $u = \cos(2x)$

$$\frac{du}{dx} = -2\sin(2x)$$

$$du = -2\sin(2x)\,dx$$

$$dx = \frac{du}{-2\sin(2x)}$$

2. Rewrite the integral substituting for $\cos(2x)$ and dx.

$$\int \sin(2x)\, e^{\cos(2x)}\, dx = \int \sin(2x)\, e^{u}\, \frac{du}{-2\sin(2x)}$$

3. Simplify the integral noting that $\dfrac{\sin(2x)}{-2\sin(2x)} = -\dfrac{1}{2}$.

$$= \int \frac{\sin(2x)}{-2\sin(2x)} e^{u}\, du$$

$$= -\frac{1}{2}\int e^{u}\, du$$

4. The integrand is written in terms of u only, so it can be integrated.

Integrate using $\int e^{x}\, dx = e^{x} + c$.

$$= -\frac{1}{2}e^{u} + c$$

5. Substitute for u so that the final answer is in terms of x only.

$$= -\frac{1}{2}e^{\cos(2x)} + c$$

6.3.2 Definite integration using non-linear substitutions

When using substitution to evaluate definite integrals, there is no need to return to an expression in terms of x, provided the terminals are expressed in terms of u.

WORKED EXAMPLE 12 Calculating a definite integration using non-linear substitutions

Calculate $\displaystyle\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2+4}}\, dt$.

THINK

1. The function is in the form $f(g(t))\, g'(t)$ where $g(t) = t^2 + 4$. Let $u = t^2 + 4$ and differentiate to determine dt.

WRITE

Let $u = t^2 + 4$

$$\frac{du}{dt} = 2t$$

$$du = 2t\, dt$$

$$dt = \frac{du}{2t}$$

2. Rewrite the integral substituting for $t^2 + 4$ and dt. Note that the integral is occurring between $t = 0$ and $t = \sqrt{5}$.

$$\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2+4}}\, dt = \int_{t=0}^{t=\sqrt{5}} \frac{t}{\sqrt{u}}\, \frac{du}{2t}$$

3. Simplify the integral noting that $\dfrac{t}{2t} = \dfrac{1}{2}$.

Determine values for u when $t = 0$ and $t = \sqrt{5}$.

$t = 0$: $u = 0^2 + 4 = 4$

$t = \sqrt{5}$: $u = \left(\sqrt{5}\right)^2 + 4 = 9$

$$\int_{t=0}^{t=\sqrt{5}} \frac{t}{\sqrt{u}}\, \frac{du}{2t} = \int_{u=4}^{u=9} \frac{t}{2t}\, \frac{1}{\sqrt{u}}\, du$$

$$= \frac{1}{2}\int_4^9 \frac{1}{\sqrt{u}}\, du$$

4. The integrand is written in terms of u only, so it can be integrated.
Rewrite the integral using power notation and integrate.

$$= \frac{1}{2}\int_4^9 u^{-\frac{1}{2}}\,du$$
$$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_4^9$$
$$= \frac{1}{2}\left[2\sqrt{u}\right]_4^9$$

5. Evaluate the definite integral.

$$= \frac{1}{2}\left(2\sqrt{9}-2\sqrt{4}\right)$$
$$= \frac{1}{2}(6-4)$$
$$= 1$$

6. State the final result.

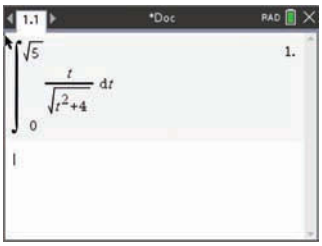
$$\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2+4}}\,dt = 1$$

TI | THINK

On a calculator page, go to MENU, Calculus, Integral and complete the entry line as:

$\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2+4}}\,dt$ and press ENTER.

DISPLAY/WRITE

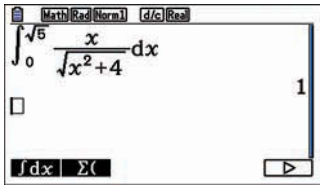


CASIO | THINK

On a Run-Matrix screen, select MATHS by pressing F4, select $\int dx$ by pressing F6 and then F1, and complete the entry line as:

$\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}\,dx$ and press EXE.

DISPLAY/WRITE



Exercise 6.3 Integration by non-linear substitutions

learnon

6.3 Exercise	6.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16	17, 18	19, 20

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. WE7 Determine:

a. $\int \frac{8x}{(x^2+16)^3}\,dx$

b. $\int \frac{5x}{\sqrt{2x^2+3}}\,dx$

c. $\int 2x\left(x^2+3\right)^4 dx$

d. $\int 2x\left(6-x^2\right)^{-3} dx$

2. WE8 Determine:

a. $\int \frac{x+2}{x^2+4x+29}\,dx$

b. $\int \frac{x^2}{x^3+9}\,dx$ for $x \neq \sqrt[3]{-9}$

c. $\int \frac{x+1}{x^2+2x+9}\,dx$

d. $\int \frac{3x}{x^2+9}\,dx$

3. Determine the integral of the following:

a. $x\left(x^2+4\right)^5$

b. $\frac{x}{x^2+4}$

c. $\frac{x}{\left(x^2+9\right)^2}$

d. $\frac{x}{\sqrt{x^2+9}}$

4. Determine the following:

a. $\int \frac{x^2}{\left(x^3+27\right)^3}\,dx$

b. $\int \frac{x^2}{\sqrt{x^3+27}}\,dx$

c. $\int \frac{x^2}{x^3+8}\,dx$ for $x \neq -2$

d. $\int x^2\left(x^3+8\right)^3\,dx$

5. Determine $f(x)$ given:

a. $f'(x) = (x-2)\left(x^2-4x+13\right)^3$

b. $f'(x) = \frac{x-2}{\left(x^2-4x+13\right)^2}$

c. $f'(x) = \frac{4-x}{\sqrt{x^2-8x+25}}$

d. $f'(x) = \frac{4-x}{x^2-8x+25}$

6. Determine the anti-derivative of the following:

a. $3x^2\left(x^3-5\right)^2$

b. $2(x+2)\left(x^2+4x\right)^{-3}$

c. $(2x+5)\sqrt{x^2+5x}$

d. $\frac{2x-3}{(x^2-3x)^4}$

7. Determine the integral of the following functions:

a. $6x^2\left(x^3-2\right)^5$

b. $x\left(4-x^2\right)^3$

c. $x^2\left(x^3-1\right)^7$

d. $(x+3)\left(x^2+6x-2\right)^4$

8. Given the following functions, determine the anti-derivative:

a. $(x+1)\left(x^2+2x+3\right)^{-4}$

b. $\frac{4x+6}{\sqrt{x^2+3x}}$

c. $\frac{2x-5}{\left(x^2-5x+2\right)^6}$

d. $\left(x^2-1\right)\sqrt{4-3x+x^3}$

9. Determine:

a. $\int x\left(x^2+1\right)^{\frac{5}{2}}\,dx$

b. $\int x\sqrt{1-x^2}\,dx$

c. $\int (2x+1)\sqrt{x+x^2-3}\,dx$

d. $\int \frac{3x^2+4x}{\sqrt{x^3+2x^2}}\,dx$

10. WE9 Determine:

a. $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

b. $\int x\cos\left(x^2\right)dx$

c. $\int \left(3x^2+5\right)\cos\left(x^3+5x\right)dx$

d. $\int (2x+3)\sin\left(x^2+3x-2\right)dx$

11. WE10 Determine:

a. $\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$, for $x \neq 0$.

b. $\int \frac{\ln(6x)}{4x}\,dx$, for $x > 0$.

c. $\int \frac{\ln(3x)}{2x}\,dx$, for $x > 0$.

d. $\int \frac{(4x-2)\ln\left(x^2-x\right)}{x^2-x}\,dx$, for $x > 1$

12. WE11 Determine:

a. $\int \cos(3x)\ e^{\sin(3x)}dx$

b. $\int \sec^2(2x)\ e^{\tan(2x)}dx$

c. $\int (6x-3)\,e^{x^2-x+3}\,dx$

d. $\int x^2 e^{x^3+2}\,dx$

13. Determine the following:

a. $\int \frac{e^{2x}}{4e^{2x}+5}\,dx$

b. $\int \frac{e^{-3x}}{\left(2e^{-3x}-5\right)^2}\,dx$

c. $\int \frac{e^{-2x}}{\left(3e^{-2x}+4\right)^3}\,dx$

d. $\int \frac{2e^{2x}+1}{\left(e^{2x}+x\right)^2}\,dx$

14. Determine the following.

a. $\int \frac{1}{x}\sin(\ln(4x))\,dx$

b. $\int \frac{1}{x}\cos(\ln(3x))\,dx$

c. $\int \frac{e^{2x}}{e^x+2}\,dx$

d. $\int \frac{e^{3x}}{e^x-1}\,dx$ for $x \neq 0$.

15. WE12 Calculate:

a. $\int_0^{2\sqrt{2}} \frac{x}{\sqrt{2x^2+9}}\,dx$

b. $\int_0^1 \frac{x}{\left(3x^2+5\right)^2}\,dx$

c. $\int_{-2}^0 4x\,e^{x^2}\,dx$

d. $\int_0^1 (4x+7)\sqrt{2x^2+7x}\,dx$

16. Determine the value of the following definite integrals:

a. $\int_3^5 \frac{3t}{\sqrt{t^2-9}}\,dt$

b. $\int_0^2 \frac{x}{4x^2+9}\,dx$

c. $\int_1^2 \frac{x}{\left(2x^2+1\right)^2}\,dx$

d. $\int_{-1}^1 \frac{t}{\sqrt{2t^2+3}}\,dt$

Complex familiar

17. **a.** A certain curve has a gradient given by $x\sin\left(x^2\right)$. Determine the equation of the particular curve which passes through the origin.

b. Calculate the value of y when $x = \frac{3}{\pi}$ given the following:

$$\frac{dy}{dx} = \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)$$

when $x = \frac{4}{\pi}, y = 0$

c. Given that $f'(x) = \frac{5-x}{x^2-10x+29}$ and $f(0) = 0$, calculate $f(1)$.

d. If a curve has a gradient $\sin(2x)\ e^{\cos(2x)}$ and passes through $\left(\frac{\pi}{4}, 0\right)$, calculate the value of y when $x = 0$.

18. Calculate the following:

a. $\displaystyle\int_4^5 \frac{3-x}{x^2-6x+34}\,dx$

b. $\displaystyle\int_2^3 \frac{2-x}{\left(x^2-4x+5\right)^2}\,dx$

c. $\displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

d. $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2\theta)\,e^{\tan(2\theta)}\,d\theta$

Complex unfamiliar

19. Using the substitution $u=\tan x$, prove that $\displaystyle\int \frac{1}{\cos^6(x)}\,dx=\tan(x)+\frac{2}{3}\tan^3(x)+\frac{1}{5}\tan^5(x)+c$.

20. A teacher is challenging her students to use $\displaystyle\int_0^{\frac{1}{2}}\left(\frac{1}{1-u}+\frac{1}{1+u}\right)du$ and a substitution with a trigonometric function to calculate $\displaystyle\int_0^{\frac{\pi}{6}}\frac{d\theta}{\cos(\theta)}$.

Determine $\displaystyle\int_0^{\frac{\pi}{6}}\frac{d\theta}{\cos(\theta)}$.

Fully worked solutions for this chapter are available online.

LESSON
6.4 Integration using the trigonometric identities

SYLLABUS LINKS

- Integrate using the trigonometric identities $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$, $\cos^2(x)=\frac{1}{2}(1+\cos(2x))$, $1+\tan^2(x)=\sec^2(x)$ and $\cot^2(x)+1=\operatorname{cosec}^2(x)$.
- Establish and use the formula $\int \sec^2(x)\,dx=\tan(x)+c$.

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6.4.1 Review of trigonometric identities

Your formula sheet contains the following double-angle identities:

$$\begin{aligned}\sin(2A) &= 2\sin(A)\cos(A)\\ \cos(2A) &= \cos^2(A)-\sin^2(A)\\ &= 1-2\sin^2(A)\\ &= 2\cos^2(A)-1\end{aligned}$$

When integrating terms involving powers of trigonometric functions, linearisation is used to simplify the expressions. By rewriting terms like $\sin^2(ax)$ and $\cos^2(ax)$ as linear combinations of trigonometric functions without powers, the integration process becomes significantly easier.

This often involves doubling the angle within the trigonometric functions. The functions being used will often be written in terms of a multiple of x. If we substitute $A = ax$ then the formulas become:

Trigonometric identities

$$\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$$
$$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$$
$$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$$
$$\sin^2(ax) + \cos^2(ax) = 1$$
$$\tan^2(ax) + 1 = \sec^2(ax)$$
$$\cot^2(ax) + 1 = \operatorname{cosec}^2(ax)$$

Note: Through choosing different trigonometric substitutions, your answers to integration questions may appear to be different from those in the answers or those generated by online calculators. Using the trigonometry identities that you studied in Unit 2 Topic 4 (chapter 9), it can be demonstrated that solutions are identical or only different by a constant term.

6.4.2 Integrating even powers of $\sin(ax)$ or $\cos(ax)$

Integrating $\sin^2(ax)$ or $\cos^2(ax)$

When integrating multiples of $\sin^2(ax)$ or $\cos^2(ax)$, the substitutions $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$ or $\cos^2(ax) = \frac{1}{2}(\cos(2ax) + 1)$ can be used.

WORKED EXAMPLE 13 Integrating $\cos^2(ax)$ and $\sin^2(ax)$

Determine the anti-derivative of the following:

a. $\cos^2\left(\frac{x}{4}\right)$ **b.** $2\sin^2\left(\frac{x}{2}\right)$

THINK

a. 1. $\cos^2\left(\frac{x}{4}\right)$ can be rewritten using $\cos^2(ax) = \frac{1}{2}(\cos(2ax) + 1)$ where $a = \frac{1}{4}$.

2. Express the question using integral notation.

3. Rewrite the integrand using the substitution.

4. Integrate noting that $\frac{d}{dx}\sin\left(\frac{x}{2}\right) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$ so $\int \frac{1}{2}\cos\left(\frac{x}{2}\right)dx = \sin\left(\frac{x}{2}\right) + c$.

WRITE

a. Use $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

$$\cos^2\left(\frac{x}{4}\right) = \frac{1}{2}\left(1 + \cos\left(\frac{2x}{4}\right)\right)$$
$$= \frac{1}{2} + \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$\int \cos^2\left(\frac{x}{4}\right)dx$$
$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{x}{2}\right)\right)dx$$
$$= \frac{x}{2} + \sin\left(\frac{x}{2}\right) + c$$

b. 1. $\sin^2\left(\frac{x}{2}\right)$ can be rewritten using $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$ where $a = \frac{1}{2}$.

2. Express the question using integral notation.

3. Rewrite the integrand using the substitution.

4. Integrate noting that $\int \cos(x)\,dx = \sin(x) + c$.

b. Use $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$

$$2\sin^2\left(\frac{x}{2}\right) = 2\left[\frac{1}{2}\left(1 - \cos\left(\frac{2x}{2}\right)\right)\right]$$
$$= 1 - \cos(x)$$

$$\int 2\sin^2\left(\frac{x}{2}\right)dx$$
$$= \int (1 - \cos(x))\,dx$$
$$= x - \sin(x) + c$$

Integrating $\sin^{2n}(ax)$ or $\cos^{2n}(ax)$

The method used for $\sin^2(ax)$ and $\cos^2(ax)$ can be used whenever there is an even power through repeated application of the double-angle identities.

WORKED EXAMPLE 14 Integrating $\cos^{2n}(ax)$

Determine $\int \cos^4(2x)\,dx$.

THINK	WRITE
1. Rewrite $\cos^4(2x)$ as a power of $\cos^2(2x)$ so that the double-angle identity can be used.	$\cos^4(2x) = \left[\cos^2(2x)\right]^2$
2. $\cos^2(2x)$ can be rewritten using the substitution $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$ where $a = 2$.	$= \left[\frac{1}{2}(\cos(4x) + 1)\right]^2$
3. Expand the integrand.	$= \frac{1}{4}(1 + \cos(4x))^2$ $= \frac{1}{4}\left(\cos^2(4x) + 2\cos(4x) + 1\right)$
4. $\cos^2(4x)$ can be rewritten using the substitution $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$ where $a = 4$.	$= \frac{1}{4}\left[\frac{1}{2}(\cos(8x) + 1) + 2\cos(4x) + 1\right]$ $= \frac{1}{4}\left[\frac{1}{2}\cos(8x) + \frac{1}{2} + 2\cos(4x) + 1\right]$ $= \frac{1}{4}\left[\frac{1}{2}\cos(8x) + 2\cos(4x) + \frac{3}{2}\right]$
5. Express the question using integral notation and rewrite the integrand.	$\int \cos^4(2x)\,dx = \frac{1}{4}\int\left(\frac{1}{2}\cos(8x) + 2\cos(4x) + \frac{3}{2}\right)dx$
6. Integrate noting that $\frac{d}{dx}\sin(8x) = 8\cos(8x)$ so $\int \cos(8x)\,dx = \frac{1}{8}\sin(8x) + c$ and $\frac{d}{dx}\sin(4x) = 4\cos(4x)$ so $\int \cos(4x)\,dx = \frac{1}{4}\sin(4x) + c$.	$= \frac{1}{4}\left[\frac{1}{2} \times \frac{1}{8}\sin(8x) + 2 \times \frac{1}{4}\sin(4x) + \frac{3x}{2}\right] + c$ $= \frac{1}{64}\sin(8x) + \frac{1}{8}\sin(4x) + \frac{3x}{8} + c$

6.4.3 Integrating $\sin^n(ax)\cos^n(ax)$, $n \geq 1$

When the powers of $\sin(ax)$ and $\cos(ax)$ are equal, then the substitution $\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$ can be useful. It sometimes necessary to then use the substitution $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$.

WORKED EXAMPLE 15 Integrating $\sin^n(ax)\cos^n(ax), n \geq 1$

Determine the following indefinite integrals:

a. $\int \sin(x)\cos(x)\,dx$

b. $\int 8\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)dx$

THINK	WRITE
a. 1. $\sin(x)\cos(x)$ can be rewritten using $\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$ where $a = 1$.	**a.** Use $\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$ $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$
2. Rewrite the integrand using the substitution.	$\int \sin(x)\cos(x)dx = \int \frac{1}{2}\sin(2x)\,dx$ $= \frac{1}{2}\int \sin(2x)\,dx$
3. Integrate noting that $\frac{d}{dx}\cos(2x) = -2\sin(2x)$ so $\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + c$.	$= \frac{1}{2}\left(-\frac{1}{2}\cos(2x)\right) + c$ $= -\frac{1}{4}\cos(2x) + c$
b. 1. Rewrite $8\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)$ as a power of $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$.	**b.** $8\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = 8\left(\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)^2$
2. $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ can be rewritten using $\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$ where $a = \frac{1}{2}$. Make the substitution and simplify.	$= 8\left(\frac{1}{2}\sin\left(\frac{2x}{2}\right)\right)^2$ $= 8 \times \frac{1}{4}\sin^2(x)$ $= 2\sin^2(x)$
3. $\sin^2(x)$ can be rewritten using $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$ where $a = 1$.	$= 2\left[\frac{1}{2}(1 - \cos(2x))\right]$ $= 1 - \cos(2x)$
4. Rewrite the integrand using the substitution.	$\int 8\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)dx = \int (1 - \cos(2x))\,dx$
5. Integrate noting that $\frac{d}{dx}\sin(2x) = 2\cos(2x)$ so $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + c$.	$= x - \frac{1}{2}\sin(2x) + c$

6.4.4 Integrating $\sin(ax)\cos^n(ax)$ or $\cos(ax)\sin^n(ax)$, $n > 1$

When the integrand involves $\sin(ax)\cos^n(ax)$ or $\cos(ax)\sin^n(ax)$, it is in the form $f(g(x))\ g'(x)$ where $g(x) = \cos(ax)$ and $g(x) = \sin(ax)$, respectively. This means that a non-linear substitution of $u = g(x)$ can be used to determine the integral.

WORKED EXAMPLE 16 Integrating $\sin(ax)\cos^n(ax)$ and $\sin^n(ax)\cos(ax)$, $n > 1$

Determine $\int \sin(3x)\cos^4(3x)\,dx$.

THINK	WRITE
1. The integrand is in the form $f(g(x))\ g'(x)$ where $g(x) = \cos(3x)$. Let $u = \cos(3x)$ and use differentiation to determine dx.	Let $u = \cos(3x)$ $\frac{du}{dx} = -3\sin(3x)$ $du = -3\sin(3x)\,dx$ $dx = \frac{du}{-3\sin(3x)}$
2. Rewrite the integral, substituting for $\cos(3x)$ and dx.	$\int \sin(3x)\cos^4(3x)\,dx$ $= \int \sin(3x)\ u^4\ \frac{du}{-3\ \sin(3x)}$
3. Simplify the integrand, noting that $\frac{\sin(3x)}{\sin(3x)} = 1$ and $-\frac{1}{3}$ can be factored in front of the integral sign.	$= \int \frac{\sin(3x)}{-3\sin(3x)} u^4\,du$ $= -\frac{1}{3}\int u^4\,du$
4. As the integrand is now expressed as a function of u only, it can be integrated.	$= -\frac{1}{3}\left(\frac{u^5}{5}\right) + c$ $= -\frac{u^5}{15} + c$
5. As the original integral was written in terms of x, substitute $u = \cos(3x)$.	$= -\frac{1}{15}\cos^5(3x) + c$

6.4.5 Integrating odd powers of $\sin(ax)$ or $\cos(ax)$

When integrating $\sin^n(ax)$ or $\cos^n(ax)$ when n is odd, or $\sin^m(ax)\cos^n(ax)$ when either m or n is odd (or they both are), then the Pythagorean Identity $\sin^2(ax)+\cos^2(ax)=1$ is used to rewrite the integrand so that it is in the form $\sin(ax)\cos^k(ax)$ or $\cos(ax)\sin^k(ax)$. It can then be integrated like the previous example.

WORKED EXAMPLE 17 Integrating odd powers of $\sin(ax)$ and $\cos(ax)$

Determine the anti-derivatives of the following:

a. $\cos^3(x)$ **b.** $\cos^4(2x)\sin^3(2x)$

THINK	WRITE
a. 1. Notice that $\cos^3(x)=\cos(x)\ \cos^2(x)$. The Pythagorean Identity can be used to rewrite $\cos^2(x)$ as $1-\sin^2(x)$.	**a.** $\cos^2(x) = 1-\sin^2(x)$ $\cos^3(x) = \cos(x)\ \cos^2(x)$ $= \cos(x)\left(1-\sin^2(x)\right)$
2. Express the question using integral notation.	$\int \cos^3(x)\,dx = \int \cos(x)\left(1-\sin^2(x)\right)dx$
3. The integrand is in the form $f(g(x))\ g'(x)$ where $g(x)=\sin(x)$. Let $u=\sin(x)$ and use differentiation to determine dx.	Let $u=\sin(x)$ $\frac{du}{dx}=\cos(x)$ $du=\cos(x)\,dx$ $dx=\frac{du}{\cos(x)}$
4. Rewrite the integral, substituting for $\sin(x)$ and dx.	$\int \cos(x)\left(1-\sin^2(x)\right)dx$ $=\int \cos(x)\left(1-u^2\right)\frac{du}{\cos(x)}$
5. Simplify the integrand, noting that $\frac{\cos(x)}{\cos(x)}=1$.	$=\int \frac{\cos(x)}{\cos(x)}\left(1-u^2\right)du$ $=\int \left(1-u^2\right)du$
6. As the integrand is now expressed as a function of u only, it can be integrated.	$=u-\frac{u^3}{3}+c$
7. As the original integral was written in terms of x, substitute $u=\sin(x)$.	$=\sin(x)-\frac{1}{3}\sin^3(x)+c$
b. 1. There is an odd power for $\sin(2x)$. Notice $\sin^3(2x)=\sin(2x)\sin^2(2x)$. The Pythagorean Identity can be used to rewrite $\sin^2(2x)$ as $1-\cos^2(2x)$.	**b.** $\sin^2(2x) = 1-\cos^2(2x)$ $\cos^4(2x)\sin^3(2x)$ $=\cos^4(2x)\sin(2x)\sin^2(2x)$ $=\cos^4(2x)\sin(2x)\left(1-\cos^2(2x)\right)$ $=\sin(2x)\left(\cos^4(2x)-\cos^6(2x)\right)$

2. Express the question using integral notation.

$$\int \cos^4(2x)\sin^3(2x)\,dx$$
$$= \int \sin(2x)\left(\cos^4(2x) - \cos^6(2x)\right)dx$$

3. The integrand is in the form $f(g(x))\ g'(x)$ where $g(x) = \cos(2x)$. Let $u = \cos(2x)$ and use differentiation to determine dx.

Let $u = \cos(2x)$
$$\frac{du}{dx} = -2\sin(2x)$$
$$du = -2\sin(2x)\,dx$$
$$dx = \frac{du}{-2\sin(2x)}$$

4. Rewrite the integral, substituting for $\cos(2x)$ and dx.

$$\int \sin(2x)\left(\cos^4(2x) - \cos^6(2x)\right)dx$$
$$= \int \sin(2x)\left(u^4 - u^6\right)\frac{du}{-2\sin(2x)}$$

5. Simplify the integrand, noting that $\frac{\sin(2x)}{\sin(2x)} = 1$ and $-\frac{1}{2}$ can be factored in front of the integral sign.

$$= \int \frac{\sin(2x)}{-2\sin(2x)}\left(u^4 - u^6\right)du$$
$$= -\frac{1}{2}\int \left(u^4 - u^6\right)du$$

6. As the integrand in now expressed as a function of u only, it can be integrated.

$$= -\frac{1}{2}\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + c$$
$$= -\frac{u^5}{10} + \frac{u^7}{14} + c$$
$$= \frac{u^7}{14} - \frac{u^5}{10} + c$$

7. As the original integral was written in terms of x, substitute $u = \cos(2x)$.

$$= \frac{1}{14}\cos^7(2x) - \frac{1}{10}\cos^5(2x) + c$$

Integrals involving odd powers

If there is an odd power of either $\sin(ax)$, $\cos(ax)$ (or both):

1. **Rewrite the function as either $\sin^{2n+1}(ax) = \sin(ax)\sin^{2n}(ax)$ or $\cos^{2n+1}(ax) = \cos(ax)\cos^{2x}(ax)$. If both functions have odd powers, only rewrite one of them.**
2. **Use the Pythagorean Identity to rewrite the even power. The integrand will now be in the form $\sin(ax)\cos^k(ax)$ or $\cos(ax)\sin^k(ax)$.**
3. **Use an appropriate substitution ($u = \cos(ax)$ or $u = \sin(ax)$) to evaluate.**

6.4.6 Integrating products of $\sin(ax)$ and $\cos(bx)$

In the products investigated so far, the integrand has involved powers of $\sin(ax)$ and $\cos(ax)$. In order to be able to write the integrand in the form $f(g(x))\ g'(x)$ it is necessary for the angles to be the same size. Therefore a different approach must be used when the angles are not the same size. In this case, it may be possible to use the double-angle identities or the product identities. The following worked example is completed both ways.

WORKED EXAMPLE 18 Integrating products of $\sin(ax)$ and $\cos(bx)$

Determine $\int \cos(x)\sin(2x)\,dx$.

THINK	WRITE
1. Method 1 The integrand involves x and $2x$. Rewrite $\sin(2x)$ using $\sin(2A) = 2\sin(A)\cos(A)$ where $A = x$.	$\sin(2A) = 2\sin(A)\cos(A)$ $\int \cos(x)\sin(2x)\,dx = \int \cos(x)2\sin(x)\cos(x)\,dx$ $= \int 2\cos^2(x)\sin(x)\,dx$
2. The integrand is in the form $f(g(x))\ g'(x)$ where $g(x) = \cos(x)$. Let $u = \cos(x)$ and use differentiation to determine dx.	Let $u = \cos(x)$ $\frac{du}{dx} = -\sin(x)$ $du = -\sin(x)\,dx$ $dx = \frac{du}{-\sin(x)}$
3. Rewrite the integral, substituting for $\cos(x)$ and dx.	$\int 2\cos^2(x)\sin(x)\,dx$ $= \int 2u^2\sin(x)\frac{du}{-\sin(x)}$
4. Simplify the integrand, noting that $\frac{\sin(x)}{\sin(x)} = 1$ and -2 can be factored in front of the integral sign.	$= \int \frac{2\sin(x)}{-\sin(x)}u^2\,du$ $= -2\int u^2\,du$
5. As the integrand is now expressed as a function of u only, it can be integrated.	$= -2\frac{u^3}{3} + c$ $= -\frac{2u^3}{3} + c$
6. As the original integral was written in terms of x, substitute $u = \cos(x)$.	$= -\frac{2}{3}\cos^3(x) + c$
1. Method 2 The integrand involves x and $2x$. Rewrite $\cos(x)\sin(2x)$ using $\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B) - \sin(A-B))$ where $A = x$ and $B = 2x$.	$\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B) - \sin(A-B))$ $\cos(x)\sin(2x) = \frac{1}{2}(\sin(x+2x) - \sin(x-2x))$ $= \frac{1}{2}(\sin(3x) - \sin(-x))$
2. Note that $\sin(-x) = -\sin(x)$.	$= \frac{1}{2}(\sin(3x) + \sin(x))$

3. We know how to integrate each of those terms, so rewrite the integral.

$$\int \cos(x)\sin(2x)\,dx = \frac{1}{2}\int (\sin(3x)+\sin(x))\,dx$$

4. Integrate noting that $\frac{d}{dx}\cos(x) = -\sin(x)$ so

$$\int \sin(x)\,dx = -\cos(x)+c \text{ and}$$

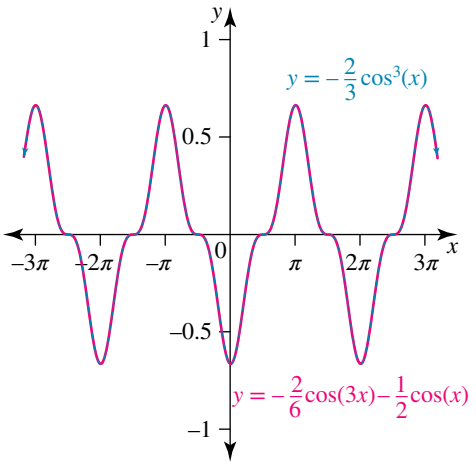
$$\frac{d}{dx}\cos(3x) = -3\sin(3x) \text{ so}$$

$$\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x)+c.$$

$$= \frac{1}{2}\left(-\frac{1}{3}\cos(3x)-\cos(x)\right)+c$$

$$= -\frac{1}{6}\cos(3x)-\frac{1}{2}\cos(x)+c$$

Note: It appears that the different methods have resulted in different results. However, it is possible to use the trigonometric identities to verify that the solutions are identical. It is also possible to graph both solutions to verify that they are the same. In both instances, the solutions may differ by a constant value, and both be valid solutions (the '$+c$' is used because there are a family of solutions that will all differentiate to give the original function, the only difference between the solutions is the constant value).



6.4.7 Integrating powers of $\tan(ax)$ or $\cot(ax)$

We know that $\frac{d}{dx}\tan(x) = \sec^2(x)$. This means that the Pythagorean Identity $\tan^2(ax)+1 = \sec^2(ax)$ can be used to rewrite powers of $\tan(ax)$ into a function that we can integrate. Note that the derivative of $\tan(x)$ can also be used to integrate $\sec^2(x)$.

Integrating $\sec^2(x)$

$$\int \sec^2(x)\,dx = \tan(x)+c$$

The quotient rule can be used to determine $\frac{d}{dx}\cot(x)$.

$$\begin{aligned}
\frac{d}{dx}(\cot(x)) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \frac{d}{dx}\left(\frac{u}{v}\right) \\
&= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \\
&= \frac{-\left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)} \\
&= \frac{-1}{\sin^2(x)} \\
&= -\operatorname{cosec}^2(x)
\end{aligned}$$

Therefore,

The derivative of cot(x)

$$\frac{d}{dx}\cot(x) = -\operatorname{cosec}^2(x)$$

The Pythagorean Identity $\cot^2(ax) + 1 = \operatorname{cosec}^2(ax)$ can be used to rewrite powers of $\cot(ax)$ into a function we can integrate.

WORKED EXAMPLE 19 Integrating powers of tan(ax) and cot(ax)

Determine the anti-derivative of each of the following integrals:

a. $\tan^2(x)$ **b.** $3\cot^2(3x)\ \operatorname{cosec}^2(3x)$

THINK	WRITE
a. 1. We know that $\frac{d}{dx}\tan(x) = \sec^2(x)$ and $\tan^2(ax) + 1 = \sec^2(ax)$. Use the identity to rewrite $\tan^2(x)$ where $a = 1$.	**a.** $\tan^2(x) = \sec^2(x) - 1$
2. Express the question using integral notation.	$\int \tan^2(x)\,dx$
3. Rewrite the integrand using the substitution.	$= \int \left(\sec^2(x) - 1\right)dx$
4. Integrate using $\int \sec^2(x)\,dx = \tan(x) + c$.	$= \tan(x) - x + c$

b. 1. As $\frac{d}{dx}\cot(3x) = -3\operatorname{cosec}^2(3x)$, the integrand is of the form $f(g(x))\ g'(x)$ where $g(x) = \cot(3x)$. Let $u = \cot(3x)$ and use differentiation to determine dx.

b. Let $u = \cot(3x)$

$$\frac{du}{dx} = -3\operatorname{cosec}^2(3x)$$

$$du = -3\operatorname{cosec}^2(3x)\,dx$$

$$dx = \frac{du}{-3\operatorname{cosec}^2(3x)}$$

2. Express the question using integral notation.

$$\int 3\cot^2(3x)\operatorname{cosec}^2(3x)\,dx$$

3. Rewrite the integral, substituting for $\cot(3x)$ and dx.

$$\int 3\cot^2(3x)\ \operatorname{cosec}^2(3x)\,dx$$

$$= \int 3\,u^2\operatorname{cosec}^2(3x)\,\frac{du}{-3\operatorname{cosec}^2(3x)}$$

4. Simplify the integrand, noting that $\frac{3\operatorname{cosec}^2(3x)}{-3\operatorname{cosec}^2(3x)} = -1$.

$$= \int \frac{3\operatorname{cosec}^2(3x)}{-3\operatorname{cosec}^2(3x)}\,u^2\,du$$

$$= \int -u^2\,du$$

5. As the integrand is now expressed as a function of u only, it can be integrated.

$$= -\frac{u^3}{3} + c$$

6. As the original integral was written in terms of x, substitute $u = \cot(3x)$.

$$= -\frac{1}{3}\cot^3(3x) + c$$

Exercise 6.4 Integration using the trigonometric identities

learnon

6.4 Exercise | **6.4 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13	14, 15, 16, 17	18, 19, 20

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Simple familiar

1. Determine the following:

a. $\frac{d}{dx}\cos(2x)$ **b.** $\frac{d}{dx}\sin(4x)$ **c.** $\frac{d}{dx}\tan(3x)$ **d.** $\frac{d}{dx}\cot(5x)$

2. WE13 Determine the anti-derivative of the following:

a. $\cos^2(x)$ **b.** $\sin^2(2x)$ **c.** $2\cos^2(4x)$ **d.** $4\sin^2(3x)$ **e.** $\cos^2(5x)$ **f.** $\sin^2(6x)$

3. Determine the following integrals:

a. $\int \cos^2\left(\frac{x}{2}\right)dx$ **b.** $\int \sin^2\left(\frac{x}{3}\right)dx$ **c.** $\int 3\cos^2\left(\frac{x}{6}\right)dx$

d. $\int 2\sin^2\left(\frac{x}{4}\right)dx$ **e.** $\int \cos^2\left(\frac{2x}{3}\right)dx$ **f.** $\int \sin^2\left(\frac{3x}{2}\right)dx$

4. **WE14** Determine the anti-derivative of the following:

a. $\cos^4(3x)$ b. $\cos^4\left(\frac{x}{2}\right)$ c. $\sin^4(x)$ d. $\sin^4(2x)$

5. **WE15a** Determine the following indefinite integrals:

a. $\int 2\sin(x)\cos(x)\,dx$ b. $\int 4\sin(2x)\cos(2x)\,dx$

c. $\int \sin(3x)\cos(3x)\,dx$ d. $\int -2\sin(4x)\cos(4x)\,dx$

6. **WE15b** Determine the following indefinite integrals:

a. $\int \sin^2(x)\cos^2(x)\,dx$ b. $\int \sin^2(2x)\cos^2(2x)\,dx$ c. $\int 2\sin^2(4x)\cos^2(4x)\,dx$

d. $\int 2\sin^2(3x)\cos^2(3x)\,dx$ e. $\int 6\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)dx$ f. $\int 4\sin^2\left(\frac{x}{3}\right)\cos^2\left(\frac{x}{3}\right)dx$

7. **WE16** Determine the following integrals:

a. $\int \sin(x)\cos^4(x)\,dx$ b. $\int \sin(2x)\cos^3(2x)\,dx$ c. $\int \sin\left(\frac{x}{2}\right)\cos^5\left(\frac{x}{2}\right)dx$

d. $\int \cos(3x)\sin^4(3x)\,dx$ e. $\int \cos\left(\frac{x}{5}\right)\sin^6\left(\frac{x}{5}\right)dx$ f. $\int \cos\left(\frac{2x}{3}\right)\sin^7\left(\frac{2x}{3}\right)dx$

8. **WE17a** Determine the anti-derivatives of the following:

a. $\sin^3(x)$ b. $\cos^3(2x)$ c. $6\sin^3(4x)$

d. $4\cos^3(3x)$ e. $\sin^3(7x)$ f. $\cos^3(6x)$

9. Determine the following:

a. $\int 3\sin^3\left(\frac{x}{2}\right)dx$ b. $\int 2\cos^3\left(\frac{x}{3}\right)dx$ c. $\int \sin^3\left(\frac{3x}{2}\right)dx$

d. $\int \cos^3\left(\frac{5x}{2}\right)dx$ e. $\int \sin^3\left(\frac{3x}{4}\right)dx$ f. $\int \cos^3\left(\frac{4x}{3}\right)dx$

10. **WE18** Determine the following:

a. $\int \sin(x)\cos(2x)\,dx$ b. $\int \cos(2x)\cos(4x)\,dx$ c. $\int \sin(3x)\cos(6x)\,dx$

d. $\int \cos(4x)\cos(8x)\,dx$ e. $\int \sin\left(\frac{x}{2}\right)\cos(x)\,dx$ f. $\int \cos\left(\frac{x}{3}\right)\cos\left(\frac{2x}{3}\right)dx$

11. **WE19** Determine the anti-derivative of each of the following integrals:

a. $1+\tan^2(2x)$ b. $1+\tan^2\left(\frac{x}{3}\right)$ c. $\tan^2(x)\sec^2(x)$

d. $\tan^3(x)\sec^2(x)$ e. $4\tan^5(2x)\sec^2(2x)$ f. $8\tan^4\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)$

12. If $f'(x)=6\sin(x)\cos^2(x)$ and $f\left(\frac{\pi}{3}\right)=0$, determine $f(x)$.

13. If $f'(x)=4\sin^2(2x)\cos^2(2x)$ and $f\left(\frac{\pi}{4}\right)=\pi$, determine $f(x)$.

Complex familiar

14. WE17b Determine the anti-derivatives of the following:

a. $\cos^2(x)\ \sin^3(x)$ **b.** $\sin^2(x)\ \cos^3(x)$ **c.** $\cos^2(2x)\sin^3(2x)$

d. $\sin^2(3x)\cos^3(3x)$ **e.** $\cos^2\left(\frac{x}{2}\right)\sin^3\left(\frac{x}{2}\right)$ **f.** $\sin^2\left(\frac{3x}{2}\right)\cos^3\left(\frac{3x}{2}\right)$

15. Determine the following:

a. $\int 4\cos^2\left(\frac{x}{3}\right)\sin^3\left(\frac{x}{3}\right)dx$ **b.** $\int -6\sin^2\left(\frac{5x}{4}\right)\cos^3\left(\frac{5x}{4}\right)dx$ **c.** $\int \sin^3(x)\ \cos^4(x)\,dx$

d. $\int \cos^3(2x)\sin^4(2x)\,dx$ **e.** $\int 2\sin^3(2x)\cos^5(2x)\,dx$ **f.** $\int -2\cos^3(3x)\sin^6(3x)\,dx$

16. Determine the following integrals:

a. $\int \tan^2(x)\ \sec^4(x)\,dx$ **b.** $\int 6\cot^2(2x)\ \operatorname{cosec}^4(2x)\,dx$ **c.** $\int 2\tan^2\left(\frac{x}{2}\right)\sec^4\left(\frac{x}{2}\right)dx$

d. $\int 3\cot^3(3x)\ \operatorname{cosec}^4(3x)\,dx$ **e.** $\int \tan^4\left(\frac{x}{5}\right)\sec^4\left(\frac{x}{5}\right)dx$ **f.** $\int 12\cot^5(6x)\ \operatorname{cosec}^6(6x)\,dx$

17. Determine $g(x)$ if $g'(x)=\sin^3\left(\frac{x}{2}\right)\cos^4\left(\frac{x}{2}\right)$ and $g(0)=-\frac{4}{35}$.

Complex unfamiliar

18. Determine the following integrals:

a. $\int \sin(x)\ \cos^n(x)\,dx$

b. $\int \cos(x)\ \sin^n(x)\,dx$

19. Determine the following integrals:

a. $\int \sin^3(x)\ \cos^n(x)\,dx$

b. $\int \cos^3(x)\ \sin^n(x)\,dx$

20. Determine the following integrals:

a. $\int \sec^2(x)\ \tan^n(x)\,dx$ for $n\in\mathbb{Z}$ with $n>1$

b. $\int \sec^4(x)\ \tan^n(x)\,dx$ for $n\in\mathbb{Z}$ with $n>1$

Fully worked solutions for this chapter are available online.

LESSON
6.5 Integration of inverse trigonometric functions

SYLLABUS LINKS

- Understand and use the inverse trigonometric functions: arcsine, arccosine and arctangent.
- Use the derivative of the inverse trigonometric functions: arcsine, arccosine and arctangent.
  - $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$
  - $\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$
  - $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$
- Integrate expressions of the form $\frac{\pm 1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$
  - $\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$
  - $\int \frac{-1}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + c$
  - $\int \frac{a}{a^2 + x^2} = \tan^{-1}\left(\frac{x}{a}\right) + c$

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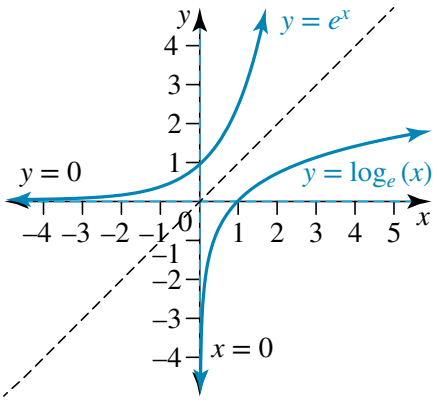
6.5.1 Inverse functions

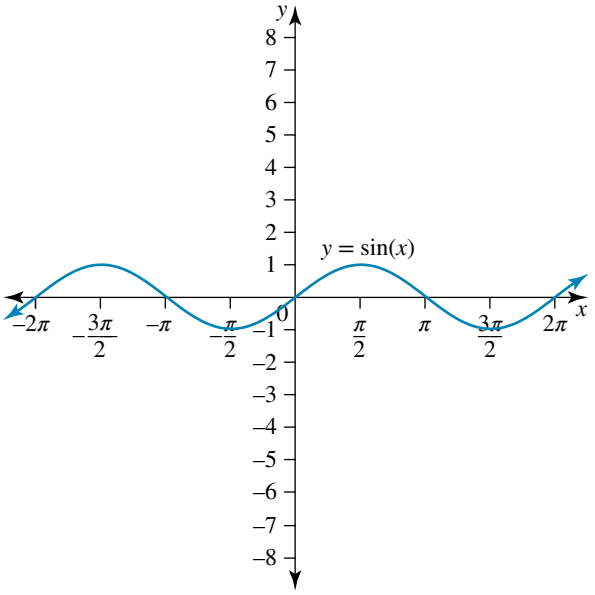
In Mathematical Methods you explored the inverse relationship between $y = a^x$ and $x = \log_a(y)$. If you graph $y = e^x$ and $y = \ln(x)$ you can see that they are reflections of each other in the line $y = x$. As both functions are one-to-one functions, the inverse is also a function.



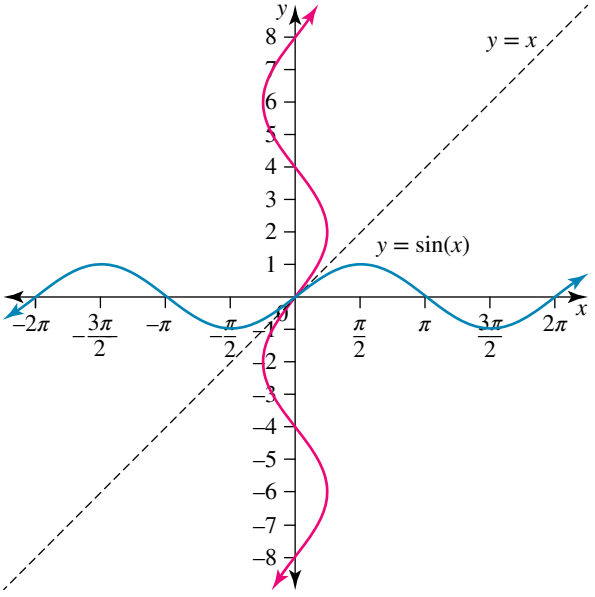
All circular functions are periodic functions. They are many-to-one functions meaning that if the domain of the circular function is not restricted to make it a one-to-one function, then the inverse of a trigonometric function is not a function.

The inverse sine function

The sine function $y = \sin(x)$, is a many-to-one function.

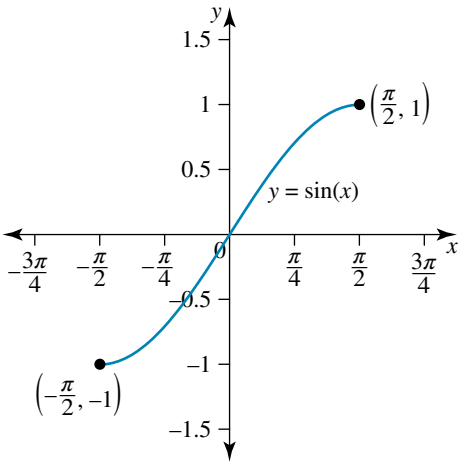


If it is reflected in the line $y = x$, the result is a one-to-many relation.

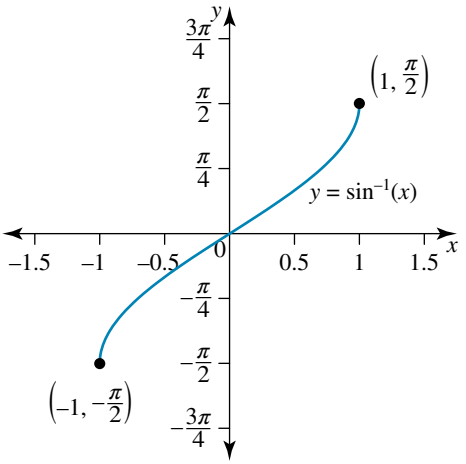


If we want the inverse to be a function, it is necessary to restrict the domain of the sine function.

Because the sine function is cyclic, there are an infinite number of possible domain restrictions, such as $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. The restriction $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is normally chosen. This will make the domain of the sine function $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range $[-1, 1]$.

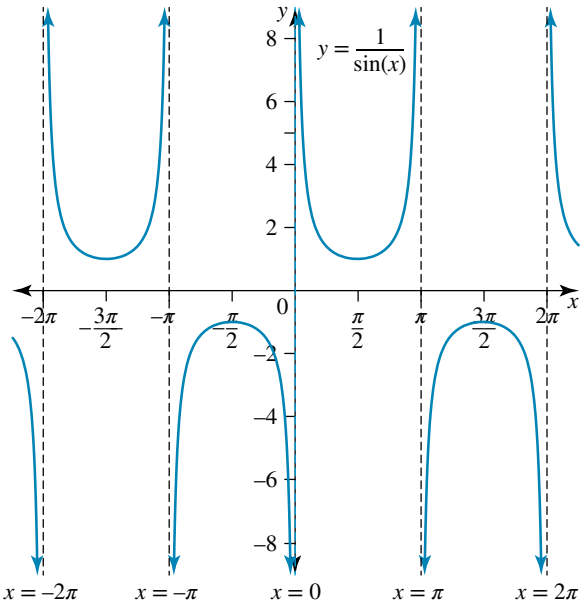


And the inverse function will be



The domain of this inverse function is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note this is different from graphing the reciprocal function $y = \frac{1}{\sin(x)}$. The graph of $y = \frac{1}{\sin(x)}$ (which you studied in Unit 2 Topic 4 (chapter 9)) looks like:



The inverse function can be written as $y = \sin^{-1}(x)$ or $y = \arcsin(x)$. Be careful not to confuse $y = \sin^{-1}(x)$ and $y = \dfrac{1}{\sin(x)} = \operatorname{cosec}(x)$.

Although there are an infinite number of solutions to $\sin(x) = \dfrac{1}{2}$, there is only one solution to $x = \sin^{-1}\left(\dfrac{1}{2}\right)$ because $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, so $\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$. Note this is also the solution that your calculator returns.

WORKED EXAMPLE 20 Calculating exact values of the inverse sine function

Calculate the following:

a. $\sin^{-1}(2)$

b. $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

c. $\sin\left(\sin^{-1}\left(\dfrac{1}{3}\right)\right)$

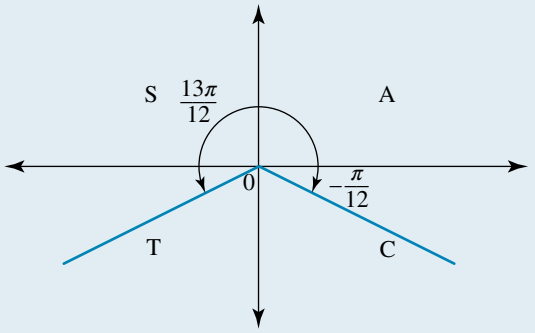
d. $\sin^{-1}\left(\sin\left(\dfrac{13\pi}{12}\right)\right)$

THINK	WRITE
a. 1. Write an equivalent statement.	**a.** Let $x = \sin^{-1}(2)$ $\sin(x) = 2$
2. The range of $\sin(x)$ is $[-1, 1]$ so there is no solution to $\sin(x) = 2$.	As $2 \notin [-1, 1]$ there is no solution to $\sin(x) = 2$.
b. 1. Write an equivalent statement.	**b.** Let $x = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ $\sin(x) = \dfrac{\sqrt{3}}{2}$
2. This relationship is from a special triangle.	$\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$
3. $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ so there will be only one solution to $x = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$.	$x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ so the only solution is $x = \dfrac{\pi}{3}$.
c. 1. Let $x = \sin^{-1}\left(\dfrac{1}{3}\right)$ and write an equivalent statement.	**c.** Let $x = \sin^{-1}\left(\dfrac{1}{3}\right)$ $\sin(x) = \dfrac{1}{3}$
2. Substitute $x = \sin^{-1}\left(\dfrac{1}{3}\right)$ into $\sin\left(\sin^{-1}\left(\dfrac{1}{3}\right)\right)$.	$\sin\left(\sin^{-1}\left(\dfrac{1}{3}\right)\right) = \sin(x)$
3. We know that $\sin(x) = \dfrac{1}{3}$. *Note:* As $\sin(x)$ and $\sin^{-1}(x)$ are inverse functions, this question could have solved by realising that $\sin\left(\sin^{-1}(x)\right) = x$ if $x \in [-1, 1]$	$\sin\left(\sin^{-1}\left(\dfrac{1}{3}\right)\right) = \dfrac{1}{3}$

d. 1. Write an equivalent statement.

2. As $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$x \neq \frac{13\pi}{12}$. $\frac{13\pi}{12}$ is in quadrant 3.



d. Let $x = \sin^{-1}\left(\sin\left(\frac{13\pi}{12}\right)\right)$

$\sin(x) = \sin\left(\frac{13\pi}{12}\right)$

$\frac{13\pi}{12}$ is in quadrant 3.

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\left(\frac{13\pi}{12}\right) = -\sin\left(\frac{\pi}{12}\right)$

$= \sin\left(-\frac{\pi}{12}\right)$

So the statement

$\sin(x) = \sin\left(\frac{13\pi}{12}\right)$ becomes,

$\sin(x) = \sin\left(-\frac{\pi}{12}\right)$.

Therefore $x = -\frac{\pi}{12}$.

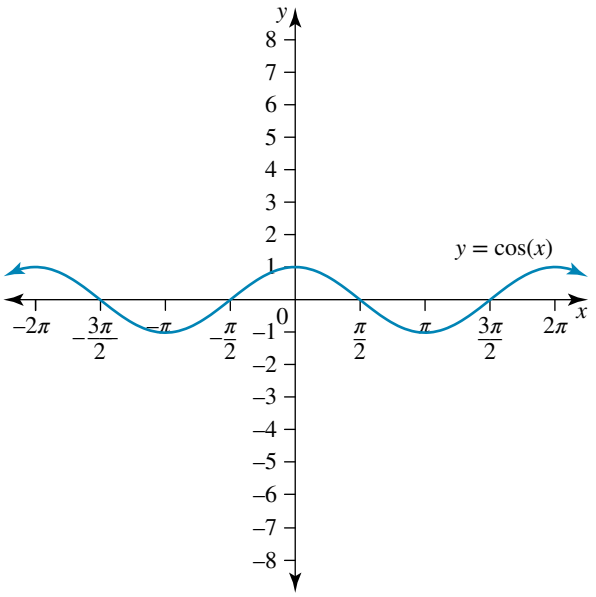
Properties of the inverse sine function

For the inverse tangent function f: $[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin^{-1}(x)$,

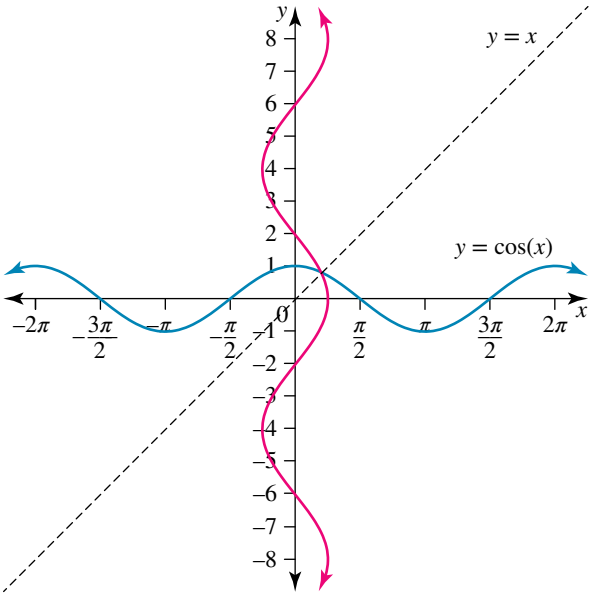
$$\sin\left(\sin^{-1}(x)\right) = x, \quad x \in [-1, 1]$$
$$\sin^{-1}(\sin(x)) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The inverse cosine function

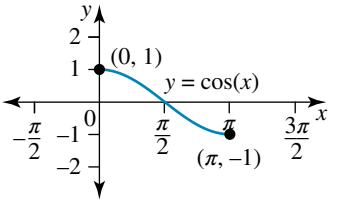
The cosine function $y = \cos(x)$, is a many-to-one function.



If it is reflected in the line $y = x$, the result is a one-to-many relation.

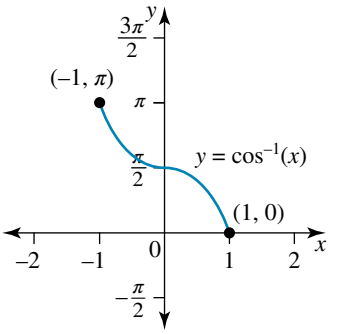


If we want the inverse to be a function, it is necessary to restrict the domain of the cosine function.



Because the cosine function is cyclic, there are an infinite number of possible domain restrictions, such as $[-\pi, 0]$, $[0, \pi]$ or $[\pi, 2\pi]$. The restriction $[0, \pi]$ is normally chosen. This will make the domain of the cosine function $[0, \pi]$ and the range $[-1, 1]$.

And the inverse function will be:



The domain of this inverse function is $[-1, 1]$ and the range is $[0, \pi]$.

Note this is different to graphing the reciprocal function $y = \dfrac{1}{\cos(x)} = \sec x$ (which you studied in Unit 2 Topic 4).

The inverse function can be written as $y = \cos^{-1}(x)$ or $y = \arccos(x)$.

Although there are an infinite number of solutions to $\cos(x) = \dfrac{\sqrt{2}}{2}$, there is only one solution to $x = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)$ because $x \in [0, \pi]$, so $\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$. Note this is also the solution that your calculator returns.

WORKED EXAMPLE 21 Calculating exact values of the inverse cosine function

Calculate the following:

a. $\cos^{-1}\left(\dfrac{3}{2}\right)$

b. $\cos^{-1}\left(\cos\left(-\dfrac{\pi}{5}\right)\right)$

THINK	WRITE
a. 1. Write an equivalent statement	**a.** Let $x = \cos^{-1}\left(\dfrac{3}{2}\right)$ $\cos(x) = \dfrac{3}{2}$

2. The range of $\cos(x)$ is $[-1, 1]$ so there is no solution to $\cos^{-1}\left(\frac{3}{2}\right)$.

As $\frac{3}{2} \notin [-1, 1]$ there is no solution to $\cos(x) = \frac{3}{2}$.

b. 1. Write an equivalent statement.

b. Let $x = \cos^{-1}\left(\cos\left(-\frac{\pi}{5}\right)\right)$

$\cos(x) = \cos\left(-\frac{\pi}{5}\right)$

2. As $x \in [0, \pi]$

$x \neq -\frac{\pi}{5}$. $-\frac{\pi}{5}$ is in quadrant 4.

$-\frac{\pi}{5}$ is in quadrant 4.

$x \in [0, \pi]$

$\cos\left(-\frac{\pi}{5}\right) = \cos\left(\frac{\pi}{5}\right)$

Therefore $x = \frac{\pi}{5}$.

Properties of the inverse cosine function

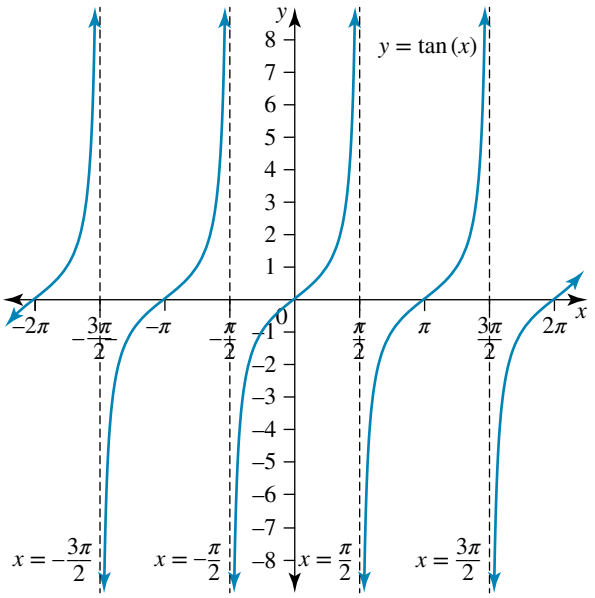
For the inverse cosine function f: $[-1, 1] \to [0, \pi]$, $f(x) = \cos^{-1}(x)$,

$$\cos\left(\cos^{-1}(x)\right) = x, \quad x \in [-1, 1]$$

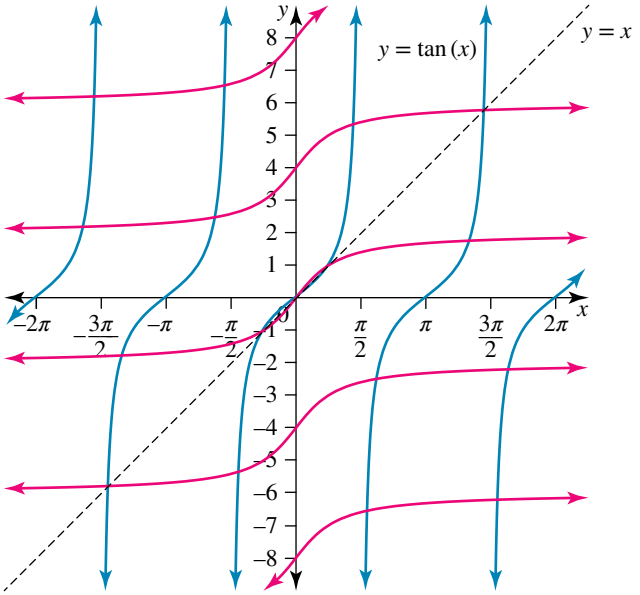
$$\cos^{-1}(\cos(x)) = x, \quad x \in [0, \pi]$$

The inverse tangent function

The tangent function $y = \tan(x)$, is a many-to-one function.

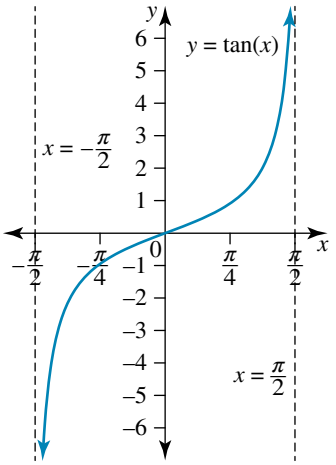


If it is reflected in the line $y = x$, the result is a one-to-many relation.

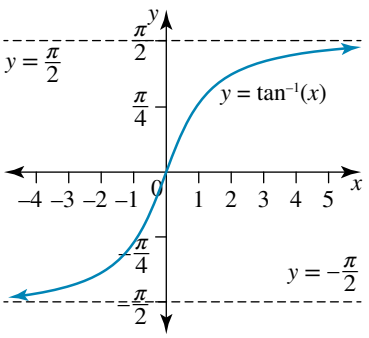


If we want the inverse to be a function, it is necessary to restrict the domain of the tangent function.

Because the tangent function is cyclic, there are an infinite number of possible domain restrictions, such as $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. The restriction $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is normally chosen. This will make the domain of the tangent function $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the range $\mathbb{R}$.



And the inverse function will be:



The domain of this inverse function is $\mathbb{R}$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The inverse function can be written as $y = \tan^{-1}(x)$ or $y = \arctan(x)$. Be careful not to confuse $y = \tan^{-1}(x)$ and $y = \frac{1}{\tan(x)} = \cot(x)$.

Although there are an infinite number of solutions to $\tan(x) = 1$, there is only one solution to $x = \tan^{-1}(1)$ because $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\tan^{-1}(1) = \frac{\pi}{4}$. Note this is also the solution that your calculator returns.

WORKED EXAMPLE 22 Calculating exact values of the inverse tan function

Calculate the following:

a. $\tan\left(\tan^{-1}(2)\right)$

b. $\tan^{-1}\left(\tan\left(\frac{13\pi}{12}\right)\right)$

THINK	WRITE
a. 1. Let $x = \tan^{-1}(2)$ and write an equivalent statement.	**a.** Let $x = \tan^{-1}(2)$ $\tan(x) = 2$
2. Substitute $x = \tan^{-1}(2)$ in $\tan\left(\tan^{-1}(2)\right)$.	$\tan\left(\tan^{-1}(2)\right) = \tan(x)$
3. We know that $\tan(x) = 2$. *Note:* As $\tan(x)$ and $\tan^{-1}(x)$ are inverse functions, this question could have solved by realising that $\tan\left(\tan^{-1}(x)\right) = x$ if $x \in \mathbb{R}$.	$= 2$
b. 1. Write an equivalent statement.	**b.** Let $x = \tan^{-1}\left(\tan\left(\frac{13\pi}{12}\right)\right)$ $\tan(x) = \tan\left(\frac{13\pi}{12}\right)$
2. As $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $x \neq \frac{13\pi}{12}$. $\frac{13\pi}{12}$ is in quadrant 3.	$\frac{13\pi}{12}$ is in quadrant 3. $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\tan\left(\frac{13\pi}{12}\right) = \tan\left(\frac{\pi}{12}\right)$ Therefore $x = \frac{\pi}{12}$.

Properties of the inverse tangent function

For the inverse tangent function $f: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $f(x) = \tan^{-1}(x)$,

$$\tan(\tan^{-1}(x)) = x, \quad x \in \mathbb{R}$$
$$\tan^{-1}(\tan(x)) = x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

6.5.2 Derivatives of the inverse trigonometric functions

It is possible to determine the derivative of the inverse trigonometric functions. In order to determine these derivatives, it is necessary to use the Pythagorean Identities. Your formula sheet lists the formulas as

Derivatives involving inverse trigonometric functions

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}, \quad x \in (-a, a)$$
$$\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}}, \quad x \in (-a, a)$$
$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}, \quad x \in \mathbb{R}$$

The inverse sine function

Let $y = \sin^{-1}(x)$. We know that $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Rearranging: $x = \sin(y)$

Differentiating: $\frac{dx}{dy} = \cos(y)$

Taking reciprocals: $\frac{dy}{dx} = \frac{1}{\cos(y)}$

The Pythagorean Identity states that $\cos^2(y) + \sin^2(y) = 1$

$$\cos^2(y) = 1 - \sin^2(y)$$

As $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos(y) \in [0, 1]$, as $x = \sin(y)$, $\cos(y) = \sqrt{1-\sin^2(y)}$
$$= \sqrt{1-x^2}$$

Therefore $\frac{dy}{dx} = \frac{1}{\cos(y)}$
$$= \frac{1}{\sqrt{1-x^2}}$$

We have demonstrated that $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$.

We can use the chain rule to determine $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)$.

Let $y = \sin^{-1}\left(\frac{x}{a}\right)$ and $u = \frac{x}{a}$

$\Rightarrow y = \sin^{-1}(u)$ with $\frac{du}{dx} = \frac{1}{a}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{\sqrt{1-u^2}} \times \frac{1}{a}$$

Replacing $u = \frac{x}{a}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}\\ &= \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}\\ &= \frac{1}{a\sqrt{\frac{a^2}{a^2}-\frac{x^2}{a^2}}}\\ &= \frac{1}{a\sqrt{\frac{a^2-x^2}{a^2}}}\\ &= \frac{1}{a\times\frac{1}{a}\sqrt{a^2-x^2}}\\ &= \frac{1}{\sqrt{a^2-x^2}}\end{aligned}$$

Therefore $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$.

WORKED EXAMPLE 23 Determining the derivative of the inverse sine function

Determine $\frac{d}{dx}\sin^{-1}(3x)$.

THINK	WRITE
1. We know that $\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$ so $\frac{x}{a} = 3x$ $a = \frac{1}{3}$	$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}, a = \frac{1}{3}$
2. Use the formula to write $\frac{d}{dx}\sin^{-1}(3x)$.	$\frac{d}{dx}\sin^{-1}(3x) = \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2-x^2}}$
3. Simplify. Note that $\sqrt{\frac{1}{9}} = \frac{\sqrt{1}}{\sqrt{9}} = \frac{1}{3}$.	$= \frac{1}{\sqrt{\frac{1}{9}-x^2}}$ $= \frac{1}{\sqrt{\frac{1}{9}-\frac{9x^2}{9}}}$ $= \frac{1}{\sqrt{\frac{1}{9}(1-9x^2)}}$ $= \frac{1}{\frac{1}{3}\sqrt{1-9x^2}}$ $= \frac{3}{\sqrt{1-9x^2}}$

The inverse cosine function

Let $y = \cos^{-1}(x)$. We know that $x \in [-1, 1]$ and $y \in [0, \pi]$.

Rearranging: $x = \cos(y)$

Differentiating: $\dfrac{dx}{dy} = -\sin(y)$

Taking reciprocals: $\dfrac{dy}{dx} = \dfrac{-1}{\sin(y)}$

The Pythagorean Identity states that $\cos^2(y) + \sin^2(y) = 1$

$$\sin^2(y) = 1 - \cos^2(y)$$

As $y \in [0, \pi]$, $\sin(y) \in [0, 1]$, as $x = \cos(y)$, $\sin(y) = \sqrt{1 - \cos^2(y)}$

$$= \sqrt{1 - x^2}$$

Therefore $\dfrac{dy}{dx} = \dfrac{-1}{\sin(y)} = \dfrac{-1}{\sqrt{1 - x^2}}$.

We have demonstrated that $\dfrac{d}{dx}\cos^{-1}(x) = \dfrac{-1}{\sqrt{1 - x^2}}$.

It is possible to use the chain rule to demonstrate that $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{x}{a}\right) = \dfrac{-1}{\sqrt{a^2 - x^2}}$ using a similar process shown earlier for the inverse sine function.

WORKED EXAMPLE 24 Determining the derivative of the inverse cosine function

Determine the derivative of $\cos^{-1}\left(\dfrac{2x}{3}\right)$.

THINK	WRITE
1. We know that $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{x}{a}\right) = \dfrac{-1}{\sqrt{a^2 - x^2}}$ so $\dfrac{x}{a} = \dfrac{2x}{3}$, $a = \dfrac{3}{2}$	$\dfrac{d}{dx}\cos^{-1}\left(\dfrac{x}{a}\right) = \dfrac{-1}{\sqrt{a^2 - x^2}}$, $a = \dfrac{3}{2}$
2. Use the formula to write $\dfrac{d}{dx}\cos^{-1}\left(\dfrac{2x}{3}\right)$.	$\dfrac{d}{dx}\cos^{-1}\left(\dfrac{2x}{3}\right) = \dfrac{-1}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$
3. Simplify. Note that $\sqrt{\dfrac{1}{4}} = \dfrac{\sqrt{1}}{\sqrt{4}} = \dfrac{1}{2}$.	$= \dfrac{-1}{\sqrt{\frac{9}{4} - x^2}}$ $= \dfrac{-1}{\sqrt{\frac{9}{4} - \frac{4x^2}{4}}}$ $= \dfrac{-1}{\sqrt{\frac{1}{4}(9 - 4x^2)}}$ $= \dfrac{-1}{\frac{1}{2}\sqrt{9 - 4x^2}}$ $= \dfrac{-2}{\sqrt{9 - 4x^2}}$

The inverse tangent function

Let $y = \tan^{-1}(x)$. We know that $x \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Rearranging: $x = \tan(y)$

Differentiating: $\frac{dx}{dy} = \sec^2(y)$

Taking reciprocals: $\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

The Pythagorean Identity states that $\sec^2(y) = 1 + \tan^2(y)$.

As $x = \tan(y)$, $\sec^2(y) = 1 + x^2$.

Therefore $\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+x^2}$.

We have demonstrated that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.

It is possible to use the chain rule to demonstrate that $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$ using a similar process shown earlier for the inverse sine function.

WORKED EXAMPLE 25 Determining the derivative of the inverse tan function

If $y = \tan^{-1}(2x)$, determine $\frac{dy}{dx}$.

THINK	WRITE
1. We know that $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$ so $\frac{x}{a} = 2x$, $a = \frac{1}{2}$	$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$, $a = \frac{1}{2}$
2. Use the formula to determine $\frac{dy}{dx}$.	$y = \tan(2x)$ $\frac{dy}{dx} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2 + x^2}$
3. Simplify. Note that in this instance you can multiply by $\frac{4}{4}$ to assist with simplifying the fraction in the denominator.	$= \frac{\frac{1}{2}}{\frac{1}{4} + x^2}$ $= \frac{\frac{1}{2}}{\frac{1}{4} + x^2} \times \frac{4}{4}$ $= \frac{2}{1+4x^2}$

6.5.3 Integration using the inverse trigonometric functions

We have established that

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$$
$$\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$$
$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$$

From these results, the following can be stated:

Anti-derivatives involving inverse trigonometric functions

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c, \quad x \in (-a, a)$$
$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c, \quad x \in (-a, a)$$
$$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c, \quad x \in \mathbb{R}$$

Integrals involving the inverse sine function

$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ can be used to determine integrals. Notice that the coefficient of x^2 is 1, so it might be necessary to use an appropriate substitution in order to use the formula.

WORKED EXAMPLE 26 Determining anti-derivatives of the inverse sine function

Determine the following integrals:

a. $\int \frac{12}{\sqrt{36 - x^2}}\,dx$

b. $\int \frac{4}{\sqrt{49 - 36x^2}}\,dx$

THINK

a. Use $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ with $a = 6$.

WRITE

a. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a = 6$

$$\int \frac{12}{\sqrt{36 - x^2}}\,dx = 12\int \frac{1}{\sqrt{6^2 - x^2}}\,dx$$
$$= 12\sin^{-1}\left(\frac{x}{6}\right) + c$$

b. 1. As we have $36x^2$, the substitution $u = 6x$ could be used so that the denominator becomes $\sqrt{49 - u^2}$. This means that the function is in the form $f(g(x))\ g'(x)$ where $g(x) = 6x$. Let $u = 6x$ and differentiate to determine dx.

b. Let $u = 6x$

$$\frac{du}{dx} = 6$$
$$du = 6\,dx$$
$$dx = \frac{du}{6}$$

2. Rewrite the integral, substituting for $6x$ and dx.

3. Use $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ with $a = 7$.

4. Substitute for u so that the final answer is in terms of x only.

$$\begin{aligned}\int \frac{4}{\sqrt{49 - 36x^2}}\,dx &= \int \frac{4}{\sqrt{49 - (6x)^2}}\,dx \\ &= \int \frac{4}{\sqrt{49 - (u)^2}}\,\frac{du}{6} \\ &= \frac{4}{6}\int \frac{1}{\sqrt{7^2 - u^2}} \sim du \\ &= \frac{2}{3}\sin^{-1}\left(\frac{u}{7}\right) + c \\ &= \frac{2}{3}\sin^{-1}\left(\frac{6x}{7}\right) + c\end{aligned}$$

Note that for part b, alternatively, instead of making the substitution, $\int \frac{4}{\sqrt{49 - 36x^2}}\,dx$ can be expressed as $k\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$ and $k\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = k\sin^{-1}\left(\frac{x}{a}\right) + c$ can then be used.

Integrals involving the inverse cosine function

$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ can also be used to determine integrals. Notice that the coefficient of x^2 is 1, so, like using the previous formula, it might be necessary to use an appropriate substitution in order to use the formula.

WORKED EXAMPLE 27 Determining anti-derivatives of the inverse cosine function

a. The gradient of a curve that passes through the origin is given by $\frac{-4}{\sqrt{81 - 25x^2}}$. Determine the equation of the curve.

b. Alternatively, $\int \frac{-4}{\sqrt{81 - 25x^2}}\,dx$ can be expressed as $k\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx$.

THINK	WRITE
a. 1. If the original curve is y, then the gradient is $\frac{dy}{dx}$.	a. $\frac{dy}{dx} = \frac{-4}{\sqrt{81 - 25x^2}}$
2. Integrate both sides to obtain an expression for y.	$y = \int \frac{-4}{\sqrt{81 - 25x^2}}\,dx$
3. As we have $25x^2$, the substitution $u = 5x$ could be used so that the denominator becomes $\sqrt{81 - u^2}$. This means that the function is in the form $f(g(x))\ g'(x)$ where $g(x) = 5x$. Let $u = 5x$ and differentiate to determine dx.	Let $u = 5x$ $\frac{du}{dx} = 5$ $du = 5\,dx$ $dx = \frac{du}{5}$

4. Rewrite the integral, substituting for $5x$ and dx.

$$\int \frac{-4}{\sqrt{81-25x^2}}\,dx = \int \frac{-4}{\sqrt{81-(5x)^2}}\,dx$$
$$= \int \frac{-4}{\sqrt{81-(u)^2}}\,\frac{du}{5}$$

5. Use $\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ with $a = 9$.

$$= \frac{4}{5}\int \frac{-1}{\sqrt{9^2-u^2}}\,du$$
$$= \frac{4}{5}\cos^{-1}\left(\frac{u}{9}\right) + c$$

6. Substitute for u so that the final answer is in terms of x only.

$$y = \frac{4}{5}\cos^{-1}\left(\frac{5x}{9}\right) + c$$

7. As the curve passes through the origin, c can be determined by substituting $x = 0$ and $y = 0$.

$$0 = \frac{4}{5}\cos^{-1}(0) + c$$
$$0 = \frac{4}{5} \times \frac{\pi}{2} + c$$
$$c = -\frac{2\pi}{5}$$

8. Substitute for c and state the equation for y.

$$y = \frac{4}{5}\cos^{-1}\left(\frac{5x}{9}\right) - \frac{2\pi}{5}$$

b. 1. Express $y = \int \frac{-4}{\sqrt{81-25x^2}}\,dx$ as $k\int \frac{-1}{\sqrt{a^2-x^2}}\,dx$

b. $$y = \int \frac{-4}{\sqrt{81-25x^2}}\,dx$$
$$= \int \frac{-4}{5\sqrt{\frac{81}{25}-x^2}}\,dx$$
$$= \frac{4}{5}\int \frac{-1}{\sqrt{\left(\frac{9}{5}\right)^2-x^2}}\,dx$$

2. Use $\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ with $a = \frac{9}{5}$.

$$y = \frac{4}{5}\int \frac{-1}{\sqrt{\left(\frac{9}{5}\right)^2-x^2}}\,dx$$
$$= \frac{4}{5}\cos^{-1}\left(\frac{5x}{9}\right) + c$$

3. As the curve passes through the origin, c can be determined by substituting $x = 0$ and $y = 0$.

$$0 = \frac{4}{5}\cos^{-1}(0) + c$$
$$= \frac{4}{5} \times \frac{\pi}{2} + c$$
$$c = -\frac{2\pi}{5}$$

4. Substitute for c and state the equation for y.

$$y = \frac{4}{5}\cos^{-1}\left(\frac{5x}{9}\right) - \frac{2\pi}{5}$$

Integrals involving the inverse tangent function

$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c$ can also be used to determine integrals. Notice that the coefficient of x^2 is 1, so, like using the previous formulas, it might be necessary to use an appropriate substitution in order to use the formula.

WORKED EXAMPLE 28 Determining anti-derivatives of the inverse tan function

Determine the anti-derivatives of the following:

a. $\frac{8}{4+x^2}$ **b.** $\frac{4}{16+9x^2}$

THINK

a. 1. Express the question using integral notation.

2. Use $\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c$ with $a=2$.

WRITE

a. $\int \frac{8}{4+x^2}\,dx$

$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c,\ a=2$

$\int \frac{8}{4+x^2}\,dx = \int \frac{4\times 2}{2^2+x^2}\,dx$

$= 4\tan^{-1}\left(\frac{x}{2}\right)+c$

b. 1. Express the question using integral notation.

2. As we have $9x^2$, the substitution $u=3x$ could be used so that the denominator becomes $16+u^2$. This means that the function is in the form $f(g(x))\ g'(x)$ where $g(x)=3x$. Let $u=3x$ and differentiate to determine dx.

3. Rewrite the integral, substituting for $3x$ and dx.

4. Use $\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right)+c$ with $a=4$.

5. Substitute for u so that the final answer is in terms of x only.

b. $\int \frac{4}{16+9x^2}\,dx$

Let $u=3x$

$\frac{du}{dx}=3$

$du = 3\,dx$

$dx = \frac{du}{3}$

$\int \frac{4}{16+9x^2}\,dx = \int \frac{4}{16+(3x)^2}\,dx$

$= \int \frac{4}{16+(u)^2}\frac{du}{3}$

$= \frac{1}{3}\int \frac{4}{4^2+u^2}\,du$

$= \frac{1}{3}\tan^{-1}\left(\frac{u}{4}\right)+c$

$= \frac{1}{3}\tan^{-1}\left(\frac{3x}{4}\right)+c$

Integrals involving completing the square

A quadratic expression, ax^2+bx+c can be expressed, by completing the square, in the form, $(px+q)^2+k$. Recognition of this can then be used to integrate expressions of the form $\frac{1}{ax^2+bx+c}$ with $a>0$ and $\Delta = b^2-4ac>0$, which will then involve the inverse tangent function.

Integrating expressions of the form $\dfrac{1}{\sqrt{ax^2+bx+c}}$ with $a>0$ and $\Delta=b^2-4ac>0$ will involve the inverse sine or cosine functions.

WORKED EXAMPLE 29 Determining integrals using completing the square

Determine the following integrals:

a. $\displaystyle\int \frac{1}{9x^2+12x+29}\,dx$

b. $\displaystyle\int \frac{1}{\sqrt{21-12x-9x^2}}\,dx$

THINK

a. 1. Use completing the square to express the denominator in the form $(px+q)^2+k$.

2. Write the denominator as the sum of two squares.

3. This means that the function is in the form $f(g(x))\ g'(x)$ where $g(x)=3x+2$. Let $u=3x+2$ and differentiate to determine dx.

4. Rewrite the integral, substituting for $3x+2$ and dx.

5. Use $\displaystyle\int \frac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\frac{x}{a}\right)+c$ with $a=5$. Note that we need the numerator to be 5 to be able to use the formula.

6. Substitute for u so that the final answer is in terms of x only.

b. 1. Use completing the square to express the denominator in the form $(px+q)^2+k$.

2. Write the denominator as the difference of two squares within the square root.

3. This means that the function is in the form $f(g(x))\ g'(x)$ where $g(x)=3x+2$. Let $u=3x+2$ and differentiate to determine dx.

WRITE

a. $9x^2+12x+29=(3x)^2+2(3x)\,2+2^2+25$

$=(3x+2)^2+25$

$$\int \frac{1}{9x^2+12x+29}\,dx=\int \frac{1}{(3x+2)^2+25}\,dx$$

Let $u=3x+2$

$\dfrac{du}{dx}=3$

$du=3\,dx$

$dx=\dfrac{du}{3}$

$$\int \frac{1}{(3x+2)^2+25}\,dx=\int \frac{1}{(u)^2+25}\,\frac{du}{3}$$

$$=\frac{1}{3}\int \frac{1}{5^2+u^2}\,du$$

$$=\frac{1}{3}\times\frac{1}{5}\int \frac{5}{5^2+u^2}\,du$$

$$=\frac{1}{15}\tan^{-1}\left(\frac{u}{5}\right)+c$$

$$=\frac{1}{15}\tan^{-1}\left(\frac{3x+2}{5}\right)+c$$

b. $21-12x-9x^2$

$=21-\left(9x^2+12x\right)$

$=21-\left((3x)^2+2(3x)\,2+2^2\right)+4$

$=25-(3x+2)^2$

$$\int \frac{1}{\sqrt{21-12x-9x^2}}\,dx=\int \frac{1}{\sqrt{25-(3x+2)^2}}\,dx$$

Let $u=3x+2$

$\dfrac{du}{dx}=3$

$du=3\,dx$

$dx=\dfrac{du}{3}$

4. Rewrite the integral, substituting for $3x+2$ and dx.

$$\int \frac{1}{\sqrt{25-(3x+2)^2}}\,dx = \int \frac{1}{\sqrt{25-(u)^2}}\frac{du}{3}$$

5. Use $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+c$ with $a=5$.

$$= \frac{1}{3}\int \frac{1}{\sqrt{5^2-u^2}}\,du$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{5}\right)+c$$

6. Substitute for u so that the final answer is in terms of x only.

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x+2}{5}\right)+c$$

Integrals involving substitutions and inverse trigonometric functions

Using the properties of indefinite integrals, $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$, we can break up complicated integrals, into two manageable integrals.

WORKED EXAMPLE 30 Determining integrals using substitution and inverse trigonometric functions

Determine $\int \frac{4x+5}{\sqrt{25-16x^2}}\,dx$.

THINK

1. $\int \frac{a}{\sqrt{25-16x^2}}\,dx$ can be integrated using the inverse sine function. $\int \frac{bx}{\sqrt{25-16x^2}}\,dx$ can be integrated using a non-linear substitution. Therefore to integrate $\int \frac{4x+5}{\sqrt{25-16x^2}}\,dx$ express the integral as the sum of two integrals.

WRITE

Let $I = \int \frac{4x+5}{\sqrt{25-16x^2}}\,dx$

$$= \int \frac{4x}{\sqrt{25-16x^2}}\,dx + \int \frac{5}{\sqrt{25-16x^2}}\,dx$$

2. Consider the first integral. The integrand is in the form $f(g(x))\ g'(x)$ where $g(x)=25-16x^2$. Let $u=25-16x^2$ and differentiate to determine dx.

Let $u = 25-16x^2$

$$\frac{du}{dx} = -32x$$

$$du = -32x\,dx$$

$$dx = -\frac{du}{32x}$$

3. Rewrite the integral, substituting for $25-16x^2$ and dx.

$$\int \frac{4x}{\sqrt{25-16x^2}}\,dx = \int \frac{4x}{\sqrt{u}}\left(-\frac{du}{32x}\right)$$

4. Simplify the integral noting that $\frac{4x}{32x} = \frac{1}{8}$.

$$= \int -\frac{4x}{32x}\frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du$$

5. Write the integrand using power notation and complete the integration.

$$= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{8}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -\frac{1}{4}u^{\frac{1}{2}} + c$$

$$= -\frac{1}{4}\sqrt{u} + c$$

6. Substitute for u so that the final answer is in terms of x only.

$$\int \frac{4x}{\sqrt{25-16x^2}}\,dx = -\frac{1}{4}\sqrt{25-16x^2} + c$$

7. Consider the second integral. This can be integrated using $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ if the substitution $u = 4x$ is used. Let $u = 4x$ and differentiate to determine dx.

Let $u = 4x$

$$\frac{du}{dx} = 4$$

$$du = 4\,dx$$

$$dx = \frac{du}{4}$$

8. Rewrite the integral, substituting for $4x$ and dx.

$$\int \frac{5}{\sqrt{25-16x^2}}\,dx = \int \frac{5}{\sqrt{25-(4x)^2}}\,dx$$

$$= \int \frac{5}{\sqrt{25-(u)^2}}\frac{du}{4}$$

9. Use $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ with $a = 5$.

$$= \frac{5}{4}\int \frac{1}{\sqrt{5^2-u^2}}\,du$$

$$= \frac{5}{4}\sin^{-1}\left(\frac{u}{5}\right) + c$$

10. Substitute for u so that the final answer is in terms of x only.

$$\int \frac{5}{\sqrt{25-16x^2}}\,dx = \frac{5}{4}\sin^{-1}\left(\frac{4x}{5}\right) + c$$

11. As we have now determined $\int \frac{4x}{\sqrt{25-16x^2}}\,dx$ and $\int \frac{5}{\sqrt{25-16x^2}}\,dx$, we can now state the original integral. It is only necessary to include one arbitrary constant.

$$I = \int \frac{4x}{\sqrt{25-16x^2}}\,dx + \int \frac{5}{\sqrt{25-16x^2}}\,dx$$

$$= -\frac{1}{4}\sqrt{25-16x^2} + \frac{5}{4}\sin^{-1}\left(\frac{4x}{5}\right) + c$$

$$= \frac{5}{4}\sin^{-1}\left(\frac{4x}{5}\right) - \frac{1}{4}\sqrt{25-16x^2} + c$$

Exercise 6.5 Integration of inverse trigonometric functions

learnon

6.5 Exercise	6.5 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13	14, 15, 16, 17, 18, 19	20

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Simple familiar

1. WE20 Calculate the following:

a. $\sin^{-1}(1.1)$ **b.** $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right)$ **c.** $\sin\left(\sin^{-1}(0.9)\right)$

d. $\sin^{-1}\left(-\frac{6}{5}\right)$ **e.** $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$ **f.** $\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

2. WE21 Calculate the following:

a. $\cos^{-1}(1.2)$ **b.** $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$ **c.** $\cos\left(\cos^{-1}\left(\frac{\pi}{6}\right)\right)$

d. $\cos^{-1}\left(\frac{4}{3}\right)$ **e.** $\cos^{-1}\left(\cos\left(\frac{11\pi}{3}\right)\right)$ **f.** $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$

3. WE22 Calculate the following:

a. $\tan^{-1}\left(\sqrt{3}\right)$ **b.** $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **c.** $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$

d. $\tan\left(\tan^{-1}\left(\frac{5}{4}\right)\right)$ **e.** $\tan^{-1}\left(\tan\left(\frac{5\pi}{3}\right)\right)$ **f.** $\tan\left(\tan^{-1}(1.1)\right)$

4. WE23 Determine:

a. $\frac{d}{dx}\sin^{-1}\left(\frac{x}{4}\right)$ **b.** $\frac{d}{dx}\sin^{-1}(5x)$

5. WE24 Determine the derivative of:

a. $\cos^{-1}\left(\frac{3x}{4}\right)$ **b.** $\cos^{-1}\left(\frac{2x-3}{5}\right)$

6. WE25 Determine $\frac{dy}{dx}$ for each of the following:

a. $y=\tan^{-1}(4x)$ **b.** $y=\tan^{-1}\left(\frac{2x-3}{5}\right)$

7. Given $f(x)=\sin^{-1}(4x)$, determine $f'\left(\frac{1}{8}\right)$.

8. Determine the gradient of the curve $y=\cos^{-1}\left(\frac{3x-4}{5}\right)$ at the point where $x=\frac{4}{3}$.

9. WE26 Determine the following integrals:

a. $\int \frac{1}{\sqrt{100-x^2}}\,dx$ **b.** $\int \frac{2}{\sqrt{4-x^2}}\,dx$ **c.** $\int \frac{12}{\sqrt{64-9x^2}}\,dx$ **d.** $\int \frac{1}{\sqrt{36-25x^2}}\,dx$

10. WE27 The gradient of a curve that passes through the origin is given by $\dfrac{-2}{\sqrt{25-16x^2}}$. Determine the equation of the curve.

11. On a certain curve the gradient is given by $\dfrac{-2}{\sqrt{4-x^2}}$. Determine the equation of the curve that passes through the point $(2, 3)$.

12. WE28 Determine the anti-derivatives of the following:

a. $\dfrac{10}{25+x^2}$ b. $\dfrac{1}{100+x^2}$ c. $\dfrac{12}{64+9x^2}$ d. $\dfrac{1}{36+25x^2}$

13. Determine the derivative of each of the following:

a. $\sin^{-1}\left(\dfrac{\sqrt{x}}{3}\right)$ b. $\sin^{-1}\left(\dfrac{3}{4x}\right)$ c. $\cos^{-1}\left(\dfrac{e^{2x}}{4}\right)$ d. $\cos^{-1}\left(\dfrac{4}{3x}\right)$ e. $\tan^{-1}\left(\dfrac{x^2}{3}\right)$ f. $\tan^{-1}\left(\dfrac{6}{5x}\right)$

Complex familiar

14. WE29 Determine the following integrals:

a. $\displaystyle\int \frac{1}{4x^2-12x+25}\,dx$ b. $\displaystyle\int \frac{1}{\sqrt{7+12x-4x^2}}\,dx$ c. $\displaystyle\int \frac{1}{25x^2-20x+13}\,dx$ d. $\displaystyle\int \frac{1}{\sqrt{12+20x-25x^2}}\,dx$

15. Determine the anti-derivative of the following:

a. $\dfrac{2}{\sqrt{5-4x-x^2}}$ b. $\dfrac{2}{x^2+4x+13}$ c. $\dfrac{6}{\sqrt{24-30x-9x^2}}$ d. $\dfrac{6}{74+30x+9x^2}$

16. Determine $f(x)$ for each of the following:

a. $f'(x)=\dfrac{x}{\sqrt{36-25x^2}}$ b. $f'(x)=\dfrac{5x}{64+25x^2}$ c. $f'(x)=\dfrac{10x}{\sqrt{49-25x^2}}$ d. $f'(x)=\dfrac{x}{49+36x^2}$

17. WE30 Determine:

a. $\displaystyle\int \frac{5-3x}{\sqrt{29-9x^2}}\,dx$ b. $\displaystyle\int \frac{3x+5}{9x^2+25}\,dx$ c. $\displaystyle\int \frac{1-4x}{\sqrt{1-16x^2}}\,dx$ d. $\displaystyle\int \frac{8+x}{1+16x^2}\,dx$

18. Determine the anti-derivative of the following:

a. $\dfrac{3x-4}{\sqrt{9-16x^2}}$ b. $\dfrac{3+4x}{9+16x^2}$ c. $\dfrac{5-2x}{\sqrt{5-2x^2}}$ d. $\dfrac{5-2x}{25+2x^2}$

19. Determine each of the following:

a. $\displaystyle\int_3^5 \frac{x-4}{x^2-6x+13}\,dx$ b. $\displaystyle\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{2x+1}{4x^2-12x+25}\,dx$ c. $\displaystyle\int_1^4 \frac{3x+2}{\sqrt{8+2x-x^2}}\,dx$ d. $\displaystyle\int_{-2}^1 \frac{2x-3}{\sqrt{12-8x-4x^2}}\,dx$

Complex unfamiliar

20. If a, b, p and q are positive real constants, determine the following:

a. $\displaystyle\int \frac{1}{\sqrt{p^2-q^2x^2}}\,dx$ b. $\displaystyle\int \frac{1}{p^2+q^2x^2}\,dx$ c. $\displaystyle\int \frac{ax+b}{\sqrt{p^2-q^2x^2}}\,dx$ d. $\displaystyle\int \frac{ax+b}{p^2+q^2x^2}\,dx$

Fully worked solutions for this chapter are available online.

LESSON
6.6 Integration by parts

SYLLABUS LINKS

- Integrate by parts.
  - $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

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6.6.1 Introduction to integration by parts

The product rule states that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Integrating both sides with respect to x gives $uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$.

Rearranging yields $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Or using different notation: If $h(x) = f(x)\,g(x)$ then the product rule says the $h'(x) = f(x)g'(x) + f'(x)g(x)$.

Integrating both sides with respect to x gives $f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$.

Rearranging yields $\int f'(x)\,g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx$.

Using either formula is known as integration by parts. The formula for integration by parts is found on your formula sheet.

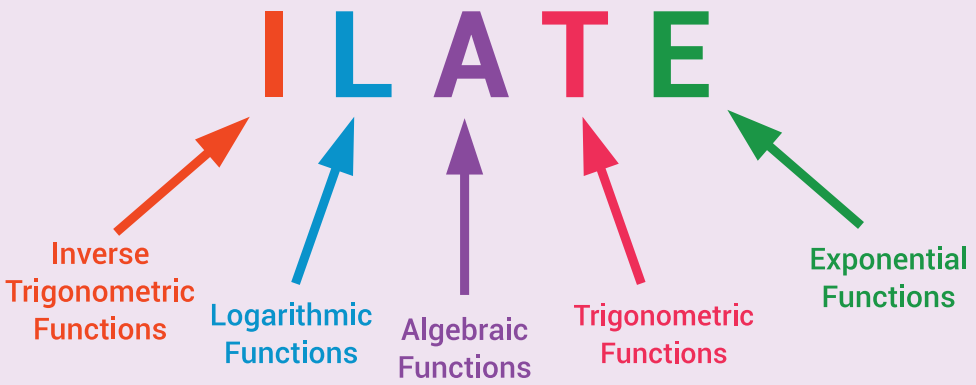
When integrating a product, integration by parts may be a suitable method. Using the integration by parts, you will need to identify u and $\frac{dv}{dx}$.

When identifying u (or $f(x)$), notice that the function you need to integrate involves $\frac{du}{dx}$ (or $f'(x)$). This means that a good choice for u (or $f(x)$) is often a polynomial, as repeated differentiation of a polynomial will eventually result in a constant.

When identifying $\frac{dv}{dx}$ (or $g'(x)$) it is necessary to be able to determine v (or $g(x)$).

TIP

When integrating by parts, to decide which function should be chosen as u (or $f(x)$), you can use the ILATE rule.



Select u to be the type of function closest to the first letter of the word ILATE.

For instance, in the case of $\int x\sin(x)\,dx$, x is an algebraic function, corresponding to the letter A, and $\sin(x)$ is a trigonometric function (corresponding to the letter T). A comes before T in ILATE, therefore, x should be chosen for u, to make the process of integration by parts as easy as possible.

WORKED EXAMPLE 31 Determining an integral using integration by parts

Determine $\int x\sin(x)\,dx$.

THINK	WRITE
1. As the integrand is a product, try to integrate by parts. We need to be able to express $\int \underbrace{x}_{A}\,\underbrace{\sin(x)}_{i}dx$ as $\int u\frac{dv}{dx}\,dx$.	$\int x\sin(x)\,dx = \int u\frac{dv}{dx}\,dx$
2. Using the ILATE rule, l $u = x$ and $\frac{dv}{dx} = \sin(x)$.	$u = x,\ \frac{dv}{dx} = \sin(x)$
3. Differentiate u to determine $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to determine v.	$\frac{du}{dx} = 1,\ v = \int \sin(x)\,dx = -\cos(x)$
4. Use $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$ and substitute for u, v and $\frac{dv}{dx}$.	$\int x\sin(x)\,dx = uv - \int v\frac{du}{dx}\,dx$
Integrate using $\int \cos(x)\,dx = \sin(x) + c$.	$= x \times -\cos(x) - \int -\cos(x) \times 1\,dx$
	$= -x\cos(x) + \int \cos(x)\,dx$
	$= -x\cos(x) + \sin(x) + c$
5. State the answer.	$\int x\sin(x)\,dx = -x\cos(x) + \sin(x) + c$

Sometimes integration by parts needs to be repeated until the $\int v\frac{du}{dx}\,dx$ or $\int f'(x)\ g(x)\,dx$ is a function that you are able to integrate. The next example will use the $f(x)$ notation.

WORKED EXAMPLE 32 Determining an integral using two successive integrations by parts

Determine $\int x^2e^x\,dx$.

THINK	WRITE
1. As the integrand is a product, try to integrate by parts. We need to be able to express $\int \underbrace{x^2}_{A}\ \underbrace{e^x}_{E}\,dx$ as $\int u\frac{dv}{dx}$.	Let $I = \int x^2e^x\,dx$ $= \int u\frac{dv}{dx}$
2. As x^2 is a polynomial, using the ILATE rule, let $u = x^2$ and $\frac{dv}{dx} = e^x$.	$u = x^2,\ \frac{dv}{dx} = e^x$
3. Differentiate u to determine $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to determine v.	$\frac{du}{dx} = 2x,\ v = \int e^x\,dx = e^x$
4. Use the integration by parts formula $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}$ and substitute for u, v and $\frac{dv}{dx}$.	$I = uv - \int v\frac{du}{dx}$ $= x^2 \times e^x - \int (2x \times e^x)\,dx$ $= x^2e^x - \int (2x\,e^x)\,dx$
5. Integration by parts is needed in order to find $\int 2x\,e^x\,dx$ where $u = 2x$ and $\frac{dv}{dx} = e^x$.	$u = 2x,$ $\frac{du}{dx} = 2,$ $\frac{dv}{dx} = e^x$ $v = e^x$ $\int (2x\,e^x)\,dx = uv - \int v\frac{du}{dx}\,dx$ $= 2x \times e^x - \int (2 \times e^x)\ dx$ $= 2xe^x - 2e^x + c$
6. Substitute for $\int 2x\,e^x\,dx$ into the original integral.	$I = x^2e^x - \int 2x\,e^x\,dx$ $= x^2e^x - (2x\,e^x - 2e^x) + c$ $= x^2e^x - 2x\,e^x + 2e^x + c$

Sometimes it isn't obvious that integration by parts is the required method. We know that $\frac{d}{dx}\ln(x) = \frac{1}{x}$, but we do not know what $\int \ln(x)\,dx$ is. By letting $u = \ln(x)$ it is possible to use integration by parts to find the integral. This can also be used to anti-differentiate $\sin^{-1}(ax)$, $\cos^{-1}(ax)$ and $\tan^{-1}(ax)$.

WORKED EXAMPLE 33 Determining the integral of the natural logarithm using integration by parts

Use integration by parts to determine $\int \ln(x)\,dx$.

THINK	WRITE
1. The integrand does not appear to be a product however, we can write $\ln(x)$ as $\ln(x) \times 1$.	$\int \ln(x)\,dx = \int (\ln(x) \times 1)\,dx$
2. State the formula for integration by parts. We need to be able to express $\int \underbrace{\ln(x)}_{\text{L}}\,\underbrace{1}_{\text{A}}\,dx$ as $\int u\frac{dv}{dx}\,dx$.	$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
3. Using the ILATE rule, l $u = \ln(x)$ and $\frac{dv}{dx} = 1$.	$u = \ln(x),\ \frac{dv}{dx} = 1$
4. Differentiate u to find $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to find v.	$\frac{du}{dx} = \frac{1}{x},\ v = x$
5. Use $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$ and substitute for u, v and $\frac{dv}{dx}$. Simplify the integrand, noting that $x \times \frac{1}{x} = 1$ and then integrate.	$\int \ln(x)\,dx = \int u\frac{dv}{dx}\,dx$ $= uv - \int v\frac{du}{dx}\,dx$ $= \ln(x) \times x - \int \left(x \times \frac{1}{x}\right)dx$ $= x\ln(x) - \int 1\,dx$ $= x\ln(x) - x + c$

Sometimes we need to integrate a product of cyclic functions. A function is cyclic if repeated differentiation will eventually bring you back to the original function (or a multiple of the original function).

WORKED EXAMPLE 34 Determining an integral of the product of cyclic functions using multiple integrations by parts

Determine $\int e^x \sin(x)\,dx$.

THINK	WRITE
1. As the integrand is a product, try to integrate by parts. Let $I = \int e^x \sin(x)\,dx$.	$I = \int e^x \sin(x)\,dx = \int u\frac{dv}{dx}\,dx$

2. As both factors are cyclic, in this case, it does not matter which function we choose to be u and which one we choose to be $\frac{dv}{dx}$.

$$u = \sin(x),\ \frac{dv}{dx} = e^x$$

3. Differentiate u to determine $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to determine v.

$$\frac{du}{dx} = \cos(x),\ v = e^x$$

4. Use $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$ and substitute for u, v and $\frac{dv}{dx}$. Label the result as equation [1] so that we can come back to it later.

$$\begin{aligned} I &= f(x)\,g(x) - \int v\frac{du}{dx}\,dx \\ &= \sin(x)e^x - \int (\cos(x)e^x)\,dx \\ &= e^x\sin(x) - \int (e^x\cos(x))\,dx \quad [1] \end{aligned}$$

5. Use integration by parts again to find $\int e^x\cos(x)\,dx$ where $u = \cos(x)$ and $\frac{dv}{dx} = e^x$.

$$u = \cos(x),\quad \frac{dv}{dx} = e^x$$
$$\frac{du}{dx} = -\sin(x),\quad v = e^x$$
$$\begin{aligned} \int e^x\cos(x)\,dx &= \int u\frac{dv}{dx}\,dx \\ &= uv - \int v\frac{du}{dx}\,dx \\ &= \cos(x)\,e^x - \int (-\sin(x))\,e^x\,dx \\ &= e^x\cos(x) + \int e^x\sin(x)\,dx \end{aligned}$$

6. Notice that we are back to the original integral $I = \int e^x\sin(x)\,dx$. Label this as equation [2] so we can use it again.

$$\int e^x\cos(x)\,dx = e^x\cos(x) + I \quad [2]$$

7. Substitute equation [2] into equation [1].

$$\begin{aligned} I &= e^x\sin(x) - (e^x\cos(x) + I) \\ &= e^x\sin(x) - e^x\cos(x) - I \end{aligned}$$

8. We now have I on both sides of the equation. Rearrange the equation to make I the subject.

$$\begin{aligned} 2I &= e^x\sin(x) - e^x\cos(x) \\ I &= \frac{1}{2}(e^x\sin(x) - e^x\cos(x)) \\ &= \frac{e^x}{2}(\sin(x) - \cos(x)) \end{aligned}$$

9. State your answer, remembering to include '$+c$'.

$$\int e^x\sin(x)\,dx = \frac{e^x}{2}(\sin(x) - \cos(x)) + c$$

Exercise 6.6 Integration by parts

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1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

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1. For each of the following, using the ILATE rule, determine which function should be chosen as u and which function should be chosen as $\frac{dv}{dx}$ for an integration by parts.

Integral	Choice for u	Choice for $\frac{dv}{dx}$
$\int (3x+1)\cos(2x)\,dx$		
$\int \ln(4x)\sin(x)\,dx$		
$\int (1-x^2)\,e^{-x}\,dx$		
$\int x^2\tan^{-1}(2x)\,dx$		

2. Use integration by parts to determine the following:

   a. $\int x\cos(x)\,dx$ (Let $u=x$)

   b. $\int x\,e^x\,dx$ (Let $u=x$)

3. Use integration by parts to determine the following:

   a. $\int x\ln(x)\,dx$ (Let $u=\ln(x)$)

   b. $\int x\sin(2x)\,dx$ (Let $u=x$)

4. WE31 Determine:

   a. $\int x\sin(4x)\,dx$

   b. $\int x\cos(3x)\,dx$

5. WE32 Determine:

   a. $\int x\,e^{-2x}\,dx$

   b. $\int x\cos\left(\frac{x}{2}\right)dx$

6. Determine:

   a. $\int x^2\cos(x)\,dx$

   b. $\int x^2\sin(x)\,dx$

7. Determine the anti-derivative of the following:

   a. $x^2\,e^{2x}$

   b. $x^2\,(x+1)^5$

8. WE33 Use integration by parts to determine:

   a. $\int \ln(2x)\,dx$

   b. $\int \sin^{-1}(x)\,dx$

9. Determine the anti-derivative of each of the following:

a. $(x-1)\sin(x)$

b. $(x+2)e^x$

10. Determine the following integrals:

a. $\int x^3 \ln(x)\,dx$

b. $\int (x+1)\ln(x)\,dx$

11. Determine:

a. $\int x^2 \sin^{-1}(x)\,dx$

b. $\int (x+1)^2 e^x\,dx$

12. Use integration by parts to determine:

a. $\int x^3 e^x\,dx$

b. $\int \frac{x}{e^x}\,dx$

Complex familiar

13. WE34 Determine:

a. $\int e^x \cos(x)\,dx$

b. $\int e^{2x} \cos(x)\,dx$

14. Determine $f(x)$ given:

a. $f'(x) = e^x \sin(2x)$

b. $f'(x) = e^{2x} \cos(3x)$

15. Determine $\int_2^4 (\ln(x))^2\,dx$ and verify the reasonableness of your answer.

16. Prove, using integration by parts, that $\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln\left(x^2+1\right) + c$.

Complex unfamiliar

17. Determine $\int x^n \ln(ax)\,dx, n \neq -1$.

18. Use integration by parts to determine $\int \frac{\ln(x)}{x}\,dx$.

19. Use integration by parts to determine $\int \sin(\ln(x))\,dx$.

20. $\forall n \in \mathbb{N}$, let $I_n = \int_0^{\pi} e^{-nx}\sin(x)\,dx$ and $J_n = \int_0^{\pi} e^{-nx}\cos(x)\,dx$.

Doing an integration by parts of I_n in two different ways, determine 2 equations between I_n and J_n and use them to calculate I_n and J_n.

Fully worked solutions for this chapter are available online.

LESSON
6.7 Integration involving partial fractions

SYLLABUS LINKS

- Use partial fractions for integration involving two distinct linear factors in the denominator, e.g. $\dfrac{2x-1}{(x+1)(x-2)}$.

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A rational function is a ratio of two functions, both of which are polynomials. For example, $\dfrac{mx+k}{ax^2+bx+c}$ is a rational function; it has a linear function in the numerator and a quadratic function in the denominator. In Worked example 29 for instance, we integrated a certain expression of this form when $a>0$ and $\Delta=b^2-4ac<0$, meaning that the quadratic function was expressed as the sum of two squares by completing the square. In this section, we examine cases when $a\neq 0$ and $\Delta=b^2-4ac>0$. This means that the quadratic function in the denominator can now be factorised into linear factors. Integrating expressions of this kind does not involve a new integration technique, just an algebraic method of expressing the integrand into its partial fractions decomposition.

6.7.1 Integration by partial fractions

Using one of the properties of indefinite integrals, $\int (f(x)\pm g(x))\,dx=\int f(x)\,dx\pm\int g(x)\,dx$, we can break up complicated integrals into two manageable integrals. Through integration, a rational expression is rearranged into the sum of rational expressions that we can integrate.

We have already observed that $\displaystyle\int \frac{f'(x)}{f(x)}\,dx=\ln|f(x)|+c$.

To integrate rational expressions that are not in this format, the rational expression is rewritten as partial fractions.

The rational expression must be written as a proper fraction first. That is, the degree of the numerator polynomial is less than the degree of the denominator polynomial.

If the denominator's linear factors are unique (each factor is different), the partial fractions are written as follows (where the original rational expression is a proper fraction).

Partial fractions with distinct linear factors

$$\frac{f(x)}{(ax+b)(cx+d)}=\frac{A}{ax+b}+\frac{B}{cx+d}$$

The rational expression $\dfrac{2x^2-5x-4}{(x+1)(x-2)x}$ would be written as $\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{x}$. This means that $\dfrac{Ax(x-2)+Bx(x+1)+C(x+1)(x-2)}{(x+1)(x-2)x}=\dfrac{2x^2-5x-4}{(x+1)(x-2)x}$.

To determine the values of A, B and C, equate the numerators and solve for A, B and C. There are two main methods that can be used to determine the values for A, B and C. These will be demonstrated in the next worked example.

WORKED EXAMPLE 35 Equating polynomials by using roots and by expanding

If $Ax(x-2)+Bx(x+1)+C(x+1)(x-2)=2x^2-5x-4$, calculate values for A, B and C.

THINK	WRITE
1. **Method 1** The values for A, B and C can be found by choosing values for x that make the factors equal 0. Let $x=0$, $x=-1$ and $x=2$.	$x=0$: $C(1)(-2)=-4$ $C=2$ $x=-1$: $A(-1)(-1-2)=2(-1)^2-5(-1)-4$ $3A=3$ $A=1$ $x=2$: $B(2)(2+1)=2(2^2)-5(2)-4$ $6B=-6$ $B=-1$
1. **Method 2** Expand $Ax(x-2)+Bx(x+1)+C(x+1)(x-2)$ so that the coefficients of x^2, x and the constant term can be equated.	$Ax(x-2)+Bx(x+1)+C(x+1)(x-2)$ $=A(x^2-2x)+B(x^2+x)+C(x^2-x-2)$ $=(A+B+C)\ x^2+(-2A+B-C)x-2C$
2. $(A+B+C)\ x^2+(-2A+B-C)x-2C=2x^2-5x-4$ Equate the coefficients.	$A+B+C=2$ [1] $-2A+B-C=-5$ [2] $-2C=-4$ [3]
3. Use equation [3] to solve for C and substitute this value into equations [1] and [2].	[3]: $-2C=-4$ $C=2$ [1]: $A+B+2=2$ $A+B=0$ [4] [2]: $-2A+B-2=-5$ $-2A+B=-3$ [5]
4. Equations [4] and [5] are in terms of A and C only, so use elimination to solve.	[4] − [5]: $3A=3$ $A=1$ [4]: $B=-1$
5. State the solutions.	$A=1, B=-1, C=2$

This means that $\dfrac{2x^2-5x-4}{(x+1)(x-2)x}=\dfrac{1}{x+1}+\dfrac{-1}{x-2}+\dfrac{2}{x}=\dfrac{1}{x+1}-\dfrac{1}{x-2}+\dfrac{2}{x}$

WORKED EXAMPLE 36 Using partial fractions for integrals of rational function with factorised denominator

For $\dfrac{x+7}{(x+2)(x-3)}$,

a. identify the partial fractions

b. determine the anti-derivative in the form $\ln|f(x)|+c$.

THINK	WRITE
a. 1. The degree of the numerator is 1.The degree of the denominator is 2. The rational expression is a proper fraction. It can be written as the sum of 2 rational numbers.	**a.** Let $\dfrac{x+7}{(x+2)(x-3)} = \dfrac{A}{x+2} + \dfrac{B}{x-3}$
2. Add the fractions together by determining a common denominator.	$\dfrac{x+7}{(x+2)(x-3)} = \dfrac{A(x-3)+B(x+2)}{(x+2)(x-3)}$
3. As the denominator are equal, equate the numerators.	$x+7 = A(x-3)+B(x+2)$
4. Solve for A and B by choosing values for x that make the factors equal 0. Let $x=3$ and $x=-2$.	$x=3$: $3+7 = A(3-3)+B(3+2)$ $10 = 5B$ $B = 2$ $x=-2$: $-2+7 = A(-2-3)+B(-2+2)$ $5 = -5A$ $A = -1$
5. Rewrite the rational expression as a sum of the partial fractions.	$\dfrac{x+7}{(x+2)(x-3)} = \dfrac{-1}{x+2} + \dfrac{2}{x-3}$
b. 1. Express the question using integral notation.	**b.** $\displaystyle\int \frac{x+7}{(x+2)(x-3)}\,dx$
2. Rewrite the integral as partial fractions.	$\displaystyle= \int \left(\frac{-1}{x+2} + \frac{2}{x-3}\right) dx$
3. Use $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c$ to determine the integral.	$\displaystyle= -\int \frac{1}{x+2}\,dx + 2\int \frac{1}{x-3}\,dx$ $= -\ln|x+2| + 2\ln|x-3| + c$ $= 2\ln|x-3| - \ln|x+2| + c$
4. Use the logarithm laws $\ln(a)+\ln(b) = \ln(ab)$ and $\ln(a)-\ln(b) = \ln\left(\dfrac{a}{b}\right)$ for $a, b > 0$.	$\displaystyle= \ln\left(\frac{(x-3)^2}{\lvert x+2\rvert}\right) + c$

TIP

In examinations, you will often be asked to express your answer in the form $\ln|f(x)|+c$. So remember the logarithm laws.

If the denominator is not expressed in factorised form, it is necessary to factorise the denominator first.

WORKED EXAMPLE 37 Using partial fractions for integrals of rational function without factorised denominator

For $\dfrac{2x-3}{x^2-3x-4}$,

a. identify the partial fractions

b. determine the anti-derivative.

THINK	WRITE
a. 1. The degree of the numerator is 1. The degree of the denominator is 2. The rational expression is a proper fraction. It can be written as the sum of 2 rational numbers. Factorise the denominator to identify the denominators of the partial fractions.	**a.** $\dfrac{2x-3}{x^2-3x-4} = \dfrac{2x-3}{(x-4)(x+1)} = \dfrac{A}{x-4} + \dfrac{B}{x+1}$
2. Add the fractions together by determining a common denominator.	$\dfrac{2x-3}{(x-4)(x+1)} = \dfrac{A(x+1)+B(x-4)}{(x-4)(x+1)}$
3. As the denominators are equal, equate the numerators.	$2x-3 = A(x+1)+B(x-4)$
4. Solve for A and B by choosing values for x that make the factors equal 0. Let $x=4$ and $x=-1$.	$x=4$: $2(4)-3 = A(4+1)+B(4-4)$ $5 = 5A$ $A = 1$ $x=-1$: $2(-1)-3 = A(-1+1)+B(-1-4)$ $-5 = -5B$ $B = 1$
5. Rewrite the rational expression as a sum of the partial fractions.	$\dfrac{2x-3}{x^2-3x-4} = \dfrac{1}{x-4} + \dfrac{1}{x+1}$
b. 1. Express the question using integral notation.	**b.** $\displaystyle\int \frac{2x-3}{x^2-3x-4}\,dx$
2. Rewrite the integral as partial fractions.	$\displaystyle= \int \left(\frac{1}{x-4} + \frac{1}{x+1}\right) dx$
3. Use $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$ to determine the integral.	$= \ln\lvert x-4\rvert + \ln\lvert x+1\rvert + c$
4. Note that logarithm laws can be used to simplify the answer.	$= \ln\lvert (x-4)(x+1)\rvert + c$

Note: The original question was in the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$ where $f(x) = x^2-3x-4$ and $f'(x) = 2x-3$, so the question could have been integrated by recognising that pattern.

Repeated factors (extension)

Note that the Specialist Mathematics syllabus only requires to use partial fractions for integration involving **two distinct linear factors** in the denominator, therefore the following is extending the required mathematical knowledge and skills for the course.

If the denominator's linear factors are repeated, the partial fractions are written as follows (where the original rational expression is a proper fraction).

Partial fractions with repeated linear factors (extension)

$$\frac{f(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

WORKED EXAMPLE 38 Using partial fractions for integrals of rational function with repeated linear factors (extension)

Determine the anti-derivative of $\dfrac{8x+10}{(x+2)^2}$.

THINK	WRITE
1. The degree of the numerator is 1. The degree of the denominator is 2. The rational expression is a proper fraction. It can be written as the sum of 2 rational numbers. The factor $(x+2)$ is repeated.	Let $\dfrac{8x+10}{(x+2)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$
2. Add the fractions together by determining a common denominator.	$\dfrac{8x+10}{(x+2)^2} = \dfrac{A(x+2)}{(x+2)^2} + \dfrac{B}{(x+2)^2}$
3. As the denominators are equal, equate the numerators.	$8x+10 = A(x+2)+B$
4. It is possible to solve for B by letting $x=-2$ and then equate the coefficients to solve for A. Alternatively, expand the right-hand side and equate the coefficients.	$8x+10 = Ax+2A+B$ Consider coefficient of x: $A=8$ Consider constant: $10 = 2A+B$ $10 = 2\times 8+B$ $B=-6$
5. Rewrite the rational expression as the sum of the partial fractions.	$\dfrac{8x+10}{(x+2)^2} = \dfrac{8}{x+2} - \dfrac{6}{(x+2)^2}$
6. Express the question using integral notation.	$\displaystyle\int \frac{8x+10}{(x+2)^2}\,dx$
7. Rewrite the integral as partial fractions.	$\displaystyle= \int \left(\frac{8}{x+2} - \frac{6}{(x+2)^2}\right) dx$

8. Use $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$ to integrate $\int \frac{8}{x+2}\,dx$. To determine $\int \frac{6}{(x+2)^2}\,dx$, rewrite the integrand to include negative powers.

$$= 8\int \frac{1}{x+2}\,dx - 6\int (x+2)^{-2}\,dx$$

$$= 8\ln|x+2| - \frac{6(x+2)^{-1}}{-1} + c$$

9. Simplify by writing with positive powers.

$$= 8\ln|x+2| + \frac{6}{x+2} + c$$

6.7.2 Non-proper rational functions

If the degree of the numerator is not smaller than the degree of the denominator, the rational function is not proper. It is necessary to use division to rewrite the function.

For example, if the rational function is the function of two quadratic functions, then it is in the form $\frac{rx^2+sx+t}{ax^2+bx+c}$, $r \neq 0, a \neq 0$. Through division, this can be rewritten as $\frac{rx^2+sx+t}{ax^2+bx+c} = q + \frac{mx+k}{ax^2+bx+c}$ where $q = \frac{r}{a}$. The proper rational expression $\frac{mx+k}{ax^2+bx+c}$ can be rewritten as partial fractions so that it can be integrated.

WORKED EXAMPLE 39 Using long division with non-proper rational function

Determine $\int \frac{2x^2+5x+3}{x^2+3x-4}\,dx$.

THINK

1. The degree of the numerator is 2. The degree of the denominator is 2. This is not a proper fraction, so use division to obtain a polynomial and a proper rational fraction.

WRITE

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2+3x-4 & 2x^2+5x+3 \\ & -(2x^2+6x-8) \\ \hline & \quad -x+11 \end{array}$$

$$\therefore \frac{2x^2+5x+3}{x^2+3x-4} = 2 + \frac{11-x}{x^2+3x-4}$$

2. $\frac{11-x}{x^2+3x-4}$ can be rewritten as partial fractions.

$$\frac{11-x}{x^2+3x-4} = \frac{11-x}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

3. Add the fractions together by determining a common denominator.

$$\frac{11-x}{(x+4)(x-1)} = \frac{A(x-1)+B(x+4)}{(x+4)(x-1)}$$

4. As the denominators are equal, equate the numerators.

$$11 - x = A(x-1) + B(x+4)$$

5. Solve for A and B by choosing values for x that make the factors equal 0. Let $x = -4$ and $x = 1$.

$x = -4$:

$$11 - (-4) = A(-4-1) + B(-4+4)$$
$$15 = -5A$$
$$A = -3$$

$x = 1$:

$$11 - 1 = A(1-1) + B(1+4)$$
$$10 = 5B$$
$$B = 2$$

6. Rewrite $\dfrac{11-x}{x^2+3x-4}$ as the sum of partial fractions.

$$\frac{11-x}{x^2+3x-4} = -\frac{3}{x+4} + \frac{2}{x-1}$$

7. Rewrite the integral.

$$\int \frac{2x^2+5x+3}{x^2+3x-4}\,dx = \int \left(2 + \frac{11-x}{x^2+3x-4}\right) dx$$
$$= \int \left(2 - \frac{3}{x+4} + \frac{2}{x-1}\right) dx$$

8. Use $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$ to determine the integral.

$$= \int 2\,dx - 3\int \frac{1}{x+4}\,dx + 2\int \frac{1}{x-1}\,dx$$
$$= 2x - 3\ln|x+4| + 2\ln|x-1| + c$$

Note that logarithm laws can be used to simplify this answer if desired.

Exercise 6.7 Integration involving partial fractions

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6.7 Exercise	6.7 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	15, 16, 17	18, 19, 20

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Simple familiar

1. WE35 Calculate values for A, B and C.
   a. $Ax + B(x-1) = 3x - 2$
   b. $A(x+2) + B(x-3) = x - 8$

2. Calculate values for A, B and C.
   a. $A(x-4) + B = 3x - 2$
   b. $A(3x+1) + B(x-2) = 5x + 4$

3. Calculate values for A, B and C.
   a. $A(x+2)(x-3) + Bx(x-3) + Cx(x+2) = 3x^2 - x + 6$
   b. $Ax(x+2) + Bx(x+3) + C(x+2)(x+3) = x^2 - 3x - 12$

4. WE36 For $\dfrac{1}{(x+1)(x+2)}$
   a. identify the partial fractions
   b. determine the anti-derivative.

5. For each of the following, determine the integral by first identifying the partial fractions.

a. $\int \frac{12}{(x-2)(x+2)}\,dx$

b. $\int \frac{6x}{(x+3)(x-1)}\,dx$

c. $\int \frac{3x}{(x-2)(x+1)}\,dx$

d. $\int \frac{12-2x}{(1-x)(3-x)}\,dx$

6. Determine the following anti-derivatives:

a. $\frac{x+20}{(x-4)(x+4)}$

b. $\frac{x+4}{x(x-2)}$

c. $\frac{7x-4}{(x-2)(x+3)}$

d. $\frac{8x-10}{(2x+1)(x-3)}$

7. Determine $f(x)$ for each of the following:

a. $f'(x)=\frac{9x-11}{(3x-2)(x+1)}$

b. $f'(x)=\frac{11-3x}{(2-x)(x+3)}$

c. $f'(x)=\frac{4-x}{x(x+2)}$

d. $f'(x)=\frac{9x+8}{(x-3)(x+4)}$

8. WE37 For $\frac{5x+10}{24-2x-x^2}$

a. identify the partial fractions

b. determine the anti-derivative.

