ACADEMIC ASSOCIATES

STUDY GUIDE

MATHEMATICS SPECIALIST

YEAR 12 ATAR COURSE – UNITS 3 & 4

REVISED EDITION

Greg Hine & Neil McNab

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WACE Study Guide

MATHEMATICS SPECIALIST

YR 12 ATAR COURSE

Gregory Hine and Neil McNab

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Your ongoing love and support is breathtaking.

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FOREWORD

The purpose of this text is to assist Year 12 students with their preparation for tests and examinations in the Specialist Mathematical course for Western Australia.

The ***Syllabus Checklist*** indicates to students which skills they must have acquired and the objectives they need to meet under each of the major headings of the course.

The ***Worked Examples*** are presented in a detailed manner, with brief notes and explanations being used to amplify the understanding for the particular question. Some of these worked examples could be used in the written notes that students are permitted to take into an examination.

The ***Problems to Solve*** section in each chapter provides students with a broad range of questions without the repetitive nature of problems usually associated with a course textbook.

The ***Trial Tests*** are an additional component to this text, and these allow students to familiarise themselves with examination questions. Suggested times are given for these tests, and students should be encouraged to adhere to these times to prepare properly for final examinations. Fully worked solutions are provided for students to receive immediate, accurate and useful feedback on their performance.

About the Units

Unit 3 of *Specialist Mathematics* is comprised of three topics: *Vectors in three dimensions*, *Complex numbers* and *Functions and sketching graphs*. The study of vectors was introduced in Unit 1 with a focus on vectors in two-dimensional space. In this unit, three-dimensional vectors are studied and both vector equations and vector calculus are introduced, with the latter extending students' knowledge of calculus from *Mathematical Methods*. Cartesian and vector equations, together with equations of planes, enable students to solve geometric problems and to solve problems involving motion in three-dimensional space. The Cartesian form of complex numbers was introduced in Unit 2, and the study of complex numbers is now broadened to working with these numbers in polar form. The study of functions and techniques of graph sketching, begun in *Mathematical Methods*, is extended and applied in sketching graphs and solving problems involving integration.

In Unit 4 of *Specialist Mathematics* students engage with three topics: *Integration and applications of integration*, *Rates of change and differential equations* and *Statistical inference*. In Unit 4, the study of differentiation and integration of functions continues, and the calculus techniques developed in this and previous topics are applied to simple differential equations, in particular in biology and kinematics. These topics demonstrate the real-world applications of the mathematics learned throughout *Specialist Mathematics*. In this unit all of the students' previous experience working with probability and statistics is drawn together in the study of statistical inference for the distribution of sample means and confidence intervals for sample means.

For the work in Units 3 and 4, access to technology to support the computational aspects of the above mentioned topics is assumed.

Dr Gregory Hine, Ph.D.

COMPLEX NUMBERS

This section extends the introduction to complex numbers and their representation in Cartesian form in *Mathematics Specialist* to include polar representation and Argand diagrams.

Syllabus Checklist

By the end of this chapter, you should be able to:

Cartesian forms

☐ review real and imaginary parts Re(z) and Im(z) of a complex number z

☐ review Cartesian form

☐ review complex arithmetic using Cartesian forms

Complex arithmetic using polar form

☐ use the modulus | z | of a complex number z and the argument Arg (z) of a non-zero complex number z and prove basic identities involving modulus and argument

☐ convert between Cartesian and polar form

☐ define and use multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these

☐ prove and use De Moivre's theorem for integral powers

The complex plane (The Argand plane)

☐ examine and use addition of complex numbers as vector addition in the complex plane

☐ examine and use multiplication as a linear transformation in the complex plane

☐ identify subsets of the complex plane determined by relations such as

$$|z - 3i| \leq 4, \frac{\pi}{4} \leq Arg(z) \leq \frac{3\pi}{4} \text{ and } |z - 1| = 2|z - i|$$

Roots of complex numbers

☐ determine and examine the n^{th} roots of unity and their location on the unit circle

☐ determine and examine the n^{th} roots of complex numbers and their location in the complex plane

Factorisation of polynomials

☐ prove and apply the factor theorem and the remainder theorem for polynomials

☐ consider conjugate roots for polynomials with real coefficients

☐ solve simple polynomial equations

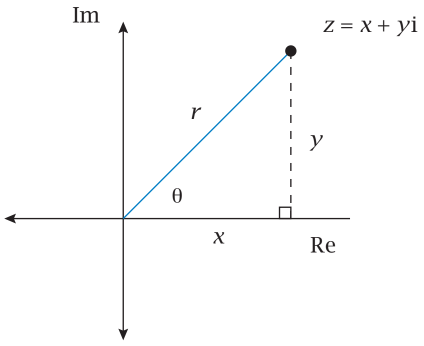
CIS FORM OF A COMPLEX NUMBER

This section essentially deals with the conversion between the Cartesian or the rectangular form of a complex number, $z = x + y\text{i}$ and its equivalent written in terms of the polar variables r and θ.

From the diagram

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$



so $z = x + y\text{i}$ becomes

$$z = r\cos\theta + \text{i}\,r\sin\theta$$

$$= r(\cos\theta + \text{i}\sin\theta)$$

$z = r\,\text{cis}\,\theta$ in abbreviated form.

Note that $\text{cis}\,\theta$ comes from $\cos\theta + \text{i}\sin\theta$

$= \text{c i s}\;\theta$

The magnitude of z, denoted by $|z|$, equals r. So by the Pythagorean Theorem

$$|z| = r = \sqrt{x^2 + y^2}$$

is a purely real quantity called the magnitude of z

The polar angle θ (of z) is called the argument of z, and denoted by $\theta = \arg z$. θ can equally well be measured in degrees or radians. Maths teachers tend to use radians and engineers tend to use degrees!

So

$$\theta = \arg z$$

$$= \tan^{-1}\left(\frac{y}{z}\right)$$

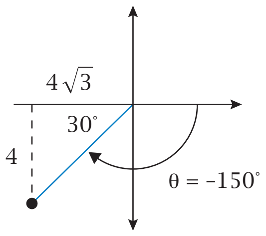
where $-180^\circ < \theta \le 180^\circ$ or

$-\pi < \theta \le \pi$.

Graphics and scientific calculators have all sorts of ways to quickly convert between $z = x + y\text{i}$ and $z = r\,\text{cis}\,\theta$. Find a way that works for you and don't forget where possible to use exact values when in radian mode.

Example i) Convert $z = -4\sqrt{3} - 4\text{i}$ to polar form.

using a calculator



$$z = 8\,\text{cis}\,(-150^\circ)$$

$$= 8\,\text{cis}\left(-\frac{5\pi}{6}\right)$$

By first principles

$$r = \sqrt{(4\sqrt{3})^2 + 4^2} = 8$$

$$\theta = \tan^{-1}\left(\frac{4}{4\sqrt{3}}\right) - 180^\circ$$

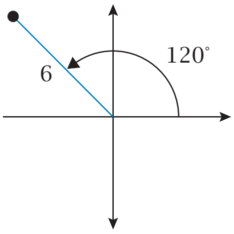
$$= -150^\circ$$

i.e. $z = 8\,\text{cis}\,(-150^\circ)$

It is important to have a sense of which quadrant a complex number lies in as this will help correct potential errors in the value of θ.

Example ii) Convert $z = 6 \text{ cis}\left(\frac{2\pi}{3}\right)$ to exact rectangular form.

Calculator methods may not give z in exact form.



$$\begin{aligned} z &= 6\left(\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}\right) \\ &= 6(\cos 120^\circ + \text{i}\sin 120^\circ) \\ &= 6(-\cos 60^\circ + \text{i}\sin 60^\circ) \\ &= 6\left(-\frac{1}{2} + \text{i}\frac{\sqrt{3}}{2}\right) \\ &= -3 + \text{i}\,3\sqrt{3} \end{aligned}$$

OPERATIONS WITH COMPLEX NUMBERS IN POLAR FORM

It will soon become apparent that when adding and subtracting complex numbers, rectangular form is best. Whereas when multiplying, dividing and raising them to powers, polar form is best.

Consider $z = r\text{ cis }\theta$ and $w = a\text{ cis }\phi$

then

$$\begin{aligned} zw &= r\text{ cis }\theta \times a\text{ cis }\phi \\ &= ra(\cos\theta + \text{i}\sin\theta)(\cos\phi + \text{i}\sin\phi) \\ &= ra(\cos\theta\cos\phi + \text{i}\cos\theta\sin\phi + \text{i}\sin\theta\cos\phi - \sin\theta\sin\phi) \\ &= ra(\cos\theta\cos\phi - \sin\theta\sin\phi + \text{i}\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= ra(\cos(\theta+\phi) + \text{i}\sin(\theta+\phi)) \\ &= ra\text{ cis }(\theta+\phi) \end{aligned}$$

So

If $z = r\text{ cis }\theta$ and $w = a\text{ cis }\phi$

then $\boldsymbol{zw} = ra\text{ cis }(\theta + \phi)$

The magnitudes are multiplied and the arguments are added.

The reciprocal of $w = a\text{ cis }\phi$ is

$$\begin{aligned} \frac{1}{w} &= \frac{1}{a\text{ cis }\phi} \\ &= \frac{1}{a} \times \frac{1}{\cos\phi + \text{i}\sin\phi} \\ &= \frac{1}{a} \times \frac{1}{\cos\phi + \text{i}\sin\phi} \times \frac{\cos\phi - \text{i}\sin\phi}{\cos\phi - \text{i}\sin\phi} \\ &= \frac{1}{a} \times \frac{\cos\phi - \text{i}\sin\phi}{\cos^2\phi + \sin^2\phi} \\ &= \frac{1}{a}(\cos\phi - \text{i}\sin\phi) \\ &= \frac{1}{a}(\cos(-\phi) + \text{i}\sin(-\phi)) \\ &= \frac{1}{a}\text{ cis }(-\phi) \end{aligned}$$

$$\frac{1}{\text{cis }\phi} = \text{cis }(-\phi)$$

is an important relationship

The quotient of $z = r \text{ cis } \theta$ and $w = a \text{ cis } \phi$ is

$$\begin{aligned} \frac{z}{w} &= \frac{r \text{ cis } \theta}{a \text{ cis } \phi} \\ &= \frac{r}{a} \text{ cis } \theta \cdot \text{cis } (-\phi) \\ &= \frac{r}{a} \text{ cis } (\theta + -\phi) \\ &= \frac{r}{a} \text{ cis } (\theta - \phi) \end{aligned}$$

If $z = r \text{ cis } \theta$ and $w = a \text{ cis } \phi$

then $\frac{z}{w} = \frac{r}{a} \text{ cis } (\theta - \phi)$

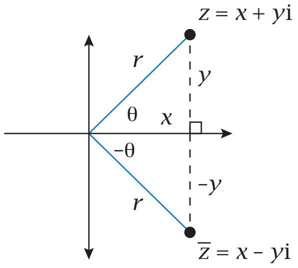
The magnitudes are divided and the angles are subtracted.
Make sure that you get the order of those values correct!

COMPLEX CONJUGATES IN POLAR FORM

For $z = x + y\text{i}$, the conjugate $\bar{z} = x - y\text{i}$ is the reflection of z in the real axis.

In polar form

$$\begin{aligned} z &= r \text{ cis } \theta \\ \bar{z} &= r \text{ cis } (-\theta) \\ &= \frac{r}{\text{cis } \theta} \end{aligned}$$



Also

$$\begin{aligned} & z\bar{z} \\ &= r \text{ cis } \theta \cdot r \text{ cis } (-\theta) \\ &= r^2 \text{ cis } (\theta + -\theta) \\ &= r^2 \text{ cis } 0^\circ \\ &= r^2 \\ &= |z|^2 \end{aligned}$$

$$\begin{aligned} \text{cis } 0^\circ &= \cos 0^\circ + \text{i} \sin 0^\circ \\ &= 1 \end{aligned}$$

So

If $z\bar{z} = |z|^2$

then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ is the reciprocal of z in terms of $\bar{z}$ and $|z|$.

Also $\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$

PROPERTIES OF COMPLEX CONJUGATES

In some of the exercises you will be required to prove the following relationships.

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

As with identities, you should start with one side and after letting $z = x + y\text{i}$ or $r \text{ cis } \theta$ etc. work downwards until you either get to the other side or more likely get to a logical stopping point. In the latter case, the other side is then processed until the same stopping point is achieved which will complete the proof.

Also, whether z and $\overline{z}$ are in Cartesian or polar form

$$z + \overline{z} = 2\,\mathrm{Re}(z) \quad \text{and} \quad z - \overline{z} = 2\mathrm{i}\,\mathrm{Im}(z)$$

These are readily proved by letting $z = x + y\mathrm{i}$ and $\overline{z} = x - y\mathrm{i}$ etc.

POLYNOMIALS IN CIS FORM

The quadratic equation $az^2 + bz + c = 0$ with real coefficients a, b and c will have solutions as a complex conjugate pair when $b^2 < 4ac$ as deduced from the quadratic formula.

If $z_1 = x + y\mathrm{i}$ and $\overline{z}_1 = x - y\mathrm{i}$ are the solutions, then the complex factors are $(z - z_1)$ and $(z - \overline{z}_1)$ which means that the above quadratic equation is equivalent to

$$(z - z_1)(z - \overline{z}_1) = 0$$

or $$(z - (x + y\mathrm{i}))(z - (x - y\mathrm{i})) = 0$$

$$(z - x - y\mathrm{i})(z - x + y\mathrm{i}) = 0$$

$$(z - x)^2 + y^2 = 0$$

$$z^2 - 2zx + x^2 + y^2 = 0 \quad \text{which when compared}$$

to $$z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \quad \text{gives}$$

$$\frac{b}{a} = -2x \quad \text{and} \quad \frac{c}{a} = x^2 + y^2$$

$$\text{i.e.} \quad x = -\frac{b}{2a} \quad \text{and} \quad y = \sqrt{\frac{c}{a} - x^2}$$

which will give $z_1 = x + y\mathrm{i}$ and $\overline{z}_1 = x - y\mathrm{i}$ in terms of a, b and c.

In polar form, if $z_1 = r \operatorname{cis} \theta$ and $\overline{z}_1 = r \operatorname{cis} (-\theta)$

then $$(z - z_1)(z - \overline{z}_1) = 0$$

$$\therefore \quad (z - r \operatorname{cis} \theta)(z - r \operatorname{cis} (-\theta)) = 0$$

$$z^2 - zr \operatorname{cis} (-\theta) - zr \operatorname{cis} \theta + r^2 = 0$$

$$z^2 - zr (\operatorname{cis} (-\theta) + \operatorname{cis} \theta) + r^2 = 0$$

$$z^2 - zr (\cos (-\theta) + \mathrm{i} \sin (-\theta) + \cos \theta + \mathrm{i} \sin \theta) + r^2 = 0$$

$$z^2 - zr (\cos \theta - \mathrm{i} \sin \theta + \cos \theta + \mathrm{i} \sin \theta) + r^2 = 0$$

$$z^2 - zr \cdot 2 \cos \theta + r^2 = 0$$

$$z^2 - 2zr \cos \theta + r^2 = 0$$

This is readily seen to be equivalent to

$$z^2 - 2zx + x^2 + y^2 = 0 \quad \text{from before as } x = r \cos \theta \text{ and } r^2 = x^2 + y^2.$$

Recent examination questions have required pupils to demonstrate the above quadratic process in terms of polar solutions.

Example Find the quadratic equation in $z^2 + bz + c = 0$ form which has solutions of $z = 3 \operatorname{cis} 120^\circ$ and $z = 3 \operatorname{cis} (-120^\circ)$.

$$(z - 3 \operatorname{cis} 120^\circ)(z - 3 \operatorname{cis} (-120^\circ)) = 0$$

$$z^2 - 3z \operatorname{cis} (-120^\circ) - 3z \operatorname{cis} 120^\circ + 9 = 0$$

$$z^2 - 3z (\operatorname{cis} (-120^\circ) + \operatorname{cis} 120^\circ) + 9 = 0$$

$$z^2 + 3z + 9 = 0 \quad \text{is the required equation}$$

Note: Here I have used $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $\mathrm{Re}(\operatorname{cis} 120^\circ) = -\frac{1}{2}$ to simplify the bracket.

LOCUS PROBLEMS

- A locus is a set of points in the complex plane each of which obeys a given rule or constraint. Most locus sketches result from recognition of the type of rule, and you should be very familiar with the types listed below.

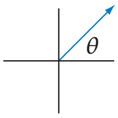
If the locus involves both z and $\overline{z}$, then you will have to substitute $z = x + y\text{i}$, $\overline{z} = x - y\text{i}$ and arrive at a Cartesian type equation (free of i's) which you can graph in the usual way.

- Assume for each case below that z is the variable, u and w are complex constants, and a is a positive real constant.

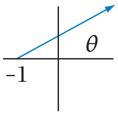
First it is essential to realise that $|z - w|$ is the length of the straight line from w to z. This is very important.

$\lvert z\rvert = a$	is a circle with its centre at the origin and radius a.
$\lvert z - w\rvert = a$	is a circle of radius a and its centre at w. For $\lvert z + 2 - \text{i}\rvert = 5$ you must first write it as $\lvert z - (-2 + \text{i})\rvert = 5$ to get the centre of $-2 + \text{i}$.
$\lvert z - w\rvert = \lvert z - u\rvert$	is the perpendicular bisector of the two points w and u which should be first plotted. Again $\lvert z - 3 + \text{i}\rvert = \lvert z + 2\text{i} + 5\rvert$ should be rewritten as $\lvert z - (3 - \text{i})\rvert = \lvert z - (-5 - 2\text{i})\rvert$ so that $3 - \text{i}$ and $-5 - 2\text{i}$ can be plotted.
$a\,\text{Re}(z) + b\,\text{Im}(z) = c$	is a straight line which is plotted as $ax + by = c$ in the usual way.

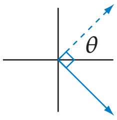
$\arg z = \theta$ is



$\arg(z + 1) = \theta$ is



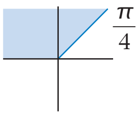
$\arg(\text{i}z) = \theta$ is



- Inequalities involve shaded regions. First get the locus as an equality and then if it was $<$ or $>$ you must now make the locus line dotted. Provided the locus does not go through the origin you should now substitute $z = 0 + 0\text{i}$ into the locus rule to see if you arrive at a true or false statement. If it's true, then shade in the region that includes $0 + 0\text{i}$ up to the solid or dotted locus line. If it's false, then shade the region on the other side of the solid or dotted locus line that does not include $0 + 0\text{i}$.

Remember that $\arg z \geq \frac{\pi}{4}$ will stop at π

i.e.



Often it is required to shade the union or intersection of 2 or more sets. At other times you will be given the sketch and then are required to provide the constraint, or combination of constraints. Sometimes a good deal of processing is required unless you recognise the type.

Example Describe the locus of $|z + 3 - 7i| = 2|z - 3 - 4i|$

Let $z = x + yi$

$$|x + yi + 3 - 7i| = 2|x + yi - 3 - 4i|$$

$$|(x + 3 + i(y - 7)| = 2|(x - 3) + i(y - 4)|$$

$$\sqrt{(x+3)^2 + (y-7)^2} = 2\sqrt{(x-3)^2 + (y-4)^2}$$

$$x^2 + 6x + 9 + y^2 - 14y + 49 = 4(x^2 - 6x + 9 + y^2 - 8y + 16)$$

$$x^2 + 6x + y^2 - 14y + 58 = 4x^2 - 24x + 4y^2 - 32y + 100$$

$$3x^2 - 30x + 3y^2 - 18y + 42 = 0 \quad \text{or} \quad 3(x^2 - 10x + y^2 - 6y + 14) = 0$$

$$x^2 - 10x + y^2 - 6y + 14 = 0 \quad \text{now complete the square for } x \text{ and } y$$

$$x^2 - 10x + 25 + y^2 - 6y + 9 + 14 = 25 + 9 \quad \text{gives} \quad (x - 5)^2 + (y - 3)^2 = 20$$

$\therefore$ The locus is the circle with centre $z = 5 + 3i$ and radius $\sqrt{20} = 2\sqrt{5}$ units.

FORMULAE AND DEFINITIONS

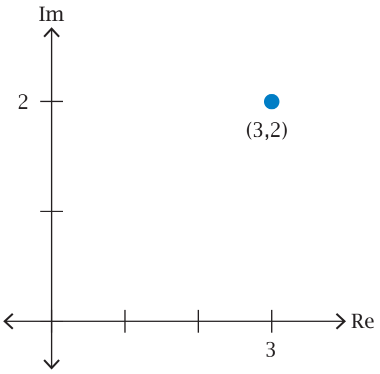
Definition

Complex Number: $Z = a + bi$
a, b real numbers
a is called the "real part"
b is called "the imaginary part"
$i = \sqrt{-1}$ or $i^2 = -1$

Cartesian form

Z, written in the form $a + bi$, is in Cartesian form since Z can be represented as an ordered pair (a,b). The ordered pair (a,b) can be plotted in the Complex Plane on an Argand diagram.

The graph shows $Z = 3 + 2i$.



Arithmetic Processses

If $Z = a + bi$ and $W = c + di$

then

(i) $Z + W = (a + bi) + (c + di) = (a + c) + (b + d)i$

(ii) $Z - W = (a + bi) - (c + di) = (a - c) + (b - d)i$

(iii) $ZW = (a + bi)(c + di)$
$= ac + adi + bci + bdi^2$
$= (ac - bd) + (ad + bc)i$

Exponential Properties

$$\text{cis}\,(\theta + \phi) = \text{cis}\,\theta \cdot \text{cis}\,\phi$$

$$\text{cis}\,(\theta - \phi) = (\text{cis}\,\theta) \div (\text{cis}\,\phi)$$

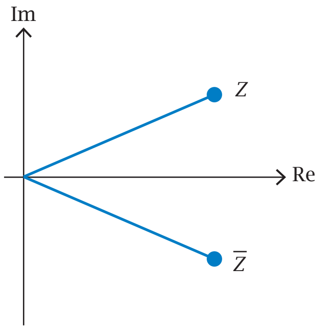
$$\text{cis}\,0 = 1$$

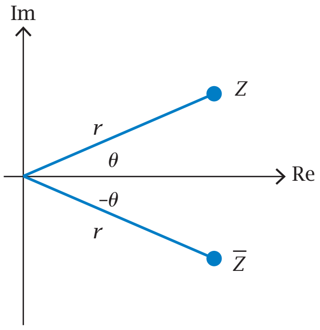
$$\text{cis}\,(-\theta) = (\text{cis}\,\theta)^{-1}$$

Conjugates

(i) Cartesian Form
$\overline{Z} = a - bi$

(ii) Polar Form
$\overline{Z} = r\,\text{cis}\,(-\theta) = r\cos\theta - i(r\sin\theta)$





Division, Reciprocals

(i) If $Z = a + bi$ and $W = c + di$

then (i) $\dfrac{Z}{W} = \dfrac{Z}{W} \cdot \dfrac{\overline{W}}{\overline{W}} = \dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di} = \dfrac{(ac+bd) + i\,(bc+ad)}{c^2 + d^2}$

(ii) $\dfrac{1}{Z} = \dfrac{1}{Z} \cdot \dfrac{\overline{Z}}{\overline{Z}} = \dfrac{1}{a+bi} \cdot \dfrac{a-bi}{a-bi} = \dfrac{a-bi}{a^2+b^2}$

(ii) If $Z = r_1\,\text{cis}\,\theta$ and $W = r_2\,\text{cis}\,\phi$

then (i) $\dfrac{Z}{W} = \dfrac{r_1\,\text{cis}\,\theta}{r_2\,\text{cis}\,\phi} = \dfrac{r_1}{r_2}\,\text{cis}\,(\theta - \phi)$

(ii) $\dfrac{1}{Z} = \dfrac{1}{r_1\,\text{cis}\,\theta} = \dfrac{\text{cis}\,0}{r_1\,\text{cis}\,\theta} = \dfrac{1}{r_1}\,\text{cis}\,(-\theta)$

Quadratic Equations

If $az^2 + bz + c = 0$

then $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

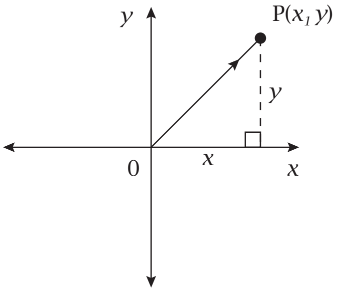
and include cases where $b^2 - 4ac < 0$ by using $i^2 = -1$

Polar coordinates are a means of specifying position in the plane by magnitude and direction. These will recur in this unit within the vectors and the complex numbers topics.

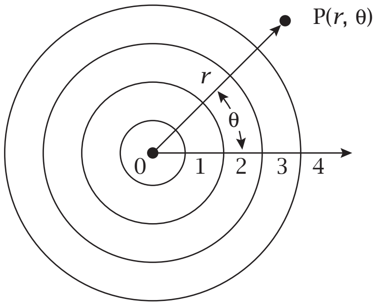
This section is the culmination of the study of complex numbers at the senior school level and links algebraic, trigonometric and geometric ideas studied in previous units.

INTRODUCTION TO THE BASICS

- The polar variables are r and θ. The polar origin 0 is the same as the origin of the $x\,y$ axes. A polar point can be represented by the ordered pair $P\,(r, \theta)$ where r is the distance the point is from the origin and θ is the polar angle which is the angle made by the ray OP and the positive x axis as shown below.



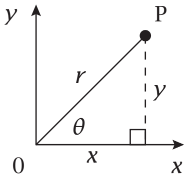
Cartesian Axes



Polar Axes

The polar angle θ is always measured from the positive x direction and it is positive in an anticlockwise direction and negative in a clockwise direction.

- The conversion formula are:



a) from Cartesian to polar

$$r^2 = x^2 + y^2 \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{i.e.} \quad r = \sqrt{x^2 + y^2}$$

b) from polar to Cartesian

$$x = r\cos\theta \quad , \quad y = r\sin\theta\,.$$

In practice the above conversions are nearly always provided as in built functions on your calculator.

Confirm that for:

a) Cartesian point (3, 4) , then Pol(3, 4) = (5, 53.13°) in degree mode and (5, 0.93) in radian mode.

b) polar point (5, $-\frac{2\pi}{3}^R$) , then Rec(5, $-\frac{2\pi}{3}^R$) in radian mode gives (–2.5, –4.33) 2 dp.

- So far there has been no mention of vectors. If point A has rectangular coordinates of (3, 4) then the position vector

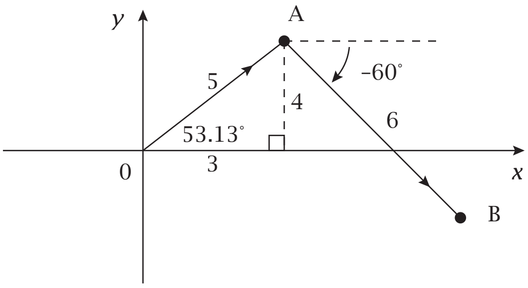
$$\begin{aligned} \mathbf{OA} &= 3\mathbf{i} + 4\mathbf{j} \\ &= <3, 4>\text{R} \end{aligned}$$

where the R indicates that the vector is in rectangular from. The point A has polar coordinates of (5, 53.13°) and the position vector for this point can be represented by

$$\begin{aligned} \mathbf{OA} &= <5, 53.13^\circ>\text{P} \\ &= <5, 0.93>\text{P} \end{aligned}$$

where the second vector needs the P to indicate a polar vector which must have its angle measured in radians as no degree symbol is present.

The displacement vector in polar terms, $\mathbf{AB} = \langle 6, -60^\circ \rangle\ P$ from A (3, 4) is shown below.



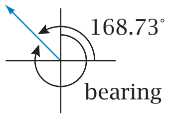
Both position and displacement vectors can be written in either rectangular or polar form

Example A scout walks on a bearing of 036.87˚ for 5 km and then walks on a bearing of 150˚ for 6 km. Find the direction and distance necessary for the scout to walk directly back to his starting point.

The diagram above matches this situation. The bearings of 036.87˚ and 150˚ have polar angles of 53.13˚ and –60˚ respectively.

$$\begin{aligned}
\mathbf{OB} &= \mathbf{OA} + \mathbf{AB} \\
\mathbf{BO} &= -(\mathbf{OA} + \mathbf{AB}) \\
&= -(\langle 5, 53.13^\circ \rangle\ P + \langle 6, -60^\circ \rangle\ P) \\
&= -(\langle 3, 4 \rangle\ R + \langle 3, -5.196 \rangle\ R) \\
&= -\langle 6, -1.196 \rangle\ R \\
&= \langle -6, 1.196 \rangle\ R \\
&= \langle 6.118, 168.73^\circ \rangle\ P
\end{aligned}$$

The polar angle of 168.73˚ has a bearing of $360^\circ - (168.73^\circ - 90^\circ)$
$= 281.27^\circ$



The scout needs to walk on a bearing of 281.27˚ for 6.12 km to reach his starting point.

The example above shows the big advantage of being able to readily convert between rectangular and polar forms on your calculator and the idea that vectors can be represented in polar as well as rectanglar form

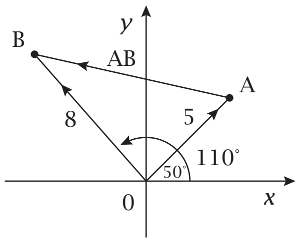
DISTANCES BETWEEN POINTS IN POLAR FORM

- When two positions are specified by their polar coordinates, the straight line distance between them is easily found by using the cosine rule.

Example Find the distance between A (5, 50˚) and B (8, 110˚).

$$|\mathbf{AB}|^2 = 5^2 + 8^2 - 2\,(5)(8)\cos(110^\circ - 50^\circ)$$

i.e. $|\mathbf{AB}| = 7$ units

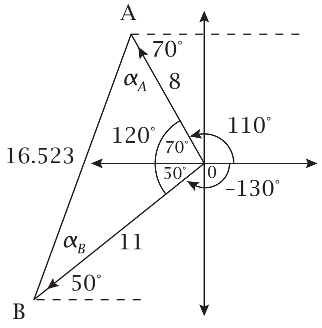


The only aspect that needs care is finding the angle between the vectors because if $\theta_1 - \theta_2 > 180^\circ$ which for $\theta_1 > 0$ can happen when $\theta_2 < 0$, then the required angle used in the cosine rule will need to be $360^\circ - (\theta_1 - \theta_2)$.

Example For $\mathbf{OA} = \langle 8, 110^\circ \rangle$ and $\mathbf{OB} = \langle 11, -130^\circ \rangle$, the required angle will be
$360^\circ - (110^\circ - -130^\circ)$
$= 120^\circ$
and $|\mathbf{AB}|^2 = 8^2 + 11^2 - 2\,(8)(11)\cos 120^\circ$
i.e. $|\mathbf{AB}| = 16.523$ 3dp

- When a straight line does not pass through the origin, its direction, with respect to the positive *x* axis, can be specified either as a positive or negative polar angle as shown below

Example For the example above find the polar angles of **AB** and **BA**.



$$\cos \alpha_A = \frac{8^2 + 16.523^2 - 11^2}{2\,(8)(16.523)}$$

$$\alpha_A = 35.21^\circ \quad \text{2dp}$$

$$\therefore \quad \alpha_B = 180^\circ - 120^\circ - 35.21^\circ$$

$$= 24.79^\circ \quad \text{2dp}$$

So the direction of **AB** is $-(\alpha_A + 70^\circ) = -105.21^\circ$

and the direction of **BA** is $\alpha_B + 50^\circ = 74.79^\circ$

Example Use rectangular to polar techniques to find the directions of AB and BA for the above example.

AB = **OB** − **OA**

= <11, −130°> P − <8, 110°> P

= <−7.071, −8.426> R − <−2.736, 7.518> R

= <−4.335, −15.944> R

= <16.52, − 105.21°> P 2dp

∴ **AB** has a polar direction of −105.21° as before.

BA = −**AB**
= −<−4.335, −15.944> R

= <4.335, 15.944> R

= <16.52, 74.79°> P 2dp

and **BA** has a polar direction of 74.79° as before.

Note that the advantage of using rec/pol methods for these types is that a diagram is not really needed, as vector notation and methods are substituted for the sine and cosine rules.

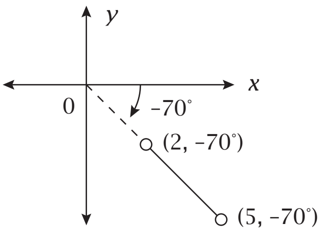
POLAR GRAPHS

- A circle of radius a with its centre at the origin is specified by the equation $r = a$ where a is the radius and a constant. If a domain for θ is also given then the graph will be an arc of the circle.

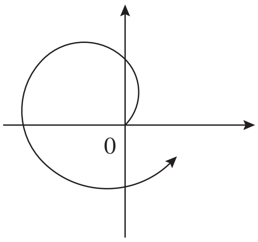
  When the circle's centre is not at the origin then the centre C needs to be specified by its polar coordinates (r_c, θ_c). Again the radius is denoted by a .

- Lines radiating from the origin have equations in the form of $\theta = \theta_1$ where θ_1 is constant. If a domain for r is given the line will become a line segment.

Example Graph $\theta = -70^\circ$, $2 < r < 5$



- Spirals are very simply represented in polar form due to their rotational nature. The general form is $r = k\theta$ where θ must be in radians. For $k > 0$ and $\theta > 0$ the graph of $r = k\theta$ spirals in an anticlockwise direction.



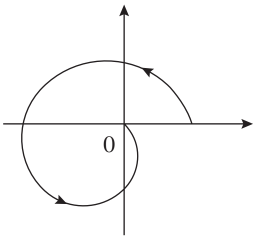
When $k < 0$ and $\theta > 0$, $r = k\theta$ will produce negative values of r. When this happens there are two choices that can be made. Either no values of $r < 0$ are graphed or the angle is increased by half a revolution i.e. π^R and $|r|$ is plotted with the new angle.

So if $r = -2\theta$ and $\theta = \frac{\pi}{3}$, then $r = -\frac{2\pi}{3}$. The new angle is $-\frac{2\pi}{3} + \pi = \frac{\pi}{3}$, and hence the point plotted is $\left(\frac{2\pi}{3}, \frac{\pi}{3}\right)$ i.e. $r = \frac{2\pi}{3}$ and $\theta = \frac{\pi}{3}$.

This latter convention is used by some calculators. This effectively means that the point is reflected about the origin and the negative r ignored. How a spiral is graphed by your calculator also depends on how the domain for θ is entered into the calculator. Using your calculator, input the domain for θ as $-2\pi \le \theta \le 0$, and set a suitably small increment for θ (sometimes called the pitch) in readiness for the example below.

Example Show how the graph of $r = -\theta$ appears using the above domain for θ.

Because the values of θ are negative and $r = -\theta$ all values of r will be positive and hence no reflections will be needed.



Notice that in this case the spiral is graphed from the outside towards the origin

CLOSEST DISTANCE WITH POLAR COORDINATES

A line drawn on the polar axes through a given point A (r_A, θ_A) with a given polar angle say α will contain a point which is closest to the origin. Other points on the line will be closest to given specified points. An example shows this best

Example A line passes through A (6, 120°) has a polar angle of –35°. Find:

a) The point on the line which is closest to the origin and its distance from the origin.

The diagram provides the important angle

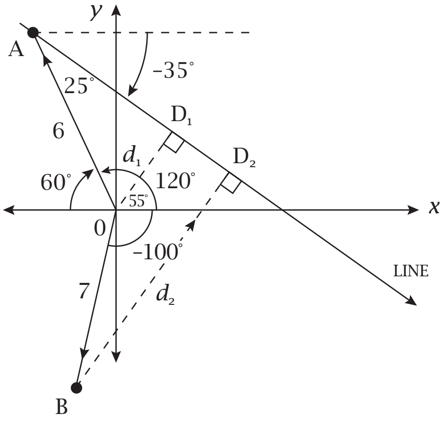
$$\angle D_1 AO = 25^\circ$$

$$\sin 25^\circ = \frac{d_1}{6}$$

$$d_1 = 6 \sin 25^\circ$$

$$= 2.536 \quad 3 \text{ d.p.}$$

Also $\angle AOD_1 = 65^\circ$
So the closest point D_1
has coordinates (2.536, 55°)
and the closest distance is
2.54 units 2 d.p.



b) The point on the line which is closest to B (7, –100˚) and the closest distance.

$$\begin{aligned}
\mathbf{AB} &= \mathbf{OB} - \mathbf{OA} \\
&= <7, -100^\circ> P - <6, 120^\circ> P \\
&= <-1.216, -6.894> R - <-3, 5.196> \quad 3 \text{ d.p.} \\
&= <1.784, -12.090> R \\
&= <12.221, -81.606^\circ> P \\
\therefore \quad \angle D_2AB &= 81.606 - 35^\circ \\
&= 46.606^\circ \\
\sin 46.606 &= \frac{d_2}{12.221} \\
d_2 &= 8.880 \quad 3 \text{ d.p.} \\
\mathbf{BD_2} &= <8.880, 55^\circ> P \text{ because } \mathbf{BD_2} \parallel \mathbf{OD_1} \\
&= <5.093, 7.274> R \\
\mathbf{OD_2} &= \mathbf{OB} + \mathbf{BD_2} \\
&= <-1.216, -6.894> + <5.093, 7.274> R \\
&= <3.877, 0.38> R \\
&= <3.90, 5.60^\circ> P
\end{aligned}$$

∴ the point on the line closest to B (7, –100˚) is (3.90, 5.60˚) and the closest distance is 8.88 units.

DE MOIVRE'S THEOREM

- As shown in Chapter 12, when a complex number in cis form or exponential form is raised to a power the result by de Moivre's theorem is

  $(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta)$

It should be noted that whenever a question involves powers of complex numbers then there is a very good chance that the best form to represent the complex number is in cis form and that most likely de Moivre's theorem will be involved.

- Using de Moivre's theorem to find the n^{th} roots of a complex number goes like this

$$\begin{aligned}
\text{If} \quad z^n &= x + yi && \text{then first change to cis form} \\
\text{ie.} \quad z^n &= r \operatorname{cis} \theta \\
z &= (r \operatorname{cis} \theta)^{\frac{1}{n}} \\
z_1 &= r^{\frac{1}{n}} \operatorname{cis} \frac{\theta}{n} && \text{is called the principal root}
\end{aligned}$$

However because $\operatorname{cis} \theta = \operatorname{cis}(\theta + 2\pi) = \operatorname{cis}(\theta + 4\pi)$ etc.

$= \operatorname{cis}(\theta + 2\pi k)$ where k is an integer

$$\begin{aligned}
\text{then} \quad z^n &= r \operatorname{cis}(\theta + 2\pi k) \\
\text{gives} \quad z &= r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta + 2\pi k}{n}\right) \\
&= r^{\frac{1}{n}} \operatorname{cis}\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) \quad \text{where } 0 \le k \le n-1 \text{ in general gives the } n \text{ roots.}
\end{aligned}$$

So $k = 0$ gives the principal root and then the other $n - 1$ roots have the same magnitude of $r^{\frac{1}{n}}$ but their arguments increase from the principal argument of $\frac{\theta}{n}$ by $\frac{2\pi}{n}$ i.e. the roots are equally spaced around a circle of radius $r^{\frac{1}{n}}$ until they meet with the principle root.

Example: If $(x+y\text{i})^{\frac{1}{5}}$ has a principle root of $2 \text{ cis } 30^\circ$, find the other roots.

Because $z^n = x + y\text{i}$

gives $z = (x + y\text{i})^{\frac{1}{5}}$, then $n = 5$ and $\frac{2\pi}{5} = \frac{360^\circ}{5}$

$= 72^\circ$.

The other 4 roots will be

$z_2 = 2 \text{ cis } (30^\circ + 72^\circ) = 2 \text{ cis } (102^\circ)$

$z_3 = 2 \text{ cis } (102^\circ + 72^\circ) = 2 \text{ cis } (174^\circ)$

$z_4 = 2 \text{ cis } (174^\circ + 72^\circ) = 2 \text{ cis } (246^\circ)$

$z_5 = 2 \text{ cis } (246^\circ + 72^\circ) = 2 \text{ cis } (318^\circ)$

Then if we do $z_6 = 2 \text{ cis } (318^\circ + 72^\circ)$

$= 2 \text{ cis } (390^\circ)$

$= 2 \text{ cis } 30^\circ$ we are back to the principal root.

Depending on the domain required which is usually $-\pi < \theta \le \pi$ or $-180^\circ < \theta \le 180^\circ$ then

$z_4 = 2 \text{ cis } 246^\circ = 2 \text{ cis } (246^\circ - 360^\circ)$

$= 2 \text{ cis } (-114^\circ)$

and $z_5 = 2 \text{ cis } 318^\circ = 2 \text{ cis } (318^\circ - 360^\circ)$

$= 2 \text{ cis } (-42^\circ)$ etc.

Example Find the identity for $\sin 3\theta$ by using de Moivre's theorem.

$$\boxed{\begin{aligned} \text{i}^2 &= -1 \\ \text{i}^3 &= -\text{i} \end{aligned}}$$

$$\text{cis}\,\theta = \cos\theta + \text{i}\sin\theta$$

$$\therefore (\text{cis}\,\theta)^3 = (\cos\theta + \text{i}\sin\theta)^3$$

$$\text{cis}\,3\theta = \cos^3\theta + 3\cos^2\theta(\text{i}\sin\theta) + 3\cos\theta(\text{i}\sin\theta)^2 + (\text{i}\sin\theta)^3$$

$$\cos 3\theta + \text{i}\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + \text{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$\therefore \sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$$

$$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ is the required identity

Note that this method always gives a 2 for 1 value as the identity for $\cos 3\theta$ is also available by equating the real rather than the imaginary parts!

FINDING IDENTITIES BY USING *Z* TRANSFORMS

- If $z = \text{cis}\,\theta$, then z is any complex number on the unit circle. The transforms are now found:

$z^n + z^{-n}$

$= (\text{cis}\,\theta)^n + (\text{cis}\theta)^{-n}$

$= \text{cis}\,n\theta + \text{cis}(-n\theta)$

$= \cos n\theta + \text{i}\sin n\theta + \cos(-n\theta) + \text{i}\sin(-n\theta)$

$= \cos n\theta + \text{i}\sin n\theta + \cos n\theta - \text{i}\sin n\theta$

$= 2\cos n\theta$

$\therefore \cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

$z^n - z^{-n}$

$= (\text{cis}\theta)^n - (\text{cis}\theta)^{-n}$

$= \text{cis}\,n\theta - \text{cis}(-n\theta)$

$= \cos n\theta + \text{i}\sin n\theta - (\cos(-n\theta) + \text{i}\sin(-n\theta))$

$= \cos n\theta + \text{i}\sin n\theta - (\cos n\theta - \text{i}\sin n\theta)$

$= \cos n\theta + \text{i}\sin n\theta - \cos n\theta + \text{i}\sin n\theta$

$= 2\text{i}\sin n\theta$

$\therefore \sin n\theta = \frac{1}{2\text{i}}\left(z^n - \frac{1}{z^n}\right)$

$\cos n\theta = \frac{1}{2}\left(z^{n} + \frac{1}{z^{n}}\right)$ and $\sin n\theta = \frac{1}{2\mathrm{i}}\left(z^{n} - \frac{1}{z^{n}}\right)$ where $z = \operatorname{cis}\theta$ are the z transforms necessary to prove identities involving powers of $\cos\theta$ and $\sin\theta$.

The left hand side of each of the above is real so the right hand side of each must also be real even though they consist of complex terms.

Example Use complex methods to prove $\sin^{2}\theta + \cos^{2}\theta = 1$.

$$\begin{aligned}
\text{LHS} &= \sin^{2}\theta + \cos^{2}\theta \\
&= \left[\frac{1}{2\mathrm{i}}(z - z^{-1})\right]^{2} + \left[\frac{1}{2}(z + z^{-1})\right]^{2} \\
&= -\frac{1}{4}(z^{2} - 2zz^{-1} + z^{-2}) + \frac{1}{4}(z^{2} + 2zz^{-1} + z^{-2}) \\
&= -\cancel{\frac{1}{4}z^{2}} + \frac{2}{4} - \cancel{\frac{1}{4}z^{-2}} + \cancel{\frac{1}{4}z^{2}} + \frac{2}{4} + \cancel{\frac{1}{4}z^{-2}} \\
&= 1 \quad = \text{RHS} \therefore \text{proven}
\end{aligned}$$

The worked example sections shows the real power of the z-transform method.

Worked Examples

1.1 A quadratic equation with real coefficients has a solution of $z = 2\operatorname{cis}(-150^{\circ})$. Find the equation by not changing the solutions to Cartesian form.

The two solutions are $z_1 = 2\operatorname{cis}(150^{\circ})$ and $z_2 = 2\operatorname{cis}(-150^{\circ})$ which form a complete conjugate pair because the quadratic has real coefficients.

$$\therefore \quad (z - 2\operatorname{cis}(150^{\circ}))(z - 2\operatorname{cis}(-150^{\circ})) = 0$$

$$z^{2} - 2z\operatorname{cis}(-150^{\circ}) - 2z\operatorname{cis}(150^{\circ}) + 4\operatorname{cis}(150^{\circ})\operatorname{cis}(-150^{\circ}) = 0$$

$$z^{2} - 2z(\operatorname{cis}(-150^{\circ}) + \operatorname{cis}(150^{\circ})) + 4\operatorname{cis}0^{\circ} = 0$$

$$z^{2} - 2z \times 2\operatorname{Re}(\operatorname{cis}150^{\circ}) + 4 = 0 \quad \text{as } \operatorname{cis}0^{\circ} = 1$$

$$z^{2} - 4z \times \cos 150^{\circ} + 4 = 0$$

$$z^{2} - 4z \times \frac{-\sqrt{3}}{2} + 4 = 0$$

$$\therefore \quad z^{2} + \left(2\sqrt{3}\right)z + 4 = 0 \quad \text{is the required equation}$$

1.2 Prove that $\overline{zw} = \bar{z}\ \bar{w}$.

Let $z = x + y\mathrm{i}$ and $w = a + b\mathrm{i}$

$$\begin{array}{llll}
\text{L.H.S.} & = \overline{zw} & \text{R.H.S.} & = \bar{z}\ \bar{w} \\
& = \overline{(x + y\mathrm{i})(a + b\mathrm{i})} & & = (x - y\mathrm{i})(a - b\mathrm{i}) \\
& = \overline{ax + bx\mathrm{i} + ay\mathrm{i} - by} & & = ax - bx\mathrm{i} - ay\mathrm{i} - by \\
& = \overline{ax - by + \mathrm{i}(bx + ay\mathrm{i})} & & = ax - by - \mathrm{i}(bx + ay) \\
& = ax - by - \mathrm{i}(bx + ay) & & = \text{L.H.S. proved}
\end{array}$$

1.3 Use a complex number method to find the exact values of $\sin 105^\circ$ and $\cos 105^\circ$.

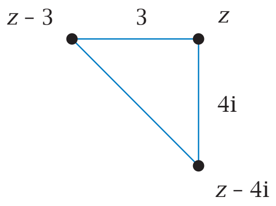
$$\begin{aligned}
\text{cis } 105^\circ &= \text{cis } (45^\circ + 60^\circ) \\
&= \text{cis } 45^\circ \times \text{cis } 60^\circ \\
&= (\cos 45^\circ + \text{i} \sin 45^\circ)(\cos 60^\circ + \text{i} \sin 60^\circ) \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\text{i}\sqrt{2}}{2}\right)\left(\frac{1}{2} + \frac{\text{i}\sqrt{3}}{2}\right) \\
4 \text{ cis } 105^\circ &= (\sqrt{2} + \text{i}\sqrt{2})(1 + \text{i}\sqrt{3}) \\
&= \sqrt{2} + \text{i}\sqrt{6} + \text{i}\sqrt{2} - \sqrt{6}
\end{aligned}$$

$$\therefore \quad 4 \cos 105^\circ + 4 \text{ i} \sin 105^\circ = \sqrt{2} - \sqrt{6} + \text{i}\,(\sqrt{2} + \sqrt{6})$$

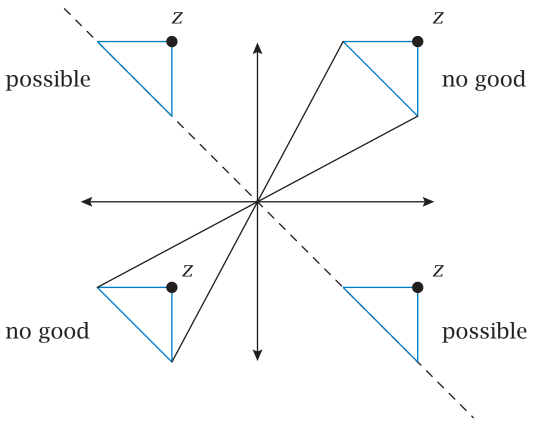
$\therefore \quad \cos 105^\circ = \dfrac{\sqrt{2} - \sqrt{6}}{4}$ and $\sin 105^\circ = \dfrac{\sqrt{2} + \sqrt{6}}{4}$ are the required exact values.

1.4 Show how to use a trial and error approach to sketch the locus of $\arg(z - 3) = \arg(z - 4\text{i})$.

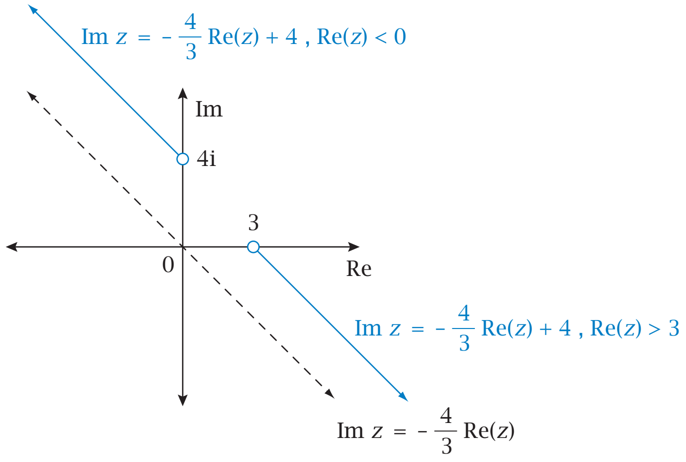
For any z on the complex plane the positions of $z - 3$ and $z - 4\text{i}$ are as shown



For the arguments of $z - 3$ and $z - 4\text{i}$ to have any chance of being the same the triangle shown must be in quadrants 2 or 4. In fact, if the positions of $z - 3$, $z - 4\text{i}$ and the origin are lined up on the same straight line, then the constraint is satisfied.



The locus of the required z values is parallel to this line and translated 4 units in the positive imaginary direction or translated 3 units in the positive real direction except for the portion of this line that passes through quadrant 1. The required locus is shown as:



The end points at 4i and 3 are not included as, in each case, one of the vertices of the triangle coincides with the origin which doesn't have an argument!

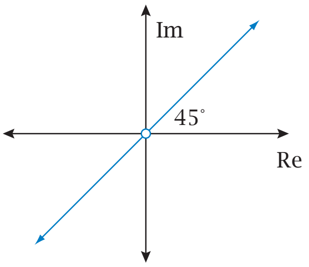
1.5 Sketch the locus of $\frac{z}{\overline{z}} = \text{i}$.

let $z = x + y\text{i}$

$$\frac{x + y\text{i}}{x - y\text{i}} = \text{i}$$

$$x + y\text{i} = x\text{i} + y$$

$$x + y\text{i} = y + x\text{i}$$



$\therefore$ $x = y$ and $y = x$ which are the same.

$\therefore$ $\text{Re}(z) = \text{Im}(z)$ is sketched as shown but the origin is not included as this point is not compatible with the $\frac{z}{\overline{z}} = \text{i}$.

1.6 Find, without the use of your calculator, the other two solutions of $2z^3 - z^2 - 14z - 12 = 0$ if one solution is $z = 1 - \text{i}\sqrt{3}$.

One other solution is $\overline{z} = 1 + \text{i}\sqrt{3}$ and hence the quadratic giving these solutions is

$$(z - (1 - \text{i}\sqrt{3}))(z - (1 + \text{i}\sqrt{3})) = 0$$

$$(z - 1 + \text{i}\sqrt{3})(z - 1 - \text{i}\sqrt{3}) = 0$$

$$(z - 1)^2 + 3 = 0$$

$$z^2 - 2z + 4 = 0$$

The other solution must be real (a cubic function always has at least one real x intercept).

$\therefore$ let $2z^3 - z^2 - 14z - 12 = (2z + a)(z^2 - 2z + 4)$

which means that $-12 = 4a$

and $a = -3$

$\therefore$ if $2z - 3 = 0$

$$z = \frac{3}{2}$$

The other two solutions are $z = \frac{3}{2}$ and $z = 1 + \text{i}\sqrt{3}$

1.7 If $w = 2 - 2i$ and $z = 1 + 3i$
then find:

(a) $w - z$
(b) w^2
(c) $w \div (\overline{z})$
(d) $\frac{1}{w}$
(e) w and z in polar form

(a) $w - z$
$= (2 - 2i) - (1 + 3i)$
$= 2 - 1 - 2i - 3i$
$= 1 - 5i$

(b) w^2
$= (2 - 2i)(2 - 2i)$
$= 4 - 4i - 4i + 4i^2$
$= -8i$

(c) $w \div (\overline{z})$

$$= \frac{2 - 2i}{1 - 3i}$$

$$= \frac{(2 - 2i)}{(1 - 3i)} \frac{(1 + 3i)}{(1 + 3i)}$$

$$= \frac{2 + 6i - 2i - 6i^2}{1 + 3i - 3i - 9i^2}$$

$$= \frac{8 + 4i}{10}$$

$$= \frac{4}{5} + \frac{2}{5} i$$

(d) $\frac{1}{w}$

$$= \frac{1}{2 - 2i}$$

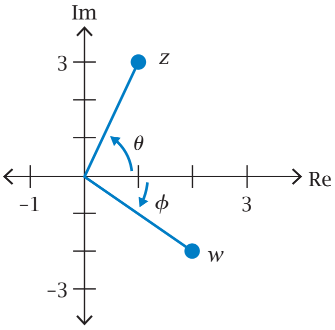
$$= \frac{1}{2 - 2i} \frac{(2 + 2i)}{(2 + 2i)}$$

$$= \frac{2 + 2i}{4 - 4i^2}$$

$$= \frac{2 + 2i}{8}$$

$$= \frac{1}{4} + \frac{1}{4} i$$

(e)



For Z: $|Z| = \sqrt{1^2 + 3^2} = \sqrt{10}$

$\tan\theta = \frac{3}{1} \therefore \theta \approx 72^\circ$

$\therefore Z = \sqrt{10}\ Cis\ 72^\circ$

For W: $|W| = \sqrt{2^2 + (-2)^2} = \sqrt{8}$

$\tan\phi = \frac{-2}{2} \therefore \phi = -45^\circ$

$\therefore W = 2\sqrt{2}\ Cis\ (-45^\circ)$

1.8 (a) Express $z = \sqrt{3} - i$ in polar form

(b) Hence find $\frac{\sqrt{3} - i}{2 - 2i}$

(a) $|Z| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$

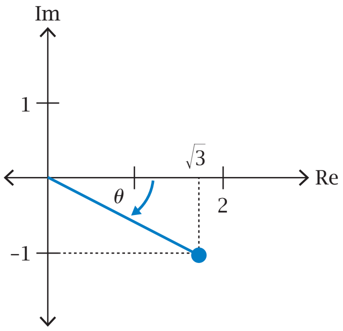
$= \sqrt{3 + 1}$

$= \sqrt{4} = 2$

$\tan\theta = -\frac{1}{\sqrt{3}}$

$\therefore \theta = -30^\circ$

$\therefore Z = 2\, Cis\left(-30^\circ\right)$



(b) $\dfrac{\sqrt{3}-i}{2-2i}=\dfrac{2Cis\left(-30^\circ\right)}{2\sqrt{2}Cis\left(-45^\circ\right)}$ ..... from Worked Examples 1.7(e) and 1.8(a)

$=\dfrac{Cis\left(-30^\circ-\left(-45^\circ\right)\right)}{\sqrt{2}}$

$=\dfrac{1}{\sqrt{2}}Cis15^\circ\ or\ \dfrac{\sqrt{2}}{2}Cis15^\circ$

NOTE: In (a), trigonometry tells us that θ = 330°. However, since $-180^\circ \le \theta \le 180^\circ$, θ is restricted to –30°.

1.9 If $Z_1 = 4Cis\frac{\pi}{6}$ and $Z_2 = 2Cis\left(\frac{-\pi}{3}\right)$ then find

(a) $Z_1 \cdot Z_2$

(b) $Z_2 \div Z_1$

(a) $Z_1.Z_2=\left(4Cis\dfrac{\pi}{6}\right)\left(2Cis\left(\dfrac{-\pi}{3}\right)\right)$

$=8Cis\left(\dfrac{\pi}{6}+\left(\dfrac{-\pi}{3}\right)\right)$

$=8Cis\left(\dfrac{-\pi}{6}\right)$

$=8\left(\cos\left(\dfrac{-\pi}{6}\right)+i\sin\left(\dfrac{-\pi}{6}\right)\right)$

$=8\left(\dfrac{\sqrt{3}}{2}+i\left(-\dfrac{1}{2}\right)\right)$

$=4\sqrt{3}-4i$

(b) $Z_2 \div Z_1=\dfrac{2Cis\left(-\dfrac{\pi}{3}\right)}{4Cis\left(\dfrac{\pi}{6}\right)}$

$=\dfrac{1}{2}Cis\left(-\dfrac{\pi}{3}-\dfrac{\pi}{6}\right)$

$=\dfrac{1}{2}Cis\left(-\dfrac{\pi}{2}\right)$

$=\dfrac{1}{2}\left(\cos\left(-\dfrac{\pi}{2}\right)=i\sin\left(-\dfrac{\pi}{2}\right)\right)$

$=\dfrac{1}{2}\left(0+i\left(-1\right)\right)$

$=-\dfrac{1}{2}i$

1.10 Solve completely, the equation $Z^3 - 3Z^2 + 4Z - 2 = 0$

Let $f(Z) = Z^3 - 3Z^2 + 4Z - 2 = 0$

Look for a real solution first, as *all* imaginary factors occur as conjugate pairs.

Now $f(1) = 1 - 3 + 4 - 2 = 0$

$\therefore\ Z - 1$ is a factor.

To find other factors:

$$\begin{array}{r} Z^2-2Z+2 \\ Z-1\overline{)Z^3-3Z^2+4Z-2} \\ \underline{Z^3-Z^2} \\ -2Z^2+4Z \\ \underline{-2Z^2+2Z} \\ 2Z-2 \\ \underline{2Z-2} \\ 0 \end{array}$$

$\therefore Z^3 - 3Z^3 + 4Z - 2 = 0$

$\Rightarrow (Z - 1)(Z^2 - 2Z + 2) = 0$

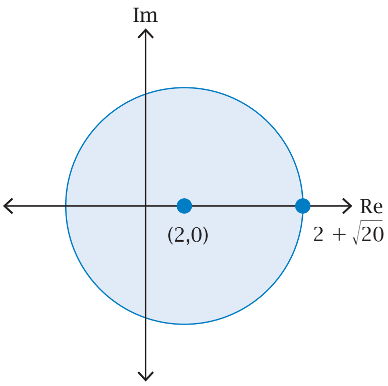
Consider $Z^2 - 2Z + 2 = 0$

$$\begin{aligned} z &= \frac{2 \pm \sqrt{4-8}}{2} && \text{using the quadratic formula} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 2i}{2} && \text{since } \sqrt{-4} = \sqrt{4i^2} = 2i \\ &= 1 \pm i \end{aligned}$$

$\therefore$ Solutions are $Z = 1$ or $Z = 1 + i$ or $Z = 1 - i$

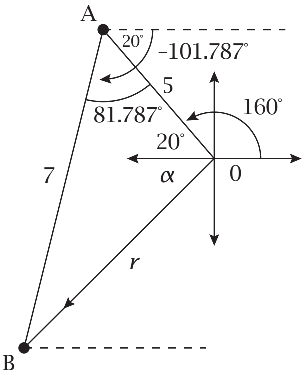
1.11 Find all complex numbers Z satisfying $|Z - 2| \le |2 + 4i|$

$$\begin{aligned} &|Z-2| \le |2+4i| \\ &\text{Let } Z = x+iy \\ &\therefore |x+iy-2| \le |2+4i| \\ &\therefore |(x-2)+iy| \le |2+4i| \\ &\therefore \sqrt{(x-2)^2+y^2} \le \sqrt{2^2+4^2} \\ &\therefore (x-2)^2+y^2 \le 2^2+4^2 \\ &\therefore (x-2)^2+y^2 \le 20 \end{aligned}$$



The locus is a circular region, centre (2,0) and radius $\sqrt{20}$.

1.12 A line segment of length 7 units has one end point at (5, 160°). If the line has its direction specified by the polar angle –101.787° find the other end point that has a negative polar angle.



$|\mathbf{OB}|^2 = 5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 81.787°$

$|\mathbf{OB}| = 8$ units

$\cos(\alpha + 20°) = \dfrac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8}$

$\alpha = 40°$

$\therefore$ The other end point has polar coordinates of (8, –140°).

1.13 A circle of radius 4 units has its centre C at (7, 50°).

(a) If A $(6, \theta_1)$ and B $(6, \theta_2)$ are on the circle, find θ_1 and θ_2.

(b) If points on the circle have $\theta = 35°$, find the possible values of r.

(a) $\cos\alpha = \dfrac{6^2 + 7^2 - 4^2}{2 \times 6 \times 7}$

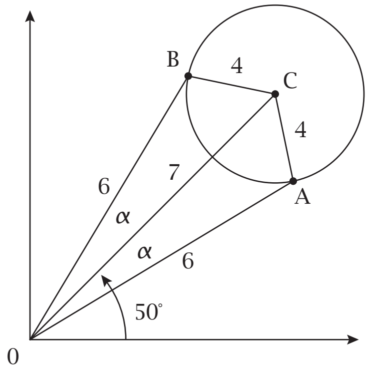
$\alpha = 34.8^\circ$

For point A, $\theta_1 = 50^\circ - 34.8^\circ$

$= 15.2^\circ$

For point B, $\theta_2 = 50^\circ + 34.8^\circ$

$= 84.8^\circ$

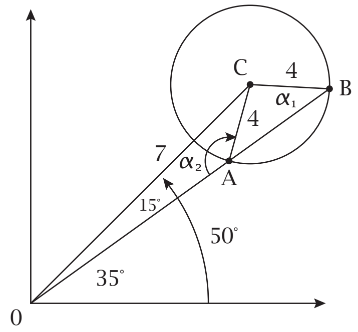


(b) $\dfrac{4}{\sin 15^\circ} = \dfrac{7}{\sin \alpha_1}$

$\alpha_1 = 26.932^\circ$ or $\alpha_2 = 180^\circ - 26.932^\circ$
$= 153.068^\circ$

$\angle OCA = 180^\circ - 15^\circ - 153.068^\circ$
$= 11.932^\circ$

$\angle OCB = 180^\circ - 15^\circ - 26.932^\circ$
$= 138.068^\circ$



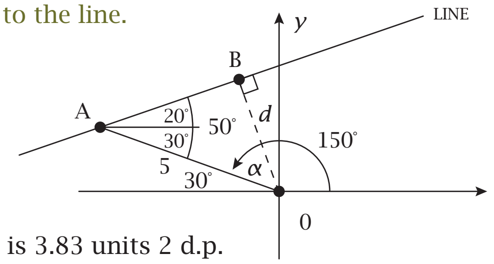
$|\mathbf{OA}|^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos 11.932^\circ$ $|\mathbf{OB}|^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos 138.068^\circ$

$|\mathbf{OA}| = 3.20$ units 2 d.p. $|\mathbf{OB}| = 10.33$ units 2 d.p.

When $\theta = 35^\circ$ the two values of r are 3.20 and 10.33 units 2 d.p.

1.14 A line passes through (5, 150°) and has a polar angle of 20°.

(a) Find the closest distance that the line comes to the origin.
(b) Find the polar coordinates of the closest point.
(c) Find the closest distance from point C (11, –50°) to the line.



(a) $\sin 50^\circ = \dfrac{d}{5}$

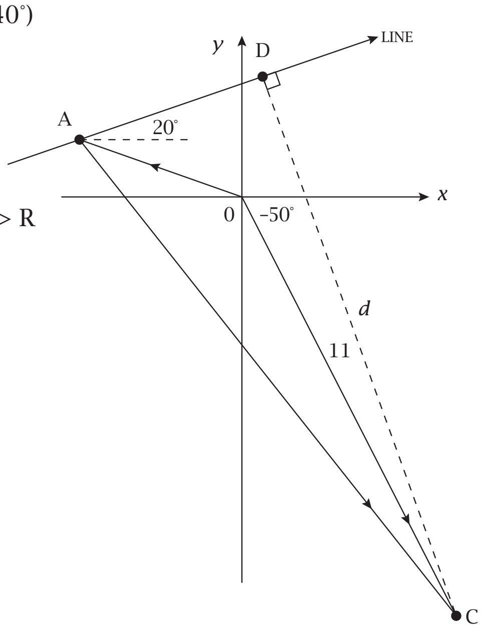
$d = 5 \sin 50^\circ$

$d = 3.83$ 2 d.p.

The closest distance that the line is to the origin is 3.83 units 2 d.p.

(b) From the diagram $\alpha = 90^\circ - 50^\circ = 40^\circ$

$\therefore$ coordinates of closest point B are $(3.83, 150^\circ - 40^\circ)$
$= (3.83, 110^\circ)$



(c) Using Vectors

$\mathbf{OA} = \langle 5, 150^\circ \rangle$ P $\mathbf{OC} = \langle 11, -50^\circ \rangle$ P

$= \langle -4.33, 2.5 \rangle$ R $= \langle 7.071, -8.426 \rangle$ R

$\mathbf{AC} = \mathbf{OC} - \mathbf{OA}$

$= \langle 7.071, -8.426 \rangle - \langle -4.33, 2.5 \rangle$ R

$= \langle 11.401, -10.926 \rangle$ R

$= \langle 15.791, -43.781^\circ \rangle$ P

$\angle DAC = 20^\circ + 43.781^\circ$

$= 63.781^\circ$

$\sin 63.781^\circ = \dfrac{d}{15.791}$

$d = 14.17$

$\therefore$ The closest distance from C (11, –50°) to the line is 14.17 units 2 d.p.

1.15 Find all values of $m \neq 0$ such that $(\sqrt{3} + i)^m - (\sqrt{3} - i)^m = 0$.

$$(2 \text{ cis } 30^\circ)^m - (2 \text{ cis } (-30^\circ))^m = 0$$

$$2^m \text{ cis } (30^\circ m) - 2^m \text{ cis } (-30^\circ m) = 0$$

$$2^m (\text{cis } (30^\circ m) - \text{cis } (-30^\circ m)) = 0$$

$$2^m (\cos (30^\circ m) + i \sin (30^\circ m) - \cos (-30^\circ m) - i \sin (-30^\circ m)) = 0 \qquad \text{but } 2^m \neq 0$$

$$\therefore \quad \cos (30^\circ m) + i \sin (30^\circ m) - \cos (30^\circ m) + i \sin (30^\circ m) = 0$$

$$2i \sin (30^\circ m) = 0$$

$$\sin (30^\circ m) = 0$$

$\therefore \quad 30m = \pm 180^\circ , \pm 360^\circ , \pm 540^\circ , \pm 720^\circ , \ldots$

$m = \pm 6 , \pm 12 , \pm 18 , \pm 24 , \ldots$ are the required values of m .

1.16 Consider $z^6 - \sqrt{2}\, z^3 + 1 = 0$ as a quadratic equation.

(a) Show that one solution for z^3 has a magnitude of 1 and an argument of –45º.

(b) Hence, or otherwise, find all the solutions for z and state them in exact polar form.

(a) Let $w = z^3$ $\qquad z^6 - \sqrt{2}\, z^3 + 1 = 0$

becomes, $\qquad w^2 - \sqrt{2}\, w + 1 = 0$

$$w = \frac{\sqrt{2} \pm \sqrt{2 - 4 \times 1 \times 1}}{2}$$

$$= \frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2} \quad = \quad \text{cis}\,(\pm 45^\circ)$$

$\therefore$ one solution for z^3 is $\text{cis}(-45^\circ)$

(b) $z^3 = \text{cis}\, 45^\circ$ $\qquad z^3 = \text{cis}(-45^\circ)$ $\qquad \dfrac{360^\circ}{3}$

$z = \text{cis}\left(\dfrac{45^\circ}{3}\right)$ $\qquad z = \text{cis}\left(\dfrac{-45^\circ}{3}\right)$ $\qquad = 120^\circ$

$z_1 = \text{cis}\, 15^\circ$ $\qquad z_4 = \text{cis}(-15^\circ)$

$z_2 = \text{cis}(15^\circ + 120^\circ)$ $\qquad z_5 = \text{cis}(-15^\circ + 120^\circ)$

$= \text{cis}\, 135^\circ$ $\qquad = \text{cis}\, 105^\circ$

$z_3 = \text{cis}(15^\circ - 120^\circ)$ $\qquad z_6 = \text{cis}(-15^\circ - 120^\circ)$

$= \text{cis}(-105^\circ)$ $\qquad = \text{cis}(-135^\circ)$

Note that once z_1, z_2 and z_3 have been found the other three solutions can be written down as their conjugates because the original equation has real coefficients.

1.17 Find the identity for $\sin^5\theta$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\sin\theta = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)$$

$$(2\mathrm{i}\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$

$$32\mathrm{i}\sin^5\theta = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 - 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 - \left(\frac{1}{z}\right)^5$$

$$= z^5 - 5z^3 + 10z - 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) - \frac{1}{z^5}$$

$$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

But $z^5 - \frac{1}{z^5} = 2\mathrm{i}\sin 5\theta,\ z^3 - \frac{1}{z^3} = 2\mathrm{i}\sin 3\theta,\ z - \frac{1}{z} = 2i\sin\theta$

$$\therefore 32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$$

$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$ is the required identity

PROBLEMS TO SOLVE

CHAPTER 1: COMPLEX NUMBERS

1. Find the real and imaginary parts of $\frac{1}{z+4}$ for $z \neq -4$ and show that $\text{Re}\left(\frac{1}{z+4}\right) = \frac{1}{8}$ when $|z| = 4$.

2. Prove that $\overline{z+w} = \overline{z} + \overline{w}$.

3. Sketch $|z - 3 + \text{i}| > |z + 5 + 5\text{i}|$.

4. Sketch $\arg z < -\frac{\pi}{4}$ and $\arg \text{i}z < -\frac{\pi}{4}$ on separate axes.

5. Find, in $a\,\text{Re}(z) + b\,\text{Im}(z) < c$ form, the locus rule for question 3.

6. Use the ordinary rules of algebra to solve $\frac{z - \text{i}}{z + \text{i}} = 1 + 3\text{i}$ for z.

7. Substitute $z = x + y\text{i}$ into $\frac{z - \text{i}}{\overline{z} - \text{i}} = 1 + 3\text{i}$ and solve for z.

8. Two solutions of a polynomial with real coefficients and highest degree 4 are $3 + 2\text{i}$ and $-5 - 3\text{i}$. Find the polynomial.

9. (a) Find by inspection a real solution of $z^3 + 8 = 0$ and hence a factor of $z^3 + 8$.
   (b) Find the quadratic factor of $z^3 + 8$.
   (c) Find the other two solutions of $z^3 + 8 = 0$ in exact form.
   (d) Change all of your solutions to exact polar form in degrees and graph them on the complex plane. Explain how the 3 solutions are grouped around the origin by mentioning magnitudes and angles.

10. Consider the set of points defined by $|z + 4 - 3\text{i}| \leq 2$.
    By sketching (and using a compass) find:
    (a) the maximum of $|z|$
    (b) the minimum of $|z|$
    (c) the maximum of $\arg z$
    (d) the minimum of $\arg z$.

11. Solve $z(2 - \text{i}) + 2\overline{z} = 3 + 4\text{i}$ for z .

12. Solve $\frac{z - \text{i}}{z + \text{i}} = \frac{1}{\text{i}} - 1$ for z .

13. (a) Show that $(z - 1 - \text{i})$ is a factor of $z^3 - 4z^2 + 6z - 4$.
    (b) Fully factorize $z^3 - 4z^2 + 6z - 4$ by showing all working.

14. Solve $z^2 + 2(\overline{z})^2 + z - \overline{z} + 9 = 0$ for z.

15. Given $(z - ai)$ is a factor of $P(z) = z^3 + z^2 + az + 1$. Find the possible values of a.

16. Sketch the region in the complex plane defined by the intersection of

$\text{Re } z \leq 2$, $\text{Im } z \leq 3$ and $\frac{-\pi}{4} \leq \text{Arg } z < \frac{3\pi}{4}$

17. Find the Cartesian equation of the locus of $|z + 3 - i| = |z - 1 + 5i|$.

18. Let $z = 1 - i$ to find:

(a) $z\bar{z}$ (b) $\frac{1}{z}$ (c) $|z|^7$ (d) $\text{Arg}(z^7)$

19. If $z = 1 + 2i$ is a solution of $z^3 + z + 10 = 0$, show how to find the other two solutions.

20. Find $\sqrt{3+4i}$.

21. Solve

(a) $x^2 + 2x + 6 = 0$

(b) $Z^2 + 5 = \frac{-4}{Z^2}$

22. Form the quadratic with roots $3 \pm 2i$ and a leading coefficient of 1.

23. Prove $\frac{6 \cos A + i6\sin A}{2 \cos A - i2\sin A} = 3Cis2A$.

24. If $Z_1 = 1 + i$ and $Z_2 = 2 - i$ then prove

(a) $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

(b) $\overline{Z_1 Z_2} = \overline{Z_1}.\overline{Z_2}$

25. If $Z = 4 - 5i$ then find in Cartesian form

(a) $\bar{Z}$

(b) $Z + \bar{Z}$

(c) $Z - \bar{Z}$

(d) $Z\bar{Z}$

(e) $\frac{Z}{\bar{Z}}$

26. Find $\frac{-2}{1 - i}$

27. Find in polar form

(a) $\dfrac{8Cis\frac{5\pi}{12}}{2Cis\frac{\pi}{6}}$

(b) $\dfrac{3Cis\frac{\pi}{12}}{12Cis\frac{\pi}{4}}$

28. (a) If $Z = 3 + 4i$ and $W = 2 - 3i$, then express in Cartesian form

i. Z^2

ii. $\overline{Z}$

iii. $iZ + 2W$

iv. $Z \div W$

v. $W\overline{W} + Z\overline{Z}$

(b) Prove $|W\overline{W}| = |W||\overline{W}|$

29. Find all complex numbers Z satisfying

$$\frac{1}{Z} + \frac{2}{\overline{Z}} = 1 + i$$

30. Sketch graphs in the complex plane to show the set of numbers Z which satisfy

(a) $|Z - 3 + i| \leq |Z + 2|$

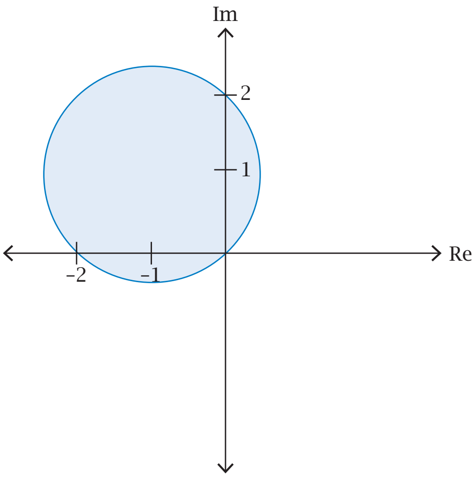
(b) $|Z| = \text{Im } Z$

(c) $0 \leq ArgZ \leq \frac{\pi}{4}$ and $\text{Re } Z \leq 2$

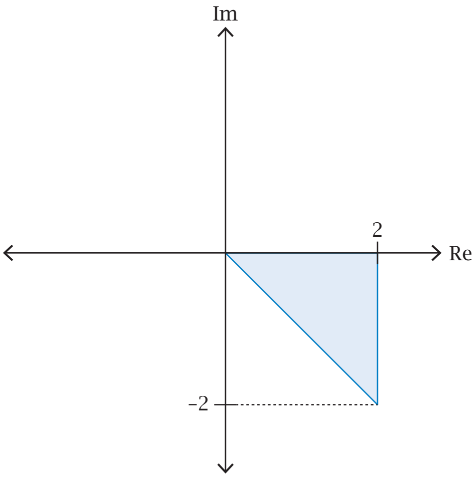
(d) $2 \leq |Z| \leq 3$

31. Determine the sets corresponding to the following shaded regions of the Argand plane.

(a)



(b)



32. If $Z = x + i\,y$

(a) find Re (Z^2)

(b) sketch the points on the Argand plane which satisfy Re $(Z^2) = 0$

33. (a) Given $W = i\,\overline{Z}$, find W if Z is

i. $2 + i$

ii. $-1 + 3\,i$

iii. $1.5 - 4\,i$

iv. $-2.5 - 2\,i$

(b) Comment on the relationship between the corresponding points represented by Z and W.

34. If $3 + i$ is one root of $x^3 - 8x^2 + 22x - 20 = 0$, then find all other roots.

35. $2i$ is a root of the equation $x^4 + 2x^3 + x^2 + 8x - 12 = 0$

Determine all the other roots.

36. Express in polar form.

(a) $2\,Cis\left(\frac{\pi}{4}\right) \times 3\,Cis\left(\frac{\pi}{3}\right)$

(b) $0.5\,Cis\left(-\frac{\pi}{2}\right) \times 2\,Cis\left(\frac{\pi}{6}\right)$

(c) $3\,Cis\left(\frac{2\pi}{3}\right) \div 2\,Cis\left(-\frac{\pi}{4}\right)$

(d) $6\,Cis\left(-\frac{\pi}{3}\right) \div 4\,Cis\left(-\frac{3\pi}{4}\right)$

(e) $\pi\,Cis\left(\frac{n\pi}{2}\right) \times 3\pi\,Cis\left(-\frac{n\pi}{3}\right)$

37. $Z_1 = 2\,Cis\frac{5\pi}{6}$ and $Z_2 = Cis\frac{2\pi}{3}$

(a) Find Z_1Z_2

(b) Convert answer (a) into Cartesian form.

(c) Change both Z_1 and Z_2 into Cartesian form.

(d) Use (c) to find Z_1Z_2 and compare this answer to (b).

38. (a) Given $W = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, find Z if $Z = W + 2$

(b) Find $|Z|$ exactly.

(c) Find arg Z exactly.

39. (a) Find in polar form

i. $1 + 2i$

ii. $(1 + 2i)^2$

iii. $(1 + 2i)^3$

iv. $(1 + 2i)^4$

(b) Plot the above (i)–(iv) on the same Argand diagram.

(c) Describe how the position of each point compares to that of the previous one.

40. The complex number Z satisfies $\frac{Z}{Z+4} = 3 + 2i$. Find

(a) Z

(b) $|Z|$

(c) arg Z

41. If $Z = \dfrac{2\,Cis\frac{\pi}{6} + 4\,Cis\frac{\pi}{2}}{\sqrt{2}\;Cis\frac{\pi}{4}}$ find

(a) Z in Cartesian form

(b) Mod Z

(c) Arg Z

42. (a) Evaluate both in polar form and cartesian form

$[\cos 60^\circ + i \sin 60^\circ][\cos(-45^\circ) + i \sin(-45^\circ)]$

(b) Use the above to show

$$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad \text{and} \quad \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(c) Find $\tan 15^\circ$

43. Find the conjugate of $\frac{2i}{3 - i}$

44. Simplify $\frac{1}{Z}\left(\frac{1}{Z+i} + \frac{1}{Z-i}\right)$

45. Two complex numbers Z and W are related by the formula $W = \frac{Z+1}{1-Z}$

(a) Find

i. W if $Z = 1.5\,i$

ii. $|W|$ if $Z = 1.5\,i$

(b) Use your graphics calculator to repeat part (a) for several values of Z, each of which represent points on the imaginary axis.

