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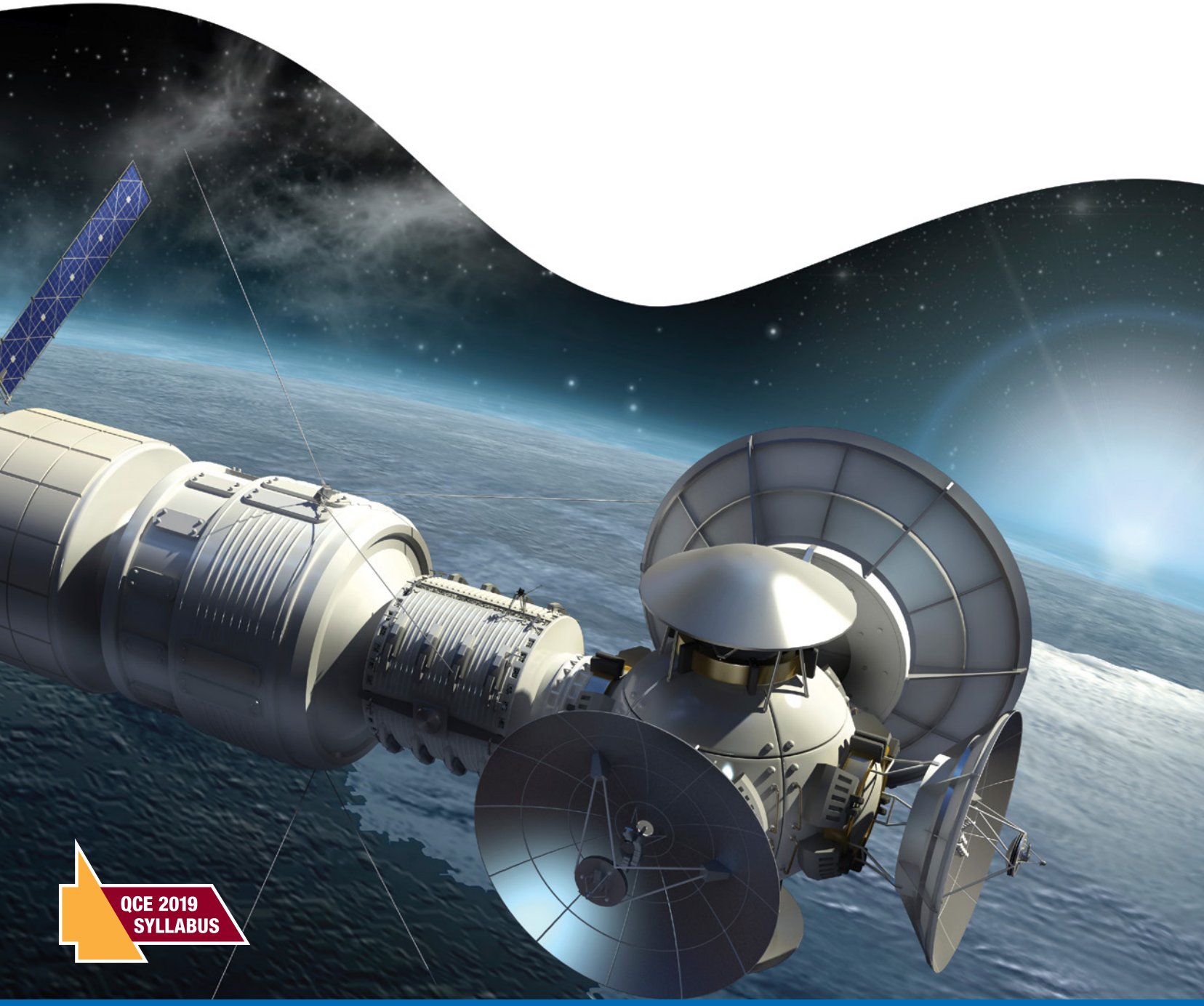
SPECIALIST MATHEMATICS

QUEENSLAND

STUDENT BOOK

12

UNITS 3 & 4



QCE 2019 SYLLABUS

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STUDENT BOOK

12

UNITS 3 & 4



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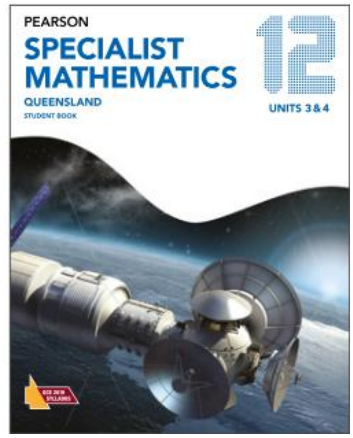
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Supporting the integrating of technology

Students are supported with the integration of technology in a number of ways. The eBook includes 'How to' user guides covering all basic functionality for the following three graphing calculators:

- TI-84 Plus CE
- TI-Nspire CX (non-CAS)
- CASIO fx-CG50AU

Throughout the student book you will find Technology worked examples strategically placed within the theory for both the TI-Nspire CX (non-CAS) and CASIO fx-CG50AU. The examples clearly demonstrate how the technology can be used effectively and efficiently for the content being covered in that chapter.

Graphing calculators are not the only technology integrated throughout the Pearson Queensland senior mathematics series. Spreadsheets, Desmos and interactive widgets have been included to provide students with the opportunity to visualise concepts, consolidate their understanding and make mathematical connections.

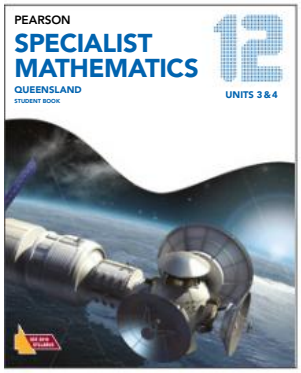
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Student book

Student book

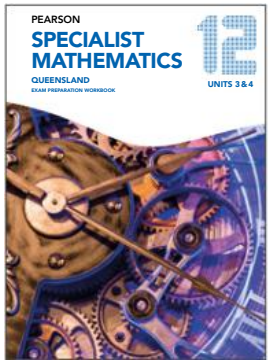
The student book has been authored by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next-generation eBook. This is an electronic textbook that students can access on any device, online or offline. It is linked to features, interactives and visual media that will help consolidate students' understanding of concepts and ideas, as well as other useful content specifically developed for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks and Making connections interactive widgets, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.



Specialist Mathematics 12
eBook



Specialist Mathematics 12
Exam preparation workbook

Exam preparation workbook

Additional component for Year 12 only

The exam preparation workbook provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson Specialist Mathematics 12 Queensland Units 3 & 4

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teacher support.

Explore further

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

Making connections

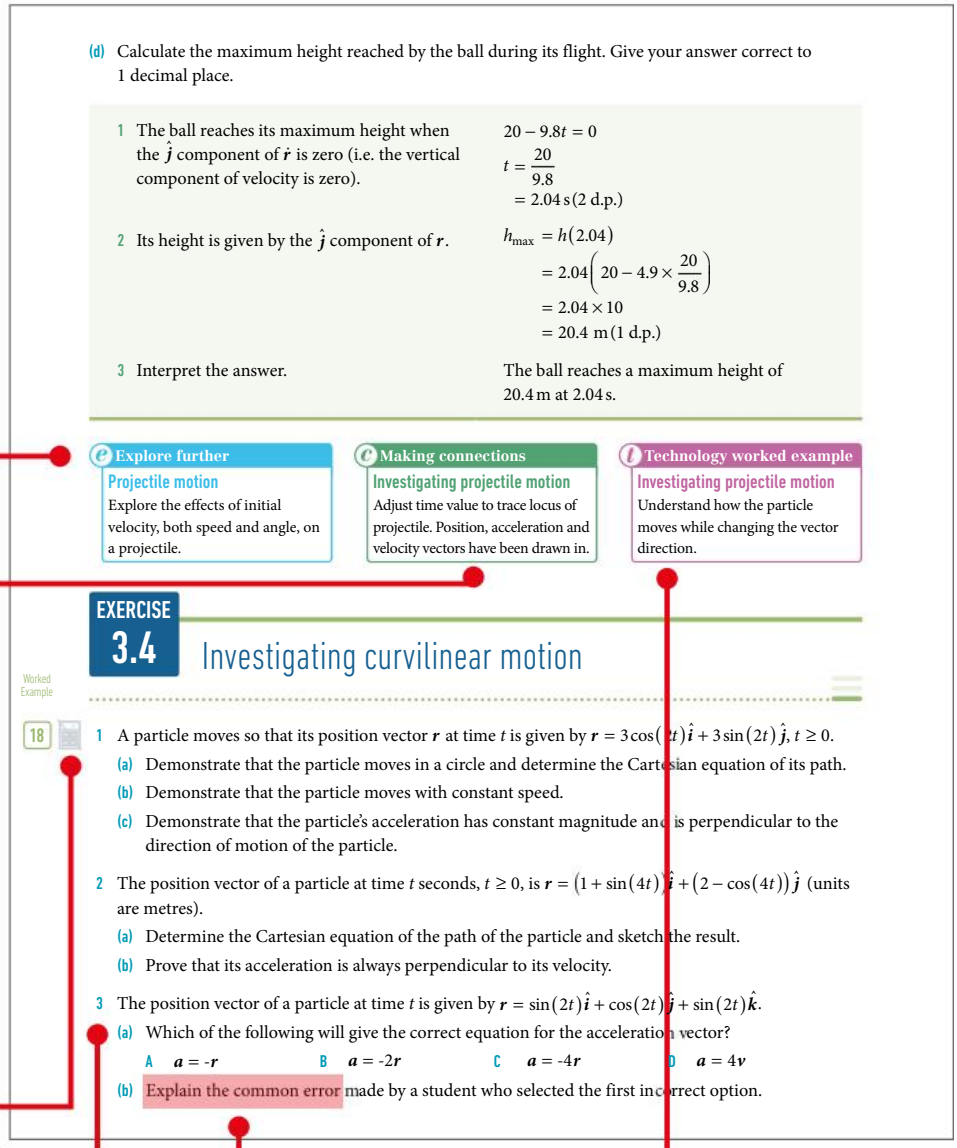
This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their understanding.

Tech-free questions

These questions are designed to provide students with the opportunity to practice algebraic manipulations to prepare them for technology-free examination papers.

Worked solutions

Fully worked solutions are provided for every question in the student textbook and can be accessed from the accompanying eBook.



(d) Calculate the maximum height reached by the ball during its flight. Give your answer correct to 1 decimal place.

1 The ball reaches its maximum height when the $\hat{j}$ component of $\dot{r}$ is zero (i.e. the vertical component of velocity is zero).

$20 - 9.8t = 0$

$t = \frac{20}{9.8}$

$= 2.04\,\text{s}\,(2\text{ d.p.})$

2 Its height is given by the $\hat{j}$ component of r.

$h_{\max} = h(2.04)$

$= 2.04\left(20 - 4.9 \times \frac{20}{9.8}\right)$

$= 2.04 \times 10$

$= 20.4\text{ m}\,(1\text{ d.p.})$

3 Interpret the answer.

The ball reaches a maximum height of 20.4 m at 2.04 s.

Explore further
Projectile motion
Explore the effects of initial velocity, both speed and angle, on a projectile.

Making connections
Investigating projectile motion
Adjust time value to trace locus of projectile. Position, acceleration and velocity vectors have been drawn in.

Technology worked example
Investigating projectile motion
Understand how the particle moves while changing the vector direction.

EXERCISE 3.4 Investigating curvilinear motion

Worked Example 18

1 A particle moves so that its position vector r at time t is given by $r = 3\cos(2t)\hat{i} + 3\sin(2t)\hat{j}$, $t \geq 0$.
(a) Demonstrate that the particle moves in a circle and determine the Cartesian equation of its path.
(b) Demonstrate that the particle moves with constant speed.
(c) Demonstrate that the particle's acceleration has constant magnitude and is perpendicular to the direction of motion of the particle.

2 The position vector of a particle at time t seconds, $t \geq 0$, is $r = (1 + \sin(4t))\hat{i} + (2 - \cos(4t))\hat{j}$ (units are metres).
(a) Determine the Cartesian equation of the path of the particle and sketch the result.
(b) Prove that its acceleration is always perpendicular to its velocity.

3 The position vector of a particle at time t is given by $r = \sin(2t)\hat{i} + \cos(2t)\hat{j} + \sin(2t)\hat{k}$.
(a) Which of the following will give the correct equation for the acceleration vector?
A $a = -r$ B $a = -2r$ C $a = -4r$ D $a = 4v$
(b) Explain the common error made by a student who selected the first incorrect option.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.

Technology worked examples

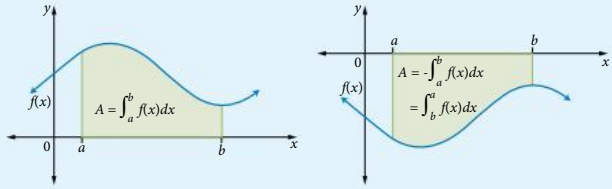
These worked examples offer support in using technology such as spreadsheets, graphing calculators and graphing software, and include technology-focused worked examples and activities.

Key information

Key information and rules are highlighted throughout the chapter.

The area bounded by a function $f(x)$, the x-axis, and the lines $x = a$ and $x = b$ (where $a < b$) is equal to:

$\int_a^b f(x)\,dx$ if the function lies above the x-axis from a to b, and $-\int_a^b f(x)\,dx$ if the function lies below the x-axis from a to b.



Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars)

The visibility of this grading helps ensure all levels of difficulty are well covered.

4 Using the chain rule to solve a related rates problem

A spherical hot air balloon develops a leak so that t minutes later the radius r metres is given by $r(t) = 24 - 3t^2$.

Assuming the balloon remains spherical (that is, $V(r) = \frac{4\pi}{3}r^3$), determine the rate at which the balloon is losing air 2 minutes after the leak commenced.

THINKING	WORKING
1 Identify the key elements in the problem then write the given information and the required rate.	Given: $V(r) = \frac{4\pi}{3}r^3$ $r(t) = 24 - 3t^2$ Required to determine: $\left.\frac{dV}{dt}\right\|_{t=2}$
2 Apply the chain rule to write the required rate of change in terms of other related rates.	Volume is an explicit parametric function of time, so $V(t) \equiv V(r(t))$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$
3 Substitute the derivatives needed into the related rates equation.	$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4\pi}{3}r^3\right) \times \frac{d}{dt}\left(24 - 3t^2\right)$ $= 4\pi\left(r^2\right) \times (-6t)$ $= 4\pi\left(24 - 3t^2\right)^2 \times (-6t)$
4 Calculate the required rate at the given time.	$\left.\frac{dV}{dt}\right\|_{t=2} = -6912\pi$
5 Interpret the solution in the context of the question and write the answer.	After 2 minutes, the balloon is losing air at a rate of 6912π cubic metres per minute.

WARNING

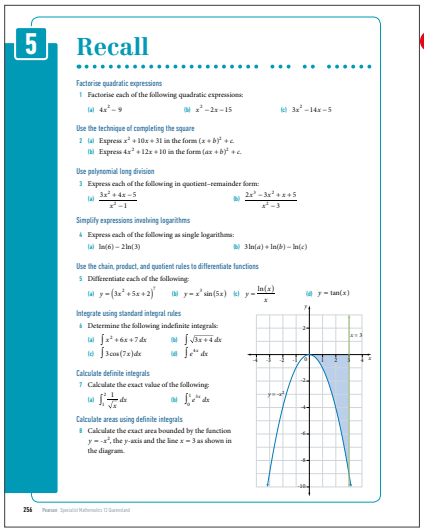
Never substitute values for changing quantities into a related rates problem until after all derivatives have been taken. Substituting too soon makes changeable variables behave like constants, with zero derivatives.

Meeting the needs of the QCE syllabus

The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.

Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.

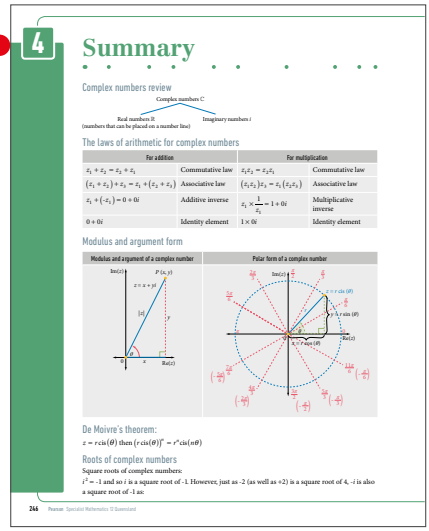

5
Recall
Factorise quadratic expressions
1 Factorise each of the following quadratic expressions:
(a) $4x^2 - 9$ (b) $x^2 - 2x - 15$ (c) $3x^2 - 14x - 5$
Use the technique of completing the square
2 (a) Express $x^2 + 10x + 31$ in the form $(x + b)^2 + c$.
(b) Express $4x^2 + 12x + 10$ in the form $(ax + b)^2 + c$.
Use polynomial long division
3 Express each of the following in quotient–remainder form:
(a) $\frac{3x^2 + 4x - 5}{x^2 - 1}$ (b) $\frac{2x^3 - 3x^2 + x + 5}{x^2 - 3}$
Simplify expressions involving logarithms
4 Express each of the following as single logarithms:
(a) $\ln(6) - 2\ln(3)$ (b) $3\ln(a) + \ln(b) - \ln(c)$
Use the chain, product, and quotient rules to differentiate functions
5 Differentiate each of the following:
(a) $y = (3x^2 + 5x + 2)^7$ (b) $y = x^3 \sin(5x)$ (c) $y = \frac{\ln(x)}{x}$ (d) $y = \tan(x)$
Integrate using standard integral rules
6 Determine the following indefinite integrals:
(a) $\int x^3 + 6x + 7\,dx$ (b) $\int \sqrt{5x + 4}\,dx$
(c) $\int 3\cos(7x)\,dx$ (d) $\int e^{4x}\,dx$
Calculate definite integrals
7 Calculate the exact value of the following:
(a) $\int_1^4 \frac{1}{\sqrt{x}}\,dx$ (b) $\int_0^{\ln 2} e^{3x}\,dx$
Calculate areas using definite integrals
8 Calculate the exact area bounded by the function $y = -x^2$, the y-axis and the line $x = 3$ as shown in the diagram.
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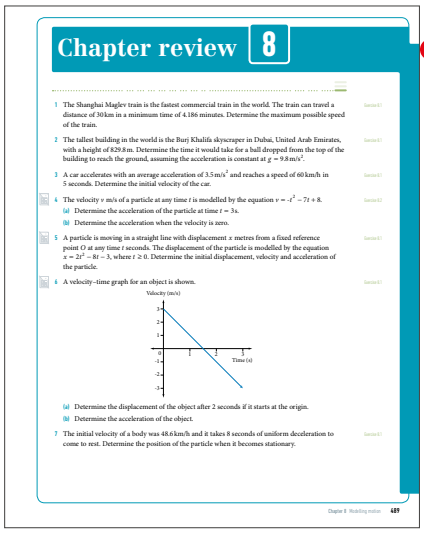
Recall

Each chapter begins with a review of assumed knowledge for the chapter.

Summary

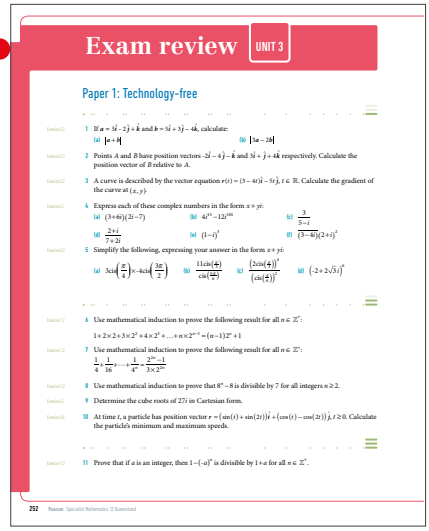
At the end of each chapter, there is a summary of the key facts and rules discussed in the chapter.


4
Summary
Complex numbers review
Complex numbers $\mathbb{C}$
Real numbers $\mathbb{R}$ (numbers that can be placed on a number line)
Imaginary numbers i
The laws of arithmetic for complex numbers
For addition
For multiplication
$z_1 + z_2 = z_2 + z_1$ Commutative law $z_1 z_2 = z_2 z_1$ Commutative law
$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ Associative law $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ Associative law
$z_1 + (-z_1) = 0 + 0i$ Additive inverse $z_1 \times \frac{1}{z_1} = 1 + 0i$ Multiplicative inverse
$0 + 0i$ Identity element $1 \times 0i$ Identity element
Modulus and argument form
Modulus and argument of a complex number
Polar form of a complex number
De Moivre's theorem:
$z = r\text{cis}(\theta)$ then $\left(r\text{cis}(\theta)\right)^n = r^n\text{cis}(n\theta)$
Roots of complex numbers
Square roots of complex numbers:
$i^2 = -1$ and so i is a square root of -1. However, just as -2 (as well as +2) is a square root of 4, $-i$ is also a square root of -1 as:
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Chapter review 8
1 The Shanghai Maglev train is the fastest commercial train in the world. The train can travel a distance of 30 km in a minimum time of 4.186 minutes. Determine the maximum possible speed of the train.
2 The tallest building in the world is the Burj Khalifa skyscraper in Dubai, United Arab Emirates, with a height of 829.8 m. Determine the time it would take for a ball dropped from the top of the building to reach the ground, assuming the acceleration is constant at $g = 9.8\,\text{m/s}^2$.
3 A car accelerates with an average acceleration of $3.5\,\text{m/s}^2$ and reaches a speed of 60 km/h in 5 seconds. Determine the initial velocity of the car.
4 The velocity v m/s of a particle at any time t is modelled by the equation $v = -t^2 - 7t + 8$.
(a) Determine the acceleration of the particle at time $t = 3$ s.
(b) Determine the acceleration when the velocity is zero.
5 A particle is moving in a straight line with displacement x metres from a fixed reference point O at any time t seconds. The displacement of the particle is modelled by the equation $x = 2t^2 - 8t - 3$, where $t \geq 0$. Determine the initial displacement, velocity and acceleration of the particle.
6 A velocity–time graph for an object is shown.
Velocity (m/s)
Time (s)
(a) Determine the displacement of the object after 2 seconds if it starts at the origin.
(b) Determine the acceleration of the object.
7 The initial velocity of a body was 48.6 km/h and it takes 8 seconds of uniform deceleration to come to rest. Determine the position of the particle when it becomes stationary.
Chapter 8 Modelling motion 489

Chapter review

Every chapter review follows the QCAA examination proportions for level of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.