9. For each of the following, determine the integral by first identifying the partial fractions.

a. $\int \frac{3x+10}{x^2+2x}\,dx$

b. $\int \frac{5x-4}{x^2-x-2}\,dx$

c. $\int \frac{x+3}{x^2+3x+2}\,dx$

d. $\int \frac{6x-1}{x^2-5x-6}\,dx$

10. Determine the following anti-derivatives:

a. $\frac{5x-7}{x^2-4x+3}$

b. $\frac{x+16}{x^2+7x+6}$

c. $\frac{7x+9}{x^2-9}$

d. $\frac{7x+1}{x^2-1}$

11. Determine $f(x)$ for each of the following:

a. $f'(x)=\frac{5x}{2x^2-3x-2}$

b. $f'(x)=\frac{16-2x}{3x^2+7x-6}$

c. $f'(x)=\frac{x+4}{2x^2-5x+2}$

d. $f'(x)=\frac{4}{4-x^2}$

12. WE38 Determine the anti-derivative of:

a. $\frac{4x+5}{(x+2)^2}$

b. $\frac{5x-26}{(x-5)^2}$

c. $\frac{2x+1}{x^2+6x+9}$

d. $\frac{6x-5}{4x^2-12x+9}$

13. WE39 Determine:

a. $\int \frac{x^2-1}{x^2+3x}\,dx$

b. $\int \frac{x^2+2x+4}{x^2-4x}\,dx$

c. $\int \frac{x^2-x}{(x+3)(x+1)}\,dx$

d. $\int \frac{x^2+x+4}{x^2-2x-3}\,dx$

14. Determine the anti-derivative of:

a. $\dfrac{x^2+3x-2}{x^2-4}$

b. $\dfrac{x^2-x+2}{x^2+2x+1}$

c. $\dfrac{2x^2-9x+7}{x^2-6x+9}$

d. $\dfrac{x-1}{x+5}$

Complex familiar

15. Determine $g(x)$ for each of the following:

a. $g'(x)=\dfrac{x^3+4x^2-x}{(x+2)(x+1)}$

b. $g'(x)=\dfrac{2x^3+x^2-5}{x^2-1}$

c. $g'(x)=\dfrac{x^3+4x-13}{x^2-4x-5}$

d. $g'(x)=\dfrac{x^3+x^2-4x}{x^2-4x+4}$

16. Determine the following integrals:

a. $\displaystyle\int \frac{5x^2+2x+17}{(x-1)(x+2)(x-3)}\,dx$

b. $\displaystyle\int \frac{x^2+18x+5}{(x+1)(x-2)(x+3)}\,dx$

17. Determine the following integrals:

a. $\displaystyle\int \frac{x^2+8x+9}{(x-1)(x+2)^2}\,dx$

b. $\displaystyle\int \frac{x^2+5x+1}{(x^2+1)(2-x)}\,dx$

Complex unfamiliar

18. If a and b are non-zero real constant, then determine each of the following:

a. $\displaystyle\int \frac{1}{b^2x^2-a^2}\,dx$

b. $\displaystyle\int \frac{x}{b^2x^2-a^2}\,dx$

19. If a, b, p and q are non-zero real constant, then determine each of the following:

a. $\displaystyle\int \frac{x}{(ax-b)^2}\,dx$

b. $\displaystyle\int \frac{1}{(px+a)(qx+b)}\,dx$

20. Determine $a, b \in \mathbb{N}$ such that $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)}\,dx=\ln\left(\sqrt{\frac{a}{b}}\right)$.

Verify the reasonableness of your answer.

Fully worked solutions for this chapter are available online.

LESSON
6.8 Review

6.8.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

- Access the chapter summary
- Review your results
- Practise exam questions

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6.8 Exercise

6.8 Exercise | 6.8 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

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- Track results and progress

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Simple familiar

1. Use the substitution $u = 2x - 1$ to determine the following integrals:

a. $\int (2x-1)^6\,dx$ **b.** $\int \frac{1}{(2x-1)^2}\,dx$ **c.** $\int \frac{1}{2x-1}\,dx$ **d.** $\int \frac{1}{\sqrt[3]{2x-1}}\,dx$

2. Use the substitution $u = x^2 + 2$ to determine the following integrals:

a. $\int \frac{x}{x^2+2}\,dx$ **b.** $\int x\cos\left(x^2+2\right)dx$ **c.** $\int 3x\,e^{x^2+2}\,dx$ **d.** $\int_0^2 \frac{4x}{\left(x^2+2\right)}\,dx$

3. Use an appropriate trigonometric identity to determine the following:

a. $\int \cos(2x)\sin(2x)\,dx$ **b.** $\int \left(\cos^2(2x)+\sin^2(2x)\right)dx$

c. $\int \cos^3(2x)\sin(2x)\,dx$ **d.** $\int \cos(2x)\sin^3(2x)\,dx$

4. Determine each of the following:

a. $\int \tan(3x)\,dx$ **b.** $\int \cot(3x)\,dx$

c. $\int \tan(3x)\sec^2(3x)\,dx$ **d.** $\int \tan^2(3x)\sec^2(3x)\,dx$

5. Solve the following for x:

a. $x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

b. $x = \tan^{-1}\left(-\sqrt{3}\right)$

c. $x = \cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

d. $x = \tan\left(\tan^{-1}(1.5)\right)$

6. Solve the following for x:

a. $x = \sin\left(\sin^{-1}\left(\frac{3}{2}\right)\right)$

b. $x = \sec\left(\sec^{-1}\left(\sqrt{2}\right)\right)$

c. $x = \cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

d. $x = \sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

7. Determine the derivative of the following:

a. $\sin^{-1}\left(\frac{x}{2}\right)$ **b.** $\tan^{-1}(3x)$ **c.** $\cos^{-1}\left(x^2\right)$ **d.** $\tan^{-1}\left(\sqrt{x}\right)$

8. Use the substitution $u = \sin^{-1}\left(\frac{x}{a}\right)$ to determine $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$.

9. Use the substitution $u = \tan^{-1}\left(\frac{x}{a}\right)$ to determine $\int \frac{a}{a^2 + x^2}\,dx$.

10. Determine the anti-derivative of the following:

a. $\frac{1}{\sqrt{16 - x^2}}$ **b.** $\frac{1}{\sqrt{1 - 16x^2}}$ **c.** $\frac{10}{\sqrt{49 - 25x^2}}$ **d.** $\frac{-2}{\sqrt{1 - 4x^2}}$

11. Use integration by parts to determine the following:

a. $\int x\sin(x)\,dx$ **b.** $\int 2x\cos(3x)\,dx$ **c.** $\int x^2\sin(x)\,dx$ **d.** $\int x^2\ln(x)\,dx$

12. Use partial fractions to determine the anti-derivative of the following:

a. $\frac{5x - 13}{(x-3)(x-2)}$ **b.** $\frac{x+4}{(x+1)^2}$ **c.** $\frac{x+1}{x^2(x-1)}$ **d.** $\frac{x^2+8}{x^2 - 5x + 6}$

Complex familiar

13. Use completing the square to rewrite the denominator and then determine the integral of $\frac{x+1}{x^2 + x + 1}$.

14. Use integration by parts to determine the anti-derivative of $\frac{\cos(3x)}{e^x}$.

15. Use integration by parts to determine $\int \tan^{-1}(x)\,dx$.

16. The gradient of a curve is given by $\frac{1}{\sqrt{4 - x^2}}$. Determine the equation of the curve that passes through the point $\left(\sqrt{3}, \pi\right)$.

Complex unfamiliar

17. Determine $\int x\,(\ln(x))^2\,dx$.

18. Determine $\int x\tan^{-1}(x)\,dx$.

19. Use the substitution $u=\sqrt{x}$ to determine $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$.

20. Determine $\int \frac{\cos(x)}{\sin^2(x)+\sin(x)-6}\,dx$.

6.8 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2020 Specialist Mathematics, Paper 1, Section 1, Q5; © State of Queensland (QCAA) 2024

Determine $\int 4x\left(3x^2+5\right)^3\,dx$

A. $\frac{1}{6}\left(3x^2+5\right)^4+c$

B. $\frac{2}{3}\left(3x^2+5\right)^4+c$

C. $2\left(3x^2+5\right)^2+c$

D. $72x^2\left(3x^2+5\right)^2+c$

Question 2 (1 mark)

Source: QCAA 2022 Specialist Mathematics, Paper 1, Section 1, Q8; © State of Queensland (QCAA) 2024

Use the substitution $u=\tan(x)$ to determine $\int \tan(x)\sec^2(x)\,dx$.

A. $\frac{1}{2}\tan(x)+c$

B. $\frac{1}{2}\tan^2(x)+c$

C. $\tan(x)+c$

D. $\tan^2(x)+c$

Question 3 (6 marks)

Source: QCAA 2022 Specialist Mathematics, Paper 1, Section 2, Q13; © State of Queensland (QCAA) 2024

a. Use partial fractions to determine $\int \frac{22}{(2x-3)(x+4)}\,dx$ **[4 marks]**

b. Use the result from Question **13a** to determine $\int_{-3}^{0} \frac{22}{(2x-3)(x+4)}\,dx$ **[2 marks]**

Express your answer in simplest form.

Question 4 (5 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 2, Q11; © State of Queensland (QCAA) 2024

Determine the following definite integrals.

a. $\int_{0}^{1} \frac{1}{1+x^2}\,dx$ **[2 marks]**

b. $\int_{0}^{\frac{\pi}{4}} 2\sin^2(x)\,dx$ **[3 marks]**

Question 5 (5 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 1, Q16; © State of Queensland (QCAA) 2024

A curve is defined by the parametric equations $x = 2\tan(\theta)$ and $y = 3\sin(2\theta)$, where $0 \le \theta < \frac{\pi}{2}$.

Given that $\frac{dy}{dx}$ can be expressed in the form $a\cos^4(\theta) + b\cos^2(\theta)$, where $a, b \in R$, determine the values of a and b.

Fully worked solutions for this chapter are available online.

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Answers

Chapter 6 Integration techniques

6.2 Integration by linear substitution

6.2 Exercise

1. C
2. B
3. B
4. a. $\frac{1}{35}(5x-9)^7+c$ b. $\frac{1}{24}(3x+4)^8+c$
   c. $\frac{1}{21}(3x-5)^7+c$ d. $-\frac{1}{12}(1-2x)^6+c$
5. $y=\frac{1}{8}\left(\sqrt{16x+25}-5\right)$
6. $\frac{11}{4}$
7. a. $\frac{1}{3}\ln|3x-5|+c$ b. $-\frac{1}{2}\ln|7-2x|+c$
   c. $\frac{2}{3}\ln|3x-1|+c$ d. $-\frac{3}{2}\ln|1-2x|+c$
8. a. $\frac{1}{21}(3x+5)^7+c$ b. $-\frac{1}{3(3x+5)}+c$
   c. $-\frac{1}{6(3x+5)^2}+c$ d. $\frac{1}{2}\sqrt[3]{(3x+5)^2}+c$
9. a. $\frac{1}{54}(6x+7)^9+c$ b. $\frac{1}{3}\sqrt{6x+7}+c$
   c. $\frac{1}{6}\ln|6x+7|+c$ d. $-\frac{1}{6(6x+7)}+c$
10. a. 2 b. 8 c. 30 d. $\frac{1}{3}$
11. a. $x=\frac{t}{2(2-5t)}$
    b. $f(x)=-\sqrt{3-2x}$
    c. $\frac{3}{2}\ln(3)$
    d. $y=\frac{1}{3}(x-9)\sqrt{2x+9}+9$
12. a. $\frac{16}{15}$ b. $\frac{76}{15}$
    c. 4 d. $\frac{5}{3}\ln\left|\frac{5}{2}\right|-1$
13. a. $\frac{1}{175}(5x-9)^7+\frac{3}{50}(5x-9)^6+c=$
    $\frac{1}{350}(10x+3)(5x-9)^6+c$
    b. $\frac{2}{9}\ln|3x-4|-\frac{5}{9(3x-4)}+c$
    c. $\frac{-(4x+7)}{8(2x+7)^2}+c$
    d. $\frac{1}{27}(3x-5)\sqrt{6x+5}+c$
14. a. $\frac{1}{72}(3x+5)^8-\frac{5}{63}(3x+5)^7+c=$
    $\frac{1}{504}(21x-5)(3x+5)^7+c$
    b. $\frac{1}{9}\ln|3x+5|+\frac{5}{9(3x+5)}+c$
    c. $\frac{-(6x+5)}{18(3x+5)^2}+c$
    d. $\frac{1}{10}(2x-5)(3x+5)^{\frac{2}{3}}+c$
15. a. $\frac{1}{360}(6x+7)^{10}-\frac{7}{324}(6x+7)^9+c=$
    $\frac{1}{3240}(54x-7)(6x+7)^9+c$
    b. $\frac{1}{27}(3x-7)\sqrt{6x+7}+c$
    c. $\frac{x}{6}-\frac{7}{36}\ln|6x+7|+c$
    d. $\frac{7}{36(6x+7)}+\frac{1}{36}\ln|6x+7|+c$
16. a. $\frac{7}{2}$ b. $\frac{47}{7}$
    c. 6 d. $\frac{333}{10}$
17. a. $\frac{2}{3a}(ax+b)^{\frac{3}{2}}+c$
    b. $\frac{2}{15a^2}(3ax-2b)(ax+b)^{\frac{3}{2}}+c$
    c. $\frac{1}{a}\ln|ax+b|+c$
    d. $\frac{x}{a}-\frac{b}{a^2}\ln|ax+b|+c$
18. a. $\frac{2}{a}\sqrt{ax+b}+c$
    b. $\frac{2}{3a^2}(ax-2b)\sqrt{ax+b}+c$
    c. $\frac{-1}{a(ax+b)}+c$
    d. $\frac{b}{a^2(ax+b)}+\frac{1}{a^2}\ln|ax+b|+c$
19. a. $\frac{ad-bc}{a^2}\ln|ax+b|+\frac{cx}{a}+k$
    b. $\frac{c}{a^2}\ln|ax+b|+\frac{bc-ad}{a^2(ax+b)}+k$
    c. $\frac{cx}{a^2}-\frac{2bc}{a^3}\ln|ax+b|-\frac{a^2d+cb^2}{a^3(ax+b)}+k$
    d. $\frac{a^2d+cb^2}{a^3}\ln|ax+b|+\frac{cx^2}{2a}-\frac{bcx}{a^2}+k$
20. $a=2$

6.3 Integration by non-linear substitutions

6.3 Exercise

1. a. $\dfrac{-2}{(x^2+16)^2}+c$ b. $\dfrac{5}{2}\sqrt{2x^2+3}+c$
   c. $\dfrac{1}{5}\left(x^2+3\right)^5+c$ d. $\dfrac{1}{2\left(6-x^2\right)^2}+c$
2. a. $\dfrac{1}{2}\ln\left|x^2+4x+29\right|+c$ b. $\dfrac{1}{3}\ln\left|x^3+9\right|+c$
   c. $\dfrac{1}{2}\ln\left|x^2+2x+9\right|+c$ d. $\dfrac{3}{2}\ln\left|x^2+9\right|+c$
3. a. $\dfrac{1}{12}\left(x^2+4\right)^6+c$ b. $\dfrac{1}{2}\ln\left|x^2+4\right|+c$
   c. $\dfrac{-1}{2\left(x^2+9\right)}+c$ d. $\sqrt{x^2+9}+c$
4. a. $\dfrac{-1}{6\left(x^3+27\right)^2}+c$ b. $\dfrac{2}{3}\sqrt{x^3+27}+c$
   c. $\dfrac{1}{3}\ln\left|x^3+8\right|+c$ d. $\dfrac{1}{12}\left(x^3+8\right)^4+c$
5. a. $\dfrac{1}{8}\left(x^2-4x+13\right)^4+c$ b. $\dfrac{-1}{2\left(x^2-4x+13\right)}+c$
   c. $-\sqrt{x^2-8x+25}+c$ d. $-\dfrac{1}{2}\ln\left|x^2-8x+25\right|+c$
6. a. $\dfrac{1}{3}\left(x^3-5\right)^3+c$ b. $-\dfrac{1}{2\left(x^2+4x\right)^2}+c$
   c. $\dfrac{2}{3}\left(x^2+5x\right)^{\frac{3}{2}}+c$ d. $-\dfrac{1}{3\left(x^2-3x\right)^3}+c$
7. a. $\dfrac{1}{3}\left(x^3-2\right)^6+c$ b. $-\dfrac{1}{8}\left(4-x^2\right)^4+c$
   c. $\dfrac{1}{24}\left(x^3-1\right)^8+c$ d. $\dfrac{1}{10}\left(x^2+6x-2\right)^5+c$
8. a. $-\dfrac{1}{6}\left(x^2+2x+3\right)^{-3}+c$
   b. $4\sqrt{x^2+3x}+c$
   c. $-\dfrac{1}{5}\left(x^2-5x+2\right)^{-5}+c$
   d. $\dfrac{2}{9}\left(4-3x+x^3\right)^{\frac{3}{2}}+c$
9. a. $\dfrac{1}{7}\left(x^2+1\right)^{\frac{7}{2}}+c$ b. $-\dfrac{1}{3}\sqrt{\left(1-x^2\right)^3}+c$
   c. $\dfrac{2}{3}\sqrt{\left(x+x^2-3\right)^3}+c$ d. $2\sqrt{x^3+2x^2}+c$
10. a. $\cos\left(\dfrac{1}{x}\right)+c$ b. $\dfrac{1}{2}\sin\left(x^2\right)+c$
    c. $\sin\left(x^3+5x\right)+c$ d. $-\cos\left(x^2+3x-2\right)+c$
11. a. $2\sin\left(\sqrt{x}\right)+c$ b. $\dfrac{1}{8}\left(\ln|6x|\right)^2+c$
    c. $\dfrac{1}{4}\left(\ln|3x|\right)^2+c$ d. $\left(\ln\left|x^2-x\right|\right)^2+c$
12. a. $\dfrac{1}{3}e^{\sin(3x)}+c$ b. $\dfrac{1}{2}e^{\tan(2x)}+c$
    c. $3e^{x^2-x+3}+c$ d. $\dfrac{1}{3}e^{x^3+2}+c$
13. a. $\dfrac{1}{8}\ln\left|4e^{2x}+5\right|+c$ b. $\dfrac{1}{6(2e^{-3x}-5)}+c$
    c. $\dfrac{1}{12\left(3e^{-2x}+4\right)^2}+c$ d. $\dfrac{-1}{e^{2x}+x}+c$
14. a. $-\cos(\ln|4x|)+c$
    b. $\sin(\ln|3x|)+c$
    c. $e^x-2\ln\left|e^x+2\right|+c$
    d. $\dfrac{1}{2}e^{2x}+e^x+\ln\left|e^x-1\right|+c$
15. a. 1 b. $\dfrac{1}{80}$ c. $2-2e^4$ d. 18
16. a. 12 b. $\dfrac{1}{4}\ln\left|\dfrac{5}{3}\right|$ c. $\dfrac{1}{18}$ d. 0
17. a. $f(x)=\dfrac{1}{2}\left(1-\cos\left(x^2\right)\right)$ b. $1-\sqrt{3}$
    c. $\dfrac{1}{2}\ln\left|\dfrac{29}{20}\right|$ d. $\dfrac{1}{2}(1-e)$
18. a. $\dfrac{1}{2}\ln\left|\dfrac{26}{29}\right|$ b. $-\dfrac{1}{4}$
    c. $2e(e-1)$ d. $\dfrac{1}{2}(e-1)$
19. Sample responses can be found in the worked solutions in the online resources.
20. $\displaystyle\int_0^{\frac{\pi}{6}}\frac{d\theta}{\cos(\theta)}=\frac{1}{2}\ln(3)$

6.4 Integration using the trigonometric identities

6.4 Exercise

1. a. $-2\sin(2x)$ b. $4\cos(4x)$
   c. $3\sec^2(3x)$ d. $-5\operatorname{cosec}^2(5x)$
2. a. $\dfrac{1}{2}x+\dfrac{1}{4}\sin(2x)+c$ b. $\dfrac{1}{2}x-\dfrac{1}{8}\sin(4x)+c$
   c. $x+\dfrac{1}{8}\sin(8x)+c$ d. $2x-\dfrac{1}{3}\sin(6x)+c$
   e. $\dfrac{1}{2}x+\dfrac{1}{20}\sin(10x)+c$ f. $\dfrac{1}{2}x-\dfrac{1}{24}\sin(12x)+c$
3. a. $\dfrac{1}{2}x+\dfrac{1}{2}\sin(x)+c$ b. $\dfrac{1}{2}x-\dfrac{3}{4}\sin\left(\dfrac{2x}{3}\right)+c$
   c. $\dfrac{3}{2}x+\dfrac{9}{2}\sin\left(\dfrac{x}{3}\right)+c$ d. $x-2\sin\left(\dfrac{x}{2}\right)+c$
   e. $\dfrac{1}{2}x+\dfrac{3}{8}\sin\left(\dfrac{4x}{3}\right)+c$ f. $\dfrac{1}{2}x-\dfrac{1}{6}\sin(3x)+c$
4. a. $\dfrac{3}{8}x+\dfrac{1}{12}\sin(6x)+\dfrac{1}{96}\sin(12x)+c$
   b. $\dfrac{3}{8}x+\dfrac{1}{2}\sin(x)+\dfrac{1}{16}\sin(2x)+c$

c. $\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$

d. $\frac{3}{8}x - \frac{1}{8}\sin(4x) + \frac{1}{64}\sin(8x) + c$

5. a. $\sin^2(x) + c$ or $-\frac{1}{2}\cos(2x) + c$

b. $-\frac{1}{2}\cos(4x) + c$ or $\sin^2(2x) + c$

c. $-\frac{1}{12}\cos(6x) + c$ or $\frac{1}{6}\sin^2(3x) + c$

d. $\frac{1}{8}\cos(8x) + c$ or $-\frac{1}{4}\sin^2(4x) + c$

6. a. $\frac{1}{8}x - \frac{1}{32}\sin(4x) + c$ b. $\frac{1}{8}x - \frac{1}{64}\sin(8x) + c$

c. $\frac{1}{4}x - \frac{1}{64}\sin(16x) + c$ d. $\frac{1}{4}x - \frac{1}{48}\sin(12x) + c$

e. $\frac{3}{4}x - \frac{3}{8}\sin(2x) + c$ f. $\frac{1}{2}x - \frac{3}{8}\sin\left(\frac{4x}{3}\right) + c$

7. a. $-\frac{1}{5}\cos^5(x) + c$ b. $-\frac{1}{8}\cos^4(2x) + c$

c. $-\frac{1}{3}\cos^6\left(\frac{x}{2}\right) + c$ d. $\frac{1}{15}\sin^5(3x) + c$

e. $\frac{5}{7}\sin^7\left(\frac{x}{5}\right) + c$ f. $\frac{3}{16}\sin^8\left(\frac{2x}{3}\right) + c$

8. a. $\frac{1}{3}\cos^3(x) - \cos(x) + c$

b. $\frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + c$

c. $\frac{1}{2}\cos^3(4x) - \frac{3}{2}\cos(4x) + c$

d. $\frac{4}{3}\sin(3x) - \frac{4}{9}\sin^3(3x) + c$

e. $\frac{1}{21}\cos^3(7x) - \frac{1}{7}\cos(7x) + c$

f. $\frac{1}{6}\sin(6x) - \frac{1}{18}\sin^3(6x) + c$

9. a. $2\cos^3\left(\frac{x}{2}\right) - 6\cos\left(\frac{x}{2}\right) + c$

b. $6\sin\left(\frac{x}{3}\right) - 2\sin^3\left(\frac{x}{3}\right) + c$

c. $\frac{2}{9}\cos^3\left(\frac{3x}{2}\right) - \frac{2}{3}\cos\left(\frac{3x}{2}\right) + c$

d. $\frac{2}{5}\sin\left(\frac{5x}{2}\right) - \frac{2}{15}\sin^3\left(\frac{5x}{2}\right) + c$

e. $\frac{4}{9}\cos^3\left(\frac{3x}{4}\right) - \frac{4}{3}\cos\left(\frac{3x}{4}\right) + c$

f. $\frac{3}{4}\sin\left(\frac{4x}{3}\right) - \frac{1}{4}\sin^3\left(\frac{4x}{3}\right) + c$

10. a. $\cos(x) - \frac{2}{3}\cos^3(x) + c$ or $-\frac{1}{6}\cos(3x) + \frac{1}{2}\cos(x) + c$

b. $\frac{1}{2}\sin(2x) - \frac{1}{3}\sin^3(2x) + c$ or $\frac{1}{4}\sin(2x) + \frac{1}{12}\sin(6x) + c$

c. $\frac{1}{3}\cos(3x) - \frac{2}{9}\cos^3(3x) + c$ or

$-\frac{1}{18}\cos(9x) + \frac{1}{6}\cos(3x) + c$

d. $\frac{1}{4}\sin(4x) - \frac{1}{6}\sin^3(4x) + c$ or $\frac{1}{8}\sin(4x) + \frac{1}{24}\sin(12x) + c$

e. $2\cos\left(\frac{x}{2}\right) - \frac{4}{3}\cos^3\left(\frac{x}{2}\right) + c$ or

$-\frac{1}{3}\cos\left(\frac{3x}{2}\right) - \cos\left(\frac{x}{2}\right) + c$

f. $3\sin\left(\frac{x}{3}\right) - 2\sin^3\left(\frac{x}{3}\right) + c$ or

$-\frac{1}{2}\cos(x) + \frac{3}{2}\cos\left(\frac{x}{3}\right) + c$

11. a. $\frac{1}{2}\tan(2x) + c$ b. $3\tan\left(\frac{x}{3}\right) + c$

c. $\frac{1}{3}\tan^3(x) + c$ d. $\frac{1}{4}\tan^4(x) + c$

e. $\frac{1}{3}\tan^6(2x) + c$ f. $\frac{16}{5}\tan^5\left(\frac{x}{2}\right) + c$

12. $f(x) = -2\cos^3(x) + \frac{1}{4}$

13. $f(x) = \frac{x}{2} - \frac{1}{16}\sin(8x) + \frac{7\pi}{8}$

14. a. $\frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + c$

b. $\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + c$

c. $\frac{1}{10}\cos^5(2x) - \frac{1}{6}\cos^3(2x) + c$

d. $\frac{1}{9}\sin^3(3x) - \frac{1}{15}\sin^5(3x) + c$

e. $\frac{2}{5}\cos^5\left(\frac{x}{2}\right) - \frac{2}{3}\cos^3\left(\frac{x}{2}\right) + c$

f. $\frac{2}{9}\sin^3\left(\frac{3x}{2}\right) - \frac{2}{15}\sin^5\left(\frac{3x}{2}\right) + c$

15. a. $\frac{12}{5}\cos^5\left(\frac{x}{3}\right) - 4\cos^3\left(\frac{x}{3}\right) + c$

b. $\frac{24}{25}\sin^5\left(\frac{5x}{4}\right) - \frac{8}{5}\sin^3\left(\frac{5x}{4}\right) + c$

c. $\frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + c$

d. $\frac{1}{10}\sin^5(2x) - \frac{1}{14}\sin^7(2x) + c$

e. $\frac{1}{8}\cos^8(2x) - \frac{1}{6}\cos^6(2x) + c$

f. $\frac{2}{27}\sin^9(3x) - \frac{2}{21}\sin^7(3x) + c$

16. a. $\frac{1}{3}\tan^3(x) + \frac{1}{5}\tan^5(x) + c$

b. $-\cot^3(2x) - \frac{3}{5}\cot^5(2x) + c$

c. $\frac{4}{3}\tan^3\left(\frac{x}{2}\right)+\frac{4}{5}\tan^5\left(\frac{x}{2}\right)+c$

d. $-\frac{1}{4}\cot^4(3x)-\frac{1}{6}\cot^6(3x)+c$

e. $\tan^5\left(\frac{x}{5}\right)+\frac{5}{7}\tan^7\left(\frac{x}{5}\right)+c$

f. $-\frac{1}{3}\cot^6(6x)-\frac{1}{2}\cot^8(6x)-\frac{1}{5}\cot^{10}(6x)+c$

17. $g(x)=\frac{2}{7}\cos^7\left(\frac{x}{2}\right)-\frac{2}{5}\cos^5\left(\frac{x}{2}\right)$

18. a. $\frac{-1}{n+1}\cos^{n+1}(x)+c$

b. $\frac{1}{n+1}\sin^{n+1}(x)+c$

19. a. $\frac{1}{n+1}\tan^{n+1}(x)+c$

b. $\frac{1}{n+3}\cos^{n+3}(x)-\frac{1}{n+1}\cos^{n+1}(x)+c$

20. a. $\frac{1}{n+1}\sin^{n+1}(x)-\frac{1}{n+3}\sin^{n+3}(x)+c$

b. $\frac{1}{n+3}\tan^{n+3}(x)+\frac{1}{n+1}\tan^{n+1}(x)+c$

6.5 Integration of inverse trigonometric functions

6.5 Exercise

1. a. Does not exist b. $-\frac{\pi}{3}$ c. 0.9
d. Does not exist e. $-\frac{\pi}{6}$ f. $\frac{1}{3}$

2. a. Does not exist b. $\frac{5\pi}{6}$ c. $\frac{\pi}{6}$
d. Does not exist e. $\frac{\pi}{3}$ f. $\frac{1}{4}$

3. a. $\frac{\pi}{3}$ b. $-\frac{\pi}{6}$ c. $\frac{\pi}{6}$
d. $\frac{5}{4}$ e. $-\frac{\pi}{3}$ f. 1.1

4. a. $\frac{1}{\sqrt{16-x^2}}$ b. $\frac{5}{\sqrt{1-25x^2}}$

5. a. $\frac{-3}{\sqrt{16-9x^2}}$ b. $\frac{-1}{\sqrt{(x+1)(4-x)}}$

6. a. $\frac{4}{1+16x^2}$ b. $\frac{5}{2x^2-6x+17}$

7. $\frac{8\sqrt{3}}{3}$

8. $-\frac{3}{5}$

9. a. $\sin^{-1}\left(\frac{x}{10}\right)+c$ b. $2\sin^{-1}\left(\frac{x}{2}\right)+c$
c. $4\sin^{-1}\left(\frac{3x}{8}\right)+c$ d. $\frac{1}{5}\sin^{-1}\left(\frac{5x}{6}\right)+c$

10. $y=\frac{1}{4}\left(2\cos^{-1}\left(\frac{4x}{5}\right)-\pi\right)$

11. $y=2\cos^{-1}\left(\frac{x}{2}\right)+3$

12. a. $2\tan^{-1}\left(\frac{x}{5}\right)+c$ b. $\frac{1}{10}\tan^{-1}\left(\frac{x}{10}\right)+c$
c. $\frac{1}{2}\tan^{-1}\left(\frac{3x}{8}\right)+c$ d. $\frac{1}{30}\tan^{-1}\left(\frac{5x}{6}\right)+c$

13. a. $\frac{1}{2\sqrt{x(9-x)}}$ b. $\frac{-3}{|x|\sqrt{16x^2-9}}$
c. $\frac{-2e^{2x}}{\sqrt{16-e^{4x}}}$ d. $\frac{4}{|x|\sqrt{9x^2-16}}$
e. $\frac{6x}{x^4+9}$ f. $\frac{-30}{25x^2+36}$

14. a. $\frac{1}{8}\tan^{-1}\left(\frac{2x-3}{4}\right)+c$

b. $\frac{1}{2}\sin^{-1}\left(\frac{2x-3}{4}\right)+c$

c. $\frac{1}{15}\tan^{-1}\left(\frac{5x-2}{3}\right)+c$

d. $\frac{1}{5}\sin^{-1}\left(\frac{5x-2}{4}\right)+c$

15. a. $2\sin^{-1}\left(\frac{x+2}{3}\right)+c$ b. $\frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$
c. $2\sin^{-1}\left(\frac{3x+5}{7}\right)+c$ d. $\frac{2}{7}\tan^{-1}\left(\frac{3x+5}{7}\right)+c$

16. a. $-\frac{1}{25}\sqrt{36-25x^2}+c$ b. $\frac{1}{10}\ln\left(64+25x^2\right)+c$
c. $-\frac{2}{5}\sqrt{49-25x^2}+c$ d. $\frac{1}{72}\ln\left(36x^2+49\right)+c$

17. a. $\frac{5}{3}\sin^{-1}\left(\frac{3x}{5}\right)+\frac{1}{3}\sqrt{25-9x^2}+c$

b. $\frac{1}{6}\ln\left(9x^2+25\right)+\frac{1}{3}\tan^{-1}\left(\frac{3x}{5}\right)+c$

c. $\frac{1}{4}\left(\sin^{-1}(4x)+\sqrt{1-16x^2}\right)+c$

d. $2\tan^{-1}(4x)+\frac{1}{32}\ln\left(1+16x^2\right)+c$

18. a. $\cos^{-1}\left(\frac{4x}{3}\right)-\frac{3}{16}\sqrt{9-16x^2}+c$

b. $\frac{1}{4}\tan^{-1}\left(\frac{4x}{3}\right)+\frac{1}{8}\ln\left(16x^2+9\right)+c$

c. $\sqrt{5-2x^2}+\frac{5\sqrt{2}}{2}\sin^{-1}\left(\frac{\sqrt{10}x}{5}\right)+c$

d. $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}x}{5}\right)-\frac{1}{2}\ln\left(2x^2+25\right)+c$

19. a. $\frac{1}{2}\ln(2)-\frac{\pi}{8}$ b. $\frac{1}{4}\ln(2)+\frac{\pi}{8}$
c. $9+\frac{5\pi}{2}$ d. $\sqrt{3}-\frac{5\pi}{3}$

20. a. $\frac{1}{q}\sin^{-1}\left(\frac{qx}{p}\right)+c$

b. $\frac{1}{q}\tan^{-1}\left(\frac{qx}{p}\right)+c$

c. $\frac{b}{q}\sin^{-1}\left(\frac{qx}{p}\right)-\frac{a}{q^2}\sqrt{p^2-q^2x^2}+c$

d. $\frac{b}{q}\tan^{-1}\left(\frac{qx}{p}\right)+\frac{a}{2q^2}\ln\left(q^2x^2+p^2\right)+c$

6.6 Integration by parts

6.6 Exercise

1. See the table at the bottom of the page*
2. a. $\cos(x)+x\sin(x)+c$

   b. $(x-1)\ e^x+c$
3. a. $\frac{1}{2}x^2\ln(x)-\frac{1}{4}x^2+c$

   b. $\frac{1}{4}\sin(2x)-\frac{1}{2}x\cos(2x)+c$
4. a. $-\frac{1}{4}x\cos(4x)+\frac{1}{16}\sin(4x)+c$

   b. $\frac{1}{3}x\sin(3x)+\frac{1}{9}\cos(3x)+c$
5. a. $-\frac{1}{4}(2x+1)\ e^{-2x}+c$

   b. $2x\sin\left(\frac{x}{2}\right)+4\cos\left(\frac{x}{2}\right)+c$
6. a. $x^2\sin(x)+2x\cos(x)-2\sin(x)+c$

   b. $-x^2\cos(x)+2x\sin(x)+2\cos(x)+c$
7. a. $\frac{1}{2}x^2e^{2x}-\frac{1}{2}x\,e^{2x}+\frac{1}{4}e^{2x}+c$

   b. $\frac{1}{6}x^2(x+1)^6-\frac{1}{21}x(x+1)^7+\frac{1}{168}(x+1)^8+c$
8. a. $x\ln(2x)-x+c$

   b. $x\sin^{-1}(x)+\sqrt{1-x^2}+c$
9. a. $-x\cos(x)+\sin(x)+\cos(x)+c$

   b. $(x+1)\ e^x+c$
10. a. $\frac{1}{4}x^4\ln(x)-\frac{1}{16}x^4+c$

    b. $\frac{1}{2}(x+1)^2\ln(x)-\frac{1}{4}x^2-x-\frac{1}{2}\ln(x)+c$
11. a. $\frac{1}{3}x^3\sin^{-1}(x)+\frac{1}{9}x^2\sqrt{1-x^2}+\frac{2}{9}\sqrt{1-x^2}+c$

    b. $\left(x^2+1\right)\ e^x+c$
12. a. $e^x\left(x^3-3x^2+6x-6\right)+c$

    b. $-e^{-x}(x+1)+c$
13. a. $\frac{e^x}{2}(\cos(x)+\sin(x))+c$

    b. $\frac{e^{2x}}{5}(2\cos(x)+\sin(x))+c$
14. a. $f(x)=\frac{e^x}{5}(\sin(2x)-2\cos(2x))+c$

    b. $f(x)=13e^{2x}(2\cos(3x)+3\sin(3x))+c$
15. $14(\ln(2))^2-12\ln(2)+4$
16. Sample responses can be found in the worked solutions in the online resources.
17. $\frac{x^{n+1}}{n+1}\left(\ln(ax)-\frac{1}{n+1}\right)+c$
18. $\frac{1}{2}\left[\ln(x)\right]^2+c, x>0$
19. $\frac{x}{2}\left[\sin(\ln(x))-\cos(\ln(x))\right]+c$
20. Sample responses can be found in the worked solutions in the online resources.

6.7 Integration involving partial fractions

6.7 Exercise

1. a. $A=1, B=2$ b. $A=-1, B=2$
2. a. $A=3, B=10$ b. $A=2, B=-1$
3. a. $A=-1, B=2, C=2$

   b. $A=2, B=1, C=-2$
4. a. $\frac{1}{x+1}-\frac{1}{x+2}$ b. $\ln\left|\frac{x+1}{x+2}\right|+c$
5. a. $\frac{3}{x-2}-\frac{3}{x+2}, 3\ln\left|\frac{x-2}{x+2}\right|+c$

   b. $\frac{9}{2(x+3)}+\frac{3}{2(x-1)}, \frac{9}{2}\ln|x+3|+\frac{3}{2}\ln|x-1|+c$

   c. $\frac{2}{x-2}+\frac{1}{x+1}, 2\ln|(x-2)|+\ln|x+1|+c$

   d. $\frac{5}{1-x}-\frac{3}{3-x}, 3\ln|3-x|-5\ln|1-x|+c$

*1.

Integral	Choice for u	Choice for $\frac{dv}{dx}$
$\int(3x+1)\cos(2x)\,dx$	$(3x+1)$	$\cos(2x)$
$\int\ln(4x)\sin(x)\,dx$	$\ln(4x)$	$\sin(x)$
$\int\left(1-x^2\right)e^{-x}\,dx$	$\left(1-x^2\right)$	e^{-x}
$\int x^2\tan^{-1}(2x)\,dx$	$\tan^{-1}(2x)$	x^2

6. a. $3\ln|x-4| - 2\ln|x+4| + c$
b. $3\ln|x-2| - 2\ln|x| + c$
c. $2\ln|x-2| + 5\ln|x+3| + c$
d. $2\ln|(2x+1)(x-3)| + c$

7. a. $f(x) = 4\ln|x+1| - \ln|3x-2| + c$
b. $f(x) = 4\ln|x+3| - \ln|2-x| + c$
c. $f(x) = 2\ln|x| - 3\ln|x+2| + c$
d. $f(x) = 5\ln|x-3| + 4\ln|x+4| + c$

8. a. $\dfrac{-2}{x+6} + \dfrac{3}{4-x}$
b. $-2\ln|x+6| - 3\ln|4-x| + c$

9. a. $5\ln|x| - 2\ln|x+2| + c$
b. $2\ln|x-2| + 3\ln|x+1| + c$
c. $2\ln|x+1| - \ln|x+2| + c$
d. $5\ln|x-6| + \ln|x+1| + c$

10. a. $4\ln|x-3| + \ln|x-1| + c$
b. $3\ln|x+1| - 2\ln|x+6| + c$
c. $2\ln|x+3| + 5\ln|x-3| + c$
d. $3\ln|x+1| + 4\ln|x-1| + c$

11. a. $\dfrac{1}{2}\ln|2x+1| + 2\ln|x-2| + c$
b. $\dfrac{4}{3}\ln|3x-2| - 2\ln|x+3| + c$
c. $2\ln|x-2| - \dfrac{3}{2}\ln|2x-1| + c$
d. $\ln\left|\dfrac{2+x}{2-x}\right| + c$

12. a. $4\ln|x+2| + \dfrac{3}{x+2} + c$
b. $5\ln|x-5| + \dfrac{1}{x-5} + c$
c. $\dfrac{5}{x+3} + 2\ln|x+3| + c$
d. $\dfrac{3}{2}\ln|2x-3| - \dfrac{2}{2x-3} + c$

13. a. $x - \dfrac{1}{3}\ln|x| - \dfrac{8}{3}\ln|x+3| + c$
b. $x + 7\ln|x-4| - \ln|x| + c$
c. $x + \ln|x+1| - 6\ln|x+3| + c$
d. $x + 4\ln|x-3| - \ln|x+1| + c$

14. a. $x + \ln|x+2| + 2\ln|x-2| + c$
b. $x - 3\ln|x+1| - \dfrac{4}{x+1} + c$
c. $2x + 3\ln|x-3| + \dfrac{2}{x-3} + c$
d. $x - 6\ln|x+5| + c$

15. a. $\dfrac{1}{2}x^2 + x - 10\ln|x+2| + 4\ln|x+1| + c$
b. $x^2 + x + 3\ln|x+1| - \ln|x-1| + c$
c. $\dfrac{1}{2}x^2 + 4x + 22\ln|x-5| + 3\ln|x+1| + c$
d. $\dfrac{1}{2}x^2 + 5x - \dfrac{4}{x-2} + 12\ln|x-2| + c$

16. a. $-4\ln|x-1| + \dfrac{33}{15}\ln|x+2| + \dfrac{34}{5}\ln|x-3| + c$
b. $2\ln|x+1| + 3\ln|x-2| - 4\ln|x+3| + c$

17. a. $2\ln|x-1| - \ln|x+2| - \dfrac{1}{x+2} + c$
b. $\ln(x^2+1) - \tan^{-1}(x) - 3\ln|2-x| + c$

18. a. $\dfrac{1}{2ab}\ln\left|\dfrac{bx-a}{bx+a}\right| + c$ b. $\dfrac{1}{2b^2}\ln|b^2x^2 - a^2| + c$

19. a. $\dfrac{1}{a^2}\ln|ax-b| - \dfrac{b}{a^2(ax-b)} + c$
b. $\dfrac{1}{aq-bp}\ln\left|\dfrac{qx+b}{px+a}\right| + c$

20. $a = 3$ and $b = 2$

6.8 Review

6.8 Exercise

1. a. $\dfrac{1}{14}(2x-1)^7 + c$ b. $-\dfrac{1}{2(2x-1)} + c$
c. $\dfrac{1}{2}\ln|2x-1| + c$ d. $\dfrac{3}{4}\sqrt{(2x-1)^2} + c$

2. a. $\dfrac{1}{2}\ln(x^2+2) + c$ b. $\dfrac{1}{2}\sin(x^2+2) + c$
c. $\dfrac{3}{2}e^{x^2+2} + c$ d. $2\ln(3)$

3. a. $-\dfrac{1}{8}\cos(4x) + c$ b. $x + c$
c. $-\dfrac{1}{8}\cos^4(2x) + c$ d. $\dfrac{1}{8}\sin^4(2x) + c$

4. a. $-\dfrac{1}{3}\ln|\cos(3x)| + c$ b. $\dfrac{1}{3}\ln|\sin(3x)| + c$
c. $\dfrac{1}{6}\tan^2(3x) + c$ d. $\dfrac{1}{9}\tan^3(3x) + c$

5. a. $\dfrac{5\pi}{6}$ b. $-\dfrac{\pi}{3}$ c. $\dfrac{\sqrt{2}}{2}$ d. 1.5

6. a. Does not exist b. $\sqrt{2}$
c. $\dfrac{7\pi}{6}$ d. $-\dfrac{\pi}{3}$

7. a. $\dfrac{1}{\sqrt{4-x^2}}$ b. $\dfrac{3}{1+9x^2}$
c. $-\dfrac{2x}{\sqrt{1-x^4}}$ d. $\dfrac{1}{2\sqrt{x}(1+x)}$

8. $8\sin^{-1}\left(\dfrac{x}{a}\right) + c$

9. $\tan^{-1}\left(\dfrac{x}{a}\right) + c$

10. a. $\sin^{-1}\left(\dfrac{x}{4}\right) + c$ b. $\dfrac{1}{4}\sin^{-1}(4x) + c$
c. $2\sin^{-1}\left(\dfrac{5x}{7}\right) + c$ d. $\cos^{-1}(2x) + c$

11. a. $-x\cos(x)+\sin(x)+c$

b. $\frac{2}{3}x\sin(3x)+\frac{2}{9}\cos(3x)+c$

c. $-x^2\cos(x)+2x\sin(x)+2\cos(x)+c$

d. $\frac{1}{3}x^3\ln(x)-\frac{1}{9}x^3+c$

12. a. $3\ln|x-2|+2\ln|x-3|+c$

b. $\ln|x+1|-\frac{3}{x+1}+c$

c. $-2\ln|x|+\frac{1}{x}+2\ln|x-1|+c$

d. $x-12\ln|x-2|+17\ln|x-3|+c$

13. $\frac{1}{2}\ln\left(x^2+x+1\right)+\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$

14. $\frac{3\sin(3x)-\cos(3x)}{10e^x}+c$

15. $x\tan^{-1}(x)-\frac{1}{2}\ln\left(1+x^2\right)+c$

16. $y=\sin^{-1}\left(\frac{x}{2}\right)+\frac{2\pi}{3}$

17. $\frac{1}{2}x^2(\ln(x))^2-\frac{1}{2}x^2\ln(x)+\frac{1}{4}x^2+c$

18. $\frac{1}{2}x^2\tan^{-1}(x)+\frac{1}{2}\tan^{-1}(x)-\frac{1}{2}x+c$

19. $-x+4\sqrt{x}-4\ln\left(1+\sqrt{x}\right)+c$

20. $\frac{1}{5}\ln\left(\frac{2-\sin(x)}{\sin(x)+3}\right)+c$

6.8 Past QCAA exam questions

1. A

2. B

3. a. $2\ln|2x-3|-2\ln|x+4|+c$

b. $2\ln\left(\frac{1}{12}\right)$

4. a. $\frac{\pi}{4}$ b. $\frac{\pi-2}{4}$

5. $a=6,\ b=-3$

7 Applications of integral calculus

LESSON SEQUENCE

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7.2 Area between a function and the axes 480
7.3 Area between functions 489
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Answers 546

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 7
Exam questions	Exam question booklet — Chapter 7
Digital documents	Learning matrix — Chapter 7 Chapter summary — Chapter 7

LESSON
7.1 Overview

Hey students! Bring these pages to life online

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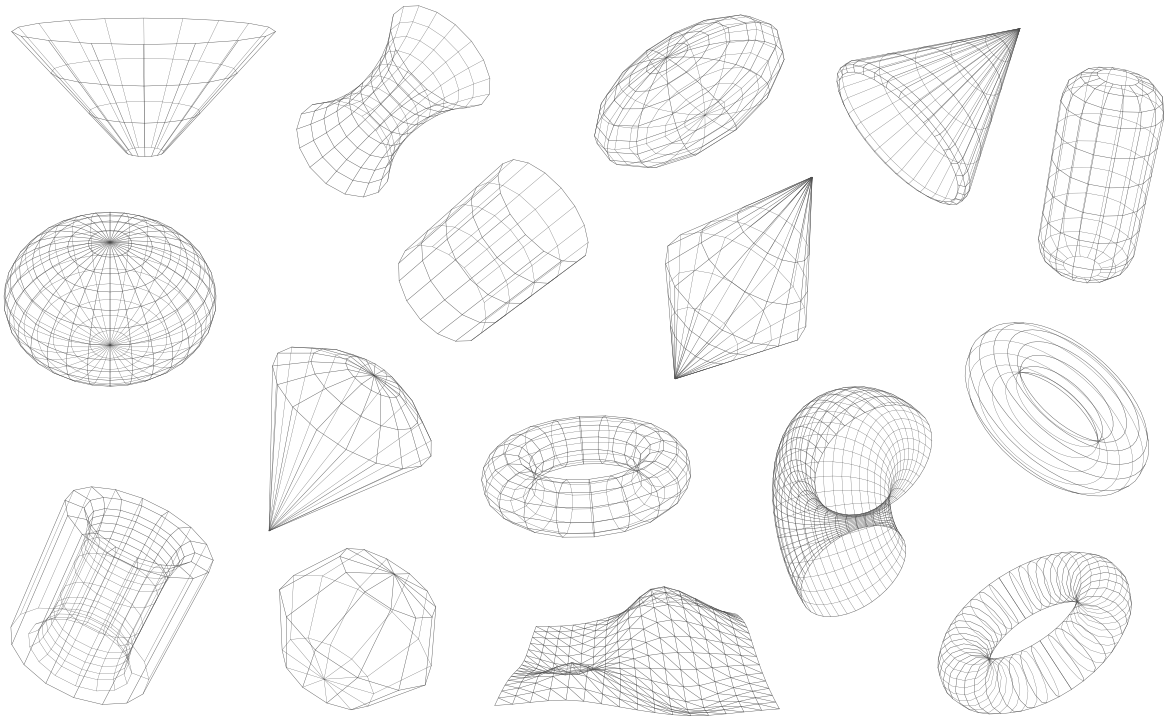
7.1.1 Introduction

In Chapter 6, you learned integration techniques that can be used to calculate the area between functions and the axes, as well as the area between two different functions. However, in some cases, algebraic techniques are insufficient for calculating the integral, requiring a method to approximate the area. In Mathematical Methods, you used the trapezoidal rule to approximate areas under curves. In this chapter, you will learn to use Simpson's rule for a more accurate approximation. This method can also be applied to estimate the area of irregular shapes. Integration can also be used to determine the volume of solids by revolving regions about an axis. Finally, we will apply integration techniques to calculate probabilities for the exponential probability distribution.



7.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
7.2	**Area between a function and the axes**	◯ Apply techniques from Unit 4 Topic 1 Sub-topic: Integration techniques to calculate areas between curves determined by functions, with and without technology.
7.3	**Area between functions**	◯ Apply techniques from Unit 4 Topic 1 Sub-topic: Integration techniques to calculate areas between curves determined by functions, with and without technology.
7.4	**Volumes of solids of revolution**	◯ Determine volumes of solids of revolution about either axis, with and without technology. • about the x-axis: $V = \pi \int_a^b [f(x)]^2 \, dx$ • about the y-axis: $V = \pi \int_a^b [f(y)]^2 \, dy$
7.5	**Volumes of revolution**	◯ Determine volumes of solids of revolution about either axis, with and without technology. • about the x-axis: $V = \pi \int_a^b [f(x)]^2 \, dx$ • about the y-axis: $V = \pi \int_a^b [f(y)]^2 \, dy$
7.6	**Approximation using Simpson's rule**	◯ Use Simpson's rule to approximate an area and the value of a definite integral, with and without technology. • $\int_a^b f(x)\,dx \approx \frac{w}{3}\left[f(x_0) + 4\left[f(x_1) + f(x_3) + \cdots\right] + 2\left[f(x_2) + f(x_4) + \cdots\right] + f(x_n)\right]$ where $w = \frac{b-a}{n}$
7.7	**Exponential probability density function**	◯ Understand and use the probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, of the exponential random variable with parameter $\lambda > 0$. • Mean: $\frac{1}{\lambda}$ • Standard deviation: $\frac{1}{\lambda}$ ◯ Model and solve problems that involve exponential random variables and associated probabilities and quantiles, with and without technology.

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LESSON
7.2 Area between a function and the axes

SYLLABUS LINKS

- Apply techniques from Unit 4 Topic 1 Sub-topic: Integration techniques to calculate areas between curves determined by functions, with and without technology.

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7.2.1 The Fundamental Theorem of Calculus

Basic integration techniques and calculating areas bounded by functions and the x-axis have been covered in the Mathematical Methods course.

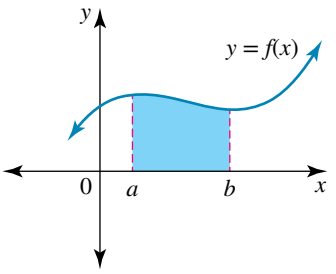
You might want to review the integration concepts seen in Mathematical Methods Unit 3, Topic 4: Introduction to integration, and in Unit 4, Topic 1: Further integration.

The area of the shaded region in the diagram is calculated by applying the **Fundamental Theorem of Calculus**.

The Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F(x)$ is the anti-derivative of $f(x)$.

When calculating a definite integral, the result can be either a positive or negative number depending on whether the area lies above or below the x-axis. It can be interpreted as the net area, $\int_a^b f(x)\,dx = A_1 - A_2$, where A_1 is the area above the x-axis and A_2 is the area below. If the area below the x-axis is greater than the area above, the define integral will be a negative number. If the two areas are equal, the definite integral is zero. For example: $\int_0^{2\pi} \sin(x)\,dx = 0$.



The definite integral is determined similarly to your work in Mathematical Methods, but these integrands require Specialist Mathematics knowledge, as seen in the following Worked examples.

WORKED EXAMPLE 1 Calculating definite integrals

Calculate the following definite integrals. Give your answers to 3 decimal places.

a. $\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{9 - x^2}}\,dx$

b. $\int_0^{0.5} \frac{x + 5}{x^2 + x - 2}\,dx$

THINK

a. 1. Recall the general anti-derivative formula for $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$.

WRITE

a. $\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{9 - x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{\frac{\pi}{4}}$

2. Substitute the upper and lower limits. $= \left(\sin^{-1}\left(\frac{\frac{\pi}{4}}{3}\right)\right) - \left(\sin^{-1}\left(\frac{0}{3}\right)\right)$

3. Calculate the definite integral. $= 0.265$

b. 1. Express the integrand as partial fractions.

b. $f(x)\ \dfrac{x+5}{x^2+x-2} = \dfrac{a}{x-1} + \dfrac{b}{x+2}$

$$= \frac{a(x+2)}{x-1} + \frac{b(x-1)}{x+2}$$

2. Equate the numerators from both sides. $x+5 = a(x+2) + b(x-1)$

3. Solve for b by letting $x = -2$.

$$x = -2:\ -2+5 = b(-2-1)$$
$$3 = -3b$$
$$b = -1$$

4. Solve for a by letting $x = 1$.

$$x = 1:\ 1+5 = a(1+2)$$
$$a = \frac{6}{3}$$
$$a = 2$$

5. Anti-differentiate the partial fractions. $l = \displaystyle\int_4^5 \frac{x+5}{x^2+x-2}\,dx = \int_4^5 \frac{2}{x-1} - \frac{1}{x+2}\,dx$

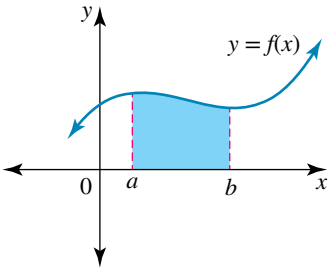
6. Substitute the upper and lower limits.

$$= [2\ln(x-1) - 1\ln(x+2)]_4^5$$
$$= (0.8267) - (0.4055)$$

7. Calculate the definite integral. $= 0.421$

7.2.2 Areas above the x-axis

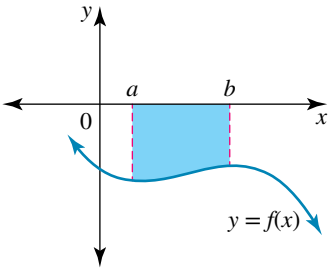
If $f(x) \geq 0$ for the region $a \leq x \leq b$, then the region is above the x-axis and $\displaystyle\int_a^b f(x)\,dx > 0$. This means that the area of the region can be found using $A = \displaystyle\int_a^b f(x)\,dx$.



7.2.3 Areas below the x-axis

If $f(x) < 0$ for the region $a \leq x \leq b$, then the region is below the x-axis and $\displaystyle\int_a^b f(x)\,dx < 0$. This means that the area of the region can be found using:

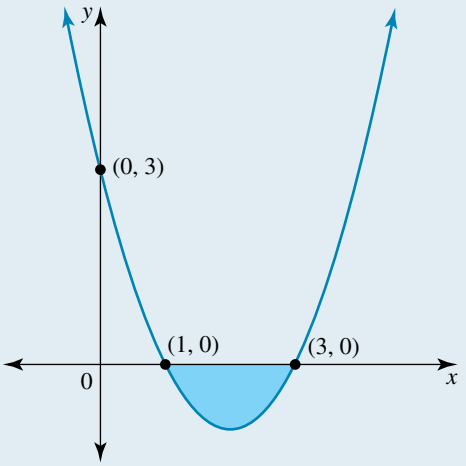
$$A = \left|\int_a^b f(x)\,dx\right|$$
$$= -\int_a^b f(x)\,dx$$
$$= \int_b^a f(x)\,dx$$



Notice how a and b have swapped positions.

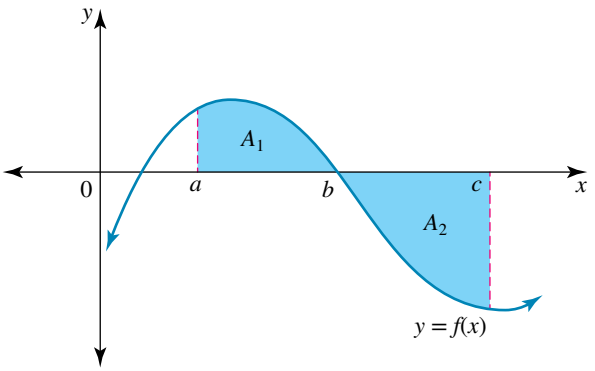
WORKED EXAMPLE 2 Determining an area below the x-axis

Determine the area of the region bounded by the curve $y = x^2 - 4x + 3$ and the x-axis.

THINK	WRITE
1. Factorise the quadratic and let $y = 0$ to determine the x-intercepts. The y-intercept can be easily found by substituting $x = 0$.	$0 = x^2 - 4x + 3$ $0 = (x - 3)(x - 1)$ The graph crosses the x-axis at $x = 1$ and $x = 3$. The y-intercept is $y(0) = 3$.
2. Sketch the graph, shading the required area.	
3. The required area is below the x-axis and will evaluate to a negative number. The area must be given by the absolute value or the negative of this definite integral.	$A = \left\lvert \int_1^3 (x^2 - 4x + 3)\,dx \right\rvert$ $= -\int_1^3 (x^2 - 4x + 3)\,dx$
4. Anti-differentiate the integrand.	$A = -\left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$
5. Evaluate the definite integral.	$A = -\left[\left(\frac{1}{3} \times 3^3 - 2 \times 3^2 + 3 \times 3\right) - \left(\frac{1}{3} \times 1^3 - 2 \times 1^2 + 3 \times 1\right)\right]$ $= \frac{4}{3}$
6. State the value of the required area.	The area is $\frac{4}{3}$ square units.

7.2.4 Areas above and below the x-axis

For regions that are combinations of areas above and below the x-axis, each area needs to be calculated by separate integrals.



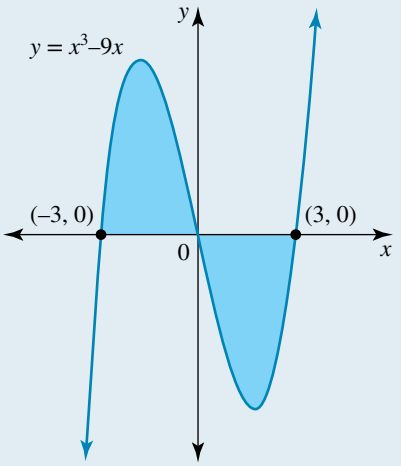
For example, from the diagram $A_1 = \int_a^b f(x)\,dx$ and $A_2 = -\int_b^c f(x)\,dx$. This means that the total Area can be found using Area $= A_1 + A_2$

$$= \int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

Note that this is not the same as finding $\int_a^c f(x)\,dx$.

WORKED EXAMPLE 3 Determining an area below and above the x-axis

Determine the area bounded by the curve $y = x^3 - 9x$ and the x-axis.

THINK	WRITE
1. Factorise the cubic and let $y = 0$ to determine the x-intercepts.	$0 = x^3 - 9x$ $0 = x(x^2 - 9)$ $0 = x(x+3)(x-3)$ The graph crosses the x-axis at $x = 0$ and $x = \pm 3$.
2. Sketch the graph, shading the required area.	
3. The required area is given by a definite integral. If we calculate $A = \int_{-3}^{3} (x^3 - 9x)\,dx$ the outcome is zero, as the positive and negative area have cancelled out.	Total area $= \int_{-3}^{0} (x^3 - 9x)\,dx + \left\lvert \int_0^3 (x^3 - 9x)\,dx \right\rvert$ Let $A_1 = \int_{-3}^{0} (x^3 - 9x)\,dx$ and $A_2 = \int_0^3 (x^3 - 9x)\,dx$, so that $A_1 > 0$ and $A_2 < 0$, but $A_1 = \lvert A_2 \rvert$ by symmetry.

4. Anti-differentiate the integrand.

$$A_1 = \int_{-3}^{0} (x^3 - 9x)\,dx$$

$$A_1 = \left[\frac{1}{4}x^4 - \frac{9}{2}x^2\right]_{-3}^{0}$$

5. Evaluate the definite integral. Note that A_2 gives a negative answer since the area is below the x-axis. When calculating the total area, we take the absolute value of any areas below the x-axis to ensure they are positive. The total area required is the sum of A_1 and $|A_2|$ or $2 \times A_1$ due to the symmetry of the two areas.

$$A_1 = \left[0 - \left(\frac{1}{4} \times (-3)^4 - \frac{9}{2} \times (-3)^2\right)\right]$$

$$A_1 = \frac{81}{4}$$

$$= 20\frac{1}{4}$$

$$A_2 = \int_{0}^{3} (x^3 - 9x)\,dx = -\frac{81}{4} = -20\frac{1}{4}$$

$$A_1 + |A_2| = 2 \times \frac{81}{4} = \frac{81}{2} = 40\frac{1}{2}$$

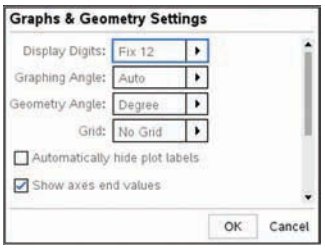
6. State the value of the required area.

The area is $40\frac{1}{2}$ square units.

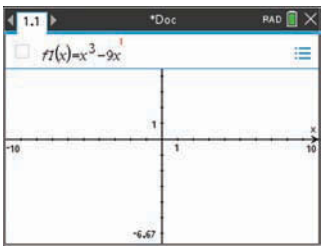
TI | THINK

1. On a Calculator page, press Menu then select Add graphs to: New document. Press MENU, 9 Settings. Change Display Digits to Float 12 and then click OK

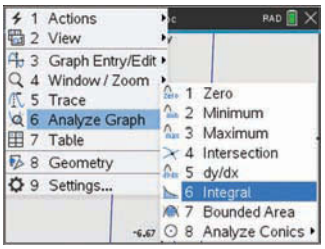
DISPLAY



2. Complete the entry line as $x^3 - 9x$ then press the ENTER button.



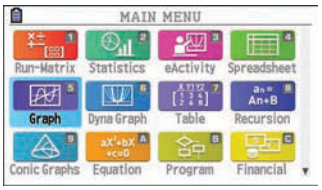
3. Press MENU then select
6: Analyze Graph
6: Integral



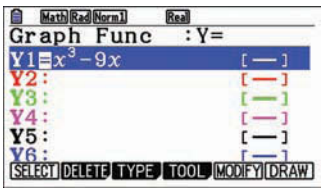
CASIO | THINK

1. On a Main Menu screen, select GRAPH

DISPLAY



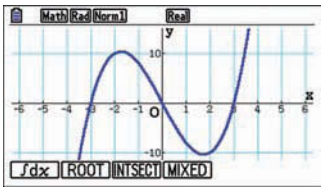
2. Complete the entry line as $x^3 - 9x$ then press the EXE or DRAW button.



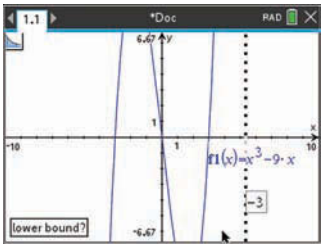
3. To calculate the definite integral, select SHIFT F5

$\int dx$

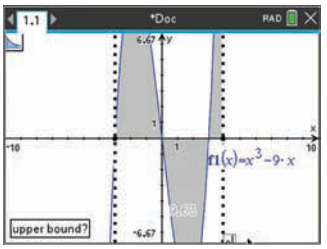
$\int dx$ (a second time)



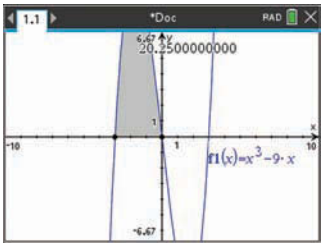
4. Select the lower bound, $x = -3$, by entering the number -3.



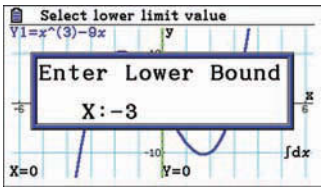
5. Select the upper bound, $x = 0$, by entering the number 0.



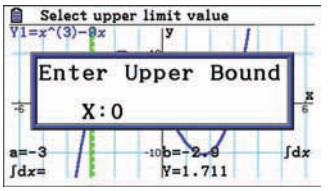
6. The answer appears on the screen. *Note:* Due to symmetry, the final answer can be determined as $2 \times 20.25 = 40.5$. Alternatively, the process can be repeated for the region, $0 \le x \le 3$.



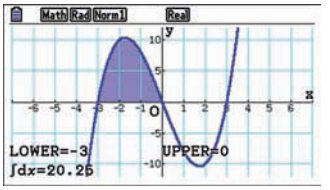
4. Select the lower bound, $x = -3$, by first pressing -3, followed by the EXE button.



5. Select the upper bound, $x = 0$, by first pressing 0, followed by the EXE button.



6. The answer appears on the screen. *Note:* Due to symmetry, the final answer can be determined as $2 \times 20.25 = 40.5$. Alternatively, the process can be repeated for the region, $0 \le x \le 3$.



When we are determining areas under functions that are not easy to sketch, the area can be calculated provided that the x-intercepts can be determined. If there are no x-intercepts in the region $[a, b]$, then the area can be found using $\left|\int_a^b f(x)\,dx\right|$. If an x-intercept occurs at c where $a < c < b$, then the area can be found using $\left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|$.

WORKED EXAMPLE 4 Calculating the area between a graph and the x-axis

Consider the curve $f(x) = \sin(x)\cos^3(x)$ for $0 \le x \le \frac{5}{2}$.

a. Use technology to determine any possible roots. Hence, define the regions above and below the x-axis.

b. Calculate the total area between the graph of $f(x)$ and the x-axis. Give your answer to 3 decimal places.

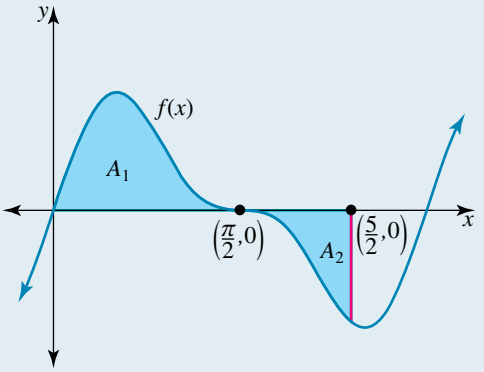
THINK

a. 1. Use a graphics calculator to sketch the curve to identify x-intercepts within the given domain.

WRITE

a. One root (x-intercept) exists within the given domain at $x = \frac{\pi}{2}$.

2. Refer to the graphics calculator sketch to identify regions above and below the x-axis.



The curve is positive for $0 \le x < \frac{\pi}{2}$ and negative for $\frac{\pi}{2} < x \le \frac{5}{2}$.

b. 1. Create an equation to define the area between the curve and the x-axis.

b. Area $= A_1 + A_2$

$$= \int_0^{\frac{\pi}{2}} f(x)\,dx + \left| \int_{\frac{\pi}{2}}^{\frac{5}{2}} f(x)\,dx \right|$$

2. Determine the anti-derivative using the substitution method.

$$u = \cos(x)$$
$$\frac{du}{dx} = -\sin(x)$$
$$dx = -\frac{du}{\sin(x)}$$

3. Rewrite the integrand in terms of u.

$$\int \sin(x)\cos^3(x)\,dx = -\int \cancel{\sin(x)} \times u^3 \times \frac{du}{\cancel{\sin(x)}}$$
$$= -\int u^3\,du$$

4. Calculate the anti-derivative and apply the substitution $u = \cos(x)$.

$$= -\frac{1}{4}u^4 + c$$
$$= -\frac{1}{4}\cos^4(x) + c$$

5. Substitute values into Area $= \int_0^{\frac{\pi}{2}} f(x)\,dx + \left| \int_{\frac{\pi}{2}}^{\frac{5}{2}} f(x)\,dx \right|$ for the upper and lower terminals.

$$\text{Area} = \left[-\frac{1}{4}\cos^4(x)\right]_0^{\frac{\pi}{2}} + \left| \left[-\frac{1}{4}\cos^4(x)\right]_{\frac{\pi}{2}}^{\frac{5}{2}} \right|$$
$$= 0.25 + |-0.10299|$$

6. State the total area.

$$= 0.353 \text{ square units.}$$

Exercise 7.2 Area between a function and the axes

learnon

7.2 Exercise	7.2 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9	10, 11, 12, 13, 14, 15, 16, 17	18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. MC Select which of the following definite integrals exists.

A. $\int_{-3}^{4} \ln(x^2-4)\,dx$

B. $\int_{0}^{4} \frac{2}{(x-1)(x+3)}\,dx$

C. $\int_{-2}^{1} \sqrt{9-x^2}\,dx$

D. $\int_{-1}^{1} \sin^{-1}(1+2x)\,dx$

2. MC Complete the following statement.

The function $\frac{2-x}{25-x^2}$ is above the x-axis $\forall x \in$:

A. $\mathbb{R}\backslash\{-5,5\}$.

B. $(-5,5)$.

C. $(-\infty,-5)\cup[2,5)$.

D. $(-5,2]\cup(5,\infty)$.

3. WE1 Calculate the following definite integrals without technology and verify your answers using technology. Give your answers to 3 decimal places.

a. $\int_{0}^{\frac{\pi}{4}} \frac{1}{\sqrt{25-x^2}}\,dx$

b. $\int_{3}^{4} \frac{x-4}{x^2+4x-12}\,dx$

4. Calculate the following definite integrals without technology and verify your answers using technology. Give your answers to 3 decimal places.

a. $\int_{2}^{5} \frac{15x^2+2}{\sqrt{5x^3+2x}}\,dx$

b. $\int_{0.25}^{0.5} \frac{12x^2+3}{(4x^3+3x)^4}\,dx$

5. WE2 Determine the area bounded by the following curves and the x-axis.

a. $y=x^2-5x+6$

b. $y=1-x^2$

c. $y=8-x^2$

d. $y=12-3x^2$

e. $y=x^2-2x-15$

f. $y=3x^2-10x-8$

6. WE3 Determine the area bounded by the following curves and the x-axis.

a. $y=x^3-4x$

b. $f(x)=x(x-2)(x-3)$

c. $h(x)=(x-1)(x+2)(x+5)$

d. $g(x)=x^3-4x^2$

7. a. WE4 Determine the area between the graph of $y=\frac{x}{x^2+2}$, the x-axis, $x=1$ and $x=2$.

b. Determine the area between the graph of $y=\frac{x}{x^2+2}$, the x-axis, $x=-1$ and $x=1$.

c. Determine the area between the graph of $y=3-e^{2x}$, the x-axis, $x=-2$ and $x=0$.

d. Determine the area between the graph of $y=3-e^{2x}$, the x-axis, $x=-1$ and $x=1$.

8. For each of the following functions, identify the maximal domain of the function and determine the area described.

a. The area between the function $y = \sqrt{2x+1}$, the coordinate axes and the line $x = 4$.

b. The area between the function $y = \dfrac{1}{3x+5}$, the coordinate axes and $x = 3$.

c. The area between the function $y = \dfrac{1}{(2x+3)^2}$, the x-axis, $x = 1$ and $x = 2$.

d. The area between the function $y = \dfrac{x}{\sqrt{16-3x}}$, the coordinate axes and $x = 5$.

9. Determine the area of the region enclosed by the curve $f(x) = \dfrac{5x-5}{x^2-25}$, $x \in [0, 1]$ and the x-axis without technology and verify your answer with technology. Give your answer to 3 decimal places.

Complex familiar

10. Apply the double-angle trigonometric identities to calculate the following definite integrals without technology and verify your answers using technology. Give your answers to 3 decimal places.

a. $\displaystyle\int_0^{\pi} 2\sin^2(4x)\,dx$

b. $\displaystyle\int_0^{\frac{\pi}{6}} 1 + \tan^2(x) + 2\sin(2x)\,dx$

11. Calculate the exact value of $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2(x)\,dx$.

12. Consider the curve $f(x) = \sin^3 x \cos x$ for $\dfrac{\pi}{3} \le x \le \dfrac{5\pi}{6}$.

a. Use technology to determine any possible roots. Hence, define the regions above and below the x-axis.

b. Calculate the total area between the graph of $f(x)$ and the x-axis. Give your answer to 3 decimal places.

13. Examine the curve defined by $f(x) = x\cos(2x)$ for $0 \le x \le 2$.

a. Use technology to determine any possible roots. Hence, define the regions above and below the x-axis.

b. Calculate the total area between the graph of $f(x)$ and the x-axis. Give your answer to 3 decimal places.

14. Use a suitable trigonometric identity to calculate the following definite integrals. $\displaystyle\int_0^{\frac{\pi}{6}} \sec^4(x)\tan^6(x)\,dx$ without technology and verify your answer using technology. Give your answer to 3 decimal places.

15. a. The graph of $y = \dfrac{x}{x^2-c}$ has vertical asymptotes at $x = \pm 2$. Identify the value of c and determine the area bounded by the curve, the x-axis and the lines $x = 3$ and $x = 5$.

b. Determine the area bounded by $y = x\cos(x^2)$, the x-axis, $x = 0$ and $x = \dfrac{\sqrt{\pi}}{2}$.

c. Determine the area bounded by $y = xe^{-x^2}$, the x-axis, $x = 0$ and $x = 2$.

d. Determine the area bounded by $y = \dfrac{x}{\sqrt{x^2+4}}$, the coordinate axes and $x = 2\sqrt{3}$.

16. a. Determine the area bounded by the curve $y = \dfrac{5}{x^2 + x - 6}$, the coordinate axes and $x = -1$.

b. Determine the area bounded by the curve $y = \dfrac{2x - 3}{4 + 3x - x^2}$, the x-axis, $x = 2$ and $x = 3$.

c. Determine the area bounded by $y = \dfrac{21}{40 - 11x - 2x^2}$, the coordinate axes and $x = -5$.

d. Determine the area bounded by $y = \dfrac{x^3 - 9x + 9}{x^2 - 9}$, the x-axis, $x = 4$ and $x = 6$.

17. Determine the area between the curve $y^2 = 4 - x$ and the y-axis.

Complex unfamiliar

18. Determine the area of the loop with the equation $y^2 = x^2(4 - x)$.

19. If $a > 0$, determine the area bounded by the curve $y^2 = a - x$ and the y-axis.

20. If $a > 0$, determine the area of the loop $y^2 = x^2(a - x)$.

Fully worked solutions for this chapter are available online.

LESSON
7.3 Area between functions

SYLLABUS LINKS

- Apply techniques from Unit 4 Topic 1 Sub-topic: Integration techniques to calculate areas between curves determined by functions, with and without technology.

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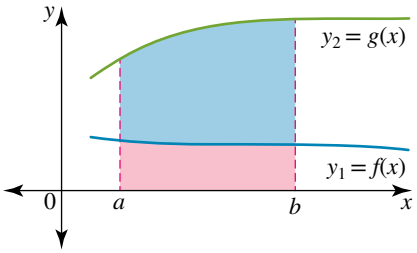
When trying to determine the area between two functions, $f(x)$ and $g(x)$, over an interval $a \le x \le b$ the method depends on whether the functions intersect or do not intersect in the interval.

7.3.1 Area between two functions that do not intersect in the required interval

If $y_1 = f(x)$ and $y_2 = g(x)$ are two continuous functions that do not intersect between $x = a$ and $x = b$, then the area between the curves can be found by subtracting the lower function from the higher function.



If we define the area between $y_2 = g(x)$, $x = a$ and $x = b$, A_1, then

$$A_1 = \int_a^b g(x)\,dx.$$

If we define the pink area between $y_1 = f(x)$, $x = a$ and $x = b$, A_2, then

$$A_2 = \int_a^b f(x)\,dx.$$

The required blue area can be found using $A = A_1 - A_2$

$$= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx.$$

By the properties of definite integrals,

$$\begin{aligned} A &= A_1 - A_2 \\ &= \int_a^b (g(x) - f(x))\,dx \\ &= \int_a^b (y_2 - y_1)\,dx \end{aligned}$$

Note: it is only necessary to evaluate one definite integral, the difference between y_1 and y_2.

Area formula

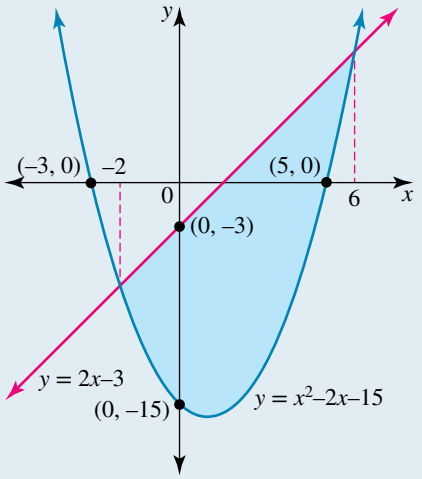
If $f(x)$ and $g(x)$ are continuous functions for $[a, b]$, and given $f(x) \geq g(x)\ \forall x \in [a, b]$ then the area of the region bounded by the curves is $A = \int_a^b [f(x) - g(x)]\,dx$.

WORKED EXAMPLE 5 Determining the area between two function that do not intersect

Determine the area between the parabola $y = x^2 - 2x - 15$ and the straight line $y = 2x - 3$.

THINK	WRITE
1. Let $x = 0$ and $y = 0$ to calculate the y-intercept and the x-intercept respectively. For the quadratic, remember to factorise and apply the null factor law.	$y = x^2 - 2x - 15$ $y = (x - 5)(x + 3)$ The parabola crosses the x-axis at $x = 5$ and $x = -3$ and crosses the y-axis at $y = -15$. The straight line crosses the x-axis at $x = \frac{3}{2}$ and crosses the y-axis at $y = -3$.
2. Determine the x-values of the points of intersection between the parabola and the straight line.	Let $y_1 = x^2 - 2x - 15$ and $y_2 = 2x - 3$. To find the points of intersection, solve $y_1 = y_2$. $x^2 - 2x - 15 = 2x - 3$ $x^2 - 4x - 12 = 0$ $(x - 6)(x + 2) = 0$ $x = 6, -2$

3. Sketch the graph of the parabola and the straight line on one set of axes, shading the required area.



4. The line $y_2 = 2x - 3$ is always above the parabola $y_1 = x^2 - 2x - 15$ in the region bounded by the line and the curve. This means that the area can be determined by calculating $\int_{-2}^{6} (y_2 - y_1)\,dx$.

$A = \int_a^b (y_2 - y_1)\,dx$ with $a = -2$, $b = 6$, $y_1 = x^2 - 2x - 15$ and $y_2 = 2x - 3$.

$y_2 - y_1 = -x^2 + 4x + 12$

$A = \int_{-2}^{6} (-x^2 + 4x + 12)\,dx$

5. Anti-differentiate the integrand.

$A = \left[-\frac{1}{3}x^3 + 2x^2 + 12x\right]_{-2}^{6}$

6. Evaluate the definite integral.

$A = \left[\left(-\frac{1}{3} \times 6^3 + 2 \times 6^2 + 12 \times 6\right) - \left(-\frac{1}{3} \times (-2)^3 + 2 \times (-2)^2 + 12 \times (-2)\right)\right]$

$A = 85\frac{1}{3}$

7. State the value of the required area between the parabola and the straight line.

The area between the straight line and the parabola is $85\frac{1}{3}$ square units.

TI | THINK

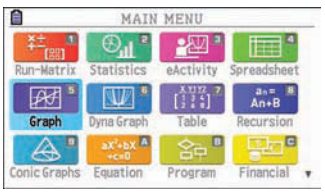
1. On a Calculator page, press MENU then select Add Graphs to: New documents

DISPLAY

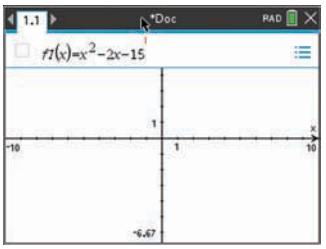
CASIO | THINK

1. On a Main Menu screen, select GRAPH

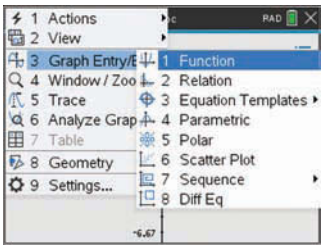
DISPLAY



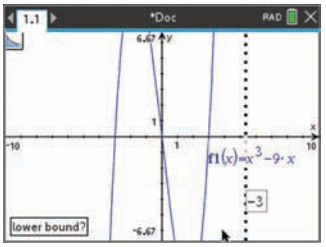
2. Complete the entry line as $x^2 - 2x - 15$ then press the ENTER button.



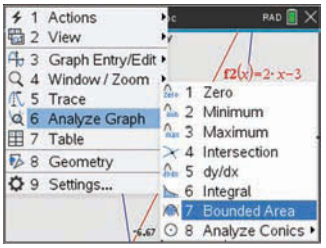
3. Press MENU then select
3: Graph Entry/Edit
1: Function



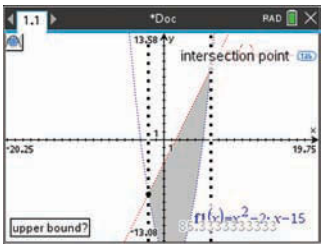
4. Complete the entry line as $2x - 3$ then press the ENTER button.
Note that you might need to press MENU, 4 Window/Zoom, 4 Zoom - Out, and select the origin as the centre, to be able to see the two intersections between the graphs.



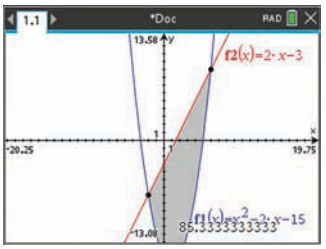
5. Press MENU then select
6: Analyze Graph
7: Bounded Area



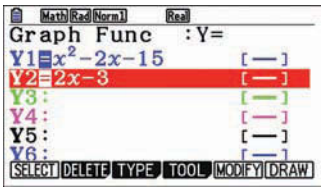
6. Select the two intersection points using the cursor. Press the ENTER button on both intersection points.



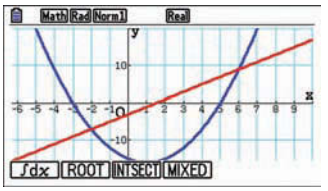
7. The answer appears on the screen as textbox next to the shaded area.



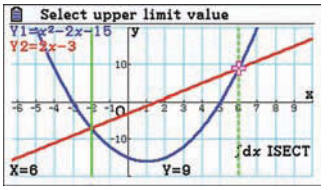
2. Complete the entry line in Y1 and Y2 as $x^2 - 2x - 15$ $2x - 3$ then press the EXE or DRAW button.



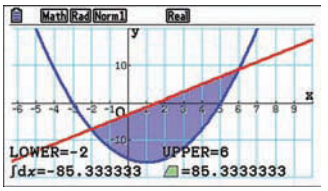
3. To calculate the area bounded by the two functions select SHIFT F5 $\int dx$ INTSECT



4. Begin by pressing the EXE button to select the lower intersection point. Press the right cursor button to select the upper intersection point. Select this intersection point by pressing the EXE button.

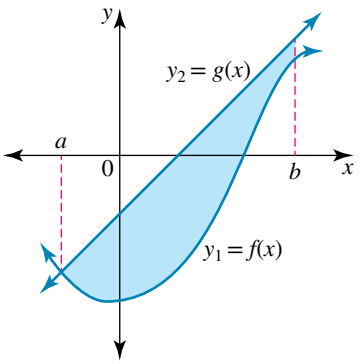


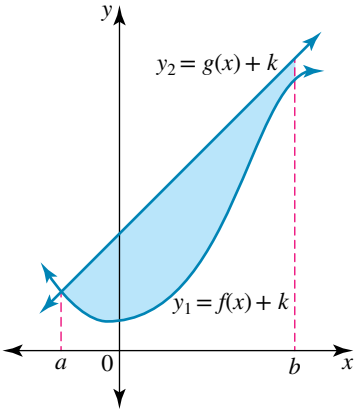
5. The answer appears on the screen.



Note: When finding areas between curves, it does not matter if some of the area is above or below the x-axis.

We can translate both curves up k units parallel to the y-axis so that the area between the curves lies entirely above the x-axis as shown.



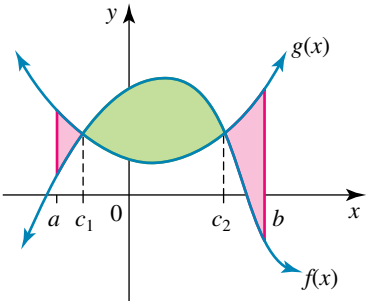


$$A = \int_a^b (g(x) + k)\,dx - \int_a^b (f(x) + k)\,dx$$
$$= \int_a^b (g(x) - f(x))\,dx$$
$$= \int_a^b (y_2 - y_1)\,dx$$

Provided that $y_2 \geq y_1$ for $a \leq x \leq b$, it does not matter if some or all of the area is above or below the x-axis, as the required area between the curves will be a positive number.

7.3.2 Area between two functions that intersect in the required interval

When determining the area between functions, the lower function is subtracted from the higher function. This means that if the functions intersect in the interval, then the lower and higher functions will change, so it is necessary to break the interval into sub-intervals.



For the graph shown, the area is found by calculating

$$\int_a^{c_1} (g(x) - f(x))\,dx + \int_{c_1}^{c_2} (f(x) - g(x))\,dx + \int_{c_2}^{b} (g(x) - f(x))\,dx$$

WORKED EXAMPLE 6 Determining the area between two functions that intersect once

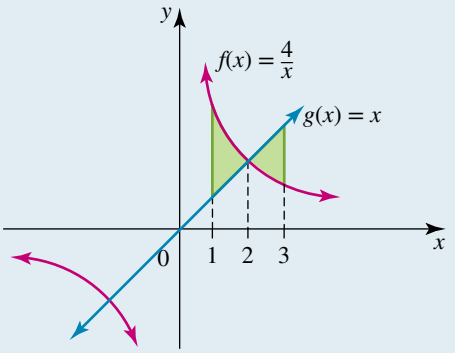
Determine the area between $f(x) = \dfrac{4}{x}$ and $g(x) = x$ from $x = 1$ to $x = 3$.

THINK	WRITE
1. State the functions.	$f(x) = \dfrac{4}{x}$ $g(x) = x$
2. Determine the x-coordinates of the point of intersection.	$\dfrac{4}{x} = x$ $x^2 = 4$ $x = \pm 2$

3. Identify if the functions intersect in the interval $1 \le x \le 3$.

$x = 2$ occurs in the interval $1 \le x \le 3$

4. Sketch $f(x)$ and $g(x)$ on the same set of axes. Shade the area between the functions, $x = 1$ and $x = 3$. Indicate that the functions intersect at $x = 2$ in the region $1 \le x \le 3$



5. Within the interval $1 \le x \le 2$, $f(x)$ is higher than $g(x)$.
Within the interval $2 \le x \le 3$, $g(x)$ is higher than $f(x)$.
State the area as the sum of two integrals for the sub-intervals.

$$A = \int_1^2 (f(x) - g(x))\,dx + \int_2^3 (g(x) - f(x))\,dx$$

$$= \int_1^2 \left(\frac{4}{x} - x\right) dx + \int_2^3 \left(x - \frac{4}{x}\right) dx$$

6. Anti-differentiate the integrands.

$$= \left[4\ln|x| - \frac{1}{2}x^2\right]_1^2 + \left[\frac{1}{2}x^2 - 4\ln|x|\right]_2^3$$

7. Evaluate the definite integrals.

$$= \left(4\ln(2) - \frac{1}{2}(2)^2\right) - \left(4\ln(1) - \frac{1}{2}(1)^2\right) + \left(\frac{1}{2}(3)^2 - 4\ln(3)\right) - \left(\frac{1}{2}(2^2) - 4\ln(2)\right)$$

$$= 4\ln(2) - 2 - 4\ln(1) + \frac{1}{2} + \frac{9}{2} - 4\ln(3) - 2 + 4\ln(2)$$

$$= 4\ln\left(\frac{2 \times 2}{3}\right) + \frac{10}{2} - 4$$

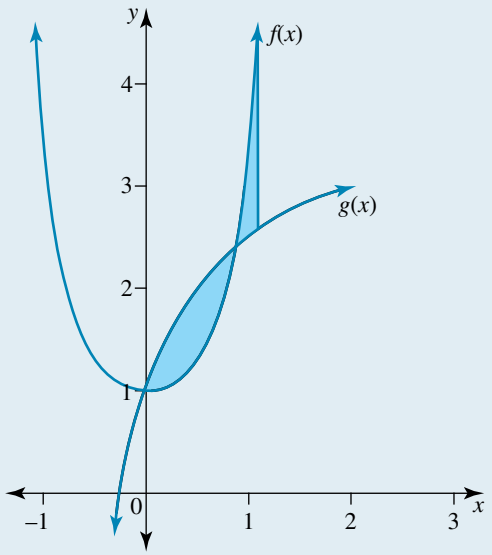
$$= 4\ln\left(\frac{4}{3}\right) + 1$$

8. State the value of the required area.

The area is $4\ln\left(\frac{4}{3}\right) + 1$ or approximately 2.15 square units.

WORKED EXAMPLE 7 Determining the area between two functions that intersect twice

Calculate the area bounded by the curves $f(x) = 1 + \tan^2(x)$ and $g(x) = \dfrac{-3}{x+1} + 4$, between the lines $x = 0$ and $x = 1.1$.



THINK	WRITE
1. Determine the points of intersection using a graphics calculator.	Intersection 1: $A = (0, 1)$ Intersection 2: $B = (0.868, 2.394)$
2. Define the two regions of area according to the domain and intersection points.	$A_1 = \text{Area } 1, \int_0^{0.868} (g(x) - f(x))\, dx$ $A_2 = \text{Area } 2, \int_{0.868}^{1.1} (f(x) - g(x))\, dx$ $\text{Area} = A_1 + A_2$
3. Apply a Pythagorean identity to rewrite $f(x)$.	$f(x) = 1 + \tan^2(x) = \sec^2(x)$, $g(x) = \dfrac{-3}{x+1} + 4$
4. Within the interval $0 \le x \le 0.868$, $g(x)$ is above $f(x)$.	$A_1 = \int_0^{0.868} [g(x) - f(x)]\, dx$ $= \int_0^{0.868} \left[\left(\dfrac{-3}{x+1} + 4\right) - \left(\sec^2(x)\right)\right] dx$
5. Anti-differentiate the integrands.	$= [-3\ln(x+1) + 4x - \tan(x)]_0^{0.868}$

6. Evaluate the definite integral to calculate the area of the first region.	$= (-3\ln(0.868+1)+4\times 0.868-\tan(0.868))$ $-(-3\ln(0+1)+4\times 0-\tan(0))$ $= 1.5974-1.1805$ $= 0.4169$
7. Within the interval $0.868 \le x \le 1.1$, $f(x)$ is above $g(x)$.	$A_2 = \int_{0.868}^{1.1} [f(x)-g(x)]\,dx$ $= \int_{0.868}^{1.1} \left[\left(\sec^2(x)\right)-\left(\frac{-3}{x+1}+4\right)\right] dx$
8. Anti-differentiate the integrands.	$= [\tan(x)+3\ln(x+1)-4x]_{0.868}^{1.1}$
9. Evaluate the definite integral to calculate the area of the second region.	$= (\tan(1.1)+3\ln(1.1+1)-4\times 1.1)$ $-(\tan(0.868)+3\ln(0.868+1)-4\times 0.868)$ $= -0.2094 - -4169$ $= 0.2074$
10. Calculate the value of the total area.	$A = A_1 + A_2$ $= 0.4169 + 0.2074$ $= 0.6243$ square units.

Therefore, when calculating areas between two functions over an interval, it must be determined whether the functions intersect within that interval. If they do, the area is broken into sub-intervals.

As with areas under curves, the graph should be sketched to identify the higher and lower function.

If the graph is not sketched, then the absolute value of the integral is used to ensure that the resulting area has a positive value.

7.3.3 Using Technology

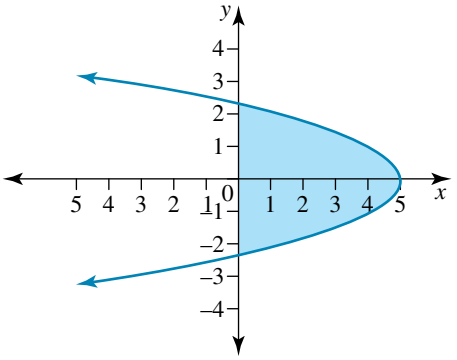
Graphics calculators could be used to either assist with the graphing or to verify the area obtained.

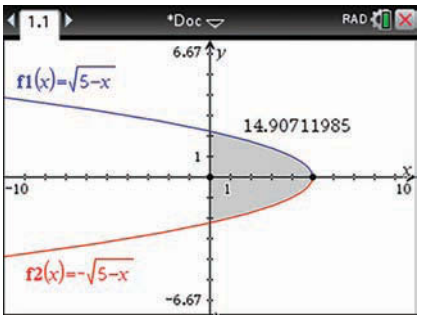
You should already be able to use your calculator to sketch graphs, determine the roots of graphs and obtain the points of intersection between graphs. These skills can also be useful when solving integration questions.

When performing integral calculations, your calculator can determine the area between a function and the x-axis over a particular domain. This can be seen in the technology section of Worked example 3.

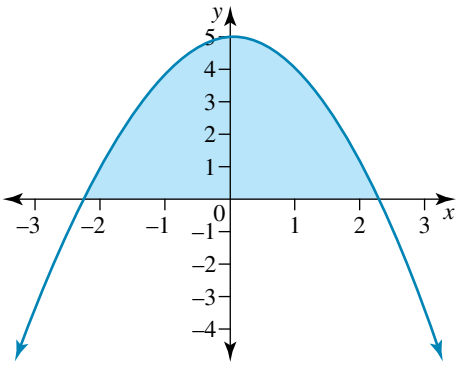
Your calculator can also determine the area bounded by a function and the x-axis (without you needing to calculate the roots of the function first) and also determine the area bounded by two functions. This is due to the inbuilt features of the calculators which can automatically determine points of interest such as roots and points of intersection. This can be seen in the technology section of Worked example 5.

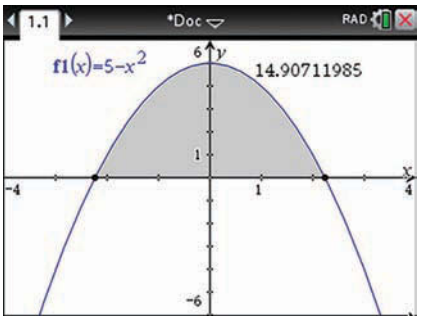
One area that requires further investigation is to note that these graphics calculators are only able to calculate definite integrals in the form $\int_{x=a}^{x=b} f(x)\,dx$. This means that if you wish to calculate the area between $x = 5 - y^2$ and the y-axis, it is necessary to rewrite the curve as $y^2 = 5 - x$ so $y = \pm\sqrt{5-x}$. Both functions would need to be graphed and the area between them calculated. As you can see, the area is approximately 14.91 square units.





Alternatively, the same area could be obtained by determining $\int_{-\sqrt{5}}^{\sqrt{5}} (5 - y^2)\, dy$. To use the calculator to do this, it is necessary to use the inverse relations (effectively swapping the x and y-axes). In this case, instead of graphing $x = 5 - y^2$ as two functions $y = \pm\sqrt{5 - x}$, determine the inverse relation and graph $y = 5 - x^2$. In this instance, the area can be found by determining the area between the function and the x-axis. As demonstrated, the result is an area of 14.91 square units.





The next worked example will be solved using technology.

WORKED EXAMPLE 8 Using technology to determine the area between functions

Use technology to determine the area bounded by the curves $x = y^2 - 2$ and $y = x^3$.

TI | THINK

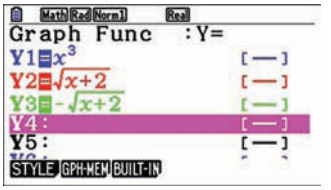
1. On a Calculator page, press MENU then select Add Graphs to: New documents

DISPLAY

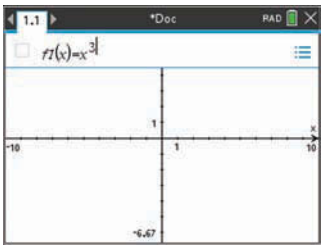
CASIO | THINK

1. On a GRAPH screen, complete the entry line in Y1, Y2 and Y3 as x^3, $\sqrt{x+2}$, $-\sqrt{x+2}$ then press the EXE or DRAW button.

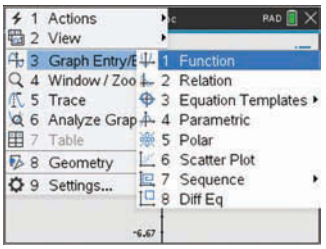
DISPLAY



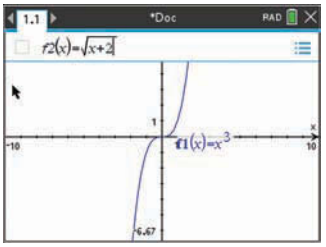
2. Complete the entry line as $f1(x) = x^3$ then press the ENTER button.



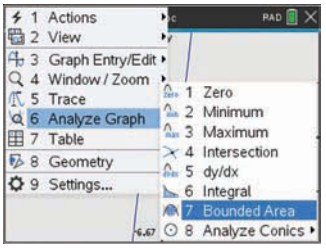
3. Press MENU then select
3: Graph Entry/Edit
1: Function



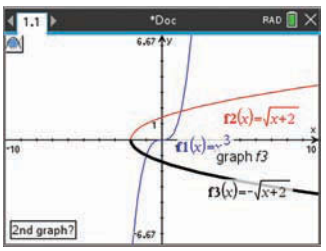
4. Complete the entry line as $\sqrt{x+2}$ then press the ENTER button. Repeat steps 3 and 4 to include $-\sqrt{x+2}$



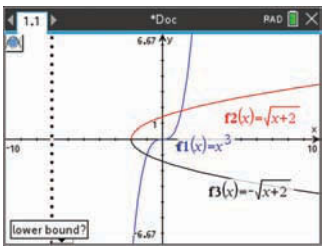
5. Press MENU then select
6: Analyze Graph
7: Bounded Area



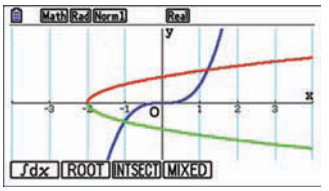
6. Use the directional cursor to select $f2(x)$ and $f3(x)$.



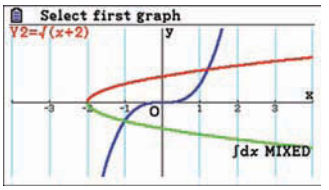
7. Enter the lower boundary by completing the entry line -2 Continue by pressing the ENTER button.



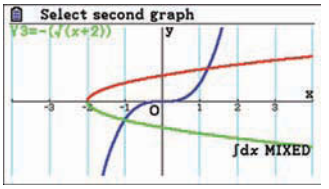
2. To calculate the area bound by Y2 and Y3, select SHIFT F5 $\int dx$ MIXED



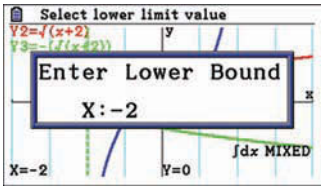
3. Use the directional cursor to select the first graph, Y2. Continue by pressing the EXE button.



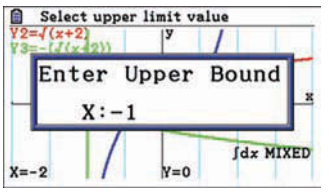
4. Use the directional cursor to select the second graph, Y3. Continue by pressing the EXE button.



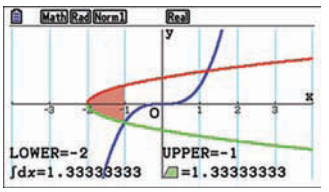
5. Enter the lower boundary by completing the entry line -2 Continue by pressing the EXE button.



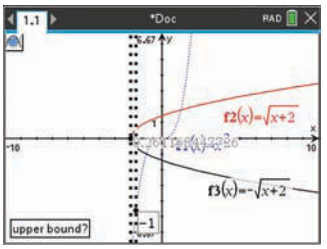
6. Enter the lower boundary by completing the entry line -1 Continue by pressing the EXE button.



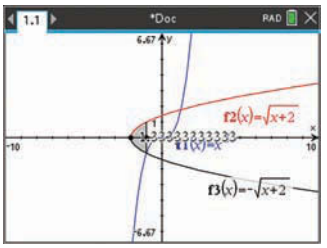
7. The answer appears on the screen.



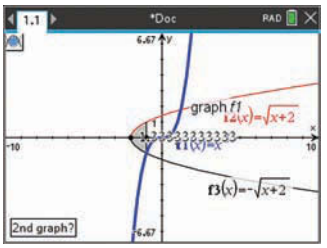
8. Enter the upper boundary by completing the entry line −1 Continue by pressing the ENTER button.



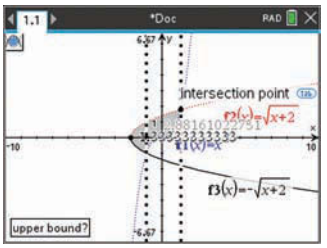
9. The answer appears on the screen.



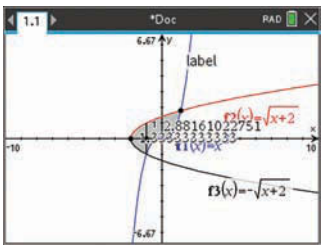
10. Press MENU then select
6: Analyze Graph
7: Bounded Area
Use the directional cursor to select $f2(x)$ $f1(x)$ and.



11. Enter the lower boundary by completing the entry line −1 Continue by pressing the ENTER button. The upper boundary can be selected by using the cursor to locate the intersection point.



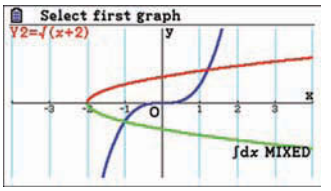
12. The answer appears on the screen.



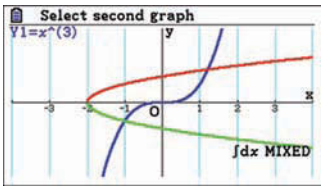
13. Determine the total area and state the solution.

Total area = 1.3333333 + 2.8816102 = 4.21 square units (correct to 2 decimal places).

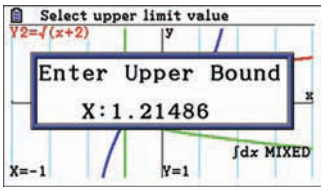
8. To calculate the area bound by Y2 and Y1, use the directional cursor to select the first graph, Y2. Continue by pressing the EXE button.



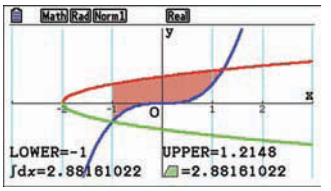
9. Use the directional cursor to select the second graph, Y1. Continue by pressing the EXE button.



10. Enter the lower and upper boundaries by completing the entry line −1 EXE 1.21486 EXE.



11. The answer appears on the screen.



Determine the total area and state the solution.

Total area = 1.3333333 + 2.8816102 = 4.21 square units (correct to 2 decimal places).

When it is necessary to identify points of intersection, but it is difficult or impossible to do so algebraically, technology can be used. Technology can also be used to determine integral calculations in similar circumstances.

Exercise 7.3 Area between functions

learnon

7.3 Exercise	7.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9	10, 11, 12, 13, 14, 15, 16, 17, 18	19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. **a.** WE5 Determine the area between the parabola $y = x^2 - 3x - 18$ and the straight line $y = 4x - 10$.

b. Determine the area between the parabola $y = x^2 - 2x - 8$ and the straight line $y = 1 - 2x$.

c. Determine the area between the parabola $y = x^2 - 2x - 35$ and the straight line $y = 4x - 8$.

d. Determine the area between the parabola $y = x^2 + 5x - 14$ and the straight line $y = 2x + 4$.

2. WE6 Determine the area between $f(x) = \dfrac{9}{x}$ and $g(x) = x$ from $x = 1$ to $x = 4$.

3. Determine the area between the functions.

a. $y = x^2$ and $y = x$

b. $y = x^3$ and $y = x$

c. $y = x^4$ and $y = x$

d. $y = x^5$ and $y = x$

4. Determine the area between the pair of curves below, over the given interval.

a. $y = \sin(x)$, $y = \cos(x)$, $0 \le x \le \pi$

b. $y = (x - 1)^2$, $y = (x + 1)^2$, $-1 \le x \le 1$

c. $y = \dfrac{1}{x}$, $y = 4x$, $\dfrac{1}{4} \le x \le 1$

d. $y = e^x$, $y = -e^x$, $-2 \le x \le 1$

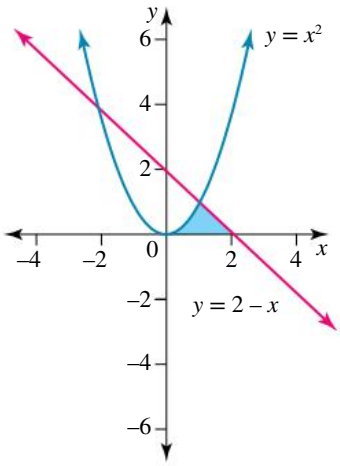
5. Determine the following areas.

a. The area between the line $2y + x - 5 = 0$ and the hyperbola $y = \dfrac{2}{x}$.

b. The area between the line $9y + 3x - 10 = 0$ and the hyperbola $y = \dfrac{1}{3x}$.

6. Determine the area of the region bounded by the curve $y = 4\sqrt{x - 1}$, the curve $y = 4\sqrt{3 - x}$ and the x-axis.

7. Calculate the area of the shaded region.



8. **a.** Use technology to determine the area of the region bounded by the curves $y = 4\ln(3 - x)$, $y = 3\sqrt{x+3}$ and the x-axis.

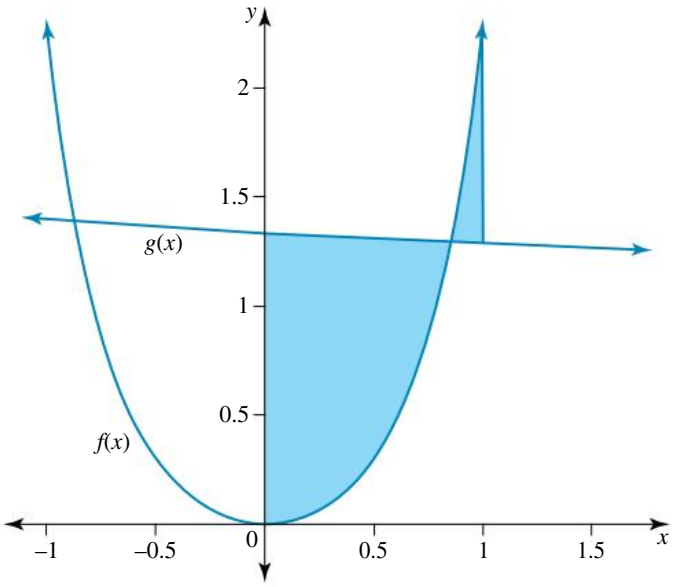
b. If the line $x = c$ divides the region found in part **a** into two regions with equal area, determine the value of c.

9. **a.** Use technology to determine the area of the region bounded by the curve $y = \ln(x)$ the line $y = 6 - x$ and the x-axis.

b. If the line $y = c$ divides the region found in part **a** into two regions with equal area, determine the value of c.

Complex familiar

10. **WE7** Calculate, without technology, the area bounded by the curves $f(x) = \tan^2(x)$ and $g(x) = \dfrac{2}{x+6} + 1$, between the lines $x = 0$ and $x = 1$. Verify your answer using technology.

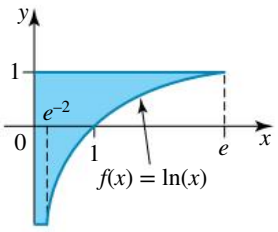


11. **WE8** Use technology to determine the area bounded by the curves $x = y^2 - 3$ and $y = x^3$. Give your answer to 2 decimal places.

12. The graphs $f(x) = \sin^2(x) + 1$ and $g(x) = \sec^2(0.5x)$ intersect at the points $A(0, 1)$ and $B\left(\dfrac{\pi}{2}, 2\right)$. Given $f(x) > g(x)$ between these points of intersection, calculate the exact area bounded by the curves between the lines $x = 0$ and $x = \dfrac{\pi}{2}$.

13. Determine the area bounded by the curves $y = 5\sin^2(2x)$, $y = 5\cos^2(2x)$ and the lines $x = 0$ and $x = \dfrac{\pi}{4}$.

14. A cross-section of a platform is shown (all measurements are in metres).
Determine the cross-sectional area of the platform.



15. Calculate, without technology, the area bounded by the curves $f(x) = x\left(\frac{1}{4}x^2 - 1\right)^3 + 2$ and $g(x) = \dfrac{4}{x^2 + 16}$, between the lines $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$. Verify your answer using technology.

16. Calculate, without technology, the area bounded by the curves $f(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$ and $g(x) = \dfrac{1}{\sqrt{9 - x^2}}$, between the lines $x = 1$ and $x = 2$. Verify your answer using technology.

17. Calculate, without technology, the area bounded by the curves $f(x) = \dfrac{4x}{\sqrt{4x^2 + 5}}$ and $g(x) = \cot^2(x) - 1$, between the lines $x = 0.8$ and $x = 2.3$. Give your answer to 1 decimal place.

18. Calculate the exact area bounded by the curves $f(x) = \cos^2(x) + 2$ and $g(x) = \csc^2(0.5x)$ between the lines $x = 0$ and $x = \dfrac{\pi}{4}$.

Complex unfamiliar

19. We know that the area of a circle with radius r is πr^2. Use the equation $x^2 + y^2 = r^2$ to justify the formula for area of a circle.

20. The area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab. Use integration to justify this formula.

Fully worked solutions for this chapter are available online.

LESSON
7.4 Volumes of solids of revolution

SYLLABUS LINKS

- Determine volumes of solids of revolution about either axis, with and without technology.
  - about the x-axis: $V = \pi \int_a^b [f(x)]^2\, dx$
  - about the y-axis: $V = \pi \int_a^b [f(y)]^2\, dy$

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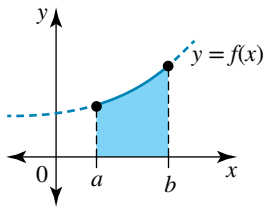
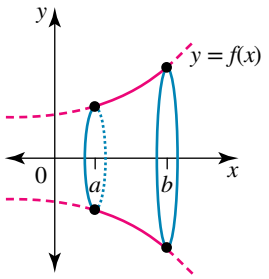
7.4.1 Rotations about the x-axis

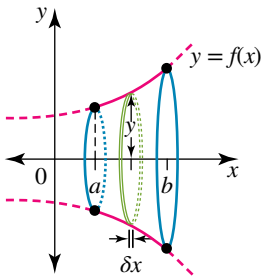
Suppose that a curve $y = f(x)$ is continuous on the closed interval $a \le x \le b$. The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ can be rotated 360° about x-axis resulting in a **solid of revolution**.



The solid generated is symmetrical about the x-axis and any vertical cross-section is a circle with a radius equal to the value of y at that point. For example, the radius of the circle at $x = a$ is $f(a)$.



A thin vertical slice could be considered to be a cylinder, with at radius y and a height δx. The volume of the cylinder is $\pi y^2 \delta x$.



To determine the volume of the solid of revolution, the volumes of all of the cylinders between $x = a$ and $x = b$ are added together and the limit as $\delta x \to 0$ determined.

$$\begin{aligned} V &= \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x \\ &= \int_a^b \pi y^2 \, dx \\ &= \pi \int_a^b \left[f(x) \right]^2 dx \end{aligned}$$

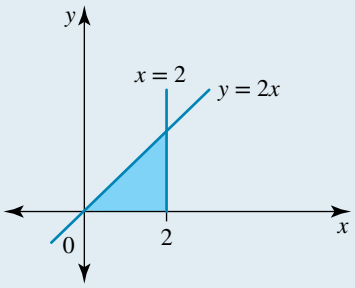
The volume of a solid of revolution about the x-axis

If a region R bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ is rotated about the x-axis, then the volume of the solid of revolution is given by $V = \pi \int_a^b [f(x)]^2 dx$

WORKED EXAMPLE 9 Determining the volume of a solid of revolution formed when a region is rotated about the x-axis

a. Sketch the graph of $y = 2x$ and identify the region bounded by the graph, the x-axis and the line $x = 2$.

b. Calculate the volume of the solid of revolution when the region is rotated about the x-axis.

THINK	WRITE
a. Sketch the lines $y = 2x$ and $x = 2$ and shade the area bounded by $y = 2x$, $x = 2$ and the x-axis.	**a.** 
b. 1. The volume generated is bounded by $x = 0$ and $x = 2$. The formula for volume of a solid formed by rotation about the x-axis is $\pi\int_b^a [f(x)]^2\,dx$.	**b.** $V = \pi\int_0^2 [f(x)]^2\,dx$
2. The integrand needs to be expressed in terms of x. Substitute $f(x) = 2x$.	$= \pi\int_0^2 (2x)^2\,dx$
3. Simplify the integrand.	$= 4\pi\int_0^2 x^2\,dx$
4. Determine the anti-derivative.	$= 4\pi\left[\frac{1}{3}x^3\right]_0^2$ $= 4\pi\left[\frac{1}{3}(2)^3 - \frac{1}{3}(0)^3\right]$
5. Evaluate the integral.	$= 4\pi\left[\frac{8}{3} - 0\right]$ $= \frac{32\pi}{3}$
6. State the volume.	The volume is $\frac{32\pi}{3}$ cubic units.

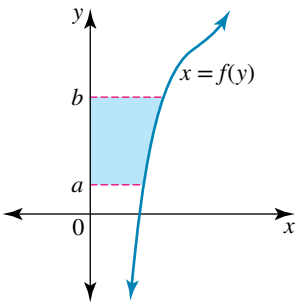
In Worked example 9, the solid generated is a cone. The height of the cone is 2 units and the radius of the base is 4 units. Using the formula for volume of a cone:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\,(4)^2\,(2) \\ &= \frac{32\pi}{3} \text{ cubic units} \end{aligned}$$

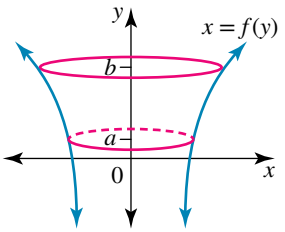
This is the same result as determined in the worked example.

7.4.2 Rotations about the y-axis

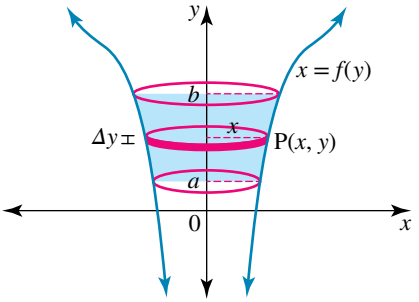
Curves can also be rotated about the y-axis. In this instance, the region is bounded by the curve $x = f(y)$, the lines $y = a$, $y = b$ and the y-axis.



The solid generated is symmetrical about the y-axis and any horizontal cross-section is a circle with a radius equal to the value of x at that point. For example, the radius of the circle at $y = a$ is $f(a)$.



A thin horizontal slice could be considered to be a cylinder, with at radius x and a height δy. The volume of the cylinder is $\pi x^2 \delta y$.



To determine the volume of the solid of revolution, the volumes of all of the cylinders between $y = a$ and $y = b$ are added together and the limit as $\delta y \to 0$ determined.

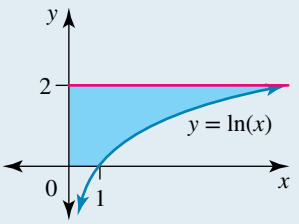
$$\begin{aligned} V &= \lim_{\delta y \to 0} \sum_{y=a}^{y=b} \pi x^2 \, \delta y \\ &= \int_a^b \pi x^2 \, dy \\ &= \pi \int_a^b \left[f(y) \right]^2 dy \end{aligned}$$

The volume of a solid of revolution about the y-axis

If a region R bounded by $x = f(y)$, $x = 0$, $y = a$ and $y = b$ is rotated about the y-axis, then the volume of the solid of revolution is given by $V = \pi \int_a^b [f(y)]^2 \, dy$

WORKED EXAMPLE 10 Determining the volume of a solid of revolution formed when a region is rotated about the y-axis

a. Sketch the region bounded by the curve $y = \ln(x)$, the x-axis, the y-axis and the line $y = 2$.
b. Calculate the volume of the solid of revolution formed when the region is rotated about the y-axis.

THINK	WRITE
a. Sketch the curve $y = \ln(x)$ and the line $y = 2$. Shade the area bounded by $y = \ln(x)$, the x-axis, the y-axis and the line $y = 2$.	a. 
b. 1. The volume generated is bounded by $y = 0$ and $y = 2$. The formula for volume of a solid formed by rotation about the y-axis is $\pi \int_a^b [f(y)]^2 \, dy$.	b. $V = \pi \int_0^2 [f(y)]^2 dy$
2. The integrand needs to be expressed in terms of y. Rearrange $y = \ln(x)$ to make x the subject.	$y = \ln(x)$ $x = e^y$ $f(y) = e^y$
3. Substitute $f(y) = e^y$ into the formula for volume.	$V = \pi \int_0^2 (e^y)^2 \, dy$
4. Simplify the integrand.	$= \pi \int_0^2 e^{2y} \, dy$
5. Determine the anti-derivative.	$= \pi \left[\frac{1}{2} e^{2y} \right]_0^2$
6. Evaluate the integral.	$= \pi \left[\frac{1}{2} e^4 - \frac{1}{2} e^0 \right]$ $= \frac{\pi}{2}(e^4 - 1)$
7. State the volume.	The volume is $\frac{\pi}{2}(e^4 - 1) \approx 84.19$ cubic units.

7.4.3 Applications

The volume of some common geometric shapes can now be found using calculus, by rotation lines or curves about the x- or y-axis and using the techniques described.

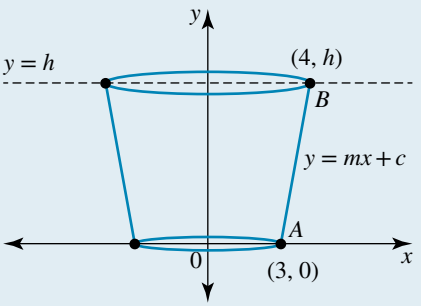
WORKED EXAMPLE 11 Determining the capacity of a glass

A drinking glass has a base diameter of 5 cm, a top diameter of 7 cm and a height of 11 cm. Determine the capacity of the glass to the nearest mL.

THINK

1. Sketch the graph and identify the area to rotate. Write the coordinates of the points A and B.

WRITE



2. Determine the gradient of the line segment AB joining the points $A\left(\frac{5}{2}, 0\right)$ and $B\left(\frac{7}{2}, 11\right)$.

$$m = \frac{11-0}{\frac{7}{2}-\frac{5}{2}} = 11$$

3. Use $y - y_1 = m(x - x_1)$ to determine the equation of the line segment AB.

$$y - 0 = 11\left(x - \frac{5}{2}\right)$$

$$y = 11x - \frac{55}{2}$$

4. The glass is formed by rotating AB about the y-axis. The formula for rotation about the y-axis is $V = \pi\int_a^b x^2\,dy$. We are interested in the region between $y = 0$ and $y = 11$.

$$V = \pi\int_a^b [f(y)]^2\,dy$$

$$= \pi\int_0^{11} x^2\,dy$$

5. The integrand needs to be expressed in terms of y. Rearrange $y = 11x - \frac{55}{2}$ to make x the subject.

$$y = 11x - \frac{55}{2}$$
$$y + \frac{55}{2} = 11x$$
$$x = \frac{1}{11}y + \frac{5}{2}$$
$$f(y) = \frac{1}{11}y + \frac{5}{2}$$

6. Substitute the equation for x into the formula for volume so that the integrand is expressed in terms of y only.

$$V = \pi\int_0^{11} [f(y)]^2\, dy$$
$$= \pi\int_0^{11} \left(\frac{1}{11}y + \frac{5}{2}\right)^2 dy$$

7. Determine the anti-derivative.

$$= \pi\left[\frac{1}{3}\left(\frac{1}{11}y + \frac{5}{2}\right)^3 \times 11\right]_0^{11}$$
$$= \frac{11\pi}{3}\left[\left(\frac{1}{11}y + \frac{5}{2}\right)^3\right]_0^{11}$$

8. Evaluate the integral.

$$= \frac{11\pi}{3}\left[\left(1 + \frac{5}{2}\right)^3 - \left(0 + \frac{5}{2}\right)^3\right]$$
$$= \frac{11\pi}{3}\left(\left(\frac{7}{2}\right)^3 - \left(\frac{5}{2}\right)^3\right)$$
$$= \frac{11\pi}{3} \times \frac{109}{4}$$
$$= \frac{1199\pi}{12}$$
$$\approx 314$$

9. State the volume in cm^3. Use $1\ cm^3 = 1$ mL to state the volume of the glass to the nearest mL.

The volume is 314 cm^3. This means that the glass can hold 314 mL.

Exercise 7.4 Volumes of solids of revolution

learnon

7.4 Exercise	7.4 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	15, 16, 17, 18, 19	20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

1. a. WE9 Sketch the graph of $y=\sqrt{x}$ and identify the region bounded by the graph, the x-axis and the line $x=4$.
   b. Calculate the volume of the solid of revolution when the region is rotated about the x-axis.

2. The area bounded by the curve $y=4-x^2$ and the coordinate axis is rotated about the x-axis to form a solid of revolution. Determine the volume of the solid formed.

3. a. WE10 Sketch the region bounded by the curve $y=\sqrt{x}$, the y-axis, the origin and the line $y=2$.
   b. Calculate the volume of the solid of revolution formed when the region is rotated about the y-axis.

4. The area bounded by the curve $y=4-x^2$ and the coordinate axes is rotated about the y-axis to form a solid of revolution. Calculate the volume of the solid.

5. Consider the line $y=3x$.
   a. Determine the volume of the solid of revolution formed by rotating the section of the line between the origin and $x=5$ about the x-axis.
   b. Determine the volume of the solid of revolution formed by rotating the section of the line between the origin and $y=5$ about the y-axis.

6. A cone is formed by rotating the line segment of $2x+3y=6$ cut off by the coordinate axes about:
   a. the x-axis
   b. the y-axis.
   Determine the volume of the solid formed in each case.

7. If the region bounded by the curve $y=\sec(2x)$, the co-ordinate axes and $x=\frac{\pi}{8}$ is rotated 360° about the x-axis, determine the volume of the solid formed.

8. If the region bounded by the curve $y=2e^{\frac{x}{2}}$, the coordinate axes and $x=2$ is rotated 360° about the x-axis, determine the volume of the solid formed.

9. WE11 A plastic bucket has a base diameter of 20 cm, a top diameter of 26 cm and a height of 24 cm. Determine the capacity of the bucket to the nearest litre.

10. Calculate the exact volumes of the solids generated by revolving the regions bounded by the lines and curves about the x-axis.

a. $y = 4\sqrt{x}$, $0 \le x \le 3$, correct to 4 decimal places.

b. $y = \dfrac{1}{\sqrt{16+x}}$, $0 \le x \le 3$, correct to 3 decimal places.

c. $y = \sqrt{\cos(2x)}, 0 \le x \le \dfrac{\pi}{6}$, correct to 3 decimal places.

11. Calculate the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x-axis.

a. $y = \dfrac{e^{4x}}{\sqrt{1+e^{8x}}}$, $0 \le x \le 1$

b. $y = \sqrt{36 - x^2}, 1 \le x \le 2$

c. $y = \dfrac{x}{8}\sqrt{25 - x^2}$, $0 \le x \le 5$

12. Consider the curve $y = 3x^2 + 4$.

a. Determine the volume of the solid formed by rotating the area between the curve, the coordinate axes and the line $x = 2$ about the x-axis.

b. Determine the volume of the solid formed by rotating the area between the curve, $y = 4$ and $y = 10$ about the y-axis.