(c) Plot all of your values of W on an Argand diagram and comment on what W describes.

46. Z and W are complex numbers given by $Z = \frac{m}{1-i}$ and $W = \frac{n}{2+i}$ where m and n are real.

Find the values of m and n if $Z + W = 3\ i$.

47. Find the value of m if $\frac{2+m}{5-12i} + \frac{m-i}{13i}$ is real.

48. If $Z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ and $W = \frac{1-Z}{1+Z}$

(a) Find W

(b) Comment on the location of W on an Argand diagram.

49. Find the complex numbers W and Z such that

$Z + 2 = iW$

and $W - 3 = iZ$

50. If $Z_1 = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$ and $Z_2 = 1 + \left(\sqrt{2} - 1\right)i$

(a) Find

i. Z_1Z_2

ii. $Z_1^2Z_2$

(b) Show that Z_2, Z_1Z_2, $Z_1^2Z_2$ form an equilateral triangle on an Argand diagram.

(c) For any complex number Z_2 explain why Z_2, Z_1Z_2, $Z_1^2Z_2$ will always form an equilateral triangle.

51. If $2 + i$ and $1 - 3i$ are two of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ find the values of a, b, c, d, and e.

52. Find the distance between points with polar coordinates $(11, -70^\circ)$ and $(8, 80^\circ)$.

53. Find the two possible polar angles for the line segment of Question 52 which specify the inclination of the line.

54. Find the equation of the spiral(s), $r = k\theta$, which pass through the point where $r = 5$ and $\theta = \frac{-2\pi}{3}$ intersect. Only consider the spirals for $k > 1$ or $k < -2$.

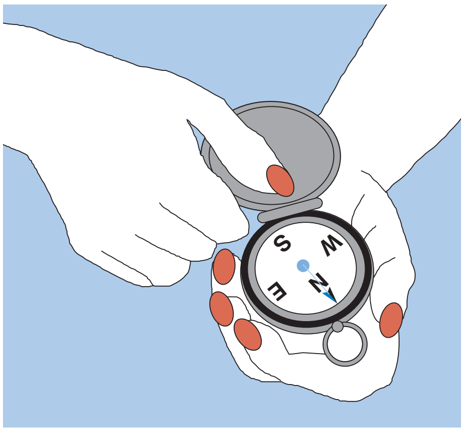
55. A line 10 cm long has a direction specified by the polar angle 100°. If one end is at $(7, 160^\circ)$ find the polar coordinates of the other end.

56. (a) Find, correct to 3 d.p., the point on the line of Question 55 which is closest to the origin.

(b) Find the point on the above line which is closest to $C\ (8, 70^\circ)$ and the closest distance.

57. (a) If (r,θ) is a variable point on line through $(8, 30^\circ)$ which has a inclination of -100° find $r = f(\theta)$.

(b) Use calculus techniques to find the closest distance that the line is to the orgin and fully justify your answer analytically.



58. If $z = 2\,\text{cis}\left(\frac{\pi}{6}\right)$ and $w = 2e^{\frac{2\pi}{3}i}$ find:

(a) w in exact Cartesian form

(b) zw in exact polar form

(c) z^{-1} in exact Cartesian form

(d) $\frac{z\bar{z}}{w}$ in exact polar form.

59. Sketch the locus of $\text{Re}(z - iz) < 2$ on the Argand plane.

60. Solve $z^4 = 1 - i$ and leave your answers in exact polar form.
i.e. $z = r\,\text{cis}\,\theta$ where $0 \leq \theta \leq 2\pi$.

61. Solve $z^3 - 4z - 15 = 0$ for z in exact Cartesian form. Do not use your calculator.

62. Find $(\sqrt{3} + i)^{-3}$ in exact form.

63. Show how to find all solutions to $z^4 = 3 - 4i$ in $x + yi$ form correct to 3 d.p.

64. If $z = \text{cis}\,\theta$ show that $z + z^{-1} = 2\cos\theta$ and $z^n + z^{-n} = 2\cos n\theta$ and then use your results to prove that $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$.

65. Simply without your calculator

$$\frac{\left(3\,\text{cis}\,\frac{3\pi}{4}\right)\left(8\,\text{cis}\,\frac{\pi}{3}\right)}{\left(2\,\text{cis}\,\frac{\pi}{6}\right)\left(6\,\text{cis}\left(\frac{-5\pi}{12}\right)\right)}$$

and express your result in polar form where $-\pi < \text{Arg}\,z \leq \pi$.

66. (a) Factorise $z^3 - z^2 - 6z + 18$ without your calculator and leave your answer in exact form.

(b) If $z_1 = \text{cis}\ \theta$, and $z_2 = \text{cis}\ \theta_2$, prove that $\overline{z}_1\ \overline{z}_2 = \text{cis}(-(\theta_1 + \theta_2))$

67. Simplify the expression and leave your answer in exact polar form

$$\left[\frac{\sqrt{3}\ \text{cis}\ \frac{3\pi}{4}}{6\ \text{cis}\ \frac{5\pi}{6}\ \text{cis}\ \frac{2\pi}{3}}\right]^{-1}$$

68. Solve $2\overline{z} + \text{i}z = 1 - 2\text{i}$ for z.

69. Show how to solve $z^3 = -8\text{i}$ for z and leave your answers in exact Cartesian form.

70. Use de Moivre's theorem to prove that
$\sin 5\theta = \sin\theta(5\cos^4\theta - 10\cos^2\theta\ \sin^2\theta + \sin^4\theta)$.

71. Solve $z^4 + 6z^2 + 25 = 0$ for z^2 and find the four roots in exact Cartesian form.

72. Solve $z^5 = -\text{i}$ for $-\pi < \theta \leq \pi$ and leave your answers in exact polar form.

73. (a) Find both of the square roots of –25 leaving them in $x + y\text{i}$ form.

(b) Hence, or otherwise, show that the solutions to $(z + 1)^2 + 25(z - 1)^2 = 0$ are $\frac{12 \pm 5\text{i}}{13}$.

2

FUNCTIONS AND CURVE SKETCHING

The study of Calculus begins with the basic concepts of functions which are explored in detail. The approach is informal and intuitive and the underlying ideas are illustrated wherever possible by graphs and sketches to provide understanding without overwhelming with technical detail.

Syllabus Checklist

By the end of this chapter, you should be able to:

Functions

- ☐ determine when the composition of two functions is defined
- ☐ determine the composition of two functions
- ☐ determine if a function is one-to-one
- ☐ find the inverse function of a one-to-one function
- ☐ examine the reflection property of the graphs of a function and its inverse

Sketching graphs

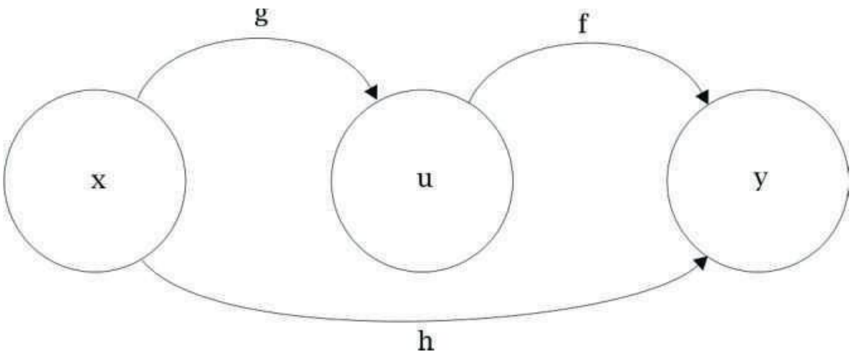
- ☐ use and apply $|x|$ for the absolute value of the real number x and the graph of $y = |x|$
- ☐ examine the relationship between the graph of

$$y = f(x) \text{ and the graphs of } y = \frac{1}{f(x)},\ y = |f(x)| \text{ and } y = f(|x|)$$

- ☐ sketch the graphs of simple rational functions where the numerator and denominator are polynomials of low degree

COMPOSITE FUNCTIONS

When x values are inputted into a function, say u = g (x) and the resulting u values are in turn inputted into another function, say y = f (u) the overall function y = h (x) is called a composite function. The **mapping** diagram will look like this:-



The substitution process and notation is as follows.

If $u = g(x)$

and $y = f(u)$

then $y = f(g(x))$

or $y = fog(x) = h(x)$

y = fog (x) is the notation when composing functions f and g with g before f.

A different composition of f and g is y = g(f(x)) = gof(x) which mostly will not be the same as y = fog(x)

Sometimes it may be necessary to include brackets

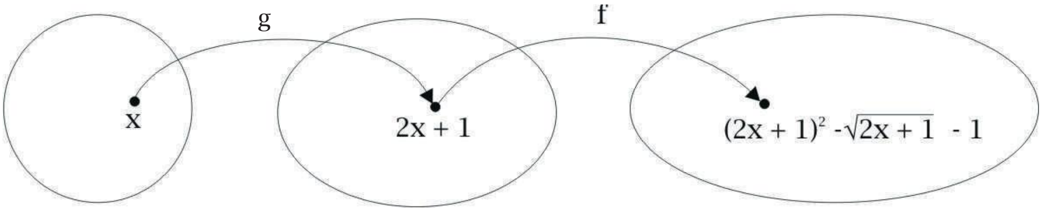
ie y=fog(x) can be written as y=(fog)(x)

Example If g(x)=2x+1 and $f(x) = x^2 - \sqrt{x} - 1$, find y = fog(x)

$$
\begin{aligned}
\text{then } y &= fog(x) \\
&= f(g(x)) \\
&= f(2x+1) \\
&= (2x+1)^2 - \sqrt{2x+1} - 1
\end{aligned}
$$

Important step

Regarding the **'important step'** it must be realised that wherever 'x' appears in the function f it is replaced by (2x+1) which is the function g. So for a value of x the **'Mapping chain'** goes: -



For most cases it is not necessary to include the variable u as originally mentioned but it could be included in this way.

If g(x) = 2x+1 and $y = f(x) = x^2 - \sqrt{x} - 1$, find y = fog(x)

$$
\begin{aligned}
&\text{Because } y = f(x) = x^2 - \sqrt{x} - 1 \\
&\quad \text{then } y = f(u) = u^2 - \sqrt{u} - 1 \text{ , now let } u = (2x+1) \\
&\quad \therefore \ y = f(2x+1) = (2x+1) - \sqrt{2x+1} - 1 \\
&\text{i.e. } fog(x) = (2x+1)^2 - \sqrt{2x+1} - 1 \quad \text{as before}
\end{aligned}
$$

This illustrates the important aspects of functions as mapping. The function rule f takes values which are in its domain, squares them and then subtracts the square root of them before finally subtracting 1 to get the y-values. It doesn't matter whether the values in the domain are called x or u!!

DOMAIN AND RANGE OF FUNCTIONS

For the purposes of this course the **domain** of a function f will be the set of values which are available to be mapped by f. Sometimes the domain will be stated explicitly as the second part of the function definition.

e.g. $y = f(x) = x^2 - 2x + 3$ for $-2 \le x \le 3$

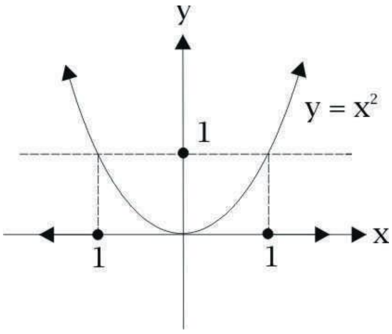
If no domain is specifically stated then it is assumed that the domain will be the **natural domain**.

Example: If $f(x) = \sqrt{x^2 - 1}$ then $x^2 - 1$ must be ≥ 0

ie $x^2 \geq 1$

and $x \geq 1$ or $x \leq -1$

is the natural domain



The range of a function is the set of values produced by the function rule from the values stated in the domain or implied from its natural domain.

Many students will automatically grab their graphics calculators when finding domains and ranges. However this method can take the unwary a long time and often a quick sketch as shown below will find the domain and thinking about how the function works will yield the range.

Example: The range for $f(x)=\sqrt{x^2-1}$ will be $y \geq 0$ as $x=\pm 1$ will give $y = 0$ and all the other values of $x > 1$ or $x < -1$ will give bigger values of y up to positive infinity.

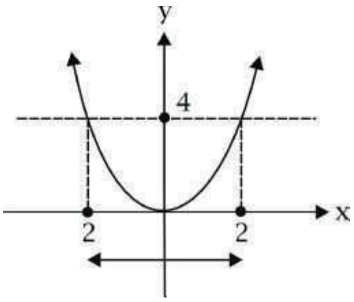
Example: Find the domain and range of $f(x)=\sqrt{4-x^2}$

Well $4 - x^2$ has to be ≥ 0 so $4 \geq x^2$ or $x^2 \leq 4$

ie $-2 \leq x \leq 2$

So D_x is $-2 \leq x \leq 2$

When $x=\pm 2$, $y=0$ and when $x=0$, $y=2$ so it seems quite likely that R_y is $0 \leq y \leq 2$.



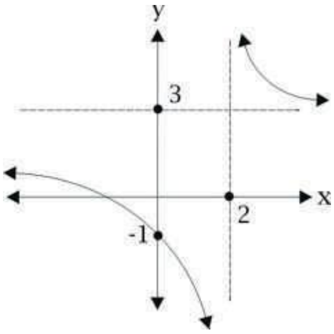
Regarding sketching functions, a common type of function is $f(x)=\dfrac{\text{linear}}{\text{linear}}$ ie $\dfrac{ax+b}{cx+b}$ and it is not hard to show that these are always shifted reciprocal types.

Example: Find the domain and range of $f(x)=\dfrac{3x+2}{x-2}$ and sketch the function.

The vertical asymptote is at $x = 2$. Let $x = 0$ gives $y = -1$ which is the y-intercept.

As x gets large $y \rightarrow \dfrac{3x}{x} = 3$.

So $y = 3$ is the horizontal asymptote.

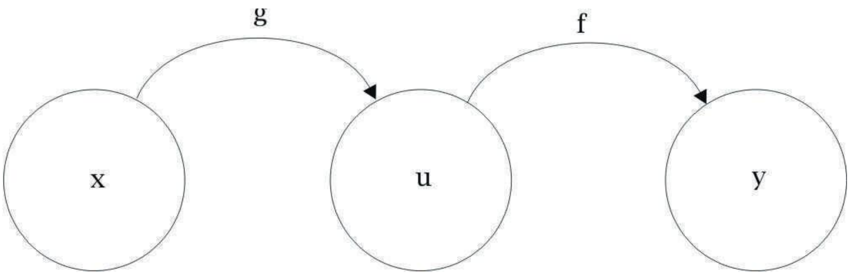


So D_x is $x \neq 2$, $x \in R$

R_y is $y \neq 3$, $y \in R$

DOMAIN AND RANGE OF COMPOSITE FUNCTIONS

When y = g(x) and y = f (x) are given and say the composite function y = fog (x) is needed. There can be problems with compatibility. The mapping diagram is



If the range of g, i.e. R_g , is not a subset or the same set as the domain of f , i.e. D_f , then the composite function y = fog (x) is said to be undefined

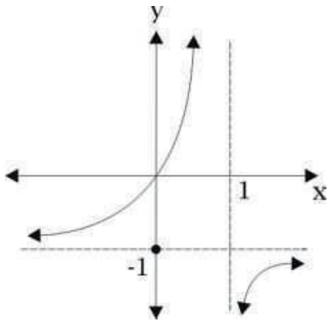
So for y = fog(x) to be defined

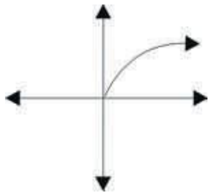
$$R_g \subseteq D_f$$

If $R_g \not\subseteq D_f$, for y = fog (x) to be defined, R_g will have to be restricted to equal or be a subset of D_f, which in turn is achieved by restricting D_g in an appropriate manner. The following layout is recommended.

Example: If $f(x)=\frac{x}{1-x}$ and $g(x)=\sqrt{x}$ find out whether y = gof (x) is defined. If it is defined, find its rule, domain and range. If it is not defined show how to restrict D_f and then find its rule, domain and range.

$$\begin{array}{c} D_f \\ x \neq 1 \end{array} \longrightarrow \boxed{f(x)=\frac{x}{1-x}} \longrightarrow \underbrace{\begin{array}{c} R_f \\ y\neq -1 \end{array} \qquad \begin{array}{c} D_g \\ x \geq 0 \end{array}} \longrightarrow \boxed{g(x)=\sqrt{x}} \longrightarrow R_g\ , y \geq 0$$





A problem exists here because $R_f \not\subseteq D_g$

So as it stands y = gof (x) is not defined.

For y = gof (x) to be defined R_f needs to be restricted to $y \geq 0$ to comply with D_g being $x \geq 0$. From the sketch of $f(x)=\frac{x}{1-x}$ for $y \geq 0$ it is easily seen that D_f must be restricted to $0 \leq x < 1$.

The rule $y = gof(x)$

$$\begin{aligned} &= g\big(f(x)\big) \\ &= g\left(\frac{x}{1-x}\right) \\ &= \sqrt{\frac{x}{1-x}} \end{aligned}$$

So D_{gof} is $0 \leq x < 1$

and R_{gof} is $y \geq 0$ ie the same as R_g

THE INVERSE OF A FUNCTION

The function y = f (x) takes x-values and maps them producing y-values. The function that takes y-values back to the x-values can be called the reverse function.

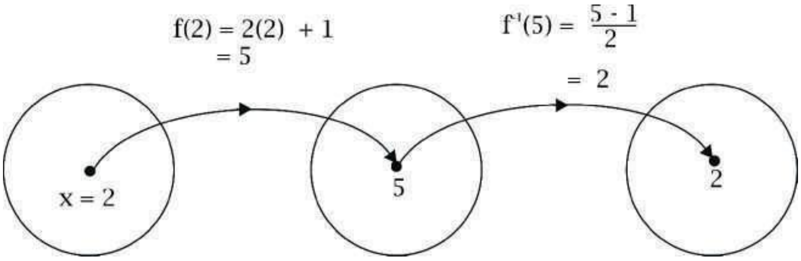
Example: If $y = f(x) = 2x + 1$

then $y - 1 = 2x$

and $x = \dfrac{y-1}{2}$

So $x = g(y) = \dfrac{y-1}{2}$ is the reverse function

If x and y are swapped the function $y = g(x) = \dfrac{x-1}{2}$ becomes the **inverse** function of y = f (x) which is notated as $y = f^{-1}(x)$ ie $f^{-1}(x) = \dfrac{x-1}{2}$. To see how f^{-1} works consider the mapping diagram below for x = 2.



This suggests the composition $f^{-1}of(x) = x$ which can be tested by

$$\begin{aligned} &f^{-1}of(x)\\ &= f^{-1}(f(x))\\ &= f^{-1}(2x+1)\\ &= \frac{2x+1-1}{2}\\ &= \frac{2x}{2}\\ &= x \quad \text{as expected}\end{aligned}$$

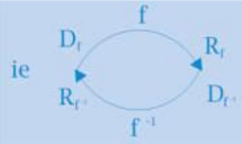
Also

$$\begin{aligned} &f\ o\ f^{-1}(x)\\ &= f(f^{-1}(x))\\ &= f\left(\frac{x-1}{2}\right)\\ &= 2\left(\frac{x-1}{2}\right) + 1\\ &= x - 1 + 1\\ &= x \quad \text{as expected}\end{aligned}$$

In conclusion then for inverse functions f and f^{-1}

$f^{-1}\ o\ f(x) = x$

$f\ o\ f^{-1}(x) = x$



And the range of f becomes the domain of f^{-1}

i.e. $R_f = D_{f^{-1}}$

and the range of f^{-1} becomes the domain of f

i.e. $R_{f^{-1}} = D_f$

The function rule for f^{-1} can either be found from the rule of f by:

- Solving y = f (x) for x, then swapping x and y
- Swapping x and y in y = f (x) and solving for y.

Conditions for f^{-1} to exist

For a relation to qualify as a function each value of x in the domain must map to only one y-value, but two or more x-values can map to the same y-value.

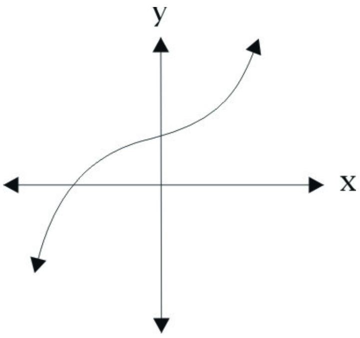
So any relation which qualifies as **many to one** is called a function.

e.g. for $f(x) = x^2$, $f(2) = 4$ and $f(-2) = 4$

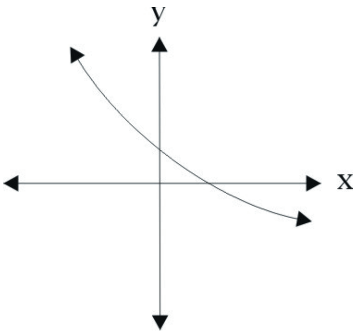
However this particular function of f cannot have an inverse as the value of 4 in the domain of f^{-1} would not know if it mapped to 2 or -2 !!

So for f to have an inverse f^{-1} it must be a **one-to-one** function

For **continuous** functions (i.e. ones which have no jumps or gaps) the one to one restriction means that to have an inverse it must be either **strictly increasing** or **strictly decreasing** as shown below.



Strictly increasing,
so has an inverse



Strictly decreasing,
so has an inverse

GRAPHS OF INVERSE FUNCTIONS

Because of the swapping of x and y when finding f^{-1} from f, the graphs of the two functions are reflections of each other about the line y = x.

Example. For $f(x)=\sqrt{x+2}$ graph f and f^{-1} on the same axes.

let $y=\sqrt{x+2}$ D_f $x \geq -2$, R_f $y \geq 0$

$y^2 = x+2$

$x = y^2 - 2 \rightarrow$ swap $y = x^2 - 2$

$\therefore f^{-1}(x) = x^2 - 2\ ,\ x \geq 0$

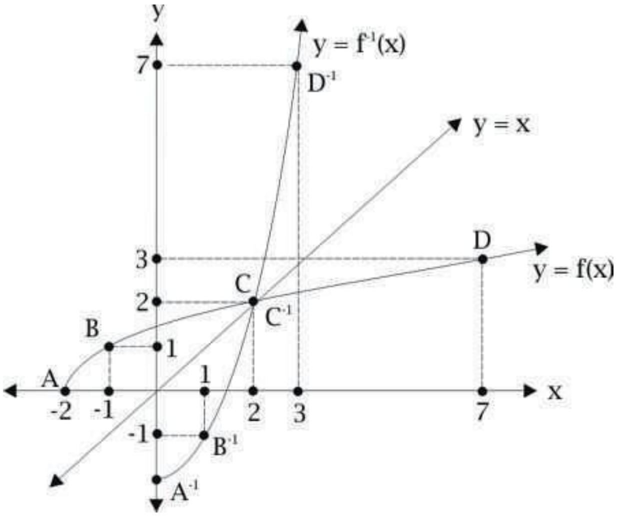
$R_f = D_{f^{-1}}$

For y = f (x)

	x	y
A	-2	0
B	-1	1
C	2	2
D	7	3

For $y = f^{-1}(x)$

	x	y
A^{-1}	0	-2
B^{-1}	1	-1
C^{-1}	2	2
D^{-1}	3	7



It should be noted that:

- If y = f (x) crosses the line y = x then so does $y = f^{-1}(x)$ at the same point.
- The graphs of y = f (x) and $y = f^{-1}(x)$ are mirror images of each other about y = x as seen by the pairs of points A and A^{-1}, B and B^{-1} etc.
- If a question asks you to algebraically solve the equation $f(x)=f^{-1}(x)$, you can always choose to solve the potentially easier equations of f (x) = x or $f^{-1}(x) = x$ as the solutions will be the same.

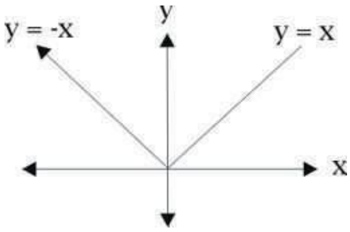
THE ABSOLUTE VALUE FUNCTION

The definition of the absolute value of x , i.e. |x| is

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

This is an example of a piecewise function

The graph of $y = |x|$ is



y = | x | has its vertex (corner "cusp") at the origin

A few important properties of the absolute value function are

- $|ab| = |a||b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|-x| = |x|$
- $|x - a| = |a - x|$ is the distance between x and a.
- $y = |ax + b| = \left|a\left(x + \frac{b}{a}\right)\right|$ has its vertex at $x = -\frac{b}{a}$ or simply equate the parenthetical function [inside] to zero and solve for x.

  e.g. $ax + b = 0$, $ax = -b$, $x = -\frac{b}{a}$ as before

Shifted absolute value types work in the usual way. Some examples are:-

$y=|x-2|$ is $y=|x|$ shifted 2 units right

$y=|x+3|$ is $y=|x|$ shifted 3 units left

$y=|2x|=2|x|$ has its vertex at the origin but the "right piece" has gradient m=2 and the "left piece" has gradient m=-2.

$y=-|3x|$ is inverted i.e. reflected in the x-axis, gradients of 3 and - 3.

$y=|x+1|-2$ shifted left 1 unit and down 2 units.

$$y=4-|2x+3|$$
$$=-|2(x+1.5)|+4$$ inverted, gradients of 2 and -2, shifted 1.5 left and up 4

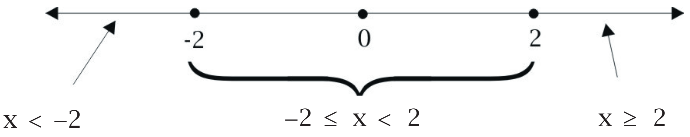
WRITING ABSOLUTE VALUE FUNCTIONS IN PIECEWISE FORM

Consider the function $y=|x-2|-\left|\frac{1}{2}x+1\right|$ and say that you were asked to rewrite it in piecewise form .

$$\text{ie}\quad y=\begin{cases}\text{piece 1, domain 1}\\ \text{piece 2, domain 2}\end{cases}$$

One way to do this is to split the x-axis into intervals, by first finding where the cross-over values are.

Step 1. Let x -2 = 0, i.e. x = 2 and let $\frac{1}{2}x+1=0$ i.e. $\frac{1}{2}x=-1$, i.e. , x = -2 which means that the function has corners (not vertices) at x = 2 and x = -2. The x-axis is split as follows.



Step 2. For $x\ge 2$, both $|x-2|$ and $\left|\frac{1}{2}x+1\right|$ are positive so the absolute value symbols are not needed.

$$\therefore\ y=x-2-\left(\frac{1}{2}x+1\right)$$
$$=x-2-\frac{1}{2}x-1$$
$$=\frac{1}{2}x-3$$

Step 3. For $-2\le x<2$, $|x-1|$ is negative so $|x-2|=-(x-2)$, but $\left|\frac{1}{2}x+1\right|$ is still positive

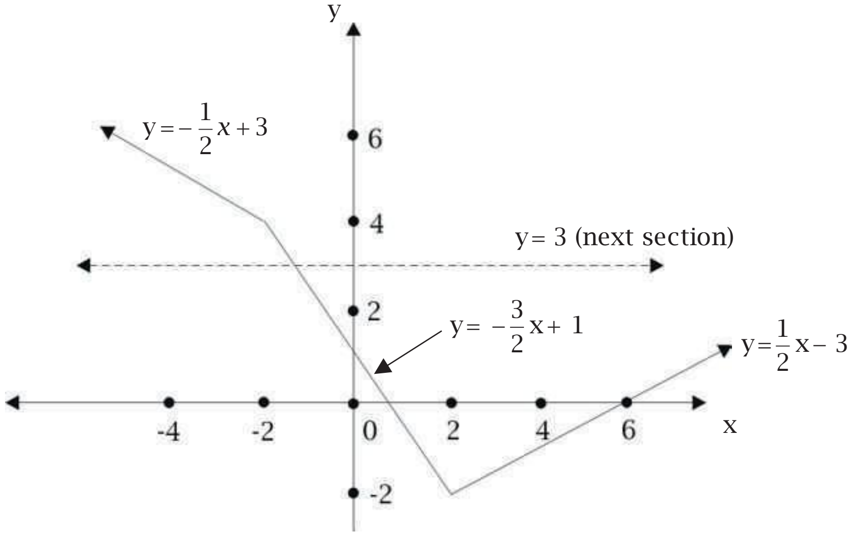
$$\therefore\ y=-(x-2)-\left(\frac{1}{2}x+1\right)$$
$$=-x+2-\frac{1}{2}x-1$$
$$=-\frac{3}{2}x+1$$

Step 4. For $x<-2$, both $|x-2|$ and $\left|\frac{1}{2}x+1\right|$ are negative

$$\therefore\ y=-(x-2)--\left(\frac{1}{2}x+1\right)$$
$$=-x+2+\frac{1}{2}x+1$$
$$=-\frac{1}{2}x+3$$

$$\therefore y = \begin{cases} \frac{1}{2}x - 3, & x \geq 2 \\ -\frac{3}{2}x + 1, & -2 \leq x < 2 \\ -\frac{1}{2}x + 3, & x < -2 \end{cases}$$ is the required piecewise function.

The graph of the function is



SOLVING ABSOLUTE VALUE EQUATIONS

- Using the previous example if a question asked you to solve the equation $|x - 2| - \left|\frac{1}{2}x + 1\right| = 3$, the graph indicates two solutions which are found by choosing the two pieces as follows:

$$-\frac{3}{2}x + 1 = 3 \quad \text{and} \quad \frac{1}{2}x - 3 = 3$$

$$1 - 3 = \frac{3}{2}x \qquad \frac{1}{2}x = 3 + 3$$

$$-2 = \frac{3}{2}x \qquad \frac{1}{2}x = 6$$

$$-\frac{4}{3} = x \qquad x = 12$$

$$x = -1\frac{1}{3} \qquad \therefore \text{ the required solutions are } x = -1\frac{1}{3},\ x = 12$$

- Without the benefit of the graph use the following method to solve $|x - 2| - \left|\frac{1}{2}x + 1\right| = 3$

Step 1. let $x \geq 2$, then $x - 2 - \left(\frac{1}{2}x + 1\right) = 3$

$$x - 2 - \frac{1}{2}x - 1 = 3$$

$$\frac{1}{2}x = 6$$

$$x = 12 \quad \text{which agrees with } x \geq 2$$

Step 2. let $-2 \le x < 2$ then $-(x-2)-\left(\frac{1}{2}x+1\right) = 3$

$$-x+2-\frac{1}{2}x-1 = 3$$

$$-\frac{3}{2}x = 3+1-2$$

$$-\frac{3}{2}x = 2$$

$$x = -\frac{4}{3} \text{ which agrees } -2 \le x < 2$$

Step 3. let $x < -2$ then $-(x-2)--\left(\frac{1}{2}x+1\right) = 3$

$$-x+2+\frac{1}{2}x+1 = 3$$

$$-\frac{1}{2}x = 3-1-2$$

$x = 0$ which doesn't agree with $x < -2$
so it is rejected

The two solutions are $x = 12$, $x = -\frac{4}{3}$ as before.

SOLVING ABSOLUTE VALUE INEQUALITIES

Example

Solve $|2x-5|+|x+3| \ge 10$

Let $x \ge 2.5$, $2x-5+x+3 \ge 10$

$3x-2 \ge 10$

P $\nearrow$ $\therefore\ x \ge 4$ so the intersection of $x \ge 2.5$
and $x \ge 4$ is $x \ge 4$

let $-3 \le x < 2.5$, $-(2x-5)+x+3 \ge 10$

$-2x+5+x+3 \ge 10$

P $\nearrow$ $-x+8 \ge 10$

$-x \ge 2$

$x \le -2$ so the required intersection is $-3 \le x \le -2$

let $x < -3$, $-(2x-5)+-(x+3) \ge 10$

$-2x+5-x-3 \ge 10$

P $\nearrow$ $-3x+2 \ge 10$

$x \le -2\frac{2}{3}$ so the required intersection is $x < -3$

What does this really mean? Well, valid solutions are:

$$\underbrace{x < -3 \text{ or } -3 \le x \le -2} \text{ or } x \ge 4$$

These two combined become $x \le -2$,

The required solution is $x \le -2$ or $x \ge 4$

A few notes here are appropriate.

- Solving absolute value inequalities is not easy and having a bit of trouble is normal!
- The 3 lines marked with a P are the piecewise equivalents to the absolute value statements over the stated domains.
- When stuck, draw graphs of $y = |2x - 5| + |x + 3|$ and $y = 10$, or better still, $y = 3x - 2$, $y = - x + 8$, $y = - 3x + 2$ and $y = 10$ to check on exactly what's going on.

TRANSFORMATIONS OF FUNCTIONS

- If the graph of $y = f(x)$ (called the **object** function) is known, how would the graph of $y = af(b(x - c)) + d$ (called the **image** function) look for various values of the constants a,b,c and d? Well, the two graphs have the same sort of shape but the image function can be translated in the x-direction, dilated in the x-direction and reflected in the y-axis and similarly but quite independently affected in the y-direction.
- The image function variables are assigned as x' and y' to distinguish them from the object function variables of x and y.

So	**Image function**	**Object function**
	$y' = af\left(b\left(x' - c\right)\right) + d$	$y = f(x)$
So	$\dfrac{y' - d}{a} = f\left(b\left(x' - c\right)\right)$	
But	$y = f(x)$	
	$\therefore\ y = \dfrac{y' - d}{a}$	$x = b\left(x' - c\right)$

And

For $y = af\left(b\left(x - c\right)\right) + d$

$$y' = ay + d \qquad x' = \frac{1}{b}x + c$$

are called the transformation equations

This means that when the image function is written as $y = af\left(b\left(x - c\right)\right) + d$ the order for x is to dilate (in the x-direction) by a scale factor of $\dfrac{1}{|b|}$ (and reflect in the y-axis if $b < 0$) **before** translating c units right for $c > 0$ or $|c|$ units left for $c < 0$.

The order for y is to dilate (in the y-direction) by a scale factor of $|a|$ (and reflect in the x-axis if $a < 0$) **before** translating d units up for $d > 0$ and $|d|$ units down for $d < 0$.

- When the image function is written in the form $y = af\left(bx - c\right) + d$ then the transformation equations are $y' = ay + d$ and $x = bx' - c$ or $x' = \dfrac{x + c}{b}$.

For $y = af\left(bx - c\right) + d$ the transformation equations are

$$y' = ay + d \quad \text{and} \quad x' = \frac{x + c}{b}$$

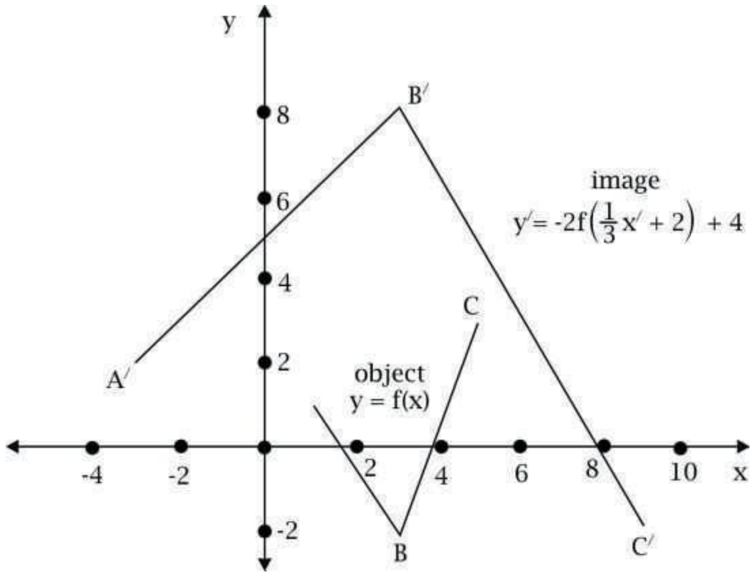
This means that the order for y is the same as above but the order for x is to translate (in the x-direction) by c units right for $c > 0$ or $|c|$ units left for $c < 0$ **before** dilating by $\dfrac{1}{|b|}$ and reflecting in the y-axis if $b < 0$.

Example Describe how the graph of $y = -2\,f\left(\frac{1}{3}(x + 6)\right) + 4$ can be found from the graph of $y = f(x)$ as above.

x first. Multiply each x-value of $y = f(x)$ by 3 (because of the dilation factor of $\frac{1}{3}$) and shift this new value left 6 units (because of the +6 in brackets) gives the new x-value.

y second. For the point you chose above on $y = f(x)$ multiply its y-value by -2 (ie a reflection as well in the x-axis) and then shift it up 4 units.

Now demonstrate for yourself that these processes work for the object function $y = f(x)$ composed of the shape ABC below.



Note. While you should be able to get the image graph using the x- and y-transformations, if you get lost each point can be individually transformed by using the transformation equations.

So for $y' = -2\,f\left(\frac{1}{3}x' + 2\right) + 4$

then $y' = -2\,y + 4$ and $\frac{1}{3}x' + 2 = x$

$$x' = 3(x - 2)$$
$$x' = 3x - 6$$

A table of values works best

$x' = 3x - 6$ ←		x	y	→ $y' = -2y + 4$
-3	A	(1	1)	2
3	B	(2	-2)	8
9	C	(3	3)	-2

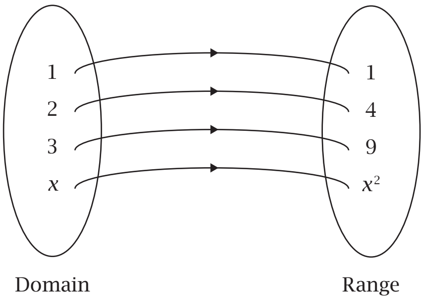
So plot $A'(-3, 2)$ $B'(3, 8)$ and $C'(9, 2)$ and join them up!!

FORMULA AND DEFINITIONS

Functions

- a function is a rule of correspondence between two sets of numbers; one called the DOMAIN of the function; the other called the RANGE of the function.

  For example $f = \{(x, y): y = x^2\}$ or $y = x^2$ or $f(x) = x^2$



$f(1) = 1$

$f(2) = 4$

.

.

.

$f(x) = x^2$

For each domain value, there is 1, and only 1, range value.

Polynomial Functions

- a function of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ is a polynomial function of degree n.

  $a_n, a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are real constants, called coefficients of the polynomial P.

- Remainder Theorem

  If $P(x)$ is divided by $x - a$, a polynomial of degree 1, then the remainder is $P(a)$.

- Factor Theorem

  If $P(x)$ is divided by $x - a$ such that $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

  This theorem is used to factorise polynomials into linear factors.

  Note: the domain and range of a polynomial function will always be the set of Real Numbers, unless stated otherwise.

Division of Polynomials

When sketching cubic or other higher order functions, it is useful to factorise polynomials so that the x-intercept(s) can be found.

To begin the factorisation process, we can divide one polynomial, $P(x)$, by another, $D(x)$. The result of this division can be expressed as:

$$P(x) = Q(x) + \frac{R(x)}{D(x)}$$

where $Q(x)$ is the quotient
$R(x)$ is the remainder
$D(x)$ is the divisor

Two common methods used to factorise polynomials are (i) long division and (ii) synthetic division. These methods are illustrated in Worked Examples 2.17 and 2.18.

The Remainder Theorem

When $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$

When $P(x)$ is divided by $(ax + b)$, the remainder is $P\left(\frac{-b}{a}\right)$

The Factor Theorem

If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$

If $P\left(\frac{-b}{a}\right) = 0$, then $(ax + b)$ is a factor of $P(x)$

Rational Functions

- A rational function R is a quotient of 2 polynomial functions.

$$R(x) = \frac{P(x)}{Q(x)} \qquad Q(x) \neq 0$$

- to find the zeroes of $R(x)$, we solve $P(x) = 0$
- to find the poles of $R(x)$, we solve $Q(x) = 0$
- the domain of $R(x)$ contains all the real numbers x, **excluding** the poles.
- the zeroes are x-intercepts and the poles are vertical asymptotes.

Piecewise Defined Functions

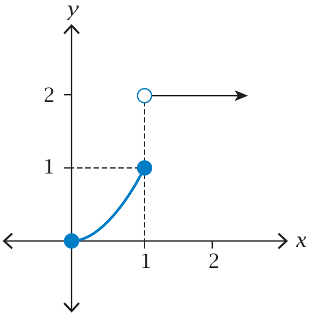
- these functions are defined by different rules over different subdomains.

Example 1:

$$R(x) = \begin{cases} x^2 \ldots 0 \leq x \leq 1 \\ 2 \ldots x > 1 \end{cases}$$

Domain $= \{x: x \geq 0\}$

Range $= \{y{:}0 \leq y \leq 1 \text{ or } y = 2\}$

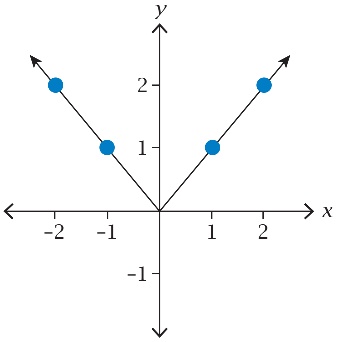


Example 2: Absolute Value Function

$$|x| = f(x) = \begin{cases} x \ldots x \geq 0 \\ -x \ldots x < 0 \end{cases}$$

Domain = {Real Numbers}

Range = {Non negative Real Numbers}

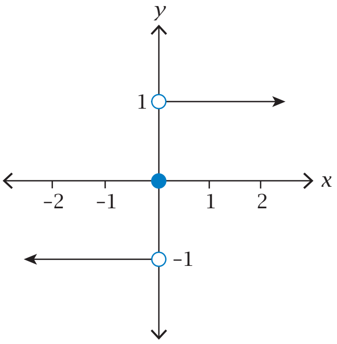


Example 3: Sign Function

$$\operatorname{sgn}(x) = \begin{cases} 1 \ldots x > 0 \\ 0 \ldots x = 0 \\ -1 \ldots x < 0 \end{cases}$$

Domain = {Real Numbers}

Range $= \{y\text{: y} = -1, 0, \text{or } 1\}$

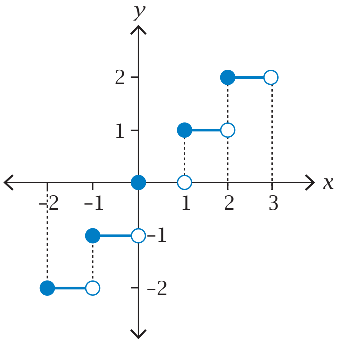


Example 4: Greatest Integer Function

$[|x|] = \text{int}(x) =$ the greatest integer less than or equal to x.

Domain = {Real Numbers}

Range = {Integers}



Limits

◇ Consider $f(x) = \dfrac{x^2 - 1}{x - 1}$

Now $f(1) = \dfrac{0}{0}$, which is undefined.

The domain of f is $Df = \{x: x \in R, x \neq 1\}$

Let x approach the value 1, written as $x \to 1$

	x below 1			x above 1		
x	0.9	0.99	0.999	1.1	1.01	1.001
y	1.9	1.99	1.999	2.1	2.01	2.001

As $x \to 1$ $\quad f(x) \to 2$
We write $\quad \lim_{x \to 1} f(x) = 2$

Note: The spirit of the Specialist Mathematics course suggests that all limit questions can be done using a calculator and a table of values. The following methods are suggested short cuts to computation and provide alternative methods.

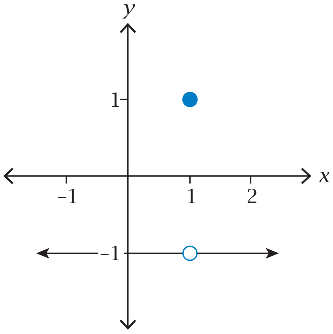
Limit methods:

1. Direct substitution $\lim_{x \to 2} (x + 6) = 2 + 6 = 8$
2. Factorisation for Rational Functions

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{(x - 1)} \qquad x \neq 1 \\ &= \lim_{x \to 1} (x + 1) \\ &= 2 \end{aligned}$$

3. Piecewise Functions: use left-hand and right-hand limits.

Example: $f(x) = \begin{cases} 1 \ \ldots \ x = 1 \\ -1 \ \ldots \ x \neq 1 \end{cases}$



$\lim_{x \to 1^+} f(x) = -1$ (right limit)

$\lim_{x \to 1^-} f(x) = -1$ (left limit)

$\therefore \lim_{x \to 1} f(x) = -1$ as the left hand limit is equal to the right hand limit.

4. Infinite limits.

(a) $\lim_{x \to \infty} \dfrac{1}{x} = 0$ as $\lim_{x \to +\infty} \dfrac{1}{x} = \lim_{x \to -\infty} \dfrac{1}{x} = 0$

(b) $\lim_{x \to \infty} \dfrac{4x^2 + x}{3x^2 - 1} = \lim_{x \to \infty} \dfrac{4 + \frac{1}{x}}{3 - \frac{1}{x^2}}$ (divide each term by the highest power of the variable)

$= \dfrac{4}{3}$

Derivatives of Functions Using Limits

(a) For a function $y = f(x)$, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

(b) General case: If $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n\right) - x^n}{h}$$ binomial expansion on $(x + h)^n$

$$= \lim_{h \to 0} \frac{h\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}\right)}{h}$$

$$= nx^{n-1}$$

Continuous functions

- A function is continuous if it has no gaps or breaks
- A function $f(x)$ is continuous at a point where $x = a$ if

(i) $f(a)$ exists

and (ii) $\lim\limits_{x \to a} f(x)$ exists

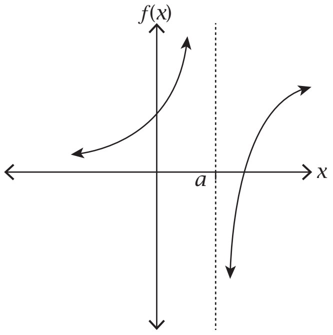
and (iii) $\lim\limits_{x \to a} f(x) = f(a)$ are all true.

Note: $\lim\limits_{x \to a} f(x)$ exists if $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$

i.e. the right hand limit equals the left hand limit.

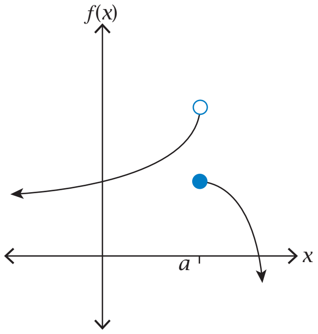
- Types of Discontinuities

(i) Infinite



discontinuous at $x = a$ since $f(a)$ does not exist

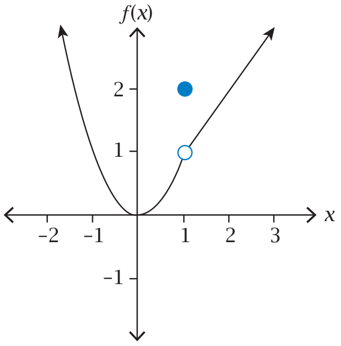
(ii) Jump



discontinuous at $x = a$ since $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

(iii) Removable

$$f(x) = \begin{cases} x \ldots x > 1 \\ 2 \ldots x = 1 \\ x^2 \ldots x < 1 \end{cases}$$



discontinuous at $x = 1$ since $\lim_{x \to 1} f(x) \neq f(1)$ i.e. $1 \neq 2$

the discontinuity can be **removed** by redefining $f(x) = \begin{cases} x \ldots x > 1 \\ 1 \ldots x = 1 \\ x^2 \ldots x < 1 \end{cases}$

Differentiability

◇ A function is differentiable if it is smooth; that is, no breaks or sharp corners.

◇ A function is differentiable at a point if it possesses a derivative at the point.

◇ For a function $f(x)$ to be differentiable at a point where $x = a$ then

(i) $f(x)$ must be continuous at $x = a$.
and
(ii) the left hand derivative at the point must equal the right hand derivative at the same point.
i.e. $f_-'(a) = f_+'(a)$
must both be true.

Worked Examples

2.1 If $= f(x) = \dfrac{x}{1-\sqrt{x}}$ and $g(x) = 9 - 2x^2$ find

(a) The domain and range of f and g.

(b) Whether y = fog (x) exists and explain why or why not.
If it does exist find D_{fog} and R_{fog}
If it doesn't exist find the minimum restrictions needed to make it exist and then find the new D_{fog} and R_{fog}.

(c) Repeat part (b) for y=gof(x).

(a) If $= f(x) = \dfrac{x}{1-\sqrt{x}}$ then $x > 0$ because $\sqrt{\text{negative numbers}}$ not allowed

and $x \neq 1$ because 0 on the denominator not allowed

$\therefore D_f$ is $x \geq 0 , x \neq 1 , x \in \mathbb{R}$

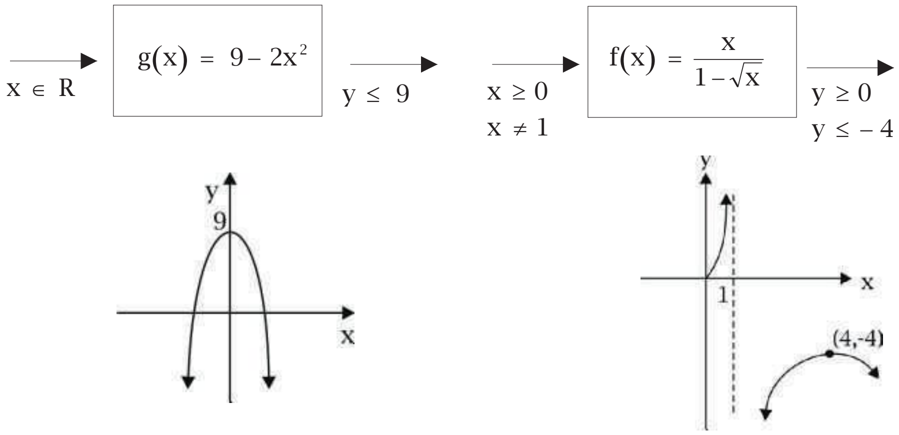
The range is not easy to get algebraically; however, the graphics calculator shows that:

R_f is $y \geq 0 , y \leq -4 , y \in \mathbb{R}$

D_g is $x \in \mathbb{R}$ and R_g is $y \leq 9$

The graphs below show the functions f and g

(b) Does y = f (g (x)) exist?



y = f (g (x)) does not exist as $R_g \not\subseteq D_f$, i.e. there are values produced by g which cannot be mapped by f. An example is g (3) = –9 and f (–9) does not exist.

If the range of g is restricted to $0 \le y \le 9$, $y \ne 1$, then f will be happy.

$$\begin{aligned} \text{But if } g(x) &\ne 1 \\ 9 - 2x^2 &\ne 1 \\ 8 &\ne 2x^2 \\ 4 &\ne x^2 \\ x &\ne \pm 2 \end{aligned}$$

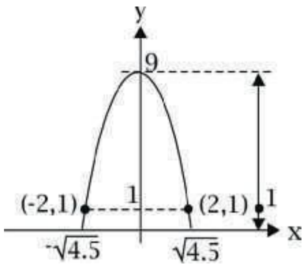
The intercepts of g are $9 - 2x^2 = 0$

$$2x^2 = 9$$

$$x = \pm\sqrt{4.5}$$

$\therefore D_{fog}$ is $-\sqrt{4.5} \le x \le \sqrt{4.5}$

$x \ne \pm 2$, $x \notin R$



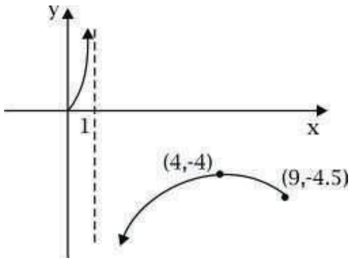
New R_g

$0 \le y \le 9$

$y \ne 1$

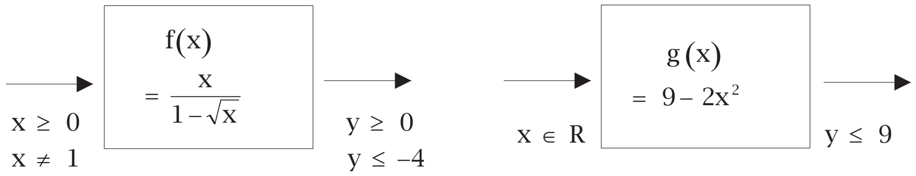
The domain that f now sees will be

$0 \le x \le 9$, $x \ne 1$. Because f (9) = –4.5, the graph of f will now be



and hence R_{fog} is unchanged from R_f, i.e. R_{fog} is $y \ge 0$, $y \le -4$, $y \notin \mathbb{R}$.

(c) Does y = g (f (x)) exist?



Because $R_f \subset D_g$ as D_g is $x \in \mathbb{R}$, y= g (f (x)) exists over the domain of f which will become the domain of gof.

$\therefore D_{gof}$ is $x \geq 0 , x \neq 1 , x \in \mathbb{R}$

The new domain for g is $x \geq 0 , x \leq -4$

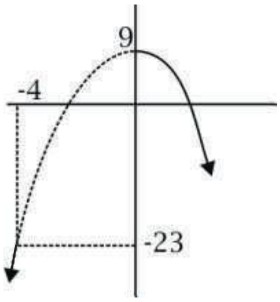
and $g(-4) = 9 - 2(-4)^2$

$= 9 - 2 \times 16$

$= -23$

So while part of g is missing, this does not affect the range and hence the range of g is the range of gof.

i.e. R_{gof} is $y \leq 9 , y \in \mathbb{R}$



2.2 If $f(x) = 3 - \sqrt{4-x}$ find $y = f^{-1}(x)$

let $y = 3 - \sqrt{4-x}$

$x = 3 - \sqrt{4-y}$ swap x and y

$\sqrt{4-y} = 3 - x$

$4 - y = (3-x)^2 = (x-3)^2$

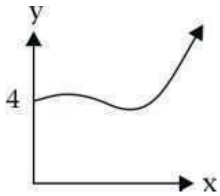
$4 - (x-3)^2 = y$

D_f	$x \geq 4$
$x = 4$	$y = 3$
$x = 3$	$y = 2$
$x = 0$	$y = 1$
R_f	$y \leq 3$

$\therefore f^{-1}(x) = 4 - (x - 3)^2, x \leq 3$

Note that the domain must be included as part of the function rule.

2.3 For $f(x) = 4 - x^2 + 2^x$ find whether or $y = f^{-1}(x)$ exists

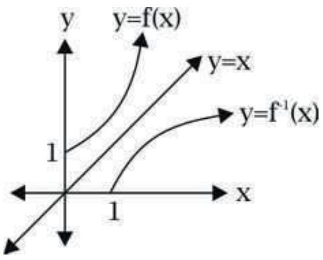


The graph of f is clearly not 1 to 1 so no $y = f^{-1}(x)$ does not exist.

2.4 Find f^{-1} for $y = \sqrt{x} + 2^x$ and graph both functions.

As x increases so does y, so f is 1 to 1 and $y = f^{-1}(x)$ exists.

Swap x and y gives $x = \sqrt{y} + 2^y$ which cannot be explicitly solved for y; however, f^{-1} is specified as $y = f^{-1}(x) : x = \sqrt{y} + 2^y$



2.5 Solve algebraically |x| (x – 4) + 1 = 0

For $x \geq 0$ $\quad x(x-4)+1 = 0$

$$x^2 - 4x + 1 = 0$$

$$x = 0.268 , x = 3.732 \quad 3 \text{ d.p.}$$

For $x < 0$ $\quad -x(x-4)+1 = 0$

$$-x^2 + 4x + 1 = 0$$

$$x = -0.236 , \; x = 4.236 \; 3 \text{ d.p.} \quad \text{reject as not} \; < 0$$

∴ Solutions are x=-0.236, 0.268, 3.732

2.6 Solve algebraically |3 – 2x| + |x – 4| < 10

Corners are where 3 - 2x = 0, x - 4 = 0
i.e. at x = 1.5, x = 4

Change the inequality to $|2x-3|+|x-4| = 10$

For $x \geq 4$, $\quad 2x-3+x-4 = 10$

$$3x = 17$$

$$x = 5\frac{2}{3} \quad \text{OK}$$

For $1.5 \leq x < 4$, $\quad 2x-3-(x-4) = 10$

$$2x-3-x+4 = 10$$

$$x = 9 \quad \text{reject}$$

For $x < 1.5$, $\quad -(2x - 3) + -(x-4) = 10$

$$-2x+3-x+4 = 10$$

$$-3x = 3$$

$$x = -1 \quad \text{OK}$$

$$\therefore x = -1 \quad \text{or} \quad x = 5\frac{2}{3}$$

Check $x = 0$ in $|3-2x|+|x-4| > 10$

$$3+4<10 \quad \text{True} \qquad \therefore \text{ Solution is } -1<x<5\frac{2}{3}$$

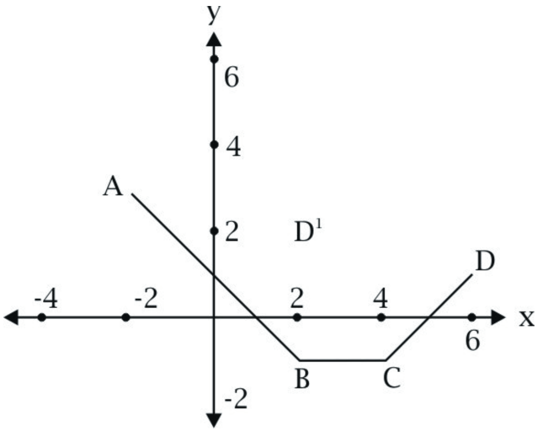
2.7 (a) A function y = f (x) is transformed to y = –2f (2x + 3) + 4. Explain what transformations have taken place.

$$y = -2f\big(2(x+1.5)\big)+4$$

x is dilated in the direction by scale factor $\times\frac{1}{2}$ and then this x is shifted 1.5 left

y is dilated in the direction by scale factor ×2 and then this y is reflected in the x-axis before being translated 4 units in the positive y-direction.

(b) If y = f (x) is shown below, draw the new function



Note that in practice it is easier to use the transformation equations.

i.e. Let y = -2f (2x + 3) + 4 be written as $y^{/} = -2f\left(2x^{/} + 3\right) + 4$

Comparing to y = f (x) gives $y^{/} = -2y + 4$

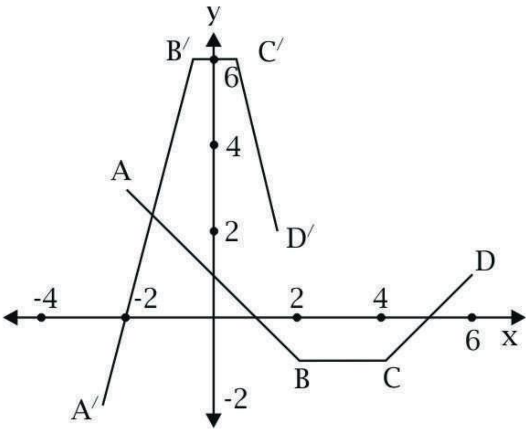
$x = 2x^{/} + 3$

or $x^{/} = \frac{x-3}{2}$

A table of values showing both the original points and transformed points is

	$x^{/} = \frac{x-3}{2}$	$\longleftarrow$ (x , y) $\longrightarrow$	$y^{/} = -2y+4$	
$A^{/}$	-2.5	A (-2 3)	-2	$A^{/}$
$B^{/}$	-0.5	B (2 -1)	6	$B^{/}$
$C^{/}$	0.5	C (4 -1)	6	$C^{/}$
$D^{/}$	1.5	D (6 1)	2	$D^{/}$

Graph these points



2.8 If $f(x) = \sqrt{4 - x^2}$ then find

(a) $f\left(\frac{1}{2}\right)$

(b) x if $f(x) = \frac{1}{2}$

(c) domain of $f(x)$

(d) range of $f(x)$

(a) $$f\left(\frac{1}{2}\right) = \sqrt{4 - \left(\frac{1}{2}\right)^2}$$
$$= \sqrt{4 - \frac{1}{4}}$$
$$= \frac{\sqrt{15}}{2}$$

(b)
$$f(x) = \frac{1}{2}$$
$$\sqrt{4 - x^2} = \frac{1}{2}$$
$$4 - x^2 = \frac{1}{4} \quad \text{(squaring)}$$
$$x^2 = \frac{15}{4}$$
$$x = \pm\frac{\sqrt{15}}{2}$$

(c) we require $4 - x^2 \geq 0$

i.e. $x^2 \leq 4$

$\therefore -2 \leq x \leq 2$

$\therefore D_f = \{x: -2 \leq x \leq 2\}$

(d) Since a $\sqrt{\ }$ is never negative,

$R_f = \{y: y \geq 0\}$

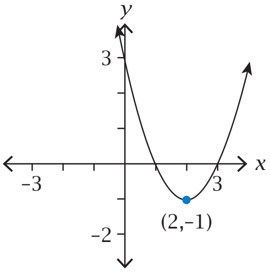
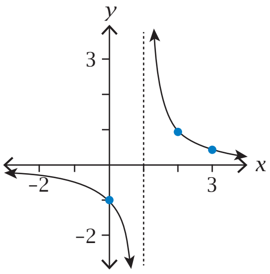
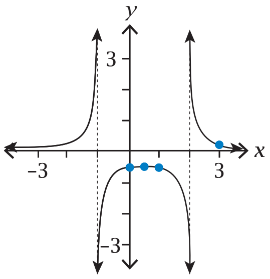
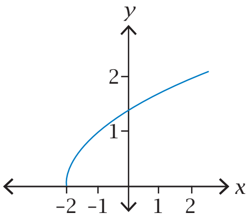
2.9 Sketch the following functions and give the domains and range in each case.

(a) $f(x) = (x - 2)^2 - 1$

(b) $g(x) = \frac{1}{x - 1}$

(c) $h(x) = \frac{1}{(x - 2)(x + 1)}$

(d) $k(x) = \sqrt{x + 2}$

	FUNCTION	DOMAIN	RANGE	SKETCH
(a)	$f(x) = (x - 2)^2 - 1$	{Real Numbers}	$\{y : y \geq -1\}$ since the turning point is (2, −1)	
(b)	$g(x) = \dfrac{1}{x - 1}$	$\{x : x \neq 1\}$ since $x - 1 \neq 0$	$\{y : y \neq 0\}$ read from graph	
(c)	$h(x) = \dfrac{1}{(x - 2)(x + 1)}$	$\{x : x \neq 2 \text{ or } x \neq -1\}$ since $x - 2 \neq 0$ and $x + 1 \neq 0$	$\left\{y : y > 0 \text{ or } y \leq -\dfrac{4}{9}\right\}$ read from graph	 Turning point $\left(\dfrac{1}{2}, -\dfrac{4}{9}\right)$
(d)	$k(x) = \sqrt{x + 2}$	$\{x : x \geq -2\}$	$\{y : y \geq 0\}$ since a $\sqrt{}$ is never negative	

2.10 Evaluate the following limits, if they exist.

(a) $\lim\limits_{x \to 2} \dfrac{x^2 + 4}{x^3 + 8}$

(b) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x^3 - 8}$

(c) $\lim\limits_{x \to \infty} \dfrac{x^2 - 4}{x^3 - 8}$

(d) $\lim\limits_{h \to 0} \dfrac{\sqrt{4 + h} - 2}{h}$

(a)

$$\lim_{x \to 2} \frac{x^2 + 4}{x^3 + 8} \quad \text{direct substitution}$$

$$= \frac{4 + 4}{8 + 8}$$

$$= \frac{1}{2}$$

(b)

$$\lim_{x\to 2}\frac{x^2-4}{x^3-8}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)} \quad \text{factorise to cancel the pole}$$

$$=\lim_{x\to 2}\frac{x+2}{x^2+2x+4}$$

$$=\frac{2+2}{4+4+4}$$

$$=\frac{1}{3}$$

(c)

$$\lim_{x\to \infty}\frac{x^2-4}{x^3-8}$$

$$=\lim_{x\to \infty}\frac{\frac{1}{x}-\frac{4}{x^3}}{1-\frac{8}{x^3}} \quad \text{divide by highest power of } x$$

$$=\frac{0}{1}$$

$$=0$$

(d)

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$$

We recognise this limit as the definition of the first derivative of $f(x)=\sqrt{x}$ when $x = 4$.

thus $f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$$\therefore \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}=f'(4) \text{ where } f(x)=\sqrt{x}$$

$$=\frac{1}{2\sqrt{4}}$$

$$=\frac{1}{4}$$

2.11 Use the limit definition to find $f'(1)$ when $f(x) = x^2 - 2x + 1$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{[(x+h)^2-2(x+h)+1]-[x^2-2x+1]}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-2x-2h+1-x^2+2x-1}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2-2h}{h}=\lim_{h\to 0}\frac{h(2x+h-2)}{h}$$

$$=\lim_{h\to 0}(2x+h-2)$$

$$=2x-2$$

$$\therefore f'(1)=2(1)-2$$

$$=0$$

2.12 Sketch the function $p\left(x\right)=\dfrac{x^2-9}{x^2-4}$

(a) zeros $x^2-9=0$

$\therefore x=\pm 3$

NOTE: Sketching involves the detailed workings of the following 5 parts.

(b) poles $x^2-4=0$

$\therefore x=\pm 2$

(c) y-intercept $x=0$

$p(0)=\dfrac{9}{4}$

(d) behaviour for x large

$$\lim_{x\to+\infty} p(x)=\lim_{x\to+\infty}\frac{1-\dfrac{9}{x^2}}{1-\dfrac{4}{x^2}}$$

$$=1$$

Similarly $\lim_{x\to-\infty} p(x)=1$

This means that there is a horizontal asymptote at $y=1$

(e) turning points $p'(x)=\dfrac{(x^2-4)2x-(x^2-9)2x}{(x^2-4)^2}$

$$=\frac{10x}{(x^2-4)^2}$$

for stationary points $p'(x)=0$

$\therefore\ 10x=0$

$\therefore\ x=0$

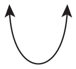
But $p(0)=\dfrac{9}{4}$

$\therefore\left(0,\dfrac{9}{4}\right)$ is a stationary point

To determine whether this is maximum or minimum, either a second derivative test or a sign diagram for p′(x) needs to be done.

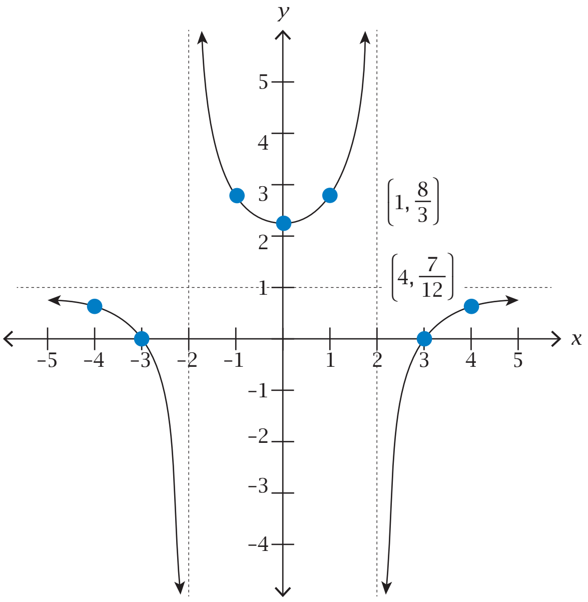
Sign Diagram:

x		0	
p′(x)	−ve	0	+ve

$\therefore$ the shape is 

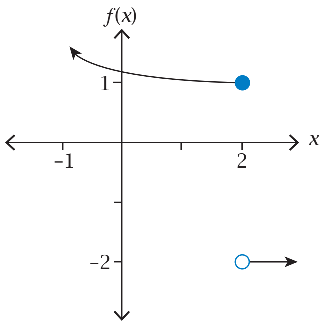
$\therefore\left(0,\dfrac{9}{4}\right)$ is a minimum turning point.

Sketch:



2.13 The following examples examine continuity and differentiability.

(a)

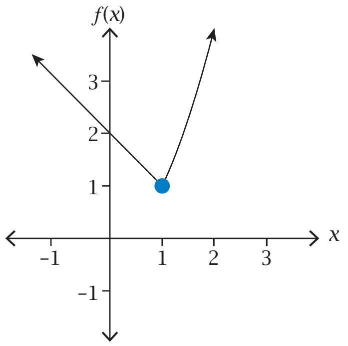


(i) $f(x)$ is not continuous at $x = 2$ since $\lim_{x\to 2^-} f(x) = 1$ and $\lim_{x\to 2^+} f(x) = -2$

i.e. $\lim_{x\to 2} f(x)$ does not exist.

(ii) As a result of (i), $f(x)$ is not differentiable at $x = 2$

(b) $f(x) = \begin{cases} 2 - x \ \ldots \ x < 1 \\ x^2 \ \ldots \ x \geq 1 \end{cases}$



(i) $f(x)$ is continuous at $x = 1$

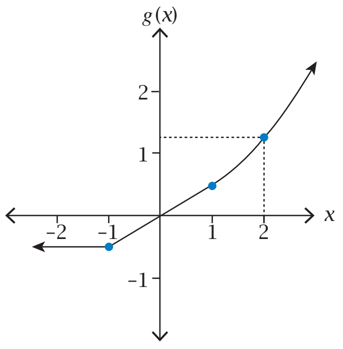
(ii) $f'_+(x) = d/dx(x^2) = 2x$ $\quad \therefore f'_+(1) = 2$

$f'_-(x) = d/dx(2 - x) = -1$ $\quad \therefore f'_-(1) = -1$

$\therefore f(x)$ is not differentiable at $x = 1$

since $f'_+(1) \neq f'_-(1)$

(c) $g(x) = \begin{cases} -\dfrac{1}{2} \ \ldots\ldots \ x < -1 \\ \dfrac{x}{2} \ \ldots \ -1 \leq x \leq 1 \\ \dfrac{x^2}{2} + \dfrac{1}{4} \ \ldots \ x > 1 \end{cases}$



At $x = -1$

(i) $g(x)$ is continuous

(ii) $g(x)$ is not differentiable

as $g'_+(x) = d/dx\left(\dfrac{x}{2}\right) = \dfrac{1}{2}$ $\quad$ i.e. $g'_+(-1) = \dfrac{1}{2}$

$g'_-(x) = d/dx\left(-\dfrac{1}{2}\right) = 0$ $\quad$ i.e. $g'_-(-1) = 0$

At $x = 1$

(i) $g(x)$ is continuous

(ii) $g(x)$ is differentiable

as $g'_+(x) = d/dx\left(\dfrac{x^2}{4} + \dfrac{1}{4}\right) = \dfrac{x}{2}$ $\quad \therefore g'_+(1) = \dfrac{1}{2}$

$g'_-(x) = d/dx\left(\dfrac{x}{2}\right) = \dfrac{1}{2}$ $\quad \therefore g'_-(-1) = \dfrac{1}{2}$

and $g'_+(l) = g'_-(l) = \dfrac{1}{2}$

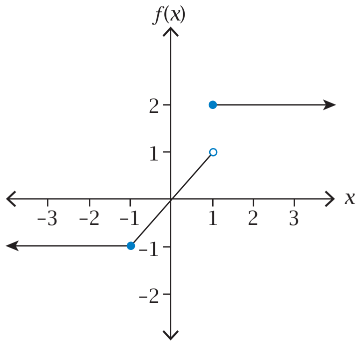
2.14 (a) Sketch $f\left(x\right)=\begin{cases}-1\,.\,.\,.\,.\,.\,.\,x<-1\\ x\,.\,.\,.-1\le x<1\\ 2\,.\,.\,.\,.\,.\,.\,.\,x\ge 1\end{cases}$

(b) Find $\lim\limits_{x\to -1} f\left(x\right)$

(c) Find $\lim\limits_{x\to 1} f\left(x\right)$

(d) Comment on the differentiability of $f(x)$

(a)



(b) $\lim\limits_{x\to -1^+} f\left(x\right)=\lim\limits_{x\to -1^+} x=-1$

$\lim\limits_{x\to -1^-} f\left(x\right)=\lim\limits_{x\to -1^-} -1=-1$

$\therefore \lim\limits_{x\to -1} f\left(x\right)=-1$

(c) $\lim\limits_{x\to 1^+} f\left(x\right)=2$ and $\lim\limits_{x\to 1^-} f\left(x\right)=1$

$\therefore \lim\limits_{x\to 1} f\left(x\right)$ does not exist

(d) $f(x)$ is differentiable everywhere except $x = -1$ and $x = 1$.

For $x = -1$, $\lim\limits_{x\to -1^-} f'\left(x\right)\ne \lim\limits_{x\to -1^+} f'\left(x\right)$

For $x = 1$, $\lim\limits_{x\to 1^-} f'\left(x\right)\ne \lim\limits_{x\to 1^+} f'\left(x\right)$

2.15 Find the values of a and c for which $h\left(x\right)=\begin{cases}ax+1\,.\,.\,.\,x\ge 1\\ cx^2-2\,.\,.\,.\,x<1\end{cases}$ is both continuous and differentiable.

The only possible point of discontinuity is at $x = 1$.

To be continuous: $\lim\limits_{x\to 1^-} h(x)=\lim\limits_{x\to 1^+} h(x)$

$\therefore \lim\limits_{x\to 1^-}(cx^2-2)=\lim\limits_{x\to 1^+}(ax+1)$

$\therefore c-2=a+1$ ①

To be differentiable: $h'_+(1)=h'_-(1)$ $\therefore \frac{d}{dx}(ax+1)=\frac{d}{dx}(cx^2-2)$ at $x = 1$

$\therefore a=2cx$ at $x = 1$

$\therefore a=2c$ ②

Solving ① and ② gives $c = -3$ and $a = -6$.

2.16 Long division

Divide the polynomial $p(x) = x^3 + 4x^2 + x - 6$ by $x + 3$ using long division.
Hence, write $p(x)$ as a product of its linear factors.

$$\begin{array}{rl} & \;x^2 \\ x+3 & \overline{\big)\,x^3+4x^2+x-6} \end{array}$$

1. Divide x into x^3 (i.e. the 1st term)

$$\begin{array}{rl} & \;x^2 \\ x+3 & \overline{\big)\,x^3+4x^2+x-6} \\ - & \underline{\;(x^3+3x^2)\;} \\ & \qquad 1x^2 \end{array}$$

2. Multiply x^2 by $(x + 3)$ and write the terms underneath. Then subtract the common terms.

$$\begin{array}{r l}
 & x^2 + 1x \\
x + 3 & \overline{\big)\, x^3 + 4x^2 + x - 6} \\
- & \underline{(x^3 + 3x^2)\ \ \downarrow} \\
 & \quad 1x^2 + x \\
- & \quad \underline{(1x^2 + 3x)} \\
 & \qquad -2x
\end{array}$$

3. Divide x into $1x^2$ (i.e. the remainder from the subtraction). Then multiply $1x$ by $(x + 3)$ and write the terms underneath. Then subtract the common terms. Also bring down 1x (the third term of the polynomial) in preparation for the subtraction. Then perform the subtraction as per Step 2.

$$\begin{array}{r l}
 & x^2 + 1x - 2 \\
x + 3 & \overline{\big)\, x^3 + 4x^2 + x - 6} \\
- & \underline{(x^3 + 3x^2)} \quad \downarrow \\
 & \quad 1x^2 + x \\
- & \quad \underline{(1x^2 + 3x)} \ \downarrow \\
 & \qquad -2x - 6 \\
 & \qquad \underline{(-2x - 6)} \\
 & \qquad\qquad 0
\end{array}$$

4. Repeat Step 3 by dividing x into $-2x$. As there is a remainder of 0, we can conclude that $(x + 3)$ is a factor of $p(x)$.

Hence, $p(x) = x^3 + 4x^2 + x - 6 = (x + 3)(x^2 + 1x - 2)$
$= (x + 3)(x + 2)(x - 1)$

2.17 Synthetic long division

Divide the polynomial $q(x) = 2x^3 + 6x^2 - 12x - 16$ by $(x - 2)$. Hence, write $q(x)$ as a product of its linear factors.

$$\begin{array}{r|rrrr}
2 & 2 & 6 & -12 & -16 \\
 & \downarrow & & & \\
\hline
 & 2 & & &
\end{array}$$

1. Write the coefficients of $q(x)$ in descending order from left to right. To the far left, write 2 (because $(x - 2)$ is being divided into $q(x)$), and then ‘bring down’ the leading coefficient of 2 beneath the line (we are really calculating 2 + 0)

$$\begin{array}{r|rrrr}
2 & 2 & 6 & -12 & -16 \\
\searrow & & \nearrow 4 & & \\
\hline
 & 2 & 10 & &
\end{array}$$

2. Multiply 2 by the 2 that was ‘brought down’, and write the answer of 4 underneath the next coefficient. Add these numbers for another number beneath the line.

$$\begin{array}{r|rrrr}
2 & 2 & 6 & -12 & -16 \\
\searrow & & 4 & \nearrow 20 & \\
\hline
 & 2 & 10 & 8 &
\end{array}$$

3. Multiply 2 by the 10 that was ‘brought down’, and write the answer of 20 underneath the next coefficient. Add these numbers for another number beneath the line.

$$\begin{array}{r|rrrr}
2 & 2 & 6 & -12 & -16 \\
\searrow & & 4 & 20 & \nearrow 16 \\
\hline
 & 2 & 10 & 8 & 0
\end{array}$$

4. Continue the process until the process has been used with all coefficients. As the final addition performed gives an answer of 0, we can conclude that $x - 2$ is a factor of $q(x)$ that gives no remainder. The numbers beneath the line now become the coefficients of a quadratic factor that has been produced (as we have just divided a cubic polynomial by a linear factor)

Hence, $2x^3 + 6x^2 - 12x - 16 = (x - 2)(2x^2 + 10x + 8)$
$= (x - 2)(2x + 8)(x + 1)$
$= 2(x - 2)(x + 4)(x + 1)$

PROBLEMS TO SOLVE

CHAPTER 2: FUNCTIONS AND CURVE SKETCHING

Try to not use your graphics calculator

1. If $f(x) = x^2 + x - 12$ and $g(x) = \sqrt{x}$
   (a) Explain why $y = f(g(x))$ exists but $y = g(f(x))$ does not exist.
   (b) Find $y = f(g(x))$ and give its domain and range.
   (c) Modify the domain of $y = f(x)$ such that $y = g(f(x))$ does exist and give the resulting domain and range of the new function.
   (d) Explain why $y = f^{-1}(x)$ is not defined and then show two ways that it will be defined and give the rules in each case.

2. (a) Explain why $x^2 + y^2 =4$ is not a function.
   (b) Solve for y and give your answer as two functions $y = f(x)$ and $y = g(x)$
   (c) Find the domain and range of f and g

3. Solve algebraically
   (a) $|2x - 5| + x = 16$
   (b) $|2x - 5| + x > 16$

4. Write $y = |4 - x| + |2x + 3|$ in piecewise form

5. (a) Find the domain and range of $f(x) = \sqrt{x - 1} + 2$
   (b) Find $y = f^{-1}(x)$

6. (a) If $f(x) = \dfrac{2x + 3}{5x - 2}$, $y = f^{-1}(x)$, and comment on the result.
   (b) If two functions have the same inverse, what can you say about their graphs in relation to the line $y = x$?