Exam review UNIT 3
Paper 1: Technology-free
1 If $\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$, calculate:
(a) $|\boldsymbol{a} + \boldsymbol{b}|$ (b) $|3\boldsymbol{a} - 2\boldsymbol{b}|$
2 Points A and B have position vectors $-2\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k}$ and $5\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$ respectively. Calculate the position vector of B relative to A.
3 A curve is described by the vector equation $\boldsymbol{r}(t) = (3 - 4t)\boldsymbol{i} - 5t\boldsymbol{j}$, $t \in \mathbb{R}$. Calculate the gradient of the curve at (x, y).
4 Express each of these complex numbers in the form $x + yi$:
(a) $(3 + 6i)(2i - 7)$ (b) $4^{15} - 12^{100}$ (c) $\frac{3}{5 - i}$
(d) $\frac{2 + i}{7 + 2i}$ (e) $(1 - i)^4$ (f) $\frac{5}{(3 - 4i)(2 + i)^2}$
5 Simplify the following, expressing your answer in the form $x + yi$:
(a) $3\text{cis}\left(\frac{\pi}{4}\right) \times 4\text{cis}\left(\frac{3\pi}{2}\right)$ (b) $\frac{11\text{cis}\left(\frac{\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{3}\right)}$ (c) $\frac{\left(2\text{cis}\left(\frac{\pi}{3}\right)\right)^3}{\left(\text{cis}\left(\frac{\pi}{4}\right)\right)^2}$ (d) $\left(-2 + 2\sqrt{3}i\right)^4$
6 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:
$1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \ldots + n \times 2^{n-1} = (n - 1)2^n + 1$
7 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:
$\frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{4^n} = \frac{2^{2n} - 1}{3 \times 2^{2n}}$
8 Use mathematical induction to prove that $8^n - 8$ is divisible by 7 for all integers $n \geq 2$.
9 Determine the cube roots of $27i$ in Cartesian form.
10 At time t, a particle has position vector $\boldsymbol{r} = \left(\sin(t) + \sin(2t)\right)\boldsymbol{i} + \left(\cos(t) - \cos(2t)\right)\boldsymbol{j}$, $t \geq 0$. Calculate the particle's minimum and maximum speeds.
11 Prove that if a is an integer, then $1 - (-a)^n$ is divisible by $1 + a$ for all $n \in \mathbb{Z}^+$.
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Exam review

Exam reviews provide cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each Unit.





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Induction and proof

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1 Recall

Factorise algebraic expressions

1 Factorise each of the following expressions.

(a) $(2k-3)^3+5(2k-3)^2$ (b) $2k^2+11k+5$ (c) $6k^4+10k^3-4k^2$

Simplify algebraic fractions

2 Express each of the following as a simplified single fraction.

(a) $\dfrac{(2k+1)(3k+2)}{6}-k^2$ (b) $\dfrac{3}{(k+1)(k+2)}+\dfrac{3}{k+2}$ (c) $\dfrac{2k-6}{k^2+2k}+\dfrac{3}{k}$

Simplify expressions involving indices

3 Use index laws to simplify the following expressions.

(a) $a\times a^k$ (b) $2\times 3^k+5\times 3^{k+1}$ (c) $7^{k+2}-7^k$

Simplify expressions involving factorials

4 Simplify:

(a) $\dfrac{(k+2)!}{k!}$ (b) $(k+1)!-k!$ (c) $\dfrac{1}{k!}-\dfrac{k^2+2k}{(k+2)!}$

Prove simple algebraic equivalences

5 Prove that $\dfrac{1}{k}+\dfrac{1}{k+1}=\dfrac{2(k+1)-1}{k(k+1)}$.

Prove simple results involving integers

6 Prove that the sum of an odd integer and an even integer is odd.

Sums and divisibility

1.1

In Unit 1 you were introduced to the nature of mathematical proof and learnt several specific techniques (such as proof by contradiction) that often allow you to prove particular types of statements.

Mathematical induction is another specific technique that allows you to prove particular types of statements. An inductive proof may be appropriate whenever the result to be established must be demonstrated for all positive integers (or at least for all positive integers greater than some particular integer). For example, you will learn how to use mathematical induction to prove that the number $6^n - 1$ is divisible by 5 for all positive integers n.

Inductive proof is extremely powerful and applicable to results in many areas of mathematics (such as combinatorics, geometry, matrices and vectors). This chapter will focus on using mathematical induction to prove results about sums and divisibility results. These two areas provide an abundance of results in which proof by induction may be used. It should help you develop a deep understanding of this technique in order to adapt it beyond these areas if necessary.

Mathematical induction should not be confused with informal *inductive reasoning* (making a likely claim based on specific examples). While inductive reasoning is useful in mathematics for discovering possible patterns and structure, it cannot ever establish the absolute truth of mathematical statements. On the other hand, mathematical induction is a specific method of proof that does just this.

In this first section you will gain familiarity with the fundamental ideas relating to sums and divisibility that will be needed later in this chapter.

Expressions for sums

Consider representing the sum of the first 40 positive odd integers. Writing out every term of this sum would be counterproductive. Instead, you could express this sum as follows:

$1 + 3 + 5 + \ldots + 77 + 79$

The ellipsis (…) indicates to continue the pattern that is apparent from the first three terms. The fact that the final number to be added is 79 can be seen by noting the following pattern:

The 1st positive odd integer is equal to 1	Determining a rule for these two elements involved the number 2 since $3 - 1 = 2$, $5 - 3 = 2$, and so on. Because all the elements are positive odd numbers, the first element is equal to $2 \times 1 - 1$ and the second element is equal to $2 \times 2 - 1$.
The 2nd positive odd integer is equal to 3	
The 3rd positive odd integer is equal to 5	5 is equal to $2 \times 3 - 1$
The 4th positive odd integer is equal to 7	7 is equal to $2 \times 4 - 1$
⋮	⋮
Thus, the 40th positive odd integer is equal to 79	79 is equal to $2 \times 40 - 1$

Suppose you wish to write an expression for the sum of the first n positive odd integers, where n is a variable used to represent any integer greater than or equal to 1. In this case, the final term to added will be $2n-1$ instead of $2\times 40-1$, and so the expression would be $1+3+5+\ldots+(2n-1)$.

An expression for a sum that depends on a variable can be determined for different values of that variable. Observe the examples below:

when $n=7$	when $n=2$	when $n=1$
the final term to be added is $2\times 7-1=13$. Thus, the value of the sum is $1+3+5+7+9+11+13=49$.	the final term to be added is $2\times 2-1=3$. Thus, the value of the sum is $1+3=4$.	the final term to be added is $2\times 1-1=1$. Thus, the value of the sum is equal to the first term, 1.

Note that since the terms being added increase by 2, and the final term to be added is $2n-1$, the second last term to be added must be $2n-3$. Thus, this sum could be rewritten as $1+3+5+\ldots+(2n-3)+(2n-1)$ to give greater clarity.

While it is common for the variable n in expressions for sums to represent the number of terms being added, this does not need be the case. For example, consider the sum $5^2+6^2+\ldots+n^2$. If $n=7$, for example, the final term in this sum is equal to 7^2. The value of the sum is obtained by calculating $5^2+6^2+7^2$, that is, adding the first 3 terms.

WARNING

In expressions for sums, the variable n need not always represent the number of terms being added.

1 Determining the value of sums

Consider the following sum: $1+3+9+27+\ldots+3^n$, where $n\in\mathbb{Z}$ and $n\geq 0$.

(a) Determine the value of this sum for $n=5$, $n=6$, $n=2$ and $n=1$.

THINKING	WORKING
1 Determine the pattern that the terms follow and substitute the value of n into the expression for the final term to obtain the final number to be added.	If $n=5$, the final term to be added is $3^5=243$. If $n=6$, the final term to be added is $3^6=729$. If $n=2$, the final term to be added is $3^2=9$. If $n=1$, the final term to be added is $3^1=3$.
2 Write the expression and calculate the value of the sum.	For $n=5$, the sum is $1+3+9+27+81+243=364$. For $n=6$, the sum is $1+3+9+27+81+243+729=1093$. For $n=2$, the sum is $1+3+9=13$. For $n=1$, the sum is $1+3=4$.

(b) Rewrite the sum including an expression for the second-last term added.

1 Deduce the terms of the sum.	$3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^n$
2 Write the sum including an expression for second-last term added.	The terms in this sum are successive powers of 3. Since the final power of 3 is n, the second-last power of 3 is $n-1$. $3^0 + 3^1 + \ldots + 3^{n-1} + 3^n$ then, $1 + 3 + 9 + 27 + \ldots + 3^{n-1} + 3^n$

2 Writing expressions for sums

Write expressions for the following sums:

(a) The sum of the first n multiples of 3, starting with the number 3, where $n = 1$.

THINKING	WORKING
1 Determine the expression for the final term.	Note that the expression must equal 3 when $n = 1$, 6 when $n = 2$, 9 when $n = 3$, and so on. Thus, the expression for the final term is $3n$.
2 Write the expression for the sum.	$3 + 6 + 9 + \ldots + 3n$

(b) The sum of the first n multiples of 3, starting with the number 6.

1 Determine the expression for the final term.	Note that the expression must equal 6 when $n = 1$, 9 when $n = 2$, 12 when $n = 3$, and so on. Thus, the expression for the final term is $3(n+1)$.
2 Write the expression for the sum.	$6 + 9 + 12 + \ldots + 3(n+1)$

Summation notation

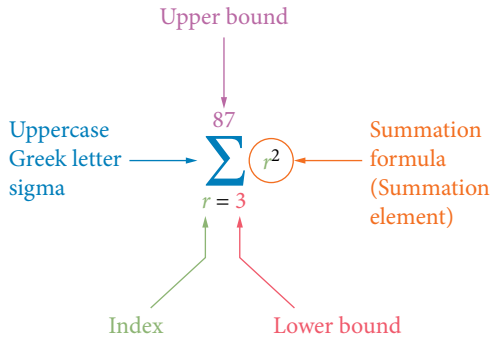
Often mathematical calculations require the addition of many variables. *Summation* is a convenient and simple form of writing the addition to give a concise expression for a sum of the values of a variable.

Consider adding together the square of every integer from 3 to 87. Recall that one way to write this sum is as follows: $3^2 + 4^2 + 5^2 + \ldots + 87^2$. This is known as expressing the sum in *expanded form*.

As a more concise alternative, this sum could be expressed using *summation notation* (also known as *sigma notation*) as follows: $\sum_{r=3}^{87} r^2$.

Summation notation involves:

- The symbol Σ, which is the uppercase Greek letter sigma, is used to mean 'the sum of …' .
- An expression in terms of a variable known as the ***index of summation***, often denoted by the letter i or r.
- A ***summation formula*** or summation element containing the index variable.
- A number underneath the sigma known as the ***lower bound***, which is the starting point of the summation.
- A number on top of the sigma known as the ***upper bound***, which is the stopping point for the summation.



$$\sum_{r=3}^{87} r^2 = 3^2 + 4^2 + 5^2 + \ldots + 87^2$$

An *expanded form* of a sum will show the value of each of its terms. $r_1 + r_2 + r_3 + \ldots + r_n$

The *summation notation* is a concise expression for a sum of a sequence of numbers. $\sum_{r=1}^{n} r_i$

When learning about subsequent ideas, having sums expressed in expanded form often brings greater clarity, but this method is less intuitive. For this reason, summation notation will be used sparingly.

3 Converting from summation notation to expanded form

Express each of the following sums in expanded form:

(a) $\sum_{r=1}^{4} 2r$

THINKING

Substitute all integer values of the index from the lower to the upper bound into the summation formula to obtain the terms of the sum.

WORKING

$$\sum_{r=1}^{4} 2r = (2\times 1)+(2\times 2)+(2\times 3)+(2\times 4)$$
$$= 2+4+6+8$$

(b) $\sum_{r=4}^{7} (3r+1)$

Substitute all integer values of the index from the lower to the upper bound into the summation formula to obtain the terms of the sum.

$$\sum_{r=4}^{7} (3r+1) = (3\times 4+1)+(3\times 5+1)+(3\times 6+1)+(3\times 7+1)$$
$$= 13+16+19+22$$

(c) $\sum_{r=2}^{n} r!$

Substitute all integer values of the index from the lower to the upper bound into the summation formula to obtain the terms of the sum.

$$\begin{aligned}\sum_{r=2}^{n} r! &= 2! + 3! + 4! + \ldots + n! \\ &= (2 \times 1) + (3 \times 2 \times 1) + (4 \times 3 \times 2 \times 1) + \ldots + n! \\ &= 2 + 6 + 24 + \ldots + n!\end{aligned}$$

(d) $\sum_{r=0}^{3} (-1)^r$

Substitute all integer values of the index from the lower to the upper bound into the summation formula to obtain the terms of the sum.

$$\begin{aligned}\sum_{r=0}^{3} (-1)^r &= (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 \\ &= 1 - 1 + 1 - 1\end{aligned}$$

Divisibility

Explore further

Summation notation

Use technology to add together a large number of terms in a given sequence.

The following important definition clarifies what it means for an integer to be divisible by some other integer:

> An integer is divisible by the integer m if the number can be expressed in the form mA for some integer A.

For example, 20 is divisible by 4 since $20 = 4 \times 5$. However, 21 is not divisible by 4 since $21 = 4 \times 5.25$ and 5.25 is not an integer.

While this may seem obvious, it allows statements involving divisibility to be expressed as equations, which are easier to work with. For example, the statement $4^n + 2$ is divisible by 3 can be rewritten as $4^n + 2 = 3A$ for some integer A.

Further, the following fact is extremely useful when proving results about divisibility:

> Adding, subtracting or multiplying two or more integers produces an integer. Likewise, raising an integer to any non-negative integer power produces an integer.

For example, if k is a non-negative integer, then a number such as $5 + 4 \times 3^k$ is also an integer. Note that dividing an integer by another integer obviously will not always produce an integer.

> **WARNING**
>
> Zero is considered neither positive nor negative, and therefore the set of *positive numbers* excludes zero but the set of *non-negative numbers* includes it.

4 Demonstrating divisibility

Demonstrate that if k is a non-negative integer, then the number $2 \times 5^{k+1} + 5^k + 77$ is divisible by 11.

THINKING	WORKING
1 To demonstrate that a number is divisible by m, demonstrate that the number can be expressed in the form mA for some integer A.	A number is divisible by 11 if and only if it is an integer multiple of 11. $2 \times 5^{k+1} + 5^k + 77$ $= 2 \times 5 \times 5^k + 5^k + 77$ $= 10 \times 5^k + 5^k + 77$ $= 5^k(10 + 1) + 77$ $= 11 \times 5^k + 11 \times 7$ $= 11 \times \left(5^k + 7\right)$
2 Prove that when a factor of m is taken out, what is left must be an integer.	$11 \times \left(5^k + 7\right)$ Since k is a non-negative integer, $5^k + 7$ is an integer.
3 Interpret your result and write the answer.	Thus, $2 \times 5^{k+1} + 5^k + 77$ is divisible by 11.

Proving sum and divisibility results for specific values of *n*

Consider the expression from earlier for the sum of the first n positive odd integers: $1 + 3 + 5 + \ldots + (2n - 1)$. Notice the following pattern that emerges when the value of this sum is determined for some small values of n:

For $n = 1$, the value of the sum is equal to 1.

For $n = 2$, the value of the sum is equal to $1 + 3 = 4$.

For $n = 3$, the value of the sum is equal to $1 + 3 + 5 = 9$.

For $n = 4$, the value of the sum is equal to $1 + 3 + 5 + 7 = 16$.

In general, it appears that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$.

Given a statement such as this, it is straightforward to prove that it is true for a specific values of n. For example, to prove that it is true for $n = 7$, substitute $n = 7$ into both the left-hand side of this statement and the right-hand side, and determine that the same result of 49 is obtained.

Statements about divisibility may similarly be proved for particular values of n.

5 Proving sum and divisibility for specific values of *n*

Prove that the following statements are true for $n = 5$:

(a) $1 + 4 + 9 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

THINKING	WORKING
1 Substitute the given value of n into both the left-hand side and right-hand side of the statement and determine that the same result is obtained.	When $n = 5$: $\text{LHS} = 1 + 4 + 9 + 16 + 25$ $= 55$ $\text{RHS} = \frac{5(5+1)(2\times 5+1)}{6}$ $= \frac{5\times 6\times 11}{6}$ $= 55$ $= \text{LHS}$
2 Interpret your result and write the answer.	Given that RHS = LHS, the statement is true for $n = 5$.

(b) $10^n - 1$ is divisible by 9

1 Substitute the given value of n into the expression.	When $n = 5$: $10^5 - 1 = 100\,000 - 1$ $= 99\,999$ $= 9\times 11\,111$
2 Interpret your result and write the answer.	When $n = 5$, the given expression can be written as an integer multiple of 9; therefore, it is divisible by 9.

EXERCISE 1.1 Sums and divisibility

Worked Example 1

1 Determine the value of each of the following sums for $n = 5$, $n = 6$, $n = 2$ and $n = 1$.

(a) $1 + 8 + 27 + \ldots + n^3$, where $n \in \mathbb{Z}^+$.

(b) $4 + 7 + 10 + \ldots + (3n + 1)$, where $n \in \mathbb{Z}^+$.

(c) $1 + 3 + 5 + \ldots + (2n + 3)$, where $n \in \mathbb{Z}$ and $n \geq -1$.

(d) $-2 + 4 - 6 + \ldots + (-1)^n 2n$, where $n \in \mathbb{Z}$ and $n \geq -1$.

(e) $1 + 2 + 2^2 + \ldots + 2^{n+2}$, where $n \in \mathbb{Z}$ and $n \geq -2$.

2 Rewrite each of the sums from question **1**, including expressions for the second-last term added.

Worked Example 2

3 Write expressions for the following sums:

(a) The sum of the reciprocals of the first n positive integers.

(b) The sum of the first n even numbers starting at 2.

(c) The sum of the first n even numbers starting at 4.

(d) The sum of the first n positive integer powers of 3.

(e) The sum of the first n non-negative integer powers of 3.

Worked Example

3

4 Express each of the following sums in expanded form:

(a) $\sum_{r=1}^{5} r^3$ **(b)** $\sum_{r=0}^{4} 2r+1$ **(c)** $\sum_{r=1}^{4} (-1)^r r^2$ **(d)** $\sum_{r=1}^{n} r!$

(e) $\sum_{r=1}^{n} (2r-1)^2$ **(f)** $\sum_{r=3}^{n} r^2$ **(g)** $\sum_{r=0}^{n-1} \frac{r+1}{r+2}$ **(h)** $\sum_{r=1}^{n} (-1)^r r^3$

(i) $\sum_{r=1}^{n} (-1)^{r+1} r^3$ **(j)** $\sum_{r=1}^{n} a^r$ **(k)** $\sum_{r=1}^{n} r^a$

4

5 Let k be a non-negative integer. Demonstrate that:

(a) $2 \times 3^{k+1} - 3^{k+1} + 21$ is divisible by 3. **(b)** $3 \times 4^k - 6 \times 4^{k+1} + 28$ is divisible by 7.

(c) $5^{k+2} - 7 \times 5^k + 12$ is divisible by 6.

5

6 Prove that the following statements are true for $n = 5$, $n = 2$, and $n = 1$:

(a) $2 + 4 + 6 + \ldots + 2n = n(n+1)$

(b) $1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$

(c) $1 \times 2 + 2 \times 5 + \ldots + n(3n-1) = n^2(n+1)$

(d) $n^3 + 2n$ is divisible by 3

(e) $3^{n-1} + 1$ is divisible by 2

(f) $3^{n-1} - 2n + 5$ is divisible by 4.

7 Consider the following sum: $3! + 4! + 5! + \ldots + n!$.

(a) What is the smallest value that n may take?

A 0 **B** 1 **C** 3 **D** 5

(b) Explain the common misconceptions that lead to each of the incorrect options in part **(a)**.

8 **(a)** If $k \in \mathbb{Z}^+$, which of the following expressions must always equal an integer that is divisible by 6?

A $6\left(k^2 + \frac{1}{2}k + 7\right)$ **B** $24 + 6^{2k-3}$ **C** $3\left(k^2 + k\right)$ **D** $2\left(2k^2 + 1\right)$

(b) Explain the common misconceptions that lead to each of the incorrect options in part **(a)**.

9 Consider the sum of the first n even numbers.

(a) Write the expanded form of this sum.

(b) Write the summation notation of this sum.

(c) By evaluating your expression for $n = 1, 2, 3, 4$ and identifying a pattern, make a conjecture (a statement that appears to be true) in the form of an equation as to the result of adding the first n even numbers.

(d) Determine if your conjecture is true for $n = 10$.

10 Let k be a positive integer and suppose that $5^k - 1$ is divisible by 4. Prove that $5^{(k+1)} - 1$ is also divisible by 4.