13. Consider the area between the curve $y = x^2 - 9$ and the x-axis.

a. If the area described is rotated 360° about the x-axis, determine the volume of the solid formed.

b. If the area described is rotated 360° about the y-axis, determine the volume of the solid formed.

14. Calculate the exact volumes of the solids formed when the following regions are rotated around the y-axis.

a. $y = 3 - \dfrac{x^2}{5}$ for $0 \le y \le 3$

b. $y = 4\sin^{-1} x$ for $0 \le y \le 2\pi$

Complex familiar

15. Calculate the volume of the solid formed by rotating $y = 100\ln\left(\dfrac{x-10}{20}\right)$, $0 \le y \le 50$ around the y-axis.

16. **a.** Determine the volume of the solid formed by rotating the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ about the x-axis.

b. Determine the volume of the solid formed by rotating the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ about the y-axis.

c. Determine the volume of the solid formed by rotating the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the x-axis.

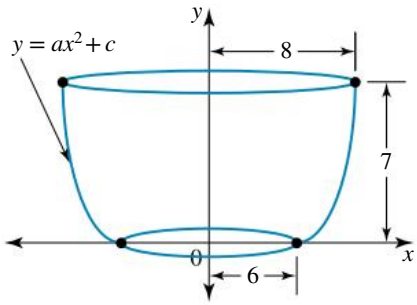
d. Determine the volume of the solid formed by rotating the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the y-axis.

e. An egg can be regarded as an ellipsoid (formed by rotating an ellipse). If the egg has a total length of 57 mm and its diameter at the centre is 44 mm, determine an equation in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ that can be rotated to form the required ellipsoid.

f. Use the equation from **e** to determine the volume of the egg in cubic millimetres.

17. A travel mug has a base diameter of 6 cm and a top diameter of 8 cm and a height of h cm. Determine its minimum height, to 1 decimal place, such that its capacity is greater than 500 cm^3.

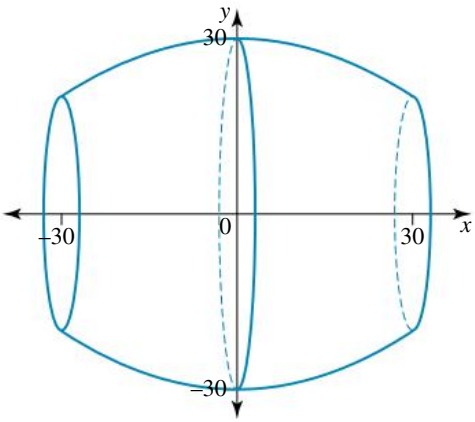
18. A soup bowl has a base radius of 6 cm, a top radius of 8 cm and a height of 7 cm. The edge of the bowl is a parabola of the form $y = ax^2 + c$. Determine the capacity of the soup bowl to the nearest mL.



19. The diagram below shows a wine barrel. The barrel has a total length of 60 cm, a total height of 60 cm at the middle and a total height of 40 cm at the ends.

a. If the upper arc can be represented by a parabola, demonstrate that its equation is given by $y = -\frac{x^2}{90} + 30$ for $-30 \le x \le 30$.

b. If the arc is rotated about the x-axis, determine the capacity of the wine barrel to the nearest litre.



Complex unfamiliar

20. A steel monument was constructed by revolving the curve $y = \dfrac{5x^2 - 30x + 1}{x - 6}$ about the x-axis, with the bounds $x = 0.8$ and $x = 5.7$. All measurements are in centimetres. The manufacturer claims that 4460 cm^3 of steel used to construct the monument. Using algebraic methods, calculate the volume of the monument to 2 decimal places. Then, assess whether the manufacturer's claim is reasonable.

Fully worked solutions for this chapter are available online.

LESSON
7.5 Volumes of revolution

SYLLABUS LINKS

- Determine volumes of solids of revolution about either axis, with and without technology.
  - about the x-axis: $V = \pi \int_a^b [f(x)]^2 \, dx$
  - about the y-axis: $V = \pi \int_a^b [f(y)]^2 \, dy$

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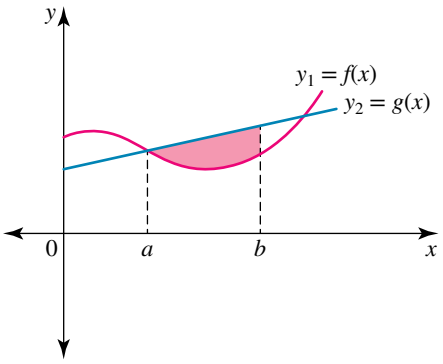
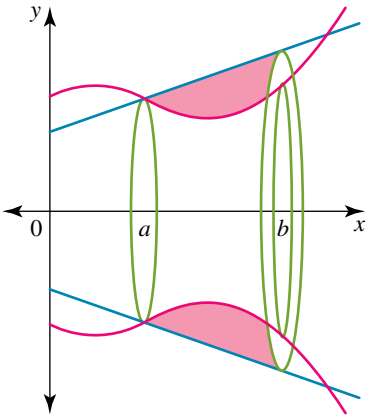
7.5.1 Rotations about the x-axis

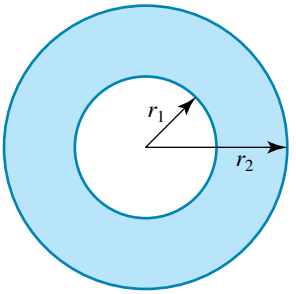
Suppose that there are two curves $y_1 = f(x)$ and $y_2 = g(x)$ that are continuous over the closed interval $a \le x \le b$ and over the interval, $g(x) \ge f(x)$.



If the area bounded by the functions and the lines $x = a$ and $x = b$ is rotated about the x-axis, the resulting object will be hollow in the middle.

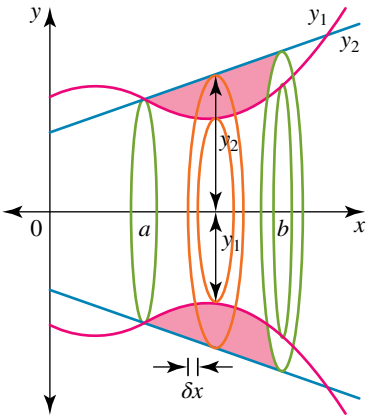


A cross-section of the shape would look like a circular washer. The inner radius, r_1, would be equal to $y_1 = f(x)$ and the outer radius, r_2, would be equal to $y_2 = g(x)$.



The area of this cross-section can be found be determining $\pi(r_2^2 - r_1^2)$.

The volume of the thin vertical slice of height δx would have a volume of $\pi(y_2^2 - y_1^2)\,\delta x$.



The volume of the solid of revolution can be found by adding the volume of the thin circular washers between $x = a$ and $x = b$ and finding the limit as $\delta x \to 0$.

$$\begin{aligned} V &= \sum_{x=a}^{x=b} \pi \left(y_2^2 - y_1^2\right) \delta x \\ &= \int_a^b \pi \left(y_2^2 - y_1^2\right) dx \\ &= \pi \int_a^b \left(\left[g(x)\right]^2 - \left[f(x)\right]^2 \right) dx \end{aligned}$$

WORKED EXAMPLE 12 Determining the volume formed by an area bounded by a curve and a horizontal line is rotated about the axes

Determine the volume formed when the area bounded by the curve $y = 4 - x^2$ and the line $y = 3$ is rotated about:

a. the x-axis **b. the y-axis.**

THINK

a. 1. Determine the point of intersection between the curve and the line.

2. Sketch the graph and identify the area to rotate about the x-axis.

3. The volume can be found using $V = \pi\int_a^b \left(\left[g(x)\right]^2 - \left[f(x)\right]^2\right) dx$. $a = -1$ and $b = 1$.

4. The integrand needs to be expressed in terms of x. In the shaded area, $4 - x^2 \geq 3$, therefore $y_2 = 4 - x^2$ and $y_1 = 3$.

5. Simplify the integrand so that it can be determined.

6. Determine the anti-derivative.

WRITE

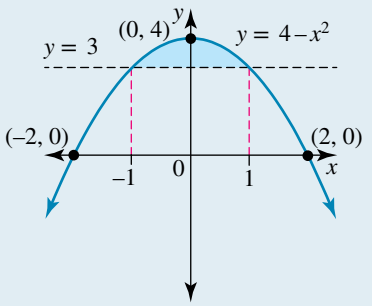
a. Let $y_1 = 4 - x^2$ and $y_2 = 3$

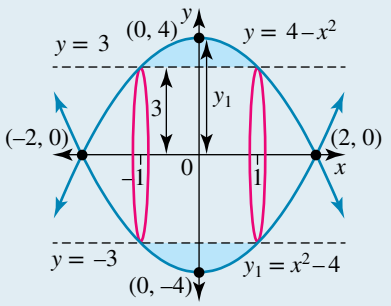
$y_1 = y_2$.

$4 - x^2 = 3$

$x^2 = 1$

$x = \pm 1$





$$V = \pi\int_{-1}^{1} \left(\left[g(x)\right]^2 - \left[f(x)\right]^2\right) dx$$

$$= \pi\int_{-1}^{1} \left(\left(4 - x^2\right)^2 - 3^2\right) dx$$

$$= \pi\int_{-1}^{1} (16 - 8x^2 + x^4 - 9)\, dx$$

$$= \pi\int_{-1}^{1} (7 - 8x^2 + x^4)\, dx$$

$$= \pi\left[7x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1}$$

7. Evaluate the integral.

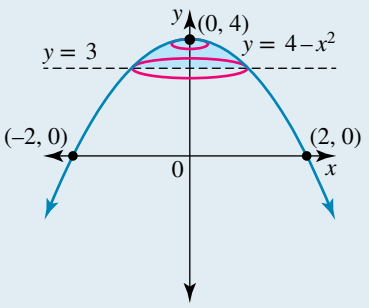
$$= \pi\left[\left(7-\frac{8}{3}+\frac{1}{5}\right)-\left(-7+\frac{8}{3}-\frac{1}{5}\right)\right]$$
$$= 2\pi\left(7-\frac{40}{15}+\frac{3}{15}\right)$$
$$= \frac{136\pi}{15}$$

8. State the value of the volume.

The volume is $\frac{136\pi}{15}$ cubic units.

b. 1. Identify the region to rotate about the y-axis.

b.



2. The volume created will be completely solid. The region is bounded by $y = 3$ and $y = 4$. The formula for volume of a solid formed by rotation about the y-axis is $\pi\int_b^a [f(y)]^2\,dy$.

$$V = \pi\int_3^4 [f(y)]^2\,dy$$

3. The integrand needs to be expressed in terms of y. Rearrange $y = 4 - x^2$ to make x^2 the subject.

$$y = 4 - x^2$$
$$x^2 = 4 - y$$
$$[f(y)]^2 = 4 - y$$

4. Substitute $[f(y)]^2 = 4 - y$ into the volume formula.

$$V = \pi\int_3^4 (4-y)\,dy$$

5. Determine the anti-derivative.

$$= \pi\left[4y - \frac{1}{2}y^2\right]_3^4$$

6. Evaluate the integral.

$$= \pi\left[\left(4\times4-\frac{1}{2}\times4^2\right)-\left(4\times3-\frac{1}{2}\times3^2\right)\right]$$
$$= \pi\left(16-8-12+\frac{9}{2}\right)$$
$$= \frac{\pi}{2}$$

7. State the volume.

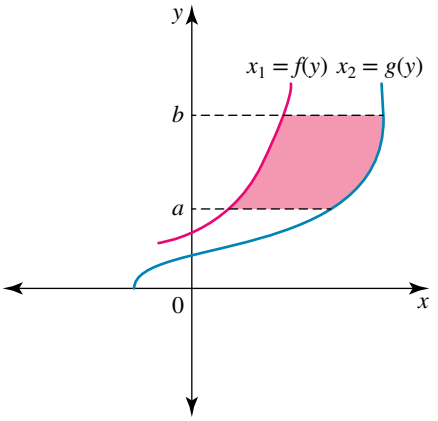
The volume is $\frac{\pi}{2}$ cubic units.

7.5.2 Rotations about the y-axis

It is also possible to rotate two curves about the y-axis. If $x_1 = f(y)$ and $x_2 = g(y)$ and $g(y) \geq f(y)$ over the interval $a \leq y \leq b$.

The solid formed will be hollow in the middle. The volume of the solid formed will be

$$V = \int_a^b \pi \left(x_2^2 - x_1^2\right) dy$$
$$= \pi \int_a^b \left(\left[g(y)\right]^2 - \left[f(x)\right]^2\right) dy$$



WORKED EXAMPLE 13 Determining the volume formed by an area bounded by a curve and the x-axis is rotated about the axes

Determine the volume formed when the area bounded by the curve $y = x^2 - 4$, the x-axis and the line $x = 3$ is rotated about:

a. the y-axis **b. the x-axis.**

THINK

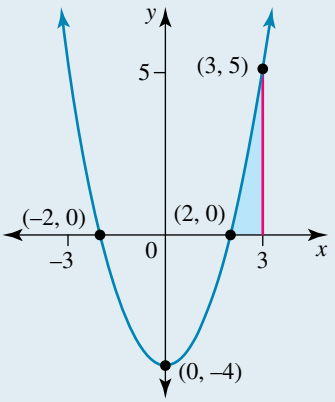
a. 1. Determine the point of intersection between the curve and the line.

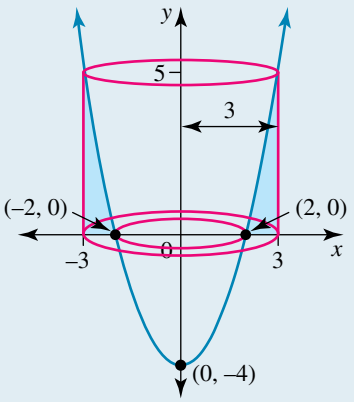
2. Sketch the graph and identify the area to rotate about the y-axis.

WRITE

a. $y = x^2 - 4$ and $x = 3$.

$y = (3)^2 - 4$
$= 9 - 4$
$= 5$





3. The volume can be found using $V=\pi\int_a^b\left(\left[g(y)\right]^2-\left[f(y)\right]^2\right)dy$. $a=0$ and $b=5$.

$$V=\pi\int_0^5\left(\left[g(y)\right]^2-\left[f(y)\right]^2\right)dy$$

4. Rearrange $y=x^2-4$ to make x^2 the subject.

$$y=x^2-4$$
$$x^2=y+4$$

5. The integrand needs to be expressed in terms of y. In the shaded area, $3\geq\sqrt{y+4}$, therefore $g(y)=3$ and $f(y)=\sqrt{y+4}$.

$$V=\pi\int_0^5\left(3^2-(y+4)\right)dy$$
$$V=\pi\int_0^5(9-(y+4))\,dy$$

6. Simplify the integrand so that it can be determined.

$$=\pi\int_0^5(5-y)\,dy$$

7. Determine the anti-derivative.

$$=\pi\left[5y-\frac{1}{2}y^2\right]_0^5$$

8. Evaluate the integral.

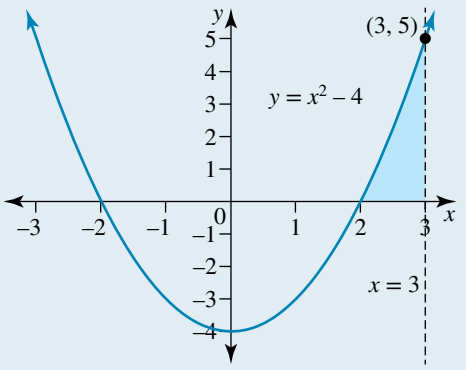
$$=\pi\left[\left(25-\frac{1}{2}\times5^2\right)-0\right]$$
$$=\frac{25\pi}{2}$$

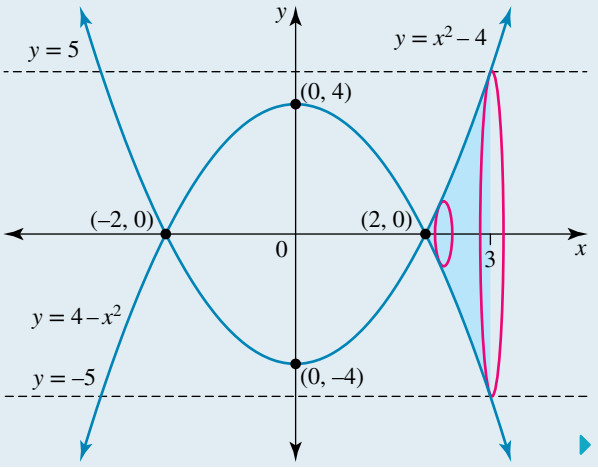
9. State the volume.

The volume is $\frac{25\pi}{2}$ cubic units.

b. 1. Identify the region to rotate about the x-axis.

b.





2. The volume created will be completely solid. The region is bounded by $x = 2$ and $x = 3$. The formula for volume of a solid formed by rotation about the x-axis is $\int_b^a \pi y^2\, dx$.

$$V = \pi \int_2^3 [f(x)]^2\, dx$$

3. Substitute $f(x) = x^2 - 4$ into the integrand so that it is expressed entirely in terms of x.

$$= \pi \int_2^3 (x^2 - 4)^2\, dx$$

4. Simplify the integrand so that it can be determined.

$$= \pi \int_2^3 (x^4 - 8x^2 + 16)\, dx$$

5. Determine the anti-derivative.

$$= \pi \left[\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x\right]_2^3$$

6. Evaluate the integral.

$$= \pi \left[\left(\frac{1}{5} \times 3^5 - \frac{8}{3} \times 3^3 + 16 \times 3\right) - \left(\frac{1}{5} \times 2^5 - \frac{8}{3} \times 2^3 + 16 \times 2\right)\right]$$

$$= \pi \left(\frac{243}{5} - 72 + 48 - \frac{32}{5} + \frac{64}{3} - 32\right)$$

$$= \frac{113\pi}{15}$$

7. State the volume.

The volume is $\frac{113\pi}{15}$ cubic units.

7.5.3 Composite figures

Volumes of revolution

If $g(x) \geq f(x)$ over the interval $a \leq x \leq b$, then the volume of the shape formed by rotating the region about the x-axis is $V = \pi \int_a^b \left([g(x)]^2 - [f(x)]^2\right) dx$.

If $g(y) \geq f(y)$ over the interval $a \leq y \leq b$, then the volume of the shape formed by rotating the region about the y-axis is $V = \pi \int_a^b \left([g(y)]^2 - [f(y)]^2\right) dy$.

It may be necessary to identify a number of smaller regions so that within each region the functions do not intersect.

WORKED EXAMPLE 14 Determining the volume of a composite figure

Determine the volume formed when the area between the curves $y = x^2$ and $y = 8 - x^2$ is rotated about:

a. the x-axis **b. the y-axis.**

THINK

a. 1. Determine the point of intersection between the two curves.

2. Identify the region to be rotated about the x-axis.

3. Over the region $-2 \le x \le 2$, $8 - x^2 \ge x^2$, therefore only one integral is required. $a = -2,\ b = 2$ and

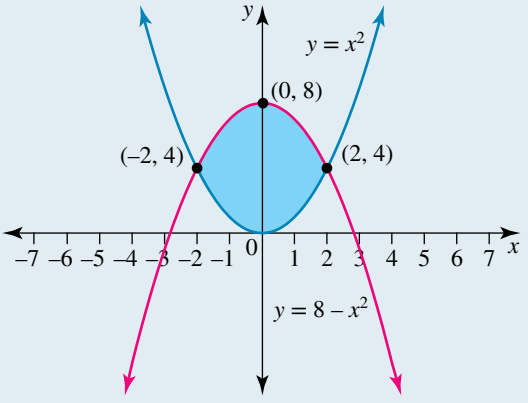
$$V = \pi \int_a^b \left([g(x)]^2 - [f(x)]^2 \right) dx.$$

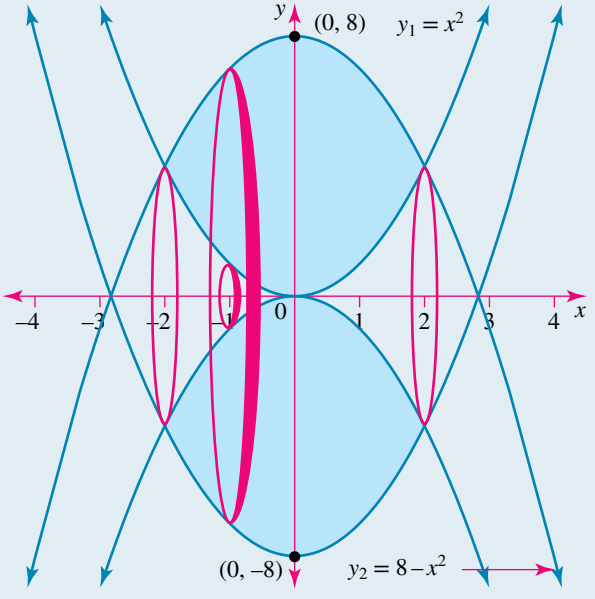
WRITE

a. Let $y_1 = x^2$ and $y_2 = 8 - x^2$.

$$y_1 = y_2$$
$$x^2 = 8 - x^2$$
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

When $x = \pm 2, y = 4$.





$$V = \pi \int_{-2}^{2} \left([g(x)]^2 - [f(x)]^2 \right) dx$$

4. As $8 - x^2 \geq x^2$, $g(x) = 8 - x^2$ and $f(x) = x^2$. Substitute in the integrand.

$$= \pi \int_{-2}^{2} \left(\left(8 - x^2\right)^2 - \left(x^2\right)^2 \right) dx$$

5. Simplify the integrand so that it can be determined.

$$= \pi \int_{-2}^{2} (64 - 16x^2 + x^4 - x^4)\, dx$$

$$= 16\pi \int_{-2}^{2} (4 - x^2)\, dx$$

6. Determine the anti-derivative.

$$= 16\pi \left[4x - \frac{1}{3}x^3 \right]_{-2}^{2}$$

7. Evaluate the integral.

$$= 16\pi \left[\left(4 \times 2 - \frac{1}{3} \times 8 \right) - \left(4 \times -2 - \frac{1}{3} \times -8 \right) \right]$$

$$= 16\pi \times 2 \left(8 - \frac{8}{3} \right)$$

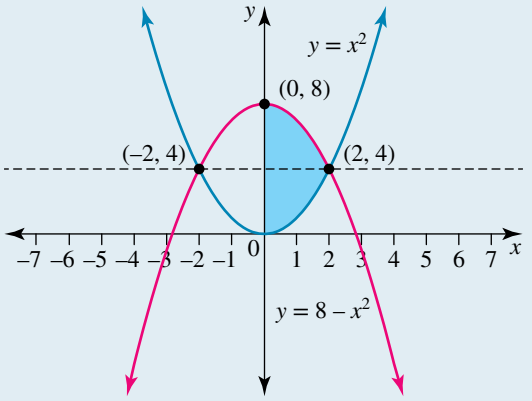
$$= \frac{512\pi}{3}$$

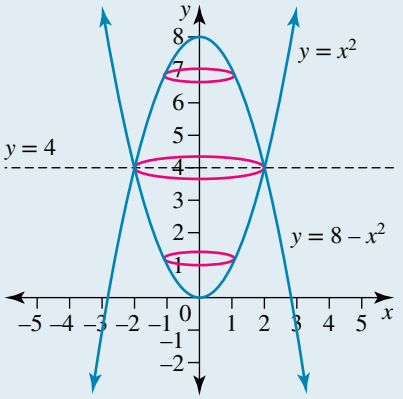
8. State the volume.

The volume is $\frac{512\pi}{3}$ cubic units.

b. 1. Sketch the graph and identify the area to rotate about the y-axis.

b.





2. For $0 \leq y \leq 4$ the region is bounded by the curve $y_1 = x^2$. For $4 \leq y \leq 8$, the region is bounded by the curve $y_2 = 8 - x^2$. In both cases, the volume can be determined using $\pi \int_a^b [f(y)]^2\, dy$.

$$V = \pi \int_0^4 [f(y)]^2\, dy + \pi \int_4^8 [g(y)]^2\, dy$$

3. The integrand needs to be expressed in terms of y.
Rearrange y_1 and y_2 for x^2 so they can be easily substituted into the formula.
$y_1 = x^2 \leftrightarrow x^2 = y_1$
$y_2 = 8 - x^2 \Rightarrow x^2 = 8 - y_2$

$$= \pi \int_0^4 y\,dy + \pi \int_4^8 (8 - y)\,dy$$

4. Determine the anti-derivative.

$$= \pi \left[\frac{1}{2}y^2\right]_0^4 + \pi \left[8y - \frac{1}{2}y^2\right]_4^8$$

5. Evaluate the integral.

$$= \pi \left(\frac{1}{2} \times 4^2 - 0\right) + \pi \left[\left(8 \times 8 - \frac{1}{2} \times 8^2\right) - \left(8 \times 4 - \frac{1}{2} \times 4^2\right)\right]$$
$$= 8\pi + \pi(64 - 32 - 32 + 8)$$
$$= 16\pi$$

6. State the value of the volume.

The volume is 16π cubic units.

Exercise 7.5 Volumes of revolution

learnon

7.5 Exercise	7.5 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	12, 13, 14, 15, 16, 17, 18	19, 20

These questions are even better in jacPLUS!
- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

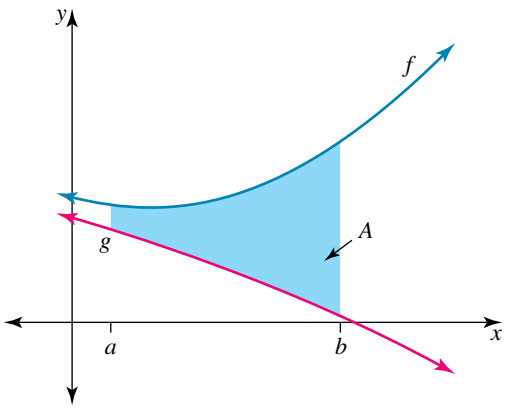
1. MC The volume formed when the area A is rotated about the x-axis is given by:

A. $\pi \int_a^b \left([g(x)]^2 - [f(x)]^2\right) dx$

B. $\pi \int_a^b \left([g(y)]^2 - [f(y)]^2\right) dy$

C. $\pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$

D. $\pi \int_a^b \left([f(y)]^2 - [g(y)]^2\right) dy$



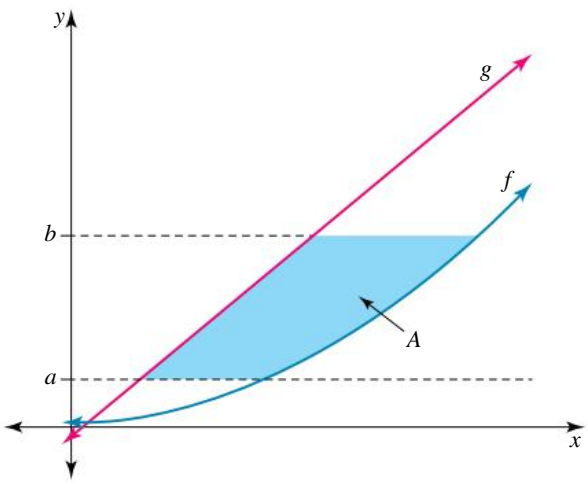
2. **MC** The volume formed when the area A is rotated about the y-axis is given by:
The volume formed when the area A is rotated about the y-axis is given by:

A. $\pi\int_a^b \left([g(x)]^2 - [f(x)]^2\right) dx$

B. $\pi\int_a^b \left([g(y)]^2 - [f(y)]^2\right) dy$

C. $\pi\int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$

D. $\pi\int_a^b \left([f(y)]^2 - [g(y)]^2\right) dy$



3. **WE12** Determine the volume formed when the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$ is rotated about:

a. the x-axis **b.** the y-axis.

4. Determine the volume formed when the area bounded by the curve $y = \sqrt{x}$ and the line $y = 2$ is rotated about:

a. the x-axis **b.** the y-axis.

5. **WE13** Determine the volume formed when the area bounded by the curve $y = x^2 - 9$, the x-axis and the line $x = 4$ is rotated about:

a. the y-axis **b.** the x-axis.

6. Determine the volume formed when the area bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ is rotated about:

a. the y-axis **b.** the x-axis.

7. Consider the area bounded by the curve $y = 16 - x^2$ and the line $y = 12$.

a. If the area is rotated about the x-axis, determine the volume of the object created.
b. If the area is rotated about the y-axis, determine the volume of the object created.

8. Consider the area bounded by the curve $y = x^2 - 16$, the x-axis and the line $x = 5$.

a. If the area is rotated about the y-axis, determine the volume of the object created.
b. If the area is rotated about the x-axis, determine the volume of the object created.

9. Let R be the region bounded by the curve $y = x^3$, the x-axis and the line $x = 3$.

a. If the region R is rotated about the y-axis, determine the volume of the object formed.
b. If the region R is rotated about the x-axis, determine the volume of the object formed.

10. The area between the curve $y = \sqrt{2x - 8}$, the x-axis and the line $x = 6$ is rotated about:

a. the y-axis **b.** the x-axis.

Determine the volume of the resulting solid of revolution in each case.

11. **WE14** Determine the volume formed when the area between the curves $y = 2x$ and $y = x^2$ is rotated about:

a. the x-axis **b.** the y-axis.

Complex familiar

12. **a.** Determine the volume formed when the area bounded by the curve $y = 4\cos^{-1}\left(\frac{x}{2}\right)$ and the lines $x = 2$ and $y = 2\pi$ is rotated about the y-axis.

b. Determine the volume formed when the area bounded by the curve $y = 2\sin^{-1}\left(\frac{x}{3}\right)$, the x-axis and the line $x = 3$ is rotated about the y-axis.

13. Determine the volume formed when the area between the curves $y = x^2$ and $y = x^3$ is rotated about:

a. the x-axis **b.** the y-axis.

14. **a.** If the area between two curves $y = 2\sqrt{x}$ and $y = \frac{x^2}{4}$ is rotated about the x-axis, determine the volume formed.

b. If the area between two curves $y = \sqrt{ax}$ and $y = \frac{x^2}{a}$ is rotated about the x-axis, determine the volume formed.

15. Determine the volume formed when the area between the curves $y = x^2$ and $y = 18 - x^2$ is rotated about:

a. the x-axis **b.** the y-axis.

16. Determine the volume obtained when the area between the curves $y = x^2 - 4x + 4$ and $y = 4 + 4x - x^2$ is rotated about the x-axis.

17. Determine the volume obtained when the area between the curves $y = 8\sin\left(\frac{\pi x}{4}\right)$ and $y = 2x^2$ is rotated about the x-axis.

18. **a.** A cylindrical hole of radius 2 cm is cut vertically through the centre of a solid sphere of cheese radius 4 cm. Determine the volume of cheese remaining.

b. A cylindrical hole of radius $\frac{a}{2}$ is cut vertically through the centre of a solid sphere of radius a. Determine the volume remaining.

Complex unfamiliar

19. **a.** Determine the volume of a torus (a doughnut shaped figure) formed when the area bounded by the circle $x^2 + (y - 8)^2 = 16$ is rotated about the x-axis.

b. Determine the volume of a torus (a doughnut shaped figure) formed when the area bounded by the circle $(x - a)^2 + y^2 = r^2$, $a > r > 0$, is rotated about the y-axis.

20. Determine the volume formed when the area bounded by the curve $y = a^2 - x^2$ and the line $y = \frac{a^2}{4}$, where $a > 0$, is rotated about

a. the x-axis **b.** the y-axis.

Fully worked solutions for this chapter are available online.

LESSON
7.6 Approximation using Simpson's rule

SYLLABUS LINKS

- Use Simpson's rule to approximate an area and the value of a definite integral, with and without technology.
- $\int_a^b f(x)\,dx \approx \frac{w}{3}\left[f(x_0) + 4\left[f(x_1) + f(x_3) + \cdots\right] + 2\left[f(x_2) + f(x_4) + \cdots\right] + f(x_n)\right]$ where $w = \frac{b-a}{n}$

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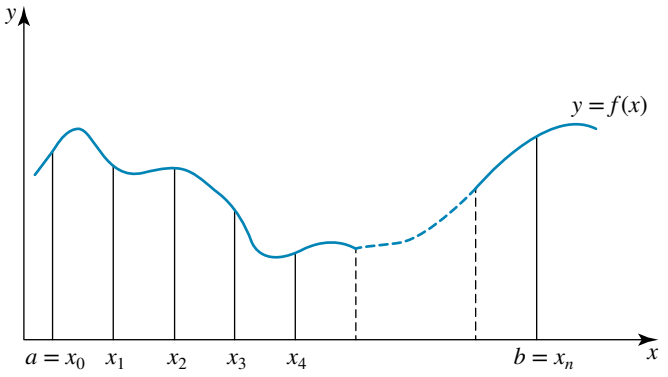
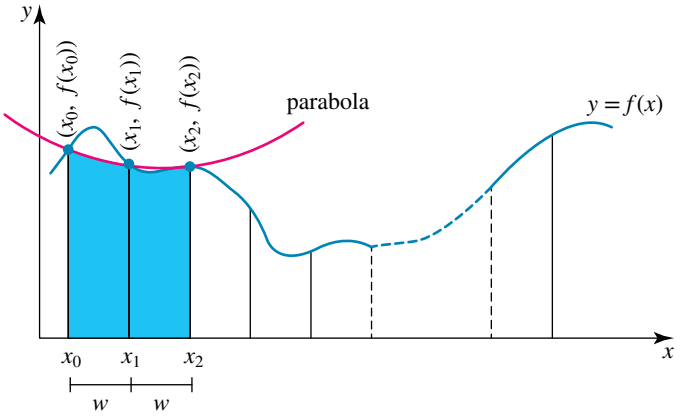
7.6.1 Simpson's rule

In Mathematical Methods, you have used the trapezoidal rule to estimate the area under a curve.

Another way to estimate the area under a curve is to use **Simpson's rule**. In this method, the region between $x_0 = a$ and $x_n = b$ is divided into n intervals, where n is an even number. The width of an interval, w, is found using $w = \frac{b-a}{n}$.

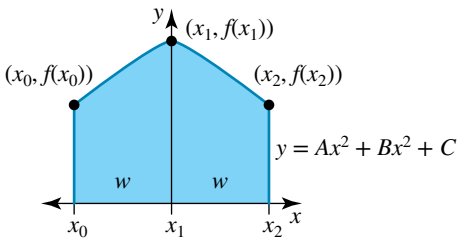


When using the trapezoidal rule, the area is approximated by fitting trapeziums across each interval. With Simpson's rule, the intervals are taken in pairs and a parabola is fitted through the three points.



Simpson's rule is considerably more accurate than the trapezoidal rule.

The derivation of this rule is an interesting exercise in algebra. Assume that the three points lie on the parabola $y = Ax^2 + Bx + C$. We can derive the formula for the area between the parabola and the x-axis over the region $x_0 \le x \le x_2$. The area under the parabola will be the same no matter where on the x-axis it is located. Therefore to simplify the algebra, assume that the parabola is centred on the y-axis.



The area under the parabola $y = Ax^2 + Bx^2 + C$ between $x_0 = -w$ and $x_2 = w$ can be found as follows:

$$\begin{aligned} A_S &= \int_{x_0}^{x_2} \left(Ax^2 + Bx + C\right) dx \\ &= \left[\frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx\right]_{-w}^{w} \\ &= \left(\frac{1}{3}Aw^3 + \frac{1}{2}Bw^2 + Cw\right) - \left(-\frac{1}{3}Aw^3 + \frac{1}{2}Bw^2 - Cw\right) \\ &= \frac{2}{3}Aw^3 + 2Cw \end{aligned}$$

The point $(x_0, f(x_0))$ lies on the parabola $(x_0 = -w)$

$$f(x_0) = Aw^2 - Bw + C \qquad [1]$$

The point $(x_1, f(x_1))$ lies on the parabola $(x_1 = 0)$

$$f(x_1) = C \qquad [2]$$

The point $(x_2, f(x_2))$ lies on the parabola $(x_2 = w)$

$$\begin{aligned} f(x_2) &= Aw^2 - Bw + C && [3] \\ [2] - [1]\colon f(x_1) - f(x_0) &= -Aw^2 + Bw && [4] \\ [3] - [2]\colon f(x_2) - f(x_1) &= Aw^2 + Bw && [5] \\ [5] - [4]\colon f(x_0) - 2f(x_1) + f(x_2) &= 2Aw^2 && [6] \end{aligned}$$

Using [2] and [6], A_s can be simplified as follows:

$$\begin{aligned} A_S &= \frac{w}{3}\left(2Aw^2\right) + 2w(C) \\ &= \frac{w}{3}\left[f(x_0) - 2f(x_1) + f(x_2)\right] + 2wf(x_1) \\ &= \frac{w}{3}\left[f(x_0) - 2f(x_1) + f(x_2) + 6f(x_1)\right] \\ &= \frac{w}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] \end{aligned}$$

Using a similar method, it can be shown that the area of the next 2 intervals is $\frac{w}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$.

The total area can be found by adding the area of each parabola together.

$$\begin{aligned}\int_a^b f(x) &= \frac{w}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]+\frac{w}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]+\ldots\frac{w}{3}\left[f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]\\ &= \frac{w}{3}\left[f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)+\ldots f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]\\ &= \frac{w}{3}\left[f(x_0)+4\left[f(x_1)+f(x_3)+\ldots\right]+2\left[f(x_2)+f(x_4)+\ldots\right]+f(x_n)\right]\end{aligned}$$

Simpson's rule

Simpson's rule for approximating $\int_a^b f(x)\,dx$ is

$$\int_a^b f(x)\,dx \approx \frac{w}{3}\left[f(x_0)+4\left[f(x_1)+f(x_3)+\ldots\right]+2\left[f(x_2)+f(x_4)+\ldots\right]+f(x_n)\right]$$

where $w=\dfrac{b-a}{n}$.

WORKED EXAMPLE 15 Using Simpson's rule with four strips

Use Simpson's rule with four strips to determine the approximate value of the definite integral $\int_0^{\frac{\pi}{2}} \sin\left(\sqrt{x}\right)dx$.

THINK

1. There are 4 strips, so $n=4$. The integration is over the interval $0\le x\le\frac{\pi}{2}$ so $a=0$ and $b=\frac{\pi}{2}$. Calculate the width of the interval.

WRITE

$$\begin{aligned}w &= \frac{b-a}{n}\\ &= \frac{\frac{\pi}{2}-0}{4}\\ &= \frac{\pi}{8}\end{aligned}$$

2. Construct a table of values for x_0 to x_4 and $f(x_0)$ to $f(x_4)$. Technology could be used to generate the $f(x_i)$ values.

n	x_n	$f(x_n)=\sin\left(\sqrt{x_n}\right)$
0	0	0
1	$\frac{\pi}{8}$	0.586
2	$\frac{2\pi}{8}=\frac{\pi}{4}$	0.775
3	$\frac{3\pi}{8}$	0.884
4	$\frac{4\pi}{8}=\frac{\pi}{2}$	0.950

3. State the formula for Simpson's rule.

$$\text{Area} \approx \frac{w}{3}\left[f(x_0)+4\left[f(x_1)+f(x_3)+\ldots\right]\right.$$
$$\left.+2\left[f(x_2)+f(x_4)+\ldots\right]+f(x_n)\right]$$

4. Write the formula to approximate the value of the required definite integral. Recall that in the formula, the first and last terms, $f(x_0)$ and $f(x_n)$ here $f(x_4)$ are not multiplied by the any factor.

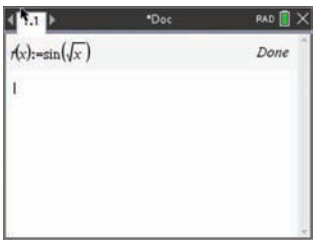
$$\text{So} \int_0^{\frac{\pi}{2}} \sin\left(\sqrt{x}\right) dx$$
$$\approx \frac{w}{3}\left[f(x_0)+4\left(f(x_1)+f(x_3)\right)+2\left(f(x_2)\right)+f(x_4)\right]$$

5. The formula for Simpson's rule groups together $f(x_0)$ and $f(x_n)$, $f(x_j)$ where j is even, and $f(x_k)$ where k is odd. Substitute for w and $f(x_i)$ in the formula.

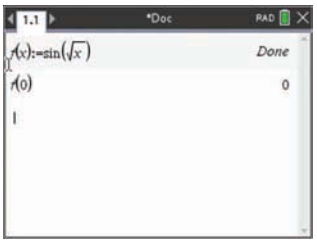
$$= \frac{1}{3} \times \frac{\pi}{8}[0+4(0.586+0.884)+2(0.775)+0.950]$$
$$= 1.097$$

TI | THINK **DISPLAY**

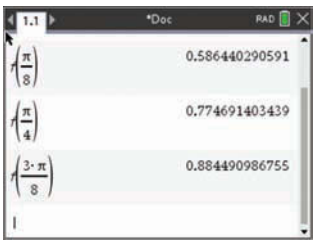
1. On a Calculator page, define the function f by completing the line $f(x) := \sin\left(\sqrt{x}\right)$ and press the ENTER button.



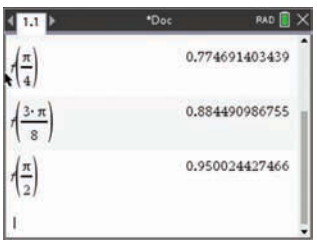
2. To generate the $f(x_n)$ values, complete the line $f(0)$ and press the ENTER button. Ensure that the setting for the angles is in radians.



3. Repeat with $f\left(\frac{\pi}{8}\right)$, $f\left(\frac{\pi}{4}\right)$ and $f\left(\frac{3\pi}{8}\right)$

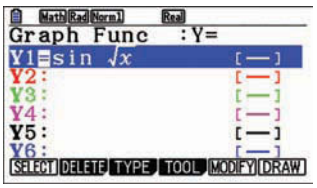


4. Calculate the last value by completing the line $f\left(\frac{\pi}{2}\right)$ and press the ENTER button

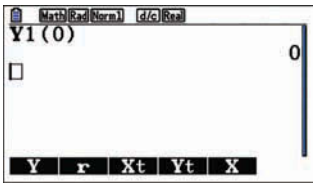


CASIO | THINK **DISPLAY**

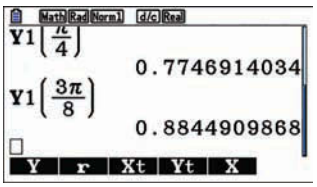
1. On a Main Menu screen, select GRAPH and complete the entry line as $\sin\left(\sqrt{x}\right)$. Ensure that the setting for the angles is in radians.



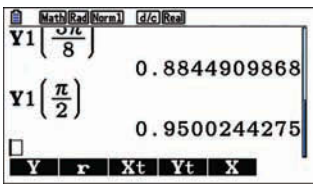
2. On a Run-Matrix screen, complete the line Y1(0) and press the EXE button.



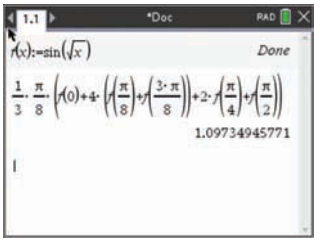
3. Repeat with Y1$\left(\frac{\pi}{8}\right)$, Y1$\left(\frac{\pi}{4}\right)$ and Y1$\left(\frac{3\pi}{8}\right)$



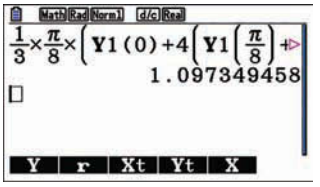
4. Calculate the last value by completing the line Y1$\left(\frac{\pi}{2}\right)$ and press the EXE button.



5. Note that to avoid keying errors when calculating the approximation, you can, after step 1, directly complete the line $\frac{1}{3} \times \frac{\pi}{8}$ $\left[f(0) + 4\left(f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right)\right) + 2f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right]$ and press the ENTER button.



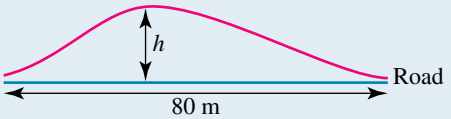
5. Note that to avoid keying errors when calculating the approximation, you can, after step 1, directly complete the line $\frac{1}{3} \times \frac{\pi}{8}$ $\left[Y1(0) + 4\left(Y1\left(\frac{\pi}{8}\right) + Y1\left(\frac{3\pi}{8}\right)\right) + 2Y1\left(\frac{\pi}{4}\right) + Y1\left(\frac{\pi}{2}\right)\right]$ and press the EXE button.



Simpson's rule can also be used to approximate areas and volumes of objects by taking measurements at regular intervals.

WORKED EXAMPLE 16 Using Simpson's rule to approximate the volume of an object

A road is to be built by cutting through a hillside as shown.



The width of the cutting is 80 m and it has a depth of 50 m. A measurement of the height, *h*, of the cutting is taken every 10 m. The results are recorded in the table below.

Measurement	Horizontal distance	Height
1	0	0
2	10	2
3	20	4.5
4	30	6
5	40	5
6	50	4
7	60	3
8	70	2.5
9	80	0

Determine the volume, in cubic metres, of rock that is to be removed to build the road.

THINK

1. Consider the shape of the cutting to be a prism and state the appropriate formula for volume.

2. State Simpson's rule to estimate the area of the cross-section.

WRITE

Volume = area of cross-section × width

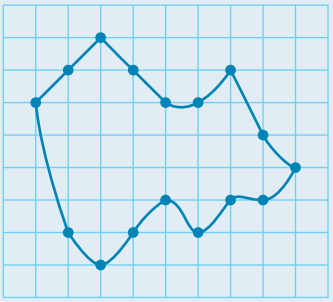
$$\text{Area} \approx \frac{w}{3}\left[f(x_0) + 4\left[f(x_1) + f(x_3) + \ldots\right] + 2\left[f(x_2) + f(x_4) + \ldots\right] + f(x_n)\right]$$

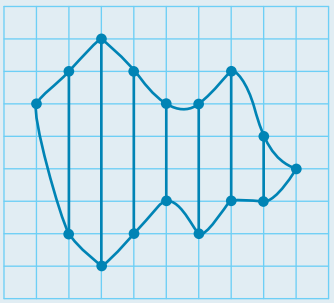
3. Calculate the width, w, of each strip by identifying the number of strips, n, and the values for a and b.	Examine the table given in the question. There are nine measurements, however, measurement 1 is the initial measurement and the first strip occurs once measurement 2 is taken. Therefore $n = 8, a = 0, b = 80$. So $w = \frac{b-a}{n} = \frac{80-0}{8} = 10$ The width of each strip is 10 m.
4. The measurement for h can be thought of as $f(x_i)$ for the formula. The first measurement is $f(x_0)$. The formula for Simpson's rule groups together $f(x_0)$ and $f(x_n)$, $f(x_j)$ where j is even, and $f(x_k)$ where k is odd.	$\text{Area} \approx \frac{10}{3}\left[f(x_0) + 4\left(f(x_1) + f(x_3) + f(x_5) + f(x_7)\right)\right.$ $\left. + 2\left(f(x_2) + f(x_4) + f(x_6)\right) + f(x_8)\right]$ $= \frac{10}{3}[0 + 4(2 + 6 + 4 + 2.5)$ $+ 2(4.5 + 5 + 3) + 0]$ $= 276.\dot{6}$ $\text{Area} \approx 276.67 \text{ m}^2$
5. Substitute the estimate for area into the formula for volume.	$\text{Volume} \approx 276.\dot{6} \times 50$ $= 13\,833.\dot{3}$
6. State the solution.	Approximately 13 833 cubic metres of rock will need to be removed.

In the examples so far, one of the sides has been a straight line. Simpson's rule can be used to estimate the area of any convex figure.

WORKED EXAMPLE 17 Using Simpson's rule to approximate the area of a convex figure

Use Simpson's rule to determine the area of the following if each major square has the dimensions 1 cm × 1 cm.



THINK	WRITE
1. The shape needs to be divided into an even number of intervals. Eight strips will fit on the grid nicely. Each interval is 1 unit wide so $w = 1$.	

2. Measure the length of each line and record the results in a table.

n	$f(x_n)$
0	0
1	5
2	7
3	5
4	3
5	4
6	4
7	2
8	0

3. State the formula for Simpson's rule to determine the area. Note that the width of each strip is 1 cm.

$$\text{Area} \approx \frac{w}{3}\left[f(x_1) + 4\left[f(x_1) + f(x) + \ldots\right] + 2\left[f(x_2) + f(x_4) + \ldots\right] + f(x_n)\right]$$

where $w = 1$.

4. The formula for Simpson's rule groups together $f(x_0)$ and $f(x_n)$, $f(x_j)$ where j is even, and $f(x_k)$ where k is odd.

$$= \frac{1}{3}\left[0 + 4(5 + 5 + 4 + 2) + 2(7 + 3 + 4) + 0\right]$$

$$= \frac{1}{3}(4 \times 16 + 2 \times 14)$$

$$= 30.\dot{6}$$

5. State the area.

The area is approximately 30.67 cm^2.

Because Simpson's rule fits a parabola across the top of 2 intervals, there must be an even number of strips, and the area will be exact for determining the area under a parabola. Perhaps unexpectedly, Simpson's rule is also exact for determining the area under a cubic function.

Technology can be used to calculate areas using Simpson's rule. In this spreadsheet, the area between the curve $f(x) = \sqrt{x}$ between $x = 0$, $x = 16$ and the x-axis with 12 strips is calculated. In the spreadsheet, the values for a, b and n are entered into cells G1, G2 and G3 respectively. The value for w is calculated using $w = \frac{b-a}{n}$. The values for x_i are determined by adding w to the previous value of x_i. For each x_i, $f(x_i)$ is calculated. In the column labelled Simpson, the appropriate multiples of the $f(x_i)$ are applied. The formulas for D3 and D4 can be entered and the pair can be dragged to fill the series. Cell D15 calculates the totals and G15 multiples the total by $\frac{w}{3}$.

	A	B	C	D	E	F	G
1	i	xi	f(xi)	Simpson		a =	0
2	0	0	0	0		b =	16
3	1	1.333333	1.154701	4.618802		n =	12
4	2	2.666667	1.632993	3.265986		w =	1.333333
5	3	4	2	8			
6	4	5.333333	2.309401	4.618802			
7	5	6.666667	2.581989	10.32796			
8	6	8	2.828427	5.656854			
9	7	9.333333	3.05505	12.2202			
10	8	10.66667	3.265986	6.531973			
11	9	12	3.464102	13.85641			
12	10	13.33333	3.651484	7.302967			
13	11	14.66667	3.829708	15.31883			
14	12	16	4	4			
15				95.71878		Area =	42.54168
16							

	A	B	C	D	E	F	G
1	i	xi	f(xi)	Simpson		a =	0
2	0	=G1	=SQRT(B2)	=C2		b =	16
3	1	=B2+G$4	=SQRT(B3)	=4*C3		n =	12
4	2	=B3+G$4	=SQRT(B4)	=2*C4		w =	=(G2-G1)/G3
5	3	=B4+G$44	=SQRT(B5)	=4*C5			
6	4	=B5+G$4	=SQRT(B6)	=2*C6			
7	5	=B6+G$4	=SQRT(B7)	=4*C7			
8	6	=B7+G$4	=SQRT(B8)	=2*C8			
9	7	=B8+G$4	=SQRT(B9)	=4*C9			
10	8	=B9+G$4	=SQRT(B10)	=2*C10			
11	9	=B10+G$4	=SQRT(B11)	=4*C11			
12	10	=B11+G$4	=SQRT(B12)	=2*C12			
13	11	=B12+G$4	=SQRT(B13)	=4*C13			
14	12	=B13+G$4	=SQRT(B14)	=C14			
15				=SUM(D2:D14)		Area =	=G4*(D15)/3
16							

Exercise 7.6 Approximation using Simpson's rule

learnon

7.6 Exercise	7.6 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	12, 13, 14	15

These questions are even better in jacPLUS!

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Simple familiar

1. WE15 Use Simpson's rule with four strips to determine the approximate value of the definite integral $\int_0^{\frac{\pi}{2}} \sin(x^2)\,dx$. Give your answer to 3 decimal places.

2. Use Simpson's rule to estimate, correct to 3 decimal places, the value of the definite integral $\int_0^1 2^{-x^2}\,dx$ using

a. 4 strips **b.** 6 strips.

3. Use Simpson's rule to estimate, correct to 3 decimal places, the value of the definite integral $\int_0^2 \frac{1}{1+x^2}\,dx$ using

a. 4 strips **b.** 6 strips.

4. **a.** Use an integration technique to determine the exact value of $I = \int_0^2 \left(-3x^4 + 12x^2 + 6\right)\,dx$.

b. Use Simpson's rule to estimate, correct to 2 decimal places, the value of I using 4 strips and compare your answer to the result from part **a**.

5. Compare the results obtained for the area of a semicircle of radius 3, represented by $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

a. using an integration technique

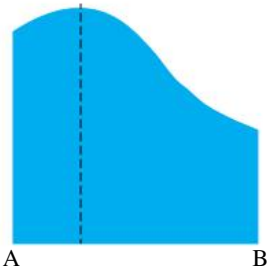
b. using Simpson's rule with 6 strips.

6. WE16 A road is to be built by cutting through a hillside. The width of the cutting is 60 m and it has a depth of 35 m. A measurement of the height, h, is taken every 10 m. The results are recorded in the table below.

Measurement	Horizontal distance (m)	Height (h) (m)
1	0	1
2	10	1.5
3	20	3
4	30	4
5	40	5.5
6	50	3
7	60	2

Use Simpson's rule to estimate the volume of rock to be removed to make this cutting.

7. Plan of Lot 22, Paradise Place, is shown.



The distance from A to B is 30 metres. Every 5 metres, a surveyor measures the distance from AB to the opposite boundary. The results are recorded below.

Distance from A (m)	0	5	10	15	20	25	30
Distance to boundary (m)	28	33	30	27	23	21	18

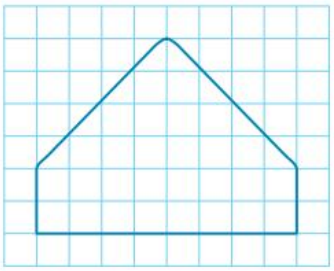
Use these measurements and Simpson's rule to determine the approximate area of the block.

8. A surveyor is using an airborne bathymetric LIDAR (laser imaging detection and ranging) to measure the depth of a river with a width of 18 m. The measures made every 1.5 m are displayed in the table below.

Width (m)	0	1.5	3	4.5	6	7.5	9	10.5	12	13.5	15	16.5	18
Depth (m)	0.49	1.2	1.8	2.5	3.6	4.3	4.7	5.2	4.9	3.7	3.3	2.4	0.84

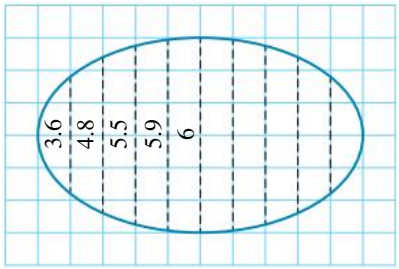
Estimate the cross-sectional are of the river. Give your answer to 1 decimal place.

9. Use Simpson's rule with four strips to determine the area of the following if each major square has the dimensions 1 cm × 1 cm.

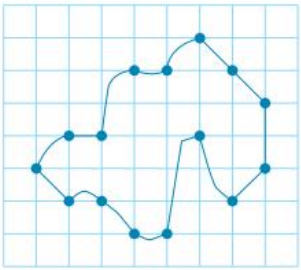


10. **a.** Use Simpson's rule with 10 strips to determine, correct to 1 decimal place, the approximate area of the following shape. The area of each major square has the dimensions 1 cm × 1 cm. The numbers shown represent the length (in cm) of the internal vertical distances.

b. Compare your answer in part **a** to the result obtained by using the formula for the area of an ellipse $A = \pi ab$, where a is half the length of the ellipse and b is half the width.

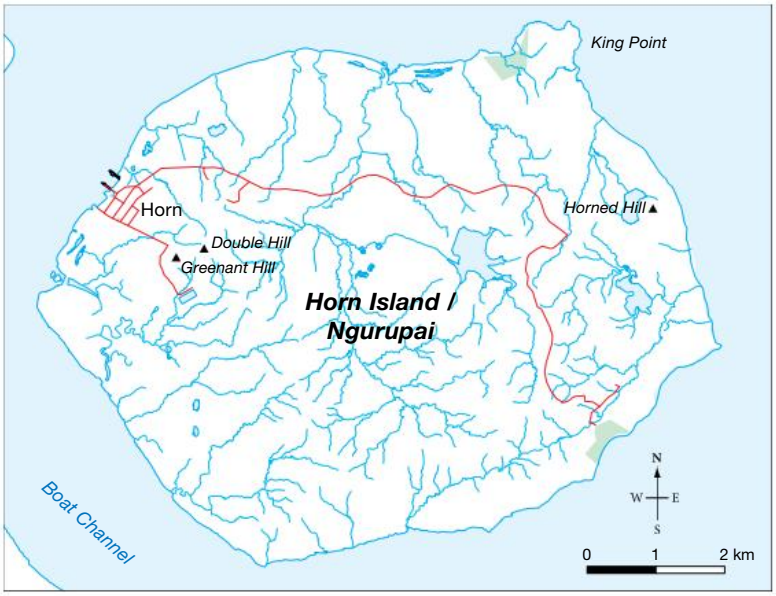


11. **WE17** Use Simpson's rule to determine the area of the following if each major square has the dimensions 1 cm × 1 cm. Give your answer to 2 decimal places.



Complex familiar

12. Use Simpson's rule to estimate the area of Horn Island.



Source: Natural Resources, Mines and Energy, Queensland Government

13. The area bounded by the curve $y = x^2$ between $x = a$ and $x = b$ and the x-axis is given by $\int_a^b x^2\,dx$.

Because Simpson's rule uses a parabola to approximate the area under a curve, this method should give an exact value for $\int_a^b x^2\,dx$.

Show that if $\int_a^b x^2\,dx$ is approximated by Simpson's rule using two strips then it gives the exact value.

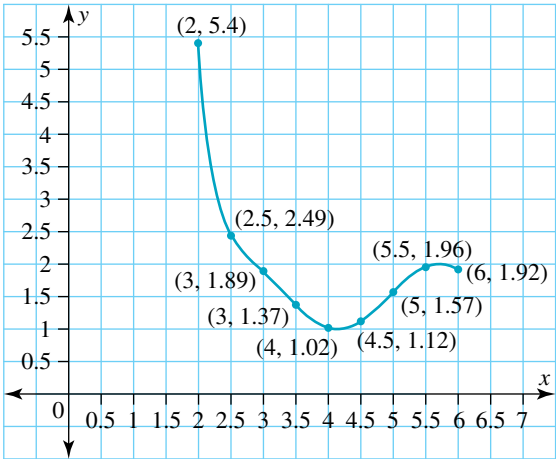
14. A sprinter recorded the first 5 seconds of their 100-metre sprint event. The recording accurately measured their sprint velocity at half second intervals shown in the table below.

Time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Velocity (m/s)	0	4.2	5.4	6.9	7.4	8.3	8.9	9.8	10.4	10.6	10.9

Apply Simpson's rule to determine an estimate of the distance travelled. Give your answer to 2 decimal places.

Complex unfamiliar

15. Consider the region shown in the diagram below where all units are in cm. Use Simpson's rule with $n = 8$ to calculate the volume of the solid obtained by rotating the region about the x-axis. Give your answer to 2 decimal places.



Fully worked solutions for this chapter are available online.

LESSON
7.7 Exponential probability density function

SYLLABUS LINKS

- Understand and use the probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, of the exponential random variable with parameter $\lambda > 0$.
  - Mean: $\frac{1}{\lambda}$
  - Standard deviation: $\frac{1}{\lambda}$
- Model and solve problems that involve exponential random variables and associated probabilities and quantiles, with and without technology.

Source: Specialist Mathematics Senior Syllabus 2024 © State of Queensland (QCAA) 2024; licensed under CC BY 4.0.

7.7.1 Exponential probability density function

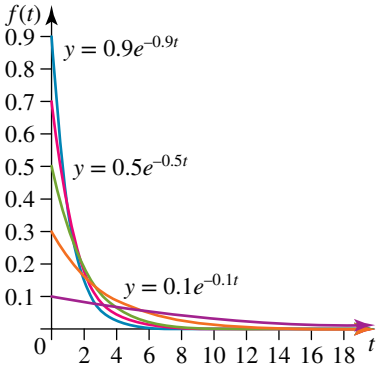
In Mathematical Methods, you explored the graph of the probability density function of the normal distribution. This particular distribution is symmetric, with a mean of μ and a standard deviation of σ. The normal distribution is one example of a probability distribution function, there are many others.

For a real function, f, to be a valid probability distribution, the following must hold:

1. f is non-negative, that is $f(t) \geq 0$ for all $t \in \mathbb{R}$
2. the area under the graph of the function is equal to 1, $\int_{-\infty}^{\infty} f(t)\,dt = 1$

The **exponential probability density function** is an example. The function is non-negative and

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,dt &= \int_{0}^{\infty} \lambda e^{-\lambda t}\,dt \\ &= \left[\frac{\lambda}{-\lambda}e^{-\lambda t}\right]_0^{\infty} \quad \text{as } \lim_{t\to\infty} e^{-t} = 0 \text{ and } e^0 = 1. \\ &= -e^{-\infty} - (-e^0) \\ &= 1\end{aligned}$$



In this distribution, t is the wait time for an event to occur and the average number of events in a time period (e.g. minute, hour, day) is λ. It can be shown that the average wait time is $\frac{1}{\lambda}$. This has been left for an exercise.

In this diagram, different values for λ are shown. In this instance, $\lambda \in \{0.1, 0.5, 0.9\}$.

Exponential probability density function

The exponential distribution with parameter λ has the probability density function,

$$f(t) = \begin{cases} \lambda e^{-2t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \text{ where } \int_0^{\infty} f(t)\,dt = 1, \text{ Mean} = \frac{1}{\lambda}, \text{ Standard deviation} = \frac{1}{\lambda}.$$

The probability that t falls within an interval is given $P(a \leq t \leq b) = \int_a^b f(t)\,dt$.

WORKED EXAMPLE 18 Using the exponential probability density function — lifetime of a component

If the average life of an LED lightbulb is 50 000 hours, determine the probability that a particular lightbulb lasts for the following:

a. between 40 000 and 70 000 hours

b. more than 60 000 hours.

THINK	WRITE
a. 1. The average life is 50 000. Therefore $\frac{1}{\lambda} = 50\,000$.	**a.** $\frac{1}{\lambda} = 50\,000$ $\lambda = \frac{1}{50\,000}$
2. As the distribution is concerned with wait times (waiting for the lightbulb to die), it is appropriate to use.	$f(t) = \begin{cases} \lambda e^{-\lambda t} t & t \geq 0 \\ 0 & t < 0 \end{cases}$ $= \begin{cases} \frac{1}{50\,000} e^{\frac{-t}{50\,000}} & t \geq 0 \\ 0 & t < 0 \end{cases}$

3. $P(40\,000 \le T \le 70\,000)$ can be determined by integrating $f(t)$ between $t = 40\,000$ and $t = 70\,000$.

$$\begin{aligned} P(40\,000 \le T \le 70\,000) &= \int_{40\,000}^{70\,000} \frac{1}{50\,000} e^{\frac{-t}{50\,000}}\, dt \\ &= \left[-\frac{50\,000}{50\,000} e^{\frac{-t}{50\,000}} \right]_{40\,000}^{70\,000} \\ &= \left[-e^{\frac{-t}{50\,000}} \right]_{40\,000}^{70\,000} \\ &= -e^{\frac{-70\,000}{50\,000}} + e^{\frac{-40\,000}{50\,000}} \\ &= -e^{-\frac{7}{5}} + e^{-\frac{4}{5}} \end{aligned}$$

4. State the solution.

$$\begin{aligned} P(40\,000 \le T \le 70\,000) &= e^{-\frac{4}{5}} - e^{-\frac{7}{5}} \\ &\approx 0.2027 \end{aligned}$$

b. 1. An important feature of a continuous distribution is that $P(T > x) \equiv P(T \ge x)$. Thus, $P(T > 60\,000) = P(T \ge 60\,000)$. Remember that $\lim_{t \to \infty} e^{-t} = 0$.

b.
$$\begin{aligned} P(T \ge 60\,000) &= \int_{60\,000}^{\infty} \frac{1}{50\,000} e^{\frac{-t}{50\,000}}\, dt \\ &= \left[-e^{\frac{-t}{50\,000}} \right]_{60\,000}^{\infty} \\ &= -e^{-\infty} + e^{\frac{-60\,000}{50\,000}} \\ &= e^{-\frac{6}{5}} \end{aligned}$$

Alternatively, the problem can be solved using the following process.
$P(T \ge 60\,000) = 1 - P(T \le 60\,000)$

$$= 1 - \int_{0}^{60\,000} \frac{1}{50\,000} e^{\frac{-t}{50\,000}} dt$$

$$\begin{aligned} P(T \ge 60\,000) &= 1 - P(T \le 60\,000) \\ &= 1 - \int_{0}^{60\,000} \frac{1}{50\,000} e^{\frac{-t}{50\,000}}\, dt \\ &= 1 - \left[-e^{\frac{-t}{50\,000}} \right]_{0}^{60\,000} \\ &= 1 - \left(-e^{\frac{-60\,000}{50\,000}} + e^{0} \right) \\ &= 1 + e^{-\frac{6}{5}} - 1 \\ &= e^{-\frac{6}{5}} \end{aligned}$$

2. State the solution.

$$\begin{aligned} P(T \ge 60\,000) &= e^{-\frac{6}{5}} \\ &\approx 0.3012 \end{aligned}$$

TIP

The following results are useful to memorise to quickly determine probabilities for exponentially distributed random variables.

$$\begin{aligned} \mathrm{P}(a \le t \le b) &= \int_a^b \lambda e^{-\lambda t}\, dt \\ &= e^{-\lambda a} - e^{-\lambda b} \\ \mathrm{P}(0 \le t \le T) &= 1 - e^{-\lambda T} \\ \mathrm{P}(t \ge T) &= -\mathrm{P}(t \le T) \\ &= e^{-\lambda T} \end{aligned}$$

WORKED EXAMPLE 19 Using the exponential probability density function — waiting time

The expected wait time for a bus is given by the function $f(t) = \begin{cases} 0.2e^{-0.2t} & t \ge 0 \\ 0 & t < 0 \end{cases}$, where t is the wait time in minutes.

a. Determine the probability that wait time is between 0 and 2 minutes.

b. Determine the probability that wait time is between 4 and 6 minutes given that they have already waited 4 minutes.

THINK	WRITE
a. The probability can be calculated by determining the area under the curve between $t = 0$ and $t = 2$.	**a.** $\mathrm{P}(0 \le T \le 2) = \int_0^2 0.2e^{-0.2t}\, dt$ $= \left[\frac{0.2}{-0.2} e^{-0.2t}\right]_0^2$ $= -e^{-0.4} + e^0$ $= 1 - e^{-0.4}$ ≈ 0.3297
b. 1. Recall the conditional probability formula $\mathrm{P}(A\|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$. $\mathrm{P}(A \cap B)$ represents the intersection of the two events $\mathrm{P}(T \ge 4)$ and $\mathrm{P}(T \le 6)$. Thus $\mathrm{P}(A \cap B) = \mathrm{P}(4 \le T \le 6)$. Determine $\mathrm{P}(4 \le T \le 6)$.	**b.** $\mathrm{P}(4 \le T \le 6) = \int_4^6 0.2e^{-0.2t}\, dt$ $= \left[\frac{0.2}{-0.2} e^{-0.2t}\right]_4^6$ $= -e^{-1.2} + e^{-0.8}$ $= e^{-0.8} - e^{-1.2}$
$\mathrm{P}(B)$ is the probability that they have already waited 4 minutes. This means that the wait will be a minimum of 4 minutes. Determine $\mathrm{P}(T \ge 4)$. Remember $\lim_{t \to \infty} e^{-t} = 0$.	$\mathrm{P}(T \ge 4) = \int_4^\infty 0.2e^{-0.2t}\, dt$ $= \left[\frac{0.2}{-0.2} e^{-0.2t}\right]_4^\infty$ $= -e^{-\infty} + e^{-0.8}$ $= e^{-0.8}$

2. To determine the probability that the wait time is between 4 and 6 minutes, given that the wait has already been 4 minutes, calculate $\frac{P(4 \le T \le 6)}{P(T \ge 4)}$.

$$\begin{aligned}\frac{P(4 \le T \le 6)}{P(T \ge 4)} &= \frac{e^{-0.8} - e^{-1.2}}{e^{-0.8}} \\ &= e^{0.8}\left(e^{-0.8} - e^{-1.2}\right) \\ &= 1 - e^{-0.4} \\ &\approx 0.3297\end{aligned}$$

In the example, the probability of a particular wait time is independent of the time that has already been spent waiting. This property of the exponential probability distribution means that the distribution can be referred to as *memoryless*.

For example, severe floods are sometimes referred to as *hundred year floods*, meaning that their likelihood of occurring is approximately once every one hundred years. This expected wait time until the next flood can be modelled using the exponential probability density function. Some people erroneously believe that once the flood has occurred, it should be about one hundred years until the next one, however, the memoryless nature of the distribution means that once a flood has occurred, the likelihood of another flood occurring within the next 10 years, for example, is the same as it was before the flood hit.

Exercise 7.7 Exponential probability density function

learnon

7.7 Exercise | **7.7 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9	10, 11, 12, 13	14, 15

These questions are even better in jacPLUS!
- Receive immediate feedback
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Simple familiar

1. Let T be an exponential random variable with a probability density function given by $f(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & t < 0 \end{cases}$, $\lambda = 2$. Determine the following:
   a. $P(T \le 2)$
   b. $P(T \ge 1)$.

2. WE18 If the average life of an electric light tube is 500 hours, determine the probability that a particular light tube will last for the following:
   a. between 400 and 600 hours
   b. more than 700 hours.

3. If a text arrives every 15 seconds on average, (this means that $\lambda = 4$ per minute), calculate the probability of waiting less than or equal to 30 seconds (that is 0.5 minutes) for a text.

4. Accidents occur at an average of 4 per week, (this means that $\lambda = 4$). Calculate the probability that at least 2 weeks will elapse between accidents.

5. **WE19** The expected wait time for a bus is given by the function $f(t) = \begin{cases} 0.2e^{-0.2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$, where t is the wait time in minutes.

   a. Determine the probability that wait time is between 0 and 5 minutes.
   b. Determine probability that a bus will arrive in the next 5 minutes given that you have already waited 10 minutes.

6. The average rate of job requests in a computer support centre is 4 per minute. It can be assumed that the number of requests per minute follows an exponential probability density function.

   a. Identify the exponential probability density function.
   b. Calculate the probability that at least 15 seconds will elapse between requests.
   c. Calculate the probability that less than 1 minute will elapse between requests.
   d. If no requests have been made in the last 30 seconds, determine the probability that a job will arrive in the next 15 seconds.

7. Suppose that the response time T of a help line (the elapse time between the end of a user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected response time equal to 5 seconds.

   a. Determine the probability, correct to 4 decimal places, that the response time is at most 10 seconds.
   b. Determine the probability, correct to 4 decimal places, that the response time is between 5 and 10 seconds.

8. Suppose that calls are received at a 24-hour hotline at a rate of 0.5 calls per day. The number of days, T, between successive calls has an exponential distribution.
   Determine the probability, correct to 4 decimal places, that more than two days elapses between calls.

9. Calls arrive at an average rate of 12 per hour. Calculate the probability, correct to 4 decimal places, that a call will occur in the next 5 minutes given that you have already waited 10 minutes for a call.

Complex familiar

10. Suppose that the time between messages from your sister has an exponential distribution with a mean time of 3 days.

    a. If you just received a message from her, determine the probability that you will receive the next message within the next 2 days.
    b. You realise that you have not received a message in at least 2 days. Determine the probability, correct to 4 decimal places, that she will not message you within the next 2 days.

11. The time between emergency patients in a country hospital follows an exponential distribution with an average of 2.1 new patients per day.

    a. Identify the exponential probability distribution function for the expected wait time between new patients.
    b. The triage nurse Cameron has just started his shift. Determine the probability, correct to 4 decimal places, that a new patient will arrive in the next 15 minutes.
    c. Cameron has nearly finished his shift. With 15 minutes remaining, determine the probability, correct to 4 decimal places, that a new patient will arrive before the shift finishes.
    d. Teagan works a regular Tuesday to Friday 10-hour shift. Determine the probability, correct to 4 decimal places, that no new patients arrive during one of her shifts.
    e. Determine the probability, correct to 4 decimal places, that no new patients arrive during one week of Teagan's shifts.
    f. Teagan has observed that in 10% of shifts, a new patient arrives in the first x hours. Determine x correct to 1 decimal place.

12. The expected wait time at a call centre on weekdays follows an exponential distribution with probability density function,

$$f(t)=\begin{cases}\lambda e^{-\lambda t}, & t\geq 0\\ 0 & , t<0\end{cases}$$

It is known that the mean weekday wait time is greater than 2 minutes and the probability of waiting between 1 and 2 minutes is 17.23%.
On the weekend, the wait time is defined by a different function, $g(t)$, where the mean time is $\frac{4}{3}$ minutes.

Calculate the median wait times for both weekday and weekend to determine the difference. Give your answer to 2 decimal places.

13. The shelf life t (in months) of an electrolytic capacitor can be modelled using the exponential distribution with parameter λ.

$$f(t)=\begin{cases}\lambda e^{-\lambda t} & \text{for } t\geq 0\\ 0 & \text{for } t<0\end{cases}$$

The probability that the capacitor still works after 4 years is 0.12.
Determine how long it takes for the probability that the capacitor still work to drop under 0.3. Give your answer to 3 decimal places.

Complex unfamiliar

14. The time t (in years) it takes for carbon-14 atoms to decay can be modelled using the exponential distribution with parameter λ.

$$f(t)=\begin{cases}\lambda e^{-\lambda t} & \text{for } t\geq 0\\ 0 & \text{for } t<0\end{cases}$$

The half-life $t_{\frac{1}{2}}$ of carbon-14 is such that $\mathrm{P}\left(0\leq t\leq t_{\frac{1}{2}}\right)=\mathrm{P}\left(t_{\frac{1}{2}}\leq t\right)$.

A Physics student affirms that, given that the half-life of carbon-14 is 5730 years, the probability that a carbon-14 atom still hasn't decayed after 15 000 years is greater than 15%. Verify the reasonableness of this claim.

15. The expected wait time at a traffic light is given by the function $f(t)=\begin{cases}\lambda e^{-\lambda t} & \text{for } t\geq 0\\ 0 & \text{for } t<0\end{cases}$, where t is the wait time in seconds. Given that the probability that the wait time is between 15 seconds and 30 seconds is 0.25, determine the probability that the wait time exceeds 90 seconds. Give your answer to 4 decimal places.

Fully worked solutions for this chapter are available online.

LESSON
7.8 Review

7.8.1 Summary

Hey students! Now that it's time to revise this chapter, go online to:

- Access the chapter summary
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7.8 Exercise

learnon

7.8 Exercise	7.8 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

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Simple familiar

1. Calculate the following definite integrals. Give your answers to 2 decimal places.

a. $\displaystyle\int_0^{\frac{\pi}{6}} \frac{4}{16+x^2}\,dx$

b. $\displaystyle\int_{0.5}^{2.2} \frac{3x-18}{x^2-2x-8}\,dx$

2. Determine the area of the region bounded by the curve $y = x^3 - 8x$ and the x-axis.

3. Determine the area of the region bounded by the curve $y = \dfrac{1}{x^2-1}$, the lines $x = 2$, $x = 3$ and the x-axis.

4. Determine the area of the region bounded by the curve $y = x^2 - 4x$ and the line $y = 2x$.

5. Determine the area of the region between the curve $y = \dfrac{4}{9-x^2}$ and the line $y = 2x$ from $x = 1$ to $x = 2$.

6. Determine the volume of the solid formed by rotating the line $y = \dfrac{x}{2}$ about the x-axis from the origin to $x = 4$.

7. Determine the volume of the solid formed by rotating the line $y = \dfrac{x}{4}$ about the y-axis from the origin to $x = 8$.

8. Determine the volume of revolution when the region bounded by the curve $y = x^2 - 6x$ and the x-axis is rotated about the x-axis.

9. Determine the volume of revolution when the region bounded by the curve $y = -5\ln(x+1) + 3$ and the coordinate axes is rotated about the y-axis. Give your answer to 2 decimal places.

10. Determine the volume of revolution when the region bounded by the curve $y = \sqrt{2 - x}$ and the coordinate axes is rotated about the x-axis.

11. Determine the volume of revolution when the region bounded by the curve $y = \sqrt{16 - x}$ and the coordinate axes is rotated about the y-axis.

12. An exponential probability density function is given by $f(t) = \begin{cases} 2.2e^{-2.2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$.

    a. Determine $P(0 \leq T \leq 2)$ correct to 4 decimal places.
    b. Determine $P(T \geq 3)$ correct to 4 decimal places.

Complex familiar

13. Determine the exact volume formed when the area bounded by the curve with the equation $y = x^4$, the x-axis and the line $x = 2$ is rotated about the:

    a. x-axis
    b. y-axis.

14. Consider the functions $f(x) = 5\sin\left(\frac{3x}{4}\right)$ and $g(x) = \frac{3x}{4}$ where $x \geq 0$. Determine, correct to 3 decimal places, the area between functions f and g.

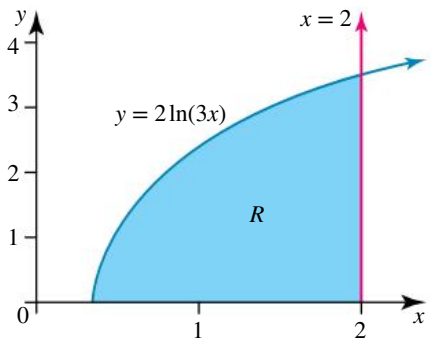
15. The area of a quadrant of a circle is represented by $\int_0^6 \sqrt{36 - x^2}\, dx$. Compare the areas obtained by using an exact calculus method and using Simpson's rule with 6 strips.

16. Kayley estimates that the average time between nuisance phone calls is 2 days. If it has been one day without a call, determine the probability that there will be a call in the next 3 hours. Give your answer correct to 4 decimal places.

Complex unfamiliar

Use the following information for questions 17–20.

The region R is bounded by the line $x = 2$, the curve $y = 2\ln(3x)$ and the x-axis.



17. Determine the exact value of the region R.

18. Determine the exact value of the volume formed when the region R is rotated about the y-axis.

19. The line $x = k$ divides the exact value of the region in half. Determine the value of k correct to 3 decimal places.

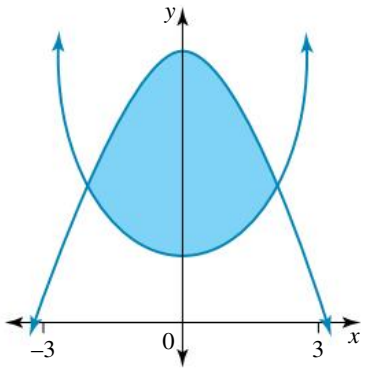
20. When the region R is rotated about the x-axis it forms a solid of revolution. Determine the volume of the solid.

7.8 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2021 Specialist Mathematics, Paper 2, Section 1, Q2; © State of Queensland (QCAA) 2024.

Determine the area of the shaded region between the graphs of the functions $y = \frac{1}{3}\sec\left(\frac{x}{3}\right)$ and $y = 2\cos\left(\frac{x}{2}\right)$, as shown.



Not to scale

A. 5.29 units^2 **B.** 5.51 units^2 **C.** 5.65 units^2 **D.** 5.71 units^2

Question 2 (1 mark)

Source: QCAA 2021 Specialist Mathematics, Paper 2, Section 1, Q4; © State of Queensland (QCAA) 2024.

The mean time that visitors spend at an art exhibition is 39 minutes and the standard deviation is 6 minutes.

Determine the approximate probability that the mean time spent at the exhibition by a random sample of 35 visitors is between 38 and 40 minutes.

A. 0.13 **B.** 0.16 **C.** 0.68 **D.** 0.84

Question 3 (4 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 2, Q14; © State of Queensland (QCAA) 2024.

At a certain location, a biologist measures the width of a river to be $12\,\text{m}$. She also records the depth of the river at regular $2\,\text{m}$ interval widths as shown.

Width (m)	0	2	4	6	8	10	12
Depth (m)	0.52	2.15	3.70	4.27	3.32	1.28	0.59

The biologist estimates the cross-sectional area of the river at this location to be $15\,\text{m}^2$.

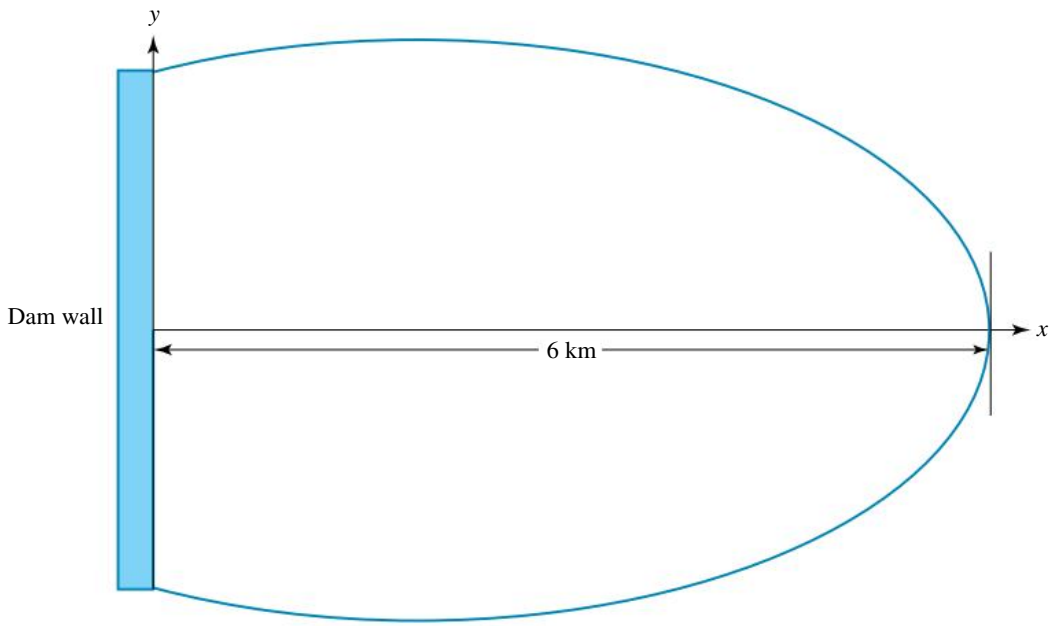
Use Simpson's rule to evaluate the reasonableness of this estimation. Justify your area calculation and decision regarding reasonableness using mathematical reasoning.

Question 4 (6 marks)

Source: QCAA 2022 Specialist Mathematics, Paper 2, Section 2, Q11; © State of Queensland (QCAA) 2024.

An aerial view of the surface of a dam, 6 km in length, is symmetrically positioned on a Cartesian plane as shown. A dam wall is located along the y-axis.

The surrounding edge of the dam can be modelled by the ellipse $\frac{(x-2)^2}{16}+\frac{y^2}{9}=1$, for $0 \le x \le 6$.



a. Use Simpson's rule with four strips to determine an approximate area of the surface of the dam. **[4 marks]**

b. Evaluate the reasonableness of this approximation. **[2 marks]**

Question 5 (7 marks)

Source: QCAA 2021 Specialist Mathematics, Paper 1, Section 2, Q17; © State of Queensland (QCAA) 2024.

The area between the graphs of the functions $y=4x$ and $y=2x^2$ is rotated about the y-axis to form a solid of revolution with a volume of V cubic units.

Determine the exact value of V.

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Answers

7 Applications of integral calculus

7.2 Area between a function and the axes

7.2 Exercise

1. $\int_{-2}^{1} \sqrt{9-x^2}\,dx$
2. $(-5, 2] \cup (5, \infty)$
3. a. 0.1577 b. -0.042
4. a. 37.132 b. 0.5798
5. a. $\frac{1}{6}$ square units b. $\frac{4}{3}$ square units
   c. $\frac{64\sqrt{2}}{3}$ square units d. 32 square units
   e. $\frac{256}{3}$ square units f. $\frac{1372}{27}$ square units
6. a. 8 square units b. $\frac{37}{12}$ square units
   c. $\frac{81}{2}$ square units d. $\frac{64}{3}$ square units
7. a. $\frac{1}{2}\ln(2)$ square units
   b. $\ln\left(\frac{3}{2}\right)$ square units
   c. $\frac{11}{2} + \frac{1}{2}e^{-4}$ square units
   d. $3\ln(3) + \frac{1}{2}(e^2 + e^{-2}) - 3$ square units
8. a. $x > -\frac{1}{2}$, $\frac{26}{3}$ square units
   b. $\mathbb{R}\backslash\left\{-\frac{5}{3}\right\}$, $\frac{1}{3}\ln\left(\frac{14}{5}\right)$ square units
   c. $\mathbb{R}\backslash\left\{-\frac{3}{2}\right\}$, $\frac{1}{35}$ square units
   d. $x < \frac{16}{3}$, 6 square units
9. 0.1007
10. a. 3.142 b. 1.077
11. $\frac{2\sqrt{3}}{3} - \frac{\pi}{6}$
12. a. The curve is positive for $\frac{\pi}{3} \le x < \frac{\pi}{2}$ and negative for $\pi < x \le \frac{5\pi}{6}$.
    b. 0.344
13. a. The curve is positive for $0 < x < \frac{\pi}{4}$ and negative for $\frac{\pi}{4} < x \le 2$.
    b. 0.143
14. 0.004
15. a. $\frac{1}{2}\ln\left(\frac{21}{5}\right)$ square units
    b. $\frac{\sqrt{2}}{4}$ square units
    c. $\frac{1}{2}(1 - e^{-4})$ square units
    d. 2 square units
16. a. $\ln\left(\frac{9}{4}\right)$ square units
    b. $\ln\left(\frac{3}{2}\right)$ square units
    c. $\ln(8)$ square units
    d. $10 + \frac{3}{2}\ln\left(\frac{7}{3}\right)$ square units
17. $\frac{32}{3}$ square units
18. $\frac{256}{15}$ square units
19. $\frac{4}{3}\sqrt{a^3}$ square units
20. $\frac{8\sqrt{a^5}}{15}$ square units

7.3 Area between functions

7.3 Exercise

1. a. $\frac{243}{2}$ square units b. 36 square units
   c. 288 square units d. $\frac{243}{2}$ square units
2. $9\ln\left(\frac{9}{4}\right) - \frac{1}{2}$ square units
3. a. $\frac{1}{6}$ square units b. $\frac{1}{2}$ square units
   c. $\frac{3}{10}$ square units d. $\frac{2}{3}$ square units
4. a. $2\sqrt{2}$ square units b. 4 square units
   c. $\frac{9}{8}$ square units d. $2e - 2e^{-2}$ square units
5. a. $\frac{15}{4} - 2\ln(4)$ square units
   b. $\frac{40}{27} + \frac{1}{3}\ln\left(\frac{1}{9}\right)$ square units
6. $\frac{16}{3}$ square units
7. $\frac{5}{6}$ square units
8. a. 15.43 square units
   b. -0.54
9. a. 4.39 square units
   b. 0.49

10. 0.903 square units

11. 7.34 square units

12. $\frac{3\pi - 8}{4}$ square units

13. $\frac{5}{2}$ square units

14. $e - e^{-2}$ m^2

15. Area $\approx$ 5.534 square units

16. Area $\approx$ 2.4 square units

17. Area $\approx$ 3.5 square units

18. Area $= \frac{7 - 5\pi}{4}$ square units

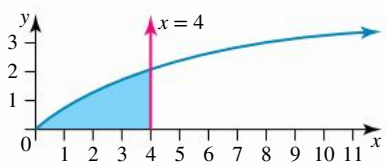
19. Sample responses can be found in the worked solutions in the online resources.

20. Sample responses can be found in the worked solutions in the online resources.

7.4 Volumes of solids of revolution

7.4 Exercise

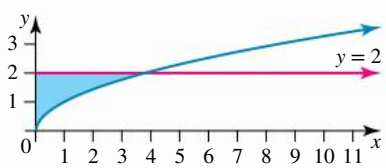
1. a.



b. 8π cubic units

2. $\frac{512\pi}{15}$ cubic units

3. a.



b. $\frac{32\pi}{5}$ cubic units

4. 8π cubic units

5. a. 375π cubic units

b. $\frac{125\pi}{27}$ cubic units

6. a. 4π cubic units

b. 6π cubic units

7. $\frac{\pi}{2}$ cubic units

8. $4\pi(e^2 - 1)$ cubic units

9. 10 L

10. a. Volume = 72 cubic units

b. Volume $= \pi \ln \frac{19}{16}$ cubic units

c. Volume $= \frac{\sqrt{3}}{4}\pi$ cubic units

11. a. Volume $\approx$ 2.8695 cubic units

b. Volume $\approx$ 105.767 cubic units

c. Volume $\approx$ 20.453 cubic units

12. a. $\frac{768\pi}{5}$ cubic units

b. 6π cubic units

13. a. $\frac{1296\pi}{5}$ cubic units

b. $\frac{81\pi}{2}$ cubic units

14. a. Volume $= \frac{45\pi}{2}$ cubic units

b. Volume $= \pi^2$ cubic units

15. Volume $\approx$ 205 191.514 cubic units

16. a. $\frac{320\pi}{3}$ cubic units

b. $\frac{400\pi}{3}$ cubic units

c. $\frac{4}{3}\pi ab^2$ cubic units

d. $\frac{4}{3}\pi a^2 b$ cubic units

e. If rotating about the x-axis: $\frac{4x^2}{57^2} + \frac{y^2}{22^2} = 1$

If rotating about the y-axis: $\frac{x^2}{22^2} + \frac{4y^2}{57^2} = 1$

f. 57 780 mm^3

17. $h = 12.9$ cm

18. 1100 mL

19. a. Sample responses can be found in the worked solutions in the online resources.

b. 136 L

20. The assertion is very reasonable, there is a negligeable difference in volumes.

7.5 Volumes of revolution

7.5 Exercise

1. $\pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx$

2. $\pi \int_a^b \left([f(y)]^2 - [g(y)]^2 \right) dy$

3. a. $\frac{704\pi}{5}$ cubic units b. 8π cubic units

4. a. 8π cubic units b. $\frac{32\pi}{5}$ cubic units

5. a. $\frac{49\pi}{2}$ cubic units b. $\frac{76\pi}{5}$ cubic units

6. a. $\frac{128\pi}{5}$ cubic units b. 8π cubic units

7. a. $\frac{4352\pi}{15}$ cubic units b. 8π cubic units

8. a. $\frac{81\pi}{2}$ cubic units b. $\frac{383\pi}{15}$ cubic units

9. a. $\frac{486\pi}{5}$ cubic units b. $\frac{2187\pi}{7}$ cubic units

10. a. $\frac{416\pi}{15}$ cubic units b. 4π cubic units

11. a. $\frac{64\pi}{15}$ cubic units b. $\frac{8\pi}{3}$ cubic units

12. a. $4\pi^2$ cubic units b. $\frac{9\pi^2}{2}$ cubic units

13. a. $\frac{2\pi}{35}$ cubic units b. $\frac{\pi}{10}$ cubic units

14. a. $\frac{96\pi}{5}$ cubic units b. $\frac{3\pi a^3}{10}$ cubic units

15. a. 1296π cubic units b. 81π cubic units

16. $\frac{512\pi}{3}$ cubic units

17. $\frac{192\pi}{5}$ cubic units

18. a. $32\sqrt{3}\pi$ cubic units b. $\frac{a^3\pi\sqrt{3}}{2}$ cubic units

19. a. 256π cubic units b. $2\pi^2 ar^2$ cubic units

20. a. $\frac{11\sqrt{3}\pi a^5}{40}$ cubic units b. $\frac{9\pi a^4}{32}$ cubic units

7.6 Approximation using Simpson's rule

7.6 Exercise

1. 0.828 square units
2. a. 0.810 square units
   b. 0.810 square units
3. a. 1.105 square units
   b. 1.107 square units
4. a. $I = 24.8$
   b. $I \approx 24.75$, Simpson's rule with 4 strips provides a good estimation of this definite integral.
5. a. $\frac{9\pi}{2}$ square units
   b. 13.995 square units which is a 1% difference
6. 6300 m^3
7. 793.3 m^2
8. 25.6 m^2
9. 32 cm^2
10. a. 46.5 cm^2
    b. Simpson's rule is accurate to about 1.3%
11. The area is approximately 21.33 cm^2.
12. *Note:* answers will vary depending on the number of strips used. The area of Horn Island is 53 km^2.
13. Exact value $= \frac{1}{3}(b^3 - a^3)$. Simpson's rule approximation $= \frac{1}{3}(b^3 - a^3)$.
14. The sprinter ran approximately 39.05 metres.
15. 46.12 cm^3

7.7 Exponential probability density function

7.7 Exercise

1. a. $1 - e^{-4}$ b. e^{-2}
2. a. $-e^{\frac{-6}{5}} + e^{\frac{-4}{5}}$ b. $e^{\frac{-7}{5}}$
3. $1 - e^{-2}$
4. e^{-8}
5. a. $1 - e^{-1}$ b. $1 - e^{-1}$
6. a. $f(t) = \begin{cases} 4e^{-4t} & t \geq 0 \\ 0 & t < 0 \end{cases}$
   b. e^{-1}
   c. $1 - e^{-4}$
   d. $1 - e^{-1}$
7. a. 0.8647 b. 0.2325
8. 0.3679
9. 0.6321
10. a. 0.4866 b. 0.5134
11. a. $f(t) = \begin{cases} 2.1e^{-2.1t} & t \geq 0 \\ 0 & t < 0 \end{cases}$
    b. 0.0216
    c. 0.0216
    d. 0.4169
    e. 0.0302
    f. 1.2041 hours ($\approx$ 1 hours 12 minutes)
12. The median wait time difference is 1.85 minutes.
13. 8.075 months
14. $P(t \geq 15\,000) \approx 16.3\%$, thus, the claim is reasonable.
15. 0.0156

7.8 Review

7.8 Exercise

1. a. 0.13 b. 2.74
2. 32 square units
3. $\frac{1}{2}\ln\left(\frac{3}{2}\right)$ square units
4. 36 square units
5. $3 + \frac{2}{3}\ln\left(\frac{2}{5}\right)$ square units
6. $\frac{16\pi}{3}$ cubic units
7. $\frac{128\pi}{3}$ cubic units
8. $\frac{1296\pi}{5}$ cubic units
9. $\frac{\pi}{2}\left(21 + 5e^{\frac{6}{5}} - 20e^{\frac{3}{5}}\right) \approx 1.82$ cubic units
10. 2π cubic units
11. $\frac{8192\pi}{15}$ cubic units

12. a. 0.9877 b. 0.0014

13. a. $\frac{512\pi}{9}$ cubic units

b. $\frac{52\pi}{3}$ cubic units

14. Area is 7.873 square units

15. Exact value: $9\pi \approx 28.2743$. Simpson's rule: 27.991 14

16. 0.0606

17. $4\ln(6) - \frac{10}{3}$ square units

18. $\frac{\pi}{9}(72\ln(6) - 35)$ cubic units

19. 1.411

20. $\frac{8\pi}{3}(3(\ln(6))^2 - 6\ln(6) + 5)$ cubic units

7.8 Past QCAA exam questions

1. D

2. C

3. Sample responses can be found in the worked solutions in the online resources.

4. a. $A_S = 29.68\ \text{km}^2$

b. Area $\approx 30.32\ \text{km}^2$

The approximation is reasonable as the error is only around 2%.

5. $V = \frac{16\pi}{3}$

8 Rates of change and differential equations

LESSON SEQUENCE

8.1 Overview 552
8.2 Implicit differentiation 554
8.3 Related rates as instances of the chain rule 563
8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$ 572
8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$ 581
8.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ 589
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8.8 Review 608
Answers 614

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

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on Resources

Solutions	Solutions — Chapter 8
Exam questions	Exam question booklet — Chapter 8
Digital documents	Learning matrix — Chapter 8 Chapter summary — Chapter 8

LESSON
8.1 Overview

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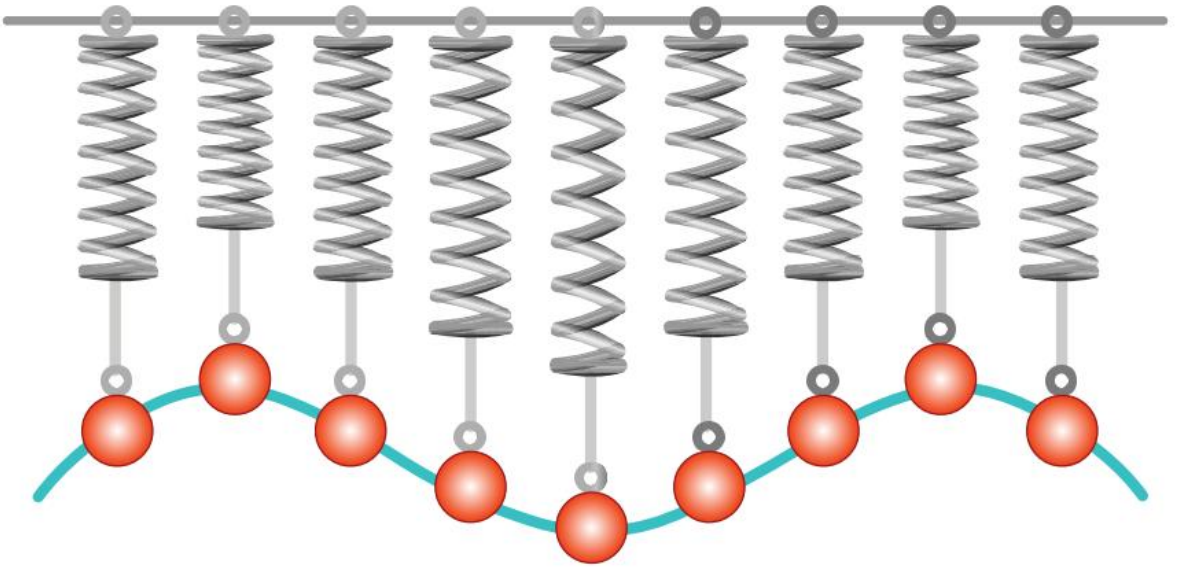
8.1.1 Introduction

The rate of change of one quantity with respect to another quantity is a measure of how the quantities are related or how a change in one quantity will effect a change in the other.

From Mathematical Methods, you have already learned that rates of change form the basis of the study of calculus. Differential equations build on the concepts already covered in calculus (chapters 6 and 7); differential and integral. Modelling the world around us using complex differential equations and sophisticated computers to solve these equations has led to an ever-increasing understanding of our world and the universe we inhabit.

Differential equations are used in many fields, including criminology, physics and biology. One long-standing use of differential equations has been in creating mathematical models which track the spread of virus among a population. As happened during the COVID pandemic, these models then help scientists and governments to predict future numbers of those affected, allowing them to plan their actions ahead of time.



8.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
8.2	**Implicit differentiation**	◯ Use implicit differentiation to determine the gradient of curves whose equations are given in implicit form.
8.3	**Related rates as instances of the chain rule**	◯ Model and solve related rates problems as instances of the chain rule including situations that involve surface area and volume of cones, pyramids and spheres, with and without technology.
8.4	**Solving differential equations of the form $\frac{dy}{dx} = f(x)$**	◯ Determine general and particular solutions of first-order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables.
8.5	**Solving differential equations of the form $\frac{dy}{dx} = g(y)$**	◯ Determine general and particular solutions of first-order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables.
8.6	**Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$**	◯ Determine general and particular solutions of first-order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables.
8.7	**Slope fields**	◯ Understand and use slope (direction or gradient) fields of a first-order differential equation.

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LESSON
8.2 Implicit differentiation

SYLLABUS LINKS

- Use implicit differentiation to determine the gradient of curves whose equations are given in implicit form.

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8.2.1 Introduction to implicit differentiation

Up until now the relationship between x and y has always been explicit: y is the dependent variable and x the independent variable, with y in terms of x written as $y = f(x)$. It is said that y depends upon x and $\frac{dy}{dx}$ can be found directly. That is because y is already isolated before we derive. For example:

If $y = x^2 + 4x + 13$, then $\frac{dy}{dx} = 2x + 4$.

If $y = 3\sin(2x) + 4e^{-2x}$, then $\frac{dy}{dx} = 6\cos(2x) - 8e^{-2x}$.

If $y = \sqrt{16 - x^2}$, then $\frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}}$.

If $y = \ln(5x + 3)$, then $\frac{dy}{dx} = \frac{5}{5x + 3}$.

There are times, however, when y is not expressed explicitly in terms of x, i.e. when y is not isolated. In these cases there is a functional dependence or a so-called implicit relationship between x and y of the form $f(x, y) = c$, where c is a constant, or $g(x, y) = 0$. For example:

$$x^2 + y^2 = 16$$
$$4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$$
$$e^{-2xy} + 3\sin(2x - 3y) = c$$

These represent curves as an implicit relation and are not graphs of functions.

In some cases it may be possible to rearrange to make y the subject, but in most cases this is simply not possible. In this implicit form an expression for $\frac{dy}{dx}$ can still be obtained, but it will be in terms of both x and y. This can be obtained by differentiating each term in turn, i.e. without needing to isolate y first before we derive.

Referring to the earlier equation of $x^2 + y^2 = 16$: $\frac{d}{dx}\left(x^2\right) = 2x$, and in general, if n is a constant, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Recall that $\frac{d}{dx}(\text{constant}) = 0$.

Consider $\frac{d}{dx}\left(y^2\right)$. To find this, use the chain rule and let $u = y^2$. Then $\frac{d}{dx}\left(y^2\right) = \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$; furthermore, in general $\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\frac{dy}{dx} = ny^{n-1}\frac{dy}{dx}$.

Implicit differentiation

This is implicit differentiation; it is just an application of the chain rule. When it is necessary to differentiate a function of y with respect to x, differentiate with respect to y and multiply by $\frac{dy}{dx}$. At the end of a problem, it is normal to rearrange the expression to isolate $\frac{dy}{dx}$ (to identify the derivative function in terms of y and x).

$$\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\ \frac{dy}{dx} = ny^{n-1}\ \frac{dy}{dx}$$

WORKED EXAMPLE 1 Using implicit differentiation with the equation of a circle

Given $x^2 + y^2 = 16$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y.

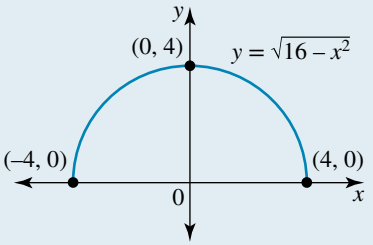
THINK	WRITE/DRAW
1. Write the equation.	$x^2 + y^2 = 16$
2. Take $\frac{d}{dx}(\ )$ of each term in turn. Remembering, when deriving y, to multiply by $\frac{dy}{dx}$.	$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$
3. Use the results above together with $\frac{d}{dx}(c) = 0$, since the derivative of a constant is zero.	$2x + 2y\frac{dy}{dx} = 0$
4. Transpose the equation to make $\frac{dy}{dx}$ the subject.	$2x = -2y\frac{dy}{dx}$
5. Cancel the common factor and divide by x. State the final result, isolating $\frac{dy}{dx}$ in terms of both x and y.	$\frac{dy}{dx} = -\frac{x}{y}$

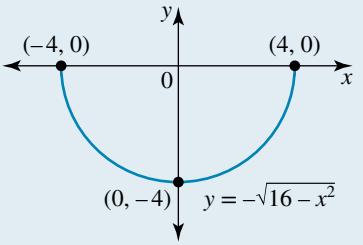
Alternate method

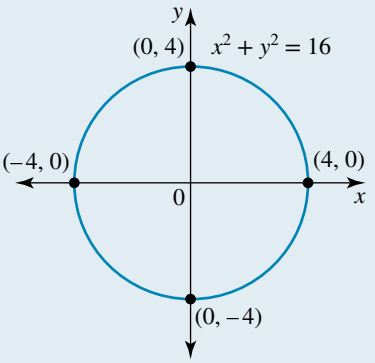
1. Write the equation.	$x^2 + y^2 = 16$
2. Rearrange to make y the subject.	$y^2 = 16 - x^2$

3. There are two branches to the relation, which is a circle with centre at the origin and radius 4. Consider the branch or top half of the circle, which by itself is a function.

$y = \pm\sqrt{16 - x^2}$







4. Rearrange to write $\frac{dy}{dx}$ in terms of x, differentiating using the chain rule.

Consider $y = \sqrt{16 - x^2}$.

$$\frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}}$$

5. However, since $y = \sqrt{16 - x^2}$, express $\frac{dy}{dx}$ in terms of both x and y as before.

$$\frac{dy}{dx} = -\frac{x}{y}$$

Further examples involving implicit differentiation

In the following worked example where $3xy$ is derived, the function contains two interdependent variables being multiplied together. It is important to note that we cannot think of either variable as a constant with respect to the other. Thus, the derivative needs to be determined using the product rule.

WORKED EXAMPLE 2 Using implicit differentiation with a product

a. Determine $\frac{d}{dx}(3xy)$.

b. Given $4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$, write down an expression for $\frac{dy}{dx}$ in terms of both x and y.

THINK	WRITE
a. 1. Write the expression.	a. $\frac{d}{dx}(3xy)$
2. Use the product rule.	$\frac{d}{dx}(3x \times y) = \frac{d}{dx}(u \times v)$ Let $u = 3x$ and $v = y$.

3. Calculate the derivatives, remembering, when deriving y, to include a factor of $\frac{dy}{dx}$.	$\frac{du}{dx} = 3$ and $\frac{dv}{dx} = \frac{d}{dy}(y)\frac{dy}{dx} = 1 \times \frac{dy}{dx} = \frac{dy}{dx}$
4. Use the product rule.	$\frac{d}{dx}(u \times v) = u\frac{dv}{dx} + v\frac{du}{dx}$
5. State the final result.	$\frac{d}{dx}(3xy) = 3x\frac{dy}{dx} + 3y$
b. 1. Write the equation.	**b.** $4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$
2. Take $\frac{d}{dx}(\ )$ of each term in turn.	$\frac{d}{dx}(4x^2) + \frac{d}{dx}(3xy) - \frac{d}{dx}(2y^2) + \frac{d}{dx}(5x) - \frac{d}{dx}(7y) + \frac{d}{dx}(8) = 0$
3. Substitute for the result from the second term from part **a** above and use implicit differentiation on each term.	$8x + 3x\frac{dy}{dx} + 3y - 4y\frac{dy}{dx} + 5 - 7\frac{dy}{dx} + 0 = 0$
4. Transpose the equation to get all terms involving $\frac{dy}{dx}$ on one side of the equation.	$8x + 3y + 5 = 7\frac{dy}{dx} + 4y\frac{dy}{dx} - 3x\frac{dy}{dx}$
5. Factor the terms involving $\frac{dy}{dx}$ on the right-hand side of the equation.	$8x + 3y + 5 = \frac{dy}{dx}(7 + 4y - 3x)$
6. Divide and state the final result, giving $\frac{dy}{dx}$ in terms of both x and y.	$\frac{dy}{dx} = \frac{8x + 3y + 5}{7 + 4y - 3x}$

Implicit differentiation with exponential or trigonometric functions

Sometimes it may be necessary to use the derivatives of exponential or trigonometric functions together with implicit differentiation techniques to obtain the required derivative.

WORKED EXAMPLE 3 Using implicit differentiation with exponential and trigonometric functions

Given $e^{-2xy} + 3\ \sin(2x - 3y) = c$, where c is a constant, determine an expression for $\frac{dy}{dx}$ in terms of both x and y.

THINK	WRITE
1. Write the equation.	$e^{-2xy} + 3\sin(2x - 3y) = c$
2. Take $\frac{d}{dx}(\ )$ of each term in turn.	$\frac{d}{dx}(e^{-2xy}) + \frac{d}{dx}(3\sin(2x - 3y)) = \frac{d}{dx}(c)$

3.	Consider just the first term and use implicit differentiation.	$\frac{d}{dx}(e^{-2xy}) = \frac{d}{dx}(e^{-u})$ where $u = 2xy$ $\frac{d}{du}(e^{-u})\frac{du}{dx}$
4.	Use the product rule.	$-e^{-u}\frac{d}{dx}(2xy) = -e^{-u}\left(2y + 2x\frac{dy}{dx}\right)$
5.	Substitute back for u.	$\frac{d}{dx}(e^{-2xy}) = -\left(2y + 2x\frac{dy}{dx}\right)e^{-2xy}$
6.	Consider just the second term and use implicit differentiation.	$\frac{d}{dx}(3\sin(2x-3y)) = \frac{d}{dv}(3\sin(v))\frac{dv}{dx}$ where $v = 2x - 3y$ so that $\frac{dv}{dx} = 2 - 3\frac{dy}{dx}$. $\frac{d}{dx}(3\sin(2x-3y)) = 3\cos(v)\left(2 - 3\frac{dy}{dx}\right)$
7.	Substitute back for v.	$\frac{d}{dx}(3\sin(2x-3y)) = 3\cos(2x-3y)\left(2 - 3\frac{dy}{dx}\right)$
8.	Substitute for the first and second terms.	$-\left(2y + 2x\frac{dy}{dx}\right)e^{-2xy} + 3\cos(2x-3y)\left(2 - 3\frac{dy}{dx}\right) = 0$
9.	Expand the brackets.	$-2ye^{-2xy} - 2xe^{-2xy}\frac{dy}{dx} + 6\cos(2x-3y) - 9\frac{dy}{dx}\cos(2x-3y) = 0$
10.	Transpose the equation to get all terms involving $\frac{dy}{dx}$ on one side of the equation.	$-2ye^{-2xy} + 6\cos(2x-3y) = 2xe^{-2xy}\frac{dy}{dx} + 9\cos(2x-3y)\frac{dy}{dx}$
11.	Factor the terms involving $\frac{dy}{dx}$ on the right-hand side of the equation.	$-2ye^{-2xy} + 6\cos(2x-3y) = (2xe^{-2xy} + 9\cos(2x-3y))\frac{dy}{dx}$
12.	Divide and state the final result, giving $\frac{dy}{dx}$ in terms of both x and y.	$\frac{dy}{dx} = \frac{6\cos(2x-3y) - 2ye^{-2xy}}{2xe^{-2xy} + 9\cos(2x-3y)}$

8.2.2 Parametric differentiation

In the previous section, we saw that whether the variables x and y are given in the form $y = f(x)$ as an explicit relationship or the form $f(x, y) = 0$ as an implicit relationship, an expression for $\frac{dy}{dx}$ can be found.

In this section, the two variables x and y are connected or related in terms of another variable, called a linking variable or a parameter. Often t or θ are used as parameters. In parametric equations, y and x will appear in separate equations, each involving the linking **parameter**.

A parameter is a variable that changes from case to case but in each particular case or instant it remains the same.

For example, an expression can be found for $\frac{dy}{dx}$; however, it will be in terms of the parameter by using the chain rule, since $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$, and noting that $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$. Alternatively, it may also be possible to eliminate the parameter from the two parametric equations and obtain an implicit relationship between the two variables x and y. The following examples will illustrate these concepts.

WORKED EXAMPLE 4 Using parametric differentiation

a. If $x = 3\cos(t)$ and $y = 4\sin(t)$, write the gradient $\frac{dy}{dx}$ in terms of t.

b. For the parametric equations $x = 3\cos(t)$ and $y = 4\sin(t)$, determine an implicit relationship between x and y, and write an expression for the gradient. Hence, verify your answer to part a.

THINK	WRITE
a. 1. Differentiate x with respect to t. The dot notation is used for the derivative with respect to t.	**a.** $x = \cos(t)$ $\frac{dx}{dt} = \dot{x} = -3\sin(t)$
2. Differentiate y with respect to t.	$y = 4\sin(t)$ $\frac{dy}{dt} = \dot{y} = 4\cos(t)$
3. Use the chain rule to determine $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \dot{y} \times \frac{1}{\dot{x}} = \frac{\dot{y}}{\dot{x}}$
4. Substitute for the derivatives.	$\frac{dy}{dx} = \frac{4\cos(t)}{-3\sin(t)}$
5. State the gradient in simplest form.	$\frac{dy}{dx} = -\frac{4}{3}\cot(t)$
b. 1. Write the parametric equations.	**b.** $x = 3\cos(t)$ [1] $y = 4\sin(t)$ [1]
2. Express the trigonometric ratios on their own.	$\cos(t) = \frac{x}{3}$ [1] $\sin(t) = \frac{y}{4}$ [2]
3. Eliminate the parameter to find the implicit relationship.	Since $\cos^2(1) + \sin^2(t) = 1$, $\frac{x^2}{9} + \frac{y^2}{16} = 1$
4. Use implicit differentiation on the implicit form.	$\frac{1}{9}\frac{d}{dx}(x^2) + \frac{1}{16}\frac{d}{dx}(y^2) = \frac{d}{dx}(1)$
5. Perform the implicit differentiation.	$\frac{2x}{9} + \frac{2y}{16}\frac{dy}{dx} = 0$
6. Transpose the equation.	$\frac{2x}{9} = -\frac{y}{8}\frac{dy}{dx}$
7. Make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} = -\frac{16x}{9y}$

8. From the implicit differentiation, substitute for the parametric equations. $\frac{dy}{dx} = -\frac{16 \times 3\cos(t)}{9 \times 4\sin(t)}$

9. Simplify to verify the given result, as above. $\frac{dy}{dx} = -\frac{4}{3}\cot(t)$

Exercise 8.2 Implicit differentiation

learnon

8.2 Exercise	8.2 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16	17, 18, 19, 20, 21, 22	23, 24, 25, 26

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Simple familiar

1. WE1 Given $x^3 + y^3 = 27$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y.

2. Given $\sqrt{x} + \sqrt{y} = 4$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y.

3. For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y.

   a. $y^2 - 2x = 3$
   b. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
   c. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
   d. $4x - 2y - 3x^2 + y^2 = 10$

4. WE2 a. Determine $\frac{d}{dx}\left(4x^2y^2\right)$.

   b. Given $9x^3 + 4x^2y^2 - 3y^3 + 2x - 5y + 12 = 0$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y.

5. Calculate the gradient of the normal to the curve $x^2 - 4xy + 2y^2 - 3x + 5y - 7 = 0$ at the point where $x = 2$ in the first quadrant.

6. For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y.

   a. $2x^2 + 3xy - 3y^2 + 8 = 0$
   b. $x^2 + x^2y^2 - 6x + 5 = 0$
   c. $x^3 - 3x^2y + 3xy^2 - y^3 - 27 = 0$
   d. $y^3 - y^2 - 3x - x^2 + 9 = 0$

7. a. Calculate the gradient of the tangent to the curve $y^2 = x^3(2 - x)$ at the point $(1, 1)$.
   b. Calculate the equation of the tangent to the curve $x^3 + 3xy + y^3 + 13 = 0$ at the point $(1, -2)$.
   c. For the circle $x^2 + 4x + y^2 - 3y + 6xy - 4 = 0$, calculate the gradient of the tangent to the circle at the point $(2, -1)$.
   d. A certain ellipse has the equation $3x^2 + 2xy + 4y^2 + 5x - 10y - 8 = 0$. Calculate the gradient of the normal to the ellipse at the point $(1, 2)$.

8. a. Determine the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point in the fourth quadrant where $x = 3$.

b. Determine the gradient of the tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at the point in the first quadrant where $x = 1$.

c. Determine the gradient of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point in the fourth quadrant where $x = 5$.

d. Determine the gradient of the normal to the curve $2\left(x^2 + y^2\right)^2 = 25\left(x^2 - y^2\right)$ at the point (3, 1).

9. WE4 a. If $x = 4\cos(t)$ and $y = 4\sin(t)$, determine the gradient $\frac{dy}{dx}$ in terms of t.

b. For the parametric equations $x = 4\cos(t)$ and $y = 4\sin(t)$, determine an implicit relationship between x and y, and write an expression for the gradient. Hence, verify your answer to part a.

10. a. Given $x = t^2$ and $y = 2t - t^4$, determine $\frac{dy}{dx}$ in terms of t.

b. For the parametric equations $x = t^2$ and $y = 2t - t^4$, express y in terms of x and determine $\frac{dy}{dx}$. Hence, verify your answer to part a.

11. For each of the following, determine $\frac{dy}{dx}$ in terms of the parameter.

a. The parabola $x = 2t^2$ and $y = 4t$

b. The ellipse $x = 3\sin(2t)$ and $y = 4\cos(2t)$

c. The rectangular hyperbola $x = 5t$ and $y = \frac{5}{t}$

12. In each of the following, a and b are constants. Determine $\frac{dy}{dx}$ in terms of the parameter.

a. $x = at^2$ and $y = 2at$

b. $x = a\cos(t)$ and $y = b\sin(t)$

c. $x = at$ and $y = \frac{a}{t}$

13. In each of the following, a and b are constants. Eliminate the parameter and determine $\frac{dy}{dx}$.

a. $x = at^2$ and $y = 2at$

b. $x = a\cos(t)$ and $y = b\sin(t)$

c. $x = at$ and $y = \frac{a}{t}$

14. A curve has the equation $x^3 - 3axy + y^3 = 0$. Determine $\frac{dy}{dx}$ for this curve, given that a is a constant.

15. A curve has the implicit equation $\left(x^2 + y^2\right)^2 = 2\left(x^2 - y^2\right)$. Determine an expression for $\frac{dy}{dx}$ in terms of x and y.

16. A curve has the equation $y^2 = \frac{x^3}{2a - x}$.
Determine $\frac{dy}{dx}$ for this curve, given that a is constant.

Complex familiar

17. WE3 Given $2xy + e^{-(x^2+y^2)} = c$, where c is a constant, determine an expression for $\frac{dy}{dx}$ in terms of both x and y.

18. If $\sin(3x + 2y) + x^2 + y^2 = c$, where c is a constant, determine $\frac{dy}{dx}$ in terms of both x and y.

19. For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y.

a. $\frac{y^2-2x}{3x^2+4y}=6x$ **b.** $\frac{x^3+8y^3}{x^2-2xy+4y^2}=x^2$ **c.** $e^{x+y}+\cos(y)-y^2=0$ **d.** $e^{xy}+\cos(xy)+x^2=0$

20. For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y.

a. $\ln(2x+3y)+4x-5y=10$ **b.** $\ln(3xy)+x^2+y^2-9=0$

c. $\frac{1}{x}+2xy+\frac{1}{y}-6=0$ **d.** $2y^3-\frac{\sin(3x)}{\sec(2y)}+x^2=0$

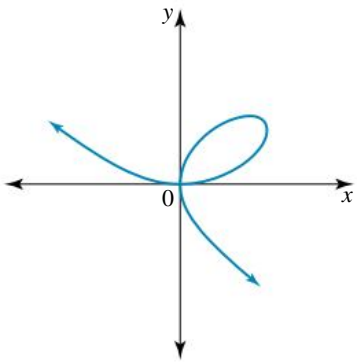
For questions 21–22, determine an expression for $\frac{dy}{dx}$ in terms of x and y.

21. The hyperbola $x=3\sec(t)$ and $y=4\tan(t)$

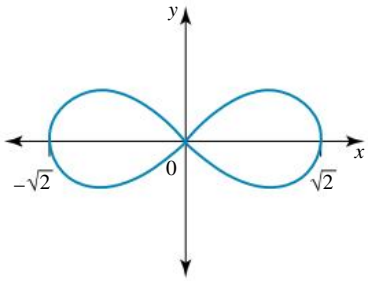
22. $x=a\sec(t)$ and $y=b\tan(t)$

Complex unfamiliar

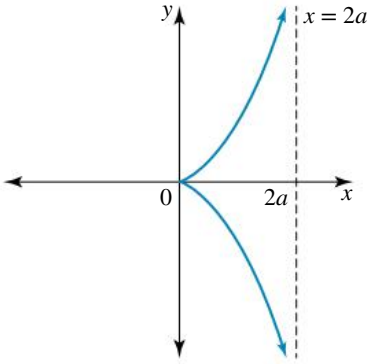
23. Rene Descartes (1596–1650) was a French mathematician and philosopher who lived in the Dutch republic. He is noted for introducing the Cartesian coordinate system in two dimensions and is credited as the father of analytical geometry, the link between algebra and geometry.
A curve called the folium of Descartes has the equation $x^3-3axy+y^3=0$, where a is a constant.
Show that the folium of Descartes can be represented by the parametric equations $x=\frac{3at}{1+t^3}$ and $y=\frac{3at^2}{1+t^3}$, and calculate the gradient of the curve at the point t.



24. A curve called a lemniscate has the implicit equation $\left(x^2+y^2\right)^2=2\left(x^2-y^2\right)$.
Show that the curve can be represented by the parametric equations $x=\frac{\sqrt{2}\cos(t)}{\sin^2(t)+1}$ and $y=\frac{\sqrt{2}\sin(t)\cos(t)}{\sin^2(t)+1}$, and determine $\frac{dy}{dx}$ in terms of t.



25. A curve known as the cissoid of Diocles has the equation $y^2=\frac{x^3}{2a-x}$, where a is constant.
Show that a parametric representation of the cissoid curve is given by the equations $x=\frac{2at^2}{1+t^2}$ and $y=\frac{2at^3}{1+t^2}$, and determine $\frac{dy}{dx}$ in terms of t.



26. A curve known as the cissoid of Diocles has the equation $y^2=\frac{x^3}{2a-x}$, where a is constant.
Show that another alternative parametric representation of the cissoid curve is given by the equations $x=2a\sin^2(t)$ and $y=\frac{2a\sin^3(t)}{\cos(t)}$, and determine $\frac{dy}{dx}$ in terms of t.

Fully worked solutions for this chapter are available online.

LESSON
8.3 Related rates as instances of the chain rule

SYLLABUS LINKS

- Model and solve related rates problems as instances of the chain rule including situations that involve surface area and volume of cones, pyramids and spheres, with and without technology.

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8.3.1 Introduction

When two or more quantities vary with time and are related by some condition, their rates of change are also related. The steps involved in solving these related rate problems are listed below.

Steps in solving a problem using related rates

1. **Define the variables.**
2. **Write down the rate that is provided in the question.**
3. **Write down any other known formulas that relate to the equation; for example, an applicable volume or area formula involving the variables of the rate.**
4. **Establish a relationship between the variables involving the chain rule and its conventions.**
5. **Write down the rate that needs to be determined.**
6. **Use a chain rule or implicit differentiation to find rate required, and answer the question. Sometimes this may mean further rearranging.**

WORKED EXAMPLE 5 Solving a related rate problem

A circular metal plate is being heated and its radius is increasing at a rate of 2 mm/s. Calculate the rate at which the area of the plate is increasing when the radius is 30 millimetres.

THINK	WRITE
1. Following the steps in solving a problem using related rates. Define the variables.	Let r millimetres be the radius of the metal at time t seconds. Let A mm^2 be the area of the plate at time t seconds.
2. The radius is increasing at a rate of 2 mm/s. Note that the units also help to determine the given rate.	$\frac{dr}{dt} = 2$ mm/s.
3. The plate is circular. Write the formula for the area of a circle, relating the variables A and r.	$A = \pi r^2$.
4. Establish a relationship involving the chain rule.	$\frac{dA}{dt} = ?$ and evaluate this rate when $r = 30$. $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$
5. Since one rate, $\frac{dr}{dt}$, is known, we need to determine $\frac{dA}{dr}$.	$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

6. Substitute the given rates into the required equations.	$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $= 2\pi r \times 2$
7. Evaluate the required rate when $r = 30$.	$\frac{dA}{dt} = 4\pi r$ when $r = 30$
8. State the final result with the required units, leaving the answer in terms of π.	$\left.\frac{dA}{dr}\right\|_{r=30} = 120\pi \text{ mm}^2/\text{s}$ The area is increasing at 120π mm^2/s.

Decreasing rates

If a quantity has a rate decreasing with respect to time, then the required rate is given as a negative quantity.