7. Sketch $y = 2|x|$ and $y = 4 - |x - 3|$ on the same axes and hence solve $2|x| + |x - 3| = 4$

8. If $y = \sqrt{x}$ explain the transformations necessary to get $y = 3\sqrt{x - 2} + 5$.

9. If $y = |x|$ explain the transformations necessary to get $y = 4 - |3x + 6|$.

10. Explain in two different ways how $y = \log5x$ has been transformed from $y = \log x$

11. Consider $f(x) = \sqrt{x - 1} - 2$ and $g(x) = \dfrac{1}{x}$, $x > 0$.
    (a) Find the domain and range of f and g.
    (b) Show why $y = g(f(x))$ does not exist.
    (c) Find the necessary restrictions such that $y = g(f(x))$ is defined and hence give the domain and range of this function.
    (d) Find $y = f^{-1}(x)$

12. Two functions f and g are such that $f(g(x)) = x$. What can you say about these functions?

13. (a) Sketch $y = \frac{4}{x+2} - 3$ and give its domain and range.
    (b) Write the function in $y = \frac{ax+b}{cx+d}$ form
    (c) Find $y = f^{-1}(x)$

14. If $f(x) = 2x - 3$ and $g(x) = x^2 - 4x$ find and simplify $y = f(g(x))$ and $y = g(f(x))$.

15. Consider the two functions f and g:

$f(x) = \sqrt{1 - x^2}$ $\qquad$ $g(x) = 2 + x^2$

(a) State the domain and range of each function.
(b) Explain why $f(g(x))$ does not exist.
(c) Find the rule for $g(f(x))$ and state its range.

16. Write $f(x) = |x + a| - |x - a|, a > 0$ in piecewise form and sketch the graph of f.

17. Explain how $f(x) = x^2$ is transformed to $y = 3(2x - 3)^2 - 5$.

18. If $f(x) = \log 2x$ and $g(x) = \frac{10^x}{2}$ show how to find $y = f(g(x))$ and $y = g(f(x))$ and comment.

19. If $g(f(x)) = 2x^2 - 13x + 14$ where $f(x) = x - 5$ find $y = g(x)$.

(Hint: First write the linear function as $y = x - 5$ and the composite function as $f(y) = 2x^2 - 13x + 14$. Now solve the linear function for x and substitute.)

20. Find the domain and range of $y = \sqrt{x^2 - 4}$ without a calculator and sketch its graph.

21. Find the relationship between any of a, b, c and d such that for $f(x) = \frac{ax+b}{cx+d}$, $f(x) = f^{-1}(x)$.

22. Solve $|2x + 4| - |5 - x| = -6$ algebraically for x and hence also solve $|2x + 4| < |5 - x| - 6$.

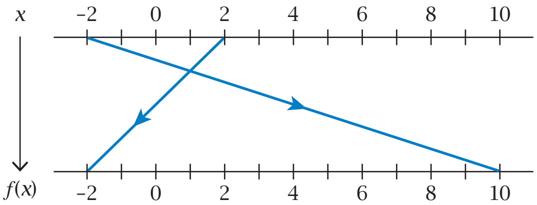
23. Explain why $\sqrt{x^2}$ is not always equal to x and then give the full piecewise definition of $\sqrt{x^2}$.

24. (a) Carefully sketch on graph paper the function $f(x) = \begin{cases} -\frac{1}{2}x & , \quad -2 \le x < 2 \\ 3x - 8 & , \quad 2 \le x \le 3 \end{cases}$
    (b) Now sketch the following on separate axes;
        i. $y = f(-x)$
        ii. $y = 2f(x + 2)$
        iii. $y = 4 - f(x - 2)$
        iv. $y = \frac{1}{2} f(1 - x)$
        v. $y = f(|x|)$
        vi. $y = |f(x)|$

25. The arrow diagram represents part of the mapping

$$f : x \to \frac{2a}{bx + 4}$$

Find the value of a and b.



26. State the domain and range of the following

(a) $f(x) = x^2 -4x + 7$

(b) $g(x) = -\sqrt{3 + x}$

(c) $m(x) = \dfrac{1}{2 - x}$

(d) $h(x) = \dfrac{2}{\sqrt{x - 2}}$

27. Find the zeroes of the polynomial $P(x) = x^4 - 6x^3 - 3x^2 + 20x - 12$.

28. Sketch

$$f(x) = \begin{cases} x + 2 & x < -2 \\ \sqrt{4 - x^2} & -2 \le x \le 2 \\ [|x|] & 2 < x \le 3 \end{cases}$$

29. Locate all zeroes, poles (vertical asymptotes) and turning point of the given function. Determine its behaviour near ±∞ and sketch its graph.

(a) $y = \dfrac{x^2 + 1}{x^2 - 1}$ (b) $y = \dfrac{2x + 1}{(x - 2)^2}$

For (b) find also point of inflection.

30. Sketch the graph of the functions

(a) $y = SGN(1 - 2x)$

(b) $y = INT(1 - x)$

31. (a) Sketch the graph of the piecewise function

$$y = \begin{cases} -3 & x < -2 \\ 1 - x^2 & -2 \le x < 0 \\ \dfrac{x}{2} - 1 & 0 \le x \end{cases}$$

(b) Use the graph to find (if they exit)

i. $\lim\limits_{x \to -2} y$

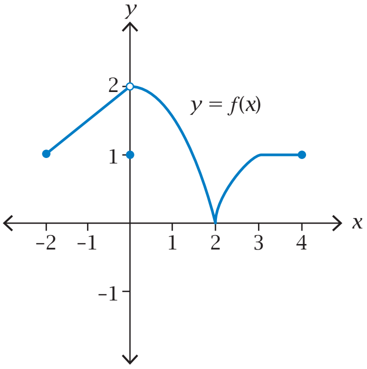
ii. $\lim\limits_{x \to -1} y$

iii. $\lim\limits_{x \to 0} y$

iv. $\lim\limits_{x \to \infty} y$

v. $\lim\limits_{x \to -\infty} y$

32. The piecewise function f has the following graph. Use the graph to find (if they exist)



(a) $\lim\limits_{x \to -1} y$

(b) $\lim\limits_{x \to 0} y$

(c) $\lim\limits_{x \to 2} y$

(d) $\lim\limits_{x \to 1} y$

33. Evaluate

(a) $\lim\limits_{x \to -2} SGN(x)$

(b) $\lim\limits_{x \to 1} [|x|] + 2$

34. Use a table of values to find $\lim\limits_{x \to -3} \dfrac{\sqrt{x+7} - 2}{x + 3}$

35. Evaluate

(a) $\lim\limits_{x \to 2} \dfrac{2x^2 - x - 6}{x^2 + 3x - 10}$

(b) $\lim\limits_{x \to \infty} \dfrac{2x^2 - x - 6}{x^2 + 3x - 10}$

36. Given
$$f(x) = \begin{cases} b - x^2 & x < b \\ 2 - 2x & x \geq b \end{cases}$$

find the value/s of b for which the $\lim\limits_{x \to b} f(x)$ exists.

37. Examine the continuity and differentiability of f at $x = -1$ and $x = 2$ if

$$f(x) = \begin{cases} -2x & x < -1 \\ x^2 + 1 & -1 \leq x < 2 \\ -x^2 + 6x - 3 & x \geq 2 \end{cases}$$

38. A function f is defined below

$$f(x) = \begin{cases} 2x + 4 & x < -1 \\ -x^2 + 1 & -1 \leq x \leq 1 \\ ax + b & x > 1 \end{cases}$$

(a) Explain why f is discontinuous at $x = -1$.

(b) Find constants a and b such that f is continuous and differentiable at $x = 1$.

39. If $f(x) = \dfrac{2}{x}$

evaluate $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

40. From first principles evaluate $\lim\limits_{h \to 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h}$

(hint: multiply by $\dfrac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}$)

41. (a) Sketch the graph of the function f defined by

$$f(x) = \begin{cases} -(x + 1)^2 + 3 & -3 \leq x < 0 \\ -2x + 2 & 0 \leq x < 1 \\ x - SGN(x) & 1 \leq x \leq 3 \end{cases}$$

(b) Use your graph to find (if they exist)

i. $f'(-3)$

ii. $f'(0)$

iii. $f'(1)$

iv. $f'(2)$
(Note f' is the derivative)

42. Let f be the function defined by $f(x) = \dfrac{x}{x-2}$

(a) State the domain and range of f.

(b) Evaluate $\lim_{x \to +\infty} f(x)$

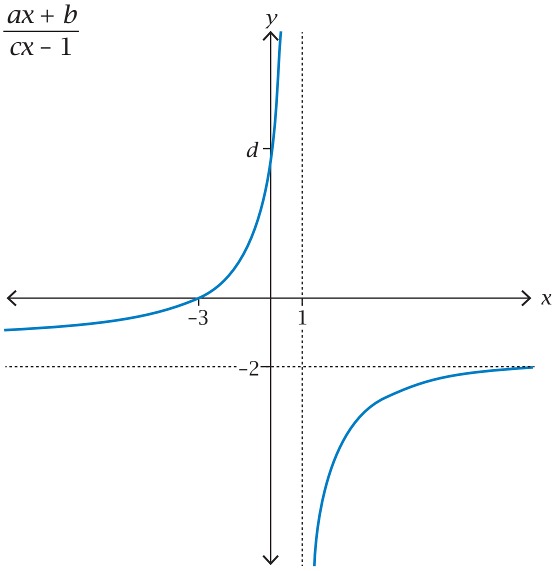
(c) Find $f(x-1)$

(d) Find $f\left(\dfrac{2}{x}\right)$

(e) Find x where $f(x-1) = f\left(\dfrac{2}{x}\right)$

43. The diagram shows the graph of $y = \dfrac{ax+b}{cx-1}$

Find the values of a, b, c and d.



44. The equation of a curve is given by $2x^3 - 7xy + 2y^3 = 0$.

Using your graphics calculator, test y values approaching –2 from above and below so as to check the continuity of the given relation at $y = -2$.

45. A function f is defined as $f(x) = \sqrt{\dfrac{9}{x^2} - 1}$

Find the domain and range of f.

46. A polynomial $P(x) = x^3 + ax^2 + cx - 10$ has a zero at $x = 2$ and is parallel to the x axis at $x = -1$. Determine the values of a and c.

47. Sketch the graph of the function

$y = x \times INT(x)$ [–2,2]

48. Give the equation of a rational function with two zeros at $x = -2$ and $x = -1$, and a pole at $x = -3$ such that the function has

(a) no horizontal asymptote

(b) 1 horizontal asymptote

49. Evalute $\lim_{x \to 0} x^x$ using a calculator.

50. Find the quotient, $Q(x)$, and the remainder, $R(x)$, when the following polynomial $P(x)$ is divided by the polynomial $D(x)$:

(a) $P(x) = x^2 - 9x - 10$; $D(x) = x + 1$

(b) $P(x) = x^2 - 4x + 5$; $D(x) = x - 3$

(c) $P(x) = x^3 + 7x^2 + 7x - 6$; $D(x) = x + 2$

(d) $P(x) = x^3 + 6x^2 + 11x + 8$; $D(x) = x + 2$

(e) $P(x) = 4x^3 - 12x^2 + 9x - 1$; $D(x) = x - 1$

(f) $P(x) = 3x^3 + 14x^2 - 7x - 10$; $D(x) = x - 3$

51. For each of the following polynomials, $P(x)$, determine $P(2)$. Hence, conclude whether $(x - 2)$ is a factor $P(x)$.

(a) $P(x) = x^2 - 5x - 7$

(b) $P(x) = x^2 - 3x + 2$

(c) $P(x) = x^3 + 6x^2 - x - 30$

(d) $P(x) = x^3 + 9x^2 - 4x - 96$

(e) $P(x) = 2x^3 + 3x^2 - 18x + 8$

(f) $P(x) = 2x^3 - 23x^2 + 58x + 35$

52. Using a 'trial and error' approach, find a number c such that $P(c) = 0$. Then without the use of a calculator completely factorise $P(x)$.

(a) $P(x) = x^3 - 2x^2 - 23x + 60$

(b) $P(x) = x^3 + 2x^2 - 41x - 42$

(c) $P(x) = x^3 + 4x^2 - 39x + 54$

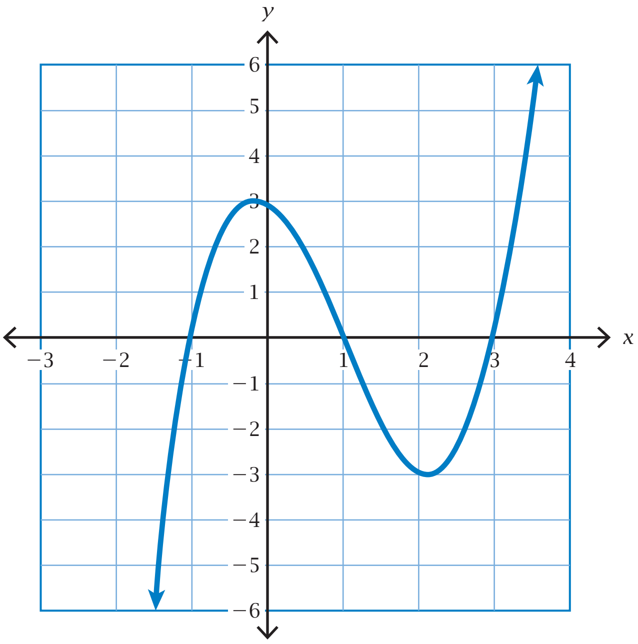
(d) $P(x) = 2x^3 + x^2 - 25x + 12$

(e) $P(x) = 3x^3 + x^2 - 22x - 24$

(f) $P(x) = 5x^3 + 13x^2 - 146x + 56$

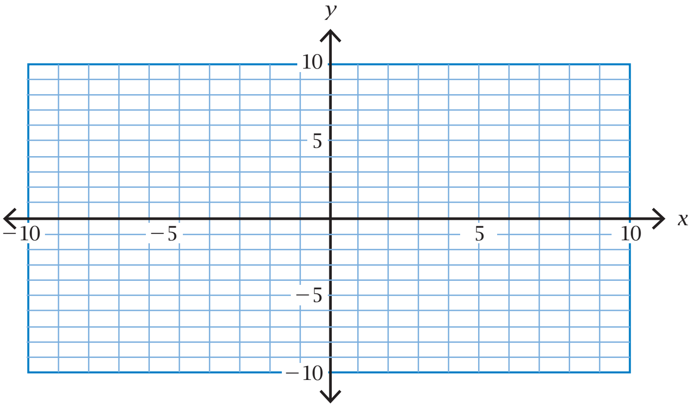
53. Given that $(x + 2)$ is a factor of the polynomial $g(x) = 2x^3 + \mathrm{b}x^2 - 23x + 14$, determine the value of b. Hence, fully factorise $g(x)$.

54. Below is a graph of the function $y = f(x)$.

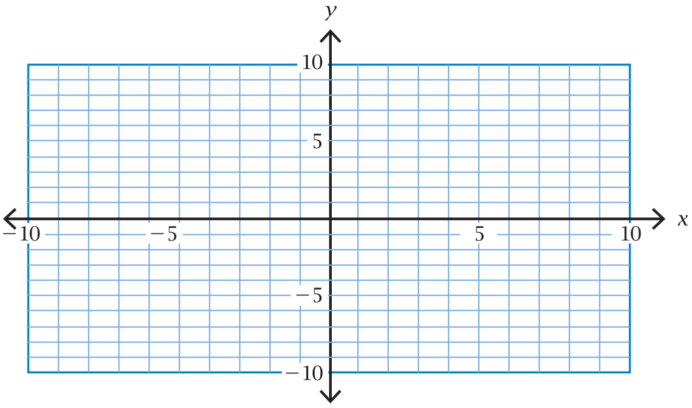


On the axes provided, sketch the graphs given below:

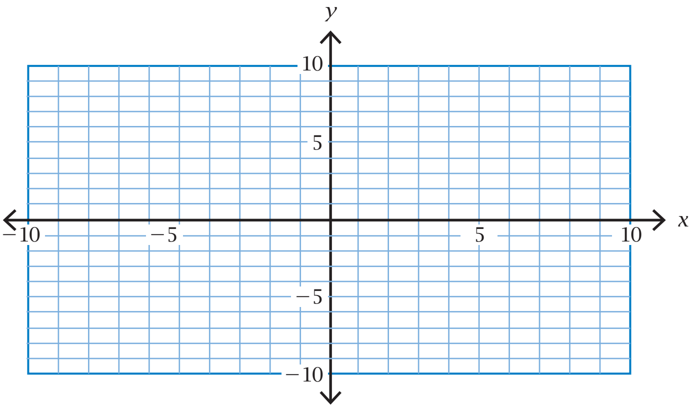
(a) $y = |f(x)|$



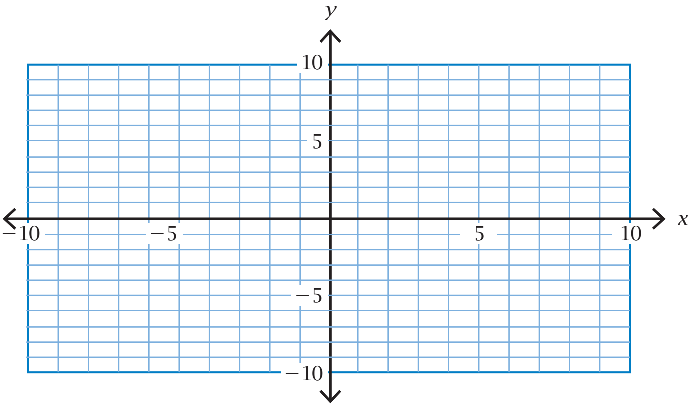
(b) $y = f(|x|)$



(c) $y = f^{-1}(x)$

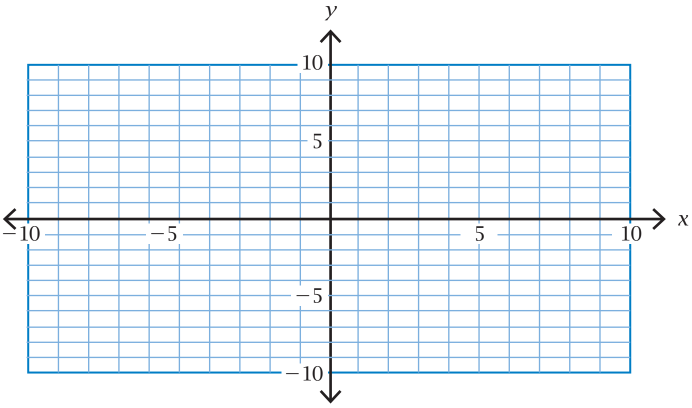


(d) $y = \dfrac{1}{f(x)}$

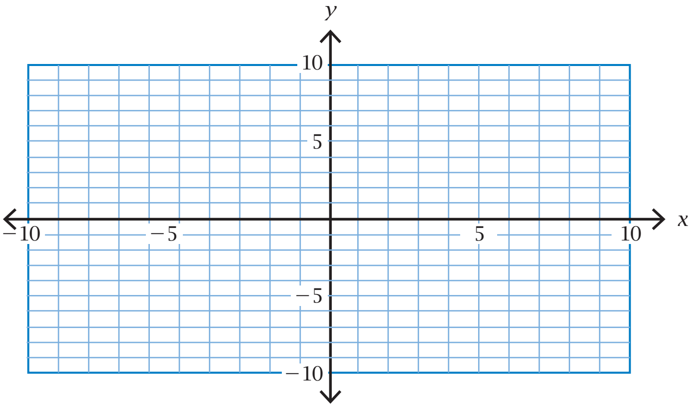


55. Sketch the graph of the following rational functions on the axes provided, clearly showing all intercepts and asymptotes.

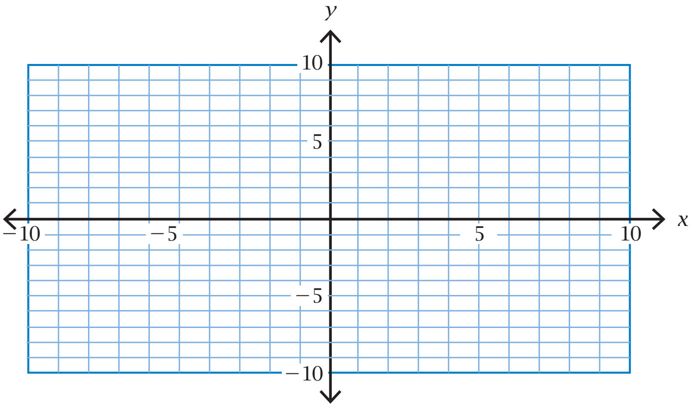
(a) $f(x) = \dfrac{x^2 - 9}{x - 3}$



(b) $g(x) = \dfrac{x^2 + 2x - 3}{x^2 - 1}$



56. On the Cartesian plane provided below, graph the function $f(x) = \dfrac{x^2 - 6x + 8}{x^3 - 4x}$ over the domain [–4, 6]. On the graph, clearly show any asymptotes, axis intercepts and overall behaviour of the function as $x \to \pm\infty$.



VECTORS IN 3-DIMENSIONS

The geometry of three-dimensional space is conceptually more difficult than the geometry of the plane. However, the transition from two to three dimensions is facilitated by the vector approach.

In this section the similarities, rather than the differences, between two- and three-dimensional geometry are emphasised. The applications studied here should have an emphasis on real-life situations.

Syllabus Checklist

By the end of this chapter, you should be able to:

The algebra of vectors in three dimensions

☐ review the concepts of vectors from Unit 1 and extend to three dimensions, including introducing the unit vectors **i**, **j** and **k**

☐ prove geometric results in the plane and construct simple proofs in 3 dimensions

Vector and Cartesian equations

☐ introduce Cartesian coordinates for three dimensional space, including plotting points and equations of spheres

☐ use vector equations of curves in two or three dimensions involving a parameter and determine a 'corresponding' Cartesian equation in the two-dimensional case

☐ determine a vector equation of a straight line and straight line segment, given the position of two points or equivalent information, in both two and three dimensions

☐ examine the position of two particles, each described as a vector function of time, and determine if their paths cross or if the particles meet

☐ use the cross product to determine a vector normal to a given plane

☐ determine vector and Cartesian equations of a plane

Systems of linear equations

☐ recognise the general form of a system of linear equations in several variables, and use elementary techniques of elimination to solve a system of linear equations

☐ examine the three cases for solutions of systems of equations – a unique solution, no solution, and infinitely many solutions – and the geometric interpretation of a solution of a system of equations with three variables

Vector calculus

☐ consider position vectors as a function of time

☐ derive the Cartesian equation of a path given as a vector equation in two dimensions, including ellipses and hyperbolas

☐ differentiate and integrate a vector function with respect to time

☐ determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration

☐ apply vector calculus to motion in a plane, including projectile and circular motion

INTRODUCTION

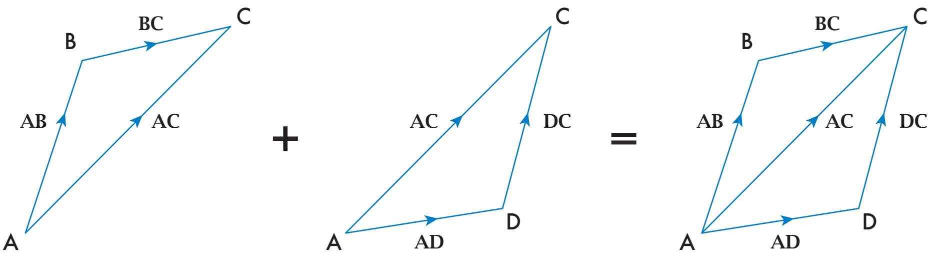
Most students studying this subject will have been fully introduced to vectors in two dimensions from their studies of Mathematics Specialist Units 1 and 2. Here the basics of two dimensional vectors will be summarised in preparation for their extension to three dimensional situations.

TWO DIMENSIONAL VECTORS – A BRIEF SUMMARY

- A vector is a quantity which needs two or more numbers in order to be fully specified. In two dimensions, vectors are given either in **component** form (sometimes called **rectangular** form) or in **polar** form which is **magnitude/direction** form.

  The vector types that we will be most concerned with are displacement, force, velocity and acceleration. Each of these vectors describe some sort of action. For example A (3, 5) and B (7, 2) are simply two points on a plane but the **displacement** vector **AB** describes the instructions necessary in order to get straight from A to B.

- Another characteristic that vectors have in common is that they are all represented diagrammatically by **arrows** and all add according to the triangle of vectors or parallelogram rule which consists of two equivalent vector triangles.



AB + **BC** = **AC**

AD + **DC** = **AC**

AD = **BC** **AB** = **DC**
AB + **AD** = **AC**

In all of these diagrams the vector **AC** is called the **resultant** vector from adding **AB** and **BC** or **AD** and **DC** or **AB** and **AD**.

The diagonal **AC** of parallelogram ABCD results from **adding** two side vectors. The other diagonal, say **BD** results from **subtracting** two side vectors.

For diagonal **BD**

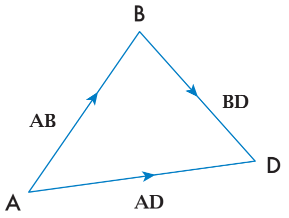
AB + **BD** = **AD**

gives **BD** = **AD** – **AB**

For diagonal **DB**

AD + **DB** = **AB**

gives **DB** = **AB** – **AD**



Note that if A is the common tail point of **AB** and **AD** then the order for B and D is opposite on either side of each equation

BD = **AD** – **AB** **DB** = **AB** – **AD**

This is how the subtraction of two vectors always works

The parallelogram ABCD shows that vectors **AB** and **DC** have the same length and are parallel which makes them **equivalent** vectors as are **AD** and **BC**.

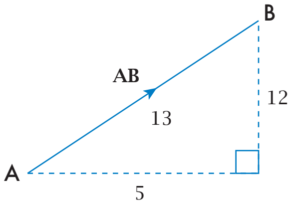
On a vector diagram we will often slide vectors around without rotating them (which is called a translation) and this property, as will be seen later, increases our options when proving and solving vector problems.

The combination of vector notation, vector diagrams consisting of triangles, parallelograms and by extension, polygons where all sides are drawn as arrows which are all linked up algebraically by vector equations is what makes vectors so versatile, powerful and interesting.

- The **magnitude** of a vector is its size which is calculated using Pythagoras' Theorem if the vector is given in component form.

  So for **AB** = $\langle 5, 12\rangle = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ the magnitude of **AB** is $|\mathbf{AB}|$

  $$\begin{aligned} \text{and} \quad |\mathbf{AB}| &= \sqrt{5^2 + 12^2} \\ &= 13 \text{ units} \\ \text{for} \quad \mathbf{CD} &= \langle 0.6, 0.8\rangle \\ |\mathbf{CD}| &= \sqrt{0.6^2 + 0.8^2} \\ &= 1 \text{ unit} \end{aligned}$$



  Any vector which has a magnitude of 1 unit is called a **unit** vector. A vector **b** has a unit vector in the same direction and is given by

  $$\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|}$$

- The **standard unit** vectors are **i** and **j** which in two dimensions are given as:

  $$\begin{aligned} \mathbf{i} &= \langle 1, 0\rangle \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbf{j} &= \langle 0, 1\rangle \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

  All other vectors in the *x*–*y* plane can be written as a linear combination of **i** and **j**.
  For **scalar multiples** of 2 and -3 we could have

  $$\begin{aligned} 2\mathbf{i} &= 2\langle 1, 0\rangle \\ &= \langle 2, 0\rangle \\ &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{aligned} \qquad \text{or} \qquad \begin{aligned} -3\mathbf{j} &= -3\langle 0, 1\rangle \\ &= \langle 0, -3\rangle \\ &= \begin{bmatrix} 0 \\ -3 \end{bmatrix} \end{aligned}$$

  When 2**i** is added to -3**j** and the resultant is called **a,** then

  $$\begin{aligned} \mathbf{a} &= 2\mathbf{i} + -3\mathbf{j} \\ &= 2\mathbf{i} - 3\mathbf{j} \\ &= \langle 2, -3\rangle \\ &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{aligned}$$

  Another vector **b** could equal 5**a**.

  $$\begin{aligned} \text{So} \quad \mathbf{b} &= 5\mathbf{a} \\ &= 5\,(2\mathbf{i} - 3\mathbf{j}) \\ &= 10\mathbf{i} - 15\mathbf{j} \end{aligned}$$

  The magnitude of **a** is scaled by a factor of 5 (i.e. $|\mathbf{b}| = 5|\mathbf{a}|$) giving the magnitude of **b**. The directions of **a** and **b** are the same. If the scalar multiple was -5 then the directions of the two vectors would be opposite.

- The conversion between vectors given in component form and polar form is introduced by considering the **displacement** vector **AB** where A (3, 5) is at the vectors tail and B (7, 2) is at the vectors tip.

  The **position** vectors for A and B are

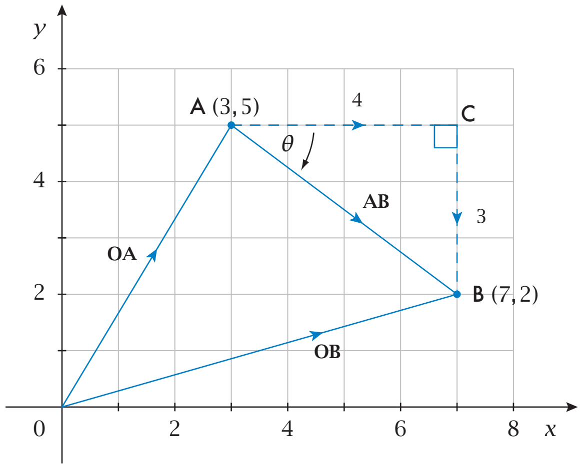
  $$\begin{aligned} \mathbf{OA} &= \langle 3, 5\rangle \\ &= \begin{bmatrix} 3 \\ 5 \end{bmatrix} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbf{OB} &= \langle 7, 2\rangle \\ &= \begin{bmatrix} 7 \\ 2 \end{bmatrix} \end{aligned}$$

Position vectors indicate the position of point's relative to the origin and hence always have their tails at the origin 0.

Because

$$\begin{aligned}\mathbf{OA} + \mathbf{AB} &= \mathbf{OB}\\ \mathbf{AB} &= \mathbf{OB} - \mathbf{OA}\\ &= \langle 7, 2\rangle - \langle 3, 5\rangle\\ &= \langle 7 - 3, 2 - 5\rangle\\ &= \langle 4, -3\rangle\end{aligned}$$

The diagram shows the vectors mentioned above.



Using Pythagoras

$$\begin{aligned}|\mathbf{AB}| &= \sqrt{4^2 + 3^2}\\ &= 5 \text{ units}\end{aligned}$$

Using trigonometry

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{3}{4}\right)\\ &= 36.87^\circ \quad \text{2dp}\end{aligned}$$

The polar angle for the vector **AB** is -36.87° because from the vector **AC** which is parallel to the *x*-axis the direction of θ is in the clockwise direction.

So

$$\mathbf{AB} = \langle 4, -3\rangle \quad \text{or} \quad 4\mathbf{i} - 3\mathbf{j} \quad \text{or} \quad \begin{bmatrix}4\\-3\end{bmatrix}$$

$|\mathbf{AB}| = 5$ units and is the magnitude of **AB**

AB has a direction of -36.87° as a polar angle and if the positive *y*-axis points North then **AB** has a bearing of $90^\circ + 36.87^\circ = 126.87^\circ$

Also students should confirm that for

$\mathbf{OA} = 3\mathbf{i} + 5\mathbf{j}$

$|\mathbf{OA}| = \sqrt{34} = 5.83$ 2dp

polar angle for **OA** = 59.04° 2dp

bearing for **OA** = 030.96° 2dp

and

$\mathbf{OB} = 7\mathbf{i} + 2\mathbf{j}$

$|\mathbf{OB}| = \sqrt{53} = 7.28$ 2dp

polar angle for **OB** = 15.95° 2dp

bearing for **OB** = 074.05° 2dp

- The vector **BA** has the same magnitude as **AB** but the **opposite** direction.
  Also $\mathbf{AB} + \mathbf{BA} = 0$ which means that $\mathbf{BA} = -\mathbf{AB}$

If

$$\begin{aligned}\mathbf{AB} &= \langle 4, -3\rangle\\ &= 4\mathbf{i} - 3\mathbf{j}\end{aligned}$$

then

$$\begin{aligned}\mathbf{BA} &= -\langle 4, -3\rangle\\ &= \langle -4, 3\rangle\end{aligned}$$

or

$$\begin{aligned}\mathbf{BA} &= -(4\mathbf{i} - 3\mathbf{j})\\ &= -4\mathbf{i} + 3\mathbf{j}\end{aligned}$$

AB and **BA** are called opposite vectors

Also $|\mathbf{BA}| = \sqrt{4^2 + 3^2}$

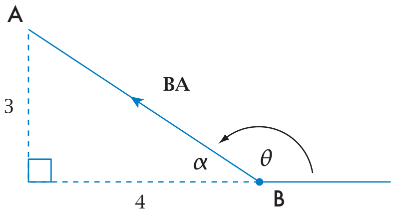
$= 5$ units

$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$

$= 36.87^\circ$ 2dp

polar angle for **BA** $= 180^\circ - 36.87^\circ$
$= 143.13^\circ$

bearing for **BA** $= 270^\circ + 36.87^\circ$
$= 306.87^\circ$



◇ The Dot Product of Two Vectors

If $\mathbf{a} = \langle a_1 , a_2\rangle$ and $\mathbf{b} = \langle b_1 , b_2\rangle$, then the dot product of **a** and **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is a scalar quantity given in two forms as

i. Rectangular form $\mathbf{a} \cdot \mathbf{b} = a_1\, b_1 + a_2\, b_2$
ii. Polar form $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ where θ is the angle between **a** and **b**.

The above forms originate from the equation $a_1\, b_1 + a_2 b_2 = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ as shown in the Mathematics Specialist (Units 1 and 2) Study Guide pp.48–49. The dot product notation serves as a way to separately use each side of this equation.

When $\theta = 90^\circ$ the polar form shows that $\mathbf{a} \cdot \mathbf{b} = 0$ which means that when $a_1\, b_1 + a_2\, b_2 = 0$ the vectors **a** and **b** are perpendicular.

When $\mathbf{a} = \mathbf{b}$ then $\theta = 0^\circ$ and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$
becomes $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|\,|\mathbf{a}| \cos 0^\circ$
i.e. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

In summary then

$\mathbf{a} \cdot \mathbf{b} = 0$
means **a** and **b**
are perpendicular

and

$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
which serves as a link between
magnitude and dot product.

It is not hard to show that

i. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
ii. $r\mathbf{a} \cdot s\mathbf{b} = rs\,(\mathbf{a} \cdot \mathbf{b})$
iii. $\mathbf{a} \cdot (\mathbf{b}+\mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
iv. $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{c}+\mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$

Example Use the dot product to prove Pythagoras' Theorem

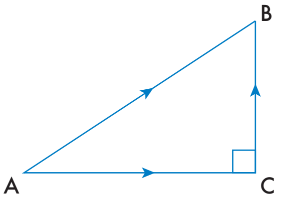
Let **AC** and **CB** be at 90° as shown

Then $\mathbf{AB} = \mathbf{AC} + \mathbf{CB}$

$\mathbf{AB} \cdot \mathbf{AB} = (\mathbf{AC} + \mathbf{CB}) \cdot (\mathbf{AC} + \mathbf{CB})$

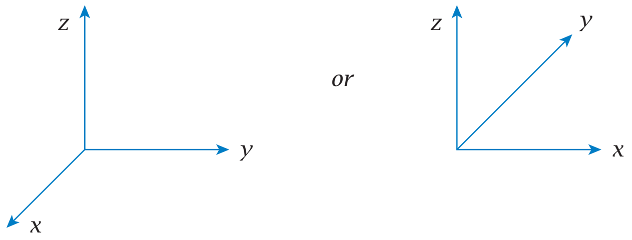
$|\mathbf{AB}|^2 = \mathbf{AC} \cdot \mathbf{AC} + 2\mathbf{AC} \cdot \mathbf{CB} + \mathbf{CB} \cdot \mathbf{CB}$

$\therefore$ $|\mathbf{AB}|^2 = |\mathbf{AC}|^2 + |\mathbf{CB}|^2$ is proved because $\mathbf{AC} \cdot \mathbf{CB} = 0$



THREE DIMENSIONAL VECTORS

The coordinate system for three dimensions usually has the *x*–*y* plane horizontal and the *z*-axis vertical as shown below.



Each axis is at 90° to each of the other axes.

Vectors in 3D are most conveniently specified in component or rectangular form:

e.g. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$= \langle 2, 3, 4\rangle$

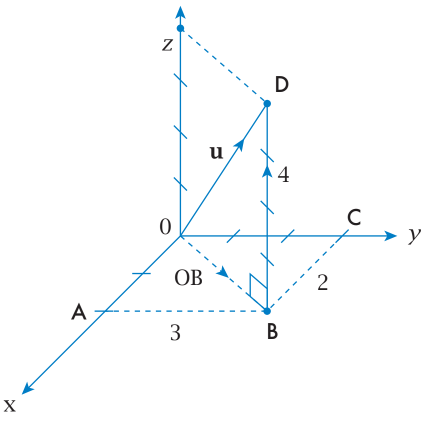
where $\mathbf{i} = \langle 1, 0, 0\rangle$

$\mathbf{j} = \langle 0, 1, 0\rangle$

and $\mathbf{k} = \langle 0, 0, 1\rangle$

are the three standard unit vectors.

To draw **u** in 3D as shown below we have chosen to use the axes as drawn on the left diagram above. First form a parallelogram OABC in the *x*–*y* plane using the lengths 2 and 3 in the positive directions and then count up 4 units in the *z*-direction from B.



$$\begin{aligned} \mathbf{OD} &= 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \\ &= \mathbf{OB} + \mathbf{BD} \\ \text{where } \mathbf{OB} &= 2\mathbf{i} + 3\mathbf{j} \\ \text{and } \mathbf{BD} &= 4\mathbf{k} \\ \text{Also } \mathbf{OB} &= \langle 2, 3, 0\rangle \\ \text{and } \mathbf{BD} &= \langle 0, 0, 4\rangle \end{aligned}$$

OD is a position vector and

$$|\mathbf{OD}| = \sqrt{|\mathbf{OB}|^2 + |\mathbf{BD}|^2} \quad \text{where} \quad |\mathbf{OB}|^2 = |\mathbf{OA}|^2 + |\mathbf{AB}|^2$$

$$\begin{aligned} \therefore \quad |\mathbf{OD}| &= \sqrt{|\mathbf{OA}|^2 + |\mathbf{AB}|^2 + |\mathbf{BD}|^2} \\ &= \sqrt{2^2 + 3^2 + 4^2} \\ &= \sqrt{29} \quad \text{units} \end{aligned}$$

In general, if $\mathbf{u} = \langle a, b, c\rangle$

then $|\mathbf{u}| = \sqrt{a^2 + b^2 + c^2}$

is the magnitude of a 3D vector.

Example (a) If $\mathbf{OA} = \langle 4, -3, 5\rangle$ and $\mathbf{OB} = \langle 9, 7, -10\rangle$

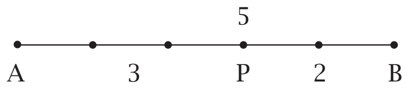
then $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$= \langle 9, 7, -10\rangle - \langle 4, -3, 5\rangle$

$= \langle 5, 10, -15\rangle$.

$\mathbf{AB}$ is the displacement required to get from A to B

(b) Find P such that $\mathbf{AP} : \mathbf{PB} = 3 : 2$.



$$\mathbf{AP} = \frac{3}{5}\,\mathbf{AB}$$

$$= \frac{3}{5}\langle 5, 10, -15\rangle$$

$$= \langle 3, 6, -9\rangle$$

$$\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$

$$= \langle 4, -3, 5\rangle + \langle 3, 6, -9\rangle$$

$$= \langle 7, 3, -4\rangle$$

$\therefore$ P is $(7, 3, -4)$

Example Two forces $\mathbf{F}_1 = \langle 3, -4, 6\rangle$ N and

$\mathbf{F}_2 = \langle -2, 8, -9\rangle$ N

act on a block. Find the single force $\mathbf{F}_3$ which will keep the system in equilibrium.

$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$

$\mathbf{F}_3 = -(\mathbf{F}_1 + \mathbf{F}_2)$

$= -(\langle 3, -4, 6\rangle + \langle -2, 8, -9\rangle)$

$= -\langle 1, 4, -3\rangle$

$= \langle -1, -4, 3\rangle$ N is the required force.

THE DOT PRODUCT IN 3D

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ it is not hard to show that

$a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

As in 2D, each side of this equation is assigned the dot product notation, $\mathbf{a} \cdot \mathbf{b}$:

$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ Rectangular Form

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$ Polar Form

The angle between $\mathbf{a}$ and $\mathbf{b}$ is then found by using

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \quad \text{where } \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

All of the properties of the dot product listed on p.79 also apply to three dimensions.

Example Find the angle between $\mathbf{a} = \langle 4, -3, 5\rangle$ and $\mathbf{b} = \langle 9, 7, -10\rangle$.

$$\cos\theta = \frac{4 \times 9 + -3 \times 7 + 5 \times -10}{\sqrt{4^2 + 3^2 + 5^2}\;\sqrt{9^2 + 7^2 + 10^2}}$$

$$= \frac{-35}{\sqrt{50}\;\sqrt{230}}$$

$$\therefore \quad \theta = 109.05^\circ \quad \text{2dp}$$

Example If $\mathbf{a} = \langle 2, -3, 4\rangle$ and $\mathbf{b} = \langle 1, 5, p\rangle$ find p which makes $\mathbf{a}$ and $\mathbf{b}$ perpendicular.

$\mathbf{a} \cdot \mathbf{b} = 0$ for perpendicular vectors

$$\therefore \quad \langle 2, -3, 4\rangle \cdot \langle 1, 5, p\rangle = 0$$
$$2 \times 1 + -3 \times 5 + 4p = 0$$
$$\therefore \quad p = 3.25$$

VECTOR EQUATION OF A PLANE

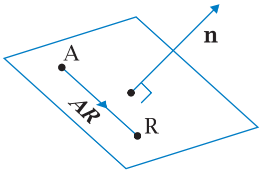
In 3D a plane is any flat surface. The direction or orientation of the plane with respect to the x–y–z coordinate system is specified by a vector $\mathbf{n}$ which is normal or at 90˚ to the plane. If the plane passes through point A and has a normal vector of $\mathbf{n}$ and R (x, y, z) is any point on the plane, then

$$\mathbf{AR} \cdot \mathbf{n} = 0$$

$(\mathbf{OR} - \mathbf{OA}) \cdot \mathbf{n} = 0$ and if $\mathbf{r} = \mathbf{OR}$ and $\mathbf{a} = \mathbf{OA}$

then $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$

$$\mathbf{r} \cdot \mathbf{n} - \mathbf{a} \cdot \mathbf{n} = 0$$
$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$



$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ is the vector equation of the plane with normal vector $\mathbf{n}$ passing through point A where $\mathbf{OA} = \mathbf{a}$.

Note: You may remember that in 2D, $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ is the normal form or scalar product form of the vector equation of a line!

Example If $\mathbf{n} = \langle 2, -4, 3\rangle$ and A $(3, 2, -1)$ is on the plane, find its vector equation and its Cartesian equation.

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$
$$\mathbf{r} \cdot \langle 2, -4, 3\rangle = \langle 3, 2, -1\rangle \cdot \langle 2, -4, 3\rangle$$
$$= 6 - 8 - 3$$

$\mathbf{r} \cdot \langle 2, -4, 3\rangle = -5$ Vector Equation

or $\langle x, y, z\rangle \cdot \langle 2, -4, 3\rangle = -5$

i.e. $2x - 4y + 3z = -5$ Cartesian Equation

VECTOR EQUATION OF A LINE IN 3D

◇ A line in 3D can only be represented parametrically. There is no scalar product form as is the case in 2D because $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ becomes the equation of a plane in 3D.

If a line passes through A (a_1, a_2, a_3), is parallel to $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{r} = \mathbf{OR}$ is any point on the line, then its vector equation is:

$$\mathbf{OR} = \mathbf{OA} + \lambda\mathbf{u}$$

or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$

where λ is the parameter.

The parametric equations follow from

$$\langle x, y, z\rangle = \langle a_1, a_2, a_3\rangle + \lambda\langle u_1, u_2, u_3\rangle$$

i.e. $x = a_1 + \lambda u_1$, $y = a_2 + \lambda u_2$, $z = a_3 + \lambda u_3$

- The equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ represents the motion of an object in a straight line when t (time) is used as the parameter and $\mathbf{v}$ (velocity) is used for the parallel vector,

$$\text{i.e.} \quad \mathbf{r} = \mathbf{a} + t\mathbf{v}$$
$$\text{where } t\mathbf{v} \text{ represents the displacement from A}$$

  This means that all of the collision and closest approach problems involving relative velocity that were fully covered in the Specialist Mathematics (Units 1 and 2) Study Guide pp.55–59 apply equally well to 3D situations.

VECTOR EQUATION OF A SPHERE

If a sphere has centre C (c_1, c_2, c_3) and radius a then the vector equation is

$$|\mathbf{r} - \mathbf{c}| = a$$

where $\mathbf{r} = \mathbf{OR}$ and R (x, y, z) is any point on the sphere's surface.

The equivalent Cartesian equation is:

$$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = a^2$$

APPLICATIONS OF VECTORS IN 3D

- If two objects A and B have velocities $\mathbf{V}_A$ and $\mathbf{V}_B$ then the relative velocity ${}_A\mathbf{V}_B$ is the velocity of object A as seen by an observer travelling with velocity $\mathbf{V}_B$. Also, it can be shown that,

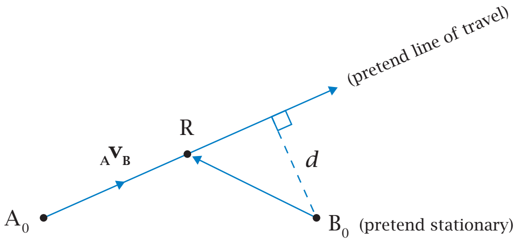
$$\begin{aligned} & {}_A\mathbf{V}_B = \mathbf{V}_A - \mathbf{V}_B \\ \text{and} \quad & {}_B\mathbf{V}_A = \mathbf{V}_B - \mathbf{V}_A \end{aligned}$$

- If, at $t = 0$, objects A and B have positions A_0 and B_0 then their initial displacement is either $\mathbf{A_0B_0}$ or $\mathbf{B_0A_0}$.

$$\text{If it can be shown that } \mathbf{A_0B_0} = t\,{}_A\mathbf{v}_B \text{ then the objects collide after } t \text{ units of time.}$$

- If the objects don't collide then there may be a time when they are closest. There are two ways to do these types which both involve the dot product in different ways. For both cases it is presumed that the initial positions are A_0 and B_0 and the constant velocities are $\mathbf{v}_A$ and $\mathbf{v}_B$. Instead of two moving objects, the situation for both cases is best modelled by pretending that one of the objects is stationary and is observing the other object moving with a relative velocity along a pretend line of travel. So first decide which object you are going to pretend is stationary, say this is B, then find ${}_A\mathbf{v}_B$.

  1st Method: i. Find $\mathbf{OR} = \mathbf{OA_0} + t\,{}_A\mathbf{v}_B$ which becomes the pretend line of travel of object A.



  ii. Now find $\mathbf{B_0R} = \mathbf{OR} - \mathbf{OB_0}$ which is the motion of A as seen by someone actually travelling with velocity $\mathbf{v}_B$ but pretending to be stationary at B.

  iii. Let $\mathbf{B_0R} \cdot {}_A\mathbf{v}_B = 0$ and solve for t which is the time A and B are closest.

  iv. Substitute t into $|\mathbf{B_0R}|$ gives the closest distance d that the objects get to each other.

2nd Method:

i. Find $\mathbf{A_0B_0}$

ii. Find the angle θ between $\mathbf{A_0B_0}$ and ${}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}}$ using

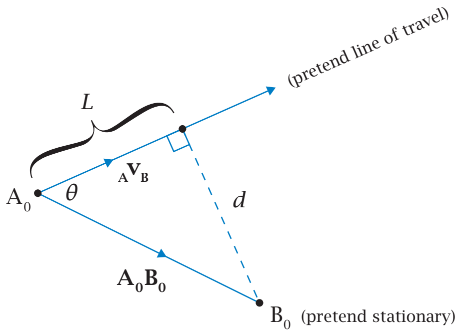
$$\cos\theta = \frac{{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} \cdot \mathbf{A_0B_0}}{|{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}}|\,|\mathbf{A_0B_0}|}$$

iii. Find $d = |\mathbf{A_0B_0}| \sin\theta$ and $L = |\mathbf{A_0B_0}| \cos\theta$

iv. Find $t = \dfrac{L}{|{}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}}|}$

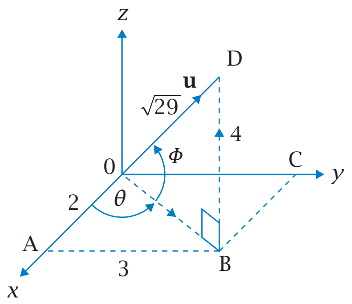
v. Now the closest distance d and the time at which it happens are both known.

The diagram associated with the above calculations is:



POLAR COORDINATES IN 3D

This topic is not specifically mentioned in the syllabus but it is included here for its application to Earth geometry. The vector $\mathbf{u} = \mathbf{OD}$ on the diagram on p.80 can be specified in polar terms, rather than by its components, by defining two angles $\theta = \angle\mathrm{BOA}$ and $\phi = \angle\mathrm{DOB}$ as shown below for $\mathbf{u} = \langle 2, 3, 4\rangle$.



$$|\mathbf{u}| = \sqrt{2^2 + 3^2 + 4^2}$$

$$= \sqrt{29} \quad \text{units}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right)$$

$$= 56.31^\circ \quad \text{2dp}$$

$$\phi = \sin^{-1}\left(\frac{4}{\sqrt{29}}\right)$$

$$= 47.97^\circ \quad \text{2dp}$$

In general for $\mathbf{u} = \langle x, y, z\rangle$

$$|\mathbf{u}| = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad , \quad -180^\circ < \theta \le 180^\circ$$

$$\phi = \sin^{-1}\left(\frac{z}{|\mathbf{u}|}\right) \quad , \quad -90^\circ \le \phi \le 90^\circ$$

Note:

1. The positive directions for θ and ϕ are shown in the diagram above.
2. The signs of x and y will determine the quadrant for θ in the xy plane which will then enable you to find the correct value of θ within the above domain.
3. When given $|\mathbf{u}|$, θ and ϕ the process of finding x, y and z is covered in the Worked Example Section and in the Exercises.
4. When doing Earth geometry, θ will correspond to longitude, ϕ will correspond to latitude and $|\mathbf{u}| = \mathrm{R}$ is the radius of the Earth.

THE VECTOR CROSS PRODUCT

The cross product of two vectors **a** and **b** is written **a** × **b**, and unlike the dot product, the result is a vector.

The vector cross product is only defined for three-dimensional vectors, and the vector **a** × **b** is perpendicular to both **a** and **b**.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$

Example If $\mathbf{a} = \langle 2, 1, -3 \rangle$ and $\mathbf{b} \langle 2, 3, -1 \rangle$, then determine $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1(-1) - (-3)(3) \\ -3(2) - (2)(-1) \\ 2(3) - (1)(2) \end{pmatrix} = \begin{pmatrix} -1 + 9 \\ -6 + 2 \\ 6 - 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix}$$

MATRICES

Systems of Equations

For a system of linear equations of the form

$$\begin{aligned} &a\,x + b\,y + c\,z = d \\ &e\,x + f\,y + g\,z = h \\ &m\,x + n\,y + p\,z = q \end{aligned}$$

$\begin{bmatrix} a & b & c \\ e & f & g \\ m & n & p \end{bmatrix}$ is called the **coefficient matrix**

$\begin{bmatrix} a & b & c & : & d \\ e & f & g & : & h \\ m & n & p & : & q \end{bmatrix}$ is called the **augmented matrix**

$\begin{bmatrix} a & b & c & : & d \\ 0 & r & s & : & t \\ 0 & 0 & u & : & v \end{bmatrix}$ is called the **echelon form**

Note: i) $u \neq 0$ implies one solution

ii) $u = 0,\ v \neq 0$ implies no solutions

iii) $u = 0,\ v = 0$ implies infinite solutions

Inverse Matrix

The inverse of a 2 × 2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is given by $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $a\,d - b\,c = \det A$

Worked Examples

3.1 For the vectors $\mathbf{a} = \mathbf{OA} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{OB} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, find:

(a) the angle between $\mathbf{a}$ and $\mathbf{b}$

(b) a vector equation of the line passing through C (1, 2, –3) and D, the midpoint of **AB**

(c) the points where the above line intersects the sphere $|\mathbf{r} - \langle 3, -4, 1\rangle| = 10$.

(a)

$$\begin{aligned}
|\mathbf{a}| &= \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \\
|\mathbf{b}| &= \sqrt{5^2 + 1^2 + 3^2} = \sqrt{35} \\
\mathbf{a} \cdot \mathbf{b} &= \langle 2, -3, 4\rangle \cdot \langle 5, 1, -3\rangle \\
&= 10 - 3 - 12 \\
&= -5
\end{aligned}$$

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \\
&= \frac{-5}{\sqrt{29}\,\sqrt{35}} \\
\theta &= 99^\circ
\end{aligned}$$

The angle between the vectors is 99°

(b)

$$\begin{aligned}
\mathbf{OD} &= \frac{1}{2}(\mathbf{OA} + \mathbf{OB}) \\
&= \frac{1}{2}(\langle 2, -3, 4\rangle + \langle 5, 1, -3\rangle) \\
&= \frac{1}{2}\langle 7, -2, 1\rangle \\
&= \langle 3.5, -1, 0.5\rangle \\
\mathbf{CD} &= \mathbf{OD} - \mathbf{OC} \\
&= \langle 3.5, -1, 0.5\rangle - \langle 1, 2, -3\rangle \\
&= \langle 2.5, -3, 3.5\rangle
\end{aligned}$$

> If A and B are the end points of a line and D is the midpoint of AB, then the position vector of D is given by
>
> $$\mathbf{OD} = \frac{1}{2}(\mathbf{OA} + \mathbf{OB})$$

Using point C (1, 2, –3) on the line

$$\mathbf{r} = \langle 1, 2, -3\rangle + t\langle 2.5, -3, 3.5\rangle$$

(c)

$$\begin{aligned}
|\mathbf{r} - \langle 3, -4, 1\rangle| &= 10 \\
|\langle x, y, z\rangle - \langle 3, -4, 1\rangle| &= 10 \\
|\langle x - 3, y + 4, z - 1\rangle| &= 10 \\
(x - 3)^2 + (y + 4)^2 + (z - 1)^2 &= 100
\end{aligned}$$

$$\mathbf{r} = \langle 1 + 2.5t, 2 - 3t, -3 + 3.5t\rangle$$ component form of the line

$$x = 1 + 2.5t, \quad y = 2 - 3t, \quad z = -3 + 3.5t$$ parametric equations

sub into sphere equation

$$\begin{aligned}
(1 + 2.5t - 3)^2 + (2 - 3t + 4)^2 + (-3 + 3.5t - 1)^2 &= 100 \\
(2.5t - 2)^2 + (6 - 3t)^2 + (-4 + 3.5t)^2 &= 100 \\
6.25t^2 - 10t + 4 + 36 - 36t + 9t^2 + 16 - 28t + 12.25t^2 &= 100 \\
27.5t^2 - 74t - 44 &= 0
\end{aligned}$$

$t = 3.192$ or -0.501 by using the quadratic formula or graphics calculator
if $t = 3.192$, intersection point is (9.0, –7.6, 8.2)
if $t = -0.501$, intersection point is (–0.3, 3.5, –4.8)

3.2 (a) Find the angle between $\mathbf{a} = \langle 5, -4, -7\rangle$ and the positive

(i) x axis — call it α_1

(ii) y axis — call it α_2

(iii) z axis — call it α_3

(b) Confirm that $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1$.

(a) $|\mathbf{a}| = \sqrt{5^2 + 4^2 + 7^2} = \sqrt{90}$

(i) x axis $\mathbf{i} = \langle 1, 0, 0\rangle$

$$\cos\alpha_1 = \frac{\mathbf{i}\cdot\mathbf{a}}{|\mathbf{i}|\,|\mathbf{a}|}$$

$$= \frac{\langle 1, 0, 0\rangle \cdot \langle 5, -4, -7\rangle}{1\sqrt{90}}$$

$$= \frac{5}{\sqrt{90}}$$

$$\therefore\quad \alpha_1 = 58.19^\circ$$

(ii) y axis $\mathbf{j} = \langle 0, 1, 0\rangle$

$$\cos\alpha_2 = \frac{\mathbf{j}\cdot\mathbf{a}}{|\mathbf{j}|\,|\mathbf{a}|}$$

$$= \frac{-4}{\sqrt{90}}$$

$$\therefore\quad \alpha_2 = 114.94^\circ$$

(iii) z axis $k = \langle 0, 0, 1\rangle$

$$\cos\alpha_3 = \frac{-7}{\sqrt{90}}$$

$$\therefore\quad \alpha_3 = 137.55^\circ$$

(b) $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3$

$= \cos^2 58.19^\circ + \cos^2 114.94^\circ + \cos^2 137.55^\circ$

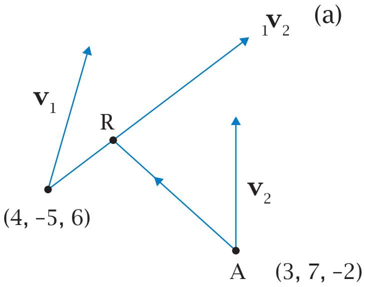
$= 1.000090604$

$= 1$ without round-off error

3.3 Two jets have velocities of $\mathbf{v}_1 = \langle 200.350, 450\rangle$ m/s and $\mathbf{v}_2 = \langle -300, -450, 250\rangle$ m/s. If their positions at $t = 0$ sec are $\mathbf{r}_1 = \langle 4, -5, 6\rangle$ km and $\mathbf{r}_2 = \langle 3, 7, -2\rangle$ km, find:

(a) the time when the jets are closest

(b) the distance between them at this time.



pretend stationary

(a)
$$\begin{aligned} {}_1\mathbf{v}_2 &= \mathbf{v}_1 - \mathbf{v}_2 \\ &= \langle 200, 350, 450\rangle - \langle -300, -450, 250\rangle \\ &= \langle 500, 800, 200\rangle \text{ m/s} \\ &= \langle 0.5, 0.8, 0.2\rangle \text{ km/s} \end{aligned}$$

If jet 2 is considered to be stationary at $\langle 3, 7, -2\rangle$ km, then jet 1 has a path (as seen by jet 2) whose equation is

$$\begin{aligned} \mathbf{r} = \mathbf{OR} &= \langle 4, -5, 6\rangle + t\langle 0.5, 0.8, 0.2\rangle \\ &= \langle 4 + 0.5t, -5 + 0.8t, 6 + 0.2t\rangle \end{aligned}$$

The vector from jet 2 to jet 1 is **AR** where A is (3, 7, -2) km. and $\mathbf{AR} = \mathbf{OR} - \mathbf{OA}$

$= \langle 4 + 0.5t, -5 + 0.8t, 6 + 0.2t\rangle - \langle 3, 7, -2\rangle$

$= \langle 1 + 0.5t, 0.8t - 12, 0.2t + 8\rangle$

Now letting $\mathbf{AR} \cdot {}_1\mathbf{v}_2 = 0$

we have
$$\begin{aligned} \langle 1 + 0.5t, 0.8t - 12, 0.2t + 8\rangle \cdot \langle 0.5, 0.8, 0.2\rangle &= 0 \\ 0.5(1 + 0.5t) + 0.8(0.8t - 12) + 0.2(0.2t + 8) &= 0 \\ 0.5 + 0.25t + 0.64t - 9.6 + 0.04t + 1.6 &= 0 \\ 0.93t &= 7.5 \\ t &= \frac{7.5}{0.93} \\ t &= 8.0645 \text{ seconds} \end{aligned}$$

So after approximately 8.06 seconds the jets are closest.

(b) Substitute $t = 8.0645$ into **AR**

gives $\quad \mathbf{AR} = \langle 1 + 0.5(8.0645), 0.8(8.0645) - 12, 0.2(8.0645) + 8\rangle$

$\quad = \langle 5.032, -5.548, 9.613\rangle$ km

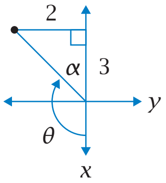
and $\quad |\mathbf{AR}| = 12.19$ km

So the closest the jets are is 12.19 km after 8.06 seconds.

3.4 (a) Find the polar equivalent for $\mathbf{u} = \langle -3, -2, 5\rangle$.

(b) Find $\mathbf{u}$ in component form if $|\mathbf{u}| = 8$, $\theta = -30^\circ$ and $\phi = -60^\circ$.

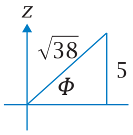
(a)



$$|\mathbf{u}| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38}$$

$$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.69^\circ$$

$$\theta = -(180^\circ - 33.69^\circ) = -146.31^\circ$$



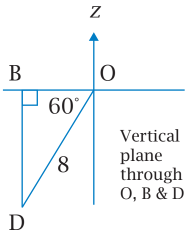
$$\sin\phi = \frac{5}{\sqrt{38}} \qquad \phi = 54.20^\circ$$

So $\mathbf{u} = \langle -3, -2, 5\rangle$ has $|\mathbf{u}| = \sqrt{38}$

$\theta = -146.31^\circ$ 2 d.p.

$\phi = 54.20^\circ$ 2 d.p.

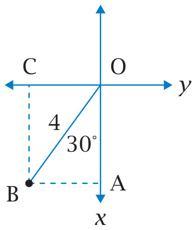
(b)



$$\sin 60^\circ = \frac{BD}{8} \qquad \therefore \quad BD = 6.93$$

So the z component = -6.93

$$\cos 60^\circ = \frac{BO}{8} \qquad \therefore \quad BO = 4$$



$$\cos 30^\circ = \frac{OA}{4} \qquad \therefore \quad OA = 3.46$$

$$\sin 30^\circ = \frac{AB}{4} \qquad \therefore \quad AB = 2$$

So $\quad$ the x component = 3.46

and $\quad$ the y component = -2

The required vector $\mathbf{u} = \langle 3.46, -2, -6.93\rangle$ 2 d.p.

Note that in each case above, two 2D diagrams have been drawn rather than one 3D diagram. The letters A, B, C and D in the 2D diagrams do correspond to those on the 3D diagram on p.78.

3.5 At $t = 0$ hrs two objects A and B are moving according to:

$A_0\,(2, -3, 4)$ km $\quad \mathbf{v_A} = \langle 0.3, -0.2, 0.1\rangle$ km/hr

$B_0\,(11.6, -6.6, 7.6)$ km $\quad \mathbf{v_B} = \langle -0.5, 0.1, -0.2\rangle$ km/hr .

Prove using a relative velocity that the objects collide and state when and where this happens.

$$\begin{aligned} {}_{\mathrm{A}}\mathbf{v}_{\mathbf{B}} &= \mathbf{v_A} - \mathbf{v_B} \\ &= \langle 0.3, -0.2, 0.1\rangle - \langle -0.5, 0.1, -0.2\rangle \\ &= \langle 0.8, -0.3, 0.3\rangle \text{ km/hr} \\ \mathbf{A_0B_0} &= \mathbf{OB_0} - \mathbf{OA_0} \\ &= \langle 11.6, -6.6, 7.6\rangle - \langle 2, -3, 4\rangle \\ &= \langle 9.6, -3.6, 3.6\rangle \text{ km} \end{aligned}$$

let $\quad \mathbf{A_0B_0} = t\,{}_{\mathrm{A}}\mathbf{v}_{\mathbf{B}}$

$\langle 9.6, -3.6, 3.6\rangle = t\,\langle 0.8, -0.3, 0.3\rangle$

$9.6 = t\,(0.8) \quad , -3.6 = t\,(-0.3) \quad , \quad 3.6 = t\,(0.3)$

$t = 12 \qquad t = 12 \qquad t = 12$

So the objects collide after 12 hours.

$\mathbf{r_A} = \mathbf{OA_0} + t\mathbf{v_A}$

$= \langle 2, -3, 4\rangle + 12\langle 0.3, -0.2, 0.1\rangle$

$= \langle 5.6, -5.4, 5.2\rangle$ km

and the objects collide at $(5.6, -5.4, 5.2)$ km

3.6 Find the vector $\mathbf{c}$ which is perpendicular to $\mathbf{a} = \langle -2, 2, 2\rangle$ and $\mathbf{b} = \langle 1, -3, -7\rangle$ and has a magnitude of $\sqrt{14}$ units.

Let $\mathbf{c} = \langle x, y, z\rangle \qquad |\mathbf{c}| = \sqrt{14} \qquad \therefore \quad x^2 + y^2 + z^2 = 14$

$\mathbf{a} \cdot \mathbf{c} = 0 \qquad \therefore \quad -2x + 2y + 2z = 0$

i.e. $\quad x - y - z = 0 \quad \rightarrow \quad x = y + z$

$\mathbf{b} \cdot \mathbf{c} = 0 \qquad \therefore \quad x - 3y - 7z = 0 \quad \rightarrow \quad x = 3y + 7z$

$$\begin{aligned} \therefore \quad y + z &= 3y + 7z \\ 2y + 6z &= 0 \\ y &= -3z \end{aligned}$$

$$\begin{aligned} x^2 + y^2 + z^2 &= 14 \\ y &= -3z \qquad x = y + z \end{aligned}$$

$$\begin{aligned} (y + z)^2 + y^2 + z^2 &= 14 \\ (-3z + z)^2 + (-3z)^2 + z^2 &= 14 \\ 4z^2 + 9z^2 + z^2 &= 14 \\ 14z^2 &= 14 \\ z &= \pm 1 \end{aligned}$$

$z = 1 \quad$ gives $\quad y = -3 \quad$ and $\quad x = -2$

$z = -1 \quad$ gives $\quad y = 3 \quad$ and $\quad x = 2$

$\therefore \quad \mathbf{c} = \langle -2, -3, 1\rangle \quad$ or $\quad \mathbf{c} = \langle 2, 3, -1\rangle$

3.7 Using the elimination method, solve the following system of equations:

$$3x + 2y - z = 19$$
$$4x - y + 2z = 4$$
$$2x + 4y - 5z = 32$$

$3x + 2y - z = 19$ → Equation ①

$4x - y + 2z = 4$ → Equation ②

$2x + 4y - 5z = 32$ → Equation ③

Equation ① × 2	$6x + 4y - 2z = 38$	
Equation ②	$4x - y + 2z = 4$	
Addition eliminates ‘z’	$10x + 3y = 42$	→ Equation ④
Equation ① × 5	$15x + 10y - 5z = 95$	
Equation ③	$2x + 4y - 5z = 32$	
Subtraction eliminates ‘z’	$13x + 6y = 63$	→ Equation ⑤
Equation ④ × 2	$20x + 6y = 84$	
Equation ⑤	$13x + 6y = 63$	
Subtraction eliminates ‘y’	$7x = 21$	
	$x = 3$	
Substitute $x = 3$ into Equation ④	$10(3) + 3y = 42$	
	$30 + 3y = 42$	
	$3y = 12$	
	$y = 4$	
Substitute $x = 3$, $y = 4$ into Equation ①	$3(3) + 2(4) - z = 19$	
	$9 + 8 - z = 19$	
	$-z = 2$	
	$z = -2$	

∴ $x = 3$, $y = 4$ and $z = -2$

3.8 Find A^{-1} given $A = \begin{bmatrix} 2 & x \\ -3y & \frac{1}{2} \end{bmatrix}$

$$\det A = 2x\,\tfrac{1}{2} - (x)(-3y)$$
$$= 1 + 3xy$$
$$A^{-1} = \frac{1}{3xy+1}\begin{bmatrix} \frac{1}{2} & -x \\ 3y & 2 \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{1}{6xy+2} & \dfrac{-x}{3xy+1} \\ \dfrac{3y}{3xy+1} & \dfrac{2}{3xy+1} \end{bmatrix}$$

3.9 Solve the following system of 3 linear equations with 3 variables.

$2x - y + z = 7$
$x + 5y - 2z = -18$
$3x + y + 2z = 4$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 7 \\ 1 & 5 & -2 & -18 \\ 3 & 1 & 2 & 4 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 3 & 25 \\ 0 & 11 & -5 & -43 \\ 0 & -14 & 8 & 58 \end{array}\right]\begin{array}{l} R_1 - R_2 \\ 2R_2 - R_1 \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 3 & 25 \\ 0 & 1 & -\frac{5}{11} & -\frac{43}{11} \\ 0 & 1 & -\frac{4}{7} & -\frac{29}{7} \end{array}\right]\begin{array}{l} R_1 \\ R_2 \div 11 \\ R_3 \div -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 3 & 25 \\ 0 & 1 & -\frac{5}{11} & -\frac{5}{11} \\ 0 & 0 & \frac{9}{77} & \frac{18}{77} \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_2 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 3 & 25 \\ 0 & 1 & -\frac{5}{11} & -\frac{43}{11} \\ 0 & 0 & 1 & 2 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \div \frac{9}{77} \end{array}$$

* use the fraction function on your calculator

Alternative solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 7 \\ 1 & 5 & -2 & -18 \\ 3 & 1 & 2 & 4 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 7 \\ 0 & -11 & 5 & 43 \\ 0 & -5 & -1 & 13 \end{array}\right]\begin{array}{l} \\ R_1 - 2R_2 \\ 3R_1 - 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 7 \\ 0 & -11 & 5 & 43 \\ 0 & 0 & 36 & 72 \end{array}\right]\begin{array}{l} \\ \\ 5R_2 - 11R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -\frac{5}{11} & -\frac{43}{11} \\ 0 & 0 & 1 & 2 \end{array}\right]\begin{array}{l} R_1 \div 2 \\ R_2 \div -11 \\ R_3 \div 36 \end{array}$$

$R_3 \Rightarrow z = 2$

$R_2 \Rightarrow y - \frac{10}{11} = -\frac{43}{11},\ y = -3$

$R_1 \Rightarrow x + \frac{3}{2} + 1 = \frac{7}{2} \quad x = 1$

from R_3, $z = 2$

from R_2 $y - \frac{5}{11}z = -\frac{43}{11}$

ie $y = -\frac{43}{11} + \frac{5}{11}.2$

$y = -3$

from R_1 $x - 6y + 3z = 25$

$x = 25 + 6\ .\ (-3) - 3(2)$

$= 25 - 18 - 6$

$x = 1$

ie $x = 1,\ y = -3,\ z = 2$

Note: There are programs available on some graphics calculators which will carry out this process step by step. e.g. GAUSELIM on Casio calculators.

3.10 For the system of equations below give the values of *k* and *m* for which the system has:

(a) 1 solution
(b) no solutions
(c) infinite solutions

$$3x + 3y + 3z = -17$$
$$-x - 2y + z = 4$$
$$2x + 5y + kz = m$$

$$\left[\begin{array}{cccc} 3 & 3 & 3 & : & -17 \\ -1 & -2 & 1 & : & 4 \\ 2 & 5 & k & : & m \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccccc} 3 & 3 & 3 & : & -17 \\ 0 & -3 & 6 & : & -5 \\ 0 & 1 & 2+k & : & 8+m \end{array}\right]\begin{array}{l} \\ R_1+3R_2 \\ 2R_2+R_3 \end{array}$$

$$\left[\begin{array}{ccccc} 3 & 3 & 3 & : & -17 \\ 0 & -3 & 6 & : & -5 \\ 0 & 0 & 12+3k & : & 19+3m \end{array}\right]\begin{array}{l} \\ \\ R_2+3R_3 \end{array}$$

(a) 1 solution $\Rightarrow$ $12 + 3k \neq 0$ i.e. $k \neq -4$

(b) 0 solutions $\Rightarrow$ $12 + 3k = 0$ and $19 + 3m \neq 0$
i.e. $k = -4$ and $m \neq -6\tfrac{1}{3}$

(c) infinite solutions $\Rightarrow$ $k = -4$ and $m = -6\tfrac{1}{3}$

3.11 For what values of p is the inverse of $\begin{bmatrix} 2 & p \\ 3 & 6 \end{bmatrix}$ not defined?

(ie. for which the matrix is singular)

The inverse of a 2 × 2 matrix is not defined if $ad - bc = 0$. Since $a = 2$, $b = p$, $c = 3$ and $d = 6$ it is not defined if $12 - 3p = 0$
ie if $p = 4$

3.12 Solve the following series of 2 equations with 2 unknowns using the inverse of a matrix.
$2m + 3n = 2$
$5m - 2n = 24$

$$\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 2 \\ 24 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$$A^{-1} = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{19} & \frac{3}{19} \\ \frac{5}{19} & \frac{-2}{10} \end{bmatrix}$$

Note: A^{-1} can be found on a graphics calculator.
Enter matrix as previously shown and use x^{-1} key.

$$\begin{bmatrix} m \\ n \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 24 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{19} & \frac{3}{19} \\ \frac{5}{19} & \frac{-2}{10} \end{bmatrix}\begin{bmatrix} 2 \\ 24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

i.e. $m = 4$ and $n = -2$

PROBLEMS TO SOLVE

CHAPTER 3: VECTORS IN 3-DIMENSIONS

1. If $\mathbf{c} = \langle 1, 0, 1 \rangle$ and $\mathbf{d} = \langle 2, -4, 0 \rangle$ and $3(\mathbf{x} + \mathbf{c}) = 2\mathbf{d} + \mathbf{x}$ find $\mathbf{x}$ in component form.

2. If $\mathbf{OP} = \langle 3, -4, 5 \rangle$ and $\mathbf{OQ} = \langle -2, -1, 3 \rangle$, find the exact magnitude of $\mathbf{PQ}$.

3. Vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + \mathbf{k}$ is perpendicular to both $\langle 1, -1, 1 \rangle$ and $\langle -2, 1, 1 \rangle$. Find a and b.

4. Find the acute angle between $\mathbf{a} = \langle 3, -1, 2 \rangle$ and $\mathbf{b} = \langle 1, -2, -2 \rangle$.

5. If A is $(-1, 4, 7)$ and B is $(3, 0, 5)$, find P if P divides $\mathbf{AB}$ internally in the ratio 3 : 1.

6. Find the vector equation of the sphere which has Cartesian equation
$x^2 + y^2 + z^2 + 2x - 4y + 6z - 11 = 0$.

7. P, Q and R are the points $(8, 3, 0)$, $(10, 4, -10)$ and $(5, k, -3)$ respectively.
   (a) Find $\mathbf{RP}$ and $\mathbf{RQ}$.
   (b) Find the value(s) of k for which ΔPQR is right angled at R.
   (c) If P, Q and R lie on the circumference of a circle, find the coordinates of the centre T.
   (d) Find the vector equation of the sphere, centre T, through P, Q and R.

8. Relative to the origin O, points A and B have position vectors $2\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$ and $6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ respectively; $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ being orthogonal * unit vectors. C is the point such that $\mathbf{OC} = 2\mathbf{OA}$ and D is the midpoint of $\mathbf{AB}$. Find:
   (a) the position vectors of C and D
   (b) a vector equation of line through C and D
   (c) the point on the line through C and D which is closest to the origin O
   (d) the shortest distance between the line through C and D and the point $(5, -2, 4)$.

9. The position vectors of the vertices of a triangle A, B and C are $\langle 6, 3, 7 \rangle$, $\langle 5, -4, 1 \rangle$ and $\langle -1, -3, 8 \rangle$ respectively. Find the angle between sides AB and BC.

10. Find the vector $\langle x, y, z \rangle$ which has a magnitude of 6 cm and makes an angle of 35° and 80° with the positive x and y axes respectively.

11. A plane is flying with the constant velocity of $\mathbf{u} = 0.2\mathbf{i} - 0.1\mathbf{j} + 0.015\mathbf{k}$ km/s, where the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ point East, North and vertically upwards respectively. The initial position of the plane is 30 km West of an airfield, at a height of 2000 m. Find:
    (a) the speed of the plane
    (b) the angle of ascent of the plane

** orthogonal is another word for "at right angles to each other"*

(c) the time at which the plane is closest to the airfield

(d) the closest distance that the plane comes to the airfield.

12. Points P, Q and R have position vectors $\langle 2, -3, 4\rangle$, $\langle 8, 0, 10\rangle$ and $\langle 6, a, b\rangle$.
Find a and b such that P, Q and R are collinear.

13. If a vector makes an angle of 40° to the positive x axis and 70° to the positive y-axis, find the angle that the vector makes with the positive z-axis.

14. Find the point where a line through $(2, 3, -1)$ and parallel to $\langle -1, 4, 2\rangle$ intersects another line through $(18, 9, 16)$ and parallel to $\langle 3, -2, 1\rangle$.

15. Find the point where the line $\mathbf{r} = \langle 2, -3, 4\rangle + t\langle -1, 3, 2\rangle$ intersects the sphere
$|\,\mathbf{r} - \langle -5, 8, 7\rangle\,| = 11$.

16. An aircraft is flying with a constant velocity of $\mathbf{u} = \langle 0.4, -0.1, 0.2\rangle$ km/s where the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ point East, North and vertically upwards respectively. Initially the aircraft is 50 km due West of the target at a height of 4 km. Find:

(a) the speed of the aircraft in km/hr

(b) the time, in minutes and seconds, that the aircraft is due South of the target

(c) the closest distance that the aircraft comes to the target and the time at which this happens to the nearest second

(d) the closest distance that the aircraft comes to a missile site located at $(20, -25, 1.5)$ km.

17. Consider two aircraft A and B flying with constant velocities in m/s and initial positions as shown below.

A at $t = 0_{\text{sec}}$ has position $(5, -2, 1)$ km, $\mathbf{v_A} = \langle -30, 50, 5\rangle$
B at $t = 0_{\text{sec}}$ has position $(-8, -4, 2.5)$ km, $\mathbf{v_B} = \langle 40, 70, 15\rangle$

Find the closest distance that these two aircraft come to each other and the time at which this happens.

18. Find the acute angle that $\mathbf{u} = \langle 4, -1, 3\rangle$
makes with:
(a) the x–y plane
(b) the y–z plane
(c) the x–z plane.

19. A plane has equation $3x - 4y + 2z = 10$.
If $\mathbf{u} = \langle -2, -3, 5\rangle$, find:

(a) the acute angle between $\mathbf{u}$ and the normal vector of the plane

(b) the acute angle between $\mathbf{u}$ and the plane.

20. For $\mathbf{u} = \langle 4, -2, 3\rangle$ find the polar coordinates $|\mathbf{u}|$, θ and ϕ correct to 2dp.

21. Find $\mathbf{u}$, in component form, which has polar coordinates of $|\mathbf{u}| = 25$, $\theta = 120^\circ$ and $\phi = -50^\circ$.

22. (a) A jet has a velocity of $\langle -200, -300, 50\rangle$ m/s. Find the angle of elevation of its flight path

(b) If at 10am the jet is at (300, 400, 5) km, find how close it comes to the control tower located at the origin and the time at which this happens.

(c) How close does the jet come to a stationary weather balloon situated at (120, 200, 2) km.

23. Use vector techniques to find the shortest distance between A (35°N, 75°W) and B (22°S, 50°E). Use $R = 6350$ km as the radius of the earth.

24. If **a** = <1, 3, 4> and **b** = <2, 7, –5>, then determine **a** × **b**.

25. Prove that the vector **a** × **b** is orthogonal to both **a** and **b**.

26. Given the points (2, 2, 1), (–4, 1, 0) and (6, 1, –5) lie in a plane, find the Cartesian equation of the plane.

27. Solve the following systems of equations using the elimination method.

(a)
$$x + y = -3$$
$$2x - 5y = -6$$

(b)
$$2x = 3(y + 1)$$
$$5x = 19 - 4y$$

(c)
$$x + 3y - 6z = 7$$
$$2x - y + 2z = 0$$
$$x + y + 2z = -1$$

(d)
$$x = 1 - y$$
$$2x = z$$
$$2z = -2 - y$$

(e)
$$6x - 25y - 8z = 8$$
$$12x - 15y + 4z = 12$$
$$3x + 5y + 2z = 0$$

28. A total of $20,000 was invested in two separate share portfolios. The low risk investment pays 5% per annum while the high risk investment pays 12% per annum interest. At the end of one year the total portfolio was worth $21,455.