1.2 Mathematical induction for sums

Previously, the statement $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ was proved to be true for some small values of n. Despite proving these particular cases, there is no guarantee that there isn't some larger value of n for which the statement is false.

A method to prove this result is true *for all* positive integer values of n is needed. Essentially, there is a list of infinitely many statements to prove – one for each value of n. Mathematical induction provides a way to accomplish this. Basically, it involves proving just the first of these infinitely many statements, and then establishing a guarantee that *if* any particular statement in the list were known to be true, then the next statement in the list would necessarily need to be true.

More formally:

To use mathematical induction to prove that a statement is true for all positive integers n, it is necessary to complete the following two steps:

STEP 1: (the *base step*) Prove that the statement is true for the first case, often $n = 1$.

STEP 2: (the *inductive step*) Let $k \geq 1$ and assume that the statement is true for $n = k$. Then prove that, given this result, the statement would have to be true for $n = k + 1$.

WARNING

Notice that step 2 does not require the statement to be true for $n = k$. You simply discover the consequences that would result if it were true and examine their implications for the case $n = k + 1$. This allows you to form statements about the case for $n = k + 1$ that would apply if and only if the $n = k$ case were true.

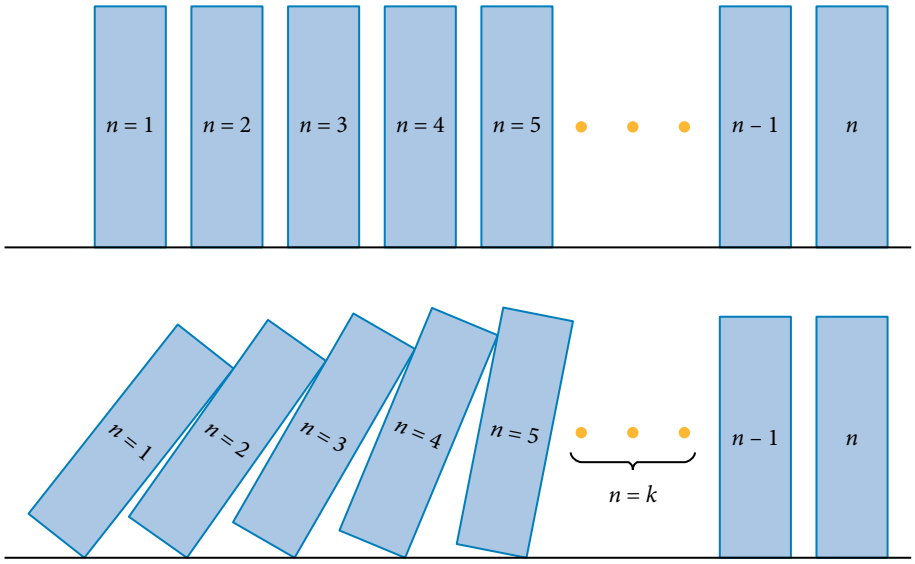
By completing step 1 and step 2 you guarantee that *if* you could be certain that the statement were true for any particular value of n then it would necessarily need to be true for the next value of n.

Statement	Conclusion
you *can* be certain that the statement is true for $n = 1$	this guarantee means that it must be true for $n = 2$.
then, you can be certain the statement is true for $n = 2$	this guarantee means that it must be true for $n = 3$.
then, you can be certain that the statement is true for $n = 3$	and now that, this guarantee means that it must be true for $n = 4$, and so on.

Completing the base and inductive steps does prove that the original statement is true for all positive integer values of n.

To help make sense of this idea, imagine lining up an infinite line of dominoes close to each other and toppling the first domino. Imagine the first domino represents the statement when $n = 1$, the second domino represents the statement when $n = 2$, and so on.

A domino toppling over means the statement has been proved for that particular value of n ($n = k$). Lining the dominoes up close to each other has the effect of guaranteeing that if any particular domino were to topple, then the next domino, $n = k + 1$, would have to topple (the inductive step). By toppling the first domino (the base step), every other domino must then topple, one at a time.



The following two worked examples outline the technical details of proving results for sums using mathematical induction.

6 Proving results for simple sums

Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$: $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

THINKING	WORKING
1 Complete the base step; that is, prove the statement for $n = 1$.	When $n = 1$: LHS $= 1$ RHS $= 1^2$ $= 1$ $=$ LHS Therefore the statement is true for $n = 1$.
2 Clarify what is being assumed in the inductive step. (To write the statement when $n = k$, substitute every occurrence of n in the original statement with k.)	Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ [1]
3 Clarify what must be proved in the inductive step. (To write the statement when $n = k + 1$, substitute every occurrence of n in the original statement with $k + 1$.)	Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate $1 + 3 + 5 + \ldots + (2(k + 1) - 1) = (k + 1)^2$

4 Complete the proof for the inductive step. As a first step, it is usually helpful to rewrite the sum to include an expression for the second-last term being added. Your goal should then be to incorporate equation [1].	$\begin{aligned}\text{LHS} &= 1+3+5+\ldots+\big(2(k+1)-1\big)\\ &= 1+3+5+\ldots+(2k-1)+\big(2(k+1)-1\big)\\ &= k^2+\big(2(k+1)-1\big) \quad \text{using [1]}\\ &= k^2+2k+2-1\\ &= k^2+2k+1\\ &= (k+1)^2\\ &= \text{RHS}\end{aligned}$
5 Deduce that completing the base and inductive steps ensures that the original statement must be true for all specified values of n.	The original statement is true for $n = 1$, assumed true for $n = k$ and proven to be true for $n = k + 1$. Therefore $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ is true for all $n \geq 1$.

7 Proving results for complex sums

Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$: $\displaystyle\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} = \frac{n}{6n+4}$

THINKING	WORKING
1 Rewrite the statement with the sum expressed in expanded form.	To prove: $\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\ldots+\dfrac{1}{(3n-1)(3n+2)}=\dfrac{n}{6n+4}$
2 Complete the base step; that is, prove the statement for $n = 1$.	When $n = 1$: $\text{LHS} = \dfrac{1}{2\times5}$ $= \dfrac{1}{10}$ $\text{RHS} = \dfrac{1}{6\times1+4}$ $= \dfrac{1}{10}$ $= \text{LHS}$ Therefore the statement is true for $n = 1$.
3 Clarify what is being assumed in the inductive step. (To write the statement when $n = k$, substitute every occurrence of n in the original statement with k.)	Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume $\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\ldots+\dfrac{1}{(3k-1)(3k+2)}=\dfrac{k}{6k+4}$ [1]

4 Clarify what must be proved in the inductive step. (To write the statement when $n = k + 1$, substitute every occurrence of n in the original statement with $k + 1$.)

Must demonstrate statement is true when $n = k + 1$. That is:

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\ldots+\frac{1}{\big(3(k+1)-1\big)\big(3(k+1)+2\big)}=\frac{k+1}{6(k+1)+4}$$

5 Complete the proof for the inductive step. As a first step, it is usually helpful to rewrite the sum to include an expression for the second-last term being added. Your goal should then be to incorporate equation [1].

$$\begin{aligned}
\text{LHS} &= \frac{1}{2\times 5}+\frac{1}{5\times 8}+\ldots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{\big(3(k+1)-1\big)\big(3(k+1)+2\big)}\\
&= \frac{k}{6k+4}+\frac{1}{\big(3(k+1)-1\big)\big(3(k+1)+2\big)} \quad \text{using [1]}\\
&= \frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}\\
&= \frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}\\
&= \frac{k(3k+5)}{2(3k+2)(3k+5)}+\frac{2}{2(3k+2)(3k+5)}\\
&= \frac{k(3k+5)+2}{2(3k+2)(3k+5)}\\
&= \frac{3k^2+5k+2}{2(3k+2)(3k+5)}\\
&= \frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}\\
&= \frac{k+1}{6k+10}\\
&= \frac{k+1}{6(k+1)+4}\\
&= \text{RHS}
\end{aligned}$$

6 Deduce that completing the base and inductive steps ensures that the original statement must be true for all specified values of n.

The original statement is true for $n = 1$, assumed true for $n = k$ and proven to be true for $n = k + 1$. Therefore $\sum_{r=1}^{n}\frac{1}{(3r-1)(3r+2)}=\frac{n}{6n+4}$ is true for all $n \in \mathbb{Z}^+$.

WARNING

There are no requirements to start a demonstration from the left-hand side (LHS) of a statement to be equal to the right-hand side (RHS), which is a common practice. It is perfectly valid to focus on the right-hand side initially.

While mathematical induction is most often used to prove a statement is true for all positive integers n, it can be adapted to prove a statement true for all integers greater than or equal to some particular integer other than 1. The only difference in the method is that the base case involves proving that the statement is true for the smallest possible value of n. For example, to prove that a statement is true for all non-negative integers, the base case would involve proving the statement for $n = 0$ rather than $n = 1$.

To use mathematical induction to prove that a statement is true for all integers $n \geq a$, where $a \in \mathbb{Z}$, it is necessary in the base step to prove that the statement is true for $n = a$.

EXERCISE 1.2 Mathematical induction for sums

Worked Example 6

1 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:
$2 + 4 + 6 + \ldots + 2n = n(n+1)$.

2 Use mathematical induction to prove the following result for all integers $n \geq 1$:
$1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(2n+1)(n+1)}{6}$

3 Use mathematical induction to prove the following result for all positive integers n:
$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{1}{3}n(2n-1)(2n+1)$

4 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:
$1 \times 2 + 2 \times 5 + \ldots + n(3n-1) = n^2(n+1)$

5 Use mathematical induction to prove the following result for all integers $n \geq 3$:
$7 + 9 + 11 + \ldots + (2n+1) = n^2 + 2n - 8$

6 Use mathematical induction to prove that the sum of the first n multiples of 3, commencing with the number 3, is equal to $\dfrac{3n(n+1)}{2}$.

7 Use mathematical induction to prove that the sum of the cubes of the first n positive integers is equal to $\dfrac{n^2(n+1)^2}{4}$.

Worked Example 7

8 Use mathematical induction to prove the following result for all integers $n \geq 1$:
$\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

9 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

10 Use mathematical induction to prove the sum of the first n positive integer powers of 3 is equal to $\frac{3^{n+1}-3}{2}$.

11 Use mathematical induction to prove the following result for all positive integers n:

$$\sum_{r=1}^{n} r(3)^r = \frac{3}{4}\left((2n-1)3^n+1\right)$$

12 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$a+ar+ar^2+\ldots+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

13 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=\frac{(n+1)!-1}{(n+1)!}$$

14 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$-2+4-8+\ldots+(-2)^n=\frac{2}{3}\left((-2)^n-1\right)$$

15 A particular statement appears to be true for all even positive integers.

(a) To prove this by induction, which of the following would be a valid way to proceed? (Assume that there exists odd positive integers for which the statement is not true.)

A Prove that the statement is true for $n = 1$; assume that the statement is true for $n = k$, where k is an integer with $k \geq 1$, and demonstrate that this would imply it were true for $n = k + 1$.

B Prove that the statement is true for $n = 4$; assume that the statement is true for $n = k$, where k is an even integer with $k \geq 4$, and demonstrate that this would imply it were true for $n = k + 2$.

C Prove that the statement is true for $n = 2$; assume that the statement is true for $n = k$, where k is an even integer with $k \geq 2$, and demonstrate that this would imply it were true for $n = k + 2$.

D Prove that the statement is true for $n = 1$; assume that the statement is true for $n = k$, where k is an even integer with $k \geq 1$, and demonstrate that this would imply it were true for $n = k + 2$.

(b) Explain the misconceptions or common errors presented in the incorrect options from part **(a)**.

16 Consider the following unusual sum: $n^1+n^2+n^3+\ldots+n^n$. When $n = k$, this sum is equal to $k^1+k^2+k^3+\ldots+k^k$.

(a) When $n = k + 1$, this sum is equal to:

A $k^1+k^2+k^3+\ldots+k^k+k^{k+1}$

B $k^1+k^2+k^3+\ldots+k^k+(k+1)^{k+1}$

C $(k+1)^1+(k+1)^2+(k+1)^3+\ldots+(k+1)^k$

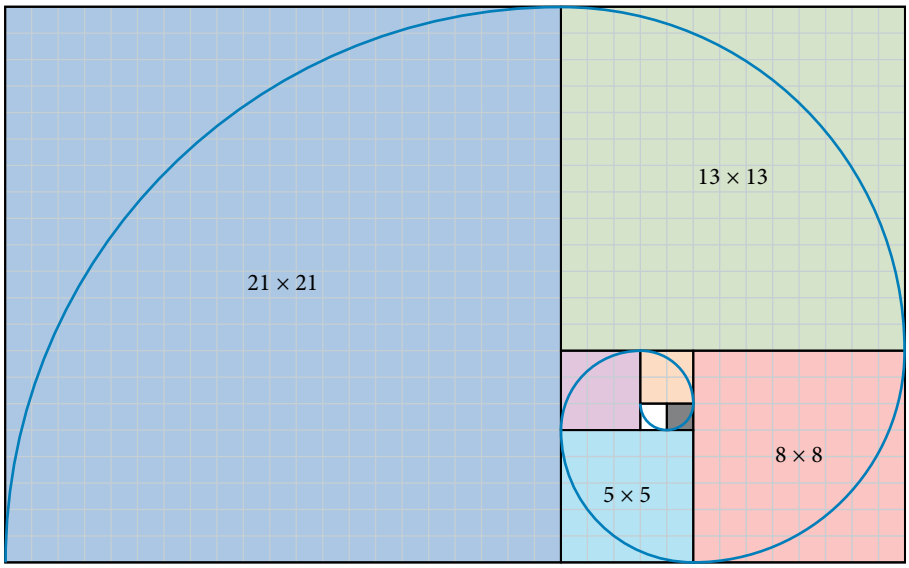
D $(k+1)^1+(k+1)^2+(k+1)^3+\ldots+(k+1)^{k+1}$

(b) Explain the common errors in each of the incorrect options in part **(a)**.

17 The Fibonacci numbers form a well-known sequence of numbers with the property that the sum of any two consecutive numbers gives the next number in the sequence. The first few Fibonacci numbers are:

$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13$, and so on.

Prove that $F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n F_{n+1}$ for all $n \in \mathbb{Z}^+$.



18 Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ with $\sin\left(\frac{1}{2}x\right) \neq 0$ and with $n \geq 0$. Prove that

$$1 + \cos(x) + \cos(2x) \ldots + \cos(nx) = \frac{\sin\left(\left(n + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2}.$$

The following trigonometric identity may be helpful:

$2\cos(A)\sin(B) = \sin(A + B) - \sin(A - B)$.

19 Prove the following result for all $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^{n} r^2 2^{n-r} = 2^{n+3} - 2^{n+1} - n^2 - 4n - 6$$

1.3 Mathematical induction for divisibility

Inductive proof is particularly effective in proving statements about divisibility. The following examples outline how to use mathematical induction to prove divisibility results.

8 Proving divisibility 1

Use mathematical induction to prove that $7^n - 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

THINKING	WORKING
1 Clarify the statement to be proved.	The statement given can be rewritten as: $7^n - 1 = 6A$, where $A \in \mathbb{Z}$.
2 Complete the base step; that is, prove the statement for the smallest possible value of n.	In this case, the smallest value of n is 1. When $n = 1$: $\text{LHS} = 7^1 - 1$ $= 6$ $= 6 \times 1$ $= \text{RHS}$ Therefore the statement is true for $n = 1$.
3 Clarify what is being assumed in the inductive step.	Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume $7^k - 1 = 6A$, where $A \in \mathbb{Z}$ [1]
4 Clarify what must be proved in the inductive step.	Must demonstrate statement is true when $n = k + 1$. That is: $7^{k+1} - 1 = 6B$, where $B \in \mathbb{Z}$:
5 Complete the proof for the inductive step.	$\text{LHS} = 7^{k+1} - 1$ $= 7 \times 7^k - 7 + 6$ $= 7\left(7^k - 1\right) + 6$ $= 7 \times 6A + 6$ $= 6(7A + 1)$ $= 6B$, where $B = 7A + 1$ $= \text{RHS}\left(A \in \mathbb{Z} \Rightarrow B = (7A + 1) \in \mathbb{Z}\right)$
6 Deduce that completing the base and inductive steps ensures that the original statement must be true for all specified values of n.	The original statement is true for $n = 1$, assumed true for $n = k$ and proven to be true for $n = k + 1$. Therefore, $7^n - 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

9 Proving divisibility 2

Use mathematical induction to prove that $2^{4n+3} + 3^{3n+1}$ is divisible by 11 for all integers $n \geq 0$.

THINKING	WORKING
1 Clarify the statement to be proved.	The statement given can be rewritten as: $2^{4n+3} + 3^{3n+1} = 11A$, where $A \in \mathbb{Z}$.
2 Complete the base step; that is, prove the statement for the smallest possible value of n.	In this case, the smallest value of n is 0. When $n = 0$: LHS $= 2^{4\times0+3} + 3^{3\times0+1}$ $= 2^3 + 3^1$ $= 11$ $= 11 \times 1$ $=$ RHS Therefore the statement is true for $n = 0$.
3 Clarify what is being assumed in the inductive step.	Let $k \geq 0$ and assume statement is true for $n = k$. That is, assume $2^{4k+3} + 3^{3k+1} = 11A$, where $A \in \mathbb{Z}$ [1]
4 Clarify what must be proved in the inductive step.	Must demonstrate statement is true when $n = k + 1$. That is: $2^{4(k+1)+3} + 3^{3(k+1)+1} = 11B$, where $B \in \mathbb{Z}$:
5 Complete the proof for the inductive step.	LHS $= 2^{4(k+1)+3} + 3^{3(k+1)+1}$ $= 2^{4k+7} + 3^{3k+4}$ $= 2^4 \times 2^{4k+3} + 3^3 \times 3^{3k+1}$ $= 2^4 \times 2^{4k+3} + 2^4 \times 3^{3k+1} - 2^4 \times 3^{3k+1} + 3^3 \times 3^{3k+1}$ $= 2^4\left(2^{4k+3} + 3^{3k+1}\right) - 16 \times 3^{3k+1} + 27 \times 3^{3k+1}$ $= 2^4 \times 11A + 11 \times 3^{3k+1}$ $= 16 \times 11A + 11 \times 3^{3k+1}$ $= 11\left(16A + 3^{3k+1}\right)$ $= 11B$, where $B = 16A + 3^{3k+1}$ $=$ RHS $(A, k \in \mathbb{Z} \Rightarrow B = 16A + 3^{3k+1} \in \mathbb{Z})$
6 Deduce that completing the base and inductive steps ensures that the original statement must be true for all specified values of n.	The original statement is true for $n = 0$, assumed true for $n = k$ and proven to be true for $n = k + 1$. Therefore $2^{4n+3} + 3^{3n+1}$ is divisible by 11 for all integers $n \geq 0$.

In Worked example 9, note that equation [1] was rearranged to obtain an expression for 2^{4k+3}, which was then substituted into the main proof for the inductive step. Alternatively, you could have rearranged equation [1] to obtain an expression for 3^{3k+1} and substituted this into the proof of the inductive step. In most situations, any substitution made using the equation should work. (It is almost certain you will need to use this equation—otherwise, the result could be proved readily without the need for induction at all).

With induction proofs in general, it is tempting to think that the trivial task of completing the base step is not important or essential. However, consider the following statement that is not true for any positive integers:

'$5n + 3$ is divisible by 5'

The inductive step for this statement works perfectly well. You are encouraged to complete the details, but essentially the expression $5n + 3$ increases by 5 as n is increased by 1, meaning that if the statement were true for some particular values of n, it would necessarily need to be true for the next value of n. The problem is that the base step cannot be completed as the statement is not true for $n = 1$. Thus, the inductive step on its own does not prove that the statement is true for any value of n.

WARNING

Completing the inductive step without the base step does not prove that a statement is true for all values of n.

EXERCISE 1.3 Mathematical induction for divisibility

Worked Example

8

1 Use mathematical induction to prove the following:

(a) $11^n - 1$ is divisible by 10 for all $n \in \mathbb{Z}^+$.

(b) $5^n + 3$ is divisible by 4 for all integers $n \geq 0$.

(c) $4^n - 7$ is divisible by 3 for all integers $n \geq 2$.

9

(d) $3^{3n} + 2^{n+2}$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

(e) $11^{n+2} + 12^{2n+1}$ is divisible by 133 for all integers $n \geq 0$.

(f) $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all integers $n \geq 0$.

(g) $3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all integers $n \geq 0$.

(h) $n^3 - n$ is divisible by 3 for all integers $n \geq 2$.

(i) the sum of the cubes of three consecutive positive integers is divisible by 9.

(j) $4^n - 3n - 1$ is divisible by 9 for all integers $n \geq 2$.

(k) $3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in \mathbb{Z}^+$.

2 Prove that $7^{2n} + 2^{3n-3} \times 3^{n-1}$ is divisible by 25 for all $n \in \mathbb{Z}^+$.

3 Prove that $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 for all integers $n \geq 0$.

4 Let x be an integer with $x \neq 1$. Prove that $x^n - 1$ is divisible by $x - 1$, for all integers $n \geq 1$.

5 Let x be a positive integer. Prove that $(x + 1)^n - nx - 1$ is divisible by x^2 for all integers $n \geq 2$.

6 Prove that $(n + 1)(n + 2)\ldots(2n)$ is divisible by 2^n for all $n \in \mathbb{Z}^+$.

Summary

Summation notation

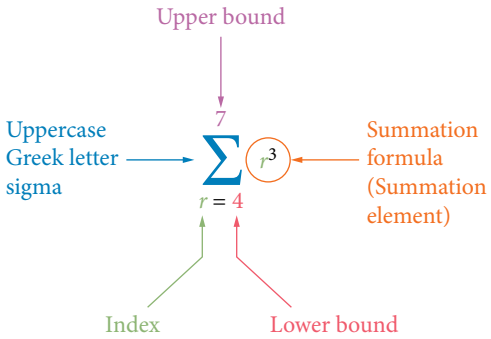
Summation notation is sometimes used to express long sums in more compact form.