WORKED EXAMPLE 6 Solving a related decreasing rate problem

A spherical balloon has a hole in it, and the balloon's volume is decreasing at a rate of 2 cm^3/s. At what rate is the radius changing when the radius of the balloon is 4 centimetres?

THINK	WRITE
1. Define the variables.	Let r centimetres be the radius of the balloon at time t seconds. Let V cm^3 be the volume of the balloon at time t seconds.
2. The volume is decreasing at a rate of 2 cm^3/s. Notice that the units also help determine the given rate.	$\frac{dV}{dt} = -2 \text{ cm}^3/\text{s}$
3. The balloon is spherical. Write the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$
4. We need to determine the rate at which the radius changing.	$\frac{dr}{dt} = ?$ and evaluate this rate when $r = 4$.
5. Form a chain rule for the required rate in terms of the given variables.	$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$
6. Since one rate, $\frac{dV}{dt}$, is known, we need to determine $\frac{dr}{dV}$.	Since $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$ and $\frac{dr}{dV} = 1/\frac{dV}{dr} = \frac{1}{4\pi r^2}$
7. Substitute the given rates into the required equations.	$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ $= \frac{1}{4\pi r^2} \times -2$ $= -\frac{1}{2\pi r^2}$

8. Evaluate the required rate when $r = 4$.

When $r = 4$, $\left.\frac{dr}{dt}\right|_{r=4} = -\frac{1}{32\pi}$.

9. State the final result in the required units, leaving the answer in terms of π.

The radius is decreasing at a rate of $\frac{1}{32\pi}$ cm/s.

Alternate method

1. Use another chain rule.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$
$$-2 = 4\pi r^2 \times \frac{dr}{dt}$$
$$-\frac{2}{4\pi r^2} = \frac{dr}{dt}$$

2. Evaluate the required rate when $r = 4$.

$$\frac{dr}{dt} = -\frac{2}{4\pi 4^2}$$
$$= -\frac{1}{32\pi} \text{ cm/s}$$

3. State the answer.

The radius is decreasing at a rate of $\frac{1}{32\pi}$ cm/s.

Relating the variables

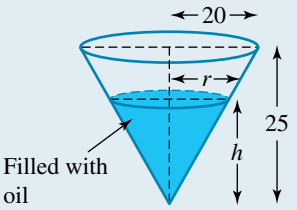
It is often necessary to express a required expression in terms of only one variable instead of two. This can be achieved by determining relationships between the variables.

WORKED EXAMPLE 7 Relating the variables

A conical funnel has a height of 25 centimetres and a radius of 20 centimetres. It is positioned so that its axis is vertical and its vertex is downwards. Oil leaks out through an opening in the vertex at a rate of 4 cubic centimetres per second. Calculate the rate at which the oil level is falling when the height of the oil in the funnel is 5 centimetres. (Ignore the cylindrical section of the funnel.)

THINK	WRITE/DRAW
1. Define the variables.	Let r cm be the radius of the oil in the funnel. Let h cm be the height of the oil in the funnel.
2. Write the rate given in the question.	$\frac{dV}{dt} = -4 \text{ cm}^3/\text{s}$
3. Write which rate is required to be determined.	$\frac{dh}{dt} = ?$ when $h = 5$
4. Make up a chain rule for the required rate, in terms of the given variables.	$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

5. Determine the relationship between the variables. The variables r and h change; however, the height and radius of the funnel are constant, and this can be used to determine a relationship between h and r.



$V = \frac{1}{3}\pi r^2 h$

Using similar triangles,

$\frac{20}{25} = \frac{r}{h}$

$r = \frac{4h}{5}$

6. We need to express the volume in terms of h only.

Substitute into $V = \frac{1}{3}\pi r^2 h$:

$V = \frac{1}{3}\pi\left(\frac{4h}{5}\right)^2 h$

$= \frac{16\pi h^3}{75}$

7. Determine the rate to substitute into the chain rule.

$\frac{dV}{dh} = \frac{16\pi h^2}{25}$

$\frac{dh}{dV} = \frac{25}{16\pi h^2}$

8. Substitute for the required rates.

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{25}{16\pi h^2} \times -4$

9. Evaluate the required rate when $h = 5$.

$\left.\frac{dh}{dt}\right|_{h=5} = \frac{25}{16\pi 5^2} \times -4 = -\frac{1}{4\pi}$

10. State the final result with the required units, leaving the answer in terms of π.

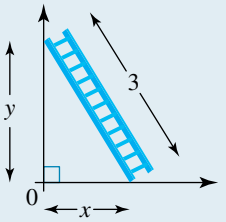
The height is decreasing at a rate of $\frac{1}{4\pi}$ cm/s.

Determining the required variables

An alternate method to solving related rate problems is to use implicit differentiation.

WORKED EXAMPLE 8 Using implicit differentiation to solve a related rate problem

A ladder 3 metres long has its top end resting against a vertical wall and its lower end on horizontal ground. The top end of the ladder is slipping down at a constant speed of 0.1 metres per second. Determine the rate at which the lower end is moving away from the wall when the lower end is 1 metre from the wall.

THINK	WRITE/DRAW
Method 1: Using the chain rule	
1. Define the variables.	Let x metres be the distance of the base of the ladder from the wall, and let y metres be the distance of the top of the ladder from the ground. 
2. The top end of the ladder is slipping down at a constant speed of 0.1 m/s.	$\frac{dy}{dt} = -0.1$ m/s
3. Apply Pythagoras' theorem.	$x^2 + y^2 = 3^2 = 9$
4. We need to determine the rate at which the lower end is moving away from the wall when the lower end is 1 m from the wall.	$\frac{dx}{dt} = ?$ when $x = 1$.
5. Construct a chain rule for the required rate in terms of the given variables.	$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$
6. Express y in terms of x.	$x^2 + y^2 = 9$ $y^2 = 9 - x^2$ $y = \pm\sqrt{9 - x^2}$ $y = \sqrt{9 - x^2}$ since $y > 0$
7. Since one rate, $\frac{dy}{dt}$, is known, we need to determine $\frac{dx}{dy}$.	$\frac{dy}{dx} = -\frac{x}{\sqrt{9 - x^2}}$ $\frac{dx}{dy} = -\frac{\sqrt{9 - x^2}}{x}$
8. Substitute the given rates into the required equation.	$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$ $= -\frac{\sqrt{9 - x^2}}{x} \times -0.1$
9. Evaluate the required rate when $x = 1$.	$\left.\frac{dx}{dt}\right\|_{x=1} = \frac{\sqrt{8}}{10} = \frac{2\sqrt{2}}{10}$
10. State the final result with the required units, giving an exact answer.	$\left.\frac{dx}{dt}\right\|_{x=1} = \frac{\sqrt{2}}{5}$ m/s The lower end is moving away from the wall at a rate of $\frac{\sqrt{2}}{5}$ m/s.

Method 2: Using implicit differentiation

The first 4 steps are identical to method 1 above. From this point, we use implicit differentiation to find the required rate.

5. Take $\frac{d}{dt}(\ )$ of each term in turn. $\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(9)$

6. Use implicit differentiation. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

7. Rearrange to make the required rate the subject. $\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$

8. Determine the appropriate values. When $x = 1$, $y = \sqrt{8}$ and $\frac{dy}{dt} = -0.1 = -\frac{1}{10}$

9. Substitute in the appropriate values of x and y. $\left.\frac{dx}{dt}\right|_{x=1} = \frac{\sqrt{8}}{10} = \frac{2\sqrt{2}}{10}$

10. State the final result as before. $\left.\frac{dx}{dt}\right|_{x=1} = \frac{\sqrt{2}}{5}$ m/s

The lower end is moving away from the wall at a rate of $\frac{\sqrt{2}}{5}$ m/s.

Exercise 8.3 Related rates as instances of the chain rule

learnon

8.3 Exercise | 8.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!
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Simple familiar

1. WE5 A circular oil slick is expanding so that its radius increases at a rate of 0.5 m/s. Determine the rate at which the area of the oil slick is increasing when the radius of the slick is 20 metres.

2. A circular disc is expanding so that its area increases at a rate of 40π cm^2/s. Determine the rate at which the radius of the disc is increasing when the radius is 10 cm.

3. a. A square has its sides increasing at a rate of 2 centimetres per second. Calculate the rate at which the area is increasing when the sides are 4 centimetres long.
   b. A stone is dropped into a lake, sending out concentric circular ripples. The area of the disturbed water region increases at a rate of 2 m^2/s. Calculate the rate at which the radius of the outermost ripple is increasing when its radius is 4 metres.

4. A spherical bubble is blown so that its radius is increasing at a constant rate of 2 millimetres per second. When its radius is 10 millimetres, calculate the rate at which:
   a. its volume is increasing
   b. its surface area is increasing.

5. WE6 A spherical basketball has a hole in it and its volume is decreasing at a rate of 6 cm^3/s. At what rate is the radius changing when the radius of the basketball is 6 cm?

6. A metal ball is dissolving in an acid bath. Its radius is decreasing at a rate of 3 cm/s. At what rate is the ball's surface area changing when the radius of the ball is 2 cm?

7. A mothball has its radius decreasing at a constant rate of 0.2 millimetres per week. Assume it remains spherical.

   a. Show that the volume is decreasing at a rate that is proportional to its surface area.
   b. If its initial radius is 30 millimetres, calculate how long it takes to disappear.

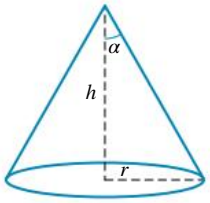
8. WE7 A conical vase has a height of 40 cm and a radius of 8 cm. The axis of the vase is vertical and its vertex is downwards. Initially it is filled with water which leaks out through a small hole in the vertex at a rate of 6 cm^3/s. Calculate the rate at which the water level is falling when the height of the water is 16 cm.

9. A cone is such that its radius is always equal to half its height. If the radius is decreasing at a rate of 2 cm/s, calculate the rate at which the volume of the cone is decreasing when the radius is 4 cm.

10. WE8 A ladder 5 metres long has its top end resting against a vertical wall and its lower end on horizontal ground. The bottom end of the ladder is pushed closer to the wall at a speed of 0.3 metres per second. Calculate the rate at which the top end of the ladder is moving up the wall when the lower end is 3 metres from the wall.

11. A kite is 30 metres above the ground and is moving horizontally away at a speed of 2 metres per second from the boy who is flying it. When the length of the string is 50 metres, at what rate is the string being released?

12. a. Show that the formula for the volume, V, of a right circular cone with height h is given by $V = \frac{1}{3}\pi h^3 \tan^2(\alpha)$, where α is the semi-vertex angle.



b. Falling sand forms a heap in the shape of a right circular cone whose semi-vertex angle is 71.57°. If its height is increasing at 2 centimetres per second when the heap is 5 centimetres high, calculate the rate at which its volume is increasing.

Complex familiar

13. The volume, V cm^3, of water in a hemispherical bowl of radius r cm when the depth of the water is h cm is given by $V = \frac{1}{3}\pi h^2(3r - h)$. A hemispherical bowl of radius 10 centimetres is being filled with water at a constant rate of cubic centimetres per second. At what rate is the depth of the water increasing when the depth is 5 cm?

14. A drinking glass is in the shape of a truncated right circular cone. When the glass is filled to a depth of h cm, the volume of liquid in the glass, V cm^3, is given by $V = \frac{\pi}{432}\left(h^3 + 108h^2 + 388h\right)$. Lemonade is leaking out from the glass at a rate of 7 cm^3/s. Calculate the rate at which the depth of the lemonade is falling when the depth is 6 cm.

15. A rubber flotation device is being pulled into a wharf by a rope at a speed of 26 metres per minute. The rope is attached to a point on the wharf 1 metre vertically above the dingy. At what rate is the rope being drawn in when the dingy is 10 metres from the wharf?

16. A car approaches the ground level of a 30-metre-tall building at a speed of 54 kilometres per hour. Calculate the rate of change of the distance from the car to the top of the building when it is 40 metres from the foot of the building.

Complex unfamiliar

17. The distance, q cm, between the image of an object and a certain lens in terms of p cm, the distance of the object from the lens, is given by $q = \dfrac{10p}{p-10}$.

a. Show that the rate of change of distance that an image is from the lens with respect to the distance of the object from the lens is given by $\dfrac{dq}{dp} = \dfrac{-100}{(p-10)^2}$.

b. If the object distance is increasing at a rate 0.2 cm/s, how fast is the image distance changing, when the distance from the object is 12 cm?

18. a. When a gas expands without a change of temperature, the pressure P and volume V are given by the relationship $PV^{1.4} = C$, where C is a constant. At a certain instant, the pressure is 1.01×10^5 pascals and the volume is cubic metres. The volume is increasing at a rate of 0.005 cubic metres per second. Calculate the rate at which the pressure is changing at this instant.

b. The pressure P and volume V of a certain fixed mass of gas during an adiabatic expansion are connected by the law $PV^n = C$, where n and C are constants. Show that the time rate of change of volume satisfies $\dfrac{dV}{dt} = -\dfrac{V}{nP}\dfrac{dP}{dt}$.

19. A jet aircraft is flying horizontally at a speed of 300 km/h at an altitude of 1 km. It passes directly over a radar tracking station located at ground level. Calculate the rate in degrees per second at which the radar beam to the aircraft is turning when the jet is at a horizontal distance of 30 kilometres from the station.

20. A helicopter is flying horizontally at a constant height of 300 metres. It passes directly over a light source located at ground level. The light source is always directed at the helicopter. If the helicopter is flying at 108 kilometres per hour, calculate the rate in degrees per second at which the light source to the helicopter is turning when the helicopter is at a horizontal distance of 0.4 kilometres from the light source.

Fully worked solutions for this chapter are available online.

LESSON
8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

SYLLABUS LINKS

- Determine general and particular solutions of first-order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables.

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8.4.1 Classification of differential equations

Any equation involving a derivative function could be considered a **differential equation**. However differential equations can be classified more specifically depending on the order and degree of the equation.

A first-order differential equation involves only the first derivative (e.g. $\frac{dy}{dx} = x^2$) whereas a second-order differential equation involves the second derivative (e.g. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 7$).

Thus, the order of the differential equation is the size of the largest derivative involved.

Separately, the **degree** of a differential equation is the highest power of the derivative (e.g. $\left(\frac{dy}{dx}\right)^2$ has a degree of 2).

Note this expression is significantly different to $\frac{d^2y}{dx^2}$, examples of differential equations are:

a. $\frac{dy}{dx} = ky$

b. $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

c. $\ddot{x} - t\dot{x} + 2x = t$

d. $\frac{d^2y}{dx^2} + n^2y = 0$

e. $x\left(\frac{dy}{dx}\right)^3 + 3\frac{dy}{dx} + 5y = 0$

f. $D_t^3x = \sqrt{x^2 + 1}$

Note that **a.** and **e.** are first order; **b.**, **c.** and **d.** are second order; and **f.** is third order. Equation **e.** has a degree of 3, whereas all the others have a degree of 1. Equations **a.**, **b.**, **c.** and **d.** are linear; **e.** and **f.** are non-linear. Note also that there are many different notations for derivatives; for example, second-order derivatives can be expressed as $\ddot{x} = \frac{d^2x}{dt^2}$ and $D_t^2x = \frac{d^2x}{dt^2}$.

8.4.2 Classifying solutions to a differential equation

The solution of a differential equation is usually obtained by the process of integration. Because the integration process produces an arbitrary constant of integration, the solutions of a differential equation are classified as follows.

A **general solution** is one which contains arbitrary constants of integration and satisfies the differential equation.

A **particular solution** is one which satisfies the differential equation and some other initial value condition, also known as a boundary value, that enables the constant(s) of integration to be found.

In general, the number of arbitrary constants of integration to be found is equal to the order of the differential equation. Throughout this course we study and solve special types of first- and second-order differential equations.

Throughout lessons 8.4 to 8.6, we focus on solving first-order differential equations. When solving these, the technique used is dependent on the functions involved within the equation.

8.4.3 Differential equations of the form $\frac{dy}{dx} = f(x)$

Direct integration

In this section we solve first-order differential equations of the form $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$. Differential equations of this form can be solved by direct integration. Hence, it is necessary to be familiar with all the integration techniques studied so far. You have seen in Mathematical Methods one way of solving this type of equation: anti-differentiating both sides gives $y = \int f(x)\,dx + c$. This is the general solution, which can be thought of as a family of curves.

However, moving forward it is more important to become accustomed to breaking up $\frac{dy}{dx}$. Until now, you have probably only seen $\frac{dy}{dx}$ written as a single term which cannot be separated. This is not the case. The 'dy' and 'dx' can be manipulated like any other part of an equation, and it is useful to do so when solving differential equations.

$$\frac{dy}{dx} = f(x)$$
$$dy = f(x)dx$$
$$\int dy = \int f(x)dx$$
$$y = \int f(x)dx$$

Remember that $\int dy = y + \text{constant}$ and $\int dx = x + \text{constant}$.

This matches what you have learnt in Mathematical Methods. If we use the given condition $x = x_0$ when $y = y_0$, we can determine the value of the constant of integration c in this particular case, which thus gives us the particular solution.

WORKED EXAMPLE 9 Using direct integration

a. Determine the general solution to $\frac{dy}{dx} + 12x = 0$.

b. Determine the particular solution of $\frac{dy}{dx} + 6x^2 = 0$, $y(1) = 2$.

THINK	WRITE
a. 1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.	a. $\frac{dy}{dx} + 12x = 0$ $\frac{dy}{dx} = -12x$

2. Separate dy from dx, ensuring that $f(x)$ is on the same side as dx.

$$dy = -12x\,dx$$

3. Anti-differentiate to obtain y.

$$\int dy = \int -12x\,dx$$
$$y = -\int 12x\,dx$$

4. Write the general solution in terms of a constant.

$$y = -6x^2 + c$$

b. 1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.

b. $\frac{dy}{dx} + 6x^2 = 0$

2. Separate dy from dx, ensuring that $f(x)$ is on the same side as dx.

$$dy = -6x^2\ \ dx$$

3. Anti-differentiate to obtain y.

$$\int dy = \int -12x\,dx$$
$$y = -\int 6x^2$$

4. Express y in terms of x with an arbitrary constant.

$$y = -2x^3 + c$$

5. Substitute and use the given conditions to determine the value of the constant.

$y(1) = 2$:
$\Rightarrow x = 1$ when $y = 2$

$$2 = -2(1)^3 + c$$
$$c = 4$$

6. Substitute back for c and state the particular solution.

$$y = 4 - 2x^3$$

Determining particular solutions

In chapter 6, linear substitutions were used to integrate linear expressions. The example presented in Worked example 10 is a review of this process.

WORKED EXAMPLE 10 Determining particular solutions

Solve the differential equation $(4-3x)^2\frac{dy}{dx} + 1 = 0$, $y(1) = 2$.

THINK	WRITE
1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$(4-3x)^2\frac{dy}{dx} + 1 = 0$
	$(4-3x)^2\frac{dy}{dx} = -1$
	$\frac{dy}{dx} = \frac{-1}{(4-3x)^2}$

2. Separate dy from dx, ensuring that $f(x)$ is on the same side as dx.

$$dy = \frac{-1}{(4-3x)^2}dx$$
$$\int dy = \int \frac{-1}{(4-3x)^2}dx$$

3. Anti-differentiate to obtain y.

$$y = \int \frac{-1}{(4-3x)^2}dx$$

4. Use index laws to express the integrand as a function to a power.

$$y = -\int (4-3x)^{-2}dx$$

5. Use a linear substitution. Express dx in terms of du by inverting both sides.

Let $u = 4 - 3x$.
$$\frac{du}{dx} = -3$$
$$\frac{dx}{du} = -\frac{1}{3}$$
$$dx = -\frac{1}{3}du$$

6. Substitute for u and dx.

$$y = -\int u^{-2}\,\frac{-1}{3}du$$

7. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$y = \frac{1}{3}\int u^{-2}\,du$$

8. Perform the integration process, using $\int u^n du = \frac{1}{n+1}u^{n+1}$.

$$y = -\frac{1}{3}u^{-1} + c$$
$$y = -\frac{1}{3u} + c$$

9. Substitute back for x.

$$y = -\frac{1}{3(4-3x)} + c$$

10. Substitute and use the given conditions to determine the value of the constant.

$y(1) = 2$
$\Rightarrow x = 1$ when $y = 2$
$$2 = -\frac{1}{3} + c$$
$$c = 2 + \frac{1}{3}$$
$$c = \frac{7}{3}$$

11. Substitute back for c, and state the particular solution. Although this is a possible answer, this result can be simplified.

$$y = \frac{-1}{3(4-3x)} + \frac{7}{3}$$

12. Form the lowest common denominator.

$$y = \frac{-1 + 7(4-3x)}{3(4-3x)}$$

13. Expand the brackets in the numerator; do not expand the brackets in the denominator.

$$y = \frac{-1 + 28 - 21x}{3(4-3x)}$$

14. Simplify and take out common factors which cancel.	$y = \dfrac{27-21x}{3(4-3x)}$ $= \dfrac{3(9-7x)}{3(4-3x)}$
15. State the final answer in simplified form. Note the maximal domain for which the solution is valid.	$y = \dfrac{9-7x}{4-3x}$ for $x \neq \dfrac{4}{3}$
16. Note that as a check, we can use the given condition to check the value of y.	Substitute $x = 1$: $y = \dfrac{9-7}{4-3} = 2$

Stating the domain for which the solution is valid

As seen in the last example, the **maximal domain** for which the solution is valid is important. When solving differential equations, unless the solution is defined for all values of x, that is, for $x \in \mathbb{R}$, we are required to state the largest subset of $\mathbb{R}$ for which the given differential equation and solution are valid.

WORKED EXAMPLE 11 Stating the domain for which the solution is valid

Solve the differential equation $\sqrt{3x-5}\,\dfrac{dy}{dx} + 6 = 0$, $y(7) = 2$, stating the largest domain for which the solution is valid.

THINK	WRITE
1. Rewrite the equation to make $\dfrac{dy}{dx}$ the subject.	$\sqrt{3x-5}\,\dfrac{dy}{dx} + 6 = 0$ $\sqrt{3x-5}\,\dfrac{dy}{dx} = -6$
2. Separate dy from dx, ensuring that $f(x)$ is on the same side as dx.	$dy = \dfrac{-6}{\sqrt{3x-5}}dx$
3. Anti-differentiate to obtain y.	$\int dy = \int \dfrac{-6}{\sqrt{3x-5}}dx$ $y = \int \dfrac{-6}{\sqrt{3x-5}}dx$
4. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$y = -6\int \dfrac{1}{\sqrt{3x-5}}dx$
5. Use index laws to express the integrand as a function to a power.	$y = -6\int (3x-5)^{-\frac{1}{2}}dx$

6. Use a linear substitution. Express dx in terms of du by inverting both sides.	Let $u = 3x - 5$. $\frac{du}{dx} = 3$ $\frac{dx}{du} = \frac{1}{3}$ $dx = \frac{1}{3}\,du$
7. Substitute for u and dx.	$y = -6\int u^{-\frac{1}{2}}\left(\frac{1}{3}du\right)$
8. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.	$y = -2\int u^{-\frac{1}{2}}du$
9. Perform the integration process using $\int u^n du = \frac{1}{n+1}u^{n+1}$.	$y = -4\sqrt{u} + c$
10. Substitute back for x.	$y = -4\sqrt{3x-5} + c$
11. Substitute and use the given conditions to determine the value of the constant.	$y(7) = 2$ $\Rightarrow x = 7$ when $y = 2$ $2 = -4\sqrt{16} + c$ $c = 18$
12. Substitute back for c and state the particular solution.	$y = 18 - 4\sqrt{3x-5}$
13. Determine the domain for which the solution is valid from the differential equation.	$\frac{dy}{dx} = \frac{-6}{\sqrt{3x-5}}$ for $3x - 5 > 0$
14. Solve the inequality for x to state the largest domain for which the solution is valid for the given differential equation. State the answer.	$3x > 5$ $x > \frac{5}{3}$ The solution $y = 18 - 4\sqrt{3x-5}$ is valid for $x > \frac{5}{3}$.

As we have said at the start of this chapter, all of the integration techniques that you have learned in Mathematical Methods and Specialist Mathematics up until this point could appear in a differential equation question. These include trigonometric and inverse trigonometric functions, integration by parts, integration by substitution, partial fractions and so on.

Solving first-order differential equations involving inverse trigonometric functions

The results $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ for $x \in (-a, a)$, $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ for $x \in (-a, a)$ and $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ for $x \in \mathbb{R}$ are used throughout this topic.

WORKED EXAMPLE 12 Solving first-order differential equations involving inverse trigonometric functions

Solve the differential equation $\sqrt{16-x^2}\,\dfrac{dy}{dx}+2=0$, $y(0)=0$, stating the largest domain for which the solution is valid.

THINK	WRITE
1. Rewrite the equation to make $\dfrac{dy}{dx}$ the subject.	$\sqrt{16-x^2}\,\dfrac{dy}{dx}+2=0$ $\sqrt{16-x^2}\,\dfrac{dy}{dx}=-2$ $y=\dfrac{-2}{\sqrt{16-x^2}}dx$
2. Separate dy from dx, ensuring that $f(x)$ is on the same side as dx.	$\int dy=\int\dfrac{-6}{\sqrt{3x-5}}dx$
3. Anti-differentiate to obtain y.	$y=\int\dfrac{-2}{\sqrt{16-x^2}}dx$
4. Perform the integration process using $\int\dfrac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\dfrac{x}{a}\right)+c$.	$y=2\cos^{-1}\left(\dfrac{x}{4}\right)+c$
5. Substitute and use the given conditions to determine the value of the constant.	$y(0)=0$ $\Rightarrow x=0$ when $y=0$ $0=2\cos^{-1}(0)+c$ $c=-2\cos^{-1}(0)$ $c=-2\times\dfrac{\pi}{2}$ $c=-\pi$
6. Substitute back for c and state the particular solution.	$y=2\cos^{-1}\left(\dfrac{x}{4}\right)-\pi$
7. Determine the domain for which the solution is valid from the differential equation.	$y=\int\dfrac{-2}{\sqrt{16-x^2}}\,dx$ $\sqrt{16-x^2}>0$ $x^2<16$
8. Solve the inequality for x to state the largest domain for which the solution and the differential equation is valid. State the answer.	$\lvert x\rvert<4$ The solution $y=2\cos^{-1}\left(\dfrac{x}{4}\right)-\pi$ is valid for $\lvert x\rvert<4$ or $x\in(-4,\ 4)$.

Exercise 8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

learnon

8.4 Exercise | 8.4 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	12, 13, 14, 15, 16, 17, 18	19, 20

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Simple familiar

1. a. WE9 Determine the general solution of $\frac{dy}{dx} + 12x^3 = 0$.

   b. Determine the particular solution of $\frac{dy}{dx} + 6x = 0$, $y(2) = 1$.

2. a. Determine the general solution of $\frac{dy}{dx} + 12\cos(2x) = 0$.

   b. Solve the differential equation $\frac{dy}{dx} + 6\sin(3x) = 0$, $y(0) = 0$, and express y in terms of x.

3. Determine the general solution to each of the following.

   a. $\frac{dy}{dx} - 4x = 3$

   b. $\frac{dy}{dx} - (3x - 5)(x + 4) = 0$

   c. $e^{2x}\frac{dy}{dx} + 6 = 2e^{4x}$

   d. $\sqrt{x^2 + 9}\,\frac{dy}{dx} - x = 0$

4. WE10 Solve the differential equation $(5 - 4x)^2\frac{dy}{dx} + 1 = 0$, $y(1) = 2$.

5. Solve the differential equation $(7 - 4x)\frac{dy}{dx} + 2 = 0$, $y(2) = 3$.

6. WE11 Solve the differential equation $\sqrt{2x - 5}\,\frac{dy}{dx} + 1 = 0$, $y(3) = 0$, stating the largest domain for which the solution is valid.

7. Solve the differential equation $\sqrt{x}\,\frac{dy}{dx} + 2 = 0$, $y(4) = 3$, expressing y in terms of x, and state the largest domain for which the solution is valid.

8. WE12 Solve the differential equation $\sqrt{64 - x^2}\,\frac{dy}{dx} - 6 = 0$, $y(4) = 0$, stating the largest domain for which the solution is valid.

9. Solve the differential equation $(16 + x^2)\frac{dy}{dx} + 4 = 0$, $y(4) = \frac{\pi}{4}$, stating the largest domain for which the solution is valid.

For questions 10–18, solve each of the differential equations given and state the maximal domain for which the solution is valid.

10. **a.** $3x\dfrac{dy}{dx} - 2x^2 = 5,\ y(1) = 3$

b. $\dfrac{dy}{dx} = 6(e^{-3x} + e^{3x}),\ y(0) = 0$

11. **a.** $\dfrac{dy}{dx} - 4\sin(2x) = 0,\ y(0) = 2$

b. $\dfrac{dy}{dx} + 6\cos(3x) = 0,\ y\left(\dfrac{\pi}{2}\right) = 5$

Complex familiar

12. **a.** $\dfrac{dy}{dx} - 8\ \sin^2(2x) = 0,\ y(0) = 0$

b. $\dfrac{dy}{dx} - 12\ \cos^2(3x) = 0,\ y(0) = 0$

13. **a.** $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4x+9}},\ y(0) = 0$

b. $\dfrac{dy}{dx} + \dfrac{1}{3-2x} = 0,\ y(2) = 1$

14. **a.** $\dfrac{dy}{dx} = \dfrac{1}{(3x-5)^2},\ y(2) = 3$

b. $\dfrac{dy}{dx} = \dfrac{8}{7-4x},\ y(2) = 5$

15. **a.** $(x^2+9)\dfrac{dy}{dx} - 3x = 0,\ y(0) = 0$

b. $\sqrt{x^2+4}\ \dfrac{dy}{dx} + x = 0,\ y(0) = 0$

16. **a.** $(x^2+6x+13)\ \dfrac{dy}{dx} - x = 3,\ y(0) = 0$

b. $(x^2-4x+9)\ \dfrac{dy}{dx} + x = 2,\ y(0) = 0$

17. **a.** $\sec(2x)\dfrac{dy}{dx} + \sin^3(2x) = 0,\ y(0) = 0$

b. $\operatorname{cosec}(3x)\dfrac{dy}{dx} + 9\ \cos^2(3x) = 0,\ y(0) = 0$

18. **a.** $\dfrac{dy}{dx} + \ln(2x) = 4,\ y\left(\dfrac{1}{2}\right) = 1$

b. $e^x\ \dfrac{dy}{dx} + x = 5,\ y(0) = 0$

Complex unfamiliar

19. Determine a differential equation with $y(x) = \dfrac{x+k}{x^2+1}, k \in \mathbb{R}$ as its general solution.

20. Determine the general solution to $\dfrac{dy}{dx} + y = y(0) + y(1)$.

Fully worked solutions for this chapter are available online.

LESSON
8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

SYLLABUS LINKS

- Determine general and particular solutions of first-order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables.

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8.5.1 Writing $\frac{dy}{dx} = g(y)$ as $\int \frac{dy}{g(y)} = \int dx$

In lesson 8.4, we solved equations where the right-hand side was a function of x. To solve these, we typically used a process where we separated the dy from the dx, ensuring the dx moved to the right-hand side to be alongside the function of x.

$$\frac{dy}{dx} = f(x)$$
$$dy = f(x)dx$$
$$\int dy = \int f(x)\,dx$$

In this lesson, we will look at situations where the right-hand side is a function of y. To solve this type of differential equation, we again separate dy from the dx. We then need to move the function of y to be on the same side as the dy.

$$\frac{dy}{dx} = g(y)$$
$$dy = g(y)\,dx$$
$$\frac{dy}{g(y)} = dx$$
$$\int \frac{dy}{g(y)} = \int dx$$

Also note that until now, when you have integrated functions in Mathematical Methods and Specialist Mathematics, you would be typically used to adding the constant of integration ($+c$) to the side involving a function of x.

While there is nothing wrong with doing this, sometimes, when solving differential equations, it can be more mathematically convenient to add this constant to the $g(y)$ side. Be aware that even though you are integrating on both sides, only one constant needs to be added (on the side of your choosing).

Determining general solutions

Finding a general solution means finding the solution in terms of an arbitrary constant.

WORKED EXAMPLE 13 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

Determine the general solution to the differential equation $\frac{dy}{dx} - 4\sqrt{y} = 0$.

THINK	WRITE
1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} - 4\sqrt{y} = 0$ $\frac{dy}{dx} = 4\sqrt{y}$
2. Separate dy from dx, ensuring that $g(y)$ is on the same side as dy.	$\frac{dy}{\sqrt{y}} = 4dx$ $\frac{dx}{dy} = \frac{1}{4\sqrt{y}}$
3. Anti-differentiate to obtain x in terms of y.	$\int \frac{dy}{\sqrt{y}} = \int 4dx$ $\int \frac{1}{4\sqrt{y}}dy = \int 4dx$
4. Use index laws to express the integrand as a power.	$\int y^{-\frac{1}{2}}dy = \int 4dx$
5. Perform the integration process using $\int u^n du = \frac{1}{n+1}u^{n+1}$. Remember that the constant of integration can be added to either side.	$2y^{\frac{1}{2}} = 4x + c$
6. Simplify.	$\sqrt{y} = 2x + \frac{c}{2}$
7. Since c is a constant, $\frac{c}{2}$ is also a constant.	Let $A = \frac{c}{2}$ $\sqrt{y} = 2x + A$
8. Square both sides and state the answer in terms of an arbitrary constant A.	$y = (2x + A)^2$

Determining particular solutions

Finding particular solutions involves solving the differential equation and expressing y in terms of x, and finding the value of the constant of integration.

WORKED EXAMPLE 14 Determining particular solutions to differential equations in the form $\frac{dy}{dx} = g(y)$

Solve the differential equation $\frac{dy}{dx} + (4-3y)^2 = 0$, $y(2) = 1$.

THINK	WRITE
1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} + (4-3y)^2 = 0$ $\frac{dy}{dx} = -(4-3y)^2$
2. Separate dy from dx, ensuring that $g(y)$ is on the same side as dy.	$\frac{dy}{(4-3y)^2} = -1dx$
3. Express x as an integral of y.	$\int \frac{1}{(4-3y)^2} dy = \int -dx$
4. Use index laws to express the integrand as a function to a power.	$\int (4-3y)^{-2} dy = -x + c$
5. Recognise the form $\int n \times u' \times u^{n-1}$ where $u = 4-3y$ and $n = -1$.	$\int -1 \times \frac{1}{3} \times -3 \times (4-3y)^{-2}\, dy = -x + c$
6. Perform the integration process using $\int u^n du = \frac{1}{n+1} u^{n+1}$.	$\frac{1}{3(4-3y)} = -x + C$
7. Substitute and use the given conditions to determine the value of the constant.	$y(2) = 1$ $\Rightarrow x = 2$ when $y = 1$ $\frac{1}{3(4-3\times 1)} = -2 + c$ $\frac{1}{3} = -l + c$ $c = \frac{7}{3}$
8. Substitute back for c.	$\frac{1}{3(4-3y)} = -x + \frac{7}{3}$ $\frac{1}{7-3x} = 4 - 3y$ $3y = 4 - \frac{1}{7-3x}$
9. Rearrange to make y the subject.	$y = \frac{4}{3} - \frac{1}{3(7-3x)}$
10. Express the right-hand side of the equation with a common denominator.	$y = \frac{4(7-3x)-1}{3(7-3x)}$ $= \frac{27-12x}{3(7-3x)}$ $= \frac{g-4x}{7-3x}$

11. Simplify.	$y = \dfrac{9-4x}{7-3x}$
12. State the final answer in a simplified form and state the maximal domain.	$y = \dfrac{9-4x}{7-3x}$ for $x \in \mathbb{R} \setminus \left\{\dfrac{7}{3}\right\}$.

Determine c or rearrange to make y the subject?

When solving these types of differential equations, it is necessary to determine the constant of integration and also rearrange to make y the subject. Sometimes the order in which we do these operations can make the processes simpler. In fact, sometimes the constant of integration takes a different form after being manipulated in the equation, as shown in the method 2 below.

WORKED EXAMPLE 15 Determining the constant of integration

Solve the differential equation $\dfrac{dy}{dx} + 4y = 0,\ y(0) = 3$.

THINK	WRITE
1. Rewrite the equation to make $\dfrac{dy}{dx}$ the subject.	$\dfrac{dy}{dx} + 4y = 0$ $\dfrac{dy}{dx} = -4y$
2. Separate dy from dx, ensuring that $g(y)$ is on the same side as dy	$\dfrac{dy}{y} = -4x$
3. Integrate both sides.	$\int \dfrac{1}{y}dy = \int -4dx$
4. Use $\int \dfrac{1}{u}du = \ln(\lvert u\rvert)$ to express x in terms of y and the constant of integration c.	$\ln\lvert y\rvert = -4x + c$
From this point forward, we have two processes to complete: determine c, and transpose the equation to make y the subject.	
Method 1 Determine c first, then transpose to make y the subject.	
5. Substitute and use the given conditions to determine the value of the constant.	$y = (0) = 3$ $\Rightarrow x = 0$ when $y = 3$ $\ln(3) = -4 \times 0 + c$ $c = \ln(3)$
6. Substitute back for c.	$\ln\lvert y\rvert = -4x + \ln(3)$ $\ln\lvert y\rvert - \ln(3) = -4x$
7. Use the logarithm laws to simplify the expression.	$\ln\left(\dfrac{\lvert y\rvert}{3}\right) = -4x$

8. Use the definition of the logarithm. $\frac{|y|}{3} = e^{-4x}$

9. Make y the subject. $|y| = 3e^{-4x}$

10. Because $e^{-4x} > 0$, the modulus is not needed. State the particular solution to the differential equation. $y = 3e^{-4x}$

Method 2

Make y the subject and then determine the constant c.

5. Rearrange to make y the subject.

$$\ln|y| = -4x + c$$
$$|y| = e^{-4x+c}$$
$$= e^{-4x} \times e^{c}$$

6. Since c is a constant, e^c is also a constant.

$$A = e^c$$
$$|y| = Ae^{-4x}$$

7. Substitute and use the given condition to determine the value of the constant.

$$y(0) = 3$$
$$\Rightarrow x = 0 \text{ when } y = 3$$
$$3 = Ae^{-0}$$
$$3 = A$$

8. Because $e^{-4x} > 0$, the modulus is not needed. Substitute for A and state the particular solution to the differential equation. $y = 3e^{-4x}$

Stating the domain for which the solution is valid

As discussed in the previous section, the solution to a differential equation should include the largest domain for which the solution is valid.

WORKED EXAMPLE 16 Stating the domain for which the solution is valid for equations in the form $\frac{dy}{dx} = g(y)$

Solve the differential equation $2\frac{dy}{dx} + \sqrt{16 - y^2} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

THINK	WRITE
1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$2\frac{dy}{dx} + \sqrt{16 - y^2} = 0$ $2\frac{dy}{dx} = -\sqrt{16 - y^2}$ $\frac{dy}{dx} = \frac{-\sqrt{16 - y^2}}{2}$
2. Invert both sides.	$\frac{dx}{dy} = \frac{-2}{\sqrt{16 - y^2}}$

3. Integrate with respect to y.

$$x = \int \frac{-2}{\sqrt{16-y^2}}\,dy$$

4. Perform the integration process using $\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$.

$$x = 2\cos^{-1}\left(\frac{y}{4}\right) + c$$

5. Substitute and use the given conditions to determine the value of the constant.

$$y(0) = 0$$
$$\Rightarrow x = 0 \text{ when } y = 0$$
$$0 = 2\cos^{-1}(0) + c$$
$$c = -2\cos^{-1}(0)$$
$$c = -2 \times \frac{\pi}{2}$$
$$c = -\pi$$

6. Substitute back for c.

$$x = 2\cos^{-1}\left(\frac{y}{4}\right) - \pi$$

7. Rewrite the equation.

$$2\cos^{-1}\left(\frac{y}{4}\right) = x + \pi$$
$$\cos^{-1}\left(\frac{y}{4}\right) = \frac{x+\pi}{2}$$

8. Take the cosine of both sides to make y the subject.

$$\frac{y}{4} = \cos\left(\frac{x}{2} + \frac{\pi}{2}\right)$$
$$y = 4\cos\left(\frac{x}{2} + \frac{\pi}{2}\right)$$

9. Expand using trigonometric compound-angle formulas.

$$y = 4\left(\cos\left(\frac{x}{2}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{x}{2}\right)\sin\left(\frac{\pi}{2}\right)\right)$$
$$= 4\left(\cos\left(\frac{x}{2}\right) \times 0 - \sin\left(\frac{x}{2}\right) \times 1\right)$$

10. State the particular solution.

$$y = -4\sin\left(\frac{x}{2}\right)$$

11. Determine the domain for which the solution is valid.

$$\cos^{-1}\left(\frac{y}{4}\right) = \frac{x+\pi}{2}$$

The range of $y = \cos^{-1}(x)$ is $[0, \pi]$, but $|y| < 4$, so

$$0 < \frac{x+\pi}{2} < \pi$$

12. Solve the inequality for x to state the largest domain for which the solution is valid. State the answer.

$$0 < x + \pi < 2\pi$$
$$-\pi < x < \pi$$

The solution $y = -4\sin\left(\frac{x}{2}\right)$ is valid for $-\pi < x < \pi$.

Exercise 8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

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8.5 Exercise | 8.5 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	12, 13, 14, 15, 16, 17, 18	19, 20

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Simple familiar

1. WE13 Determine the general solution to the differential equation $\sqrt{y}\,\frac{dy}{dx} + 4 = 0$.

2. Determine the general solution to the differential equation $\frac{dy}{dx} - \tan(2y) = 0$.

3. Determine the general solution to each of the following.

   a. $\frac{dy}{dx} = \frac{y^2}{4}$ b. $\frac{dy}{dx} = y + 4$ c. $\frac{dy}{dx} = \frac{y}{4}$ d. $\frac{dy}{dx} = \frac{4}{y^2}$

4. WE14 Solve the differential equation $\frac{dy}{dx} + (5 - 4y)^2 = 0$, $y(1) = 2$.

5. Solve the differential equation $\frac{dy}{dx} + 4y - 7 = 0$, $y(0) = 3$.

6. WE15 Solve the differential equation $\frac{dy}{dx} + 3y = 0$, $y(0) = 5$.

7. Given the differential equation $\frac{dy}{dx} - 5y = 0$, $y(0) = 3$, express y in terms of x.

8. Solve the following differential equations, expressing y in terms of x.

   a. $\frac{dy}{dx} + 5y = 0$, $y(0) = 4$ b. $\frac{dy}{dx} - 3y = 0$, $y(1) = 2$

9. Solve the following differential equations, expressing y in terms of x.

   a. $\frac{dy}{dx} + 2y = 5$, $y(0) = 3$ b. $\frac{dy}{dx} - 3y + 4 = 0$, $y(0) = 2$

For questions 10–18, solve each of the differential equations given and state the maximal domain for which the solution is valid.

10. a. $\frac{dy}{dx} = \sqrt{y}$, $y(1) = 4$ b. $\frac{dy}{dx} = y^2$, $y(1) = 3$

11. a. $\frac{dy}{dx} = 4e^{2y}$, $y(2) = 0$ b. $\frac{dy}{dx} + 6e^{3y} = 0$, $y(1) = 0$

Complex familiar

12. WE16 Solve the differential equation $\sqrt{64 - y^2} - 6\frac{dy}{dx} = 0$, $y(0) = 8$, stating the largest domain for which the solution is valid.

13. Solve the differential equation $16 + y^2 - 4\frac{dy}{dx} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

14. **a.** $\frac{dy}{dx} = (5-2y)^2$, $y(1) = 3$

b. $\frac{dy}{dx} + (7-3y)^2 = 0$, $y(3) = 2$

15. **a.** $\frac{dy}{dx} + 6\operatorname{cosec}\left(\frac{y}{2}\right) = 0$, $y\left(\frac{1}{3}\right) = 0$

b. $\frac{dy}{dx} = 2\sec(2y)$, $y\left(\frac{1}{8}\right) = \frac{\pi}{12}$

16. **a.** $\frac{dy}{dx} - \sqrt{4y+9} = 0$, $y(0) = 0$

b. $\frac{dy}{dx} - 4y^2 = 9$, $y(0) = 0$

17. **a.** $\frac{dy}{dx} + 4y = y^2$, $y(0) = 3$

b. $\frac{dy}{dx} - 3y = y^2$, $y(0) = 6$

18. **a.** $\frac{dy}{dx} + 7y = y^2 + 12$, $y(0) = 0$

b. $\frac{dy}{dx} - 6y - y^2 = 8$, $y(0) = 0$

Complex unfamiliar

19. **a.** If k and y_0 are constants, solve the differential equation $\frac{dy}{dx} + ky = 0$, $y(0) = y_0$.

b. Given that a, b and c are non-zero real constants, solve the differential equations:

i. $\frac{dy}{dx} + ay = b$, $y(0) = c$

ii. $\frac{dy}{dx} + ay = by^2$, $y(0) = c$.

20. **a.** Given that a and b are non-zero real constants, solve the differential equations:

i. $\frac{dy}{dx} = (ay+b)^2$, $y(0) = 0$

ii. $\frac{dy}{dx} = b^2y^2 + a^2$, $y(0) = 0$.

b. If a and b are constants with $a > b > 0$:

i. solve the differential equation $\frac{dy}{dx} = (y+a)(y+b)$, $y(0) = 0$

ii. find $\lim_{x\to\infty} y(x)$.

Fully worked solutions for this chapter are available online.

LESSON
8.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$

SYLLABUS LINKS

- Determine general and particular solutions of first-order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables.

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8.6.1 Separation of variables

Differential equations of the form $\frac{dy}{dx} = f(x)g(y)$, $y(x_0) = y_0$ are called variables separable equations, as it is possible to separate all the x terms onto one side of the equation and all the y terms onto the other side of the equation.

For $\frac{dy}{dx} = f(x)g(y)$, divide both sides by $g(y)$, since $g(y) \neq 0$. This gives

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrate both sides of the equation with respect to x.

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx$$

Remember, as was stated in lesson 8.5, you can choose which side to place the constant of integration, after the integration part of the problem has been completed.

$$\text{Thus, } \int \frac{1}{g(y)}dy + c_1 = \int f(x)\,dx + c_2.$$

$$\int \frac{1}{g(y)}dy = \int f(x)\,dx + c, \text{ since } c = c_2 - c_1.$$

After performing the integration, an implicit relationship between x and y is obtained. However, in specific cases it may be possible to rearrange to make y the subject.

WORKED EXAMPLE 17 Using separation of variables

Determine the general solution to the differential equation $\frac{dy}{dx} = \frac{x+4}{y^2+4}$.

THINK	WRITE
1. Write the differential equation.	$\frac{dy}{dx} = \frac{x+4}{y^2+4}$

2. Separate the variables and integrate both sides. $\int (y^2+4)\,dy = \int (x+4)\,dx$

3. Perform the integration and add the constant on one side only. $\frac{1}{3}y^3 + 4y = \frac{1}{2}x^2 + 4x + c$

4. The general solution is given as an implicit equation, as in this case it is impossible to solve this equation explicitly for y. $\frac{1}{3}y^3 + 4y - \frac{1}{2}x^2 - 4x = c$

Determining particular solutions

Finding particular solutions involves solving the differential equation, expressing y in terms of x where possible, and finding the value of the constant of integration.

WORKED EXAMPLE 18 Determining particular solutions for equations in the form $\frac{dy}{dx} = f(x)g(y)$

Solve the differential equation $\frac{dy}{dx} + y = 6x^2y$, $y(0) = 1$.

THINK	WRITE
1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} + y = 6x^2y$ $\frac{dy}{dx} = 6x^2y - y$
2. Factor the RHS.	$\frac{dy}{dx} = y(6x^2 - 1)$
3. Separate the variables and integrate both sides.	$\int \frac{1}{y}dy = \int (6x^2 - 1)dx$
4. Perform the integration and add in the constant on one side only.	$\ln(\lvert y\rvert) = 2x^3 - x + c$
Method 1	
5. Substitute and use the given conditions to determine the value of the constant.	$y(0) = 1$ $\Rightarrow x = 0$ when $y = 1$ $\ln(\lvert 1\rvert) = 0 + c$ $c = 0$
6. Substitute back for c and use definition of a logarithm to state the solution explicitly as y in terms of x. Note that the modulus is not needed, as $e^{2x^3 - x} > 0$.	$\ln(\lvert y\rvert) = 2x^3 - x$ $y = e^{2x^3 - x}$

Alternate method

Make y the subject before determining c.

$$\begin{aligned}\ln|y| &= 2x^3 - x + c\\ |y| &= e^{2x^3-x+c}\\ &= e^{2x^3-x}e^c\\ &= Ae^{2x^3-x}\\ 1 &= Ae^0\\ A &= 1\\ y &= e^{2x^3-x}\end{aligned}$$

Stating the domain for which the solution is valid

As previously stated, when solving differential equations it is necessary to state the largest domain for which the solution is valid.

WORKED EXAMPLE 19 Determining the domain for which solutions to equations in the form $\frac{dy}{dx} = f(x)g(y)$ are valid

Solve the differential equation $\frac{dy}{dx} + 2x\sqrt{16-y^2} = 0$, $y(0) = 4$, stating the largest domain for which the solution is valid.

THINK	WRITE
1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.	$\frac{dy}{dx} + 2x\sqrt{16-y^2} = 0,\ y(0) = 4$ $\frac{dy}{dx} = -2x\sqrt{16-y^2}$
2. Separate the variables and integrate both sides.	$\int \frac{-1}{\sqrt{16-y^2}}\,dy = \int 2x\,dx$
3. Perform the integration and add the constant on one side only.	$\cos^{-1}\left(\frac{y}{4}\right) = x^2 + c$
4. Substitute and use the given conditions to determine the value of the constant.	$y(0) = 4$ $\Rightarrow x = 0$ when $y = 4$ $\cos^{-1}(1) = c$ $c = 0$
5. Substitute back for c.	$\cos^{-1}\left(\frac{y}{4}\right) = x^2$
6. Take the cosine of both sides to make y the subject.	$\frac{y}{4} = \cos(x^2)$ $y = 4\cos(x^2)$

7. Determine the domain for which the solution is valid.

$\cos^{-1}\left(\frac{y}{4}\right) = x^2$

The range of $y = \cos^{-1}(x)$ is $[0, \ \pi]$, but $x \neq 0$ and $\frac{1}{\sqrt{16 - y^2}}$ is defined for $|y| < 4$, so $0 < x^2 < \pi$.

8. Solve the inequality for x to state the largest domain for which the solution is valid. State the answer.

The solution $y = 4\cos(x^2)$ is valid for $0 < x < \sqrt{\pi}$.

Exercise 8.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$

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8.6 Exercise | 8.6 Exam question on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6	7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17	18, 19, 20

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Simple familiar

1. WE17 Determine the general solution to the differential equation $\frac{dy}{dx} = \frac{x+2}{y^3+8}$.

2. Obtain an implicit relationship of the form $f(x, \ y) = c$ for each of the following differential equations.

a. $\frac{dy}{dx} = \frac{x^2+4}{y^2+4}$ b. $\frac{dy}{dx} = \frac{xy}{y^2+4}$ c. $\frac{dy}{dx} = \frac{x^2y^2}{y^2+4}$ d. $\frac{dy}{dx} = \frac{xy^2e^{x^2}}{y^3+8}$

For questions 3–6, solve each of the given differential equations and express y in terms of x.

3. a. $\frac{dy}{dx} - \frac{y^2}{x} = 0, \ y(1) = 1$ b. $\frac{dy}{dx} + 12y^2\sin(4x) = 0, \ y(\pi) = 1$

4. a. $\frac{dy}{dx} + \frac{x}{y} = 0, \ y(1) = 2$ b. $\frac{dy}{dx} + 6y^2x^2 = 0, \ y(1) = 3$

5. a. $\frac{dy}{dx} + 18x^3y^2 = 0, \ y(-1) = 2$ b. $\frac{dy}{dx} - \frac{y^2}{x^2} = 0, \ y(1) = 2$

6. a. $\frac{dy}{dx} = y^2e^{2x}, \ y(0) = 1$ b. $\frac{dy}{dx} + 12x^5y^2 = 0, \ y(1) = 2$

Complex familiar

7. Obtain an implicit relationship of the form $f(x, y) = c$ for $\frac{dy}{dx} = \frac{y^2+4}{x^2y^2}$.

8. WE18 Solve the differential equation $\frac{dy}{dx} - y = 3x^2y$, $y(0) = 1$.

9. Given the differential equation $\frac{dy}{dx} + y^2 = 2xy^2$, $y(2) = 1$, express y in terms of x.

10. WE19 Solve the differential equation $\frac{dy}{dx} - 2x\sqrt{64 - y^2} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

11. Solve the differential equation $2\frac{dy}{dx} - x(16 + y^2) = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

For questions 12–17, solve each of the given differential equations and express y in terms of x.

12. a. $\frac{dy}{dx} + y = 3x^2y$, $y(0) = 1$ b. $\frac{dy}{dx} + 6x^2y^2 = y^2$, $y(-1) = 2$

13. a. $\frac{dy}{dx} + 2xy^2 = y^2$, $y(1) = 2$ b. $\frac{dy}{dx} + 8x^3y^4 = y^4$, $y(0) = 1$

14. a. $x\frac{dy}{dx} + 2y = y^2$, $y(1) = 1$ b. $x\frac{dy}{dx} - 4y = y^2$, $y(1) = 1$

15. a. $(4 + x^2)\frac{dy}{dx} - 2xy = 0$, $y(0) = 1$ b. $\frac{y^2 + 4}{x^2 + 9} - \frac{y}{x}\frac{dy}{dx} = 0$, $y(0) = 2$

16. a. $\frac{dy}{dx} - x(25 + y^2) = 0$, $y(0) = 0$ b. $\frac{dy}{dx} + 4x\sqrt{25 - y^2} = 0$, $y(0) = 5$

17. For each of the following, use the substitution $v = \frac{y}{x}$ to show that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, and hence reduce to a separable differential equation and determine the solution.

a. $x\frac{dy}{dx} + 3y = 4x$, $y(2) = 1$ b. $x\frac{dy}{dx} - y = 4x$, $y(1) = 2$

Complex unfamiliar

18. Solve the differential equation $\frac{dy}{dx} = \frac{6}{y}\tan(3x)$, $y(0) = 1$.

19. Determine the general solution to the differential equation $\frac{dy}{dx} = \frac{y^2 + x^2}{2y}$.

Hint: Use a quadratic substitution.

20. Solve the differential equation $\frac{1}{x}\cos(y)\frac{dy}{dx} = 2$, $y(0) = 1$, $y(0) = -\frac{\pi}{6}$.

Hint: Use a trigonometric substitution.

Fully worked solutions for this topic are available online.

LESSON
8.7 Slope fields

SYLLABUS LINKS

- Understand and use slope (direction or gradient) fields of a first-order differential equation.

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8.7.1 Introduction

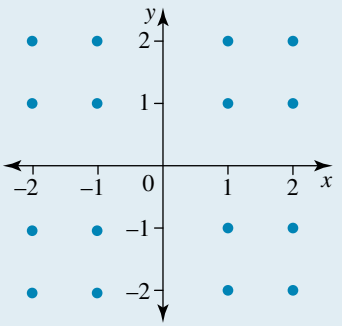
Slope fields, also known as direction fields, are a tool for graphically visualising the solutions to a first-order differential equation. A slope field is simply a graph on the Cartesian coordinate system showing short line segments that represent the slopes of the possible solutions at each point.

Type 1, $\frac{dy}{dx} = f(x)$

Type 1 first-order differential equations are of the form $\frac{dy}{dx} = f(x)$. In this section you will draw the slope fields for a simple differential equation of this type.

WORKED EXAMPLE 20 Sketching a slope field for $\frac{dy}{dx} = f(x)$

Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{x}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on the grid shown.



THINK	WRITE/DRAW
1. Make some observations about the slope.	When $x = 0$ (i.e. along the y-axis), the slope $\frac{dy}{dx}$ is zero. The y-values do not affect the slope.
2. Determine the slope at appropriate values of x.	When $x = 1$, the slope is $\frac{dy}{dx} = \frac{1}{2}$. A slope of $\frac{1}{2}$ makes an angle of $\tan^{-1}\left(\frac{1}{2}\right) \approx 27°$ with the positive x-axis.
3. Consider another x-value.	When $x = 2$, the slope is $\frac{dy}{dx} = 1$. A slope of 1 makes an angle of $\tan^{-1}(1) = 45°$ with the positive x-axis.

4. Consider the negative values of x.

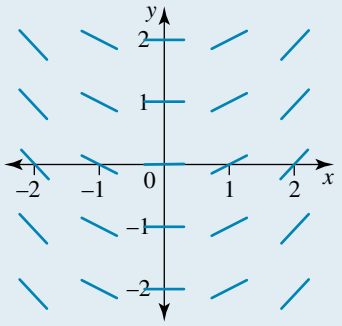
When $x = -1$, the slope is $\frac{dy}{dx} = -\frac{1}{2}$. A slope of $-\frac{1}{2}$ makes an angle of $\tan^{-1}\left(-\frac{1}{2}\right) \approx 153°$ with the positive x-axis.

5. Consider the final x-value.

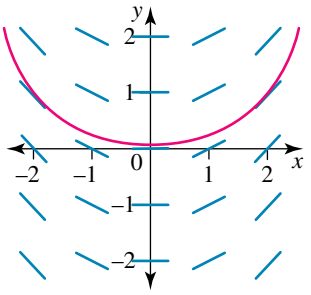
When $x = -2$, the slope is $\frac{dy}{dx} = -1$. A slope of -1 makes an angle of $\tan^{-1}(-1) = 135°$ with the positive x-axis.

6. Draw short line segments with the slopes found at each of the points.

The slope field in this case is symmetrical about the y-axis.



Note that we obtain these graphical solutions to a general solution of a first-order differential equation; that is, we do not necessarily require a particular condition.



In general, the solutions to the differential equation $\frac{dy}{dx} = \frac{x}{2}$ are a family of curves of the form $y = \frac{x^2}{4} + c$. However, if we joined some of the slopes shown in the answer to Worked example 14 and drew a particular curve passing through the origin, the result would represent the solution curve $y = \frac{x^2}{4}$, which is the particular solution to the differential equation $\frac{dy}{dx} = \frac{x}{2}$, $y(0) = 0$.

To develop the pink curve, mark the origin and then use the surrounding slopes that are already marked to guide your sketch, as the tangent to the curve, at any point, is parallel to the closest slope. In this case, the curve is 'pushed' upwards as you move left and right from the origin. The slope field simply represents tangents to the solution curve at the points.

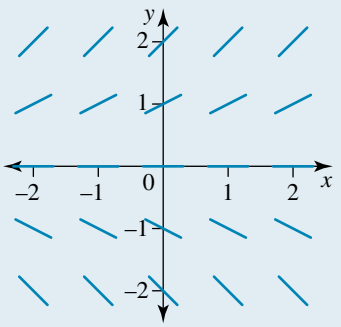
Type 2, $\frac{dy}{dx} = f(y)$

Type 2 first-order differential equations are of the form $\frac{dy}{dx} = f(y)$.

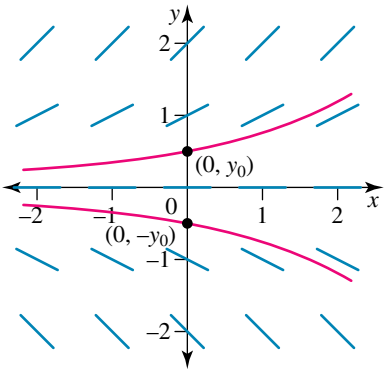
WORKED EXAMPLE 21 Sketching a slope field for $\frac{dy}{dx} = f(y)$

Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{y}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on a grid.

THINK	WRITE/DRAW
1. Make some observations about the slope.	When $y = 0$ (i.e. along the x-axis), the slope is zero. The x-values do not affect the slope.
2. Determine the slope at appropriate values of y.	When $y = 1$, the slope is $\frac{dy}{dx} = \frac{1}{2}$. A slope of $\frac{1}{2}$ makes an angle of $\tan^{-1}\left(\frac{1}{2}\right) \approx 27°$ with the positive x-axis.
3. Consider another y-value.	When $y = 2$, the slope is $\frac{dy}{dx} = 1$. A slope of 1 makes an angle of $\tan^{-1}(1) = 45°$ with the positive x-axis.
4. Consider the negative values of y.	When $y = -1$, the slope is $\frac{dy}{dx} = -\frac{1}{2}$. A slope of $-\frac{1}{2}$ makes an angle of $\tan^{-1}\left(-\frac{1}{2}\right) \approx 153°$ with the positive x-axis.
5. Consider the final y-value.	When $y = -2$, the slope is $\frac{dy}{dx} = -1$. A slope of -1 makes an angle of $\tan^{-1}(-1) = 135°$ with the positive x-axis.
6. Draw short line segments with the slopes found at each of the points.	The slope field in this case is symmetrical about the x-axis.



Note that, if we joined the slope field lines in Worked example 15 and drew curves, they would represent the curve $y = y_0 e^{\frac{x}{2}}$, which is the particular solution to the differential equation $\frac{dy}{dx} = \frac{y}{2}$, $y(0) = y_0$. The two curves shown have $y_0 > 0$ and $y_0 < 0$.



Type 3, $\frac{dy}{dx} = f(x, y)$

Type 3 first-order differential equations are of the form $\frac{dy}{dx} = f(x, y)$.

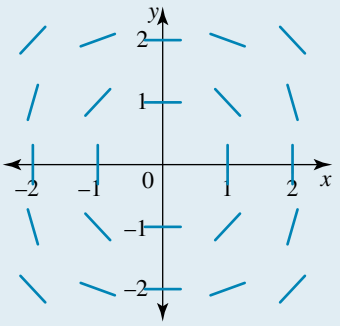
WORKED EXAMPLE 22 Sketching a slope field for $\frac{dy}{dx} = f(x, y)$

Sketch the slope field for the differential equation $y\frac{dy}{dx} + x = 0$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on a grid.

THINK	WRITE/DRAW
1. Rearrange the differential equation to make $\frac{dy}{dx}$ the subject.	$y\frac{dy}{dx} + x = 0$ $y\frac{dy}{dx} = -x$ $\frac{dy}{dx} = -\frac{x}{y}$
2. We have to evaluate the slope at each of the 25 points on the grid. Rather than substituting point by point, draw up a table.	Substitute $x = 2$ and $y = 2$: the slope $\frac{dy}{dx} = -\frac{2}{2} = -1$.
3. Complete the values. Note that an undefined slope is one that is parallel to the y-axis.	

x y	**−2**	**−1**	**0**	**1**	**2**
−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
−1	−2	−1	0	1	2
0	Undef.	Undef.	Undef.	Undef.	Undef.
1	2	1	0	−1	−2
2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	−1

4. Draw the slope field by drawing short line segments with the slope found at each point.



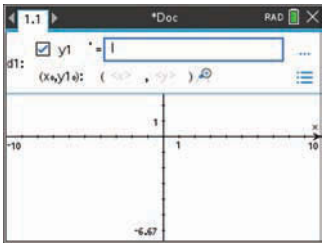
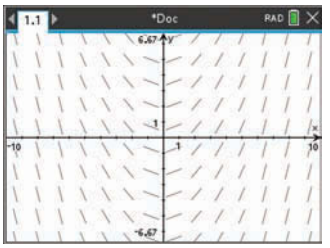
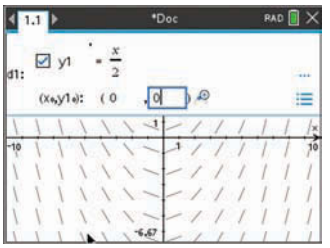
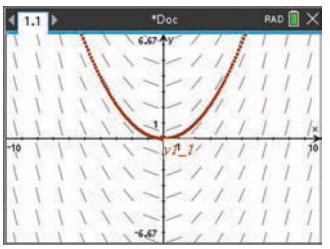
In the answer for Worked example 22, if the line segments were joined, the field would look like a series of circles of varying radii centred at the origin. In fact, if $x^2 + y^2 = r^2$, using implicit differentiation gives $2x + 2y\frac{dy}{dx} = 0$ or $x + y\frac{dy}{dx} = 0$.

8.7.2 Sketching slope fields using technology

Certain software packages such as GeoGebra, Desmos and various CAS capable technologies can produce slope fields. The TI-nspire CX calculator also has the capability to produce a slope field.

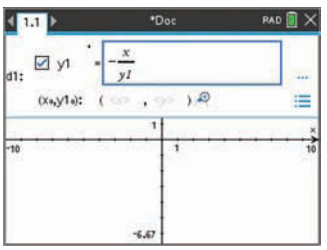
Consider Worked example 20 where a slope field was drawn for the differential equation $\frac{dy}{dx} = \frac{x}{2}$.

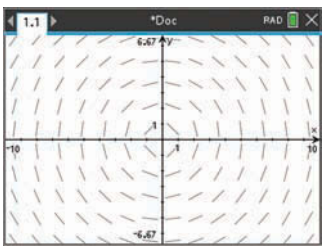
Using the TI-nspire CX calculator we can produce a slope field by completing the steps below.

1. Open a Graphs page.	
2. Slope fields can be graphed by selecting, MENU 3: Graph Entry/Edit 8: Diff Eq	
3. Complete the entry line as $\frac{x}{2}$. Followed by ENTER.	
4. A particular solution can be drawn if the initial conditions are known.	
5. Assume the initial conditions for the differential equation are $(0, 0)$.	
Press TAB to view the entry line and insert 0 and 0 into initial conditions coordinate template as shown. Press ENTER to see the particular solution.	

Note: If the differential equation is of the form $\frac{dy}{dx} = f(y)$ or $\frac{dy}{dx} = f(x, y)$ the y variable must be entered as y followed by the function number (rather than simply y).

For example, to sketch a slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ you should enter $-\frac{x}{y1}$ into the entry line for $y1' =$ as shown.





8.7.3 Interpreting a slope field

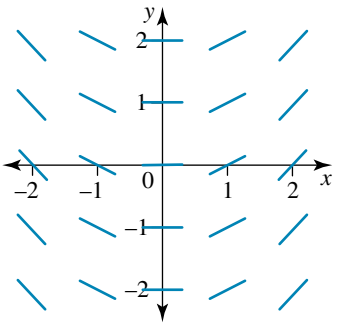
Often you will be asked to analyse a given slope field.

In these types of questions, it is common to be asked to match a differential equation (or its solution) to a slope field diagram or vice versa.

When beginning such questions, *it is important to strategically choose specific points to pay attention to*.

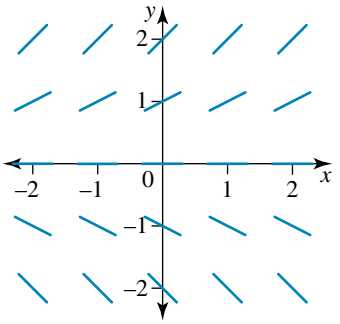
Specific points you may choose to pay attention to include:

- Points on an x or y axis
- Points in specific quadrants (e.g. in quadrant 2 where we know x is negative and y is positive)
- Locations where the slope is flat (i.e. $\frac{dy}{dx} = 0$)
- Locations where no slopes is given, this may help with domains



Lastly, when analysing diagrams, you may notice a vertical or horizontal pattern similar to Worked examples 20 or 21 which give clues to the variables involved:

- In Worked example 20 you saw the pattern of slopes where the differential equation only depended on x: $\frac{dy}{dx} = f(x)$
- In Worked example 21 you saw the pattern of slopes where the differential equation only depended on y: $\frac{dy}{dx} = f(y)$



Note that we are not solving the differential equation when we analyse the slope field. But if we are given a solution, we can use implicit differentiation to generate the family of curves being presented.

WORKED EXAMPLE 23 Interpreting a slope field

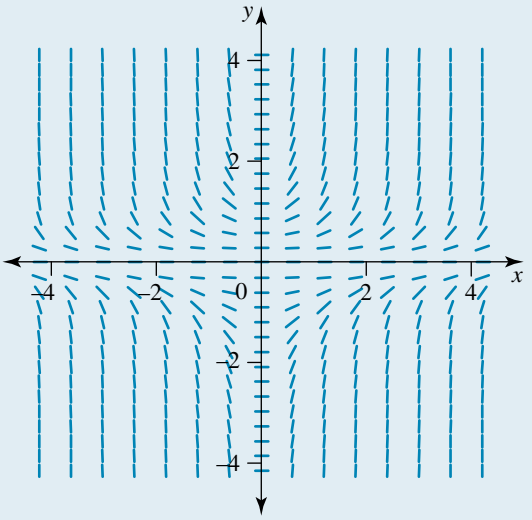
The differential equation that best represents the slope field shown is:

A. $\frac{dy}{dx} = \frac{y^2}{x}$

B. $\frac{dy}{dx} = \frac{x^2}{y}$

C. $\frac{dy}{dx} = yx^2$

D. $\frac{dy}{dx} = y^2x$



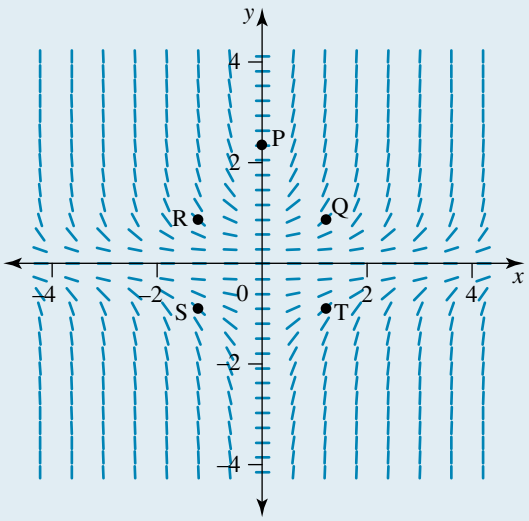
THINK

1. Consider the y-axis.

WRITE

Consider P, a point along the x-axis, that is, when $x = 0$: the slopes are all zero. Option A is incorrect.

2. Consider the first quadrant.	Consider a point in the first quadrant, such as Q(1, 1) the slopes are 1. This satisfies B, C and D.
3. Consider the second quadrant.	Consider a point in the second quadrant, such as R(−1, 1), the slopes are −1. Option C is incorrect.
4. Consider the third quadrant.	Consider a point in the third quadrant, such as S(−1, −1), the slopes are 1. Options B and D are still valid.
5. Consider the fourth quadrant.	Consider a point in the fourth quadrant, such as T(1, −1), the slopes are 1. Option B is incorrect.
6. State the result.	Option D, $\frac{dy}{dx} = y^2x$, is the only differential equation that is represented by this slope field.



Exercise 8.7 Slope fields

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8.7 Exercise | **8.7 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14	15, 16, 17, 18, 19	20

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For questions 1–6, sketch the fields for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$.

1. WE20 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{x}$.

2. Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

3. WE21 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{y}$.

4. Sketch the slope field for the differential equation $\frac{dy}{dx} = 2\sqrt{y}$.

5. WE22 Sketch the slope field for the differential equation $y\frac{dy}{dx} - x = 0$.

6. Sketch the slope field for the differential equation $\frac{dy}{dx} = xy$.

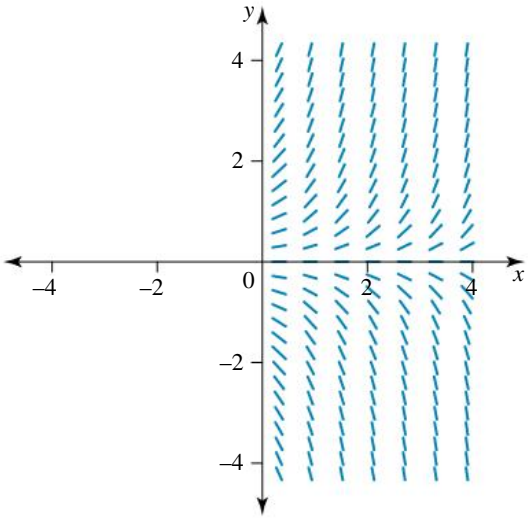
7. WE23 MC The differential equation that best represents the slope field shown is:

A. $\frac{dy}{dx} = \sqrt{xy}$

B. $\frac{dy}{dx} = x\sqrt{y}$

C. $\frac{dy}{dx} = y\sqrt{x}$

D. $\frac{dy}{dx} = \frac{1}{\sqrt{xy}}$



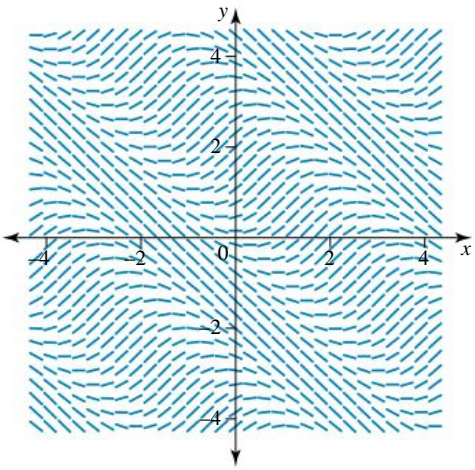
8. MC The differential equation that best represents the slope field shown is:

A. $\frac{dy}{dx} = \cos(x + y)$

B. $\frac{dy}{dx} = \cos(x - y)$

C. $\frac{dy}{dx} = \sin(x - y)$

D. $\frac{dy}{dx} = \sin(x + y)$



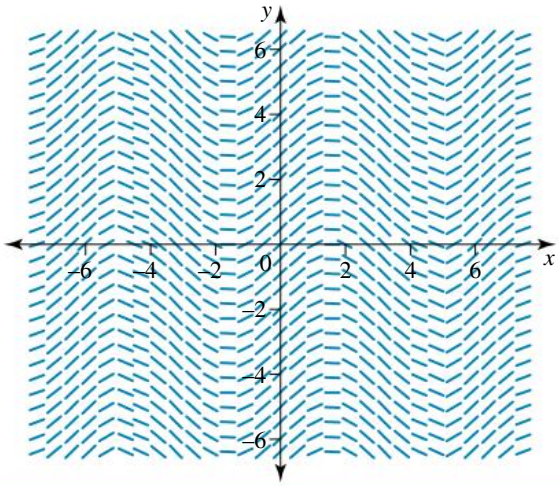
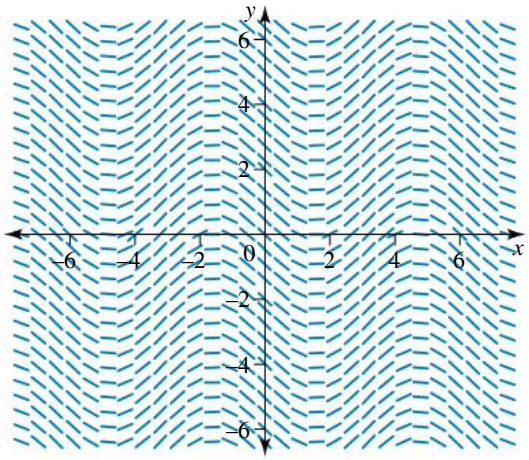
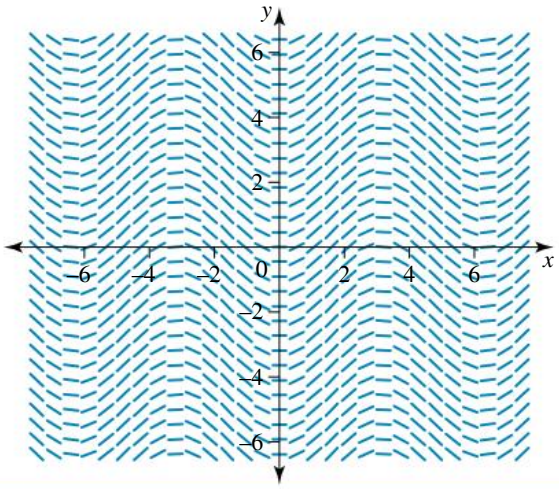
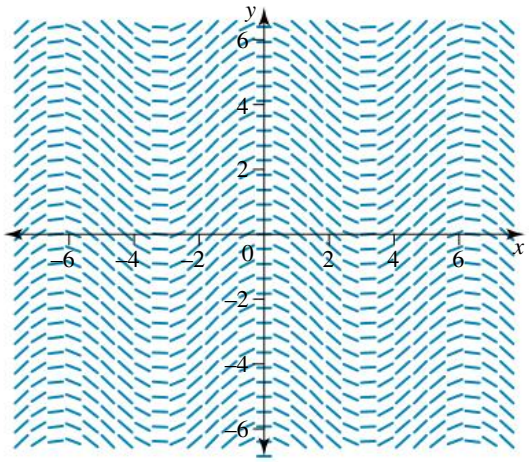
9. Match the four differential equations to the four slope fields shown below.

a. $\frac{dy}{dx} + \sin(x) = 0$

b. $\frac{dy}{dx} + \cos(x) = 0$

c. $\frac{dy}{dx} - \sin(x) = 0$

d. $\frac{dy}{dx} - \cos(x) = 0$

Slope 1	Slope 2
	
Slope 3	**Slope 4**
	

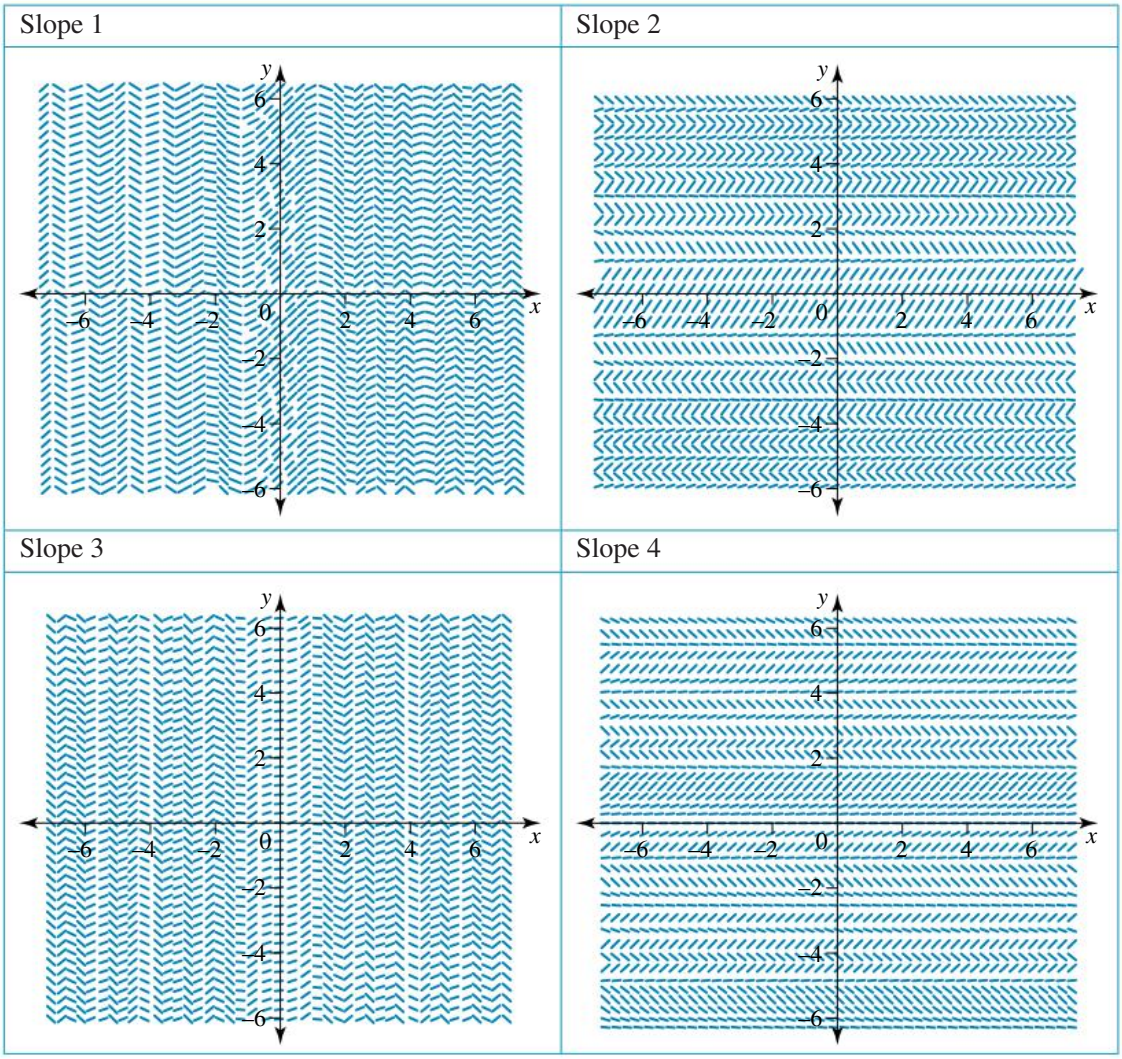
10. Match the four differential equations to the four slope fields shown on the next page.

a. $\dfrac{dy}{dx} = \sin(x^2)$

b. $\dfrac{dy}{dx} = \cos(x^2)$

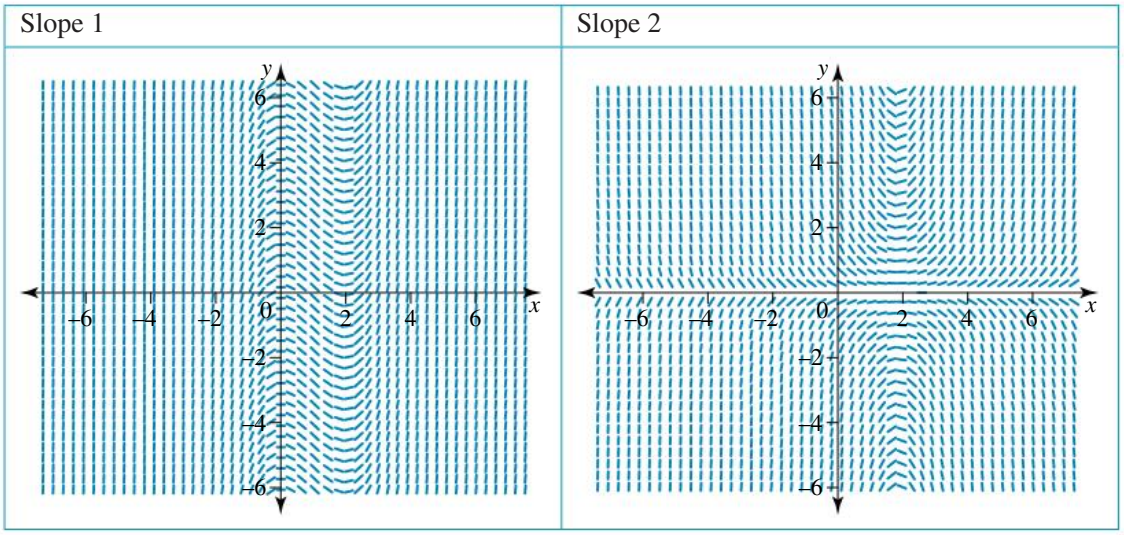
c. $\dfrac{dy}{dx} = \sin(y^2)$

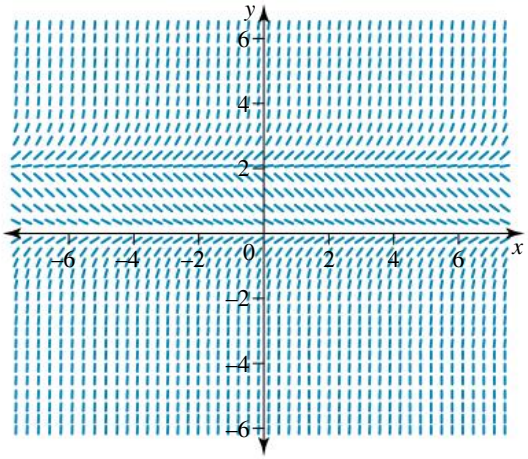
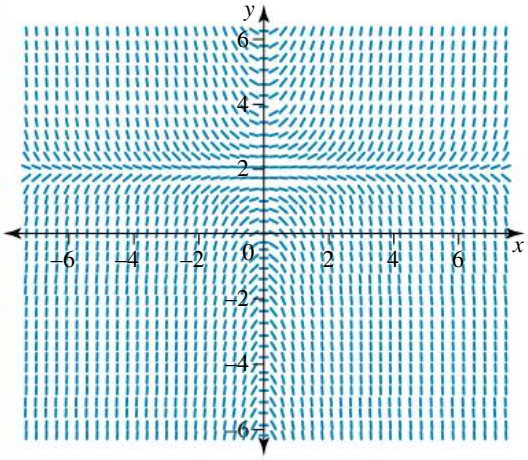
d. $\dfrac{dy}{dx} = \cos(y^2)$



11. Match the four differential equations to the four slope fields shown below.

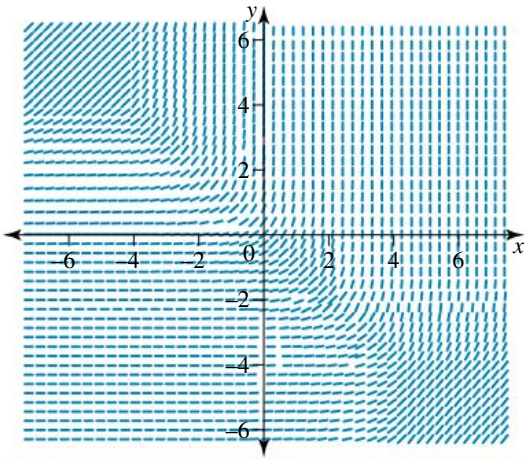
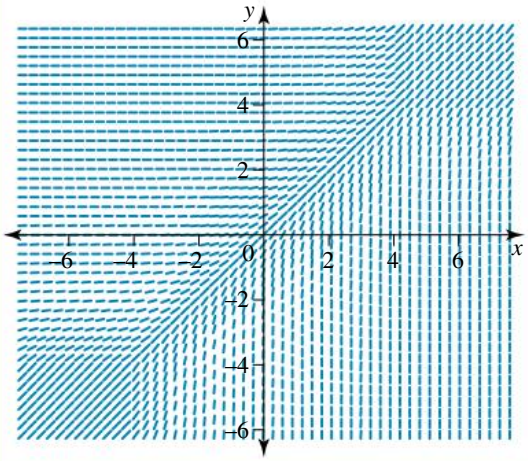
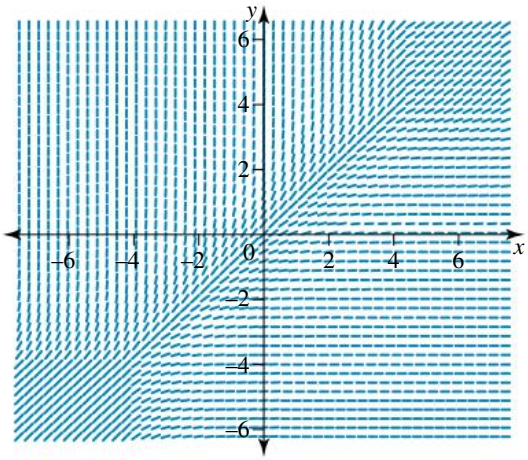
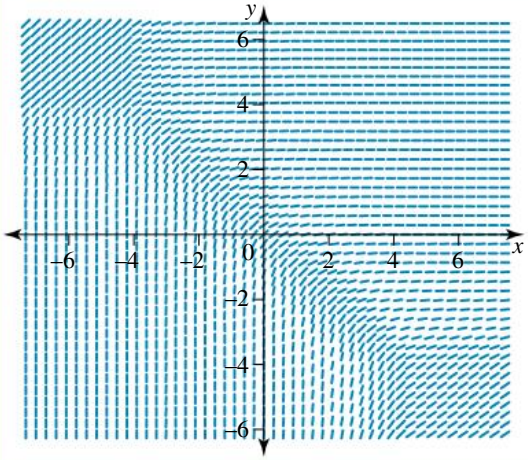
a. $\dfrac{dy}{dx} = x(x-2)$ **b.** $\dfrac{dy}{dx} = y(y-2)$ **c.** $\dfrac{dy}{dx} = y(x-2)$ **d.** $\dfrac{dy}{dx} = x(y-2)$



Slope 3	Slope 4
	

12. Match the four differential equations to the four slope fields shown below.

a. $\dfrac{dy}{dx} = e^{x-y}$ **b.** $\dfrac{dy}{dx} = e^{y-x}$ **c.** $\dfrac{dy}{dx} = e^{x+y}$ **d.** $\dfrac{dy}{dx} = e^{-(x+y)}$

Slope 1	Slope 2
	
Slope 3	**Slope 4**
	

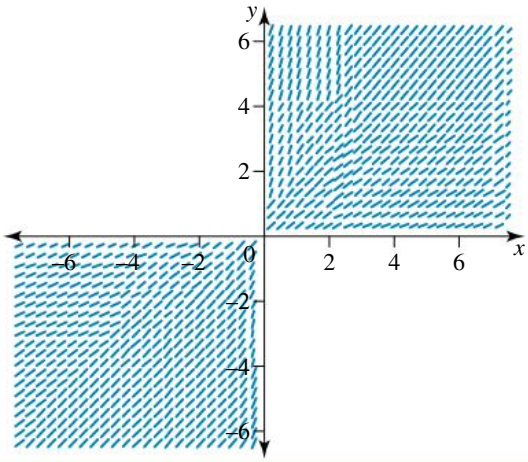
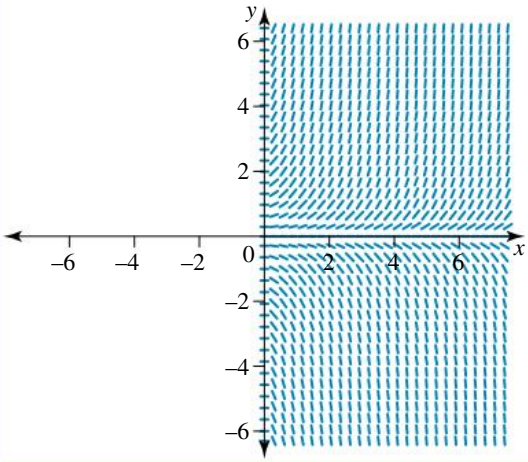
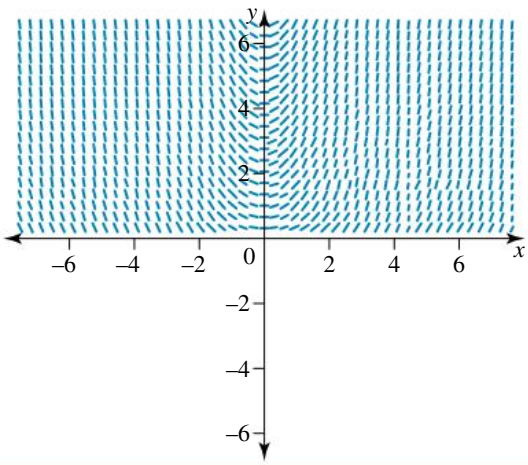
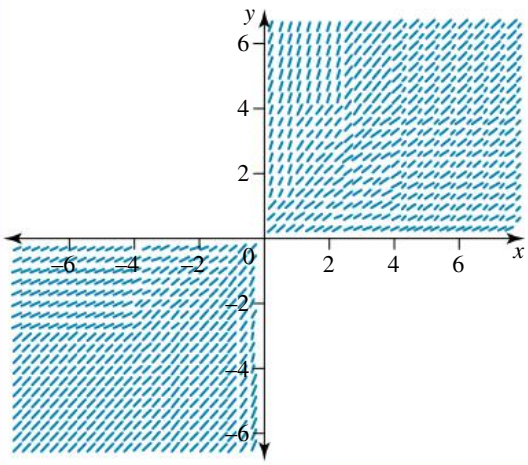
13. Match the four differential equations to the four slope fields shown below.

a. $\dfrac{dy}{dx} = \sqrt{\dfrac{y}{x}}$

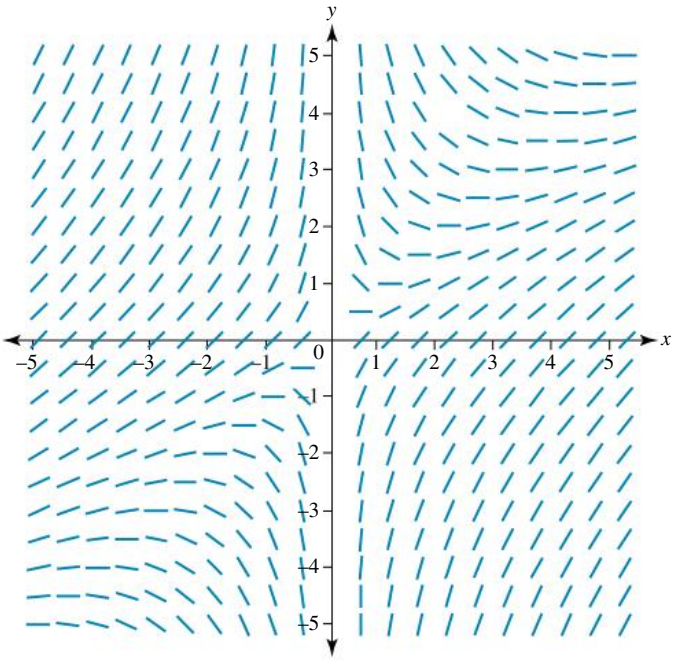
b. $\dfrac{dy}{dx} = \sqrt{\dfrac{x}{y}}$

c. $\dfrac{dy}{dx} = y\sqrt{x}$

d. $\dfrac{dy}{dx} = x\sqrt{y}$

Slope 1	Slope 2
	
Slope 3	**Slope 4**
	

14. The slope field for the differential equation $\frac{dy}{dx} = \alpha + \frac{\beta y}{x}$, where α and β are constants, is shown.



On the same diagram, represent the particular solutions $y(1) = -1$, $y(2) = 4$, $y(1) = -1$ and $y(-2) = -4$.

Complex Familiar

For questions 15–18, sketch the fields for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$.

15. a. Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{(x-1)^2}$.

b. i. Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{1}{x^2+1}$.

ii. Solve the differential equation $\frac{dy}{dx} = \frac{1}{x^2+1}$, $y(0) = 0$ and sketch the solution curve on the diagram for part **i**.

16. a. Sketch the slope field for the differential equation $\frac{dy}{dx} = y^2$.

b. i. Sketch the slope field for the differential equation $\frac{dy}{dx} = y^2 + 1$.

ii. Solve the differential equation $\frac{dy}{dx} = y^2 + 1$, $y(0) = 0$ and sketch the solution curve on the diagram for part **i**.

17. Sketch the slope fields for each of the following differential equations.

a. $\frac{dy}{dx} = \frac{-x}{y+1}$

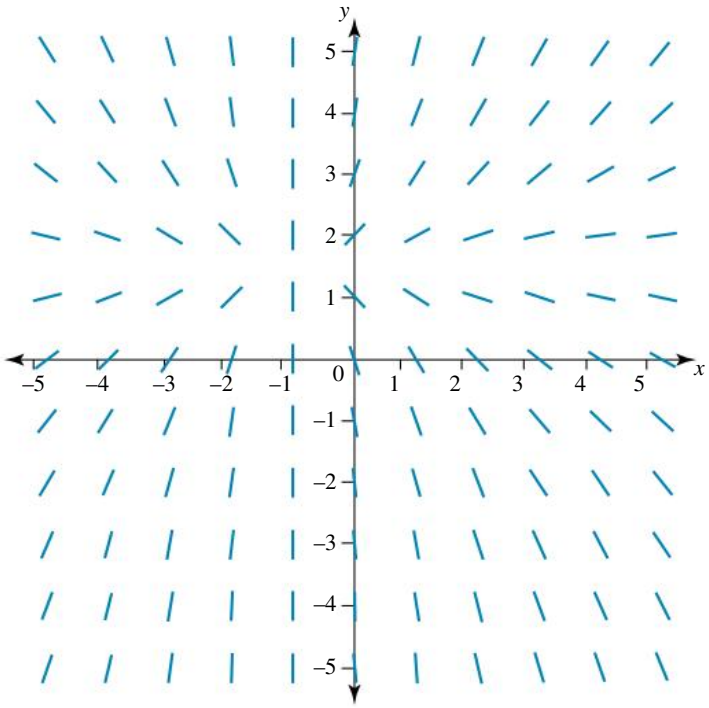
b. $\frac{dy}{dx} = \frac{x-2}{y}$

18. Sketch the slope fields for each of the following differential equations.

a. $\frac{dy}{dx} = (y-1)\cos\left(\frac{\pi x}{2}\right)$

b. $\frac{dy}{dx} = y^2 \sin\left(\frac{\pi x}{2}\right)$

19. The slope field for the differential equation $\frac{dy}{dx} = \frac{2y-3}{x+1}$ is shown.



a. Use the slope field to sketch the solution curve for $\frac{dy}{dx} = \frac{2y-3}{x+1}, y(-5) = -3$.

b. Verify your answer.

Complex Unfamiliar

20. a. Sketch the slope field for $\frac{dy}{dx} = x + y$.

b. Given that $y = Ae^{kx} + Bx + C$ is a solution of the initial value problem $\frac{dy}{dx} = x + y, y(0) = 0$, determine the values of A, B, C and k.

c. Show that the solution curve has a turning point at $(0, 0)$.

d. On the slope field, sketch the particular solution to the differential equation $\frac{dy}{dx} = x + y$ that passes through the point $(0, 0)$.

Fully worked solutions for this topic are available online.

LESSON
8.8 Review

8.8.1 Summary

Hey students! Now that it's time to revise this topic, go online to:

- Access the chapter summary
- Review your results
- Practise exam questions

Find all this and MORE in jacPLUS

8.8 Exercise

learnon

8.8 Exercise	8.8 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

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Simple Familiar

1. Show that the gradient of the curve $2x^2 + 3y^2 - 4xy - 9 = 0$ at the point $(-3, -1)$ is equal to $\frac{4}{3}$.

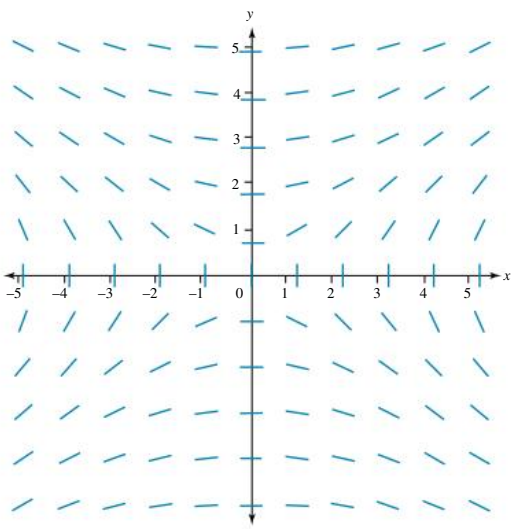
2. An ice block has the shape of a cube and has its volume decreasing at a rate of 2 cm^3/min. Determine the rate at which each of the sides are decreasing in cm/min, when the sides are 2 cm.

3. Complete the following.
   a. Write an expression for the gradient in terms of both x and y for the curve $2x^3 - 6x^2y^2 + 3y^3 - 2 = 0$.
   b. Determine the gradient in terms of x and y if $4x^4 - 7xy - 3y^4 - 8 = 0$.
   c. Calculate the coordinates of the points on the curve $2x^2 + 2xy + y^2 = 8$ where the tangent to the curve is horizontal.

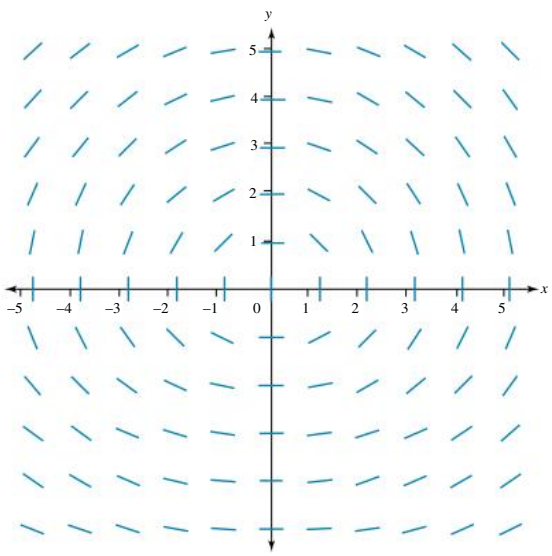
4. Determine the general solution of the differential equation $\frac{dy}{dx} + 36x = 0$.

5. **MC** The slope field that represents the differential equation $\frac{dy}{dx} = \frac{2y}{x}$ is

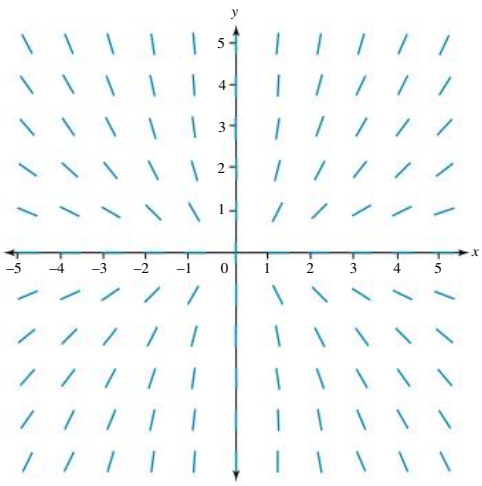
A.



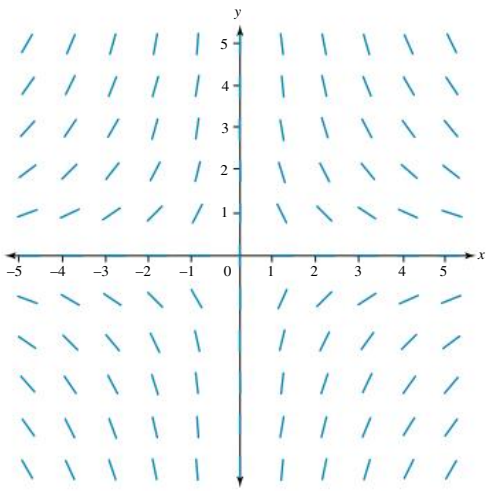
B.



C.



D.



6. Determine the general solution of the differential equation $3\frac{dy}{dx} + y = 0$ (let A be the constant).

7. Determine the general solution of the differential equation $\frac{dy}{dx} + 3x^2y^2 = 0$ (let A be the constant).

8. Determine the particular solution of the differential equation $\frac{dy}{dx} = y^2 + 4,\ y(0) = 0$.

9. Calculate particular solutions to each of the following differential equations, stating the maximal domain, where appropriate.

a. $\frac{dy}{dx} - x^2 = 4,\ y(1) = 2$ **b.** $x^2\frac{dy}{dx} + 4 = 2x,\ y(1) = 2$ **c.** $(5x+3)^2\frac{dy}{dx} + 4 = 0,\ y(-1) = 2$

10. Solve the following:

a. $\left(x^2+9\right)\frac{dy}{dx} + 3 = 0,\ y(0) = 2$ **b.** $\sqrt{9+x^2}\ \frac{dy}{dx} + 3x = 0,\ y(0) = 2$ **c.** $\left(9-x^2\right)\frac{dy}{dx} + 3 = 0,\ y(0) = 2$

11. Solve the following:

a. $\frac{dy}{dx} + 9y^2 = 0,\ y(0) = 1$ **b.** $\frac{dy}{dx} + \frac{9}{y} = 0,\ y(0) = 1$ **c.** $\frac{dy}{dx} + 9y^3 = 0,\ y(0) = 1$

12. Solve the following:

a. $\frac{dy}{dx} + 4y = 7,\ y(0) = 2$ **b.** $\frac{dy}{dx} + (5 - 2y)^2 = 0,\ y(1) = 2$ **c.** $\frac{dy}{dx} + \sqrt{9 - y^2} = 0,\ y(0) = 3$

Complex familiar

13. **a.** Determine the equation of the normal to the curve $x^3 - 4xy + y^3 - 15 = 0$ at the point $(2, -1)$.

b. For the ellipse $4x^2 + 4xy + 9y^2 - 6y - 2x - 23 = 0$, determine the equation of the tangent at the point where $x = 2$ in the first quadrant.

c. Determine the coordinates of the points on the curve $9x^2 - 54x + 16y^2 + 64y + 1 = 0$ where the tangent to the curve is parallel to the y-axis.

14. While blowing up a balloon a child observes that the radius of the balloon increase at a rate of 2 cm/s. Assuming it remains spherical, when the radius is 6 cm, calculate:

a. the rate at which the volume is increasing

b. that rate at which its surface area is increasing.

15. Solve the following:

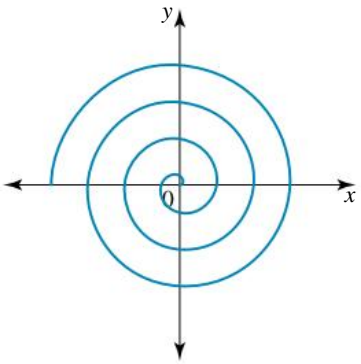
a. $\frac{dy}{dx} - y^2 = 4,\ y(0) = 0$ **b.** $\frac{dy}{dx} - y^2 = 4y,\ y(0) = 2$ **c.** $16\frac{dy}{dx} + e^{4y} = 0,\ y(0) = 0$

16. Solve the following:

a. $\frac{dy}{dx} - 6x^2y^2 = 0,\ y(1) = 1$ **b.** $\frac{dy}{dx} = 4x^3e^{-y},\ y(0) = 0$ **c.** $\frac{dy}{dx} - y^2 = 4xy^2,\ y(1) = 1$

Complex unfamiliar

17. A spiral curve has the parametric equations $x = t\cos(t)$ and $y = t\sin(t)$.



a. Show that $\frac{dy}{dx} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$.

b. Show that the curve has the implicit equation $x^2 + y^2 = \left(\tan^{-1}\left(\frac{y}{x}\right)\right)^2$.

c. Verify that $\frac{dy}{dx} = \frac{x^3 + xy^2 + y\tan^{-1}\left(\frac{y}{x}\right)}{x\tan^{-1}\left(\frac{y}{x}\right) - yx^2 - y^3}$.

18. A man 2 metres tall is walking at 1.5 m/s. He passes under a light source 6 metres above the ground. Calculate:

a. the rate at which his shadow is lengthening

b. the speed at which the end point of his shadow is increasing

c. the rate at which his head is receding from the light source when he is 8 metres from the light.

19. A particular solution to the differential equation $\frac{dy}{dx} = \frac{y}{\tan(3x)}$, where $0 < x < \frac{\pi}{3}$ and $y > 1$ passes through the point $\left(\frac{\pi}{6}, 2\right)$.

Determine this solution in the form $y = f(x)$. Leave your answer in simplified form.

20. Use the substitution $v = \frac{y}{x}$ to show that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Hence, reduce the differential equation $x\frac{dy}{dx} + ay = bx$ to a separate differential equation and determine its general solution for the cases when:

a. $a = -1$ **b.** $a \neq -1$

8.8 Past QCAA exam questions

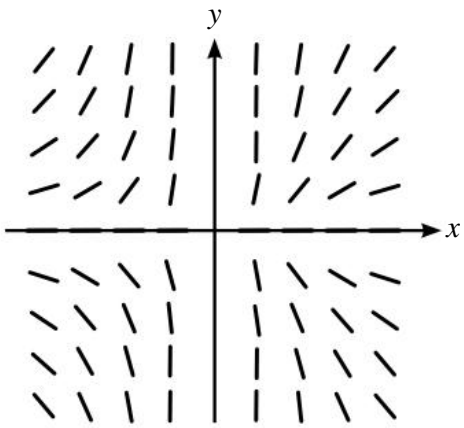
Question 1 (1 mark)

Source: QCAA 2020 Specialist Mathematics, Paper 1, Section 1, Q7; © State of Queensland (QCAA) 2024.

The diagram shows a slope field.

The differential equation represented by the slope field is

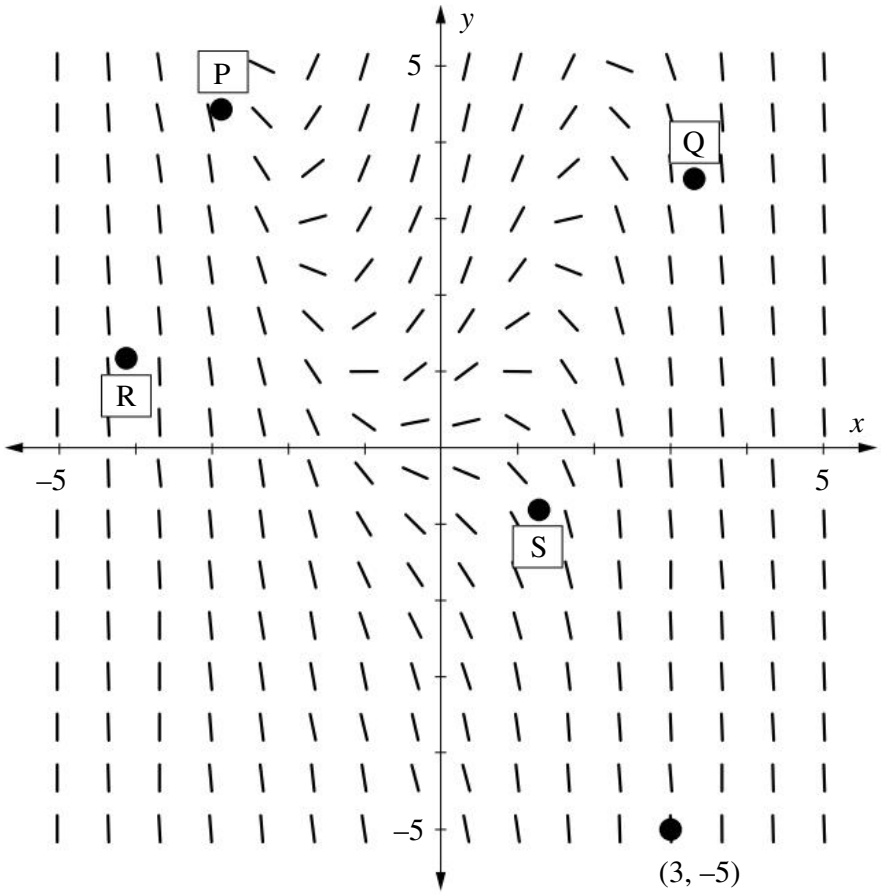
A. $\frac{dy}{dx} = \frac{5y}{x}$ **B.** $\frac{dy}{dx} = \frac{5y^2}{x}$ **C.** $\frac{dy}{dx} = \frac{5y}{x^2}$ **D.** $\frac{dy}{dx} = \frac{5y^2}{x^2}$



Question 2 (1 mark)

Source: QCAA 2021 Specialist Mathematics, Paper 1, Section 1, Q9; © State of Queensland (QCAA) 2024.

The slope field for the differential equation $\frac{dy}{dx} = y - x^2$ is shown.



The solution curve to the differential equation that passes through the point $(3, -5)$ also passes through point

A. P
B. Q
C. R
D. S

Question 3 (5 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 2, Q16; © State of Queensland (QCAA) 2024.

A curve is defined by the parametric equations $x = 2\tan(\theta)$ and $y = 3\sin(2\theta)$, where $0 \le \theta < \frac{\pi}{2}$

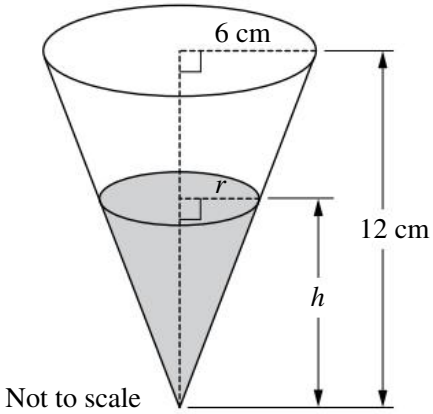
Given that $\frac{dy}{dx}$ can be expressed in the form $a\cos^4(\theta) + b\cos^2(\theta)$, where $a, b \in R$, determine the values of a and b.

Question 4 (8 marks)

Source: QCAA 2021 Specialist Mathematics, Paper 2, Section 2, Q15; © State of Queensland (QCAA) 2024.

Water is poured into a cone-shaped cup at a rate of $2\,\text{cm}^3\,\text{s}^{-1}$. The cup has a height of $12\,\text{cm}$ and a radius of $6\,\text{cm}$, as shown.

As the cup fills, the ratio of the height of the water h to the surface radius of the water r remains constant.



a. Given that $h = 2r$, determine a function for the volume of water in the cup, V, in terms of h. Express your answer in simplified form. **[1 mark]**

b. Use the results from Question 15a) to show that the rate at which the height of water in the cup is increasing with respect to time is given by $\dfrac{8}{\pi h^2}$. **[3 marks]**

c. Determine the rate at which the height of water in the cup is increasing with respect to time when the volume of water in the cup reaches half of the total capacity of the cup. **[4 marks]**

Question 5 (6 marks)

Source: QCAA 2023 Specialist Maths, Paper 1, Section 2, Q18; © State of Queensland (QCAA) 2024.

A particular solution to the differential equation $\dfrac{dy}{dx} = \dfrac{x}{(x^2+1)\tan(y)}$, where $x \geq 0$ and $-\dfrac{\pi}{2} < y \leq 0$, passes through the origin.

Determine this solution in the form $x = f(y)$. Leave your answer in simplified form.

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Answers

Chapter 8 Rates of change and differential equations

8.2 Implicit differentiation

8.2 Exercise

1. $-\frac{x^2}{y^2}$

2. $-\sqrt{\frac{y}{x}}$

3. a. $\frac{1}{y}$ b. $-\frac{9x}{4y}$
   c. $\frac{9x}{16y}$ d. $\frac{3x-2}{y-1}$

4. a. $8x^2y\frac{dy}{dx}+8xy^2$ b. $\frac{27x^2+8xy^2+2}{5+9y^2-8x^2y}$

5. $-\frac{9}{11}$

6. a. $\frac{4x+3y}{6y-3x}$ b. $\frac{3-x-xy^2}{x^2y}$
   c. 1 d. $\frac{2x+3}{y(3y-2)}$

7. a. 1 b. $x-5y-11=0$
   c. $-\frac{2}{7}$ d. $\frac{8}{15}$

8. a. $3x-4y-25=0$ b. $-\frac{\sqrt{3}}{2}$
   c. $\frac{4}{5}$ d. $\frac{13}{9}$

9. a. $-\cot(t)$
   b. Sample responses can be found in the worked solutions in the online resources.

10. a. $\frac{1-2t^3}{t}$
    b. Sample responses can be found in the worked solutions in the online resources.

11. a. $\frac{1}{t}$ b. $-\frac{4}{3}\tan(2t)$ c. $-\frac{1}{t^2}$

12. a. $\frac{1}{t}$ b. $-\frac{b}{a}\cot(t)$ c. $-\frac{1}{t^2}$

13. a. $\frac{1}{t}$ b. $-\frac{b}{a}\cot(t)$
    c. $-\frac{1}{t^2}$

14. $\frac{x^2-ay}{ax-y^2}$

15. $-\frac{x(x^2+y^2-1)}{y(x^2+y^2+1)}$

16. $\frac{y^2+3x^2}{2y(2a-x)}$

17. $\frac{y-xe^{-(x^2+y^2)}}{ye^{-(x^2+y^2)}-x}$

18. $-\frac{3\cos(3x+2y)+2x}{2(y+\cos(3x+2y))}$

19. a. $\frac{27x^2+12y+1}{y-12x}$ b. $\frac{4x^3-6x^2y-3x^2+8xy^2}{24y^2+2x^3-8x^2y}$
    c. $\frac{e^{x+y}}{2y+\sin(y)-e^{x+y}}$ d. $\frac{y\sin(xy)-ye^{xy}-2x}{xe^{xy}-x\sin(xy)}$

20. a. $\frac{2+8x+12y}{15y+10x-3}$ b. $-\frac{y(2x^2+1)}{x(2y^2+1)}$
    c. $\frac{y^2(2x^2y-1)}{x^2(1-2xy^2)}$ d. $\frac{3\cos(3x)\cos(2y)-2x}{6y^2+2\sin(3x)\sin(2y)}$

21. $\frac{4}{3}\operatorname{cosec}(t)$

22. $\frac{b}{a}\operatorname{cosec}(t)$

23. $\frac{t(t^3-2)}{2t^3-1}$

24. $\frac{2-3\cos^2(t)}{\sin(t)(2+\cos^2(t))}$

25. $\frac{t(t^2+3)}{2}$

26. $\frac{\sin(t)(2\cos^2(t)+1)}{2\cos^3(t)}$

8.3 Related rates as instances of the chain rule

8.3 Exercise

1. 20π m^2/s
2. 2 cm/s
3. a. 16 cm^2/s b. $\frac{1}{4\pi}$ m/s
4. a. 800π mm^3/s b. 160π mm^2/s
5. Decreasing at $\frac{1}{24\pi}$ cm/s
6. Decreasing at 48π cm^2/s
7. a. Sample responses can be found in the worked solutions in the online resources.
   b. 150 weeks
8. Decreasing at $\frac{225}{128\pi}$ cm/s
9. Decreasing at 64π cm^3/s
10. 0.225 m/s
11. $\frac{8}{5}$ m/s
12. a. Sample responses can be found in the worked solutions in the online resources.
    b. 450π cm^3/s

13. $\frac{1}{25\pi}$ cm/s

14. $\frac{27}{16\pi}$ cm/s

15. 25.87 m/s

16. 12 m/s

17. a. Sample responses can be found in the worked solutions in the online resources.

b. Decreases by 5 cm/s

18. a. Decreases by 31 562.5 Pa/s

b. Sample responses can be found in the worked solutions in the online resources.

19. 0.005°/s

20. −2.1°/s

8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

8.4 Exercise

1. a. $y = c - 3x^4$ b. $y = 13 - 3x^2$

2. a. $y = c - 6\sin(2x)$ b. $y = 2(\cos(3x) - 1)$

3. a. $y = 2x^2 + 3x + c$

b. $y = x^3 + \frac{7}{2}x^2 - 20x + c$

c. $y = e^{2x} + 3e^{-2x} + c$

d. $y = \sqrt{x^2 + 9} + c$

4. $y = \frac{9x - 11}{4x - 5}, x \in \mathbb{R} \setminus \left\{\frac{5}{4}\right\}$

5. $y = 3 + \frac{1}{2}\ln(|7 - 4x|), x \in \mathbb{R} \setminus \left\{\frac{7}{4}\right\}$

6. $y = 1 - \sqrt{2x - 5}, x > \frac{5}{2}$

7. $y = 11 - 4\sqrt{x}, x > 0$

8. $y = 6\sin^{-1}\left(\frac{x}{8}\right) - \pi, x \in (-8, 8)$

9. $y = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{4}\right), x \in \mathbb{R}$

10. a. $y = \frac{1}{3}[5\ln(|x|) + x^2 + 8], x \in \mathbb{R} \setminus \{0\}$

b. $y = 2(e^{3x} - e^{-3x}), x \in \mathbb{R}$

11. a. $y = 4 - 2\cos(2x), x \in \mathbb{R}$

b. $y = 3 - 2\sin(3x), x \in \mathbb{R}$

12. a. $y = 4x - \sin(4x), x \in \mathbb{R}$

b. $y = 6x + \sin(6x), x \in \mathbb{R}$

13. a. $y = \frac{1}{2}\left(\sqrt{4x + 9} - 3\right), x > -\frac{9}{4}$

b. $y = 1 + \frac{1}{2}\ln(|2x - 3|), x \in \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$

14. a. $y = \frac{10x - 17}{3x - 5}, x \in \mathbb{R} \setminus \left\{\frac{5}{3}\right\}$

b. $y = 5 - 2\ln(|7 - 4x|), x \in \mathbb{R} \setminus \left\{\frac{7}{4}\right\}$

15. a. $y = \frac{3}{2}\ln\left(\frac{x^2 + 9}{9}\right), x \in \mathbb{R}$

b. $y = 2 - \sqrt{x^2 + 4}, x \in \mathbb{R}$

16. a. $y = \frac{1}{2}\ln\left(\frac{x^2 + 6x + 13}{13}\right), x \in \mathbb{R}$

b. $y = \ln\left(\frac{3}{\sqrt{x^2 - 4x + 9}}\right), x \in \mathbb{R}$

17. a. $y = -\frac{1}{8}\sin^4(2x), x \in \mathbb{R}$

b. $y = \cos^3(3x) - 1, x \in \mathbb{R}$

18. a. $y = 5x - x\ln(2x) - \frac{3}{2}, x > 0$

b. $y = (x - 4)e^{-x} + 4, x \in \mathbb{R}$

19. $\frac{dy}{dx} = \frac{1 - 2xy}{x^2 + 1}$

20. $y(x) = -k(1 + e^{-1}) + ke^{-x}, k \in \mathbb{R}$

8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

8.5 Exercise

1. $y = \sqrt[3]{(c_2 - 6x)^2}$

2. $y = \frac{1}{2}\sin^{-1}(Be^{2x})$

3. a. $y = \frac{4}{c - x}$ b. $y = Ae^x - 4$

c. $y = Ae^{\frac{x}{4}}$ d. $y = \sqrt[3]{12x + A}$

4. $y = \frac{15x - 13}{12x - 11}, x \in \mathbb{R} \setminus \left\{\frac{11}{12}\right\}$

5. $y = \frac{1}{4}(7 + 5e^{-4x})$

6. $y = 5e^{-3x}$

7. $y = 3e^{5x}$

8. a. $y = 4e^{-5x}$ b. $y = 2e^{3x-3}$

9. a. $y = \frac{1}{2}(5 + e^{-2x})$ b. $y = \frac{2}{3}(2 + e^{3x})$

10. a. $y = \frac{1}{4}(x + 3)^2, x \in \mathbb{R}$

b. $y = \frac{3}{4 - 3x}, x \in \mathbb{R} \setminus \left\{\frac{4}{3}\right\}$

11. a. $y = -\frac{1}{2}\ln(17 - 8x), x < \frac{17}{8}$

b. $y = -\frac{1}{3}\ln(18x - 17), x > \frac{17}{18}$

12. $y = 8\cos\left(\frac{x}{6}\right), -6\pi < x < 0$

13. $y = 4\tan(x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

14. a. $y = \frac{5x - 8}{2x - 3}, x \in \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$

b. $y = \frac{7x - 23}{3x - 10}, x \in \mathbb{R} \setminus \left\{\frac{10}{3}\right\}$

15. a. $y = 2\cos^{-1}(3x), x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

b. $y = \frac{1}{2}\sin^{-1}(4x), x \in \left(-\frac{1}{4}, \frac{1}{4}\right)$

16. a. $y = x^2 + 3x, x \in \mathbb{R}$

b. $y = \frac{3}{2}\tan(6x), -\frac{\pi}{12} < x < \frac{\pi}{12}$

17. a. $y = \frac{12}{3 + e^{4x}}, x \in \mathbb{R}$

b. $y = \frac{6e^{3x}}{3 - 2e^{3x}}, x \in \mathbb{R} \setminus \left\{\frac{1}{3}\ln\left(\frac{3}{2}\right)\right\}$

18. a. $y = \frac{12(e^x + 1)}{4e^x + 3}, x \in \mathbb{R}$

b. $y = \frac{4(1 - e^{2x})}{e^{2x} - 2}, x \in \mathbb{R} \setminus \left\{\ln\left(\sqrt{2}\right)\right\}$

19. a. $y = y_0 e^{-kx}$

b. i. $y = \left(c - \frac{b}{a}\right)e^{-ax} + \frac{b}{a}$

ii. $y = \frac{ac}{(a - bc)e^{ax} + bc}$

20. a. i. $y = \frac{b^2 x}{1 - abx}, x \neq \frac{1}{ab}$

ii. $y = \frac{a}{b}\tan(abx)$

b. i. $y = \frac{ab(1 - e^{-(a-b)x})}{ae^{-(a-b)x} - b}$

ii. $-a$

8.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$

8.6 Exercise

1. $\frac{y^4}{4} + 8y - \frac{x^2}{2} - 2x = c$

2. a. $\frac{1}{3}y^3 + 4y - \frac{1}{3}x^3 - 4x = c$

b. $\frac{1}{2}y^2 + 4\ln(|y|) - \frac{1}{2}x^2 = c$

c. $y - \frac{4}{y} - \frac{1}{3}x^3 = c$

d. $\frac{1}{2}y^2 - \frac{8}{y} - \frac{1}{2}e^{x^2} = c$

3. a. $y = \frac{1}{1 - \ln(|x|)}, x \in \mathbb{R} \setminus \{-e, 0, e\}$

b. $y = \frac{1}{4 - 3\cos(4x)}$

4. a. $y = \sqrt{5 - x^2}, x \in [-5, 5]$

b. $y = \frac{3}{6x^3 - 5}, x \in \mathbb{R} \setminus \left\{\sqrt[3]{\frac{5}{6}}\right\}$

5. a. $y = \frac{2}{9x^4 - 8}, x \in \mathbb{R} \setminus \left\{\sqrt[4]{\frac{8}{9}}\right\}$

b. $y = \frac{2x}{2 - x}, x \in \mathbb{R} \setminus \{2\}$

6. a. $y = \frac{2}{3 - e^{2x}}, x \in \mathbb{R} \setminus \left\{\ln\left(\sqrt{3}\right)\right\}$

b. $y = \frac{2}{4x^6 - 3}, x \in \mathbb{R} \setminus \left\{\sqrt[6]{\frac{3}{4}}\right\}$

7. $y - 2\tan^{-1}\left(\frac{y}{2}\right) + \frac{1}{x} = c$

8. $y = e^{x^3 + x}$

9. $y = \frac{1}{3 + x - x^2}, x \in \mathbb{R} \setminus \left\{\frac{1 \pm \sqrt{13}}{2}\right\}$

10. $y = 8\sin(x^2), 0 < x < \frac{\sqrt{2\pi}}{2}$

11. $y = 4\tan(x^2), 0 < x < \frac{\sqrt{2\pi}}{2}$

12. a. $y = e^{x^3 - x}, x \in \mathbb{R}$

b. $y = \frac{2}{4x^3 - 2x + 3}, 4x^3 - 2x + 3 \neq 0$

13. a. $y = \frac{2}{2x^2 - 2x + 1}, x \in \mathbb{R}$

b. $y = \frac{1}{\sqrt[3]{6x^4 - 3x + 1}}, 6x^4 - 3x + 1 \neq 0$

14. a. $y = \frac{2}{1 - x^2}, x \in \mathbb{R} \setminus \{\pm 1\}$

b. $y = \frac{4x^4}{5 - x^4}, x \in \mathbb{R} \setminus \left\{\pm\sqrt[4]{5}\right\}$

15. a. $y = \frac{1}{4}(x^2 + 4), x \in \mathbb{R}$

b. $y = \frac{2}{3}\sqrt{2x^2 + 9}, x \in \mathbb{R}$

16. a. $y = 5\tan\left(\frac{5x^2}{2}\right), x \in \left(-\frac{\sqrt{5\pi}}{5}, \frac{\sqrt{5\pi}}{5}\right)$

b. $y = 5\cos(2x^2), x \in \left(-\frac{\sqrt{2\pi}}{2}, \frac{\sqrt{2\pi}}{2}\right)$

17. a. $y = x - \frac{8}{x^3}, x \in \mathbb{R} \setminus \{0\}$

b. $y = 2x\left(1 + 2\ln\left(|x|\right)\right), x \in \mathbb{R} \setminus \{0\}$

18. $y = \sqrt{1 - 4\ln|\cos(3x)|}$

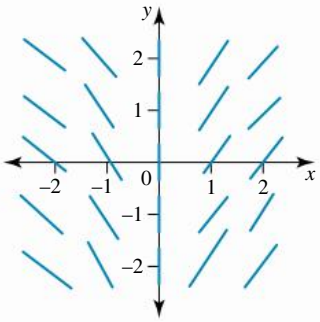
19. $y^2\left(y^2 - 2\right) = \frac{2}{3}x^3 + k, k \in \mathbb{R}$

20. $y = \sin^{-1}\left(x^2 - \frac{1}{2}\right)$

8.7 Slope fields

8.7 Exercise

1.