(a) Determine two equations in two unknowns for the situation above.

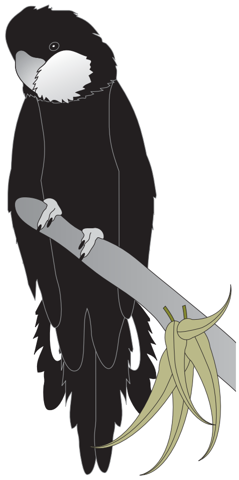
(b) Using the elimination method, determine how many shares were purchased in each portfolio?

29. An elite athlete is placed on a diet where the daily amount of vitamins, proteins and carbohydrates is carefully monitored.

She must have twice as many units of carbohydrates as vitamins. Vitamin units per day must exceed protein units by two. The daily intake of all three types must be exactly 38 units.

By forming three equations in three unknowns and by using the elimination method, determine how many units of each type she will consume per day?

30. Bird seed is made up of three main ingredients: maize, wheat and sunflower seeds. Each type of seed contains a number of units of protein, fat and fibre necessary for the well-being of the birds. The breakdown of each is given below:



	Protein	**Fat**	**Fibre**
Maize	0.4	0.2	0.2
Wheat	0.2	0.3	0.1
Sunflower	0.25	0.4	0.3

How much of each seed is required to make bird seed containing exactly 11 units of protein, 14 units of fat and 9 units of fibre?

31. Find the inverse of the following.

(a) $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 2x \\ -x & 3y \end{bmatrix}$

32. Under what conditions is each of these matrices singular?

(a) $\begin{bmatrix} 2 & 3 \\ -4 & a \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 2b \\ 3b & 6 \end{bmatrix}$

(c) $\begin{bmatrix} c & 2c \\ -3 & 3c \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ d & d^2 \end{bmatrix}$

33. Find the matrix X such that the following is true $X\begin{bmatrix} -4 & 5 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

34. (a) Solve the following series of equations using Gaussian elimination.

i. $2x - 3y = 7$
$3x + 5y = -18$

ii. $5p - 3q = -35$
$4p + 2q = -6$

(b) What values of b and c would result in the following system of equations having infinite solutions?

$x + b\,y = -2$
$2x - (b + 1)y = c$

35. Consider $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, find

(a) A^{-1}

(b) $[AB]^{-1}$

(c) $[A^2]^{-1}$

(d) $AA^{-1}B^{-1}B$

36. Consider $P = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$ and $Q = \begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$

(a) Find the matrix product PQ.

(b) What special relationship exists between P and Q?

37. (a) Find $P = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix}^2$.

(b) What special relationship exists between this matrix and its inverse?

38. Use the inverse of a matrix to solve

$2p - 3q = -19$
$-7p + 7q = 49$

39. (a) Solve the following series of equations with 3 unknowns using matrices.

i. $2a + 3b + 4c = 8$
$3a + 5b - 3c = 13$
$5a + 2b - 4c = 9$

ii. $2p - 3q + r = 12$
$7p - 2q + 2r = 26$
$-2p + 3q - 3r = -22$

(b) For what value of p will the following system of equations have no solution?

$5x + 2y - 3z = 6$

$\frac{5}{16}x + p\,y - \frac{3}{16}z = 9$

$x + y + z = 6$

40. Solve for A; $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} A = \begin{bmatrix} 37 \\ -33 \end{bmatrix}$

41. Use matrix inverse to solve the following:

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} X = \begin{bmatrix} -5 & -3 \\ -1 & 15 \end{bmatrix}$

(b) $P \begin{bmatrix} 2 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -11 \\ -9 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{2}{3} & 1 \end{bmatrix} X = \begin{bmatrix} -7\frac{3}{4} \\ -3\frac{2}{3} \end{bmatrix}$

(d) $M \begin{bmatrix} 2 & 5 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & -3 \\ -\frac{5}{3} & -\frac{2}{3} \end{bmatrix}$

42. Consider a 2 × 2 matrix A that satisfies $A^2 - 3A + I = 0$. Show that $A^{-1} = 3I - A$.

43. Consider a matrix B satisfying $B^2 - 3B + I = 0$.

(a) Show that $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies this equation.

(b) Show that $B^{-1} = 3I - B$ and hence find the inverse of B.

44. Consider a matrix C that satisfies $C^2 - bC - 2I = 0$, where b is an integer.

(a) Show that $C = bI + 2C^{-1}$

(b) Given that $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ satisfies the equation, find b.

(c) Find C^{-1} using two different methods.

45. The inverse of a 3 × 3 matrix is very difficult to find. Consider the matrix equation $D^3 - 4D + I = 0$ where D is a 3 × 3 square matrix

(a) Find an expression for D using the above equation.

(b) Find an expression for D^{-1} using the above equation.

(c) Show that $D = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ satisfies the expression in (b).

(d) Use your answer to part (b) to find the inverse of matrix D.

46. Consider $X = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -2 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$.

Show that $X^{-1} = Y$

47. Solve the following series of equations using two different methods.

$-2a - 3b = 2.5$
$3a - 7b = -9.5$

48. Solve for X, $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} X = \begin{bmatrix} -2 & -1 \\ -4 & -3 \end{bmatrix}$

49. Find the inverse of $\begin{bmatrix} 1 & a \\ m & n \end{bmatrix}$ and hence find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

50. Consider $A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix}$

Write down the matrices AB, A^{-1} and B^{-1} and use these results to find $(AB)^{-1}$

51. Solve the following system of equations using your calculator and matrix inverses.

$2x + 3y + z = 4$
$3x - 4y + 2z = -19$
$-4x + 5y - 3z = 24$

52. Given $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & -1 \\ -1 & 2 & 2 & 1 \\ 5 & -2 & 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 15 & 11 & -7 & 6 \\ 2 & 36 & 4 & -14 \\ -7 & -15 & 23 & 12 \\ 25 & -31 & 13 & 10 \end{bmatrix}$

(a) Find AB from your calculator.

(b) Use your result from part (a) to solve the following system of equations.

$a + b - c + 2d = 8$
$2a + b + c - d = -4$
$-a + 2b + 2c + d = -4$
$5a - 2b + 3c - 2d = -5$

4

VECTOR CALCULUS

Syllabus Checklist

By the end of this chapter, you should be able to:

☐ consider position vectors as a function of time

☐ derive the Cartesian equation of a path given as a vector equation in two dimensions, including ellipses and hyperbolas

☐ differentiate and integrate a vector function with respect to time

☐ determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration

☐ apply vector calculus to motion in a plane, including projectile and circular motion.

FORMULAE AND DEFINITIONS

◇ A vector function $\underset{\sim}{f}(x)$ can be expressed as a sum of scalar component functions $f_1(x)$ and $f_2(x)$. x is a scalar variable.

i.e. $\underset{\sim}{f}(x) = f_1(x)\underset{\sim}{i} + f_2(x)\underset{\sim}{j}$

◇ Limits:

$$\lim_{x\to a} \underset{\sim}{f}(x) = \lim_{x\to a} f_1(x)\underset{\sim}{i} + \lim_{x\to a} f_2(x)\underset{\sim}{j}$$

◇ Continuity:

A vector function $\underset{\sim}{f}(x)$ is continuous if $\lim_{x\to a} \underset{\sim}{f}(x) = \underset{\sim}{f}(a)$, or alternatively, if both of its component functions $f_1(x)$ and $f_2(x)$ are continuous.

◇ Differentiation Properties:

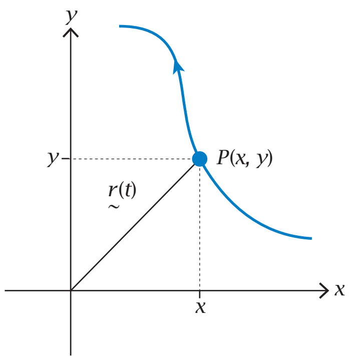
If $\underset{\sim}{f}(x) = f_1(x)\underset{\sim}{i} + f_2(x)\underset{\sim}{j}$ then $\underset{\sim}{f}'(x) = f'_1(x)\underset{\sim}{i} + f'_2(x)\underset{\sim}{j}$

◇ Integration Properties:

If $\underset{\sim}{f}(x) = f_1(x)\underset{\sim}{i} + f_2(x)\underset{\sim}{j}$ then $\int \underset{\sim}{f}(x)dx = [\int f_1(x)dx]\underset{\sim}{i} + [\int f_2(x)dx]\underset{\sim}{j}$

◇ Curvilinear Motion:

Consider a particle moving on a curve. Let P be any point on the curve, with a position vector $\underset{\sim}{r}$ where $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$.



Now x and y vary, depending on the position of P on the curve. We assume x and y are the same functions of a third scalar parameter t, usually called time.
$\therefore x = f(t)$ and $y = g(t)$
so that $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j}$

$x = f(t)$ and $y = g(t)$ are called the parametric equations for the motion. These equations can be solved simultaneously, to eliminate the third variable t, giving y in terms of x.
i.e. the cartesian equation of the curve.

- For a particle undergoing curvilinear motion:

  a) $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j}$ gives the position vector for any point on the curve at time t.

  b) $\underset{\sim}{v}(t) = \dfrac{d\underset{\sim}{r}}{dt} = f'(t)\underset{\sim}{i} + g'(t)\underset{\sim}{j}$ gives the instantaneous velocity vector, at the point, on the curve at time t.

  c) $\underset{\sim}{a}(t) = \dfrac{d\underset{\sim}{v}}{dt} = \dfrac{d^2\underset{\sim}{r}}{dt^2} = f''(t)\underset{\sim}{i} + g''(t)\underset{\sim}{j}$ gives the instantaneous acceleration vector, at the point, on the curve at time t.

  d) $\underset{\sim}{r}$, $\underset{\sim}{v}$, and $\underset{\sim}{a}$ are related by the integration formulae

     i) $\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)dt$

     ii) $\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)dt$

  e) i) the displacement of a particle over an interval $t_1 \le t \le t_2$ is given by $\underset{\sim}{r}(t_2) - \underset{\sim}{r}(t_1)$.

     ii) at time $t = t_1$, the distance a particle is from the origin is given by

     $$\left|\underset{\sim}{r}(t_1)\right| = \sqrt{\left[f(t_1)\right]^2 + \left[g(t_1)\right]^2}$$

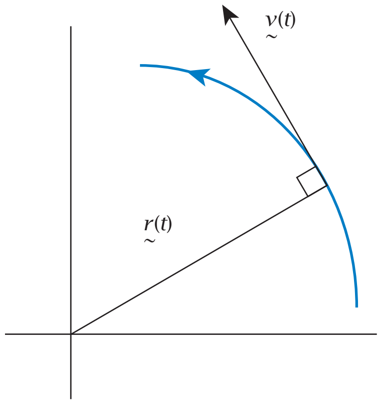
     iii) at time $t = t_1$, the speed a particle has attained is given by

     $$\left|\underset{\sim}{v}(t_1)\right| = \sqrt{\left[f'(t_1)\right]^2 + \left[g'(t_1)\right]^2}$$

- For a particular instant t:

  The instantaneous velocity vector $\underset{\sim}{v}(t)$ is always perpendicular to the position vector $\underset{\sim}{r}(t)$.

  This can be proven by showing that the dot product $\underset{\sim}{r}(t)\cdot\underset{\sim}{v}(t) = 0$



Worked Examples

4.1 Consider $\underset{\sim}{r}$, the position vector from the origin to a point $P(x,y)$, moving on a circle, centre (0,0) and radius r.

Now $\dfrac{x}{r} = Cos\,\theta$ and $\dfrac{y}{r} = Sin\,\theta$

$\therefore x = r\,Cos\,\theta$ and $y = r\,Sin\,\theta$

$\therefore \underset{\sim}{r} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$

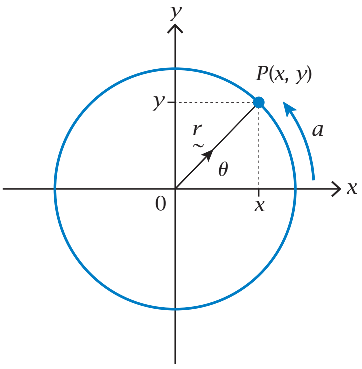
$= r\,Cos\,\theta\,\underset{\sim}{i} + r\,Sin\,\theta\,\underset{\sim}{j}$

Further, the arc length $a = r\theta$

$$\therefore \frac{da}{dt} = r\frac{d\theta}{dt} \quad (r \text{ constant})$$

$\dfrac{da}{dt}$ is the speed the object moves along the circle.

$\dfrac{d\theta}{dt}$ is the angular speed, notated w.



Suppose $\dfrac{da}{dt}$ is constant. $\therefore$ w is constant.

As $w = \dfrac{d\theta}{dt}$

$\therefore d\theta = wdt$

$\therefore \int d\theta = \int wdt$

$\therefore \theta = wt + c$ as w is a constant

We assume $\theta = 0$ when $t = 0$ $\therefore c = 0$

Hence $\theta = wt$

$\therefore \underset{\sim}{r} = r\, Cos\, wt\, \underset{\sim}{i} + r\, Sin\, wt\, \underset{\sim}{j}$

We can see that the position vector $\underset{\sim}{r}$ is indeed a vector function of time t.

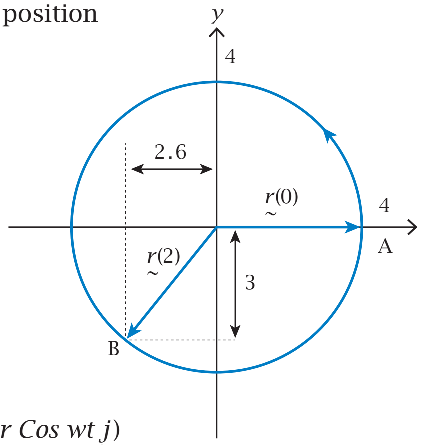
i.e. $\underset{\sim}{r}(t) = r\, Cos\, wt\, \underset{\sim}{i} + r\, Sin\, wt\, \underset{\sim}{j}$

Note 1. If $w = 2$ radians/sec, $r = 4$m then $\underset{\sim}{r}(t) = 4\, Cos\, 2t\, \underset{\sim}{i} + 4\, Sin\, 2t\, \underset{\sim}{j}$

Now at $t = 0$ $\underset{\sim}{r}(0) = 4\, Cos\, 0\, \underset{\sim}{i} + 4\, Sin\, 0\, \underset{\sim}{j} = 4\, \underset{\sim}{i}$

and at $t = 2$ $\underset{\sim}{r}(2) = 4\, Cos\, 4\, \underset{\sim}{i} + 4\, Sin\, 4\, \underset{\sim}{j} \approx -2.6\, \underset{\sim}{i} - 3.0\, \underset{\sim}{j}$

The diagram shows the position of the particle at A, with position vector $\underset{\sim}{r}(0)$ and at B, with position vector $\underset{\sim}{r}(2)$.



Note 2. If $\underset{\sim}{r}(t) = r\, Cos\, wt\, \underset{\sim}{i} + r\, Sin\, wt\, \underset{\sim}{j}$

then $\underset{\sim}{v}(t) = \dfrac{d\,\underset{\sim}{r}}{dt} = -wr\, Sin\, wt\, \underset{\sim}{i} + wr\, Cos\, wt\, \underset{\sim}{j}$

and $\underset{\sim}{a}(t) = \dfrac{d\,\underset{\sim}{v}}{dt} = -w^2 r\, Cos\, wt\, \underset{\sim}{i} - w^2 r\, Sin\, wt\, \underset{\sim}{j}$

Therefore

(i) $\underset{\sim}{r}(t) \cdot \underset{\sim}{v}(t) = (r\, Cos\, wt\, \underset{\sim}{i} + r\, Sin\, wt\, \underset{\sim}{j}) \cdot (-\, wr\, Sin\, wt\, \underset{\sim}{i} + wr\, Cos\, wt\, \underset{\sim}{j})$

$= -wr^2 Cos\, wt\, Sin\, wt + wr^2\, Sin\, wt\, Cos\, wt$

$= 0$

this proves, that no matter the values of w or t, the position vector $\underset{\sim}{r}(t)$ and the velocity vector $\underset{\sim}{v}(t)$ will always be perpendicular.

(ii) $\underset{\sim}{a}(t) = -w^2 rCos\, wt\, \underset{\sim}{i} - w^2 r\, Sin\, wt\, \underset{\sim}{j}$

$= -w^2 (rCos\, wt\, \underset{\sim}{i} + r\, Sin\, wt\, \underset{\sim}{j})$

$= -w^2\, \underset{\sim}{r}(t)$

this means that the acceleration is in the opposite direction to the position vector … i.e. directed to the centre.

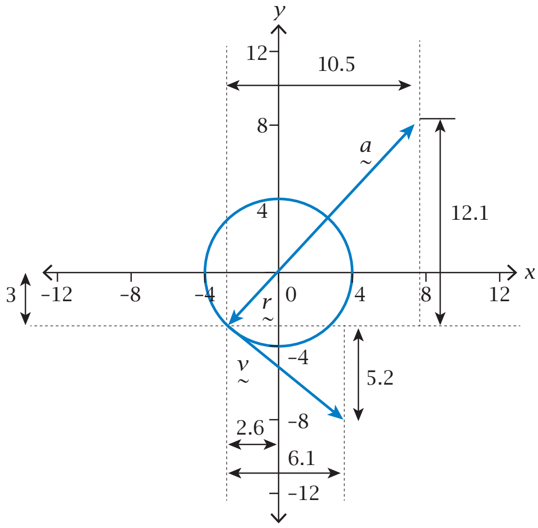
(iii) the relationship between $\underset{\sim}{r}(t)$, $\underset{\sim}{v}(t)$, $\underset{\sim}{a}(t)$ can be shown by the following, relating to Note 1 in Example 5.1.

$\underset{\sim}{r}(t) = 4\, Cos\, 2t\, \underset{\sim}{i} + 4\, Sin\, 2t\, \underset{\sim}{j}$ $\therefore \underset{\sim}{r}(2) = -2.6\, \underset{\sim}{i} - 3.0\, \underset{\sim}{j}$

$\underset{\sim}{v}(t) = 8\, Sin\, 2t\, \underset{\sim}{i} + 8\, Cos\, 2t\, \underset{\sim}{j}$ $\therefore \underset{\sim}{v}(2) = 6.1\, \underset{\sim}{i} - 5.2\, \underset{\sim}{j}$

$\underset{\sim}{a}(t) = -16\, Cos\, 2t\, \underset{\sim}{i} - 16\, Sin\, 2t\, \underset{\sim}{j}$ $\therefore \underset{\sim}{a}(2) = +10.5\, \underset{\sim}{i} + 12.15\, \underset{\sim}{j}$

The following diagram, shows these vectors, to scale for the point P when $t = 2$.



(iv) $\underset{\sim}{v}(t) = -8\ Sin\ 2t\ \underset{\sim}{i} + 8\ Cos\ 2t\ \underset{\sim}{j}$

$$\therefore\ \text{Speed} = \left|\underset{\sim}{v}(t)\right| = \sqrt{(-8\,Sin\ 2t)^2 + (8\,Cos\ 2t)^2}$$
$$= \sqrt{64\,Sin^2 2t + 64\,Cos^2 2t}$$
$$= \sqrt{64\left(Sin^2 2t + Cos^2 2t\right)}$$
$$= \sqrt{64}$$
$$= 8$$

This shows that the speed is independent of time and is the constant value 8 m/sec.

4.2 If $\underset{\sim}{r}(t) = (t - 1)\underset{\sim}{i} + t^2\ \underset{\sim}{j}$ then

(a) sketch the path of the motion

(b) find the Cartesian equation of the motion.

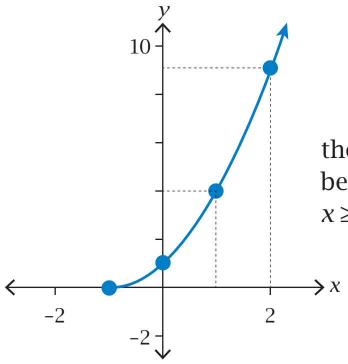
(a) $\underset{\sim}{r}(t) = (t - 1)\underset{\sim}{i} + t^2\ \underset{\sim}{j}$

$\underset{\sim}{r}(0) = -\underset{\sim}{i}$

$\underset{\sim}{r}(1) = \underset{\sim}{j}$

$\underset{\sim}{r}(2) = \underset{\sim}{i} + 4\ \underset{\sim}{j}$

$\underset{\sim}{r}(3) = 2\ \underset{\sim}{i} + 9\ \underset{\sim}{j}$



the path appears to be parabolic, for $x \geq -1$

(b) the parametric equations are $x = t - 1$ ...... ①

$y = t^2$ ...... ②

Eliminiate t ① $\Rightarrow t = x + 1$

$\therefore\ y = (x + 1)^2$

i.e. $\underset{\sim}{r}(t) = (t - 1)\underset{\sim}{i} + t^2\ \underset{\sim}{j}$ defines the path of a parabola with Cartesian equation $y = (x + 1)^2$

4.3 A curve is defined by $\underset{\sim}{r}(t) = (t + 1)\underset{\sim}{i} + (4t - t^2)\,\underset{\sim}{j} \ldots\ldots t \geq 0$

Find

(a) the initial position

(b) the distance from the origin when $t = 5$

(c) the Cartesian equation of the path

(d) the time when the particle lies on the x axis

(e) the maximum distance the particle is from the x axis

(f) the initial speed.

(a) $\underset{\sim}{r}(0) = 1\underset{\sim}{i} + 0\,\underset{\sim}{j}$

$\therefore$ particle is at the point (1,0)

(b) $\underset{\sim}{r}(5) = 6\,\underset{\sim}{i} - 5\,\underset{\sim}{j}$

$$\therefore \left|\underset{\sim}{r}(5)\right| = \sqrt{6^2 + (-5)^2}$$

$$= \sqrt{61}$$

$\therefore$ the particle is $\sqrt{61}$ units from the origin

(c) $x = t + 1 \ldots\ldots\ldots$ ①

$y = 4t - t^2 \ldots\ldots$ ②

① $\Rightarrow t = x - 1$

$\therefore\ y = 4(x - 1) - (x - 1)^2$

$y = 4x - 4 - x^2 + 2x - 1$

$y = -x^2 + 6x - 5$

$\therefore$ a parabola

(d) On the x axis $y = 0$

$\therefore\ j$ component of $\underset{\sim}{r}$ is 0

$\therefore\ 4t - t^2 = 0$

$\therefore\ t = 0$ or 4 seconds

(e) maximise y component of $\underset{\sim}{r}$

$\therefore$ maximise $y = 4t - t^2$

Now $\dfrac{dy}{dt} = 4 - 2t$

and $\dfrac{dy}{dt} = 0 \Rightarrow t = 2$

Since $\dfrac{d^2y}{dt^2} = -2$ *is less than* 0.

y is maximised

then $4t - t^2$

$= 4(2) - (2)^2$

$= 4$

$\therefore$ maximum distance from x axis is 4 units

(f) $\underset{\sim}{v}(t) = \dfrac{d}{dt}\left(\underset{\sim}{r}(t)\right) = \underset{\sim}{i} + (4 - 2t)\,\underset{\sim}{j}$

$\underset{\sim}{v}(0) = \underset{\sim}{i} + 4\,\underset{\sim}{j}$

$\therefore$ initial speed $= \left|\underset{\sim}{v}(0)\right| = \sqrt{1^2 + 4^2}$

$= \sqrt{17}$ units/sec

4.4 If $\underset{\sim}{r}(t) = \frac{\sin(t-2)}{t-2}\underset{\sim}{i} + \frac{t^2-4}{t-2}\underset{\sim}{j}$

Find

(a) $\lim\limits_{t\to 2}\underset{\sim}{r}\left(t\right)$

(b) discuss the continuity of $\underset{\sim}{r}(t)$

(a) $$\lim_{t\to 2}\underset{\sim}{r}\left(t\right) = \lim_{t\to 2}\frac{\sin\left(t-2\right)}{t-2}\underset{\sim}{i} + \lim_{t\to 2}\frac{t^2-4}{t-2}\underset{\sim}{j}$$

Let $x = t-2$

$$= \lim_{x\to 0}\frac{\sin x}{x}\underset{\sim}{i} + \lim_{t\to 2}\frac{\left(t-2\right)\left(t+2\right)}{\left(t-2\right)}\underset{\sim}{j}$$

$$= \underset{\sim}{i} + 4\underset{\sim}{j}$$

(b) $\underset{\sim}{r}(t)$ is not continuous at $t = 2$ as $\underset{\sim}{r}(2)$ does not exist.

4.5 Given $\underset{\sim}{r}''(t) = 2\underset{\sim}{i},\underset{\sim}{r}'(1) = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{r}(1) = 2\underset{\sim}{i}$, find $\underset{\sim}{r}(3)$.

$\underset{\sim}{r}''(t) = 2\underset{\sim}{i}$

$\therefore\ \underset{\sim}{r}'(t) = \int 2\underset{\sim}{i}\ dt$

$= 2t\,\underset{\sim}{i} + \underset{\sim}{c}$ $\qquad\underset{\sim}{c}$ is a constant vector

But $\underset{\sim}{r}'(1) = \underset{\sim}{i} + \underset{\sim}{j}$

$\therefore\ \underset{\sim}{i} + \underset{\sim}{j} = 2\,\underset{\sim}{i} + \underset{\sim}{c}$

$\therefore\ \underset{\sim}{c} = -\,\underset{\sim}{i} + \underset{\sim}{j}$

Hence $\underset{\sim}{r}'(1) = 2t\,\underset{\sim}{i} + (-\underset{\sim}{i} + \underset{\sim}{j})$

$= (2t-1)\underset{\sim}{i} + \underset{\sim}{j}$

Now $\underset{\sim}{r}(t) = \int\underset{\sim}{r}'(t)\ dt$

$= \int[(2t-1)\underset{\sim}{i} + \underset{\sim}{j}]dt$

$= (t^2-t)\underset{\sim}{i} + t\,\underset{\sim}{j} + \underset{\sim}{c}$

But $\underset{\sim}{r}(1) = 2\underset{\sim}{i}$

$\therefore\ 2\underset{\sim}{i} = 0\,\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{c}$

$\therefore\ \underset{\sim}{c} = 2\,\underset{\sim}{i} - \underset{\sim}{j}$

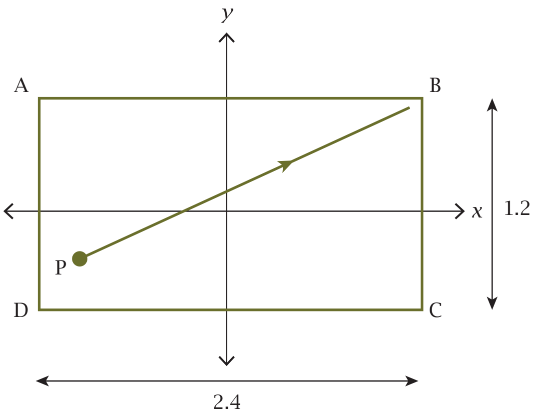
$\therefore\ \underset{\sim}{r}(t) = (t^2-t)\underset{\sim}{i} + t\,\underset{\sim}{j} + 2\,\underset{\sim}{i} - \underset{\sim}{j}$

$= (t^2-t+2)\underset{\sim}{i} + (t-1)\underset{\sim}{j}$

It follows $\underset{\sim}{r}(3) = 8\underset{\sim}{i} + 2\underset{\sim}{j}$

the particle is at the point (8,2) when $t = 3$.

4.6 Bob hits a pool ball, located at *P* in the direction of the pocket *B*, on a pool table 2.4 m long and 1.2 m wide. The *x*- and *y*-axes, as shown on the diagram, bisect the table lengthwise and widthwise respectively. *P* is 40 cm in from edges *AD* and *DC*.



(a) Find the co-ordinates of point *P*.

(b) Find an expression for the position vector for the ball if it is known that the ball is hit with an initial velocity vector of $1.8\,\underset{\sim}{i} - 2.4\,\underset{\sim}{j}$ m/s, producing a constant acceleration of $6.4\,\underset{\sim}{j}$ m/s^2.

(c) Will the ball hit the edge of the table or go in the pocket *B*?

(d) Find the time when this happens.

(e) Find the speed the ball is travelling when this happens.

(a) At P: $x = -1.2 + 0.4 = -0.8$

$y = -0.6 + 0.4 = -0.2$

$\therefore\ P = (-0.8, -0.2)$

(b) $\underset{\sim}{a}(t) = 6.4\,\underset{\sim}{j}$

$\therefore\ \underset{\sim}{v}(t) = \int 6.4\,\underset{\sim}{j}\ dt$

$= 6.4t\,\underset{\sim}{j} + \underset{\sim}{c}$ ($\underset{\sim}{c}$ is a constant vector)

But $\underset{\sim}{v}(0) = 1.8\,\underset{\sim}{i} - 2.4\,\underset{\sim}{j}\ \therefore\ \underset{\sim}{c} = 1.8\,\underset{\sim}{i} - 2.4\,\underset{\sim}{j}$

$\therefore\ \underset{\sim}{v}(t) = 6.4t\,\underset{\sim}{j} + 1.8\,\underset{\sim}{i} - 2.4\,\underset{\sim}{j}$

$= 1.8\,\underset{\sim}{i} + (6.4t - 2.4)\underset{\sim}{j}$

Now $\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)dt = 1.8t\,\underset{\sim}{i} + (3.2t^2 - 2.4t)\underset{\sim}{j} + \underset{\sim}{c}$

Since $P = (-0.8, -0.2)$, $\underset{\sim}{r}(0) = -0.8\,\underset{\sim}{i} - 0.2\,\underset{\sim}{j}\ \therefore\ \underset{\sim}{c} = -0.8\,\underset{\sim}{i} - 0.2\,\underset{\sim}{j}$

$\therefore\ \underset{\sim}{r}(t) = 1.8t\,\underset{\sim}{i} + (3.2t^2 - 2.4t)\underset{\sim}{j} + (-0.8\,\underset{\sim}{i} - 0.2\,\underset{\sim}{j})$

$= (1.8t - 0.8)\,\underset{\sim}{i} + (3.2t^2 - 2.4t - 0.2)\underset{\sim}{j}$

(c) If the ball hits the pocket

$$\begin{array}{ll}
1.8t - 0.8 = 1.2 \text{ and } & 3.2t^2 - 2.4t - 0.2 = 0.6 \\
\therefore\ 1.8t = 2 & 3.2t^2 - 2.4t - 0.8 = 0 \\
t \approx 1.1 & t = \dfrac{2.4 \pm \sqrt{5.76 + 10.24}}{6.4} \\
 & = \dfrac{2.4 \pm 4}{6.4} \\
 & = 1 \quad \text{or} \quad -\cancel{0.25} \ \text{ignore}
\end{array}$$

The solutions are not consistent. $\therefore$ the ball does not go into the pocket and must hit an edge of the table.

(d) Since $y = 0.6$ before $x = 1.2$ in time, the ball must hit edge *AB* before edge *BC*. This happens when $y = 0.6$

i.e. $3.2t^2 - 2.4t - 0.2 = 0.6$

i.e. $t = 1$

(e) $\underset{\sim}{v}(t) = 1.8\,\underset{\sim}{i} + (6.4t - 2.4)\underset{\sim}{j}$

$\therefore\ \underset{\sim}{v}(1) = 1.8\,\underset{\sim}{i} + 4\underset{\sim}{j}$

$$\begin{aligned}
\therefore\ \text{Speed } &= \left|\underset{\sim}{v}(1)\right| = \sqrt{(1.8)^2 + 4^2} \\
&= \sqrt{3.24 + 16} \\
&= \sqrt{19.24} \\
&\approx 4.4 \text{ m/s (1 d.p.)}
\end{aligned}$$

Note 1. How far from the pocket does the ball hit?

When $t = 1$, x-component of $\underset{\sim}{r}(t) = 1.8t - 0.8$

$= 1$

$\therefore$ distance $= 1.2 - 1 = 0.2$ m

i.e. the ball hits the side of the table, on edge *AB*, but 20 cm from *B*.

Note 2. What is the ball's speed when it crosses the *y*-axis?

It crosses the *y*-axis when the *x*-component of $\underset{\sim}{r}(t) = 0$

$\therefore\ 1.8t - 0.8 = 0$

$t = \dfrac{0.8}{1.8} = \dfrac{4}{9}$

Now $\underset{\sim}{v}(t) = 1.8\,\underset{\sim}{i} + (6.4t - 2.4)\,\underset{\sim}{j}$

$\therefore$ Speed $= \left|\underset{\sim}{v}(t)\right| = \sqrt{(1.8)^2 + (6.4t - 2.4)^2}$

When $t = \dfrac{4}{9}$

$$\left|\underset{\sim}{v}\left(\frac{4}{9}\right)\right| = \sqrt{(1.8)^2 + \left[(6.4)\frac{4}{9} - 2.4\right]^2}$$

$= 1.85$ m/s (2 d.p.)

Note 3. This example is designed to show the component techniques of Vector Calculus, not the logistics of real-life examples. It is dubious that the pool ball could be hit with a constant acceleration. It is equally dubious that the path of the ball will be parabolic, as is the case in this example.

PROBLEMS TO SOLVE

CHAPTER 4: VECTOR CALCULUS

1. The parametric equations for a particle undergoing curvilinear motion as $x = 2t$ and $y = t^2$.

   (a) Sketch the path of the particle $0 \leq t \leq 3$

   (b) Find the Cartesian equation of the path and identify its shape.

2. A curve is defined by $\underset{\sim}{r}(t) = 2t\,\underset{\sim}{i} + (4t - t^2)\,\underset{\sim}{j}\ t \geq 0$. Find

   (a) the initial position and speed

   (b) the distance from the origin when $t = 5$

   (c) the Cartesian equation of the path

   (d) when the particle lies on the x-axis

   (e) the maximum distance above the x-axis

3. (a) Evaluate

   i. $\lim\limits_{t\to 0}\left(4t^2\,\underset{\sim}{i} + \dfrac{\sin 2t}{t}\,\underset{\sim}{j}\right)$

   ii. $\lim\limits_{t\to\infty}\left[\left(\dfrac{1}{t}\right)\underset{\sim}{i} + \left(\dfrac{2+3t^2}{4-5t^2}\right)\underset{\sim}{j}\right]$

   (b) Prove $\lim\limits_{t\to 2}\left[t\,\underset{\sim}{i} - 3t^2\,\underset{\sim}{j}\right]\bullet\left[(t-1)\,\underset{\sim}{i} + 2t\,\underset{\sim}{j}\right] = \lim\limits_{t\to 2}\left[t\,\underset{\sim}{i} - 3t^2\,\underset{\sim}{j}\right]\bullet\lim\limits_{t\to 2}\left[(t-1)\,\underset{\sim}{i} + 2t\,\underset{\sim}{j}\right]$

   Note: $\bullet$ is dot product.

4. If $\underset{\sim}{r}(t) = t\,\underset{\sim}{i} + 2\sqrt{t+1}\,\underset{\sim}{j}$, find $\dfrac{d}{dt}\left|\underset{\sim}{r}(t)\right|$

5. If $\underset{\sim}{r}(t) = e^{2t}\,\underset{\sim}{i} + \ln t\,\underset{\sim}{j}$, show $\underset{\sim}{r}(t)\bullet\underset{\sim}{v}(t) = \left|\underset{\sim}{r}(t)\right| \times \dfrac{d}{dt}\left|\underset{\sim}{r}(t)\right|$

6. Find

   (a) $\int\left[t^2\,\underset{\sim}{i} + (2t+5)\,\underset{\sim}{j}\right]dt$

   (b) $\int_0^1\left(2t^2\,\underset{\sim}{i} + e^t\,\underset{\sim}{j}\right)dt$

7. If $\underset{\sim}{v}(t) = 3t^2\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}$,

   (a) find:

   i. $\int_0^\pi \underset{\sim}{v}(t)\,dt$

   ii. $\left|\int_0^\pi \underset{\sim}{v}(t)\,dt\right|$

   (b) interpret the results from (a)

8. A particle moves according to $\underset{\sim}{r}(t) = 3 \cos 2t\,\underset{\sim}{i} + 3 \sin 2t\,\underset{\sim}{j}$. Find:

(a) the velocity function

(b) the acceleration function

(c) the magnitude of the acceleration

(d) the dot products

i. $\underset{\sim}{v} \cdot \underset{\sim}{r}$

ii. $\underset{\sim}{v} \cdot \underset{\sim}{a}$

(e) interpret the results in (d).

9. A particle moving on a curve has position vector $t^{\frac{3}{2}}\,\underset{\sim}{i} + (t - 4)\underset{\sim}{j}$. Find the time(s) when the velocity vector is perpendicular to the position vector.

10. The acceleration of a particle is $2\underset{\sim}{i} + 6t\,\underset{\sim}{j}$. When $t = 0$, $\underset{\sim}{v}(0) = -\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = -\underset{\sim}{i}$. Find:

(a) the velocity vector $\underset{\sim}{v}(t)$

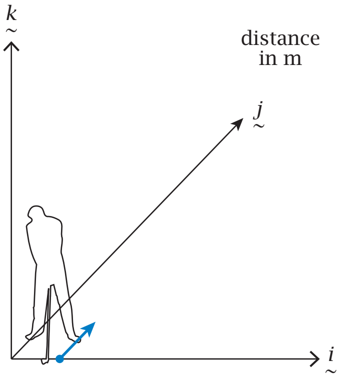
(b) the position vector $\underset{\sim}{r}(t)$

(c) the position of the particle when it reaches its minimum height.

11. (a) Show that the curve with parametric equations $x = (t - 1)^2$ and $y = t(t - 1)(2 - t)$ crosses itself at the point (1,0).

(b) Find the angle between the two tangent lines at the point (1,0).

12. A golfer chips a ball with initial velocity vector $40\,\underset{\sim}{j} + 4.9\,\underset{\sim}{k}$ and an acceleration vector of $2\underset{\sim}{i} - \underset{\sim}{j} - 9.8\,\underset{\sim}{k}$. Find when and where the ball hits the ground (Ball is hit in $\underset{\sim}{j}$ direction).



13. A cyclist is moving around an elliptical track so that her position vector at time t is given by $\underset{\sim}{r}(t) = \frac{5}{2} \sin 2t\,\underset{\sim}{i} + 2 \cos 2t\,\underset{\sim}{j}$. Find:

(a) $\underset{\sim}{v}(t)$

(b) show that her speed is given by $\sqrt{25 - 9 \sin^2 2t}$

(c) when her speed is a maximum $t > 0$.

14. A moving particle has position vector $\underset{\sim}{r}(t) = \cos 2t\,\underset{\sim}{i} - 2 \sin 2t\,\underset{\sim}{j}$ $\quad t \geq 0$. Find:

(a) the particle's original position

(b) the initial speed

(c) the maximum distance the particle achieves from its origin

(d) the first time after $t = 0$ when this occurs.

15. For a position vector $\underset{\sim}{r}(t) = (t - 4)\,\underset{\sim}{i} + \dfrac{1}{(4 - t)}\,\underset{\sim}{j}$

(a) Sketch the curvilinear path in the domain $0 \le t \le 4$

(b) Evaluate

i. $\underset{\sim}{r}(2)$

ii. $\underset{\sim}{v}(2)$

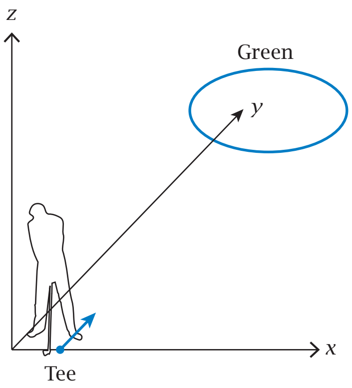
iii. $\underset{\sim}{a}(2)$

(c) Sketch on a diagram, to scale, the vectors in (b) (Use 2 cm = 1 unit).

(d) Find the Cartesian equation of the path and identify its shape.

(e) Evaluate $\int_0^3 \underset{\sim}{v}(t)dt$ and interpret the result.

16. A golf hole is 300 m long. A golfer hits the ball with an initial velocity vector $40\,\underset{\sim}{j} + 19.6\,\underset{\sim}{k}$. ($\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ are the unit vectors in the x, y, z directions respectively as shown on the diagram.)



Find:

(a) the initial speed

(b) if the acceleration vector is $\underset{\sim}{a}(t) = 4\,\underset{\sim}{i} - 9.8\,\underset{\sim}{k}$, find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(c) when does the ball hit the ground?

(d) how far is it from the tee at the time found in (c)?

(e) what was the maximum height of the ball during flight?

17. The acceleration vector of a moving particle at time t is given by $\underset{\sim}{a}(t) = (e^{t} + e^{-t})\,\underset{\sim}{j}$.

(a) Find $\underset{\sim}{v}(t)$ given initial velocity of $2\,\underset{\sim}{i}$ m/s.

(b) Determine, in its simplest form, the speed of the particle at any time t.

18. The position vector of a particle at time t is given by $\underset{\sim}{r}(t) = 2\cos\pi t\,\underset{\sim}{i} + 2\sin 2\pi t\,\underset{\sim}{j}$

(a) Show algebraically that the Cartesian path travelled is given by $y = \pm x\sqrt{4 - x^2}$

(b) Use your graphics calculator to sketch the path of the particle.

19. The Viking lander craft launched a missile with a speed of 140 m/s. On Mars, the acceleration due to gravity is $-3.7\,\underset{\sim}{j}$ m/s^2.

(a) Find an expression for $\underset{\sim}{r}(t)$ if the missile was fired from the origin at ground level.

(b) If the surface was flat and the missle landed 2800 m from the launch craft, determine the angle/s it could have been fired at.

(c) What is the maximum height that the missile reached?

20. A radar at a theme park measures a rider's speed on a slide at a point (0, 22.5), i.e. 22.5 metres above an origin, and again at a point at the bottom of the slide (d, 0) where d is the horizontal distance travelled. If the velocity vector $\underset{\sim}{v}(t)$ at any time t seconds is given by

$$\underset{\sim}{v}(t) = 15\,(0.5\,e^{-2t} + 1)\,\underset{\sim}{i} + (20\,t - 30)\,\underset{\sim}{j}$$

Find:

(a) $\underset{\sim}{r}(t)$, the position vector of the rider at any time t;

(b) the speed when the rider passes the bottom measuring point.

21. A projectile is fired at a target which lies flat on the ground in front of the launch vehicle. The position vector $\underset{\sim}{r}(t)$ at time t seconds is given by

$$\underset{\sim}{r}(t) = 30\,t\,\underset{\sim}{i} + (2.2 + 14.7\,t - 4.9\,t^2)\,\underset{\sim}{j}$$

The origin is the point on the ground directly below the projectile's release point. Distances are measured in metres.

(a) Find the height from which the projectile is fired.

(b) What is the maximum height above the ground that the projectile reaches?

The target is 90 m from the launch vehicle. Assume the projectile is fired directly in line with the target.

(c) How far from the target does the projectile hit the ground?

22. The parametric equations of the acceleration of a body are $x = 2\cos t$ and $y = 3\sin t$. Find an expression, in non-parametric form, for its velocity $\underset{\sim}{v}$, given that at $t = \frac{\pi}{4}$ the object is stationary.

23. The orbit of a planet around its sun is given by the position vector

$$\underset{\sim}{r}(t) = \cos\frac{t\pi}{200}\,\underset{\sim}{i} - 2\sin\frac{t\pi}{200}\,\underset{\sim}{j}$$

where t is time measured in Earth days and distance is in appropriate astronomical units.

(a) Find $\underset{\sim}{r}(0)$, $\underset{\sim}{r}(400)$ and hence the length of the planet's year.

(b) Show that the distance of the planet from its sun is $d = \sqrt{1 + 3\sin^2\frac{t\pi}{200}}$

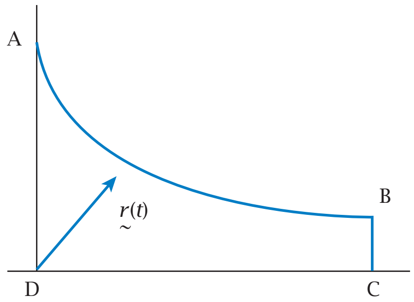
(c) At what time during the planet's year, is it a maximum distance from its sun?

(d) Find its orbiting speed then.

(e) Show the acceleration vector is a scalar multiple of the position vector.

(f) Let $x = \cos\frac{t\pi}{200}$ and $y = -2\sin\frac{t\pi}{200}$. Hence use these equations to show that the Cartesian equation of the path is defined as $4x^2 + y^2 = 4$.

24. The diagram shows a slide AB where DC is level ground with C being directly below B, the end of the slide, and D being the origin, directly below point A, the start of the slide. A person is pushed down the slide and a stopwatch indicates the journey takes exactly 2 seconds. If the position vector $\underset{\sim}{r}(t)$ of the person moving on the slide is given by $\underset{\sim}{r}(t) = te^{0.5t}\,\underset{\sim}{i} + (4 - e^{0.6t})\,\underset{\sim}{j}$ (distance measured in metres) then find:



(a) BC, the height of the end of the slide above the ground level.

(b) AD, the height of the start of the slide above ground level.

(c) DC, the horizontal displacement of the person sliding.

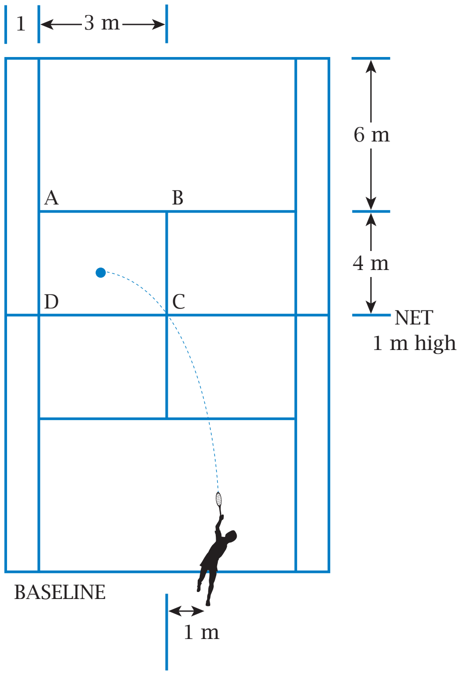
(d) the speed of the person

i. initially

ii. at the end of the journey.

25. A modified tennis court is shown. A server stands on the base line and 1 m in from the centre line. She must serve into the 3 m × 4 m rectangle ABCD, serving the ball over a 1 m high net. She hits the ball from a height of 1.9 m with initial velocity vector $\underset{\sim}{v}(0) = 10\underset{\sim}{i} + 4.9\underset{\sim}{k}$ and imparts spin on the ball, resulting in an acceleration vector $\underset{\sim}{a}(t) = -4\underset{\sim}{j} - 9.8\underset{\sim}{k}$ where $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ are the unit vectors in the forward, right and upward directions and the server is taken as the origin.

Will the serve be good?



RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

5

This section commences with a cursory review of techniques used commonly in differential calculus. Then, these techniques will be applied in solving trigonometric, rectilinear motion, simple harmonic motion, and related rates problems.

Syllabus Checklist

By the end of this chapter, you should be able to:

Applications of differentiation

- ☐ use implicit differentiation to determine the gradient of curves whose equations are given in implicit form
- ☐ examine related rates as instances of the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
- ☐ apply the increments formula $\delta y \approx \frac{dy}{dx}\delta x$ to differential equations
- ☐ solve simple first order differential equations of the form $\frac{dy}{dx} = f(x)$; differential equations of the form $\frac{dy}{dx} = g(y)$; and, in general, differential equations of the form $\frac{dy}{dx} = f(x)\,g(y)$, using separation of variables
- ☐ examine slope (direction or gradient) fields of a first order differential equation
- ☐ formulate differential equations, including the logistic equation that will arise in, for example, chemistry, biology and economics, in situations where rates are involved

Modelling motion

- ☐ consider and solve problems involving motion in a straight line with both constant and non-constant acceleration, including simple harmonic motion and the use of expressions, $\frac{dv}{dt}$, $v\frac{dv}{dx}$ and, $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ for acceleration

DIFFERENTIATION FROM FIRST PRINCIPLES

When the derivative function is needed from $y = f(x)$ we use the first principles definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which was demonstrated in *Mathematics Methods* to find $\frac{d}{dx}(\sin x)$. When $y = f(x)$ is a polynomial, the above definition is quite straightforward to use.

Example Use the first principles definition to find $f'(x)$ when $f(x) = 4 - 3x^2 + 5x$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{4 - 3(x+h)^2 + 5(x+h) - (4 - 3x^2 + 5x)}{h} \\ &= \lim_{h \to 0} \frac{\cancel{4} - \cancel{3x^2} - 6xh - 3h^2 + \cancel{5x} + 5h - \cancel{4} + \cancel{3x^2} - \cancel{5x}}{h} \\ &= \lim_{h \to 0} \frac{-6xh - 3h^2 + 5h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(-6x - 3h + 5)}{\cancel{h}} \\ &= \lim_{h \to 0} (-6x - 3h + 5) \\ &= -6x + 5 \quad \text{as expected.} \end{aligned}$$

PRODUCT, QUOTIENT AND CHAIN RULES

You should know that the product rule is

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \quad \text{or} \quad \frac{d}{dx}(f(x)\,g(x))$$

$$= vu' + uv' \qquad\qquad = g(x)\,f'(x) + f(x)\,g'(x)$$

and the quotient rule is

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

The two notations for the chain rule are:

If $y = f(g(x))$ and if $y = f(u)$ and $u = g(x)$

then $\frac{dy}{dx} = f'(g(x))\,g'(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

is Newton's notation. is Leibinitz notation.

Any combination of the above is possible with polynomials, exponential, logarithmic and trigonometric functions. Some of these have been covered in previous chapters and previous study guides.

EQUATIONS OF TANGENTS

To find the equation of the tangent line to the curve $y = f(x)$ at a given point say A (x_1, y_1) the instantaneous gradient to the curve at A is needed; this will be $f'(x_1)$. The equation of the tangent line $y = mx + c$ is found from the substitution of $y = y_1$, $x = x_1$ and $m = f'(x_1)$.

$$\begin{aligned} \text{i.e.} \quad y_1 &= f'(x_1)\,x_1 + c \\ c &= y_1 - f'(x_1)\,x_1 \\ \text{and hence} \quad y &= mx + c \quad \text{becomes} \\ y &= f'(x_1)\,x + y_1 - f'(x_1)\,x_1 \\ \text{or} \quad y &= f'(x_1)(x - x_1) + y_1 \\ \text{or} \quad y - y_1 &= f'(x_1)(x - x_1) \end{aligned}$$

In general then

> The equation of the tangent line at (x_1, y_1) on $y = f(x)$ is
>
> $$y - y_1 = f'(x_1)(x - x_1)$$
>
> $$\text{or} \qquad y = f'(x_1)(x - x_1) + y_1$$

This equation then can be written in any of the linear forms if needed.

$$\text{i.e.} \quad y = mx + c$$

$$\text{or} \quad ax + by = c \quad \text{or} \quad ax + by + c = 0$$

$$\text{or} \quad \frac{x}{a} + \frac{y}{b} = 1 \quad \text{etc.}$$

IMPLICIT DIFFERENTIATION

- This very powerful differentiation process follows from the chain rule.

  If $u = g(f(x))$, then $\frac{du}{dx} = g'(f(x)) \cdot f'(x)$.

  Now let $y = f(x)$ so then

  $$u = g(y) \text{ gives } \frac{du}{dx} = g'(y) \cdot \frac{dy}{dx}$$

  So in summary, when $u = g(y)$ and $y = f(x)$

  $$\text{then } \frac{d}{dx}(g(y)) = g'(y) \cdot \frac{dy}{dx}$$

  In actual practice $y = f(x)$ is often a relation rather than a function but the function notation is still used to describe the rule.

  When doing an implicit differentiation we will often need to recognize the product and quotient rules. For example x^2y^3 is a product because y is a function of x.

  Examples The following show how it all works.

  (a) $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

  (b) $\frac{d}{dx}(xy) = 1 \cdot y + x \cdot \frac{dy}{dx} = y + x\frac{dy}{dx}$

  (c) $\frac{d}{dx}(2y^3 + \frac{x}{y^2}) = 6y^2\frac{dy}{dx} + \frac{1 \cdot y^2 - x \cdot 2y \cdot \frac{dy}{dx}}{y^2}$

  $= (6y^2 + 1 - \frac{2x}{y})\frac{dy}{dx}$

  (d) $\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$

  (e) $\frac{d}{dx}(\cos xy) = -\sin xy \cdot \frac{d}{dx}(xy)$

  $= -\sin xy \cdot (1 \cdot y + x \cdot \frac{dy}{dx})$

  (f) $\frac{d}{dx}(\ln y) = \frac{1}{y} \cdot \frac{dy}{dx}$

- Consider the circle equation $x^2 + y^2 = 1$. Here the relation between x and y is said to be defined **implicitly.** If the equation is solved for y is

  $x^2 + y^2 = 1$

  then $y^2 = 1 - x^2$

  and $y = \pm\sqrt{1 - x^2}$ is still a relation. However, $y = f_1(x) = \sqrt{1 - x^2}$ and $y = f_2(x) = -\sqrt{1 - x^2}$ are the two functions making up the relation. The functions f_1 and f_2 are said to be defined **explicitly**.

- Consider a rule involving x and y written as L.H.S. = R.H.S.. Then differentiating both sides with respect to x gives

$$\frac{d}{dx}(\text{L.H.S.}) = \frac{d}{dx}(\text{R.H.S.})$$

or differentiating both sides with respect to time t gives

$$\frac{d}{dt}(\text{L.H.S.}) = \frac{d}{dt}(\text{R.H.S.})$$

where x and y both depend on t. This will be recommended procedure for related rates in Specialist Mathematics.

Example Find $\frac{dy}{dx}$ for $x^2 + y^2 = 1$.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}$$

For $f_1(x) = \sqrt{1-x^2}, \ \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$

For $f_2(x) = -\sqrt{1-x^2}, \ \frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$

PARAMETRIC EQUATIONS

The vector equation of a straight line, as previously studied, was an example of parametric equations with parameter t which often represented time.

$x = x_1 + at, \quad y = y_1 + bt$

In general, $x = f(t)$ and $y = g(t)$. Finding $\frac{dy}{dx}$ involves the chain rule as follows

$$\frac{dx}{dt} = f'(t), \quad \frac{dy}{dt} = g'(t)$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

DIFFERENTIAL EQUATIONS AND SEPARATION OF VARIABLES

Consider $\frac{dy}{dx} = \frac{x}{y}$ which is a differential equation. The problem is to find a function $y = f(x)$ which is the solution of this equation. Imagine that we are allowed to separate $\frac{dy}{dx}$ and cross multiply to give:

$$y\,dy = x\,dx$$

$$\therefore \quad \int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

$$\text{or} \quad y^2 = x^2 + c_2$$

$$\text{i.e.} \quad y = \sqrt{x^2 + c} \quad \text{or} \quad y = -\sqrt{x^2 + c}$$

are two functions which satisfy the differential equation. The method above is called the **separation of variables** method. This will work when a differential equation can be rearranged into $f(y)dy = g(x)\,dx$ form so that the integration process is allowed.

SIMPLE HARMONIC MOTION

- When an object moves in a straight line (say the x-axis) and at any time its acceleration is related to its position x by the equation $a = -n^2 x$ (for some constant n), the motion is called **simple harmonic**.

Because $a = \frac{d^2x}{dt^2}$, the differential equation is written as

$$\frac{d^2x}{dt^2} = -n^2 x \quad \text{or} \quad \frac{d^2x}{dt^2} + n^2 x = 0$$

The solution of this differential equation is a function $x = f(t)$ which is found, as will be shown later, to be either of

$$\begin{aligned} x(t) &= A \sin (nt + \alpha) \\ \text{or} \quad x(t) &= A \cos (nt + \alpha) \\ \text{or} \quad x(t) &= C \sin nt + D \cos nt \end{aligned}$$

Example Show that $x(t) = B \cos (nt + \alpha)$ is a solution of $a = -n^2 x$.

$$\begin{aligned} \text{LHS} &= a \\ &= \frac{d^2x}{dt^2} \\ &= \frac{d^2}{dt^2}(B\cos(nt + \alpha)) \\ &= \frac{d}{dt}(-nB\sin(nt + \alpha)) \\ &= -n^2 B\cos(nt + \alpha) \\ &= -n^2 x \\ &= \text{RHS proved} \end{aligned}$$

- When the differential equation $a = -n^2 x$ is processed the first result found is the velocity v as a function of position x as follows.

$$\begin{aligned} a &= -n^2 x \\ \frac{dv}{dt} &= -n^2 x \\ \frac{dv}{dx} \cdot \frac{dx}{dt} &= -n^2 x \\ \frac{dv}{dx} \cdot v &= -n^2 x \\ v\,dv &= -n^2 x\,dx \\ \int v\,dv &= \int -n^2 x\,dx \end{aligned}$$

$$\frac{v^2}{2} = -n^2 \frac{x^2}{2} + c_1$$

$$v^2 = -n^2 x^2 + c_2$$

Now let $v = 0$ when $x = A$

$$\begin{aligned} 0 &= -n^2 A^2 + c_2 \\ c_2 &= n^2 A^2 \\ \therefore v^2 &= -n^2 x^2 + n^2 A^2 \\ v^2 &= n^2 (A^2 - x^2) \end{aligned}$$

$v^2 = n^2 (A^2 - x^2)$ or $v = \pm n\sqrt{A^2 - x^2}$ are the equations relating the velocity of an object to its position.

For any value of x the object will either have a positive velocity if it is travelling in the positive x direction, or a negative velocity if it is travelling in the negative x-direction. Either way its speed will be the same.

- The velocity/position equation is now processed further to produce the required $x = f(t)$ function.

$$\begin{aligned}
\text{Let } v &= n\sqrt{A^2 - x^2} \\
\frac{dx}{dt} &= n\sqrt{A^2 - x^2} \\
\frac{dx}{\sqrt{A^2 - x^2}} &= ndt \\
\int \frac{dx}{\sqrt{A^2 - x^2}} &= \int ndt \\
\int \frac{dx}{\sqrt{A^2 - x^2}} &= nt + c & \text{Now let } x &= A\sin\theta \\
\int \frac{A\cos\theta d\theta}{\sqrt{A^2 - A^2\sin^2\theta}} &= nt + c & \frac{dx}{d\theta} &= A\cos\theta \\
\int \frac{A\cos\theta d\theta}{\sqrt{A^2(1 - \sin^2\theta)}} &= nt + c & dx &= A\cos\theta d\theta \\
\int \frac{\cancel{A}\cos\theta d\theta}{\cancel{A}\sqrt{\cos^2\theta}} &= nt + c \\
\int \frac{\cos\theta}{\cos\theta} d\theta &= nt + c \\
\int d\theta &= nt + c \\
\theta &= nt + c \\
\text{But } x &= A\sin\theta \\
\therefore\ x &= A\sin(nt + c)
\end{aligned}$$

If the substitution had been $x = A\cos\theta$, then $x = A\cos(nt + c)$ and either of the above functions can be written as $x = C\sin nt + D\cos nt$.

$x = A\sin(nt + \alpha)$ or $x = A\cos(nt + \alpha)$ or $x = C\sin nt + D\cos nt$ are the equations giving the position as a function of time which shows that **simple harmonic motion** is **oscillatory** in nature.

- The function $x = A\sin(nt + \alpha)$ shows that x has extreme positions of $|A|$ and $-|A|$ and that the period T, which is the time needed to complete one cycle is:

$$T = \frac{2\pi}{n} \text{ seconds.}$$

The constant α determines where the object is at $t = 0$ and in which direction it is travelling.

The constant n is called the angular or **radial frequency** and its units are radians/sec.

If one cycle takes $T = \frac{2\pi}{n}$ seconds then $\frac{1}{T} = \frac{n}{2\pi}$ cycles will be completed in one second which is called the **frequency of oscillation** f.

$$\text{So } f = \frac{1}{T} = \frac{n}{2\pi} \text{ cycles/sec}$$

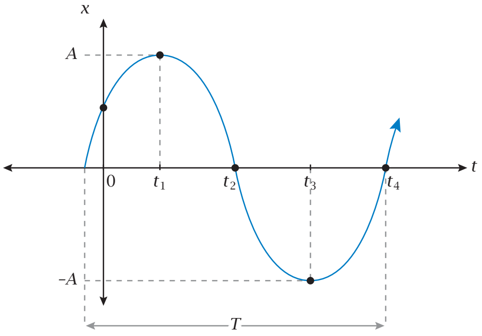
$$\text{and } n = 2\pi f \text{ radians/sec .}$$

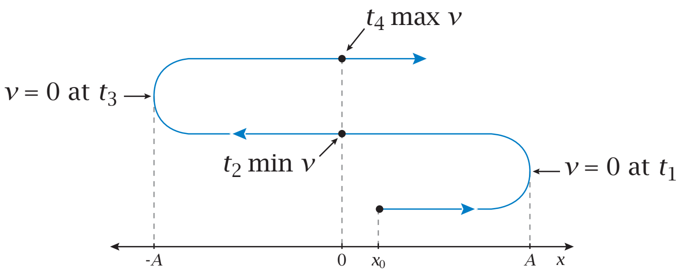
- Once $x = f(t)$ is known, then obviously $v = f'(t)$ and $a = f''(t)$ are all known.

The value of $|A|$ is called the **amplitude** and at $\pm|A|$ the object stops and "turns around" (or "returns", or "changes direction"), that is at these moments the velocity $v = 0$.

The maximum speed $|v|$ occurs mid way between $x = |A|$ and $x = -|A|$ (i.e. at $x = 0$). This means that the maximum and minimum velocities also happen when the object is at the centre of motion depending on which direction it is going at the time.

- If for $x = A \sin(nt + \alpha)$, and $A > 0$, $\alpha > 0$ then $x(0) = A \sin \alpha$ and the two diagrams shown are often useful when sorting out particular types of problems.





The quarter period $\dfrac{T}{4} = t_2 - t_1 = t_3 - t_2 = t_4 - t_3$ etc., is the time taken for the object to move from the initial position to an extreme position or vice versa.

As with rectilinear motion, the following apply to S.H.M.:

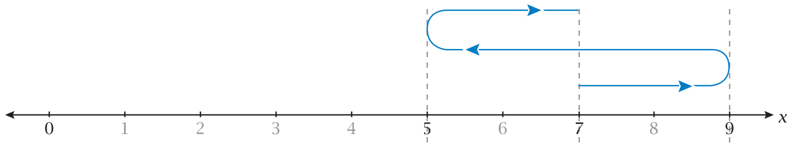
$$\text{displacement} = \int_a^b v(t)\, dt \ \text{ and distance} = \int_a^b |v(t)|\, dt \ .$$

- If the motion of an object is simple harmonic about a centre $x = c$ rather than $x = 0$ then $a = -n^2 x$ becomes

$$a = -n^2 (x - c) \ \text{ for S.H.M. about } x = c$$

This doesn't cause any problems because in practice it is straightforward to create a new origin with a change of variable.

For example, if the motion was S.H.M. from a maximum of $x = 9$ to a minimum of $x = 5$, then

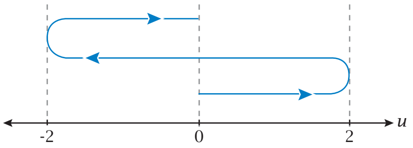


The centre of motion c is

$$c = 5 + \frac{9-5}{2} \quad \text{or} \quad c = \frac{5+9}{2}$$

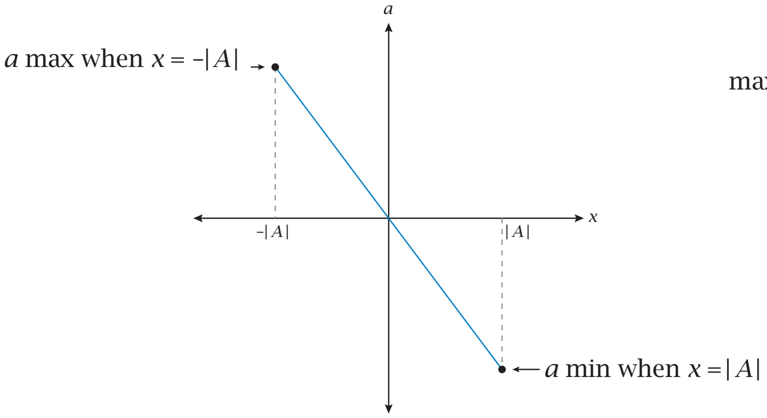
$$= 7 \qquad\qquad = 7$$

If $u = x - 7$ then the u axis will be

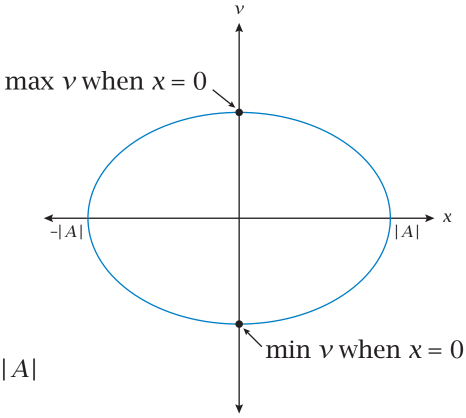


The value of A is obviously 2 and $a = -n^2 (x - 7)$ becomes $a = -n^2 u$. Any u values computed are simply converted back to x's by $x = u + 7$.

- The graphs of $a = -n^2 x$ and $v^2 = n^2\,(A^2 - x^2)$ are interesting. The first is a straight line with gradient $-n^2$ and the second is a circle for $n = 1$ and an ellipse for $n \neq 1$.



$a = -n^2 x$



$v^2 = n^2\,(A^2 - x^2)$

or $x^2 + \dfrac{1}{n^2}v^2 = A^2$

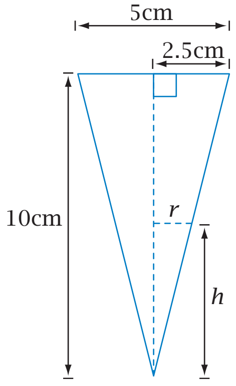
RELATED RATES

- As water flows into an inverted cone at a constant rate, the rate that the height of the water increases will slow down as the height increases. Finding out how rates (mostly changing with time) are related is best achieved through implicit differentiation.

Example Water flows into the cone shown at 3 cm³/sec. Find

a) the rate at which the water is rising when the radius of the water surface is 2 cm

b) the rate at which the circular water area is increasing when the water's height is 7 cm.



a) Volume of water $V = \dfrac{\pi r^2 h}{3}$

$\dfrac{dv}{dt} = 3\text{cm}^3/\text{sec}$, find $\dfrac{dh}{dt}$ when $r = 2\text{cm}$

By similar $\triangle$'s

$$\frac{2.5}{r} = \frac{10}{h}$$

i.e. $h = 4r$

and $\dfrac{dh}{dt} = 4\dfrac{dr}{dt}$

$$V = \frac{\pi r^2 h}{3}, \qquad \text{substitute} \quad h = 4r$$

$$= \frac{\pi r^2 \cdot 4r}{3}$$

$$V = \frac{4\pi}{3}r^3 \qquad \text{now } \frac{d}{dt} \text{ of both sides gives}$$

$$\frac{dV}{dt} = 3 \cdot \frac{4\pi}{3}r^2 \cdot \frac{dr}{dt} \qquad \text{but } \frac{dv}{dt} = 3\text{cm}^3/\text{sec}$$

$$3 = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{3}{4\pi r^2} \qquad \text{which shows how } \frac{dr}{dt} \text{ is reducing as } r \text{ increases}$$

$$\frac{dh}{dt} = 4 \cdot \frac{3}{4\pi r^2}$$

$$= \frac{3}{\pi r^2} \qquad \text{and when } r = 2\text{cm}$$

$$\frac{dh}{dt} = \frac{3}{\pi 2^2}$$

$$= 0.239\,\text{cm}/\text{sec}$$

$\therefore$ When the radius of the water is 2 cm the height is increasing at 0.24 cm/sec 2 d.p.

b) Area of water surface $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

From above $\frac{dr}{dt} = \frac{3}{4\pi r^2}$, $\frac{dA}{dt} = 2\pi r \cdot \frac{3}{4\pi r^2}$

$$= \frac{3}{2r}$$

$h = 4r,\ 7 = 4r,\ r = \frac{7}{4}$ $\quad = \frac{3}{2\left(\frac{7}{4}\right)}$

$$= 0.857\,\text{cm}^2/\text{sec}$$

$\therefore$ When the water's height is 7 cm, the surface area of the water is increasing at 0.86 cm^2/sec 2 d.p.

◇ In the above example it was necessary to find the equation linking the radius and the height, that is, $h = 4r$. After substitution the volume became a function of r only which is advised for simplicity. Finding $\frac{dh}{dt}$ was easy as $h = 4r$ led to $\frac{dh}{dt} = \frac{4dr}{dt}$. Of course the substitution could have been $r = \frac{h}{4}$ etc.

The example below is slightly different in that the equation linking θ and x is defined implicitly. Due to this, implicit differentiation with respect to t is very powerful.

Example At a certain instant in time the direct distance from a 2 m tall boy and his model plane is 120 m. The plane is due East of the boy and it is travelling due west at a speed of 20 m/s and at an altitude of 74 m. Find the rate at which the angle of elevation of the plane (as seen by the boy) is changing at that instant.

Let x be horizontal distance that the plane is from the boy.

So $\frac{dx}{dt} = -20\text{m/s}$

$$\tan\theta = \frac{72}{x}$$

$$\tan\theta = 72x^{-1}$$

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(72x^{-1})$$

$$\frac{1}{\cos^2\theta} \cdot \frac{d\theta}{dt} = \frac{-72}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-72}{x^2} \cdot \frac{dx}{dt} \cdot \cos^2\theta$$

$$= \frac{-72}{96^2}(-20)(0.8^2)$$

$$= 0.1\,{}^{R}\!/_{\text{sec}}$$

$$= 5.73\,{}^{o}\!/_{\text{sec}}$$

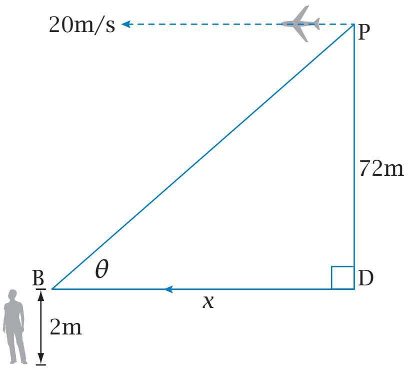
When BP = 120m

$$x = \sqrt{120^2 - 72^2}$$

$$= 96\text{m}$$

and $\cos\theta = \frac{96}{120}$

$$= 0.8$$



The angle of elevation is increasing at 0.1^R/sec or about 5.73°/sec at that instant.

RECTILINEAR MOTION

Motion of an object along a straight line, say the x axis, will mostly have its position x as a function of time t. You are either given $x = f(t)$ or you will need to find it. The following details are important.

i) Velocity $v = \frac{dx}{dt}$, acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

ii) From $a = g(t)$, $v = \int a\,dt$, $x = \int v\,dt$ where the constants of integration are calculated as you go.

iii) If the object is travelling in the positive x-direction, then $v > 0$. If $v < 0$, then the object is travelling in the negative x-direction. If $v = 0$ then the object has stopped. The object's speed $= |v|$.

iv) **Positive acceleration** is either
when the object has $v > 0$ and is speeding up or
when the object has $v < 0$ and is slowing down.

Negative acceleration is either
when the object has $v > 0$ and is slowing down or
when the object has $v < 0$ and is speeding up.

> Alternatively the object is speeding up when the signs of a and v are the same and it is slowing down when the signs of a and v are opposite.

v) Velocity/time graphs are very useful because the derivative (slope) of the curve at any time is the object's acceleration and the area under the curve is the distance travelled. Displacement follows by combining distances travelled in the positive x-direction (i.e. when $v > 0$) with distances travelled in the negative x-direction (i.e. when $v < 0$).

vi) When finding information from a $v - t$ graph it is best to use time intervals (t_1, t_2 etc.) as shown below as equations are easily written from $\text{slope} = \frac{\text{rise}}{\text{run}}$, and areas are found from triangles, rectangles and trapeziums.

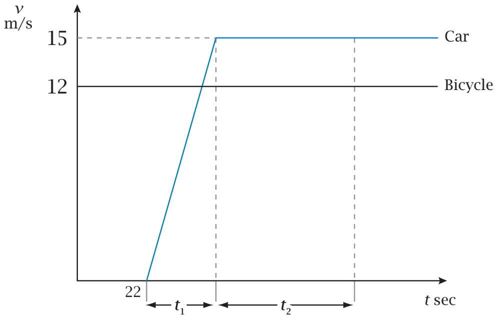
Example A bicycle travelling at 43.2 km/hr passes a stationary car. Twenty two seconds later the car follows the bicycle and uniformly accelerates for 210 m until it reaches a speed of 54 km/hr which then remains constant. Find:

a) the acceleration of the car before it gets to 54 km/hr

b) the time taken for the car to overtake the bicycle

c) the distance the car travels until it overtakes the bicycle.

(a) $43.2\,\text{km/hr} = \frac{43.2}{3.6}\,\text{m/s} = 12\,\text{m/s},\ 54\,\text{km/hr} = \frac{54}{3.6}\,\text{m/s} = 15\,\text{m/s}$



$210 = \frac{15 \times t_1}{2}$ $\qquad a = \frac{15}{28}\,\text{m/s}^2$ is the acceleration of the car

$t_1 = 28\,\text{sec}$

(b) Distance travelled by bicycle = Distance travelled by car

$$12 \times (22 + 28 + t_2) = 210 + 15 \times t_2$$

$$600 + 12t_2 = 210 + 15t_2$$

$$600 - 210 = 3t_2$$

$$t_2 = 130\,\text{sec}$$

The car overtook the bike 130 sec after the car started to follow it.

(c) $\text{Dist} = 210 + 15 \times t_2 \quad , \quad t_2 = 130$
$= 2160\text{ m}$

The car travelled 2.16 km before overtaking the bicycle.

INCREMENTAL CHANGE

If $y = f(x)$ represents a function, then

i) the instantaneous rate of change of the function $f(x)$ with respect to the variable x is given by $\frac{dy}{dx}$ or $f'(x)$

ii) further, $\frac{dy}{dx}$ or $f'(x)$ gives the gradient of the curve formed by the graph of $y = f(x)$ at any point (x,y).

: If $y = f(x)$ represents a function then the notation δy indicates a small change or increment in the variable y. If y increases, δy is poisitive. If y decreases, δy is negative.

: $\delta y \approx \left(\frac{dy}{dx}\right)\delta x$ gives an approximation to the change in y, given a change (δx) in x. The approximation is valid provided δx is **small**.

: $y + \delta y = y + \left(\frac{dy}{dx}\right)\delta x$ is called the tangent approximation to y. It provides an approximation to the new value of a variable y, after the variable x undergoes a small change δx.

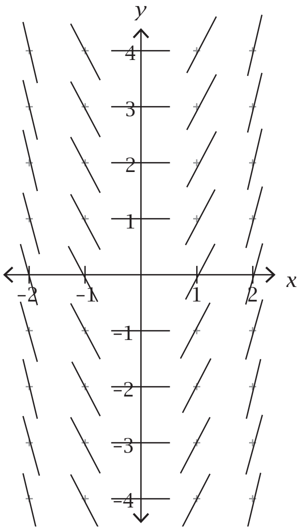
: $\frac{\delta y}{y}$ gives the relative change in y.

: $100 \bullet \frac{\delta y}{y}$ gives the percentage change in y.

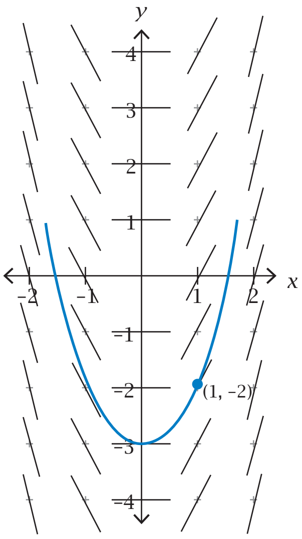
SLOPE FIELDS

Problems that begin with the derivative are differential equations. Instead of using anti-differentiation to determine a function, we can use a graphical method which examines the slope fields (or direction fields) of the function.

Example Suppose we are given the following slope fields sketched on the Cartesian plane.



By examining the direction, shape, and overall pattern of these slopes, we can develop a general idea of what the curve (and the relation) graphic is. For the given example the curve appears to be a parabola, which would imply that the relation is quadratic. If we are given a coordinate (initial condition) of (1, –2), then we are able to sketch a specific relation



of $y = x^2 - 3$.

We can also analyse the gradient of the 'slopes' within the slope fields by substituting values of x and y:

$y' = 2x$		
x	y	y'
0	0	0
0	1	0
0	2	0
0	3	0
1	0	2
1	1	2
2	0	4
−1	0	−2
−2	0	−4

Plug 0 into y' for x

Plug 1 into y' for x

An analysis of these slopes is consistent with a gradient function of $y' = 2x \ \forall x, x \geq 0$ and $y' = -2x \ \forall x, x < 0$.

GROWTH AND DECAY

If the rate of change is proportional to the amount present, the change can be modelled by the differential equation:

$$\frac{dy}{dt} = \text{ky} \qquad ①$$

To determine the original equation, separate the variables and antidifferentiate:

$$\frac{1}{y}\,dy = kdt \qquad \text{(divide both sides by } y\text{)}$$

$$\int \frac{1}{y}\,dy = \int kdt \qquad \text{(antidifferentiate both sides)}$$

$$\ln|y| = kt + C$$

$$e^{\ln|y|} = e^{kt+C} \qquad \text{(exponentiate both sides)}$$

$$|y| = e^{C} \bullet e^{kt}$$

$$y = \pm e^{C} \bullet e^{kt} \ \Rightarrow \ y = A_0 e^{kt} \qquad ②$$

LOGISTIC GROWTH MODEL

Real-life populations do not increase forever, as there are some limiting factors to consider (e.g. food, living space). With ② in mind, a more realistic model can be developed which accounts for such limiting factors. This model is the logistic growth model, where:

- the growth rate, r, is proportional to both the size of P (population) and the amount by which y falls short of the maximal size $(k - P)$
- k is the maximum population (or carrying capacity).

Thus we have $\frac{dP}{dt} = rP(k - P)$ ③

To solve this equation, we separate the variables and antidifferentiate

$$\int \frac{dP}{P(k-P)} = \int r dt$$

$$\int \left(\frac{1}{kP} + \frac{1}{k(k-P)}\right) dP = \int r dt$$

$$\ln|P| - \ln|k - P| = rkt + C$$

$$\frac{P}{k-P} = e^{rkt + C}$$

$$P = (k - P)e^{rkt + C}$$

$$P + Pe^{rkt + C} = ke^{rkt + C}$$

$$P(1 + e^{rkt + C}) = ke^{rkt + C}$$

$$P = \frac{ke^{rkt + C}}{1 + e^{rkt + C}}$$

$$P = \frac{k}{1 - e^{-rkt + C}}$$

$$P = \frac{k}{1 - e^{C}e^{-rkt + C}}$$

$$P_0 = \frac{k}{1 - e^{C}} \qquad (t = 0 \text{ for initial } P)$$

$$e^{C} = 1 - \frac{k}{P_0}$$

$$P = \frac{k}{1 - (1 - \frac{k}{P_0})e^{-rkt}}$$

and $P = \frac{P_0 k}{P_0 + (k - P_0)e^{-rkt}}$, where $P_0 = P(0)$ ③

All solutions of the logistical differential equation can be expressed in the general form:

$$y = \frac{L}{1 + be^{-kt}}$$ ④

where:
- L is the carrying capacity
- $b = \frac{K - P_0}{P_0}$
- k is the growth rate

Worked Examples

5.1 A particle in S.H.M. has a maximum displacement (x) of 6 and a maximum velocity (v) of 4. If $x = 2$ and $v < 0$ when $t = 0$ find $x = f(t)$.