An *expanded form* of a sum will show the value of each of its digits.

$$r_1 + r_2 + r_3 + \ldots + r_n$$

The *summation notation* is a concise expression for a sum of a sequence of numbers.

$$\sum_{r=r_1}^{n} r_i$$



To express a sum such as $\sum_{r=4}^{7} r^3$ in expanded form, substitute every integer from 4 to 7 in place of r in the expression following the Σ symbol, and add these terms together. That is, $\sum_{r=4}^{7} r^3 = 4^3 + 5^3 + 6^3 + 7^3$.

Proving divisibility

An integer is divisible by m if the number can be expressed in the form mA for some integer A. For example, 12 is divisible by 4 since $12 = 4 \times 3$ and 3 is an integer. However, 13 is not divisible by 4 since $13 = 4 \times 3.25$ and 3.25 is not an integer.

Mathematical induction

Mathematical induction is a method of proof applicable to statements involving an integer variable. It allows such statements to be proved for every integer value of this variable greater than some particular integer (most commonly 1).

To use mathematical induction to prove that a statement is true for all positive integers n, it is necessary to complete the following two steps:

STEP 1: (the *base step*)	Prove that the statement is true for the first case, often $n = 1$.
STEP 2: (the *inductive step*)	Let $k \geq 1$ and assume that the statement is true for $n = k$; then prove that, given this result, the statement would have to be true for $n = k + 1$.

To use mathematical induction to prove that a statement is true for all integers $n \geq a$, where $a \in \mathbb{Z}$, it is necessary in the base step to prove that the statement is true for $n = a$.

Adding, subtracting, or multiplying two or more integers produces an integer. Likewise, raising an integer to any non-negative integer power produces an integer.

Chapter review 1

Exercise 1.1

1 Prove that the following statement is true for $n = 5$, $n = 2$ and $n = 1$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Exercise 1.1

2 Express each of the following sums in expanded form:

(a) $\sum_{r=2}^{5}\sqrt[3]{r}$

(b) $\sum_{r=0}^{n}(-1)^r r!$

Exercise 1.1

3 **(a)** Write an expression in expanded form for the sum of the first n integers, starting with 7, that have a remainder of 2 when divided by 5.

(b) Express the sum from part **(a)** using summation notation.

Exercise 1.1

4 Consider the following sums:

$2+4+6+\ldots+2n$ and $n+(n+1)+(n+2)+\ldots+2n$

Which of the following statements is true?

A The second-last term in both sums is $2n-1$.

B The second-last term in both sums is $2(n-1)$.

C The second-last term in the first sum is $2(n-1)$, while the second-last term in the second sum is $2n-1$.

D The second-last term in the first sum is $2n-1$, while the second-last term in the second sum is $2(n-1)$.

Exercise 1.2

5 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$1+2+3+\ldots+n=\frac{n}{2}(n+1)$$

Exercise 1.2

6 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^{n}\frac{r^2}{(2r-1)(2r+1)}=\frac{r(r+1)}{2(2r+1)}$$

Exercise 1.2

7 Use mathematical induction to prove that the sum of the first n positive integer powers of 5 is equal to $\frac{5}{4}\left(5^n-1\right)$.

Exercise 1.2

8 Use mathematical induction to prove the following result for all positive integers n:

$$a+(a+d)+(a+2d)+\ldots+\left(a+(n-1)d\right)=\frac{n}{2}\left(2a+(n-1)d\right)$$

Exercise 1.2

9 Use mathematical induction to prove the following result for all positive integers n:

$$3\times 2^0+5\times 2^1+7\times 2^2+9\times 2^3+\ldots+(2n+1)2^{n-1}=1+(2n-1)2^n$$

Exercise 1.3

10 Use mathematical induction to prove that 9^n-1 is divisible by 8 for all $n \in \mathbb{Z}^+$.

Exercise 1.3

11 Use mathematical induction to prove that $5^n+2\times 11^n$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

Exercise 1.3

12 Use mathematical induction to prove that n^3+2n is divisible by 3 for all $n \in \mathbb{Z}^+$.

13 Use mathematical induction to prove that the product of any two consecutive positive integers is even. Exercise 1.3

14 Use mathematical induction to prove that $3^n - 1 - 2n$ is divisible by 4 for all integers $n \geq 2$. Exercise 1.3

15 The famous mathematician Srinivasa Ramanujan proved that: Exercise 1.2

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}} = 3.$$

Generalise this result by proving that:

$$\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\ldots}}}}} = n+1 \text{ for all integers } n \geq 2.$$

You may make use of Ramanujan's result in your proof.

16 Prove that $7^n - 4^n - 3^n$ is divisible by 12 for all integers $n \geq 2$. Exercise 1.3

17 Prove that $2^n - (-1)^n - 3$ is divisible by 6 for all integers $n \geq 3$. Exercise 1.3





2

Vectors and matrices

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Chapter review 142

2 Recall

Simplify surds

1 Express each of the following in simplest surd form $a\sqrt{b}$, where a and b are positive integers.

(a) $\sqrt{63}$ (b) $\sqrt{54}$ (c) $\sqrt{162}$ (d) $\sqrt{363}$

Calculate the exact distance between two points on a Cartesian plane

2 Calculate the exact distance between the following pairs of coordinate points.

(a) $C(-4,5)$ and $D(2,3)$ (b) $E(5,-3)$ and $F(4,9)$

(c) $G(-4,-6)$ and $H(3,5)$ (d) $I(-7,1)$ and $J(3,-6)$

Use technology to determine values of trigonometric ratios

3 Use technology to calculate the following trigonometric ratios, correct to 4 decimal places.

(a) $\cos(65°)$ (b) $\sin(156°)$ (c) $\sin(235°)$

(d) $\cos(128°)$ (e) $\tan(35°)$ (f) $\tan(118°)$

Determine exact values of trigonometric ratios

4 Calculate the exact values of the following trigonometric ratios.

(a) $\cos(60°)$ (b) $\sin(120°)$ (c) $\tan(315°)$

(d) $\sin(240°)$ (e) $\tan(150°)$ (f) $\cos(315°)$

Use technology to determine angles given trigonometric ratios

5 Use technology to calculate the value of the angle θ, in degrees, correct to 2 decimal places.

(a) $\sin(\theta)=0.3675$ (b) $\cos(\theta)=-0.9821$ (c) $\sin(\theta)=-0.4389$

(d) $\cos(\theta)=0.1256$ (e) $\tan(\theta)=1.4562$ (f) $\tan(\theta)=-0.6989$

Determine angles given trigonometric ratios without using technology

6 Solve the following equations for θ, in degrees, where $0° \leq \theta \leq 360°$, listing the solutions in ascending order.

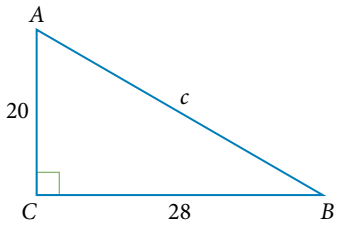
(a) $\sin(\theta)=\frac{\sqrt{3}}{2}$ (b) $\cos(\theta)=\frac{\sqrt{2}}{2}$ (c) $\tan(\theta)=\sqrt{3}$

(d) $\cos(\theta)=-\frac{1}{2}$ (e) $\tan(\theta)=-1$ (f) $\sin(\theta)=-\frac{\sqrt{2}}{2}$

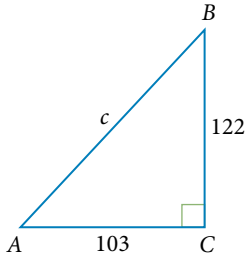
Solve right-angled triangles using trigonometry and Pythagoras' theorem

7 For right-angled triangles ΔABC, calculate the unknown side lengths and angles correct to 2 decimal places:

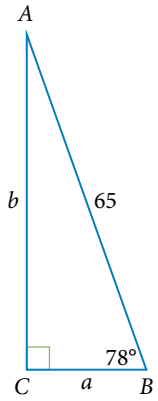
(a)



(b)



(c)



Solve non right-angled triangles using the sine rule

8 Solve ΔABC given that:

(a) $a = 39$, $b = 27$ and $A = 56°$

(b) $a = 56$, $c = 67$ and $C = 80°$

Solve non right-angled triangles using the cosine rule

9 Solve ΔABC given that:

(a) $a = 16$, $b = 19$ and $C = 65°$

(b) $b = 67$, $c = 54$ and $A = 96°$

Add and subtract vectors in component form

10 If $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$, calculate:

(a) $\boldsymbol{a} + \boldsymbol{b}$

(b) $5\boldsymbol{a} - 3\boldsymbol{b}$

Calculate determinants of 2×2 matrices

11 Calculate the determinant of the following matrices:

(a) $\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 5 & 2 \end{bmatrix}$

2.1 Vectors review

Scalar quantities and vector quantities

A scalar quantity can be completely described by its magnitude along with an appropriate unit such as time, length, speed, mass, temperature or density. Mathematically, a scalar quantity is just a number.

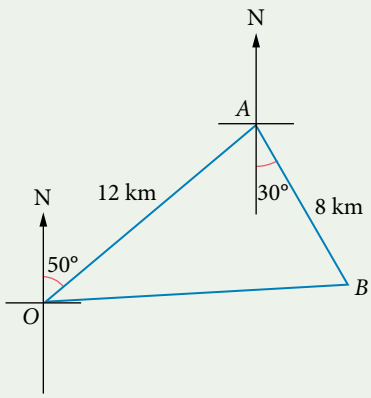
A vector quantity can only be completely described by specifying both its magnitude and direction. Examples of quantities that need to be specified in this way include:

- displacement (change in position of an object)
- velocity (rate of change of displacement or position with respect to time)
- acceleration (rate of change of velocity with respect to time)
- force (an action that alters the state of motion of an object)
- momentum (the product of an object's mass and velocity).

Problems involving vectors often require the use of trigonometric ratios, Pythagoras' theorem (for problems involving right-angled triangles), or the sine and cosine rules (for problems involving non-right-angled triangles).

1 Calculating displacement

From starting point O, a hiking group walk for 12 km on a bearing of N50°E to a camp site, A. The next day, they head off in the direction S30°E to another camp site, B, a distance of 8 km away. Calculate the group's final position relative to O, expressing your answers correct to 2 decimal places.

THINKING	WORKING
1 Sketch a diagram to illustrate the situation.	
2 Use properties of angles and triangles to determine $\angle OAB$.	$\angle OAB = 50° + 30°$ $= 80°$
3 Calculate the distance from O to B using the cosine rule.	$\overrightarrow{OB}^2 = \overrightarrow{OA}^2 + \overrightarrow{AB}^2 - 2 \times \overrightarrow{OA} \times \overrightarrow{AB} \times \cos(\angle OAB)$ $= 12^2 + 8^2 - 2 \times 12 \times 8 \times \cos(80°)$ $= 174.659\ldots$ $\therefore \overrightarrow{OB} = \sqrt{174.659\ldots}$ $= 13.22 \text{ km}$ (2 d.p.)

4 To specify the group's final position relative to O, its direction from O needs to be determined. Use the sine rule to calculate $\angle AOB$ in ΔOAB.	$\frac{\sin(\angle AOB)}{8} = \frac{\sin(80°)}{13.22}$ $\sin(\angle AOB) = \frac{\sin(80°)}{13.22} \times 8$ $\angle AOB = \sin^{-1}(0.5959\ldots)$ $\angle AOB = 36.58°$ (2 d.p.)
5 Determine the direction of the group relative to O.	The bearing from O is $\mathrm{N}(50° + 36.58°)\mathrm{E} = \mathrm{N}86.58°\mathrm{E}$.
6 Determine the group's final position relative to O.	The group's displacement from O is 13.2 km on a bearing of N86.58°E.

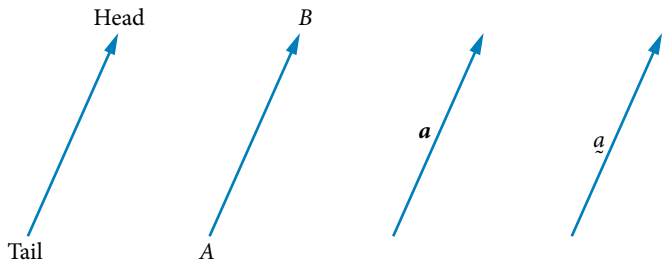
Vector algebra

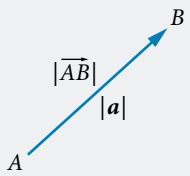
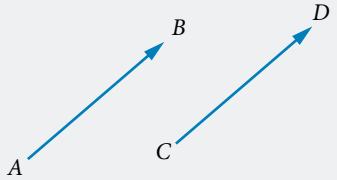
Vectors and vector notation

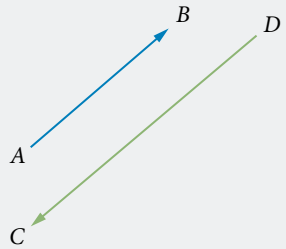
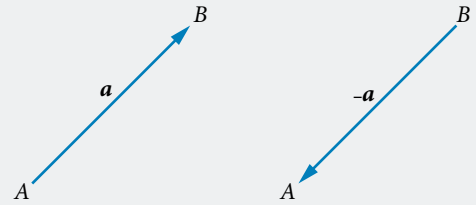
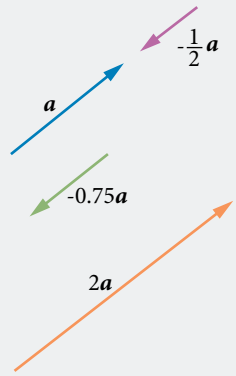
Vectors are denoted by a directed line segment. The length of the directed line segment represents the magnitude and the direction is indicated by its orientation.

The vector $\overrightarrow{AB}$ is defined where A is the initial point (the 'tail' of the vector) and B is the terminal point (the 'head' of the vector).

Often the vector will be labelled using a bold italic letter, such as $\boldsymbol{a}$. However, this notation can be impractical for handwritten solutions, so often the tilde notation $\underset{\sim}{a}$ (as seen in the worked solutions for this chapter) is used. This notation is useful for distinguishing vectors from scalars.



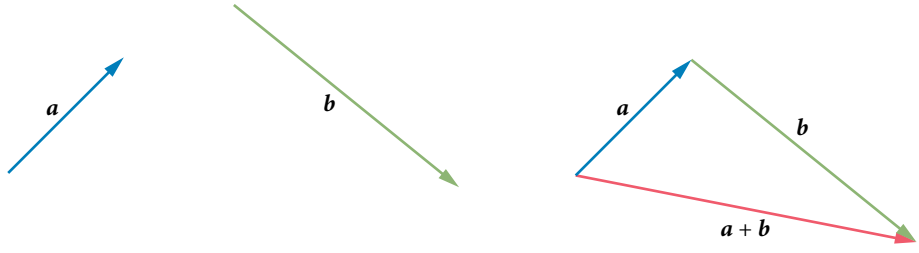
The magnitude of a vector	The magnitude of the vector is denoted by $\lvert\overrightarrow{AB}\rvert = \lvert\boldsymbol{a}\rvert$ and is a scalar quantity.	
Equality of vectors	Two vectors are said to be equal if and only if they have the same magnitude and the same direction.	

Parallel vectors	Two vectors are parallel vectors if they have the same or opposite direction.	
Negative of a vector	If $\boldsymbol{a}$ is the vector $\overrightarrow{AB}$, then $-\boldsymbol{a}$ is the vector $\overrightarrow{BA}$. This means that the negative of a vector has the same magnitude as the original, but the direction is reversed. That is, $-\overrightarrow{AB} = \overrightarrow{BA}$ or $-(\boldsymbol{a}) = -\boldsymbol{a}$	
The zero vector	The zero vector is a vector of zero magnitude and neutral direction. It is represented as a point and is denoted $\mathbf{0}$. For example, $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$.	
Scalar multiplication of a vector	Multiplying a vector by a scalar (a number), k, where $k \in \mathbb{R} \setminus \{0\}$, results in a vector with k times the magnitude that is parallel to the original vector. For example, the vector $2\boldsymbol{a}$ is parallel to vector $\boldsymbol{a}$ but its magnitude is two times as great, whereas the vector $-\frac{1}{2}\boldsymbol{a}$ is a vector parallel to $\boldsymbol{a}$ (although in the opposite direction) and with half the magnitude of $\boldsymbol{a}$.	

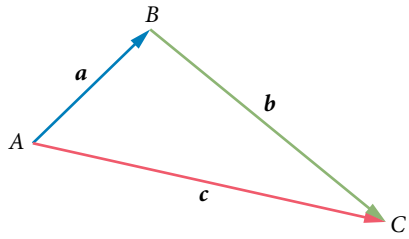
Addition of vectors

The triangle rule for the addition of vectors

To add two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the vectors are placed with the head of $\boldsymbol{a}$ at the tail of $\boldsymbol{b}$. The vector sum, also known as the resultant vector, is a single vector joining the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$. This vector addition forms a triangle, hence the *triangle rule*, and is sometimes called the *head-to-tail* rule for vector addition.



Also, consider ΔABC below. In this triangle, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ or $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}$.



Making connections

Triangle rule for addition of vectors

Move points C and D to observe the change in the resultant vector, $\boldsymbol{u}+\boldsymbol{v}$.

This method is not limited to the addition of two vectors and may be extended to any number of vectors in space (sometimes referred to as the *polygon* of vectors).

Making connections

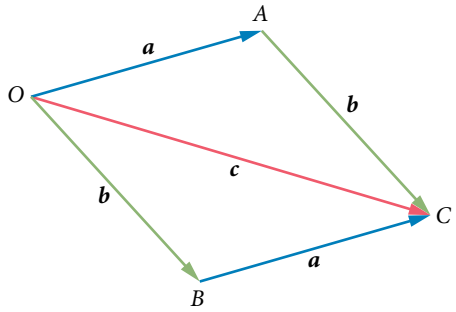
Parallelogram rule for addition of vectors

Move points C and D to observe the change in the resultant vector, $\boldsymbol{u}+\boldsymbol{v}$.

The parallelogram rule for addition

When two vectors are represented by directed line segments with the same initial point, rather than trying to move one of the vectors, a parallelogram can be drawn. The diagonal from the initial point of the parallelogram represents the sum of the vectors.

Vectors $\overrightarrow{OA}=\boldsymbol{a}$ and $\overrightarrow{OB}=\boldsymbol{b}$ generate the parallelogram $OACB$, whose diagonal, $\overrightarrow{OC}$ denotes the sum of $\overrightarrow{OA}$ and $\overrightarrow{OB}$:
$\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$ or $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}$.



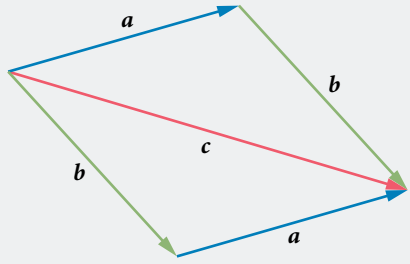
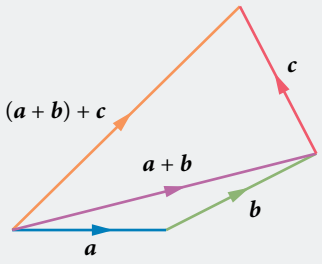
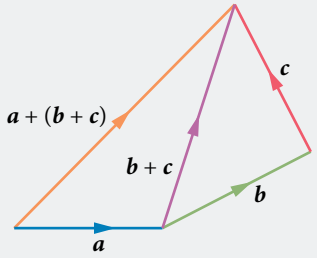
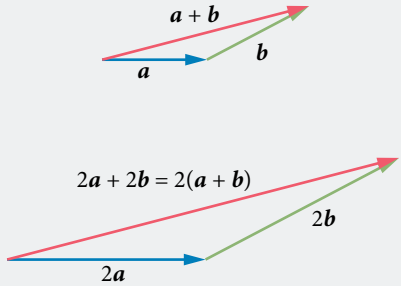
$\overrightarrow{OC}$ is also the sum of $\overrightarrow{OA}$ and $\overrightarrow{AC}$ or $\overrightarrow{OB}$ and $\overrightarrow{BC}$.

Since $\overrightarrow{BC}$ is parallel to and equal in magnitude to $\overrightarrow{OA}$, then $\overrightarrow{BC}=\boldsymbol{a}$.

Since $\overrightarrow{AC}$ is parallel to and equal in magnitude to $\overrightarrow{OB}$, then $\overrightarrow{AC}=\boldsymbol{b}$.

Therefore, $\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$ or $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}$ and $\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}$ or $\boldsymbol{b}+\boldsymbol{a}=\boldsymbol{c}$.

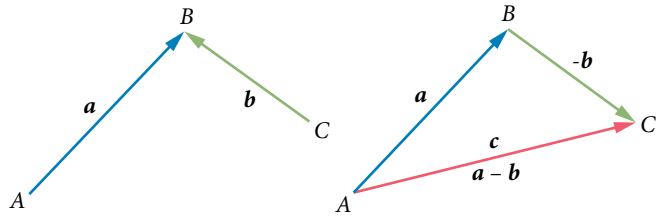
The diagram shows that $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}=\boldsymbol{c}$, which is known as the parallelogram rule.

The commutative law for addition	The associative law for addition	The distributive law
The order in which two vectors are added is not important; i.e. $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$.	The sum of three vectors does not depend on which pair of vectors is added first; i.e. $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$.	If $m\in\mathbb{R}$, then $m(\boldsymbol{a}+\boldsymbol{b})=m\boldsymbol{a}+m\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors. For example, $2(\boldsymbol{a}+\boldsymbol{b})=2\boldsymbol{a}+2\boldsymbol{b}$.
	 	