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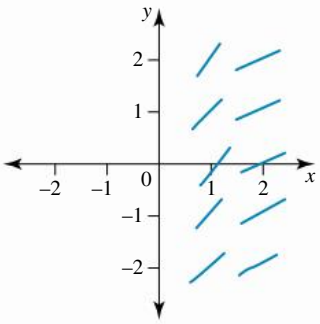

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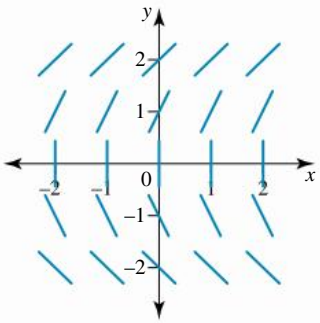

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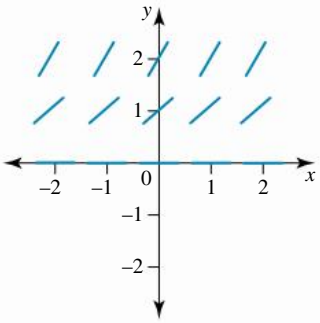

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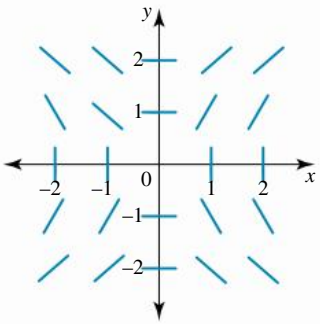

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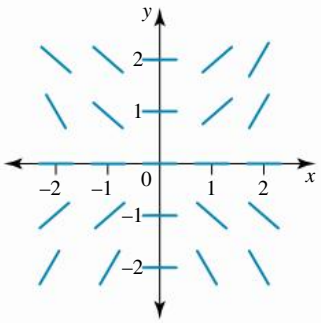


7. C

8. D

9. **a.** Slope 4 **b.** Slope 2
c. Slope 3 **d.** Slope 1

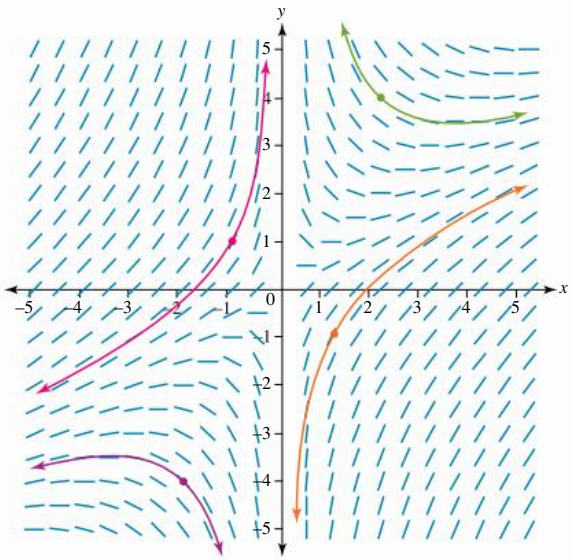
10. **a.** Slope 3 **b.** Slope 1
c. Slope 4 **d.** Slope 2

11. **a.** Slope 1 **b.** Slope 3
c. Slope 2 **d.** Slope 4

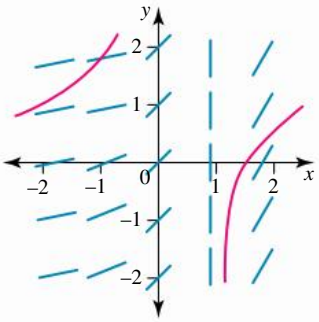
12. **a.** Slope 2 **b.** Slope 3
c. Slope 1 **d.** Slope 4

13. **a.** Slope 1 **b.** Slope 4
c. Slope 2 **d.** Slope 3

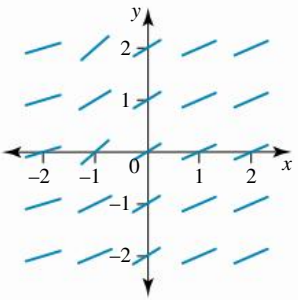
14.



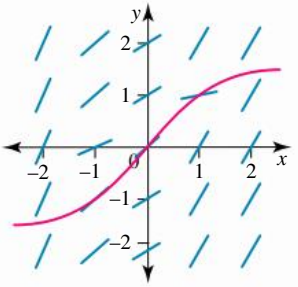
15. **a.**



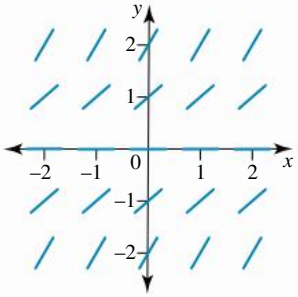
b. i.



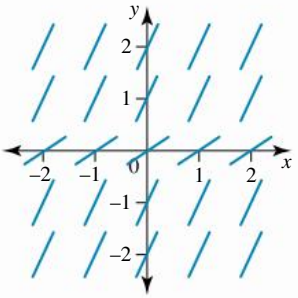
ii. $y = \tan^{-1}(x)$



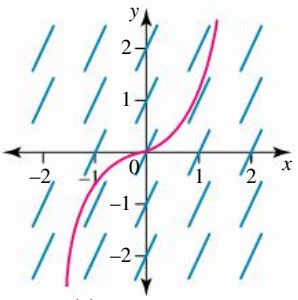
16. a.



b. i.

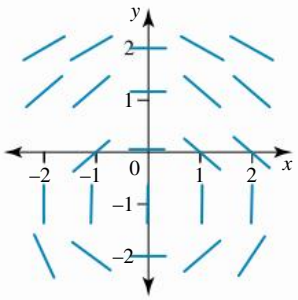


ii.

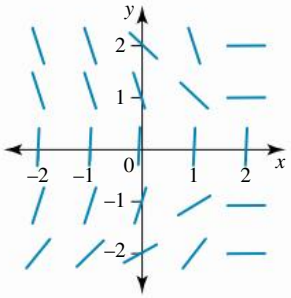


$y = \tan(x)$

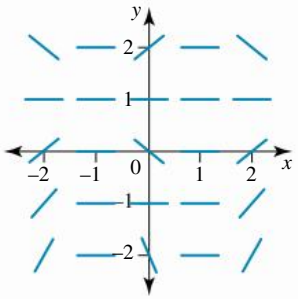
17. a.



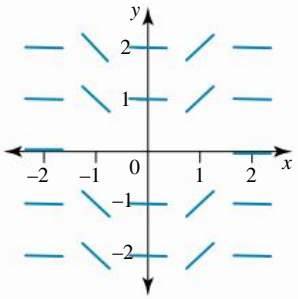
b.



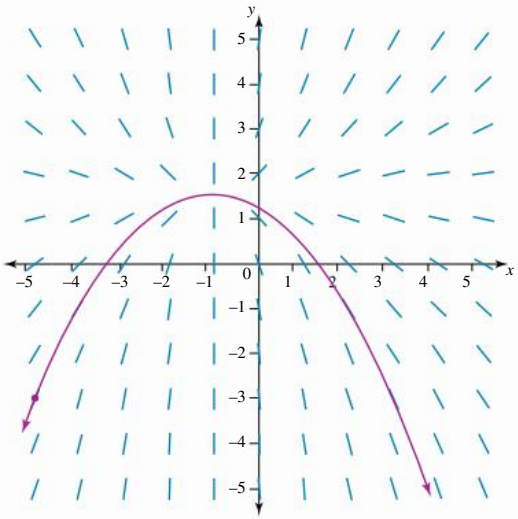
18. a.



b.

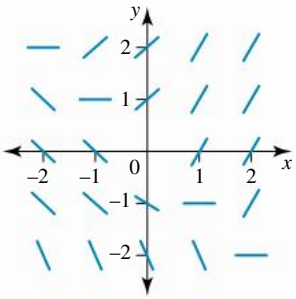


19. a.



b. The solution to $\frac{dy}{dx} = \frac{2y-3}{x+1}$, $y(-5) = -3$ is the inverted parabola $y = \frac{3}{2} + -\frac{9}{16}(x+1)^2$, with vertex $\left(-1, \frac{3}{2}\right)$, which does correspond to our sketch.

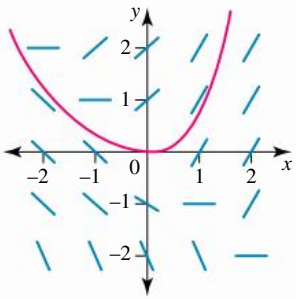
20. a.



b. $A = 1, B = -1, C = -1, k = 1$

c. Sample responses can be found in the worked solutions in the online resources.

d.



8.8 Review

8.8 Exercise

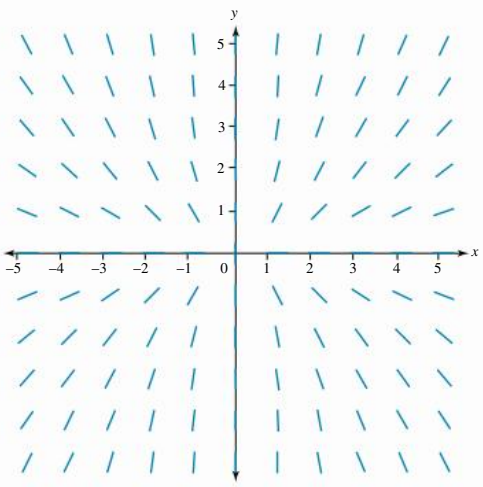
1. Sample responses can be found in the worked solutions in the online resources.

2. Decrease by $\frac{1}{6}$

3. a. $\frac{2x(x-2y^2)}{y(4x^2-3y)}$ **b.** $\frac{16x^3-7y}{7x+12y^3}$ **c.** $(2, -4)$, $(-2, 4)$

4. $y = c - 18x^2$

5.



6. $y = Ae^{-\frac{x}{3}}$

7. $y = \frac{1}{A + x^3}$

8. $y = 2\tan(2x)$

9. a. $y = 4x + \frac{1}{3}(x^3 - 7)$ $x \in \mathbb{R}$

b. $y = 2\ln(|x|) - 2 + \frac{4}{x}$ $x \in \mathbb{R}\backslash\{0\}$

c. $y = \frac{12x+8}{5x+3}$ $x \in \mathbb{R}\backslash\left\{-\frac{3}{5}\right\}$

10. a. $y = 2 - \tan^{-1}\left(\frac{x}{3}\right)$

b. $y = 11 - 3\sqrt{x^2+9}$

c. $y = 2 + \frac{1}{2}\ln\left(\left|\frac{x-3}{x+3}\right|\right)$, $x \in \mathbb{R}\backslash\{-3, 3\}$

11. a. $y = \frac{1}{9x+1}$, $x \in \mathbb{R}\backslash\left\{-\frac{1}{9}\right\}$

b. $y = \sqrt{1-18x}$, $x < \frac{1}{18}$

c. $y = \frac{1}{\sqrt{18x+1}}$, $x > -\frac{1}{18}$

12. a. $y = \frac{1}{4}(7 + e^{-4x})$

b. $y = \frac{5x-7}{2x-3}$, $x \in \mathbb{R}\backslash\left\{\frac{3}{2}\right\}$

c. $y = 3\cos(x)$, $0 \le x \le \pi$

13. a. $16y + 5x + 6 = 0$

b. $10y + 9x - 28 = 0$

c. $(7, -2)$, $(-1, -2)$

14. a. 288π cm^3/s **b.** 96π cm^2/s

15. a. $y = 2\tan(2x)$, $-\frac{\pi}{4} < x < \frac{\pi}{4}$

b. $y = \frac{4e^{4x}}{3 - e^{4x}}$, $x \in \mathbb{R}\backslash\left\{\frac{1}{4}\ln(3)\right\}$

c. $y = \frac{1}{4}\ln\left(\frac{4}{x+4}\right)$, $x > -4$

16. a. $y = \frac{1}{3 - 2x^3}$

b. $y = \ln(x^4 + 1)$

c. $y = \frac{1}{4 - x - 2x^2}$

17. a–c. Sample responses can be found in the worked solutions in the online resources.

18. a. 0.75 m/s **b.** 2.25 m/s **c.** 1.6 m/s

19. $y = 2\sqrt[3]{\sin(3x)}$

20. a. $y = x(c + b\ln(|x|))$, $x \in \mathbb{R}\backslash\{0\}$

b. $y = \frac{bx}{a+1} + \frac{c}{x^a}$

8.8 Past QCAA exam questions

1. C

2. A

3. $a = 6, b = -3$

4. a. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$= \frac{\pi h^3}{12}$

b. $\frac{dV}{dh} = \frac{\pi h^2}{4}$

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$2 = \frac{\pi h^2}{4}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{8}{\pi h^2}$

c. Half capacity $= \frac{1}{2}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{6}\pi(6)^2(12)$

$\approx 226.195 \text{ cm}^3$

Solving $\frac{\pi h^3}{12} = 226.195$

$\therefore h \approx 9.52 \text{ cm}$

$\frac{dh}{dt} \approx \frac{8}{\pi(9.52)^2} \approx 0.03$

Required rate is 0.03 cm s^{-1}

5. Sample responses can be found in the worked solutions in the online resources.

9 Applications of first-order differential equations

LESSON SEQUENCE

9.1 Overview 622
9.2 Growth and decay 623
9.3 Other applications of first-order differential equations 634
9.4 Bounded growth and Newton's law of cooling 641
9.5 Chemical reactions and dilution problems 648
9.6 The logistic equation 659
9.7 Review 672
Answers 678

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 9
Exam questions	Exam question booklet — Chapter 9
Digital documents	Learning matrix — Chapter 9 Chapter summary — Chapter 9

LESSON
9.1 Overview

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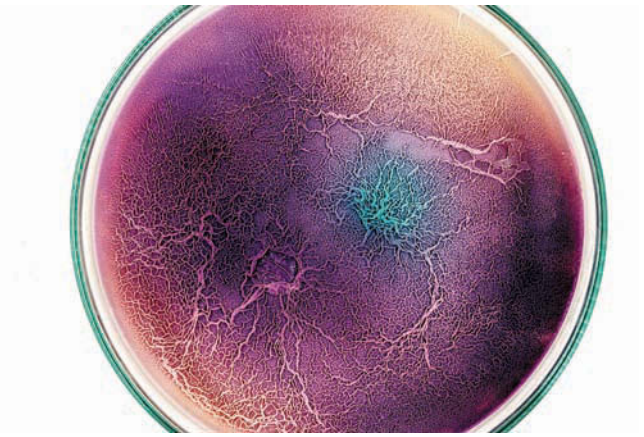
9.1.1 Introduction

Differential equations originated with the development of calculus by Isaac Newton (1643–1727) and Gottfried Leibniz (1646–1716). As seen in Chapter 8, a differential equation is a mathematical expression that relates a physical quantity to its rate of change. Differential equations are used to represent real-world applications. In this chapter, we will focus on exponential growth or decay of a population or substance (such as bacterial growth or decay of radioactive elements), asymptomatic growth and decay, where there is an equilibrium value, such as Newton's law of cooling.



9.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
9.2	**Growth and decay**	◯ Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.
9.3	**Other applications of first-order differential equations**	◯ Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.
9.4	**Bounded growth and Newton's law of cooling**	◯ Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.
9.5	**Chemical reactions and dilution problems**	◯ Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.
9.6	**The logistic equation**	◯ Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.

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LESSON
9.2 Growth and decay

SYLLABUS LINKS

- Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.

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9.2.1 Introduction

In Chapter 8, methods of solving first-order differential equations were discussed. In this chapter some applications of first-order differential equations are explored. There are many applications of differential equations in business and science; in fact, they are useful whenever a rate of change needs to be considered.

In this lesson, we are looking at exponential growth and decay, which applies to quantities changing rapidly, when the rate of change is proportional to the quantity present at any time.

In many modelling situations, the rate of change of a variable y is proportional to the size of y. That is, we express this proportionality as

$$\frac{dy}{dx} \propto y$$

Since y is a function of x, we can write the relationship in the form

$$\frac{dy}{dx} = ky$$

where k is a positive constant known as the **constant of proportionality**.

This **differential equation**, known as the law of natural growth or uninhibited growth, is given by

$$\frac{dy}{dx} = ky \text{ with } k > 0.$$

WORKED EXAMPLE 1 Solving $\frac{dy}{dx} = ky, y(0) = y_0$

Solve the differential equation $\frac{dy}{dx} = ky, y(0) = y_0$, where k and y_0 are positive constants.

THINK	WRITE
1. Separate the variables.	$\frac{1}{y}dy = k\,dx$
2. Integrate the LHS with respect to y and the RHS with respect to x.	$\int \frac{1}{y}\,dy = \int k\,dx$
	$\ln\lvert y\rvert = kx + c$
3. Make y the subject, where $y > 0$.	$y = e^{kx+c}$
4. Apply the first index law to rewrite the RHS.	$y = e^{kx} \times e^{c}$

5. Let $A = e^c$ to reach a general solution.	$y = Ae^{kx}$
6. Use $y(0) = y_0$ to determine A.	$y_0 = Ae^{k\times 0}$
	$y_0 = A$
7. State the particular solution.	$y = y_0e^{kx}$

Thus, there is a function, $y = y_0e^{kx}$, whose derivative is proportional to itself. This result gives rise to several applications studied in the following sections.

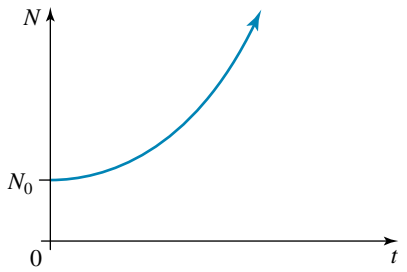
9.2.2 The law of natural growth

The **law of natural growth** states that the rate of increase of population is proportional to the current population at that time. Let $N = N(t)$ represent the population number of a certain quantity at a time t, where $t \geq 0$. This leads to the equation

$$\frac{dN}{dt} \propto N$$

$$\Rightarrow \frac{dN}{dt} = kN, \text{ where } k \text{ is a positive constant.}$$

Assuming that the initial population number is $N(0) = N_0$, the solution of this differential equation is given by $N(t) = N_0e^{kt}$. This is obtained by simply substituting N for y and t for x from the Worked example 1.



This equation has been found to model population growth, although it applies over only a limited time frame.

Be aware that it does not account for specific events that can alter population size, such as war, migration or famine.

Note that mathematically, N is treated as a continuous variable. However, when these equations are used to model populations, N is really a discrete quantity, the number of individuals in the population. In these situations, the final answer may need to be rounded to the nearest whole number.

Law of natural growth

If a quantity $y(t)$ grows at a rate proportional to its size, it satisfies the differential equation $\frac{dy}{dt} = ky$ where k is the positive growth constant.

Given $y(0) = y_0$, the particular solution is $y(t) = y_0e^{kt}$.

By convention, k represents a rate per unit time, but this is often left unstated.

The unit of time for t (years, days, hours etc.) is generally specified explicitly, otherwise, there will be an implicit best option. For instance, in Worked example 2, it is implicit that the unit of time most convenient to use is years.

WORKED EXAMPLE 2 Using the law of natural growth

The population of a certain town increases at a rate proportional to the current population. At the start of 2009 the population was 250 000, and at the start of 2017 the population was 750 000.

a. Express the population number in millions, N, in terms of t, the time in years after the start of 2009.

b. Use $N(t)$ to predict the population at the start of 2025.

c. Determine when the population reaches 5 million.

THINK	WRITE
a. 1. Begin with the proportionality statement, then follow with the associated differential equation. The equation was not solved formally, with the general solution for $N(t)$ being applied.	a. $\frac{dN}{dt} \propto N$ $\therefore \frac{dN}{dt} = kN$ $\Rightarrow N = N_0 e^{kt}$ $\frac{1}{4} = N_0 e^0$ $N_0 = \frac{1}{4}$
2. In the year 2009, that is, at $t = 0$, the population was 250 000, or one-quarter of a million. Use this to determine the value of N_0.	Substitute $N_0 = \frac{1}{4}$: $N(t) = \frac{1}{4}e^{kt}$
3. In the year 2017 ($t = 8$), the population was 750 000, which is three-quarters of a million. Use this to determine the value of k.	Substitute $t = 8$ and $N = \frac{3}{4}$: $\frac{3}{4} = \frac{1}{4}e^{8k}$
4. Solve the equation for k by taking the natural logarithms of both sides.	$e^{8k} = 3$ $8k = \ln(3)$ $k = \frac{1}{8}\ln(3)$
5. Express the population number in millions at a time t years after 2009.	$N(t) = \frac{1}{4}e^{\frac{t}{8}\ln(3)}$
b. 1. Determine the population in 2025, that is, when $t = 16$.	b. Substitute $t = 16$: $N(16) = \frac{1}{4}e^{\frac{16}{8}\ln(3)}$ $N(16) = \frac{1}{4}e^{2\ln(3)}$ $= \frac{1}{4}e^{\ln(9)}$ $= \frac{1}{4} \times 9$ $= 2.25$
2. State the result.	The population in 2025 is predicted to be 2.25 million.
c. 1. Determine the year by finding the value of t when $N = 5$.	c. $5 = \frac{1}{4}e^{\frac{t}{8}\ln(3)}$

2. Solve the equation for t by taking the natural logarithms of both sides.	$20 = e^{\frac{t}{8}\ln(3)}$ $\frac{t}{8}\ln(3) = \ln(20)$ $t = \frac{8\ln(20)}{\ln(3)}$ ≈ 21.815
3. State the result.	The population reaches 5 million late in the year 2030.

9.2.3 The law of natural decay

When a quantity $y = y(t)$ decreases at a rate proportional to the quantity present, it is considered an exponential decay model and $y(t)$ satisfies the differential equation $\frac{dy}{dt} = -ky$, where k is a positive constant.

A solution to this differential equation can be calculated using a similar method to the natural growth example. But in this case, a slightly different approach is used to determine the particular solution.

WORKED EXAMPLE 3 Solving $\frac{dy}{dx} = -ky, y(0) = y_0$

Solve the differential equation $\frac{dy}{dx} = -ky, y(0) = y_0$, where k and y_0 are positive constants.

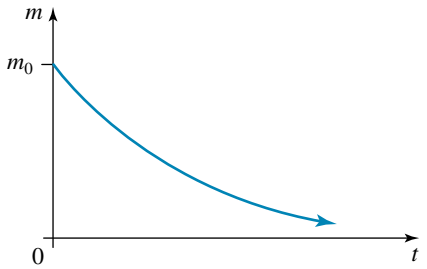
THINK	WRITE
1. Separate the variables.	$\frac{1}{y}dy = -k\,dx$
2. Integrate the LHS with respect to y and the RHS with respect to x.	$\int \frac{1}{y}\,dy = \int -k\,dx$ $\ln\lvert y\rvert = -kx + c$
3. Use $y(0) = y_0$ where $y_0 > 0$ to calculate the constant of integration, c.	$\ln\lvert y_0\rvert = -k \times 0 + c$ $c = \ln y_0$
4. Substitute $c = \ln(y_0)$, rearrange and simplify the natural logarithms.	$\ln(y) = -kx + \ln(y_0)$ $\ln(y) - \ln(y_0) = -kx$ $\ln\frac{y}{y_0} = -kx$
5. State the particular solution.	$\frac{y}{y_0} = e^{-kx}$ $y = y_0 e^{-kx}$

9.2.4 Radioactive decay and half-lives

Chemical experiments have shown that when a radioactive substance decays, it changes into another element or an isotope of the same element.

Scientists have determined that the rate of decay is proportional to the mass of the radioactive substance present at that time.

Let $m = m(t)$ represent the mass of the radioactive substance at a time, t, where $t \geq 0$.



Then

$$\frac{dm}{dt} \propto -m$$

$$\Rightarrow \frac{dm}{dt} = -km, \text{ where } k \text{ is a positive constant.}$$

For $m(t) = m_0 e^{-kt}$, where $k > 0$, as $t \to \infty, m \to 0$.

An infinite time is required for all of the radioactive material to disintegrate. For this reason, the rate of disintegration is often measured in terms of the **half-life** of the radioactive material. The half-life, T, is the time it takes for half of the original mass to disintegrate.

The value of the half-life T is obtained when $m = \frac{1}{2}m_0$.

Law of natural decay

If a quantity $m(t)$ decays at a rate proportional to its size, it satisfies the differential equation $\frac{dm}{dt} = -kt$ where k is the positive decay constant.

Given $m(0) = m_0$, the particular solution is $m(t) = m_0 e^{-kt}$.

WORKED EXAMPLE 4 Using the law of natural decay

The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. Initially 30 milligrams of a radioactive substance is present. After 2 hours the experimenter observes that 20% has disintegrated. Determine the amount remaining after a further 3 hours, correct to 3 decimal places.

THINK	WRITE
1. Let $m = m(t)$ represent the amount of the radioactive substance present in milligrams (mg) after a time t hours.	$\frac{dm}{dt} \propto -m$ $\therefore \frac{dm}{dt} = -km$ $\Rightarrow m = m_0 e^{-kt}$ [1]
2. Initially, when $t = 0$, the mass is 30 mg. Use this to determine the value of m_0.	$m_0 = m(0) = 30$
3. After 2 hours, 20% has disintegrated, so 80% remains of the initial 30 mg. Use this to determine the amount left after 2 hours.	When $t = 2$, $m = 0.8 \times m_0$: $m(2) = 0.8 \times 30$ $= 24$

4. Substitute $m_0 = 30$, $t = 2$ and $m = 24$ into the law of exponential decay, $m = m_0 e^{-kt}$, where $m_0 = m(0)$.	Substitute $m_0 = 30$, $t = 2$ and $m = 24$ into [1]: $24 = 30e^{-2k}$
5. Solve the equation for k by taking the natural logarithms of both sides.	$e^{-2k} = \frac{24}{30}$ $-2k = \ln\left(\frac{4}{5}\right)$ $k = -\frac{1}{2}\ln\left(\frac{4}{5}\right)$
6. Express the mass remaining in milligrams after t hours.	$m(t) = 30e^{\frac{t}{2}\ln\left(\frac{4}{5}\right)}$
7. Determine the amount remaining after a further 3 hours, that is, when $t = 5$.	Substitute $t = 5$: $m(t) = 30e^{\frac{5}{2}\ln\left(\frac{4}{5}\right)}$ ≈ 17.173
8. State the result.	17.173 mg remains after 5 hours.

WORKED EXAMPLE 5 Calculating the half-life of a radioactive isotope for different initial masses

The mass in grams of a certain radioactive isotope after t years decays exponentially according to $m(t) = m_0 e^{-0.00013t}, t \geq 0$.
Calculate to the nearest year the half-life T of this isotope for the following initial masses.
a. $m(0) = 12$ **grams**
b. $m(0) = 3.7$ **kilograms**

THINK	WRITE
a. 1. Given $m(0) = 12, m_0 = 12$.	**a.** $m_0 = 12$ $\therefore m(t) = 12e^{-0.00013t}$
2. Let the mass equal 6 grams to calculate the half-life time T.	Let $m(T) = 6$ $6 = 12e^{-0.00013T}$
3. Solve for time t.	$\frac{6}{12} = e^{-0.00013t}$ $\frac{1}{2} = e^{-0.00013t}$ $\ln\left(\frac{1}{2}\right) = -0.00013t$ $\frac{-\ln 2}{-0.00013} = t$ $t = 5331.9$ years
State the half-life of the isotope.	The half-life of the radioactive isotope is approximately 5332 years.

b. 1. Given $m(0) = 3.7, m_0 = 3.7$.

b. $m_0 = 3.7$

$\therefore m(t) = 3.7e^{-0.00013t}$

2. Let the mass equal 1.85 kilograms to calculate the half-life time T.

Let $m(T) = 1.85$

$$1.85 = 3.7e^{-0.00013t}$$

3. Solve for time t.

$$\frac{1.85}{3.7} = e^{-0.00013t}$$

$$\frac{1}{2} = e^{-0.00013t}$$

$$\ln\left(\frac{1}{2}\right) = -0.00013t$$

$$\frac{-\ln(2)}{-0.00013} = t$$

$$t = 5331.9 \text{ years}$$

State the half-life of the isotope.

The half-life of the radioactive isotope is approximately 5332 years.

The half-life T of a radioactive isotope with a decay constant k can be expressed in terms of k.

Given $m(t) = m_0e^{-kt}$, $m(0) = m_0$ and $m(T) = \frac{1}{2}m_0$:

$$\begin{aligned} m_0e^{-kT} &= \frac{1}{2}m_0 \\ e^{-kT} &= \frac{1}{2}\frac{\cancel{m_0}}{\cancel{m_0}} \\ e^{-kT} &= \frac{1}{2} \\ e^{kT} &= 2 \\ kT &= \ln(2) \\ T &= \frac{1}{k}\ln(2) \end{aligned}$$

Notice that this half-life formula does not depend upon the initial mass m_0 and is thus independent of the time when observations began. Half-lives for naturally occurring radioactive substances can range from a few minutes (e.g. francium-223) to billions of years (e.g. iodine-129).

Half-life

The half-life, T of a radioactive isotope is the time it takes for half of the original mass to disintegrate.

For a radioactive isotope with a decay constant k, $m(t) = m_0e^{-kt}$ and $T = \frac{1}{k}\ln(2)$.

T is expressed with the same units of time as the variable t.

WORKED EXAMPLE 6 Calculating the half-life of a radioactive isotope

a. Calculate the half-life T of an isotope given $m(t) = m_0e^{-0.1386t}$, where t is in years. Give your answer to 3 decimal places.

b. It takes 2 years for 30% of a particular substance to disintegrate.
Determine the half-life of the substance, correct to 3 decimal places.

THINK	WRITE
a. 1. Identify the decay constant k.	$m(t) = m_0e^{-0.1386t}$ $k = 0.1386$
2. Use $T = \frac{1}{k}\ln(2)$.	$T = \frac{1}{k}\ln(2)$ $= \frac{1}{0.1386}\ln(2)$ $= 5.001$ years
3. State the answer.	The half-life T of this isotope is 5.001 years.
b. 1. Let $m = m(t)$ represent the amount of the radioactive substance present after time t in years.	**b.** $m(t) = m_0e^{-kt}$
2. Since 30% has disintegrated after 2 years, 70% remains.	When $t = 2$ years, $m = 0.7m_0$.
3. Substitute $t = 2$ years and $m = 0.7m_0$ to determine k.	$0.7m_0 = m_0e^{-k\times 2}$ $0.7 = e^{-2k}$ $-2k = \ln(0.7)$ $k = -\frac{\ln(0.7)}{2}$
4. Use $T = \frac{1}{k}\ln(2)$ to determine the half-life.	$T = \frac{1}{k}\ln(2)$ $= -\frac{2}{\ln(0.7)}\ln(2)$ $= 3.8867\cdots$ years
5. State the answer.	The half-life of the substance is 3.887 years.

Exercise 9.2 Growth and decay

learnon

9.2 Exercise | 9.2 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18	19, 20	21, 22

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Simple familiar

1. WE1 Solve the differential equation $\frac{dy}{dx} = \frac{1}{25}y, y(0) = 2$.

2. WE2 The population of a city increases at a rate proportional to the current population. In 2015 the population was 1.79 million, and in 2019 the population was 2 million.
   a. Express the population number, N, in terms of t, the time in years after 2015.
   b. Calculate the predicted population in 2030.
   c. Determine when the population reaches 5 million.

3. WE3 Solve the differential equation $\frac{dy}{dx} = -0.008y, y(0) = \frac{1}{2}$.

4. The number of frogs in a colony increases at a rate proportional to the current number. After 15 months there are 297 frogs present, and this number grows to 523 after 26 months. Determine the initial number of frogs present in the colony.

5. a. The rate of growth of a population of insects is proportional to its present size. A colony of insects initially contains 600 individuals and is found to contain 1300 after 2 weeks. Determine the number of insects after a further 3 weeks.
   b. Initially there are 10 000 fish in a reservoir. The number of fish in the reservoir grows continuously at a rate of 4% per year. Determine the number of fish in the reservoir after 3 years.

6. The number of possums in a certain area grows at a rate proportional to the current number. In 4 months the number of possums has increased from 521 to 678. Determine the number of possums in the area after a further 5 months.

7. The population of Australia in 2013 was approximately 23 131 000. In 2018 it had grown to 24 992 000.
   a. Calculate the predicted population in 2025.
   b. Determine when the population will first exceed 30 million.

8. In 2011 the world population reached 7 billion. In 2018 the world population was approximately 7 632 819 325.
   a. Calculate the predicted population in 2024.
   b. Determine when the population will reach 10 million.

9. **WE4** The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. Initially 80 milligrams of a radioactive substance is present, and after 1 hour it is observed that 10% has disintegrated. Determine the amount remaining after a further 2 hours. Give your answer to 2 decimal places.

10. **WE5** The mass in grams of a certain radioactive isotope after t years decays exponentially according to $m(t) = m_0 e^{-0.004t}, t \geq 0$.
Calculate to the nearest year the half-life T of this isotope for the following initial masses.

a. $m(0) = 4.50$ grams

b. $m(0) = 1.80$ kilograms

11. The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. After 2 hours, 64 milligrams of a radioactive substance is present, and after 4 hours, 36 milligrams is present. Determine the initial amount of the substance present. Give your answer to 2 decimal places.

12. **WE6** **a.** Calculate the half-life T of an isotope given $m(t) = m_0 e^{-0.425t}$, where t is in hours. Give your answer to 2 decimal places.

b. For a radioactive substance, it takes 3 years for 15% to disintegrate. Determine the half-life of the substance. Give your answer to 2 decimal places.

13. An isotope of radium has a half-life of 1601 years. Calculate what percentage remains after 1000 years.

14. **a.** The rate of decay of a radioactive substance is proportional to the amount of the substance present at any time. Determine how much of the radioactive material remains after 40 000 years, given that 1 gram decomposes to half a gram in 22 652 years. Give your answer to 2 decimal places.

b. If 15% of a radioactive element disintegrates in 6 years, determine the half-life of this element. Give your answer to 2 decimal places.

15. Iodine-131 is present in radioactive waste and decays exponentially to form a substance that is not radioactive. The half-life of iodine-131 is 8 days. If 120 milligrams is considered a safe level, determine how long it will take 2 grams of iodine-131 to decay to a safe level. Give your answer to 2 decimal places.

16. Strontium-90 is an unpleasant radioactive isotope that is a byproduct of a nuclear explosion. Strontium-90 has a half-life of 28.9 years. Calculate the number of years that elapse before 60% of the strontium-90 in a sample has decayed. Give your answer to 1 decimal place.

17. Marie Curie (1867–1934) was a Polish-born French scientist. She coined the term 'radioactivity' and was also the first female scientist to win a Nobel Prize. Unfortunately she died of radiation poisoning as a result of her experiments. In 1944 the element Curium was named after her.

a. Curium-243 has a half-life of 29.1 years. It decays into plutonium-239 through alpha decay. Determine the percentage of curium-243 that remains in a sample after 10 years.

b. Plutonium-239 is a silvery metal that is used for the production of nuclear weapons. If 3 micrograms of plutonium-239 decomposes to 1 microgram in 38 213 years, determine the half-life of plutonium-239 to the nearest year.

18. Uranium-238 has a half-life of 4.468 billion years. Estimate the percentage of the original amount of uranium-238 that remains in the universe, assuming we began with that original amount 13.8 billion years ago. Give your answer to 2 decimal places.

Complex familiar

19. A biologist is investigating two different types of bacteria. Both types grow at a rate proportional to the number present. Initially 900 type A bacteria are present, and after 10 hours 3000 are present. The number of type B bacteria doubles after 4 hours, and there are 2500 type B bacteria after 8 hours. Determine the time at which equal numbers of type A and type B bacteria are present. Give your answer to 3 decimal places.

20. Determine the year when the population of Brisbane will first exceed the population of Melbourne.
The table below shows the comparative population in millions of Brisbane and Melbourne, in certain years.
Assume the growth rates are proportional to the current population and these growth rates continue.

City	2001	2009
Melbourne	3.339	3.996
Brisbane	1.609	2.004

Complex unfamiliar

21. Cobalt-60 is a radioisotope produced in a nuclear reactor and used, among other things, in the sterilisation of medical equipment. A Cobalt-60 source needs to be replaced when at least 95% of its initial mass has decayed.
A hospital bought a new Cobalt-60 source in 2023, and in 2026, one-third of the initial mass has already decayed.
Determine in what year the hospital will need to buy a new source and show your working.

22. In the 1950s, W.F. Libby and others at the University of Chicago devised a method of estimating the age of organic material based on the decay rate of Carbon-14, a radioactive isotope of carbon. Carbon-14 dating can be used on objects ranging from a few hundred years old to 40 000 years old. Carbon-14 obeys the law of radioactive decay. If Q denotes the amount of Carbon-14 present at time t years, then $\frac{dQ}{dt} = -kQ$.
Q_0 represents the amount of Carbon-14 present at time t_0 when decay starts (when death occurs) and the half-life of Carbon-14 is 5730 years.
In residual amounts of carbon-14 it is possible to measure $\frac{Q}{Q_0}$ for some wood and plant remains, and hence determine the elapsed time since the death of these remains, that is, the period during which decay has been taking place. This technique of radiocarbon dating is of great value to archaeologists.
For a particular specimen, $\frac{Q}{Q_0} = \frac{2}{7}$.
Determine the age of the specimen to the nearest year.

Fully worked solutions for this chapter are available online.

LESSON
9.3 Other applications of first-order differential equations

SYLLABUS LINKS

- Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.

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9.3.1 Miscellaneous types

There are many other applications of first-order differential equations that are similar to the growth and decay models.

Investments and money matters

When an amount of money known as the principal, $\$P$, is invested with continuously compounding interest at $R\%$, the rate at which the accumulated amount grows is proportional to the amount invested. In fact, the proportionality constant is the interest, so $\frac{dP}{dt} = Pr$, where t is the time ($t \geq 0$) and $r = \frac{R}{100}$.

Pressure and height in the atmosphere

The rate of decrease of pressure in the atmosphere, P, with respect to the height above sea level, h, is proportional to the pressure at that height. For a proportionality constant k (such that $k > 0$), $\frac{dP}{dh} = -kP$.

Drug disappearance in the body

For many drugs, the rate of disappearance from the body is proportional to the amount, D, of the drug still present in the body at time t. If k is the proportionality constant ($k > 0$), $\frac{dD}{dt} = -kD$.

Light intensity and depth

When a beam of light passes through a medium, it loses its intensity as it penetrates more deeply. If I is the intensity of light at depth x, the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth. If k is the proportionality constant ($k > 0$), then $\frac{dI}{dx} = -kI$.

Other applications

There are also applications related to electrical circuits. Applications to kinematics, dynamics and variable forces are considered in Chapter 10.

WORKED EXAMPLE 7 Using first-order differential equations for investments

When money is invested in a bank at a constant rate of $R\%$ with continuously compounding interest, the accumulated amount $\$Pt$ years after the start of the investment satisfies the differential equation $\frac{dP}{dt} = Pr$.

a. Assuming an initial investment of $\$P_0$, solve the differential equation to determine the amount $\$P$ after a time t years.

b. Determine the amount to which \$10 000 will grow in 6 years if invested at 4.5%.

THINK	WRITE
a. 1. To solve the differential equation, invert both sides. The initial investment is $\$P_0$.	**a.** $\frac{dP}{dt} = Pr, P(0) = P_0$ $\frac{dt}{dP} = \frac{1}{Pr}$
2. Because r is a constant, it can be taken outside the integral sign.	$t = \frac{1}{r}\int \frac{1}{P}\,dP$
3. Integrate.	$t = \frac{1}{r}\ln(\lvert P\rvert) + c$ $= \frac{1}{r}\ln(P) + c,\ \text{as } P > 0$
4. Substitute $t = 0$, $P = P_0$.	$0 = \frac{1}{r}\ln(P_0) + c$ $c = -\frac{1}{r}\ln(P_0)$
5. Rewrite the equation, substituting $c = -\frac{1}{r}\ln(P_0)$, and use logarithm laws to simplify.	$t = \frac{1}{r}\ln(P) - \frac{1}{r}\ln(P_0)$ $t = \frac{1}{r}\ln\left(\frac{P}{P_0}\right)$
6. Rearrange and write as an exponential.	$rt = \ln\left(\frac{P}{P_0}\right)$ $\frac{P}{P_0} = e^{rt}$
7. Solve for P and state the solution of the differential equation.	$P = P_0 e^{rt}$
b. 1. Substitute $P_0 = 10\,000$ and $r = 4.5\% = \frac{4.5}{100}$.	**b.** $P(t) = 10\,000e^{\frac{4.5t}{100}}$
2. Determine the amount after 6 years.	$P(6) = 10\,000e^{\frac{4.5 \times 6}{100}}$ $= 13\,099.64$
3. State the total amount.	The principal has grown to \$13 099.64 after 6 years.

9.3.2 Other population models

In the natural growth model, the growth rate of a population is proportional to the current population. However, population models can have other growth rates. For example, the growth rate may be proportional to a power of the current population other than 1. In these models, the same approaches can be used as for natural growth.

WORKED EXAMPLE 8 Using first-order differential equations for other population models

The population of a certain town increases at a rate proportional to the square root of the current population. In 2009 the population was 250 000, and in 2017 the population was 640 000. Express the population number, N, in terms of t, the time in years after 2009. Calculate the predicted population in 2025.

THINK	WRITE
1. Let $N = N(t)$ represent the population number of the town after 2009.	$\frac{dN}{dt} = k\sqrt{N},\ N(0) = 250\,000$
2. To solve the differential equation, invert both sides and write in index notation.	$\frac{dt}{dN} = \frac{1}{k\sqrt{N}}$ $\frac{dt}{dN} = \frac{1}{k}N^{-\frac{1}{2}}$
3. Integrate both sides with respect to N and multiply through by k.	$\int 1dt = \frac{1}{k}\int N^{-\frac{1}{2}}dN$ $kt = \int N^{-\frac{1}{2}}dN$
4. Integrate.	$kt = 2N^{\frac{1}{2}} + c$ $kt = 2\sqrt{N} + c$
5. Substitute $t = 0$ and $N = 250\,000$ into the equation.	$0 = 2\sqrt{250\,000} + c$ $c = -1000$
6. Substitute $c = -1000$.	$kt = 2\sqrt{N} - 1000$
7. Substitute $t = 8$ and $N = 640\,000$ to solve for k.	$8k = 2\sqrt{640\,000} - 1000$ $8k = 600$ $k = 75$
8. Substitute $k = 75$ into the equation $kt = 2\sqrt{N} - 1000$ and rearrange to make N the subject.	$75t = 2\sqrt{N} - 1000$ $2\sqrt{N} = (75t + 1000)$ $\sqrt{N} = \frac{1}{2}(75t + 1000)$
9. State the particular solution of the differential equation.	$N(t) = \frac{1}{4}(75t + 1000)^2$
10. Substitute $t = 16$.	$N(16) = \frac{1}{4}(75 \times 16 + 1000)^2$ $= 1\,210\,000$
11. State the final result.	The predicted population in 2025 is 1 210 000.

9.3.3 Population models with regular removal

Another model that can limit but not bound the population is to remove some of the population at regular intervals. This has applications for situations such as people emigrating from a country or leaving a certain city yearly. The growth rate k still represents the excess of births over deaths and is often given as a percentage.

WORKED EXAMPLE 9 Using first-order differential equations for a population model with removal

A farm initially has 200 sheep. The number of sheep on the farm grows at a rate of 10% per year. Each year the farmer sells 15 sheep.

a. Write the differential equation modelling the number of sheep, N, on the farm after t years.

b. Solve the differential equation to express the number of sheep, N, on the farm after t years.

c. Hence, determine the number of sheep on the farm after 5 years.

THINK	WRITE
a. 1. Let $N(t)$ be the number of sheep on the farm at time t years. The growth rate is 10%, and 15 are sold each year.	**a.** $k = 0.1$ $\frac{dN}{dt} = 0.1N - 15$ $\frac{dN}{dt} = \frac{N}{10} - 15$
2. Write the differential equation in simplest form, along with the initial condition. The initial number of sheep is 200.	$\frac{dN}{dt} = \frac{N-150}{10}, N(0) = 200$
b. 1. To solve the differential equation, invert both sides.	**b.** $\frac{dt}{dN} = \frac{10}{N-150}$
2. Integrate both sides.	$t = \int \frac{10}{N-150}\,dN$
3. Perform the integration.	$t = 10\ln(\lvert N-150\rvert) + c$ $= 10\ln(N-150) + c$, as $N > 150$
4. Substitute $t = 0$ and $N = 200$.	$0 = 10\ \ln(200-150) + c$ $c = -10\ \ln(50)$
5. Substitute $c = -10\ \ln(50)$ and use log laws to simplify.	$t = 10\ \ln(N-150) - 10\ln(50)$ $t = 10\,[\ln(N-150) - \ln(50)]$ $\frac{t}{10} = \ln\left(\frac{N-150}{50}\right)$
6. Rearrange and solve for N.	$\frac{N-150}{50} = e^{\frac{t}{10}}$ $N - 150 = 50e^{\frac{t}{10}}$

7. State the solution of the differential equation.

$N = 150 + 50e^{\frac{t}{10}}$

$N = 50\left(3 + e^{\frac{t}{10}}\right)$

c. 1. Substitute $t = 5$.

c. $N(5) = 50\left(3 + e^{\frac{1}{2}}\right) \approx 232.44$

2. State the number of sheep present after 5 years. Note that we always round down, and express the answer as an integer.

There are 232 sheep present after 5 years.

Exercise 9.3 Other applications of first-order differential equations

learnon

9.3 Exercise	9.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

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Simple familiar

1. WE7 When money is invested in a bank at a constant rate of R% with continuously compounding interest, the accumulated amount \$$P$ at a time t years after the start of the investment satisfies the differential equation $\frac{dP}{dt} = P \times r$. Calculate the initial investment required to reach a target of \$10 000 in 2 years, if the interest rate is 5%. Give your answer to the nearest cent.

2. The price of houses in a certain area grows at a rate proportional to their current value. In 2002 a particular house was purchased for \$315 000; in 2018 the house was sold for \$1 260 000. Calculate the value of the house in 2010. Give your answer to the nearest dollar.

3. a. The rate of decrease of air pressure, P, with respect to the height above sea level, h, is proportional to the pressure at that height. The constant of proportionality is k such that $k > 0$. If P_0 is the air pressure at sea level, solve the differential equation to determine the pressure at height h.

   b. If the air pressure at sea level is 76 centimetres of mercury, and the pressure at a height of 1 kilometre above sea level is 62.2235 centimetres of mercury, determine the pressure at a height of 2 kilometres above sea level. Give your answer to 2 decimal places.

4. a. For many drugs, the rate of disappearance from the body is proportional to the amount of the drug still present in the body. If $D = D(t)$ is the amount of the drug present at time t and k is the constant of proportionality where $k > 0$, solve the differential equation expressing D in terms of t given the initial dosage of D_0.

   b. The elimination time for alcohol (the time it takes for a person's body to remove alcohol from their bloodstream) is measured as $T = \frac{1}{k}$. The process varies from person to person, so the value of k varies. For one person the elimination time is 3 hours. For this person, determine how long it will take the excess level of alcohol in their bloodstream to be reduced from 0.10% to the legal adult driving level of 0.049%. Give your answer to 2 decimal places.

5. When light passes through a medium, it loses its intensity as it penetrates deeper. If I is the intensity of light at depth x, the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth, so that $\frac{dI}{dx} = -kI$, where k is a positive constant.

   a. Solve the differential equation to show that $I = I_0 e^{-kx}$.
   b. If 5% of light is lost in penetrating 25% of a glass slab, determine the percentage is lost in penetrating the whole slab. Give your answer to 1 decimal place.

6. When a rope is wrapped around a pole, the rate of change of tension, T (in newtons), in the rope with respect to the angle, θ (in radians), is proportional to the tension at that instant. The constant of proportionality is the coefficient of friction, μ.

   a. Given that $\frac{dT}{d\theta} = \mu T$ and the tension when the angle is zero is T_0, show that $T = T_0 e^{\mu\theta}$.
   b. A man holds one end of a rope with a pull of 80 newtons. The rope goes halfway around a tree, and the coefficient of friction in this case is 0.3. Calculate the force the other end can sustain. Give your answer to 2 decimal places.

7. a. The variation of resistance, R ohms, of a copper conductor with temperature T degrees Celsius satisfies the differential equation $\frac{dR}{dT} = \alpha R$.

      i. If the resistance of copper is R_0 at 0 °C, show that $R = R_0 e^{\alpha T}$.
      ii. If $\alpha = 0.004$ per degree Celsius and the resistance at 60 °C is 40 ohms, determine the resistance at 30 °C. Give your answer to 2 decimal places.

   b. The charge, Q units, on a plate of a condenser t seconds after it starts to discharge is proportional to the charge at that instant. If the charge is 500 units after a half-second and falls to 250 units after 1 second, determine:

      i. the original charge, to the nearest unit.
      ii. the time needed for the charge to fall to 125 units. Give your answer to 1 decimal place.

8. WE9 A certain area initially contains 320 rabbits. The number of rabbits in the area grows at a rate of 25% per year. Each year 40 rabbits are culled to try to limit the population.

   a. Write the differential equation modelling the number of rabbits in the area, N, after t years.
   b. Solve the differential equation to express the number of rabbits in the area after t years.
   c. Hence, determine the number of rabbits in the area after 8 years.

9. Another area initially contains N_0 rabbits. The number of rabbits in this area grows at a rate of 20% per year. Each year K rabbits are culled to limit the population. After 10 years, there are $6K$ rabbits present. Show that $N_0 = K\left(5 + \frac{1}{e^2}\right)$.

10. A farm initially has 200 cows. The number of cows on the farm grows at a rate of 5% per year. Each year the farm sells five cows.

    a. Write the differential equation modelling the number of cows, N, on the farm after t years.
    b. Solve the differential equation to express the number of cows on the farm after t years.
    c. Hence, determine the number of cows on the farm after 3 years.

11. Koalas on a plantation have a population growth rate of 10%. Initially 400 koalas are present.

a. Every year 50 koalas are sold to zoos. Calculate how many years elapse before the number of koalas is reduced to 200.

b. After the population has reached 200, determine the maximum number of koalas the managers of the plantation can sell to zoos each year so that the number of koalas at the plantation will increase above 200.

12. Solve the differential equation $\frac{dN}{dt} = kN^{\alpha}$, $N(0) = N_0$ for when:

a. $\alpha = \frac{1}{2}$

b. $\alpha = -1$

c. $\alpha = \frac{3}{2}$

d. $\alpha = 2$.

Complex familiar

13. WE8 The population of a country increases at a rate proportional to the square root of the current population. In 2013 the population was 4 million, and in 2018 the population was 9 million. Express the population number, N, in terms of t, the time in years after 2013. Determine the predicted population in 2028.

14. The number of insects in a colony increases at a rate inversely proportional to the current number. At first there are 20 insects present, and this number grows to 80 after 5 months. Determine the number of insects present in the colony after a further 11 months.

15. A certain country has a 4% population growth rate, and additionally every year 10 000 immigrate to the country. If in 2019 the population of the country was half a million, estimate the population in 2024.

16. Assume Australia has a population growth rate of 2.5%, and a net overseas migration of 445 000 people arriving each year. If there were 27.2 million people in Australia in 2024, predict when the population will reach 35 million. Give your answer to the nearest year.

Complex unfamiliar

17. a. In an electrical circuit consisting of a resistance of R ohms and an inductance of L henries, the current, i amperes, after a time t seconds satisfies the differential equation $L\frac{di}{dt} + Ri = 0$. The initial current is i_0. Solve the differential equation to determine the current at any time t.

b. In an electrical circuit consisting of a resistance of R ohms and a capacitance of C farads, the charge, Q coulombs, at a time t seconds decays according to the differential equation $\frac{dQ}{dt} + \frac{Q}{RC} = 0$. Assuming the initial charge is Q_0, solve the differential equation to determine the charge at any time t after discharging.

18. a. Show that the solution of the differential equation $\frac{dN}{dt} = kN - c$, where k and c are positive constants and $N(0) = N_0$, is given by $N(t) = \left(N_0 - \frac{c}{k}\right)e^{kt} + \frac{c}{k}$.

b. Determine how N changes depending on the value of N_0 compared to $\frac{c}{k}$.

19. The number of bacteria in a culture increases at a rate proportional to the number of bacteria present in the culture at any time. By natural increase, the culture will double in ln(8) days. If the initial number of bacteria present is N_0 and bacteria are removed from the colony at a constant rate of Q per day, determine how the number of changes depending on the value of N_0 compared to $3 - Q$.

20. Liam has set up a small fish farm in a lake with 200 fish initially at the start of summer. The fish population size increases proportionally to its current size. Without any external pressure, the population of fish would double every 10 days. However, every day, two fish escape the farm and three die of natural causes. Liam wants to permit people fishing on his farm with two potential goals: slightly increasing his fish population, or having at least 50 fish remaining after 15 days. Calculate how many fish can be captured each day for each potential goal.

Fully worked solutions for this chapter are available online.

LESSON
9.4 Bounded growth and Newton's law of cooling

SYLLABUS LINKS

- Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.

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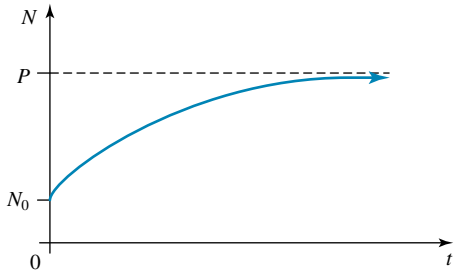
9.4.1 Bounded growth models

If $N = N(t)$ is the population, then the law of exponential growth states that $\frac{dN}{dt} = kN$, where k is a positive constant. Assuming that the initial population number is $N_0 = N(0)$, the solution of this differential equation is given by $N = N_0 e^{kt}$. This model predicts that, as time increases, the population number increases without bounds; that is, as $t \to \infty$, $N \to \infty$. Usually the growth of a quantity is not unlimited but is instead bounded; that is, the quantity approaches some maximum value or equilibrium value, and growth cannot occur above this value. This is called bounded growth. In such cases, the growth rate is proportional to the difference between the population number and the equilibrium value.

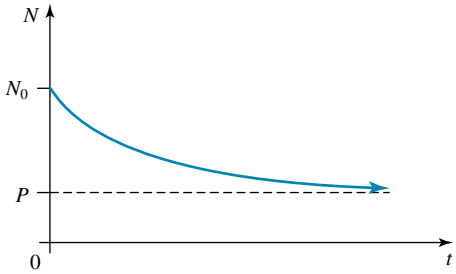


The differential equation to model this type of behaviour is $\frac{dN}{dt} = k(P - N)$, where $k > 0$ and $P > 0$.

P is called the carrying capacity or the equilibrium value.

Assuming an initial condition $N_0 = N(0)$, if $N_0 < P$, then as $t \to \infty$, $N \to P$ from below.

However, if $N_0 > P$, then as $t \to \infty$, $N \to P$ from above.



Bounded growth models

If a quantity $y(t)$ grows at a rate proportional to the difference between its upper bound P and its current size y, it satisfies: $\frac{dy}{dt} = k(P - y)$ where $k > 0$ is the growth constant, and P is the carrying capacity. Given $y(0) = y_0$, the solution is: $y(t) = P - (P - y_0)e^{-kt}$.

WORKED EXAMPLE 10 Using a bounded growth model

The mass of medium-sized dogs are found to follow a bounded growth model. The initial mass of a typical newborn puppy is 0.5 kilograms, and after 16 weeks its mass has increased to 4.5 kilograms. It is known that the dog's mass will never exceed 15 kilograms.

a. Determine the mass of the dog after 1 year.

b. Sketch the graph of mass versus time for this model.

THINK	WRITE/DRAW
a. 1. Let $m = m(t)$ be the mass in kilograms of the dog after t weeks. The equilibrium value is $P = 15$ and the initial mass is 0.5 kilograms. State the differential equation to model the mass.	a. $\frac{dm}{dt} = k(P - m)$ $\frac{dm}{dt} = k(15 - m), m(0) = 0.5$
2. To solve this differential equation, invert both sides.	$\frac{dt}{dm} = \frac{1}{k(15 - m)}$
3. Separate the variables and multiply by the constant k.	$t = \frac{1}{k}\int \frac{1}{15 - m}dm$ $kt = \int \frac{1}{15 - m}dm$
4. Perform the integration.	$kt = -\ln(\lvert 15 - m\rvert) + c$ $kt = -\ln(15 - m) + c$ as $0.5 < m < 15$
5. The initial mass of a newborn puppy is 0.5 kilograms. Use this to determine the value of c.	$m(0) = 0.5$: $0 = -\ln(15 - 0.5) + c$ $c = \ln(14.5)$
6. Substitute $c = \ln(14.5)$ and simplify.	$kt = -\ln(15 - m) + \ln(14.5)$ $kt = \ln\left(\frac{14.5}{15 - m}\right)$
7. After 16 weeks the mass is 4.5 kilograms. Use this to determine the value of k.	$m(16) = 4.5$: $16k = \ln\left(\frac{14.5}{15 - 4.5}\right)$ $k = \frac{1}{16}\ln\left(\frac{29}{21}\right)$
8. Leave the equation in terms of k and use the definition of the logarithm to rearrange.	$kt = \ln\left(\frac{14.5}{15 - m}\right)$ $e^{kt} = \frac{14.5}{15 - m}$
9. Rearrange to make m the subject.	$(15 - m)e^{kt} = 14.5$ $15 - m = 14.5e^{-kt}$ $m = 15 - 14.5e^{-kt}$
10. State the particular solution of the differential equation.	$m(t) = 15 - 14.5e^{-\frac{t}{16}\ln\left(\frac{29}{21}\right)}$

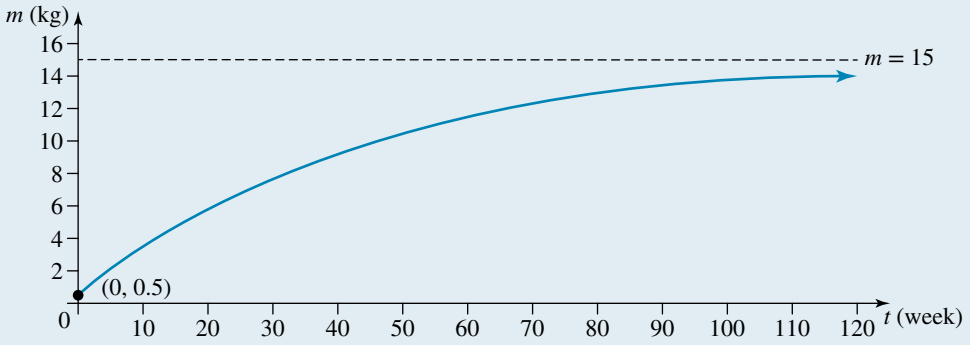
11. Now determine the mass of the dog after one year, that is $t = 52$ weeks.

$$m(52) = 15 - 14.5e^{-\frac{52}{16}\ln\left(\frac{29}{21}\right)}$$
$$\approx 9.92$$

12. State the final result.

The mass of the dog after one year is 9.92 kilograms.

b. Sketch the graph of m versus t, including the upper bound $m = 15$ and the coordinates at $t = 0$.



9.4.2 Newton's law of cooling

Newton's law of cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the surrounding medium. Let T denote the temperature of a body after a time t and T_m denote the temperature of the surrounding medium, which is assumed to be constant. Newton's law of cooling can be written as $\frac{dT}{dt} \propto (T - T_m)$ or $\frac{dT}{dt} = k(T - T_m)$, where k is a constant of proportionality. If the body is cooling, then $k < 0$.

Newton's law of cooling

The rate of change of an objects temperature $T(t)$ is proportional to the difference between its current temperature and the ambient temperature T_m. Therefore, $\frac{dT}{dt} = k(T - T_m)$, where $k < 0$ is the cooling constant. Given $T(0) = T_0$, the solution is: $T(t) = (T_0 - T_m)e^{-kt} + T_m$.

WORKED EXAMPLE 11 Using Newton's law of cooling

A metal ball is heated to a temperature of 200 °C and is then placed in a room that is maintained at a constant temperature of 30 °C. After 5 minutes the temperature of the ball has dropped to 150 °C. Assuming Newton's law of cooling applies, determine:

a. the equation for the difference between the temperature of the ball and the temperature of the room

b. the temperature of the ball after a further 10 minutes

c. how long it will take for the temperature of the ball to reach 40 °C.

THINK	WRITE
a. 1. Let T denote the temperature of the ball after a time t minutes. The constant room temperature is $T_m = 30$ °C.	a. $\frac{dT}{dt} = k(T - T_m)$ $\frac{dT}{dt} = k(T - 30)$

2. Separate the variables.

$$\frac{dT}{T-30} = kdt$$
$$\frac{1}{T-30}dT = kdt$$

3. Anti-differentiate both sides.

$$\int \frac{1}{T-30}\,dT = \int k\,dt$$
$$\ln|T-30| = kt + c$$

4. Substitute values for the initial condition to determine the constant of integration.

$$\ln|200-30| = k \times 0 + c$$
$$\ln(170) = c$$

5. Apply logarithm laws to make T the subject of the equation.

$$\ln|T-30| = kt + \ln(170)$$
$$\ln|T-30| - \ln(170) = kt$$
$$\ln\left|\frac{T-30}{170}\right| = kt$$
$$e^{kt} = \frac{T-30}{170}$$
$$T = 170e^{kt} + 30$$

6. Substitute $t = 5, T = 150$ to determine the cooling constant k.

$$150 = 170e^{k\times 5} + 30$$

7. Rearrange and use the definition of the logarithm.

$$\frac{150-30}{170} = e^{5k}$$
$$5k = \ln\left(\frac{12}{17}\right)$$

8. Solve for k.

$$k = \frac{1}{5}\ln\left(\frac{12}{17}\right)$$
$$k = -0.0697$$

9. Write the particular solution of the differential equation.

$$T(t) = 170e^{-0.0697t} + 30$$

b. 1. Substitute $t = 15$ to calculate the temperature 10 minutes later.

b. $T(10) = 170e^{-0.0697\times 15} + 30$
$= 89.79\ °\text{C}$

c. 1. To determine when the temperature of the ball reaches 40 °C, use $T = 40$.

c. $40 = 170e^{-0.0697t} + 30$

2. Solve the equation for t, using the definition of the logarithm.

$$\frac{40-30}{170} = e^{-0.0697t}$$
$$-0.0697t = \ln\left(\frac{1}{17}\right)$$
$$t = \ln\left(\frac{1}{17}\right) \div -0.0697$$
$$t \approx 40.67\ \text{min}$$

3. State the required result.

The temperature of the ball reaches 40 °C after 40.67 minutes.

Exercise 9.4 Bounded growth and Newton's law of cooling

learnon

9.4 Exercise | 9.4 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

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Simple familiar

1. WE10 The mass of a bird is found to follow a bounded growth model. The initial mass of a baby bird is 30 grams, and after 10 weeks its mass is 100 grams. If it is known that the mass of the bird will never exceed 200 grams, determine the mass of the bird after 30 weeks. Give your answer to 1 decimal place.

2. The number of birds in a flock follows a bounded growth model. Initially there are 200 birds in the flock. After 5 months the number has grown to 800. The number of birds in the flock can never exceed 3000. Calculate how many months pass for the flock to reach 1500 birds.

3. **a.** The mass of cats are found to follow a bounded growth model. The initial mass of a typical newborn cat is 0.1 kilograms, and after 30 weeks the cat's mass has increased to 4 kilograms. If it is known that the cat's mass will never exceed 5 kilograms, determine the mass of the cat after 40 weeks. Give your answer to 1 decimal place.

b. The mass of toy poodle dogs are found to follow a bounded growth model. The initial mass of a puppy toy poodle is 0.2 kilograms, and after 14 weeks its mass has increased to 1.3 kilograms. If it is known that the dog's mass will never exceed 4.5 kilograms, determine the mass of the toy poodle after 1 year. Give your answer to 1 decimal place.

4. The number of fish in a lake follows a bounded growth model. Initially there are 50 fish in the lake, and after 10 months the number of fish has increased to 500. The number of fish in the lake can never exceed 1000.

a. Determine the number of fish in the lake after 20 months.

b. Calculate the time in months for the fish population to reach 900. Give your answer to 1 decimal place.

5. A woman is on a diet. Her initial mass is 84 kilograms, and she knows that her mass will always be above 70 kilograms, which is her ideal healthy weight. After 10 weeks she has lost 7 kilograms. Assuming a bounded decay model, determine her total mass loss after 20 weeks. Give your answer to 1 decimal place.

6. WE11 A warm can of soft drink at a temperature of 25 °C is chilled by placing it in a refrigerator at a constant temperature of 2 °C. If the temperature of the drink falls to 22 °C in 5 minutes, determine how long it will take for the temperature of the drink to reach 13 °C. Give your answer to 2 decimal places.

7. An iron is preheated to a temperature of 250 °C. After 6 minutes the temperature of the iron is 210 °C. Brent is ironing in a room that is kept at a constant temperature of 20 °C. Brent knows that it is best to iron a cotton T-shirt when the temperature of the iron is between 195 °C and 205 °C. Determine the duration of time Brent has to iron the T-shirt. Give your answer to 2 decimal places.

8. a. A cold can of soft drink is taken from a refrigerator at a temperature of 3 °C and placed in a sunroom at a temperature of 30 °C. If after 2 minutes the temperature of the can is 4 °C, determine its temperature after a further 3 minutes. Give your answer to 2 decimal places.

   b. A body at an unknown temperature is placed in a room that has a temperature of 18 °C. If after 10 minutes the body has a temperature of 22 °C, and after a further 10 minutes its temperature is 20 °C, determine the initial temperature of the body, to the nearest degree.

9. A mother is giving her baby a bath. The bathtub contains hot water that is initially at a temperature of 42 °C in a room where the temperature is constant at 20 °C. The water cools, and after 2 minutes its temperature is 40 °C. The mother knows that babies like to be bathed when the temperature of the water is between 34 °C and 38 °C, otherwise the bath water is either too hot or too cold. Calculate the duration of time the baby should stay in the bath. Give your answer to 2 decimal places.

10. The temperature of a room is 25 °C. A thermometer which was in the room is taken outdoors and in 5 minutes it reads 15 °C. Five minutes later the thermometer reads 10 °C. Calculate the temperature outdoors.

11. A frozen chicken should be thawed before cooking. A freezer is maintained at a constant temperature of −18 °C. A chicken is taken out of the freezer at 8:00 am and placed on a kitchen bench. At 11:00 am the temperature of the chicken is 0 °C, and at 2:00 pm its temperature is 10 °C. At 5:00 pm the chicken has thawed and is placed in an oven which has been preheated to a temperature of 200 °C. At 6:00 pm the temperature of the chicken in the oven is 70 °C. Determine the temperature of the chicken when it is removed from the oven at 7:00 pm. Give your answer to 2 decimal places.

12. a. An RL series circuit consists of a resistance, R ohms, and an inductance, L henries, connected to a voltage source, E volts. The rise of current, i amperes, after a time t seconds satisfies the differential equation $L\dfrac{di}{dt} + Ri = E$.

   i. Assuming the initial current is zero, determine the current at any time t.

   ii. Show that the time required for the current to reach half its ultimate value is given by $\dfrac{L}{R}\ln\left(\dfrac{1}{2}\right)$.

   b. A capacitor of C farads is charged by applying a steady voltage of E volts through a resistance of R ohms. The potential difference, v volts between the plates satisfies the differential equation $RC\dfrac{dv}{dt} + v = E$. Assuming the initial voltage is zero, determine the voltage v at any time t.

Complex familiar

13. Show that the particular solution of the differential equation $\frac{dN}{dt} = k(P - N)$, $N(0) = N_0 > 0$, where $P > 0$, is given by $N(t) = P + (N_0 - P)e^{-kt}$. Consider both cases when $N_0 > P$ and $N_0 < P$.

14. **a.** Show that the solution of the differential equation $\frac{dT}{dt} = k(T - T_m)$, where k and T_m are positive constants and $T(0) = T_0$, is given by $T(t) = T_m + (T_0 - T_m)e^{kt}$.

b. Police discover a dead body in a hotel room at 8:00 am. At that time its temperature is 30.4 °C. One hour later the temperature of the body is 29.9 °C. The room was kept at a constant temperature of 17 °C overnight. If normal body temperature is 37 °C, determine the time of death.

15. A refrigerator is kept at a constant temperature of 3 °C. A baby's bottle is taken from the fridge, but the bottle is too cold to give to the baby. The mother places the bottle in a saucepan full of boiling water (maintained at 95 °C) for 90 seconds. When the bottle is removed from the saucepan, its temperature is 45 °C.

a. Let $H = H(t)$ be the temperature of the bottle at a time t minutes while it is being heated. If $\frac{dH}{dt} = k_1(H - T_h)$, where $k_1 < 0$ for $0 \le t \le t_1$, state the values of t_1 and T_h.

b. The mother knows that the bottle should be at a temperature of 35 °C for the baby to drink it. She realises that the bottle is now unfortunately too hot to give to the baby, so she places it back in the fridge to cool. She takes the bottle out of the fridge after 45 seconds. It is now at the correct temperature to give to the baby. Let $C = C(t)$ be the temperature of the bottle at a time t minutes after it is placed back in the fridge. If $\frac{dC}{dt} = k_2(C - T_c)$, where $k_2 < 0$ for $0 \le t \le t_2$, state the values of t_2 and T_c.

c. Determine the ratio of $\frac{k_1}{k_2}$. Give your answer to 2 decimal places.

16. Let $\$P_0$ be the initial amount of money borrowed from a lender at an annual interest rate of r%. Assume that the interest is compounded continuously and that the borrower makes equal payments of $\$m$ per year to reduce the amount borrowed. If $\$P$ is the amount of money owing at time t, the differential equation modelling the repayments is given by $\frac{dP}{dt} = rP - m$.

a. If the loan is completely paid off after a time T years, show that $\frac{m}{m - rP_0} = e^{rT}$.

b. If Jared borrows \$300 000 for a house loan at 6% per annum and pays off the loan after 20 years, determine his monthly repayments and the total amount of interest paid. Give your answers to the nearest cent.

Complex unfamiliar

Use the following for questions 17–19.

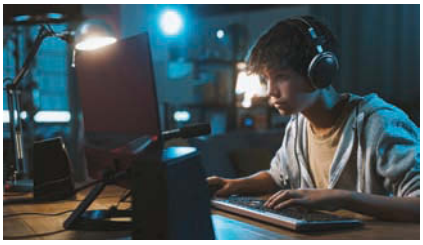
When $\$P$ is the amount of money owing at time t, the differential equation modelling the repayments is given by $\frac{dP}{dt} = rP - m$ where $r\%$ is the annual interest rate and $\$m$ are the equal payments per year, and when $\$P_0$ is the initial amount of money borrowed, then the particular solution is $P(t) = P_o e^{rt} + \frac{m}{r}\left(1 - e^{rt}\right)$.

17. Sharon can afford to pay \$350 a month over 5 years at 8% per annum compounded continuously. Calculate the maximum amount she can borrow for a car loan. Give your answers to the nearest cent.

18. Ashley wants to buy a block of land and will have to borrow \$120 000 to do it. He can afford to make monthly repayments of \$1200 on the loan, compounded continuously at 6.3%. Calculate how long will it take Ashley to pay off the loan. Give your answer to 1 decimal place.

19. Ryan pays off a loan of \$8000 by paying back \$150 per month over 6 years. Assuming the interest is compounded continuously, determine the annual interest rate. Give your answer to 2 decimal places.

20. Jamie is playing on their computer in a room with a constant temperature of 25 °C.
By the time Jamie decides to stop playing, the temperature of the microprocessor is 75 °C. They turn off their computer and the microprocessor starts to cool down at a rate k in min^{-1}. It takes 25 minutes for the temperature of the microprocessor to reach 40 °C. Determine how long it takes for its temperature to reach 35 °C. Give your answer to the nearest minute.



Fully worked solutions for this chapter are available online.

LESSON
9.5 Chemical reactions and dilution problems

SYLLABUS LINKS

- Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.

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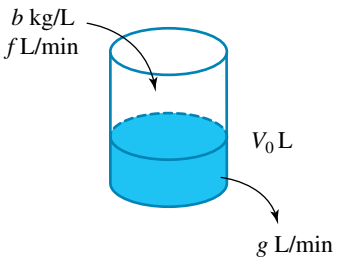
9.5.1 Input–output mixing problems

Consider a tank that initially holds V_0 litres of a solution in which a kilograms of salt have been dissolved. Another solution containing b kilograms of salt per litre is poured into the tank at a rate of f litres per minute. The well-stirred mixture leaves the tank at a rate of g litres per minute. The problem is to determine the amount of salt in the tank at any time t.



Let Q denote the amount of salt in kilograms in the tank at a time t minutes. The time rate of change of Q is equal to the rate at which the salt flows into the tank minus the rate at which the salt flows out of the tank; that is,

Differential equation for mixing problems

$$\frac{dQ}{dt} = \text{inflow rate} - \text{outflow rate}$$

We know b kilograms of salt per litre is being poured into the tank at a rate of f litres per minute; multiplying these together gives the inflow rate of the salt as bf kilograms per minute.

To determine the outflow rate, first we determine the volume $V(t)$ of the tank at time t. The initial volume is V_0, but f litres per minute flow in, while g litres per minute flow out. Therefore, the volume at time t minutes is given by $V(t) = V_0 + (f - g)t$. To get the required units, we multiply the solution's outflow rate of g litres per minute by the mass and divide by the volume to get the outflow rate of the salt as $\dfrac{gQ}{V_0 + (f-g)t}$ kilograms per minute. Therefore, the differential equation is

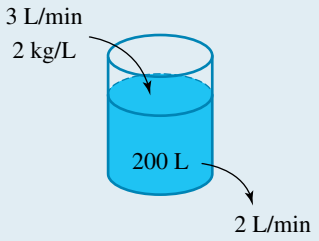
$$\frac{dQ}{dt} = bf - \frac{Qg}{V_0 + (f-g)t}, \text{ and } Q(0) = a.$$

(Note that concentration, c, is the mass divided by the volume, $c = \dfrac{\text{mass}}{\text{volume}}$.)

When $g \neq f$, this equation cannot be solved by the techniques of integration studied in this course. However, if we are given a solution, differentiation and substitution can be used to verify the solution.

WORKED EXAMPLE 12 Using first-order differential equations for mixing problems

A tank has a capacity of 300 litres and contains 200 litres of water in which 100 kilograms of salt have been dissolved. A salt solution of concentration 2 kilograms per litre is poured into the tank at a rate of 3 litres per minute, and the well-stirred mixture flows out at a rate of 2 litres per minute.



a. State the differential equation for Q, the amount of salt in kilograms in the tank after t minutes.

b. Verify that $Q = 2(200 + t) + \dfrac{C}{(200+t)^2}$ is a general solution of the differential equation.

c. Determine the value of C.

d. Determine the concentration of salt in kilograms per litre when the tank overflows.

THINK	WRITE
a. 1. The initial volume in the tank is 200 litres; 3 litres per minute flows in and 2 litres per minute flows out. Let $V(t)$ be the volume in litres at time t minutes.	a. $V(t) = 200 + (3-2)t$ $= 200 + t$
2. The inflow rate is the concentration × the inflow volume rate.	Inflow rate $= 3 \text{ L/min} \times 2 \text{ kg/L} = 6 \text{ kg/min}$
3. The output flow rate is the outflow concentration × the outflow volume rate.	Outflow rate $= \dfrac{2Q}{V(t)} \text{ kg/min}$ $= \dfrac{2Q}{200+t} \text{ kg/min}$

4. State the required differential equation along with the initial condition.

$$\frac{dQ}{dt} = \text{inflow rate} - \text{output rate}$$
$$= 6 - \frac{2Q}{200+t} \quad [1]$$
$$Q(0) = 100$$

b. 1. To verify the given solution, first write the solution in index form.

b. $Q = 2(200+t) + \dfrac{C}{(200+t)^2}$ [2]

$$= 2(200+t) + C(200+t)^{-2}$$

2. Now differentiate the given solution as the left-hand side.

$$\text{LHS} = \frac{dQ}{dt}$$
$$= 2 - 2C(200+t)^{-3}$$
$$= 2 - \frac{2C}{(200+t)^3}$$

3. Substitute for Q into the right-hand side of [1].

$$\text{RHS} = 6 - \frac{2Q}{200+t}$$
$$= 6 - \frac{2}{200+t}\left[2(200+t) + \frac{C}{(200+t)^2}\right]$$

4. Expand the right-hand side.

$$= 6 - \frac{2 \times 2(200+t)}{200+t} - \frac{2C}{(200+t)^3}$$

5. Simplify the right-hand side. Because the left-hand side is equal to the right-hand side, we have verified that the given solution does satisfy the differential equation.

$$= 6 - 4 - \frac{2C}{(200+t)^3}$$
$$= 2 - \frac{2C}{(200+t)^3}$$

c. 1. To determine the value of C, use the given initial conditions.

c. Substitute $t = 0$ and $Q = 100$ into [2].

2. Substitute the given values in the given solution.

$$100 = 2 \times 200 + \frac{C}{200^2}$$

3. Solve for C.

$$100 = 400 + \frac{C}{200^2}$$
$$\frac{C}{200^2} = -300$$
$$C = -12\,000\,000$$

d. 1. Determine the time when the tank overflows. The capacity of the tank is 300 litres.

d. $300 = 200 + t$

The tank overflows when $t = 100$ minutes.

2. Determine the value Q of when the tank overflows.

Substitute for C:

$$Q(t) = 2(200+t) - \frac{12\,000\,000}{(200+t)^2}$$
$$Q(100) = 600 - \frac{12\,000\,000}{300^2}$$
$$= 466\frac{2}{3} \text{ kg}$$

3. Determine the concentration when the tank overflows.

$$\text{Concentration} = \frac{\text{mass}}{\text{volume}}$$

$$\frac{Q(100)}{V(100)} = \frac{466\frac{2}{3}}{300}$$

$$\approx 1.556 \text{ kg/litre}$$

4. State the answer.

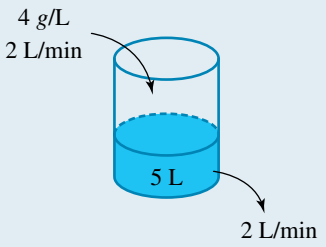
When the tank overflows, the concentration of salt in the solution is 1.556 kg/litre.

9.5.2 Equal input and output flow rates

In the special case when the inflow rate is equal to the outflow rate, that is, when $f = g$ so the volume remains constant, the differential equation can be solved using the separable integration techniques learnt in a previous chapter. In fact, in this case the differential equation is just another application of a bounded growth or decay model.

WORKED EXAMPLE 13 Using first-order differential equations for mixing problems with equal input and output flows

A tank contains 5 litres of water in which 10 grams of salt have been dissolved. A salt solution containing 4 grams per litre is poured into the tank at a rate of 2 litres per minute, and the mixture is kept uniform by stirring. The mixture leaves the tank at a rate of 2 litres per minute.



a. Set up the differential equation for the amount of salt, Q grams, in the tank at time t minutes.
b. Solve the differential equation to determine Q at any time t.
c. Calculate how much salt is in the tank after 5 minutes. Determine its concentration at that time.
d. Show that the salt solution can never exceed 20 grams.
e. Sketch the graph of Q versus t.

THINK	WRITE/DRAW
a. 1. The inflow rate is equal to the outflow rate. The volume in the tank remains constant at 5 litres.	a. $V(t) = 5$
2. The inflow rate is inflow concentration × the inflow volume rate. The output flow rate is the outflow concentration × the outflow volume rate.	$\frac{dQ}{dt} = \text{inflow rate} - \text{output rate}$ $= 4 \times 2 - \frac{2Q}{V(t)}$
3. State the differential equation.	$\frac{dQ}{dt} = 8 - \frac{2Q}{5}$
4. State the differential equation in simplest form, along with the initial condition.	$\frac{dQ}{dt} = \frac{2(20-Q)}{5}, Q(0) = 10$
b. 1. To solve the differential equation, invert both sides.	b. $\frac{dt}{dQ} = \frac{5}{2(20-Q)}$
2. Use the separation of variables technique, taking the constant factors to one side.	$\int \frac{2}{5} dt = \int \frac{1}{22-Q} dQ$

3. Perform the integration.

$$\frac{2t}{5} = -\ln\left(|20 - Q|\right) + c$$

4. Since $Q(0) = 10$, the modulus signs are not needed. Use the initial condition to determine the constant of integration.

Substitute $t = 0$ when $Q = 10$:

$$0 = -\ln(20 - 10) + c$$
$$c = \ln(10)$$

5. Substitute back for C and simplify using log laws.

$$\frac{2t}{5} = -\ln|20 - Q| + \ln(10)$$
$$\frac{2t}{5} = \ln\left|\frac{10}{20 - Q}\right|$$

6. Use the definition of the logarithm.

Remove the absolute value sign as $e^{\frac{2t}{5}} > 0$

$$e^{\frac{2t}{5}} = \frac{10}{20 - Q}$$

7. Invert both sides.

$$\frac{20 - Q}{10} = e^{-\frac{2t}{5}}$$

8. Rearrange and solve for Q.

$$20 - Q = 10e^{-\frac{2t}{5}}$$
$$Q = 20 - 10e^{-\frac{2t}{5}}$$

9. State the solution of the differential equation in simplest form.

$$Q(t) = 10\left(2 - e^{-\frac{2t}{5}}\right)$$

c. Determine the concentration after 5 minutes.

c. Substitute $t = 5$:

$$Q(5) = 10(2 - e^{-2})$$
$$= 18.646 \text{ grams}$$

The concentration is

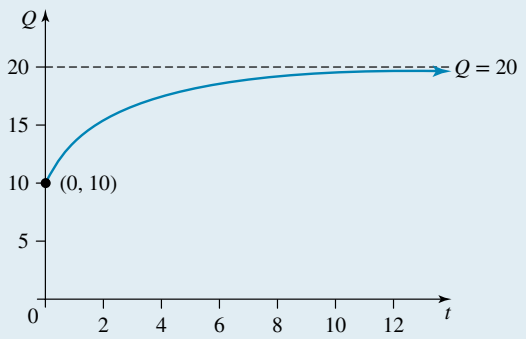
$$c = \frac{Q}{V}$$
$$= \frac{18.646}{5}$$
$$= 3.73 \text{ grams per litre}$$

d. Determine the limit as t approaches infinity.

d. As $t \to \infty$, $e^{-\frac{2t}{5}} \to 0$, $\therefore Q \to 20$.
Therefore, the amount of salt can never exceed 20 grams.

e. The graph starts at $Q(0) = 10$ and approaches the horizontal asymptote of $Q = 20$.

e.



9.5.3 Chemical reaction rates

In a chemical reaction, the rate at which a new substance is formed is proportional to the unused amounts of the reacting substances. Differential equations can be set up and solved to model the amount of the new substance as it forms over time.

WORKED EXAMPLE 14 Using first-order differential equations with chemical rates

In a biomolecular chemical reaction between 3 grams of substance A and 6 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B respectively. A and B combine in equal parts to form substance X. Initially no X is present, but after 3 minutes, 3 grams of X have formed.

a. Set up the differential equation for the amount of substance X at time t minutes.

b. Determine the amount of X present after a further 3 minutes.

THINK	WRITE
a. 1. Let $x = x(t)$ be the amount in grams of substance X formed at time t minutes.	**a.** $[\mathrm{A} + \mathrm{B} \rightarrow \mathrm{X}]$
2. The velocity of the reaction is proportional to the product of the unused amounts of A and B, so $\frac{x}{2}$ is used from both A and B.	$\frac{dx}{dt} \propto \left(3 - \frac{x}{2}\right)\left(6 - \frac{x}{2}\right)$ $\frac{dx}{dt} = k(6-x)(12-x), x(0) = 0$ where k is a constant to be found.
b. 1. To solve the differential equation, invert both sides.	**b.** $\frac{dx}{dt} = \frac{1}{k(6-x)(12-x)}$
2. Take the constant k to the other side and use separation of variables.	$kt = \int \frac{1}{(6-x)(12-x)} dx$
3. To determine the integral on the right-hand side, we need to use partial fractions.	$\frac{1}{(6-x)(12-x)} = \frac{A}{6-x} + \frac{B}{12-x}$
4. Add the partial fractions and simplify.	$\frac{1}{(6-x)(12-x)} = \frac{A(12-x) + B(6-x)}{(6-x)(12-x)}$ $= \frac{12A + 6B - x(A+B)}{(6-x)(12-x)}$
5. Equate the coefficients and solve for A and B.	$12A + 6B = 1 \quad [1]$ $A + B = 0 \Rightarrow A = -B \quad [2]$ $A = \frac{1}{6}, B = -\frac{1}{6}$
6. Substitute back for the partial fraction decomposition.	$kt = \frac{1}{6}\int \left(\frac{1}{6-x} - \frac{1}{12-x}\right) dx$
7. Perform the integration.	$kt = -\frac{1}{6}\ln(\lvert 6-x \rvert) + \frac{1}{6}\ln(\lvert 12-x \rvert) + c$
8. Take out the common factors and use log laws.	$kt = \frac{1}{6}\ln\left(\left\lvert \frac{12-x}{6-x} \right\rvert\right) + c$

9. Since $x(0) = 0$ and $0 \le x < 6$, the modulus signs are not needed. Use the initial condition to determine the constant of integration.

$$0 = \frac{1}{6}\ln(2) + c$$
$$c = -\frac{1}{6}\ln(2)$$

10. Substitute back for c and use log laws again.

$$kt = \frac{1}{6}\ln\left(\frac{12-x}{6-x}\right) - \frac{1}{6}\ln(2)$$
$$kt = \frac{1}{6}\ln\left(\frac{12-x}{2(6-x)}\right) \qquad [3]$$

11. We can now determine the value of k by substituting $t = 3$ when $x = 3$.

Substitute $t = 3$ when $x = 3$:
$$3k = \frac{1}{6}\ln\left(\frac{9}{6}\right)$$
$$3k = \frac{1}{6}\ln\left(\frac{3}{2}\right)$$
$$k = \frac{1}{18}\ln\left(\frac{3}{2}\right) \qquad [4]$$

12. Attempt to make x the subject, but leave the result in terms of k, since k has a known value.

From [3],
$$6kt = \ln\left(\frac{12-x}{2(6-x)}\right)$$

13. Use the definition of a logarithm.

$$\frac{12-x}{2(6-x)} = e^{6kt}$$

14. Invert both sides.

$$\frac{12-2x}{12-x} = e^{-6kt}$$

15. Remove the denominator.

$$12 - 2x = (12-x)e^{-6kt}$$

16. Expand the brackets.

$$12 - 2x = 12e^{-6kt} - xe^{-6kt}$$

17. Transfer the x to one side and factorise.

$$12 - 12e^{-6kt} = 2x - xe^{-6kt}$$
$$12\left(1 - e^{-6kt}\right) = x\left(2 - e^{-6kt}\right)$$

18. State the solution of the differential equation.

$$x = x(t) = \frac{12\left(1 - e^{-6kt}\right)}{2 - e^{-6kt}}$$
where $k = \frac{1}{18}\ln\left(\frac{3}{2}\right)$

19. Determine the value of x after a further 3 minutes. When $t = 6$

$$x(6) = \frac{12\left(1 - e^{-36k}\right)}{2 - e^{-36k}}$$
From [4], $36k = 2\ln\left(\frac{3}{2}\right)$

20. Use exact values.

$$36k = \ln\left(\frac{9}{4}\right)$$
$$e^{-36k} = \frac{4}{9}$$

21.	Evaluate.	$x(6) = \dfrac{12\left(1-\frac{4}{9}\right)}{2-\frac{4}{9}} = \dfrac{30}{7}$
22.	State the final result. Note that as $t \to \infty$, $x \to 12$.	After 6 minutes, $4\frac{2}{7}$ grams of X have formed.

Exercise 9.5 Chemical reactions and dilution problems

learnon

9.5 Exercise | **9.5 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9	10,11,12,13, 14, 15, 16, 17	18, 19, 20

These questions are even better in jacPLUS!
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Simple familiar

1. WE12 A vessel initially contains 200 litres of a salt solution with a concentration of 0.1 kg/litre. The salt solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt of concentration 1.5 kg/litre is added to the vessel at a rate of 2 litres per minute. The contents of the vessel are kept well stirred.

a. Set up the differential equation for Q, the amount of salt in kilograms in the vessel after t minutes.

b. Verify that $Q = \frac{3}{2}(200-t) + C(200-t)^3$ is a general solution of the differential equation.

c. Determine the value of C.

d. Determine the concentration of salt after 100 minutes. Give your answer to 2 decimal places.

2. A vat initially contains 50 litres of a sugar solution. More sugar solution containing b grams per litre is poured into the vat at a rate of 6 litres per minute, and simultaneously the well-stirred mixture leaves the vat at a rate of 3 litres per minute.

a. Set up the differential equation for Q, the amount of sugar in grams in the vat after t minutes.

b. Verify that $Q = 2(50+3t) + \dfrac{C}{50+3t}$ is a general solution of the differential equation, and determine the value of b.

3. a. A tank initially contains 50 litres of water. A chemical solution is drawn off at a rate of 5 litres per minute, and at the same time a mixture containing the chemical at a concentration of 3 grams per litre is added to the tank at a rate of 4 litres per minute. The contents of the tank are kept well stirred.

i. Set up the differential equation for Q, the amount of the chemical in grams in the tank after t minutes.

ii. Verify that $Q(t) = 3(50-t) + C(50-t)^5$ is a general solution of the differential equation.

b. A container initially contains 30 litres of water. A brine solution of concentration 4 grams per litre is added to the container at a rate of 1 litre per minute. The well-stirred mixture is drawn off at 2 litres per minute.

i. Set up the differential equation for Q, the amount of brine in grams in the container after t minutes.

ii. Verify that $Q(t) = \dfrac{2t}{15}(30-t)$ is the particular solution of the differential equation.

4. a. A vessel initially contains 20 litres of water. A sugar solution with a concentration of 3 grams per litre is added at a rate of 2 litres per minute. The well-stirred mixture is drawn off at 1 litre per minute.

   i. Set up the differential equation for Q, the amount of sugar in grams in the vessel after t minutes.

   ii. Verify that $Q(t) = \dfrac{3t(t+40)}{20+t}$ is the particular solution of the differential equation.

   b. A trough initially contains 20 litres of water. A dye solution with a concentration of 4 grams per litre is added to the trough at a rate of 2 litres per minute. The well-stirred mixture is sent down a drain at a rate of 3 litres per minute.

   i. Set up the differential equation for Q, the amount of dye in grams in the trough after t minutes.

   ii. Verify that $Q(t) = \dfrac{t}{100}(t-40)(t-20)$ is a particular solution of the differential equation.

5. a. A vessel initially contains 64 litres of a salt solution with a concentration of 2 grams per litre. The solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt at a concentration of 4 grams per litre is added to the vessel at a rate of 5 litres per minute. The contents of the vessel are kept well stirred.

   i. Set up the differential equation for Q, the amount of salt in grams in the vessel after t minutes.

   ii. Verify that $Q = 4(64+2t) + C(64+2t)^{-\frac{3}{2}}$ is a general solution of the differential equation.

   iii. Determine the value of C.

   iv. Determine the concentration when the volume of the vessel is 100 litres.

   b. A vat initially contains 40 litres of water. A brine solution containing b grams per litre is poured into the vat at a rate of 2 litres per minute, and simultaneously the well-stirred mixture leaves the vat at a rate of 5 litres per minute. If Q is the amount of brine in grams in the vessel after a time t minutes and a general solution of the differential equation is $Q = 3(40-3t) + C(40-3t)^{\frac{5}{3}}$, determine the value of b.

6. WE13 A tank initially contains 20 litres of a salt solution that has a concentration of 0.25 grams per litre. The salt solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt of concentration 4 grams per litre is added to the tank at a rate of 3 litres per minute. The contents of the tank are kept well stirred.

   a. Set up the differential equation for Q, the amount of salt in grams in the tank after t minutes.

   b. Solve the differential equation to determine Q at any time t.

   c. Determine the concentration of salt after 1.6 minutes. Give your answer to 2 decimal places.

   d. Show that the amount of salt can never exceed 80 grams.

   e. Sketch the graph of Q versus t.

7. A vat contains 15 litres of pure water. A sugar solution containing 5 grams per litre is poured into the vat at a rate of 4 litres per minute, and simultaneously the well-stirred mixture leaves the vat at the same rate.

   a. Set up the differential equation for the amount of sugar, Q grams, in the vat at time t minutes.

   b. Solve the differential equation to determine Q at any time t.

   c. Determine when the sugar concentration is 4 grams per litre. Give your answer to 2 decimal places.

8. a. A sink contains 50 litres of water for washing dishes, in which 50 grams of detergent have been dissolved. A dishwashing solution containing 3 grams per litre of detergent is poured into the sink at a rate of 4 litres per minute and the mixture is kept uniform. The mixture leaves the sink down the drain at the same rate. Determine the concentration of detergent after 5 minutes. Give your answer to 3 decimal places.

b. A tank contains 600 litres of water in which 30 kilograms of salt have been dissolved. Water is poured into the tank at a rate of 5 litres per minute, and the mixture is kept uniform by stirring. The mixture leaves the tank at a rate of 5 litres per minute.

i. Determine the amount of salt in kilograms in the tank at any time t minutes.
ii. Determine the amount of salt in kilograms after 2 hours. Give your answer to 2 decimal places.
iii. Calculate how long it will take before there are 15 kilograms of salt in the tank. Give your answer to 2 decimal places.

9. a. A 10-litre urn is full of boiling water. Caterers pour the entire contents of a 300-gram jar of coffee into the urn and mix them thoroughly. While the caterers are making cups of coffee, the coffee is drawn out of the urn at a rate of 0.2 litres per minute, and at the same time boiling water is added at the same rate.

i. Lilly likes to drink her coffee when the concentration of the coffee is 25 grams per litre. Calculate how long after the process starts she should wait to get her cup of coffee. Give your answer to 2 decimal places.
ii. When the concentration of coffee in the urn falls below 15 grams per litre, more coffee must be added to the urn. Determine how long after the process starts more coffee must be added. Give your answer to 2 decimal places.

b. The urn now contains hot water at a temperature of 93 °C. Water is poured from the urn to make a cup of coffee. The coffee cup is placed in a room where the temperature is constant at 17 °C. After 1 minute the temperature of the coffee is 88 °C. Lilly likes to drink her coffee when its temperature is between 50 °C and 65 °C. Calculate how long Lilly has to drink the coffee. Give your answer to 1 decimal place.

Complex familiar

10. WE14 In a chemical reaction between 1 gram of substance A and 3 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X, and initially no X is present. After 3 minutes, 1 gram of X has formed.

a. Set up the differential equation to determine the amount of substance X at time t minutes.
b. Solve the differential equation to determine the amount of substance X at time t minutes.
c. Determine the exact amount of X present after a further 3 minutes.

11. In a chemical reaction between 4 grams of substance A and 4 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X. Initially, no X is present, but after 2 minutes, 3 grams of X has formed.

a. Set up the differential equation for the amount of substance X at time t minutes.
b. Solve the differential equation to determine the amount of X in grams present after a time t minutes.
c. Calculate the additional time required for the system to reach 6 grams of X.
d. Determine the maximum quantity of X that can eventually be produced.

12. In a chemical reaction between 2 grams of substance A and 4 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X. Initially, no X is present, and after 2 minutes, 1 gram of X has formed.

a. Set up the differential equation to determine the amount of substance X at time t minutes.
b. Solve the differential equation to determine the amount of substance X at time t minutes.
c. Determine the exact amount of X present after a further 2 minutes.
d. Calculate the ultimate amount of substance X that can eventually be formed.

13. In a chemical reaction between 5 grams of substance A and 2 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form a substance X. Initially, no X is present, and after 3 minutes, 2 grams of X have formed.

a. Calculate how long it will take before 3 grams of X have formed. Give your answer to 2 decimal places.
b. Determine exactly how much of X has formed after 6 minutes.
c. Calculate the ultimate amount of substance X that can eventually be formed.

14. Many chemical reactions follow Wilhelmy's Law, which states that the velocity of the reaction is proportional to the concentration of the reacting substance. In such a reaction containing a grams of a reagent, the amount of substance X transformed, x grams, after a time t minutes is given by $\frac{dx}{dt} = k(a - x)$, where a and k are both positive constants.

a. If initially there is no X present, show that $x(t) = a(1 - e^{-kt})$.
b. If $a = 5$ and after 4 minutes 2 grams of X is present, determine the amount of X present after 10 minutes. Give your answer to 2 decimal places.

15. In a chemical reaction, equal amounts of A and B combine to form substance X. Initially there are b grams of both A and B present, and no X is present. If x grams of X have formed after t minutes, the velocity of the reaction is proportional to the product of the unused amounts of A and B.

a. Given that $\frac{dx}{dt} = k\left(b - \frac{x}{2}\right)^2$ where b and k are both positive constants, show that $x(t) = \frac{2b^2kt}{2 + bkt}$.
b. If $b = 5$ and after 4 minutes 2 grams of X is present, determine the amount of X present after 10 minutes. Give your answer to 2 decimal places.
c. Determine the ultimate amount of substance X that can eventually be formed.

16. A vat initially contains V_0 litres of water. A brine solution of concentration b grams per litre is added to the vat at a rate of f litres per minutes. The well-stirred mixture is drawn off at the same rate.

a. Set up the differential equation for Q, the amount of brine in grams in the vat after t minutes.
b. Use integration to show that $Q(t) = bV_0\left(1 - e^{-\frac{ft}{V_0}}\right)$ is a particular solution of the differential equation.

17. A container initially contains V_0 litres of a brine solution in which q_0 grams of brine have been dissolved. A brine solution of concentration b grams per litre is added to the container at a rate of f litres per minute. The well-stirred mixture leaves the container at the same rate.

a. Set up the differential equation for Q, the amount of brine in grams in the container after t minutes.
b. Use integration to show that $Q(t) = bV_0 + (q_0 - bV_0)e^{-\frac{ft}{V_0}}$ is a particular solution of the differential equation.

Complex unfamiliar

18. In a trimolecular chemical reaction, three substances, A, B and C, react to form a single substance X. The rate of the reaction is proportional to the product of the unreacted amounts of A, B and C. Substances A, B and C combine in equal parts to form substance X, and initially no X is present.

Given that $\frac{dx}{dt} = k\left(a - \frac{x}{3}\right)^3$ where a is the initial amount of all substances A, B and C present and $k > 0$, show that $t = \frac{3x(6a - x)}{2a^2k(3a - x)^2}$ for $0 \le x < 3a$.

19. In a chemical reaction between a grams of substance A and b grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. Substances A and B combine in equal parts to form substance X, and initially no X is present. Given that $\frac{dx}{dt} = k\left(a - \frac{x}{2}\right)\left(b - \frac{x}{2}\right)$ where $k > 0$ is a constant, x is the amount of substance X formed at a time t minutes, and $a > b > 0$, show that $x(t) = \dfrac{2ab\left(1 - e^{-\frac{(a-b)kt}{2}}\right)}{a - be^{-\frac{(a-b)kt}{2}}}$ and deduce that the limiting amount of X present is $2b$.

20. A tank with a capacity of 500 litres contains 300 litres of water in which of salt has been dissolved. Starting at $t = 0$, brine containing 0.2 kilogram of salt per litre is poured into the tank at a rate of 10 litres per minute. The mixture is kept uniform by stirring. Additionally, mixture flows out at a rate of 5 litres per minute. Determine whether the tank will overflow before the concentration of salt reaches 0.20 kg/L.

Fully worked solutions for this chapter are available online.

LESSON
9.6 The logistic equation

SYLLABUS LINKS

- Model and solve problems using provided differential equations, including the logistic equation, Newton's law of cooling and radioactive decay, with and without technology.

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9.6.1 Introduction

In previous sections we considered growth when it is directly proportional to itself; that is, the law of natural growth, $\frac{dN}{dt} = kN$; and also when the growth rate is proportional to the difference between the number and the equilibrium value; that is, bounded growth, $\frac{dN}{dt} = k(P - N)$. In this section we consider growth rates that are proportional to the product of both of these: $\frac{dN}{dt} \propto N(P - N)$.

9.6.2 Logistic growth

Logistic growth describes how the number N in a population grows rapidly when small but slows as it approaches an environment's carrying capacity, P, due to limited resources. Unlike uninhibited exponential growth, logistic growth considers environmental constraints that restrict the population to P, the maximum sustainable size. The logistic differential equation can be written in the form $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$. If the constants are relabelled, the simplified linear model is $\frac{dN}{dt} = k(P - N)$.

Logistic growth model

If a population $N(t)$ grows at a rate proportional to both its size and the remaining capacity for growth, it satisfies:

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right),$$

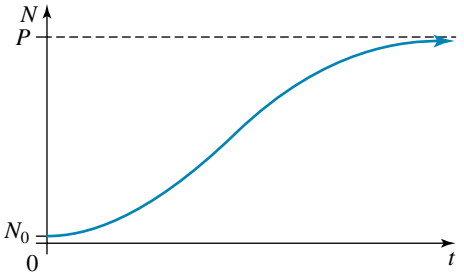
where k is the growth constant, and P is the carrying capacity.

Given $N(0) = N_0$ and assuming $k > 0$ and $P > N_0 > 0$, it can be shown that:

$$N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}} \text{ or equivalently, } N(t) = \frac{P}{1 + \left(\frac{P}{N_0} - 1\right)e^{-kt}}.$$

In this relationship, as $t \to \infty$, $N(t) \to P$.

The graph of the logistic equation always has the S shape shown below.



The growth pattern modelled by this type of differential equation is called logistic growth. It was devised by the Belgian mathematician Pierre François Verhulst (1804–1849). His idea was a response to the work of the English scholar Thomas Robert Malthus, who published a paper in 1798, *An Essay on the Principle of Population*, predicting unlimited population growth. Verhulst disagreed with Malthus's model and used his own model to show the characteristics of bounded population growth.

WORKED EXAMPLE 15 Using a logistic growth model

A rumour is spreading in a neighbourhood that contains 800 people. The rumour spreads at a rate proportional to the product of the number who have heard the rumour and the number who have not yet heard the rumour. At first only two people in the area have heard the rumour, but after 4 days, 15% of the residents of the neighbourhood have heard it.

a. Let $N = N(t)$ be the number who have heard the rumour after t days. Write and solve the differential equation.

b. Determine the number of residents in the neighbourhood who have heard the rumour after 5 more days.

c. Calculate the time in days when 50% of the residents in the area have heard the rumour.

d. Sketch the graph.

THINK	WRITE/DRAW
a. 1. The growth rate is proportional to the product of N and $(P - N)$.	a. The maximum number is $P = 800$. $\frac{dN}{dt} \propto N(800 - N)$
2. Write the differential equation and the initial condition.	$\frac{dN}{dt} = kN(800 - N), N(0) = 2$
3. To solve the differential equation, invert both sides.	$\frac{dt}{dN} = \frac{1}{kN(800 - N)}$

4. Use the separation of variables.

$$kt = \int \frac{1}{N(800 - N)}\, dN$$

5. Use partial fractions on the integrand on the right-hand side.

$$\frac{1}{N(800 - N)} = \frac{A}{N} + \frac{B}{800 - N}$$

6. Add the partial fractions.

$$\frac{1}{N(800 - N)} = \frac{A(800 - N) + BN}{N(800 - N)}$$
$$= \frac{N(B - A) + 800A}{N(800 - N)}$$

7. Write equations that can be solved to determine the values of A and B.

From the coefficient of N:
$B - A = 0 \quad [1]$
From the term independent of N:
$800A = 1 \quad [2]$

8. Solve the equations and state the values of A and B.

From [1], $B = A$.
From [2], $A = \frac{1}{800}$.
$A = B = \frac{1}{800}$

9. Substitute for A and B in the integration equation.

$$kt = \frac{1}{800}\int \left(\frac{1}{N} + \frac{1}{800 - N}\right) dN$$

10. Perform the integration.

$$800kt = \ln(|N|) - \ln(|800 - N|) + c$$

11. Because $2 \le N < 800$, the modulus signs are not needed.

$$800kt = \ln(N) - \ln(800 - N) + c$$

12. Use the initial condition to determine the constant of integration.

Substitute $N = 2$ when $t = 0$:
$0 = \ln(2) - \ln(800 - 2) + c$

13. Solve for c and apply log laws.

$$c = \ln(798) - \ln(2)$$
$$= \ln\left(\frac{798}{2}\right)$$
$$= \ln(399)$$

14. Substitute back for c and use log laws again.

$$800kt = \ln(N) - \ln(800 - N) + \ln(399)$$
$$= \ln\left(\frac{399N}{800 - N}\right)$$

15. After 4 days, 15% of the residents have heard the rumour. This is 120 people. Use this to determine k.

Substitute $N = 120$ when $t = 4$:

$$3200k = \ln\left(\frac{399 \times 120}{800 - 120}\right)$$
$$k = \frac{1}{3200}\ln\left(\frac{1197}{17}\right)$$
$$800k = \frac{1}{4}\ln\left(\frac{1197}{17}\right)$$

16. Use the definition of the logarithm but leave the result in terms of k, as k has a known value.

$$\frac{399N}{800-N} = e^{800kt}$$

17 Rearrange to make N the subject.

$$399N = (800 - N)e^{800kt}$$
$$800 - N = 399Ne^{-800kt}$$
$$800 = N + 399Ne^{-800kt}$$

18. Take out the common factor of N.

$$800 = N(1 + 399e^{-800kt})$$

19. State the solution of the differential equation.

$$N(t) = \frac{800}{1 + 399e^{-\frac{t}{4}\ln\left(\frac{1197}{17}\right)}}$$

b. 1. Determine the value of N after 5 more days (i.e. when $t = 9$).

b. Substitute $t = 9$:

$$N(9) = \frac{800}{1 + 399e^{-\frac{9}{4}\ln\left(\frac{1197}{17}\right)}}$$
$$\approx 778.3767$$

2. State the required number, rounding down.

778 people have heard the rumour.

c. 1. Determine t when 50% of 800 have heard the rumour; that is, when $N = 400$. It is easier to use an earlier equation.

c. $$800kt = \frac{1}{4}\ln\left(\frac{1197}{17}\right)$$
$$= \ln\left(\frac{399N}{800 - N}\right)$$

Substitute $N = 400$:

$$\frac{t}{4}\ln\left(\frac{1197}{17}\right) = \ln\left(\frac{399 \times 400}{800 - 400}\right)$$

2. Solve for t.

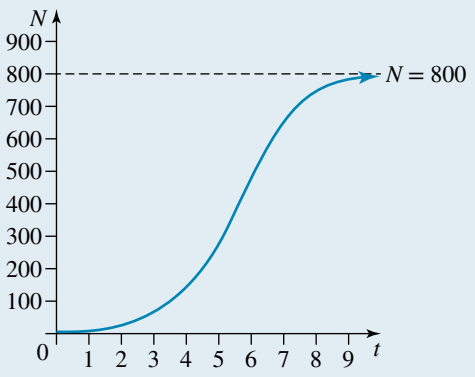
$$t = \frac{4\ln\left(\frac{399\times400}{800-400}\right)}{\ln\left(\frac{1197}{17}\right)}$$
$$= \frac{4\ln(399)}{\ln\left(\frac{1197}{17}\right)}$$
$$\approx 5.63$$

3. State the answer.

After 5.63 days, 50% of the residents have heard the rumour.

d. Sketch the graph. (Note that as $t \to \infty$, $N \to 800$.)

d.

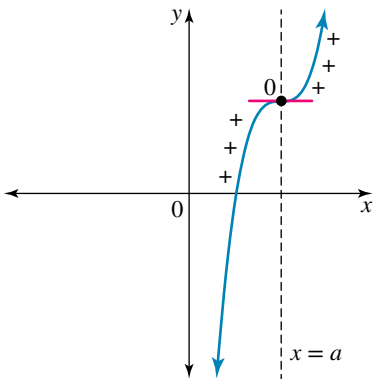


9.6.3 Points of inflection

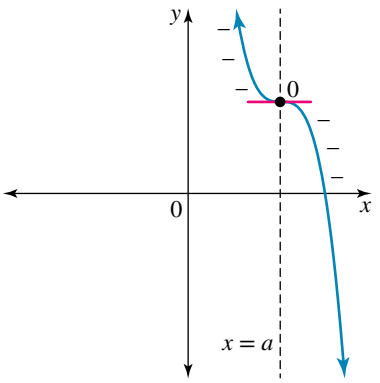
From your studies in Mathematical Methods, recall that a stationary point is defined as a point on the curve at which the slope of the tangent to the curve is zero; that is, where the gradient $\frac{dy}{dx} = f'(x) = 0$. This concept was used to determine the location of any turning points. To distinguish between a local maximum and a minimum, we use the sign test or the second derivative test.

On a continuous curve, between a local maximum and minimum there is another critical point called an inflection point. At such a point the curve changes from being concave to convex or vice versa. The tangent to the curve at such a point crosses the graph. The rate of change of the gradient is zero at a point of inflection. Mathematically, this can be interpreted as $\frac{d^2y}{dx^2} = f''(x) = 0$. There are two types of inflection points.

Horizontal stationary point of inflection

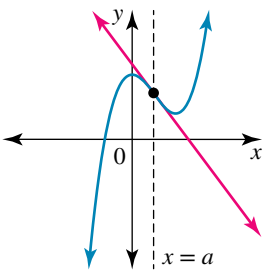


$x < a,\ f'(x) > 0$
$x > a,\ f'(x) = 0$
$x > a,\ f'(x) > 0$
$f''(a) = 0$



$x < a,\ f'(x) < 0$
$x = a,\ f'(x) = 0$
$x > a,\ f'(x) < 0$
$f''(a) = 0$

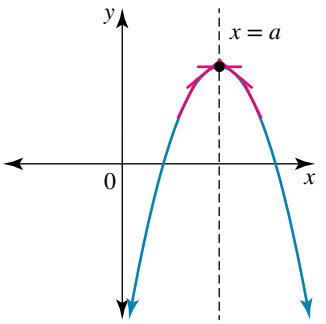
Inflection point



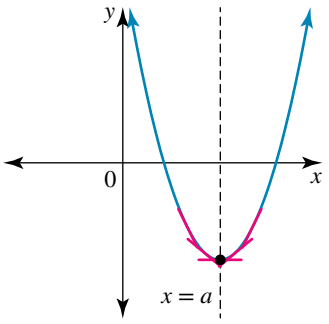
At $x = a$,
$f'(a) \neq 0$
$f''(a) = 0$

Concavity

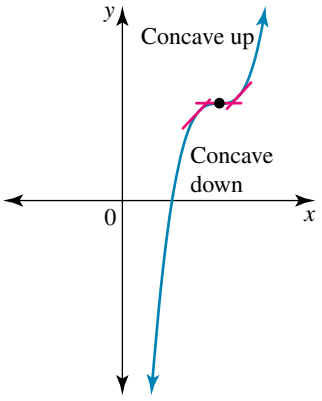
A non-stationary point of inflection is where the tangent to the curve moves from being above the curve to below the curve or vice versa. Curves are also defined as being **concave up** or **concave down**. When a tangent is above the curve at each point, then the derivative function is decreasing and $f''(x) < 0$.

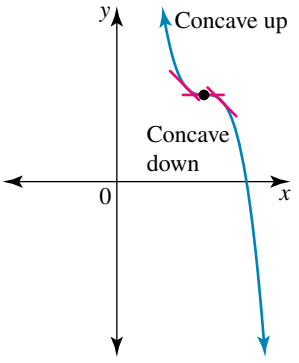


If the tangent to the curve is below the curve at each point, then the derivative function is increasing and $f''(x) > 0$.



At a point of inflection, the tangent will pass through the curve, and the curve will change from concave down to concave up or vice versa on either side of the point of inflection.

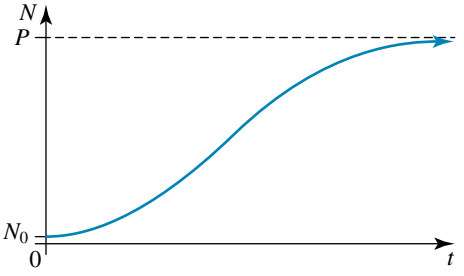




An interesting feature of a logistic curve is that it contains a point of inflection corresponding to the point of maximum growth rate. The mathematics and reasoning used to determine the location of this point of inflection will be examined in the following section.

9.6.4 Analysis of the logistic solution

The differential equation for the logistic curve in general is given by $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$. For an initial number $N(0) = N_0$, assuming $k > 0$ and $P > N_0 > 0$, the solution is given by $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}} = \frac{P}{1 + \left(\frac{P}{N_0} - 1\right)e^{-kt}}$ for $t \geq 0$.



The graph of the logistic curve always has this so-called S-shaped curve. The graph is relatively flat at the ends, so the gradient is close to zero as $N \to 0$ and $N \to P$. These lines, $N = 0$ and $N = P$, are in fact horizontal asymptotes.

Because $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$, we can write $\frac{dN}{dt} = \frac{k}{P}(PN - N^2)$. Taking the derivative again with respect to t, because k and P are constants, and using implicit differentiation or the chain rule, then

$$\begin{aligned}\frac{d^2N}{dt^2} &= \frac{d}{dt}\left(\frac{dN}{dt}\right)\\ &= \frac{k}{P}\frac{d}{dt}(PN - N^2)\\ &= \frac{k}{P}\frac{d}{dt}(PN - N^2)\frac{dN}{dt}\\ &= \frac{k}{P}(P - 2N)\frac{dN}{dt}\end{aligned}$$

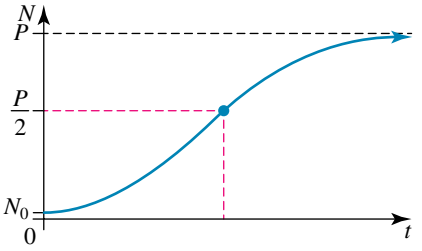
There is a point on the graph at which the gradient or rate of change of growth is a maximum. Assume $k > 0$, $N_0 > 0$ and $\frac{dN}{dt} > 0$. This means that in the equation $\frac{d^2N}{dt^2} = \frac{k}{P}(P - 2N)\frac{dN}{dt}$, the expressions $\frac{k}{P}$ and $\frac{dN}{dt}$ are both positive.

To keep the two sides of the equation equal, if $\frac{d^2N}{dt^2} > 0$, then $P - 2N > 0$.

Therefore, $0 < N < \frac{P}{2}$ is where the curve is concave up.

Conversely, if $\frac{d^2N}{dt^2} < 0$, then $P - 2N < 0$.

Therefore, $\frac{P}{2} < N < P$ is where the curve is concave down.



Recall that if the second derivative on a curve is zero, there is a point of inflection. For a logistic curve, the point of inflection occurs when $\frac{d^2N}{dt^2} = 0$; that is, for $N = \frac{P}{2}$, or halfway up the curve.

The point of inflection is the point of maximum growth rate, or where the value is increasing most rapidly. It can be shown in general that when $N = \frac{P}{2}$, the corresponding value for t is given by $\frac{1}{k}\ln\left(\frac{P}{N_0} - 1\right)$.

WORKED EXAMPLE 16 Analysing a logistic solution

A logistic equation has the solution $y(x) = \frac{200}{1 + 99e^{-\frac{x}{2}}}$.

a. Using differentiation, show that the solution satisfies the differential equation $\frac{dy}{dx} = \frac{y(200 - y)}{400}$.

b. Show that $\frac{d^2y}{dx^2} = \frac{y(100 - y)(200 - y)}{80000}$.

c. Hence determine the coordinates of the point of inflection.

d. Sketch the graph of y versus x.

THINK	WRITE/DRAW
a. 1. Write the given solution in index form.	a. $y = 200\left(1 + 99e^{-\frac{x}{2}}\right)^{-1}$
2. Use the chain rule for differentiation.	$\frac{dy}{dx} = 200 \times -1 \times -\frac{1}{2} \times 99e^{-\frac{x}{2}}\left(1 + 99e^{-\frac{x}{2}}\right)^{-2}$

3. Simplify the expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{100 \times 99e^{-\frac{x}{2}}}{\left(1 + 99e^{-\frac{x}{2}}\right)^2}$$

4. We need to express x in terms of y.

$$\frac{200}{1 + 99e^{-\frac{x}{2}}} = y \qquad [1]$$

$$\frac{1}{1 + 99e^{-\frac{x}{2}}} = \frac{y}{200} \qquad [2]$$

5. Express the exponential in terms of y.

$$1 + 99e^{-\frac{x}{2}} = \frac{200}{y}$$

$$99e^{-\frac{x}{2}} = \frac{200}{y} - 1$$

$$99e^{-\frac{x}{2}} = \frac{200 - y}{y} \qquad [3]$$

6. Rewrite the expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{100 \times 99e^{-\frac{x}{2}}}{\left(1 + 99e^{-\frac{x}{2}}\right)^2}$$

7. Replace $99e^{-\frac{x}{2}}$ with $\frac{200 - y}{y}$.

Substitute [3] into $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{100 \times \left(\frac{200-y}{y}\right)}{\left(1 + \frac{200-y}{y}\right)^2}$$

8. Simplify.

$$\frac{dy}{dx} = \frac{y(200 - y)}{400}$$

b. 1. To determine the second derivative, differentiate with respect to x again. However, since $\frac{dy}{dx}$ is in terms of y, use implicit differentiation.

b.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) \times \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{y(200 - y)}{400}\right) \times \left(\frac{y(200 - y)}{400}\right)$$

2. Expand the brackets.

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{1}{400}(200 - y^2)\right) \times \left(\frac{y(200 - y)}{400}\right)$$

3. Take the derivative and leave the constant factor.

$$\frac{d^2y}{dx^2} = \frac{1}{400}(200 - 2y) \times \left(\frac{y(200 - y)}{400}\right)$$

4. Take out the common factor of 2 and simplify.

$$\frac{d^2y}{dx^2} = \frac{y(100 - y)(200 - y)}{80\,000}$$

c. 1. The inflection point occurs when the second derivative is zero.

c. When $\frac{d^2y}{dx^2} = 0$, $y = 0, 100, 200$. However $y \neq 0$, so the graph never crosses the x-axis, and $y \neq 200$. The lines and $y = 0$ are horizontal asymptotes. Therefore, $y = 100$ is the only possible result.

2. Determine the corresponding x-value.

Substitute $y = 100$:

$$100 = \frac{200}{1 + 99e^{-\frac{x}{2}}}$$

3. Solve for x.

$$1 + 99e^{-\frac{x}{2}} = 2$$

$$99e^{-\frac{x}{2}} = 1$$

$$e^{-\frac{x}{2}} = \frac{1}{99}$$

$$e^{\frac{x}{2}} = 99$$

4. Use the definition of the logarithm.

$$\frac{x}{2} = \ln(99)$$

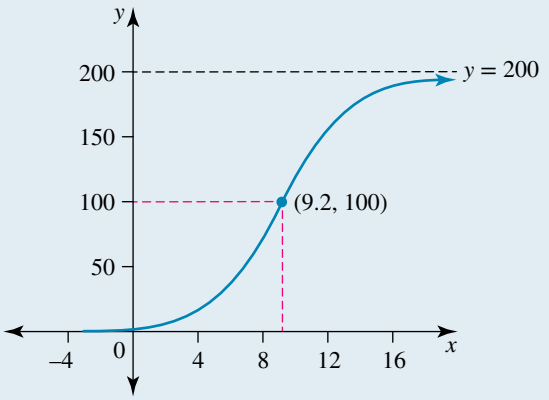
$$x = 2\ln(99)$$

5. State the coordinates of the inflection point.

$$(2\ln(99),\ 100) \approx (9.2,\ 100)$$

d. Sketch the graph.

d. For $y = \frac{200}{1 + 99e^{-\frac{x}{2}}}$, $y \to 200$ as $x \to \infty$ and the inflection point is at $(2\ln(99), 100) \approx (9.2, 100)$.