Let $x = 6\sin(nt + \alpha)$ as $A = 6$

and $v = 6n\cos(nt + \alpha)$

$$V_{\max} = 6n = 4$$

$$\therefore n = \frac{4}{6} = \frac{2}{3}$$

$$x = 6\sin\left(\frac{2}{3}t + \alpha\right)$$

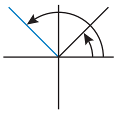
Let $t = 0$, $2 = 6\sin\alpha$

$$\sin\alpha = \frac{1}{3}$$

$$\alpha = 0.3398 \text{ (4 d.p.)}$$

or $\alpha = \pi - 0.3398$

$= 2.8018$ (4 d.p.)



For $\alpha = 0.3398$

$$x = 6\sin\left(\frac{2}{3}t + 0.3398\right)$$

$$v = 4\cos\left(\frac{2}{3}t + 0.3398\right)$$

$v(0) > 0$ reject

For $\alpha = 2.8018$

$$x = 6\sin\left(\frac{2}{3}t + 2.8018\right)$$

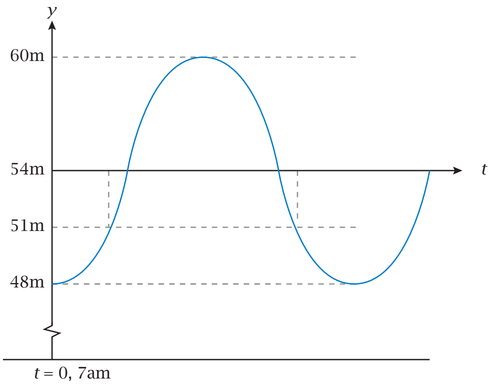
$$v = 4\cos\left(\frac{2}{3}t + 2.8018\right)$$

$v(0) < 0$ ok

$\therefore x = 6\sin\left(\frac{2}{3}t + 2.8018\right)$ is the required function.

Letting $t = 0$ does confirm that $x = 2$

5.2 A harbour has high water mark of 60 m and a low water mark of 48 m. The height of the water varies in S.H.M. with a period of 12 hours. If the low water mark is at 7 a.m. and a ship needs at least 51 m of water to be able to enter the harbour, find the earliest time that the ship can enter the harbour and the maximum length of time it can stay.



$$\text{centre of motion} = \frac{48 + 60}{2}$$

$$= 54 \text{ m}$$

$$\text{let } y = u + 54$$

$$a = -n^2 u$$

$$T = 12 = \frac{2\pi}{n}$$

$$n = \frac{2\pi}{12}$$

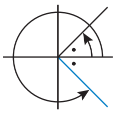
$$n = \frac{\pi}{6}$$

So $u = -6\cos\frac{\pi}{6}t$ models the water level

When $y = 51$, $u = -3$

So $-3 = -6\cos\frac{\pi}{6}t$

and $\cos\frac{\pi}{6}t = \frac{1}{2}$



$$RA = \cos^{-1}0.5$$

$$= \frac{\pi}{3}$$

$$\therefore \frac{\pi}{6}t = \frac{\pi}{3} \quad \text{or} \quad \frac{\pi}{6}t = 2\pi - \frac{\pi}{3}$$

$$t = 2 \quad \text{or} \quad t = 10$$

So the ship can enter the harbour at 7 a.m. + 2 hours, that is 9 a.m. and stay a maximum time of $10 - 2 = 8$ hours. This means that it must leave before 7 a.m. + 10 hours, that is, 5 p.m.

5.3 The velocity of an object v is related to its position x by $v^2 = 4(9 - x^2)$. Prove that the object is moving in S.H.M..

$$v^2 = 4(9 - x^2)$$

$$\frac{d}{dt}v^2 = \frac{d}{dt}4(9 - x^2)$$

$$\cancel{2}\,v\frac{dv}{dt} = 4(-\cancel{2}\,x)\frac{dx}{dt}$$

$$\cancel{v}\frac{dv}{dt} = -4x \cdot \cancel{v}$$

$$a = -2^2 x \qquad \text{i.e. SHM with } n = 2 \text{ proved}$$

5.4 An object moves according to $x = 7\sin\left(3t - \frac{\pi}{4}\right)$. Show that $v = \pm 3\sqrt{49 - x^2}$.

$$x = 7\sin\left(3t - \frac{\pi}{4}\right)$$

$$v = \frac{dx}{dt} = 7(3)\cos\left(3t - \frac{\pi}{4}\right)$$

$$v^2 = 21^2\cos^2\left(3t - \frac{\pi}{4}\right)$$

$$v^2 = 21^2\left(1 - \sin^2\left(3t - \frac{\pi}{4}\right)\right)$$

$$v^2 = 21^2\left(1 - \left(\frac{x}{7}\right)^2\right)$$

$$v^2 = 3^2 \cdot 7^2\left(\frac{49 - x^2}{7^2}\right)$$

$$v^2 = 9(49 - x^2)$$

$$v = \pm 3\sqrt{49 - x^2} \qquad \text{as required}$$

5.5 A weight bouncing on a spring moves according to the equation $y = 1.5 - 0.6\sin\frac{3}{2}t$ m where y is the vertical distance of the weight from the floor and t is measured in seconds. Find the values of y when the speed of the weight is $\frac{3}{\sqrt{20}}$ m/s.

Let the centre of motion be $y = 1.5$ so $u = -0.6\sin\frac{3}{2}t$

$$n = \frac{3}{2}$$

$$A = |-0.6| = \frac{3}{5}$$

$$\left(\frac{3}{\sqrt{20}}\right)^2 = \left(\frac{3}{2}\right)^2\left(\left(\frac{3}{5}\right)^2 - u^2\right)$$

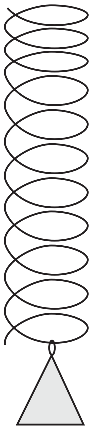
$$\frac{9}{20} = \frac{9}{4}\left(\frac{9}{25} - u^2\right)$$

$$\frac{1}{5} = \frac{9}{25} - u^2$$

$$u^2 = \frac{9}{25} - \frac{1}{5}$$

$$u^2 = \frac{4}{25}$$

$$u = \pm\frac{2}{5}$$

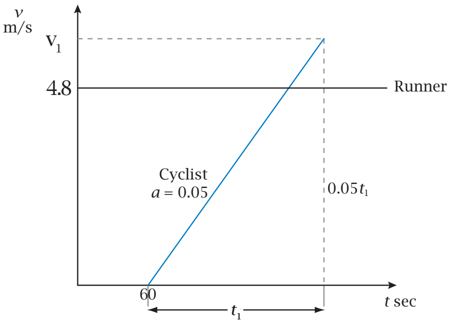


So $y = 1.5 \pm \frac{2}{5}$

$= 1.9 \quad \text{or} \quad 1.1$

When the weight is either 1.9 m or 1.1 m from the floor its speed is $\frac{3}{\sqrt{20}}$ m/s.

5.6 A minute after a marathon runner moving at 17.28 km/hr passes a stationary cyclist, the cyclist sets off in the same direction with a constant acceleration of 0.05 m/s². Find the distance travelled by the runner before being overtaken by the cyclist.



$$17.28\text{ km/hr} = \frac{17.28}{3.6}\text{ m/s}$$

$$= 4.8\text{ m/s}$$

$$\frac{v_1}{t_1} = 0.05$$

$$\therefore v_1 = 0.05t_1$$

distance travelled by runner = distance travelled by cyclist

$$4.8(t_1 + 60) = \frac{1}{2}t_1 \times 0.05t_1$$

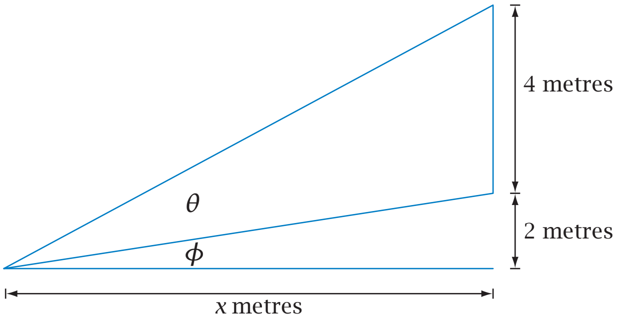
$$4.8t_1 + 288 = 0.025t_1^2$$

$$0.025t_1^2 - 4.8t_1 - 288 = 0$$

$$\text{gives } t_1 = 240\text{ sec}$$

∴ The runner will have travelled 4.8 (240 + 60) = 1.44 km before being overtaken by the cyclist.

5.7 A picture 4 metres high is hung vertically on a wall with its base 2 metres above the level of an observer's eye. When an observer stands x metres from the wall, the angle of elevation of the base of the picture is ϕ and the angle subtended by the picture at the observer's eye is θ as shown below.



(a) Express $\tan\phi$ and $\tan(\phi + \theta)$ in terms of x.

(b) Show that $\tan\theta = \dfrac{4x}{x^2 + 12}$

(c) If the observer is walking at 0.8 m/s towards the picture, find the rate of change of θ when they are 8 m from the picture.

(a) $\tan\phi = \dfrac{2}{x}$ $\tan(\phi + \theta) = \dfrac{6}{x}$

(b) $$\tan(\phi+\theta) = \frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta}$$

$$\frac{6}{x} = \frac{\frac{2}{x}+\tan\theta}{1-\frac{2}{x}\tan\theta}$$

now multiply top and bottom of RHS by x

$$\frac{6}{x} = \frac{2+x\tan\theta}{x-2\tan\theta}$$

$$6x-12\tan\theta = 2x+x^2\tan\theta$$

$$4x = x^2\tan\theta+12\tan\theta$$

$$4x = \tan\theta(x^2+12)$$

$$\tan\theta = \frac{4x}{x^2+12}$$ as required.

(c) $$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{4x}{x^2+12}\right)$$

$$\frac{1}{\cos^2\theta}\cdot\frac{d\theta}{dt} = \frac{4(x^2+12)-4x\cdot 2x}{(x^2+12)^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{48-4x^2}{(x^2+12)^2}\cdot\frac{dx}{dt}\cdot\cos^2\theta$$

$$= \frac{48-4\times 8^2}{(8^2+12)^2}(-0.8)(0.9216...)^2$$

$$= 0.0245\,{}^{R}\!/_{\text{sec}}\quad \text{4dp}$$

when $x = 8$ $\quad \frac{dx}{dt} = -0.8$

$$\tan\theta = \frac{4\times 8}{8^2+12}$$

$$\theta = 0.3985...^{R}$$

$$\cos\theta = 0.9216...$$

So, at that instant, θ is increasing at approximately 0.0245^{R}/sec.

5.8 Find the equations of the tangents to $f(x) = \frac{1}{x}+x-1$ which pass through the point $(3, -6)$. Give the points of contact in exact form.

Let the tangent equations be $y - -6 = m(x-3)$
i.e. $y = m(x-3)-6$

$$f(x) = x^{-1}+x-1$$

$$f'(x) = -\frac{1}{x^2}+1$$

$$\therefore\quad m = -\frac{1}{x^2}+1 = 1-\frac{1}{x^2}$$

Because the tangent lines touch f, equate the curve and line functions

$$\frac{1}{x}+x-1 = m(x-3)-6$$

i.e. $$\frac{1}{x}+x-1 = \left(1-\frac{1}{x^2}\right)(x-3)-6$$

Solver gives $x = 0.5$ or $x = -0.75$

For $x = 0.5$, $\quad m = 1-\frac{1}{0.5^2}$

$$= -3$$

and $f(0.5) = 1\frac{1}{2}$

$\therefore\quad y = -3(x-3)-6$
or $\quad y = -3x+3$ is the tangent making contact at $(0.5, 1.5)$

For $x = -0.75$,

$$m = -\frac{7}{9}$$

and $f(-0.75) = -3\frac{1}{12}$

$y = -\frac{7}{9}(x-3)-6$ is the tangent

making contact at $\left(-\frac{3}{4}, -3\frac{1}{12}\right)$

5.9 If $\tan\theta = \dfrac{x}{x^2+2}$, find x when $\dfrac{d\theta}{dx} = 0$.

$\dfrac{d}{dx}$ of both sides gives

$$\frac{1}{\cos^2\theta}\cdot\frac{d\theta}{dx} = \frac{1\cdot(x^2+2) - x\cdot 2x}{(x^2+2)^2}$$

$$= \frac{2-x^2}{(x^2+2)^2}$$

let $\dfrac{d\theta}{dx} = 0$ gives $\dfrac{2-x^2}{(x+2)^2} = 0$

i.e. $2 - x^2 = 0$

$\therefore\ x = \sqrt{2}$ or $x = -\sqrt{2}$

5.10 If $y = \sin^{-1} x$ find $\dfrac{dy}{dx}$ and hence an integral

$$\sin y = x \longrightarrow \sin^2 y = x^2$$

$$1 - \cos^2 y = x^2$$

$$\cos^2 y = 1 - x^2$$

$$\cos y = \sqrt{1-x^2}$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}x$$

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$$

5.11 Solve $\dfrac{dy}{dx} = \dfrac{3x(x+2)}{2y-1}$ given that $x = 1$ when $y = 2$.

Separate variables $(2y-1)\,dy = 3x(x+2)\,dx$

$$\int(2y-1)\,dy = 3\int(x^2+2x)\,dx$$

$$y^2 - y = 3\left(\frac{x^3}{3} + x^2\right) + c$$

Sub $x = 1,\quad y = 2$ $\qquad 2^2 - 2 = 3\left(\dfrac{1}{3} + 1\right) + c$

$$c = -2$$

$\therefore\ y^2 - y = x^3 + 3x^2 - 2$ is the required solution

5.12 Find $\dfrac{dy}{dx}$ for the function y defined by the integral as

$$y = \int_{\sqrt{x}}^{x^2} e^{t^2}\,dt$$

Let a be such that $\sqrt{x} < a < x^2$ and split up the function like this

$$y = \int_{\sqrt{x}}^{a} e^{t^2}\,dt + \int_{a}^{x^2} e^{t^2}\,dt$$

$$y = -\int_{a}^{x^{\frac{1}{2}}} e^{t^2}\,dt + \int_{a}^{x^2} e^{t^2}\,dt$$

$$\frac{dy}{dx} = -e^{(x^{\frac{1}{2}})^2}\cdot\frac{1}{2}x^{-\frac{1}{2}} + e^{(x^2)^2}\cdot 2x$$

$$= 2xe^{x^4} - \frac{1}{2\sqrt{x}}e^{x}$$

5.13 Using function notation, the incremental formula can be expressed:

$\delta y \approx f'(x) \times \delta x$

Example: If $y = x^4 - 5x$, estimate the change in y as x changes from 2 to 2.01.

Let $y = x^4 - 5x$

Then $\frac{dy}{dx} = 4x^3 - 5$ and $\frac{dy}{dx}_{|x=2} = 27$

Now $\delta y \approx \frac{dy}{dx} \times \delta x$

and $\delta y \approx 27 \times 0.01 \approx 0.27$

In other words, y increases by approximately 0.27 as x increases from 2 to 2.01 (this can be checked via $[2.01^4 - 5 \times 2.01] - [2^4 - 5 \times 2] = 0.2724081$

5.14 Oftentimes we are interested in the amount by which a numerical quantity changes as a fraction of the total amount.

We can also look at the size or proportion of the change itself, relative to the total amount. This is called *small percentage change* or *relative change*.

If a variable x changes by an amount δx, the relative change in x is in the ratio $\frac{\delta x}{x}$. This is often given as a percentage, and we can use the incremental formula to assist us.

Example: The radius of a circle increases by 3%. Estimate the change in the area of the circle.

If $A = \pi r^2$ then $\frac{dA}{dr} = 2\pi r$ and because $\frac{\delta A}{\delta r} \approx \frac{dA}{dr}$ we can say that $\frac{\delta A}{\delta r} = 2\pi r$

So $\delta A \approx 2\pi r \times \delta r$

Dividing both sides by A gives $\frac{\delta A}{A} \approx \frac{2\pi r \times \delta r}{\pi r^2}$

$$\frac{\delta A}{A} \approx \frac{2\pi r}{r} \approx 2(0.03) \approx 0.06$$

This represents a 6% increase in the circle's area when the radius increases by 3%.

5.15 The idea of small incremental change arrives to us through an understanding of the formula:

$y + \delta y \approx y + \frac{dy}{dx} \times \delta x$

This formula gives an approximation of δy, the change in y due to a small change in x. In other words, by adding the estimate of δy to the 'old' value y, we get an estimate for the new value $y + \delta y$.

We call this a tangent approximation because it is equivalent to using the tangent lines as an approximation to the graph of y against x. Using function notation, the tangent approximation is:

$f(x + h) \approx f(x) + f'(x) \times h$

As an example, we could look at how to approximate $\sqrt{103}$.

To start with, $\sqrt{103}$ is close to $\sqrt{100}$, which is 10.

The function we're dealing with is $y = \sqrt{x}$, $y + \delta y = \sqrt{103}$, and $\delta x = 3$.

Plugging these values into the formula gives:

$\sqrt{103} \approx \sqrt{100} + \frac{1}{2\sqrt{100}} \times 3 \approx 10.15$ (2 d.p.)

PROBLEMS TO SOLVE

CHAPTER 5: RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

1. At 1.00 a.m. on the first of January, 2016, the depth of the water at the end of a jetty in a particular port could be described by:

$$h = 10 + 0.5\cos\frac{\pi t}{9} \quad m$$

where h is the depth of the water, and t is time in hours after 1.00 a.m.

(a) At what time of day was it low tide?

(b) How many hours were there between high and low tide?

(c) Prove that the tide moved in simple harmonic motion.

2. The depth D m of water in a port a time t hours after midnight is given by the function

$$D = 2\sin\frac{1}{2}\left(t + \frac{\pi}{3}\right) + 10$$

(a) What is the water depth at midnight?

(b) How soon after midnight will it first be high tide?
(give your answer to the nearest minute)

(c) What time elapses between high and low tide?

(d) Find the rate at which the water level is changing at 6 a.m.

3. A particle, moving on an x axis graduated in metres, has, after t seconds, a position given by

$$x = 2\cos\left(3t + \frac{\pi}{6}\right)$$

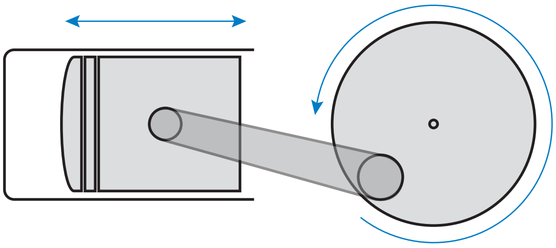
(a) State the amplitude and period of the motion.

(b) Express its velocity and acceleration as functions of t.

(c) What was the particle's initial position and in which direction did it first move?

(d) What time elapsed before the particle was next at its initial position and what was its velocity then?

4. The head of a piston moves with simple harmonic motion of amplitude $\frac{\sqrt{3}}{10}$ about a mean position of 0.



How far from 0 is the head of the piston when it is travelling at a speed equal to half of its maximum speed?

5. A buoy floating in the sea moves vertically with a velocity of $v = 3\cos 2t + 4\sin 2t$ m/s.

(a) Find its displacement between the times $t = 0$ and $t = 3$ seconds.

(b) If, at $t = 0$, the buoy is 10 m above the seabed, show that the buoy is moving with simple harmonic motion.

(c) Find the first time after $t = 0$ that the buoy is at its maximum height.

6. A particle is moving such that its position x (metres) at time t (seconds) is given by

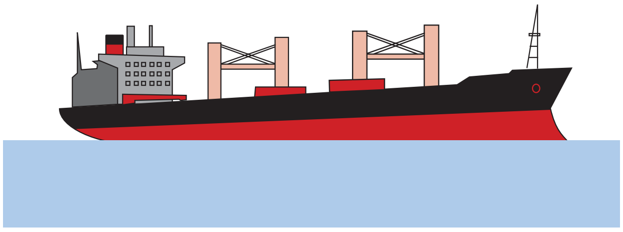
$$x = 3 \sin (2t + 5)$$

(a) Find the period of the motion.

(b) Find the maximum acceleration of the particle.

(c) Find the speed of the particle when $x = 2$ m .

7. On a certain day, the depth of water in a bay at high tide at 3 p.m. was 17 m. At low tide 6 hours earlier, the depth was 11 m. If the level of the water fluctuated between these depths in simple harmonic motion, determine the earliest time at which a ship requiring 15 m of water could have entered the bay on that day.



8. (a) The volume of a cylinder is constant at 50 cm^3 but both the height and the radius are changing.

Prove that $\frac{dh}{dt} = -\frac{100}{\pi r^3}\frac{dr}{dt}$

(b) At the instant when the radius is 5 cm, the height is decreasing at 3 cm/sec. Find the rate at which the radius is changing at this instant.

9. A train starts from rest and moves with constant acceleration until it reaches a speed of 15 ms^{-1} . It continues at this speed for a time after which it is brought to rest by constant retardation. The total time taken is 22 seconds and the distance travelled is 240 metres. If the time for the retardation is half that for the acceleration:

(a) sketch the velocity-time graph

(b) find the time the train takes to accelerate to its top speed

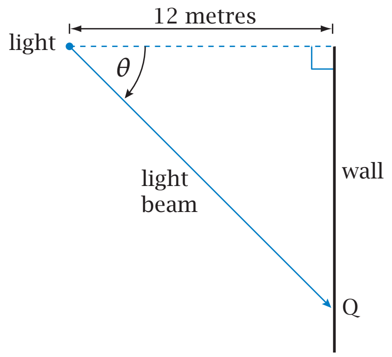
(c) find how far the train travels at the top speed of 15 ms^{-1}.

10. Car A and car B are stationary and alongside each other on an open country road. Car A moves off with an acceleration of 1 ms^{-2} . This is maintained for thirty seconds after which it continues to move with constant speed.

Thirty seconds after A starts, B sets off in pursuit of A with constant acceleration of 2 ms^{-2}, until it draws level with A. Using a velocity-time graph, or otherwise, find:

(a) the time taken by B in pursuit to draw level with A
(b) the distance travelled by B, correct to the nearest metre.

11. A police car has a rotating light mounted on top of its roof. The light completes one revolution every 20 seconds. The light is 12 m from the nearest point on a straight brick wall, and a point Q on the wall is exactly 13 m from the light source. How fast is the spot of light moving along the wall when it passes point Q?



12. If $6x - xy + y^3 = 9$ find $\dfrac{dy}{dx}$.

13. In the theory of relativity, the mass M of a particle moving at velocity v is given by

$M = \mathrm{m}\left(1 - \dfrac{v^2}{\mathrm{c}^2}\right)^{-\frac{1}{2}}$ where m is the mass at rest and c is the speed of light.

At what rate is the mass (M) changing when the particle's velocity is $\frac{1}{2}$c and the rate of change of velocity is 0.01c per second?

14. A stationary police motor cycle is passed by a car travelling at 80 kmh^{-1}. The motor cycle starts in pursuit 10 seconds later moving with a constant acceleration for 300 metres and reaches a speed of 90 kmh^{-1} which it maintains.

(a) Draw a velocity time diagram of the car and the motor cycle.

(b) Find the time it takes the police cycle to catch the car.

(c) Determine the distance the police motor cycle travels before catching up with the car.

15. A particle moves along the x axis so that its acceleration is given by $a = \dfrac{1}{(t+1)^2}$ where $t \geq 0$ is the time in seconds.

The particle passes through the origin at $t = 0$.

(a) If the particle was observed at $t = 1$ to be at $x = \ln\frac{1}{2}$, find the initial speed and the direction of the motion.

(b) Will the particle pass through the origin again during its subsequent motion?

(c) Where will the particle be when t is very large? Give reasons for your answer.

16. A particle is travelling in a straight line in such a way that at time t seconds its displacement from its initial position at the origin is x, and $\dfrac{dx}{dt} = \dfrac{6t}{3 + t^2}$ $t \geq 0$.

(a) Express x in terms of t.

(b) Find the acceleration when $t = 1$.

(c) Show that the particle is always moving in the same direction.

(d) Find the distance travelled by the particle in the first second of its motion.

(e) When will the particle be twice this distance (in (d) above) from its initial position?

17. (a) Evaluate $\int_0^1 \sqrt{4-t^2}\, dt$ (substitute $t = 2\sin\theta$).

An **exact answer** is required.

(b) A particle is first observed at time $t = 0$ and its position at this point taken as its initial position. The particle moves in a straight line so that its velocity, $v(t)$, at time t is given by

$$v(t) = \begin{cases} \sqrt{4-t^2} & \text{for } 0 \le t \le 1 \\ \dfrac{4-t}{\sqrt{3}} & \text{for } t > 1 \end{cases}$$

i. Sketch the velocity - time graph.

ii. Find the distance travelled by the particle from its initial position until it comes to rest. An **exact answer** is required.

iii. If the particle returns to its initial position at $t = T$, calculate T correct to two decimal places.

18. Differentiate the following with respect to x

(a) $y = x^2 \ln(\sin x)$

(b) $y = 2^{x^2} + e^{\cos x}$

19. Find $\dfrac{dy}{dx}$ when $x = 2$ if $2xy^2 - 3x^2y = 16$.

20. Find each of the following derivatives. Do not spend much time simplifying your answers.

(a) $\dfrac{dy}{dx}$ where $y = \dfrac{\cos 2x}{(6x+3)^2}$

(b) $\dfrac{dy}{dx}$ where $y = \log_8 e^{4x}$

(c) $f'(x)$ if $f(x) = \int_{-2x}^{2x} \sin^2 2t\, dt$

21. A curve has an equation given by $x^3 + 2xy^2 + y^3 = 1$.

(a) Find the coordinates of the points on the curve which have $x = 1$.

(b) Find the equations of the tangents to the curves where $x = 1$.

22. If $y = x^{\frac{4}{x}}$ find $\dfrac{dy}{dx}$.

23. Find $\dfrac{dy}{dx}$ for:

(a) $y = \sin(\cos x)$

(b) $y^2 + xy + x^3 = 12$

(c) $y = \dfrac{\ln(x+1)}{x+1}$.

24. Find $\dfrac{d}{dx}(x^2 e^x)$ and use your answer to determine $\int 4x\, e^x (x+2)\, dx$.

25. Find $\dfrac{dy}{dx}$ for $y^2 + 2xy + 3 = 0$.

26. Find $y = f^{-1}(x)$ for $f(x) = \dfrac{e^{x-1}}{2}$.

27. Ben has a young son who was recently ill with a fever. He noticed that after being given a dose of penicillin the child's temperature increased, peaked and then decreased. Ben approximated the child's temperature above 37°C by the function

$T(x) = x\,e^{-0.2x}$ where $x > 0$

where x refers to the time in hours after 7.00pm. Using this model find:

(a) the rate of change of temperature and show that for $0 < x < 5$. the temperature was increasing

(b) the maximum temperature and the time this occurred

(c) and hence, use the above information sketch the graph of

$y = x\,e^{-0.2x}$ for $x > 0$.

28. Find the equation of the tangent to the curve

$e^y + y \ln x = 3$ at the point $P(1, \ln 3)$.

29. Estimate $\sqrt{50}$ using calculus techniques (to 2 d.p.).

30. The radius and volume of a spherical balloon are decreasing as air leaks out. Estimate the change in the volume as the radius changes from 5 m to 4.95 m.

31. An old clock has an oscillating pendulum of length L. The period (T) of oscillation (in minutes) is governed by the equation: $T = k\sqrt{L}$, where k is a constant. Determine how much the period would change if the length of the pendulum was shortened by 2%.

32. The radius of a sphere is measured as 20 cm, to an accuracy of ±0.04 cm. Using calculus techniques, estimate the possible error in the volume of the sphere.

33. A cone of fixed height, h, has a variable radius. If V represents the volume of the cone, find:

(a) δV in terms of r if $\delta r = 0.5$

(b) δV if $\frac{\delta r}{r} = 1\%$

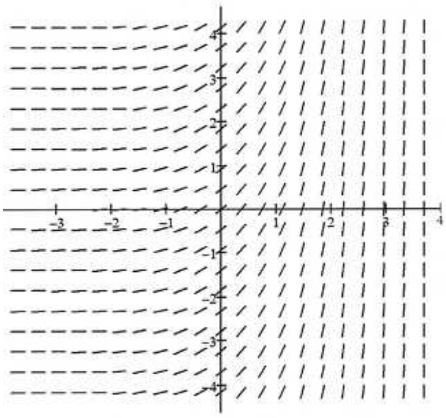
34. A spherical ball has its radius increased by 1%. Determine the approximate percentage increase in the volume of the ball.

35. The escape velocity V at the surface of a planet of radius R on which acceleration due to gravity g is given by $V = \sqrt{2gR}$. Find the approximate percentage increase in V when the radius increases by 9% if acceleration due to gravity remains constant.

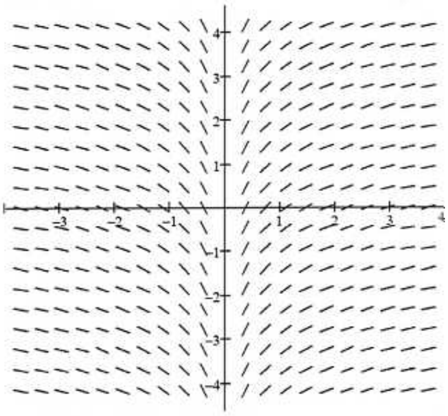
36. For the Graphs A and B, match the slope field with the function it appears to be modelling

$y = \ln|x|$
$y = e^x$

A

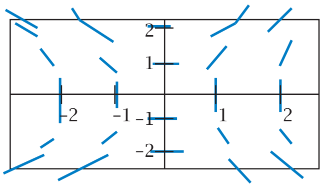


B

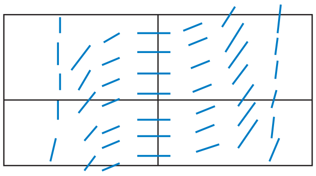


37. Match the slope fields below with their differential equations.

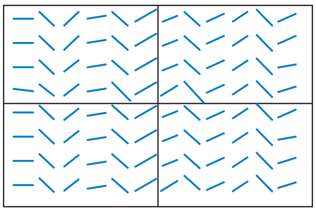
A



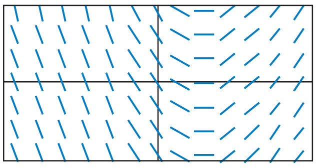
B



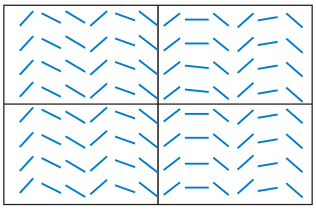
C



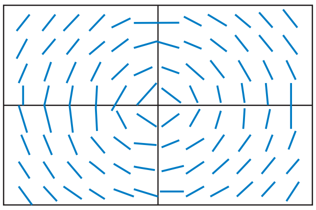
D



E



F



(1) $\frac{dy}{dx} = \cos x$

(2) $\frac{dy}{dx} = \frac{-x}{y}$

(3) $\frac{dy}{dx} = \frac{x^2}{2}$

(4) $\frac{dy}{dx} = \sin x$

(5) $\frac{dy}{dx} = \pm\sqrt{x^2+4}$

(6) $\frac{dy}{dx} = x - 1$

38. Commencing with the logistic growth model (Equation ③ from Notes), prove that there is a point of inflection for this model at:

$$\left(\frac{\ln|\frac{k}{P_0} - 1|}{rk}, \frac{k}{2}\right).$$

39. The population of a termite colony is modelled by:

$\frac{dP}{dt} = \frac{1}{400}P(250 - P)$, where t is in months.

(a) Identify the (i) maximum population of the colony; (ii) growth rate of the colony.

(b) If there are 72 termites present initially, develop a logistic equation to model the growth of the colony's population $P(t)$.

(c) Determine the population of the colony after 15 months.

40. A farmer releases 30 emus into a large paddock. After 4 years, the emu population has grown to 74. The farmer will be unable to support more than 570 emus. The growth rate of the emu population p is modelled by:

$\frac{dp}{dt} = kp\left(1 - \frac{p}{570}\right)$, $30 \le p \le 570$ where t is the number of years.

(a) Develop an equation of general form $P = \frac{L}{1 + be^{-kt}}$ where values have been determined for L, k and b.

(b) After how many years will the emu population reach 90% of the largest supportable population? (Assuming the growth rate remains constant.)

6

INTEGRATION AND APPLICATIONS OF INTEGRATION

Finding derivatives and integrals of elementary functions, which partly commenced in *Mathematics Methods* is extended here to include a study of the trigonometric functions. The starting point is the basic trigonometric approximation $\sin x \approx x$ for small values of x.

This section reviews differentiation of exponential and logarithmic functions and uses a more formal approach to the integration and differentiation of logarithmic and exponential functions.

Syllabus Checklist

By the end of this chapter, you should be able to:

Integration techniques

☐ integrate using the trigonometric identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 - \cos 2x)$ and $1 + \tan^2 x = \sec^2 x$

☐ use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))g'(x)$

☐ establish and use the formula $\int \frac{1}{x} dx = \ln|x| + c$ for $x \neq 0$

☐ use partial fractions where necessary for integration in simple cases

Applications of integral calculus

☐ calculate areas between curves determined by functions

☐ determine volumes of solids of revolution about either axis

☐ use technology with numerical integration

ESTABLISHING THE MAIN TRIGONOMETRIC LIMIT

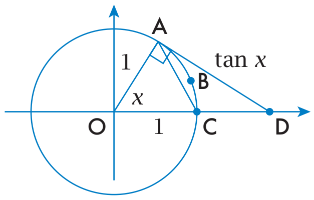
The starting trigonometric limit, $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$, is needed when finding $\frac{d}{dx}(\sin x)$.

This limit can be found from a table of values that examines the value of $\frac{\sin x}{x}$ as x tends to 0 in radian mode from the positive and negatives sides.

There are also various geometrical ways to prove the limit, one of which is shown below and the others are investigated in the exercises.

Consider the unit circle showing angle x^R in triangles OAC and OAD where AD is tangent to the circle at A.

Then $\quad \tan x = \frac{AD}{OA}$ or $AD = OA \tan x = \tan x$ as $|OA| = 1$.



$$\begin{array}{llclcll}
\text{Then} & \text{Area } \Delta\text{OAC} & < & \text{Area Sector OABC} & < & \text{Area OAD} & \\
\text{i.e.} & \frac{1}{2}(1^2)\sin x & < & \frac{1}{2}(1^2)x & < & \frac{1}{2}(1)\tan x & \times 2 \\
 & \sin x & < & x & < & \tan x & \\
 & \frac{1}{\sin x} & > & \frac{1}{x} & > & \frac{\cos x}{\sin x} & \times \sin x \\
 & 1 & > & \frac{\sin x}{x} & > & \cos x & \\
 & \cos x & < & \frac{\sin x}{x} & < & 1 & \\
\therefore & \lim\limits_{x\to 0}\cos x & < & \lim\limits_{x\to 0}\frac{\sin x}{x} & < & \lim\limits_{x\to 0} 1 &
\end{array}$$

Because $\lim\limits_{x\to 0}\cos x = 1$ and $\lim\limits_{x\to 0} 1 = 1$ it follows by the sandwich or squeeze concept that the middle limit $\lim\limits_{x\to 0}\frac{\sin x}{x}$ must also equal 1.

$$\therefore \quad \boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{where } x \text{ is in radians}}$$

PROVING THE SECOND MAIN TRIGONOMETRIC LIMIT

We also will need the value of $\lim\limits_{x\to 0}\frac{1-\cos x}{x}$ which can be proved algebraically as shown below

$$\begin{aligned}
& \lim_{x\to 0}\frac{1-\cos x}{x} \\
&= \lim_{x\to 0}\frac{1-\cos x}{x} \times \frac{1+\cos x}{1+\cos x} \\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} \\
&= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} \\
&= \lim_{x\to 0}\frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} \\
&= \lim_{x\to 0}\frac{\sin x}{x} \cdot \lim_{x\to 0}\frac{\sin x}{1+\cos x} \\
&= 1 \cdot 0 \\
&= 0
\end{aligned}$$

$$\therefore \quad \boxed{\lim_{x\to 0}\frac{1-\cos x}{x} = 0}$$

LIMIT THEOREMS

When limits are found algebraically the following limit theorems are often needed which will not be proven.

i. $\lim\limits_{x\to a}(f \pm g) = \lim\limits_{x\to a} f \pm \lim\limits_{x\to a} g$

ii. $\lim\limits_{x\to a}(fg) = \lim\limits_{x\to a} f \cdot \lim\limits_{x\to a} g$

iii. $\lim\limits_{x\to a}\left(\frac{f}{g}\right) = \frac{\lim\limits_{x\to a} f}{\lim\limits_{x\to a} g}, \ \lim\limits_{x\to a} g \neq 0$

iv. $\lim\limits_{x\to a}\left(f^n\right) = \left(\lim\limits_{x\to a} f\right)^n$

THE DERIVATIVE OF $y = \sin x$ FROM FIRST PRINCIPLES

When $\quad f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ is applied to $f(x) = \sin x$

we get
$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \sin h \cos x}{h} \\
&= \lim_{h \to 0} \frac{\sin x}{1} \cdot \frac{(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{\cos x}{1} \\
&= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x
\end{aligned}
$$

$$
\therefore \quad \boxed{\begin{aligned} \text{for} \quad f(x) &= \sin x \\ f'(x) &= \cos x \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} \text{for} \quad f(x) &= \cos x \\ f'(x) &= -\sin x \end{aligned}}
$$

Proof of the second derivative will be a part of the exercises.

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Now that the two standard trig derivatives are known almost any combination is now possible through the use of the product, quotient and chain rules.

Example If $f(x) = \tan x$ find $y = f'(x)$

$$
\begin{aligned}
f(x) &= \tan x \\
&= \frac{\sin x}{\cos x} \\
f'(x) &= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\
f'(x) &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}
$$

quotient rule

$$
\therefore \quad \boxed{\begin{aligned} \text{for} \quad y &= \tan x \\ \frac{dy}{dx} &= \frac{1}{\cos^2 x} \end{aligned}} \quad \text{and by the chain rule} \quad \boxed{\begin{aligned} \text{for} \quad y &= \tan f(x) \\ \frac{dy}{dx} &= \frac{f'(x)}{\cos^2 f(x)} \end{aligned}}
$$

Example If $y = \sin^5(\ln x)$ find $\dfrac{dy}{dx}$.

then $\dfrac{dy}{dx} = \underbrace{5\sin^4(\ln x)}_{\text{w.r.t power 5}} \cdot \underbrace{\cos(\ln x)}_{\text{w.r.t sin}} \cdot \underbrace{\frac{1}{x}}_{\text{w.r.t inside}}$ by the chain rule

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

- If $y = \sin ax$ then by the chain rule $\frac{d}{dx}(\sin ax) = a \cos ax$.

  $\therefore \quad \int a \cos ax \, dx = \sin ax$

  or $\quad a \int \cos ax \, dx = \sin ax$

  i.e. $\quad \boxed{\int \cos ax \, dx = \frac{\sin ax}{a} + c}$

  and $\quad \boxed{\int \sin ax \, dx = -\frac{\cos ax}{a} + c}$

  and $\quad \boxed{\int \frac{1}{\cos^2 ax} dx = \frac{\tan ax}{a} + c}$

- Finding $\int \sin^2 x \, dx$ and $\int \cos^2 x \, dx$

  involves the use of the double angle cosine identity. If $\quad \cos 2x = 2\cos^2 x - 1$

  $$= 1 - 2\sin^2 x$$

  then $\quad \cos^2 x = \frac{\cos 2x + 1}{2}$ and $\sin^2 x = \frac{1 - \cos 2x}{2}$.

  $$\therefore \quad \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx$$

  $$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

  $$= \frac{1}{2} (x - \frac{\sin 2x}{2}) + c$$

  $$= \frac{x}{2} - \frac{\sin 2x}{4} + c$$

  and $\quad \int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$

- Often functions to be integrated are in the form

  $$\int (f(x))^n \cdot f'(x) \, dx$$

  That is, a power function with the derivative of the inside function on the outside of the term.

  Consider $\quad y = \frac{(f(x))^{n+1}}{n+1}$ then $\frac{dy}{dx} = \frac{(n+1)(f(x))^n}{n+1} \cdot f'(x)$ by the chain rule

  $$= (f(x))^n \cdot f'(x)$$

  $$\therefore \quad \boxed{\int (f(x))^n f'(x) \, dx = \frac{(f(x))^{n+1}}{n+1} + c}$$

  This general form means that you should always be looking to find the derivative of the "inside" function "on the outside" but sometimes they are quite deceptive!

Example Find $\int \sin x \cos x\, dx$

The derivative of the first function $\sin x$ is $\cos x$.

$$\therefore \int \sin x \cos x\, dx = \frac{\sin^2 x}{2} + c .$$

Alternatively, $\int \sin x \cos x\, dx$

$$= \frac{1}{2} \int \sin 2x\, dx$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c$$

$$= -\frac{\cos 2x}{4} + c .$$

Students now often say that $\frac{\sin^2 x}{2}$ doesn't look anything like $-\frac{\cos 2x}{4}$? However, the functions only differ by a constant which means that you must not assume that the value of c at the end of each integral is the same (see Problems to Solve, Chapter 6, q.5).

CANCELLATION EQUATIONS

Interestingly, the cancellation equations for the sine function and the inverse sine function become:

- $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$

The cancellations equations for the cosine function and inverse cosine function are:

- $\cos^{-1}(\cos x) = x$ for $0 \leq x \leq \pi$
- $\cos(\cos^{-1} x) = x$ for $-1 \leq x \leq 1$

The cancellation equations for the tangent function and inverse tangent function are:

- $\tan^{-1}(\tan x) = x$ for $-\infty \leq x \leq \infty$
- $\tan(\tan^{-1} x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

DIFFERENTIATION

To differentiate the inverse sine function, we commence with $y = \sin^{-1}x$.

$\sin y = x$	Take sine of both sides
$\cos y \frac{dy}{dx} = 1$	Differentiate implicitly with respect to x
$\frac{dy}{dx} = \frac{1}{\cos y}$	Rearrange for $\frac{dy}{dx}$

Now, because $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos y \geq 0$. Through the Pythagorean Theorem we have:

$$\cos^2 y = 1 - \sin^2 y$$

and $\cos^2 y = 1 - x^2$ (because $x = \sin y$)

$$\therefore \cos y = \sqrt{1 - x^2}$$

We can conclude that: $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$

$$\therefore \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}} \text{ for } -1 < x < 1$$

The derivatives for $y = \cos x$ and $y = \tan x$ are as follows:

- $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

INVERSE TRIGONOMETRIC FUNCTIONS

Looking at the function $y = \sin x$, you will notice that it is not a one-to-one function (it doesn't pass the horizontal line test!). As such, it does not have an inverse unless the domain is restricted from $-\infty < x < \infty$. Let's restrict the domain of $y = \sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ so that an inverse function can be determined.

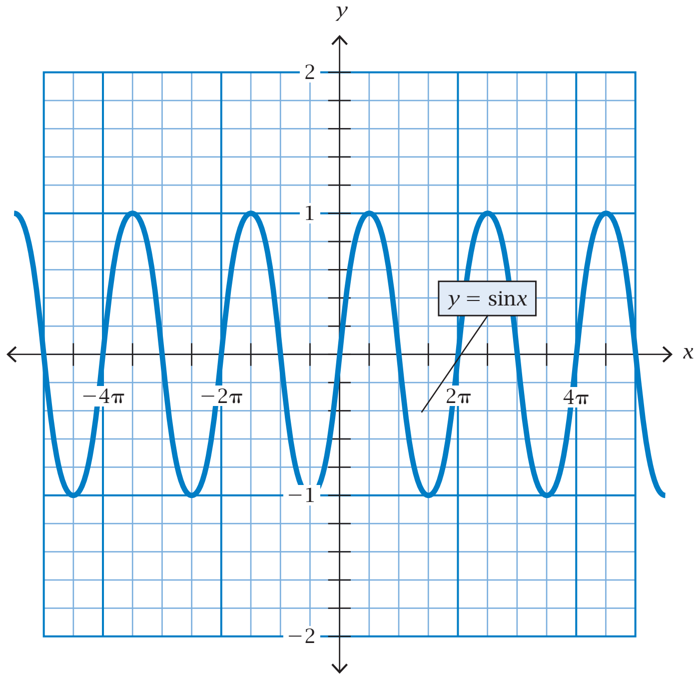


Figure 1: $y = \sin x$

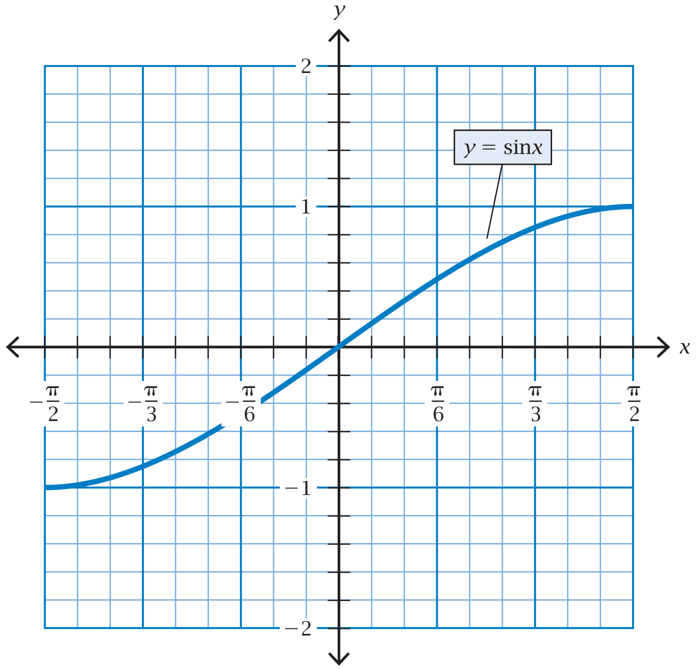


Figure 2: $y = \sin x$; $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

The definition of an inverse function states that if $f(y) = x$, then $f^{-1}(x) = y$ and vice versa. Consequently we can assert that if $y = \sin x$ then $\sin^{-1} y = x$. Alternatively, if $y = \sin^{-1} x$ then $\sin y = x$.

Because inverse functions are written as y in terms of x we shall refer to the inverse function of $y = \sin x$ as: $y = \sin^{-1} x = \arcsin x$ for $-1 \leq x \leq 1$.

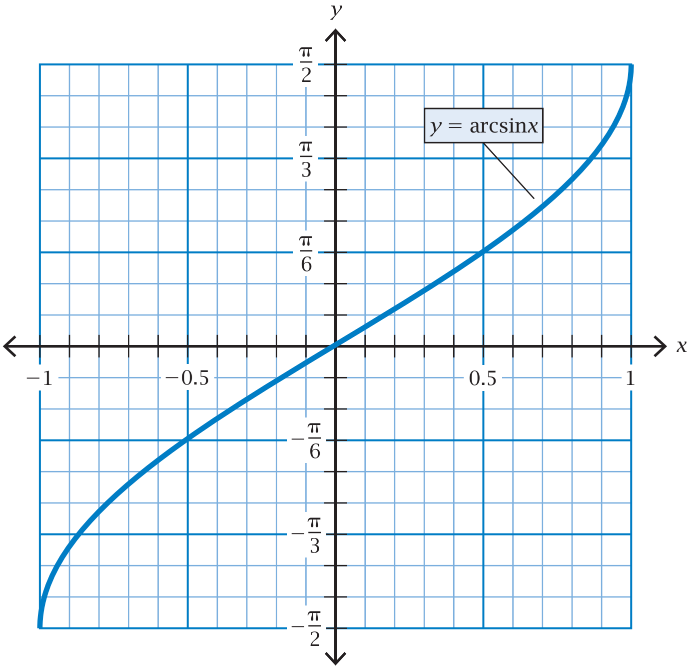


Figure 3: $y = \sin^{-1} x = \arcsin x$

USING SUBSTITUTION WITH TRIGONOMETRIC INTEGRATION

Sometimes a substitution (i.e. a change of variable) will make an integration easier. These are best demonstrated by examples.

Example 1 Find $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} \, dx$$

$$= \int -\frac{du}{u}$$

$$= -\ln|u| + c$$

$$= -\ln|\cos x| + c$$

let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$du = -\sin x \, dx$$

$$\sin x \, dx = -du$$

Example 2 Find $\int \frac{\sin(\ln 2x)}{2x} \, dx$

$$= \frac{1}{2} \int \sin(\ln 2x) \cdot \frac{dx}{x}$$

$$= \frac{1}{2} \int \sin u \, du$$

$$= -\frac{1}{2} \cos u + c$$

$$= -\frac{1}{2} \cos(\ln 2x) + c$$

let $u = \ln 2x$

$$\frac{du}{dx} = \frac{2}{2x}$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

Example 3 When you are asked to show all working to compute a definite integral using a substitution, it is best to change the limits as well.

Find exactly $\int_{\frac{2\pi}{3}}^{2\pi} \sin^3 \frac{x}{2}\, dx$ by letting $u = \cos \frac{x}{2}$

$$= \int_{\frac{2\pi}{3}}^{2\pi} \sin^2 \frac{x}{2} \cdot \sin \frac{x}{2}\, dx$$

$$= \int_{\frac{2\pi}{3}}^{2\pi} \left(1 - \cos^2 \frac{x}{2}\right) \text{Sub} \sin \frac{x}{2}\, dx$$

$$= \int_{0.5}^{-1} (1 - u^2)\,(-2du)$$

$$= -2\int_{0.5}^{-1} (1 - u^2)\, du$$

$$= 2\int_{-1}^{0.5} (1 - u^2)\, du$$

$$= 2\left(u - \frac{u^3}{3}\right)_{-1}^{0.5}$$

$$= 2\left[\left(0.5 - \frac{0.5^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right)\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{24} + 1 - \frac{1}{3}\right]$$

$$= 2\frac{1}{4}$$

$$\frac{du}{dx} = -\frac{1}{2} \sin \frac{x}{2}$$

$$-2du = \sin \frac{x}{2}\, dx$$

when $x = \frac{2\pi}{3}$, $u = \cos \frac{\pi}{3}$

$= 0.5$

when $x = 2\pi$, $u = \cos \pi$

$= -1$

INTEGRAL PROPERTIES

As demonstrated in some of the previous examples you should be familiar with, and often use, the following:

(a) $\int a\, f(x)\, dx = a \int f(x)\, dx$ taking out the constant of multiplication is often a good idea

(b) $\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

(c) $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$ where $a < b < c$

(d) $-\int_a^b f(x)\, dx = \int_b^a f(x)\, dx$.

INTRODUCTION

You will remember that for $f(x) = e^x$ and $g(x) = \ln x$

then $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$

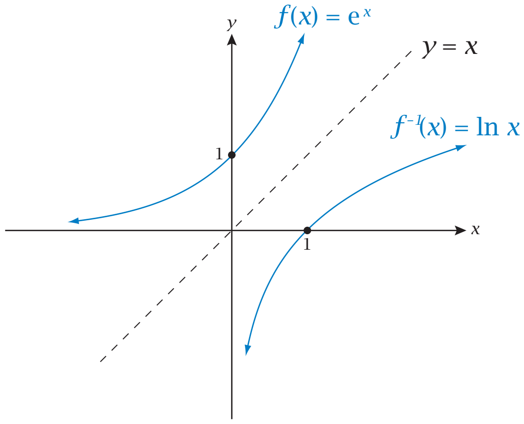
$$\begin{aligned} (f\ o\ g)(x) &= f(g(x)) \\ &= f(\ln x) \\ &= e^{\ln x} \\ &= x \end{aligned} \qquad \begin{aligned} (g\ o\ f)(x) &= g(f(x)) \\ &= g(e^x) \\ &= \ln e^x \\ &= x \ln e \\ &= x \end{aligned}$$

which means that f and g are inverse functions.

That is, if $f(x) = e^x$ then $f^{-1}(x) = \ln x$

and if $g(x) = \ln x$ then $g^{-1}(x) = e^x$.

The inverse relationship is shown graphically by the fact that $f(x) = e^x$ and $f^{-1}(x) = \ln x$ are symmetrical about the line $y = x$ as shown below.



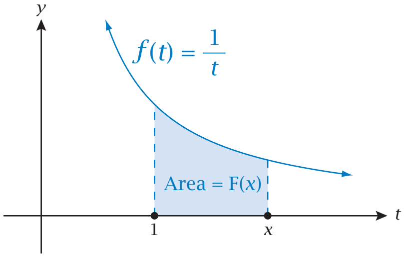
Similarly, for other bases, (e.g. base 10)

if $f(x) = 10^x$ then $f^{-1}(x) = \log_{10} x$ etc.

INVESTIGATION OF $\int_1^x \frac{1}{t}\, dt$

Much information can be found out about the function $F(x) = \int_1^x \frac{1}{t}\, dt$

which is the area under $f(t) = \frac{1}{t}$ as shown below,



By using the fundamental theorem as follows:

(a) When $x = 1$, $F(1) = \int_1^1 \frac{1}{t}\, dt = 0 \quad \therefore\ F(1) = 0$.

(b) When $x > 1$, $F(x) > 0$ i.e. $y = F(x)$ is above the x axis.

(c) When $x < 1$, $F(x) < 0$ i.e. $y = F(x)$ is below the x axis.

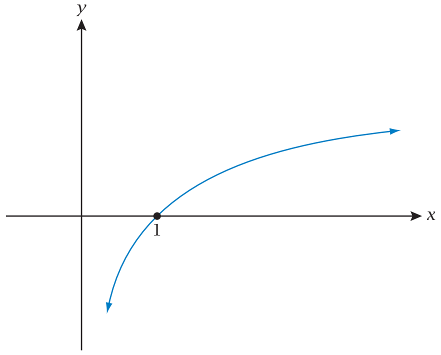
(d) Because $F'(x) = \frac{1}{x}$, $F'(x) = 0$ has no solutions and hence $y = F(x)$ has no stationary points.

(e) For $x > 0$, $F'(x) = \frac{1}{x}$ is > 0 i.e. $y = F(x)$ always has a positive slope so it is an increasing function.

(f) For $x > 0$, $F''(x) = -\frac{1}{x^2}$ is < 0 for all x so $y = F(x)$ is always concave downwards.

(g) Also for $x \to 0^+$ i.e. if we consider $\lim_{x \to 0^+} F(x)$ it appears that the limit is tending to $-\infty$.

From all of the information above, the graph of $y = \text{F}(x)$ has to be as shown below.



MORE PROPERTIES OF $\text{F}(x) = \int_1^x \frac{1}{t}\,dt$

Now if you haven't been convinced that this function is of course $\text{F}(x) = \ln x$, the log laws are now shown to hold as follows:

$$\text{If} \quad \text{F}(x) = \int_1^x \frac{1}{t}\,dt$$

$$\text{then} \quad \text{F}(bx) = \int_1^{bx} \frac{1}{t}\,dt = \int_1^{x} \frac{1}{t}\,dt + \int_x^{bx} \frac{1}{t}\,dt \quad \text{by the integral property on p.132.}$$

The first integral is obviously $\text{F}(x)$ and using the substitution $u = \frac{t}{x}$ in the second integral gives:

$$\int_x^{bx} \frac{1}{t}\,dt$$

$$= \int_1^{b} \frac{du}{u}$$

$$= \text{F}(b)$$

$$\text{let} \quad u = \frac{t}{x}\ , \ t = xu$$

$$\frac{du}{dt} = \frac{1}{x}\ , \ dt = xdu$$

$$\text{then} \quad \frac{dt}{t} = \frac{xdu}{xu}$$

$$\frac{dt}{t} = \frac{du}{u}$$

$$\text{when} \ t = x\ , \ u = 1$$

$$\text{when} \ t = bx\ , \ u = b$$

So, in conclusion, $\text{F}(bx) = \text{F}(x) + \text{F}(b)$

or $\text{F}(bx) = \text{F}(b) + \text{F}(x)$ which is the **first** log law.

The proof of the **second** law $\text{F}\left(\frac{x}{b}\right) = \text{F}(x) - \text{F}(b)$ is set as a problem for you in the Problems to Solve section. The **third** law $\text{F}(x^n) = n\,\text{F}(x)$ is proved in Worked Example 7.9.

In conclusion then

$$\text{F}(x) = \ln x = \int_1^x \frac{1}{t}\,dt$$

$$\text{and} \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Note that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ will be proven in Mathematics Methods (Unit 4).

THE CHANGE OF BASE FORMULA

Consider the logarithm of a to the base b. This is, $\log_b a$ and let it equal x.

$$\text{If} \quad \log_b a = x$$

$$\text{then} \quad b^x = a \qquad \text{Now write } b \text{ and } a \text{ as powers of base } c$$

$$\therefore \quad (c^{\log_c b})^x = c^{\log_c a}$$

$$c^{x\log_c b} = c^{\log_c a} \qquad \text{Equate Indices}$$

$$x\log_c b = \log_c a$$

$$x = \frac{\log_c a}{\log_c b}$$

$$\therefore \quad \log_b a = \frac{\log_c a}{\log_c b}$$

is called the change of base rule.

Most of the time the new base c will be chosen as either $c = 10$ or $c = \mathrm{e}$ because $\log_{10} x$ and $\log_{\mathrm{e}} x = \ln x$ are available on your calculator.

$$\therefore \quad \log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$$

Example Find the value of $\log_8 5$.

Either $\log_8 5 = \dfrac{\log 5}{\log 8}$ or $\log_8 5 = \dfrac{\ln 5}{\ln 8}$

$= 0.774$ 3dp $\qquad = 0.774$ 3dp

DIFFERENTIATING LOGARITHMIC FUNCTIONS

◇ If $y = \ln x$ also if $y = \ln f(x)$

then $\dfrac{dy}{dx} = \dfrac{1}{x}$ then, by the chain rule, $\dfrac{dy}{dx} = \dfrac{1}{f(x)} \cdot f'(x)$

In general,

$$\text{if} \quad y = \ln f(x)$$

$$\text{then} \quad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Again, something to be stressed in Mathematics Methods is that if $f(x)$ is a complicated function from which $f'(x)$ is not straightforward, then $\ln f(x)$ should be first broken down using the various log laws **before** $\dfrac{dy}{dx}$ is attempted.

Example If $y = \ln\left[\dfrac{(2x+3)^{\frac{1}{2}}}{(4-x^2)^3}\right]$ use the second log law first

$= \ln(2x+3)^{\frac{1}{2}} - \ln(4-x^2)^3$ now use the third log law

$$= \frac{1}{2}\ln(2x+3) - 3\ln(4-x^2)$$

$$\text{then} \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{2}{2x+3} - 3 \cdot \frac{-2x}{4-x^2}$$

$$= \frac{1}{2x+3} + \frac{6x}{4-x^2}$$

- For $y = \log_b x$ use the change of base rule

$$\text{i.e. } y = \frac{\ln x}{\ln b} = \frac{1}{\ln b} \cdot \ln x$$

$$\text{then } \frac{dy}{dx} = \frac{1}{\ln b} \cdot \frac{1}{x}$$

$$= \frac{1}{x \ln b}$$

$$\therefore \quad \boxed{\text{For } y = \log_b x \quad \frac{dy}{dx} = \frac{1}{x \ln b}} \quad \text{and} \quad \boxed{\text{For } y = \log_b f(x) \quad \frac{dy}{dx} = \frac{f'(x)}{(\ln b)\, f(x)}}$$

- For $\boxed{y = e^x \quad \frac{dy}{dx} = e^x}$ and $\boxed{y = e^{f(x)} \quad \frac{dy}{dx} = f'(x)\, e^{f(x)}}$

And for $y = a^x$ use $a = e^{\ln a}$

$$y = (e^{\ln a})^x$$

$$y = e^{x \ln a}$$

$$\frac{dy}{dx} = (\ln a)\, e^{x \ln a}$$

$$= (\ln a)\, a^x$$

$$\boxed{\text{If } y = a^x \quad \text{then } \frac{dy}{dx} = (\ln a) a^x}$$

$$\text{If } y = a^{f(x)} \qquad \text{then let} \qquad u = f(x)$$

$$y = a^u \qquad \frac{du}{dx} = f'(x)$$

$$\frac{dy}{du} = (\ln a)\, a^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (\ln a) a^u \cdot f'(x)$$

$$= (\ln a) a^{f(x)} \cdot f'(x)$$

$$\therefore \quad \boxed{\text{If } y = a^{f(x)} \quad \text{then } \frac{dy}{dx} = (\ln a)\, f'(x) \cdot a^{f(x)}}$$

INTEGRATING EXPONENTIAL FUNCTIONS AND OTHERS

- From above, $\frac{d}{dx}(e^{f(x)}) = f'(x)\, e^{f(x)}$ from which it follows that $\int f'(x)\, e^{f(x)}\, dx = e^{f(x)} + c$.

$$\therefore \quad \boxed{\int k\, f'(x)\, e^{f(x)}\, dx = k\, e^{f(x)} + c}$$

Sometimes these types are disguised and some initial processing is needed to write the function in the required form before integration

Example Find $\int e^{3x^2} \cdot e^{4.5x} \cdot (4x + 3)\, dx$

$$= \int e^{3x^2 + 4.5x} (4x + 3)\, dx$$

$$= \frac{2}{3} \int e^{3x^2 + 4.5x} \cdot \frac{3}{2} (4x + 3)\, dx$$

$$= \frac{2}{3} \int e^{3x^2 + 4.5x} \cdot (6x + 4.5)\, dx$$

$$= \frac{2}{3}\, e^{3x^2 + 4.5x} + c$$

◇ Because for $y = a^{f(x)}$, $\frac{dy}{dx} = (\ln a)\, f'(x) \cdot a^{f(x)}$

Therefore $\int (\ln a)\, f'(x)\, a^{f(x)}\, dx = a^{f(x)} + c$.

$$\therefore \quad \boxed{\int f'(x)\, a^{f(x)}\, dx = \frac{a^{f(x)}}{\ln a} + c}$$

◇ If $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$, it follows that

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c}$$

Example $\int \tan x\, dx$

$= \int \frac{\sin x}{\cos x}\, dx$

$= -\int \frac{(-\sin x)}{\cos x}\, dx$ arrange for the derivative of the bottom to be on the top

$= -\ln|\cos x| + c$

As before, you will need to be on the lookout for rational functions.

That is, those in the form of $\frac{g(x)}{h(x)}$ where the derivative of $h(x)$ looks very like $g(x)$.

AREA BETWEEN CURVES

◇ If $y = f(x)$ for $f(x) > 0$ between $x = a$ and $x = b$, then the area under the curve is given by:

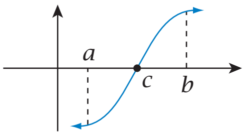
$$\text{Area} = \int_a^b f(x)\, dx$$

If $f(x) < 0$, then the value of the integral will be negative and the sign will have to be ignored as area can't be < 0.

$$\text{In general,} \quad \text{Area} = \left| \int_a^b f(x)\, dx \right|$$

◇ If $y = f(x)$ crosses the x axis between a and b at say c,

then the area will have to be computed from the diagram as:



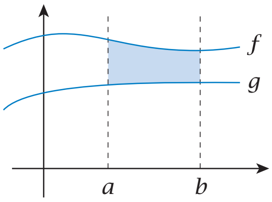
$$\text{Area} = -\int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

The negative outside the first integral makes that area positive. Of course the value of c will have to be computed as an x intercept of f. If no working is required, then simply find

Area $= \int_a^b |f(x)|\, dx$ using your graphics function.

◇ The area between two functions f and g:

i.e.

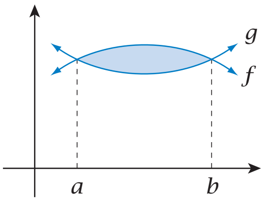


is Area $= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

$= \int_a^b (f(x) - g(x))\, dx$.

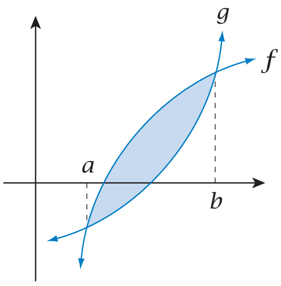
The situation could also be as below where the points a and b have to be found first.

i.e.



$$\text{Area} = \int_a^b (f(x) - g(x))\, dx\,.$$

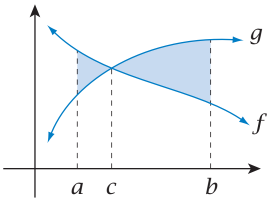
If f and g cross the x axis like this:



then as before,

$$\text{Area} = \int_a^b (f(x) - g(x))\, dx$$

- ◇ If f and g intersect each other at $x = c$, then:



$$\text{Area} = \int_a^c (f(x) - g(x))\, dx + \int_c^b (g(x) - f(x))\, dx\,.$$

In general, the more positive function should be placed first in the subtraction.

- ◇ If no working is required then the area between f and g, irrespective of how many times they cross each other or the x axis within $a \le x \le b$, is

$$\text{Area} = \int_a^b |\, f(x) - g(x)\,|\, dx$$

As before utilise the integral function on your calculator.

INTEGRATION INVOLVING SUBSTITUTIONS

- ◇ Again these types have been introduced previously in relation to trigonometric functions. Here the method is formally presented as a change of variable to simplify indefinite and definite integrals.
- ◇ If $\int f(x)\, dx$ is needed and it seems rather obvious that some part of $f(x)$ could be let equal to say $u = g(x)$, then by a series of steps it may be possible to write the integral in terms of u, which hopefully is easier to process.

The steps are:

i. find $\frac{du}{dx} = g'(x)$

ii. rearrange $\int f(x)\, dx$ into the form $\int h(x) \cdot \frac{du}{dx} \cdot dx$

iii. use $u = g(x)$ and/or $x = g^{-1}(u)$ to write the integral of ii) as $\int h(g^{-1}(u))\, du$ which is now entirely in terms of u

iv. Substitute back after the integration is done.

The following examples show the technique.

① $$\int x\sqrt{1+x^2}\,dx \qquad \text{let} \qquad u = 1 + x^2$$

$$= \frac{1}{2}\int \sqrt{1+x^2}\;2x\,dx \qquad \frac{du}{dx} = 2x$$

$$= \frac{1}{2}\int \sqrt{1+x^2}\;\frac{du}{dx}\cdot dx \quad *$$

Note: In practice it is easier to separate the variables and write $du = 2x\,dx$. This then avoids the line marked*.

$$= \frac{1}{2}\int \sqrt{u}\;du$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\;du$$

$$= \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}u^{\frac{3}{2}} + c \qquad \text{Now substitute back}$$

$$= \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$$

Note that because the "derivative of the inside (i.e. $1 + x^2$)" is on the "outside", the x term (except for the constant of $\frac{1}{2}$) in the above integral, falls into the shape of:

$$\int (f(x))^n \cdot f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + c$$

and students are encouraged to recognize and use this quicker way. The next example does not fall into this category.

② $$\int x^3\sqrt{1+x^2}\,dx \qquad \text{let} \qquad u = 1 + x^2, \quad x^2 = u - 1$$

$$= \frac{1}{2}\int x^2\sqrt{1+x^2}\;2x\,dx \qquad \frac{du}{dx} = 2x$$

$$= \frac{1}{2}\int (u-1)\sqrt{u}\;du \qquad du = 2x\,dx$$

$$= \frac{1}{2}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$$

$$= \frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{1}{5}\sqrt{(1+x^2)^5} - \frac{1}{3}\sqrt{(1+x^2)^3} + c$$

◇ If a definite integral involves a substitution you are advised to change the x limits to u limits and not go back to the x variable at all.

③ Evaluate $\int_{\sqrt{3}}^{\sqrt{8}} x^3 \sqrt{1+x^2}\, dx$ exactly by letting $u = 1 + x^2$

The process is as above with the addition of the limit change step as follows:

for $x = \sqrt{3}$, $u = 4$

for $x = \sqrt{8}$, $u = 9$.

Now when x's are replaced by u's we have, after a few steps, as above

$$\int_{\sqrt{3}}^{\sqrt{8}} x^3 \sqrt{1+x^2}\, dx = \frac{1}{2}\int_4^9 (u-1)\sqrt{u}\, du \quad \text{which after a few more steps gives}$$

$$= \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_4^9 \quad \text{which finally leads to the answer}$$

$$= 35\frac{13}{15}$$

TOTAL CHANGE

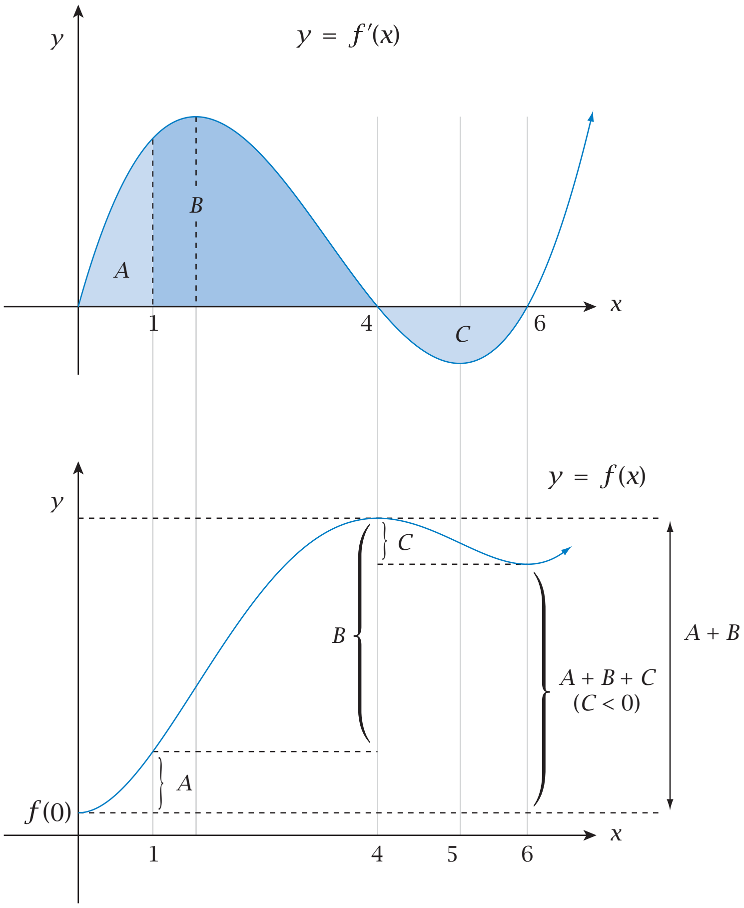
It is well known that the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is computed from the definite integral $\int_a^b f(x)\, dx$. If the function crosses the x axis between a and b then you will need to be careful about whether you are finding the difference between the area above the axis and the area below the x axis instead of the total area. When the function being integrated is itself a derivative; e.g. $\int f'(x)\, dx$, then the result is $f(x) + c$. This makes more sense when a practical application is involved. For example, when the derivative is the velocity of an object $v(t) = \frac{dx}{dt}$ then

$$x(t) = \int v(t)\, dt = \int \frac{dx}{dt} \cdot dt$$

In order to find the function $x(t)$, a point on the x t curve needs to be known which enables the constant of integration say c to be found. If a point is not specified and c cannot be found then the following definite integral gives the total change in x i.e. Δx which will be the displacement of the object between times t_1 and t_2.

$$\Delta x = \int_{t_1}^{t_2} v(t)\, dt$$

This means that the areas under a derivative function $\frac{dy}{dx}$ correspond to changes in y i.e. Δy on the $y = f(x)$ function. The example below shows how this works.



If a definite integral of f' equals the total change in y, that is Δy between x_1 and x_2,

then $$A = \int_0^1 f'(x)\,dx > 0 \qquad A + B = \int_0^4 f'(x)\,dx > 0$$

$$B = \int_1^4 f'(x)\,dx > 0 \qquad A + B - |C| = \int_0^6 f'(x)\,dx \quad \text{or} \quad = A + B + C \quad \text{where} \quad C < 0$$

$$C = \int_4^6 f'(x)\,dx = y(6) - y(4) < 0\,.$$

Integration by Parts

There are some functions for which integration is difficult. If the integrand is the product of 2 functions of x, then the technique **integration by parts** is useful. The basis of this technique is drawn from the product rule used in differential calculus.

The product rule states for a function $f(x) = uv$

then $f'(x) = udv + vdu$

Putting this process in reverse, we integrate both sides of the differentiated equation with the idea that "the sum of the integrals = the integral of the sums".

So $$\int f'(x)\,dx = \int udv + \int vdu$$

i.e. $$uv = \int udv + \int vdu$$

Rearranging gives $$\int udv = uv - \int vdu$$

Worked Example 14 outlines how this process works. Worked Example 15 illustrates how sometimes the process has to be used twice before the correct integral can be determine.

Integration: Inverse Trigonometric Forms

Theory

Using the derivatives of inverse trigonometric functions as a guide, we can see that:

- ◇ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$
- ◇ $\int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c$
- ◇ $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

Taking these a little further, we can also obtain:

- ◇ $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
- ◇ $\int -\frac{1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$
- ◇ $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Partial Fractions

Theory

If we are given an integral expressed as a fraction, and the common rules (e.g. logarithmic integration) or techniques (e.g. substitution) cannot be applied then the method of partial fractions may be applied.

For instance, if we are asked to determine $\int \frac{x-4}{x^2-5x+6}\,dx$ we can re-express the integrand such that it is the sum of two fractions.

Beginning with $\frac{x-4}{x^2-5x+6}$ we rewrite with the denominator factorised.

So $\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-2)(x-3)}$

Rewriting the fraction as the sum of partial fractions we obtain:

$\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$, where A and B are constants.

We can do this as the degree of the numerator is lower than the degree of the denominator.
To find the values of A and B, we add the fractions on the RHS of the equation to achieve:

$$\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)}{(x-2)(x-3)} + \frac{B(x-2)}{(x-2)(x-3)}$$

Because the denominators of the LHS and RHS fractions are equal, we can equate the numerators.

In other words, $x - 4 = A(x - 3) + B(x - 2)$.

Expanding the brackets gives: $x - 4 = Ax - 3A + Bx - 2B$

Equating the coefficients of like terms we can obtain the following equations:

$1 = A + B$ and $-4 = -3A - 2B$

Solving these equations simultaneously we can obtain $A = 2$, $B = -1$.

Therefore, $\frac{x-4}{(x-2)(x-3)} = \frac{2}{x-2} - \frac{1}{x-3}$

Consequently we can continue the original integral as such:

$$\begin{aligned}\int \frac{x-4}{x^2-5x+6}\,dx &= \int \frac{2}{x-2}\,dx - \int \frac{1}{x-3}\,dx \\ &= 2\ln(x-3) - \ln(x-3) + c\end{aligned}$$

With a partial fractions approach, there are various forms to consider. In general, the forms can be detected by the types of factors found in the denominator. Some of these are indicated below:

1. If the degree of the numerator is greater or equal to that of the denominator: Take the preliminary step of dividing the denominator into the numerator (e.g. using long division).

$$\frac{x^2-1}{x^2-16} = 1 + \frac{A}{x+4} + \frac{B}{x-4}$$

2. If the denominator has more than two linear factors: Include a term that corresponds to each factor.

$$\frac{-5x+1}{(x-3)(x+4)} = \frac{A}{x+4} + \frac{B}{x-3}$$

3. If there is a repeated linear factor: Include extra terms in the partial fraction expansion.

$$\frac{3x+5}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

4. After factorising as far as possible, there remains an irreducible quadratic factor ($ax^2 + bx + c$, for which there are no real solutions) in the denominator. The corresponding partial fraction will be of the form: $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants. To integrate this term, complete the square for the denominator and then use the formula:

$$\int \frac{1}{a^2+x^2}\,dx = \tfrac{1}{a}\tan^{-1}\left(\tfrac{x}{a}\right) + c$$

THE FUNDAMENTAL THEOREM OF CALCULUS

If a function is defined as an integral such as

$$y = \mathrm{F}(x) = \int_a^x f(t)\,dt \qquad \text{where } a \text{ is constant}$$

then $\frac{dy}{dx} = f(x)$ which can be written as

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ is the first version of the fundamental theorem of calculus.

Reference should be made to your text book or to the Mathematics Methods Study Guide (Units 3 and 4), for the derivation.

The second version of the fundamental theorem is

$$\int_a^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) \quad \text{where } \mathrm{F}(x) \text{ is an antiderivative of } f(x).$$

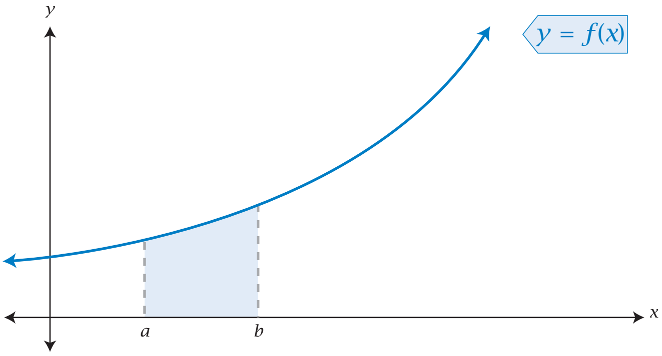
The first version has an extension as follows

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x) \quad \text{which can be shown by using the chain rule.}$$

VOLUMES OF SOLIDS OF REVOLUTION

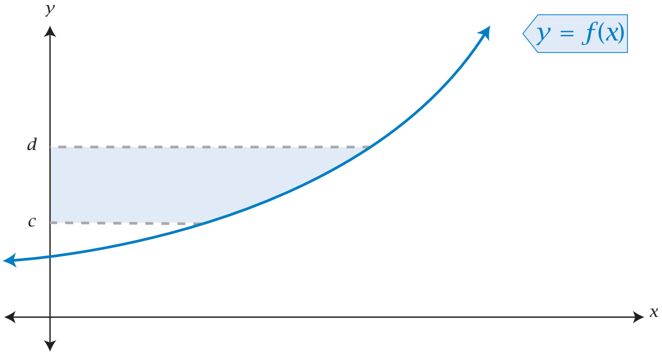
The volume, V, of a solid of revolution is formed when the area bounded by $y = f(x)$, $x = a$ and $x = b$ is rotated about the x axis.

$$V = \pi \int_a^b y^2 \, dx$$



The volume, V, of a solid of revolution is formed when the area bounded by $y = f(x)$, $y = c$, $y = d$ and $x = 0$ is rotated about the y axis.

$$V = \pi \int_c^d x^2 \, dy \text{ where } x = f^{-1}(y)$$



Worked Examples

6.1 Find $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\sin 5x}$ algebraically.

First write the function as $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} \cdot \dfrac{x}{\sin 5x}$

$$= \lim_{x \to 0} \frac{2 \sin 2x}{2x} \cdot \lim_{x \to 0} \frac{5x}{5 \sin 5x}$$

$$= 2 \lim_{2x \to 0} \frac{\sin 2x}{2x} \cdot \frac{1}{5} \lim_{5x \to 0} \frac{5x}{\sin 5x}$$

Now let $y = 2x$ and $u = 5x$

$$= \frac{2}{5} \lim_{y \to 0} \frac{\sin y}{y} \cdot \lim_{u \to 0} \frac{u}{\sin u}$$

$$= \frac{2}{5} \cdot 1 \cdot 1$$

$$= \frac{2}{5}$$

Note that $\lim\limits_{x \to 0} \dfrac{x}{\sin x} = 1$ (see Problems to Solve, 7.3)

6.2 Find $\int \sin^5 x \, dx$.

For these types, if the power is odd the best way is to use $\int (f(x))^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + c$.

So

$$
\begin{aligned}
& \int \sin^5 x \, dx \\
= & \int \sin^4 x \sin x \, dx \\
= & \int (1 - \cos^2 x)^2 \sin x \, dx \\
= & \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx \\
= & \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x) \, dx \\
= & -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + c
\end{aligned}
$$

Because $\frac{d}{dx}(\cos x) = -\sin x$, the signs had to be changed over for both the second and third terms.