Subtraction of vectors

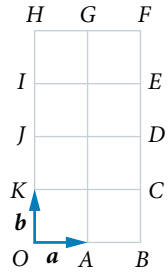
The subtraction of a vector is defined as the addition of a negative vector. Given two vectors $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{CB} = \boldsymbol{b}$, as $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$, to subtract $\boldsymbol{b}$ from $\boldsymbol{a}$ you add $-\boldsymbol{b}$ to $\boldsymbol{a}$: $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{c}$.

If a vector is subtracted from itself, the result is a zero vector: $\boldsymbol{a} - \boldsymbol{a} = 0$.



2 Vector arithmetic

In the diagram, the line segments $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$ etc. have the same length. If $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OK} = \boldsymbol{b}$, express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



(a) $\overrightarrow{OB}$

THINKING	WORKING
Identify vectors in terms of the given vectors $\boldsymbol{a}$ or $\boldsymbol{b}$, where possible.	$\overrightarrow{OB} = 2\boldsymbol{a}$

(b) $\overrightarrow{OI}$

Identify vectors in terms of the given vectors $\boldsymbol{a}$ or $\boldsymbol{b}$, where possible.	$\overrightarrow{OI} = 3\boldsymbol{b}$

(c) $\overrightarrow{AC}$

Identify vectors in terms of the given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, writing as the sum or difference of two vectors.	$\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$

(d) $\overrightarrow{CG}$

Identify vectors in terms of the given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, writing as the sum or difference of two vectors.	$\overrightarrow{CG} = -\boldsymbol{a} + 3\boldsymbol{b}$

(e) $\overrightarrow{FO}$

Identify vectors in terms of the given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, writing as the sum or difference of two vectors.

$\overrightarrow{FO} = -2\boldsymbol{a} - 4\boldsymbol{b}$

Vectors in two dimensions

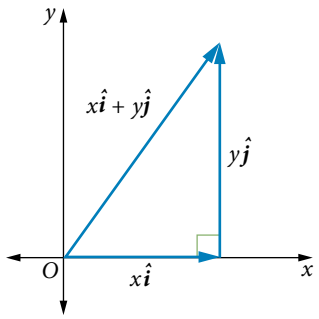
Unit vectors

A vector of magnitude 1 is called a *unit vector*.

A unit vector is denoted using a 'hat' as $\hat{\boldsymbol{a}}$, where $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$ (and $|\boldsymbol{a}|$ is the magnitude of $\boldsymbol{a}$).

Vectors in component form

The vector $\hat{\boldsymbol{i}}$ is defined to be a unit vector parallel to the x-axis, and the vector $\hat{\boldsymbol{j}}$ is defined to be the corresponding unit vector parallel to the y-axis. As these vectors are defined as having a magnitude of 1, it is not necessary to include the 'hat' (as seen in the worked solutions for this chapter where unit vectors use the tilde notation without a 'hat'). However, unit vectors will include the 'hat' in this text for clarity.



Any vector on the Cartesian plane can be expressed in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.

The form $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ is called Cartesian form, *component form* or $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}$ *form* of a vector. The vector $\boldsymbol{r}$ may also be represented in column matrix form as $\begin{bmatrix} x \\ y \end{bmatrix}$ or as Cartesian coordinates (x,y).

The magnitude and direction of a vector in component form

If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, then $|\boldsymbol{r}| = \sqrt{x^2 + y^2}$

and $\tan(\theta) = \dfrac{y}{x}$

where θ is the angle made by the vector $\boldsymbol{r}$ with the positive x-axis.

A vector can be defined in the Cartesian plane by expressing it in the form $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, or by stating its magnitude and direction.

3 Unit vectors in Cartesian or component form

Determine a unit vector $\hat{\boldsymbol{r}}$ that is parallel to the vector $\boldsymbol{r} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$.

THINKING	WORKING
1 Determine the magnitude of the vector. If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, then the magnitude is found by $\lvert\boldsymbol{r}\rvert = \sqrt{x^2 + y^2}$.	$\boldsymbol{r} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ $\lvert\boldsymbol{r}\rvert = \sqrt{4^2 + 3^2}$ $= \sqrt{25}$ $= 5$
2 The unit vector $\boldsymbol{r}$ is defined by $\hat{\boldsymbol{r}} = \dfrac{\boldsymbol{r}}{\lvert\boldsymbol{r}\rvert}$	$\hat{\boldsymbol{r}} = \frac{1}{5}\left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$ $= 0.6\hat{\boldsymbol{i}} + 0.8\hat{\boldsymbol{j}}$

Polar form of a vector

When expressing a vector in polar form, the magnitude and direction (from the positive x-axis) of the vector are stated directly.

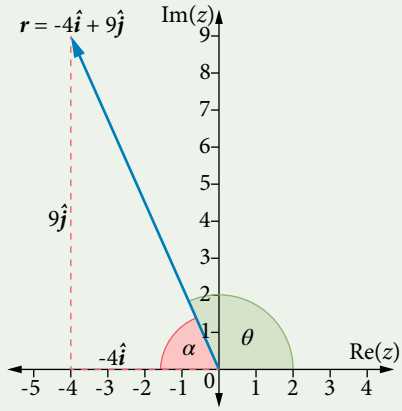
t Technology worked example
Calculating unit vectors

If vector $\boldsymbol{r}$ has a magnitude of $\lvert\boldsymbol{r}\rvert$ and forms an angle of θ (called the azimuth angle) with the positive x-axis, where $0° \leq \theta < 360°$, then the vector can be written in polar form as $\boldsymbol{r} = \left(\lvert\boldsymbol{r}\rvert, \theta\right)$.

WARNING
To avoid confusing polar and Cartesian form, the angle symbol can be added to give $\boldsymbol{r} = \left(\lvert\boldsymbol{r}\rvert, \angle\theta\right)$.

4 Expressing vectors in polar form

Express the vector $\boldsymbol{r} = -4\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}$ in polar form, stating your answer to 2 decimal places.

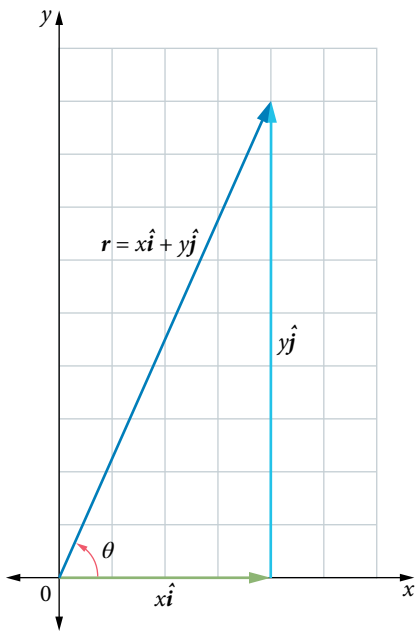
THINKING	WORKING
1 Sketch the position vector, where θ is the angle made by the vector $\boldsymbol{r}$ and the positive x-axis.	

2 If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, then use $\lvert\boldsymbol{r}\rvert = \sqrt{x^2 + y^2}$ to determine the magnitude of the vector.	$\boldsymbol{r} = -4\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}$ $\lvert\boldsymbol{r}\rvert = \sqrt{(-4)^2 + 9^2}$ $= \sqrt{97}$ ≈ 9.85 The length of vector $\boldsymbol{r}$ is 9.85 units.
3 If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, use the rule $\tan(\alpha) = \frac{y}{x}$ to determine α, the angle made by the vector $\boldsymbol{r}$ with the x-axis.	$\tan(\alpha) = -\frac{9}{4}$ $\alpha = \tan^{-1}\left(-\frac{9}{4}\right)$ $\alpha = -66.04°$ The angle made by vector $\boldsymbol{r}$ and the positive x-axis is -66.04°, that is 66.04° with the negative x-axis.
4 Express the angle in the range $0° \leq \theta < 360°$.	Angle θ is in the 2nd quadrant, therefore $\theta = 180° - 66.04°$ $= 113.96°$
5 Express in polar form.	$\boldsymbol{r} = (9.85, 113.96°)$

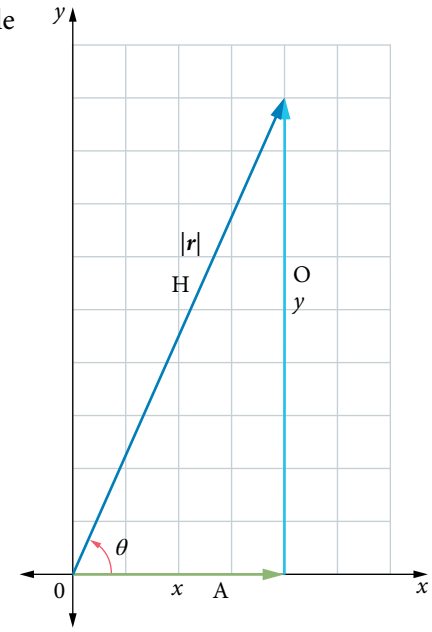
WARNING

When calculating the direction of vectors in component form, it is important to check the quadrant in which the vector lies. This should be done by drawing a sketch of the diagram, which will enable you to check the reasonableness of your solutions.

Consider the diagram showing vector $\boldsymbol{r}$.



The magnitude of vector $\boldsymbol{r}$ is given by $|\boldsymbol{r}|$ and the vector makes an angle of θ with the positive x-axis. Using the trigonometric ratios and the labelled diagram showing the magnitude of each component,



the vertical component of the vector $\boldsymbol{r}$ can be expressed as:

$$\sin(\theta) = \frac{\text{O}}{\text{H}}$$
$$\sin(\theta) = \frac{y}{|r|}$$
$$y = |r|\sin(\theta)$$

and the horizontal component of the vector $\boldsymbol{r}$ can be expressed as:

$$\cos(\theta) = \frac{\text{A}}{\text{H}}$$
$$\cos(\theta) = \frac{x}{|r|}$$
$$x = |r|\cos(\theta)$$

Therefore $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ can be expressed as $\boldsymbol{r} = |r|\cos(\theta)\hat{\boldsymbol{i}} + |r|\sin(\theta)\hat{\boldsymbol{j}}$.

5 Expressing vectors in component form

(a) $\boldsymbol{r}$ is a vector of magnitude 12 units and makes a direction of 30° with the positive x-axis. Express $\boldsymbol{r}$ in component form $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

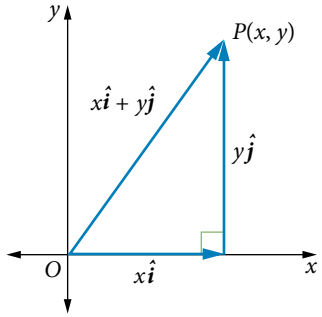
THINKING	WORKING
Recall that if a vector $\boldsymbol{r}$ of magnitude $\|\boldsymbol{r}\|$ makes an angle of θ with the positive x-axis, then $\boldsymbol{r} = \|\boldsymbol{r}\|\cos(\theta)\hat{\boldsymbol{i}} + \|\boldsymbol{r}\|\sin(\theta)\hat{\boldsymbol{j}}$	Since $\|\boldsymbol{r}\| = 12$ and $\theta = 30°$, then $\boldsymbol{r} = 12\cos(30°)\hat{\boldsymbol{i}} + 12\sin(30°)\hat{\boldsymbol{j}}$ $= 12 \times \frac{\sqrt{3}}{2}\hat{\boldsymbol{i}} + 12 \times \frac{1}{2}\hat{\boldsymbol{j}}$ $= 6\sqrt{3}\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$

(b) $\boldsymbol{r}$ is defined in polar form as $\boldsymbol{r} = (25.35, 172°)$. Express $\boldsymbol{r}$ in component form $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

1 Identify the quadrant in which the vector lies.	The vector is in the 2nd quadrant, hence the $\hat{\boldsymbol{i}}$ component is negative while the $\hat{\boldsymbol{j}}$ component is positive.
2 Recall that if a vector $\boldsymbol{r}$ of magnitude $\|\boldsymbol{r}\|$ makes an angle of θ with the positive x-axis, then $\boldsymbol{r} = \|\boldsymbol{r}\|\cos(\theta)\hat{\boldsymbol{i}} + \|\boldsymbol{r}\|\sin(\theta)\hat{\boldsymbol{j}}$.	Since $\|\boldsymbol{r}\| = 25.35$ and $\theta = 172°$, then $\boldsymbol{r} = 25.35\cos(172°)\hat{\boldsymbol{i}} + 25.35\sin(172°)\hat{\boldsymbol{j}}$ $= -25.10\hat{\boldsymbol{i}} + 3.53\hat{\boldsymbol{j}}$

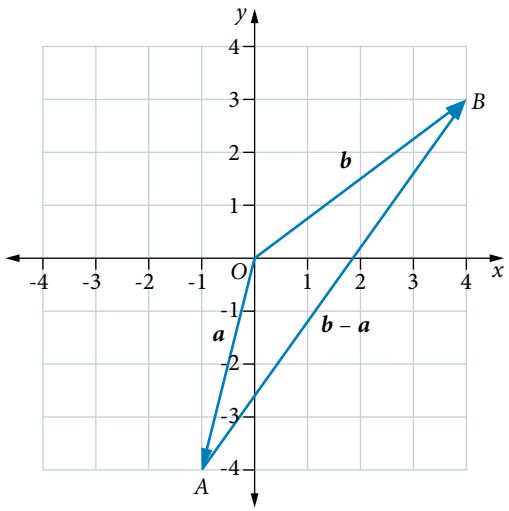
Position vectors

If P is the point $P(x, y)$, then $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. The vector $\overrightarrow{OP}$ is the *position vector* of P relative to O.



Relative position vectors

You have already looked at position vectors that represent the position of one point in relation to the origin. A *relative position vector* represents a point's position in relation to another point.



If point A is defined by the position vector $\boldsymbol{a}$ and point B is defined by the position vector $\boldsymbol{b}$, then the position vector of B relative to A is:

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\boldsymbol{a} + \boldsymbol{b}\\ &= \boldsymbol{b} - \boldsymbol{a}\end{aligned}$$

Likewise, the position vector of A relative to B is:

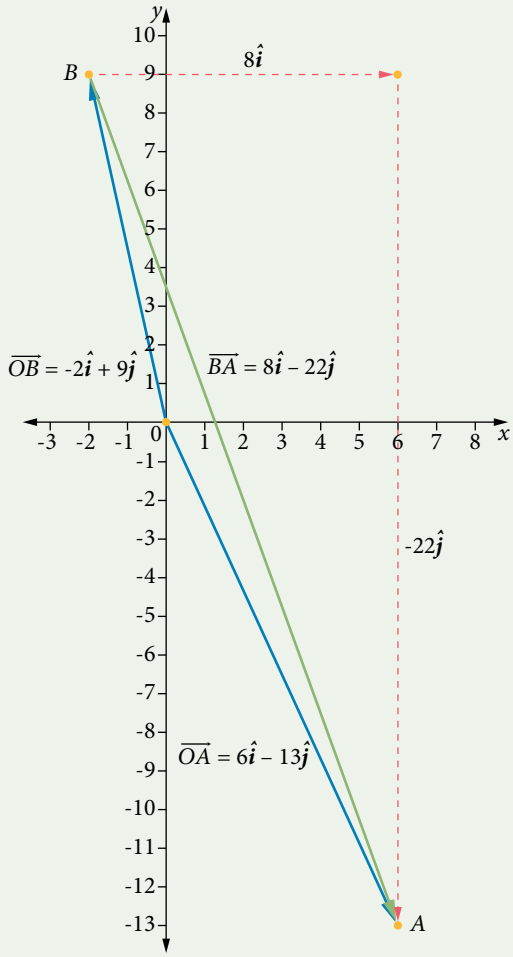
$$\begin{aligned}\overrightarrow{BA} &= \overrightarrow{BO} + \overrightarrow{OA}\\ &= -\boldsymbol{b} + \boldsymbol{a}\\ &= \boldsymbol{a} - \boldsymbol{b}\end{aligned}$$

6 Determining relative position vectors in component form

The position vector of point A on the Cartesian plane is $\overrightarrow{OA} = 6\hat{\boldsymbol{i}} - 13\hat{\boldsymbol{j}}$ and the position vector of point B is $\overrightarrow{OB} = -2\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}$. Calculate the position vector of A relative to B.

THINKING	WORKING
1 The position vector of A relative to B is $\overrightarrow{BA}$. Recall the rule to calculate $\overrightarrow{BA}$.	$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$ $\therefore \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$
2 Substitute the components and simplify.	$\overrightarrow{BA} = \left(6\hat{\boldsymbol{i}} - 13\hat{\boldsymbol{j}}\right) - \left(-2\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}\right)$ $= 8\hat{\boldsymbol{i}} - 22\hat{\boldsymbol{j}}$

3 Interpret the answer and evaluate the reasonableness of the answer with a graph.



Position vector A relative to position vector B is given by $\overrightarrow{BA} = 8\hat{i} - 22\hat{j}$.

Addition and subtraction of vectors in component form

Given two vectors $\boldsymbol{a} = x_1\hat{i} + y_1\hat{j}$ and $\boldsymbol{b} = x_2\hat{i} + y_2\hat{j}$, then the vectors $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{a}-\boldsymbol{b}$ are defined as follows:

$$\begin{aligned}\boldsymbol{a}+\boldsymbol{b} &= \left(x_1\hat{i} + y_1\hat{j}\right) + \left(x_2\hat{i} + y_2\hat{j}\right)\\ &= \left(x_1 + x_2\right)\hat{i} + \left(y_1 + y_2\right)\hat{j}\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}-\boldsymbol{b} &= \left(x_1\hat{i} + y_1\hat{j}\right) - \left(x_2\hat{i} + y_2\hat{j}\right)\\ &= \left(x_1 - x_2\right)\hat{i} + \left(y_1 - y_2\right)\hat{j}\end{aligned}$$

If $\boldsymbol{a} = x_1\hat{i} + y_1\hat{j}$ and $\boldsymbol{b} = x_2\hat{i} + y_2\hat{j}$, then

$\boldsymbol{a}+\boldsymbol{b} = \left(x_1 + x_2\right)\hat{i} + \left(y_1 + y_2\right)\hat{j}$ and $\boldsymbol{a}-\boldsymbol{b} = \left(x_1 - x_2\right)\hat{i} + \left(y_1 - y_2\right)\hat{j}$

Scalar multiplication of vectors in component form

Multiplying a vector $\boldsymbol{a}$ by a scalar (a number) k, where $k \in \mathbb{R}\setminus\{0\}$, results in a vector with k times the magnitude that is parallel to the original vector $\boldsymbol{a}$.

$k\boldsymbol{a} = k(x\hat{i} + y\hat{j}) = kx\hat{i} + ky\hat{j}$

If $\boldsymbol{a} = x_1\hat{i} + y_1\hat{j}$, then $k\boldsymbol{a} = kx\hat{i} + ky\hat{j}$.

7 Addition, subtraction and scalar multiplication of vectors in component form

For vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -5\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$, calculate each of the following:

(a) $\boldsymbol{a}+\boldsymbol{b}$

THINKING	WORKING
1 Write the sum as the sum of the vectors in component form.	$\boldsymbol{a}+\boldsymbol{b}=\left(3\hat{\boldsymbol{i}}+8\hat{\boldsymbol{j}}\right)+\left(-5\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}\right)$
2 Group the coefficients of the components together and simplify.	$\boldsymbol{a}+\boldsymbol{b}=(3+(-5))\hat{\boldsymbol{i}}+(8+(-4))\hat{\boldsymbol{j}}$ $=-2\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}$

(b) $\boldsymbol{b}-\boldsymbol{c}$

1 Write the calculation as the difference of the vectors in component form.	$\boldsymbol{b}-\boldsymbol{c}=\left(-5\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}\right)-\left(\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}\right)$
2 Group the coefficients of the components together and simplify.	$\boldsymbol{b}-\boldsymbol{c}=(-5-1)\hat{\boldsymbol{i}}+(-4-(-2))\hat{\boldsymbol{j}}$ $=-6\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}$

(c) $3\boldsymbol{a}+5\boldsymbol{c}$

1 Write the sum as the sum of the vectors in component form.	$3\boldsymbol{a}+5\boldsymbol{c}=3\left(3\hat{\boldsymbol{i}}+8\hat{\boldsymbol{j}}\right)+5\left(\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}\right)$
2 Calculate the scalar multiplication by expanding the brackets.	$=9\hat{\boldsymbol{i}}+24\hat{\boldsymbol{j}}+5\hat{\boldsymbol{i}}-10\hat{\boldsymbol{j}}$
3 Group the coefficients of the components together and simplify.	$=(9+5)\hat{\boldsymbol{i}}+(24-10)\hat{\boldsymbol{j}}$ $=14\hat{\boldsymbol{i}}+14\hat{\boldsymbol{j}}$

Technology worked example

Addition, subtraction and scalar multiplication of vectors

EXERCISE 2.1 Vectors review

1 From his house, Frederick rode his bicycle 300 m east and then 400 m north to Hubert's house. Calculate the displacement of Hubert's house from Frederick's house, stating your answers to the nearest whole number.

2 Three cyclists are riding in a reserve. Roslyn rides 500 m west, turns, and then rides 400 m east. Francois rides 600 m south, turns and then rides 800 m west. Gabrielle rides 300 m north, turns and then rides 500 m in a direction N30°E. Round answers to 2 decimal places, where necessary.

Worked Example

(a) Determine the final displacement of each cyclist from their starting point.

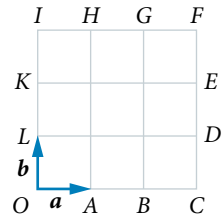
(b) Determine the total distance travelled by each cyclist.