Exercise 9.6 The logistic equation

learnon

9.6 Exercise | 9.6 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5	6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20	21

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Simple familiar

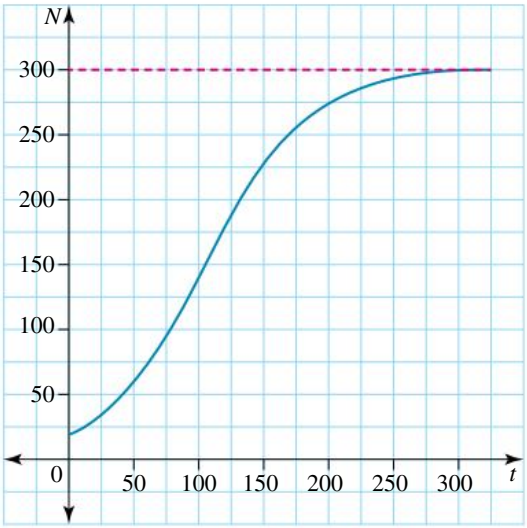
1. A cold virus is spreading in a kindergarten with 47 kids. The virus spreads at a rate proportional to the product of the number who have caught the virus and the number who have not yet caught the virus. On the first day, three kids have the virus.
Write a differential equation representing this situation, defining the terms used.

2. A rumour is spreading in a school. The number of people who have heard the rumour after t days is given by $N(t) = \dfrac{426}{1 + 141e^{-kt}}$, $k > 0$.
Determine the number of students in the school and how many students have heard the rumour initially.

3. Consider the equation $N(t) = \dfrac{1540}{1 + 109e^{-kt}}$.
Given that $N(25) = 55$, determine the exact value of k.

4. A logistic equation has the solution $N(t) = \dfrac{1890}{1 + 269e^{-0.5t}}$ where t is in hours.
Determine t such that $N(t) \geq 210$ and give your answer to the nearest integer. Show your working.

5. Consider the graph of a logistic curve and answer the following.



a. Determine the carrying capacity.
b. Estimate the coordinates of the point of inflection.

Complex familiar

6. **WE15** A rumour is spreading in an elderly nursing home which contains 100 pensioners. The rumour spreads at a rate proportional to both the number who have heard the rumour and the number who have not yet heard the rumour. At first, 4 pensioners in the home have heard the rumour, and after 2 weeks, 20 of the pensioners have heard it.
   - **a.** Let $N = N(t)$ be the number who have heard the rumour after t weeks. Write and solve the differential equation.
   - **b.** Determine the number of pensioners in the home who have heard the rumour after 3 more weeks.
   - **c.** Determine how long it is before 50 pensioners in the home have heard the rumour. Give your answer to 2 decimal places.

7. An infection is spreading in another elderly nursing home which contains 80 pensioners. The infection spreads at a rate proportional to both the number who have the infection and the number who have not yet caught the infection. At first, only two pensioners in the home have the infection. After 3 days, 25% of the pensioners have the infection.
   - **a.** Let $N = N(t)$ be the number who have the infection after t days. Write and solve the differential equation.
   - **b.** Determine the number of pensioners in the home who have the infection after 4 more days.
   - **c.** Determine how long it is before 60% of the pensioners in the home have the infection. Give your answer to 2 decimal places.

8. **WE16** A logistic equation has the solution $y(x) = \dfrac{500}{1 + 9e^{-3x}}$.
   - **a.** Using differentiation, show that it satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{3y(500 - y)}{500}$.
   - **b.** Show that $\dfrac{d^2y}{dx^2} = \dfrac{9y(250 - y)(500 - y)}{125\,000}$.
   - **c.** Hence, determine the exact coordinates of the point of inflection.
   - **d.** Sketch the graph of y versus x.

9. A logistic equation has the solution $y(x) = \dfrac{400}{1 + 199e^{-2x}}$.
   - **a.** Using differentiation, show that it satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{y(400 - y)}{200}$.
   - **b.** Show that $\dfrac{d^2y}{dx^2} = \dfrac{y(200 - y)(400 - y)}{20\,000}$.
   - **c.** Hence, determine the exact coordinates of the point of inflection.
   - **d.** Sketch the graph of y versus x.

10. A logistic equation has the solution $y(x) = \dfrac{600}{1 + 99e^{-\frac{x}{3}}}$.
    - **a.** Using differentiation, show that it satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{y(600 - y)}{1800}$.
    - **b.** Show that $\dfrac{d^2y}{dx^2} = \dfrac{y(300 - y)(600 - y)}{1\,620\,000}$.
    - **c.** Hence, determine the exact coordinates of the point of inflection.
    - **d.** Sketch the graph of y versus x.

11. An area of the ocean initially contains 500 fish. The area can support a maximum of 4000 fish. The number of fish in the area grows continuously at a rate of 8% per year; that is, the constant of proportionality is 8%.

a. Write the differential equation for the number of fish, $N(t)$, in the area at a time t years, assuming a logistic growth model.

b. Solve the differential equation to determine the number of fish in the area after t years.

c. Hence, determine the number of fish in the area after 5 years.

d. Determine after how many years the number of fish will reach 3000. Give your answer to 2 decimal places.

12. In a movie, people in a town are turned into zombies by a touch from an existing zombie. The town contains 360 people, and initially there is only one zombie. After 3 days, there are 60 zombies. The number of zombies in the town grows at a rate proportional to both the number of zombies and the number of people in the town who are not yet zombies.

a. Write the differential equation for the number of zombies, N, after t days.

b. Determine after how many days 75% of the people in the town will become zombies. Give your answer to the nearest day.

13. A rumour is spreading in a school with 2000 students. The rumour spreads at a rate proportional to both the number who have heard the rumour and the number who have not yet heard it. Initially, only two students have heard the rumour, but after 3 days, 10% of the school has heard it.

a. Write the differential equation for the number of students, N, in the school after t days who have heard the rumour, assuming a logistic growth model.

b. Determine after how many days half of the students have heard the rumour.

c. Identify the number of days at which the rumour is spreading most rapidly.

14. A logistic equation has the solution $N(t) = \dfrac{500}{1 + 9e^{-kt}}$ where N is the population number of a city in millions at a time t years and k is a positive real constant. Determine the population size at which the population is increasing most rapidly.

15. The number of children who contract the common cold from a kindergarten is found to be proportional to those who have the cold and those yet to get the cold. Initially, only one child from the kindergarten has the common cold. After 2 days, 5 children have the cold, and after a further 2 days, 15 children have the cold.

a. Determine how many children from the kindergarten will eventually get the cold.

b. Identify after how many days the cold is spreading most rapidly. Give your answer to the nearest day.

c. Calculate how many children from the kindergarten have the cold after 6 days.

d. Sketch the graph of the number of children with the cold against the time in days.

16. A disease is spreading through an area with a population of 10 000. Initially, four people in the area have the disease. After 4 days, 50 people have the disease. The disease is spreading at a rate proportional to both the number of people who have the disease and the number of people who do not yet have the disease. Determine how many people in the area have the disease after 2 weeks.

17. A landscaper has built a series of small ponds in a client's garden. At $t = 0$, they introduce 30 striped marsh frogs. Assume a logistic growth model for the frog population. It is observed that after 2 months, there are 78 frogs. The habitat has a carrying capacity of 300 such frogs.
Predict how long it will take for the population of striped frog marsh frogs to reach 95% of the carrying capacity. Give your answer to 2 decimal places.

18. At the start of the school year, 6 out of the 92 students in year 12 are playing tabletop role-playing games (TTRPGs) regularly. The interest for TTRPGs grows at a rate proportional to both the number of students already playing regularly and the number who are not yet playing regularly. After 4 weeks, 18 students are playing TTRPGs regularly. Determine how long it will take for at least half of the students in year 12 to play TTRPGs regularly. Give your answer to the nearest week.

19. Lucas and Ashley have decided to save a maximum of \$3000 for a trip overseas next year. They open a saving account and transfer \$600 on it initially. They plan for the amount of money on this saving account to grow at a rate proportional to the amount already saved and the remaining amount to save for their goal. Determine the logistic equation that would enable them to have saved two-thirds of their maximum goal in 9 months.

20. Jaime's local library has a collection of 100 fantasy novels,12 of which Jaime has already read by the time she is 12. She plans to read each remaining book once. She reads books at a rate proportional to the number of books she has already read and the number of books she has yet to read.
Estimate how many books Jaime should have read by age 13 if she wants to reach her intermediate goal of having read 80 books by age 15. Give your answer to the nearest integer.

Complex unfamiliar

21. A project is being discussed to remedy an issue with invasive carps in a lake by encouraging harvesting of carps by local fishermen. The number $N(t)$ of invasive carps in the lake at time t weeks is represented by $N(t) = \dfrac{5000}{1 + ae^{bt}}$ where a and b are positive constants.
The project will be considered a success if the population of carp reaches a threshold of 100 carps in 8 weeks, and if there are less than 2000 carps remaining after 4 weeks.
Determine the values of a and b for the project to be successful.

Fully worked solutions for this chapter are available online.

LESSON
9.7 Review

9.7.1 Summary

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9.7 Exercise

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Simple familiar

1. The number of rabbits in a certain area increases at a rate proportional to the number present. If the number of rabbits doubles in 4 months, calculate how many months elapse before the number of rabbits triples. Give your answer to 1 decimal place.

2. A radioactive substance has a half-life of 5 years. After 3 years, calculate how much of the substance has disintegrated.

3. When light passes through a medium it loses its intensity as it penetrates depths. The rate of loss of intensity with respect to the depth is proportional to the intensity at that depth. If 5% of light is lost in penetrating 40% of a glass slab, calculate the percentage lost penetrating the whole slab (give your answer correct to 1 decimal place).

4. A container initially holds 10 litres of water. A salt solution containing 5 kilograms per litre is poured into the tank at a rate of 6 litres per minute and the solution is kept uniform by stirring. If the mixture flows out at a rate of 4 litres per minute and Q kilograms is the amount of salt in the tank at time t minutes, devise an expression for $\frac{dQ}{dt}$.

5. A cake is taken out of an oven, which has been kept at a constant temperature of 180 °C, and placed on a table where the temperature is 18 °C. Using Newton's law of cooling, construct a differential equation where the temperature T °C, of the cake at a time t minutes which satisfies these conditions.

6. Let $W(t)$ represent the number of wolves in a certain area at time t months. Initially there are 200 wolves present and the number of wolves is increasing at a rate directly proportional to the product of the number of wolves and to $200 - W(t)$. The wolves are also being killed off at 20 per month. If k is a positive constant, create the differential equation for the number of wolves present.

7. If a logistic equation has as its solution $N(t) = \dfrac{200}{1 + 99e^{-kt}}$ where $k > 0$, determine an expression for $\dfrac{dN}{dt}$.

8. The population of a city increases at a rate proportional to the current population. In 2018 the population was 1.12 million, and in 2020 the population was 1.67 million.
   a. Construct a differential equation of N in terms of t.
   b. Solve the differential equation to express the population number, N, in terms of t, the time in years after 2018.
   c. Use the model to calculate an estimate of the population in 2029.
   d. Determine when the population would reach 4 million.

9. A simple electric circuit contains a battery supplying a constant voltage V volts and a current of I amperes at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries. According to Kirchhoff's laws, the differential equation for this simple circuit is,

$$L\frac{dI}{dt} + RI = V$$

   Consider a simple electric circuit under the following conditions,
   - a battery with a constant voltage of $V = 60$ volts
   - resistance $R = 12$ ohms
   - the inductance is $L = 4$ henries
   - the switch is closed at $t = 0$ seconds, hence $I(0) = 0$.

   a. Show that the simplified differential equation for the current in the circuit is given by $\dfrac{dI}{dt} = 15 - 3I$.
   b. Determine a particular (specific) solution to the differential equation.

10. The number of mice in a certain area grows at a rate proportional to the current number.
   Initially there are four mice present and after 3 months the number of mice has grown to 32.
   a. Calculate the number of mice in the area after 1 year.
   b. Calculate the time in months when 1000 mice are present.

11. The rate of disintegration of a radioactive element is proportional to the amount of the element remaining at any time. If 30% disintegrates in 3 years, calculate the half-life. Give your answer to 2 decimal places.

12. When light passes through a medium it loses its intensity as it penetrates depths. If I is the intensity of light at depth x, the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth, so that $\dfrac{dI}{dx} = -kI$, where k is a positive constant. Solve the differential equation to show that $I = I_0 e^{-kx}$.

Complex familiar

13. A hospital patient is receiving a drug at a constant rate of R milligrams per hour through a drip. At a time t hours the amount of the drug in the patient is x milligrams. The rate of loss of the drug from the patient is proportional to the amount of the drug in the patient. Initially there are no amounts of the drug in the patient. Given that $\frac{dx}{dt} = R - kx$ where k is a given constant, solve the differential equation to express x in terms of t, k and R.

14. A lake has a capacity to hold a maximum of 50 ducks. Initially there are four ducks in the lake. The number of ducks in the lake grows at a rate proportional to the difference between the maximum number and the current number of ducks in the lake. After 4 months there are 16 ducks in the lake.

a. Develop and solve the differential equation for the number of ducks n in the lake after t months.

b. Calculate how many ducks are in the lake after 8 months.

15. A gas BBQ is used for cooking and is outdoors where the surrounding temperature is constant at 15 °C. When the gas is ignited the BBQ heats up at a rate 25 °C per minute, but also cools according to Newton's law of cooling. When the temperature of the BBQ grill is 75 °C, the temperature of the BBQ is increasing at a rate of 45 °C per minute.

a. Show that differential equation representing the temperature T °C of the grill on the BBQ, at a time t minutes after the gas has been ignited, satisfies $\frac{dT}{dt} = \frac{60 + T}{3}$.

b. Solve the differential equation.

c. Determine how long it takes the grill of the BBQ to reach a suitable cooking temperature of 185 °C. Give your answer to 2 decimal places.

16. A tank contains 20 litres of water. A salt solution of concentration 2 grams per litre is poured into the tank at a rate of 6 litres per minute, and the well-stirred mixture flows out at a rate of 4 litres per minute.

a. Develop the differential equation for Q the amount of salt in grams in the tank after a time t minutes.

b. Verify, by differentiation that $Q = \frac{4t\left(t^2 + 30t + 300\right)}{(t + 10)^2}$ is a solution of the differential equation.

Complex unfamiliar

17. A water fountain has a volume of 800 litres. Someone has dissolved 4 kilograms of detergent into the fountain. To remove the detergent concentration, fresh water is added to the fountain at a rate of 4 litres per minute and the solution is kept uniform by stirring. The mixture leaves the fountain at a rate of 4 litres per minute. Create and solve the differential equation for Q the amount of the detergent in kilograms in the fountain at a time t minutes after the fresh water started to be added. Determine when there is only 200 grams of detergent in the fountain.

18. Let $N(t)$ be the population number in an environment, which can support a maximum number of P. The population grows at a rate proportional to both the current population and the difference between the maximum number P and the current population. Initially the population number is $\frac{P}{10}$, and initially is growing at a rate $\frac{3P}{100}$.

a. Show that the differential equation is given by $\frac{dN}{dt} = \frac{N(P-N)}{3P}$, $N(0) = \frac{P}{10}$.

b. Solve the differential expressing N in terms of t.

c. Show that the population is increasing most rapidly when $t = 3\ln(9)$.

19. A biologist is studying the growth of two different strands of bacteria. She has two different experimental models, and both start at exactly the same time.

Type 1

The number of type 1 bacteria follows a logistic growth rate. Initially, there are 50 type 1 bacteria present, and it is known that the number of these bacteria cannot rise above 1000. The number of type 1 bacteria is increasing most rapidly before 10 hours and the rate slows down after 10 hours.

Type 2

The number of type 2 bacteria follows the law of natural growth. Initially, there are 25 type 2 bacteria present, and after 18 hours there are 800 type 2 bacteria present.

a. Develop and solve the differential equation for the number of type 1 bacteria present at t hours after the experiment starts.

b. Develop and solve the differential equation for the number of type 2 bacteria present at t hours after the experiment starts.

c. Determine after how many hours the number of type 1 and type 2 bacteria are equal. Give your answer to 1 decimal place. Calculate the actual number of each type at that time.

d. Sketch the graphs for both bacteria on the same set of axes.

20. Given the general logistic equation $\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right)$, $N(0) = N_0$, where k is a positive constant and $P > N_0 > 0$:

a. show, using integration, that $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$

b. show that $\lim_{n \to \infty} N(t) = P$

c. show that $\frac{d^2N}{dt^2} = k^2N\left(1 - \frac{2N}{P}\right)\left(1 - \frac{N}{P}\right)$ and hence that there is a point of inflection at $\left(\frac{1}{k}\ln\left(\frac{P}{N_0} - 1\right), \frac{P}{2}\right)$

d. verify, using differentiation, that $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$ is a solution of the following:

$$\frac{dN}{dt} = kN\left(1 - \frac{N}{P}\right),\ N(0) = N_0.$$

9.7 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 1, Q7; © State of Queensland (QCAA) 2024.

The differential equation for which the solution is a logistic equation of the form $y = \dfrac{a}{b + Ce^{-at}}$ where a, b and c are constants is

A. $\dfrac{dy}{dt} = 0.25(1 - 0.01t)$

B. $\dfrac{dy}{dt} = 0.25(1 - 0.01y)$

C. $\dfrac{dy}{dt} = 0.25t(1 - 0.01t)$

D. $\dfrac{dy}{dt} = 0.25y(1 - 0.01y)$

Question 2 (1 mark)

Source: QCAA 2022 Specialist Mathematics, Paper 2, Section 1, Q3; © State of Queensland (QCAA) 2024.

Determine the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{\sin(2x)}{\cos(2x)}$ given $y = 0$ when $x = \dfrac{\pi}{5}$.

A. $y = -2\ln|\cos(2x)| - 2.35$

B. $y = -2\ln|\cos(2x)| + 2.35$

C. $y = -\dfrac{1}{2}\ln|\cos(2x)| - 0.59$

D. $y = -\dfrac{1}{2}\ln|\cos(2x)| + 0.59$

Question 3 (1 mark)

Source: QCAA 2022 Specialist Mathematics, Paper 2, Section 1, Q8; © State of Queensland (QCAA) 2024.

Determine the gradient of the tangent to the curve $y^2 - 3x = 5$ at the point $(1, 2\sqrt{2})$.

A. 0.41

B. 0.53

C. 1.06

D. 8.49

Question 4 (9 marks)

Source: QCAA 2020 Specialist Mathematics, Paper 2, Section 2, Q12; © State of Queensland (QCAA) 2024.

For a certain experiment, the number of yeast cells, N, after t hours in a test tube can be modelled by the differential equation

$$\frac{dN}{dt} = \frac{1}{1000}N(1000 - N) \text{ for } t \geq 0$$

a. Given $\frac{1000}{N(1000-N)} = \frac{1}{N} + \frac{1}{1000-N}$, show that the general solution of the differential equation can be expressed as **[2 marks]**

$$\ln\left|\frac{N}{1000-N}\right| = t + c$$

A scientist commenced this experiment at $9{:}00$ am on a certain day and observed that 100 yeast cells were present at this time.

b. Show that the solution of the differential equation can be expressed as **[3 marks]**

$$N = \frac{1000}{1 + 9e^{-t}}$$

c. Determine the time of day when 900 yeast cells were present.
The scientist predicted that the number of yeast cells would eventually exceed 1200. **[2 marks]**

d. Evaluate the reasonableness of the scientist's prediction. **[2 marks]**

Question 5 (6 marks)

Source: QCAA 2021 Specialist Mathematics, Paper 1, Section 2, Q18; © State of Queensland (QCAA) 2024.

This differential equation can be used to determine the current I (amperes) at time t (seconds) with voltage V (volts) in an electric circuit containing a resistance R (ohms):

$$k\frac{dI}{dt} + RI = V$$

where k, R and V are positive constants and $t \geq 0$.

Assuming that there is no current in the electric circuit initially, show that the size of the current can never be greater than $\frac{V}{R}$.

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Answers

Chapter 9 Applications of first-order differential equations

9.2 Growth and decay

9.2 Exercise

1. $y = 2e^{\frac{1}{25}x}$
2. a. $N = 1.79 \times 10^6 e^{0.0277t}$
   b. 2.71 million
   c. 2052
3. $y = \frac{1}{2}e^{-0.008x}$
4. 137
5. a. 4146 b. 11 275
6. 942
7. a. 27 851 607 b. 2029
8. a. 8 220 578 135 b. 2039
9. 58.31 mg
10. a. 173 years b. 173 years
11. 113.78 mg
12. a. 1.63 hours b. 12.80 years
13. 65%
14. a. 0.29 g b. 25.59 years
15. 32.47 days
16. 38.2 years
17. a. 78.8% b. 24 110 years
18. 11.76%
19. 6.180 hours
20. 2147
21. The hospital will need to buy a new cobalt-60 source in 2045.
22. 10 356 years

9.3 Other applications of first-order differential equations

9.3 Exercise

1. \$9048.37
2. \$630 000
3. a. $P(h) = P_0 e^{kh}, k > 0$
   b. 50.94 cm Hg
4. a. $D(t) = D_0 e^{-kt}$
   b. 2.08 hours
5. a. Sample responses can be found in the worked solutions in the online resources.
   b. 18.5%
6. a. Sample responses can be found in the worked solutions in the online resources.
   b. 205.31 newtons
7. a. i. Sample responses can be found in the worked solutions in the online resources.
   ii. 35.48 ohms
   b. i. 1000 units
   ii. 1.5 seconds
8. a. $\frac{dN}{dt} = \frac{N - 160}{4}, N(0) = 320$
   b. $N(t) = 160\left(1 + e^{\frac{t}{4}}\right)$
   c. 1342
9. Sample responses can be found in the worked solutions in the online resources.
10. a. $\frac{dN}{dt} = \frac{N - 100}{20}, N(0) = 200$
    b. $N(t) = 100\left(1 + e^{\frac{t}{20}}\right)$
    c. 216
11. a. 11 years b. 19
12. a. $N(t) = \left(\frac{1}{2}kt + \sqrt{N_0}\right)^2$
    b. $N(t) = \sqrt{2kt + N_0^2}$
    c. $N(t) = \frac{4N_0}{\left(2 - kt\sqrt{N_0}\right)^2}$
    d. $N(t) = \frac{N_0}{1 - ktN_0}$
13. 25 million
14. 140
15. 666 052
16. 2030
17. a. $i(t) = i_0 e^{-\frac{Rt}{L}}$
    b. $Q(t) = Q_0 e^{-\frac{t}{RC}}$
18. a. Sample responses can be found in the worked solutions.
    b. If $N_0 > \frac{c}{k}$, N increases, if $N_0 < \frac{c}{k}$, N decreases and if $N_0 = \frac{c}{k}$, N remains stable.
19. If $N_0 > 3Q$, N increases, if $N_0 < 3Q$, N decreases and if $N_0 = 3Q$, N remains stable.
20. 8 fish per day for the population to slightly increase. Between 9 and 14 fish per day for at least 50 fish remaining after 15 days.

9.4 Bounded growth and Newton's law of cooling

9.4 Exercise

1. 165.4 g
2. 13 months
3. a. 4.4 kg b. 3.1 kg
4. a. 736 b. 35.1 months
5. 10.5 kg
6. 26.34 min
7. 1.75 min
8. a. 5.43 °C b. 26 °C
9. 5.27 min

10. 5 °C

11. 108.37 °C

12. a. i. $i(t) = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$

ii. Sample responses can be found in the worked solutions in the online resources.

b. $v(t) = E\left(1 - e^{-\frac{t}{RC}}\right)$

13. Sample responses can be found in the worked solutions in the online resources.

14. a. Sample responses can be found in the worked solutions in the online resources.

b. 9:28 pm

15. a. $t_1 = \frac{3}{2}, T_h = 95$

b. $t_2 = \frac{3}{4},\ T_c = 3$

c. 1.12

16. a. Sample responses can be found in the worked solutions in the online resources.

b. Monthly repayments: \$2146.52, total interest: \$215 164.60

17. \$17 308.20

18. 11.8 years

19. 10.56% per annum

20. 33 minutes

9.5 Chemical reactions and dilution problems

9.5 Exercise

1. a. $\frac{dQ}{dt} = 3 - \frac{3Q}{200 - t}, Q(0) = 20$

b. Sample responses can be found in the worked solutions in the online resources.

c. $-\frac{7}{200\,000}$

d. 1.15 kg/L

2. a. $\frac{dQ}{dt} = 6b - \frac{3Q}{50 + 3t}$

b. 2

3. a. i. $\frac{dQ}{dt} = 12 - \frac{5Q}{50 - t}$

ii. Sample responses can be found in the worked solutions in the online resources.

b. i. $\frac{dQ}{dt} = 4 - \frac{2Q}{30 - t}, Q(0) = 0$

ii. Sample responses can be found in the worked solutions in the online resources.

4. a. i. $\frac{dQ}{dt} = 6 - \frac{Q}{20 + t}, Q(0) = 0$

ii. Sample responses can be found in the worked solutions in the online resources.

b. i. $\frac{dQ}{dt} = 8 - \frac{3Q}{20 - t}, Q(0) = 0$

ii. Sample responses can be found in the worked solutions in the online resources.

5. a. i. $\frac{dQ}{dt} = 20 - \frac{3Q}{64 + 2t}, Q(0) = 128$

ii. Sample responses can be found in the worked solutions in the online resources.

iii. -2^{16}

iv. 3.34 g/L

b. 3

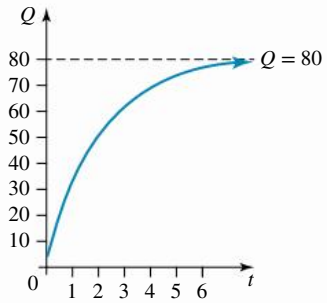
6. a. $\frac{dQ}{dt} = \frac{3(80 - Q)}{20}, Q(0) = 5$

b. $Q(t) = 5\left(16 - 15e^{-\frac{3t}{20}}\right)$

c. 1.05 g/L

d. Sample responses can be found in the worked solutions in the online resources.

e.



7. a. $\frac{dQ}{dt} = \frac{4(75 - Q)}{15}, Q(0) = 0$

b. $Q(t) = 75\left(1 - e^{-\frac{4t}{15}}\right), t \to \infty, Q \to 75$

c. 6.04 min

8. a. 1.659 g/L

b. i. $30e^{-\frac{t}{120}}$ ii. 11.04 kg iii. 83.18 min

9. a. i. 9.12 min ii. 34.66 min

b. 5.5 min

10. a. $\frac{dx}{dt} = k(2 - x)(6 - x), x(0) = 0$

b. $x(t) = \frac{6(1 - e^{-4kt})}{3 - e^{-4kt}}$ where $k = \frac{1}{12}\ln\left(\frac{5}{3}\right)$

c. $\frac{16}{11}$ g

11. a. $\frac{dx}{dt} = k(8 - x)^2, x(0) = 0$

b. $x(t) = \frac{24t}{3t + 10}$

c. 8 minutes (for 10 minutes total after the start of the reaction)

d. 8 g

12. a. $\frac{dx}{dt} = k(4 - x)(8 - x), x(0) = 0$

b. $x(t) = \frac{8(1 - e^{-4kt})}{2 - e^{-4kt}}$ where $k = \frac{1}{8}\ln\left(\frac{7}{6}\right)$

c. $\frac{52}{31}$ g

d. 8 g

13. a. 6.57 minutes **b.** $\frac{26}{9}$ g **c.** 4 g

14. a. Sample responses can be found in the worked solutions.
b. 3.61 g

15. a. Sample responses can be found in the worked solutions.
b. 3.85 g
c. 10 g

16-18. Sample responses can be found in the worked solutions in the online resources.

20. The concentration in salt reaches 0.20 kg/L before the tank overflows.

9.6 The logistic equation

9.6 Exercise

1. $\frac{dN}{dt} = kN(47 - N), N(0) = 3$

2. 426 and 3 respectively.

3. $k = -\frac{1}{25}\ln\left(\frac{27}{109}\right)\left(\text{or}\frac{1}{25}\ln\left(\frac{109}{27}\right)\right)$

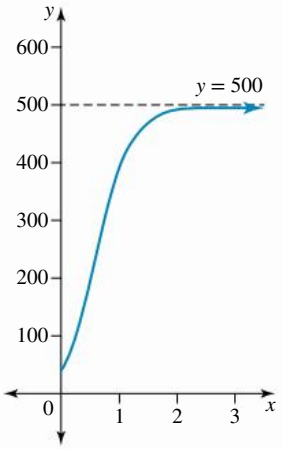
4. $t \geq 7$ hours

5. a. 300 **b.** (105, 150)

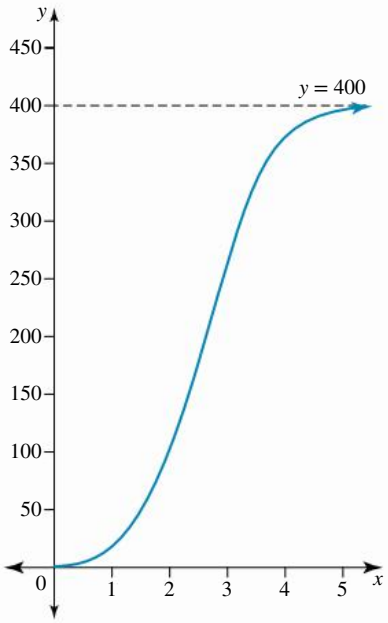
6. a. $N(t) = \frac{100}{1 + 24e^{-0.8959t}}$
b. 78
c. 3.55 weeks

7. a. $N(t) = \frac{80}{1 + 39e^{-0.855t}}$
b. 72
c. 4.76 days

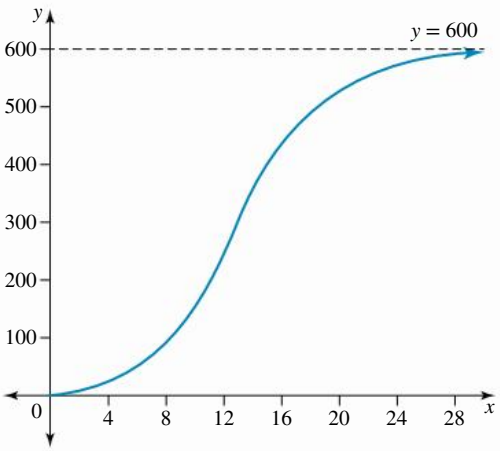
8. a-b. Sample responses can be found in the worked solutions in the online resources.
c. $\left(\frac{1}{3}\ln(9), 250\right)$
d.



9. a-b. Sample responses can be found in the worked solutions in the online resources.
c. $\left(\frac{1}{2}\ln(199), 200\right)$
d.



10. a-b. Sample responses can be found in the worked solutions in the online resources.
c. (3 ln(99), 300)
d.



11. a. $\frac{dN}{dt} = \frac{N(4000 - N)}{50\,000}, N(0) = 500$
b. $N(t) = \frac{4000}{1 + 7e^{-\frac{2t}{5}}}$
c. 2054
d. 7.61 years

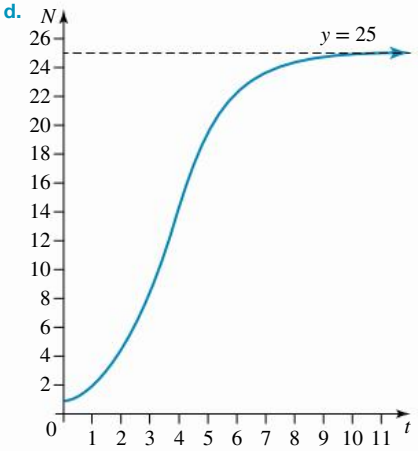
12. a. $\frac{dN}{dt} = kN(360 - N), N(0) = 1$
b. 5 days (4.9) days

13. a. $\frac{dN}{dt} = kN(2000 - N), N(0) = 2$
b. 5 days (4.4 days)
c. 5 days (4.4 days)

14. 250

15. a. 25
b. 4 days (3.55 days)
c. 22

d.



16. 7374

17. It takes approximately 8.93 months for the number of frogs to reach 95% of the carrying capacity.

18. It will take 9 weeks for at least 50% of the students to be playing TTRPGs regularly.

19. $A(t) = \dfrac{3000}{1 + 4e^{-\frac{\ln 8}{9}t}}$

20. Jaime should have read 30 books by age 13 to reach her intermediate goal.

21. $a \geq \dfrac{9}{4 \times 49}$ and $b = \dfrac{1}{8}\ln\left(\dfrac{49}{a}\right)$

9.7 Review

9.7 Exercise

1. $\dfrac{\ln(81)}{\ln(2)} \approx 6.3$ months

2. 34% has disintegrated.

3. 12%

4. $\dfrac{dQ}{dt} = 30 - \dfrac{4Q}{10 + 2t}$

5. $\dfrac{dT}{dt} = k\ (T - 18)$, $T(0) = 180$ where $k < 0$

6. $\dfrac{dW}{dt} = kW\ (200 - W)\ - 20$, $W(0) = 200$

7. $\dfrac{dN}{dt} = kN\left(1 - \dfrac{N}{200}\right)$, $N(0) = 2$

8. a. $\dfrac{dN}{dt} = kN$, where $N =$ number of people

   b. $N(t) = 1120000e^{0.2t}$

9. a. Sample responses can be found in the worked solutions.

   b. $I(t) = -5e^{-3t} + 5$

10. a. $4 \times 2^{12} = 16\ 384$

    b. $t = \log_2(250) \approx 8$ months

11. $T_{\frac{1}{2}} = \dfrac{-3\ln(2)}{\ln(0.7)} \approx 5.83$ years

12. Sample responses can be found in the worked solutions in the online resources.

13. $x(t) = \dfrac{R}{k}\left(1 - e^{-kt}\right)$

14. a. $n(t) = 2\left(25 - 23e^{-kt}\right)$, $k = \dfrac{1}{4}\ln\left(\dfrac{23}{17}\right)$

    b. $\dfrac{572}{23} \approx 24$ ducks

15. a. Sample responses can be found in the worked solutions in the online resources.

    b. $T(t) = 15\left(5e^{\frac{t}{3}} - 4\right)$

    c. $t = 3\ln\left(\dfrac{49}{15}\right) \approx 3.55$ minutes

16. a. $\dfrac{dQ}{dt} = 12 - \dfrac{2Q}{10 + t}$, $Q(0) = 0$

    b. Sample responses can be found in the worked solutions in the online resources.

17. $Q(t) = 4e^{-\frac{t}{200}}$, 10 hours

18. a. Sample responses can be found in the worked solutions in the online resources.

    b. $N(t) = \dfrac{P}{1 + 9e^{-\frac{t}{3}}}$

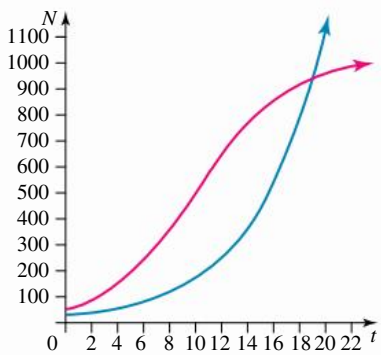
    c. Sample responses can be found in the worked solutions in the online resources.

19. a. $n(t) = \dfrac{1000}{1 + 19e^{-\frac{t}{10}\ln(19)}}$

    b. $m(t) = 25e^{\frac{t}{18}\ln(32)}$

    c. 18.8 h, 930 bacteria of each type

    d.



20. a-d Sample responses can be found in the worked solutions in the online resources.

9.7 Past QCAA exam questions

1. D

2. C

3. B

4. a. $t + c = \ln(|N|) - \ln(|1000 - N|) = \ln\left(\left|\dfrac{N}{1000 - N}\right|\right)$

   b. Sample responses can be found in the worked solutions in the online resources.

   c. 1:24 pm

   d. The scientist is clearly wrong.

5. Sample responses can be found in the worked solutions in the online resources.

10 Modelling motion

LESSON SEQUENCE

10.1 Overview 684
10.2 Equilibrium situations under concurrent forces 686
10.3 Non-equilibrium situations under concurrent forces 705
10.4 Forces that depend on time 714
10.5 Forces that depend on velocity 726
10.6 Forces that depend on displacement 744
10.7 Review 760
Answers 764

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 10
Exam questions	Exam question booklet — Chapter 10
Digital documents	Learning matrix — Chapter 10 Chapter summary — Chapter 10

LESSON
10.1 Overview

Hey students! Bring these pages to life online

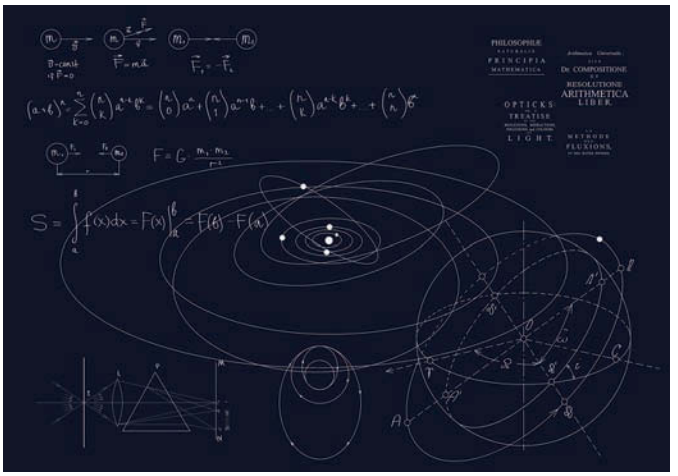
Engage with interactivities

Answer questions and check results

Track your progress

Find all this and MORE in jacPLUS

10.1.1 Introduction



In the seventeenth century, two incredible thinkers developed our understanding of the motion of objects. Galileo Galilei (1564–1642) was a famous Italian scientist and mathematician. He believed that the natural world could be best understood using mathematics, and through his efforts, the study of nature moved from a descriptive and qualitative form to one which was more mathematical and experimental. He began to record observations.

He also made many contributions to the study of motion. There are many situations of practical significance that involve motion under constant acceleration. The common example of bodies in free fall towards Earth's surface, regardless of their size, mass or composition, involves motion under constant acceleration — that of acceleration due to gravity.

It is said that Galileo illustrated this property by dropping various sized cannon balls from the top of the Leaning Tower of Pisa. Provided the effects of air resistance are minimised (i.e. the object should not be too large in area or fall too far) all bodies will fall with constant acceleration. The acceleration due to gravity depends on the distance from the centre of Earth, it varies slightly at different places on Earth's surface — being greater at the poles than at the equator and less at high altitudes. It is usual to state the constant of acceleration due to gravity as $9.8\ \text{m/s}^2$ unless stated otherwise.

Sir Isaac Newton (1642–1726) was a scientist, mathematician and astronomer. In 1687, he published his work, called Principia, which stated the three universal laws of motion and defined the law of universal gravitation. When he was being celebrated for his achievements, he said that 'If I have seen further [than others], it is by standing on the shoulders of giants'. His writings led to many more advancements in the understanding of our universe.

As we continue our study of the area of mathematics that deals with forces and motion, it can be appreciated that the arrangement of forces being exerted on objects can become very complex. In our course of study, we will investigate how to model motion.

10.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
10.2	**Equilibrium situations under concurrent forces**	◯ Understand and use momentum, constant force, non-constant force, resultant force, action and reaction.
10.3	**Non-equilibrium situations under concurrent forces**	◯ Understand and use momentum, constant force, non-constant force, resultant force, action and reaction. ◯ Understand and use motion of a body in non-equilibrium situations under concurrent forces.
10.4	**Forces that depend on time**	◯ Understand and use the expressions $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to represent the acceleration of an object moving in a straight line.
10.5	**Forces that depend on velocity**	◯ Understand and use the expressions $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to represent the acceleration of an object moving in a straight line.
10.6	**Forces that depend on displacement**	◯ Understand and use the expressions $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to represent the acceleration of an object moving in a straight line. ◯ Model and solve problems that involve motion in a straight line with both constant and non-constant acceleration, including simple harmonic motion, vertical motion under gravity with and without air resistance, and motion of a body in non-equilibrium situations on a smooth inclined plane (excluding situations with pulleys and connected bodies). • If $\frac{d^2x}{dt^2} = -\omega^2 x$ then $x = A\sin(\omega t + \alpha)$ or $x = A\cos(\omega t + \beta)$ • $v^2 = \omega^2\left(A^2 - x^2\right)$ • $T = \frac{2\pi}{\omega}$ • $f = \frac{1}{T}$

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LESSON
10.2 Equilibrium situations under concurrent forces

SYLLABUS LINKS

- Understand and use momentum, constant force, non-constant force, resultant force, action and reaction.

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Statics is the study of the **equilibrium** of a particle under the action of concurrent coplanar forces. A review of Unit 1, Topics 3 and 4 (Vectors in the plane and Algebra of vectors in two dimensions) might be useful here.

While not assessable, understanding statics concepts helps the understanding of the modelling of motion, and dynamics, the study of the causes of the changes in motion, which is assessable. Vector methods are used to analyse equilibrium and non-equilibrium situations.

10.2.1 Definitions

Particle

A particle is a body or an object with a mass such that all of the mass acts through one point. Large objects such as cars, trains or desks are still regarded as point particles for the purposes of this chapter. It will always be assumed that a particle is uniform or a point particle, so that in problems a square or a rectangle can be drawn to represent the object.

Forces

A force can be defined as a push or a pull. A force is an action which tries to alter the state of the motion of a particle. For example, pushing a car with a certain force may move the car, but if the push is not strong enough, it will not move.

A force has both magnitude and direction and is a vector quantity. The magnitude of the vector $\underset{\sim}{F}$ is expressed as F (i.e. without the vector tilde). The standard unit of force is the newton, N. This is named after Sir Isaac Newton (1642–1727) who postulated several laws of motion.

Weight force

The mass and weight of a particle are different. The mass of a particle is defined as the amount of substance contained in the body. The mass of a particle is constant and is usually measured in kilograms or grams. Mass is a scalar quantity. Weight is the force with which gravity pulls on a mass. The weight of a particle is a variable quantity and changes with the change in position and location due to the acceleration of the gravity acting on it. Think of a particle on Earth compared to the Moon. The weight (force) of the particle is reduced because gravity is weaker there, but the mass is still the same.

Weight is a vector quantity and is defined to be the downwards force exerted in a gravitational field on a particle. For a particle of mass m kg, this weight force is m kgwt or mg newtons and always acts vertically downwards, where $|\underset{\sim}{g}| = 9.8\ \text{m/s}^2$ is the acceleration due to gravity in Earth's gravitational field.

A **weight force** acts on all bodies near Earth's surface. The units for a weight force can be either kilogram-weight or newtons; the International System of Units (SI) for force, is the newton. The weight force is generally denoted $\underset{\sim}{w}$. It can also be written as $\boldsymbol{w}$.

The conversion is 1 kgwt $= g$N

Tension

A taut string has a tension in it denoted by T.

The tension is the same at all points in the string and this tension is unaltered if the string passes over a smooth hook or pulley.

Note: In this discussion, only light strings are considered, so that the mass of the string can be ignored. Also, strings are assumed to be inextensible, so that when the string is pulled tight it is taut.

10.2.2 Equilibrium

A system of concurrent coplanar forces acting on a particle is a set of two-dimensional forces all passing through the same point. The equilibrium of a particle means that the vector sum of all the forces acting on the particle is zero. The forces cancel each other out; the particle is said to be in static equilibrium, and the acceleration of the particle is zero.

When a particle is in equilibrium and is acted upon by only two forces, $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$, then $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 = \underset{\sim}{0}$ and $\underset{\sim}{F}_1 = -\underset{\sim}{F}_2$, so the two forces act in a straight line but in opposite directions. If a particle is in equilibrium and has two forces acting on it that are not in a straight line, then there must be another, or third, force that balances these forces. So when a particle is acted upon by three forces, $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$, and the particle is in equilibrium, the vector sum of these three forces must be zero: $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$. In this case these three concurrent forces can be represented as the sides of a closed triangle, and the vector sum when the forces are placed head to tail is zero. In statics, we use the notation $\sum \underset{\sim}{F} = \underset{\sim}{0}$.

Trigonometry and Pythagoras' theorem can be used to solve many force problems.

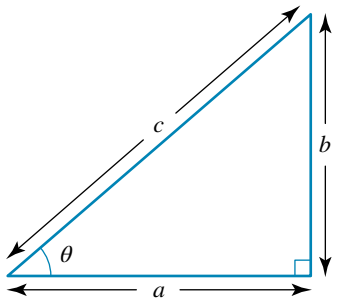
Throughout this chapter answers should be given as exact values where appropriate; otherwise, they should be rounded correct to 2 decimal places. For calculations involving inexact values, retain the decimal places throughout the working, otherwise the final answer will be affected by the rounding process.

Trigonometry (right-angled triangles)

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$



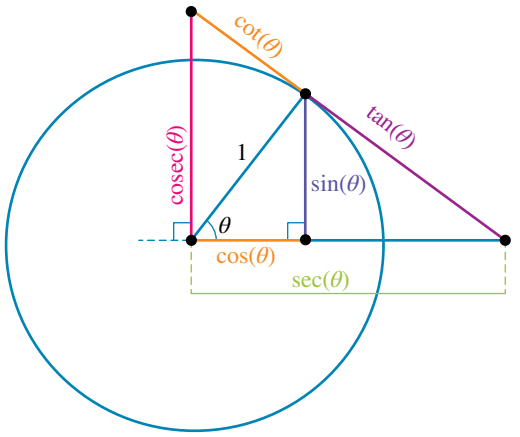
Pythagoras' theorem

$$c^2 = a^2 + b^2$$

$$\text{cosec}(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

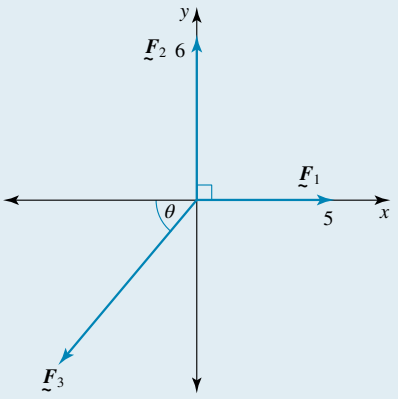
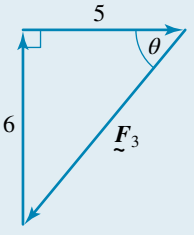
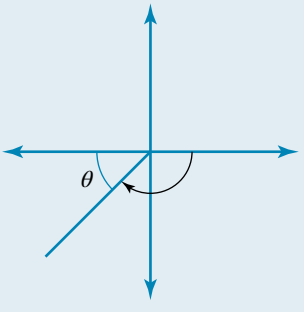


θ	0	$30°\left(\frac{\pi}{6}\right)$	$45°\left(\frac{\pi}{4}\right)$	$60°\left(\frac{\pi}{3}\right)$	$90°\left(\frac{\pi}{3}\right)$	$180°\ (\pi)$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0

Note that you can review Unit 2 Topic 4 (Chapter 9 of the Units 1 and 2 title of this series) to review Pythagorean identities and how to determine exact values of reciprocal trigonometric functions.

WORKED EXAMPLE 1 Simple equilibrium using Pythagoras' theorem

A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 5 newtons; another force acts at right angles to the first force and has a magnitude of 6 newtons. Determine the third force.

THINK	WRITE/DRAW
1. Draw a force diagram with the three forces acting through a point. Since the particle is in equilibrium, the third force, $\underset{\sim}{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.	
2. Because the particle is in equilibrium, the vector sum of three forces is zero $\sum \underset{\sim}{F} = \underset{\sim}{0}$ and they form a closed triangle.	Redraw the diagram. 
3. Use Pythagoras' theorem to determine the magnitude of the third force.	$\lvert\underset{\sim}{F}_3\rvert^2 = 6^2 + 5^2$ $= 61$
4. State the magnitude of the third force.	$\lvert\underset{\sim}{F}_3\rvert = \sqrt{61}$
5. Calculate the direction of the third force.	$\tan(\theta) = \dfrac{6}{5}$
6. Solve for θ.	$\theta = \tan^{-1}\left(\dfrac{6}{5}\right)$ $= 50.19°$
7. Consider the direction in the context of the question. $\theta = 180° - 50.19°$ $= 129.81°$	
8. State the final result.	The third force has a magnitude of $\sqrt{61}$ newtons and acts at an angle of 129.81° clockwise from the horizontal force, with the force of 5 newtons.

10.2.3 Angles other than right angles

If the forces are not at right angles, problems can be solved by using the sine and cosine rules, or by resolving the vectors in useful directions.

Trigonometry (angles other than right angles)

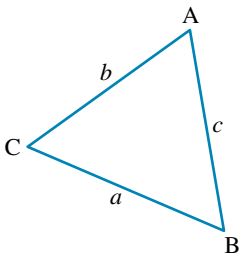
Sine rule

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$



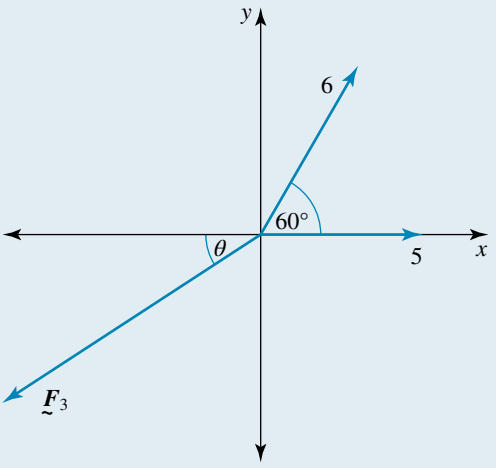
WORKED EXAMPLE 2 Simple equilibrium using sine and cosine rules

A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 5 newtons; the second force, of magnitude 6 newtons, acts at 60° with the first force. Determine the third force by the sum of vectors.

THINK

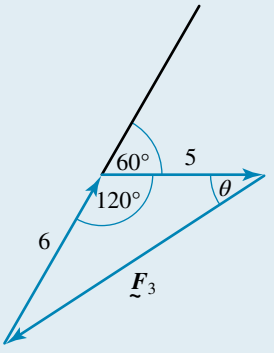
1. Draw a force diagram with the three forces acting through a point. Because the particle is in equilibrium, the third force, $\underset{\sim}{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.

WRITE/DRAW



2. Because the particle is in equilibrium, the vector sum of three forces is zero $\sum \underset{\sim}{F} = \underset{\sim}{0}$ and they form a closed triangle.

Redraw the diagram.



3. Use the cosine rule to determine the magnitude of the third force.

$$|\underset{\sim}{F}_3|^2 = 6^2 + 5^2 - 2 \times 6 \times 5\cos(120°)$$
$$= 91$$

4. State the magnitude of the third force.

$$|\underset{\sim}{F}_3| = \sqrt{91}$$

5. To calculate the direction of the third force, use the sine rule.

$$\frac{F_3}{\sin(120°)} = \frac{6}{\sin(\theta)}$$

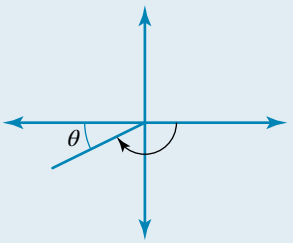
6. Solve for $\sin(\theta)$.

$$\sin(\theta) = \frac{6\sin(120°)}{F_3}$$
$$= \frac{3\sqrt{3}}{\sqrt{91}}$$

7. Calculate the angle θ.

$$\theta = \sin^{-1}\left(\frac{3\sqrt{3}}{\sqrt{91}}\right)$$
$$\approx 33°$$

8. Consider the direction in the context of the question.
$\theta = 180° - 33°$
$= 147°$



9. State the final result.

The third force has a magnitude of $\sqrt{91}$ newtons and acts at an angle of 147° clockwise from the horizontal force, with the force of 5 newtons.

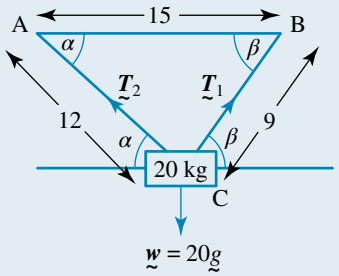
WORKED EXAMPLE 3 Equilibrium using sine and cosine rules

A particle of mass 20 kg is supported by two strings of lengths 12 cm and 9 cm. The ends of the string are attached to two fixed points 15 cm apart on the same horizontal level. Determine the tensions in the strings by the sum of vectors.

THINK

1. Draw the force diagram. Let the two fixed points be A and B and the position of the particle be point C. Let $\angle ABC = \alpha$ and $\angle ABC = \beta$. Let the tension in the 12 cm string be T_2 and let the tension in the 9 cm string be T_1. Include the weight force, $20g$. Note that all the forces in the diagram are in newtons.

WRITE/DRAW



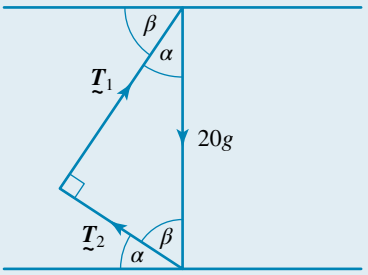
2. The triangle is a right-angled triangle.

Recognise the Pythagorean triad (9, 12, 15), $12^2 + 9^2 = 15^2$, ΔACB is a right-angled triangle. $\angle ACB = 90°$ and $\alpha + \beta = 90°$.

3. Determine the trigonometric ratios. $\sin(\beta) = \cos(\alpha) = \frac{12}{15}$ and $\cos(\beta) = \sin(\alpha) = \frac{9}{15}$

4. Since the particle is in equilibrium, the vector sum of three forces is zero $\sum \underset{\sim}{F} = \underset{\sim}{0}$ and they form a closed triangle. Note that the lengths of the strings do not represent the magnitudes of the tensions in the strings; however, the tensions act along the line and direction of the strings.

Draw the triangle of forces.



5. Use the sine rule. $\frac{T_1}{\sin(\beta)} = \frac{T_2}{\sin(\alpha)} = \frac{20g}{\sin(90°)}$

6. Since $\sin(90°) = 1$, calculate the tension in the shorter string. $T_1 = 20g\sin(\beta)$

7. Substitute for the trigonometric ratio and evaluate the tension in the shorter string.

$$\begin{aligned} T_1 &= \frac{20g \times 12}{15} \\ &= 16g \\ &= 16 \times 9.8 \\ &= 156.8 \end{aligned}$$

8. Calculate the tension in the longer string. $T_2 = 20g\sin(\alpha)$

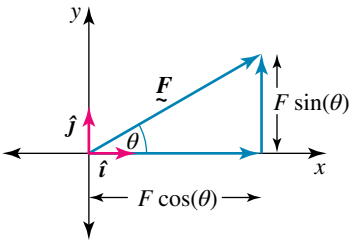
9. Substitute for the trigonometric ratio and evaluate the tension in the longer string.

$$\begin{aligned} T_2 &= \frac{20g \times 9}{15} \\ &= 12g \\ &= 12 \times 9.8 \\ &= 117.6 \end{aligned}$$

10. State the final result. The tension in the 9 cm string is 156.8 newtons and the tension in the 12 cm string is 117.6 newtons.

10.2.4 Resolving force vectors

Consider a force $\underset{\sim}{F}$ making an angle of θ with the positive x-axis.



Resolving a vector means splitting the vector up into its horizontal and vertical components. Because $\hat{\imath}$ is a unit vector in the x-direction and $\hat{\jmath}$ is a unit vector in the y-direction, $\underset{\sim}{F} = F\cos(\theta)\hat{\imath} + F\sin(\theta)\hat{\jmath}$.

In two dimensions, the equilibrium of a particle means that the vector sum of all the resolved horizontal components of the forces is equal to zero $\sum \underset{\sim}{F} = \underset{\sim}{0}$. In the $\hat{\imath}$ direction, $\sum F_x\hat{\imath} = 0\hat{\imath}$. The vector sum of all the resolved vertical components of the forces in a perpendicular direction is equal to zero, from the $\hat{\jmath}$ component: $\sum F_y\hat{\jmath} = 0\hat{\jmath}$. Now we will repeat the first three worked examples, but this time, we will solve them using the method of resolution of force vectors.

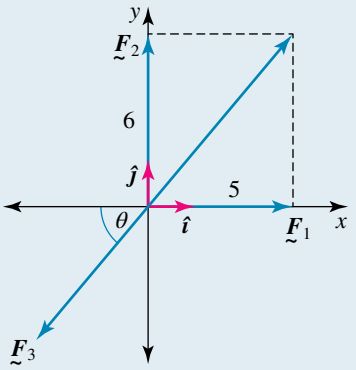
WORKED EXAMPLE 4 Simple equilibrium by resolving force vectors

A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 5 newtons; another force acts at right angles to the first force and has a magnitude of 6 newtons. By resolving the forces, determine the third force.

THINK

1. Draw a force diagram with the three forces acting through a point. Since the particle is in equilibrium, the third force, $\underset{\sim}{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.

WRITE/DRAW



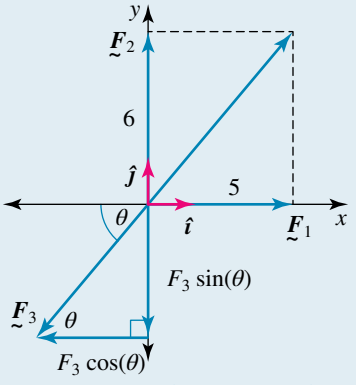
2. Express the given horizontal force in terms of $\hat{\imath}$.

$\underset{\sim}{F}_1 = 5\hat{\imath}$

3. Express the given vertical force in terms of $\hat{\jmath}$.

$\underset{\sim}{F}_2 = 6\hat{\jmath}$

4. Resolve the force $\underset{\sim}{F}_3$ into its horizontal and vertical components.



$\underset{\sim}{F}_3 = -F_3 \cos(\theta)\hat{\imath} - F_3 \sin(\theta)\hat{\jmath}$

5. Because the particle is in equilibrium $\sum \underset{\sim}{F} = \underset{\sim}{0}$ and by addition of vectors, $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$.

$$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$$
$$(5 - F_3 \cos(\theta))\,\hat{\imath} + (6 - F_3 \sin(\theta))\,\hat{\jmath} = \underset{\sim}{0}$$
$$= 0\hat{\imath} + 0\hat{\jmath}$$

6. By resolving horizontally, the $\hat{\imath}$ component must be zero.

$$5 - F_3 \cos(\theta) = 0$$
$$F_3 \cos(\theta) = 5 \qquad [1]$$

7. By resolving vertically, the $\hat{\jmath}$ component must be zero.

$$6 - F_3 \sin(\theta) = 0$$
$$F_3 \sin(\theta) = 6 \qquad [2]$$

8. To determine F_3 eliminate θ from [1] and [2].

Square both equations.
$[1]^2 : F_3^2 \cos^2(\theta) = 25$
$[2]^2 : F_3^2 \sin^2(\theta) = 36$

9. Now add these equations, since $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$F_3^2 \cos^2(\theta) + F_3^2 \sin^2(\theta) = 25 + 36$$
$$F_3^2(\cos^2(\theta) + \sin^2(\theta)) = 61$$
$$F_3^2 = 61$$

10. State the magnitude of the third force.

$$F_3 = \sqrt{61}$$

11. To calculate θ, the direction of the third force, divide the equations to eliminate F_3.

$$\frac{[2]}{[1]} \Rightarrow \tan(\theta) = \frac{6}{5}$$

12. Solve for θ.

$$\theta = \tan^{-1}\left(\frac{6}{5}\right)$$
$$= 50.19°$$

13. State the final result. Note that the angles are measured anticlockwise from the positive $\hat{\imath}$ directions.

The third force has a magnitude of $\sqrt{61}$ newtons and acts at an angle of $-129.81°$ with the force of 5 newtons.

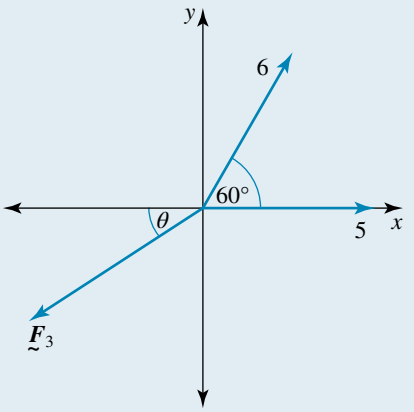
WORKED EXAMPLE 5 Equilibrium by resolving force vectors (1)

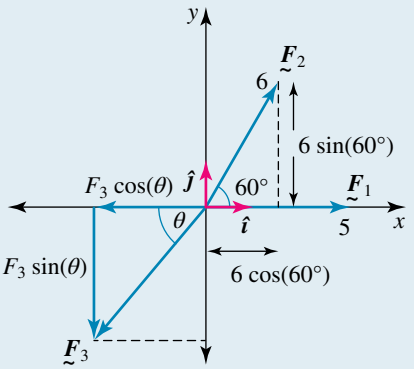
A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 5 newtons; the second force, of magnitude 6 newtons, acts at 60° with the first force. By resolving the forces into components, determine the third force.

THINK

1. Draw a force diagram and resolve the forces into their horizontal and vertical components.

WRITE/DRAW





2. Express the given horizontal force $\underset{\sim}{F}_1$ in terms of $\hat{\boldsymbol{\imath}}$.

$$\underset{\sim}{F}_1 = 5\,\hat{\boldsymbol{\imath}}$$

3. $\underset{\sim}{F}_2$ has components $F_2\cos(60°)$ horizontally to the right and $F_2\sin(60°)$ vertically upwards.

$$\begin{aligned}\underset{\sim}{F}_2 &= 6\cos(60°)\hat{\boldsymbol{\imath}} + 6\sin(60°)\hat{\boldsymbol{\jmath}}\\ &= 3\hat{\boldsymbol{\imath}} + 3\sqrt{3}\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

4. Resolve the force $\underset{\sim}{F}_3$ into its horizontal and vertical components, noting that both are in the opposite directions to $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.

$$\underset{\sim}{F}_3 = -F_3\cos(\theta)\hat{\boldsymbol{\imath}} - F_3\sin(\theta)\hat{\boldsymbol{\jmath}}$$

5. Since the particle is in equilibrium, use addition of vectors.

$$\begin{aligned}\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 &= \underset{\sim}{0}\\ (8 - F_3\cos(\theta))\hat{\boldsymbol{\imath}} + (3\sqrt{3} - F_3\sin(\theta))\hat{\boldsymbol{\jmath}} &= \underset{\sim}{0}\\ &= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}\end{aligned}$$

6. By resolving horizontally, the $\hat{\boldsymbol{\imath}}$ component must be zero.

$$\begin{aligned}8 - F_3\cos(\theta) &= 0\\ F_3\cos(\theta) &= 8 \quad [1]\end{aligned}$$

7. By resolving vertically, the $\hat{\boldsymbol{\jmath}}$ component must be zero.

$$\begin{aligned}3\sqrt{3} - F_3\sin(\theta) &= 0\\ F_3\sin(\theta) &= 3\sqrt{3} \quad [2]\end{aligned}$$

8. To determine F_3 eliminate θ from [1] and [2].

Square both equations.

$[1]^2$: $F_3^2\cos^2(\theta) = 64$

$[2]^2$: $F_3^2\sin^2(\theta) = 27$

9. Now add these equations, since $\cos^2(\theta) + \sin^2(\theta) = 1$.

$$\begin{aligned}F_3^2\cos^2(\theta) + F_3^2\sin^2(\theta) &= 64 + 27\\ F_3^2(\cos^2(\theta) + \sin^2(\theta)) &= 91\\ F_3^2 &= 91\end{aligned}$$

10. State the magnitude of the force.

$$F_3 = \sqrt{91}$$

11. To calculate θ, the direction of the third force, divide the equations to eliminate F_3.

$$\frac{[2]}{[1]} \Rightarrow \tan(\theta) = \frac{3\sqrt{3}}{8}$$

12. Solve for θ.

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{3\sqrt{3}}{8}\right)\\ &= 33°\end{aligned}$$

13. State the final result.

The third force has a magnitude of $\sqrt{91}$ newtons and acts at an angle of 147° with the force of 5 newtons.

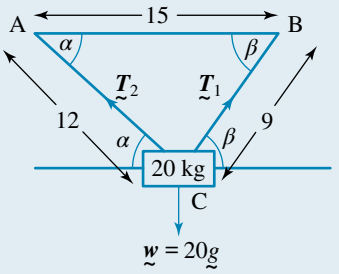
WORKED EXAMPLE 6 Equilibrium by resolving force vectors (2)

A particle of mass 20 kg is supported by two strings of lengths 12 cm and 9 cm. The ends of the string are attached to two fixed points 15 cm apart on the same horizontal level. By resolving the forces into components, determine the tensions in the strings.

THINK

1. Draw the force diagram. Let the two fixed points be A and B and the position of the particle be point C. Let $\angle BAC = \alpha$ and $\angle ABC = \beta$. Let the tension in the 12 cm string be T_2 and let the tension in the 9 cm string be T_1. Include the weight force $20g$. Note that all the forces in the diagram are in newtons.

WRITE/DRAW



2. The triangle is a right-angled triangle.

Recognise the Pythagorean triad $(9, 12, 15)$, $12^2 + 9^2 = 15^2$, ΔACB is right angled. $\angle ACB = 90°$ and $\alpha + \beta = 90°$.

3. Determine the trigonometric ratios.

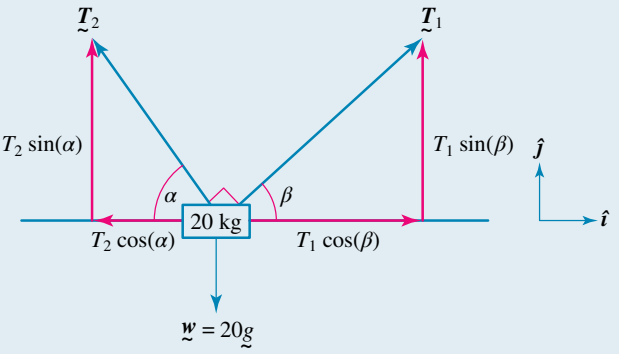
$\sin(\beta) = \cos(\alpha) = \dfrac{12}{15}$ and

$\cos(\beta) = \sin(\alpha) = \dfrac{9}{15}$

4. The particle is in equilibrium so the sum of the three forces is zero.

$$\sum \underset{\sim}{F} = \underset{\sim}{0}$$
$$= \underset{\sim}{T}_1 + \underset{\sim}{T}_2 + 20\underset{\sim}{g}$$

5. Set positive $\hat{\imath}$ to the right and positive $\hat{\jmath}$ up.
6. $\underset{\sim}{T}_1$ has components $T_1\cos(\beta)\,\hat{\imath}$ and $T_1\sin(\beta)\,\hat{\jmath}$. $\underset{\sim}{T}_2$ has components $-T_2\cos(\alpha)\,\hat{\imath}$ and $T_2\sin(\alpha)\,\hat{\jmath}$.



7. Resolving horizontally, equate the forces in the $\hat{\imath}$ direction.

$$(T_1\cos(\beta) - T_2\cos(\alpha))\,\hat{\imath} = 0\hat{\imath}$$
$$T_1\cos(\beta) - T_2\cos(\alpha) = 0 \qquad [1]$$

8. Resolving vertically, equate the forces in the $\hat{\jmath}$ direction.

$$(T_2\sin(\alpha) + T_1\sin(\beta) - 20g)\,\hat{\jmath} = 0\hat{\jmath}$$
$$T_2\sin(\alpha) + T_1\sin(\beta) - 20g = 0 \qquad [2]$$

9. Substitute for the trigonometric ratios into [1] and [2].

$$\frac{9T_1}{15} - \frac{12T_2}{15} = 0 \qquad [1]$$
$$\frac{9T_2}{15} + \frac{12T_1}{15} - 20g = 0 \qquad [2]$$

10. From [1], express T_2 in terms of T_1.

$$T_2 = \frac{3T_1}{4}$$

11. Simplify [2].

$$9T_2 + 12T_1 = 300g$$

12. Substitute for T_2 and solve for T_1.

$$9\left(\frac{3T_1}{4}\right) + 12T_1 = 300g$$
$$\frac{T_1(27+48)}{4} = 300g$$
$$75T_1 = 1200g$$

13. State the value of T_1.

$$T_1 = \frac{1200g}{75}$$
$$= 16g$$
$$= 16 \times 9.8$$
$$= 156.8$$

14. Substitute for T_1 and evaluate T_2.

$$T_2 = \frac{3T_1}{4}$$
$$= \frac{3}{4} \times 16g$$
$$= 12g$$
$$= 12 \times 9.8$$
$$= 117.6$$

15. State the final result.

The tension in the 9 cm string is 156.8 newtons and the tension in the 12 cm string is 117.6 newtons.

Note: The shorter string, the one closest to the vertical, carries more tension.

10.2.5 Resolving all the forces

When there are more than three forces acting, resolve the forces.

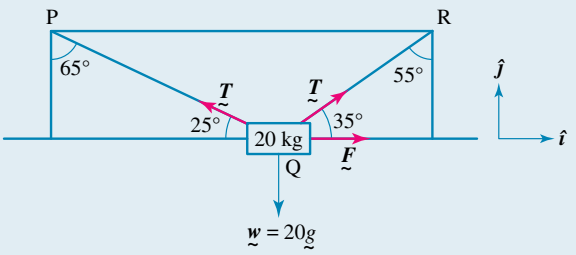
WORKED EXAMPLE 7 Equilibrium with more than three forces

A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of 20 kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of F newtons until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 65° and 55° to the vertical respectively. Determine the value of F and the tension in the strings.

THINK

1. Because the hook is smooth, the tension is unaltered when passing over it, so let the tension in the strings PQ and QR both be T. We need to determine the values of T and F. Note that all the forces in the diagram are in newtons, and positive $\hat{\imath}$ and $\hat{\jmath}$ directions are specified.

WRITE/DRAW

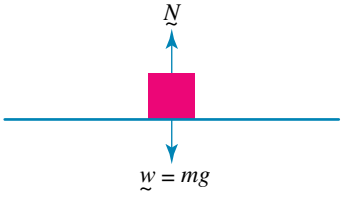


2. The system is in equilibrium.	$\sum \underset{\sim}{F} = \underset{\sim}{0}$ $2\underset{\sim}{T} + \underset{\sim}{F} + \underset{\sim}{w} = \underset{\sim}{0}$
3. The tension in the string QR has a $\hat{\imath}$ $T\cos(35°)$ and a $\hat{\jmath}$ $T\sin(35°)$.	$\overrightarrow{QR} = T\cos(35°)\hat{\imath} + T\sin(35°)\hat{\jmath}$
4. The tension in the string PQ has a $\hat{\imath}$ component $-T\cos(25°)$ and a $\hat{\jmath}$ component $T\sin(25°)$.	$\overrightarrow{PQ}$: $T\cos(25°)\hat{\imath} + T\sin(25°)\hat{\jmath}$
5. Resolve the forces in the $\hat{\imath}$ direction.	$(F + T\cos(35°) - T\cos(25°)\hat{\imath} = 0\hat{\imath}$ [1]
6. Resolve the forces in the $\hat{\jmath}$ direction.	$(T\sin(35°) + T\sin(25°) - 20g)\hat{\jmath} = 0\hat{\jmath}$ [2]
7. Use [2] to factor and solve for T.	$T(\sin(35°) + \sin(25°)) = 20g$ $T = \dfrac{20g}{\sin(35°) + \sin(25°)}$ ≈ 196.75
8. State the value of the tension in the strings.	The tension in the strings PQ and QR is 196.75 newtons.
9. Use [1] to determine F.	$F = T\cos(25°) - T\cos(35°)$ $= T(\cos(25°) - \cos(35°))$
10. Substitute for T.	$F = \dfrac{20g(\cos(25°) - \cos(35°))}{(\sin(35°) + \sin(25°))}$
11. State the value of F.	$F \approx 17.15$ N

10.2.6 Action and reaction forces

If we think of a computer resting on a desk, the weight force is not the only force acting on the computer, for if the desk were not there, the computer would fall vertically downwards.

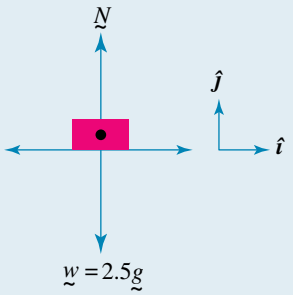
Newton's third law of motion states that for every action there is an equal and opposite reaction. That is, if a body exerts a force by resting on a desk, then the desk exerts a force of equal magnitude but in the opposite direction. This type of force is called the **normal reaction** and will generally be denoted by $\underset{\sim}{N}$. The upward force is the reaction of the desk on the computer or object.



The normal reaction is always perpendicular to the surface and only acts when a body is in contact with another surface.

WORKED EXAMPLE 8 Normal reaction on a flat plane

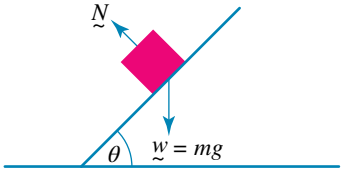
An object is in equilibrium, sitting on a bench, and is acted upon by two forces, the weight force and the normal reaction. Given that the mass of the object is 2.5 kg, calculate the normal reaction being exerted by the bench on the object.

THINK	WRITE
1. Draw the force diagram showing the two forces and designating the $\hat{\imath}$ and $\hat{\jmath}$ directions.	
2. The system is in equilibrium, so $\sum \underset{\sim}{F} = \underset{\sim}{0}$.	$\underset{\sim}{N} + \underset{\sim}{w} = \underset{\sim}{0}$
3. Resolve the forces in the $\hat{\jmath}$ direction.	$(N - 2.5g)\hat{\jmath} = 0\hat{\jmath}$ $N = 2.5g$ $\approx 24.5\text{ N}$
4. State the value of $\underset{\sim}{N}$.	The normal reaction is 24.5 N upwards.

10.2.7 Inclined planes

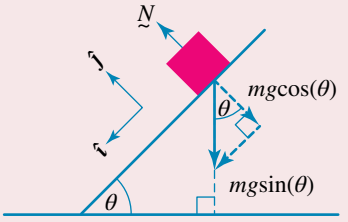
Consider a particle of mass m kg on a smooth plane inclined at an angle of θ to the horizontal. A smooth plane means that there are no frictional forces acting. The particle is being acted on by two external forces: the normal reaction N, which is perpendicular to the plane, and the weight force, which acts vertically downwards.

When dealing with problems involving inclined planes, it is usually easier to resolve parallel and perpendicular to the plane rather than horizontally and vertically.



Weight force on an inclined plane

The weight force has components $mg\sin(\theta)$ parallel to and down the plane and $mg\cos(\theta)$ perpendicular to the plane.

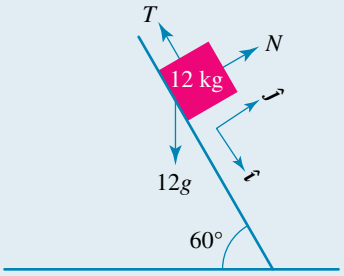
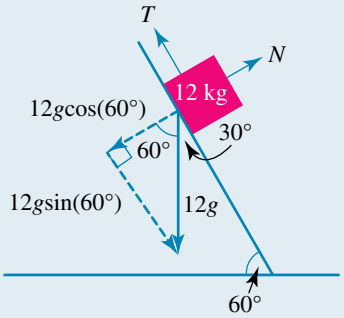


Note that if these were the only forces acting on the particle, then it could not be in equilibrium. In that situation the particle would slide down the plane.

It makes sense, in the case of an inclined plane, to set up the positive directions of the $\hat{\imath}$ and $\hat{\jmath}$ components of the force vectors in terms of parallel and perpendicular to the plane respectively. This makes most of the forces easier to analyse. In simple problems, the weight force ($\underset{\sim}{w}$) may be the only force that needs to be resolved in $\hat{\imath}$ and $\hat{\jmath}$ components.

WORKED EXAMPLE 9 Simple inclined plane in equilibrium

A suitcase of mass 12 kg is maintained in equilibrium on a smooth plane inclined at 60° to the horizontal by a light strap. Calculate the tension in the strap and the reaction of the plane if the strap is parallel to and acting up the line of greatest slope.

THINK	WRITE/DRAW
1. Draw the diagram and mark in the forces on the suitcase. Let T be the tension in the strap, acting up the line of the slope. $12g$ is the weight force and N is the normal reaction. Note that the normal reaction is perpendicular to the plane, in the $\hat{\jmath}$ direction, and the $\hat{\imath}$ direction is parallel to the plane. All forces in the diagram are in newtons.	
2. Resolve the weight force into its components parallel and perpendicular to the plane.	 $\underset{\sim}{w} = (12g\sin(60°))\hat{\imath} - (12g\cos(60°))\hat{\jmath}$
3. The suitcase is in equilibrium so the sum of the three forces is zero.	$\sum \underset{\sim}{F} = \underset{\sim}{0}$ $\underset{\sim}{w} + \underset{\sim}{T} + \underset{\sim}{N} = \underset{\sim}{0}$
4. Resolve the forces parallel to the plane. The tension force is up the plane and the component of the weight force acts down the plane.	$(12g\sin(60°) - T)\hat{\imath} = 0\hat{\imath}$ [1]
5. Resolve the forces perpendicular to the plane. The normal reaction and the component of the weight force act perpendicular to the plane.	$(N - 12g\cos(60°))\hat{\jmath} = 0\hat{\jmath}$ [2]
6. Use [1] to solve for T.	From [1]: $T = 12g\sin(60°)$ $= 12g \times \frac{\sqrt{3}}{2}$ $\approx 101.84\text{ N}$
7. State the value of T.	The tension in the strap is approximately 101.84 N.

8. Use [2] to solve for N.	From [2]: $N = 12g\cos(60°)$ $= 12g \times \frac{1}{2}$ $= 6g$ $\approx 58.80\,\text{N}$
9. State the value of N.	The normal reaction is approximately 50.80 N.

Resolve all the forces

When two or more forces have components, it is a common mistake to forget to determine the resolved components of *all* the forces both parallel and perpendicular to the plane.

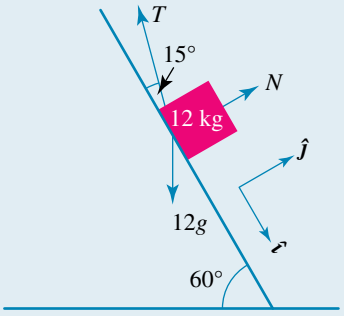
WORKED EXAMPLE 10 Inclined plane in equilibrium

A suitcase of mass 12 kg is maintained in equilibrium on a smooth plane inclined at 60° to the horizontal by means of a strap. Calculate the tension in the strap and the reaction of the plane if the strap is inclined at an angle of 15° to the plane.

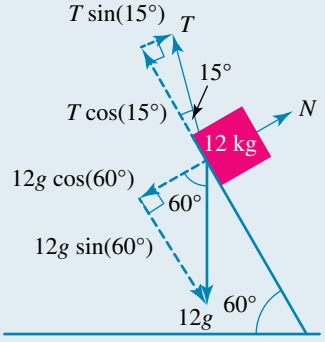
THINK

WRITE/DRAW

1. Draw the diagram and mark in the forces on the suitcase. Let T be the tension in the strap, acting at an angle to the plane. $12g$ is the weight force and N is the normal reaction. All forces in the diagram are in newtons.
Set the positive direction of the $\hat{\imath}$ component parallel to the plane, and the positive direction of the $\hat{\jmath}$ component perpendicular to the plane.



2. Resolve the weight force and the tension into their components parallel and perpendicular to the plane.



3. The suitcase is in equilibrium so the sum of the three forces is zero.

$$\sum \underset{\sim}{F} = \underset{\sim}{0}$$

$$\underset{\sim}{w} + \underset{\sim}{T} + \underset{\sim}{N} = \underset{\sim}{0}$$

4. Resolve the forces parallel the plane. The component of the tension force is up the plane and the component of the weight force acts down the plane.

$$(12g\sin(60°) - T\cos(15°))\hat{\imath} = 0\hat{\imath} \qquad [1]$$

5. Resolve the forces perpendicular to the plane. The normal reaction and the component of the tension act up and the component of the weight force acts down, perpendicular to the plane.	$(N + T\sin(15°) - 12g\cos(60°))\hat{\boldsymbol{j}} = 0\hat{\boldsymbol{j}}$ [2]
6. Use [1] to solve for T.	From [1]: $T\cos(15°) = 12g\sin(60°)$ $T = \dfrac{12g\sin(60°)}{\cos(15°)}$
7. State the value of T.	The tension in the strap is approximately 105.44 newtons.
8. Use [2] to solve for N.	From [2]: $N = 12g\cos(60°) - T\sin(15°)$ $= 12g \times \dfrac{1}{2} - 105.44 \times \sin(15°)$
9. State the value of N.	The normal reaction is approximately 31.51 N.

Note that, in Worked examples 9 and 10, the normal reaction and tension forces have different values.

Exercise 10.2 Equilibrium situations under concurrent forces

learnon

10.2 Exercise | **10.2 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!
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Simple familiar

1. WE1 A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 7 newtons; another force acts at right angles to this force and has a magnitude of 8 newtons. Determine the third force.

2. A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of F newtons; another force acts at right angles to this force and has a magnitude of $2F$ newtons. Determine the third force.

3. **a.** A body is in equilibrium under the action of three forces, $\underset{\sim}{F}_1 = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$, $\underset{\sim}{F}_2 = x\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ and $\underset{\sim}{F}_3 = 6\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. Determine the values of x and y.

b. A body is in equilibrium under the action of three forces, $\underset{\sim}{F}_1 = -2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$, $\underset{\sim}{F}_2 = 6\hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}}$ and $\underset{\sim}{F}_3 = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. Determine the values of x and y and the magnitude of the force $\underset{\sim}{F}_3$.

4. WE2 & WE5 A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 7 newtons; a second force of magnitude 8 newtons acts at 30° with the first force. Determine the third force

a. by the sum of vectors

b. by resolving the forces into components.

5. A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of F newtons; a second force of magnitude $2F$ newtons acts at 45° with the first force. Determine the third force

   a. by the sum of vectors
   b. by resolving the forces into components.

6. WE3 & WE6 A particle of mass 10 kg is supported by two strings of lengths 5 cm and 12 cm. The ends of the string are attached to two fixed points 13 cm apart on the same horizontal level. Determine the tensions in the strings

   a. by the sum of vectors
   b. by resolving the forces into components.

7. A particle of mass 5 kg is supported by two strings of lengths 7 cm and 24 cm. The ends of the string are attached to two fixed points 25 cm apart on the same horizontal level. Determine the tensions in the strings

   a. by the sum of vectors
   b. by resolving the forces into components.

8. WE4 A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 7 newtons; another force acts at right angles to this force and has a magnitude of 8 newtons. By resolving the forces, determine the third force.

9. WE7 A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of 5 kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of F newtons, until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 60° and 30° to the vertical respectively. Calculate the value of F and the tension in the strings.

10. WE8 An object is in equilibrium, sitting on a bench, and is acted upon by two forces, the weight force and the normal reaction. Given that the mass of the object is 1.6 kg, calculate the normal reaction being exerted by the bench on the object. Give your answer to 1 decimal place.

11. WE9 A parcel of mass 5 kg is maintained in equilibrium on a smooth plane inclined at 40° to the horizontal by means of a force acting up and parallel to the plane. Calculate the magnitude of this force and the reaction of the plane on the parcel.

12. WE10 A parcel of mass 5 kg is maintained in equilibrium on a smooth plane inclined at 40° to the horizontal by means of a force acting at an angle of 20° to the plane. Calculate the magnitude of this force and the reaction of the plane.

Complex familiar

13. a. A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 30 newtons; the second force, with magnitude 60 newtons, acts at 40° with the first force. Determine the third force.
    b. A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 120 newtons; the second force, with magnitude 50 newtons, acts at an angle of 140° clockwise with the first force. Determine the third force.

14. a. A particle of mass 4 kg is supported by a string AB attached to a fixed point A and is pulled aside from the vertical by a horizontal force of F newtons acting on the particle at B. If AB makes an angle of 35° with the vertical in the equilibrium position, calculate:

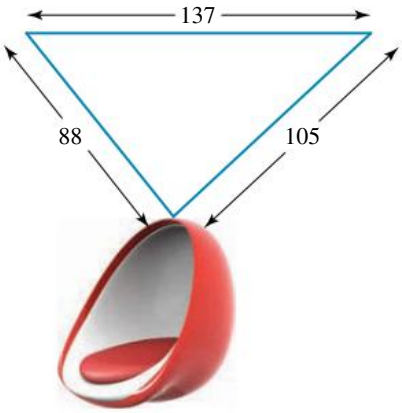
       i. the value of F
       ii. the magnitude of the tension in the string.

    b. A child of mass m kg is sitting on a swing that is held in equilibrium by a horizontal force of 200 newtons applied by his father. At this time, the rope attached to the swing makes an angle of 40° with the vertical. Calculate the tension in the rope and the mass of the child.

15. a. A painting of mass 600 g is to be hung on a wall. It is supported by a string whose ends are attached to two fixed points on the wall at the same horizontal level a distance of 80 cm apart. A hook is attached to the painting. When the painting is hanging in equilibrium, the hook is 30 cm directly below the midpoint of the two fixed points. Calculate the tension in the string around the hook.

b. A string of length $2L$ has its ends fixed to two points at the same horizontal level a distance of L apart. A bead of mass m kg is threaded on the string and hangs at the midpoint of the string. Calculate the tension in the string.

16. a. An egg chair is hanging by two inextensible cables of lengths 88 cm and 105 cm. The other ends of the cable are attached to the ceiling at a distance of 137 cm apart. The shorter cable has a tension of T_1 newtons and the longer cable has a tension of T_2 newtons.



i. Show that $T_2 = \dfrac{88\,T_1}{105}$.

ii. The longer cable is strong, but the shorter cable will break if the tension in it exceeds 980 newtons. The egg chair has a mass of 10 kg, and a person of mass m kg sits in it. Determine the maximum value of m for which the egg chair remains in equilibrium.

b. A downlight of mass 250 g is hanging from two cables of lengths 48 cm and 55 cm. The other ends of the cables are attached to the ceiling a distance of 73 cm apart. Show that the tensions in the 48 cm and 55 cm cables have magnitudes of $\dfrac{55g}{292}$ and $\dfrac{48g}{292}$ newtons respectively.

Complex unfamiliar

17. a. A particle of mass 10 kg is attached to one end of a string, and the other end of the string is attached to a fixed point. When in equilibrium, the mass is pulled aside by a horizontal force of $2T$ newtons and the tension in the string is $3T$ newtons. The string makes an angle of θ with the vertical. Determine the angle θ and the value of T.

b. A particle of mass $2m$ kg is attached to one end of a string, and the other end of the string is attached to a fixed point. When in equilibrium, the mass is pulled aside by a horizontal force of $3mg$ newtons and the tension in the string is T newtons. The string makes an angle of θ with the vertical. Determine the angle θ and express T in terms of m.

18. a. A string ABC is tied to two fixed points, A and C, at the same horizontal level. A mass of 10 kg is suspended at B. When in equilibrium, the strings AB and BC make angles of θ and 70° with AC, and the magnitudes of the tensions in the strings AB and BC are T and $2T$ newtons respectively. Determine the angle θ and the value of T.

b. A string PQR is tied to two fixed points, P and R, at the same horizontal level. A mass of 4 kg is suspended at Q. When in equilibrium, the strings PQ and QR make angles of 50° and θ with PR, and the magnitude of the tension in the string PQ is 12 newtons. Determine the angle θ and the tension in the string QR.

19. a. A string is tied to two points at the same horizontal level. A mass of m kg is suspended from a smooth hook and is pulled across horizontally by a force of 10 newtons. When the string makes angles of 55° and 35° to the vertical, the system is in equilibrium. Determine the tension in the string and the mass, m.

b. A string is tied to two points at the same horizontal level. A mass of m kg is suspended from a smooth hook and is pulled across horizontally by a force of F newtons. When in equilibrium, the string makes angles of α and β to the horizontal, where $\beta > \alpha$. Show that

$$\frac{F}{mg} = \frac{\cos(\alpha) - \cos(\beta)}{\sin(\alpha) + \sin(\beta)}$$

20. a. A string PQR is tied to two fixed points P and R at the same horizontal level. A mass of 5 kg is suspended at Q. When in equilibrium, the strings PQ and QR make angles of α and β with PR respectively. The magnitude of the tensions in the strings PQ and QR are 30 and 35 newtons respectively. Determine the angles α and β.

b. A string ABC is tied to two fixed points A and C at the same horizontal level. A mass of $7m$ kg is suspended at B. When in equilibrium, the strings AB and BC make angles of α and β with AC respectively. The magnitude of the tensions in the strings AB and BC are 5 mg and 4 mg newtons respectively. Determine the angles α and β.

Fully worked solutions for this chapter are available online.

LESSON
10.3 Non-equilibrium situations under concurrent forces

SYLLABUS LINKS

- Understand and use momentum, constant force, non-constant force, resultant force, action and reaction.
- Understand and use motion of a body in non-equilibrium situations under concurrent forces.

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10.3.1 Newton's laws of motion

Dynamics is the study of the causes of the motion of particles.

The study of dynamics is based upon Newton's three laws of motion. Sir Isaac Newton (1642–1727) worked in many areas of mathematics and physics. It is believed that he first started studying the effects of gravity after watching an apple fall from a tree.

He wondered why the apple fell, and he wanted to calculate the speed at which it hit the ground. He was also curious about the stars and the movement of the planets in the skies: he asked, were they falling too? He developed his theory of gravitation in, 1666, and in 1686 he published his three laws of motion in the book *Principia Mathematica Philosophiae Naturalis*. These laws can be stated in modern English as follows.

Newton's first law of motion: Every body continues in a state of rest or uniform motion in a straight line unless acted upon by an external force.

Newton's second law of motion: The resultant force acting on a body is proportional to the rate of change of momentum. (This is often restated as $F = ma$, due to some work that was done later.)

Newton's third law of motion: For every action there is an equal but opposite reaction.

10.3.2 Momentum

The concept of **momentum** is fundamental to Newton's second law of motion. The momentum $\underset{\sim}{p}$ of a body of mass m moving with velocity $\underset{\sim}{v}$ is defined by

$$\text{momentum} = \text{mass} \times \text{velocity}$$
$$\underset{\sim}{p} = m\underset{\sim}{v}$$

The standard unit for momentum is kilogram metres/second (kg m/s), so some conversion may be needed if values are given in different units.

Note that momentum is a vector quantity.

WORKED EXAMPLE 11 Converting units and determining momentum

Determine the magnitude of the momentum of a car of mass 1.4 tonnes moving at 72 km/h.

THINK	WRITE
1. Determine the mass of the car in SI units; that is, express the mass in kg.	$m = 1.4$ tonnes $m = 1400$ kg
2. Convert the speed into m/s. Multiply by 1000 and divide by 60 and 60 again, since there are 60 minutes in an hour and 60 seconds in a minute.	$v = 72$ km/h $v = \dfrac{72 \times 1000}{60 \times 60}$ $= 20$ m/s
3. Calculate the product of the mass and speed.	$p = mv$ $= 1400 \times 20$
4. State the magnitude of the momentum.	The magnitude of the momentum is 28 000 kg m/s.

10.3.3 Resultant force, $\underset{\sim}{R} = m\underset{\sim}{a}$

Newton's second law also involves the resultant force. Let $\underset{\sim}{R} = \sum \underset{\sim}{F}$ be the vector resultant sum of all the forces acting on a body. If the resultant force acting on a body is proportional to the rate of change of momentum, then $\underset{\sim}{R} \propto \dfrac{d}{dt}(\underset{\sim}{p})$. Taking k as the proportionality constant, $\underset{\sim}{R} = k\dfrac{d}{dt}(\underset{\sim}{p})$.

Now substituting for the momentum, $\underset{\sim}{R} = k\dfrac{d}{dt}(m\underset{\sim}{v})$. Differentiating using the product rule gives $\underset{\sim}{R} = k\left[m\dfrac{d\underset{\sim}{v}}{dt} + \underset{\sim}{v}\dfrac{dm}{dt}\right]$. But since the mass is constant, $\dfrac{dm}{dt} = 0$, we have $\underset{\sim}{R} = km\dfrac{d\underset{\sim}{v}}{dt} = km\underset{\sim}{a}$, since the acceleration $\underset{\sim}{a} = \dfrac{d\underset{\sim}{v}}{dt}$.

We can make $k = 1$ if we use correct and appropriate units. If the mass m is measured in kg, the forces in N and the acceleration in m/s^2, then $k = 1$. So a force of 1 newton is defined as the force which, when applied to a mass of 1 kilogram, produces an acceleration of 1 m/s^2.

Newton's laws imply that a force is needed to make an object move. By considering the forces acting on an object we can analyse its motion.

Momentum and resultant force

$$\text{momentum} = \text{mass} \times \text{velocity}$$
$$\underset{\sim}{p} = m\underset{\sim}{v}$$
$$(\text{kg m/s}) = (\text{kg})\ (\text{m/s})$$
$$\text{resultant force} = \text{mass} \times \text{acceleration}$$
$$\underset{\sim}{R} = m\underset{\sim}{a}$$
$$(\text{N}) = (\text{kg})\ \left(\text{m/s}^2\right)$$

Note: When a situation is in equilibrium, the resultant force is equal to zero, $\underset{\sim}{R} = \underset{\sim}{0}$, therefore the acceleration is also equal to zero, $\underset{\sim}{a} = \underset{\sim}{0}$.

This relates to statics where $\sum \underset{\sim}{F} = \underset{\sim}{0}$.

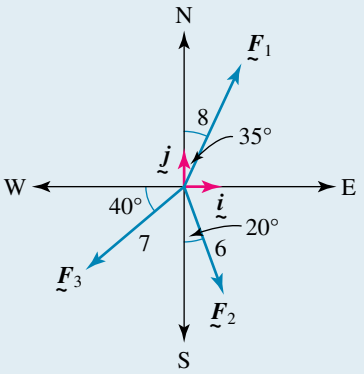
WORKED EXAMPLE 12 Determining the resultant force

A particle of mass 2 kg on a smooth horizontal plane is acted upon by three forces in that horizontal plane. $\underset{\sim}{F}_1$ has a magnitude of 8 newtons and acts in direction N35°E, $\underset{\sim}{F}_2$ has a magnitude of 6 newtons and acts in direction *S*20°E, and $\underset{\sim}{F}_3$ has a magnitude of 7 newtons and acts in direction *S*50°W. If $\hat{\imath}$ is a unit vector of magnitude 1 newton acting east and $\hat{\jmath}$ is a unit vector of magnitude 1 newton acting north, determine the magnitude of the acceleration of the particle.

THINK

1. Draw the force diagram.

WRITE/DRAW



2. Resolve the force $\underset{\sim}{F}_1$ into its components.

$$\underset{\sim}{F}_1 = 8\sin(35°)\hat{\imath} + 8\cos(35°)\hat{\jmath}$$

3. Resolve the force $\underset{\sim}{F}_2$ into its components.

$$\underset{\sim}{F}_2 = 6\sin(20°)\hat{\imath} - 6\cos(20°)\hat{\jmath}$$

4. Resolve the force $\underset{\sim}{F}_3$ into its components.

$$\underset{\sim}{F}_3 = -7\cos(40°)\hat{\imath} - 7\sin(40°)\hat{\jmath}$$

5. Determine the resultant force, which is the vector sum of all the forces acting on the body.

$$\underset{\sim}{R} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$$
$$\underset{\sim}{R} = (8\sin(35°)\hat{\imath} + 8\cos(35°)\hat{\jmath})$$
$$+ (6\sin(20°)\hat{\imath} - 6\cos(20°)\hat{\jmath})$$
$$- (7\cos(40°)\hat{\imath} + 7\ \sin(40°)\hat{\jmath})$$

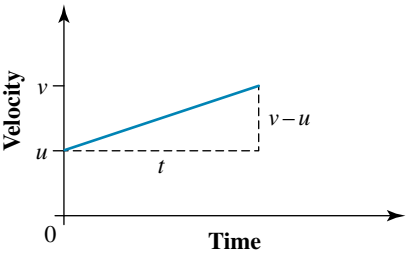
6.	Use the properties of vector addition.	$\underset{\sim}{R} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$ $\underset{\sim}{R} = (8\sin(35°) + 6\sin(20°) - 7\cos(40°))\hat{\boldsymbol{\imath}}$ $+ (8\cos(35°) - 6\cos(20°) - 7\sin(40°))\hat{\boldsymbol{\jmath}}$
7.	Evaluate the resultant force.	$\underset{\sim}{R} = 1.278\hat{\boldsymbol{\imath}} - 3.584\hat{\boldsymbol{\jmath}}$
8.	Use Newton's second law of motion to calculate the acceleration, which is a vector.	$\underset{\sim}{R} = m\underset{\sim}{a}$ Since the mass $m = 2$, $\underset{\sim}{R} = 1.278\hat{\boldsymbol{\imath}} - 3.584\hat{\boldsymbol{\jmath}} = 2\underset{\sim}{a}$ $\underset{\sim}{a} = 0.639\hat{\boldsymbol{\imath}} - 1.792\hat{\boldsymbol{\jmath}}$
9.	Calculate the magnitude of the acceleration.	$\lvert\underset{\sim}{a}\rvert = \sqrt{0.639^2 + 1.792^2}$ $= 1.90$
10.	State the result. Remember to include the units.	The magnitude of the acceleration is. 1.90 m/s^2.

10.3.4 Acceleration due to constant force

If an object is moving due to a constant force, then its acceleration is constant. Rectilinear motion is motion in a straight line, either horizontally or vertically.

When we consider the motion of an object in a straight line with uniform acceleration, a number of concepts can be reviewed.

The diagram on the next page, represents the motion of an object with initial velocity u and final velocity v after t seconds.



The gradient of the line is calculated by $\frac{v-u}{t}$. On a velocity–time graph, the gradient is the acceleration of the object. Hence, $a = \frac{v-u}{t}$.

The acceleration is also the derivative of the velocity function with respect to time, just as the velocity is the derivative of the displacement function with respect to time. A negative acceleration indicates a retardation or deceleration.

The area under a velocity–time graph gives the displacement of the object.

In solving constant acceleration problems it is important to determine the resultant force that is causing the acceleration. It is also important to check units. All quantities must be converted to compatible units such as m and m/s.

Remember that to convert km/h to m/s, we multiply by $\frac{1000}{3600}$ or $\frac{5}{18}$; to convert m/s to km/h, we multiply by $\frac{18}{5}$ or 3.6.

Various notations can be used to describe forces acting on objects in modelling motion problems, most commonly:

$\underset{\sim}{w}$	Weight force
$\underset{\sim}{N}$	Normal reaction
$\underset{\sim}{R}$	Resultant force
$\underset{\sim}{F}_{\mathrm{R}}$	Resistance force
F_{T}	Tractive force, tensile force or thrust force

In each case, define the force (F) with an appropriate subscript for the context of the problem.

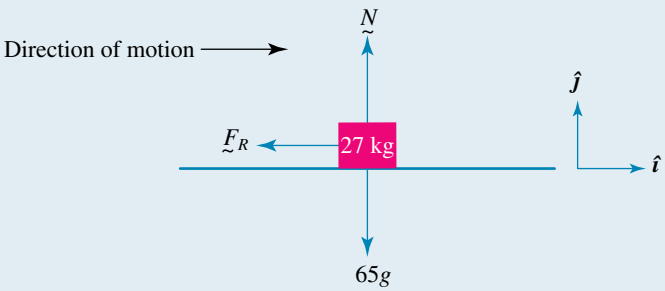
WORKED EXAMPLE 13 Constant force (horizontal)

A boy of mass 25 kg on a skateboard of mass 2 kg is freewheeling on a level track. His speed decreases from 10 m/s to 5 m/s as over 20 seconds. Determine the magnitude of the resistant force, which is assumed to be constant.

THINK

1. Draw a force diagram. The only horizontal force acting on the boy and the skateboard is the resistance force $\underset{\sim}{F}_{\mathrm{R}}$, which opposes the direction of motion.
   Show the positive vector directions.

WRITE/DRAW



2. Calculate the acceleration, a, of the boy on the skateboard.

$$\underset{\sim}{a} = \frac{5-10}{20}\hat{\imath}$$
$$= -0.25\hat{\imath}\ \mathrm{m/s^2}$$

The acceleration is negative as he is slowing down.

3. Determine the resistance force using Newton's second law.

$$\underset{\sim}{R} = m\underset{\sim}{a}$$
$$m = 27 \text{ and } \underset{\sim}{a} = -0.25\hat{\imath}$$
$$R = 27 \times -0.25\hat{\imath}$$
$$= -6.75\hat{\imath}$$

4. State the magnitude of the resistance force.

The magnitude of the resistance force is 6.75 newtons.

Newton's third law has already been considered when including the normal reaction on particles when they are at rest on, or in contact with, a surface. Note that only the external forces acting on a body will be considered in this course, and the effects of compression or the rotational aspects of bodies when forces are applied to them will be ignored.

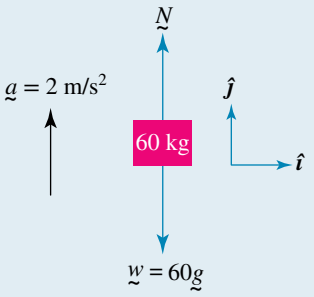
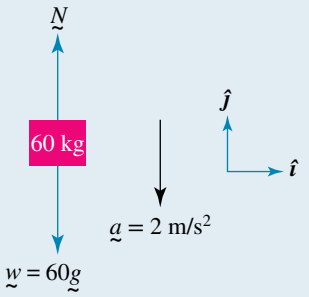
WORKED EXAMPLE 14 Constant force (vertical)

A 60 kg student stands in a lift. Calculate the reaction of the lift floor on the student if the lift is:

a. accelerating upwards at 2 m/s^2

b. accelerating downwards at 2 m/s^2

c. moving with constant velocity.

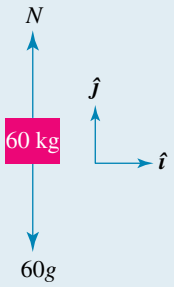
THINK	WRITE/DRAW
a. 1. Draw a force diagram, including the direction of motion. Let N newtons be the reaction of the lift floor on the student when the lift travels upwards. Show the positive vector directions	**a.** 
2. Resolve the forces in the direction of motion using Newton's second law.	The lift is accelerating at $2\hat{\boldsymbol{j}}$ m/s^2 and $m = 60$ kg. $\underset{\sim}{R} = m\underset{\sim}{a}$ $(N - 60g)\hat{\boldsymbol{j}} = 60 \times 2\hat{\boldsymbol{j}}$
3. Calculate the magnitude of the reaction force.	$N = 60g + 60 \times 2$ $= 60(9.8 + 2)$ $= 708$
4. State the value of the reaction force.	The reaction force is 708 N.
b. 1. Draw a force diagram, including the direction of motion. Let N newtons now be the reaction of the lift floor on the student when the lift travels downwards. This value of N is different from the case in part **a**.	**b.** 
2. Resolve the forces in the direction of motion using Newton's second law.	Consider the downwards direction as positive. The lift is accelerating downwards at $2\hat{\boldsymbol{j}}$ m/s^2. $\underset{\sim}{R} = m\underset{\sim}{a}$ $(60g - N)\hat{\boldsymbol{j}} = 60 \times 2\hat{\boldsymbol{j}}$
3. Calculate the magnitude of the reaction force.	$N = 60g - 60 \times 2$ $= 60(9.8 - 2)$ $= 468$
4. State the value of the reaction force.	The reaction force is 468 N.

c. 1. When the lift moves with constant velocity, its acceleration is zero.

$$\underset{\sim}{a} = \underset{\sim}{0}$$
$$\therefore \underset{\sim}{R} = \underset{\sim}{0}$$

c. Lift is in equilibrium situation.

$$\sum \underset{\sim}{F} = \underset{\sim}{0}$$
$$\sum F_y = 0$$
$$= N - 60g$$
$$N = 60g$$
$$= 60 \times 9.8$$
$$= 588$$



2. State the value of the reaction force in this case.

The reaction force is 588 N.

Exercise 10.3 Non-equilibrium situations under concurrent forces

learnon

10.3 Exercise	10.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

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Simple familiar

1. WE11 Determine the magnitude of the momentum of a truck of mass 6 tonnes moving at 96 km/h.

2. Determine the magnitude of the momentum of the Earth as it orbits the Sun. Take the mass of the Earth to be 5.972×10^{24} kg and the orbital speed to be 30 km/s.

3. WE12 A particle of mass 3 kg on a smooth horizontal plane is acted upon by three forces in that horizontal plane. $\underset{\sim}{F}_1$ has a magnitude of 9 newtons and acts in direction N55°E, $\underset{\sim}{F}_2$ has a magnitude of 7 newtons and acts in direction S40°E, and $\underset{\sim}{F}_3$ has a magnitude of 6 newtons and acts in direction S55°W. If $\hat{\boldsymbol{\imath}}$ is a unit vector of magnitude 1 newton acting east and $\hat{\boldsymbol{\jmath}}$ is a unit vector of magnitude 1 newton acting north, determine the magnitude of the acceleration of the particle.

4. A particle of mass 4 kg on a smooth horizontal plane is acted upon by four forces in that horizontal plane. $\underset{\sim}{F}_1$ has a magnitude of 12 newtons and acts due north, $\underset{\sim}{F}_2$ has a magnitude of 8 newtons and acts in direction S60°W, $\underset{\sim}{F}_3$ has a magnitude of 8 newtons and acts in direction S60°E, and $\underset{\sim}{F}_4$ has a magnitude of 10 newtons and acts due east. If $\hat{\boldsymbol{\imath}}$ is a unit vector of magnitude 1 newton acting east and $\hat{\boldsymbol{\jmath}}$ is a unit vector of magnitude 1 newton acting north, determine the magnitude of the acceleration of the particle.

5. **WE13** A car of mass 1200 kg moves on a horizontal road and brakes. Its speed decreases from 16 m/s to 6 m/s while it moves a distance of 25 m. Calculate the magnitude of the resistant breaking force which is assumed to be constant.

6. A woman is riding a bicycle and is moving at a speed of 6 km/h. She freewheels, coming to rest after 5 seconds. If the magnitude of the constant resistance force is 20 newtons, calculate the mass of the woman and the bicycle.

7. **WE14** An 80 kg man stands in a lift. Calculate the reaction of the lift floor on the man if the lift is:
   - **a.** accelerating upwards at 3 m/s^2
   - **b.** accelerating downwards at 3 m/s^2
   - **c.** moving with constant velocity.

8. A girl stands in a lift. When the lift accelerates up at 1.5 m/s^2, the reaction of the lift floor on the girl is 734.5 N. When the lift accelerates down at 1.5 m/s^2, calculate the reaction of the lift floor on the girl.

9. **a.** Calculate the resultant force acting on a body of mass 3 kg moving with an acceleration of 2 m/s^2.
   - **b.** Calculate the magnitude of the acceleration when a mass of 2.5 kg is acted upon by a force of $\underset{\sim}{F} = 3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ N.
   - **c.** Calculate the magnitude of the acceleration when a mass of 2 kg is acted upon by two forces of $\underset{\sim}{F}_1 = 3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{F}_2 = -7\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}}$ N.

10. **a.** A particle of mass 2 kg is acted upon by two forces. One force has a magnitude of $3\sqrt{2}$ N acting in the north-west direction; the other force has magnitude of $4\sqrt{2}$ N acting in the south-west direction. Determine the magnitude of the acceleration of the particle.
    - **b.** A particle of mass m kg is acted upon by two forces. One force has a magnitude of b N acting in the north-east direction; the other force has magnitude of c N acting in the south-east direction. Determine the magnitude of the acceleration acting on the body.

11. **a.** A package of mass 4 kg lies on the floor of a lift that is accelerating upwards at 2 m/s^2. Calculate the reaction of the lift floor on the package.
    - **b.** A parcel of mass 5 kg rests on the floor of a lift. If the reaction between the box and the floor is 48 N, calculate the downwards acceleration of the lift.

12. A man of mass 75 kg stands in a lift. Determine the reaction between his feet and the floor when the lift is:
    - **a.** ascending with a constant acceleration of 0.5 m/s^2
    - **b.** descending with a constant acceleration of 0.5 m/s^2
    - **c.** moving with a constant speed of 0.5 m/s.

Complex familiar

13. A particle of mass 5 kg on a smooth horizontal plane is acted upon by three forces. $\underset{\sim}{F}_1$ has a magnitude of 9 N and acts in direction N55°E, $\underset{\sim}{F}_2$ has a magnitude of 8 N and acts in direction S40°E, and $\underset{\sim}{F}_3$ has a magnitude of 7 N and acts in direction S55°W. Determine the magnitude of the acceleration of the particle.

14. A body on a smooth horizontal plane is acted upon by four forces. $\underset{\sim}{F}_1$ has a magnitude of 6 N and acts in direction N30°E, $\underset{\sim}{F}_2$ has a magnitude of $8\sqrt{2}$ N and acts in direction S45°E, $\underset{\sim}{F}_3$ has a magnitude of 4 N and acts in direction N60°W, and $\underset{\sim}{F}_4$ has a magnitude of $5\sqrt{2}$ N and acts in direction S45°W. If $\hat{\boldsymbol{\imath}}$ is a unit vector of magnitude 1 N acting east and $\hat{\boldsymbol{\jmath}}$ is a unit vector of magnitude 1 N acting north, calculate the resultant force acting on the body.

15. **a.** A constant force of 12 newtons acts on a body of mass 4 kg originally at rest. Calculate:

i. the speed of the body after 3 seconds

ii. the distance travelled by the body in 3 seconds.

b. Calculate the average resistance force exerted by the brakes on a car of mass 1.2 tonnes during braking if its speed decreases from 60 km/h to rest:

i. in a time of 1.5 seconds

ii. in travelling a distance of 15 metres.

16. A boy stands in a lift. When the lift accelerates upwards with a constant acceleration of a m/s^2, the reaction of the floor on the boy is 502.5 N. When the lift accelerates downwards with a constant acceleration of a m/s^2, the reaction of the floor on the boy is 477.5 N. Determine the mass of the boy and the value of a.

Complex unfamiliar

17. A child of mass 40 kg is on a 2 kg sled on horizontal snow-covered ground. The child and sled are being pulled along the snow by a force of 100 N. Determine the acceleration of the child and the magnitude of the normal reaction of the child on the snow if the force is:

a. horizontal

b. at an angle of 35° to the horizontal.

18. A parcel of mass m kg is on a smooth slope inclined at an angle of θ to the horizontal. When a force of 25 N is applied acting up and parallel to the slope, the parcel accelerates up the slope at 0.8 m/s^2. When the force is increased to 30 N acting up and parallel to the slope, the parcel accelerates up the slope at 1.8 m/s^2. Determine the values of m and θ.

19. A package of mass m kg is on a smooth horizontal surface. When a force of 5 newtons is applied to the parcel acting at an angle of α to the horizontal, the parcel accelerates at 0.5 m/s^2. When the force is increased to 10 N and is acting at an angle of 2α to the horizontal, the package accelerates at 0.9 m/s^2. Determine the values of m and α.

20. A package of mass 5 kg is on a smooth slope inclined at an angle of θ to the horizontal. When a force of 40 N is applied acting at an angle of α to the slope, the package accelerates up the slope at 1.2 m/s^2. When the force is increased to 50 N, the package accelerates at 3.1 m/s^2. Determine the values of θ and α.

Fully worked solutions for this chapter are available online.

LESSON
10.4 Forces that depend on time

SYLLABUS LINKS

- Understand and use the expressions $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to represent the acceleration of an object moving in a straight line.

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10.4.1 Setting up the equations of motion

In this lesson we consider the motion of an object of constant mass m moving in a straight line and subjected to a system of forces. In previous lessons, the forces F have been constant, and since $\underset{\sim}{R} = m\underset{\sim}{a}$, the acceleration a was also constant. However, if the forces are dependent upon time, t, velocity, v, or displacement, x, then the acceleration is not constant. In situations where the forces are not constant, a differential equation must be set up and solved.

Note: The notation used for acceleration in this lesson is two dots above x, that is, $\ddot{x}$ or $\ddot{y}$ or $\ddot{r}$.

The dot above a variable represents differentiation with respect to time, for example, $\dot{x} = \frac{dx}{dt}$.

If the resultant force, $R = F(t)$, depends upon time, t, then by Newton's second law of motion, $m\ddot{x} = m\frac{dv}{dt} = F(t)$.

Because the mass is constant, the acceleration, $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \ddot{x}$, is a function of time, t, where v is the velocity. Thus,

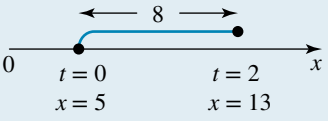
$$a = \ddot{x} = \frac{dv}{dt} = \frac{F(t)}{m}$$

This equation is called the equation of motion. It can be solved by integrating both sides with respect to t to obtain the velocity as $v = \frac{1}{m}\int F(t)\,dt$. A constant of integration can be found using an initial condition for v and t. Because $v = \frac{dx}{dt}$, we can integrate again to express the displacement, x, in terms of t. This is another application of solving a second-order differential equation.

WORKED EXAMPLE 15 Using non-constant acceleration

An object of mass 2 kilograms moves in a straight line and is acted upon by a force of $12t - 24$ newtons, where t is the time in seconds and $t \geq 0$. Initially the object is 5 metres to the right of the origin, and it comes to rest after 2 seconds. Calculate the distance travelled by the object in the first 2 seconds.

THINK	WRITE/DRAW
1. Use Newton's second law of motion.	$m\ddot{x} = F(t)$ where $m = 2$ and $F(t) = 12t - 24$
2. Formulate the differential equation to be solved (the equation of motion).	$2\ddot{x} = 12t - 24$ $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 6t - 12$

3. Integrate both sides with respect to t.	$v = \int (6t - 12)\, dt$
4. Perform the integration using basic integration techniques, adding in the first constant of integration.	$v = 3t^2 - 12t + c_1$
5. Use the given condition to calculate the first constant of integration.	When $t = 2$, the object is at rest, so $v = 0$. Substitute to determine c_1: $0 = 12 - 24 + c_1$ $c_1 = 12$
6. Substitute back for the constant of integration.	$v = \dfrac{dx}{dt} = 3t^2 - 12t + 12$
7. Integrate both sides again with respect to t.	$x = \int (3t^2 - 12t + 12)\, dt$
8. Perform the integration, adding in a second constant of integration.	$x = t^3 - 6t^2 + 12t + c_2$
9. Use the given initial condition to calculate the second constant of integration.	Initially, when $t = 0$, $x = 5$. Substitute to determine c_2: $5 = 0 + c_2$ $c_2 = 5$
10. Substitute back for the constant of integration.	$x = x(t) = t^3 - 6t^2 + 12t + 5$
11. Determine the displacement at the required time.	Substitute $t = 2$: $x(2) = 8 - 24 + 24 + 5$ $= 13$
12. State the required result.	The object moves from $t = 0$, $x = 5$ to $t = 2$, $x = 13$. The distance travelled is 8 metres. 

10.4.2 Integrals involving trigonometric functions

Recall the integrals involving basic trigonometric functions and exponential functions:

$$\int \sin(kx)\, dx = -\frac{1}{k}\cos(kx)$$

$$\int \cos(kx)\, dx = \frac{1}{k}\sin(kx)$$

$$\int e^{kx}\, dx = \frac{1}{k}e^{kx}.$$

WORKED EXAMPLE 16 Using trigonometric functions

A particle of mass 5 kilograms moves back and forth along the x-axis and is subjected to a force of $-20\sin(2t) - 20\cos(2t)$ newtons at time t seconds. If its initial velocity is 2 metres per second and it starts from a point 3 metres to the left of the origin, express the displacement x metres in terms of t.

THINK	WRITE
1. Use Newton's second law of motion.	$m\ddot{x} = F(t)$ where $m = 5$ and $F(t) = -20\sin(2t) - 20\cos(2t)$
2. Formulate the equation of motion to be solved.	$5\ddot{x} = -20\sin(2t) - 20\cos(2t)$ $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -4\sin(2t) - 4\cos(2t)$
3. Integrate both sides with respect to t.	$x = \int(-4\sin(2t) - 4\cos(2t))\,dt$
4. Perform the integration using the results $\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx)$ and $\int \cos(kx)\,dx = \frac{1}{k}\sin(kx)$ with $k = 2$. Note that only one constant of integration is required.	$v = 2\cos(2t) - 2\sin(2t) + c_1$
5. Use the given initial condition to calculate the first constant of integration.	Initially, when $t = 0$, $v = 2$: $2 = 2\cos(0) - 2\sin(0) + c_1$ $c_1 = 0$
6. Substitute back for the constant of integration.	$v = \frac{dx}{dt} = 2\cos(2t) - 2\sin(2t)$
7. Integrate both sides again with respect to t.	$x = \int(2\cos(2t) - 2\sin(2t))\,dt$
8. Perform the integration, adding in a second constant of integration.	$x = \sin(2t) + \cos(2t) + c_2$
9. Use the given initial condition to calculate the second constant of integration.	Initially, when $t = 0$, $x = -3$ (to the left): $-3 = \sin(0) + \cos(0) + c_2$ $c_2 = -4$
10. Substitute back for the constant of integration. This is the required result.	$x = \sin(2t) + \cos(2t) - 4$

10.4.3 Horizontal rectilinear motion

When a driver of a car travelling at some initial speed applies the brakes, the braking force for a short period of time is a resistance force that acts opposite to the direction of motion. The force is a function of the time during which the brakes are applied. This can be used to model the equation of motion during this time and determine, for example, the distance travelled while braking.

TIP

In the International System of Units, the unit of time is the second (s), the unit of length is the metre (m) and the unit of mass is the kg.

The units for the magnitude of velocity and acceleration are m/s and m/s^2 respectively.

The common derived units useful in this chapter, and their corresponding SI units are summarised in the table.

Quantity	Derived unit	Unit in terms of SI units
Force	N (newton)	$kg\,m\,s^{-2}$
Momentum	N s	$kg\,m\,s^{-1}$
Torque	N m	$kg\,m^2\,s^{-2}$

If different units are provided (e.g. km/h or cm/s^2), you will need to convert to SI units.

WORKED EXAMPLE 17 Horizontal linear acceleration

A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is 32 000 – 12 800*t* newtons, where *t* is the time in seconds after the driver applies the brakes. Calculate:

a. the time after which the speed of the car has been reduced to 57.6 km/h

b. the distance travelled in this time.

THINK

a. 1. Use Newton's second law of motion. The braking force acts opposite to the direction of motion.

WRITE/DRAW

a. $m\ddot{x} = -F(t)$ where $m = 1600$ and $F(t) = 32\,000 - 12\,800t$

2. Formulate the equation of motion to be solved.

$$1600\ddot{x} = 12\,800t - 32\,000$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 8t - 20$$

3. Integrate both sides with respect to *t*.

$$v = \int (8t - 20)dt$$

4. Perform the integration using the basic integration techniques, adding in the first constant of integration.

$$v = 4t^2 - 20t + c_1$$

5. Use the given initial conditions to calculate the first constant of integration. We need to use correct units.

The initial speed is 90 km/h. Convert km/h to m/s: $\frac{90 \times 1000}{60 \times 60} = 25$ m/s

Initially, when $t = 0$, $v = 25$:

$$25 = 0 + c_1$$

$$c_1 = 25$$

6. Substitute back for the constant of integration.

$$v = 4t^2 - 20t + 25$$

7. Determine the braking time.

The final speed is 57.6 km/h.
Convert km/h to m/s:

$$\frac{57.6 \times 1000}{60 \times 60} = 16 \text{ m/s}$$

Determine t when $v = 16$.

8. The earlier time is the one required as this will be the first instance the car's speed will be reduced from 90 km/h to 57.6 km/h before coming to rest.

$$\begin{aligned} 16 &= 4t^2 - 20t + 25 \\ 4t^2 - 20t + 9 &= 0 \\ (2t-1)(2t-9) &= 0 \\ t &= \frac{1}{2}, \frac{9}{2} \\ t &= \frac{1}{2} \end{aligned}$$

9. State the required result.

After 0.5 seconds, the car's speed has been reduced from 90 to 57.6 km/h.

b. 1. Express the velocity in terms of time, t.

b. $v = \dfrac{dx}{dt} = 4t^2 - 20t + 25$

2. Integrate both sides again with respect to t.

$$x = \int (4t^2 - 20t + 25)dt$$

3. Perform the integration using basic integration techniques, adding in a second constant of integration.

$$x = \frac{4t^3}{3} - 10t^2 + 25t + c_2$$

4. Use the given initial condition to calculate the second constant of integration.

Since we want to determine the distance travelled from first braking, assume that when $t = 0$, $x = 0$, so that $c_2 = 0$.

5. Express the displacement travelled in terms of time.

$$x = x(t) = \frac{4t^3}{3} - 10t^2 + 25t$$

6. Determine the distance travelled, in metres, while the car is braking, as in these cases the distance travelled is the displacement. Make the substitution $t = 0.5$ seconds.

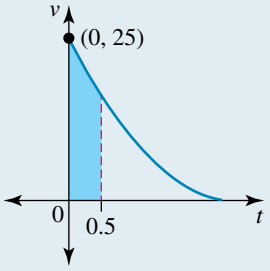
$$\begin{aligned} \text{Distance travelled} &= x(t) \\ &= \frac{4 \times (0.5)^3}{3} - 10 \times 0.5^2 + 25 \times 0.5 \\ &= 10\frac{1}{6} \end{aligned}$$

7. State the final result.

The distance travelled while the car is braking is $10\frac{1}{6}$ metres.

8. An alternate method to calculate the distance travelled can be used.

Since the distance travelled is the area under the velocity–time graph, this distance is given by the definite integral.



$$D = \int_0^{\frac{1}{2}} (4t^2 - 20t + 25)dt$$

$$= \int_0^{\frac{1}{2}} (2t - 5)^2 dt$$

$$= \left[\frac{1}{6}(2t - 5)^3\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{6}(-4)^3 - \frac{1}{6}(-5)^3$$

$$= 10\frac{1}{6}$$

Note: The alternate method may be used as a check in complicated problems, when it is not clear if the object travels in the same direction in the time interval.

10.4.4 Motion on inclined planes

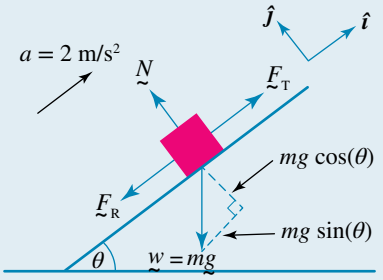
When a particle is moving along a slope, the forces are resolved parallel and perpendicular to the slope. For inclined plane problems, consider the $\hat{\boldsymbol{\imath}}$ direction as parallel to the slope. Often, the positive direction if given in the direction of motion Newton's second law of motion states that $\sum F_x = ma$. That is, the vector sum of all the resolved forces in the direction of motion is equal to ma. The vector sum of all the resolved forces in the direction perpendicular to the motion in the $\hat{\boldsymbol{\jmath}}$ direction, $\sum F_y$, is equal to zero, because the particle is not moving in that direction $\therefore \sum F_y = 0$.

Slopes

When a particle moves up or down a slope, the inclination of the slope is often stated as the vertical distance raised over the horizontal distance travelled. So a slope of 1 in n means that $\tan(\theta) = \frac{1}{n}$, where θ is the inclination of the slope to the horizontal.

WORKED EXAMPLE 18 Constant force on an inclined plane (1)

A boy of mass 70 kg rides a 10 kg bicycle up a slope of 1 in 5. If the resistance to the motion is one-quarter of the weight force, calculate the tractive force of the tyres when he ascends with an acceleration of 2 m/s^2.

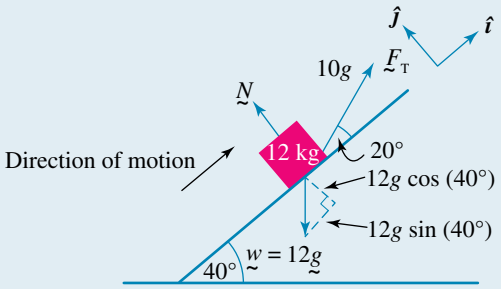
THINK	WRITE/DRAW
1. Draw a force diagram. Let F_R newtons be the tractive force exerted by the tyres that makes the bike go up the hill. Consider the $\hat{\imath}$ direction as the positive direction going up and parallel to the slope.	
2. Define the angle of the inclined plane.	$\tan(\theta) = \frac{1}{5}$ $\theta = \tan^{-1}\left(\frac{1}{5}\right)$
Note that we can use trigonometric identities to determine the exact values of $\sin(\theta)$ and $\cos(\theta)$. Refer to the Trigonometry (right-angled triangles) box in Lesson 10.2 if necessary.	$\text{cosec}^2(\theta) = \cot^2(\theta) + 1$ $\cos^2(\theta) + \sin^2(\theta) = 1$ $\sin(\theta) = \frac{1}{\sqrt{26}}$ $\cos(\theta) = \frac{5}{\sqrt{26}}$
3. Calculate the magnitude of the resistance force F_R.	$\underset{\sim}{F}_R = \frac{\underset{\sim}{w}}{4}$ $F_R = \frac{mg}{4}$ $= 20g$
4. Resolve the forces in the $\hat{\imath}$ direction.	$\underset{\sim}{R} = \underset{\sim}{F}_T - \underset{\sim}{F}_R + \underset{\sim}{w} + \underset{\sim}{N}$ $R = F_T - F_R - mg\sin(\theta)$ $= F - 20g - \frac{80g}{\sqrt{26}}$
5. Use Newton's second law to equate.	$\underset{\sim}{R} = m\underset{\sim}{a}$ $80 \times 2 = F_T - 20g - \frac{80g}{\sqrt{26}}$
6. Evaluate and determine the value of the required force.	$F_T = 160 + 20g + \frac{80g}{\sqrt{26}}$ $= 509.76$
7. State the result, including units.	The tractive force is 509.76 N.