> It is a good idea to differentiate each term of your answer and confirm that you do indeed get each answer above. If the sign is wrong then simply swap it over.

6.3 If $y = 2\cos^2 x \tan\sqrt{x}$ find $\frac{dy}{dx}$.

By the chain and product rules and $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$

$$\frac{dy}{dx} = 2\cos x\,(-\sin x)\tan\sqrt{x} + 2\cos^2 x \cdot \frac{1}{\cos^2\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\cos^2 x}{\sqrt{x}\cos^2\sqrt{x}} - 2\sin x\cos x\tan\sqrt{x}$$

6.4 Find the exact value of $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{3} + h\right) - \frac{1}{2}}{h}$.

Because $\cos\frac{\pi}{3} = \frac{1}{2}$ this limit fits the shape of a first principles derivative where $f(x) = \cos x$

$$f'(x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} \qquad x = \frac{\pi}{3},\ \cos\frac{\pi}{3} = \frac{1}{2}$$

$$f'\left(\frac{\pi}{3}\right) = \lim_{h\to 0}\frac{\cos\left(\frac{\pi}{3} + h\right) - \cos\frac{\pi}{3}}{h}$$

So $f(x) = \cos x$ and the limit is the value of $f'\left(\frac{\pi}{3}\right)$

$$
\begin{aligned}
f'(x) &= -\sin x \\
f'\left(\frac{\pi}{3}\right) &= -\sin\frac{\pi}{3} \\
&= \frac{-\sqrt{3}}{2} \qquad \therefore \quad \lim_{h\to 0}\frac{\cos\left(\frac{\pi}{3}+h\right) - \frac{1}{2}}{h} = -\frac{\sqrt{3}}{2}
\end{aligned}
$$

6.5 Find $\lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$ by using a first principles derivative.

The alternative first principles derivative for finding the derivative of a function f at $x = a$ is

$$f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}$$

The required limit is matched up with the above definition, giving:

$f(x) = \cos x$ and $a = \frac{\pi}{2}$.

Confirmation is required that $f(a) = 0$.

$$\text{i.e.}\quad f(a) = f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$\therefore\quad \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\frac{\pi}{2}} = f'\left(\frac{\pi}{2}\right) \quad , \quad \text{for } f(x) = \cos x\ .$$

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f'\left(\frac{\pi}{2}\right) &= -\sin\frac{\pi}{2} \\ &= -1 \end{aligned}$$

$$\therefore\quad \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{x-\frac{\pi}{2}} = -1$$

6.6 Find $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$ by letting $x = 4\sin\theta$.

This type demonstrates that some integrals take a fair bit of processing. The thing is to be sure of your technique and check each line as you go. Also, as shown below, it is often advisable to do much of the simplifying outside the integral.

$$\begin{aligned} &\int \frac{x^2}{\sqrt{16-x^2}}\,dx \\ &= \int \frac{16\sin^2\theta}{4\cos\theta}\cdot 4\cos\theta\,d\theta \\ &= 16\int \sin^2\theta\,d\theta \\ &= 16\int \frac{1-\cos 2\theta}{2}\,d\theta \\ &= \frac{16}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c \\ &= 8\left[\theta - \frac{2\sin\theta\cos\theta}{2}\right] + c \\ &= 8\left[\theta - \sin\theta\cos\theta\right] + c \\ &= 8\left[\sin^{-1}\left(\frac{x}{4}\right) - \frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4}\right] + c \\ &= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x}{2}\sqrt{16-x^2} + c \end{aligned}$$

$$\begin{aligned} &\text{let } x = 4\sin\theta \\ &\frac{dx}{d\theta} = 4\cos\theta \\ &dx = 4\cos\theta\,d\theta \\ &\sin^2\theta = \frac{1-\cos 2\theta}{2} \end{aligned}$$

$$\begin{aligned} &\sqrt{16-x^2} \\ &= \sqrt{16-16\sin^2\theta} \\ &= \sqrt{16(1-\sin^2\theta)} \\ &= 4\sqrt{\cos^2\theta} \\ &= 4\cos\theta \end{aligned}$$

$$\begin{aligned} x &= 4\sin\theta \\ \frac{x}{4} &= \sin\theta\ ,\ \theta = \sin^{-1}\left(\frac{x}{4}\right) \\ \cos\theta &= \sqrt{1-\sin^2\theta} \\ &= \sqrt{1-\left(\tfrac{x}{4}\right)^2} \\ &= \sqrt{1-\tfrac{x^2}{16}} \\ &= \sqrt{\tfrac{16-x^2}{16}} \\ &= \frac{\sqrt{16-x^2}}{4} \end{aligned}$$

6.7 Show two different ways to find $\frac{d}{dx}(x^x)$

(i) let

$$\begin{aligned} y &= x^x \\ \ln y &= \ln x^x \\ \ln y &= x\ln x \\ \frac{d}{dx}(\ln y) &= \frac{d}{dx}(x\ln x) \\ \frac{1}{y}\frac{dy}{dx} &= 1\cdot\ln x + x\cdot\frac{1}{x} \\ \therefore\quad \frac{dy}{dx} &= y(\ln x + 1) \\ &= x^x(\ln x + 1) \\ &= (\ln x + 1)\,x^x \end{aligned}$$

(ii)

$$\begin{aligned} y &= x^x \quad \text{let } x = e^{\ln x} \\ y &= (e^{\ln x})^x \\ y &= e^{x\ln x} \\ \frac{dy}{dx} &= (1\cdot\ln x + x\cdot\frac{1}{x})\,e^{x\ln x} \\ &= (\ln x + 1)\,x^x \end{aligned}$$

as before.

6.8 Evaluate the following integral and then check your answer by differentiation.

$$\int e^{x^3}(3x^3 + 1)\ dx$$

The derivative of x^3 is $3x^2$ so try factorising out x from the bracket

$$\int e^{x^3} \cdot x\,(3x^2 + \frac{1}{x})\ dx \qquad \text{now let } x = e^{\ln x}$$

$$= \int e^{x^3} \cdot e^{\ln x}(3x^2 + \frac{1}{x})\ dx$$

$$= \int e^{x^3+\ln x}(3x^2 + \frac{1}{x})\ dx \qquad \text{is now in the required form}$$

$$= e^{x^3+\ln x} + c$$

$$= xe^{x^3} + c$$

check $\dfrac{d}{dx}(xe^{x^3} + c)$

$$= 1 \cdot e^{x^3} + x \cdot 3x^2\, e^{x^3}$$

$$= e^{x^3}(3x^3 + 1)$$

Note that if you can identify $e^{x^3} + 3x^3\, e^{x^3}$ as being the result of differentiating xe^{x^3}, then so much the better!!

6.9 If $F(x) = \displaystyle\int_1^x \frac{1}{t}\ dt$, use the substitution $u = t^{\frac{1}{n}}$ to prove that $F(x^n) = n\,F(x)$.

$$F(x^n) = \int_1^{x^n} \frac{1}{t}\ dt$$

$$= \int_1^{x} n\,\frac{du}{u}$$

$$= n\int_1^{x} \frac{du}{u}$$

$$= n\,F(x) \quad \text{as required}$$

let $u = t^{\frac{1}{n}}$

$$\frac{du}{dt} = \frac{1}{n}\, t^{\frac{1}{n}-1} \qquad t = 1\ ,\ u = 1$$

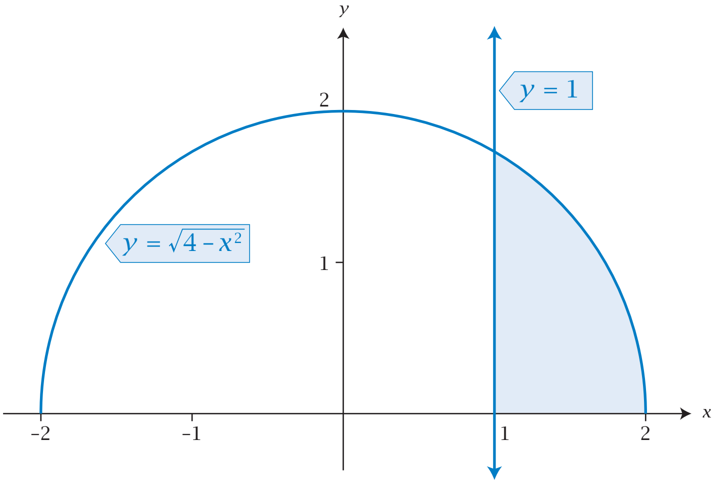
$$= \frac{1}{n} \cdot t^{\frac{1}{n}} \cdot \frac{1}{t} \qquad t = x^n\ ,\ u = (x^n)^{\frac{1}{n}} = x$$

$$\frac{du}{dt} = \frac{1}{n} \cdot u \cdot \frac{1}{t}$$

$$\therefore\ \frac{dt}{t} = n\,\frac{du}{u}$$

6.10 Volumes of Solids of Revolution

The line $y = 1$ cuts off a region from the semicircle $y = \sqrt{4 - x^2}$ which is rotated about the x axis. Determine the volume of the solid formed.



Volume

$$= \int_1^2 \pi y^2 \, dx \qquad\qquad y = \sqrt{4 - x^2}$$

$$y^2 = 4 - x^2$$

$$= \int_1^2 \pi (4 - x^2) \, dx$$

$$= \pi \left[4x - \frac{x^3}{3} \right]_1^2$$

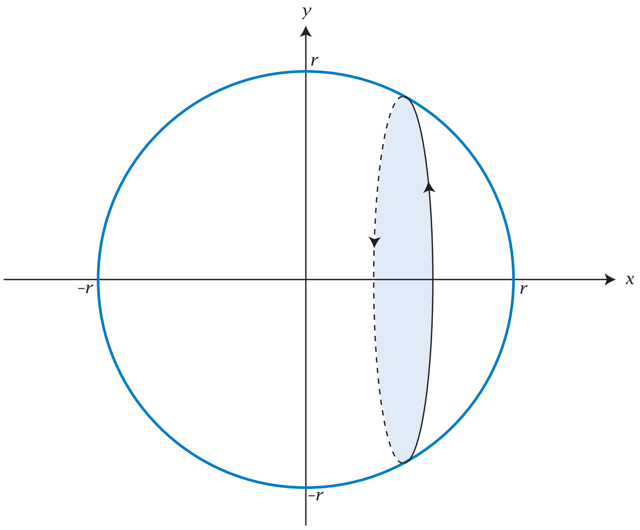
$$= \pi \left(\left[4(2) - \frac{(2)^3}{3} \right] - \left[4(1) - \frac{(1)^3}{3} \right] \right)$$

$$= \frac{5\pi}{3} \text{ cubic units}$$

6.11 Mensuration Formula

Volume of a Sphere

A Sphere may be generated by rotating the circle $x^2 + y^2 = r^2$ about the x axis.



$$v = \pi \int_{-r}^{r} y^2 \, dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2) \, dx$$

$$= 2\pi \int_0^r (r^2 - x^2) \, dx$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$= 2\pi \left(r^3 - \frac{r^3}{3} \right)$$

$$= 2\pi \left(\frac{2r^3}{3} \right)$$

$$= \frac{4\pi r^3}{3}$$

6.12 Find the total area enclosed between $y_1 = x^2 - 4$ and $y_2 = -x^2 - 2x$.

First determine any points of intersection to assist with a sketch of these functions and the area they enclose:

let $y_1 = y_2$, $x^2 - 4 = -x^2 - 2x$ (equating)

$2x^2 + 2x - 4 = 0$ (collecting like terms)

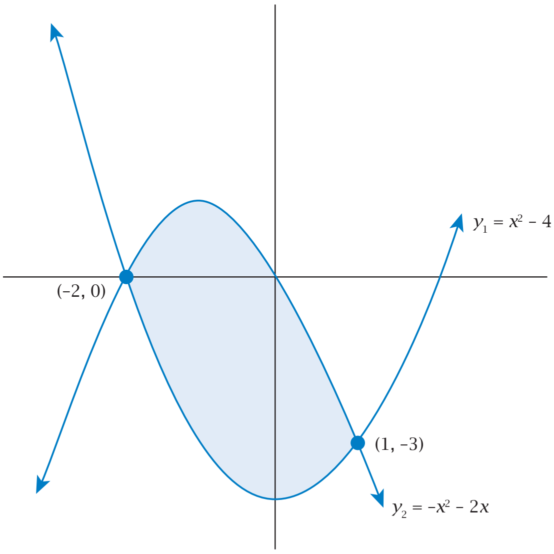
$x^2 + x - 2 = 0$ (÷ 2 to simplify)

$(x + 2)(x - 1) = 0$ (factorise)

$x = -2,\ x = 1$ (solve)

and points $(-2, 0)$ & $(1, -3)$ (re-substitution)

With our knowledge of the critical features of parabolas, and these points of intersection, we can produce a sketch.



Area enclosed $= \int_{-2}^{1}(-x^2 - 2x) - (x^2 - 4)dx$ (determine integral)

$= \int_{-2}^{1}(-2x^2 - 2x + 4)dx$ (simplify)

$= \left[\frac{-2x^3}{3} - x^2 + 4x\right]_{-2}^{1}$ (anti-differentiate)

$= 9$ units^2 (evaluate)

6.13 Find the derivative of the following functions:

(a) $y = \sin^{-1}(3x + 4)$

(b) $y = \cos^{-1}(e^{2x})$

(a) If $y = \sin^{-1}(3x + 4)$

Then $\sin y = 3x + 4$ Take the sine of both sides

$\frac{d}{dx}(\sin y) = \frac{d}{dx}(3x + 4)$ Differentiate both sides with respect to x

$\cos y \frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{3}{\cos y}$ Rearrange for a differential equation of form $\frac{dy}{dx} =$

$\frac{dy}{dx} = \frac{3}{\sqrt{1 - \sin^2 y}}$ Substitute $\sqrt{1 - \sin^2 y}$ for $\cos y$

$\frac{dy}{dx} = \frac{3}{\sqrt{1 - (3x + 4)^2}}$ Substitute $3x + 4$ for $\sin y$

$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{-9x^2 - 24x - 15}}$ Expand and simplify for final answer

(b) If $y = \cos^{-1}(e^{2x})$

Then $\cos y = e^{2x}$	Take the cosine of both sides
$\frac{d}{dx}(\cos y) = \frac{d}{dx}(e^{2x})$	Differentiate both sides with respect to x
$-\sin y \frac{dy}{dx} = 2e^{2x}$	
$\frac{dy}{dx} = \frac{-2e^{2x}}{\sin y}$	Rearrange for a differential equation of form $\frac{dy}{dx} =$
$\frac{dy}{dx} = \frac{-2e^{2x}}{\sqrt{1-\cos^2 y}}$	Substitute $\sqrt{1-\cos^2 y}$ for $\sin y$
$\frac{dy}{dx} = \frac{-2e^{2x}}{\sqrt{1-\left(e^{2x}\right)^2}}$	Substitute e^{2x} for $\cos y$
$\therefore \frac{dy}{dx} = \frac{-2e^{2x}}{\sqrt{1-e^{4x}}}$	Expand and simplify for final answer

6.14 Find $\int xe^x dx$

If $\int xe^x dx$ Let $u = x$ and $du = 1\ dx$

Let $dv = e^x dx$ and $v = e^x$

Following the rule $\int udv = uv - \int vdu$

We substitute to obtain

$$\int xe^x dx = xe^x - \int e^x (1) dx$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + c$$

6.15 Determine $\int x^2 e^{3x} dx$

If $\int x^2 e^{3x} dx$ Let $u = x^2$ and $du = 2xdx$

Let $dv = e^{3x} dx$ and $v = \frac{1}{3} e^{3x}$

Following the rule $\int udv = uv - \int vdu$

We substitute to obtain

$$\int x^2 e^{3x} dx = x^2\left(\tfrac{1}{3}e^{3x}\right) - \int \left(\tfrac{1}{3}e^{3x}\right)2xdx$$
$$= \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{3}\int xe^{3x} dx$$

Since the integral on the RHS contains 2 functions of x, we need to use the integration by parts process again.

So, if $\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int xe^{3x} dx$ Let $u = x$ and $du = 1\ dx$

Let $dv = e^{3x} dx$ and $v = \frac{1}{3} e^{3x}$

$$\int x^2 e^{3x} dx = \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{3}\left(uv - \int vdu\right)$$
$$= \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{3}\left(x\left(\tfrac{1}{3}e^{3x}\right) - \int \tfrac{1}{3}e^{3x}(1)dx\right)$$
$$= \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{3}\left(\tfrac{1}{3}\right)\left(xe^{3x} - \int e^{3x} dx\right)$$
$$= \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{3}e^{3x} + c$$

PROBLEMS TO SOLVE

CHAPTER 6: INTEGRATION AND APPLICATIONS OF INTEGRATION

1. Use the first principles definition of the derivative to find $f'(x)$ for $f(x) = \cos 3x$.

2. Show that $\int \cos^2 ax\, dx = \frac{\sin 2ax}{4a} + \frac{x}{2} + c$.

3. Use a table of values to prove that

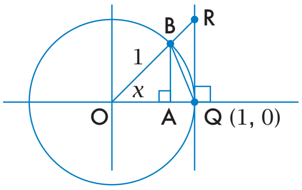
$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

4. Prove that (a) $\lim_{x \to 0} \frac{\tan x}{x} = 1$ and (b) $\lim_{x \to 0} \frac{x}{\tan x} = 1$.

5. As mentioned in the notes $\int \sin x \cos x\, dx$ gives either $\frac{\sin^2 x}{2} + c_1$ or $-\frac{\cos 2x}{4} + c_2$.

   Find the relationship between c_1 and c_2 .

6. Consider the diagram shown.



   Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

   by using the following inequality, Area ΔOQR > Area Sector OBQ > Area ΔOBQ .

7. Use an algebraic method to find the value of $\lim_{t \to -1} \frac{\sin (t+1)}{1 - t^2}$.

8. Show all working to find the exact value of $\int_3^6 \frac{x^2}{\sqrt{36 - x^2}}\, dx$ by letting $x = 6 \sin \theta$.

9. Use a substitution to find $\int \frac{\sin 2x}{(2 + \cos^2 x)^3}\, dx$.

10. If $y = \tan^3(2\, f(x))$, find $\frac{dy}{dx}$.

11. Show evidence to find the value of

$$\lim_{x \to 0} \frac{1 + x - \cos x}{\sin x}$$

12. Find $f'(x)$ if $f(x) = 5 \cos^3(4x^2 + 3)$

13. Choose a suitable method to find the exact value of:

(a) $\lim\limits_{x\to 0} \dfrac{\tan x}{\sin 2x}$ (b) $\lim\limits_{x\to 0} \dfrac{x\sqrt{12}}{\sin 3x}$.

14. Find the exact value of $\lim\limits_{\theta\to 0} \dfrac{\sin^2\theta}{\theta \cdot \tan\theta}$.

15. If x is in radians, then we know $\lim\limits_{x^R\to 0} \dfrac{\sin x^R}{x^R} = 1$.

Now let x^R and θ° be measures of the same sized angle. Use the fact that $\sin x^R = \sin\theta^\circ$ and a suitable substitution to prove that

$$\lim_{\theta^\circ\to 0} \frac{\sin\theta^\circ}{\theta^\circ} = \frac{\pi}{180^\circ} .$$

16. Prove that:

(a) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 7x} = \dfrac{2}{7}$

(b) $\lim\limits_{x\to 0} \dfrac{1 - \sin^2 x\cos x - \cos^2 x}{x^2\sin x} = 0$.

17. Find $\lim\limits_{x\to\infty} x\sin\left(\frac{1}{x}\right)$ by using the substitution $h = \dfrac{1}{x}$.

18. Find the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x}$

(b) $\lim\limits_{x\to 0} \dfrac{\tan^2 x}{1 - \cos x}$.

19. If $\dfrac{d^2y}{dx^2} = \cos 2x - \sin x$ and when $x = \dfrac{\pi}{2}$, $y = \dfrac{5}{4}$ and $\dfrac{dy}{dx} = 2$, find y in terms of x.

20. Evaluate:

(a) $\lim\limits_{x\to 0} \dfrac{3\sin 2x}{x^2 - x}$

(b) $\lim\limits_{\theta\to\pi} \dfrac{\sin\theta}{1 + \cos\theta}$.

21. Use a suitable method to find $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$.

22. Find $\dfrac{dy}{dx}$ for:

(a) $y = 4\cos^3\sqrt{x}$

(b) $\dfrac{d^2y}{dx^2} = 6x^3 + \dfrac{7}{\cos^2 x}$.

23. Show all working to find:

(a) $\displaystyle\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} |\sin t|\, dt$

(b) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4 - x^2}}\, dx$ (let $x = 2\sin\theta$)

(c) $\displaystyle\int \sqrt{9 - x^2}\, dx$ (let $x = 3\cos\theta$) .

24. In an electrical circuit, the voltage $e(t)$ and the current $i(t)$ at time t are given by the formulae: $e(t) = 160 \sin t$ and

$$i(t) = 2 \sin\left(t - \frac{\pi}{6}\right).$$

The average power is defined to be $\frac{1}{T}\int_0^T e(t) \times i(t)\, dt$ where T is the period of both the voltage and the current.

Determine T and calculate the exact average power.
(Hint, $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$)

25. Find $\frac{dy}{dx}$ for:

(a) $y = (\sin x)\, e^{\cos x}$

(b) $y = \tan^4(2x)$

(c) $y = \log_5 e^{\sin 4x}$.

26. Find the exact values of:

(a) $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{1 + 4x^2}\, dx$ (let $x = \frac{1}{2}\tan\theta$)

(b) $\int_0^{\frac{\pi}{3}} \cos^2\theta \sin^3\theta\, d\theta$ by letting $u = \cos\theta$.

27. If $F(x) = \int_1^x \frac{1}{t}\, dt$, show that $F\left(\frac{x}{b}\right) = F(x) - F(b)$.

Do this by splitting the integral limits $1 \le t \le \frac{x}{b}$ into $1 \le t \le x$ and $x \le t \le \frac{x}{b}$, then use the substitution $u = \frac{x}{t}$ in the second integral.

28. Find $\frac{d}{dx}(x \ln x)$ and use your result to find $\int \ln x\, dx$.

29. Find $\frac{d}{dx}\left(\frac{\ln x}{x}\right)$ and use your result to find $\int \frac{\ln x}{x^2}\, dx$.

30. By changing the base find:

(a) $\frac{d}{dx}(7^x)$

(b) $\int 7^x\, dx$.

31. Find:

(a) $\int \frac{3x + 1}{6x^2 + 4x - 7}\, dx$

(b) $\int e^{x^2} e^x (x + 0.5)\, dx$

(c) $\int (2 - \sin^2 x) \tan x\, dx$

(d) $\int \frac{\sin(\ln x)}{x}\, dx$

(e) $\dfrac{d}{dx}(e^{\cos x})$

(f) $\dfrac{d}{dx}(x^{\cos x})$.

32. Find $\displaystyle\int \frac{6e^{\frac{1}{x}}}{x^2}\,dx$ by letting $u = \dfrac{1}{x}$.

33. Find $\displaystyle\int \left(e^{\sin x}\right)^{\frac{1}{\cos x}} \cos^{-2} x\,dx$.

34. Find: (a) $\dfrac{d}{dt}\left(e^{3t}\ln e^{-2t}\right)$

(b) $f'(3)$ exactly if $f(x) = \log_3 x^2$.

35. Find $\displaystyle\int \frac{(1+\ln x)^2}{x}\,dx$ by letting $u = \ln x$.

36. (a) Find $\displaystyle\int \frac{e^{-x}}{1-e^{-x}}\,dx$ by letting $u = 1 - e^{-x}$.

(b) Find $\displaystyle\int \frac{dx}{1+e^{-x}}$ by letting $u = 1 + e^{x}$.

37. Determine $\displaystyle\frac{d}{dx}\int_3^{e^{2x}} \ln\sqrt{t}\,dt$

38. Evaluate:

(a) $\displaystyle\int \frac{dx}{x(\ln x)^3}$ let $u = \ln x$

(b) $\displaystyle\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$ let $x = \sin\theta$

39. (a) Write $\dfrac{x^2}{x-2}$ in the form $ax + b + \dfrac{c}{x-2}$.

(b) Now find $\displaystyle\int \frac{x^2}{x-2}\,dx$ by using your answer from (a).

(c) Find $\displaystyle\int \frac{x^2}{x-2}\,dx$ by letting $u = x - 2$ and explain why your answer is slightly different looking compared to that of your answer of (b).

40. (a) Find the exact area enclosed between the lines $y = 0$, $x = 2$, $y = 12 - 2x$ and the curve $4y = x^2$ by integrating with respect to x.

(b) Find the same enclosed area as in part (a) but this time by integrating with respect to y. Note that the area between $y = f(x)$ and the y axis between y_1 and y_2 is $\displaystyle\int_{y_1}^{y_2} x\,dy = \int_{y_1}^{y_2} f^{-1}(y)\,dy.$

41. (a) Simplify $1 - \dfrac{e^x}{1+e^x}$ by expressing it as a single fraction.

(b) Hence or otherwise find $\displaystyle\int \frac{1}{1+e^x}\,dx.$

42. Find:

(a) $\int \frac{5 - 2x \sin 3x}{\mathrm{x}} \, dx$

(b) $\int \tan x \sin x \cos x \, dx$.

43. The function f is defined by $f(\theta) = \cos^6\theta \sin^3\theta$.

(a) Explain why $f(\theta) \geq 0$ over the interval $0 \leq \theta \leq \pi$.

(b) Find exactly the area bounded by the curve $y = \cos^6 x \sin^3 \theta$ and the θ axis between $\theta = 0$ and $\theta = \pi$. (Hint: Use the substitution $u = \cos \theta$)

44. Find $\frac{d}{dx}(x^2 \mathrm{e}^x)$ and use your answer to evaluate $\int 4x \, \mathrm{e}^x (x + 2) \, dx$.

45. Find the following integrals:

(a) $\int \mathrm{e}^{x^2 + \ln x} \, dx$

(b) $\int \cos^3 2x \tan 2x \, dx$.

46. Find the exact value of $\int_2^{\mathrm{e}} \frac{1}{x \ln\sqrt{x}} \, dx$ by using the obvious substitution. If it's not obvious get the substitution from the answers and then continue!

47. Find the limit below by expressing it as a derivative.

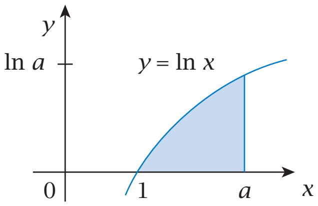
$$\lim_{h \to 0} \frac{\ln(\frac{\pi}{3} + h) - \ln(\frac{\pi}{3})}{h}$$

48. Simplify $\mathrm{e}^{\ln(2x - 3)}$ for $2x - 3 > 0$.

49. The shaded area in the diagram can be represented by:

$$\int_1^a \ln x \, dx = [F(x)]_1^a$$

$$= \mathrm{F}(a) - \mathrm{F}(1)$$



By calculating the area between the curve and the y axis, or otherwise, find an expression for this area.

50. Find the volume of revolution of the solid formed when the area below the curve $y = e^x$ between $x = 0$ and $x = 3$ is rotated about the line $y = -1$, one revolution.

51. The area in the first quadrant enclosed by the graphs of $y = 0$, $x = 0$, $x + y = 2$ and $y = e^{x-1}$ is rotated about the x axis. Determine the volume of revolution.

52. The region bounded by the lines $x = k$ and $x = 1$ and the curve $y^2 = 4x^3 - 4x$ is rotated about the x axis 180º. The volume formed is 9π. Determine the value of k where k is a positive integer.

53. The area enclosed by the line $y = mx$ and the parabola $y = x^2$ is 24.813.
Find the value of m (m is positive).

54. Verify that each of the following is a probability density function.

(a) $f(x) = \dfrac{1}{2\sqrt{x}},\ (0 \le x \le 1)$

(b) $f(x) = 2x,\ (0 \le x \le 1)$

(c) $f(x) = \dfrac{1}{x \ln 2},\ (1 \le x \le 2)$

55. Find the exact value of the following over the domain $0 \le x \le 2\pi$:

(a) $\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

(b) $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

(c) $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

(d) $\sin^{-1}(-1)$

(e) $\tan^{-1}(1)$

(f) $\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$

(g) $\sin^{-1}\left(\sin\left(\dfrac{\pi}{4}\right)\right)$

(h) $\cos^{-1}\left(\cos\left(\dfrac{\pi}{2}\right)\right)$

(i) $\tan^{-1}\left(\tan\left(\dfrac{3\pi}{2}\right)\right)$

(j) $\cos^{-1}\left(\sin\left(\dfrac{\pi}{6}\right)\right)$

(k) $\sin(\cos^{-1}(1))$

(l) $\cos^{-1}(\tan^{-1}(0))$

56. Following the steps (on p.142) in determining that $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$, prove that $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$.

57. Find the derivative of the following functions:

(a) $y = \sin^{-1}(2x - 1)$

(b) $y = \cos^{-1}(x^2)$

(c) $y = (\tan^{-1}(x))^2$

(d) $y = \cos^{-1}(e^{3x})$

(e) $f(x) = 2x^2 \times \sin^{-1}(3x)$

(f) $f(x) = \ln 2x \times \arccos x$

(g) $g(x) = \arcsin(\tan x)$

(h) $g(x) = \dfrac{1}{\sin^{-1} x}$

58. Determine the gradient of the function $y = \arctan(\ln x)$ at $x = e$.

59. Find the derivative of $\sin^{-1}(x + y) = x^2 - 1$.

60. What is the equation of the tangent to the curve $y = x \times \arccos(x)$ at the point $\left(\dfrac{1}{\sqrt{2}}, \dfrac{\sqrt{2}}{8}\right)$?

61. Use the integration by parts technique to evaluate the following integrals:

(a) $\int x \sin(5x)\, dx$

(b) $\int x\sqrt{x+1}\, dx$

(c) $\int x \cos x dx$

(d) $\int \ln x\, dx$ Yes, there are two functions of x here ☺

(e) $\int x^2 e^{-1} dx$

(f) $\int \sin^2 x dx$

(g) $\int x^2 \cos(3x)\, dx$

(h) $\int (\ln x)^2\, dx$

62. Evaluate the following using integration by parts: $\int x \tan^{-1}(x)\, dx$

63. Determine the following integrals:

(a) $\int \dfrac{1}{\sqrt{64 - x^2}}\, dx$

(b) $\int -\dfrac{1}{\sqrt{9 - x^2}}\, dx$

(c) $\int \dfrac{1}{\sqrt{4 + x^2}}\, dx$

(d) $\int \dfrac{3}{\sqrt{16 - x^2}}\, dx$

(e) $\int \dfrac{2}{\sqrt{25 + 16x^2}}\, dx$

(f) $\int -\dfrac{5}{\sqrt{4 - x^2}}\, dx$

(g) $\int \dfrac{dx}{5 + 9x^2}$

(h) $\int -\dfrac{3}{\sqrt{1 - 9x^2}}\, dx$

64. Verify each of the following rules by differentiating:

(a) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

(b) $\int -\frac{1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$

(c) $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

65. (a) $\int \frac{x+6}{(x+1)(x-4)}\,dx$

(b) $\int \frac{x+4}{(x-1)(x+7)}\,dx$

(c) $\int \frac{x+3}{(x-1)^2}\,dx$

(d) $\int \frac{3x-1}{(x+2)^2}\,dx$

(e) $\int \frac{4x-5}{x(x-3)^2}\,dx$

(f) $\int \frac{x^2 - x + 6}{x^3 + 3x}\,dx$

66. Find the areas of the regions enclosed between the following curves (produce a sketch first, showing intersection points):

(a) $y_1 = x^2$ and $y_2 = -x^2 + 4x$

(b) $y_1 = \sqrt{x}$, and $y_2 = \frac{1}{x^2}$, and $x = 4$

(c) $y_1 = 2\sin x$ and $y_2 = \sin 2x$ across $0 \le x \le \pi$

STATISTICAL INFERENCE

Syllabus Checklist

By the end of this chapter, you should be able to:

Sample means

☐ examine the concept of the sample mean $\overline{X}$ as a random variable whose value varies between samples where X is a random variable with mean μ and the standard deviation σ

☐ simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate properties of the distribution of $\overline{X}$ across samples of a fixed size n, including its mean μ its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where μ and σ are the mean and standard deviation of X), and its approximate normality if n is large

☐ simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate the approximate standard normality of $\frac{\overline{X}-\mu}{s/\sqrt{n}}$ for large samples ($n \geq 30$), where s is the sample standard deviation

Confidence intervals for means

☐ examine the concept of an interval estimate for a parameter associated with a random variable

☐ examine the approximate confidence interval $\left(\overline{X} - \frac{zs}{\sqrt{n}}, \overline{X} + \frac{zs}{\sqrt{n}}\right)$ as an interval estimate for the population mean μ, where z is the appropriate quantile for the standard normal distribution

☐ use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain μ

☐ use $\bar{x}$ and s to estimate μ and σ to obtain approximate intervals covering desired proportions of values of a normal random variable, and compare with an approximate confidence interval for μ

FORMULAE AND DEFINITIONS

Population

A population in statistics represents **all** measurements or objects being studied.

Sample

A sample is a subset or portion of the population.

All items produced by a production process is the **population** while a quality control selection is the **sample**.

- The sample mean $\overline{X}$ is used to estimate the population mean μ
- The sample standard deviation s is used to estimate the population standard deviation σ

MEAN AND STANDARD DEVIATION OF THE SAMPLE MEAN

X is a random variable and thus has a probability distribution known as a sampling distribution.

Let $\bar{X}$ be the mean of a sample of size n from a population having a mean μ and standard deviation σ. The mean and standard deviation of X are

$\mu_{\bar{x}} = \mu$ ← *population mean*

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ ← *population standard deviation*

SAMPLING DISTRIBUTION OF $\bar{X}$

If a population has a distribution with parameters $N(\mu, \sigma)$ then the sample mean $\bar{X}$ of size n has a distribution with parameters $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

CENTRAL LIMIT THEOREM

The Central Limit Theorem states that given a distribution with mean μ and standard deviation σ the sampling distribution of the mean $\bar{X}$ approaches a normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ when n is large.

i.e. $\bar{X}$ is approximately $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

i.e. $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

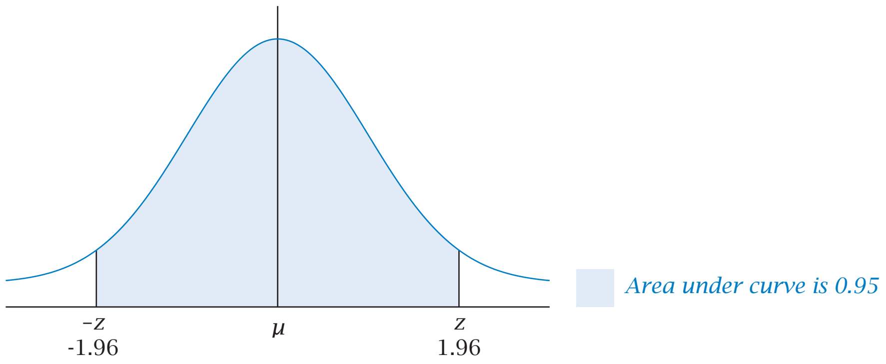
CONFIDENCE INTERVALS

A **confidence interval** determined from data obtained from a sample estimates an unknown parameter. The confidence interval also indicates how accurate the estimate is. A confidence interval consists of two parts – an interval calculated from the sample data and a confidence level.

A **confidence level** determines the probability that the interval will contain the unknown parameter. The three most common confidence intervals are 90%, 95% and 99%. If a 95% confidence interval is used then there is a 95% confidence that the population mean is contained within that interval when the values are normally distributed in the population.

The value of z for the confidence interval of 95% is 1.96. This is obtained by using the standard normal distribution as shown in the diagram below:

95% Confidence Interval



Using the central limit theorem approximately 95% of the sample means will be within ±1.96 standard errors of the population mean when n is large.

i.e. $\bar{X} \pm 1.96\left(\frac{\sigma}{n}\right)$

Common z scores and associated confidence intervals are

Confidence Interval	90%	95%	99%
z score	1.645	1.960	2.576

CONFIDENCE INTERVAL FOR A POPULATION MEAN

For a sample of size n from a population having an unknown μ and a known σ the confidence interval is: $\overline{X} - z\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + z\frac{\sigma}{\sqrt{n}}$

Where z is the standard score for a 90%, 95% or 99% confidence interval.

CALCULATING SAMPLE SIZE

To calculate the size of the sample use the following formula:

$$n = \left(\frac{z\sigma}{w}\right)^2$$

n = sample size

z = standard score for a 90%, 95% or 99% confidence interval

σ = population standard deviation

w = width of confidence interval

Note: Always round n up to the next whole number.

Worked Examples

7.1 (a) A company wishes to estimate the average age of its employees. From past information it was found the standard deviation was 2.3 years. A sample of 40 employees is selected and the mean is calculated as 27.4 years. Find the 95% confidence interval of the population mean.

(b) How large a sample should the company use to be 95% sure that the sample is within 1 year of the sample mean of 27.4 years.

(a) The z score for a 95% confidence interval is 1.96.

Using the formula

$$\text{mean of sample (27.4)} \rightarrow \overline{X} - \overset{\text{z score (1.96)}}{z}\frac{\overset{\text{standard deviation of population (2.3)}}{\sigma}}{\underset{\text{number in sample (40)}}{\sqrt{n}}} \leq \mu \leq \overline{X} + z\frac{\sigma}{\sqrt{n}}$$

$$27.4 - 1.96\left(\frac{2.3}{\sqrt{40}}\right) \leq \mu \leq 27.4 + 1.96\left(\frac{2.3}{\sqrt{40}}\right)$$

$$26.687 \leq \mu \leq 28.113$$

ie. With a 95% confidence, the average age of employees lies between 26.607 and 28.113 years.

(b) To find the sample size use the formula

$$n = \left(\frac{z\sigma}{w}\right)^2$$

n = ? sample size

z = 1.96 standard score for a 95% confidence interval

σ = 2.3 population standard deviation

w = 1 width of confidence interval

$$n = \left(\frac{1.96 \times 2.3}{1}\right)^2$$

$n \approx 20.322$

The required sample size is 21 employees.

7.2 Use the information supplied in Example 13.1.

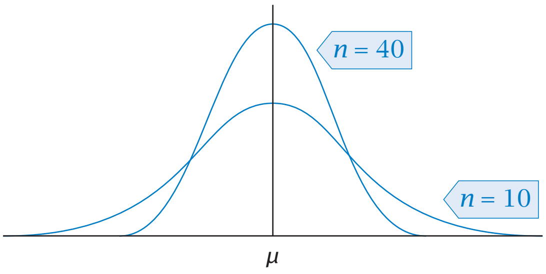
If the sample was reduced to 10 employees will the 95% confidence interval of the population mean be wider or narrower?

Interval:

$$27.4 - 1.96\left(\frac{2.3}{\sqrt{10}}\right) \leq \mu \leq 27.4 + 1.96\left(\frac{2.3}{\sqrt{10}}\right)$$

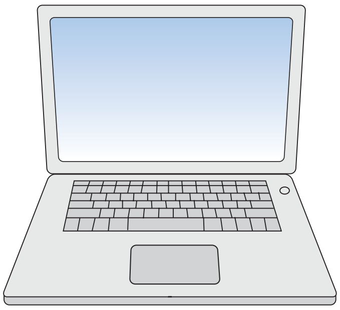
$$25.974 \leq \mu \leq 28.826$$

The 95% confidence interval is **wider**. This can be seen in the diagram below:



7.3 A survey of 50 adults finds the average age of a person's laptop computer to be 4.5 years. Assuming the distribution is normal with a population standard deviation of 1.2 years, find the 99% confidence interval of the population mean.

$\bar{X}$ = 4.5 years	mean
σ = 1.2 years	population standard deviation
z = 2.576	standard score for a 99% confidence interval
n = 50	sample size



Using confidence interval formula

$$\bar{X} - z\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z\frac{\sigma}{\sqrt{n}}$$

$$4.5 - 2.576\left(\frac{1.2}{\sqrt{50}}\right) \leq \mu \leq 4.5 + 2.576\left(\frac{1.2}{\sqrt{50}}\right)$$

$$4.063 \leq \mu \leq 4.937$$

7.4 Heights of women are normally distributed with a mean of 162cm and a standard deviation of 3.7cm.

What is the probability that a sample of 7 women have an average height of less than 160cm?

As the population is normally distributed the sample mean

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

$\mu_{\bar{X}} = 162 \longleftarrow$ *population mean*

$\sigma_{\bar{X}} = \dfrac{3.7}{\sqrt{7}}$ $\longleftarrow$ *population standard deviation* / $\longleftarrow$ *size of sample*

$$\therefore \bar{X} \sim N\left(162, \frac{3.7}{\sqrt{7}}\right)$$

Using calculator

$P(X < 160) \approx 0.0763$

PROBLEMS TO SOLVE

CHAPTER 7: STATISTICAL INFERENCE

1. A sample of 1500 people is selected from a large population with a mean of 200 and a standard deviation of 50. If the sample is approximately normal and the population mean $\mu = 200$, determine the sample standard deviation?

2. An Australia wide survey on 'Children's Television Viewing Habits' found that 62% of the 1000 randomly selected children watched TV between the hours of 4 and 5 p.m. on weekdays.

   (a) The company conducting the survey announced a margin of error of ±4% for 95% confidence in its conclusions. What is the 95% confidence interval that children watch TV between the hours of 4 and 5 p.m. on weekdays?

   (b) Why cannot we conclude that 62% of all children watch TV between the hours of 4 and 5 p.m. on weekdays?

   (c) What does '95% confidence' mean?

3. (a) A statistician working for a telemarketing company informs the bosses that data collected from the population had a standard deviation of 19.95. He assured them that with a confidence interval of 99% and to be within 5 units of the mean, the required sample would have to be 106. Was he correct?

   (b) If the sample size is reduced to 50, what happens to the width of the confidence interval? Justify your answer.

4. The confidence level is $\overline{X} \pm z\frac{\sigma}{\sqrt{n}}$

   The margin of error or width is: $z\frac{\sigma}{\sqrt{n}}$

   A small margin of error is usually preferred.

   A sample of 2000 male scores gave a $\overline{X} = 150$ and a $\sigma = 40$.

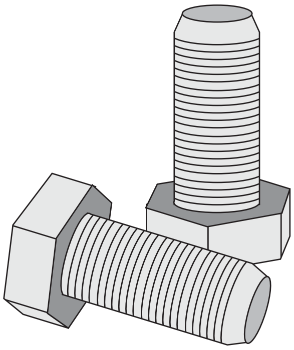
   (a) Give the 95% confidence interval for the population mean μ.

   (b) If the sample size is altered to 200 and all other statistics remain the same, determine the 95% confidence interval for the population mean μ.

   (c) If the sample size is increased to 3000, $\overline{X} = 150$ and $\sigma = 40$ determine the 95% confidence interval.

   (d) How does altering the sample size affect the margin of error?

5. A company is asked to make bolts with lengths accurate to within ± 0.002 with a 95% confidence level. If $\sigma = 0.004$, determine how many bolt lengths must be averaged to satisfy this request?



6. A random sample of 30 students from schools in the South West region of Perth gave the following percentage results for a Maths competition.

76	51	46	83	97
58	62	79	28	44
81	70	66	59	72
53	51	58	73	49
38	95	88	60	50
42	86	71	63	51

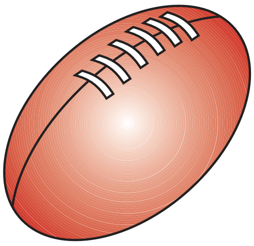
(a) Assuming the distribution is close to normal and the standard deviation of marks in this competition for the population is 12, give a 99% confidence interval for the mean score in the population.

(b) The 30 students results are from one school in the South West region. Is the 99% confidence interval above valid for the population? Explain.

7. If the 90% confidence interval for a sample of 10 observations based on a population standard deviation of 0.05 with a $\overline{X} = 0.927$ is 0.901 to 0.953. Describe the effect on the width of the confidence interval when the confidence level is increased?

8. A survey was conducted via the internet. It asked participants who logged on to a football website: 'What fine should a football player receive for major indiscretions?' The statistics from 1256 respondents were $\overline{X} = \$4212$ and a standard deviation $s = \$1004$.

As this is a large sample, s is close to the unknown population σ. The website supervisor calculated the 95% confidence interval to be \$4156.47 to \$4267.53.



(a) Is the confidence interval correct? Explain.

(b) Does this conclusion represent the view of the population? Explain.

9. IQ results from a population are normally distributed with $X \sim N(\mu,\ 9.5^2)$. A random sample of 25 gives a sample mean of $\overline{X} = 126.5$. Determine the 90% confidence interval for the population mean μ.

10. The average consumption of Coca-Cola (ml) per week was $\overline{X} = 1125$ ml. The survey was conducted for 1400 people. Assuming $\sigma = 50$ find an approximate 95% confidence interval for the population mean μ.

11. A survey was sent to 760 teachers in independent schools. The question asked 'Are teachers satisfied with curriculum support available for the new Year 11 and 12 courses?'

The survey responses required the use of a five point scale:

① meaning – highly satisfactory to ⑤ meaning – highly unsatisfactory.

Altogether 270 responses were received.

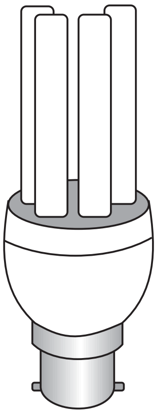
(a) What problems were associated with this survey and what was the intended population?

(b) If the 'satisfaction' data was $\overline{X} = 3.25,\ \sigma = 1.4$, give the 95% confidence interval for the mean of the population.

(c) If the 'curriculum support documents' data was $\overline{X} = 2.1,\ \sigma = 1.4$, give the 99% confidence interval for the mean of the population.

(d) Are the measurements normally distributed? Explain.

12. It is known that the life time of manufactured light bulbs are normally distributed with a mean life of 2500 hours and a standard deviation of 220 hours. If a bulb is selected at random, determine the probability that:



(a) the life of the bulb is at least 2620 hours?

(b) the life of the bulb is between 2400 and 2650 hours?

A quality control inspector makes a random selection of 81 bulbs.

(c) If the sample mean is 2480 hours, determine a 95% confidence interval for the mean life of the bulbs.

(d) Determine the probability the sample mean will lie between 2499.5 and 2500.3 hours.

(e) Determine the size of the sample if the inspector is to be 99% sure that the sample mean is within 100 hours of 2500 hours.

13. Dhufish are caught off the coast of North-West Australia. Their weights are normally distributed with a mean of 20 kg and a standard deviation of 2.7 kg.

A dhufish is randomly selected. Determine the probability that:

(a) the dhufish weighs less than 19.5 kg?

(b) the dhufish weighs more than 17 kg given it weighs less than 19.5 kg?

Fisherman Joe makes a random selection of 32 dhufish.

(c) If the sample mean weight is 19.2 kg, determine a 99% confidence interval for the mean weight of the dhufish.

(d) Determine the probability that the sample mean lies between 17.5 kg and 20.2 kg.

(e) What size sample is required to ensure that the sample mean is within 2 kg of 20 kg?

TRIAL TEST 1: COMPLEX NUMBERS I

Calculators NOT allowed

Time Allowed: 60 minutes

Total Marks: 60

1. If $z = -2 + 5i$ and $w = 6 - 3i$ find without your calculator

(a) zw

(b) $3w - 4z$

(c) $(\overline{w})^2$

(d) $|z|$

(e) $\frac{w}{z}$

[12]

2. Find the exact complex solutions of $z^2 + 10z + 41 = 0$.

[4]

3. A quadratic equation with real coefficients has one of its roots as $z = 7 - 2i$.
Find the equation.

[4]

4. Express $Z = -1 - \sqrt{3i}$ in polar form.

[3]

5. If $Z_1 = 5Cis\frac{\pi}{6}$ and $Z_2 = 2Cis\,\frac{\pi}{12}$, then prove $Z_1Z_2 = 5\sqrt{2}(1+i)$

[4]

6. If $Z = 1 + i$ then show $\left(\frac{Z}{\bar{Z}}\right)^2 = -1$

[3]

7. If $Z = \frac{1}{2 - 3i}$ then express Z in Cartesian form. Hence find $Z.\bar{Z}$.

[3]

8. Find the distance between the complex numbers $W = \sqrt{2} - i$ and $Z = 1 - i\sqrt{2}$.

[3]

9. Find Z if $Z\bar{Z} + 2Z = \frac{1}{4} + i$

[4]

10. (a) Change the complex equation $|Z - i| = |Z - 1|$ into its Cartesian equivalent.

(b) Hence identify the locus of all points Z satisfying the equation in (a).

[5]

11. Z is a complex number. Sketch the region given by

$\text{Re}(Z) < 1$ and $\text{Im}(Z) > -2$

and $1 < |Z| < 3$

and $-\frac{5\pi}{12} \leq ArgZ \leq \frac{2\pi}{3}$

[6]

12. Simplify $(2 - 2i\sqrt{3})^5$

[4]

13. (a) Find the cube roots of $-8i$.

(b) Plot them on an Argand Diagram.

[5]

TRIAL TEST 2:
COMPLEX NUMBERS II

Calculators allowed

Time Allowed: 60 minutes

Total Marks: 60

1. The locus of points z in the complex plane is determined by the constraint $z\overline{z} + z + \overline{z} = 24$. Demonstrate how to find the nature of this locus.

[5]

2. Show that $|a + z|^2 = a^2 + a(z + \overline{z}) + |z|^2$ for real a.

[4]

3. Separately sketch the following in the Argand plane.

(a) $|1 - z| < |z + \mathrm{i} - 2|$

(b) $\text{Arg}\,(z + 1) \geq \frac{\pi}{4}$

[5]

4. Use the ordinary rules of algebra to solve $\frac{z - i}{z + i} = 1 + i$ for z.
(i.e. don't substitute $z = x + yi$)

[3]

5. A straight line drawn on the polar axes makes an angle of 30° with the positive x axis and passes through point A (6, 160°). B is also on the line and is 15 cm away from A. Find the two possible locations of B.

[11]

6. If $z = \text{cis}\ \theta$, show that $\frac{z - z^{-1}}{i\,(z + z^{-1})} = \tan \theta$.

[4]

7. z and w are complex numbers such that $z = 1 - i$ and $w = 2\ \text{cis}\left(\frac{\pi}{6}\right)$.
Find the following:

(a) $|\,z\,|$

(b) w in exact Cartesian form

(c) $\overline{z}w$ in exact polar form

(d) $\frac{w}{z^2}$ in exact Cartesian form

(e) z^5 in exact polar form.

[10]

8. If $z = 4 \text{ cis } \theta$ and $z = A \text{ cis}\left(\frac{2\pi}{3}\right)$ are two of the four roots of z^4, find:

(a) exactly, all values of A and θ, where $-\pi \le \theta \le \pi$.

(b) z^4 in exact polar form

[5]

9. Use de Moivre's rule to find the exact value of $(1 + i)^5 - (1 - i)^5$.

[5]

10. (a) Show how to solve $w^3 = -2 + 2\text{i}$ for one solution of w in exact cartesian form.

(b) Use your answer above to solve the equation below for one solution of z without substitution of $z = x + y\text{i}$.

$$(z - 1)^3 = -2(z + 1)^3 + 2\text{i}(z + 1)^3$$

[8]

TRIAL TEST 3:
FUNCTIONS AND CURVE SKETCHING

Calculators allowed

Time Allowed:	67 minutes
Total Marks:	67

1. (a) If $g(x) = x^2 - x + 1$ and $f(g(x)) = 2x^2 - 2x + 1$ find $y = f(x)$.

(b) If $g(f(x)) = x^2 - 7x + 13$ and $g(x) = x^2 - x + 1$ find $y = f(x)$.

[8]

2. (a) Write $y = | 4x + 3 | - | 5 - x |$ in piecewise form.

[5]

(b) Solve $|4x - 3| = |5 - x| + 8$ by using your answer from (a).

[4]

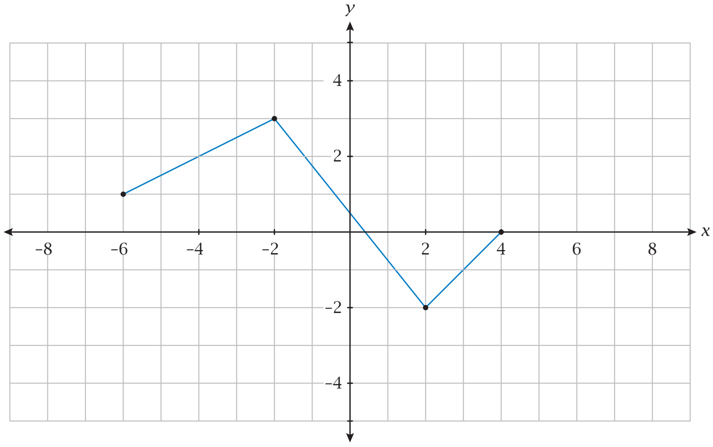
(c) Solve $|4x - 3| > |5 - x| + 8$ by using your answer from (b).

[2]

3. (a) The function $y = f(x)$ is first dilated in the x-direction by a scale factor of $\frac{1}{2}$ and then shifted 3 units left. Also in the y-direction the graph is translated down 2 units then reflected in the x-axis and then dilated by scale factor 2. Write the new function in terms of $y = f(x)$.

(b) Describe how the graph of $y = -2f(3x + 2) - 1$ can be found by transforming the graph of $y = f(x)$.

(c) On the same axes as $y = f(x)$ is drawn below, draw the graph of $y = -\frac{1}{3}f(2x + 1) - 1$. On your graph mark in the coordinates of the end points and corners.



[11]

4. State the domain and range of the following, using any notation:

(a) $f(x) = 1 - x$

(b) $h(x) = \frac{1}{x + 1}$

(c) $m(x) = \sqrt{x^2 - 9}$

[7]

5. Find the zeroes of the polynomial

$P(x) = x^4 + 6x^3 + 9x^2 - 4x - 12$

[6]

6. Locate all zeros, poles and turning points of the function

$$y = \frac{-3}{(x-2)(2x+1)}$$

Determine its behaviour near ±∞ and sketch its graph.

[7]

7. A function f is defined as follows.

$$f(x) = \begin{cases} 2 & x \leq 0 \\ \dfrac{x^2}{x-1} & 0 < x < 1 \\ 3 - x & x \geq 1 \end{cases}$$

find

(a) $\lim\limits_{x \to 0^-} f(x)$

(b) $\lim\limits_{x \to 0^+} f(x)$

(c) $\lim\limits_{x \to 0} f(x)$

(d) $\lim\limits_{x \to 5} f(x)$

(e) $\lim\limits_{x \to 1^+} f(x)$

(f) $\lim\limits_{x \to 1} f(x)$

[6]

8. Evaluate the following limits if they exist.

(a) $\lim\limits_{x \to 3} \dfrac{2x^2 - 5x - 3}{x^2 - 5x + 6}$

[2]

(b) $\lim\limits_{x \to \infty} \dfrac{3x^2 - 2}{(x + 1)^2}$

[2]

9. Consider

$$h(x) = \begin{cases} -(x + 2)^2 + 2 & -4 \le x < -1 \\ \dfrac{1}{x + 2} & -1 \le x < 0 \\ x - 1 & 0 \le x \le 2 \end{cases}$$

(a) Sketch $h(x)$

(b) State the reason for h having a discontinuity at $x = 0$.

(c) With reasoning, state why h is not differentiable at $x = -1$.

[8]

TRIAL TEST 4:
VECTORS IN 3-DIMENSIONS I

Time Allowed:	50 minutes
Total Marks:	50

1. If $\mathbf{u} = \langle -2, 3, 1\rangle$ and $\mathbf{v} = \langle 3, 1, -5\rangle$ find:

(a) $4\mathbf{u} - 3\mathbf{v}$

(b) the size of the angle between $\mathbf{u}$ and $\mathbf{v}$

(c) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

(d) the acute angle between $\mathbf{v}$ and the x axis

(e) the acute angle between $\mathbf{u}$ and the $x - y$ plane.

[10]

2. Find the value(s) of a if $\langle 6, a, -4\rangle$ and $\langle -1, 2a, a\rangle$ are perpendicular.

[3]

3. (a) Find the equation of the plane through A $(2, -3, 4)$ which has a normal vector of $\langle -1, 5, 3\rangle$.

(b) Find the equation of the line through $\langle 16, -17, -8\rangle$ and parallel to $\langle -2, 3, 1\rangle$.

(c) Hence, find where the above line and plane intersect.

[6]

4. A and B have position vectors of ⟨2, 3, –1⟩ and ⟨–1, 15, 5⟩ respectively. Find point C such that AB : AC = 3 : 5.

[6]

5. Find the equation of the plane which passes through A (3, 2, 6) , B (1, –3, 10) and C (10, 0, 5). Do this by first finding **AB** and **AC** , then use these to find $\mathbf{n} = \langle 1, p, q \rangle$, the normal direction of the plane.

[7]

6. A jet travelling at $\langle -200, 150, 0.5\rangle$ km/hr passes through point $(50, -20, 3.8)$ km at 2 p.m. one day. At the same time a light plane is at $(-238, 460, 4.52)$ km travelling at $\langle -80, -50, 0.2\rangle$ km/hr.

(a) If the aircraft continue as above, prove that the planes will collide and find the time and place of the collision.

(b) After two hours of travelling the air traffic controller predicts that there could be a close encounter and orders the jet to change its velocity to $\langle -220, 130, 0.8\rangle$ km/hr. Find the closest distance that the two aircraft come to each other after this change.

[18]

TRIAL TEST 5:
VECTOR CALCULUS

Time Allowed: 50 minutes
Total Marks: 50

1. $\underset{\sim}{r}(t) = (5t\underset{\sim}{i} - 8t^2\underset{\sim}{j})$ metres represents the position of a particle at time t in seconds.

(a) Find expressions for the velocity and acceleration.

(b) Find the speed when $t = 2.5$ seconds.

(c) What curve decribes the motion?

[7]

2. Evaluate $\lim\limits_{t \to 0}\left[\dfrac{2\sin t}{t}\underset{\sim}{i} - \sqrt{4-t}\,\underset{\sim}{j}\right]$

[2]

3. If $\underset{\sim}{r}(t) = \sin 2t\,\underset{\sim}{i} + 2\sin^2 t\,\underset{\sim}{j}$ then show that the distance from the origin is related to the speed by the formula $|\sin t||\underset{\sim}{v}(t)|=|\mathrm{r}(\mathrm{t})|$

[7]

4. If $\underset{\sim}{r}(t) = te^t\,\underset{\sim}{i} + t\ln t\,\underset{\sim}{j}$ find $\underset{\sim}{r}'(t)$.

[3]

5. If $\underset{\sim}{r}'(t) = 2e^{2t}\,\underset{\sim}{i} - \sin t\,\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = 3\underset{\sim}{i}$, find $\underset{\sim}{r}(t)$.

[4]

6. A particle is moving along a curve such that $\frac{dx}{dt} = \sin t$ and $\frac{dy}{dt} = \frac{1}{t+1}$.
When $t = 0$ the position vector of the particle is $2\underset{\sim}{i}$.
Find the position vector at time t.

[5]

7. If $\underset{\sim}{r}(t) = 2\cos 3t\,\underset{\sim}{i} + 5\sin 3t\,\underset{\sim}{j}$

(a) use $\sin^2 3t + \cos^2 3t = 1$ to show that the cartesian equation of the path is

$$\frac{y^2}{25} + \frac{x^2}{4} = 1$$

(b) what path is traced out?

(c) find an expression for the velocity vector $\underset{\sim}{v}(t)$

(d) show that the maximum speed is 15 units/s when $\cos^2 3t = 1$

[9]

8. A projectile is shot from a gun at an angle of elevation of 30° with muzzle speed 130 m/s. The acceleration due to gravity is $-9.8\underset{\sim}{j}$ m/s^2. Find:

(a) the position vector of the projectile at any time

(b) the maximum height

(c) the total flight time

(d) the speed of the projectile at impact.

[13]

TRIAL TEST 6:
RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

Time Allowed: 75 minutes
Total Marks: 75

1. Show how to find the derivative of $\cos 2x$ from first principles if it is known that $\lim_{h\to 0} \frac{\cos 2h - 1}{h} = 0$.

[5]

2. Find $\frac{dy}{dx}$ for each function below.

(a) $y = \sin 2x \cos 3x$

(b) $y = \cos(\sin x)$

(c) $y = \int_{\frac{\pi}{4}}^{\cos x} \sin^2 t \, dt$

(d) $y = \sqrt{\cos(\sin^2 x)}$

[8]

3. Find $\frac{dy}{dx}$ for each of the following:

(a) $y = 3(5)^x$

(b) $\frac{d^2y}{dx^2} = \frac{e^x}{1 - e^x}$

(c) $\int_2^{3x} t^{-1}\, dt$

(d) $y = \frac{\log x}{x}$.

[9]

4. (a) If $f(x) = \int_0^x \frac{1 - t}{1 + t}\, dt$, find the following:

i. x where $f'(x) = 0$

ii. $\lim_{x \to \infty} f'(x)$

iii. $f''(x)$

iv. $f(0)$.

(b) Explain the physical feature of each of the parts i. to iv. above in regard to the sketch of $y = f(x)$. Hence sketch the graph.

(c) Find the exact coordinates of the stationary point of $y = f(x)$ by first expressing $\frac{1 - t}{1 + t}$ in the form $\frac{a}{t + 1} + b$.

[17]

5. Find the derivative of $y = x - x^3$ from first principles.

[4]

6. Find the exact value of m and the contact point(s) of where $y = mx$ is tangent to $f(x) = \frac{1}{x} + x - 1$.

[6]

7. The velocity v m/s of an object depends on the time t seconds according to

$$v = \frac{10}{\sqrt{t+1}} + \frac{t}{2}$$

The object's position when $t = 3$ is $x = 5$ m. Find, to the nearest second, how long it takes to reach $x = 100$ m.

[5]

8. Find the exact value of $\int_3^6 \frac{x^2}{\sqrt{36 - x^2}}\,dx$.

[9]

9. A police car has a rotating light mounted on top of its roof. The light completes one revolution every 2 seconds. The light is 24 m from the nearest point P on a straight wall, and a point Q further along the wall (at the same height as P) is exactly 26 m from the light source. Find how fast the spot of light is moving along the wall when it passes the point Q. Give your answer in m/s correct to 1 d.p.

[6]

10. (a) If v is velocity, x is position and t is time show that

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dt}$$

(b) An object is moving along the x axis. Its velocity v is given by $v^2 = 10x - x^2$. Use the above result to find the acceleration of the object when $x = 4$.

[6]

TRIAL TEST 7:
INTEGRATION AND APPLICATIONS OF INTEGRATION

Calculators allowed

Time Allowed:	70 minutes
Total Marks:	70

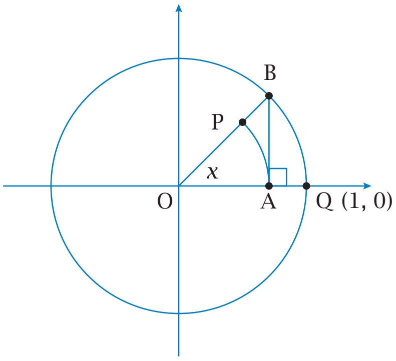
1. Consider the unit circle shown.

   Use the fact that

   length of arc AP < length of line AB < length of arc QB

   to establish that
   $x \cos x < \sin x < x$

   and hence that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.



[5]

2. Find the following

   (a) $\int \sin^3 2t \, dt$

(b) $\int \frac{\sin^3 \theta}{\cos^4 \theta} \, d\theta$, let $u = \cos \theta$

[7]

3. Find the following integrals

(a) $\int \frac{8x + 2}{(2x - 3)(x + 2)} \, dx$

(b) $\int (e^x + x^e) \, dx$

(c) $\int \frac{(x + 2) \, e^{x^2}}{e^{5-4x}} \, dx$

(d) $\int \ln x \, dx$ after finding $\frac{d}{dx} (x \ln x)$

[10]

4. Find the exact area enclosed by $y = e^{-x}$, $y = \frac{x}{e}$ and $x = 0$ by

(a) integrating with respect to x .

(b) integrating with respect to y if $\int \ln y \, dy = y \ln y - y + c$.

[10]

5. Kate dropped a cork into the sea while out boating one day. The waves of a passing ship caused the cork to move in simple harmonic motion.

The distance between the bottom of the sea and the cork varied between 1.8 metres and 2.2 metres and the period of motion was 3 seconds.

Assume that, at the time $t = 0$ seconds, the cork was 2 metres from the bottom of the sea and is moving in a downward direction.

(a) Draw a sketch to represent the position of the cork with respect to the bottom of the sea, labelling clearly the horizontal and vertical axes.

(b) Obtain an expression for the position of the cork, x at any time, t seconds.

(c) Prove that the motion of the cork is simple harmonic in nature.

(d) Calculate the maximum velocity of the cork and find the first time at which this occurred.

(e) Kate was able to retrieve the cork from the sea after 11 seconds. Find the total distance the cork moved while it was in the sea.

[14]

6. An athlete with a body temperature of 37˚C stepped into an ice bath at 1˚C as part of his recovery program. In 5 minutes the athletes temperature was 36˚ and he knew that if his temperature dropped below 35˚ there could be serious consequences. If Newtons law of cooling applies (which states that the rate of cooling is proportional to the difference between his body temperature and the water temperature), find the maximum time the athlete can stay in the icy water.

[6]

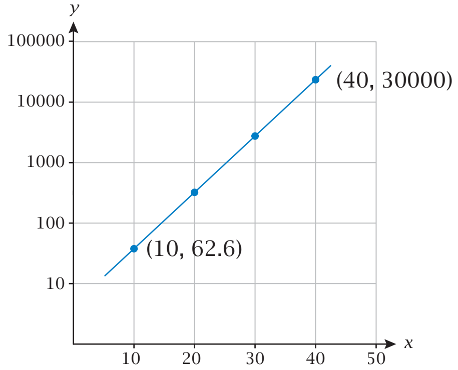
7. Find $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$ by first letting $u = e^{2x}$ and then $u = \sin\theta$.

[7]

8. A population P_0 grows at 5% per year for 10 years and then declines at an instantaneous rate of 2% per year. Find, as an exact value, how long it takes to reach its original population of P_0.

[5]

9. Data gathered by a scientist is graphed on the logarithmic scale as shown. Find the rule in the form $y = A(10^{mx})$.



[7]

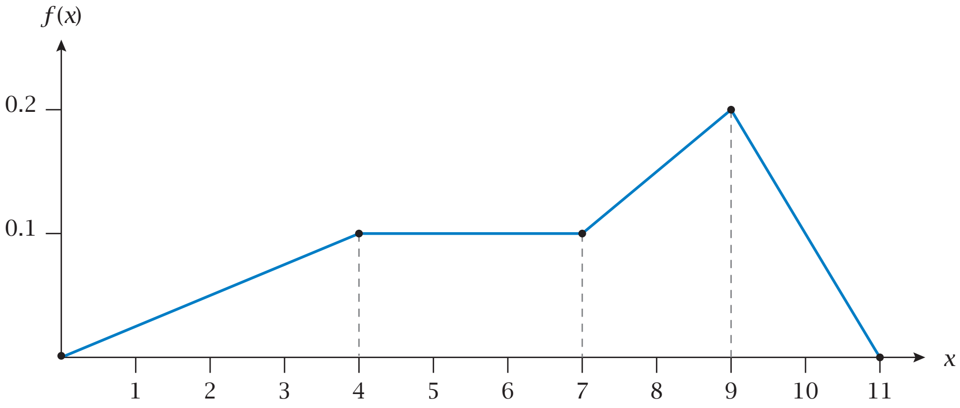
TRIAL TEST 8:
STATISTICAL INFERENCE

Calculators allowed

Time Allowed: 40 minutes

Total Marks: 40

1. A graph of a probability density function for a continuous random variable is shown below.



Determine

(a) $P(X \leq 4)$

(b) $P(4.5 \leq X \leq 6.5)$

(c) $P(7 \leq X \leq 9 \mid X \geq 4)$

(d) the upper quartile q where $P(X \leq q) = 0.75$

[7]

2. Simone takes anywhere from 14 to 26 minutes to travel from home to work each day dependent upon the road conditions.

Let Y be the time taken to travel to work each day

(a) Determine the probability density function Y.

(b) Find the probability that it takes Simone less than 18 minutes to travel to work.

(c) Find the probability that it takes between 16 and 25 minutes for Simone to travel to work.

(d) Determine the median travelling time.

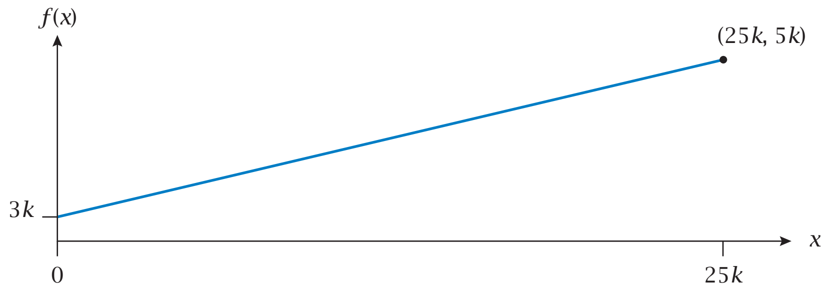
(e) Simone starts work at 9.00 am. If she leaves home at 8.45 am everyday, determine the probability that she will be late?

(f) Calculate the probability that Simone will be late on Thursday and Friday.

(g) Determine the probability that out of 5 working days, Simone will be late on at least 3 days.

[9]

3. For the random variable X, the probability density function $0 \leq x \leq 25k$ is graphed below.



(a) Determine the value of k.

(b) Calculate the value of t where $P(X \leq t) = 0.5$

[7]

4. The weights of trout in a certain lake are normally distributed with a mean of 1.8 kg and a standard deviation of 0.28 kg.

(a) If a trout is selected at random determine the probability that it weighs more than 1.6 kg given it weighs less than 1.9 kg.

(b) If 20% of the trout in the lake are underweight, determine the actual weight of an underweight trout?

A fishing and wildlife officer makes a random selection of 25 trout.

(c) If the sample mean weight is 1.75 kg, determine a 95% confidence interval for the mean weight of the trout.

(d) Determine the probability the sample mean lies between 1.62 kg and 1.93 kg.

(e) What size sample is required to be 99% sure the sample mean is within 0.4 kg of 1.8 kg?

[10]

5. A new drug, Type 1 in liquid form is designed to help overweight people lose weight. It was found the weight loss was normally distributed with a mean of 30 kg and a standard deviation of 7.2 kg.

Find the probability that

(a) the weight loss was less than 25 kg.

(b) the weight loss was between 27 kg and 32 kg.

(c) If the weight loss was less than 15 kg, Drug Type 2 was administered. If 1000 individuals use the initial drug, Type 1, how many will need to switch to Drug Type 2?

The doctors require 90% of all individuals to have a weight loss greater than 25 kg.

(d) If the current standard deviation is maintained, determine the new mean.

(e) If the original mean is maintained, determine the new standard deviation.

[7]

ANSWERS

CHAPTER 1: *Complex Numbers*

1. Let $z = x + yi$

$$\mathrm{Re}\left(\frac{1}{z+4}\right) = \frac{x+4}{(x+4)^2 + y^2}$$

$$\mathrm{Im}\left(\frac{1}{z+4}\right) = -\frac{y}{(x+4)^2 + y^2}$$

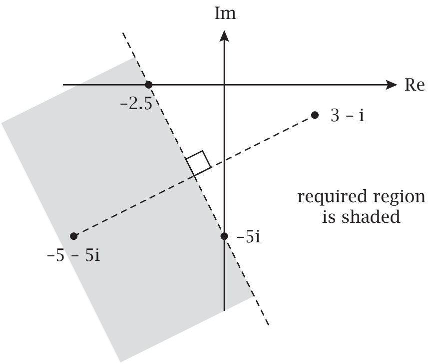
Now let $|z| = 4$

i.e. substitute $x^2 + y^2 = 16$

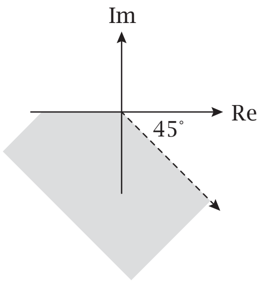
into Re to show it $= \frac{1}{8}$ easy

2. Let $z = x + yi$ in L.H.S. and R.H.S. to easily complete the proof

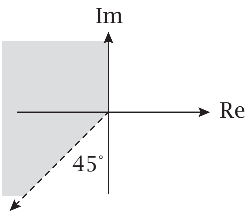
3.



4. $\mathrm{Arg}\ z < -\frac{\pi}{4}$



$\mathrm{Arg}\ iz < -\frac{\pi}{4}$



5. $2\ \mathrm{Re}\ z + \mathrm{Im}\ z < -5$

6. $z = -\frac{2}{3} - i$

7. $z = -\frac{2}{3} - i$

8. $z^4 + 4z^3 - 13z^2 - 74z + 442$

9. *(a)* Solution is $z = -2$
factor is $(z + 2)$

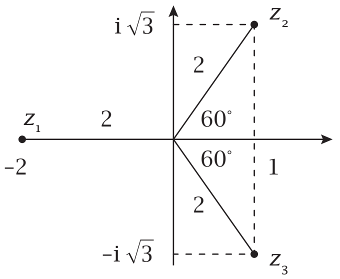
(b) $z^2 - 2z + 4$

(c) $z = 1 \pm i\sqrt{3}$

(d) $z_1 = -2 = 2 \text{ cis } 180^\circ$

$z_2 = 1 + i\sqrt{3} = 2 \text{ cis } 60^\circ$

$z_3 = 1 - i\sqrt{3} = 2 \text{ cis } (-60^\circ)$



The 3 solutions are all on the circle with its centre at the origin and radius = 2 and they are separated from each other by 120°.

10. *(a)* $\max |2| = 7$

(b) $\min |2| = 3$

(c) $\max \mathrm{Arg}\ z = 166.71^\circ$

(d) $\min \mathrm{Arg}\ z = 119.55^\circ$

11. $z = -4 + 19i$

12. $z = \frac{2}{5} - \frac{1}{5} i$

13. *(a)* let $P(z) = z^3 - 4z^2 + 6z - 4$

If $(z - 1 - i) = (z - (1 + i))$ is a factor then $z = 1 + i$ is a solution of $P(z) = 0$ Sub $1 + i$ into the cubic does give zero.

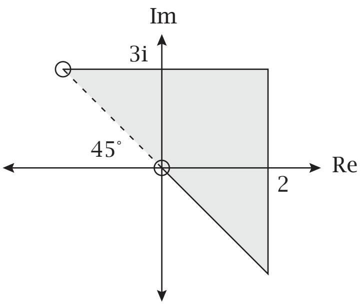
$\therefore (z - 1 - i)$ is a factor

(b) $(z - 2)(z - 1 - i)(z - 1 + i)$

14. $z = 1 \pm 2i$

15. $a = 1$

16.



17. $y = \frac{2}{3}x - \frac{4}{3}$

18. (a) 2

(b) $\frac{1}{2} + \frac{1}{2}\text{i}$

(c) $8\sqrt{2}$

(d) $\frac{\pi}{4}$

19. $z = 1 - 2\text{i}$ is also a solution

i.e. $(z - (1 + 2\text{i}))(z - (1 - 2\text{i})$

$= (z - 1 - 2\text{i})(z - 1 + 2\text{i})$

$= (z - 1)^2 + 4$

$= z^2 - 2z + 5$ is the quadratic factor

let $(z^2 - 2z + 5)(z + \text{a}) = z^3 + z + 10$

$\therefore$ $a = 2$ and the other two solutions are
$z = 1 - 2\text{i}$ and $z = -2$

20. $2 + i$ or $-2 - i$

21. (a) $-1 \pm i\sqrt{5}$

(b) $\pm 2i$ or $\pm i$

22. $x^2 - 6x + 13 = 0$

25. (a) $4 + 5i$

(b) 8

(c) $-10i$

(d) 41

(e) $-\frac{1}{41}(9 + 40i)$

26. $-1 - i$

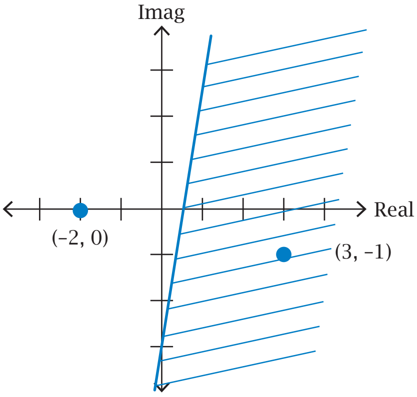
27. (a) $4\,Cis\,\frac{\pi}{4}$

(b) $\frac{1}{4}Cis\left(-\frac{\pi}{6}\right)$

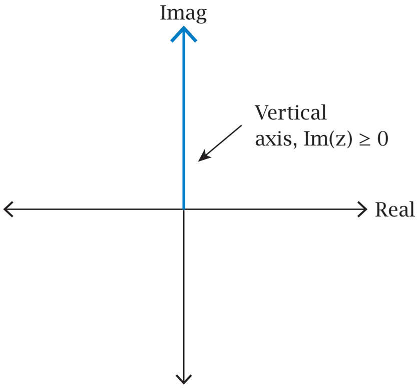
28. (a) i. $-7 + 24i$
ii. $3 - 4i$
iii. $-3i$
iv. $-\frac{6}{13} + \frac{17i}{13}$
v. 38

29. $Z = \frac{3}{10} + \frac{9}{10}i$

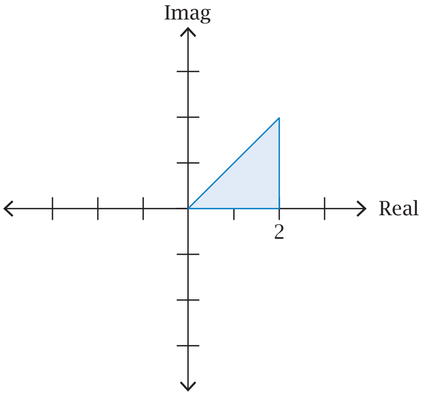
30. (a)



(b)



(c)



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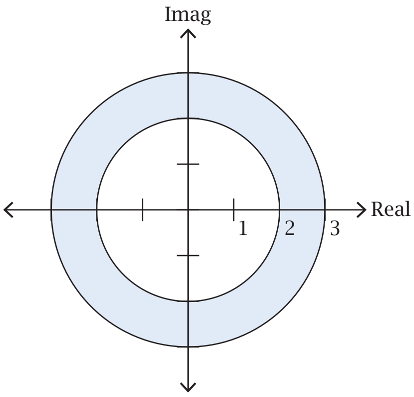
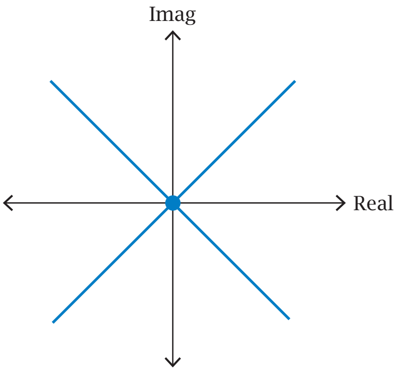


31. (a) $\{Z : |\,Z - (-1 + i)\,| \leq \sqrt{2}\}$

(b) $\{Z : \text{Re}\,(Z) \leq 2$ and $-\frac{\pi}{4} \leq Arg\,(Z) \leq 0\}$

32. (a) $x^2 - y^2$

(b)



33. (a) i. $1 + 2i$
ii. $3 - i$
iii. $-4 + 1.5i$
iv. $-2 - 2.5i$

(b) Symmetric with respect to the line $y = x$.