1

3 Sarah and Jake are orienteering in the state forest. They travel from the starting point O for a distance of 8 km on a bearing of N20°E until reaching checkpoint A. After a rest, they then travel in the direction S77°E until they reach checkpoint B, a distance of 8 km from A. What is their final position relative to their starting position O? Specify answers correct to 2 decimal places.

2

4 In the diagram, the line segments $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{CD}$ etc. have the same length.



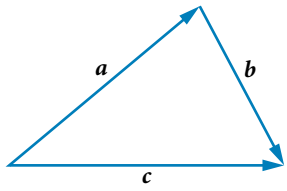
If $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OL} = \boldsymbol{b}$, express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) $\overrightarrow{OC}$ (b) $\overrightarrow{OK}$ (c) $\overrightarrow{AC}$ (d) $\overrightarrow{AD}$

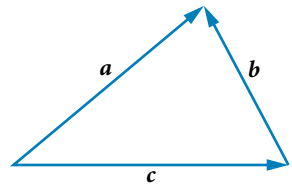
(e) $\overrightarrow{CG}$ (f) $\overrightarrow{FO}$ (g) $\overrightarrow{KC}$ (h) $\overrightarrow{EI}$

5 Express $\boldsymbol{c}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ for the following diagrams.

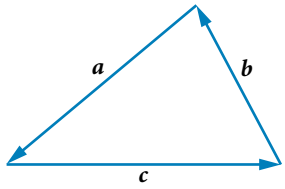
(a)



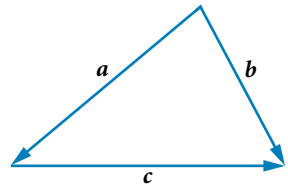
(b)



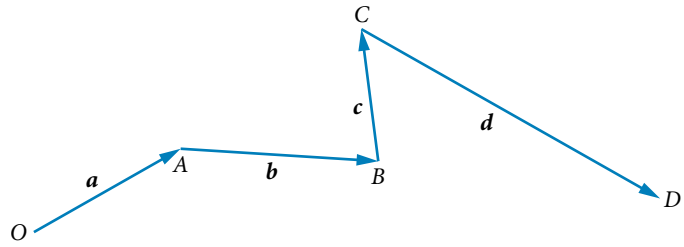
(c)



(d)

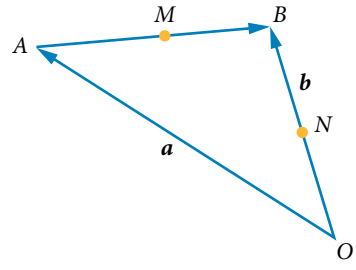


6 Express the following line segments in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ for the diagram shown.



(a) $\overrightarrow{OB}$ (b) $\overrightarrow{OC}$ (c) $\overrightarrow{DA}$ (d) $\overrightarrow{DO}$

7 In ΔOAB shown, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. M and N are the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{OB}$ respectively. Express $\overrightarrow{OM}$, $\overrightarrow{AN}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

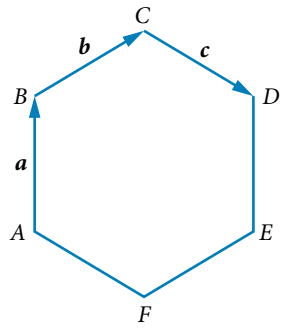


8 $ABCDEF$ is a regular hexagon with $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{BC} = \boldsymbol{b}$ and $\overrightarrow{CD} = \boldsymbol{c}$.
Express each of the following in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

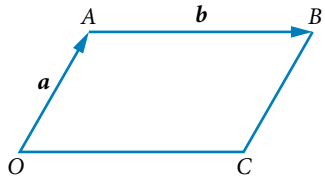
(a) $\overrightarrow{AC}$ (b) $\overrightarrow{FA}$

(c) $\overrightarrow{AD}$ (d) $\overrightarrow{AE}$

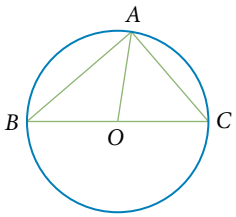
(e) $\overrightarrow{FC}$



9 $OABC$ is a parallelogram with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$. Express $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



10 A is a point on a circle with diameter BOC.
If $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$, then express $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



Worked Example

11 Determine a unit vector parallel to each of the following vectors: 3

(a) $\boldsymbol{a} = 8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ (b) $\boldsymbol{b} = 24\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$

12 Express the vector $\boldsymbol{c} = -5\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$ in polar form, giving your answer to 2 decimal places. 4

13 Express the following vectors in component form, $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. Answer in exact form where appropriate. 5

(a) $\boldsymbol{r}$ is a vector of magnitude 5 units and makes a direction of 60° with the positive x-axis.

(b) $\boldsymbol{r} = (25, 45°)$

(c) $\boldsymbol{r}$ is a vector of magnitude 33.21 units and makes a direction of 121° with the positive x-axis.

14 For each of the points P whose coordinates are given, determine the position vector $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ and hence determine a unit vector in the direction of $\overrightarrow{OP}$. 6

(a) $P(3, -2)$ (b) $P(12, 5)$ (c) $P(-2, 2)$ (d) $P(-3, -7)$

15 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$, calculate: 7

(a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $3\boldsymbol{a} + 2\boldsymbol{c}$ (c) $-3\boldsymbol{a} - 4\boldsymbol{b}$ (d) $-2\boldsymbol{a} - 4(\boldsymbol{b} - 2\boldsymbol{c})$

16 Given that $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, calculate:

(a) the magnitudes of the three vectors $|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and $|\boldsymbol{c}|$ respectively.

(b) the unit vectors $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{c}}$.

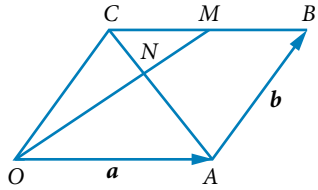
17 **(a)** Which one of the following is a unit vector parallel to $\boldsymbol{r} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$?

A $\frac{1}{5}\left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$ **B** $\frac{3}{5}\hat{\boldsymbol{i}} + \frac{4}{5}\hat{\boldsymbol{j}}$ **C** $\frac{1}{5}\left(-3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$ **D** $-3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

(b) What could be common errors in selecting the incorrect options in part **(a)**?

18 If $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, determine a simplified expression for $3\boldsymbol{a} + 5\boldsymbol{b}$.

19 $OABC$ is a parallelogram with M being the midpoint of BC. N is a point on $\overrightarrow{OM}$ such that $\overrightarrow{ON} = \frac{2}{3}\overrightarrow{OM}$.



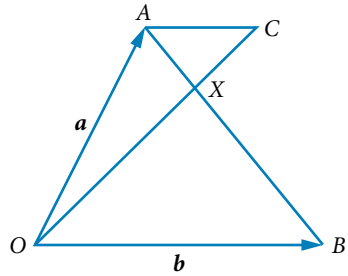
(a) If $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$, express each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(i) $\overrightarrow{OM}$ **(ii)** $\overrightarrow{ON}$ **(iii)** $\overrightarrow{CN}$ **(iv)** $\overrightarrow{CA}$

(b) Demonstrate that A, N and C are collinear by proving that $\overrightarrow{CN} = \frac{1}{3}\overrightarrow{CA}$.

20 Two triangles OAB and OAC are drawn. Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

X is a point on $\overrightarrow{AB}$ such that $\overrightarrow{XA} = \frac{1}{2}\overrightarrow{BX}$ and C is a point such that $\overrightarrow{XC} = \frac{1}{2}\overrightarrow{OX}$.



(a) Express each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

(i) $\overrightarrow{BA}$

(ii) $\overrightarrow{BX}$

(iii) $\overrightarrow{OX}$

(iv) $\overrightarrow{XC}$

(v) $\overrightarrow{AC}$

(b) Explain how part **(a)** demonstrates that $\overrightarrow{AC}$ is parallel to $\overrightarrow{OB}$ and half its length.

Vectors in three dimensions

2.2

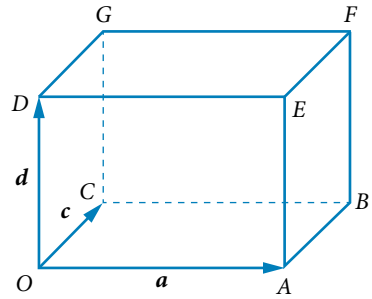
Vectors in three dimensions (the $\hat{i}, \hat{j}, \hat{k}$ system)

It is often necessary to work in three-dimensional space, so the Cartesian plane needs to be extended into a third dimension. This third dimension is usually denoted using a z-axis. Three mutually perpendicular axes, the x-, y- and z-axes, are used to locate points in three-dimensional space. Thus, the three-dimensional Cartesian coordinates of a point in space will be (x, y, z).

The vector rules and conventions relating to the two-dimensional Cartesian plane are easily applicable to the three-dimensional space.

8 Vector arithmetic in three dimensions

$OABCGDEF$ is a cuboid with $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OC} = \boldsymbol{c}$ and $\overrightarrow{OD} = \boldsymbol{d}$.



Express each of the following vectors in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

(a) $\overrightarrow{DE}$

THINKING	WORKING
Identify equal vectors and write in terms of given vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.	$\overrightarrow{DE} = \overrightarrow{OA}$ $= \boldsymbol{a}$

(b) $\overrightarrow{BC}$

Identify equal vectors and write in terms of given vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.	$\overrightarrow{BC} = -\overrightarrow{OA}$ $= -\boldsymbol{a}$

(c) $\overrightarrow{OB}$

Write as the sum or difference of two or more vectors and use vector addition to write in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.	$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ $= \boldsymbol{a} + \boldsymbol{c}$

(d) $\overrightarrow{AC}$

Write as the sum or difference of two or more vectors and use vector addition to write in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

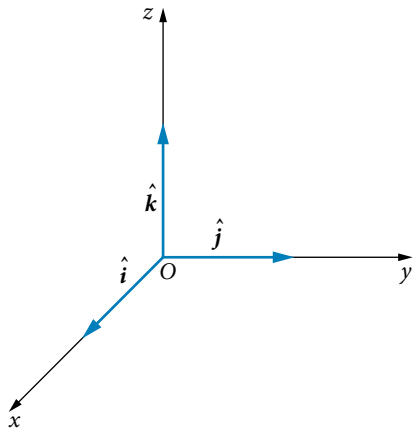
$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= -\overrightarrow{OA} + \overrightarrow{OC} \\ &= -\boldsymbol{a} + \boldsymbol{c} \\ &= \boldsymbol{c} - \boldsymbol{a}\end{aligned}$$

(e) $\overrightarrow{DB}$

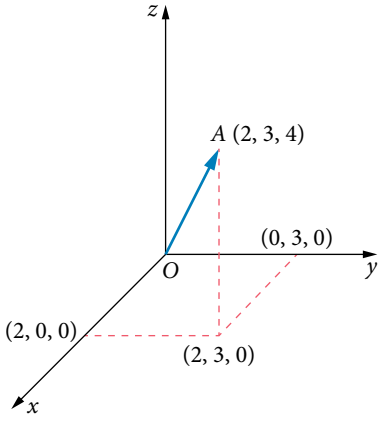
Write as the sum or difference of two or more vectors and use vector addition to write in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

$$\begin{aligned}\overrightarrow{DB} &= \overrightarrow{DO} + \overrightarrow{OB} \\ &= -\overrightarrow{OD} + \boldsymbol{a} + \boldsymbol{c} \\ &= -\boldsymbol{d} + \boldsymbol{a} + \boldsymbol{c} \\ &= \boldsymbol{a} + \boldsymbol{c} - \boldsymbol{d}\end{aligned}$$

The unit vectors in the positive direction of the x-, y- and z-axes, $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ respectively, are shown in the diagram.



For example, the point $A(2, 3, 4)$ can be represented in space as shown in the following diagram:



In three-dimensional space, a third unit vector $\hat{\boldsymbol{k}}$ is parallel to the z-axis and the position vector of a point with coordinates (x, y, z) will be $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$.

Vectors in three dimensions are of the form $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ or in column matrix form $\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

The position vector of the coordinate point $A(2, 3, 4)$ shown would be written as $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ or $\overrightarrow{OA} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

The magnitude of a vector in three-dimensional space

The magnitude of vector $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ is $|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$.

An alternative way of expressing this formula is:

If $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$, then $|\boldsymbol{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$.

Unit vectors in three-dimensional space

If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then $\hat{\boldsymbol{r}} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})$ or $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$.

An alternative way of expressing this formula is:

If $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$, then $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}}\left(a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}\right)$.

9 Magnitude of a vector in three dimensions and unit vectors

For each of the following vectors calculate the exact magnitude of the vector and a unit vector parallel to the given vector.

(a) $\boldsymbol{a} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

THINKING	WORKING
1 The magnitude of a vector is determined from $\|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$, where $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$.	$\|\boldsymbol{a}\| = \sqrt{1^2 + 2^2 + 2^2}$ $= \sqrt{9}$ $= 3$
2 The unit vector $\hat{\boldsymbol{r}}$ is defined by $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}$.	$\hat{\boldsymbol{a}} = \frac{1}{3}\left(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)$

(b) $\boldsymbol{b} = 5\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

1 The magnitude of a vector is determined from $\|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$, where $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$.	$\|\boldsymbol{b}\| = \sqrt{5^2 + (-7)^2 + (-1)^2}$ $= \sqrt{75}$ $= 5\sqrt{3}$
2 The unit vector $\hat{\boldsymbol{r}}$ is defined by $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}$. Rationalise the denominator where appropriate.	$\hat{\boldsymbol{b}} = \frac{1}{5\sqrt{3}}\left(5\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$ $= \frac{\sqrt{3}}{15}\left(5\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$

Calculations involving vectors in three-dimensional space are carried out in a similar way to those involving vectors in the two-dimensional plane.

Addition and subtraction of vectors in component form

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then

$$\begin{aligned}\boldsymbol{a}+\boldsymbol{b} &= \left(x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}\right) + \left(x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}\right)\\ &= (x_1 + x_2)\hat{\boldsymbol{i}} + (y_1 + y_2)\hat{\boldsymbol{j}} + (z_1 + z_2)\hat{\boldsymbol{k}}\end{aligned}$$

Similarly,

$$\begin{aligned}\boldsymbol{a}-\boldsymbol{b} &= \left(x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}\right) - \left(x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}\right)\\ &= (x_1 - x_2)\hat{\boldsymbol{i}} + (y_1 - y_2)\hat{\boldsymbol{j}} + (z_1 - z_2)\hat{\boldsymbol{k}}\end{aligned}$$

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then

$\boldsymbol{a}+\boldsymbol{b} = (x_1 + x_2)\hat{\boldsymbol{i}} + (y_1 + y_2)\hat{\boldsymbol{j}} + (z_1 + z_2)\hat{\boldsymbol{k}}$

$\boldsymbol{a}-\boldsymbol{b} = (x_1 - x_2)\hat{\boldsymbol{i}} + (y_1 - y_2)\hat{\boldsymbol{j}} + (z_1 - z_2)\hat{\boldsymbol{k}}$

Scalar multiplication of vectors in component form

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$, then for $k \in \mathbb{R}$,

$$\begin{aligned}k\boldsymbol{a} &= k\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}\right)\\ &= kx\hat{\boldsymbol{i}} + ky\hat{\boldsymbol{j}} + kz\hat{\boldsymbol{k}}\end{aligned}$$

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$, then

$k\boldsymbol{a} = kx\hat{\boldsymbol{i}} + ky\hat{\boldsymbol{j}} + kz\hat{\boldsymbol{k}}$ for $k \in \mathbb{R}$.

10 Algebraic operations with three-dimensional vectors in component form

If $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$, calculate each of the following:

(a) $\boldsymbol{a}+\boldsymbol{b}$

THINKING	WORKING
1 Write the sum as the sum of the vectors in component form.	$\boldsymbol{a}+\boldsymbol{b} = \left(4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right) + \left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}\right)$
2 Group the coefficients of the components together and simplify.	$= (4+3)\hat{\boldsymbol{i}} + (-5+4)\hat{\boldsymbol{j}} + (3-6)\hat{\boldsymbol{k}}$ $= 7\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

(b) $3\boldsymbol{a} - 2\boldsymbol{b}$

1 Write the calculation as the difference of the vectors in component form.	$3\boldsymbol{a} - 2\boldsymbol{b} = 3\left(4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right) - 2\left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}\right)$
2 Perform the multiplication first.	$= 12\hat{\boldsymbol{i}} - 15\hat{\boldsymbol{j}} + 9\hat{\boldsymbol{k}} - 6\hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}$
3 Group the coefficients of the components together and simplify.	$= (12-6)\hat{\boldsymbol{i}} + (-15-8)\hat{\boldsymbol{j}} + (9+12)\hat{\boldsymbol{k}}$ $= 6\hat{\boldsymbol{i}} - 23\hat{\boldsymbol{j}} + 21\hat{\boldsymbol{k}}$

Technology worked example
Vector arithmetic in three dimensions

Relative position vectors in three dimensions

Recall that if point A is defined by the position vector $\boldsymbol{a}$ and point B is defined by the position vector $\boldsymbol{b}$, then the position vector of B relative to A is $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \boldsymbol{b} - \boldsymbol{a}$. Likewise, the position vector of A relative to B is $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \boldsymbol{a} - \boldsymbol{b}$.

In three-dimensional space, if the position vectors of points A and B are $\overrightarrow{OA} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then vector $\overrightarrow{AB}$ can be calculated:

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}} - \left(x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}\right) \\ &= \left(x_2 - x_1\right)\hat{\boldsymbol{i}} + \left(y_2 - y_1\right)\hat{\boldsymbol{j}} + \left(z_2 - z_1\right)\hat{\boldsymbol{k}}\end{aligned}$$

11 Calculating relative position vectors in three dimensions

A is the point with coordinates $(2, -3, 1)$ and B is the point $(-3, 4, -2)$.

(a) Determine the position vectors of points A and B.

THINKING	WORKING
The position vector of a point P with coordinates (x, y, z) is $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$. Write the position vectors for the points.	Since $A = (2, -3, 1)$, then $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$. Since $B = (-3, 4, -2)$, then $\overrightarrow{OB} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$.

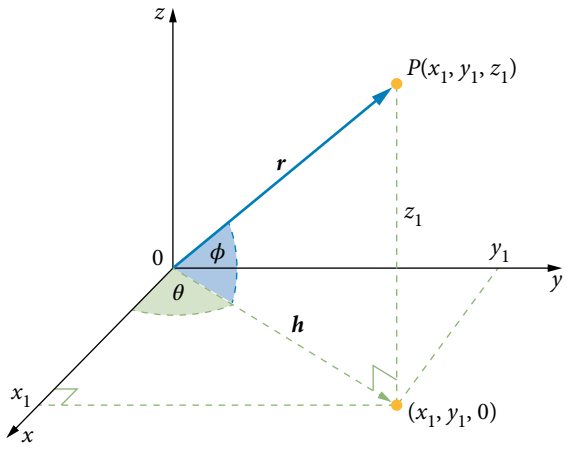
(b) Hence, calculate the vector $\overrightarrow{AB}$.

Calculate the vector $\overrightarrow{AB}$, the vector between the two points A and B.	$\begin{aligned}\overrightarrow{AB} &= \left(x_2 - x_1\right)\hat{\boldsymbol{i}} + \left(y_2 - y_1\right)\hat{\boldsymbol{j}} + \left(z_2 - z_1\right)\hat{\boldsymbol{k}} \\ &= (-3 - 2)\hat{\boldsymbol{i}} + \left(4 - (-3)\right)\hat{\boldsymbol{j}} + (-2 - 1)\hat{\boldsymbol{k}} \\ &= -5\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\end{aligned}$

Polar form of vectors in three dimensions

Recall that if a vector $\boldsymbol{r}$ in two dimensions has a magnitude of $|\boldsymbol{r}|$ and forms an angle of θ with the positive x-axis, where $0° \leq \theta < 360°$, then the vector can be written in polar form as $\boldsymbol{r} = \left(|\boldsymbol{r}|, \theta°\right)$.

Extending this to a three-dimensional coordinate system, it is necessary to use two angles to define the direction of a vector in polar form. The *azimuth* angle θ is measured anticlockwise from the x-axis onto the projection of the vector to the xy-plane ($0° \leq \theta < 360°$). The *altitude* angle ϕ is the angle from the xy-plane to the vector ($-90° \leq \phi < 90°$).



If you are given a vector in polar form, the x, y and z components of vector $\boldsymbol{r}$ can be calculated using trigonometric ratios:

$\sin(\phi)=\frac{z_1}{|\boldsymbol{r}|}$ so $z_1=|\boldsymbol{r}|\sin(\phi)$

$\cos(\phi)=\frac{h}{|\boldsymbol{r}|}$ so $h=|\boldsymbol{r}|\cos(\phi)$

$\sin(\theta)=\frac{y_1}{h}=\frac{y_1}{|\boldsymbol{r}|\cos(\phi)}$ so $y_1=|\boldsymbol{r}|\cos(\phi)\sin(\theta)$

$\cos(\theta)=\frac{x_1}{h}=\frac{x_1}{|\boldsymbol{r}|\cos(\phi)}$ so $x_1=|\boldsymbol{r}|\cos(\phi)\cos(\theta)$

If you are given a vector in component form, the magnitude, altitude and azimuth can be calculated:

$|\boldsymbol{r}|=\sqrt{(x_1)^2+(y_1)^2+(z_1)^2}$

$\tan(\theta)=\frac{y_1}{x_1}$ so $\theta=\tan^{-1}\left(\frac{y_1}{x_1}\right)$

$\sin(\phi)=\frac{z_1}{|\boldsymbol{r}|}$ so $\phi=\sin^{-1}\left(\frac{z_1}{|\boldsymbol{r}|}\right)$

The vector $\boldsymbol{r}=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$ with magnitude $|\boldsymbol{r}|=\sqrt{(x_1)^2+(y_1)^2+(z_1)^2}$ can be expressed in polar form as $\boldsymbol{r}=(|\boldsymbol{r}|,\theta,\phi)$, where θ is the azimuth angle of $\boldsymbol{r}$ $(0°\le\theta<360°)$ and ϕ is the altitude angle of $\boldsymbol{r}$ $(-90°\le\phi<90°)$.