10.4.5 Resolution of all the forces

When two or more forces have components, it is important to remember to determine the resolved components of *all* the forces parallel and perpendicular to the plane.

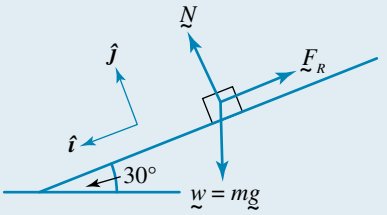
WORKED EXAMPLE 19 Constant force on an inclined plane (2)

A certain strap on a suitcase will break when the tension in it exceeds 10g. Calculate the greatest acceleration with which a 12 kg suitcase can be dragged up a smooth plane inclined at an angle of 40° to the horizontal when the strap is inclined at 20° to the plane.

THINK	WRITE/DRAW
1. Draw the force diagram. Let $\underset{\sim}{F}_T$ newtons be the tension in the strap. Consider the positive $\hat{\imath}$ direction as going up the slope.	
2. Calculate the component of the tension force up and parallel to the plane.	The $\hat{\imath}$ component of the tension in the strap is $10g\ \cos(20°)\ \hat{\imath}$.
3. Calculate the component of the weight force parallel to the plane.	The $\hat{\imath}$ component of the weight force is $-12g\ \sin(40°)\ \hat{\imath}$.
4. Use Newton's second law to calculate the acceleration.	The suitcase moves up the plane with an acceleration of a. $\underset{\sim}{R} = m\underset{\sim}{a}$ $10g\ \cos(20°) - 12g\ \sin(40°) = 12a$
5. Resolve the forces in the $\hat{\imath}$ direction.	$\underset{\sim}{R} = (10g \cos 20° - 12g \sin 40°)\,\hat{\imath}$
6. Resolve all the forces, perpendicular to the plane. However, this equation is not needed in this case.	$10g\ \sin(20°) + N - 12g\ \cos(40°)\hat{\jmath} = 0$
7. Solve to determine the maximum acceleration.	$a = \frac{9.8}{12}(10\cos(20°) - 12\sin(40°))$ $= 1.37$
8. State the acceleration that occurs when the tension is $10g$ newtons.	The acceleration is $1.37\ \text{m/s}^2$.

WORKED EXAMPLE 20 Modelling motion on an inclined plane

A child with mass 36 kg starts to slide down a 4-m long playground slide, which has a slope of 30°. The resistance to motion has a magnitude of 70 N.
Determine how long it takes the child to reach the bottom of the slide and the speed at which the child is travelling when they reach the bottom.

THINK	WRITE
1. Draw the force diagram. Consider the positive $\hat{\imath}$ direction as going down the slope.	
2. Calculate the $\hat{\imath}$ component of the weight force.	$mg\sin\theta = 36g\sin(30°)$ $= 36g \times \frac{1}{2}$ $= 18g$
3. Use Newton's second law to calculate the acceleration.	$m\ddot{x} = \sum F(t)$ $(36\ddot{x})\hat{\imath} = (18g - 70)\hat{\imath}$ $\ddot{x} = \frac{18g - 70}{36}$ $= \frac{1}{2}g - \frac{35}{18}$
4. Integrate to determine the velocity.	$v(t) = \dot{x}(t)$ $= \int \left(\frac{1}{2}g - \frac{35}{18}\right) dt$ $= \left(\frac{1}{2}g - \frac{35}{18}\right) t + c_1$
5. Use the initial condition to calculate the constant of integration. The child is initially at rest, their initial velocity is zero.	$v(0) = 0$ $= c_1$ $v(t) = \left(\frac{1}{2}g - \frac{35}{18}\right) t$
6. Integrate to determine the displacement.	$x(t) = \int \dot{x}(t)\, dt$ $= \int \left(\frac{1}{2}g - \frac{35}{18}\right) t\, dt$ $= \frac{1}{2}\left(\frac{1}{2}g - \frac{35}{18}\right) t^2 + c_2$

7. Use the initial condition to calculate the second constant of integration.

$$x(0) = 0$$
$$= c_2$$
$$x(t) = \frac{1}{2}\left(\frac{1}{2}g - \frac{35}{18}\right)t^2$$
$$\approx 1.47t^2$$

8. Determine the time when $x = 4$.

$$1.4777\cdots t^2 = 4$$
$$t = \pm 1.6452\cdots$$

9. Justify the decision for the value of t to use.

We disregard negative time in this context.
$t = 1.6452\cdots$ s.

10. Substitute the value for time into the velocity equation.

$$v(1.6452\cdots) = \left(\frac{1}{2}g - \frac{35}{18}\right) \times 1.6452\cdots$$
$$= 4.8625\cdots$$
$$\approx 4.86 \text{ m/s}$$

11. State the answers.

The child will take approximately 1.65 seconds to reach the bottom of the slide, and they will be travelling at approximately 4.86 m/s.

Exercise 10.4 Forces that depend on time

learnon

10.4 Exercise | **10.4 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16	17, 18, 19, 20, 21, 22	23, 24

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Simple familiar

1. WE15 An object of mass 3 kilograms moves in a straight line and is acted upon by a force of $6t - 18$ newtons, where t is the time in seconds and $t \geq 0$. Initially the object is 4 metres to the right of the origin, and it comes to rest after 3 seconds. Calculate the distance travelled by the object in the first 3 seconds.

2. A particle of mass 5 kilograms moves in a straight line and is acted upon by a force of $60t - 60$ newtons, where t is the time in seconds. Initially the particle is at the origin, and after 1 second the particle is 52 metres to the left of the origin. Calculate when the particle first comes to rest.

3. WE16 A particle of mass 8 kilograms moves back and forth along the x-axis and is subjected to a force of $-16\sin\left(\frac{t}{3}\right)$ newtons at time t seconds. If its initial velocity is 6 m/s and the particle starts at the origin, express the displacement, x metres, in terms of t.

4. A particle of mass 8 kilograms moves back and forth along the x-axis and is subjected to a force of $-16\cos\left(\frac{t}{2}\right)$ newtons at time t seconds. If initially the particle is at rest and starts from a point 8 metres from the origin, calculate the displacement of the particle after a time of $\frac{\pi}{2}$ seconds.

5. **WE17** A car of mass 1500 kg is moving along a straight road at a speed of 72 km/h when the driver brakes. The braking force is $180\,000t - 75\,000$ newtons, where t is the time in seconds after the driver applies the brakes. Calculate:
   a. the time after which the speed of the car has been reduced to 36 km/h
   b. the distance travelled in this time.

6. A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $\frac{320}{(t+2)^3}$ kN, where t is the time in seconds after the driver applies the brakes. Calculate:
   a. the time taken for the speed of the car to be reduced to 57.6 km/h
   b. the distance travelled in this time.

7. A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $6 - 18t$ newtons, where t is the time in seconds and $t \geq 0$. Initially the body is at rest at the origin. Express x in terms of t.

8. A particle of mass 500 grams is moving along the x-axis and at time t seconds is subject to a force of $4e^{2t} - 2$ newtons. Initially the body is at rest at the origin. Express x in terms of t.

9. A particle of mass 5 kilograms moves in a straight line and is acted upon by a force of $30t - 40$ newtons, where t is the time in seconds. Initially the particle is 3 metres to the right of the origin and moving to the right with a speed of 2 m/s. Determine its displacement after 2 seconds.

10. A particle of mass 2 kilograms is moving along the x-axis and at time t seconds is subject to a force of $48t - 12$ newtons. If after 1 second its displacement is 4 metres and its initial velocity is 1 m/s, determine its displacement at any time t seconds.

11. A particle of mass 2 kilograms moves in a straight line so that at time t seconds it is acted upon by a force of $8 - 8e^{-0.1t}$ newtons. If initially it is moving away from the origin with a velocity of 12 metres per second, calculate how far it has travelled in the first 5 seconds.

12. **WE18** A car of mass 1200 kg ascends a hill of slope 1 in 4. If the resistance to the motion is one-fifth of the weight force, calculate the tractive force of the tyres when the car ascends the hill with an acceleration of 2 m/s^2. Give your answer to the nearest newton.

13. A truck of mass 1.5 tonnes is moving at a speed of 36 km/h down a hill of slope 1 in 6. The driver applies the brakes and the truck comes to rest after 2 seconds.
    Calculate the braking force, which is assumed to be constant, ignoring any other resistance forces.

14. **WE19** A parcel of mass 5 kg is pushed up a slope inclined at an angle of 40° to the horizontal by a force of F N that is inclined at an angle of 15° to the plane. The parcel accelerates at 2 m/s^2 up the slope.
    Calculate the value of F.

15. A crate of mass 8 kg sits on a smooth plane inclined at an angle of 25° to the horizontal. It is acted upon by a force of F N that is inclined at an angle of 20° to the plane.
    a. If the crate accelerates at 1 m/s^2 up the slope, calculate the value of F.
    b. Calculate the maximum value of F that can be applied without the crate leaving the plane.

16. WE20 A teenager with mass 55 kg starts to slide down a 6-m long playground slide, which as a slope of 20°. The resistance to the motion has a magnitude of 50 N.
Determine how long it takes the teenager to reach the bottom of the slide and the speed at which the teenager is travelling when they reach the bottom. Give your answers to 2 decimal places.

Complex familiar

17. A bus of mass 6 tonnes moves in a straight line between two stops. The force acting on the bus as it moves between the two stops is given by $2000 - \frac{200t}{9}$ newtons, where t is the time in seconds after it leaves the first stop. Calculate the distance between the two stops and the time it takes to travel between them.

18. a. A particle of mass 4 kilograms moves back and forth along the x-axis and is subjected to a force of $-144\cos(3t)$ newtons at time t seconds. If initially it is at rest 2 metres from the origin, calculate the furthest distance it reaches from the origin.

b. A particle of mass 2 kilograms moves back and forth along the x-axis and is subjected to a force of $-6\sin\left(\frac{t}{2}\right)$ newtons at time t seconds. If its initial velocity is 6 m/s and the particle starts from a point 4 metres from the origin, determine the greatest distance it reaches from the origin.

19. A bullet of mass 15 g is fired vertically upwards with an initial speed of 49 m/s from a height of 2 m above the ground. It is subjected only to the gravitational force. Calculate:

i. its height in metres above the ground at any time, t seconds
ii. the greatest height above ground level that the bullet reaches.

20. A car of mass 1500 kg moves in a straight line. When travelling at 60 km/h the driver applies the brakes. The braking force is $50t$ kN, where t is the time in seconds after the driver applies the brakes. Calculate the distance travelled until the car comes to rest.

21. a. A child of mass 40 kg is on a 2 kg sled which is being pulled up a snow-covered slope by a force of 300 N. The slope is inclined at an angle of 40° to the horizontal. Determine the acceleration of the child and the magnitude of the normal reaction of the child on the snow if the force is:

i. parallel to the slope
ii. acting at an angle of 25° to the slope.

b. A child of mass 40 kg is on a 2 kg sled sliding down a snow-covered slope which is inclined at an angle of 40° to the horizontal. Given that the initial speed is zero and that the friction is negligible, calculate the speed attained when the child and sled have travelled a distance of 20 m down the slope.

22. A particle of mass 3 kilograms is acted upon by a horizontal force of $29.4e^{-0.2t}$ newtons at a time t seconds. Its initial velocity is zero and it starts 20 metres to the right of the origin. Determine the displacement at time t.

Complex unfamiliar

23. A body of mass m kg is moving in a straight line path on a horizontal table by a force of be^{-kt} newtons, where b and k are positive constants. If its initial speed is U m/s, show that after a time t seconds its displacement is given by $x = Ut + \frac{b}{mk}\left[t + \frac{1}{k}\left(e^{-kt} - 1\right)\right]$ metres.

24. A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $\frac{59\,040\sqrt{2}}{\sqrt{(369t + 128)^3}}$ kN where t is the time in seconds after the driver applies the brakes. After a time T seconds, the speed of the car has been reduced to 57.6 km/h, and in this time the car has travelled D metres. Determine the values of T and D for this situation.

Fully worked solutions for this chapter are available online.

LESSON
10.5 Forces that depend on velocity

SYLLABUS LINKS

- Understand and use the expressions $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to represent the acceleration of an object moving in a straight line.

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10.5.1 Introduction

Note that this Lesson and Lesson 10.6 do not use the same type of displacement, velocity and acceleration relationships that would be familiar to Physics students, nor do they use the ones seen in Methods and Specialist so far, when time is the independent variable.

In this Lesson and the following, when forces depends on velocity or displacement instead of depending on time, position, velocity and acceleration are not linked in terms of the time t.

Unit 4 Topics 1 and 3 (Integration techniques and Rates of change and differential equations), chapters 6 and 8 in this series, explore the solving techniques and integration methods, like partial fractions, useful in this chapter.

When the resultant force acting on the body depends upon the velocity v, using Newton's second law of motion gives $ma = m\ddot{x} = F(v)$, where $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt}$.

Additionally, using the chain rule, $\frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v\frac{dv}{dx}$.

Moreover,

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{1}{2} \times 2v\frac{dv}{dx} \\ &= \ddot{x}\end{aligned}$$

Thus, there is more than one approach when applying Newton's second law of motion, depending on whether your aim is to determine quantities such as velocity in terms of time or in terms of displacement.

TIP

When forces are dependent on velocity, when applying Newton's second law of motion, if you are unsure whether to use the rate of change with respect to time t or with respect to displacement x, consider the quantities known and the quantity you aim to determine.

Are the known quantities defined in terms of time or in terms of displacement?

Do the quantities you aim to determine need to be defined in terms of time or in terms of displacement?

Asking yourself those questions should help you select whether to use $\ddot{x} = \frac{d^2x}{dt^2}$, $\ddot{x} = \frac{dv}{dt}$, $\ddot{x} = v\frac{dv}{dx}$ or $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

10.5.2 Setting up the equations of motion

If the force $F = F(v)$ acting on a body of mass m depends upon the velocity, v, then by Newton's second law of motion, $m\ddot{x} = F(v)$. Because $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v\frac{dv}{dx}$ by the chain rule, there are two possible approaches, depending on whether you want to define the velocity in terms of time or in terms of displacement.

For instance, using $\ddot{x} = \frac{dv}{dt}$ and inverting $\frac{dv}{dt} = \frac{F(v)}{m}$ gives $\frac{dt}{dv} = \frac{m}{F(v)}$.

You can then separate the variables and integrate to determine t as a function of v (or v as a function of t by rearranging the expression).

$$\begin{aligned}\frac{dt}{dv} &= \frac{m}{F(v)} \\ dt &= \frac{m}{F(v)}dv \\ \int dt &= \int \frac{m}{F(v)}\,dv \\ t &= m\int \frac{1}{F(v)}\,dv\end{aligned}$$

Remember to use initial conditions to calculate the constant of integration.

Knowing t as a function of v is useful to determine the time taken for the speed to reach a specific value (see Worked example 22a).

Once $v(t)$ is known, use $\dot{x} = \frac{dv}{dt}$ and integrate to determine $x(t)$.

Alternatively, using $\ddot{x} = v\frac{dv}{dx}$ and inverting Newton's second law of motion gives $\frac{dx}{dv} = \frac{mv}{F(v)}$.

You can then separate the variables and integrate to determine x as a function of v:

$$dx = \frac{mv}{F(v)}dv$$
$$\int dx = \int \frac{mv}{F(v)}\,dv$$
$$x = m\int \frac{v}{F(v)}\,dv$$

$x(v)$ can be rearranged to give $v(x)$.

When forces are dependent on displacement (Lesson 10.6), you will practise once again how and when to use these different expressions for the acceleration $\ddot{x}$.

WORKED EXAMPLE 21 Velocity dependent force (1)

A body of mass 5 kg moving in a straight line is opposed by a force of 10 $(v + 3)$ newtons, where v is the velocity in m/s. Initially the body is moving at 3 m/s and is at the origin. Show that the displacement x at time t is given by $x = 3(1 - t - e^{-2t})$.

THINK	WRITE
1. Use Newton's second law of motion. The force opposes the direction of motion.	$m\ddot{x} = -F(v)$ where $m = 5$ and $F(v) = 10\,(v + 3)$
2. Formulate the equation of motion to be solved. As the aim is to express the displacement as a function of time, use $\ddot{x} = \frac{dv}{dt}$ to first determine the velocity as a function of time.	$5\ddot{x} = -10(v + 3)$ $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -2(v + 3)$
3. Invert both sides or separate the variables.	$\frac{dt}{dv} = -\frac{1}{2\,(v + 3)}$ $-2\,\frac{dt}{dv} = \frac{1}{v + 3}$
4. Integrate both sides with respect to v.	$-2t = \int \frac{1}{v + 3}dv$
5. Perform the integration, placing the first constant of integration on one side of the equation.	$-2t = \ln(\lvert v + 3\rvert) + c_1$
6. Use the given initial conditions to calculate the first constant of integration.	Initially, when $t = 0$, $v = 3$: $0 = \ln(6) + c_1$ $c_1 = -\ln(6)$
7. Substitute back for the first constant of integration.	$-2t = \ln(\lvert v + 3\rvert) - \ln(6)$

8. Use the laws of logarithms.	$-2t = \ln\left(\frac{\lvert v+3\rvert}{6}\right)$
9. Use the definition of logarithms to solve for v. The modulus signs are not needed since $e^{-2t} > 0$.	$\frac{v+3}{6} = e^{-2t}$ $v+3 = 6e^{-2t}$ $v = 6e^{-2t} - 3$
10. Given that the velocity as a function of time is now known, use $v = \frac{dx}{dt}$ to determine the displacement as a function of time.	$v = \frac{dx}{dt} = 6e^{-2t} - 3$
11. Integrate both sides with respect to t.	$x = \int (6e^{-2t} - 3)\,dt$
12. Perform the integration, placing the second constant of integration on one side of the equation.	$x = -3e^{-2t} - 3t + c_2$
13. Use the given initial conditions to calculate the second constant of integration.	Initially, when $t = 0$, $x = 0$ (it's at the origin): $0 = -3 + c_2$ $c_2 = 3$
14. Substitute back for the second constant of integration.	$x = -3e^{-2t} - 3t + 3$
15. Factorise and the result is shown.	$x = 3(1 - t - e^{-2t})$

10.5.3 Horizontal rectilinear motion

When a driver of a car travelling at some initial speed applies the brakes, the braking force for a short period of time is a resistance force that opposes the direction of motion and is a function of the speed as the brakes are applied. The equation of motion can be modelled during this time, and the distance travelled while braking can be determined.

WORKED EXAMPLE 22 Velocity dependent force (2)

A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $6400\sqrt{v}$ newtons where v is the speed in m/s after the driver applies the brakes. Calculate:

a. the time taken for the speed of the car to be reduced to 57.6 km/h

b. the distance travelled in this time.

THINK	WRITE
a. 1. Use Newton's second law of motion. The braking force opposes the direction of motion.	**a.** $m\ddot{x} = -F_R(v)$ where $m = 1600$ and $F_R(v) = 6400\sqrt{v}$
2. Formulate the equation of motion to be solved. As the aim is to determine the time taken, use $\ddot{x} = \frac{dv}{dt}$ to determine t in terms of v.	$1600\ddot{x} = -6400\sqrt{v}$ $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -4\sqrt{v}$

3. Invert both sides or separate the variables.

$$\frac{dt}{dv} = -\frac{1}{4\sqrt{v}}$$

$$-4\frac{dt}{dv} = \frac{1}{\sqrt{v}} = v^{-\frac{1}{2}}$$

4. Integrate both sides with respect to v.

$$-4t = \int v^{-\frac{1}{2}} dv$$

5. Perform the integration, placing the first constant of integration on one side of the equation.

$$\begin{aligned} -4t + c_1 &= 2v^{\frac{1}{2}} \\ &= 2\sqrt{v} \end{aligned}$$

6. Use the given initial conditions to calculate the first constant of integration. We need to use correct units.

90 km/h = 25 m/s
Initially, when $t = 0$, $v = 25$:

$$\begin{aligned} c_1 &= 2\sqrt{25} \\ &= 10 \end{aligned}$$

7. Substitute back for the constant of integration.

$$10 - 4t = 2\sqrt{v}$$

8. Determine the braking time.

57.6 km/h = 16 m/s
Calculate t when $v = 16$:

$$\begin{aligned} 10 - 4t &= 2\sqrt{16} = 8 \\ 4t &= 2 \\ t &= \frac{1}{2} \end{aligned}$$

9. State the required result.

After 0.5 seconds, the car's speed has been reduced from 90 to 57.6 km/h.

b. 1. The aim is to determine the displacement when a specific speed is reached, thus x as a function of v is required. Use the alternate form of the equation for the acceleration, $\ddot{x} = v\frac{dv}{dx}$ to determine v in terms of x, then x in terms of v.

b.

$$\ddot{x} = v\frac{dv}{dx} = -4\sqrt{v}$$

$$\begin{aligned} \frac{dv}{dx} &= -\frac{4\sqrt{v}}{v} \\ &= -\frac{4}{\sqrt{v}} \end{aligned}$$

2. Invert both sides or separate the variables.

$$-4\frac{dx}{dv} = v^{\frac{1}{2}}$$

$$-\int 4\, dx = \int v^{\frac{1}{2}} dv$$

3. Perform the integration, placing the second constant of integration on one side of the equation.

$$-4x + c_2 = \frac{2}{3}v^{\frac{3}{2}}$$

4. Use the given initial condition to calculate the second constant of integration.	When $v = 25$, $x = 0$: $c_2 = \frac{2}{3}\sqrt{25^3}$ $= \frac{250}{3}$
5. Substitute back for the second constant of integration.	$\frac{250}{3} - 4x = \frac{2}{3}v^{\frac{3}{2}}$
6. Determine the distance travelled.	Solve for x when $v = 16$: $4x = \frac{250}{3} - \frac{2}{3}\left(\sqrt{16}\right)^3$ $4x = \frac{250 - 128}{3}$ $x = \frac{61}{6}$
7. State the distance travelled while braking.	The distance travelled while braking is $10\frac{1}{6}$ metres.

10.5.4 Resistance forces

When a body moves, the drag force is in general proportional to some power of the velocity. That is, the total air resistance and drag forces can be expressed as $F_R(v) = kv^n$; typical values of n are $1, 2, 3, 4, 5, \frac{1}{2}, \frac{3}{2}$ and so on. Using these and generalising from the last example, some general expressions can be derived.

WORKED EXAMPLE 23 Resistance force dependent on velocity

A car is moving along a straight road at a velocity of U m/s when the driver brakes. The braking force in newtons is proportional to the fourth power of the velocity, v, where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2}U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{9U}{14}$.

THINK	WRITE
1. Use Newton's second law of motion. The braking force opposes the direction of motion.	Let the mass of the car be m and the proportionality constant be k. $F_R \propto v^4$ then $F_R(v) = kv^4$ and $m\ddot{x} = -F_R(v) = -kv^4$ $\ddot{x} = -\lambda v^4$ where $\lambda = \frac{k}{m}$ is one single constant.
2. First obtain a relationship between v and t. Thus use $\ddot{x} = \frac{dv}{dt}$.	Use $\ddot{x} = \frac{dv}{dt} = -\lambda v^4$.

3. Invert both sides and separate the variables.

$$\frac{dt}{dv} = -\frac{1}{\lambda v^4}$$

4. Integrate both sides with respect to v.

$$\int -\lambda dt = \int v^{-4} dv$$

5. Perform the integration, placing the first constant of integration on one side of the equation.

$$-\lambda t + c_1 = -\frac{1}{3}v^{-3}$$

6. Use the given initial conditions to calculate the first constant of integration. We are using correct units.

Initially, when $t = 0$, $v = U$:

$$c_1 = -\frac{1}{3}U^{-3}$$

$$c_1 = -\frac{1}{3U^3}$$

7. Substitute back for the constant of integration.

$$-\lambda t - \frac{1}{3U^3} = -\frac{1}{3v^3}$$

$$\lambda t + \frac{1}{3U^3} = \frac{1}{3v^3}$$

8. Obtain a relationship between the parameters.

When $t = T$, $v = \frac{1}{2}U$:

$$\lambda T + \frac{1}{3U^3} = \frac{1}{3\left(\frac{1}{2}U\right)^3}$$

$$= \frac{8}{3U^3}$$

9. Simplify this relationship and express λ in terms of T and U.

$$\lambda T = \frac{8}{3U^3} - \frac{1}{3U^3}$$

$$\lambda T = \frac{7}{3U^3}$$

$$\lambda = \frac{7}{3TU^3}$$

10. Next, obtain a relationship between v and x. To do this, use $a = v\frac{dv}{dx}$.

Use $\ddot{x} = v\frac{dv}{dx} = -\lambda v^4$ so that

$$\frac{dv}{dx} = -\lambda v^3$$

11. Invert both sides or separate the variables.

$$\frac{dx}{dv} = -\frac{1}{\lambda v^3}$$

12. Integrate both sides with respect to v.

$$\int -\lambda dx = \int v^{-3} dv$$

13. Perform the integration, placing the second constant of integration on one side of the equation.

$$-\lambda x + c_2 = -\frac{1}{2}v^{-2}$$

14. Use the given initial conditions to calculate the second constant of integration.

Initially (when $t = 0$) $x = 0$ and $v = U$:

$$c_2 = -\frac{1}{2}U^{-2}$$

$$c_2 = -\frac{1}{2U^2}$$

15. Substitute back for the second constant of integration.

$$-\lambda x - \frac{1}{2U^2} = -\frac{1}{2v^2}$$

$$\lambda x + \frac{1}{2U^2} = \frac{1}{2v^2}$$

16. Obtain a relationship between the parameters.

When $x = D$, $v = \frac{1}{2}U$:

$$\lambda D + \frac{1}{2U^2} = \frac{1}{2\left(\frac{1}{2}U\right)^2}$$

$$= \frac{2}{U^2}$$

17. Simplify this relationship and express λ in terms of D and U.

$$\lambda D = \frac{2}{U^2} - \frac{1}{2U^2}$$

$$\lambda D = \frac{3}{2U^2}$$

$$\lambda = \frac{3}{2DU^2}$$

18. Eliminate λ by equating the two expressions for λ.

$$\lambda = \frac{7}{3TU^3} = \frac{3}{2DU^2}$$

19. Simplify the resulting expression.

$$7 \times 2DU^2 = 9TU^3$$

$$14DU^2 = 9TU^3$$

$$\frac{D}{T} = \frac{9U^3}{14U^2}$$

$$\frac{D}{T} = \frac{9U}{14}$$

Hence $\frac{D}{T} = \frac{9U}{14}$ as required.

10.5.5 Vertical motion under gravity

When a body moves vertically, its weight force must be considered as part of its equation of motion.

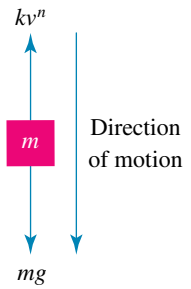
In Unit 3 Topic 4 vector calculus, we investigated vertical motion under gravity with a constant acceleration. Review in particular Worked examples 29 and 30, which considered that the only force acting on the objects was gravity, resulting in a vertical acceleration of $-9.8\,\text{m/s}^2$.

In other examples, vertical displacement was not described as a quadratic with the coefficient of t^2 being -4.9. These other situations recognise that in the real world, when objects are large and falling a long distance, other forces besides the weight force act on an object which is moving vertically.

So far, you must be used to air resistance being considered negligible (as in Worked example 24), however, you now have the mathematical tools and skills to take air resistance into account and you will see how to do it in Worked example 25.

Downwards motion

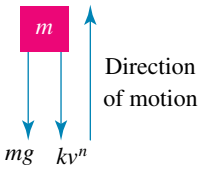
Consider a body of mass m moving vertically downwards. The forces acting on the body are its weight force, which acts vertically downwards, and the force of air resistance, which opposes the direction of motion and acts vertically upwards.



The resistance force is proportional to some power of the body's velocity. Considering downwards as the positive direction, the body's equation of motion is given by $m\ddot{x} = mg - kv^n$. As the body falls, it reaches a so-called **terminal** or **limiting velocity**, v_T. This value can be obtained from $v_T = \lim_{t\to\infty} v(t)$, or as it is a constant speed when the acceleration is zero, $\ddot{x} = 0$. Thus, the terminal velocity satisfies $mg - kv_T^n = 0$.

Upwards motion

Consider a body of mass m moving vertically upwards. The forces acting on the body are its weight force, which acts vertically downwards, and the force of air resistance, which opposes the direction of motion and also acts vertically downwards.



The resistance force is proportional to some power of the body's velocity. Considering upwards as the positive direction, the body's equation of motion is given by $m\ddot{x} = -mg - kv^n = -(mg + kv^n)$.

Typical values of n are $1, 2, 3, 4, 5, \frac{1}{2}$ and $\frac{3}{2}$.

If the object that is falling is very small in area, or is only falling a short distance, the object will have minimal air resistance and will fall with a constant acceleration of $g\,\text{m/s}^2$ downwards.

WORKED EXAMPLE 24 Vertical motion with no air resistance

A lead pellet falls from a height of 0.5 m. Determine the time to fall, assuming no air resistance.

THINK	WRITE
1. Consider down to be the negative direction of the y-axis (the $-\hat{\boldsymbol{j}}$ component).	$\ddot{y} = -9.8\hat{\boldsymbol{j}}$
2. Determine an equation for $\dot{y}$ by integrating $\ddot{y}$.	$\dot{y} = \int \ddot{y}\,dt$ $= -9.8t + c_1$

3. Use the initial condition to define the constant of integration. The pellet is initially at rest, its initial velocity is zero.

$$\dot{y}(0) = 0$$
$$= -9.8 \times 0 + c_1$$
$$c_1 = 0$$

4. Determine an equation for y by integrating $\dot{y}$.

$$y(t) = \int \dot{y}\, dt$$
$$= -4.9t^2 + c_2$$

5. Use the initial condition to determine the second constant of integration.

$$y(0) = 0.5$$
$$= -4.9 \times 0 + c_2$$
$$c_2 = 0.5$$
$$y(t) = -4.9t^2 + 0.5$$

6. Determine the time when $y = 0$.

$$-4.9t^2 + 0.5 = 0$$
$$t = \pm\sqrt{\frac{0.5}{4.9}}$$

We disregard the negative time in this context.

$$t = \sqrt{\frac{0.5}{4.9}}$$
$$\approx 0.32\text{ s}$$

7. State the time it takes for the pellet to fall.

It takes approximately 0.32 s for the pellet to fall.

WORKED EXAMPLE 25 Vertical motion with air resistance

A drone is used to drop medical supplies.
Air resistance, proportional to the first power of velocity, affects the dropped package.

a. Use a vector calculus approach to model the vertical velocity of the package.

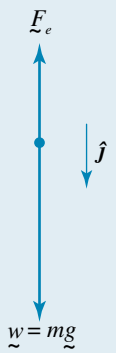
b. Determine an expression for the terminal velocity of the package.

THINK

a. 1. Draw a force diagram and define the positive $\hat{\boldsymbol{j}}$ component. The air resistance is in the direction opposite to the direction of motion.

WRITE

a.



2. Use Newton's second law to define the acceleration of the package.

$$m\underset{\sim}{a} = \underset{\sim}{w} + \underset{\sim}{F}_R$$

3. Resolve forces in the $\hat{\boldsymbol{j}}$ direction to determine an equation for the acceleration.

$$ma = mg - kv$$
$$\frac{dv}{dt} = \frac{mg - kv}{m}$$

4. Determine t in terms of v by integrating.

$$\frac{dt}{dv} = \frac{m}{mg - kv}$$
$$dt = \frac{mdv}{mg - kv}$$
$$\int dt = \int \frac{m}{mg - kv}\, dv$$
$$t = -\frac{m}{k} \ln |mg - kv| + c \text{ or } t = -\frac{m}{k} \ln |A\,(mg - kv)|$$

Method 1

5. Use the initial condition to determine the constant of integration c. The package is dropped from rest, its initial velocity is zero.

$$t = -\frac{m}{k} \ln |mg - kv| + c$$
When $t = 0$, $v = 0$
$$0 = -\frac{m}{k} \ln |mg| + c$$
$$c = \frac{m}{k} \ln |mg|$$
$$t = -\frac{m}{k} \ln |mg - kv| + \frac{m}{k} \ln |mg|$$
$$= \frac{m}{k} \ln \left| \frac{mg}{mg - kv} \right|$$

6. Rearrange to determine V in terms of t.

$$\frac{kt}{m} = \ln \left| \frac{mg}{mg - kv} \right|$$
$$e^{\frac{kt}{m}} = \frac{mg}{mg - kv}$$
$$mg - kv = mge^{-\frac{kt}{m}}$$
$$v = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$$

Method 2

7. Use the initial condition to determine the constant of integration A.

$$-\frac{kt}{m} = \ln |A\,(mg - kv)|$$
$$e^{-\frac{kt}{m}} = A\,(mg - kv)$$
When $t = 0$, $v = 0$
$$e^0 = A\,(mg)$$
$$A = \frac{1}{mg}$$

8. Rearrange to determine V in terms of t.

$$e^{-\frac{kt}{m}} = \frac{mg - kv}{mg}$$
$$= 1 - \frac{kv}{mg}$$
$$\frac{kv}{mg} = 1 - e^{-\frac{kt}{m}}$$
$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$$

b. 1. The terminal velocity is the maximum speed. As $v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$, v has a horizontal asymptote when $t \to \infty$.

b. $\lim_{t \to \infty} v(t) = \frac{mg}{k}$

2. State the answer.

The maximum velocity is $\frac{mg}{k}$.

WORKED EXAMPLE 26 Air resistance proportional to v^2

A large brick of mass 5 kg is accidentally dropped from a high-rise construction site. As it falls vertically downwards it is retarded by a force of $0.01v^2$ newtons, where v m/s is the speed of the brick at a time t seconds after it was dropped. It has travelled a distance of x metres in this time.

a. Show that while the brick is falling, its equation of motion is given by $\ddot{x} = \frac{4900 - v^2}{500}$.

b. Show that $x = 250 \ \ln\left(\frac{4900}{4900 - v^2}\right)$.

c. Calculate the magnitude of the momentum of the brick after it has fallen a distance of 100 m.

d. Show that $t = \frac{25}{7} \ \ln\left(\frac{70 + v}{70 - v}\right)$.

e. Calculate the terminal velocity of the brick.

f. Calculate the time taken for the brick to fall a distance of 100 m.

THINK

a. 1. Use Newton's second law of motion.

2. Formulate the equation of motion to be solved. Simplify and the required result is shown.

WRITE

a. $m = 5, k = 0.01$
$m\ddot{x} = mg - kv^2$

$$5\ddot{x} = 5 \times 9.8 - 0.01v^2$$
$$5\ddot{x} = 49 - \frac{v^2}{100}$$
$$5\ddot{x} = \frac{4900 - v^2}{100}$$
$$\ddot{x} = \frac{4900 - v^2}{500}$$

b. 1. To obtain a relationship between v and x, use $\ddot{x} = v\dfrac{dv}{dx}$ and invert both sides.

b. $\ddot{x} = v\dfrac{dv}{dx} = \dfrac{4900 - v^2}{500}$

$$\frac{dx}{dv} = \frac{500v}{4900 - v^2}$$

2. Integrate both sides with respect to v.

$$x = 500\int \frac{v}{4900 - v^2}\,dv$$
$$= -\frac{500}{2}\int \frac{-2v}{4900 - v^2}\,dv$$

3. Perform the integration, placing the first constant of integration on one side of the equation.

$$x = -250\ln\left(4900 - v^2\right) + c_1$$

4. Use the given initial conditions to calculate the first constant of integration.

Since the brick was dropped when $t = 0$, $x = 0$ and $v = 0$:

$$0 = -250\ln(4900) + c_1$$
$$c_1 = 250\ln(4900)$$

5. Substitute back for the first constant of integration and take out common factors.

$$x = -250\ln(4900 - v^2) + 250\ln(4900)$$
$$= 250\left[\ln(4900) - \ln(4900 - v^2)\right]$$

6. Use log laws to show the required result.

$$x = 250\ln\left(\frac{4900}{4900 - v^2}\right)$$

c. 1. To calculate the momentum, we first need to determine the speed.

c. When $x = 100$, $v = ?$

$$100 = 250\ln\left(\frac{4900}{4900 - v^2}\right)$$

2. Use the definition of the logarithm and transpose to make v the subject.

$$e^{0.4} = \frac{4900}{4900 - v^2}$$
$$4900 - v^2 = 4900e^{-0.4}$$
$$v^2 = 4900(1 - e^{-0.4})$$

3. Calculate the speed when the brick has fallen this required distance.
In this case choose the positive value of the velocity as the brick is travelling in the same direction as the direction of motion.

$$v = 70\sqrt{(1 - e^{-0.4})}$$
$$v = 40.19243 \text{ m/s}$$

4. Determine the magnitude of the momentum.

$$p = mv$$
$$= 5 \times 40.19243$$
$$= 200.962$$

The momentum of the brick is 200.96 kg m/s.

d. 1. To obtain a relationship between v and t, use $\ddot{x} = \dfrac{dv}{dt}$ and invert both sides.

d. $$\frac{dv}{dt} = \frac{4900 - v^2}{500}$$
$$\frac{dt}{dv} = \frac{500}{4900 - v^2}$$

2. Integrate both sides with respect to v.

$$t = \int \frac{500}{4900 - v^2}\, dv$$

3. To determine this integral, use partial fractions. Express the integrand into its partial fractions decomposition.

$$\begin{aligned}\frac{500}{4900 - v^2} &= \frac{A}{70 - v} + \frac{B}{70 + v} \\ &= \frac{A(70 + v) + B(70 - v)}{(70 - v)(70 + v)} \\ &= \frac{70(A + B) + v(A - B)}{4900 - v^2}\end{aligned}$$

4. Calculate the values of the constants A and B.

Equate the coefficients:

$v: \ A - B = 0 \Rightarrow A = B$

constant term: $70(A + B) = 500$

Since $A = B \Rightarrow 70\,(A + A) = 500$

$$\begin{aligned}140\,A &= 500 \\ A &= \frac{500}{140} \\ A &= \frac{25}{7} \\ \therefore A = B &= \frac{25}{7}\end{aligned}$$

5. Express the integrand in a form for which we can perform the integration.

$$\begin{aligned}t &= \int \frac{500}{4900 - v^2} dv \\ &= \frac{25}{7} \int \left(\frac{1}{70 + v} + \frac{1}{70 - v} \right) dv\end{aligned}$$

6. Perform the integration, placing the second constant of integration on one side of the equation.

$$t = \frac{25}{7} \left(\ln\left(|70 + v|\right) - \ln\left(|70 - v|\right) + c_2 \right)$$

7. Use the given initial condition to calculate the second constant of integration.

When $x = 0$, $t = 0$, $v = 0$:

$$0 = \frac{25}{7}(\ln(70) - \ln(70) + c_2)$$

$$c_2 = 0$$

8. Substitute back for the second constant of integration and use log laws again. The required result is shown.

$$t = \frac{25}{7} \left(\ln\left(|70 + v|\right) - \ln\left(|70 - v|\right) \right)$$

$$t = \frac{25}{7} \ln\left(\left| \frac{70 + v}{70 - v} \right| \right)$$

But since $0 \le v < 70$, the modulus signs are not needed.

$$t = \frac{25}{7} \ln\left(\frac{70 + v}{70 - v} \right)$$

e. The terminal velocity can be found when the acceleration is zero.

e. $\dfrac{4900 - v^2}{500} = 0$

$v^2 = 4900$

$v_T = \sqrt{4900}$

$v_T = 70$

The terminal velocity is 70 m/s.

f. 1. Determine the time taken for the brick to fall the required distance.

f. $t = ?$ when

$v = 40.19243$

$t = \dfrac{25}{7} \ln\left(\dfrac{70 + 40.19243}{70 - 40.19243}\right)$

$= 4.67$

2. State the time to fall the required distance.

The time taken to fall is 4.67 s.

3. An alternate method to determine the time is to numerically evaluate a definite integral.

$t = \displaystyle\int_0^{40.19243} \frac{500}{4900 - v^2}\, dv = 4.67$

Exercise 10.5 Forces that depend on velocity

learnon

10.5 Exercise	10.5 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	12, 13, 14	15, 16, 17, 18, 19, 20

These questions are even better in jacPLUS!

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Simple familiar

1. WE21 A body of mass 2 kg moving in a straight line is opposed by a force of $v - 8$ newtons, where v is the velocity in m/s. Initially the body is at rest at the origin.

Show that the displacement at time t is given by $x = 8\left(t + 2\left(e^{-\frac{t}{2}} - 1\right)\right)$.

2. A yacht of mass 500 kg is sailing in a straight line at a speed of 57.6 km/h when the driver disengages the engine. The resistance force is $400\sqrt{v}$ newtons, where v m/s is the speed of the yacht at a time t seconds. Calculate:

a. the time taken for the speed of the yacht to be reduced to 14.4 km/h

b. the distance travelled in this time.

3. A body of mass 6 kg moving in a straight line is acted upon by a resistance force of $2(v-6)$ newtons, where v is the velocity in m/s. Initially the body is moving at 12 m/s and is at the origin. Show that the displacement at time t is given by $x = 6\left(t + 3\left(1 - e^{-\frac{t}{3}}\right)\right)$.

4. **WE22** A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is $320\sqrt{v^3}$ newtons, where v is the speed in m/s after the driver applies the brakes. Calculate:
   a. the time taken for the speed of the car to be reduced to 57.6 km/h
   b. the distance travelled in this time.

5. A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is $\frac{369}{100}v^3$ newtons, where v is the speed in m/s after the driver applies the brakes. After a time T seconds, the speed of the car has been reduced to 57.6 km/h and in this time the car has travelled a distance of D metres. Determine the values of T and D for this situation.

6. **WE23** A car moves along a straight road at a speed of U m/s when the driver brakes. The resistance braking force in newtons is proportional to the cube of the velocity v, where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2}U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{2U}{3}$.

7. A car moves along a straight road at a speed of U m/s when the driver brakes. The resistance braking force in newtons is proportional to the square root of the velocity v cubed, where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2}U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{\sqrt{2}U}{2}$.

8. **WE24** A small rock falls from a height of 8.7 m. Determine the time to fall, assuming no air resistance and give your answer to 2 decimal places.

9. **WE25** A bearded vulture is dropping the femur of a sheep from a great height to break it and access the marrow.
   a. Air resistance, proportional to the velocity, affects the dropped bone. Use a vector calculus approach to model the vertical velocity of the bone.
   b. Determine an expression for the terminal velocity of the bone.

10. WE26 A skydiver of mass 90 kg falls vertically from rest from a plane. While falling vertically downwards he is retarded by a force of $0.1v^2$ newtons, where v m/s is his speed at a time t seconds after falling a distance of x metres.

a. Show that while the skydiver is falling, his equation of motion is given by $\ddot{x} = \dfrac{8820 - v^2}{900}$.

b. Show that $x = 450 \ln\left(\dfrac{8820}{8820 - v^2}\right)$.

c. Determine the magnitude of the momentum of the skydiver after he has fallen a distance of 150 metres.

d. Determine the terminal velocity of the skydiver.

e. Calculate the time taken for the skydiver to fall a distance of 150 metres.

11. **a.** A body of mass 2 kg moving in a straight line is acted upon by a resistance force of $4v$ newtons, where v is the velocity in m/s. Initially the body is moving at 1 m/s and is at the origin. Show that the displacement at time t is given by $x = \dfrac{1}{2}(1 - e^{-2t})$.

b. A body of mass 3 kg moving in a straight line is acted upon by a resistance force of $9(v + 4)$ newtons, where v is the velocity in m/s. Initially the body is moving at 2 m/s and is at the origin. Show that the displacement at time t is given by $x = 2(1 - e^{-3t} - 2t)$.

Complex familiar

12. A body of mass m kg is moving in a straight line path on a horizontal surface and is acted upon by a resistive force that is proportional to its speed, the constant of proportionality being k. If its initial speed is U m/s, show that:

a. its speed, v m/s, at a time t seconds satisfies $v = Ue^{-\frac{kt}{m}}$

b. its displacement, x m, after a time t seconds is given by $x = \dfrac{mU}{k}\left(1 - e^{-\frac{kt}{m}}\right)$

c. its speed, v m/s, after moving a distance x m is given by $v = U - \dfrac{kx}{m}$.

13. A sports car of mass 800 kg is moving along a level road at a speed of 57.6 km/h when the driver applies the brakes. The braking force is $80v^{\frac{3}{2}}$ newtons, where v m/s is the speed of the car at a time t seconds. After a time T s, the speed of the car is 14.4 km/h, and in this time it has travelled a distance of D m. Calculate the values of:

a. T

b. D.

14. A body of mass m kg falls from rest in a gravitational field and is subject to a force of air resistance in newtons that is proportional to its speed, the constant of proportionality being k.

a. Show that the speed at any time, t seconds, is given by $v = \dfrac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$.

b. Deduce that the limiting (or terminal speed) is given by $\dfrac{mg}{k}$.

c. Show that when the speed is half the terminal speed, the distance of the body below the point of projection is given by $\dfrac{0.193m^2g}{k^2}$ metres.

Complex unfamiliar

15. A body of mass m kg falls vertically. While falling vertically downwards the particle is retarded by a force of kv^2 newtons, where k is a positive constant and v m/s is the speed at a time t seconds. Show that the terminal speed is given by $v_T = \sqrt{\frac{mg}{k}}$ and that $t = \frac{v_T}{2g} \ln\left(\frac{v_T + v}{v_T - v}\right)$.

16. A block of mass m kg is moving in a straight line path on a smooth horizontal surface and is acted upon by a resistive force in newtons that is proportional to the square of its speed, the constant of proportionality being k. If its initial speed is U m/s, show that:

a. its speed, v m/s, after moving a distance of x metres is given by $v = Ue^{-\frac{kx}{m}}$

b. its speed, v m/s, at a time t seconds satisfies $v = \frac{mU}{m + kUt}$.

17. During a snow storm, a small block of ice of mass 10 grams falls from the sky.
As it falls vertically downwards it is retarded by a force of $0.002v^2$ newtons, where v m/s is its speed at a time t seconds after falling a distance of x m.

a. Show that while the ice block is falling, its equation of motion is given by $\ddot{x} = \frac{49 - v^2}{5}$.

b. Show that $v = 7\sqrt{1 - e^{-0.4x}}$.

c. Show that $v = \frac{7\left(1 - e^{-2.8t}\right)}{1 + e^{-2.8t}}$ and hence determine the terminal speed of the ice block.

d. Show that $x = 5 \ln\left(\frac{e^{1.4t} + e^{-1.4t}}{2}\right)$.

18. A ball of mass m kg is projected vertically upwards from ground level with an initial speed of U m/s. While travelling upwards or downwards it is subjected to a force of air resistance equal to kv^2 newtons, where k is a positive constant and v is its velocity in m/s.

a. Show that the ball reaches a maximum height of $\frac{m}{2k} \ln\left(1 + \frac{kU^2}{mg}\right)$ m.

b. Show that the time required for the ball to reach its maximum height is given by $\sqrt{\frac{m}{kg}} \tan^{-1}\left(U\sqrt{\frac{k}{mg}}\right)$ s.

c. Show that the ball returns to its original point with a speed of $U\sqrt{\frac{mg}{mg + kU^2}}$ m/s.

19. A train of mass m kg is moving in a straight line and is acted upon by a force of air resistance that is equal to $a + bv$ newtons, where a and b are positive constants and v m/s is its speed at any time t seconds. If its initial speed is U m/s and it travels a distance of D m before coming to rest in a time of T s, show that:

a. $T = \frac{m}{b} \ln\left(1 + \frac{bU}{a}\right)$

b. $D = \frac{1}{b}(mU - aT)$.

20. A motor car of mass m kg is travelling with a speed of v m/s along a level section of road when the brakes are applied. The resistance to the motion of the car is given by $a + bv^2$ newtons, where a and b are positive constants. Show that, with the engine disengaged, the brakes will bring the car from an initial speed of U m/s to rest in a time T s, and that the car will travel a distance of D m, where

$$T = \frac{m}{\sqrt{ab}} \tan^{-1}\left(U\sqrt{\frac{b}{a}}\right) \text{ and } D = \frac{m}{2b} \ln\left(1 + \frac{bU^2}{a}\right).$$

Fully worked solutions for this chapter are available online.

LESSON
10.6 Forces that depend on displacement

SYLLABUS LINKS

- Understand and use the expressions $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to represent the acceleration of an object moving in a straight line.
- Model and solve problems that involve motion in a straight line with both constant and non-constant acceleration, including simple harmonic motion, vertical motion under gravity with and without air resistance, and motion of a body in non-equilibrium situations on a smooth inclined plane (excluding situations with pulleys and connected bodies).
  - If $\frac{d^2x}{dt^2} = -\omega^2 x$ then $x = A\sin(\omega t + \alpha)$ or $x = A\cos(\omega t + \beta)$
  - $v^2 = \omega^2\left(A^2 - x^2\right)$
  - $T = \frac{2\pi}{\omega}$
  - $f = \frac{1}{T}$

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10.6.1 Setting up the equations of motion

When the resultant force acting on the body depends upon the displacement x, using Newton's second law of motion gives $ma = m\ddot{x} = F(x)$, thus the acceleration is a function of the displacement.

So far, we have used $\ddot{x} = \frac{d^2x}{dt^2}$, $\ddot{x} = \frac{dv}{dt}$, $\ddot{x} = v\frac{dv}{dx}$ to obtain relationships between time, displacement, velocity and acceleration. We have not yet used $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

Integrating both sides of $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{F(x)}{m}$ with respect to x gives $\frac{1}{2}v^2 = \int \frac{F(x)}{m}\,dx$, which provides a relationship between v and x. Remember to use initial conditions to calculate the constant of integration.

The following worked example displays the use of $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to determine v^2 and thus v. In that case, to determine where the particle comes to rest, we need to solve $v(x) = 0$ for x, that is, we first need to determine $v(x)$. By using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, v^2 and thus v can easily be determined.

Note that $a = v\dfrac{dv}{dx}$ can also be used.

WORKED EXAMPLE 27 Force dependent on displacement

A particle of mass 3 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\dfrac{-(15-6x)}{x^3}$ newtons and the particle is at rest at a distance of 5 m to the right of the origin. Determine where else the particle comes to rest.

THINK	WRITE
1. Use Newton's second law of motion.	$m\ddot{x} = F(x)$ where $m = 3$ and $F(x) = \dfrac{-(15-6x)}{x^3}$
2. Formulate the equation of motion to be solved. The particle comes to rest when $v = 0$, thus the aim is to determine $v(x)$. To do this, use $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, which will provide v^2 and thus v. Simplify by dividing through by 3.	$3\ddot{x} = \dfrac{-(15-6x)}{x^3}$ $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{-(5-2x)}{x^3}$
3. Integrate both sides with respect to x.	$\dfrac{1}{2}v^2 = \displaystyle\int \left(\dfrac{-(5-2x)}{x^3}\right) dx$
4. Express the integrand in index notation.	$\dfrac{1}{2}v^2 = \displaystyle\int \left(-5x^{-3} + 2x^{-2}\right) dx$
5. Perform the integration, placing the constant of integration on one side of the equation.	$\dfrac{1}{2}v^2 = \dfrac{5}{2}x^{-2} - 2x^{-1} + c$
6. Write the expression with positive indices.	$\dfrac{1}{2}v^2 = \dfrac{5}{2x^2} - \dfrac{2}{x} + c$
7. Use the given initial conditions to calculate the constant of integration.	When $v = 0$, $x = 5$: $0 = \dfrac{5}{2(5)^2} - \dfrac{2}{5} + c$ $c = \dfrac{2}{5} - \dfrac{1}{10} = \dfrac{3}{10}$
8. Substitute back for the constant of integration.	$\dfrac{1}{2}v^2 = \dfrac{5}{2x^2} - \dfrac{2}{x} + \dfrac{3}{10}$
9. Form a common denominator.	$\dfrac{1}{2}v^2 = \dfrac{25 - 20x + 3x^2}{10x^2}$
10. Factorise the quadratic in the numerator.	$v^2 = \dfrac{(3x-5)(x-5)}{5x^2}$
11. Express the velocity in terms of x.	$v = \dfrac{1}{\lvert x \rvert}\sqrt{\dfrac{(3x-5)(x-5)}{5}}$
12. Determine the values of x when the particle comes to rest.	If $v = 0$, then $(3x-5)(x-5) = 0$, so $x = 5$ or $x = \dfrac{5}{3}$ As we were given $x = 5$ when $v = 0$, the required solution is $x = \dfrac{5}{3}$.
13. State the final result.	The particle comes to rest $\dfrac{5}{3}$ metres from the origin.

Alternate method

1. Alternatively, $a = v\dfrac{dv}{dx}$ can be used in Newton's second law of motion.

$$3a = \frac{-(15-6x)}{x^3}$$
$$a = v\frac{dv}{dx}$$
$$= \frac{-(5-2x)}{x^3}$$

2. Separate the variables.

$$v\,dv = \frac{-(5-2x)}{x^3}\,dx$$

3. Integrate both sides.

$$\int v\,dv = \int \frac{-(5-2x)}{x^3}\,dx$$
$$= \int \left(\frac{2}{x^2} - \frac{5}{x^3}\right) dx$$
$$\frac{1}{2}v^2 = -\frac{2}{x} + \frac{5}{2x^2} + c$$
$$= \frac{5-4x}{2x^2} + c$$

4. Use the given initial conditions to calculate the constant of integration.

When $v = 0$, $x = 5$.
$$0 = \frac{5-20}{50} + c$$
$$c = \frac{3}{10}$$

5. Substitute back for the constant of integration.

$$\frac{1}{2}v^2 = \frac{5-4x}{2x^2} + \frac{3}{10}$$

6. Rearrange to express v^2, form a common denominator and factorise the numerator.

$$v^2 = \frac{5-4x}{x^2} + \frac{3}{5}$$
$$= \frac{25-20x+3x^2}{5x^2}$$
$$= \frac{(3x-5)(x-5)}{5x^2}$$

7. Express the velocity v in terms of the displacement x.

$$v = \frac{1}{|x|}\sqrt{(3x-5)(x-5)}$$

8. Determine the values of x when $v = 0$.

$x = 5$ or $x = \dfrac{5}{3}$.
$x = 5$ is the position given, thus the other position the particle comes to rest is $x = \dfrac{5}{3}$.

9. State the answer.

The particle comes to rest $\dfrac{5}{3}$ m from the origin.

10.6.2 Differentiating and rearranging

We have seen in Worked example 27 that both and $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and $a = v\frac{dv}{dx}$ can also be used, whether you select to use one expression or the other will impact the working involved, but not the final result.

WORKED EXAMPLE 28 Determining the acceleration, given $v(x)$

A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{9 - 4x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = -4x$.

THINK	WRITE
1. Differentiate using the chain rule.	$v = \sqrt{9-4x^2} = (9-4x^2)^{\frac{1}{2}}$ $\frac{dv}{dx} = \frac{1}{2} \times -8x \times (9-4x^2)^{-\frac{1}{2}}$ $= \frac{-4x}{\sqrt{9-4x^2}}$
2. Use an expression for the acceleration.	$a = v\frac{dv}{dx}$ $a = \sqrt{9-4x^2} \times \frac{-4x}{\sqrt{9-4x^2}}$ $a = -4x$
Alternate method	
1. Alternatively, as the velocity is the square root of an expression depending upon the displacement x, using $a = \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ simplifies the working.	$v^2 = 9 - 4x^2$ $\frac{1}{2}v^2 = \frac{9}{2} - 2x^2$ $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $= -4x$

Expressing x in terms of t

In some cases it may be possible to rearrange and express the displacement x in terms of t by solving for v and using $v = \frac{dx}{dt}$.

WORKED EXAMPLE 29 Determining $x(t)$ given $F(x)$

A body of mass 5 kg moves in a straight line and is retarded by a force of $20x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 1 m to the right of the origin and the initial velocity of the body is 2 m/s. Express x in terms of t where t is the time in seconds.

THINK	WRITE
1. Use Newton's second law of motion. The braking force opposes the direction of motion.	$m\ddot{x} = F(x)$ where $m = 5$ and $F(x) = -20x$
2. Formulate the equation of motion to be solved.	$5\ddot{x} = -20x$ $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$
3. Integrate both sides with respect to x.	$\frac{1}{2}v^2 = \int -4x\,dx$
4. Perform the integration, placing the constant of integration on one side of the equation.	$\frac{1}{2}v^2 = -2x^2 + c_1$
5. Use the given initial conditions to calculate the first constant of integration.	Initially, when $t = 0$, $v = 2$ and $x = 1$: $2 = -2 + c_1$ $c_1 = 4$
6. Substitute back for the first constant of integration.	$\frac{1}{2}v^2 = -2x^2 + 4$
7. Rearrange and solve for v.	$v^2 = 8 - 4x^2$ $v^2 = 4(2 - x^2)$ $v = \pm 2\sqrt{2 - x^2}$
8. Initially, when $t = 0$, $x = 1$ and $v = 2$, $2 > 0$; therefore, we can take the positive root only.	$v = \frac{dx}{dt} = 2\sqrt{2 - x^2},\ 0 \le x \le \sqrt{2}$
9. Invert both sides.	$\frac{dt}{dx} = \frac{1}{2\sqrt{2 - x^2}}$
10. Integrate both sides with respect to x.	$t = \frac{1}{2}\int \frac{1}{\sqrt{2 - x^2}}\,dx$
11. Perform the integration.	$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + c_2$
12. Use the given initial conditions to calculate the second constant of integration.	When $t = 0$, $v = 2$ and $x = 1$: $0 = \frac{1}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + c_2$ $c_2 = -\frac{\pi}{8}$
13. Substitute back for the second constant of integration.	$t = \frac{1}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{\pi}{8}$

14. Rearrange to make x the subject.

$$\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) = 2t + \frac{\pi}{4}$$
$$\frac{x}{\sqrt{2}} = \sin\left(2t + \frac{\pi}{4}\right)$$

15. Expand using compound-angle formula $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.

$$\frac{x}{\sqrt{2}} = \sin(2t)\cos\left(\frac{\pi}{4}\right) + \cos(2t)\sin\left(\frac{\pi}{4}\right)$$
$$\frac{x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin(2t) + \frac{1}{\sqrt{2}}\cos(2t)$$

16. State the final result.

$$x(t) = \sin(2t) + \cos(2t)$$

Alternate method

1. Alternatively, $a = v\dfrac{dv}{dx}$ can be used in Newton's second law of motion.

$$5a = -4x$$
$$a = v\frac{dv}{dx}$$
$$= -4x$$

2. Separate the variables.

$$v\,dv = -4x\,dx$$

3. Integrate both sides.

$$\int v\,dv = \int -4x\,dx$$
$$\frac{1}{2}v^2 = -2x^2 + c$$

4. Use the given initial conditions to calculate the constant of integration.

When $t = 0$, $v = 2$ and $x = 1$
$$2 = -2 + c$$
$$c = 4$$

5. Substitute back for the constant of integration.

$$\frac{1}{2}v^2 = -2x^2 + 4$$

6. Rearrange to express v.

$$v^2 = 4\left(2 - x^2\right)$$
$$v = \pm 2\sqrt{2 - x^2}$$

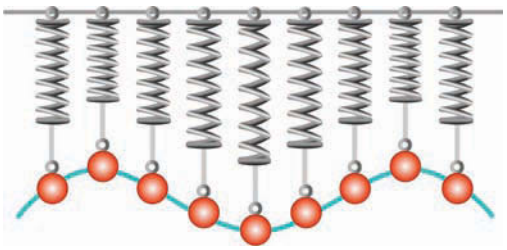
7. Initially, when $t = 0$, $v = 2 > 0$ therefore we can eliminate the negative square root.

$$v = 2\sqrt{2 - x^2}, 0 \le x \le \sqrt{2}$$

The remaining steps are the same as with the previous method.

10.6.3 Simple harmonic motion

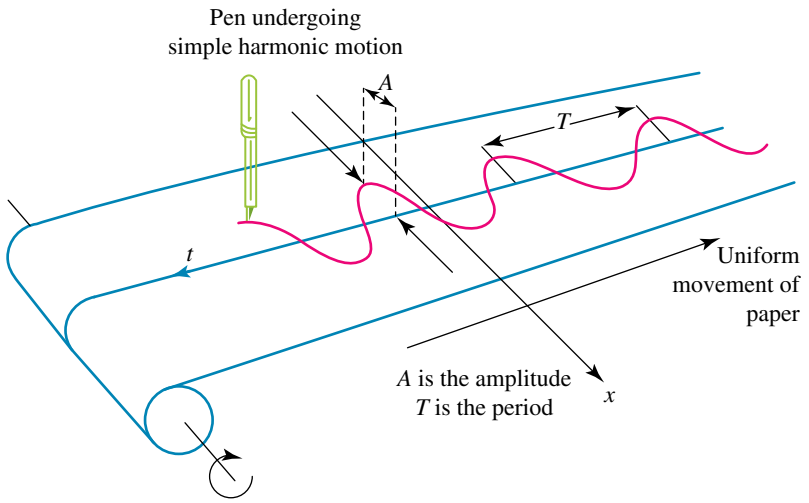
In many situations, objects **oscillate** in a straight line between two fixed points. The motion repeats itself in cycles of equal time intervals or periods. During the motion the particle is subjected to an acceleration which is always of opposite sign to the displacement. This type of motion is known as **simple harmonic motion**.



Examples of this type of motion include the swing of a pendulum, the vertical movement of a mass attached to a vertical spring, a bungee jumper (although only for a short time)

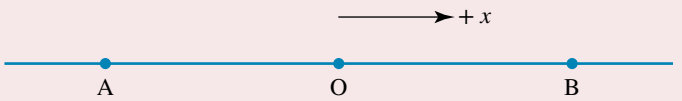
and a vibrating guitar string. Generally these motions, though periodic, are complex, but in many cases the oscillations are simple and can be mathematically modelled using trigonometric functions.

Consider the diagram where a pen is executing simple harmonic motion in a line perpendicular to the uniform motion of a reel of paper. The trace left by the pen records the displacement in the x-direction about a mean position (the time axis) as a function of time.



Simple harmonic motion

If a particle moves periodically in a straight line between two points A and B as shown below, so that its acceleration is directed towards a fixed point on that line and is proportional to the displacement from the fixed point, then the particle is said to move with simple harmonic motion. Points A and B are the furthest or extreme positions from the centre.



Mathematically this translates to

$$\frac{d^2x}{dt^2} \propto -x$$

$$\frac{d^2x}{dt^2} = -kx, \text{ where } k > 0$$

To make it easier when taking the square root in later calculations, we replace the positive constant k with ω^2 giving,

$$\ddot{x} = \frac{d^2x}{dt^2} = -\omega^2 x.$$

Simple harmonic motion is an example of a force that depends on displacement defining the motion of a particle.

Note that at the origin, when $x = 0$, the acceleration and velocity are at their maximum. The velocity v is at its minimum at the turning points, A and B, when $|x|$ is at its maximum.

As acceleration is expressed in terms of displacement,

$$\begin{aligned}
\frac{d^2x}{dt^2} &= -\omega^2 x \\
\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= -\omega^2 x \\
\frac{1}{2}v^2 &= -\omega^2 \int x\,dx \\
\frac{1}{2}v^2 &= -\frac{1}{2}\omega^2 x^2 + c
\end{aligned}$$

At the point A, let $x = A$, as this is the point at which the particle changes direction, $v = 0$.

$$\begin{aligned}
0 &= -\frac{1}{2}\omega^2 A^2 + c \\
c &= \frac{1}{2}\omega^2 A^2 \\
\frac{1}{2}v^2 &= -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega^2 A^2 \\
v^2 &= \omega^2 A^2 - \omega^2 x^2 \\
v^2 &= \omega^2 (A^2 - x^2)
\end{aligned}$$

Therefore $v = -\omega\sqrt{(A^2 - x^2)}$ if the particle is moving in the negative x direction and $v = \omega\sqrt{(A^2 - x^2)}$ if the particle is moving in the positive x direction.

To establish an expression for the displacement of the particle at time t, we can choose either the positive square root above or the negative square root as an expression for v.

Let's take the positive square root.

$$\begin{aligned}
v &= \omega\sqrt{(A^2 - x^2)} \\
\frac{dx}{dt} &= \omega\sqrt{(A^2 - x^2)} \\
\frac{dt}{dx} &= \frac{1}{\omega\sqrt{(A^2 - x^2)}},\ -A < x < A \\
t &= \frac{1}{\omega}\int \frac{dx}{\sqrt{(A^2 - x^2)}} \\
\omega t + \alpha &= \sin^{-1}\left(\frac{x}{A}\right)
\end{aligned}$$

where α is the constant of integration.

Rearranging for x gives

$x = A\sin(\omega t + \alpha)$

or $x = A\cos(\omega t + \beta)$ using the complementary trigonometric identities.

Note: Selecting $v = -\omega\sqrt{(A^2 - x^2)}$ in the beginning will result in the same equation.

If $x = 0$ when $t = 0$ (the particle is moving through the mean position at $t = 0$), then

$\omega(0) + \alpha = \sin^{-1}\left(\frac{0}{A}\right) \Rightarrow \alpha = 0$. Hence,

$$\omega t = \sin^{-1}\left(\frac{x}{\mathrm{A}}\right)$$
$$\therefore x = A\sin(\omega t)$$

If $x = A$ when $t = 0$ (the particle is at the furthest/extreme point when $t = 0$), then

$$\omega(0) + \alpha = \sin^{-1}\left(\frac{A}{A}\right)$$
$$\Rightarrow \alpha = \sin^{-1}(1) = \frac{\pi}{2}$$
$$x = A\sin\left(\omega t + \frac{\pi}{2}\right)$$
$$\therefore x = A\cos(\omega t)$$

Each complete oscillation (period) occurs in T seconds.

$$T = \frac{2\pi}{\omega}$$

The frequency (f) of simple harmonic motion is the number of oscillations or complete cycles made per second. So, $f = \frac{1}{T}$.

Simple harmonic motion formulas

If a particle moves with simple harmonic motion, then $\ddot{x} = \frac{d^2x}{dt^2} = -\omega^2 x$, where the velocity of the particle is given by the equation $v^2 = \omega^2(A^2 - x^2)$, $-A < x < A$.

If $\frac{d^2x}{dt^2} = -\omega^2 x$, then $x = A\sin(\omega t + a)$ or $x = A\cos(\omega t + \beta)$.

If $x(0) = 0$ then $x = A\sin(\omega t)$ and if $x(0) = A$ then $x = A\cos(\omega t)$ where A is the amplitude or furthest distance from the centre.

$T = \frac{2\pi}{\omega}$ gives the period of the motion and $f = \frac{1}{T}$ gives the frequency per second.

WORKED EXAMPLE 30 Simple harmonic motion 1

An object is moving with simple harmonic motion where the period of motion of the object is 4 seconds. If the velocity of the object when passing through $x = 0$ is 10 m/s calculate the amplitude and the acceleration at the furthest position.

THINK	WRITE
1. Write the rule for the period, T and the value of T.	$T = \frac{2\pi}{\omega}$ $T = 4$
2. Substitute $T = 4$ into the rule and solve for ω.	$4 = \frac{2\pi}{\omega}$ $\omega = \frac{2\pi}{4}$ $= \frac{\pi}{2}$
3. Write the rule for the velocity and the value of v when $x = 0$.	$v^2 = \omega^2(A^2 - x^2)$ $v(0) = 10$
4. Substitute the values of v and x into the formula and solve for A.	$10^2 = \left(\frac{\pi}{2}\right)^2 (A^2 - O^2)$ $100 = A^2 \times \frac{\pi^2}{4}$ $A^2 = \frac{400}{\pi^2}$ $A = \pm\frac{20}{\pi}$
5. Amplitude is always positive.	$A = \frac{20}{\pi}$, $A > 0$
6. Write the rule for acceleration and substitute into the rule the value of ω.	$\frac{d^2x}{dt^2} = -\omega^2 x$ $= -\frac{\pi^2}{4}x$
7. At the extreme position, $x = \pm\frac{20}{\pi}$.	When $x = \frac{20}{\pi}$ $\frac{d^2x}{dt^2} = -\frac{\pi^2}{4}x$ $= -\frac{\pi^2}{4} \times \frac{20}{\pi}$ $= -5\pi$ When $x = -\frac{20}{\pi}$ $\frac{d^2x}{dt^2} = -\frac{\pi^2}{4}x$ $= -\frac{\pi^2}{4} \times -\frac{20}{\pi}$ $= 5\pi$
8. Write the answer.	The amplitude is $\frac{20}{\pi}$ and the acceleration at the furthest points is $\pm 5\pi$ m/s^2.

WORKED EXAMPLE 31 Simple harmonic motion 2

An object moving with simple harmonic motion has a velocity of 8 m/s and 6 m/s respectively when it is 3 m and 4 m from the centre. Calculate:

a. the length of the path
b. the period
c. the time taken to move between the two points.

THINK	WRITE
a. 1. Write the formula for velocity.	**a.** $v^2 = \omega^2(A^2 - x^2)$
2. When $v = 8, x = 3$. When $v = 6, x = 4$. Substitute the values for v and x into the rule to obtain two equations.	$64 = \omega^2(A^2 - 9)$ [1] $36 = \omega^2(A^2 - 16)$ [2]
3. Divide equation [1] by equation [2].	$\frac{64}{36} = \frac{\omega^2(A^2 - 9)}{\omega^2(A^2 - 16)}$
4. Cancel down both fractions and cross multiply. Rearrange the equation and solve for A.	$16(A^2 - 16) = 9(A^2 - 9)$ $16A^2 - 256 = 9A^2 - 81$ $7A^2 = 175$ $A^2 = 25$ $A = \pm 5,$ $A = 5, A > 0$
5. Write the answer.	The amplitude is 5 and as the length of the path is $2A$, the length of the path is 10 m.
b. 1. Using equation (1), substitute $A = 5$ into this equation.	**b.** $64 = \omega^2(A^2 - 9)$ $64 = \omega^2(25 - 9)$ $64 = 16\,\omega^2$ $\omega^2 = 4$ $\omega = \pm 2$ $\therefore\ \omega = 2,\ \ \omega > 0$
2. Write the formula for T and substitute $\omega = 2$.	$T = \frac{2\pi}{2}$ $= \pi$ s
c. 1. Write the formula for x, assuming when $t = 0,\ \ x = 0$.	**c.** $x = 5\sin(2t)$ if $x(0) = 0$
2. Substitute $x = 3$ and solve for t.	$3 = 5\sin(2t)$ $t = \frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right) = 0.322$

Substitute $x = 4$ and solve for t.	$4 = 5\sin(2t)$ $t = \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right) = 0.464$
3. Calculate the difference to determine the time taken to move between the two points.	$0.464 - 0.322 = 0.142$ s The object takes 0.142 s to move between the two points.

WORKED EXAMPLE 32 Recognising simple harmonic motion

The speed of a particle moving along the x-axis is given by $v^2 = 48 + 32x - 16x^2$ where x is measured in metres.

a. Show that the motion is simple harmonic.

b. Determine the centre of motion, the amplitude and the period.

THINK	WRITE
a. 1. Recall that a simple harmonic motion means that the acceleration is proportional to the displacement from a fixed point. Thus, to show that the motion is simple harmonic, an expression for the acceleration in terms of x is required. As the speed is in terms of v^2, use $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.	**a.** $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $v^2 = 48 + 32x - 16x^2$ $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}(24 + 16x - 8x^2)$ $\frac{d^2x}{dt^2} = 16 - 16x$ $= -16(x - 1)$
2. A particle exhibiting simple harmonic motion satisfies the equation of the form $\frac{d^2x}{dt^2} = -\omega^2 x$, where the centre of motion is $x = 0$.	$\frac{d^2x}{dt^2} = -16(x - 1)$ $\frac{d^2x}{dt^2} = -4^2(x - 1)$ This defines simple harmonic motion about the point $x = 1$.
b. 1. The centre of motion is at the point where the acceleration is zero.	**b.** $-16(x - 1) = 0$ $x = 1$ The centre of motion is at $x = 1$.
2. To determine the amplitude the extreme points, need to be found. The velocity is zero when the particle has reached one of the extreme points. Solve the equation for $v = 0$.	Solve the equation $v^2 = 48 + 32x - 16x^2$, for $v = 0$. $48 + 32x - 16x^2 = 0$ $3 + 2x - x^2 = 0$ $x^2 - 2x - 3 = 0$ $(x - 3)(x + 1) = 0$ $x = 3$ or $x = -1$

3. The total distance travelled by the particle in one oscillation is the sum of the magnitudes of the extreme points.	$3 + \lvert -1 \rvert = 4\text{ m}$
4. The amplitude of the motion is one half of the total distance.	Amplitude $= \frac{4}{2} = 2$
5. Determine the value of ω from the acceleration equation and substitute into the formula for the period.	$\omega^2 = 16$ from the acceleration equation $\omega = +4,\ \omega > 0$ $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}\text{ s}$
6. Write the answer.	Centre of motion: $x = 1$ Amplitude: $A = 2\text{ m}$ Period: $T = \frac{\pi}{2}\text{ s}$

Exercise 10.6 Forces that depend on displacement

learnon

10.6 Exercise | **10.6 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19	20, 21, 22, 23, 24, 25	26, 27, 28, 29, 30

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Simple familiar

1. WE27 A particle of mass 2 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{3-x}{x^3}$ newtons and the particle is at rest at a distance of 1 m from the origin. Determine where else the particle comes to rest.

2. WE28 A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{4 + 9x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 9x$.

3. A body is moving in a straight line. Its velocity, v m/s, is given by $v = x^4$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 4x^7$.

4. WE29 A body of mass 8 kilograms moves in a straight line and is retarded by a force of $2x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 8 metres from the origin and at rest. Express x in terms of t where t is the time in seconds.

5. A body of mass 3 kg moves in a straight line and is retarded by a force of $\frac{x}{3}$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin and the initial velocity of the body is 6 m/s. Express x in terms of t where t is the time in seconds.

6. a. A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $12x - 18$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 2 metres from the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.

   b. A body of mass 2 kilograms moves in a straight line and is acted upon by a force of $18x - 30$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 2 metres from the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.

7. a. A body of mass 2 kilograms moves in a straight line and is acted upon by a force of $2x - 2$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.

   b. A body of mass 3 kilograms moves in a straight line and is retarded by a force of $12x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 8 metres per second. Express x in terms of t where t is the time in seconds.

8. a. A particle of mass 4 kg is subjected to a force of $\frac{28x - 24}{x^3}$ newtons, where x is its displacement in metres from a fixed origin. If the particle is at rest at a distance of 3 m from the origin, determine where else the particle comes to rest.

   b. A particle of mass 2 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{20x - 16}{x^3}$ newtons, and the particle is at rest at a distance of 2 m from the origin. Determine where else the particle comes to rest.

9. WE30 An object moving with simple harmonic motion has a speed of 5 m/s when passing through its centre of motion. Calculate the speed and acceleration when it is 1.5 m from its centre given that the period is π seconds.

10. An object is moving with simple harmonic motion with amplitude 8 m. If the period of motion is 8 seconds, determine how long it takes to travel 5 m from its centre of motion (mean position).

11. The displacement x at time t of a point moving in a straight line is given by $x = A\sin(\omega t + \alpha)$. Write an expression for x if initially $v = 0$ when $x = -5$.

12. The acceleration of an object moving with simple harmonic motion when 3 metres from the centre of motion is $6\ \text{m/s}^2$. If the amplitude of the object is 5 metres, calculate the speed of the particle at the centre and then when it is 2 metres from the centre of motion.

13. A particle is moving with simple harmonic motion of period $\frac{2\pi}{3}$ seconds and the maximum velocity is 9 m/s. Calculate the amplitude and the velocity when the particle is 2 m from the centre of motion.

14. A particle moving with simple harmonic motion starts from rest at $x = 5$ and after 2 seconds reaches $x = 2.5$. Calculate:

   a. an expression for the displacement at time $t \geq 0$
   b. the speed at $x = 0$
   c. the amplitude, frequency and period
   d. the maximum speed
   e. the maximum acceleration.

15. WE31 A particle moving with simple harmonic motion has speeds of $\sqrt{5}$ m/s and 2 m/s respectively at distances of 1 m and 2 m from the centre of motion. Calculate:

   a. the length of the path of the object
   b. the period
   c. the time spent moving between the points.

16. A particle moving in simple harmonic motion with period 2π seconds is 2 metres from the centre and moving away with a speed of $4\sqrt{2}$ m/s. Calculate the time taken for the particle to reach the centre.

17. WE32 The speed of a particle moving along the x-axis is given by $v^2 = 40 + 30x - 10x^2$.
   a. Show that the motion is simple harmonic.
   b. Determine the mean position, the amplitude and the period.

18. A cork bobs up and down with the waves, rising and falling a total of 2 metres about its mean position. Calculate its greatest velocity and acceleration if the period of the motion is 6 seconds.

19. A particle of mass m kg moves so that its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{1+bx}{x^3}$ newtons and the particle is at rest when $x = 1$ and also when $x = -\frac{1}{5}$. Calculate the value of the constant b.

Complex familiar

20. An object moves with simple harmonic motion. If its acceleration at distance d from the mean position is m, prove that the period of the motion is $2\pi\sqrt{\frac{d}{m}}$.

21. a. A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = x^n$, where n is a constant. Show that its acceleration, a m/s^2, is given by $a = nx^{2n-1}$.
   b. A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = \sqrt{b - n^2x^2}$, where n and b are constants. Show that its acceleration, a m/s^2, is given by $a = -n^2x$.
   c. A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = e^{nx} + e^{-nx}$, where n is a constant. Show that its acceleration, a m/s^2, is given by $a = n(e^{2nx} - e^{-2nx})$.
   d. A body is moving in a straight line. Its velocity, v m/s, at time t seconds is given by $v = b - ne^{-nt}$, where n is a constant. Show that its acceleration, a m/s^2, is given by $a = -n(v - b)$.

22. a. A body of mass 16 kg moves in a straight line and is retarded by a force of $64x$ newtons, where x is its displacement in metres from a fixed origin. If the initial velocity of the body is 16 m/s, determine where it comes to rest.
   b. A car of mass m kg is travelling at a speed of U m/s along a straight level road when the driver applies the brakes. The braking force is kx newtons, where x is the distance travelled in metres after the driver applies the brakes and k is a positive constant. Show that when the car comes to rest it has travelled a distance of $U\sqrt{\frac{m}{k}}$ metres.

23. a. A body is moving in a straight line. Its velocity, v m/s, is given by $v = x^2$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 2x^3$.
   b. A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{16 - 25x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = -25x$.
   c. A body is moving in a straight line. Its velocity, v m/s, is given by $v = e^{2x} + e^{-2x}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s^2, is given by $a = 2\left(e^{4x} - e^{-4x}\right)$
   d. A body is moving in a straight line. Its velocity, v m/s, at time t seconds is given by $v = 3 - 2e^{-2t}$. Show that its acceleration, a m/s^2, is given by $a = 2(3 - v)$.

24. If the displacement of a moving point at time t is given by $x = a\cos(nt) + b\sin(nt)$, prove that the motion is simple harmonic. If $a = 3,\ b = 4,\ n = 2$, determine the period and use technology to calculate the amplitude, maximum speed and maximum acceleration. All distances are in metres and the unit of time is seconds.

25. **a.** A body of mass 500 grams moves in a straight line and is retarded by a force of $8x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at rest 3 metres from the origin. Express x in terms of t where t is the time in seconds.

b. A block of mass m kilograms moves back and forth along a straight line track. It is subjected to a force that opposes the motion and whose magnitude is proportional to its distance from the origin, O, the constant of proportionality being k. The block starts from rest when its displacement from O is a metres. Show that if its speed is v metres per second at any time t seconds and displacement x metres, then $v^2 = \frac{k}{m}(a^2 - x^2)$ and $x = a\cos\left(\sqrt{\frac{k}{m}}\,t\right)$.

Complex unfamiliar

26. **a.** A block of mass m kg moves along a horizontal table top and is subjected to a resistance force of $\frac{a+bx}{x^3}$ newtons, where a and b are constants. If initially the block is at rest at a displacement of a metres, show that the block next comes to rest again when the displacement is $\frac{-a}{2b+1}$ metres.

b. A particle of mass m kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $ma(ax+b)$ newtons, where a and b are non-zero real constants. Initially the particle is $\frac{b}{a}$ m from the origin, moving with a speed of $2b$ m/s. Show that $x = \frac{b}{a}(2e^{at} - 1)$.

27. A particle of mass m kg moves in a straight line and is subjected to a force that opposes its motion. The magnitude of the opposing force is inversely proportional to the cube of the distance from a fixed point, O. The constant of proportionality is k. The particle starts from rest when its distance from O is a m.

a. Show that its speed, v m/s, is given by $v = \frac{\sqrt{k\,a^2 - x^2}}{\sqrt{m}\,ax}$, where x m is its distance from O at a time t seconds.

b. Obtain a relationship between t and x, and hence show that when x has the value of $\frac{\sqrt{7}a}{4}$ m, a time of $\frac{3a^2}{4}\sqrt{\frac{m}{k}}$ s has elapsed.

28. **a.** A particle of mass 4 kg moves in a straight line and is retarded by a force $\frac{32}{x^2}$ newtons, where x is its distance in metres from a fixed point, O. If the particle starts from rest when its distance from O is 4 m, determine the time taken to travel to the origin. Give your answer correct to 2 decimal places.

b. A particle of mass m kg moves in a straight line against a central force, that is one whose magnitude is inversely proportional to the square of the distance from a fixed point. O. The constant of proportionality is k. If the particle starts from rest when its distance from O is a metres, show that the speed, v m/s, at a distance x m from O satisfies $v = \sqrt{\frac{2k(a-x)}{max}}$.

29. A particle of mass 4 kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $32x(x^2 - 9)$ newtons and initially is 2 m from the origin, moving with a speed of 10 m/s.

a. Express x in terms of t.

b. As t approaches infinity, the particle approaches a fixed position. Determine that position.

30. If the speeds of a point moving with simple harmonic motion at distances 2 m and 3 m from the mean position are 5 m/s and 4 m/s respectively, calculate the period, the amplitude, the maximum speed and the maximum acceleration.

Fully worked solutions for this chapter are available online.

LESSON
10.7 Review

10.7.1 Summary

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10.7 Exercise

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10.7 Exercise | 10.7 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

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Simple familiar

1. A particle is acted upon by two forces, one of magnitude 10 newtons acting west, and another of magnitude 8 newtons acting in the direction N30° E. Determine the magnitude of the resultant force in newtons, acting on the particle.

2. A box of mass 5 kg is on horizontal ground. A rope making an angle of 30° with the horizontal pulls on the box with a force of $5g$ N. If the box moves with an acceleration of a m/s^2 along the ground and the normal reaction of the box on the ground is N newtons, determine the values of a and N.

3. A block of mass m kg, rests on a smooth plane inclined at an angle of θ to the horizontal. It is supported by a force of F kg wt acting at angle of α with the plane. Write an expression for $\frac{F}{mg}$ in terms of the angles θ and α.

4. A constant force of 10 newtons acts on a mass of 5 kg initially moving at 1 m/s. After the mass has moved a distance of 20 metres in a straight line, determine the magnitude of the momentum in kg m/s.

5. A particle of mass $m = 5$ kg is acted upon by three forces. One force has a magnitude of $10\sqrt{2}$ newtons acting in the north-east direction. Another force has magnitude of 10 newtons acts in the east direction and the third force has magnitude of 15 newtons acts south. Determine the acceleration of the particle.

6. A mass of m kg is supported by two light inextensible strings; one string has a length of L and has a tension of T_1, the other string has a length of $2L$ and has a tension of T_2. The other ends of the string are attached to a ceiling at a distance of $\sqrt{5}L$ apart. Write an expression for $\frac{T_1}{T_2}$.

7. A man of mass 90 kg is standing in a lift. The lift is moving downwards with an acceleration equal to $0.5\ \text{m/s}^2$. Calculate the reaction of the lift floor on the man in newtons.

8. A body of mass 2 kg is acted upon by a variable force of $4t$ newtons at a time, t seconds, when its velocity is v m/s. If the body moves in a straight line with an initial velocity of u m/s, then express v in terms of u and t.

9. A body of mass 3 kg is acted upon by a variable force of $3v$ newtons, and has an acceleration of $a\ \text{m/s}^2$, when its velocity is v m/s at a time t seconds. The initial velocity of the body is u m/s. Express v in terms of u and t.

10. An object of mass 4 kg is acted upon by a variable force so that its velocity v m/s is given by $v^2 = 4\left(4 - x^2\right)$, where x metres is its displacement from the origin. Initially the object is at rest 2 metres from the origin. Calculate the maximum force in newtons acting on the body.

11. An object moving with simple harmonic motion passes through two points P and Q which are 22 cm apart, with the same velocity having taken 2 seconds to pass between them. After 2 more seconds it returns to Q. Determine the period and the amplitude of the motion.

12. A car of mass m kg is travelling on a level roadway. The engine exerts a constant propulsive force of P newtons and the total resistance to the motion of the car is kv^4 newtons, where k is positive constant and v is its speed in m/s. If the car moves from rest, determine an expression for the distance travelled in metres until it obtains a speed of V m/s.

Complex familiar

13. A particle of mass 3 kilograms moves back and forth along the x-axis and is subjected to a force of $-48\cos(2t) - 36\sin(2t)$ newtons at time t seconds. If its initial velocity is 6 m/s and the particle starts from a point 4 metres to the right of the origin, calculate its displacement after $\frac{\pi}{8}$ seconds.

14. A body of mass 6 kg is held on a smooth plane inclined at an angle of 30° to the horizontal by a force of magnitude T newtons.
    a. Calculate the value of T if T acts up and parallel to the slope.
    b. Calculate the value of T if T acts at an angle of 30° to the slope.

15. A particle of mass 4 kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle experiences a retarding force of $8e^{-x}$ newtons and the particle speed is initially 2 metres per second. Express x in terms of t.

16. A particle of mass 3 kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $\frac{51x - 36}{x^3}$ newtons and the particle is at rest at a distance of 3 metres from the origin. Determine where else the particle comes to rest.

Complex unfamiliar

17. Complete the following.
    a. A particle can be supported on a smooth plane whose inclination is θ by either a force P newtons parallel to the plane or a force Q newtons horizontally. Show that $\tan(\theta) = \frac{\sqrt{Q^2 - P^2}}{P}$.
    b. A particle of mass 2.5 kg moves up a plane inclined at an angle of 30° to the horizontal with an acceleration of $a\ \text{m/s}^2$ when a force of F newtons acting parallel to the plane is applied. When the force F newtons acts at an angle of 30° to the plane, the mass moves up the plane with an acceleration of $\frac{a}{5}\ \text{m/s}^2$. Determine the exact value of F.

18. A cyclist of mass 75 kg is riding a 15 kg bike along a level road at a speed of 57.6 km/h when she begins to freewheel. The resistance force is $\frac{672}{\sqrt{v}}$ newtons, where v m/s is her speed after she begins to freewheel. When her speed has decreased to 14.4 km/h she has travelled a distance of D metres in a time of T seconds. Determine the values of D and T.

19. A high-speed train of mass m kg starts from rest and moves horizontally along a straight-line track. The engine of the train exerts a constant force of F newtons in the direction of motion, but the train is subjected to a resistance force of kv^2 newtons, where v is its speed in m/s. The terminal velocity of the train is 270 km/h.

a. Show that the equation of motion of the train is given by:

$$m\ddot{x} = F\left(1 - \left(\frac{v}{75}\right)^2\right)$$

b. Write in terms of F and m, the distance travelled by the train to reach a speed of 180 km/h.

c. In terms of F and m, calculate the time taken for the train to reach a speed of 180 km/h.

20. A cyclist of mass m kg travelling at a speed U m/s reaches a level section of road and begins to freewheel. He observes that after travelling a distance of D metres in time T seconds along this section of road his speed has fallen to V m/s, where $U > V > 0$. If during the freewheeling his retardation is proportional to the square root of the speed, show that $\frac{D}{T} = \frac{1}{3}\left(U + \sqrt{UV} + V\right)$.

10.7 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 1, Q1; © State of Queensland (QCAA) 2024.

The acceleration (ms^{-2}) of an object moving with simple harmonic motion is modelled by $a = -2.95x$, where x is its displacement (m) from the origin.

Determine the period of the motion in seconds.

A. 1.72 **B.** 2.13 **C.** 2.95 **D.** 3.66

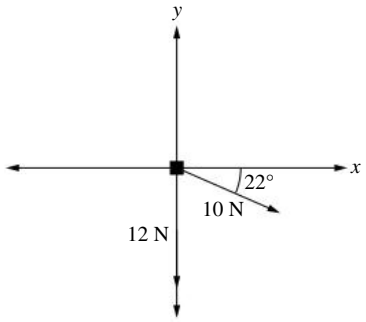
Question 2 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 1, Q6; © State of Queensland (QCAA) 2024.

Two coplanar forces of magnitudes 12 N and 10 N act on an object in the directions shown.

Determine the magnitude of the resultant force acting on the object.

A. 12.41 N
B. 15.55 N
C. 15.69 N
D. 18.27 N



Not to scale

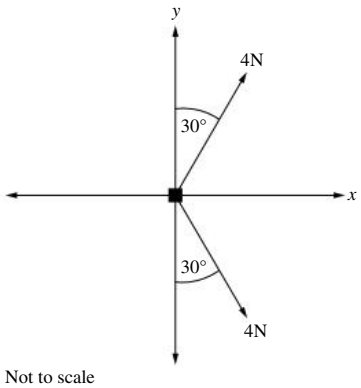
Question 3 (1 mark)

Source: QCAA 2022 Specialist Mathematics, Paper 1, Section 1, Q7; © State of Queensland (QCAA) 2024.

Two forces act concurrently on a $2\,\text{kg}$ object placed at the origin.

The magnitude of the acceleration of the object is

A. $2\,\text{m s}^{-2}$
B. $2\sqrt{3}\,\text{m s}^{-2}$
C. $4\,\text{m s}^{-2}$
D. $4\sqrt{3}\,\text{m s}^{-2}$



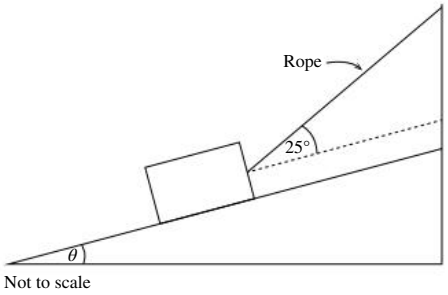
Question 4 (6 marks)

Source: QCAA 2022 Specialist Mathematics, Paper 2, Section 2, Q16; © State of Queensland (QCAA) 2024.

An object with a mass of $12\,\text{kg}$ lies on a frictionless inclined plane. A rope is attached to the object at an angle of $25°$ above the plane, as shown.

The force of the rope, $\boldsymbol{T}\,\text{N}$, prevents the object from moving. When the rope is detached, the object moves down the plane with an acceleration of $5.6\,\text{m s}^{-2}$.

Determine the magnitude of $\boldsymbol{T}$.



Question 5 (7 marks)

Source: QCAA 2022 Specialist Mathematics, Paper 2, Section 2, Q19; © State of Queensland (QCAA) 2024.

A research organisation plans to use a drone to drop a scientific instrument vertically from a stationary position above the ocean surface. The acceleration (m s^{-2}) of the falling instrument can be modelled by $9.8 - 0.1v$ where v is its velocity (m s^{-1}).

In order for the instrument sensors to activate, its speed as it hits the ocean surface must reach at least $20\,\text{m s}^{-1}$. However, if it hits with a speed above $50\,\text{m s}^{-1}$, the sensors will be damaged.

Determine the range of the drone's flying height above the ocean surface to ensure that the sensors are activated but not damaged.

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Answers

Chapter 10 Modelling motion

10.2 Equilibrium situations under concurrent forces

10.2 Exercise

1. $\sqrt{113}$ N acting at angle of 131.19° with the 7 newton force
2. $\sqrt{5}F$ N acting at angle of 116.57° with the F newton force
3. a. $x = -9, y = -3$
   b. $x = -4, y = 3, |\underset{\sim}{F_3}| = 5$
4. a. 14.49 N acting at angle of 163.98° with the 7 newton force
   b. 14.49 N acting at angle of 163.98° with the 7 newton force
5. a. 2.80F N acting at angle of 149.64° with the F newton force
   b. 2.80F N acting at angle of 149.64° with the F newton force
6. a. 90.46 N in the 5 cm string and 37.69 N in the 12 cm string
   b. 90.46 N in the 5 cm string and 37.69 N in the 12 cm string
7. a. 47.04 N in the 7 cm string and 13.72 N in the 24 cm string
   b. 47.04 N in the 7 cm string and 13.72 N in the 24 cm string
8. $\sqrt{113}$ N acting at angle of 131.19° with the 7 newton force
9. $49\left(2-\sqrt{3}\right)$ N, $49\left(\sqrt{3}-1\right)$ N
10. 15.7 N upwards
11. 31.50 N, 37.54 N
12. 33.52 N, 26.07 N
13. a. 85.19 N acting at angle of 153.08° with the 30 newton force
    b. 87.79 N acting at angle of 158.53° with the 120 newton force
14. a. i. $F = 27.45$ N
    ii. $T_{AB} = 47.85$ N
    b. 311.14 N, 24.32 kg
15. a. 4.9 N b. $\dfrac{\sqrt{3}mg}{3}$ N
16. a. i. Sample responses can be found in the worked solutions in the online resources.
    ii. 120.48 kg
    b. Sample responses can be found in the worked solutions in the online resources.
17. a. 41.81°, 43.83 N
    b. 56.31°, $\sqrt{13}\,mg$ N
18. a. 46.84°, 37.56 N
    b. 75.58°, 30.98 N
19. a. 40.72 N, 5.79 kg
    b. Sample responses can be found in the worked solutions in the online resources.
20. a. $\alpha = 44.92°, \beta = 52.63°$
    b. $\alpha = 55.95°, \beta = 45.58°$

10.3 Non-equilibrium situations under concurrent forces

10.3 Exercise

1. 160 000 kg m/s
2. 1.79×10^{29} kg m/s
3. 2.62 m/s^2
4. $\dfrac{1}{2}\sqrt{29}$ m/s^2
5. 5280 N
6. 60 kg
7. a. 1024 N b. 544 N c. 784 N
8. 539.5 N
9. a. 6 N b. 2 m/s^2 c. 2.5 m/s^2
10. a. $\dfrac{5\sqrt{2}}{2}$ m/s^2 b. $\dfrac{1}{m}\sqrt{b^2+c^2}$ m/s^2
11. a. 47.2 N b. 0.2 m/s^2
12. a. 772.5 N b. 697.5 N c. 735 N
13. 1.68 m/s^2
14. $\left(6-2\sqrt{3}\right)\hat{\imath} + \left(3\sqrt{3}-11\right)\hat{\jmath}$ N
15. a. i. 9 m/s ii. 13.5 m
    a. i. $13\,333\frac{1}{3}$ N ii. $11111\frac{1}{9}$ N
16. $m = 50$ kg, 0.25 m/s^2
17. a. 2.38 m/s^2, 411.60 N
    b. 1.95 m/s^2, 354.24 N
18. $m = 5$ kg, $\theta = 25.38°$
19. $m = 9.67$ kg, $\alpha = 14.75°$
20. $\theta = 40.77°, \alpha = 18.19°$

10.4 Forces that depend on time

10.4 Exercise

1. 9 m
2. 4 s
3. $18\sin\left(\dfrac{t}{3}\right)$
4. $4\sqrt{2}$ m
5. a. 1 s b. 25 m
6. a. $\dfrac{1}{2}$ s b. 10 m
7. $x(t) = t^2 - t^3$
8. $x(t) = 2(e^{2t} - t^2 - 2t - 1)$
9. 1 m to the left of the origin
10. $x = 4t^3 - 3t^2 + t + 2$
11. 67.39 m
12. 7692 N
13. 9950 N

14. 42.96 N

15. a. 43.77 N b. 207.75 N

16. It takes 2.2 s and the speed is 5.4 m/s.

17. 1800 m, 180 s

18. a. 6 m to the left of the origin

b. 16 m

19. i. $49t - 4.9t^2 + 2$ ii. 124.5 m

20. 11.11 m

21. a. i. 0.84 m/s^2, 315.30 N

ii. 0.17 m/s^2, 188.52 N

b. 15.87 m/s

22. $x = 49t + 245e^{-0.2t} - 225$

23. Sample responses can be found in the worked solutions in the online resources.

24. $T = \frac{1}{2}, D = \frac{400}{41}$

10.5 Forces that depend on velocity

10.5 Exercise

1. Sample responses can be found in the worked solutions in the online resources.

2. a. 5 s b. $46\frac{2}{3}$ m

3. Sample responses can be found in the worked solutions in the online resources.

4. a. $\frac{1}{2}$s b. 10 m

5. $T = \frac{1}{2}, D = \frac{400}{41}$

6. Sample responses can be found in the worked solutions in the online resources.

7. Sample responses can be found in the worked solutions in the online resources.

8. 1.33 s

9. a. $v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)$

b. $\frac{mg}{k}$

10. a. Sample responses can be found in the worked solutions in the online resources.

b. Sample responses can be found in the worked solutions in the online resources.

c. 4500.17 kg m/s

d. $42\sqrt{5}$ m/s

e. 5.69 s

11. a. $x = \frac{1}{2}\left(1 - e^{-2t}\right)$ b. $x = 2\left(1 - e^{-3t} - 2t\right)$

12. Sample responses can be found in the worked solutions in the online resources.

13. a. 5 s b. 40 m

14–20 Sample responses can be found in the worked solutions in the online resources.

10.6 Forces that depend on displacement

10.6 Exercise

1. 3 m to the left of the origin

2. Sample responses can be found in the worked solutions in the online resources.

3. Sample responses can be found in the worked solutions in the online resources.

4. $x = 8\cos\left(\frac{t}{2}\right)$

5. $x = 18\sin\left(\frac{t}{3}\right)$

6. a. $x = \frac{1}{2}\left(3 + e^{2t}\right)$ b. $x = \frac{1}{3}\left(5 + e^{3t}\right)$

7. a. $x = 1 - e^t$ b. $x = 4\sin(2t)$

8. a. $\frac{1}{2}$ m b. $\frac{1}{2}$ m

9. $V = 4$ m/s, $a = -6$ m/s^2

10. $t = \frac{4}{\pi}\sin^{-1}\left(\frac{5}{8}\right)$

11. $x = 5\cos(\omega t)$

12. $5\sqrt{2}$ m/s; $\sqrt{42}$ m/s

13. 3 m; $3\sqrt{5}$ m/s

14. a. $5\cos\left(\frac{\pi t}{6}\right)$ b. $\frac{5\pi}{6}$

c. $A = 5, T = 12, f = \frac{1}{12}$ d. $\frac{5\pi}{6}$

e. $\frac{5\pi^2}{36}$

15. a. 8 m b. $2\pi\sqrt{3}$ c. 0.47 s

16. 2.80 s

17. a. Sample responses can be found in the worked solutions in the online resources.

b. $x = \frac{3}{2}, A = 2.5, T = \frac{\pi\sqrt{10}}{5}$

18. $v = \frac{\pi}{3}$ m/s, $a = \frac{\pi^2}{9}$ m/s^2

19. 2

20. Sample responses can be found in the worked solutions in the online resources.

21. Sample responses can be found in the worked solutions in the online resources.

22. a. 8 m

b. Sample responses can be found in the worked solutions in the online resources.

23. Sample responses can be found in the worked solutions in the online resources.

24. π s; 5 m; 10 m/s; 20 m/s^2

25. a. $x = 3\cos(4t)$

b. Sample responses can be found in the worked solutions in the online resources.

26. Sample responses can be found in the worked solutions in the online resources.

27. Sample responses can be found in the worked solutions in the online resources.

28. a. 3.14 s

b. Sample responses can be found in the worked solutions in the online resources.

29. a. $x = \dfrac{3\left(5 - e^{-12t}\right)}{5 + e^{-12t}}$ b. 3 m

30. $T = \dfrac{2\pi\sqrt{5}}{3}$, $A = \dfrac{\sqrt{161}}{3}$, $v = \sqrt{\dfrac{161}{5}}$, $a = \dfrac{3\sqrt{161}}{5}$

10.7 Review

10.7 Exercise

1. $2\sqrt{21}$

2. $a = \dfrac{\sqrt{3}\,g}{2}$ and $N = \dfrac{5g}{2}$

3. $\dfrac{F}{mg} = \dfrac{\sin(\theta)}{\cos(\alpha)}$

4. 45 kg m/s

5. $\underset{\sim}{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

6. $\dfrac{T_1}{T_2} = 2$

7. 837 N

8. $v = u + t^2$

9. $v = ue^t$

10. 32

11. 8 s; $11\sqrt{2}$ cm

12. $\displaystyle\int_0^V \frac{mv}{P - kv^4}\,dv$

13. $\dfrac{7\sqrt{2}}{2}$ m

14. a. $3g$ N b. $2g\sqrt{3}$ N

15. $x = 2\ln(t + 1)$

16. 0.4 m from the origin

17. a. Sample responses can be found in the worked solutions in the online resources.

b. $F = \dfrac{98}{5\sqrt{3} - 2}$ N

18. $T = 5$, $D = 53\dfrac{1}{7}$

19. a. Sample responses can be found in the worked solutions in the online resources.

b. $\dfrac{5625\,m}{2F}\ln\left(\dfrac{9}{5}\right)$ m

c. $\dfrac{75\,m}{2F}\ln(5)$ s

20. Sample responses can be found in the worked solutions in the online resources.

10.7 Past QCAA exam questions

1. D

2. D

3. A

4. $T = 74.15$ N

5. The range of the drone's flying height above the ocean surface should be between 23.7 m and 199.5 m.

11 Statistical inference

LESSON SEQUENCE

11.1 Overview 768
11.2 Sampling distributions 770
11.3 Confidence intervals 785
11.4 Margin of error, level of confidence and sample size 795
11.5 Applications and other distributions 801
11.6 Review 817
Answers 823

Fully worked solutions for this chapter are available online.

EXAM PREPARATION

Access exam-style questions in every lesson, available online and access past QCAA questions in every review.

on Resources

Solutions	Solutions — Chapter 11
Exam questions	Exam question booklet — Chapter 11
Digital documents	Learning matrix — Chapter 11 Chapter summary — Chapter 11

LESSON
11.1 Overview

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11.1.1 Introduction

Confidence intervals, first introduced by the Polish mathematician Jerzy Neyman (1894–1981) in a paper published in 1937, provide a range of values within which a population parameter is likely to lie, based on sample data. Neyman's work on experiments and statistics eventually formed the foundation for modern hypothesis testing and is still used today by agencies such as the Federal Drug Authority (FDA) to test new medicines.

Statistical inference refers to the theory, methods and practices for making judgments about population parameters, typically based on random samples. For example, a governing body, such as a department of health, may wish to make inferences about the average sleep time of a given population. Since obtaining data from the entire population is often impractical, sample means are calculated, and these sample means are then used to construct a confidence interval estimate for the population mean, denoted as μ. The part of statistics concerned with the drawing of conclusions from data is called inferential statistics.

11.1.2 Syllabus links

Lesson	Lesson title	Syllabus links
11.2	**Sampling distributions**	◯ Understand the concept of the sample mean $\overline{X}$ as a random variable whose value varies between samples where X is a random variable with mean μ and the standard deviation σ. ◯ Use repeated random sampling data from a variety of distributions and a range of sample sizes to examine properties of the distribution of $\overline{X}$ across samples of a fixed size n, including its mean μ, its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where μ and σ are the mean and standard deviation of X) and its approximate normality if n is large. ◯ Recognise and use the link between the normal distribution of the sample mean and the statistical notation $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. ◯ Model and solve problems that involve sample means, with and without technology.

11.3	**Confidence intervals**	◯ Model and solve problems that involve sample means, with and without technology. ◯ Understand the concept of an interval estimate for a parameter associated with a random variable. ◯ Understand and use the approximate confidence interval $\left(\bar{x}-z\frac{s}{\sqrt{n}}, \bar{x}+z\frac{s}{\sqrt{n}}\right)$, as an interval estimate for μ, the population mean, where z is the appropriate quantile for the standard normal distribution. ◯ Understand and use the approximate margin of error. • $E=z\frac{s}{\sqrt{n}}$ ◯ Understand and use the concept that there are variations in confidence intervals between samples and that most but not all confidence intervals contain μ. ◯ Use $\bar{x}$ and s to estimate μ and σ, to obtain approximate intervals covering desired proportions of values of a normal random variable and compare with an approximate confidence interval for μ. ◯ Model and solve problems that involve interval estimates for sample means, with and without technology.
11.4	**Margin of error, level of confidence and sample size**	◯ Understand and use the approximate confidence interval $\left(\bar{x}-z\frac{s}{\sqrt{n}}, \bar{x}+z\frac{s}{\sqrt{n}}\right)$, as an interval estimate for μ, the population mean, where z is the appropriate quantile for the standard normal distribution. ◯ Model and solve problems that involve sample means, with and without technology. ◯ Understand and use the relationship between margin of error, level of confidence and sample size. ◯ Understand and use the concept that there are variations in confidence intervals between samples and that most but not all confidence intervals contain μ. ◯ Use $\bar{x}$ and s to estimate μ and σ, to obtain approximate intervals covering desired proportions of values of a normal random variable and compare with an approximate confidence interval for μ. ◯ Model and solve problems that involve interval estimates for sample means, with and without technology.
11.5	**Applications and other distributions**	◯ Use repeated random sampling data from a variety of distributions and a range of sample sizes to examine the approximate standard normality of $\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$ for large samples $(n \geq 30)$, where s is the sample standard deviation (Central limit theorem). ◯ Model and solve problems that involve sample means, with and without technology. ◯ Model and solve problems that involve interval estimates for sample means, with and without technology.

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LESSON
11.2 Sampling distributions

SYLLABUS LINKS

- Understand the concept of the sample mean $\overline{X}$ as a random variable whose value varies between samples where X is a random variable with mean μ and the standard deviation σ.
- Use repeated random sampling data from a variety of distributions and a range of sample sizes to examine properties of the distribution of $\overline{X}$ across samples of a fixed size n, including its mean μ, its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where μ and σ are the mean and standard deviation of X) and its approximate normality if n is large.
- Recognise and use the link between the normal distribution of the sample mean and the statistical notation $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.
- Model and solve problems that involve sample means, with and without technology.

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11.2.1 Review of statistical terminology and concepts

Inferential vs. descriptive statistics

Before proceeding, note that section 11.2.1 serves as a review, reinforcing methods you have encountered previously. However, sections 11.2.2 and 11.2.3 introduce new material that builds upon these concepts.

Statistics is a mathematical discipline that helps us interpret data and make informed decisions. Up until now, much of your studies have focused on **descriptive statistics**. In this branch, data is organised and displayed in various ways, and summarised using measures such as the mean, median or standard deviation.

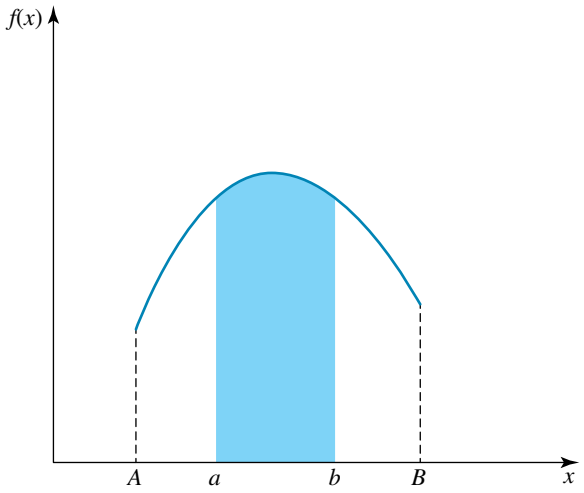
Inferential statistics aims to draw conclusions about an entire population based on data from a representative sample. By collecting and analysing suitable samples, we can make generalisations about the population. Before diving into inferential methods, we will first review key statistical terms and concepts.

Key statistical terminology and concepts

1. **Variable: A characteristic or attribute that can take on different values across observations.**
2. **Data: A collection of recorded observations or measurements of one or more variables.**
3. **Univariate data set: A data set that consists of observations on a single characteristic.**
4. **Random variable: A numerical variable, typically denoted by X, whose value is determined by the outcome of a chance experiment. It assigns a numerical value to each possible outcome of the experiment.**
5. **Discrete random variable: A random variable that can take specific, separate values, often resulting from counting. The possible values are distinct points along the number line.**
6. **Continuous random variable: A random variable that can take any value within a given range, often resulting from measurement. Its set of possible values includes entire intervals along the number line.**
7. **Population: The complete set of all elements or individuals of interest in a study.**
8. **Sample: A subset of the population selected for analysis. It is crucial that the sample be chosen randomly to minimise bias in the results.**
9. **Parameter: A numerical characteristic of the population (such as the mean).**
10. **Statistic: An estimate of a parameter from the values (observations) of a sample.**

Continuous probability distributions

In Mathematical Methods, you have studied continuous random variables. A random variable X is continuous if its possible values form an interval, meaning any number between A and B is possible. As a general rule, variables that are counted are discrete, while those that are measured are continuous.



$P(a \le X \le b) = \int_a^b f(x)\,dx$

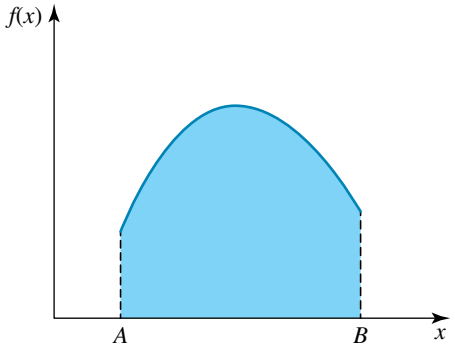
For a continuous random variable X, if X can take any value in the interval $[A, B]$, then the probability that X falls within this range is 1, i.e., $P(a \le X \le b) = 1$.

We can define a **probability density function** (pdf) for X such that the probability of X taking values between a and b is given by:

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Since the total probability over the entire interval must equal 1, we have:

$$\int_A^B f(x)\,dx = 1$$



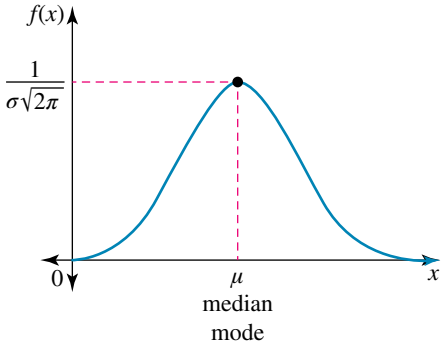
$P(A \le X \le B) = 1$

It's important to distinguish that X represents the random variable itself, while x represents a particular value that X can take.

Examples of continuous random variables include the normal, exponential and uniform distributions, among others.

The normal distribution

You have already studied the normal distribution in Mathematical Methods, so this will serve as a quick review. The **normal distribution** is symmetric, with a mean of μ and a standard deviation of σ.



Its probability density function is given by $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$.

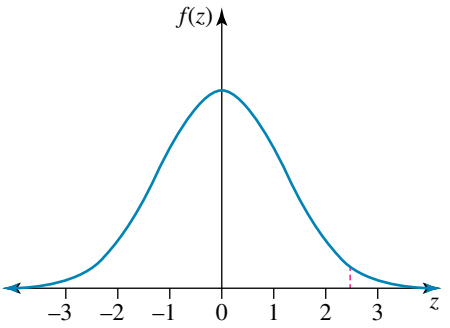
The random variable X that follows a normal distribution mean μ and variance σ^2, is denoted as:

$$X \sim N(\mu, \sigma^2)$$

If $\mu = 0$ and $\sigma = 1$, the distribution is called the **standard normal distribution**.

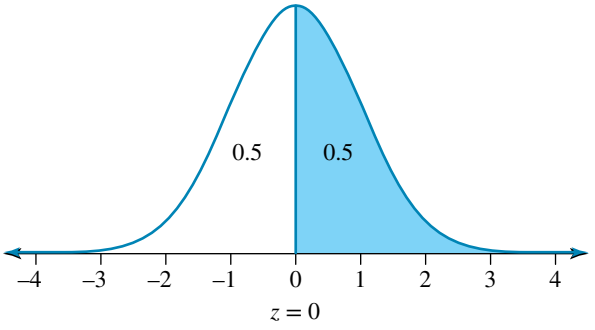
A random variable following a standard normal distribution is denoted by Z. The probability that Z takes values between z-scores z_1 and z_2 is calculated as:

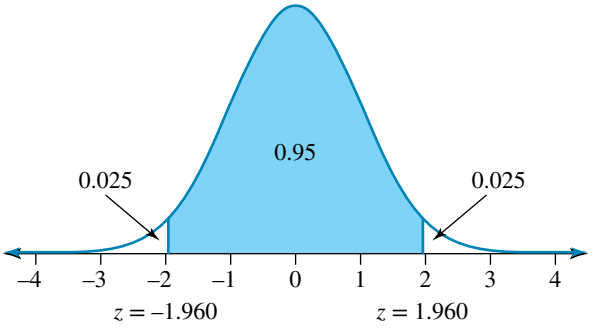
$$P(z_1 \le z \le z_2) = \int_{z_1}^{z_2} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} \right) dz$$

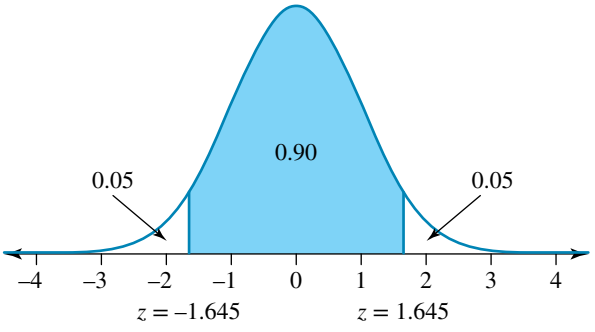


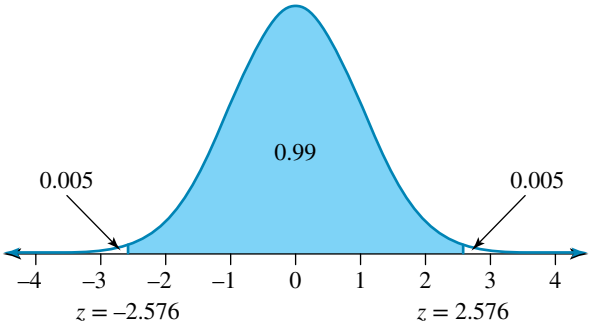
However, it is not necessary to compute this integral manually. The values have been tabulated and are readily available in standard normal distribution tables or via a graphics calculator.

Additionally, if the area under the curve (between the curve and the x-axis) is known, the inverse normal function can be used to determine the z-score corresponding to that area. In this chapter we are primarily engaging in the middle 90%, 95% or 99% of the distribution.









11.2.2 Population parameters and estimation

In a statistical investigation, the entire set of individuals or elements within a group is called the **population**. In mathematics, a population can refer to any group, not just people. The distribution of a population is summarised by specific values known as **parameters**. For a distribution of a random variable X, the key parameters include:

1. The **expected value** of the distribution, $E(X)$, which corresponds to the **population mean** μ, i.e., $E(X) = \mu$.
2. The **standard deviation** σ describes how data are spread out from the mean. Since the standard deviation and mean share the same units, they are highly useful for analysing a distribution.
3. The **variance** σ^2 is the square of the standard deviation and thus has different units. The variance, also denoted by $\text{Var}(X)$, is important for describing the spread of the data and plays a central role in many statistical analyses.

For very large populations, or when obtaining data for the entire population is impractical or expensive, samples are used to estimate population parameters. The mean and standard deviation are the most common parameters that statisticians aim to infer. Naturally, we then consider the **sample mean** $\bar{x}$ and **sample standard deviation** s.

In the following section, we focus on calculating the means of different samples. As we will see, the observed value of a statistic $\bar{x}$ will vary from sample to sample, a phenomenon known as **sampling variability**.

11.2.3 Sampling variability

In this section, we will explore sampling variability using a practical example of daily exercise time. Consider a small town where the entire population consists of 25 people. The data below shows their exercise time in minutes. The population mean is $\mu = 14.9$ and the standard deviation $\sigma = 4.1$.

$$7.9, 8.7, 9.6, 10.1, 10.4, 11.4, 11.7, 12.1, 12.4, 13.3, 13.9, 14.1, 15.2,$$
$$15.4, 15.7, 16.2, 16.6, 17.3, 17.7, 18.3, 19.2, 19.5, 21.1, 21.8, 22.9$$

To investigate how sample statistics can vary, we will take 10 different samples, each consisting of 4 individuals, from this population. By calculating and comparing the means, $\bar{x} = \frac{\sum x}{n}$, of these samples, we aim to illustrate sampling variability and observe how the sample means can differ from one another.

Activity 1: Ten samples, sample size $n = 4$

The Department of Health is investigating how many hours per week people in the town's population exercise.

Ten random samples of size $n = 4$ were collected from this population, and their sample means were calculated and summarised in the table.

Sample #	Sample data points	Sample mean $\bar{x}$
1	19.5, 12.1, 8.7, 17.7	14.5
2	12.4, 15.2, 11.7, 17.3	14.2
3	8.7, 10.1, 12.4, 19.2	12.6
4	10.1, 14.1, 18.3, 13.3	14
5	7.9, 12.1, 21.8, 15.2	14.3
6	15.7, 17.3, 16.6, 19.5	17.3
7	13.3, 12.4, 18.3, 16.2	15.1
8	13.9, 8.7, 15.4, 21.8	15
9	9.6, 19.5, 12.1, 22.9	16
10	17.3, 15.2, 12.4, 13.3	14.6

The sample means range from 12.6 to 17.3, demonstrating variability between samples. This variation occurs because the sample mean depends on which four individuals are selected. The average value of the sample means is $\mu_{\bar{x}} \approx 14.8$, which is not too dissimilar to the population mean, $\mu = 14.9$.

Given that these sample sizes are relatively small, consider whether the variability of the sample means would change significantly if ten samples of $n = 8$ or $n = 16$ were taken. Activities 2 and 3 will explore the impact of increasing the sample size on this variability.

WORKED EXAMPLE 1 Investigating sampling variability

Tara is interested in examining the variability of sample means of the same size. The study is based on a continuous random variable where X is the number of hours a person spends on an electronic device. The population is relatively small with $n = 30$.

1.8, 2.7, 3.0, 3.5, 4.1, 5.2, 5.5, 5.8, 6.3, 6.9, 7.2, 7.4, 7.7, 8.0, 8.3, 8.6, 8.8, 9.4, 10.0, 10.2, 10.8, 11.4, 11.9, 12.5, 12.8, 13, 13.3, 13.6, 14.4

Tara has collected four random samples of size five shown below.

Sample 1	Sample 2	Sample 3	Sample 4
2.7, 5.5, 8.3, 10.8, 12.8	5.2, 6.9, 8.0, 10.2, 13.3	3.0, 7.2, 7.4, 11.4, 13.6	4.1, 5.2, 8.0, 12.8, 14.4

a. Use these samples to calculate the associated sample mean.
b. Describe the variation of the means by comparing them to the population mean, $\mu = 8.4$.

THINK	WRITE
a. Recall the formula for the sample mean, $\bar{x} = \frac{\sum x}{n}$ and substitute values from Sample 1.	$\bar{x}_1 = \frac{\sum x}{n}$ $= \frac{2.7 + 5.5 + 8.3 + 10.8 + 12.8}{5}$
Calculate the mean to 1 decimal place.	$\bar{x}_1 = 8.0$
Repeat this process for samples 2, 3 and 4.	$\bar{x}_2 = \frac{5.2 + 6.9 + 8.0 + 10.2 + 13.3}{5}$ $= 8.7$ $\bar{x}_3 = \frac{3.0 + 7.2 + 7.4 + 11.4 + 13.6}{5}$ $= 8.5$ $\bar{x}_4 = \frac{4.1 + 5.2 + 8.0 + 12.8 + 14.4}{5}$ $= 8.9$
b. Summarise the four means and compare them to the population.	$\bar{x}_1 = 8.0, \bar{x}_2 = 8.7, \bar{x}_3 = 8.5, \bar{x}_4 = 8.9$ The sample means represent the variability of sampling. Sample mean 1 is below the population mean $\mu = 8.4$, whereas sample means 2, 3 and 4 are above. If the average of the sample means was used to estimate the population mean, it is calculated as: $\mu_{\bar{x}} = \frac{8.0 + 8.7 + 8.5 + 8.9}{4}$ ≈ 8.6

11.2.4 Sampling distribution of $\bar{x}$

In Worked example 1, we observed variability in the sample means. If four more samples were taken, we would likely see similar but differing values of $\bar{x}$. Repeated simulations would generally result in sample means clustering around the population mean, μ. This clustering illustrates the central idea that the expected value of the sample mean random variable, $E(\bar{X})$, equals the population mean, μ.

To further illustrate how the sample mean varies across different samples, we will revisit the weekly exercise data from Activity 1 and explore the impact of increasing the sample size.

Activity 2: Ten samples, sample size $n = 8$

In a second investigation, the Department of Health collected ten new random samples, this time with a larger sample size of $n = 8$.

The sample means were again calculated and summarised in a separate table for comparison.

Sample #	Sample data points	Sample mean $\bar{x}$
1	18.3, 11.4, 19.5, 21.1, 14.1, 12.1, 9.6, 13.3	14.9
2	12.4, 19.2, 21.1, 15.4, 7.9, 19.5, 14.1, 12.1	15.2
3	19.5, 10.4, 18.3, 22.9, 9.6, 12.4, 15.4, 15.7	15.5
4	16.6, 15.7, 18.3, 15.2, 12.1, 17.7, 10.4, 7.9	14.2
5	9.6, 21.8, 12.1, 18.3, 15.4, 12.4, 15.2, 16.6	15.2
6	13.9, 19.5, 17.7, 17.3, 9.6, 16.2, 21.8, 11.7	16.0
7	18.3, 14.1, 11.4, 10.4, 21.8, 16.6, 12.4, 15.2	15.0
8	10.1, 16.6, 17.7, 9.6, 19.2, 11.4, 10.4, 21.8	14.6
9	22.9, 13.9, 7.9, 12.4, 9.6, 14.1, 21.8, 15.2	14.7
10	19.5, 12.1, 15.2, 11.7, 13.3, 17.3, 19.2, 12.4	15.1

The sample means range from 14.2 to 16.0 showing that variability exists between samples. The average value of the sample means is $\mu_{\bar{x}} \approx 15.0$, which is very close to the population mean, $\mu = 14.9$.

Recall that in Activity 1, the range of sample means was 12.6 to 17.3. What conclusions can you begin to draw as the sample size increases?

Activity 3: Ten samples, sample size $n = 16$

For the third investigation, ten random samples of size $n = 16$ were collected from the town's population, and their sample means were recorded for analysis.