34. $3 - i$, 2

35. $-2i$, -3, 1

36. (a) $6\,Cis\left(\frac{7\pi}{12}\right)$

(b) $Cis\left(-\frac{\pi}{3}\right)$

(c) $1.5\,Cis\left(\frac{11\pi}{12}\right)$

(d) $1.5\,Cis\left(\frac{5\pi}{12}\right)$

(e) $3\pi^2\,Cis\left(\frac{n\pi}{6}\right)$

(add any suitable restriction on $\frac{n\pi}{6}$, e.g. $-\pi < n \le \pi$)

37. (a) $2\,Cis\left(-\frac{\pi}{2}\right)$

(b) $-2i$

(c) $Z_1 = -\sqrt{3} + i, Z_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

(d) $-2i$, SAME

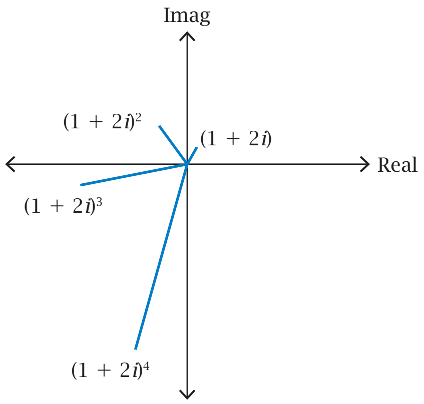
38. (a) $Z = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

(b) $\sqrt{3}$

(c) $\frac{\pi}{6}$

39. (a) i. $\sqrt{5}\ Cis\ 63.4^\circ$
ii. $5\ Cis\ 126.8^\circ$
iii. $5\sqrt{5}\ Cis\ (-169.7^\circ)$
iv. $25\ Cis\ (-106.3^\circ)$

(b)



(c) Magnitude multiplied by $\sqrt{5}$
Argument rotation of 63.4° (1 d.p.)

40. (a) $Z = -5 + i$

(b) $\sqrt{26}$

(c) 2.94 (2 d.p.) or 168.7°

41. (a) $3.366025404 + 1.633974596\ i$

(b) $\sqrt{14}$ or 3.741657387

(c) 0.4519249911

42. (a) $\cos 15^\circ + i \sin 15^\circ, \frac{\sqrt{3}+1}{2\sqrt{2}} + i\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

(c) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

43. $-\frac{1}{5} - \frac{3}{5}i$

44. $\frac{2}{Z^2 + 1}$

45. (a) i. $-\frac{5}{13} + \frac{12}{13}i$
ii. 1

(c) Circle centre origin, radius 1.

46. $m = 4, n = -5$

47. 24

48. (a) $\frac{-4i}{8 + 4\sqrt{3}}$

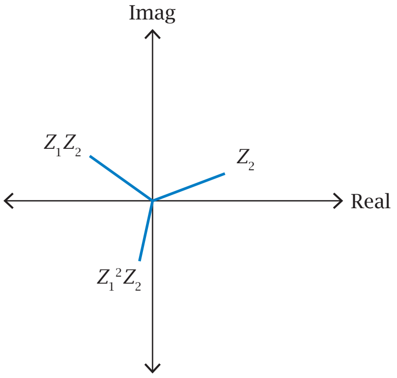
(b) W is a point on the imaginary axis.

49. $W = \frac{3}{2} - i, Z = -1 + \frac{3}{2}i$

50. (a) i. $\sqrt{4 - 2\sqrt{2}}\ Cis\frac{19\pi}{24}$

ii. $\sqrt{4 - 2\sqrt{2}}\ Cis\left(-\frac{13\pi}{24}\right)$

(b)



(c) Since magnitude of Z_1 is one, a product with Z_1 will cause a rotation of $\frac{2\pi}{3}$ on the argument and a product of Z_1^2 a further rotation of $\frac{2\pi}{3}$.

51. $a = 1, b = -6, c = 23, d = -50, e = 50$

52. dist = 18.37

53. 97.42° and -82.58°

54. $r = \frac{15}{4\pi}\theta \quad , \quad r = -\frac{15}{2\pi}\theta$

55. (8.89, -123.00°) or (14.80, 124.18°)

56. *(a)* (6.062, -170°)

(b) (9.21, 141.19°) and dist = 10.06 2dp

57. *(a)* $r = f(\theta) = \frac{8 \sin 130^\circ}{\sin(80^\circ - \theta)}$

(b) $\frac{dr}{d\theta} = \frac{8 \sin 130^\circ \cos(80^\circ - \theta)}{\sin^2(80^\circ - \theta)}$

let $\frac{dr}{d\theta} = 0$ i.e. $\cos(80^\circ - \theta) = 0$

$\theta = -10^\circ$

$r_{\min} = f(-10^\circ) = 6.13$ 2dp

sign test confirms min

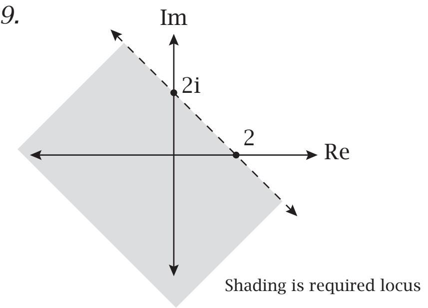
58. *(a)* $-1 + i\sqrt{3}$

(b) $4 \text{ cis}\left(\frac{5\pi}{6}\right)$

(c) $\frac{\sqrt{3}}{4} - \frac{i}{4}$

(d) $2 \text{ cis}\left(-\frac{2\pi}{3}\right)$

59.



60. $z_1 = 2^{\frac{1}{8}} \text{ cis } \frac{7\pi}{16}$

$z_2 = 2^{\frac{1}{8}} \text{ cis } \frac{15\pi}{16}$

$z_3 = 2^{\frac{1}{8}} \text{ cis } \frac{23\pi}{16} = 2^{\frac{1}{8}} \text{ cis}\left(\frac{-9\pi}{16}\right)$

$z_4 = 2^{\frac{1}{8}} \text{ cis } \frac{31\pi}{16} = 2^{\frac{1}{8}} \text{ cis}\left(\frac{-\pi}{16}\right)$

61. $z_1 = 3 + 0i$

$z_2 = -\frac{3}{2} + \frac{i\sqrt{11}}{2}$

$z_3 = -\frac{3}{2} - \frac{i\sqrt{11}}{2}$

62. $0 - \frac{1}{8}i$

63. $z_1 = 1.455 - 0.344i$

$z_2 = 0.344 + 1.455i$

$z_3 = -1.455 + 0.344i$

$z_4 = -0.344 - 1.455i$

64. $z + z^{-1} = \text{cis } \theta + \frac{1}{\text{cis } \theta}$

$= \text{cis } \theta + \text{cis } (-\theta)$

$= \cos \theta + i \sin \theta + \cos(-\theta) + i \sin(-\theta)$

$= \cos \theta + i \sin \theta + \cos \theta - i \sin \theta$

$= 2 \cos \theta$ as required

Same for $z^n + z^{-n} = 2 \cos n\theta$

L.H.S. $= 8 \cos^4 \theta$, $\cos \theta = \frac{z + z^{-1}}{2}$

$$= 8\left(\frac{z + z^{-1}}{2}\right)^4$$

$$= \frac{8}{16}(z^4 + 4z^3 \cdot z^{-1} + 6z^2 z^{-2} + 4z z^{-3} + z^{-4})$$

$$= \frac{1}{2}(z^4 + z^{-4} + 4z^2 + 4z^{-2} + 6)$$

$$= \frac{1}{2}(z^4 + z^{-4}) + 2(z^2 + z^{-2}) + 3$$

but $z^4 + z^{-4} = 2 \cos 4\theta$

and $z^2 + z^{-2} = 2 \cos 2\theta$

$$\begin{aligned} \text{L.H.S.} &= \frac{1}{2}\cdot 2\cos 4\theta + 2.2\cos 2\theta + 3 \\ &= \cos 4\theta + 4\cos 2\theta + 3 \\ &= \text{R.H.S.} \end{aligned}$$

65. $2 \text{ cis } \left(\frac{-2\pi}{3}\right)$

66. *(a)* $z^3 - z^2 - 6z + 18 = 0$

Cubic must have at least one real root. Check factors of 18 and by trial and error $z = -3$ gives 0

$\therefore\ z + 3$ is a factor

$(z+3)(z^2 + bz + 6) = z^3 - z^2 - 6z + 18$

$bz^2 + 3z^2 = -z^2$ (just doing z^2 terms)

$\therefore\ b = -4$

$z^2 - 4z + 6 = 0$

$$z = \frac{4 \pm \sqrt{16 - 4\times 1\times 6}}{2}$$

$$= \frac{4 \pm \sqrt{-8}}{2}$$

$$= 2 \pm i\sqrt{2}$$

$\therefore\ (z+3)(z - 2 - i\sqrt{2})(z - 2 + i\sqrt{2})$ is the required factorisation

(b) Proof

67. $2\sqrt{3} \text{ cis}\left(\frac{3\pi}{4}\right)$

68. $z = \frac{4}{3} + \frac{5}{3}i$

69. $z_1 = \sqrt{3} - i$, $z_2 = 2i$, $z_3 = -\sqrt{3} - i$

70. See p.14 for method

71. $z^4 + 6z^2 + 25$

$$\begin{aligned} z^2 &= \frac{-6 \pm \sqrt{36 - 4\times 1\times 25}}{2} \\ &= -3 + 4i\ ,\ -3 - 4i \\ &= 5 \text{ cis } 126.8698976^\circ \\ z &= \sqrt{5} \text{ cis } 63.4349488^\circ \\ &= 1 + 2i \\ z_1 &= 1 + 2i \rightarrow z_3 = 1 - 2i \\ z_2 &= -(1 + 2i) \\ &= -1 - 2i \rightarrow z_4 = -1 + 2i \end{aligned}$$

72.

$$\begin{aligned} z_1 &= \text{cis}\left(\frac{3\pi}{10}\right) \\ z_2 &= \text{cis}\left(\frac{7\pi}{10}\right) \\ z_3 &= \text{cis}\left(-\frac{9\pi}{10}\right) \\ z_4 &= \text{cis}\left(-\frac{\pi}{2}\right) \\ z_5 &= \text{cis}\left(-\frac{\pi}{10}\right) \end{aligned}$$

73. *(a)*

$$\begin{aligned} z^2 &= -25 \\ z &= 0 \pm 5i \end{aligned}$$

(b) $(z+1)^2 + 25(z-1)^2 = 0$

$$\begin{aligned} (z+1)^2 &= -25(z-1)^2 \\ \frac{(z+1)^2}{(z-1)^2} &= -25 \\ \left(\frac{z+1}{z-1}\right)^2 &= -25 \\ \therefore\ \frac{z+1}{z-1} &= \pm 5i \end{aligned}$$

$$\begin{aligned} z + 1 &= 5i\,(z - 1) \\ z + 1 &= 5iz - 5i \\ z - 5iz &= -1 - 5i \\ z(1 - 5i) &= -1 - 5i \\ z &= \frac{-1 - 5i}{1 - 5i} \\ z &= \frac{-1 - 5i}{1 - 5i} \times \frac{1 + 5i}{1 + 5i} \end{aligned}$$

etc. gives

$$z_1 = \frac{12 - 5i}{13}$$

or

$$\begin{aligned} z + 1 &= -5i\,(z - 1) \\ z + 1 &= -5iz + 5i \\ z + 5iz &= -1 + 5i \\ z(1 + 5i) &= -1 + 5i \\ z &= \frac{-1 + 5i}{1 + 5i} \end{aligned}$$

etc. gives $z_2 = \frac{12 + 5i}{13}$

CHAPTER 2: *Functions and Curve Sketching*

1. *(a)* y = f (g(x)) exists because

R_g $y \geq 0$ is a subset of D_f which is $x \in R$ which means that all values in D_g can be mapped by y = f (g(x)).

y = g(f (x)) does not exist because R_f $y \geq -12.25$ is not a subset of D_g which is $x \geq 0$ which means there are values in D_f which cannot be mapped by y = g(f (x)).

(b) $y = f(g(x)) = x + \sqrt{x} - 12$
D_{fog} is $x \geq 0 \quad x \in R$
R_{fog} is $y \geq -12 \quad y \in R$

(c) $y = g(f(x)) = g(x^2 + x - 12)$
$= \sqrt{x^2 + x - 12}$
D_{fog} is $x \geq 3$ or $x \leq -4, x \in R$
R_{fog} is $y \geq 0, y \in R$.

(d) $y = f(x)$ is not a 1 to 1 function and hence $y = f^{-1}(x)$ is not defined.
Restrict D_f to $x \geq -\frac{1}{2}$ then
$f^{-1}(x) = -0.5 + \sqrt{x + 12.25}$
or restrict D_f to $x \leq -\frac{1}{2}$ then
$f^{-1}(x) = -0.5 - \sqrt{x + 12.25}$

2. (a) One x can give two values of y so it is not a function.

(b) $f(x) = \sqrt{4 - x^2}$ $g(x) = -\sqrt{4 - x^2}$

(c) $D_f \quad -2 \leq x \leq 2, x \in R$
$D_g \quad -2 \leq x \leq 2, x \in R$
$R_f \quad 0 \leq y \leq 2, y \in R$
$R_g \quad -2 \leq y \leq 0, y \in R$

3. (a) $x = 7, x = -11$

(b) $x < -11, x > 7$

4. $$y = \begin{cases} 3x - 1, x \geq 4 \\ x + 7, -1.5 \leq x < 4 \\ -3x + 1, x < -1.5 \end{cases}$$

5. (a) $D_f \quad x \geq 1, x \in R$
$R_f \quad y \geq 2, y \in R$

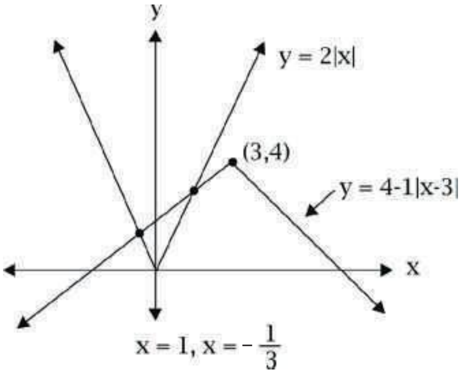
(b) $f^{-1}(x) = (x - 2)^2 + 1, x \geq 2$

6. (a) $f^{-1}(x) = \frac{2x + 3}{5x - 2}$

$y = f^{-1}(x)$ is the same function as $y = f(x)$.

(b) The graph of both functions have $y = x$ as their line of symmetry.

7.



8. Each x of $\sqrt{x}$ is moved 2 units in the positive x direction. Each y is dilated in the y direction with scale factor × 3 and then translated 5 units in the positive y direction.

9. Each x is either shifted 6 units in the negative x direction and then dilated by scale factor $\times {}^1/_3$ or each x is dilated by scale factor $\times {}^1/_3$ and then shifted 2 units in a negative x direction.
Each y is reflected in the x axis and then shifted 4 units in the positive y direction.

10. For $y = \log 5x$ x is dilated in the x direction by ${}^1/_5$.
For $y = \log 5 + \log x$ y is shifted by log5 in the positive y direction.

11. (a) $D_f \quad x \geq 1$ $\qquad D_g \quad x > 0$
$R_f \quad y \geq -2$ $\qquad R_g \quad y > 0$

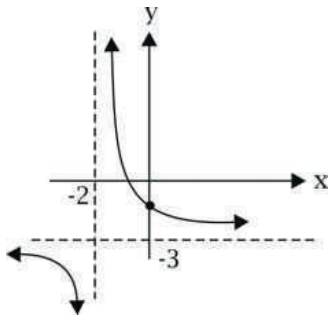
(b) $R_f \not\subset D_g$

(c) Restrict
D_f to $x > 5$ and then
D_{fog} is $x > 5, x \in R$
R_{fog} is $y > 0, y \in R$

(d) $f^{-1}(x) = (x + 2)^2 + 1, x \geq -2$.

12. $g(x) = f^{-1}(x), f(x) = g^{-1}(x)$
These are the inverses of each other.

13. (a)



D $x \neq -2, x \in R$
R $y \neq -3, y \in R$

(b) $y = \frac{-3x - 2}{x + 2}$

(c) $f^{-1}(x) = \frac{-2x - 2}{x + 3}$

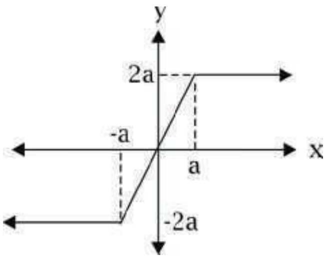
14. $y = f(g(x)) = 2x^2 - 8x - 3$
$y = g(f(x)) = 4x^2 - 20x + 21$

15. (a) $D_f \quad -1 \leq x \leq 1, x \in R$
$R_f \quad 0 \leq y \leq 1, y \in R$
$D_g \quad x \in R$
$R_g \quad y \geq 2, y \in R$

(b) $R_g \not\subset D_f$

(c) $g(f(x)) = 3 - x^2, -1 \leq x \leq 1$
$R_{gof} \quad 2 \leq y \leq 3, y \in R$

16. $f(x) = \begin{cases} 2a, x \geq a \\ 2x, -a \leq x < a \\ -2a, x < a \end{cases}$

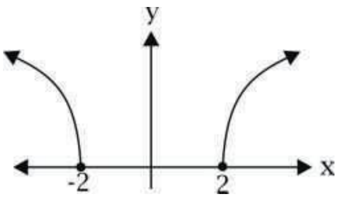


17. x is shifted 3 units in the positive x direction and is then dilated by scale factor × ½ (or x is dilated by SF × ½ and then shifted 1.5 units in the positive x direction).
y is dilated by SF × 3 and then relected in the x axis before being shifted 5 units in the negative y direction.

18. $f(g(x)) = g(f(x)) = x$, f and g are inverses of each other.

19. $g(x) = 2x^2 + 7x - 1$.

20. D $x \geq 2, x \leq -2, x \in R$
R $y \geq 0, y \in R$



21. $a = -d$

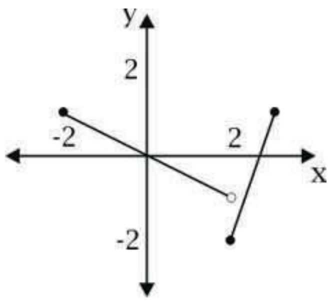
22. $x = -\frac{5}{3}, x = -3$
$-3 < x < -\frac{5}{3}$

23. for $x = -2$, $\sqrt{x^2} = 2 \neq -2$

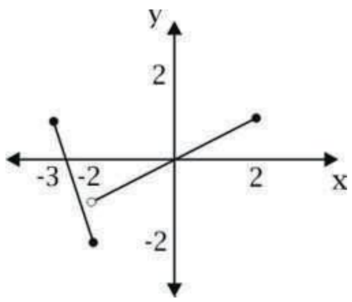
$\sqrt{x^2} = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

$= |x|$

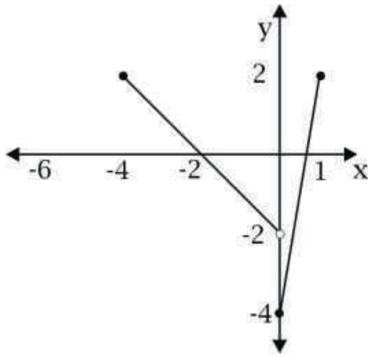
24. *(a)*



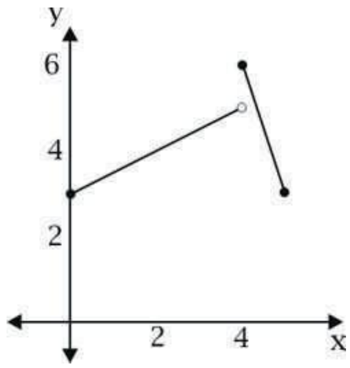
(b) *i.*



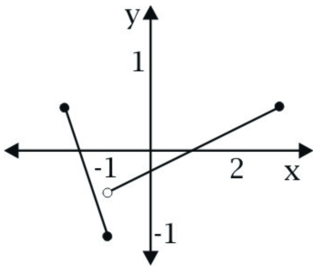
ii.



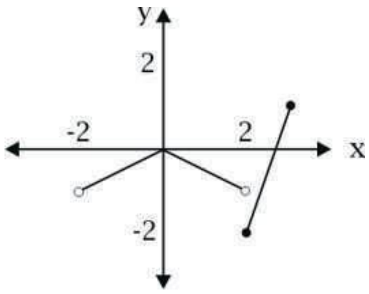
iii. The graphed solution shows the transformed function in full.
Note that the original function is defined for the domain $-2 \leq x \leq 3$.
A vertical line has been drawn in to indicate where the right-hand limit would be (see graph).



iv.

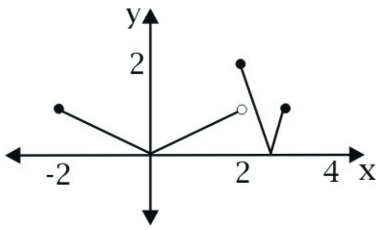


v.



Note that because the domain of f is $-2 \leq x \leq 3$ there will be no image for $-3 \leq x \leq -2$.

vi.

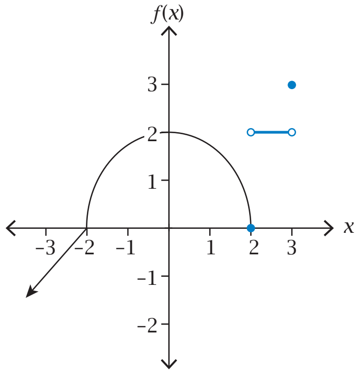


25. $a = -10$, $b = 3$

26. (a) D_f {reals}, $R_f\{y|y \geq 3\}$
    (b) $D_g\{x|x \geq -3\}$, $R_g\{y|y \leq 0\}$
    (c) $D_m\{x|x \neq 2\}$, $R_m\{y|y \neq 0\}$
    (d) $D_h\{x|x > 2\}$, $R_h\{y|y > 0\}$

27. 1, -2, 6 (1 occurs twice)

28.



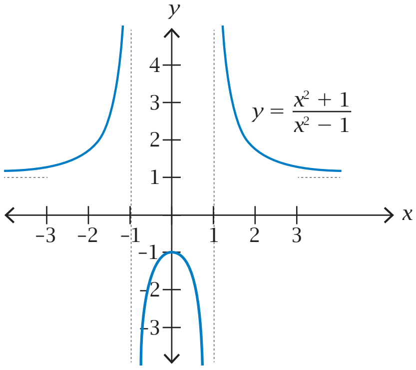
29. (a) (0,1)

Poles $x = \pm 1$

Turning point (0,–1)

As $x \to -\infty$ $y \to 1^+$

$x \to \infty$ $y \to 1^+$



(b) $\left(\frac{1}{2}, 0\right)$ $\left(0, \frac{1}{4}\right)$

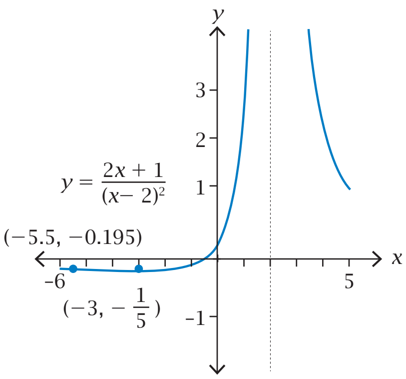
Pole at $x = 2$

Turning point $\left(-3, -\frac{1}{5}\right)$

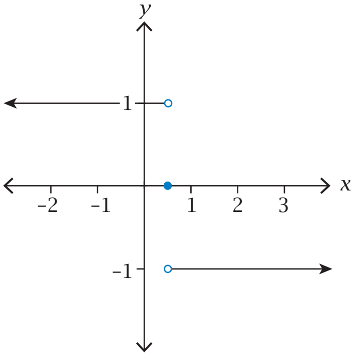
Point of Inflection (-5.5, -0.195)

As $x \to -\infty$ $y \to 0^-$

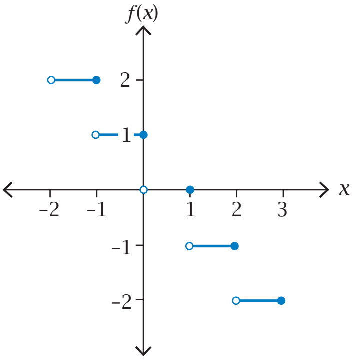
$x \to \infty$ $y \to 0^+$



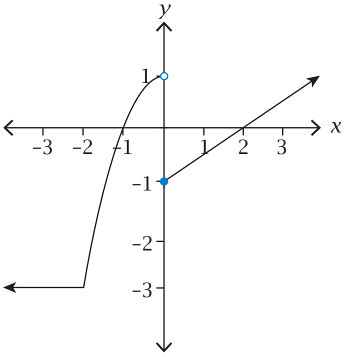
30. (a)



(b)



31. (a)



(b) i. -3
    ii. 0
    iii. does not exist
    iv. does not exist
    v. -3

32. (a) 1.5
    (b) 2
    (c) 0
    (d) 1.5

33. *(a)* -1

(b) does not exist

34. 0.25

35. *(a)* 1

(b) 2

36. 1, 2

37. Continuous and differentiable at $x = -1$
Continuous but not differentiable at $x = 2$

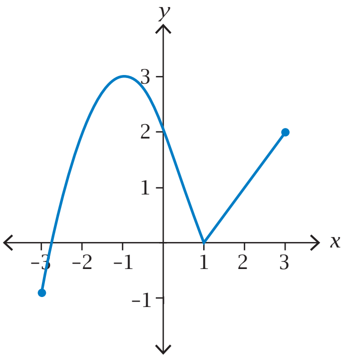
38. *(a)* $\lim\limits_{x \to -1^-} f = 2, \ \lim\limits_{x \to -1^+} f = 0$ since unequal
discontinuous at $x = -1$

(b) $a = -2, b = 2$

39. $-\dfrac{2}{x^2}$

40. $\dfrac{1}{2\sqrt{x}}$

41. *(a)*



(b)
i. does not exist
ii. -2
iii. does not exist
iv. 1

42. *(a)* $D_f = \{x : x \neq 2\}, R_f = \{y : y \neq 1\}$

(b) 1

(c) $f(x-1) = \dfrac{x-1}{x-3}$

(d) $f\left(\dfrac{2}{x}\right) = \dfrac{1}{1-x}$

(e) $-1, 2$

43. $a = -2, \ b = -6, \ c = 1, \ d = 6$

44.

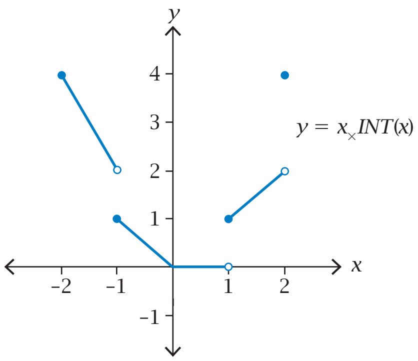
y	x
-2.01	1.00850854
-2.001	1.000850086
-2	1
-1.999	0.9991500859
-2.99	0.9915086103

When $y = -2, \ x = 1$
$\lim\limits_{y \to -2^+} x = 1 = \lim\limits_{y \to -2^-} x$
Curve continuous at $y = -2$

45. $Df\{x|-3 \leq x \leq 3 \ , x \neq 0\}, Rf\{y|y \geq 0\}$

46. $a = 1 \ , c = -1$

47.



48. *(a)* $y = \dfrac{(x+2)(x+1)}{(x+3)}$

(b) $y = \dfrac{(x+2)(x+1)}{(x+3)^2}$

49. $\lim\limits_{x \to -0^+} x^x = 1, \ \lim\limits_{x \to -0^-} x^x$ does not exist.
$\lim\limits_{y \to -0} x^x$ does not exist.

50. *(a)* $Q(x) = (x+1)(x-10); R(x) = 0$

(b) $Q(x) = (x-1); R(x) = 2$

(c) $Q(x) = (x^2 + 5x - 3); R(x) = 0$

(d) $Q(x) = (x^2 + 4x + 3); R(x) = 2$

(e) $Q(x) = (4x^2 - 8x + 1); R(x) = 0$

(f) $Q(x) = (3x^2 + 23x + 62); R(x) = 176$

51. *(a)* $P(2) = -13, \ \therefore \ (x-2)$ is not a factor

(b) $P(2) = 0, \ \therefore \ (x-2)$ is a factor

(c) $P(2) = 0, \ \therefore \ (x-2)$ is a factor

(d) $P(2) = -60, \ \therefore \ (x-2)$ is not a factor

(e) $P(2) = 0, \ \therefore \ (x-2)$ is a factor

(f) $P(2) = 75, \ \therefore \ (x-2)$ is not a factor

52. *(a)* $c = 4, -5, 3; P(x) = (x-4)(x+5)(x-3)$

(b) $c = -7, 6, -1; P(x) = (x-6)(x+7)(x+1)$

(c) $c = -9, 2, 3; P(x) = (x-2)(x+9)(x-3)$

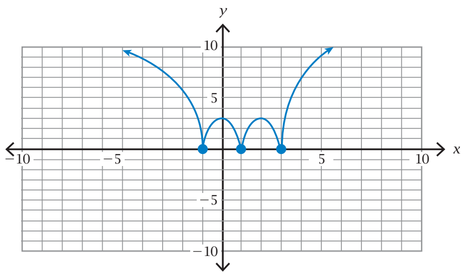
(d) $c = -4, \frac{1}{2}, 3; P(x) = (2x-1)(x+4)(x-3)$

(e) $c = -2, -\frac{4}{3}, 3; P(x) = (3x+4)(x+2)(x-3)$

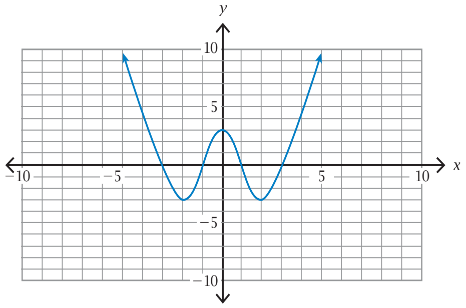
(f) $c = -7, \frac{2}{5}, 4; P(x) = (5x-2)(x+7)(x-4)$

53. $b = -11, g(x) = 2x^3 - 11x^2 - 23x + 14$
$= (2x-1)(x-7)(x+2)$

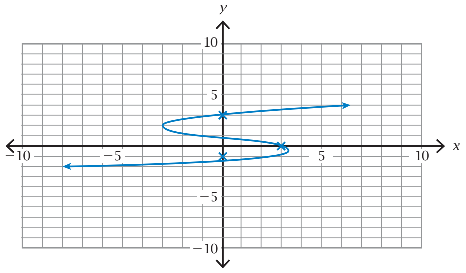
54. *(a)*



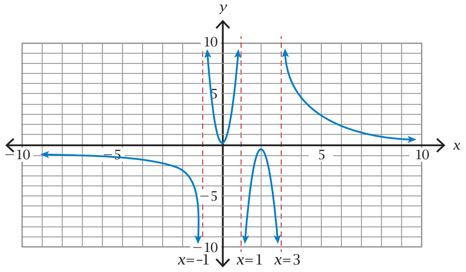
(b)



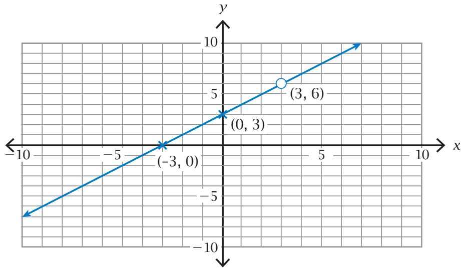
(c)



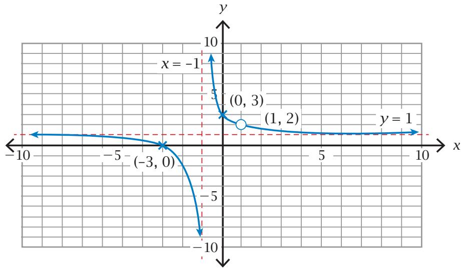
(d)



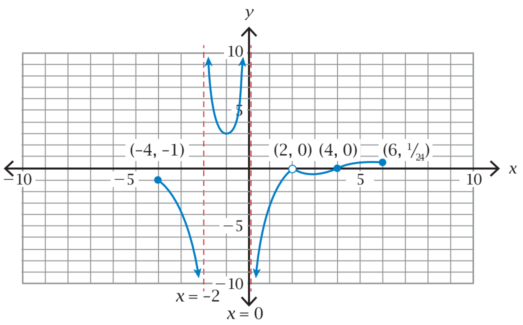
55. *(a)* $f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x + 3)\cancel{(x - 3)}}{\cancel{x - 3}} = x + 3$



(b) $g(x) = \frac{x^2 + 2x - 3}{x^2 - 1} = \frac{(x + 3)\cancel{(x - 1)}}{(x + 1)\cancel{(x - 1)}} = \frac{x + 3}{x + 1}$



56.



$f(x) = \frac{x^2 - 6x + 8}{x^3 - 4x} = \frac{(x - 2)(x - 4)}{x(x - 2)(x + 2)}$

$\frac{x - 2}{x - 2}$ implies a hole discontinuity at $x = 2$

–x-intercept @ $y = 0$
let $x^2 - 6x + 8 = 0$
$(x - 2)(x - 4) = 0$
$\therefore x = 2, x = 4$

but a hole discontinuity @ $x = 2$ implies $x = 4$ is an x-intercept

– no y-intercept as $f(x) \neq 0$
– vertical asymptote @ $x = -2$, $x = 0$

CHAPTER 3: *Vectors in 3-dimensions*

1. $\mathbf{x}$ = <0.5, -4, -1.5>

2. $|\mathbf{PQ}| = \sqrt{38}$

3. $a = 2$, $b = 3$

4. $\theta = 84.89^\circ$

5. P is (2, 1, 5.5)

6. $|\mathbf{r} - \langle -1, 2, -3 \rangle| = 5$

7. *(a)* $\mathbf{RP} = \langle 3, 3 - k, 3 \rangle$
$\mathbf{RQ} = \langle 5, 4 - k, -7 \rangle$

(b) $k = 1$ or $k = 6$

(c) $T = (9, 3.5, -5)$

(d) $|\mathbf{r} - \langle 9, 3.5, -5 \rangle| = \sqrt{26.25}$

8. *(a)* $\mathbf{OC} = \langle 4, 18, -12 \rangle$
$\mathbf{OD} = \langle 4, 6, 0 \rangle$

(b) A possible vector equation through C and D is $\mathbf{r} = \langle 4, 6 - 6t, 6t \rangle$. Remember yours will probably look different to this one but still be correct.

(c) closest point is (4, 3, 3)

(d) shortest distance = 3 units

9. $\theta = 60^\circ$

10. $<x, y, z> = <4.91, 1.04, 3.28>$

11. *(a)* speed = 0.244 km/s

(b) $\theta = 3.84^\circ$

(c) $t = 113.49$ sec

(d) closest distance = 13.94 km

12. $a = -1$, $b = 8$

13. angle with positive z axes is 57.03°

14. intersection point (–3, 23, 9)

15. two points are
(–3.87, 14.60, 15.73) and
(1.29, –0.88, 5.41)

16. *(a)* speed = 1650 km/hr

(b) time = 2 min 5 sec

(c) time = 91 sec (nearest sec)
closest distance = 27.58 km

(d) closest distance = 35.29 km

17. closest distance = 3.65 km
when $t = 173$ sec

18. *(a)* 36.04°

(b) 51.67°

(c) 11.31°

19. *(a)* 61.19°

(b) 28.81°

20. $|\mathbf{u}| = \sqrt{29}$

$\theta = 26.56^\circ$

$\phi = 33.85^\circ$

21. $\mathbf{u} = <-8.03, 13.92, -19.15>$

22. *(a)* 7.90°

(b) 78.58 km after 22 min 37 sec

(c) 55.42 km

23. A has $\theta = -75^\circ$, $\phi = 35^\circ$
B has $\theta = 50^\circ$, $\phi = -22^\circ$
The unit vectors for **OA** and **OB** where O is the centre of the earth are
$\widehat{\mathbf{OA}} = <0.212, -0.791, 0.574>$
$\widehat{\mathbf{OB}} = <0.596, 0.710, -0.375>$
Angle between **OA** and **OB** is 130.58°.
Shortest distance ≈ 14472 km

24. $-43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$

25. If $(\mathbf{a}\times\mathbf{b}) = \mathbf{a}$ then $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{a} = 0$

$$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{a} = \left[\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\right]\bullet\mathbf{a} = \begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix}\bullet\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$$
$$= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1$$
$$= 0 \text{ as required}$$

26. For the equation of a plane, we need a point and a vector normal to the plane.

Vector normal to the plane can be found by determining the cross product of two vectors:

From (2, 2, 1) to (–4, 1, 0) : <–4–2, 1–2, 0–1>
= <–6, –1, –1>

From (2, 2, 1) to (6, 1, –5) : <6–2, 1–2, –5–1>
= <4, –1, –6>

The cross product of <–6, –1, –1> and <4, –1, –6> is: <–5, –40, 10>

Equation of the plane:
$-5(x - 2) - (-40)(y - 2) - 10(z - 10) = 0$
$0 = -5x + 10 + 40y - 80 - 10z + 100$
$0 = -5x + 40y - 10z + 30$

27. *(a)*

$$\begin{array}{rcll} x+y &=& -3 & \text{①} \\ 2x-5y &=& -6 & \text{②} \\ \hline 2x+2y &=& -6 & \text{①} \times 2 = \text{③} \\ 2x-5y &=& -6 & \text{②} \\ \hline 7y &=& 0 & \text{③} - \text{②} \\ y &=& 0 & \\ \therefore\ x &=& -3 & \end{array}$$

Solution: (–3, 0)

(b)

$$\begin{array}{rcll} 2x &=& 3(y+1) & \\ 5x &=& 19-4y & \\ \therefore\ 2x &=& 3y+3 & \\ 2x-3y &=& 3 & \text{①} \\ 5x+4y &=& 19 & \text{②} \\ \hline 8x-12y &=& 12 & \text{①} \times 4 \quad \text{③} \\ 15x+12y &=& 57 & \text{②} \times 3 \quad \text{④} \\ \hline 23x &=& 69 & \text{③} + \text{④} \\ x &=& 3 & \\ \therefore\ y &=& 1 & \end{array}$$

Solution: (3, 1)

(c) $x + 3y - 6z = 7$ ①

$2x - y + 2z = 0$ ②

$x + y + 2z = -1$ ③

$2y - 8z = 8$ ① - ③ = ④

$7y - 14z = 14$ ① × 2 - ② = ⑤

$14y - 56z = 56$ ④ × 7 = ⑥

$14y - 28z = 28$ ⑤ × 2 = ⑦

$-28z = 28$ ⑥ - ⑦

$z = -1$

$\therefore y = 0$

$\therefore x = 1$

Solution is $(1, 0, -1)$

(d) $x = 1 - y$

$2x = z$

$2z = -2 - y$

$\therefore x + y = 1$ ①

$2x - z = 0$ ②

$y + 2z = -2$ ③

$2y + z = 2$ ① × 2 - ② = ④

$y + 2z = -2$ ③

$3y = 6$ ④ × 2 - ③

$\therefore y = 2$

$\therefore x = -1$

$\therefore z = -2$

Solution: $(-1, 2, -2)$

(e) $6x - 25y - 8z = 8$ ①

$12x - 15y + 4z = 12$ ②

$3x + 5y + 2z = 0$ ③

$-35y - 12z = 8$ ① - ③ × 2 = ④

$-35y - 20z = 4$ ① × 2 - ② = ⑤

$8z = 4$ ④ - ⑤

$z = \frac{1}{2}$

$\therefore y = -\frac{2}{5}$

$\therefore x = \frac{1}{3}$

Solution: $\left(\frac{1}{3}, -\frac{2}{5}, \frac{1}{2}\right)$

28. Let x = low risk

Let y = high risk

(a) $\therefore x + y = 20\,000$

$\therefore 1.05x + 1.12y = 21\,455$

(b) $\therefore x = 13\,500$

$y = 6\,500$

13 500 shares purchased in the low risk investment.

6 500 shares purchased in the high risk investment.

29. Let vitamins $= v$

proteins $= p$

carbohydrates $= c$

Equations are: $v + p + c = 38$

$c = 2v$

$v = p + 2$

Solving using elimination gives: $v = 10$

$p = 8$

$c = 20$

Vitamin units: 10 per day

Protein units: 8 per day

Carbohydrate units: 20 per day

30. Let maize $= m$

wheat $= w$

sunflower $= s$

$\therefore$ Equations are:

$0.4m + 0.2w + 0.25s = 11$

$0.2m + 0.3w + 0.4s = 14$

$0.2m + 0.1w + 0.3s = 9$

Using elimination results in

$m = 7.5$

$w = 15$

$s = 20$

$\therefore$ Require 7.5 units of maize

15 units of wheat

20 units of sunflower seeds

31. *(a)* $\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{3y}{12y+2x^2} & -\frac{2x}{12y+2x^2} \\ \frac{x}{12y+2x^2} & \frac{4}{12y+2x^2} \end{bmatrix}$, for $12y + 2x^2 \neq 0$

32. (a) $a = -6$

(b) $b = \pm\sqrt{2}$

(c) $c = 0$ or $c = -2$

(d) $d = 0$ or $d = -1.5$

33. $X = \begin{bmatrix} -\frac{7}{5} & -\frac{6}{5} \\ -\frac{16}{5} & -\frac{13}{5} \end{bmatrix}$

34. (a) i. $x = -1, y = -3$

ii. $p = -4, q = 5$

(b) $b = -\frac{1}{3}, c = -4$

35. (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$

(d) I

36. (a) $I_{3\times3}$

(b) $P^{-1} = Q$

37. (a) $I_{3\times3}$

(b) The matrix P, and its inverse are equal.

38. $p = -2, q = 5$

39. (a) i. $a = 1, b = 2, c = 0$

ii. $p = 2, q = -1, r = 5$

(b) $p = \frac{1}{8}$

40. $A = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$

41. (a) $X = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$

(b) $P = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

(c) $X = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$

(d) $M = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & 1 \end{bmatrix}$

42. Justification

43. (a) Justification

(b) $B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

44. (a) Justification

(b) $b = 5$

(c) $C^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

45. (a) $D = \frac{1}{4}(D^3 + I)$

(b) $D^{-1} = 4I - D^2$

(c) Justification

(d) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

46. Justification

47. $a = -2, b = \frac{1}{2}$

48. $X = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

49. (a) $\frac{1}{n - am}\begin{bmatrix} n & -a \\ -m & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

50. (a) $AB = \begin{bmatrix} 3 & -11 \\ -3 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, B^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{15} \\ 0 & \frac{1}{5} \end{bmatrix}$

(b) $\left(AB\right)^{-1} = \begin{bmatrix} -\frac{1}{30} & -\frac{11}{30} \\ -\frac{1}{10} & -\frac{1}{10} \end{bmatrix}$

51. $x = -3, y = 3, z = 1$

52. (a) $AB = \begin{bmatrix} 74 & 0 & 0 & 0 \\ 0 & 74 & 0 & 0 \\ 0 & 0 & 74 & 0 \\ 0 & 0 & 0 & 74 \end{bmatrix}$

(b) $AB = 74\, I$

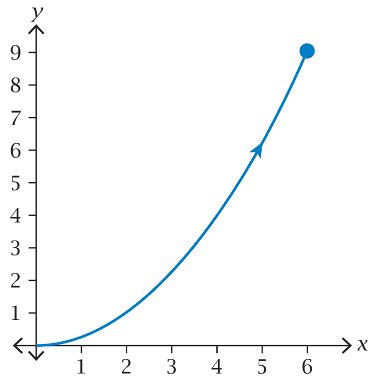
$\Rightarrow A^{-1} = \frac{B}{74}$

$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ c \end{bmatrix} = \frac{B}{74}\begin{bmatrix} 8 \\ -4 \\ -4 \\ -5 \end{bmatrix}$

i.e. $a = 1, b = -1, c = -2, d = 3$

CHAPTER 4: *Vector Calculus*

1. *(a)*



(b) $y = \frac{x^2}{4}$ parabola

2. *(a)* origin, $2\sqrt{5}$

(b) $5\sqrt{5}$

(c) $y = 2x - \frac{x^2}{4}$

(d) 0,4

(e) 4

3. *(a)* *i.* $2\underset{\sim}{j}$

ii. $-\frac{3}{5}\underset{\sim}{j}$

4. 1

6. *(a)* $\frac{t^3}{3}\underset{\sim}{i} + (t^2 + 5t)\underset{\sim}{j} + \underset{\sim}{c}$

(b) $\frac{2}{3}\underset{\sim}{i} + (e{-}1)\underset{\sim}{j}$

7. *(a)* *i.* $\pi^3\underset{\sim}{i}$

ii. π^3

(b) *i.* gives the displacement vector $\underset{\sim}{r}(\pi) - \underset{\sim}{r}(0)$

ii. gives the distance travelled between 2 points

8. *(a)* $\underset{\sim}{v}(t) = -6 \sin 2t\,\underset{\sim}{i} + 6 \cos 2t\,\underset{\sim}{j}$

(b) $\underset{\sim}{a}(t) = -12 \cos 2t\,\underset{\sim}{i} - 12 \sin 2t\,\underset{\sim}{j}$

(c) 12

(d) *i.* 0
ii. 0

(e) velocity vector perpendicular to position vector, velocity vector perpendicular to acceleration vector.

9. $\frac{4}{3}$

10. *(a)* $\underset{\sim}{v}(t) = 2t\,\underset{\sim}{i} + (3t^2 - 1)\underset{\sim}{j}$

(b) $\underset{\sim}{r}(t) = (t^2 - 1)\,\underset{\sim}{i} + (t^3 - t)\underset{\sim}{j}$

(c) $\left(-\frac{2}{3}, -\frac{2}{3\sqrt{3}}\right)$

11. *(b)* 90°

12. $t = 1$, 1 m right, 39.5 m forward

13. *(a)* $\underset{\sim}{v}(t) = 5 \cos 2t\,\underset{\sim}{i} - 4 \sin 2t\,\underset{\sim}{j}$

(c) $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$, etc

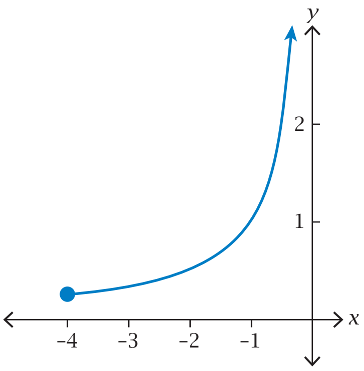
14. *(a)* $\underset{\sim}{i}$

(b) 4

(c) 2

(d) $\frac{\pi}{4}$ sec or 0.7853 sec (4 d.p.)

15. *(a)*

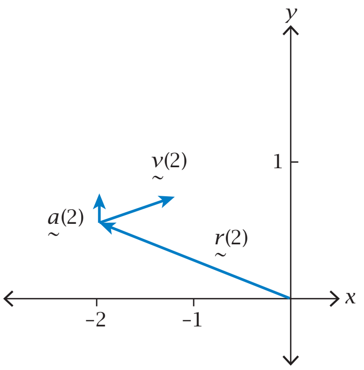


(b) *i.* $-2\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j}$

ii. $\underset{\sim}{i} + \frac{1}{4}\underset{\sim}{j}$

iii. $\frac{1}{4}\underset{\sim}{j}$

(c)



(d) $xy = -1$, hyperbola

(e) $3\underset{\sim}{i} + \frac{3}{4}\underset{\sim}{j}$, change in displacement $t = 0, 3$.

16. *(a)* 44.5 m/s

(b) $\underset{\sim}{v}(t) = 4t\,\underset{\sim}{i} + 40\,\underset{\sim}{j} + (19.6 - 9.8t)\underset{\sim}{k}$

$\underset{\sim}{r}(t) = 2t^2\,\underset{\sim}{i} + 40t\,\underset{\sim}{j} + (19.6t - 4.9t^2)\underset{\sim}{k}$

(c) 4 s

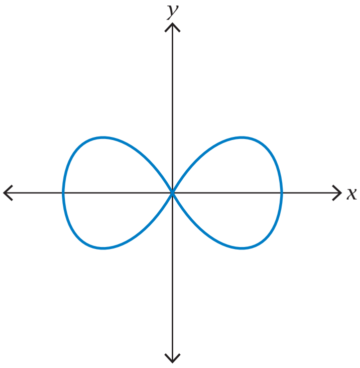
(d) 163.2 m

(e) 19.6 m

17. *(a)* $\underset{\sim}{v}(t) = 2\,\underset{\sim}{i} + (e^t - e^{-t})\,\underset{\sim}{j}$

(b) $e^t + e^{-t}$ m/s

18. *(b)*



19. *(a)* $\underset{\sim}{r}(t) = 140\,t\cos\alpha\,\underset{\sim}{i} + (140\,t\sin\alpha - 1.85\,t^2)\,\underset{\sim}{j}$

(b) 15.95°, 74.05°

(c) 200.01 m @ 10.4 s

20. *(a)* $\underset{\sim}{r}(t) = 15\,(-0.25\,e^{-2t} + t + 0.25)\underset{\sim}{i} + (10\,t^2 - 30\,t + 3.75)\,\underset{\sim}{j}$

(b) 15.37 m/s

21. *(a)* 2.2 m

(b) 13.225 m

(c) 4.3 m past target

22. $\underset{\sim}{v}(t) = \left(2\sin t - \sqrt{2}\right)\underset{\sim}{i} + \left(\frac{3\sqrt{2}}{2} - 3\cos t\right)\underset{\sim}{j}$

23. *(a)* $\underset{\sim}{i}$, $\underset{\sim}{i}$, 400 Earth days

(c) 100, 300 Earth days

(d) $\frac{\pi}{200}$ units/day

(f) $4x^2 + y^2 = 4$

24. *(a)* 0.68 m

(b) 3 m

(c) 5.44 m

(d) *i.* 1.17 m/s
ii. 5.79 m/s

25. Ball hits the ground after 1.3 seconds, clears the net by 90 cm and is 13 m long and is 3.4 m wide. The serve is good.

CHAPTER 5: *Rates of Change and Differential Equations*

1. *(a)* 10am

(b) 9hrs

(c) Let the centre of motion be at $h = 10$ m.

Let $y = 0.5\cos\left(\frac{\pi t}{9}\right)$ and show

$$\frac{d^2y}{dt^2} = -\left(\frac{\pi}{9}\right)^2 y \quad \text{i.e. S.H.M. proved}$$

2. *(a)* 11 m

(b) 2:06 am

(c) 6 hrs 17 min

(d) dropping at ≈ 93 cm/hr

3. *(a)* amp = 2 , period = $\frac{2\pi}{3}$

(b) $v = -6\sin\left(3t + \frac{\pi}{6}\right)$

$a = -18\cos\left(3t + \frac{\pi}{6}\right)$

(c) Initial position is $\sqrt{3}$ m moving in the negative x direction

(d) $t = \frac{5\pi}{9}$ sec , $v = 3$ m/s

4. 0.15 units

5. *(a)* -0.3395 m

(b) $x = \frac{3}{2}\sin 2t - 2\cos 2t + c$

$t = 0$, $x = 10$ gives $c = 12$

$x = \frac{3}{2}\sin 2t - 2\cos 2t + 12$

let $x = u + 12$

$u = \frac{3}{2}\sin 2t - 2\cos 2t$

$\frac{du}{dt} = 3\cos 2t + 4\sin 2t$

$\frac{d^2u}{dt^2} = -6\sin 2t + 8\cos 2t$

$= -4\,(1.5\sin 2t - 2\cos 2t)$

$= -4u$

$= -2^2u$ i.e. S.H.M.

about a centre of $x = 12$ m

(c) max height is when $v = 0$

i.e. $0 = 3\cos 2t + 4\sin 2t$

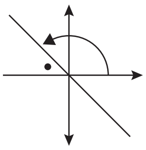
gives $\tan 2t = -0.75$

Reference angle = $\tan^{-1} 0.75$

$= 0.6435$

$2t = \pi - 0.6435$

$t = 1.249$ sec



6. *(a)* P = π sec

(b) $a_{\max} = 12$ ms^{-2}

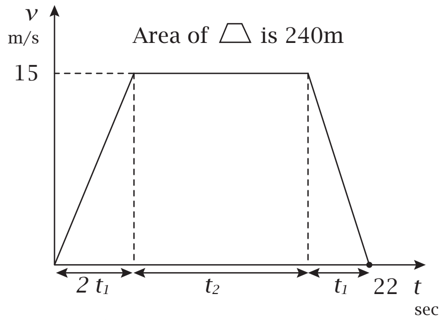
(c) speed = $\sqrt{20}$ m/s

7. 12.39 am

8. (a) Easy

(b) $3.75\,\pi$ cm/sec

9. (a)



(b) $240 = \frac{1}{2} \times 15\,(t_2 + 22)$

$t_2 = 10$

$2\,t_1 + t_2 + t_1 = 22$

$t_1 = 4 \quad 2t_1 = 8$

It takes 8 sec to get to top speed

(c) 150 m

10. (a) 40.98 sec

(b) 1679 m

11. $v = 4.42$ m/s 2dp

12. $\frac{y - 6}{3y^2 - x}$

13. 0.0077 m

14. (a) you did it!

(b) 198 seconds after the car first passed the bike

(c) 4.4 km

15. (a) Initial speed 1 unit/sec moving in negative direction

(b) No

(c) A long way in the negative x direction

16. (a) $x = 3\ln\left(1 + \frac{t^2}{3}\right)$

(b) $a = \frac{3}{4}$

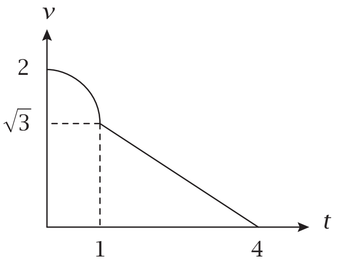
(c) $v = \frac{6t}{3 + t^2}$ is always > 0 for $t > 0$ so velocity is always positive and the motion is always in the positive x direction.

(d) $x\,(1) = 0.863$

(e) at $t = 1.528$ 3dp

17. (a) $\frac{\sqrt{3}}{2} + \frac{\pi}{3}$

(b) i.



ii. $2\sqrt{3} + \frac{\pi}{3}$

iii. 7.95 sec 2dp

18. (a) $2x\ln(\sin x) + \frac{x^2}{\tan x}$

(b) $2^{x^2}2x\ln 2 - \sin x\, e^{\cos x}$

19. $\frac{dy}{dx} = \frac{6xy - 2y^2}{4xy - 3x^2}$

when $x = 2 \quad y = 4$ or -1

for $(2, 4)$, $\frac{dy}{dx} = \frac{4}{5}$

for $(2, -1)$, $\frac{dy}{dx} = \frac{7}{10}$

20. (a) $\frac{-2\sin 2x\,(6x+3)^2 - (\cos 2x)(2)(6x+3)(6)}{(6x+3)^4}$

(b) $\frac{4}{\ln 8}$

(c) $4\sin^2 4x$

21. (a) $(1, 0)$ and $(1, -2)$

(b) differentiate w.r.t. x gives

$$3x^2 + 2y^2 + 4xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

sub $(1, -2)$ gives $\frac{dy}{dx} = -\frac{11}{4}$

Tangent equation is $y + 2 = -\frac{11}{4}(x - 1)$

$\frac{dy}{dx} = \frac{-3x^2 - 2y^2}{4xy + 3y^2}$, $\frac{dx}{dy} = \frac{4xy + 3y^2}{-3x^2 - 2y^2}$

at $(1, 0)$, $\frac{dy}{dx}$ is not defined

at $(1, 0)$, $\frac{dx}{dy} = 0$

∴ at $(1, 0)$ the tangent is parallel to the y axis.

Tangent equation is $x = 1$

22. $4\,(1 - \ln x)\,x^{\frac{4}{x} - 2}$ fully simplified

23. (a) $-\sin x\cos(\cos x)$

(b) $-\frac{3x^2 + y}{x + 2y}$

(c) $\dfrac{1 - \ln(x+1)}{(x+1)^2}$

24. derivative is $x\,e^x(x+2)$

integral is $4x^2e^x + c$

25. $-\dfrac{y}{x+y}$

26. $f^{-1}(x) = \ln 2x + 1$

27. *(a)* $T'(x) = e^{-0.2x}(1 - 0.2x)$

$e^{-0.2x} > 0$ for all x

$1 - 0.2x > 0$ for $x < 5$

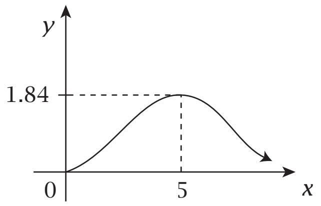
$\therefore\ T'(x) > 0$ for $0 < x < 5$

$\therefore\ T(x)$ is increasing for $0 < x < 5$

(b) $x = 5$, i.e. 12 am

Max T 38.84°C

(c)



28. $3y + x\ln 3 = 4\ln 3$

29. 7.07

30. 15.71 m^3

31. A decrease of ≈ 1%

32. 64 πcm^3

33. *(a)* $\dfrac{\pi rh}{3}$

(b) 0.02 V

34. ≈ 3%

35. 4.5%

36. Graph A: $y = e^x$
Graph B: $y = \ln|x|$

37. A = 5, B = 3, C = 4, D = 6, E = 1, F = 2

38. Let $\dfrac{k}{2} = \dfrac{k}{1 + \left(\dfrac{k}{P_0} - 1\right)e^{-rkt}}$

$$2 = 1 + \left(\frac{k}{P_0} - 1\right)e^{-rkt}$$

$$1 = \left(\frac{k}{P_0} - 1\right)e^{-rkt}$$

$$e^{rkt} = \frac{k}{P_0} - 1$$

$$\ln|\,e^{rkt}| = \ln|\frac{k}{P_0} - 1|$$

$$rkt = \ln|\frac{k}{P_0} - 1|$$

$$t = \frac{\ln|\frac{k}{P_0} - 1|}{rk} \text{ as required}$$

39. *(a)* *i.* 250

ii. 0.0025

(b) $P(t) = \dfrac{250}{1 + 2.47e^{-0.625t}}$

(c) $P(15) = 249.95 \approx 250$ termites

40. *(a)* $L = 570$, $b = 189$, $k = 0.835$ (3 d.p.)

$P = \dfrac{570}{1 + 189e^{-0.835t}}$

(b) t = 8.91 years ≈ 8 years 11 months

CHAPTER 6: *Integration and Applications of Integration*

1. Your answer should be: $f'(x) = -3\sin 3x$

2. $$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2ax = 2\cos^2 ax - 1$$

$$\cos^2 ax = \frac{\cos 2ax + 1}{2}$$

$$\int \cos^2 ax\,dx$$

$$= \frac{1}{2}\int (\cos 2ax + 1)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin 2ax}{2a} + x\right]$$

$$= \frac{\sin 2ax}{4a} + \frac{x}{2} + c$$

3.

x	$\frac{x}{\sin x}$
1	1.188395
0.1	1.001669
0.01	1.000017
0.001	1.0000002
–1	1.188395
–0.1	1.001669
–0.01	1.000017
–0.001	1.0000002

$$\therefore \lim_{x \to 0} \frac{x}{\sin x} = 1$$

4. (a) $\lim_{x \to 0} \frac{\tan x}{x}$

$$= \lim_{x \to 0} \frac{\sin x}{x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 1$$

(b) = 1 similar proof to above

5. $\frac{\sin^2 x}{x} + c_1 = \frac{-\cos 2x}{4} + c_2$

$$\frac{\sin^2 x}{2} + c_1 = \frac{-[1 - 2\sin^2 x]}{4} + c_2$$

$$\frac{\sin^2 x}{2} + c_1 = \frac{-1}{4} + \frac{\sin^2 x}{2} + c_2$$

$$c_1 = c_2 - \frac{1}{4}$$

6. $\frac{1}{2}\text{OQ} \cdot \text{QR} > \frac{1}{2}\text{OQ}^2\, x > \frac{1}{2}\text{OQ} \cdot \text{AB} \times 2$ and $\text{OQ} = 1$

gives $\text{QR} > x > \text{AB}$

$\tan x > x > \sin x$ and now proceed as on p.125

7. $\lim_{t+1 \to 0} \frac{\sin(t+1)}{1+t} \cdot \frac{1}{1-t} = \frac{1}{2}$

8. Use the substitution to first get

$$36 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \, d\,\theta.$$

Now use $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to get the final answer of $6\pi + \frac{9\sqrt{3}}{2}$.

9. Let $u = 2 + \cos^2 x$, then show that

$$dx = -\frac{du}{\sin 2x}.$$

Final answer is $\frac{1}{2(2 + \cos^2 x)} + c$.

10. $y = \tan^3(2\,f(x))$

$$\frac{dy}{dx} = 3\tan^2(2\,f(x)) \cdot \frac{1}{\cos^2(2\,f(x))} \cdot 2\,f'(x)$$

11. Split the function up as $\frac{x}{\sin x} + \frac{1 - \cos x}{\sin x}$ and process separately. The second limit is found by multiplying top and bottom by $1 + \cos x$. Final answer is 1

12. $f(x) = 5\cos^3(4x^2 + 3)$

$$f'(x) = 15\cos^2(4x^2 + 3) \cdot -\sin(4x^2 + 3) \cdot 8x$$

$$= -120\sin(4x^2 + 3)\cos^2(4x^2 + 3)$$

13. (a) $\frac{1}{2}$

(b) $\frac{2\sqrt{3}}{3}$

14. 1

15. $\lim_{\theta^\circ \to 0} \frac{\sin \theta^\circ}{\theta^\circ}$ let $\theta^\circ = \frac{180}{\pi} x^R$

$$= \lim_{x^R \to 0} \frac{\sin x^R}{\frac{180^\circ}{\pi} x^R}$$

$$= \frac{\pi}{180^\circ} \lim_{x^R \to 0} \frac{\sin x^R}{x^R}$$

$$= \frac{\pi}{180^\circ} \cdot 1$$

$$= \frac{\pi}{180^\circ} \text{ as required}$$

16. (a) rewrite the function as

$$\frac{\sin 2x}{2x} \cdot \frac{7x}{\sin 7x} \cdot \frac{2}{7}$$

(b) substitute $\sin^2 x$ for $1 - \cos^2 x$ and factorise out $\sin^2 x$ before cancelling. Now rewrite as

$$\frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \text{ etc.}$$

17. If $h = \frac{1}{x}$ then $x = \frac{1}{h}$ and as $x \to \infty$, $h \to 0$.

So the limit becomes $\lim_{h \to 0} \frac{\sin h}{h} = 1$

18. (a) 0

(b) 2

19. $y = \sin x - \frac{\cos 2x}{4} + 2x - \pi$

20. (a) –6

(b) does not exist

21. 2

22. (a) $\frac{-6\sin\sqrt{x}\cos^2\sqrt{x}}{\sqrt{x}}$

(b) $\frac{3x^4}{2} + 7\tan x + c$

23. *(a)* 2

(b) $\frac{2\pi - 3\sqrt{3}}{6}$

(c) $-\frac{9}{2}\cos^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9 - x^2} + c$

24. $80\sqrt{3}$ units

25. *(a)* $e^{\cos x}(\cos x - \sin^2 x)$

(b) $\frac{8\tan^3 2x}{\cos^2 2x}$

(c) $\frac{4\cos 4x}{\ln 5}$

26. *(a)* $\frac{\pi}{24}$

(b) $\frac{47}{480}$

27. $F\left(\frac{x}{b}\right) = \int_1^{\frac{x}{b}} \frac{1}{t}\,dt$

$= \int_1^{x} \frac{1}{t}\,dt + \int_x^{\frac{x}{b}} \frac{1}{t}\,dt$

For $\int_x^{\frac{x}{b}} \frac{1}{t}\,dt$ let $u = \frac{x}{t}$

$= \int_1^{b} -\frac{du}{u}$ $\qquad u = xt^{-1}$

$= -\int_1^{b} \frac{du}{u}$ $\qquad \frac{du}{dt} = -\frac{x}{t^2}$

$= -\frac{1}{t} \cdot \frac{x}{t}$

$\frac{du}{dt} = -\frac{1}{t} \cdot u$

$-\frac{du}{u} = \frac{dt}{t}$

$t = x\ ,\ u = 1$

$t = \frac{x}{b}\ ,\ u = b$

$\therefore\ F\left(\frac{x}{b}\right) = \int_1^{x} \frac{1}{t}\,dt + -\int_1^{b} \frac{du}{u}$

$F\left(\frac{x}{b}\right) = F(x) - F(b)$ proved

28. $\frac{d}{dx}(x\ln x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$= \ln x + 1$

$\therefore \int (\ln x + 1)\,dx = x\ln x + c$

$\int \ln x\,dx + \int 1\,dx = x\ln x + c$

If $\int \ln x\,dx + x = x\ln x$

then $\int \ln x\,dx = x\ln x - x + c$

29. $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1 - \ln x}{x^2}$ gives after a few lines

$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x} - \frac{\ln x}{x} + c$

30. *(a)* $\frac{d}{dx}\left(e^{\ln 7}\right)^x$

$= \frac{d}{dx}\ e^{x\ln 7}$

$= (\ln 7)\ e^{x\ln 7}$

$= (\ln 7)\ 7^x$

(b) $\frac{7^x}{\ln 7} + c$

31. *(a)* $\frac{1}{4}\int \frac{12x + 4}{6x^2 + 4x - 7}\,dx$

$= \frac{1}{4}\ln|6x^2 + 4x - 7| + c$

(b) $\frac{1}{2}\int e^{x^2 + x}(2x + 1)\,dx$

$= \frac{1}{2}\,e^{x^2 + x} + c$

(c) let $\sin^2 x = 1 - \cos^2 x$ and expand gives

$\int (\tan x + \tan x\cos^2 x)\,dx$, let $\tan x = \frac{\sin x}{\cos x}$

$= \int \tan x\,dx + \int \sin x\cos x\,dx$

$= -\ln|\cos x| + \frac{1}{2}\int \sin 2x\,dx$

$= -\ln|\cos x| - \frac{1}{4}\cos 2x + c$

(d) let $u = \ln x$

integral $= \int \sin u\,du$

$= -\cos u + c$

$= -\cos(\ln x) + c$

(e) $-\sin x\,e^{\cos x}$

(f) $\frac{d}{dx}\left(e^{\ln x \cdot \cos x}\right)$

$= \left(\frac{1}{x}\cos x - \ln x\sin x\right) e^{\ln x \cdot \cos x}$

$= \left(\frac{1}{x}\cos x - \ln x\sin x\right) x^{\cos x}$

32. $-6\,e^{\frac{1}{x}} + c$

33. $e^{\tan x} + c$

34. *(a)* $-2e^{3t}(1 + 3t)$

(b) $\frac{2}{3\ln 3}$

35. $\frac{(1 + \ln x)^3}{3} + c$ or $\ln x + (\ln x)^2 + \frac{(\ln x)^3}{3} + c$

36. (a) $\ln |1 - e^{-x}| + c$

(b) $\ln (1 + e^x) + c$

37. $2x\, e^{2x}$

38. (a) $-\frac{1}{2(\ln x)^2} + c$

(b) $\tan \theta + c$

39. (a) $x + 2 + \frac{4}{x - 2}$

(b) $\frac{x^2}{2} + 2x + 4 \ln | x - 2 | + c$

(c) $\frac{(x - 2)^2}{2} + 4\,(x - 2) + 4 \ln | x - 2 | + d$

Algebraically the answers are
the same because $c = d - 6$

40. (a) $A = \int_2^4 \frac{x^2}{4}\, dx + \int_4^6 (12 - 2x)\, dx$

$= 8\frac{2}{3}$ units^2

(b) $A = \int_0^4 \left(6 - \frac{1}{2}y\right) dy - \int_1^4 2y^{\frac{1}{2}}\, dy - \int_0^1 2dy$

$= 8\frac{2}{3}$ units^2 as before

41. (a) $\frac{1}{1 + e^x}$

(b) $x - \ln (1 + e^x) + c$

42. (a) $5 \ln | x | + \frac{2 \cos 3x}{3} + c$

(b) $\frac{x}{2} - \frac{1}{4} \sin 2x + c$

43. (a) $(\cos \theta)^6 \geq 0$ since the power is even

$\sin \theta \geq$ for $0 \leq \theta \leq \pi$

$\therefore (\sin \theta)^3 \geq 0$ for $0 \leq \theta \leq \pi$

$\therefore \cos^6 \theta \sin^3 \theta \geq 0$ for $0 \leq \theta \leq \pi$

(b) $\frac{4}{63}$ units^2

44. Derivative is $x\, e^x (x + 2)$

Integral is $4x^2 e^x + c$

45. (a) $\frac{1}{2} e^{x^2} + c$

(b) $-\frac{1}{6} \cos^3 2x + c$

46. Let $u = \sqrt{x}$ ans = $-2 \ln (\ln 2)$

47. $\frac{3}{\pi}$

48. $2x - 3$

49. $y = \ln x$

$x = e^y$

$$\begin{aligned} \text{Area} &= a \ln a - \int_0^{\ln a} x\, dy \\ &= a \ln a - \int_0^{\ln a} e^y\, dy \\ &= a \ln a - [\,e^y]_0^{\ln a} \\ &= a \ln a - (e^{\ln a} - e^0) \\ &= a \ln a - a + 1 \end{aligned}$$

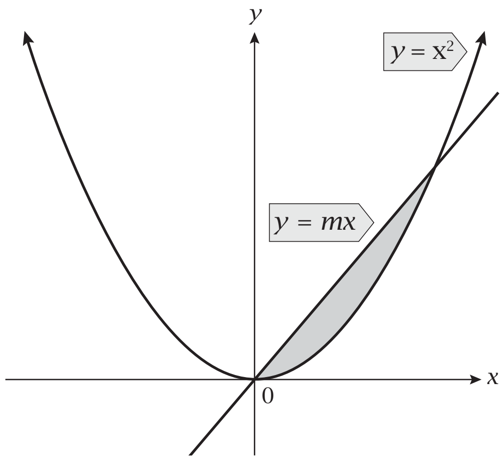
50. Volume $= \pi \int_0^3 (e^x + 1)^2\, dx$

≈ 761.476 units^3

51.
$$\begin{aligned} v &= \pi \int_0^1 (e^{2x-2})\, dx + \frac{\pi}{3} \\ &= \frac{\pi}{2} [e^{2x-2}]_0^1 + \frac{\pi}{3} \\ &= \frac{\pi}{2} [1 - e^{-2}] + \frac{\pi}{3} \\ &= \frac{\pi}{6} \left[3 - \frac{3}{e^2} + 2\right] \\ &= \frac{\pi}{6} \left[\frac{5e^2 - 3}{e^2}\right] \end{aligned}$$

52.
$$\begin{aligned} V_x &= \pi \int_1^k y^2\, dx \\ 9\pi &= \pi \int_1^k (4x^3 - 4x)\, dx \\ [x^4 - 2x^2]_1^k &= 9 \\ (k^4 - 2k^2) - (1 - 2) &= 9 \\ k^4 - 2k^2 - 8 &= 0 \\ (k^2 - 4)(k^2 + 2) &= 0 \\ \therefore k^2 - 4 &= 0 \\ \therefore k &= 2 \end{aligned}$$

53.



$$y = x^2 \text{ and } y = mx$$

$$\therefore x^2 - mx = 0$$

$$x(x - m) = 0$$

$$\therefore x = 0, x = m \quad \textit{(points of intersection)}$$

$$\int_0^m (mx - x^2)\,dx = 24.813$$

$$\left[\frac{mx^2}{2} - \frac{x^3}{3}\right]_0^m = 24.813$$

$$\frac{m^3}{2} - \frac{m^3}{3} = 24.813$$

$$m^3 = 148.878$$

$$m = 5.3$$

54. (a) $\int_0^1 \frac{1}{2\sqrt{x}}\,dx = \left[\sqrt{x}\right]_0^1 = 1 \quad \therefore$ pdf

(b) $\int_0^1 2\,x\,dx = \left[x^2\right]_0^1 = 1 \quad \therefore$ pdf

(c) $\int_1^2 \frac{1}{x\ln 2}\,dx = \left[\frac{\ln x}{\ln 2}\right]_1^2 = \frac{\ln 2}{\ln 2} - 0 = 1 \quad \therefore$ pdf

55. (a) $\frac{\pi}{4}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$

(d) $\frac{3\pi}{2}$

(e) $\frac{\pi}{4}$

(f) $\frac{3\pi}{4}$

(g) $\frac{\pi}{4}$

(h) $\frac{\pi}{2}$

(i) $\frac{3\pi}{2}$

(j) $\frac{\pi}{3}$

(k) 0

(l) 1

56. Proof

57. (a) $\frac{2}{\sqrt{-4x^2 + 4x}}$

(b) $\frac{-2x}{\sqrt{1 - x^4}}$

(c) $\frac{2\tan^{-1} x}{1 + x^2}$

(d) $\frac{-3e^{3x}}{\sqrt{1 - e^{6x}}}$

(e) $4x\sin^{-1}(3x) + \frac{6x^2}{\sqrt{1 - 9x^2}}$

(f) $\frac{\arccos x}{x} - \frac{\ln 2x}{\sqrt{1 - x^2}}$

(g) $\frac{\sec^2 x}{\sqrt{1 - \tan^2 x}}$

(h) $\frac{-1}{\left[\sin^{-1} x\right]^2 \sqrt{1 - x^2}}$

58. $\frac{1}{2e}$

59. $\frac{dy}{dx} = 2x\cos\left(x^2 - 1\right) - 1$

60. $y = \left(\frac{\pi}{4} - 1\right)x + \frac{\sqrt{2}(5 - \pi)}{8}$

61. (a) $-\frac{1}{5}x\cos(5x) + \frac{1}{25}\sin(5x) + c$

(b) $\frac{2x}{3}(x + 1)^{\frac{3}{2}} - \frac{4}{15}(x + 1)^{\frac{5}{2}} + c$

(c) $x\sin x + \cos x + c$

(d) $x\ln x - x + c$

(e) $-e^{-x}(x^2 + 2x + 2) + c$

(f) $\frac{1}{2}(x - \sin x\cos x) + c$

(g) $\frac{1}{3}x^2\sin(3x) + \frac{2}{9}x\cos(3x) - \frac{2}{27}\sin(3x) + c$

(h) $x(\ln x)^2 - 2x\ln x + 2x + c$

62. $\frac{1}{2}x^2\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + c$

63. (a) $\sin^{-1}\left(\frac{x}{8}\right) + c$

(b) $\cos^{-1}\left(\frac{x}{3}\right) + c$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

(d) $3\sin^{-1}\left(\frac{x}{4}\right) + c$

(e) $\frac{1}{10}\tan^{-1}\left(\frac{4x}{5}\right) + c$

(f) $5\cos^{-1}\left(\frac{x}{2}\right) + c$

(g) $\frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x}{\sqrt{5}}\right) + c$

(h) $3\cos^{-1}(3x) + c$

64. *(a)* Proof

(b) Proof

(c) Proof

65. *(a)* $2\ln|x-4| - \ln|x+1| + c$

(b) $\dfrac{3\ln|x+7|}{8} + \dfrac{5\ln|x-1|}{8} + c$

(c) $\ln|x-1| - \dfrac{4}{x-1} + c$

(d) $3\ln|x+2| + \dfrac{7}{x+2} + c$

(e) $\frac{5}{9}\ln|x-3| - \frac{5}{9}\ln x - \dfrac{7}{3x-9} + c$

(f) $-\dfrac{1}{2}\ln\left(x^2+3\right) - \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{x}{\sqrt{3}}\right) + 2\ln x + c$

66. *(a)* $\frac{8}{3}$ units^2

(b) $\frac{47}{12}$ units^2

(c) 4 units^2

CHAPTER 7: *Statistical Inference*

1. Sample standard deviation: $\dfrac{\sigma}{\sqrt{n}}$

$= \dfrac{50}{\sqrt{1500}}$

≈ 1.291

2. *(a)* 95% confidence interval is 91% to 99%.

(b) The results of the sample will not be the same as the population.

(c) The results are correct 95% of the time.

3. *(a)* Width $= z \cdot \dfrac{\sigma}{\sqrt{n}}$

$= 2.58 \times \dfrac{19.95}{\sqrt{106}}$

$= 4.999 \approx 5$

Yes, he was correct.

(b) Width $= z \cdot \dfrac{\sigma}{\sqrt{n}}$

$= 2.58 \times \dfrac{19.95}{\sqrt{50}}$

≈ 7.28

The confidence interval is wider.

4. *(a)* $\overline{X} \pm z\dfrac{\sigma}{\sqrt{n}}$

$150 \pm 1.96\dfrac{(40)}{\sqrt{2000}}$

150 ± 1.753

(b) $\overline{X} \pm z\dfrac{\sigma}{\sqrt{n}}$

$150 \pm 1.96\dfrac{(40)}{\sqrt{200}}$

150 ± 5.544

(c) $\overline{X} \pm z\dfrac{\sigma}{\sqrt{n}}$

$150 \pm 1.96\dfrac{(40)}{\sqrt{3000}}$

150 ± 1.431

(d) The margins of error are:
$\pm 1.753,\ \pm 5.544,\ \pm 1.431$

These decrease as the sample size n increases.

5. Sample size $n = \left(\dfrac{z\sigma}{w}\right)^2$

$n = \left(\dfrac{1.96(0.004)}{0.002}\right)^2$

$n \approx 15.366$

Sample size required is 16.

6. *(a)* $CI = \overline{X} \pm z \cdot \dfrac{\sigma}{\sqrt{n}}$

$= 63.3 \pm 2.576\dfrac{(12)}{\sqrt{30}}$

$\approx 63.3 \pm 5.644$

(b) If the results are from one school only this does not represent a random sample and hence the confidence interval may not be valid.

7. Confidence level: 95%

$CI = \overline{X} \pm z\dfrac{\sigma}{\sqrt{n}}$

$= 0.927 \pm 1.96\dfrac{(0.05)}{\sqrt{10}}$

$\approx 0.927 \pm 0.031$

Interval is 0.896 to 0.958

Confidence level: 99%

$CI = \overline{X} \pm z\dfrac{\sigma}{\sqrt{n}}$

$= 0.927 \pm 2.576\dfrac{(0.05)}{\sqrt{10}}$

$\approx 0.927 \pm 0.041$

Interval is 0.886 to 0.968

When the confidence level is increased, the confidence interval widens.

8. (a) $CI = \overline{X} \pm z\frac{\sigma}{\sqrt{n}}$

$$= 4212 + 1.96\frac{(1004)}{\sqrt{1256}}$$

Interval: \$4156.47 to \$4267.53

Confidence interval is correct.

(b) This does not represent the view of the population. The sample is not random but rather a voluntary response.

9. $X \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

$X \sim N\left(\mu, \frac{9.5}{\sqrt{25}}\right)$

Confidence interval:

$\overline{X} \mp z\frac{\sigma}{\sqrt{n}}$

$= 126.5 \pm 1.645\frac{(9.5)}{\sqrt{25}}$

$= 126.5 \pm 3.1255$

i.e. $123.3745 \le \mu \le 129.6255$

10. The large sample indicates that the central limit theorem applies and hence the sample represents an approximate normal distribution.

$CI: \overline{X} \pm z\frac{\sigma}{\sqrt{n}}$

$= 1125 \pm 1.96\frac{(50)}{\sqrt{1400}}$

$\approx 1125 \pm 2.619$

<u>Interval:</u> $1122.381 \le \mu \le 1127.619$

11. (a) Problems associated with this survey:

- Survey only sent to independent school teachers.
- Intended population was all teachers.
- Low survey responses.