The components of $\boldsymbol{r}$ can be determined:

$x_1=|\boldsymbol{r}|\cos(\phi)\cos(\theta)$

$y_1=|\boldsymbol{r}|\cos(\phi)\sin(\theta)$

$z_1=|\boldsymbol{r}|\sin(\phi)$

The azimuth and altitude can be determined:

$\theta=\tan^{-1}\left(\frac{y_1}{x_1}\right)$

$\phi=\sin^{-1}\left(\frac{z_1}{|\boldsymbol{r}|}\right)$

12 Polar form of vectors in three dimensions

(a) Express the vector $\boldsymbol{u}=(10,135°,30°)$ in component form.

THINKING	WORKING
1 Determine the necessary information from the vector in the form $(|\boldsymbol{r}|,\theta,\phi)$.	$|\boldsymbol{u}|=10,\ \theta=135°,\ \phi=30°$

2 Use trigonometry to calculate the vector components, expressing your answers in exact form where possible.

$$x_1 = |\boldsymbol{u}|\cos(\phi)\cos(\theta)$$
$$= 10\cos(30°)\cos(135°)$$
$$= 10 \times \frac{\sqrt{3}}{2} \times \frac{-1}{\sqrt{2}}$$
$$= -\frac{5\sqrt{6}}{2}$$
$$y_1 = |\boldsymbol{u}|\cos(\phi)\sin(\theta)$$
$$= 10\cos(30°)\sin(135°)$$
$$= 10 \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$
$$= \frac{5\sqrt{6}}{2}$$
$$z_1 = |\boldsymbol{u}|\sin(\phi)$$
$$= 10\sin(30°)$$
$$= 10 \times \frac{1}{2} = 5$$

3 Interpret your result.

Therefore, $\boldsymbol{u} = -\frac{5\sqrt{6}}{2}\hat{\boldsymbol{i}} + \frac{5\sqrt{6}}{2}\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

(b) Express the vector $\boldsymbol{v} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ in polar form.

1 Determine the necessary information from the vector.

$x_1 = 3, y_1 = 4, z_1 = -2$

2 Calculate the magnitude of the vector.

$$|\boldsymbol{v}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}$$
$$= \sqrt{3^2 + 4^2 + (-2)^2}$$
$$= \sqrt{29}$$

3 Use trigonometry to calculate the altitude and azimuth, taking care to express angles in the correct quadrant.

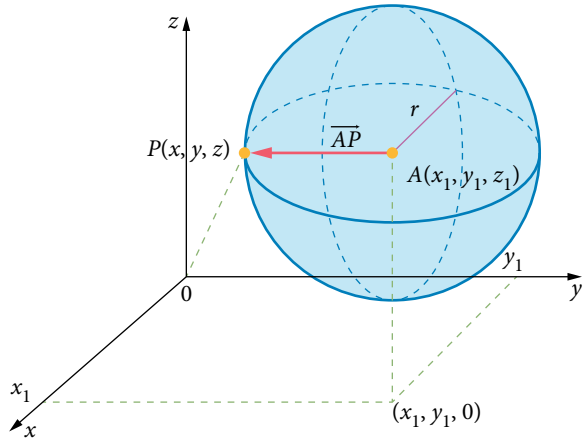
$$\theta = \tan^{-1}\left(\frac{y_1}{x_1}\right)$$
$$= \tan^{-1}\left(\frac{4}{3}\right)$$
$$= 53.13° \text{ (2 d.p.)}$$
$$\phi = \sin^{-1}\left(\frac{z_1}{|\boldsymbol{v}|}\right)$$
$$= \sin^{-1}\left(\frac{-2}{\sqrt{29}}\right)$$
$$= -21.80° \text{ (2 d.p.)}$$

4 Write the answer in polar form ($|\boldsymbol{v}|$, θ, ϕ).

Therefore, $\boldsymbol{v} = \left(\sqrt{29}, 53.13°, -21.80°\right)$ (2 d.p.)

Equation of a sphere

A sphere is defined as the set of points in three-dimensional space that are located a certain distance r (the radius) from a fixed point (the centre). Consider the general point $P(x, y, z)$ that lies on a sphere of radius r with centre $A(x_1, y_1, z_1)$.



Now, $\overrightarrow{AP} = (x - x_1)\hat{\boldsymbol{i}} + (y - y_1)\hat{\boldsymbol{j}} + (z - z_1)\hat{\boldsymbol{k}}$ and the magnitude of $\overrightarrow{AP}$ will be equal to the length of the radius r. Therefore, using the definition of magnitude in three dimensions,
$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$.

The Cartesian equation for a sphere with centre $A(x_1, y_1, z_1)$ and radius r is $r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$.

13 Equation of a sphere

(a) Determine the Cartesian equation of a sphere with centre $(4, -1, 2)$ and radius 6 units.

THINKING	WORKING
1 Write the general form of the Cartesian equation of a sphere.	$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$
2 Substitute the given values for the coordinates of the centre and the radius of the sphere.	$(6)^2 = (x - (4))^2 + (y - (-1))^2 + (z - (2))^2$
3 Simplify the equation. There is no need to expand the brackets.	The equation of the sphere is $(x - 4)^2 + (y + 1)^2 + (z - 2)^2 = 36$.

(b) Determine the centre and radius of the sphere with Cartesian equation $(x + 1)^2 + (y - 3)^2 + (z + 2)^2 = 4$.

1 Write the general form of the Cartesian equation of a sphere.	$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$
2 Rearrange the given equation to match the standard form.	$(2)^2 = (x - (-1))^2 + (y - (3))^2 + (z - (-2))^2$
3 Interpret your result.	The centre of the sphere is $(-1, 3, -2)$ and the length of the radius is 2 units.

Making connections

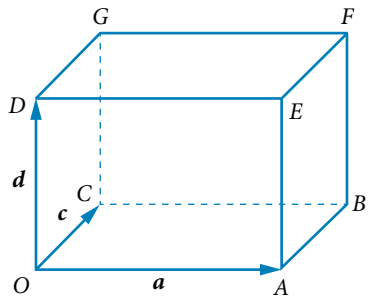
Components of a vector *v* in spherical coordinates

Observe Cartesian coordinates of a vector in 3D space by moving the sliders.

EXERCISE 2.2 Vectors in three dimensions

Worked Example

1 $OABCDEFG$ is a cuboid with $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OC} = \boldsymbol{c}$ and $\overrightarrow{OD} = \boldsymbol{d}$. 8



Express each of the following vectors in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

(a) $\overrightarrow{CB}$ (b) $\overrightarrow{EF}$ (c) $\overrightarrow{FG}$ (d) $\overrightarrow{EA}$

(e) $\overrightarrow{OG}$ (f) $\overrightarrow{AC}$ (g) $\overrightarrow{GA}$ (h) $\overrightarrow{BD}$

2 For each of the following vectors calculate the magnitude of the vector and a unit vector parallel to the given vector. 9

(a) $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ (b) $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

(c) $\boldsymbol{c} = -5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ (d) $\boldsymbol{d} = -2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

3 If $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, calculate each of the following: 10

(a) $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ (b) $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ (c) $\boldsymbol{a} + \boldsymbol{b}$

(d) $4\boldsymbol{a} + 2\boldsymbol{b}$ (e) $2\boldsymbol{c} - 2\boldsymbol{b} + 3\boldsymbol{a}$ (f) $4(\boldsymbol{a} + \boldsymbol{c}) - 3\boldsymbol{b}$

4 A is the point with coordinates (1, 3, -2) and B is the point (-4, -2, -1). 11

(a) Determine the position vectors of points A and B.

(b) Hence, calculate the vector $\overrightarrow{AB}$.

5 (a) Given the position vector of a point P with real coordinates (x, y, z), which of the following is a false statement?

A $P(x, y, z)$ can be defined by two angles called azimuth θ and altitude ϕ.

B Where O is the origin of the vector $\overrightarrow{OP}$, then $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$.

C If $|\overrightarrow{OP}| < 0$, then x, y or z are non-positive values.

D $|\overrightarrow{OP}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$

(b) Explain the common error associated with the false statement.

6 For each of the points P whose coordinates are given, determine the position vector $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ and hence determine a unit vector in the direction of $\overrightarrow{OP}$.

(a) $P(2, 2, 3)$ (b) $P(-3, 5, 2)$ (c) $P(-2, -5, 1)$ (d) $P(-3, 6, -5)$

7 Given that $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = -\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$, calculate:

(a) the magnitudes of the three vectors $|\boldsymbol{a}|$, $|\boldsymbol{b}|$, and $|\boldsymbol{c}|$ respectively.

(b) the unit vectors $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{c}}$.

8 The position vectors of points A and B are $3\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}-9\hat{\boldsymbol{k}}$ respectively. Calculate the magnitude of $\overrightarrow{AB}$ and hence determine a unit vector parallel to $\overrightarrow{AB}$.

9 The position vectors of points A, B and C are $\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$, $2\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+5\hat{\boldsymbol{k}}$ and $4\hat{\boldsymbol{i}}-9\hat{\boldsymbol{j}}+11\hat{\boldsymbol{k}}$ respectively.

(a) Determine vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Demonstrate that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors.

10 If $\boldsymbol{a}=6\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$, calculate $\boldsymbol{a}-\boldsymbol{b}$.

Worked Example

11 The position vectors of the points A, B, C and D are $5\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$, $6\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$, $3\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ and $2\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}$ respectively. Demonstrate that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

12

12 (a) Express the vector $\boldsymbol{u}=(50,300°,-45°)$ in component form.

(b) Express the vector $\boldsymbol{v}=-5\hat{\boldsymbol{i}}+5\hat{\boldsymbol{j}}+10\hat{\boldsymbol{k}}$ in polar form.

13

13 Determine the Cartesian equation of the sphere with the following centre and radius.

(a) Centre (4, 0, 1) and radius 6.

(b) Centre (-3, -1, 5) and radius $\sqrt{3}$.

14 Determine the centre and radius of the spheres with the following Cartesian equations:

(a) $(x-1)^2+(y-2)^2+z^2=4$

(b) $11=(x+5)^2+(y-1)^2+(z+2)^2$

15 In ΔOAB, $\overrightarrow{OA}=2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ and $\overrightarrow{OB}=-\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$.

(a) Determine $\overrightarrow{AB}$.

(b) Demonstrate that ΔOAB is isosceles.

(c) Determine point M, the midpoint of $\overrightarrow{AB}$.

(d) Determine $\overrightarrow{OM}$.

(e) Demonstrate that M is equidistant from the three vertices of the triangle.

16 Given that $\boldsymbol{a}=2\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}}$, $\boldsymbol{b}=-2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$, $\boldsymbol{c}=3\hat{\boldsymbol{i}}-5\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}}$ and $\boldsymbol{d}=-23\hat{\boldsymbol{i}}+7\hat{\boldsymbol{j}}+48\hat{\boldsymbol{k}}$, calculate x, y and $z\in\mathbb{R}$ such that $\boldsymbol{d}=x\boldsymbol{a}+y\boldsymbol{b}+z\boldsymbol{c}$.

17 If $\boldsymbol{a}=\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$, $\boldsymbol{b}=-2\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=-3\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$, determine the values of y and z, such that $\boldsymbol{a}+y\boldsymbol{b}+z\boldsymbol{c}$ is parallel to the x-axis.

18 Determine the area of the circle of intersection of the spheres $x^2+y^2+z^2=25$ and $(x-3)^2+y^2+z^2=16$.

19 At a certain time of the evening and at a certain fixed place on Earth, an observer looks up into the night sky. The star Sirius, which is 8.591 light years from Earth, has an azimuth of 205°43′56″ (measured clockwise from north) and an altitude of 33°33′22″. At the same time, the star Procyon, which is 11.402 light years from Earth, has an azimuth of 196°37′14″ (measured clockwise from north) and is 58°30′27″ above the horizon.

Use this information to calculate the distance between the stars Sirius and Procyon.

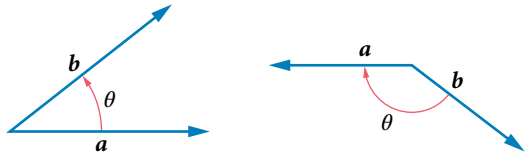
Scalar product of vectors and projections of vectors

2.3

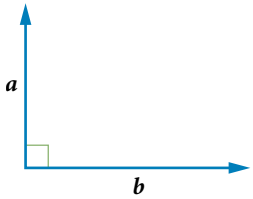
Scalar product of vectors in three dimensions

The *scalar product* (also called the *dot product*) is a way of multiplying two vectors. The scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ is read as 'a dot b' and is the product of three scalar quantities, $|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and the cosine of the smaller angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. It is a scalar quantity.

If θ is the smaller angle between the positive directions of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the scalar (dot) product is $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$.



Recall that the scalar (dot) product of perpendicular vectors is zero.



If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\left(\frac{\pi}{2}\right) = 0$.

Unit vectors in three dimensions $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are mutually perpendicular, so $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{i}} = \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{j}} = 0$.

Also, using the property of parallel vectors, $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$.

In component form, if $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then

$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= \left(x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}\right) \cdot \left(x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}\right) \\ &= (x_1x_2)\left(\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}}\right) + (x_1y_2)\left(\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}}\right) + (x_1z_2)\left(\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{k}}\right) \\ &\quad + (y_1x_2)\left(\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{i}}\right) + (y_1y_2)\left(\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}}\right) + (y_1z_2)\left(\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}}\right) \\ &\quad + (z_1x_2)\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{i}}\right) + (z_1y_2)\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{j}}\right) + (z_1z_2)\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}\right) \\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

WARNING

The use of column vector notation is used below, please note that this is not the same as matrix multiplication as the resultant matrix is undefined.

For vectors in three dimensions, if $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2$.

An alternative way of expressing this formula is:

If $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$.

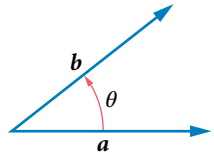
14 Calculating the scalar product of vectors in three dimensions

Calculate the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ given $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Create an equation showing the scalar product of the two vectors.	$\boldsymbol{a} \cdot \boldsymbol{b} = \left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\right) \cdot \left(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$
2 Multiply the coefficients of the like components and sum together.	$\boldsymbol{a} \cdot \boldsymbol{b} = 2 \times 1 + 1 \times 4 + (-3) \times 3$ $= 2 + 4 - 9$ $= -3$

Calculating the angle between two vectors in three dimensions

Recall that the scalar product can be used to calculate the angle between two vectors. Let θ be the angle between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in the diagram.



By the definition of the dot product, $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$.

$$|\boldsymbol{a}||\boldsymbol{b}|\cos(\theta) = \boldsymbol{a} \cdot \boldsymbol{b}$$

$$\cos(\theta) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$$

When working in Cartesian form, you also know that if $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2$.

Therefore,

$$|\boldsymbol{a}||\boldsymbol{b}|\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$$

$$\cos(\theta) = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\boldsymbol{a}||\boldsymbol{b}|}$$

If θ is the angle between vectors $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then $\cos(\theta) = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{x_1x_2 + y_1y_2 + z_1z_2}{|\boldsymbol{a}||\boldsymbol{b}|}$.

An alternative way of expressing this formula is:

If θ is the angle between vectors $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$, then $\cos(\theta) = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}$.

15 Calculating the angle between two vectors in three dimensions

Calculate the angle, in degrees correct to 2 decimal places, between vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Calculate the scalar product of the vectors, $\boldsymbol{a} \cdot \boldsymbol{b}$.	$\boldsymbol{a} \cdot \boldsymbol{b} = \left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right) \cdot \left(4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\right)$ $= 3 \times 4 + (-2) \times 1 + 1 \times (-3)$ $= 12 - 2 - 3$ $= 7$

2 Calculate the magnitude of each of the vectors.	$\lvert\boldsymbol{a}\rvert = \sqrt{3^2 + (-2)^2 + 1^2}$ $= \sqrt{14}$ $\lvert\boldsymbol{b}\rvert = \sqrt{4^2 + 1^2 + (-3)^2}$ $= \sqrt{26}$
3 Substitute into the rule $\cos(\theta) = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert}$ and simplify.	$\theta = \cos^{-1}\left(\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert}\right)$ $= \cos^{-1}\left(\dfrac{7}{\sqrt{14}\sqrt{26}}\right)$ $= 68.48°$ (2 d.p.)

Technology worked example

Calculating the dot product and angles between vectors

16 Parallel and perpendicular vectors

Consider the vectors $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$, $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}} - \frac{3}{2}\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$. Demonstrate that $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ but is perpendicular to $\boldsymbol{c}$.

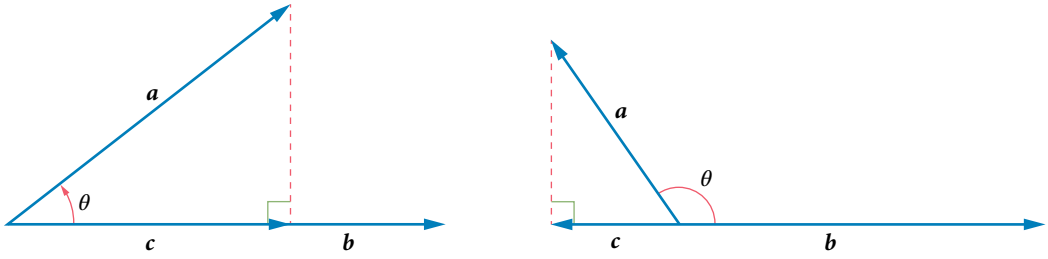
THINKING	WORKING
1 Two vectors are parallel if one vector is a scalar multiple of the other. Express $\boldsymbol{a}$ as a scalar multiple of $\boldsymbol{b}$.	$\boldsymbol{b} = -2\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}} - \frac{3}{2}\hat{\boldsymbol{k}}$ $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ $= -2\left(-2\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}} - \frac{3}{2}\hat{\boldsymbol{k}}\right)$ $= -2b$ Thus, $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors.
2 Two vectors are perpendicular if their dot product is equal to zero. Calculate the dot product.	$\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ $\boldsymbol{a} \cdot \boldsymbol{c} = 4 \times (-2) + (-1) \times 4 + 3 \times 4$ $= -8 - 4 + 12$ $= 0$ Thus, $\boldsymbol{a}$ and $\boldsymbol{c}$ are perpendicular.

Projection of vectors

Any vector in three dimensions can be expressed in the form $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, where $x\hat{\boldsymbol{i}}$, $y\hat{\boldsymbol{j}}$ and $z\hat{\boldsymbol{k}}$ are the *vector projections* of $\boldsymbol{r}$, and x, y and z are the *scalar projections* of $\boldsymbol{r}$ parallel to the x, y and z axes respectively. The dot products $\boldsymbol{r} \cdot \hat{\boldsymbol{i}}$, $\boldsymbol{r} \cdot \hat{\boldsymbol{j}}$ and $\boldsymbol{r} \cdot \hat{\boldsymbol{k}}$ give the scalar projections of $\boldsymbol{r}$ in the x, y and z directions respectively.

In the following diagrams, vector $\boldsymbol{c}$ is the vector projection of vector $\boldsymbol{a}$ in the direction of vector $\boldsymbol{b}$.

Scalar projection



The magnitude of $\boldsymbol{c}$, $|\boldsymbol{c}|$ is called the *scalar projection* of vector $\boldsymbol{a}$ onto vector $\boldsymbol{b}$.

The scalar projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is equal to the scalar product of $\boldsymbol{a}$ and unit vector $\hat{\boldsymbol{b}}$.

The scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

The scalar projection of $\boldsymbol{b}$ on $\boldsymbol{a}$ is $\boldsymbol{b} \cdot \hat{\boldsymbol{a}} = \dfrac{\boldsymbol{b} \cdot \boldsymbol{a}}{|\boldsymbol{b}|} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

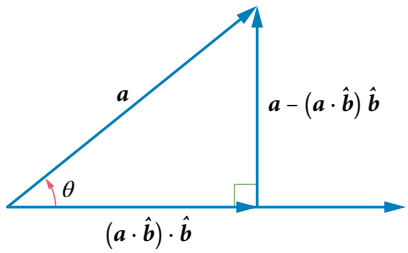
Vector projection

Vector $\boldsymbol{c}$ in each of the diagrams above is the *vector projection* of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

Now

$\boldsymbol{c} = |\boldsymbol{c}|\hat{\boldsymbol{b}}$

$= \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$



The *vector projection* of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ (sometimes referred to as the *vector rejection* of $\boldsymbol{a}$ from $\boldsymbol{b}$) is

$\boldsymbol{a} - \boldsymbol{c} = \boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$

The vector projection or component of $\boldsymbol{a}$ on $\boldsymbol{b}$ is $\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ (or the vector rejection of $\boldsymbol{a}$ from $\boldsymbol{b}$) is $\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

17 Scalar and vector projections

If $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, calculate:

(a) the scalar projection of $\boldsymbol{b}$ on $\boldsymbol{a}$.