Sample #	Sample data points	Sample mean $\bar{x}$
1	18.3, 11.7, 8.7, 22.9, 16.2, 19.5, 13.3, 15.4, 17.7, 12.1, 12.4, 10.1, 21.8, 17.3, 9.6, 7.9	14.7
2	17.7, 10.1, 11.7, 18.3, 19.5, 13.3, 19.2, 21.8, 15.7, 7.9, 15.2, 16.6, 17.3, 12.1, 8.7, 11.4	14.8
3	16.6, 7.9, 17.7, 11.7, 16.2, 8.7, 15.4, 17.3, 13.3, 10.1, 11.4, 12.4, 22.9, 21.1, 21.8, 13.9	14.9

Sample #	Sample data points	Sample mean $\bar{x}$
4	13.3, 15.7, 21.8, 14.1, 11.4, 17.3, 19.2, 12.4, 12.1, 10.1, 18.3, 15.2, 8.7, 13.9, 15.4, 19.5	14.9
5	9.6, 17.3, 19.5, 7.9, 10.4, 11.7, 13.3, 18.3, 10.1, 14.1, 21.1, 11.4, 15.7, 17.7, 19.2, 16.6	14.6
6	17.7, 7.9, 19.5, 12.4, 14.1, 18.3, 11.7, 13.9, 9.6, 10.4, 13.3, 19.2, 21.8, 22.9, 11.4, 17.3	15.1
7	15.7, 11.4, 10.1, 13.3, 11.7, 19.5, 9.6, 14.1, 15.4, 19.2, 21.8, 17.7, 21.1, 12.4, 16.2, 7.9	14.8
8	21.8, 10.1, 10.4, 17.7, 15.4, 8.7, 16.6, 11.7, 22.9, 12.1, 19.5, 15.7, 13.9, 7.9, 16.2, 13.3	14.6
9	21.8, 19.5, 16.2, 11.7, 17.3, 9.6, 11.4, 22.9, 13.3, 7.9, 21.1, 14.1, 10.4, 13.9, 12.1, 15.4	14.9
10	7.9, 10.1, 14.1, 15.4, 17.3, 16.6, 19.5, 22.9, 10.4, 11.4, 13.9, 12.4, 12.1, 21.8, 16.2, 8.7	14.4

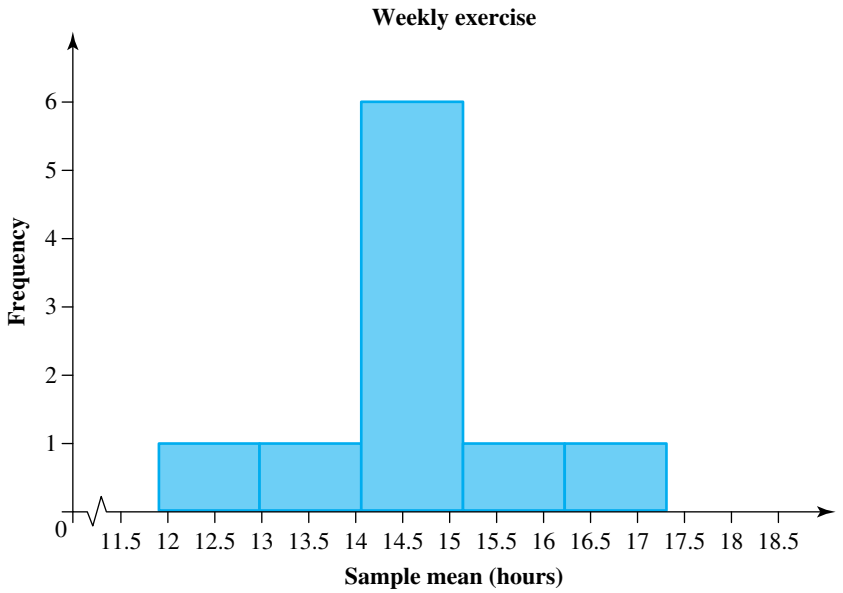
The sample means range from 14.4 to 15.1, indicating less variability between samples. The average of the sample means is $\mu_{\bar{x}} \approx 14.8$, which is very close to the population mean, $\mu = 14.9$.

Remember, in Activity 1 (with a sample size of $n = 4$), the range of sample means was 12.6 to 17.3. In Activity 2 (with $n = 8$), the range narrowed to 14.2 to 16.0. Notice how the range shrinks as the sample size increases. What does this suggest about variability in larger samples?

In all three Activities, the sample means varied but remained approximately centred around the population mean, μ. The variability of the observed means $\bar{x}$ can be modelled by the random variable $\bar{X}$ known as a **sampling distribution**. Histograms are provided below, displaying the sampling distributions of means for 10 samples of sizes $n = 4$, $n = 8$ and $n = 16$.

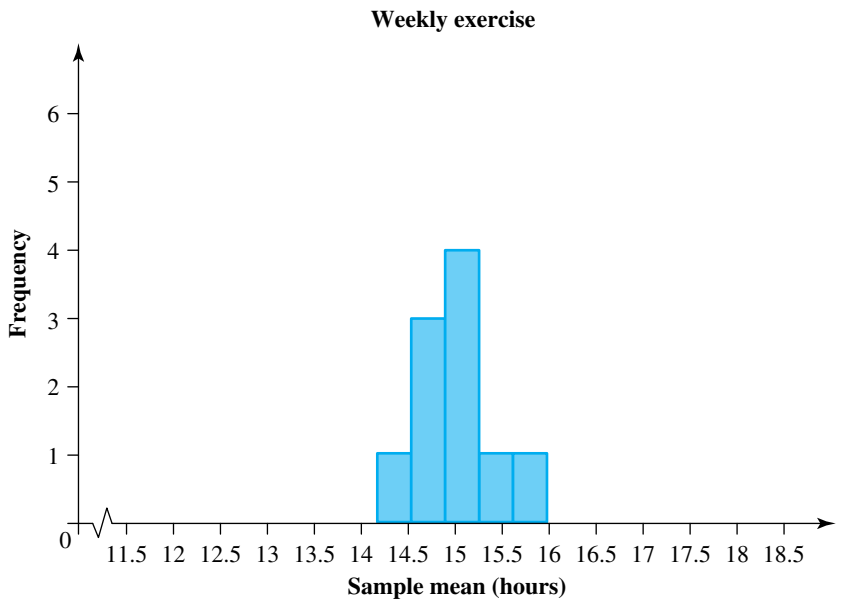
Activity 1:

- 10 samples
- Sample size, $n = 4$
- $\mu_{\bar{x}} = 14.7$



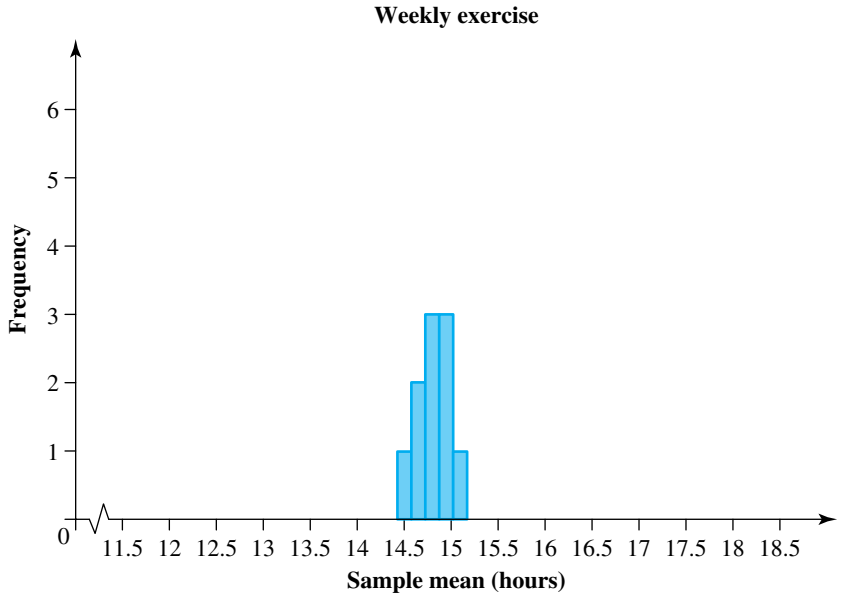
Activity 2:

- 10 samples
- Sample size, $n = 8$
- $\mu_{\bar{x}} = 15.0$



Activity 3:

- 10 samples
- Sample size, $n = 16$
- $\mu_{\bar{x}} = 14.8$



The histograms describe the sampling distribution of $\bar{x}$ as the sample size n increases. Based on these graphs we can make numerous observations:

1. All histograms resemble the normal distribution.
2. The sample means are centred close to the population mean, $\mu = 14.9$.
3. As the sample size n increases, the spread of the sampling distribution decreases.

Sample means are called **unbiased estimators** since on average they produce the value we are trying to estimate. The average of the observed sample means, denoted by $\mu_{\bar{x}}$, is summarised as follows:

Activity 1	Activity 2	Activity 3
$\mu_{\bar{x}} = 14.8$	$\mu_{\bar{x}} = 15.0$	$\mu_{\bar{x}} = 14.8$

While the means of individual samples may vary, the long run average of the sample means is the population mean, i.e., $\mu_{\bar{x}} = \mu$. The following section describes the empirical properties of the sampling distribution of sample means.

11.2.5 Sampling distribution of means

A **sampling distribution** is the theoretical probability distribution of a random variable.

X, represents a sample statistic (such as a mean), and describes how X varies across all possible samples of a given size from a population.

The **sampling distribution of the sample mean** is created by repeatedly taking samples from a population, calculating their sample means, and observing how these means vary.

Sampling distribution properties

- **To make inferences about the population mean μ, samples are taken from a finite population $X_1, X_2, \ldots, X_N$, where X is a random variable measuring a particular characteristic (e.g. annual income).**
- **In a simple random sample, each member of the population has the same chance of being selected, and observations are independent.**
- **The sample mean $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$ is the arithmetic mean of the observed random variables.**
- **$\bar{X}$ can be thought of as the mean of *all possible sample means of the same size* from the population.**
- **Hence, the sample mean $\bar{X}$ is itself a random variable with a theoretical mean μ or μ_X.**
- **To illustrate this, consider a small population of five data points: 23, 42, 12, 21 and 11, where $\mu = 21.8$ and $\sigma \approx 11.16$.**
- **The table below shows all possible samples and their corresponding means for $n = 2$.**

Data points	23, 42	23, 12	23, 21	23, 11	42, 12	42, 21	42, 11	12, 21	12, 11	21, 11
$\bar{x}$	32.5	17.5	22	17	27	31.5	26.5	16.5	11.5	16

- **These combinations show wide variability in $\bar{x}$ values, which would narrow with larger sample sizes.**
- **Examining the *average of these sample means* (denoted by $\mu_{\bar{x}}$), we find $\mu_{\bar{x}} = 21.8$.**
- **This matches the population mean, illustrating that while individual sample means vary, the mean of the sample means approximates the population mean: $\mu_{\bar{x}} \approx \mu_X$.**
- **The central limit theorem tells us that as the sample size increases, the distribution of sample means becomes increasingly normal.**

A summary of terminology and symbols are shown below.

Population		Sample		Sampling distribution $\overline{X}$	
μ	Population mean	$\overline{x}$	Sample mean	$\mu_{\overline{X}}$	Average of observed sample means
σ	Standard deviation	s	Standard deviation	$E(\overline{X})$	Expected value
N	Total size	n	Sample size	$\sigma_{\overline{X}}$	Standard deviation

WORKED EXAMPLE 2 Calculating expected value and standard deviation

800 students are studying an Engineering course at University of Queensland. The age of this population of students is normally distributed with a mean $\mu = 23.5$ and standard deviation, $\sigma = 2.1$.

a. Samples of size 5 are selected and the means determined. Calculate the expected value and standard deviation of the distribution of $\overline{X}$.

b. If the sample size was increased to 36, determine the effect this would have on the expected value and standard deviation of the distribution of sample means.

THINK	WRITE
a. The expected value of the distribution of sample means is equal to the population mean.	$\mu_{\overline{X}} = \mu$ $= 23.5$
Write down the formula for standard deviation of the distribution of sample means.	$\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$
Calculate the standard deviation for $n = 5$.	$\sigma_{\overline{X}} = \dfrac{2.1}{\sqrt{5}}$ ≈ 0.94
b. The expected value is not dependent on sample size.	$\mu_{\overline{X}} = \mu$ $= 23.5$
Write down the formula for standard deviation of the distribution of sample means.	$\sigma_{\overline{X}} = \dfrac{\sigma}{\sqrt{n}}$
Calculate the standard deviation for $n = 36$.	$\sigma_{\overline{X}} = \dfrac{2.1}{\sqrt{36}}$ $= 0.35$
Write your conclusions.	Increasing the sample size does not change the mean of the distribution. The standard deviation of the distribution is reduced.

WORKED EXAMPLE 3 Calculating mean and standard deviation

A wildlife researcher is studying the weight (kg) of male red kangaroos in a specific region. It is known that the weights of these kangaroos are normally distributed. A researcher repeatedly took samples of size 100 and calculated the empirical average of the sample means to be 73 kg, with a standard deviation of 0.9 kg for these sample means.
Use this information to make the following inferences about population parameters.

a. The mean weight μ of male red kangaroos.
b. The standard deviation σ of their weights.

THINK	WRITE
a. The sampling distribution is in the form $\overline{X} = N\left(\mu, \frac{\sigma^2}{n}\right)$. Based on the central limit theorem, $\mu_{\overline{x}} = \mu$.	$\mu_{\overline{x}} = \mu$ $= 73\text{ kg}$
b. To calculate the variance, let $\frac{\sigma^2}{n} = 0.81$ where $n = 100$ and solve for σ.	$\frac{\sigma^2}{100} = 0.81$ $\sigma^2 = 0.81 \times 100$ $\sigma = \sqrt{81}$ $\sigma = 9\text{ kg}$
Conclude by stating the calculated values for the population mean and standard deviation.	Based on these inferences, the key parameters of the population are $\mu = 73$ kg and $\sigma = 9$ kg.

WORKED EXAMPLE 4 Calculating probabilities for a normal distribution

A car battery manufacturer claims that the average lifespan of their batteries is normally distributed with $\mu = 4.2$ and $\sigma = 0.7$ years. Customers would be concerned if the average lifespan is shorter than 4 years. To investigate this concern, an independent testing agency selects a random sample of 50 batteries. Calculate the probability that a batteries lifespan is less than 4 years.

THINK	WRITE
State the formula for $\sigma_{\overline{X}}$.	$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$
Calculate the sample standard deviation for $n = 50$.	$\sigma_{\overline{X}} = \frac{0.7}{\sqrt{50}}$ ≈ 0.1
Since $n = 50$ is sufficiently large, the Central Limit Theorem applies	$\mu_{\overline{X}} = \mu$ $= 4.2$
Calculate the z value for 4 years.	$z = \frac{X - \mu}{\sigma}$ $= \frac{4 - 4.2}{0.1}$ $= -2$

Use technology or z-score tables to calculate the probability.	$P(X < 4) = P(z < -2)$ ≈ 0.023
State your conclusion.	The probability that a batteries lifespan is less than 4 years is 2.3%.

Exercise 11.2 Sampling distributions

learnon

11.2 Exercise | **11.2 Exam questions** on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	16, 17, 18, 19, 20	–

These questions are even better in jacPLUS!

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Simple familiar

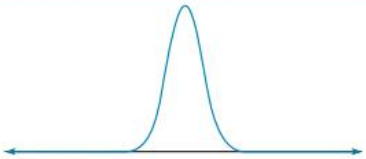
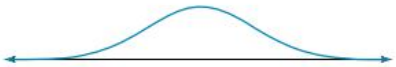
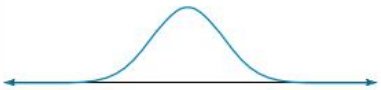
1. Classify $\bar{x}, \sigma, \sigma_{\bar{x}}, s, \mu$ as either being associated with a sample or population.

2. MC Identify the correct statement about the sampling distribution $\bar{X}$.
   - A. The distribution $\bar{X}$ is exponential in shape.
   - B. $\bar{X}$ is always normal regardless of the sample size.
   - C. The value of $\bar{x}$ remains constant whereas variation exists with μ.
   - D. The value of $\bar{x}$ varies between samples and the expected value $E(\bar{X}) = \mu$.

3. MC Identify the correct statement about the sampling distribution $\bar{X}$.
   - A. It consists of varying values of $\bar{x}$.
   - B. The spread of the distribution decreases as n decreases.
   - C. It is approximately normal if $n \leq 10$.
   - D. It is a discrete random variable.

4. MC According to a recent census, the average annual cost of house insurance is \$1200, with a standard deviation of \$300. A mean of \$1150 is calculated from a random sample of 64 households. Based on this data, identify the correct statement.
   - A. $X = 1200, \mu = 300$
   - B. $\bar{x} = 1200, s = 64$
   - C. $\sigma = 1200, \bar{x} = 1150$
   - D. $\mu = 1200, \bar{x} = 1150$

5. The graphs below represent three distributions of $\bar{x}$ based on the same population.

Distribution 1	Distribution 2	Distribution 3
		

Match each distribution with one of the following properties.

- Population distribution of $\bar{x}$.
- Sampling distribution of $\bar{x}$, $n = 10$.
- Sampling distribution of $\bar{x}$, $n = 40$.

6. **MC** A random variable X is normally distributed with a mean of 81 and a standard deviation of 14. Select the respective expected value and standard deviation of the distribution of $\bar{X}$ from a repeated sample of size 49.

A. 81 and 7
B. 81 and 2
C. 9 and 2
D. 81 and 14

7. **WE1** Kellie is interested in examining the variability of sample means of the same size. The study is based on a continuous random variable where X is the number of minutes a person spends on a mobile phone. The population is relatively small with $n = 25$.
Kellie has collected four random samples of size five shown below.

Sample 1	Sample 2	Sample 3	Sample 4
43.9	66.3	41.3	66.3
57.5	85.7	91.2	76.9
46.6	51.5	93.0	94.9
82.7	54.9	78.4	88.8
54.9	91.2	96.8	43.9

a. Use the samples to calculate the related sample means. Give your answers to 1 decimal place.
b. Calculate the average of the sample means, $\mu_{\bar{x}}$. Give your answer to 1 decimal place.
c. Describe the variation of the means by comparing $\mu_{\bar{x}}$ and $\mu = 73.0$.

8. Tanya is studying the variability of sample means for the birth weight (grams) of a native animal. She has collected four random samples of size five.

Sample 1	Sample 2	Sample 3	Sample 4
47	108	94	40
87	50	112	106
57	54	90	90
78	87	82	80
61	82	47	61

a. Use the samples to calculate the associated sample means. Give your answers to the nearest integer.
b. Calculate the average of the sample means, $\mu_{\bar{x}}$. Give your answer to 1 decimal place.
c. Describe the variation of the means by comparing $\mu_{\bar{x}}$ and $\mu = 78.5$.

9. For a particular random variable X, five samples were collected to make inferences about the population mean.

Sample 1	Sample 2	Sample 3	Sample 4	Sample 5
$\bar{x} = 217.8$	$\bar{x} = 225.7$	$\bar{x} = 218.0$	$\bar{x} = 203.5$	$\bar{x} = 219.4$

a. Calculate the average value of the sample means, $\mu_{\bar{x}}$. Give your answer to 1 decimal place.
b. Hypothetically, the population mean could be $\mu = 200$, $\mu = 215$ or $\mu = 230$. Using your previous answer, determine which value is the most plausible.

10. WE2 1000 employees work at a tech company in Brisbane. The annual salary of this population is normally distributed with a mean $\mu = \$75\,000$ and standard deviation, $\sigma = \$8500$.

a. Samples of size 4 are selected, and the means are calculated. Determine the expected value and standard deviation of the distribution of $\bar{X}$.
b. If the sample size is increased to 25, calculate the effect on the expected value and standard deviation of the distribution of sample means.

11. 750 passengers travel daily through an airport. The time spent in the airport is normally distributed with a mean $\mu = 95$ minutes and a standard deviation $\sigma = 12$ minutes.

a. Samples of size 6 are selected, and the means are calculated. Determine the expected value and standard deviation of the distribution of $\bar{X}$.
b. If the sample size is increased to 36, calculate the effect on the expected value and standard deviation of the distribution of sample means.

12. WE3 A marine biologist is studying the length (in metres) of adult blue whales in a certain region. It is known that the lengths of these whales are normally distributed. A researcher repeatedly took samples of size 64 and calculated the empirical average of the sample means to be 27.5 metres, with a standard deviation of 1.44 metres.

Use this information to make the following inferences about population parameters. Give your answers to 1 decimal place.

a. The mean length μ of adult blue whales.
b. The standard deviation σ of their lengths.

13. A botanist is studying the height (in centimetres) of maple trees in a specific forest. It is known that the heights of these trees are normally distributed. Repeated samples of size 81 were collected, with the empirical average of the sample means being 150.0 cm, with a standard deviation of 4.84 cm.
Use this information to make the following inferences about population parameters. Give your answers to 1 decimal place.

a. The mean height μ of maple trees.

b. The standard deviation σ of their heights.

14. WE4 A smartphone manufacturer claims that the average battery life of their devices is normally distributed with a mean of 12 hours and a standard deviation of 2.5 hours. Customers may not be satisfied if the average battery life is shorter than 11 hours.
To investigate this concern, an independent testing agency selects a random sample of 60 smartphones. Calculate the probability, correct to 5 decimal places, that this smartphone's battery life is less than 11 hours. Show your working.

15. In a certain town, the average number of residents per household is 3.94, with a standard deviation of 1.58. A random sample of 40 households is selected from the town. Calculate the probability, correct to 3 decimal places, that the mean number of residents per household in this sample will exceed 4. Show your working.

Complex familiar

16. The commute time to work for employees at a certain company is assumed to be normally distributed, with a population mean of 35.6 minutes and a standard deviation of 6.2 minutes.
Commute times are collected from a random sample of 100 of these employees and used to calculate a sample mean, $\bar{x}_1$, in minutes.

a. Determine $P\left(\bar{x}_1 \leq 34\right)$, correct to 4 decimal places.

b. Given $P(\bar{x}_1 > w) = 0.85$, determine the value of w, correct to 2 decimal places.

c. Commute times are collected from a second random sample of the company's employees and used to calculate a second sample mean, $\bar{x}_2$, in minutes. Given $P(\bar{x}_2 \leq 34) \approx 0.4$, Evaluate the reasonableness of the number of employees in the second sample.

17. Repeated random samples were collected to investigate the distribution of $\bar{X}$. Assess the normality of the distribution.

$\bar{x}_1 = 15.4$	$\bar{x}_2 = 14.9$	$\bar{x}_3 = 15.6$	$\bar{x}_4 = 15.8$	$\bar{x}_5 = 14.9$
$\bar{x}_6 = 15.0$	$\bar{x}_7 = 15.3$	$\bar{x}_8 = 15.7$	$\bar{x}_9 = 15.1$	$\bar{x}_{10} = 15.5$
$\bar{x}_{11} = 15.4$	$\bar{x}_{12} = 15.3$	$\bar{x}_{13} = 14.8$	$\bar{x}_{14} = 15.7$	$\bar{x}_{15} = 15.0$

18. Repeated random samples were collected to investigate the distribution of $\bar{X}$. Assess the normality of the distribution.

$\bar{x}_1 = 21.8$	$\bar{x}_2 = 22.9$	$\bar{x}_3 = 23.2$	$\bar{x}_4 = 23.6$	$\bar{x}_5 = 23.8$
$\bar{x}_6 = 23.9$	$\bar{x}_7 = 24.2$	$\bar{x}_8 = 24.6$	$\bar{x}_9 = 24.6$	$\bar{x}_{10} = 24.8$
$\bar{x}_{11} = 25.0$	$\bar{x}_{12} = 25.4$	$\bar{x}_{13} = 25.6$	$\bar{x}_{14} = 26.5$	$\bar{x}_{15} = 26.8$

19. In a semi-rural region of southwest Queensland, the shire comprises of 40 parcels of land designated for residential use. The land area (in square metres) of these residential parcels is provided below.

27.4, 28.8, 30.1, 31.4, 32.7, 34, 35.8, 37.6, 39.6, 41.5, 43.4, 45.3, 47.5, 49.6,
51.8, 54.2, 56.6, 59, 61.4, 63.2, 65.0, 67.4, 69.6, 71.7, 73.9, 76, 78.2, 80.4,
82.5, 84.7, 86.2, 87.7, 89.4, 91.0, 92.4, 93.7, 95.1, 96.3, 97.6, 100.3

a. Use technology to simulate 10 random samples of size, $n = 15$.

b. Calculate the respective sample means.

c. Use technology to sketch the distribution of sample means.

20. In a mining company located in a remote area of Queensland, there are 40 employees working across various roles. The ages (in years) of all workers at the company are recorded in the dataset below.

$$17, 18, 19, 20, 20, 21, 23, 24, 25, 26, 28, 29, 30, 32, 33, 35, 36, 38, 40, 41,$$
$$42, 44, 45, 46, 48, 49, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65.$$

a. Use technology to simulate 10 random samples of size, $n = 20$.
b. Calculate the respective sample means.
c. Use technology to sketch the distribution of sample means.

Fully worked solutions for this chapter are available online.

LESSON
11.3 Confidence intervals

SYLLABUS LINKS

- Model and solve problems that involve sample means, with and without technology.
- Understand the concept of an interval estimate for a parameter associated with a random variable.
- Understand and use the approximate confidence interval $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$, as an interval estimate for μ, the population mean, where z is the appropriate quantile for the standard normal distribution.
- Understand and use the approximate margin of error.
  - $E = z\frac{s}{\sqrt{n}}$
- Understand and use the concept that there are variations in confidence intervals between samples and that most but not all confidence intervals contain μ.
- Use $\bar{x}$ and s to estimate μ and σ, to obtain approximate intervals covering desired proportions of values of a normal random variable and compare with an approximate confidence interval for μ.
- Model and solve problems that involve interval estimates for sample means, with and without technology.

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11.3.1 Point estimate

As discussed earlier, inferential statistics is concerned with drawing conclusions from data. For example, a sample of 400 values might suggest that the sample mean of 7.1 hours is a reasonable prediction for the number of hours sleep typical of a 17-year-old. The sample mean $\bar{x}$ in this case is known as a **point estimate**. This single value is used to estimate a population mean, μ.

Point estimate

A point estimate is a single value that approximates the population parameter.

In the weekly exercise activities the first sample from each of the three studies could be used as a point estimate to make an inference about the population mean.

$n = 4, \bar{x}_1 = 14.5$ $\quad$ $n = 8, \bar{x}_1 = 14.9$ $\quad$ $n = 16, \bar{x}_1 = 14.7$

However, because sample means can vary, it is insufficient to say that the population mean is simply equal to the point estimate. A more accurate method is to determine an interval estimate of μ known as a **confidence interval**.

11.3.2 Confidence interval for a population mean

A confidence interval provides a range of values within which the population mean is likely to fall, offering a more reliable estimate than a single value. Based on the sample $\bar{x} = 7.1$ hours, plausible values of μ, the actual mean sleep hours are between 6.95 and 7.25. This one-sample confidence interval for μ is written as (6.95, 7.25) hours.

Recall the small-town population of 25, where the true population mean of the daily exercise was $\mu = 14.9$. The table below shows how the point estimates from the three Activities were used to develop 90% confidence intervals for μ. These intervals provide an estimate of the population mean.

Sample size	Sample mean	Sample standard deviation	90% Confidence interval for μ
$n = 4$	$\bar{x}_1 = 14.5$	$s = 5.0$	(10.39, 18.61)
$n = 8$	$\bar{x}_1 = 14.9$	$s = 4.2$	(12.46, 17.34)
$n = 16$	$\bar{x}_1 = 14.7$	$s = 4.7$	(12.77, 16.63)

In most cases, the population mean μ is unknown, and the confidence interval provides plausible values for μ. As sample size increases, the width of the confidence interval narrows, improving the precision of the estimate. In the daily exercise time study, the population mean $\mu = 14.9$ minutes was known due to an unusually small population. Note that all three confidence intervals include the true value of μ.

Approximate confidence interval

The approximate confidence interval for μ is $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$ where,

$\bar{x}$ = sample mean

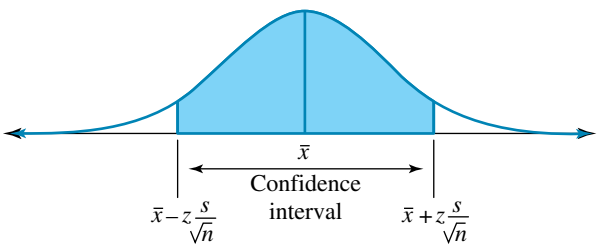
s = sample standard deviation

n = sample size

z = z-score confidence level.

Notice that the interval will be $z\frac{s}{\sqrt{n}}$ units either side of the sample mean $\bar{x}$, where s is the sample standard deviation, n is the sample size and z is the z score for the level of confidence we are interested in.

It is important to remember that the sample mean is always at the centre of the confidence interval.



Earlier in this chapter, we observed that the middle 90% of the standard normal distribution lies within 1.645 standard deviations of the mean. That is $P(-1.645 \le Z \le 1.645) = 0.9$.

Let us construct a rule for a 90% confidence interval.

1. Recall that a z-value is calculated as,

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

2. A 90% confidence then lies between,

$$-1.645 \leq \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} \leq 1.645$$

3. Manipulating these inequalities, the confidence interval of a population mean is,

$$\bar{x} - 1.645\left(\frac{s}{\sqrt{n}}\right) \leq \mu \leq \bar{x} + 1.645\left(\frac{s}{\sqrt{n}}\right)$$

The value of z varies depends on the desired level of confidence. Typical levels of confidence and associated z-values that are used when making calculations in this textbook are:

% Confidence level	z-score
90	1.65
95	1.96
99	2.58

Note: When μ and σ are unknown, and if the sample size n is large, the distribution of sample means $\bar{X}$ approaches $\bar{X} \sim N\left(\mu, \frac{s}{\sqrt{n}}\right)$ due to the Central Limit Theorem, where s is the sample standard deviation. To calculate a z-score estimate of a population mean based on s, we assume $s \approx \sigma$. This assumption is not too critical as the sample size n increases. In more advanced inferential studies that are above the scope of this course, when μ and σ are unknown, and small sized sample means and standard deviations are used to calculate interval estimates, the T_{n-1} distribution, $\left(\bar{x} \pm (t \text{ critical value}) \frac{s}{\sqrt{n}}\right)$ is applied, reflecting the added uncertainty in estimating σ.

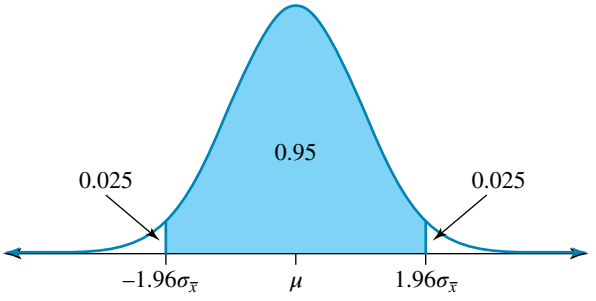
WORKED EXAMPLE 5 Calculating a 90% confidence interval

After surveying the 50 people in your school, you find that they plan to spend an average of $25.42 per week on takeaway food. The standard deviation for your sample was $2.20. Use these statistics to calculate a 90% confidence interval for the average amount that will be spent on takeaway food this week.

THINK	WRITE
1. 50 people were surveyed. This is sample size.	$n = 50$
2. State the sample mean and standard deviation.	$\bar{x} = 25.42$ $s = 2.20$
3. A 90% level of confidence is required. Select the appropriate z-score.	$z = 1.65$
4. Use the formula $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$ and substitute suitable values.	$\bar{x} - z\frac{s}{\sqrt{n}} = 25.42 - 1.65 \times \frac{2.20}{\sqrt{50}}$ $= 24.91$ $\bar{x} + z\frac{s}{\sqrt{n}} = 25.42 + 1.65 \times \frac{2.20}{\sqrt{50}}$ $= 25.93$

5. State the confidence interval numerically.	$(24.91, 25.93)$
Conclude by interpreting the confidence interval.	Based on this sample the 90% confidence interval for the amount spent on takeaway this week is between \$24.91 and \$25.93.

In Worked example 5, the interval $(24.91, 25.93)$ was in relation to a 90% **confidence level**. The other common levels of are 95% ($z = 1.96$) and 99% ($z = 2.58$). For a 95% confidence interval, 95% of the distribution is in the middle area of the distribution. This means the combined tails contain 5%.



WORKED EXAMPLE 6 Calculating a 95% confidence interval estimate of a population mean

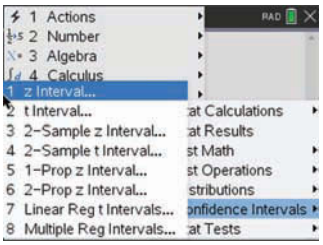
An academic study conducted at a university reported from a sample of 200 students, the average money spent on textbooks per semester was \$478.24. The sample standard deviation was \$51.45. Calculate a 95% confidence interval estimate of the population mean for the average money spent on textbooks each semester.

THINK	WRITE
1. 200 people were surveyed. This is sample size.	$n = 200$
2. State the sample mean and standard deviation.	$\bar{x} = 478.24$ $s = 51.45$
3. A 95% level of confidence is required. Select the appropriate z-score.	$z = 1.96$
4. Use the formula $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$ and substitute suitable values.	$\bar{x} - z\frac{s}{\sqrt{n}} = 478.24 - 1.96 \times \frac{51.45}{\sqrt{200}}$ $= 471.11$ $\bar{x} + z\frac{s}{\sqrt{n}} = 478.24 + 1.96 \times \frac{51.45}{\sqrt{200}}$ $= 485.37$
5. State the confidence interval numerically.	$(471.11, 485.37)$
6. Conclude by interpreting the confidence interval.	Using this sample, the 95% confidence interval for the amount spent on textbooks each semester ranges from \$471.11 to \$485.37.

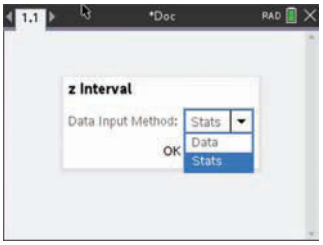
TI | THINK

1. On a Calculator page, press MENU then select
   6: Statistics
   6: Confidence Intervals
   1: z Interval

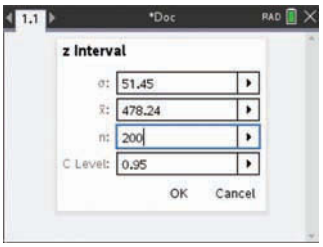
WRITE



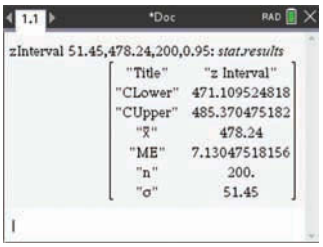
2. To set the appropriate data input method, select Stats Press the OK button.



3. Complete the entry lines:
   σ: 51.45
   $\bar{x}$: 478.24
   n: 200
   C Level: 0.95



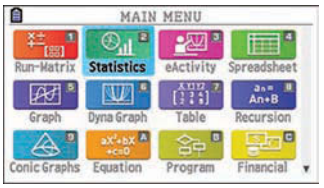
4. Press the ENTER or OK button.
   The answer appears on the screen.



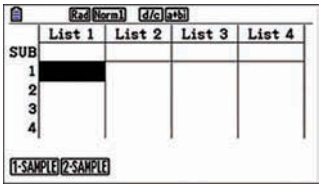
CASIO | THINK

1. On a Main Menu screen, select Statistics

WRITE

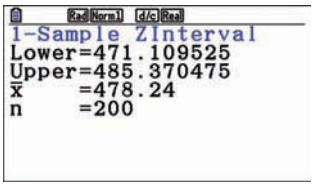


2. On a Statistics screen, select INTR z 1-SAMPLE



3. Complete the entry lines:
   Data: Variable
   C-Level: 0.95
   σ: 51.45
   $\bar{x}$: 478.24
   n : 200

4. Press the EXE button. The answer appears on the screen.



11.3.3 Interpreting confidence levels

The confidence level is a crucial concept when interpreting confidence intervals, as it reflects the success rate of the method used to estimate a population parameter. For example, a 90% confidence level means that in the long run, 90% of the intervals constructed using this method will contain the true population mean. This does not mean that any specific interval has a 90% chance of containing the mean. Instead, it indicates that if we repeatedly took random samples and computed confidence intervals, 90% of those intervals would capture the actual population mean.

To illustrate this concept, let's revisit the weekly exercise activity for sample size $n = 4$. Using a 90% confidence level, we have calculated ten confidence interval estimates for the population mean, $\mu = 14.9$ hours. The results are summarised below.

Sample #	$\bar{x}$	s	90% confidence interval (hours)	Contains population mean
1	14.5	5.0	(10.4, 18.6)	Yes
2	14.2	2.6	(12.1, 16.3)	Yes
3	12.6	4.7	(8.7, 16.5)	Yes
4	14.0	3.4	(11.2, 16.8)	Yes
5	14.3	5.9	(9.4, 19.2)	Yes
6	17.3	1.6	(16.0, 18.6)	No
7	15.1	2.7	(12.9, 17.3)	Yes
8	15.0	5.4	(10.6, 19.4)	Yes
9	16.0	6.2	(10.9, 21.1)	Yes
10	14.6	2.2	(12.8, 16.4)	Yes

Notice that the sixth sample's interval (16, 18.6) hours does not contain the population mean. As expected with a 90% confidence level, 9 out of 10 intervals include the mean. This demonstrates that if we repeatedly took random samples, 90% of the confidence intervals would capture the true population mean. Note, the tenth confidence interval (12.8, 16.4) does not imply that 90% of all daily exercise durations are between 12.8 and 16.4 hours.

WORKED EXAMPLE 7 Making assumptions and interpreting a 99% confidence interval

At Precision Sports, consistency in manufacturing baseballs is critical to ensure they are neither too lively nor too dead. To maintain quality, a sample of baseballs are tested by dropping them from a standard height and measuring their bounce height (cm). The calculated 99% confidence interval for μ cm was (54.5, 55.9).

a. State the assumptions necessary for the confidence interval to be considered a valid estimate of the population mean.

b. Describe how this interval estimate should be interpreted.

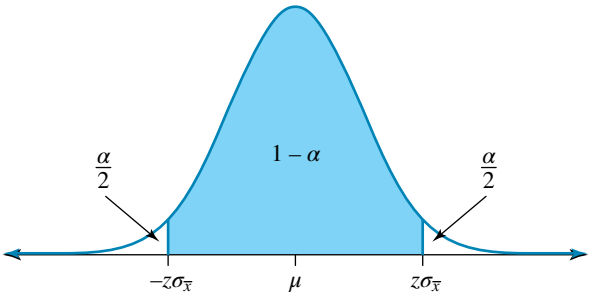
THINK	WRITE
a. 1. Consider the normality of the population and make a relevant assumption.	a. Assumption 1: The population distribution is normal.
2. The size of the sample was not provided. Consider the central limit theorem.	Assumption 2: The sample was sufficiently large ($n \geq 30$). If so, the sampling distribution of $\bar{x}$ is well approximated by a normal curve.
3. Sampling techniques were not described; hence an assumption can be made.	Assumption 3: The sample is an unbiased random sample from the population of interest.
b. 1. Interpret the confidence interval, (54.5, 55.9).	b. Based on this sample, plausible values of μ are between 54.5 and 55.9 cm.
2. Provide an explanation of the confidence level.	The method to produce this estimate is successful in capturing the actual population mean 99% of the time.

11.3.4 Other levels of confidence

It is possible to determine the z-score for any confidence interval using the inverse normal distribution.

The curve's symmetry means the area in each tail is equal. If the total tail area is α, the middle area is $(1-\alpha)$, and each tail has an area of $\frac{\alpha}{2}$.

Therefore, the z-score corresponding to the tail area $\frac{\alpha}{2}$ is used for a $(1-\alpha)$ confidence interval.



WORKED EXAMPLE 8 Using mean and standard deviation to determine an 85% confidence interval

Arabella samples 102 people and finds, with a standard deviation of 0.8, that on average they eat 5.2 cups of vegetables per day. Use these statistics to state an 85% confidence interval for the average daily vegetable consumption.

THINK	WRITE
1. There are 102 people in the sample. This is the sample size. The sample mean is 5.2. The sample standard deviation is 0.8.	$n = 102$ $\bar{x} = 5.2$ $s = 0.8$
2. For an 85% confidence interval, determine the area of each tail.	85% confidence interval $1-\alpha = 0.85$ $\alpha = 0.15$ $\frac{\alpha}{2} = 0.075$ 15% will be in the tails 7.5% in each tail
3. Use the inverse normal to determine the z-score with a tail area of 0.075.	$z_{0.075} = 1.44$
4. The confidence interval is $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$. Use technology to calculate the values of $\bar{x} - z\frac{s}{\sqrt{n}}$ and $\bar{x} + z\frac{s}{\sqrt{n}}$.	$\bar{x} - z\frac{s}{\sqrt{n}} = 5.2 - 1.44 \times \frac{0.8}{\sqrt{102}}$ $= 5.09$ $\bar{x} + z\frac{s}{\sqrt{n}} = 5.2 + 1.44 \times \frac{0.8}{\sqrt{102}}$ $= 5.31$
5. Identify the 85% confidence interval.	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right) = (5.09, 5.31)$
6. State the answer.	The 85% confidence interval for the number of cups of vegetables consumed per day is (5.09, 5.31).

Exercise 11.3 Confidence intervals

learnon

11.3 Exercise	11.3 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18	19	20

These questions are even better in jacPLUS!

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- Access sample responses
- Track results and progress

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Simple familiar

1. The 90% confidence interval (45.2, 53.2) was calculated from a sample of size 100.
   a. Calculate the sample mean. Give your answer to 1 decimal place.
   b. Describe how the confidence interval would change if a 99% level of confidence was selected instead.

2. A 95% confidence interval $(110, q)$ was calculated from a sample mean, $\bar{x} = 116$. Calculate the unknown value q.

3. The time taken to complete a psychological task when starting employment at a company is normally distributed, with a mean time of $\mu = 38$ minutes. A human resource officer records the times for 40 random samples of new employees. From these samples, 40 separate 90% confidence intervals are calculated. Describe how many of these intervals would be expected to contain μ.

4. The scores on a test are assumed to follow a normal distribution. Researchers use results from a random sample of scores to calculate a confidence interval for the population mean. Upon review, they wish to achieve a narrower confidence interval, prompting the collection of a second sample. Assuming the sample standard deviations are equal, describe the steps they must take to achieve this goal.

5. A study was conducted on two different types of fertilisers, measuring the growth of plants over a six-week period. The resulting 95% confidence intervals for the mean height of the plants are (30.5, 32.1) and (29.4, 31.6). Assuming the two sample standard deviations are equal, determine which confidence interval reflects the larger sample size. Justify your reasoning.

6. After a friend surveyed the 130 people in your community, they found that they planned to spend an average of \$425 when buying Christmas presents for their immediate family. The standard deviation for their sample was \$72. A 99% confidence interval estimate on the mean money spent was calculated.

$$\left(425 - 1.96 \times \frac{\sqrt{72}}{120}, 425 + 1.96 \times \frac{72}{\sqrt{120}}\right)$$

Identify the errors your friend made and describe the changes required to calculate an accurate confidence interval.

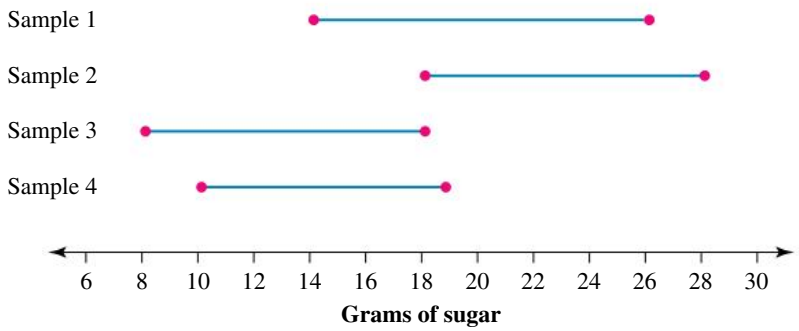
7. **WE7** At Greenhouse Organics, maintaining the quality of their fertiliser is essential for optimal plant growth. To ensure consistency, a sample of fertiliser batches is tested by measuring their nutrient content (in grams per kilogram). The calculated 99% confidence interval for nutrient content was (25.3, 27.1) grams per kilograms.

   a. State the assumptions necessary for the confidence interval to be considered a valid estimate of the population mean.
   b. Describe how this interval estimate should be interpreted.

8. At a northern coast Fishery Farm, monitoring the weight of whiting is crucial for assessing fish health and market readiness. A sample of 50 whiting was weighed, resulting in a calculated 90% confidence interval for their weight of (1.2, 1.5) kilograms.

   a. State the assumptions necessary for the confidence interval to be considered a valid estimate of the population mean.
   b. Describe how this interval estimate should be interpreted.

9. The mass of sugar in a fast-food soft serve cone is claimed to be normally distributed with a mean of 22 grams. A random sample of 50 of these soft serve cones is found to have a mean of 20 grams. Identify the sample that represents an approximate confidence interval for μ based on this sample.



10. **WE5** After surveying the 50 people in your school, you find that they plan to spend an average of \$352.61 per year on new clothes. The standard deviation for your sample was \$48.79. Use these statistics to calculate a 90% confidence interval ($z = 1.65$) for the average amount that will be spent on new clothes this year. Show your working.

11. After conducting a survey among 80 employees at a data company, you discover they plan to spend an average of \$620.50 per year on professional development. The sample standard deviation is \$70.25. Use these statistics to calculate a 90% confidence interval ($z = 1.65$) for the average amount that will be spent on professional development this year. Show your working.

12. Among the 90 Queenslanders surveyed, the average expenditure on holidays this year was \$2874, with a standard deviation of \$729. Use these statistics to determine a 90% confidence interval ($z = 1.65$), to 1 decimal place, for the average holiday spending of Queenslanders. Show your working.

13. **WE6** A survey of 150 airline passengers revealed that the average weight of baggage taken on a plane was 22.4 kilograms, with a sample standard deviation of 6.2. Calculate a 95% confidence interval ($z = 1.96$) estimate for the population mean weight of baggage carried by passengers. Give your answers to 1 decimal place.

14. A study analysing the fat content in meals from a particular fast-food chain found that a sample of 100 meals had an average fat content of 32.5 grams, with a sample standard deviation of 8.3 grams. Calculate a 95% confidence interval estimate for the population mean fat content in grams for meals from this fast-food chain. Give your answers to 1 decimal place.

15. A survey of 180 school students found that the average time spent on electronic devices for non-school-related activities was 94 minutes per day, with a sample standard deviation of 27 minutes.

a. Calculate a 95% confidence interval estimate for the population mean time spent on electronic devices by students for non-academic purposes. Give your answers to 1 decimal place.

b. Calculate a 99% confidence interval estimate for the population mean time spent on electronic devices by students for non-academic purposes. Give your answers to 1 decimal place.

16. A sample of 90 AAA batteries was tested to evaluate their average lifespan. The results showed an average lifetime of 1978 minutes, with a standard deviation of 347 minutes.

a. Calculate a 90% confidence interval estimate for the population mean time. Give your answers to 2 decimal places.

b. Calculate a 95% confidence interval estimate for the population mean time. Give your answers to 2 decimal places.

17. WE8 An engineering firm measures a sample of 90 metal components and finds that their average length is 24.5 centimetres, with a standard deviation of 1.2 centimetres. Use these statistics to determine an 85% confidence interval for the average length of the metal components produced by the company. Give your answers to 2 decimal places.

18. A school administers a selective entry aptitude test to 150 applicants and finds that the average score is 82.5, with a standard deviation of 10.3. Use these statistics to determine an 80% confidence interval for the average score of all applicants taking the test. Give your answers to 1 decimal place.

Complex familiar

19. The federal minister in the education department recently chose a random sample of 30 newly graduated university students and queried each person about the time (in months) it took to complete his or her degree.

20.3, 22.2, 24.3, 26.6, 28.1, 30.7, 32.6, 34.7, 37.3, 39.4, 39.8, 42.0, 44.6, 44.7, 46.0, 47.4, 48.6, 50.0, 50.1 51.9, 54.2, 56.5, 56.9, 58.4, 59.8, 60.1 61.2, 62.6, 65.2, 67.9

Use the sample data to calculate a 95% confidence interval to estimate the population mean time spent at university. Give your answers to 2 decimal places.

Complex unfamiliar

20. An assessor at a motor vehicle insurance company chose a random sample of 30 claims and the associated repair cost in dollars.

1562, 1652, 1748, 1856, 1869, 1922, 2042, 2132, 2228, 2348, 2377, 2444, 2564, 2684, 2687, 2750, 2816, 2870, 2936, 3020, 3128, 3236, 3321, 3379, 3452, 3522, 3540, 3626, 3641, a.

The sample standard deviation of repair costs is \$665.16 with (2464.08, 2940.12) being the 95% confidence interval.

A second confidence interval, at b% confidence, was calculated as (2578.81, 2825.39). Determine the values for a and b. Give your answers to the nearest integer.

Fully worked solutions for this chapter are available online.

LESSON
11.4 Margin of error, level of confidence and sample size

SYLLABUS LINKS

- Understand and use the approximate confidence interval $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$, as an interval estimate for μ, the population mean, where z is the appropriate quantile for the standard normal distribution.
- Model and solve problems that involve sample means, with and without technology.
- Understand and use the relationship between margin of error, level of confidence and sample size.
- Understand and use the concept that there are variations in confidence intervals between samples and that most but not all confidence intervals contain μ.
- Use $\bar{x}$ and s to estimate μ and σ, to obtain approximate intervals covering desired proportions of values of a normal random variable and compare with an approximate confidence interval for μ.
- Model and solve problems that involve interval estimates for sample means, with and without technology.

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11.4.1 Margin of error

Recall that the formula for a confidence interval is given by $\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}$. This formula can be expressed as $\bar{x} \pm z\frac{s}{\sqrt{n}}$.

In principle, a confidence interval can be explained in four simple steps:

1. It is centred at the point estimate $\bar{x}$.
2. The **standard error** $\frac{s}{\sqrt{n}}$ and **level of confidence** z are used to calculate the **margin of error**, $E = z\frac{s}{\sqrt{n}}$.
3. The margin of error is added to the sample mean to calculate the upper boundary value of the confidence interval: $\bar{x} + z\frac{s}{\sqrt{n}}$.
4. The margin of error is subtracted from the sample mean to calculate the lower boundary value of the confidence interval: $\bar{x} - z\frac{s}{\sqrt{n}}$.

The margin of error quantifies the uncertainty associated with a sample estimate, indicating how closely the estimate reflects the true population parameter. It defines the range within which the true value is likely to fall, offering a measure of precision for statistical inferences.

Properties of the margin of error

1. **Confidence Level: Higher confidence levels increase the margin of error because a larger z value is used in the formula. For example, a 95% confidence interval has a higher z value than a 90% confidence interval, resulting in a wider interval.**
2. **Sample Size (n): A larger sample size decreases the margin of error due to the inverse relationship in the formula: Margin of Error $= z\frac{\sigma}{\sqrt{n}}$. As ($n$) increases, the denominator ($\sqrt{n}$) grows, reducing**

the overall margin of error. This leads to a narrower confidence interval and a more precise estimate.

3. **Standard Deviation (σ): Greater variability in the data, reflected by a larger σ, increases the margin of error. Since σ appears in the formula $z\frac{\sigma}{\sqrt{n}}$ a larger standard deviation results in a wider confidence interval, making the estimate less precise.**

WORKED EXAMPLE 9 Using 95% confidence interval and sample size to calculate mean, standard deviation and 90% confidence interval

A sample of 120 batteries was tested to determine their average lifetime. The 95% confidence interval was (2218.7, 2281.3).

a. **Calculate the sample mean.**
b. **Calculate the margin of error for the 95% confidence interval.**
c. **Calculate the sample standard deviation, s.**
d. **Determine a 90% confidence interval for average lifetime of batteries.**

THINK	WRITE
a. The sample mean is centred about the upper and lower values of the interval. Hence, calculate the average.	$\bar{x} = \frac{2218.7 + 2281.3}{2}$ $= 2250$
b. The margin of error is the difference between either confidence interval boundary and the sample mean.	$E = 2281.3 - 2250$ $= 31.3$
c. State the formula for margin of error.	$E = z\frac{s}{\sqrt{n}}$
Substitute $E = 31.3$, $z = 1.96$ & $n = 120$ into the formula.	$31.3 = 1.96\frac{s}{\sqrt{120}}$
Apply inverse operations and solve for s.	$\frac{31.3}{1.96} = \frac{s}{\sqrt{120}}$ $\sqrt{120} \times \frac{31.3}{1.96} = s$ $s = 174.94$
d. Recall the z-score for a 90% confidence interval.	90% level of confidence: $z = 1.65$
Calculate the margin of error using the sample statistics.	$E = 1.645\frac{174.94}{\sqrt{120}}$ $= 26.27$
Determine the confidence interval using $\bar{x} = 2250$.	$\bar{x} \pm z\frac{s}{\sqrt{n}} = 2250 \pm 26.27$ $= (2223.7, 2276.3)$

11.4.2 Sample sizes

Careful considerations will be made by a statistician when attempting to make a valid inference of μ.

i. the level of confidence.
ii. the level of preciseness in the width of the confidence interval.
iii. the sample size.

The width of a confidence interval is affected by the margin of error, E. Using symmetry, the width between the upper and lower boundaries is $2 \times E$.

Before collecting data, the statistician can determine the sample size n required for a desired margin of error E.

By manipulating the formula $E = z\frac{s}{\sqrt{n}}$, the sample size needed is $n = \left(\frac{z \times s}{E}\right)^2$.

Sample sizes for different confidence levels

Level of confidence	**90%**	**95%**	**99%**
Sample size	$n = \left(\frac{1.65 \times s}{E}\right)^2$	$n = \left(\frac{1.96 \times s}{E}\right)^2$	$n = \left(\frac{2.58 \times s}{E}\right)^2$

WORKED EXAMPLE 10 Determining sample size required for specific error of margin

A company is gathering data on monthly spending for video streaming subscriptions. Given the assumed population standard deviation of $\sigma = \$14$ determine the sample size needed to achieve a margin of error of less than \$1 at a 95% confidence level.

THINK	WRITE
1. State the formula for n at 95% confidence.	$n = \left(\frac{1.96 \times s}{E}\right)^2$
2. Assume the sample standard deviation is equal to the population value.	$s = \sigma = 14$ $E = 1$
Substitute values into the formula.	$n = \left(\frac{1.96 \times 14}{1}\right)^2$
3. Calculate the sample size.	$n = 752.95$
To reduce the margin of error, to be less than 1, the value is rounded up.	A sample size of at least 753 is required.

WORKED EXAMPLE 11 Calculating sample size given standard deviation and 95% confidence interval

A sample has a standard deviation of 1.45, and the 95% confidence interval is calculated to be (20.15, 20.85). Calculate the sample size.

THINK	WRITE
1. The sample mean is centred about the upper and lower values of the interval. Hence, calculate the average.	$\bar{x} = \frac{20.15 + 20.85}{2}$ $= 20.5$
2. State the formula for a 95% confidence interval.	$\bar{x} \pm 1.96 \times \frac{s}{\sqrt{n}}$
3. Equate the upper value 20.85 with the formula.	$20.85 = \bar{x} + 1.96\frac{s}{\sqrt{n}}$
4. Substitute the values, $\bar{x} = 20.5$ and $s = 1.45$	$20.85 = 20.5 + 1.96 \times \frac{1.45}{\sqrt{n}}$
5. Rearrange and solve for n.	$\frac{20.85 - 20.5}{1.96} = \frac{1.45}{\sqrt{n}}$ $\frac{5}{28} \times \sqrt{n} = 1.45$ $n = \left(\frac{1.45 \times 28}{5}\right)^2$ $n = 65.93$
6. Conclude by stating the appropriate sample size.	Rounded to the nearest whole number, the sample size is 66.

Exercise 11.4 Margin of error, level of confidence and sample size

learnon

11.4 Exercise | 11.4 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	16, 17	18

These questions are even better in jacPLUS!

- Receive immediate feedback
- Access sample responses
- Track results and progress

Find all this and MORE in jacPLUS

Simple familiar

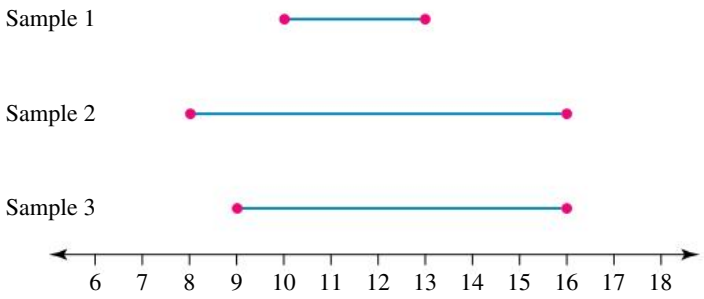
1. For the following confidence intervals, calculate the width W of the confidence interval and the margin of error E.
   a. (172, 182)
   b. (44, 56)
   c. (12.2, 15.8)
2. Calculate the integer value p from the confidence interval $(24, p)$ if it was based on a sample mean, $\bar{x} = 33$.

3. The interval estimate (120, 130) was determined at a 95% confidence level. If a 99% level of confidence was selected instead, identify the valid confidence interval from the options below. Justify your answer.

Option 1: (122.2, 129.8) Option 2: (118.5, 131.5) Option 3: (117.5, 130.5)

4. Consider a random sample of size n_1 collected from a normally distributed variable. The sample mean $\bar{x}$ and standard deviation s were applied to calculate a margin of error for a confidence interval estimate of μ. A second sample of size n_2 was taken with equivalent values of $\bar{x}$ and s. Determine n_2 in terms of n_1 if the margin of error is reduced by 50%.

5. Three random samples of different sizes were taken to estimate a certain population mean with a known standard deviation. A 90% confidence interval was calculated for each sample.



Explain which sample used the smallest sample size.

6. WE9 A sample of 140 televisions was tested to assess their average lifetime in months. The 95% confidence interval was (628.9, 651.1).
   a. Calculate the sample mean.
   b. Calculate the margin of error for the 95% confidence interval. Give your answer to 1 decimal place.
   c. Calculate the sample standard deviation, s. Give your answer to 1 decimal place.
   d. Determine a 90% confidence interval for average lifetime of televisions. Give your answer to 1 decimal place.

7. Reaction time is measured by having participants hold their thumb and forefinger 3 cm apart at the top of a ruler. When the ruler is dropped, the distance is recorded at the point where the participant grabs it, indicating their reaction time.
A sample of 250 from Queensland Year 12 students were tested, and the resulting 95% confidence interval was (4.6, 5.0) cm.
   a. Calculate the sample mean. Give your answer to 1 decimal place.
   b. Calculate the margin of error for the 95% confidence interval. Give your answer to 1 decimal place.
   c. Calculate the sample standard deviation, s. Give your answer to 2 decimal places.
   d. Determine a 90% confidence interval for average reaction distance for Year 12 students from Queensland. Give your answer to 1 decimal place.

8. WE10 A fitness study is collecting data on the resting heart rate (in beats per minute, BPM) of teenagers before exercise. Given the assumed population standard deviation of 8 BPM, determine the sample size required to achieve a margin of error of less than 2 BPM at a 95% confidence level. Give your answer to the nearest integer.

9. A health study is investigating the average sleep duration (in hours per night) of Australian adults aged 18–21. Given the assumed population standard deviation of 1.6 hours, determine the sample size required to achieve a margin of error of less than 0.4 hours at a 99% confidence level. Give your answer to the nearest integer.

10. WE11 A sample has a standard deviation of 50 and the 95% confidence interval is calculated to be $(393.9, 406.1)$. Calculate the sample size. Give your answer to the nearest integer.

11. A sample has a standard deviation of 7.5 and the 99% confidence interval is calculated to be $(32.52, 37.48)$. Calculate the sample size. Give your answer to the nearest integer.

12. A random sample of n people is required to calculate an unknown % confidence interval. Calculate the unknown level of confidence if the margin of error is 1.2, $n = 60$ and $s = 4.5$.

13. In a recent study, a sample of 50 employees revealed that they take an average of 12.5 days of sick leave per year, with a standard deviation of 3.6 days.
    a. Calculate, correct to 1 decimal place, a 95% confidence interval for the average number of sick leave days.
    b. If the company wants to reduce the margin of error to ± 1 day, determine what sample size is needed.

14. In a recent study, a sample of 50 Australian school graduates attending Schoolies revealed that they spend an average of \$850 on accommodation with a standard deviation of \$200.
    a. Calculate a 95% confidence interval for the average amount of money spent on accommodation by Schoolies attendees. Give your answers to the nearest cent.
    b. If the researchers want to reduce the margin of error to $\pm$\$50, determine what sample size is needed. Give your answer to the nearest integer.

15. The weights of all students at a college were measured. A mean weight of 71.2 kg was calculated from this data. A random sample of 35 students from this school was selected. The mean weight of this sample was 72.1 kg with a standard deviation of 3.1 kg. Determine the smallest confidence level that could be chosen from 85%, 90%, 95% and 99%, to calculate a confidence interval that contains μ based on this sample. Explain your reasoning.

Complex familiar

16. The weights of a population of wombats are known to follow a normal distribution. A random sample of wombats is taken, and their weights are measured. The sample has a mean weight of 26.2 kg and a standard deviation of 2.7 kg. A 90% confidence interval calculated for the population mean weight is $(25.6, 26.8)$ kg. Based on this data, determine, correct to 1 decimal place, a 95% confidence interval for the population mean. Show your working.

17. The prices for Chemistry tutoring sessions in a city are known to follow a normal distribution. A random sample of tutoring prices is taken, and the prices are recorded. The sample has a mean price of \$45.80 per hour and a standard deviation of \$8.50 per hour. A 90% confidence interval calculated for the population mean price is (\$43.72, \$47.88). Based on this data, determine, correct to 2 decimal place, a 99% confidence interval for the population mean price. Show your working.

Complex unfamiliar

18. The time high school students spend traveling on public transport each day is normally distributed with a mean time of μ minutes. A deputy principal measured the commuting time of all students using public transport in form class G6. From this data, an approximate 95% confidence interval for μ was calculated as $(16.2, 18.2)$ minutes.
The deputy repeated the process for form class H2, which had the same number of students, resulting in an approximate 95% confidence interval for μ of $(18.4, 20.0)$ minutes.
Using the same data for both classes at a new confidence level of x%, the deputy observed that the upper bound of the confidence interval for G6 now equalled the lower bound of H2. Determine the value of x to 1 decimal place. Show your working.

Fully worked solutions for this chapter are available online.

LESSON
11.5 Applications and other distributions

SYLLABUS LINKS

- Use repeated random sampling data from a variety of distributions and a range of sample sizes to examine the approximate standard normality of $\frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}}$ for large samples $(n \geq 30)$, where s is the sample standard deviation (Central limit theorem).
- Model and solve problems that involve sample means, with and without technology.
- Model and solve problems that involve interval estimates for sample means, with and without technology.

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11.5.1 Sampling other distributions

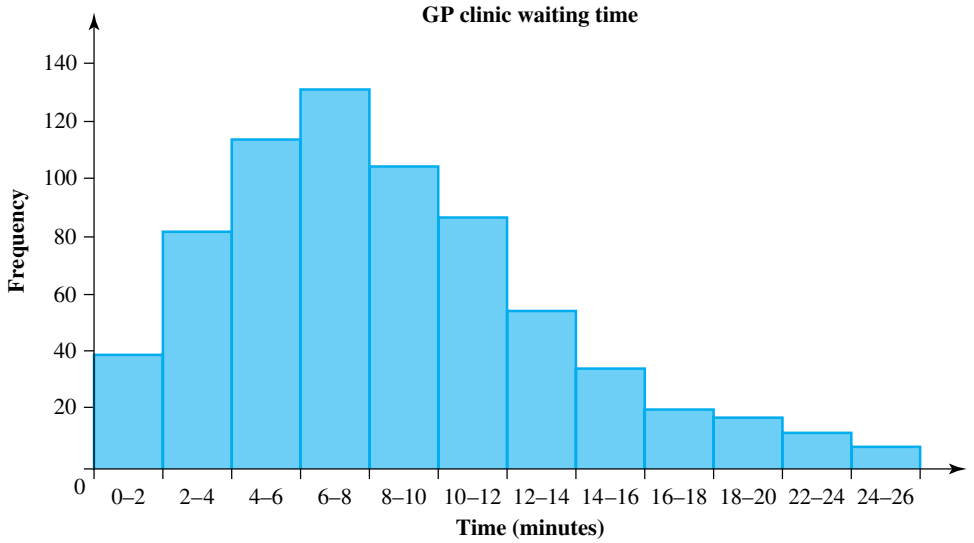
When investigating distributions that are not normal, it is still possible to simulate random sampling to examine how sample means behave.

Sample size and normal distribution approximation

If the sample size is sufficiently large (typically $n \geq 30$), the distribution of the sample means will approximate a normal distribution, even if the population itself is skewed or non-normal.

This allows the application of formulas for confidence intervals based on the sample mean $\overline{x}$ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

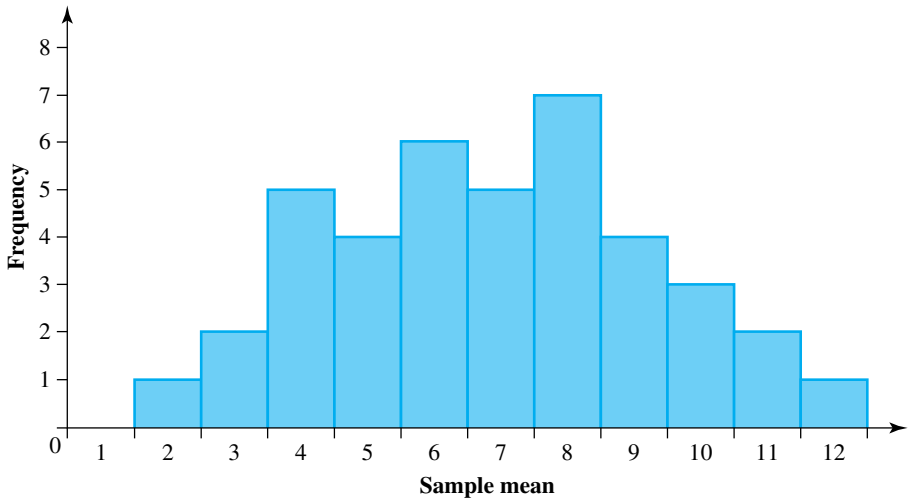
The frequency histogram on the right represents the wait time for patients at a medical general practice.



The distribution is positively skewed, making the mean value harder to identify visually. However, the calculated average wait time is 8.5 minutes. If we treat this data as the population, we will take 40 random samples with varying sample sizes to observe the shape of the sampling distribution.

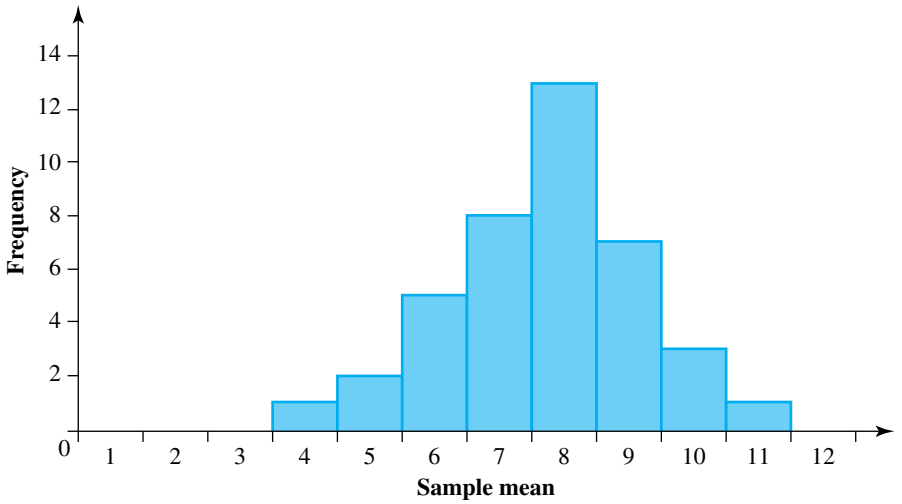
Simulation 1:

- 40 random samples
- Sample size $n = 10$



Simulation 2:

- 40 random samples
- Sample size $n = 30$



In both simulations, a common feature is that the histogram is centred around the known mean of 8.5 minutes. As the sample size n increases, the variability (or spread) decreases, consistent with the central limit theorem.

WORKED EXAMPLE 12 Simulating sampling for distributions

Consider a call centre that receives an average of 6 calls per hour. Since the calls are independent, the time between successive calls is expected to follow an exponential distribution. On a specific day, 40 calls were received, and the time intervals between these calls are recorded in the following dataset.

17.4, 12.9, 26.0, 12.4, 13.7, 8.9, 29.2, 12.3, 17.7, 11.2, 9.9, 32.1, 15.1, 7.9, 2.0, 4.7, 10.2, 14.5, 6.8, 48.1, 1.6, 2.4, 6.2, 17.7, 7.4, 0.3, 3.5, 5.6, 10.9, 7.0, 9.5, 2.5, 5.5, 5.5, 7.6, 10.8, 0.8, 9.2, 4.8, 12.3, 0.6, 32.2, 3.4, 6.5, 17.7, 8.0, 5.8, 26.4, 11.6, 1.5

a. Use technology to simulate 10 random samples of size, $n = 20$.
b. Calculate the respective sample means.
c. Use technology to sketch the distribution of sample means.

THINK

a. Excel can be used to select random samples from a list. Insert the data into column A. In column B use the formula '=RAND()'. To select random values from column C, use '=INDEX($A$2: $A$51, RANK.EQ(B2, $B$2: $B$51) + COUNTIF($B$2: B2, B2) − 1, 1)'

WRITE

a. Random sample 1:
{0.6, 6.8, 9.9, 1.5, 5.8, 2.4, 11.6, 15.1, 13.7, 10.2, 8.9, 6.5, 14.5, 17.7, 32.1, 2, 4.8, 0.8, 3.5, 12.3, 9.5, 10.9, 2.5, 48.1, 1.6}

b. 1. Calculate the sample mean $\bar{x} = \dfrac{\sum x}{n}$.

b. $\bar{x}_1 = \dfrac{\sum x}{n}$

$= \dfrac{0.6 + 6.8 + 9.9 + \ldots + 1.6}{20}$

$= 10.8$

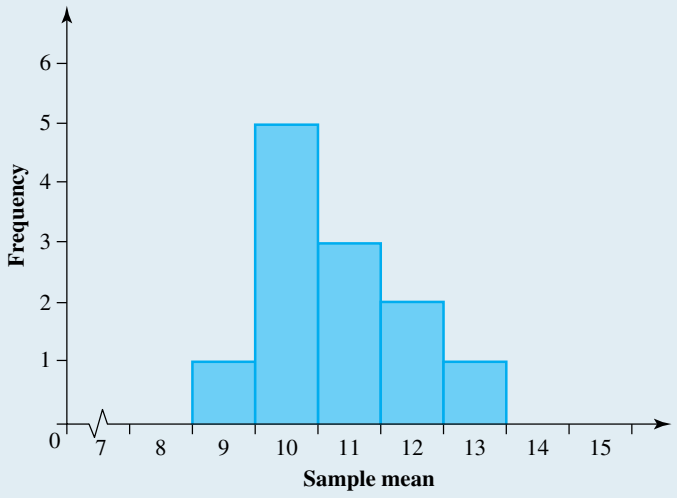
2. Repeat this process for the other nine random samples.

$\bar{x}_1 = 10.8, \bar{x}_2 = 11.1, \bar{x}_3 = 12.4, \bar{x}_4 = 10.9, \bar{x}_5 = 10.8$

$\bar{x}_6 = 9.2, \bar{x}_7 = 11.2, \bar{x}_8 = 12.6, \bar{x}_9 = 10.9, \bar{x}_{10} = 11.3$

c. Use technology to create a histogram that shows the distribution of the sample means.

c.



WORKED EXAMPLE 13 Simulating sampling for distributions that are not normal

A population is distributed so that the $P(X \le b) = \int_0^b 0.5e^{-0.5x}dx$.

a. The population mean can be found by calculating $\mu = \int_0^\infty 0.5xe^{-0.5x}dx$. Determine the population mean.

b. The population variance can be found by calculating $\sigma^2 = \int_0^\infty 0.5(x-\mu)^2e^{-0.5x}dx$. Determine the population standard deviation.

c. Use technology to model the selection of 10 random samples of size 40.

d. For each sample, use technology to calculate the 90% confidence interval for the population mean.

e. Use technology to sketch the population distribution and the distribution of sample means.

f. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.

THINK	WRITE
a. 1. Use integration by parts to determine the integral. Let $u = x$ and $\frac{dv}{dx} = 0.5e^{-0.5x}$. Determine $\frac{du}{dx}$ and v.	**a.** $u = x$ $\quad \frac{dv}{dx} = 0.5e^{-0.5x}$ $\frac{du}{dx} = 1$ $\quad v = -e^{-0.5x}$
2. Determine the integral using the formula $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$.	$\mu = \int_0^\infty 0.5xe^{-0.5x}dx$ $= \int u\frac{dv}{dx}dx$ $= uv - \int v\frac{du}{dx}dx$ $= \left[-xe^{-0.5x}\right]_0^\infty - \int_0^\infty \left(-e^{-0.5x}\right)dx$ $= \left[-xe^{-0.5x}\right]_0^\infty + \int_0^\infty e^{-0.5x}dx$ $= 0 + \left[-2e^{-0.5x}\right]_0^\infty$ $= 0 + 2$ $= 2$
3. State the population mean.	$\mu = 2$

b. 1. Using the given formula determine the variance. Substitute $\mu = 2$ into the integrand.

b. $$\sigma^2 = \int_0^\infty 0.5(x-\mu)^2 e^{-0.5x}dx$$
$$= \int_0^\infty 0.5(x-2)^2 e^{-0.5x}dx$$

2. Use integration by parts to determine the integral. Let $u = (x-2)^2$ and $\frac{dv}{dx} = 0.5e^{-0.5x}$

$$u = (x-2)^2 \qquad \frac{dv}{dx} = 0.5e^{-0.5x}$$
$$\frac{du}{dx} = 2(x-2) \qquad v = -e^{-0.5x}$$

3. Determine the integral.

$$\sigma^2 = \int_0^\infty 0.5(x-2)^2 e^{-0.5x}dx$$
$$= \int u\frac{dv}{dx}dx$$
$$= uv - \int v\frac{du}{dx}dx$$
$$= \left[-(x-2)^2 e^{-0.5x}\right]_0^\infty - \int_0^\infty -2(x-2)e^{-0.5x}dx$$
$$= 0 + 2^2e^0 + 2\int_0^\infty (x-2)e^{-0.5x}dx$$
$$= 4 + 2\int_0^\infty (x-2)e^{-0.5x}dx$$

4. Use integration by parts to complete the integral.

$$u = x-2 \qquad \frac{dv}{dx} = e^{-0.5x}$$
$$\frac{du}{dx} = 1 \qquad v = -2e^{-0.5x}$$
$$\sigma^2 = 4 + 2\left(\left[-2(x-2)e^{-0.5x}\right]_0^\infty - \int_0^\infty -2e^{-0.5x}dx\right)$$
$$= 4 + 2\left(0 + 2(-2)(1) + \left[-4e^{-0.5x}\right]_0^\infty\right)$$
$$= 4 + 2(-4 + 0 + 4)$$
$$= 4$$

5. The standard deviation is denoted by σ. Determine the standard deviation by taking the square root of the variance.

The standard deviation $\sigma = \sqrt{4} = 2$.

c. 1. Use a random number generator to generate 40 random numbers to represent 40 possible random probabilities. In Excel the command =RAND() will achieve this. *Note:* not all 40 values have been listed.

c. Sample Numbers generated (a total of 40 numbers are generated for each sample):

Random #
0.038742
0.42023
0.283778
0.430065
0.654831
0.141399
0.923711
0.990582
0.240471
0.231727
⋮

2. Let a particular random number generated be a. The random number generated could be thought of $P(x \le b) = a$. This means that the area under the curve $0.5e^{-0.5x}$ between $x = 0$ and $x = b$ is a. Solve $\int_0^b 0.5e^{-0.5x}dx = a$ for b.

$$\begin{aligned} a &= \int_0^b 0.5e^{-0.5x}dx \\ &= \left[-e^{-0.5x}\right]_0^b \\ &= -e^{-0.5b} + e^0 \\ &= 1 - e^{-0.5b} \\ 1 - a &= e^{-0.5b} \\ -0.5b &= \ln(1-a) \\ b &= -2\ln(1-a) \end{aligned}$$

3. Use technology to calculate $-2\ln(1-a)$ for each random number a that is generated. These values represent the value chosen to form the sample.

Sample data (a total of 40 numbers are generated for each sample).

Random #	Sample
0.038742	0.079024
0.42023	1.090246
0.283778	0.66753
0.430065	1.124464
0.654831	2.127445
0.141399	0.304903
0.923711	5.146463
0.990582	9.330223
0.240471	0.550112
0.231727	0.527219
⋮	⋮

d. 1. In order to determine the confidence interval, the mean must be determined. Calculate the sample mean of these new values.

d. $\bar{x} = \dfrac{0.079024 + 1.090246 + 0.66753 + \ldots}{40}$

≈ 2.27

2. For a 90% confidence interval, $z_{0.05} = 1.64$. Calculate the confidence interval $\left(\bar{x} - z\dfrac{s}{\sqrt{n}}, \bar{x} + z\dfrac{s}{\sqrt{n}}\right)$. As the population standard deviation of 2 is known, use that. Use technology to calculate the values.

$$\begin{aligned}\bar{x} - z\frac{s}{\sqrt{n}} &= 2.27 - 1.64 \times \frac{2}{\sqrt{40}}\\ &= 2.27 - 0.52\\ &= 1.75\end{aligned}$$

$$\begin{aligned}\bar{x} + z\frac{s}{\sqrt{n}} &= 2.27 + 1.64 \times \frac{2}{\sqrt{40}}\\ &= 2.27 + 0.52\\ &= 2.79\end{aligned}$$

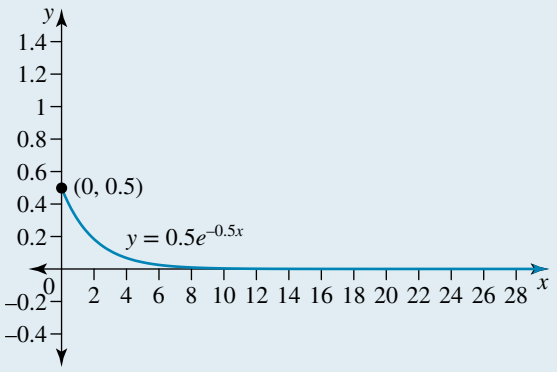
The confidence interval is (1.75, 2.79).

3. Repeat steps **c. 1.** through to **d. 2.** to generate 9 more samples and calculate the confidence intervals.

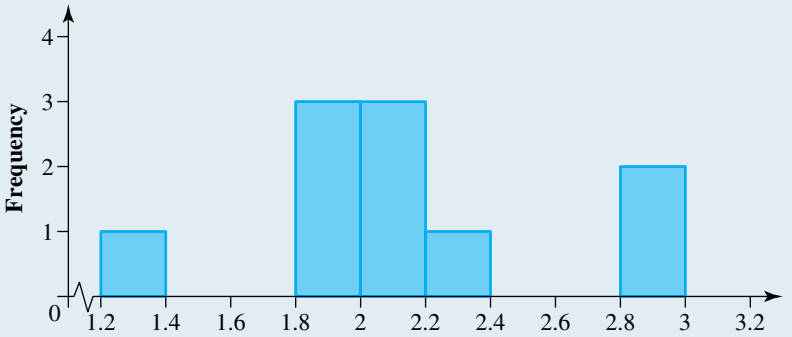
Sample	Mean	Lower limit	Upper limit
1	2.271128	1.752515	2.789742
2	2.100404	1.581791	2.833835
3	2.18984	1.671226	2.92327
4	1.821862	1.303248	2.555292
5	1.366244	0.84763	2.099674
6	2.998408	2.479794	3.731838
7	2.156755	1.638142	2.890186
8	2.911023	2.392409	3.644453
9	1.82039	1.301776	2.55382
10	1.889885	1.371272	2.623316

e. 1. Use technology to sketch the probability density function $0.5e^{-0.5x}$.

e.



2. Use technology to sketch the distribution of the sample means. For example, this can be done using the statistics option in GeoGebra.



f. Identify the samples where the 90% confidence intervals contain the population mean of 2. Comment on the distributions using your understanding of confidence intervals.

f. Confidence intervals from samples 1, 2, 3, 4, 5, 7, 9 and 10 contain the population mean. As this is a 90% confidence interval, 90% of all possible samples should contain the population mean. From the 10 samples taken, 8 of them contain the population mean, a result close to the expected one. The average of all of the sample means is 2.15 which is very close to the population mean of 2. While the population is negatively skewed, the distribution of the sample means is beginning to look like the normal distribution with a mean of 2.

11.5.2 Normal distributions

Normal distributions

If the population is believed to be normally distributed, then the means of samples of any size *n* will be normally distributed, with the average value of the sample means being the same as that of the population.

To explore this concept further, we will simulate samples from a normal distribution. This allows us to observe how sample means behave and compare them to the population mean. Randomly sampling from a normal distribution can be modelled using the inverse normal option, which generates values based on a given mean and standard deviation.

WORKED EXAMPLE 14 Simulating sampling for normal distributions

A population is normally distributed with a mean of 50 and a standard deviation of 5.

a. Use technology to model the selection of 10 random samples of size 20.

b. For each sample, use technology to calculate the 90% confidence interval for the population mean.

c. Use technology to sketch the population distribution and the distribution of sample means.

d. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.

THINK

a. 1. To model creating a sample of size 20, use a random number generator to generate 20 random numbers between 0 and 1. In Excel, =RAND() will result in a number between 0 and 1.

WRITE

a.

Random #
0.85583
0.553912
0.567864
0.566725

Random #
0.104497
0.83723
0.412178
0.980235
0.283082
0.666841
0.811105
0.659588
0.637259
0.275291
0.665903
0.824447
0.619637
0.693842
0.324964
0.505637

2. The random numbers generated can be thought of as an area under the normal curve between $-\infty$ and the element chosen to be in the sample. Use the inverse normal option to convert the random number to an element in a normal distribution with a mean of 50 and a standard deviation of 5. In Excel, NORM.INV$(a, \mu, \sigma) = b$ will return the population value for a population with mean μ and standard deviation σ where $P(Z \leq b) = a$. Generate the sample values.

Random #	InvNormal
0.85583	55.30885
0.553912	50.67776
0.567864	50.85469
0.566725	50.84022
0.104497	43.71832
0.83723	54.91569
0.412178	48.89027
0.980235	60.29314
0.283082	47.13146
0.666841	52.15603
0.811105	54.40988
0.659588	52.05669
0.637259	51.75571
0.275291	47.01555
0.665903	52.14314
0.824447	54.66222
0.619637	51.52264
0.693842	52.53386
0.324964	47.73068
0.505637	50.07065

3. Calculate the sample mean of these new values.

$$\bar{x} = \frac{55.30885 + 50.67776 + \ldots + 50.07065}{20}$$
$$\approx 51.43$$

4. For a 90% confidence interval, $z_{0.05} = 1.645$. Calculate the confidence interval $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$. As the population standard deviation is known, use that. Use technology to calculate the values.

$$\bar{x} - z\frac{s}{\sqrt{n}} = 51.43 - 1.645 \times \frac{5}{\sqrt{20}}$$
$$= 51.43 - 1.84$$
$$= 49.59$$
$$\bar{x} + z\frac{s}{\sqrt{n}} = 51.43 + 1.645 \times \frac{5}{\sqrt{20}}$$
$$= 51.43 + 1.84$$
$$= 53.27$$

The confidence interval is (49.59, 53.27).

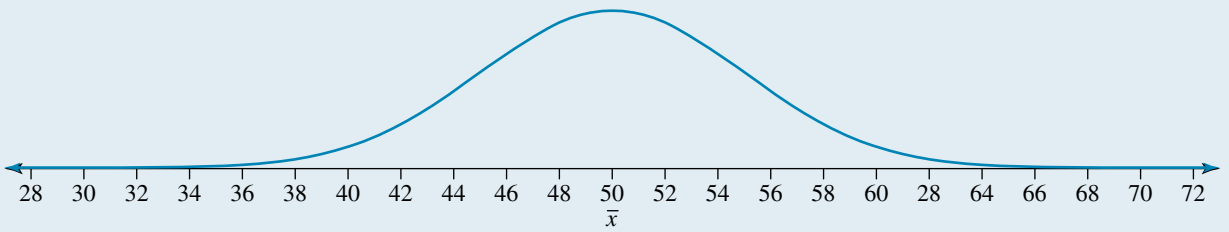
b. Repeat steps **a. 1.** through to **a. 4.** to generate 9 more samples and calculate the confidence intervals.

b.

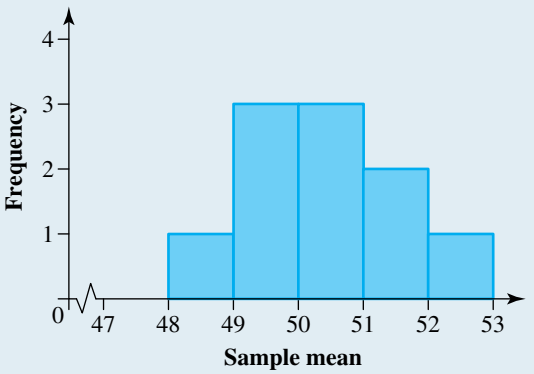
Sample	Mean	Lower limit	Upper limit
1	51.43437	49.6008	53.26795
2	49.08482	47.25124	50.91839
3	49.30787	47.4743	51.14145
4	51.66994	49.83637	53.50352
5	50.52729	48.69371	52.36087
6	48.0063	46.17272	49.83987
7	52.59666	50.76308	54.43023
8	50.53343	48.69985	52.367
9	50.84502	49.01145	52.6786
10	49.86264	48.02907	51.69622

c. 1. As the normal distribution is a known distribution, software like the probability option of GeoGebra can be used to create a graph of the normal distribution with a mean of 50 and a standard deviation of 5.

c.



2. Use technology to sketch the distribution of the sample means. For example, this can be done using the statistics option in GeoGebra.



d. Identify the samples where the 90% confidence intervals contain the population mean of 50. Comment on the distributions using your understanding of confidence intervals.

d. Samples 1, 2, 3, 4, 5, 8, 9 and 10 contain the population mean. As this is a 90% confidence interval, 90% of all possible samples should contain the population mean. From the 10 samples taken, 8 of them contain the population mean, a result close to the expected one. The average of all of the sample means is 50.39 which is very close to the population mean of 50. While the mean of the distribution of sample means is very close to the mean of the population, the spread of the distribution is smaller than that of the original population.

11.5.3 Confirming normality

In Worked examples 13 and 14, sketches of the sample mean distributions demonstrated that they are approximately normal. Another way to confirm normality is by using a **normal probability plot**.

To construct this plot, first rank the sample means. The cumulative probability p_i for each rank-ordered data point x_i is calculated as $p_i = \frac{i - 0.5}{n}$, where i is the rank and n is the total number of data points. Next, compute the corresponding z-score z_i for each p_i. If the x_i data is approximately normal, a scatterplot of z_i against x_i will show a roughly linear relationship.

WORKED EXAMPLE 15 Confirming normality

Use a normal probability plot to confirm that the sample means generated in Worked example 14 are approximately normally distributed.

THINK

1. The sample means need to be rank ordered. Sort the means from smallest to largest and allocate a rank order value to each.

WRITE

Means (rank ordered)	Rank
48.0063	1
49.08482	2
49.30787	3
49.86264	4
50.52729	5
50.53343	6
50.84502	7
51.4343	8
51.66994	9
52.59666	10

2. Calculate the cumulative probability for each point of data using $p_i = \dfrac{i - 0.5}{n}$.

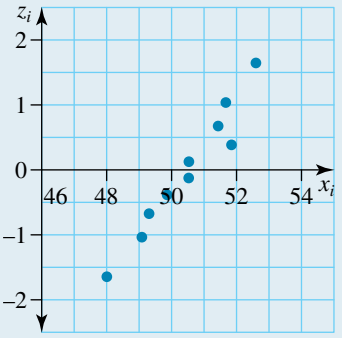
Means (x_i)	Rank (i)	p_i
48.0063	1	0.05
49.08482	2	0.15
49.30787	3	0.25
49.86264	4	0.35
50.52729	5	0.45
50.53343	6	0.55
50.84502	7	0.65
51.4343	8	0.75
51.66994	9	0.85
52.59666	10	0.95

3. Calculate the standard inverse normal for each p_i value. This can be completed in Excel using =NORM.INV(p_i, μ, σ).

=NORM.INV(0.05, 0, 1) $= -1.64485$
=NORM.INV(0.15, 0, 1) $= -1.03643$
⋮

Means (x_i)	Rank (i)	p_i	z_i
48.0063	1	0.05	−1.64485
49.08482	2	0.15	−1.03643
49.30787	3	0.25	−0.67449
49.86264	4	0.35	−0.38532
50.52729	5	0.45	−0.12566
50.53343	6	0.55	0.125661
50.84502	7	0.65	0.38532
51.4343	8	0.75	0.67449
51.66994	9	0.85	1.036433
52.59666	10	0.95	1.644854

4. Sketch z_i against x_i. If the x_i values are approximately normal, then the scatterplot should be approximately linear.



As the scatterplot is approximately linear, the distribution of sample means is approximately normal.

Exercise 11.5 Applications and other distributions

learnon

11.5 Exercise	11.5 Exam questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7	8, 9, 10, 11, 12	–

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Simple familiar

1. An accounting firm tracks the time taken by employees to complete various client tasks throughout the day. The time to complete each task is independent, and the time between completing successive tasks is expected to follow a normal distribution. On a particular day, 40 tasks were completed, and the time intervals (in minutes) between the completion of these tasks are recorded in the following dataset:

27.4, 28.8, 30.1, 31.4, 32.7, 34, 35.8, 37.6, 39.6, 41.5, 43.4, 45.3, 47.5, 49.6, 51.8, 54.2, 56.6, 59, 61.4, 63.2, 65, 67.4, 69.6, 71.7, 73.9, 76, 78.2, 80.4, 82.5, 84.7, 86, 87.7, 89.4, 91, 92.4, 93.7, 95, 96.3, 97.6, 99, 100.3

a. Use technology to simulate 10 random samples of size, $n = 20$.
b. Calculate the respective sample means.
c. Use technology to sketch the distribution of sample means.

2. A study was conducted on the amount of money spent by 17-year-olds in Toowoomba on their first car. The amounts spent (in dollars) were recorded for 40 individuals, and the data is as follows:

1596, 1684, 1772, 1860, 1948, 2036, 2156, 2276, 2404, 2532, 2660, 2788, 2932, 3076, 3220, 3380, 3540, 3700, 3860, 3980, 4100, 4260, 4404, 4548, 4692, 4836, 4980, 5124, 5268, 5412, 5500, 5612, 5724, 5836, 5924, 6012, 6100, 6188, 6276, 6364, 6452

a. Use technology to simulate 10 random samples of size, $n = 20$.
b. Calculate the respective sample means.
c. Use technology to sketch the distribution of sample means.

3. WE12 Consider a call centre that receives an average of 8 calls per hour. Since the calls are independent, the time between successive calls is expected to follow an exponential distribution. On a specific day, 40 calls were received, and the time intervals between these calls are recorded in the following dataset.

2.4, 3.6, 4.8, 10.8, 16.2, 17.4, 18, 18, 19.2, 20.4, 24, 24, 26.4, 28.8, 31.2, 34.8, 34.8, 36, 42, 45.6, 50.4, 56.4, 60, 61.2, 70.8, 75.6, 88.8, 94.8, 96, 97.2, 98.4, 99.6, 100.8, 102, 108, 109.2, 112.8, 114, 114

a. Use technology to simulate 10 random samples of size, $n = 20$.
b. Calculate the respective sample means.
c. Use technology to sketch the distribution of sample means.

4. A study was conducted on the annual incomes of residents in regional town. The data, which is known to follow a positively skewed distribution, represents the annual incomes (in dollars) of 40 individuals and is recorded as follows:

5222, 12 871, 13 324, 14 011, 15 677, 16 139, 21 505, 22 210, 23 005, 24 873, 25 547, 26 789, 27 456, 28 045, 29 343, 30 780, 31 070, 32 720, 33 054, 34 730, 35 120, 36 046, 37 590, 38 376, 39 121, 40 357, 41 246, 42 113, 43 788, 44 911, 45 005, 46 912, 47 647, 48 361, 49 008, 51 800, 52 950, 53 053, 54 031, 65 025

a. Use technology to simulate 10 random samples of size, $n = 20$.
b. Calculate the respective sample means.
c. Use technology to sketch the distribution of sample means.

5. The amount of money in dollars spent by people at a takeaway shop was recorded at Maroochydore. The data is positively skewed.

3.2, 3.3, 5.2, 5.4, 5.5, 8.6, 11.1, 12.3, 12.7, 12.9, 14.3, 15.0, 15.3, 15.3, 16.1, 18.5, 21.1, 21.4, 22.7, 23.3, 24.9, 25.2, 26.2, 26.6, 26.7, 26.8, 27.5, 31.9, 32.0, 32.2, 33.4, 34.1, 36.0, 37.2, 37.4, 37.5, 41.2, 45.5, 46.1, 46.4, 47.2, 47.3, 51.0, 51.2, 55.5, 56.5, 62.6, 65.6, 68.6, 74.3

a. Use technology to simulate 10 random samples of size, $n = 20$.
b. Calculate the respective sample means.
c. Use technology to sketch the distribution of sample means.

6. An investigation was conducted on weekly exercise time, measured in hours. Ten sample means were calculated and are ranked in the table shown.

Means $\bar{x}_i$	Rank	z_i
11.9	1	−1.547
14	2	−1
14.2	3	−0.655
14.3	4	−0.375
14.5	5	−0.123
14.6	6	0.123
15	7	0.375
15.1	8	0.655
16	9	1
17.3	10	1.547

Normal scores (z_i) can be calculated using a variety of formulae and are dependent on sample size, in this instance $n = 10$.
Assess the normality of the distribution of sample means by sketching a scatterplot of z_i vs. $\bar{x}_i$

7. The dataset relates to the amount spent by a 17-year-old on their first car. Ten samples were collected and ranked in the table shown.

Means $\bar{x}_i$	Rank	z_i
44.6	1	−1.547
45.6	2	−1
47.8	3	−0.655
48.4	4	−0.375
48.7	5	−0.123
52.8	6	0.123
53.6	7	0.375
54.5	8	0.655
56.4	9	1
58	10	1.547

Assess the normality of the distribution of sample means by sketching a scatterplot of z_i vs. $\bar{x}_i$.

Complex familiar

8. WE14, 15 A population is normally distributed with a mean of 16 and a standard deviation of 2.
   a. Use technology to model the selection of 10 random samples of size 20.
   b. For each sample, use technology to calculate the 90% confidence interval for the population mean.
   c. Use technology to sketch the population distribution and the distribution of sample means.
   d. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
   e. Use a normal probability plot to confirm that the sample means generated are approximately normally distributed.

9. A population is normally distributed with a mean of 72 and a standard deviation of 7.
   a. Use technology to model the selection of 10 random samples of size 20.
   b. For each sample, use technology to calculate the 90% confidence interval for the population mean.
   c. Use technology to sketch the population distribution and the distribution of sample means.
   d. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
   e. Use a normal probability plot to confirm that the sample means generated are approximately normally distributed.

10. X is the height of Emperor penguins and $X \sim N(122, 53.29)$.
   a. Use technology to model the selection of 10 random samples of size 15.
   b. For each sample, use technology to calculate the 95% confidence interval for the population mean.
   c. Use technology to sketch the population distribution and the distribution of sample means.
   d. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
   e. Use a normal probability plot to confirm that the sample means generated are approximately normally distributed.

11. WE13 A population is distributed so that the $P(X \leq b) = \int_0^b 4e^{-4x}dx$.
   a. The population mean can be found by calculating $\mu = \int_0^\infty 4xe^{-4x}dx$. Determine the population mean.
   b. The population variance can be found by calculating $\sigma^2 = \int_0^\infty 4(x-\mu)^2e^{-4x}dx$. Determine the population standard deviation.
   c. Use technology to model the selection of 10 random samples of size 40.
   d. For each sample, use technology to calculate the 90% confidence interval for the population mean.
   e. Use technology to sketch the population distribution and the distribution of sample means.
   f. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.

12. A population is distributed so that the $P(X \leq b) = \int_0^b 0.25e^{-0.25x}dx$.
   a. The population mean can be found by calculating $\mu = \int_0^\infty 0.25xe^{-0.25x}dx$. Determine the population mean.
   b. The population variance can be found by calculating $\sigma^2 = \int_0^\infty 0.25(x-\mu)^2e^{-0.25x}dx$. Determine the population standard deviation.
   c. Use technology to model the selection of 10 random samples of size 40.
   d. For each sample, use technology to calculate the 90% confidence interval for the population mean.
   e. Use technology to sketch the population distribution and the distribution of sample means.
   f. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.

Fully worked solutions for this chapter are available online.

LESSON
11.6 Review

11.6.1 Summary

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- Review your results
- Practise exam questions

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11.6 Exercise

learnon

11.6 Exercise	11.6 QCAA questions on

Simple familiar	Complex familiar	Complex unfamiliar
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13, 14, 15, 16	17, 18, 19, 20

These questions are even better in jacPLUS!

- Receive immediate feedback
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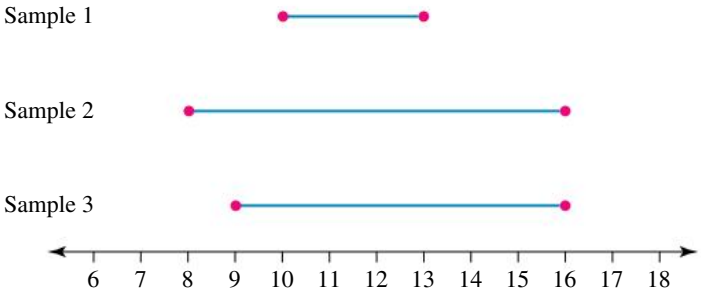
Simple familiar

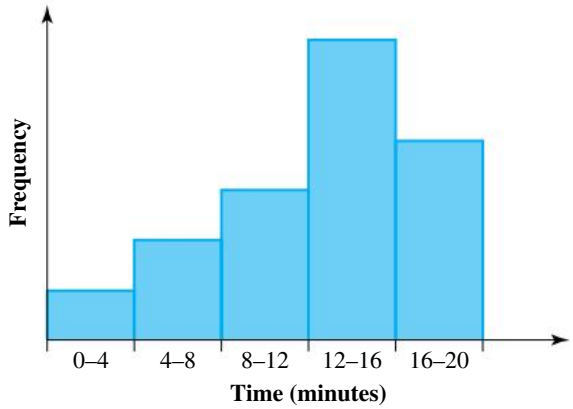
1. At Healthy Life Nutrition, ensuring consistency in the amount of protein per energy bar is crucial for maintaining product quality. A sample of energy bars is tested, and the protein content (in grams) is measured. The calculated 99% confidence interval for the average protein content is (18.5, 19.9) grams.
   a. State two assumptions necessary for the confidence interval to be considered a valid estimate of the population mean.
   b. Describe how this interval estimate should be interpreted.

2. A coffee company claims that its coffee beans have a mean caffeine content of 150 mg with a standard deviation of 8 mg. A random sample of 50 coffee beans is tested for caffeine content. Calculate the standard deviation of the sample mean. Give your answer to 2 decimal places.

3. The battery life of a certain brand of smartphone is claimed to be normally distributed with a mean of μ hours. A random sample of 50 smartphones is found to have a mean battery life of 12 hours. Identify the sample that represents an approximate confidence interval for the population mean based on this sample.

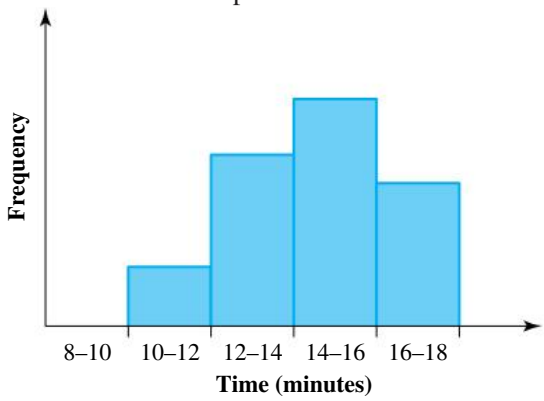


4. A random variable is drawn from a population with the distribution shown in the diagram.

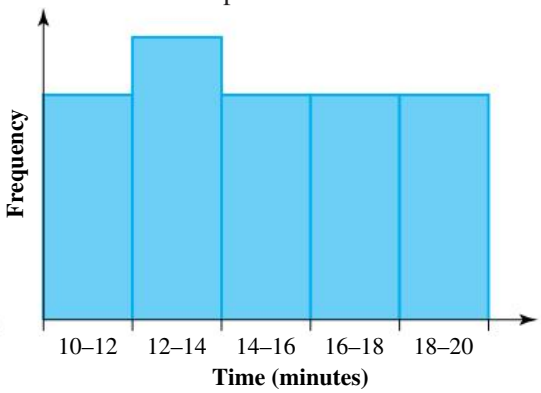


Various samples of size 20 were randomly selected from this distribution and the sample means, $\bar{x}$ were graphed. Identify the histogram that is the most plausible distribution of the sample means.

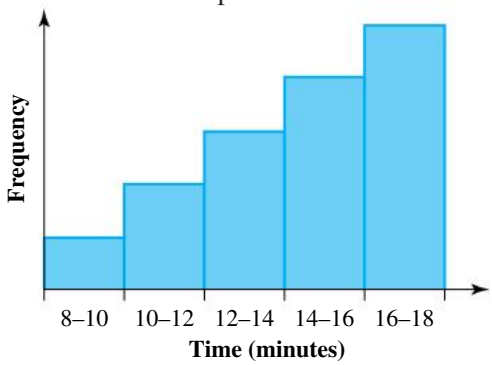
Option 1



Option 2



Option 3



5. Jamie is interested in exploring the variability of sample means for a specific size. The study focuses on a continuous random variable X, representing the number of hours a student spends studying each week. The population is moderately small with $n = 30$. Jamie has gathered two random samples of size six, presented below.

Sample 1	Sample 2
4.7, 4.8, 5.1, 5.2, 5.4, 6.1	4.2, 4.3, 5.0, 5.4, 5.5, 7.4

a. Use these samples to calculate the associated sample means. Give your answers to 1 decimal place.

b. Describe the variation of the means by comparing them to the population mean, $\mu = 5.1$.

6. A total of 640 students are enrolled in a Business program at the University of Melbourne. The ages (years) of this student population are normally distributed with a mean $\mu = 22.8$ and a standard deviation $\sigma = 2.5$.

   a. Samples of size 16 are selected and their means are calculated. Determine the expected value and standard deviation of the distribution of $\overline{X}$.
   b. If the sample size is increased to 144, analyses the impact this change would have on the expected value and standard deviation of the distribution of sample means.

7. A potion manufacturer in a magical realm guarantees that the longevity of the elixir potion has an average effect duration that follows a normal distribution with $\mu = 4.2$ and $\sigma = 0.7$ hours. However, villagers have expressed concern that the effects might wear off before reaching 4 hours. To investigate this claim, a council of wizards selects a random sample of 65 elixirs. Based on this sample, calculate the probability that the duration of elixir is less than 4 hours. Give your answer to 3 decimal places.

8. Repeated random samples were collected to investigate the distribution of $\overline{X}$.

$\overline{x}_1 = 20.6$	$\overline{x}_2 = 23.6$	$\overline{x}_3 = 25.5$	$\overline{x}_4 = 29.4$
$\overline{x}_6 = 55.9$	$\overline{x}_7 = 59.3$	$\overline{x}_8 = 66.4$	$\overline{x}_9 = 68.2$
$\overline{x}_{11} = 78.4$	$\overline{x}_{12} = 81.4$	$\overline{x}_{13} = 86.9$	$\overline{x}_{14} = 95.2$

   Assess the normality of the distribution by creating a frequency histogram of the sample means.

9. In Queensland, learner drivers are required to complete 100 hours of supervised driving before taking their practical test. However, an analysis shows that a sample of 40 learner drivers recorded an average of only 89.5 hours of supervised driving. The time spent (in hours) among these drivers is assumed to follow a normal distribution, and the recorded hours are as follows:

   70.5, 72.3, 75.1, 76.4, 78.0, 79.2, 80.0, 81.5, 82.0, 82.9, 83.3, 84.5, 85.2, 85.9, 86.4, 87.1, 87.8, 88.6, 89.2, 90.0, 91.1, 91.8, 92.4, 93.0, 93.7, 94.1, 95.0, 95.5, 96.2, 97.0, 97.6, 98.3, 98.7, 99.1, 99.5, 99.8, 100.0, 100.2, 100.5, 100.7.

   a. Use technology to simulate 10 random samples of size, $n = 20$.
   b. Calculate the respective sample means.
   c. Use technology to sketch the distribution of sample means.

10. After polling 50 members of a local theatres group, you find that they intend to spend an average of \$32.15 per week on takeout meals during rehearsals. The standard deviation for your sample is \$3.10. Using these statistics, calculate a 90% confidence interval for the average amount that theatre members will spend on takeout food this week. Give your answer to the nearest cent.

11. An internal study conducted by a company that manufactures electronic devices with an average length of 18 cm reported that, from a sample of 200 products, the average cost of materials per device was \$48.24. The sample standard deviation was \$5.45. Calculate a 95% confidence interval estimate of the population mean for the average cost of materials used in producing these electronic devices. Give your answer to the nearest cent.

12. A research team is collecting data on the age at which individuals first start using electronic devices for learning. Given an assumed population standard deviation of $\sigma = 2.88$ years, determine the sample size needed to achieve a margin of error of less than 1 year at a 95% confidence level. Show your working.

Complex familiar

13. A sample has a standard deviation of 2.6, and the 95% confidence interval is calculated to be $(41.4, 43.9)$. Calculate the sample size. Give your answer to the nearest integer.

14. This dataset pertains to the registration fees for sporting clubs in the Darling Downs region. Ten samples of fees were collected and are ranked in the table provided.

Means $\bar{x}_i$	Rank	z_i
200.54	1	−1.547
222.05	2	−1
237.64	3	−0.655
240.3	4	−0.375
241.5	5	−0.123
255.25	6	0.123
260.56	7	0.375
265.18	8	0.655
275.04	9	1
288.12	10	1.547

Evaluate the normality of the sample distribution.

15. A sample of 120 female tennis matches at Wimbledon was analysed to determine their average duration. The 95% confidence interval was $(98.3, 106.5)$ minutes. Calculate a 90% confidence interval for the average duration of a female Wimbledon match. Give your answer to 1 decimal place.

16. The total amount spent by all families visiting Movie World was recorded. A mean expenditure of \$155 was calculated from this data. A random sample of 35 families was selected, revealing a mean spending of \$160 with a standard deviation of \$20. Determine the smallest confidence level that could be chosen from 85%, 90%, 95% and 99% to calculate a confidence interval that contains the population mean based on this sample. Explain your reasoning.

Complex unfamiliar

17. If a 95% confidence interval for the mean temperature of sea water at Rainbow Beach in January is $(25.4, 26.8)$ degrees Celsius, determine a 99% confidence interval for the average temperature. Give your answer to 1 decimal place.

18. From a sample of 40 students, a 95% confidence interval for the population mean is determined to be $(176.2, 186.2)$. After adding 25 more students to the sample, the new sample mean increases to 183.0. Assuming the population standard deviation is known, calculate a 99% confidence interval for the population mean using the entire sample. Give your answer to 1 decimal place.

19. The mass of a certain species of koalas is known to be normally distributed with a mean mass of μ kg and standard deviation of σ kg. When one of the koalas is randomly selected, the probability that its mass is greater than 30.5 kg is 0.178.
When a sample of 15 koalas is randomly selected, the probability that the sample mean is less than 27.8 kg is 0.091.
A 95% approximate confidence interval of μ is calculated using a random sample of n koalas that has a sample mean mass of 29.7 kg and the sample standard deviation equal to σ.
Determine the possible range of values that n could have been, given that the confidence interval did not contain μ.

20. In analysing traffic flow, headway is the time between one car passing a fixed point and the next car beginning to pass the point. If X is the headway time between two consecutive variables in minutes, the probability density function of X can be obtained using

$$f(x) = \begin{cases} 0.25e^{-0.25(x-0.5)}, x \geq 0.5 \\ 0, x < 0.5 \end{cases}$$

where $\mu = \int_{0.5}^{\infty} x\, f(x)\, dx$ and $\sigma^2 = \int_{0.5}^{\infty} (x-\mu)^2 f(x)\, dx$.

a. Calculate the expected population mean and the population standard deviation.
b. Use technology to model the selection of 10 random samples of size 40 and determine the 90% confidence interval for each sample.
c. Use technology to sketch the normal probability plot for the distribution of sample means. Comment on the normality of the distribution.

11.6 Past QCAA exam questions

Question 1 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 1, Section 1, Q5; © State of Queensland (QCAA) 2024.

A confidence interval for a parameter is a range of values within which the

A. sample estimate of the parameter always lies.
B. sample estimate of the parameter never lies.
C. parameter always lies.
D. parameter never lies.

Question 2 (1 mark)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 1, Q2; © State of Queensland (QCAA) 2024.

The standard deviation for the scores of 1000 students completing an entry test at a certain university is 13.

A researcher takes repeated random samples of the test results, with each sample comprising 40 scores, and calculates the mean score for each sample.

Determine the standard deviation of the distribution of the sample mean scores.

A. 3.08 **B.** 2.06 **C.** 0.41 **D.** 0.33

Question 3 (4 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 2, Section 2, Q13; © State of Queensland (QCAA) 2024.

The wait time for customers put on hold when calling complaint departments is assumed to be normally distributed. A company claims that the mean wait time for their customers is 7.6 minutes.

The following data represents the wait time (minutes) from a random sample of 12 customers who called the complaint department of this company.

8.3	12.7	9.1	7.3	10.3	5.4	8.5	10.7	6.9	12.5	7.2	11.9

a. Determine the mean of this data. **[1 mark]**

The standard deviation of this data is calculated to be 2.384 minutes.

b. Use an approximate 95% confidence interval for the mean to evaluate the reasonableness of the company's claim. Justify your decision using mathematical reasoning. **[3 marks]**

Question 4 (6 marks)

Source: QCAA 2020 Specialist Mathematics, Paper 2, Section 2, Q13; © State of Queensland (QCAA) 2024.

Data records show that the speeds of cars at a particular location on a highway are normally distributed with a mean of $98.7\,\text{km h}^{-1}$ and a standard deviation of $4.1\,\text{km h}^{-1}$. The speed limit at this location is $100\,\text{km h}^{-1}$.

A police officer plans to record the speeds of 20 randomly selected cars at this location.

a. Determine the expected number of cars in the sample that will be travelling within $\pm 1\,\text{km h}^{-1}$ of the population mean. **[2 marks]**

b. Determine the probability that the mean speed of this sample will exceed the speed limit. **[2 marks]**

There is a 5% probability that the mean speed of this sample will exceed k.

c. Determine the value of k. **[2 marks]**

Question 5 (7 marks)

Source: QCAA 2023 Specialist Mathematics, Paper 2,Section 2, Q19; © State of Queensland (QCAA) 2024.

The height of Year 9 students at a school is assumed to be normally distributed with a population mean height of μ cm.

A teacher at the school measured the height of all the students in her Year 9 class. This data was used to calculate an approximate 95% confidence interval for μ of $(163.7, 166.9)$ cm.

The teacher repeated the procedure using data from another Year 9 class. Although this class had the same number of students, its data produced an approximate 95% confidence interval for μ of $(167.8, 172.4)$ cm.

Using the same data, the teacher recalculated the approximate confidence intervals for μ for each class using a confidence level of $x\%$. She observed that the upper bound of the confidence interval from her Year 9 class now equalled the lower bound of the confidence interval from the other Year 9 class.

Determine the value of x. Give your answer rounded to one decimal place.

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Answers

Chapter 11 Statistical inference

11.2 Sampling distributions

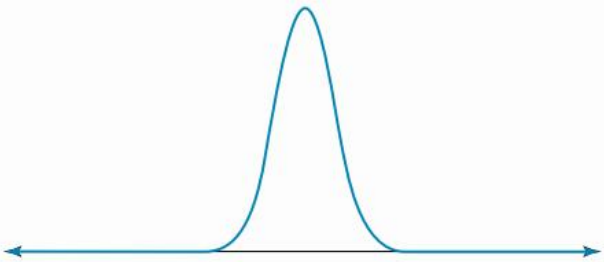
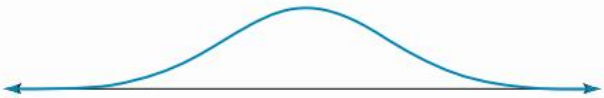
11.2 Exercise

1. Population: σ, μ
   Sample: $\bar{x}, \sigma_{\bar{x}}, s$
2. D
3. A
4. D
5. Distribution 1 matches with property 'Sampling distribution of $\bar{x}$, $n = 40$.'
   Distribution 2 matches with property 'Population distribution of $\bar{x}$.'
   Distribution 3 matches with property 'Sampling distribution of $\bar{x}$, $n = 10$.'
6. B
7. a.

Sample 1	Sample 2	Sample 3	Sample 4
$\bar{x} = 51.1$	$\bar{x} = 69.9$	$\bar{x} = 80.1$	$\bar{x} = 74.2$

   b. 70.3
   c. Sample responses can be found in the worked solutions in the online resources.
8. a.

Sample 1	Sample 2	Sample 3	Sample 4
$\bar{x} = 66$	$\bar{x} = 76$	$\bar{x} = 85$	$\bar{x} = 75$

   b. 75.7
   c. Sample responses can be found in the worked solutions in the online resources.
9. a. 216.9 b. $\mu = 215$
10. a. 75 000, 4250
    b. 75 000, 1700
11. a. 95, 4.90 b. 95, 2
12. a. 27.5 cm b. 9.6 cm
13. a. 150 cm b. 19.8 cm
14. 0.000 97
15. 0.405
16. a. 0.0049 b. 34.96 c. 1
17. 
    There are moderate signs of normality.
18. 
    There are strong signs of normality.
19. a–c Sample responses can be found in the worked solutions in the online resources.
20. a–c Sample responses can be found in the worked solutions in the online resources.

11.3 Confidence intervals

11.3 Exercise

1. a. 49.2 b. It would widen.
2. 122
3. 36
4. Increasing the sample size will reduce the margin of error. Secondly, choosing a lower level of confidence will result in a smaller z-score, resulting in a smaller margin of error.
5. (30.5, 32.1)
6. Breakdowns of errors.
   Correct confidence interval:
   $$\left(425 - 2.58 \times \frac{72}{\sqrt{130}}, 425 + 2.58 \times \frac{72}{\sqrt{130}}\right)$$
7. a. The population distribution is normal and sample is ($n \geq 30$).
   b. μ is between 25.3 and 27.1 grams at 99% confidence.
8. a. The population distribution is normal and sample is ($n \geq 30$).
   b. μ is between 1.2 and 1.5 kg at 90% confidence.
9. Sample 1
10. (341.2, 364.0) dollars
11. (607.5, 633.5) dollars
12. (2747.2, 3000.8) dollars
13. (21.4, 23.4) kg
14. (30.9, 34.1) g
15. a. (90.06, 97.94) minutes
    b. (88.81, 99.19) minutes
16. a. (1917.65, 2038.35) minutes
    b. (1906.30, 2049.70) minutes
17. (24.32, 24.68) cm
18. (81.4, 83.6)
19. (40.60, 50.34) months
    The plausible values for the mean time spent at university is (40.60, 50.34) months. The 95% confidence level is associated with the process used to calculate this estimate.
20. $a = 3711$
    $b = 69$

11.4 Margin of error, level of confidence and sample size

11.4 Exercise

1. a. $W = 10$ and $E = 5$
   b. $W = 12$ and $E = 6$
   c. $W = 3.6$ and $E = 1.8$
2. 42
3. Option 2: (118.5,131.5)
4. $n_2 = 4n_1$
5. Sample 2.

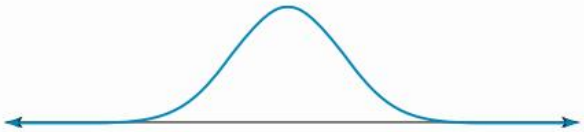
6. a. 640
   b. 11.1
   c. 67.0
   d. (630.7, 649.3) months
7. a. 4.8 b. 0.2 c. 1.61
   d. (4.6, 5.0) cm
8. 62
9. 107
10. 259
11. 61
12. 96%
13. a. (11.50, 13.50) days
    b. 50
14. a. (794.57, 905.43) dollars
    b. 62
15. The smallest CI is 95%.
16. (25.5, 26.9) kg
17. (42.55, 49.05) dollars
18. Approximately 98%

11.5 Applications and other distributions

11.5 Exercise

1–5 Results will vary between simulations.
Sample responses can be found in the worked solutions in the online resources.

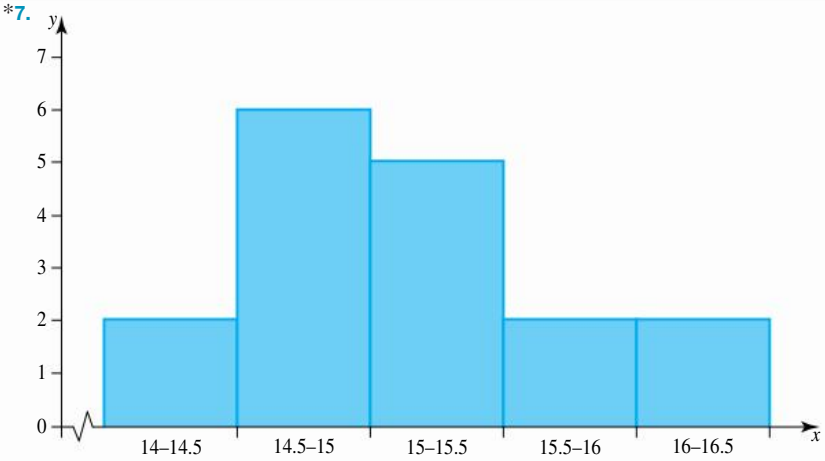
6.



The normal probability plot indicates that the distribution of sample means is approximately normal.

7. See the figure at the bottom of the page*
8. a–e Answers will vary. Distribution of sample means should have a mean of 16. Normal probability plot should be approximately linear.
9. a–e Answers will vary. Distribution of sample means should have a mean of 72. Normal probability plot should be approximately linear.
10. a–e Answers will vary. Distribution of sample means should have a mean of 122. Normal probability plot should be approximately linear.
11. a. 0.25
    b. 0.25
    c–d Sample responses can be found in the worked solutions in the online resources.

*7.



e. Population
See the figure at the bottom of the page*
Distribution of sample means will vary.

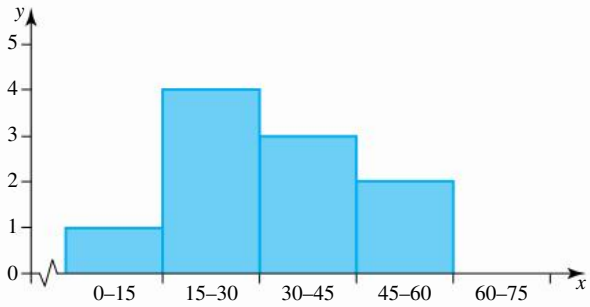
f. The mean of the sample means should be approximately equal to 0.25 and the distribution of sample means should be approximately normal.

12. a. 4

b. 4

c–d Sample responses can be found in the worked solutions in the online resources.

e. Population



Distribution of sample means will vary.

f. The mean of the sample means should be approximately equal to 4 and the distribution of sample means should be approximately normal.

11.6 Review

11.6 Exercise

1. a. Assumption 1: The population distribution is normal and $n \geq 30$.

b. Values for μ are between 18.5 and 19.9 grams at 99%.

2. 1.13

3. Interval 2.

4. Option 1.

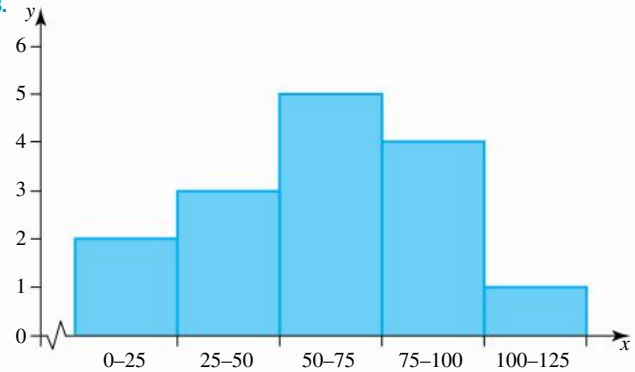
5. a. 5.2 and 5.3

b. Sample means approximate population mean.

6. a. 22.8 and $\frac{5}{8}$ **b.** 22.8 and $\frac{5}{24}$

7. 0.011

8.



The frequency histogram indicates the distribution of sample means in normally distributed.

9. a. Samples will vary across simulations.

b. Sample means will vary according to **a**.

c. Frequency histograms will vary according to **a.** and **b**.

10. (31.43, 32.87) dollars

11. (47.48, 49.00) dollars

12. $n \geq 32$

13. 17

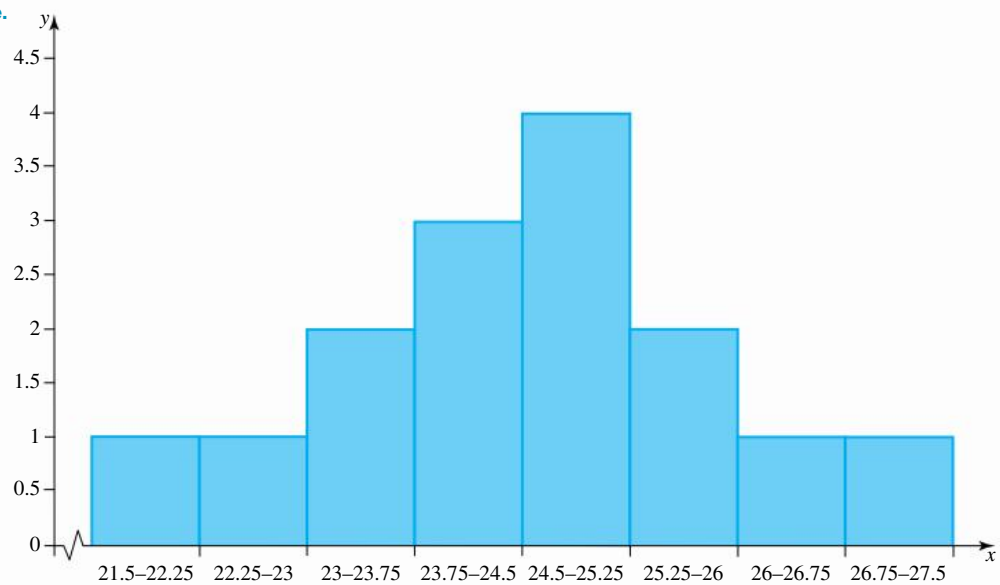
14. Sampling distribution demonstrates a strong sign of normality.

15. (98.6, 105.4) minutes

16. 90% confidence level.

17. (25.2, 27.0) °C

*11. e.



18. (177.8, 188.2) °C

19. $n \geq 13$

20. a. $\mu = 4.5$ minutes and $\sigma^2 = 4$ minutes

b. Sample responses can be found in the worked solutions in the online resources. To generate the sample elements. If a is a random number generated. $b = 0.5 - 4\ln(1 - a)$ where b is an element in the sample.

c. Sample responses can be found in the worked solutions in the online resources. The mean of the sample means should be approximately 4.5.

11.6 Past QCAA exam questions

1. A

2. B

3. a. $\bar{x} = 9.233$

b. Sample responses can be found in the worked solutions in the online resources.

5. a. 4 cars b. 0.078 c. 100.21 km h^{-1}

6. $x = 98.4\%$