(b) $CI: \overline{X} \pm z\frac{\sigma}{\sqrt{n}}$

$= 3.25 \pm 1.96\frac{(1.4)}{\sqrt{270}}$

$\approx 3.25 \pm 0.167$

Interval: 3.083 to 3.417

(c) $CI: \overline{X} \pm z\frac{\sigma}{\sqrt{n}}$

$= 2.1 \pm 2.576\frac{(1.4)}{\sqrt{270}}$

$\approx 2.1 \pm 0.219$

Interval: 1.881 to 2.319

(d) If the sample is randomly selected then as n is large the central limit theorem applies. The use of confidence intervals based on the normal distribution is justified.

12. $X \sim$ life of the bulbs in hours

$X \sim N(2500,\ 220^2)$

(a) $P(X \ge 2620) \approx 0.2927$

(b) $P(2400 \le X \le 2650) \approx 0.4276$

(c) $$\overline{X} - z\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z\frac{\sigma}{\sqrt{n}}$$

$$2480 - 1.96 \times \frac{220}{\sqrt{81}} \le \mu \le 2480 + 1.96 \times \frac{220}{\sqrt{81}}$$

$$2432.089 \le \mu \le 2527.911$$

(d) $\overline{X} \sim N\left(2500, \left(\frac{220}{\sqrt{81}}\right)^2\right)$

$P(2499.5 \le \overline{X} \le 2500.3) \approx 0.0131$

(e) Sample size $n = \left(\frac{z \times \sigma}{w}\right)^2$

$$= \left(\frac{2.576 \times 220}{100}\right)^2$$

$$= 32.12$$

$$\approx 33$$

13. $X \sim$ weight of dhufish in kg

$X \sim N(20,\ 2.7^2)$

(a) $P(X < 19.5) \approx 0.4265$

(b) $P(X > 17 \mid X < 19.5) \approx 0.6876$

(c) $$\overline{X} - z\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z\frac{\sigma}{\sqrt{n}}$$

$$19.2 - 2.576 \times \frac{2.7}{\sqrt{32}} \le \mu \le 19.2 + 2.576 \times \frac{2.7}{\sqrt{32}}$$

$$17.970 \le \mu \le 20.430$$

(d) $\overline{X} \sim N\left(20, \left(\frac{2.7}{\sqrt{32}}\right)^2\right)$

$P(17.5 \le \overline{X} \le 20.2) \approx 0.6624$

(e) Sample size $n = \left(\frac{z \times \sigma}{w}\right)^2$

$$= \left(\frac{2.576 \times 2.7}{2}\right)^2$$

$$\approx 12.094$$

Sample size required is 13 dhufish.

SOLUTIONS TO TRIAL TESTS

TRIAL TEST 1: *Complex Numbers I*

1\. (a) zw

$= (-2 + 5i)(6 - 3i)$

$= -12 + 6i + 30i + 15$ ✓

$= 3 + 36i$ ✓

(b) $3w - 4z$

$= 3(6 - 3i) - 4(-2 + 5i)$

$= 18 - 9i + 8 - 20i$ ✓

$= 26 - 29i$ ✓

(c) $(\overline{w})^2$

$= (\overline{6 - 3i})^2$

$= (6 + 3i)^2$ ✓

$= 36 + 36i - 9$ ✓

$= 27 + 36i$ ✓

(d) $|z| = |-2 + 5i|$

$= \sqrt{(-2)^2 + 5^2}$

$= \sqrt{29}$ ✓

(e) $\dfrac{w}{z} = \dfrac{6 - 3i}{-2 + 5i} \times \dfrac{-2 - 5i}{-2 - 5i}$ ✓

$= \dfrac{-12 - 30i + 6i - 15}{4 + 25}$ ✓

$= \dfrac{-27 - 24i}{29}$ ✓

$= \dfrac{-27}{29} - \dfrac{24i}{29}$ ✓ [12]

2\.

$z^2 + 10z + 41 = 0$

$z^2 + 10z + 25 + 16 = 0$ ✓

$(z + 5)^2 = -16$ ✓

$z + 5 = \pm 4i$ ✓

$z = -5 \pm 4i$ ✓ [4]

3\. $z = 7 \pm 2i$ are the roots

$(z - (7 + 2i))(z - (7 - 2i)) = 0$ ✓

$(z - 7 - 2i)(z - 7 + 2i) = 0$ ✓

$(z - 7)^2 + 4 = 0$ ✓

$z^2 - 14z + 49 + 4 = 0$

$z^2 - 14z + 53 = 0$ ✓ [4]

4\. $|Z| = \sqrt{(-1)^2 + (-\sqrt{3})^2}$ $\qquad \tan\theta = \sqrt{3}$

$= 2$ ✓ $\qquad \theta = -\dfrac{2\pi}{3}$ ✓

$Z = 2Cis\left(-\dfrac{2\pi}{3}\right)$ ✓ [3]

5\. $Z_1Z_2 = 5Cis\dfrac{\pi}{6} \times 2Cis\dfrac{\pi}{12}$

$= 10Cis\left(\dfrac{\pi}{6} + \dfrac{\pi}{12}\right)$ ✓

$= 10Cis\dfrac{\pi}{4}$ ✓

$= 10\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ ✓

$= 10\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)$ ✓

$= \dfrac{10}{\sqrt{2}}(1 + i)$

$Z_1Z_2 = 5\sqrt{2}(1 + i)$ [4]

6\. $Z = 1 + i \qquad \bar{Z} = 1 - i$

$\left(\dfrac{Z}{\bar{Z}}\right)^2 = \left(\dfrac{1 + i}{1 - i}\right)^2$

$= \left(\dfrac{1 + i}{1 - i} \times \dfrac{1 + i}{1 + i}\right)^2$ ✓

$= \left(\dfrac{1 + i + i + i^2}{1 - i^2}\right)^2$ ✓

$= \left(\dfrac{2i}{2}\right)^2$ ✓

$= i^2$

$= -1$ [3]

7\. $Z = \dfrac{1}{2 - 3i}$

$= \dfrac{1}{2 - 3i} \times \dfrac{2 + 3i}{2 + 3i}$ ✓

$= \dfrac{2 + 3i}{13}$

$Z = \dfrac{2}{13} + \dfrac{3}{13}i$ ✓

$Z\bar{Z} = \left(\dfrac{2}{13} + \dfrac{3}{13}i\right)\left(\dfrac{2}{13} - \dfrac{3}{13}i\right)$

$= \dfrac{4}{169} - \dfrac{9}{169}i^2$ ✓

$= \dfrac{13}{169}$

$Z\bar{Z} = \dfrac{1}{13}$ ✓ [3]

8.
$$\begin{aligned}\left|W-Z\right| &= \left|\sqrt{2}-i-\left(1-i\sqrt{2}\right)\right| \quad ✓\\ &= \left|\left(\sqrt{2}-1\right)+\left(\sqrt{2}-1\right)i\right|\\ &= \sqrt{\left(\sqrt{2}-1\right)^2+\left(\sqrt{2}-1\right)^2} \quad ✓\\ &= \sqrt{2\left(\sqrt{2}-1\right)^2}\\ &= \sqrt{2}\left(\sqrt{2}-1\right) \quad ✓\end{aligned}$$

[3]

9. Let $Z = a + bi$

$$Z\bar{Z} + 2Z = \frac{1}{4} + i$$

$$(a+bi)(a-bi)+2(a+bi)=\frac{1}{4}+i$$

$$a^2+b^2+2a+2bi=\frac{1}{4}+i \quad ✓$$

Compare real, imaginary

$$a^2+b^2+2a=\frac{1}{4} \quad ①$$

$$2b = 1 \quad ②$$

$$b=\frac{1}{2} \quad ✓$$

sub $b=\frac{1}{2}$ into ①

$$a^2+\left(\frac{1}{2}\right)^2+2a=\frac{1}{4}$$

$$a^2+\frac{1}{4}+2a=\frac{1}{4}$$

$$a^2+2a=0$$

$$a(a+2)=0$$

$$a=0,-2 \quad ✓$$

$$Z=-2+\frac{i}{2} \text{ or } Z=\frac{i}{2} \quad ✓$$

[4]

10. (a)
$$|Z-i|=|Z-1|$$

$$|x+yi-i|=|x+yi-1| \quad ✓$$

$$|x+i(y-1)|=|(x-1)+yi|$$

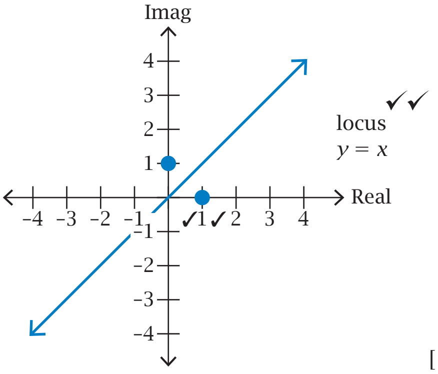
$$x^2+(y-1)^2=(x-1)^2+y^2 \quad ✓$$

$$x^2+y^2-2y+1=x^2-2x+1+y^2$$

$$2x-2y=0$$

$$x-y=0 \quad ✓$$

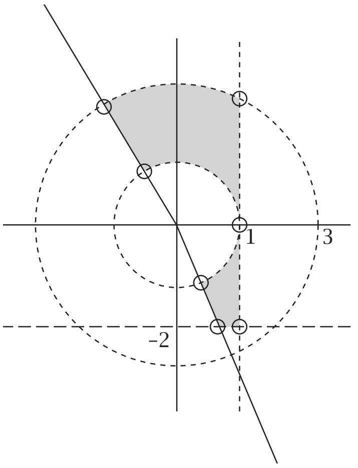
(b)



[5]

11. $Re(z) < 1$ $\quad Im(z) > -2$

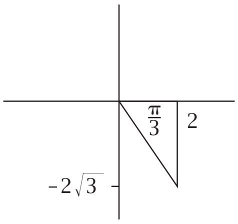
$1 < |z| < 3$ $\quad -\frac{5\pi}{12} \le Arg(z) \le \frac{2\pi}{3}$

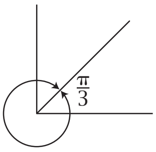


✓✓✓✓✓✓

[6]

12.
$$\begin{aligned}&\left(2-2\sqrt{3}\,i\right)^5\\ &= \left[4\,cis\left(-\frac{\pi}{3}\right)\right]^5\\ &= 4^5\,cis\left(-\frac{5\pi}{3}\right) \quad ✓✓✓✓\\ &= 1024\,cis\left(\frac{\pi}{3}\right)\end{aligned}$$





$$\sqrt{2^2+4(3)}=\sqrt{16}$$
$$=4$$

[4]

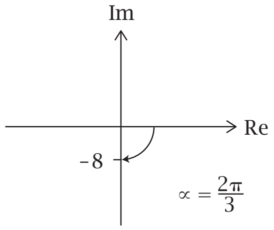
13. (a)
$$z^3=-8i$$

$$z^3=8\,cis\left(-\frac{\pi}{2}\right)$$

$$z_1=2\,cis\left(-\frac{\pi}{6}\right) \quad ✓✓✓$$

$$z_2=2\,cis\left(\frac{\pi}{2}\right)$$

$$z_3=2\,cis\left(-\frac{5\pi}{6}\right)$$

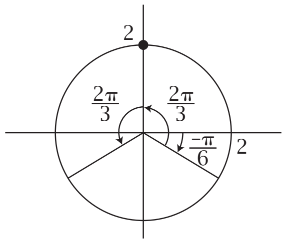


$$-\frac{1}{6}+\frac{4}{6}=\frac{3}{6}$$

$$-\frac{1}{6}+\frac{4}{6}=-\frac{5}{6}$$

[5]

(b) ✓✓



TRIAL TEST 2: *Complex Numbers II*

1. $z\bar{z} + z + \bar{z} = 24$, let $z = x + yi$
$\bar{z} = x - yi$

$(x + yi)(x - yi) + x + yi + x - yi = 24$ ✓

$x^2 + y^2 + 2x = 24$ ✓

$x^2 + 2x + 1 + y^2 = 24 + 1$ ✓

$(x + 1)^2 + y^2 = 25$ ✓

The locus is a circle of radius 5 units with centre $z = -1 + 0i$. ✓ [5]

2. $|a + z|^2 = a^2 + a(z + \bar{z}) + |z|^2$

let $z = x + yi$

L.H.S. $= |a + z|^2$

$= |a + x + yi|^2$

$= (a + x)^2 + y^2$ ✓

R.H.S. $= a^2 + a(z + \bar{z}) + |z|^2$

$= a^2 + a(x + yi + x - yi) + x^2 + y^2$ ✓

$= a^2 + 2ax + x^2 + y^2$ ✓

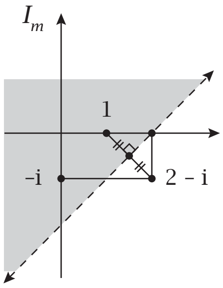
$= (a + x)^2 + y^2$ ✓

= L.H.S. proved [4]

3. (a) $|1 - z| < |z + i - 2|$

$|z - 1| < |z - (2 - i)|$ ✓

✓



Test $z = 0 + 0i$ True ✓

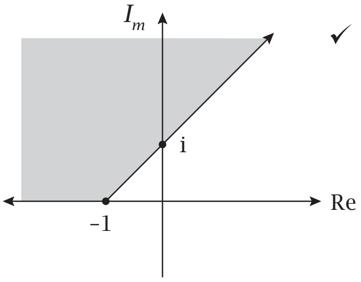
Required region is shaded

(b) The required locus is the locus of

$\text{Arg } z \geq \frac{\pi}{4}$ ✓

shifted one unit to the left

✓



Required region is shaded [5]

4. $\frac{z - i}{z + i} = 1 + i$

$z - i = (z + i)(1 + i)$

$z - i = z + zi + i - 1$ ✓

$1 - 2i = zi$

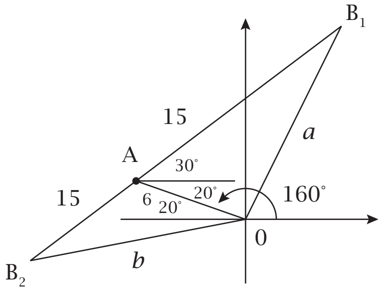
$z = \frac{1 - 2i}{i}$ ✓

$z = \frac{(1 - 2i) \times i}{i \times i}$

$= \frac{i + 2}{-1}$

$= -2 - i$ ✓ [3]

5.



✓✓

$a^2 = 6^2 + 15^2 - 2 \times 6 \times 15 \times \cos 50^\circ$ ✓

$a = 12.054$ cm 3dp ✓

$\cos \angle AOB_1 = \frac{6^2 + a^2 - 15^2}{2 \times 6 \times a}$ ✓

$\therefore \angle AOB_1 = 107.585^\circ$ ✓

$160^\circ - 107.585^\circ$

$= 52.415^\circ$

$\therefore B_1$ is $(12.054, 52.415^\circ)$ 3dp ✓

$\angle OAB_2 = 180^\circ - 50^\circ$
$= 130^\circ$

$b^2 = 6^2 + 15^2 - 2 \times 6 \times 15 \times \cos 130^\circ$ ✓

$b = 19.409$ cm 3dp ✓

$\cos \angle AOB_2 = \frac{6^2 + b^2 - 15^2}{2 \times 6 \times b}$ ✓

$\therefore \angle AOB_2 = 36.301^\circ$

and $360^\circ - 160^\circ - 36.301^\circ$

$= 163.699^\circ$ ✓

$\therefore B_2$ is $(19.409, -163.699^\circ)$ 3dp [11]

6. L.H.S. $= \frac{z - z^{-1}}{i\,(z + z^{-1})}$

$= \frac{\text{cis } \theta - \text{cis } (-\theta)}{i\,(\text{cis } \theta + \text{cis } (-\theta))}$ ✓

$= \frac{\cos \theta + i \sin \theta - \cos (-\theta) - i \sin (-\theta)}{i\,(\cos \theta + i \sin \theta + \cos (-\theta) + i \sin (-\theta))}$ ✓

$= \frac{\cos \theta + i \sin \theta - \cos \theta + i \sin \theta}{i\,(\cos \theta + i \sin \theta + \cos \theta - i \sin \theta)}$ ✓

$= \frac{2i \sin \theta}{2i \cos \theta}$ ✓

$= \tan \theta =$ R.H.S. [4]

7. (a) $|z| = \sqrt{2}\ \checkmark$

(b)
$$\begin{aligned} w &= 2\text{ cis }\frac{\pi}{6} \\ &= 2\cos\frac{\pi}{6} + 2\text{i}\sin\frac{\pi}{6}\ \checkmark \\ &= 2\cdot\frac{\sqrt{3}}{2} + 2\text{i}\cdot\frac{1}{2} \\ &= \sqrt{3} + \text{i}\ \checkmark \end{aligned}$$

(c)
$$\begin{aligned} \overline{z} &= 1 + \text{i} = \sqrt{2}\text{ cis }\frac{\pi}{4}\ \checkmark \\ \overline{z}w &= \sqrt{2}\text{ cis }\frac{\pi}{4}\cdot 2\text{ cis }\frac{\pi}{6} \\ &= \overset{\checkmark}{2\sqrt{2}}\ \text{ cis}\left(\frac{5\pi}{12}\right)\ \checkmark \end{aligned}$$

(d)
$$\begin{aligned} z^2 &= \left(\sqrt{2}\text{ cis}\left(\frac{-\pi}{4}\right)\right)^2 \\ &= 2\text{ cis}\left(\frac{-\pi}{2}\right)\ \checkmark \\ \frac{w}{z^2} &= \frac{2\text{ cis }(\frac{\pi}{6})}{2\text{ cis }(\frac{-\pi}{2})} \\ &= \text{cis}\left(\frac{2\pi}{3}\right)\ \checkmark \end{aligned}$$

(e)
$$\begin{aligned} z^5 &= \left(\sqrt{2}\text{ cis}\left(\frac{-\pi}{4}\right)\right)^5\ \checkmark \\ &= (\sqrt{2})^5\text{ cis}\left(\frac{-5\pi}{4}\right) \\ &= 4\sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)\ \checkmark \end{aligned}$$

[10]

8. (a)
$$\begin{aligned} A &= 4\ \checkmark \\ \frac{2\pi}{4} &= \frac{\pi}{2} \\ \frac{2\pi}{3} - \frac{\pi}{2} &= \frac{\pi}{6} \\ \frac{\pi}{6} - \frac{\pi}{2} &= -\frac{\pi}{3}\ \checkmark \\ -\frac{\pi}{3} - \frac{\pi}{2} &= -\frac{5\pi}{6} \\ \therefore\ \theta &= \frac{\pi}{6},\ \frac{2\pi}{3},\ -\frac{\pi}{3},\ -\frac{5\pi}{6}\ \checkmark \end{aligned}$$

(b)
$$\begin{aligned} z &= \left(4\text{ cis }\frac{\pi}{6}\right)^4 \\ &= 4^4\text{ cis}\left(\frac{4\pi}{6}\right)\ \checkmark \\ &= 256\text{ cis}\left(\frac{2\pi}{3}\right)\ \checkmark \end{aligned}$$

[5]

9. $(1 + \text{i})^5 - (1 - \text{i})^5$

$$\begin{aligned} &(\sqrt{2}\text{ cis }45^\circ)^5 - (\sqrt{2}\text{ cis }(-45^\circ))^5\ \checkmark \\ &= 2^{\frac{5}{2}}\text{ cis }225^\circ - 2^{\frac{5}{2}}\text{ cis }(-225^\circ) \\ &= 2^{\frac{5}{2}}\ (\text{cis }225^\circ - \text{cis }(-225^\circ))\ \checkmark \\ &= 2^{\frac{5}{2}}(\cos 225^\circ + \text{i}\sin 225^\circ \\ &\qquad - (\cos(-225^\circ) + \text{i}\sin(-225^\circ)))\ \checkmark \\ &= 2^{\frac{5}{2}}(\cos 225^\circ + \text{i}\sin 225^\circ \\ &\qquad - \cos 225^\circ + \text{i}\sin 225^\circ)\ \checkmark \\ &= 2^{\frac{5}{2}}(2\text{ i}\sin 225^\circ) \\ &= 2^{\frac{7}{2}}\cdot\text{i}\left(-\frac{1}{\sqrt{2}}\right)\ \checkmark \\ &= -8\text{i} \end{aligned}$$

[5]

10. (a)
$$\begin{aligned} w^3 &= -2 + 2\text{i} \\ &= \sqrt{8}\text{ cis }135^\circ \\ &= 2^{\frac{3}{2}}\text{ cis }135^\circ\ \checkmark \\ w &= \left(2^{\frac{3}{2}}\text{ cis }135^\circ\right)^{\frac{1}{3}} \\ &= 2^{\frac{1}{2}}\text{ cis }45^\circ\ \checkmark \\ &= 1 + \text{i}\ \checkmark \end{aligned}$$

(b)
$$\begin{aligned} (z - 1)^3 &= -2(z + 1)^3 + 2\text{i}\,(z + 1)^3 \\ &= (z + 1)^3\,(-2 + 2\text{i})\ \checkmark \\ \frac{(z - 1)^3}{(z + 1)^3} &= -2 + 2\text{i} \\ \left(\frac{z - 1}{z + 1}\right)^3 &= -2 + 2\text{i}\ \checkmark \\ \therefore\ \frac{z - 1}{z + 1} &= \overset{\checkmark}{1 + \text{i}}\ \text{ as }\ (1 + \text{i})^3 = -2 + 2\text{i} \\ z - 1 &= (1 + \text{i})(z + 1) \\ z - 1 &= z + 1 + \text{i}z + \text{i}\ \checkmark \\ -2 - \text{i} &= \text{i}z \\ z &= \frac{-2 - \text{i}}{\text{i}} \times \frac{\text{i}}{\text{i}} \\ &= \frac{-2\text{i} + 1}{-1} \\ z &= -1 + 2\text{i}\ \checkmark \end{aligned}$$

[8]

TRIAL TEST 3: *Functions and Curve Sketching*

1\. *(a)* $g(x) = x^2 - x + 1$

$f(g(x)) = 2x^2 - 2x + 1$

$= 2(x^2 - x + 1) - 1$ ✓

$\therefore\ f(x) = 2x - 1$ ✓

or let $y = g(x)$

$f(y) = 2x^2 - 2x + 1$ ①

$y = x^2 - x + 1$

$2y = 2x^2 - 2x + 2$

$2y - 2 = 2x^2 - 2x$ ②

Substitute ② into ① for $2x^2 - 2x$

$f(y) = 2y - 2 + 1$

$= 2y - 1$

$\therefore\ f(x) = 2x - 1$ as before

(b) $g(f(x)) = x^2 - 7x + 13$

$g(x) = x^2 - x + 1$

let $y = f(x)$

$g(y) = x^2 - 7x + 13$ ✓

$g(y) = y^2 - y + 1$

$\therefore\ y^2 - y + 1 = x^2 - 7x + 13$

$y^2 - y + \frac{1}{4} + \frac{3}{4} = x^2 - 7x + (\frac{7}{2})^2 - (\frac{7}{2})^2 + 13$

$(y - \frac{1}{2})^2 + \frac{3}{4} = (x - \frac{7}{2})^2 + \frac{3}{4}$ ✓✓

$(y - \frac{1}{2})^2 = (x - \frac{7}{2})^2$ ✓

$y - \frac{1}{2} = \pm x - \frac{7}{2}$

$y - \frac{1}{2} = x - \frac{7}{2}$ or $y - \frac{1}{2} = -x + \frac{7}{2}$

$\therefore\ f(x) = x - 3$ ✓ or $f(x) = -x + 4$ ✓

[8]

2\. *(a)* $y = |4x + 3| - |x - 5|$

for $x \geq 5$,
$y = 4x + 3 - x + 5 = 3x + 8$ ✓

for $-\frac{3}{4} \leq x < 5$, ✓
$y = 4x + 3 - -(x - 5)$
$= 5x - 2$ ✓

for $x < -\frac{3}{4}$,
$y = -(4x + 3) - -(x - 5)$
$= -4x - 3 + x - 5$
$= -3x - 8$ ✓

$$\therefore\ y = \begin{cases} 3x + 8, & x \geq 5 \\ 5x - 2, & -\dfrac{3}{4} \leq x < 5 \\ -3x - 8, & x < -\dfrac{3}{4} \end{cases}$$ ✓

(b) $|4x - 3| = |5 - x| + 8$

$|4x - 3| - |5 - x| = 8$

$|4x - 3| - |x - 5| = 8$

let $3x + 8 = 8 \quad x \geq 5$

$x = 0$ reject ✓

let $5x - 2 = 8 \quad -\frac{3}{4} \leq x < 5$

$x = 2$ ok ✓

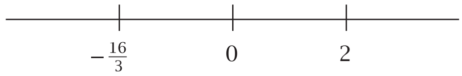
let $-3x - 8 = 8 \quad x < -\frac{3}{4}$

$-3x = 16$

$x = -\frac{16}{3}$ okay ✓

$\therefore$ Solutions are $x = 2,\ x = -\frac{16}{3}$ ✓

(c)



$|4x - 3| > |5 - x| + 8$

test at $x = 0$ $\quad |-3| > |5| + 8$? ✓

$3 > 13$ False

$\therefore\ x < -\frac{16}{3}$ or $x > 2$ ✓ are the solutions

[11]

3\. *(a)* x first $\quad y = f(2(x + 3))$ ✓

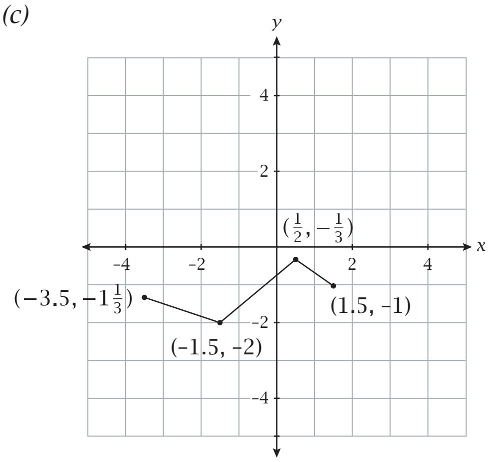
y next $\quad y = -2(f(2x + 6) - 2)$ ✓

or $y = -2f(2x + 6) + 4$

or $y = -2f(2(x + 3)) + 4$

(b) Translate two units in the negative x direction then dilate parallel to x axis with scale factor $\frac{1}{3}$ ✓. Dilate parallel to y axis scale factor 2, reflect in the x axis and translate down one unit. ✓

(c)



✓✓✓✓ 1 each point

✓✓✓ for coordinates

[11]

4\. *(a)* $D_f\{\text{Reals}\}$ ✓ , $R_f\{\text{Reals}\}$ ✓

(b) $D_h\{x|x \neq -1\}$ ✓ , $R_h\{y|y \neq 0\}$ ✓

(c) $x^2 - 9 \geq 0$ ✓

$(x-3)(x+3) \geq 0$

$x \geq 3$ or $x \leq -3$

$D_m\{x|x \leq -3 \text{ or } x \geq 3\}$ ✓ , $R_m\{y|y \geq 0\}$ ✓

[7]

5\. try $x = 1$ $P(x) = x^4 + 6x^3 + 9x^2 - 4x - 12$

$P(1) = (1)^4 + 6(1)^3 + 9(1)^2 - 4(1) - 12$

$= 0$

1 is a zero of the polynomial ✓

try $x = -2$ $P(-2) = (-2)^4 + 6(-2)^3 + 9(-2)^2 - 4(-2) - 12$

$= 0$

−2 is a zero of the polynomial ✓

$(x-1)(x+2)$ are factors

$x^2 + x - 2$ is a factor ✓

$$\begin{array}{r}
x^2+5x+6 \\
x^2+x-2\overline{\big)\,x^4+6x^3+9x^2-4x-12} \\
\underline{x^4+x^3-2x^2} \\
5x^3+11x^2-4x \\
\underline{5x^3+5x^2-10x} \\
6x^2+6x-12 \\
\underline{6x^2+6x-12} \checkmark
\end{array}$$

$x^4 + 6x^3 + 9x^2 - 4x - 12 = (x-1)(x+2)(x^2 + 5x + 6)$

$= (x-1)(x+2)(x+2)(x+3)$ ✓

zeroes are 1, −3, −2 (Repeat −2) ✓ [6]

6\. y intercept $x = 0$

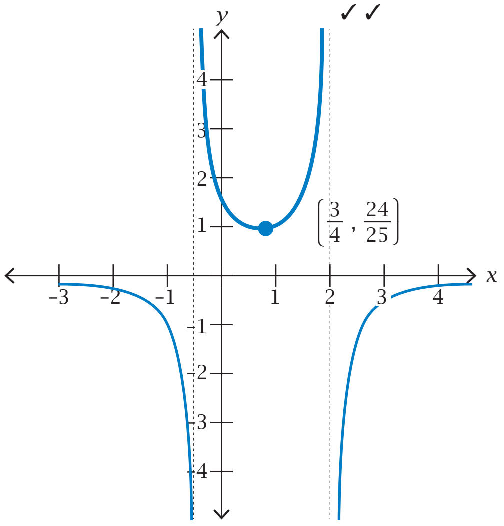
$$y = \frac{-3}{(-2)(1)}$$

$$y = \frac{3}{2}$$

$\left(0, \frac{3}{2}\right)$ ✓

Poles $x = 2,\ x = -\frac{1}{2}$ ✓

✓✓



Turning Point

$$y = \frac{-3}{2x^2 - 3x - 2}$$

$$y = -3(2x^2 - 3x - 2)^{-1}$$

$$y' = \frac{3(4x-3)}{(2x^2 - 3x - 2)^2} \checkmark$$

$$y' = 0 \Rightarrow 4x - 3 = 0$$

$$x = \frac{3}{4}$$

T.P. $\left(\frac{3}{4}, \frac{24}{25}\right)$ ✓

Other Points

$(1, 1)$ $(-1, -1)$ $\left(-2, -\frac{1}{4}\right)$ as $x \to \infty$ $y \to 0^-$

$\left(3, -\frac{3}{7}\right)$ $\left(4, -\frac{1}{6}\right)$ as $x \to -\infty$ $y \to 0^-$ ✓ [7]

7\. *(a)* 2 ✓

(b) 0 ✓

(c) does not exist ✓

(d) −2 ✓

(e) 2 ✓

(f) does not exist ✓ [6]

8\. *(a)* $\lim\limits_{x\to 3} \dfrac{2x^2 - 5x - 3}{x^2 - 5x + 6}$

$$= \lim_{x\to 3} \frac{(2x+1)(x-3)}{(x-2)(x-3)} \checkmark$$

$$= \lim_{x\to 3} \frac{2x+1}{x-2}$$

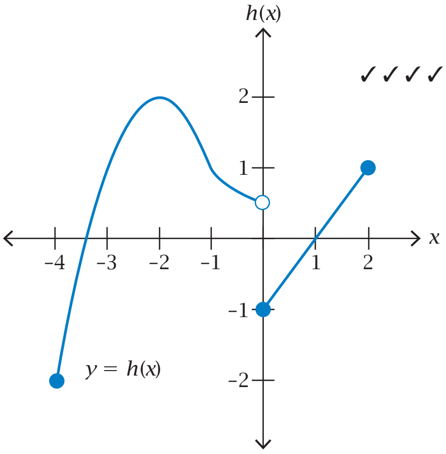
$$= 7 \qquad \checkmark$$

(b) $\lim\limits_{x\to\infty} \dfrac{3x^2-2}{(x+1)^2}$

$= \lim\limits_{x\to\infty} \dfrac{3-\dfrac{2}{x^2}}{\left(1+\dfrac{1}{x}\right)^2}$ ✓

$= 3$ ✓ [6]

9\. *(a)*



✓✓✓✓

(b) $\lim\limits_{x\to0^-} y = \dfrac{1}{2}$

$\lim\limits_{x\to0^+} y = -1$ ✓

since $\lim\limits_{x\to0^-} y \neq \lim\limits_{x\to0^+} y$

discontinous at $x = 0$ ✓

(c) $h(x) = -(x+2)^2 + 2$, $h(x) = \dfrac{1}{x+2}$

$h'(x) = -2(x+2)$, $h'(x) = \dfrac{-1}{(x+2)^2}$

$h'_-(-1) = -2$ ✓ $h'_+(-1) = -1$

Since $h'_- \neq h'_+$ not differentiable at $x = -1$. ✓ [8]

TRIAL TEST 4: *Vectors in 3-Dimensions I*

1\. *(a)* $4\mathbf{u} - 3\mathbf{v}$

$= 4\langle -2, 3, 1\rangle - 3\langle 3, 1, -5\rangle$

$= \langle -8, 12, 4\rangle - \langle 9, 3, -15\rangle$

$= \langle -17, 9, 19\rangle$ ✓✓

(b) $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$

$= \dfrac{\langle -2,3,1\rangle \cdot \langle 3,1,-5\rangle}{\sqrt{2^2+3^2+1}\,\sqrt{3^2+1+25}}$ ✓

$= \dfrac{-6+3-5}{\sqrt{14}\sqrt{35}}$ ✓

$= -0.3614 \ldots$

$\theta = 111.19^\circ$ ✓

(c) $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v})$

$= \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v}$

$= |\mathbf{u}|^2 - |\mathbf{v}|^2$ ✓

$= 14 - 35$

$= -21$ ✓

(d) $\mathbf{v} = \langle 3, 1, -5\rangle$ $\mathbf{i} = \langle 1, 0, 0\rangle$

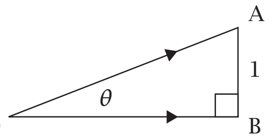
$\cos\theta = \dfrac{3}{\sqrt{35}}$ $\theta = 59.53^\circ$ ✓

(e) The $x - y$ plane means $z = 0$

Let $\mathbf{u} = \mathbf{OA} = \langle -2, 3, 1\rangle$ and

$\mathbf{OB} = \langle -2, 3, 0\rangle$

Then ΔBOA is



✓

$\sin\theta = \dfrac{1}{|\mathbf{OA}|} = \dfrac{1}{\sqrt{14}}$

$\theta = 15.50^\circ$ ✓ [10]

2\. $\langle 6, a, -4\rangle \cdot \langle -1, 2a, a\rangle = 0$

$-6 + 2a^2 - 4a = 0$ ✓

$2a^2 - 4a - 6 = 0$

$a^2 - 2a - 3 = 0$

$(a-3)(a+1) = 0$

$a = 3$ or $a = -1$ ✓✓ [3]

3\. *(a)* $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$

$\mathbf{r}\cdot\langle -1, 5, 3\rangle = \langle 2, -3, 4\rangle\cdot\langle -1, 5, 3\rangle$ ✓

$\langle x, y, z\rangle\cdot\langle -1, 5, 3\langle = -2 - 15 + 12$

$-x + 5y + 3z = -5$

$x - 5y - 3z = 5$ ✓

(b) $\mathbf{r} = \langle 16, -17, -8\rangle + t\langle -2, 3, 1\rangle$

$= \langle 16 - 2t, 3t - 17, t - 8\rangle$ ✓

(c) $x = 16 - 2t,\ y = 3t - 17,\ z = t - 8$

$x - 5y - 3z = 5$

$16 - 2t - 5(3t - 17) - 3(t - 8) = 5$ ✓

$16 - 2t - 15t + 85 - 3t + 24 = 5$

$-20t = -120$

$t = 6$ ✓

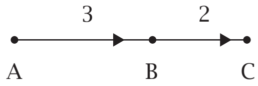
$\therefore\ x = 16 - 2(6) = 4$

$y = 3(6) - 17 = 1$

$z = 6 - 8 = -2$

The point of intersection is (4, 1, -2) ✓ [6]

4. $AB : AC = 3 : 5$



✓

$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$
$= \langle -1, 15, 5\rangle - \langle 2, 3, -1\rangle$
$= \langle -3, 12, 6\rangle$ ✓
$\frac{1}{3}\mathbf{AB} = \langle -1, 4, 2\rangle$ ✓
$\mathbf{BC} = 2\langle -1, 4, 2\rangle$
$= \langle -2, 8, 4\rangle$ ✓
$\mathbf{OC} = \mathbf{OB} + \mathbf{BC}$
$= \langle -1, 15, 5\rangle + \langle -2, 8, 4\rangle$ ✓
$= \langle -3, 23, 9\rangle$

$\therefore$ C is $(-3, 23, 9)$ ✓ [6]

5. $\mathbf{AB} = \langle 1, -3, 10\rangle - \langle 3, 2, 6\rangle$
$= \langle -2, -5, 4\rangle$ ✓

$\mathbf{AC} = \langle 10, 0, 5\rangle - \langle 3, 2, 6\rangle$
$= \langle 7, -2, -1\rangle$ ✓

$\mathbf{AB}\cdot\mathbf{n} = 0\ ,\ \langle -2, -5, 4\rangle\cdot\langle 1, p, q\rangle = 0$
$-2 - 5p + 4q = 0$
$-5p + 4q = 2 \quad (1)$ ✓

$\mathbf{AC}\cdot\mathbf{n} = 0\ ,\ \langle 7, -2, -1\rangle\cdot\langle 1, p, q\rangle = 0$
$7 - 2p - q = 0$
$2p + q = 7 \quad (2)$ ✓

Solving (1) and (2) gives
$p = 2 \quad , \quad q = 3$ ✓
$\therefore\ \mathbf{n} = \langle 1, 2, 3\rangle$
$\mathbf{r}\cdot\langle 1, 2, 3\rangle = \langle 3, 2, 6\rangle \cdot \langle 1, 2, 3\rangle$ ✓

$\therefore\ x + 2y + 3z = 25$ is the equation of the plane ✓ [7]

6. *(a)* let J = jet and P = plane
At 2pm let $t = 0$ and the initial displacement **JP** is

$\mathbf{JP} = \mathbf{OP} - \mathbf{OJ}$

$= \langle -238, 460, 4.52\rangle - \langle 50, -20, 3.8\rangle$
$= \langle -288, 480, 0.72\rangle$ ✓

also ${}_j\mathbf{v}_p = \mathbf{v}_j - \mathbf{v}_p$

$= \langle -200, 150, 0.5\rangle - \langle -80, -50, 0.2\rangle$
$= \langle -120, 200, 0.3\rangle$ ✓

For collision $\mathbf{JP} = t\,{}_j\mathbf{v}_p$

i.e. $\langle -288, 480, 0.72\rangle = t\langle -120, 200, 0.3\rangle$ ✓

$-288 = t(-120) \qquad 480 = 200t$

$t = \frac{-288}{-120} \qquad t = \frac{480}{200}$

$t = 2.4 \qquad t = 2.4$

$0.72 = t(0.3)$

$t = \frac{0.72}{0.3}$ ✓✓

$t = 2.4$

So $\mathbf{JP} = 2\cdot 4\ {}_j\mathbf{v}_p$ which means that the aircraft do collide at 4.24pm. ✓

$\mathbf{r}_j = \langle 50, -20, 3.8\rangle + 24\langle -200, 150, 0.5\rangle$
$= \langle -430, 340, 5\rangle$

So the aircraft would collide at $(-430, 340, 5)$ km ✓

(b) at $t = 2$

$\mathbf{r}_j = \langle 50, -20, 3.8\rangle + 2\langle -200, 150, 0.5\rangle$

$= \langle -350, 280, 4.8\rangle$ km ✓ and

$\mathbf{r}_p = \langle -238, 460, 4.52\rangle + 2\langle -80, -50, 0.2\rangle$

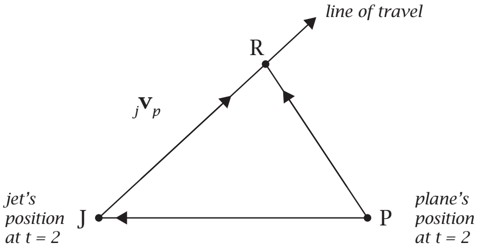
$= \langle -398, 360, 4.92\rangle$ km ✓

Consider that the plane is at rest, so find

${}_j\mathbf{v}_p = \mathbf{v}_j - \mathbf{v}_p$

$= \langle -220, 130, 0.8\rangle - \langle -80, -50, 0.2\rangle$

$= \langle -140, 180, 0.6\rangle$ ✓



$\mathbf{PJ} = \mathbf{OJ} - \mathbf{OP}$

$= \langle -350, 280, 4.8\rangle - \langle -398, 360, 4.92\rangle$

$= \langle 48, -80, -0.12\rangle$ ✓

$\mathbf{PR} = \mathbf{PJ} + {}_j\mathbf{v}_p$

$= \langle 48, -80, -0.12\rangle + t\langle -140, 180, 0.6\rangle$ ✓

$= \langle 48 - 140t, 180t - 80, 0.6t - 0.12\rangle$ ✓

Let $\mathbf{PR}\cdot{}_j\mathbf{v}_p = 0$

$\langle 48 - 140t, 180t - 80, 0.6t - 0.12\rangle \bullet$

$\langle -140, 180, 0.6\rangle = 0$ ✓

$-140(48 - 140t) + 180(180t - 80)$

$+ 0.6(0.6t - 0.12) = 0$ ✓

$-6720 + 19600t + 32400t - 14400$

$+ 0.36t - 0.072 = 0$

$52000.36t = 21120.072$

$t = \frac{21120.072}{52000.36}$

$= 0.4062$ ✓✓

at $t = 0.4062$

$|\mathbf{PR}| = |\langle -8.868, -6.884, 0.1237\rangle|$

$= 11.23$ ✓

So the closest distance that the aircraft now come to each other is 11.23 km.

[18]

TRIAL TEST 5: *Vector Calculus*

1. $\underset{\sim}{r}(t) = 5t\,\underset{\sim}{i} - 8t^2\,\underset{\sim}{j}$

(a) $\underset{\sim}{v}(t) = 5\,\underset{\sim}{i} - 16\,t\,\underset{\sim}{j}$ ✓ $\underset{\sim}{a}(t) = -16\,\underset{\sim}{j}$ ✓

(b)
$$\left|\underset{\sim}{v}(t)\right| = \sqrt{(5)^2 + (-16t)^2} \quad ✓$$
$$= \sqrt{25 + 256t^2}$$
$$\left|\underset{\sim}{v}(2.5)\right| = \sqrt{25 + 256(2.5)^2}$$
speed = 40.3 m/s (1 d.p.) ✓

(c) $x = 5t$ $\quad y = -8t^2$ ✓

$t = \frac{x}{5}$ $\quad y = -8\left(\frac{x}{5}\right)^2$

$y = -\frac{8x^2}{25}$ ✓

parabolic reflection ✓ [7]

2. $$\lim_{t \to 0}\left[\frac{2\sin t}{t}\underset{\sim}{i} - \sqrt{4-t}\,\underset{\sim}{j}\right]$$
$= 2\,\underset{\sim}{i} - 2\,\underset{\sim}{j}$ ✓✓ [2]

3. $$\underset{\sim}{r}(t) = \sin 2t\,\underset{\sim}{i} + 2\sin^2 t\,\underset{\sim}{j}$$
$$\left|\underset{\sim}{r}(t)\right| = \sqrt{(\sin 2t)^2 + (2\sin^2 t)^2} \quad ✓$$
$$= \sqrt{\sin^2 2t + 4\sin^4 t}$$
$$= \sqrt{4\sin^2 t\cos^2 t + 4\sin^4 t} \quad ✓$$
$$= \sqrt{4\sin^2 t(\cos^2 t + \sin^2 t)}$$
$$= \sqrt{4\sin^2 t}$$
$$= 2\left|\sin t\right| \quad ✓$$
$$\underset{\sim}{v}(t) = 2\cos 2t\,\underset{\sim}{i} + 4\sin t\cos t\,\underset{\sim}{j} \quad ✓$$
$$= 2\cos 2t\,\underset{\sim}{i} + 2\sin 2t\,\underset{\sim}{j}$$
$$\left|\underset{\sim}{v}(t)\right| = \sqrt{(2\cos 2t)^2 + (2\sin 2t)^2} \quad ✓$$
$$= \sqrt{4\cos^2 2t + 4\sin^2 2t}$$
$$= 2 \quad ✓$$

i.e. $\left|\sin t\right|\left|\underset{\sim}{v}(t)\right| = \left|\underset{\sim}{r}(t)\right|$

$\left|\sin t\right|2 \quad = 2\left|\sin t\right|$ ✓ [7]

4. $$\underset{\sim}{r}(t) = te^t\,\underset{\sim}{i} + t\ln t\,\underset{\sim}{j}$$
$$\underset{\sim}{r}'(t) = \left(e^t + te^t\right)\underset{\sim}{i} + \left(\ln t + t \times \frac{1}{t}\right)\underset{\sim}{j} \quad ✓✓$$
$$= \left(e^t + te^t\right)\underset{\sim}{i} + (\ln t + 1)\underset{\sim}{j} \quad ✓$$
[3]

5. $$\underset{\sim}{r}'(t) = 2e^{2t}\,\underset{\sim}{i} - \sin t\,\underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = \int\left(2e^{2t}\,\underset{\sim}{i} - \sin t\,\underset{\sim}{j}\right)dt \quad ✓$$
$$= e^{2t}\,\underset{\sim}{i} + \cos t\,\underset{\sim}{j} + \underset{\sim}{c} \quad ✓$$
$$\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{c}$$
$$3\,\underset{\sim}{i} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{c}$$
$$\underset{\sim}{c} = 2\,\underset{\sim}{i} - \underset{\sim}{j} \quad ✓$$
$$\underset{\sim}{r}(t) = e^{2t}\,\underset{\sim}{i} + \cos t\,\underset{\sim}{j} + 2\,\underset{\sim}{i} - \underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = \left(2 + e^{2t}\right)\underset{\sim}{i} + (\cos t - 1)\underset{\sim}{j} \quad ✓$$
[4]

6. $$\underset{\sim}{v}(t) = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} \quad ✓$$
$$= \sin t\,\underset{\sim}{i} + \frac{1}{t+1}\underset{\sim}{j}$$
$$\underset{\sim}{r}(t) = \int\left(\sin t\,\underset{\sim}{i} + \frac{1}{t+1}\underset{\sim}{j}\right)dt \quad ✓$$
$$= -\cos t\,\underset{\sim}{i} + \ln(t+1)\underset{\sim}{j} + \underset{\sim}{c} \quad ✓$$
$$\underset{\sim}{r}(0) = -\underset{\sim}{i} + \underset{\sim}{c}$$
$$2\,\underset{\sim}{i} = -\underset{\sim}{i} + \underset{\sim}{c}$$
$$\underset{\sim}{c} = 3\,\underset{\sim}{i} \quad ✓$$
$$\underset{\sim}{r}(t) = -\cos t\,\underset{\sim}{i} + \ln(t+1)\underset{\sim}{j} + 3\,\underset{\sim}{i}$$
$$\underset{\sim}{r}(t) = (3 - \cos t)\underset{\sim}{i} + \ln(t+1)\underset{\sim}{j} \quad ✓$$
[5]

7. $$\underset{\sim}{r}(t) = 2\cos 3t\,\underset{\sim}{i} + 5\sin 3t\,\underset{\sim}{j}$$

(a) $x = 2\cos 3t$ $\qquad y = 5\sin 3t$

$\frac{x}{2} = \cos 3t$ ✓ $\qquad \frac{y}{5} = sin3t$ ✓

$$\sin^2 3t + \cos^2 3t = 1$$
$$\left(\frac{y}{5}\right)^2 + \left(\frac{x}{2}\right)^2 = 1 \quad ✓$$
$$\frac{y^2}{25} + \frac{x^2}{4} = 1$$

(b) ellipse ✓

(c) $\underset{\sim}{v}(t) = -6\sin 3t\,\underset{\sim}{i} + 15\cos 3t\,\underset{\sim}{j}$ ✓

(d)
$$\left|\underset{\sim}{v}(t)\right| = \sqrt{(-6\sin 3t)^2 + (15\cos 3t)^2} \quad ✓$$
$$= \sqrt{36\sin^2 3t + 225\cos^2 3t} \quad ✓$$
$$= \sqrt{36\sin^2 3t + 36\cos^2 3t + 189\cos^2 3t}$$
$$= \sqrt{36 + 189\cos^2 3t} \quad ✓$$
when $\cos^2 3t = 1$
$$\text{speed} = \sqrt{36 + 189}$$
$$= \sqrt{225}$$
$= 15$ unit/s ✓ [9]

8. *(a)*

$$\underset{\sim}{v}(0) = 130\cos 30^\circ\ \underset{\sim}{i} + 130\sin 30^\circ\ \underset{\sim}{j}$$

$$= 65\sqrt{3}\ \underset{\sim}{i} + 65\ \underset{\sim}{j} \quad \checkmark$$

$$\underset{\sim}{a}(t) = -9.8\ \underset{\sim}{j}$$

$$\underset{\sim}{v}(t) = \int -9.8\ \underset{\sim}{j}\ dt$$

$$= -9.8t\ \underset{\sim}{j} + \underset{\sim}{c} \quad \checkmark$$

$$\underset{\sim}{v}(0) = \underset{\sim}{c}$$

$$65\sqrt{3}\ \underset{\sim}{i} + 65\ \underset{\sim}{j} = \underset{\sim}{c} \quad \checkmark$$

$$\underset{\sim}{v}(t) = -9.8t\ \underset{\sim}{j} + 65\sqrt{3}\ \underset{\sim}{i}\ \_65\ \underset{\sim}{j}$$

$$\underset{\sim}{v}(t) = 65\sqrt{3}\ \underset{\sim}{i} + (65 - 9.8t)\ \underset{\sim}{j} \quad \checkmark$$

$$\underset{\sim}{r}(t) = \int \left[65\sqrt{3}\ \underset{\sim}{i} + (65 - 9.8t)\ \underset{\sim}{j}\right] dt$$

$$= 65\sqrt{3}t\ \underset{\sim}{i} + \left(65t - 4.9t^2\right)\underset{\sim}{j} + \underset{\sim}{d} \quad \checkmark\ \checkmark$$

$$\underset{\sim}{r}(0) = \underset{\sim}{d}$$

$$0 = \underset{\sim}{d} \quad \checkmark$$

$$\underset{\sim}{r}(t) = 65\sqrt{3}t\ \underset{\sim}{i} + \left(65t - 4.9t^2\right)\underset{\sim}{j} \quad \checkmark$$

(b) max. height $\underset{\sim}{j}$ component $\underset{\sim}{v} = 0$

$$65 - 9.8t = 0$$

$$t = \frac{65}{9.8} \quad \checkmark$$

$$\text{height} = 65 \times \frac{65}{9.8} - 4.9\left(\frac{65}{9.8}\right)^2$$

$$= 215.6 \text{ m (1 d.p.)} \quad \checkmark$$

(c) since symmetrical total time $= 2 \times \dfrac{65}{9.8}$

$$= 13.3 \text{ s (1 d.p.)} \checkmark$$

(d)

$$\left|\underset{\sim}{v}(t)\right| = \sqrt{\left(65\sqrt{3}\right)^2 + (65 - 9.8t)^2} \quad \checkmark$$

$$\left|\underset{\sim}{v}\left(\frac{65}{4.9}\right)\right| = \sqrt{\left(65\sqrt{3}\right)^2 + \left(65 - 9.8 \times \frac{65}{4.9}\right)^2}$$

impact speed $= 130$ m/s ✓ [13]

TRIAL TEST 6: *Rates of Change and Differential Equations*

1. Let $f(x) = \cos 2x$, $\cos 2(x + h) = \cos (2x + 2h)$

$$f'(x) = \lim_{h\to 0} \frac{\cos 2(x+h) - \cos 2x}{h} \quad \checkmark$$

$$= \lim_{h\to 0} \frac{\cos 2x \cos 2h - \sin 2x \sin 2h - \cos 2x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\cos 2x(\cos 2h - 1)}{h} - \frac{\sin 2x \sin 2h}{h}\right] \quad \checkmark$$

$$= \cos 2x \lim_{h\to 0} \left(\frac{(\cos 2h - 1)}{h}\right) - \sin 2x \lim_{h\to 0} \frac{\sin 2h}{h} \quad \checkmark$$

$$= \cos 2x \cdot 0 - \sin 2x \cdot \lim_{2h\to 0} \frac{2\sin 2h}{2h} \quad \checkmark$$

$$= 0 - 2\sin 2x \cdot 1$$

$= -2\sin 2x$ as expected ✓ [5]

2. *(a)*

$$y = \sin 2x \cos 3x$$

$$\frac{dy}{dx} = 2\cos 2x \cos 3x - 3\sin 3x \sin 2x \quad \checkmark\ \checkmark$$

(b)

$$y = \cos(\sin x)$$

$$\frac{dy}{dx} = -\sin(\sin x) \cdot \cos x \quad \checkmark$$

(c)

$$y = \int_{\frac{\pi}{4}}^{\cos x} \sin^2 x\, \mathrm{d}x$$

$$\frac{dy}{dx} = \sin^2(\cos x) \cdot -\sin x \quad \checkmark\ \checkmark$$

(d)

$$y = \sqrt{\cos(\sin^2 x)}$$

$$= \cos^{\frac{1}{2}}(\sin^2 x)$$

$$\frac{dy}{dx} = \frac{1}{2}\cos^{-\frac{1}{2}}(\sin^2 x) \cdot \quad \checkmark\checkmark$$

$$(-\sin(\sin^2 x) \cdot 2\sin x \cos x)$$

$$= -\frac{\sin(\sin^2 x)\sin x \cos x}{\sqrt{\cos(\sin^2 x)}} \quad \checkmark$$

[8]

3. *(a)*

$$y = 3(5)^x$$

$$= 3(e^{\ln 5})^x \quad \checkmark$$

$$= 3e^{x \ln 5}$$

$$\frac{dy}{dx} = 3(\ln 5)5^x \quad \checkmark$$

(b)

$$\frac{d^2y}{dx^2} = \frac{e^x}{1 - e^x}$$

$$= -\frac{-e^x}{1 - e^x} \quad \checkmark$$

$$\frac{dy}{dx} = -\ln|1 - e^x| + c \quad \checkmark$$

(c)

$$y = \int_2^{3x} \frac{1}{t}\, dt$$

$$\frac{dy}{dx} = \frac{1}{3x} \cdot 3 \quad \checkmark$$

$$= \frac{1}{x} \quad \checkmark$$

(d)

$$y = \frac{\log x}{x} \quad \checkmark$$

$$= \frac{\ln x}{x \ln 10} = \frac{1}{\ln 10}\left(\frac{\ln x}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{\ln 10}\left(\frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2}\right) \quad \checkmark$$

$$= \frac{1}{\ln 10}\left(\frac{1 - \ln x}{x^2}\right) \quad \checkmark$$

[9]

4. *(a)* i.

$$f(x) = \int_0^x \frac{1 - t}{1 + t}\, dx$$

$$f'(x) = \frac{1 - x}{1 + x} \quad \checkmark$$

$$0 = \frac{1 - x}{1 + x}$$

$x = 1$ is where $f'(x) = 0$ ✓

ii. $\lim\limits_{x\to\infty} \dfrac{1-x}{1+x} = -1$ ✓

iii.
$$\begin{aligned} f''(x) &= \frac{-1(1+x) - 1(1-x)}{(1+x)^2} \\ &= \frac{-1 - x - 1 + x}{(1+x)^2} \\ &= \frac{-2}{(1+x)^2} \ ✓ \end{aligned}$$

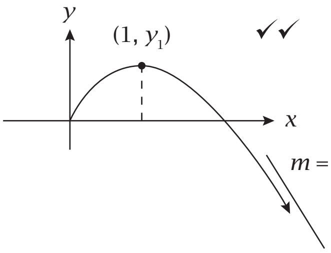
iv.
$$\begin{aligned} f(0) &= \int_0^0 \frac{1-t}{1+t}\,dt \\ &= 0 \ ✓ \end{aligned}$$

(b) *i.* means the stationary point is at $(1, y_1)$ ✓ where y_1 is not known yet

ii. means the derivative asymptotes to -1 as $x \to \infty$ ✓

iii. $f''(x) < 0$ for all x means that the graph is always concave down and so the stationary point at $(1, y_1)$ is a local maximum. ✓

iv. means that the graph passes ✓ through (0, 0)



(c)
$$f(1) = \int_0^1 \frac{1-t}{1+t}\,dt$$

$$\begin{aligned} \frac{1-t}{1+t} &= \frac{a}{1+t} + b \\ &= \frac{a + b(1+t)}{1+t} \ ✓ \end{aligned}$$

$\therefore\ 1 - t = a + b + bt$

$b = -1 \qquad 1 = a + b$ ✓
$\qquad\qquad\ \ a = 2$

$$\therefore\ \frac{1-t}{1+t} = \frac{2}{1+t} - 1 \ ✓$$

$$\begin{aligned} f(1) &= \int_0^1 \frac{1-t}{1+t}\,dt \\ &= \int_0^1 \left(\frac{2}{1+t} - 1\right) dt \ ✓ \\ &= [2\ln(1+t) - t]_0^1 \ ✓ \\ &= 2\ln 2 - 1 \end{aligned}$$

$\therefore$ the local maximum is at $(1,\ 2\ln 2 - 1)$ ✓ [17]

5. $f(x) = x - x^3$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{x + h - (x+h)^3 - (x - x^3)}{h} \ ✓ \\ &= \lim_{h\to 0} \frac{x + h - x^3 - 3x^2h - 3xh^2 - h^3 - x + x^3}{h} \ ✓ \\ &= \lim_{h\to 0} \frac{h - 3x^2h - 3xh^2 - h^3}{h} \\ &= \lim_{h\to 0} \frac{h(1 - 3x^2 - 3xh - h^2)}{h} \ ✓ \\ &= \lim_{h\to 0} (1 - 3x^2 - 3xh - h^2) \\ &= 1 - 3x^2 \quad \text{as expected} \ ✓ \end{aligned}$$

[4]

6. $f(x) = x^{-1} + x - 1$

$f'(x) = -\dfrac{1}{x^2} + 1$ ✓

$\therefore\ -\dfrac{1}{x^2} + 1 = m$ ✓ and $mx = \dfrac{1}{x} + x - 1$ ✓

$-\dfrac{1}{x} + x = mx$ and $mx = \dfrac{1}{x} + x - 1$

$$\begin{aligned} \therefore\ -\frac{1}{x} + x &= \frac{1}{x} + x - 1 \\ 1 &= \frac{1}{x} + \frac{1}{x} \\ \frac{2}{x} &= 1 \quad \text{i.e. } x = 2 \ ✓ \end{aligned}$$

$$\begin{aligned} \text{so} \qquad m &= -\frac{1}{2^2} + 1 \\ &= \frac{3}{4} \ ✓ \\ f(2) &= \frac{1}{2} + 2 - 1 \\ &= 1.5 \\ \therefore\ m &= \frac{3}{4} \end{aligned}$$

and the contact point is (2, 1.5) ✓ [6]

7. $v = 10\,(t+1)^{-\frac{1}{2}} + \dfrac{t}{2}$

$$x = \int v\,dt = \frac{10(t+1)^{\frac{1}{2}}}{\frac{1}{2}} + \frac{t^2}{4} + c \ ✓$$

i.e. $x = 20\sqrt{t+1} + \dfrac{t^2}{4} + c$

$t = 3 \quad 5 = 20\sqrt{4} + \dfrac{9}{4} + c$
$x = 5$

gives $c = -37.25$ ✓✓

$$\begin{aligned} x &= 20\sqrt{t+1} + \frac{t^2}{4} - 37.25 \\ 100 &= 20\sqrt{t+1} + \frac{t^2}{4} - 37.25 \ ✓ \\ t &= 15.1 \text{ sec} \end{aligned}$$

So after 15 seconds the object is very close to a position of $x = 100$ m. ✓ [5]

8. $x = 6 \sin \theta$

$$\frac{dx}{d\theta} = 6 \cos \theta \checkmark$$

$$\sqrt{36 - x^2}$$

$$= 6 \cos \theta \checkmark$$

$$x = 3 \quad , \quad \theta = \frac{\pi}{6} \checkmark$$

$$x = 6 \quad , \quad \theta = \frac{\pi}{2}$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta$$

$$\sin^2 \theta = \frac{1 - \cos^2 \theta}{2} \checkmark$$

$$\int_3^6 \frac{x^2}{\sqrt{36 - x^2}} dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{36 \sin^2 \theta}{6 \cos \theta} \cdot 6 \cos \theta \, d\theta \checkmark$$

$$= 36 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \checkmark$$

$$= 18 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta \checkmark$$

$$= 18 \left[\theta - \frac{\sin 2\theta}{2} \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

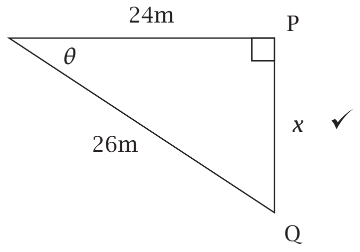
$$= 18 \left[\left(\frac{\pi}{2} - 0 \right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) \right] \checkmark$$

$$= 18 \left[\frac{\pi}{3} + \frac{\sqrt{3}}{4} \right]$$

$$= 6\pi + \frac{9}{2} \sqrt{3} \checkmark$$

[9]

9.



$$\frac{d\theta}{dt} = \frac{2\pi}{2} \checkmark$$

$$= \pi \text{ R/sec}$$

$$\tan \theta = \frac{x}{24} \checkmark$$

$$\frac{d}{dt} (\tan \theta) = \frac{d}{dt} \left(\frac{x}{24} \right)$$

$$\frac{1}{\cos^2 \theta} \cdot \frac{d\theta}{dt} = \frac{1}{24} \frac{dx}{dt} \checkmark$$

$$\frac{dx}{dt} = \frac{24}{\cos^2 \theta} \cdot \frac{d\theta}{dt}$$

$$= \frac{24\pi}{\left(\frac{24}{26}\right)^2} \checkmark$$

$$= 88.5 \text{ m/s.}$$

The light spot passes Q at 88.5 m/s at that instant ✓

[6]

10. (a) $$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{dv}{dt}$$

$$\text{L.H.S.} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

$$= \frac{1}{2} \cdot 2v \cdot \frac{dv}{dx} \checkmark$$

$$= v \frac{dv}{dx} \checkmark$$

$$= \frac{dx}{dt} \cdot \frac{dv}{dx} \checkmark$$

$$= \frac{dv}{dt} = \text{R.H.S. shown}$$

(b) $$v^2 = 10x - x^2$$

$$\frac{1}{2} v^2 = \frac{1}{2} (10x - x^2) \checkmark$$

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{d}{dx} \frac{1}{2} (10x - x^2)$$

$$\frac{dv}{dt} = \frac{1}{2} (10 - 2x) \checkmark$$

$$a = 5 - x$$

$$x = 4 \ , \quad a = 5 - 4$$

$$a = 1 \checkmark$$

[6]

TRIAL TEST 7: *Integration and Applications of Integration*

1. Remembering that arc length $l = r\,\theta$
and $AB = \sin x$,
$OA = \cos x$

$$OA \cdot x < \sin x < 1 \cdot x \checkmark$$

$x \cdot \cos x < \sin x < x$ shown. Now ÷ by x

$$\cos x < \frac{\sin x}{x} < 1 \checkmark$$

$$\lim_{x \to 0} \cos x < \lim_{x \to 0} \frac{\sin x}{x} < \lim_{x \to 0} 1 \checkmark$$

$$1 < \lim_{x \to 0} \frac{\sin x}{x} < 1 \checkmark$$

$$\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1 \checkmark$$

[5]

2. (a) $$\int \sin^3 2t \, dt$$

$$= \int \sin^2 2t \cdot \sin 2t \, dt$$

$$= \int (1 - \cos^2 2t) \sin 2t \, dt \checkmark$$

$$= \int (\sin 2t - \cos^2 2t \sin 2t) \, dt \checkmark$$

$$= -\frac{\cos 2t}{2} + \frac{\cos^3 2t}{6} + c \checkmark$$

(b) let $u = \cos\theta$

$$\frac{du}{d\theta} = -\sin\theta$$

$$du = -\sin\theta\, d\theta \checkmark \qquad [\,7\,]$$

$$\int \frac{\sin^3\theta}{\cos^4\theta}\, d\theta$$

$$= -\int \frac{\sin^2\theta}{\cos^4\theta} \cdot -\sin\theta\, d\theta$$

$$= -\int \frac{1-\cos^2\theta}{\cos^4\theta} \cdot -\sin\theta\, d\theta \checkmark$$

$$= -\int \frac{1-u^2}{u^4} \cdot du$$

$$= -\int (u^{-4} - u^{-2})\, du \checkmark$$

$$= -\int \left[\frac{u^{-3}}{-3} - \frac{u^{-1}}{-1}\right] + c$$

$$= \frac{1}{3u^3} - \frac{1}{u} + \mathrm{c}$$

$$= \frac{1}{3\cos^3\theta} - \frac{1}{\cos\theta} + c \checkmark \qquad [\,7\,]$$

3. *(a)* $$\int \frac{8x+2}{2x^2+x-6}\, dx$$

$$= 2\int \frac{4x+1}{2x^2+x-6}\, dx \checkmark$$

$$= 2\ln|2x^2+x-6| + c \checkmark$$

(b) $$\int (\mathrm{e}^x + x^{\mathrm{e}})\, dx$$

$$= \mathrm{e}^x + \frac{x^{\mathrm{e}+1}}{\mathrm{e}+1} + c \checkmark$$

(c) $$\int \frac{(x+2)\mathrm{e}^{x^2}}{\mathrm{e}^{5-4x}}\, dx$$

$$= \int (x+2)\mathrm{e}^{x^2+4x-5}\, dx \checkmark$$

$$= \frac{1}{2}\int (2x+4)\mathrm{e}^{x^2+4x-5}\, dx \checkmark$$

$$= \frac{1}{2}\mathrm{e}^{x^2+4x-5} + c \checkmark$$

(d) $$\frac{d}{dx}(x\ln x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

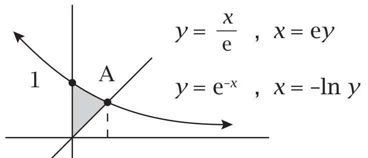
$$= \ln x + 1 \checkmark$$

$$\int (\ln x + 1)dx = x\ln x \checkmark$$

$$\int \ln x\, dx + x = x\ln x \checkmark$$

$$\int \ln x\, dx = x\ln x - x + c \checkmark \qquad [\,10\,]$$

4. *(a)*



✓✓

A is $\left(1, \frac{1}{\mathrm{e}}\right)$

$$\text{Area} = \int_0^1 (\mathrm{e}^{-x} - \frac{x}{\mathrm{e}})dx \checkmark$$

$$= \left[-\mathrm{e}^{-x} - \frac{x^2}{2\mathrm{e}}\right]_0^1 \checkmark$$

$$= \left(-\frac{1}{\mathrm{e}} - \frac{1}{2\mathrm{e}}\right) - (-1 - 0)$$

$$= 1 - \frac{3}{2\mathrm{e}} \checkmark$$

(b) $$\mathrm{A} = \int_0^{\frac{1}{\mathrm{e}}} \mathrm{e}y\, dy + \int_{\frac{1}{\mathrm{e}}}^1 -\ln y\, dy \checkmark$$

$$= \left[\frac{\mathrm{e}y^2}{2}\right]_0^{\frac{1}{\mathrm{e}}} - [y\ln y - y]_{\frac{1}{\mathrm{e}}}^1 \checkmark$$

$$= \left(\frac{\mathrm{e}}{2} \cdot \frac{1}{\mathrm{e}^2} - 0\right)$$

$$- \left((1\ln 1 - 1) - \left(\frac{1}{\mathrm{e}}\ln \mathrm{e}^{-1} - \frac{1}{\mathrm{e}}\right)\right)$$

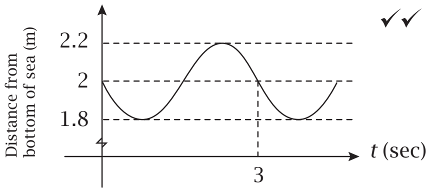
$$= \frac{1}{2\mathrm{e}} - \left(0 - 1 - \left(-\frac{1}{\mathrm{e}} - \frac{1}{\mathrm{e}}\right)\right) \checkmark$$

$$= \frac{1}{2\mathrm{e}} - \left(-1 + \frac{2}{\mathrm{e}}\right)$$

$$= \frac{1}{2\mathrm{e}} + 1 - \frac{2}{\mathrm{e}}$$

$$= 1 - \frac{3}{2\mathrm{e}} \checkmark\checkmark \qquad [\,10\,]$$

5. *(a)*



✓✓

(b) $\mathrm{P} = \frac{2\pi}{n} = 3$, $n = \frac{2\pi}{3}$ ✓

$x = -0.2 \sin\left(\frac{2\pi}{3}t\right) m$ ✓

where x is the distance from the mean position of 2 m.

(c) $$x = -0.2\sin\left(\frac{2\pi}{3}t\right)$$ ✓

$$\frac{dx}{dt} = -0.2\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}t\right)$$ ✓

$$\frac{d^2x}{dt^2} = 0.2\left(\frac{2\pi}{3}\right)^2\sin\left(\frac{2\pi}{3}t\right)$$

$$= -\left(\frac{2\pi}{3}\right)^2\left(-0.2\sin\left(\frac{2\pi}{3}t\right)\right)$$ ✓

$$= -n^2x \quad \text{where } n = \frac{2\pi}{3}$$ ✓

$\therefore$ S.H.M. is shown

(d) $v = -0.2\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}t\right)$

$\therefore \quad v_{\max} = 0.2 \times \frac{2\pi}{3}$ ✓

$= \frac{0.4\pi}{3}$

at $t = \frac{p}{2} = \frac{3}{2}$ sec ✓

(e) During each cycle the cork moves

$4 \times 0.2 = 0.8$ m ✓

$11 = 3.5 \times 3 + 0.5$ sec ✓

$\therefore$ Dist $= 3.5 \times 0.8 + 0.2\sin\left(\frac{2\pi}{3} \times 0.5\right)$ ✓

$= 2.8 + 0.1732$

$= 2.9732$ m 4dp ✓ [14]

6. Let T be the athletes temperature

then $\frac{dT}{dt} = k(T - 1)$ ✓

$$\frac{dT}{T-1} = kdt$$

$$\int\frac{dT}{T-1} = \int kdt$$

$\ln(T-1) = kt + c$ ✓

$T - 1 = e^{kt+c}$

$T = e^{kt} \cdot e^{c} + 1$

$T = Ae^{kt} + 1$ ✓

$t = 0$, $T = 37$, $37 = A + 1$
$A = 36$ ✓

$T = 36e^{kt} + 1$

$t = 5$, $36 = 36e^{5k} + 1$
$T = 36$

$$\frac{35}{36} = e^{5k}$$

$$5k = \ln\left(\frac{35}{36}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{35}{36}\right)$$

$= -0.00563$ ✓

$T = 35$, $35 = 36e^{-0.00563t} + 1$

$t = 10.15$ min ✓

so any longer than 10 min 9 seconds in the icy water is not healthy [6]

7. $\int\frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$ let $u = e^{2x}$

$= \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}}$ ✓ $u^2 = e^{4x}$ ✓

$= \frac{1}{2}\int\frac{\cos\theta}{\cos\theta}d\theta$ $\frac{du}{dx} = 2e^{2x}$

$= \frac{1}{2}\int d\theta$ ✓ $\frac{1}{2}du = e^{2x}dx$ ✓

$= \frac{1}{2}\theta + c$ ✓ $u = \sin\theta$

$= \frac{1}{2}\sin^{-1}u + c$ $du = \cos\theta\, d\theta$

$= \frac{1}{2}\sin^{-1}e^{2x} + c$ ✓ $\sqrt{1-u^2} = \cos\theta$ ✓

[7]

8. $P_1 = P_0(1.05)^t$

$t = 10$, $P_1 = P_0(1.05)^{10}$ ✓

$P_2 = P_0(1.05)^{10}e^{-0.02t}$ ✓

let $P_2 = P_0$, $P_0 = P_0(1.05)^{10}e^{-0.02t}$

$e^{0.02t} = 1.05^{10}$ ✓

$0.02t = \ln 1.05^{10}$

$t = \frac{10\ln 1.05}{0.02}$ ✓

$= 500\ln 1.05$

So after $500\ln 1.05$ years the population is back down to P_0 ✓ [5]

9. gradient of line $m = \frac{\log 30000 - \log 62.6}{40 - 10}$ ✓

$= 0.089352$ 6dp ✓

$\log y = mx + c$

$\log y = 0.089352x + c$, sub (10, 62.6) ✓

$\log 62.6 = 0.089352(10) + c$ ✓

$c = 0.90305$ ✓

$\log y = 0.089352x + 0.90305$

$y = 10^{0.089352x + 0.90305}$ ✓

$y = 8 \times 10^{0.089352x}$

is the required rule ✓ [7]

TRIAL TEST 8: *Statistical Inference*

1. (a) $P(X \le 4) = \frac{1}{2}(4)(0.1)$

$= 0.2$ ✓

(b) $P(4.5 \le X \le 6.5) = 2 \times 0.1$

$= 0.2$ ✓

(c) $P(7 \le X \le 9 \mid X \ge 4) = \dfrac{\frac{0.1 + 0.2}{2} \times 2}{0.8}$ ✓

$= \frac{0.3}{0.8}$

$= \frac{3}{8}$ ✓

(d) $P(X \le q) = 0.75$

$P(X \le 7) = 0.5$ ✓

$\therefore \int_7^q (0.05x - 0.25)\,dx = 0.25$ ✓

$\left(\frac{0.05q^2}{2} - 0.25q\right) - \left(\frac{0.05(7)^2}{2} - 0.25(7)\right) = 0.25$

$0.025q^2 - 0.25q + 0.275 = 0$

$\therefore q \approx 8.7417$ ✓

[7]

2. (a) $P(Y = y) = \begin{cases} \frac{1}{12}, & 14 \le y \le 26 \\ 0, & \text{otherwise} \end{cases}$ ✓✓

(b) $P(Y < 18) = \frac{1}{3}$ ✓

(c) $P(16 \le Y \le 25) = \frac{3}{4}$ ✓

(d) Median travelling time = 20 minutes. ✓

(e) $P(Y \ge 15) = \frac{11}{12}$ ✓

(f) $P(\text{late on Thurs and Fri}) = \frac{11}{12} \times \frac{11}{12}$

$= \frac{121}{144}$ ✓

(g) $X \sim B\left(5, \frac{11}{12}\right)$ ✓

$P(X \ge 3) \approx 0.9949$ ✓

[9]

3. (a) $\frac{3k + 5k}{2} \times 25k = 1$ ✓

$4k \times 25k = 1$ ✓

$100k^2 = 1$

$k = \frac{1}{10}$ ✓

(b) $f(x) = 0.08x + 0.3$ ✓

$\int_0^t (0.08x + 0.3)\,dx = 0.5$

$\left[\frac{0.08x^2}{2} + 0.3x\right]_0^t = 0.5$ ✓

$0.04t^2 + 0.3t - 0.5 = 0$ ✓

$t \approx 1.4039$ ✓

[7]

4. $X \sim N(1.8,\ 0.28^2)$ ✓

(a) $P(X \ge 1.6 \mid X < 1.9) \approx \frac{0.40198}{0.6395}$

≈ 0.6286 ✓

(b) $P(X < a) = 0.2$ ✓

$\therefore a \approx 1.5643$ ✓

Actual weight is 1.5643 kg

(c) $\overline{X} - z\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z\frac{\sigma}{\sqrt{n}}$

$1.75 - 1.96 \times \frac{0.28}{\sqrt{25}} \le \mu \le 1.75 + 1.96 \times \frac{0.28}{\sqrt{25}}$ ✓

$1.6402 \le \mu \le 1.8598$ ✓

(d) $\overline{X} \sim N\left(1.8,\ \left(\frac{0.28}{\sqrt{n}}\right)^2\right)$ ✓

$P(1.62 \le \overline{X} \le 1.93) \approx 0.9892$ ✓

(e) Sample size $n = \left(\frac{z \times \sigma}{w}\right)^2$

$= \left(\frac{2.56 \times 0.28}{0.4}\right)^2$ ✓

$= 3.211$

Sample size required is 4 trout. ✓ [10]

5. $X \sim N(30,\ 7.2^2)$

(a) $P(X < 25) \approx 0.2437$ ✓

(b) $P(27 \le X \le 32) \approx 0.27095$ ✓

(c) $P(X < 15) \approx 0.01861$

Number of patients $\approx 1000 \times 0.01861$

$= 18.61$

Approximately 19 patients. ✓

(d) $P(X > 25) = 0.9 \qquad \mu = ?$

$z = \frac{x - \mu}{\sigma}$

$-1.28156 = \frac{25 - \mu}{7.2}$ ✓

New mean: $\mu \approx 34.227$ kg ✓

(e) $z = \frac{z - \mu}{\sigma}$ $\sigma = ?$

$-1.28156 = \frac{25 - 30}{\sigma}$ ✓

New standard deviation:

$\sigma = 3.901\,\text{kg}$ ✓ [7]

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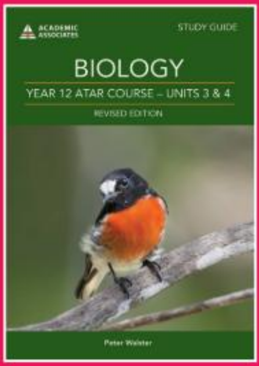
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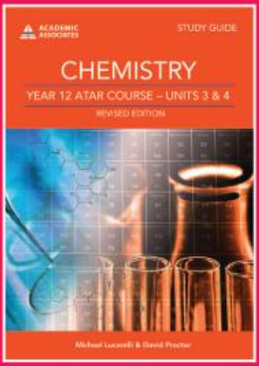
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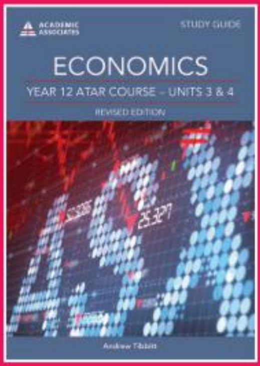
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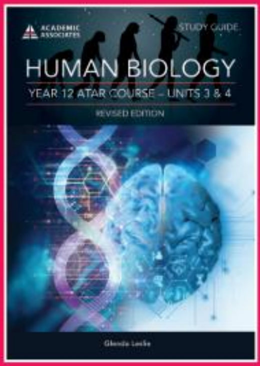
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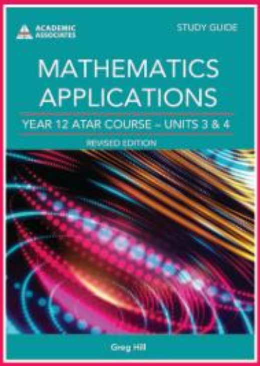
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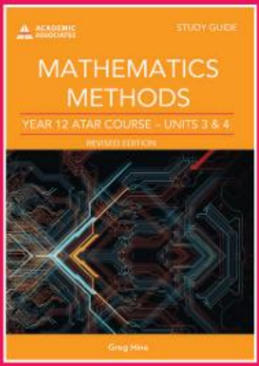
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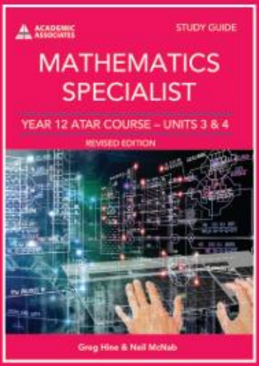
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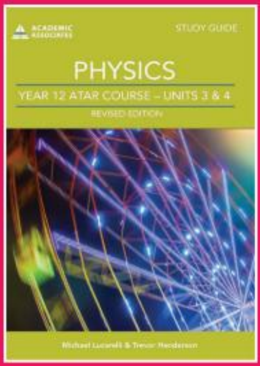
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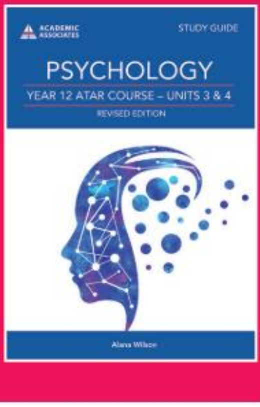
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