THINKING	WORKING
1 Calculate the scalar product $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b}$.	$\boldsymbol{a} \cdot \boldsymbol{b} = (-\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \cdot (6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k})$ $= (-1) \times 6 + 1 \times (-3) + (-1) \times 2$ $= -6 - 3 - 2$ $= -11$

2 Calculate the magnitude of $\boldsymbol{a}$.

$$\begin{aligned}|\boldsymbol{a}| &= \sqrt{(-1)^2 + 1^2 + (-1)^2} \\ &= \sqrt{3}\end{aligned}$$

3 The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is equal to the scalar product of $\boldsymbol{b}$ and unit vector $\hat{\boldsymbol{a}}$: $\boldsymbol{b} \cdot \hat{\boldsymbol{a}} = \dfrac{\boldsymbol{b} \cdot \boldsymbol{a}}{|\boldsymbol{a}|}$

$$\begin{aligned}\boldsymbol{b} \cdot \hat{\boldsymbol{a}} &= \frac{\boldsymbol{b} \cdot \boldsymbol{a}}{|\boldsymbol{a}|} \\ &= \frac{-11}{\sqrt{3}} \\ &= -\frac{11\sqrt{3}}{3}\end{aligned}$$

(b) the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

1 Calculate the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$.

Since $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = -11$

2 Calculate the magnitude of $\boldsymbol{b}$.

$$\begin{aligned}|\boldsymbol{b}| &= \sqrt{6^2 + (-3)^2 + 2^2} \\ &= \sqrt{49} \\ &= 7\end{aligned}$$

3 The scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is equal to $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

$$\begin{aligned}\boldsymbol{a} \cdot \hat{\boldsymbol{b}} &= \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} \\ &= \frac{-11}{7}\end{aligned}$$

4 The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is equal to $\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$

$$\begin{aligned}\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}} &= \left(\frac{-11}{7}\right)\frac{\boldsymbol{b}}{|\boldsymbol{b}|} \\ &= \left(\frac{-11}{7}\right)\left(\frac{6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}}{7}\right) \\ &= -\frac{11}{49}\left(6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)\end{aligned}$$

(c) the vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is equal to $\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

$$\begin{aligned}\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}} &= -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} - \left[-\frac{11}{49}\left(6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)\right] \\ &= \left(-1 + \frac{66}{49}\right)\hat{\boldsymbol{i}} + \left(1 - \frac{33}{49}\right)\hat{\boldsymbol{j}} + \left(-1 + \frac{22}{49}\right)\hat{\boldsymbol{k}} \\ &= \frac{17}{49}\hat{\boldsymbol{i}} + \frac{16}{49}\hat{\boldsymbol{j}} - \frac{27}{49}\hat{\boldsymbol{k}} \\ &= \frac{1}{49}\left(17\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}} - 27\hat{\boldsymbol{k}}\right)\end{aligned}$$

EXERCISE 2.3 Scalar product of vectors and projections of vectors

Worked Example

14

1 Calculate the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ given:

(a) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

(b) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$

(c) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

(d) $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

(e) $\boldsymbol{a} = -5\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} - 9\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

(f) $\boldsymbol{a} = 8\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

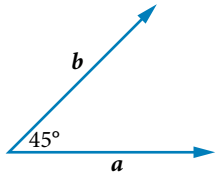
2 If the vectors $\boldsymbol{e} = 7\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\boldsymbol{f} = -3\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ are perpendicular, determine the value of z.

16

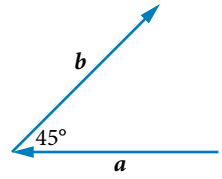
3 Consider the vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 4\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}} - 14\hat{\boldsymbol{k}}$. Demonstrate that $\boldsymbol{a}$ is parallel to $\boldsymbol{c}$ but is perpendicular to $\boldsymbol{b}$.

4 Given the following vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, with magnitudes 2 units and 3 units respectively, calculate the following scalar products:

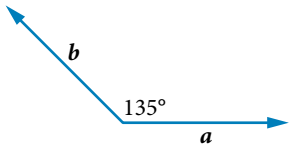
(a)



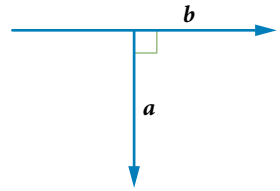
(b)



(c)



(d)



5 If $\boldsymbol{a} = \hat{\boldsymbol{i}} - \sqrt{2}\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - \sqrt{2}\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, calculate the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

6 Which one of the following vectors is parallel to $3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$?

A $\frac{1}{3\sqrt{2}}\left(3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}\right)$

B $2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

C $9\hat{\boldsymbol{i}} - 15\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}$

D $\frac{1}{5\sqrt{2}}\left(3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}\right)$

7 Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors on the same plane. Vector $\boldsymbol{u}$ is the vector projection of $\boldsymbol{a}$ parallel to $\boldsymbol{b}$, and vector $\boldsymbol{v}$ is the vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$. Which of the following statements is accurate?

A $\boldsymbol{a} = \boldsymbol{u} - \boldsymbol{v}$ B $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{v}$ C $\boldsymbol{b} = \boldsymbol{u} - \boldsymbol{v}$ D $\boldsymbol{u} \cdot \boldsymbol{v} = 1$

15

8 Calculate the angle, correct to the nearest degree, between each of the following pairs of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

(a) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

(b) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

(c) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

9 The vertices A, B, C and D of a quadrilateral have position vectors $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\overrightarrow{OB} = -2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, $\overrightarrow{OC} = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\overrightarrow{OD} = \hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$ respectively. Demonstrate that the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ of the quadrilateral are perpendicular to each other.

10 The position vectors of points A, B and C are $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ and $-2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ respectively. Demonstrate that the triangle ABC is right-angled.

11 Two vectors are defined by $\boldsymbol{a} = 2p\hat{\boldsymbol{i}} + 5p\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = p\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + p^2\hat{\boldsymbol{k}}$.

(a) Determine a simplified expression for the dot product, $\boldsymbol{a} \cdot \boldsymbol{b}$.

(b) Determine the value(s) of p for which vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

Worked Example 17

12 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$, calculate:

(a) the scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$.

(b) the vector projection of $\boldsymbol{a}$ on $\boldsymbol{b}$.

(c) the vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

13 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$, calculate:

(a) the scalar projection of $\boldsymbol{b}$ on $\boldsymbol{a}$.

(b) the vector projection of $\boldsymbol{b}$ on $\boldsymbol{a}$.

(c) the vector projection of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$.

14 If $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 8\hat{\boldsymbol{k}}$, resolve $\boldsymbol{a}$ into two components, one parallel to $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

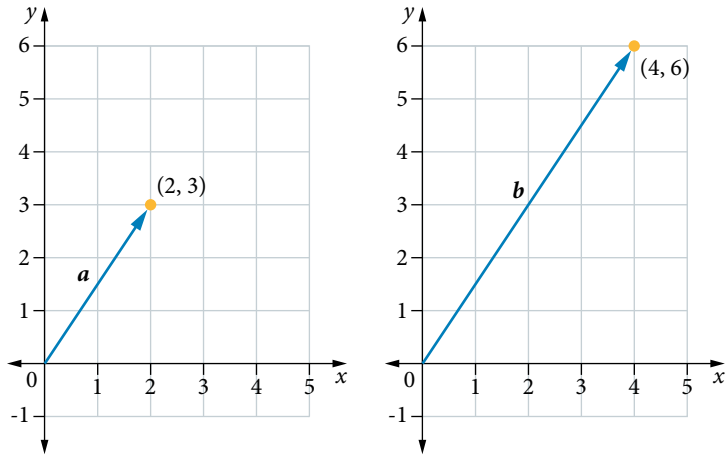
15 Given that $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$, determine two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{u}$ is parallel to $\boldsymbol{b}$, and $\boldsymbol{v}$ is perpendicular to $\boldsymbol{b}$.

16 A triangular shade sail is being constructed in a playground such that the vertices of the triangle are given by the three-dimensional coordinates $A(0,0,2)$, $B(4,1,3)$ and $C(-2,6,5)$ relative to an arbitrary origin on the ground. The design specifications state that each of the internal angles of the shade sail must be no less than 30° and no greater than 90°. Determine whether this particular shade sail meets the required specifications.

2.4 Linearly dependent and independent vectors

Vectors in a set are *linearly dependent* if one vector in the set can be expressed as a linear combination of other vectors in the set. If no vector can be expressed in such a way, then the vectors in the set are *linearly independent*.

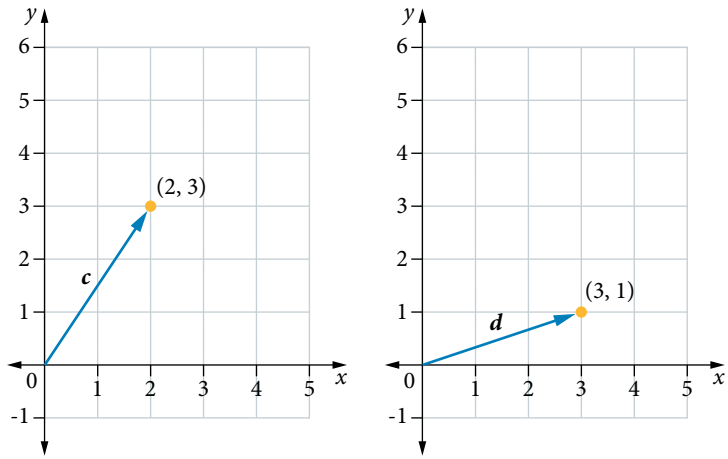
Consider a set of two vectors, $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$:



These two vectors are linearly dependent as one vector can be expressed as a scalar multiple of the other, creating the linear equation:

$\boldsymbol{b} = 2\boldsymbol{a}$ or $\boldsymbol{a} = \frac{1}{2}\boldsymbol{b}$

Now consider a different set of two vectors, $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{d} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$:



In this case, there is no possible way of expressing one vector as a scalar multiple of the other. Therefore, vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ are said to be linearly independent.

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be *linearly dependent* if and only if there are two scalars, x and y, such that $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ or $\boldsymbol{a} = -\frac{y}{x}\boldsymbol{b}$ (that is, one vector is a non-zero scalar multiple of the other). This implies that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors.

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be *linearly independent* if and only if the only solution to $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ is $x = y = 0$. This implies that $\boldsymbol{a}$ and $\boldsymbol{b}$ will be non-parallel vectors.

18 Two linearly dependent and independent vectors

Determine whether the following pairs of vectors are linearly independent.

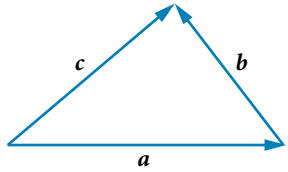
(a) $\boldsymbol{a} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

THINKING	WORKING
1 Write as a vector sum $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$.	$\boldsymbol{a} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ Thus, $x\left(6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\right) + y\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) = \boldsymbol{0}$
2 Simplify by collecting like components on the LHS.	$\left(6x + 2y\right)\hat{\boldsymbol{i}} + \left(-4x + y\right)\hat{\boldsymbol{j}} = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$
3 Write two equations in x and y.	Thus, $6x + 2y = 0$ [1] and $-4x + y = 0$ [2]
4 Solve simultaneously for x and y.	$x = 0$ and $y = 0$
5 Write a solution.	Thus, the only solution is $0\boldsymbol{a} + 0\boldsymbol{b} = \boldsymbol{0}$.
6 Interpret whether the vectors are linearly independent.	Since $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ when $x = 0$ and $y = 0$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.

(b) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

1 Write as a vector sum $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ or $\boldsymbol{a} = -\frac{y}{x}\boldsymbol{b}$.	$\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ Thus, $x\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right) + y\left(6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right) = \boldsymbol{0}$ or $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} = -\frac{y}{x}\left(6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)$
2 Determine whether one vector can be written in terms of the other.	$3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} = \frac{1}{2}\left(6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)$
3 Solve for x and y by equating coefficients.	Thus, $\frac{y}{x} = -\frac{1}{2}$ or $y = -\frac{1}{2}x$
4 Write a solution.	Thus, a solution is $\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} = \boldsymbol{0}$.
5 Interpret the answer.	Since $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ when $y = -\frac{1}{2}x$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.

When the set is extended to include three or more vectors in two dimensions, or in a plane, then each vector can be expressed as a combination of the other two.



Three non-parallel vectors in the Cartesian plane are always linearly dependent.

19 Three linearly dependent vectors

Demonstrate that the vectors $\boldsymbol{a} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{c} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ are linearly dependent.

THINKING	WORKING
1 Write as a vector sum $x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$ and interpret.	$x\left(6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\right) + y\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) + z\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right) = \boldsymbol{0}$ If vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent, then the equation is true for some values of x, y and z other than $x = y = z = 0$.
2 Simplify by collecting like components on the LHS.	$(6x + 2y + 3z)\hat{\boldsymbol{i}} + (-4x + y + 2z)\hat{\boldsymbol{j}} = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$
3 Write two equations in terms of x, y and z.	$6x + 2y + 3z = 0$ [1] $-4x + y + 2z = 0$ [2]
4 Express a variable as a function of one of the other variables, say z as a function of x.	From [2], $y = 4x - 2z$ [3]. Substituting [3] into [1] gives: $6x + 2(4x - 2z) + 3z = 0$ $14x - z = 0$ Thus, $z = 14x$.
5 Express the other variable as a function of one of the other variables, say y as a function of x.	Substituting into [3] gives $y = 4x - 2(14x)$ Thus, $y = -24x$.
6 Interpret whether the vectors are linearly independent or not.	Thus, since y and z can be expressed as functions of x, the vectors are linearly dependent.

Note that in Worked example 19 there is no unique combination of values of x, y and z that will satisfy the vector equation $x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$. However, a specific set can be found by substitution. If $x = 1$, then, from the equations, $y = -24x = -24$ and $z = 14x = 14$. These values can be tested to verify that the solution is correct.

$$x\left(6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\right) + y\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) + z\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right) = \boldsymbol{0}$$
$$1\left(6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\right) - 24\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) + 14\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right) = \boldsymbol{0}$$
$$6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - 48\hat{\boldsymbol{i}} - 24\hat{\boldsymbol{j}} + 42\hat{\boldsymbol{i}} + 28\hat{\boldsymbol{j}} = \boldsymbol{0}$$
$$\boldsymbol{0} = \boldsymbol{0}$$

This demonstrates that the three vectors are linearly dependent, that is, the vectors depend on each other to make the equation $xa + yb + zc = 0$ true.

The property of linear independence can be used to solve vector equations.

20 Solving vector equations using the property of linear independence

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent vectors, determine the values of x and y if $(2x^2 - 12)\boldsymbol{a} + (6y + 3)\boldsymbol{b} = \boldsymbol{0}$.

THINKING	WORKING
1 Recall the definition of linearly independent vectors.	Two vectors are linearly independent if and only if they are not parallel. Therefore the only way that $m\boldsymbol{a} + n\boldsymbol{b} = \boldsymbol{0}$ is if $m = 0$ or $n = 0$.
2 Let the coefficients be equal to 0 and solve for x and y.	$2x^2 - 12 = 0$ $2x^2 = 12$ $x^2 = 6$ $x = \pm\sqrt{6}$ $6y + 3 = 0$ $6y = -3$ $y = -\frac{1}{2}$
3 Interpret the answer.	If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent then for $(2x^2 - 12)\boldsymbol{a} + (6y + 3)\boldsymbol{b} = \boldsymbol{0}$, $x = \pm\sqrt{6}$ and $y = -\frac{1}{2}$.

EXERCISE 2.4 Linearly dependent and independent vectors

Worked Example

1 Determine whether the following pairs of vectors are linearly dependent or linearly independent. 18

(a) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}, \boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

(b) $\boldsymbol{a} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}, \boldsymbol{b} = -2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

(c) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}, \boldsymbol{b} = 6\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}$

(d) $\boldsymbol{a} = -2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}, \boldsymbol{b} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

(e) $\boldsymbol{a} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \frac{3}{2}\hat{\boldsymbol{k}}, \boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$

(f) $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}, \boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

(g) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}, \boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

(h) $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}, \boldsymbol{b} = -3\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$

2 Which of the following statements is *not* true?

A The unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are linearly independent.

B The unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are linearly dependent.

C The vectors $\boldsymbol{u}$ and $-3\boldsymbol{u}$ are linearly dependent.

D The vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}, \boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = 5\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ are linearly dependent.

3 Demonstrate that the following sets of vectors are linearly dependent. 19

(a) $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}, \boldsymbol{b} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{c} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

(b) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}, \boldsymbol{b} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

4 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent vectors, determine the values of x and y if: 20

(a) $(5x - 3)\boldsymbol{a} + (2y + 1)\boldsymbol{b} = \boldsymbol{0}$

(b) $(3x^2 - 5)\boldsymbol{a} + (2y^2 - 5y + 2)\boldsymbol{b} = \boldsymbol{0}$

5 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent vectors and $\left(2x^2 - 5x + 2\right)\boldsymbol{a} + \left(3y - 5\right)\boldsymbol{b} = \boldsymbol{0}$, determine the values of x and y.

6 If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent vectors, determine the values of x, y and z, where $x, y, z \in \mathbb{R}$, if:

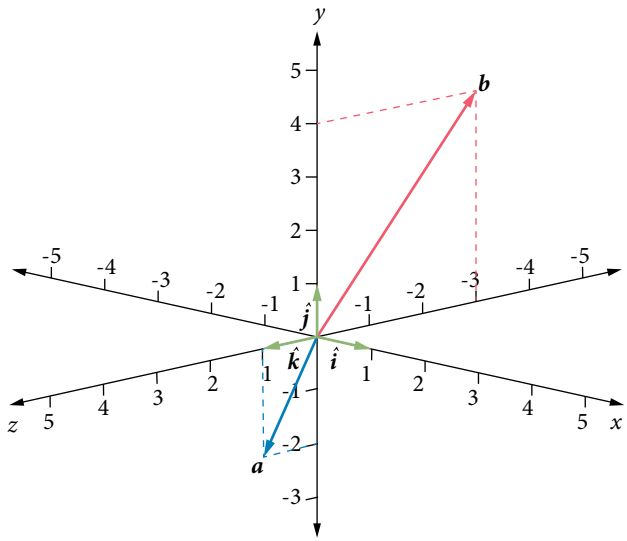
(a) $(x-3)\boldsymbol{a} + (y+1)\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$

(b) $(2x+1)\boldsymbol{a} + 3y\boldsymbol{b} + (z+3)\boldsymbol{c} = \boldsymbol{0}$

(c) $\left(3x^2 - 5x - 2\right)\boldsymbol{a} + \left(2y^2 - 18\right)\boldsymbol{b} + \left(z^3 + 8\right)\boldsymbol{c} = \boldsymbol{0}$

(d) $\left(x^2 - 16\right)\boldsymbol{a} + \left(y^2 - 12\right)\boldsymbol{b} + \left(3z^2 + 10z + 8\right)\boldsymbol{c} = \boldsymbol{0}$

7 (a) Two vectors are defined as $\boldsymbol{a} = 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$. Which third vector could be added to create a set of three vectors that are linearly dependent?



A $\boldsymbol{c} = 5\hat{\boldsymbol{i}}$

B $\boldsymbol{c} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

C $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{k}}$

D $\boldsymbol{c} = -6\hat{\boldsymbol{j}}$

(b) Explain why each of the other options in part **(a)** is incorrect.

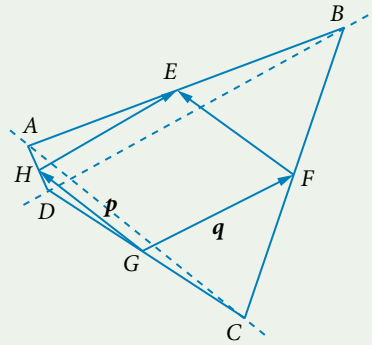
Vector proofs of geometric theorems

2.5

Many geometric theorems can be proven using vectors in the plane. Your knowledge of three-dimensional vectors will also allow you to solve more complicated theorems.

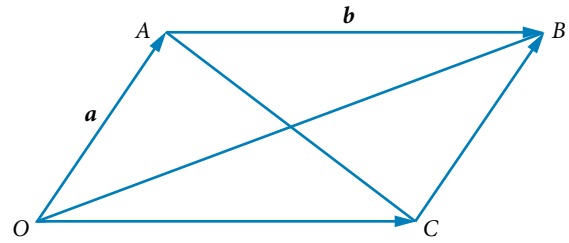
21 Proving that the midpoints of the sides of a quadrilateral join to form a parallelogram

Prove that the lines joining the midpoints *EFGH* of a quadrilateral *ABCD* form a parallelogram.

THINKING	WORKING
1 Determine what it is that you need to prove.	For the midpoints of the quadrilateral to join to form a parallelogram, you need to prove that opposite side lengths are parallel and equal in length, that is $\overrightarrow{GH} = \overrightarrow{FE} = \boldsymbol{p}$ and $\overrightarrow{GF} = \overrightarrow{HE} = \boldsymbol{q}$. 
2 Prove that $\overrightarrow{GH} = \overrightarrow{FE} = \boldsymbol{p}$.	$\boldsymbol{p} = \overrightarrow{GH}$ $= \overrightarrow{GD} + \overrightarrow{DH}$ $= \frac{1}{2}\overrightarrow{CD} + \frac{1}{2}\overrightarrow{DA}$ $= \frac{1}{2}\left(\overrightarrow{CD} + \overrightarrow{DA}\right)$ $= \frac{1}{2}\overrightarrow{CA}$ $\overrightarrow{FE} = \overrightarrow{FB} + \overrightarrow{BE}$ $= \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{BA}$ $= \frac{1}{2}\left(\overrightarrow{CB} + \overrightarrow{BA}\right)$ $= \frac{1}{2}\overrightarrow{CA}$ $= \boldsymbol{p}$
3 Prove that $\overrightarrow{GF} = \overrightarrow{HE} = \boldsymbol{q}$.	$\boldsymbol{q} = \overrightarrow{GF}$ $= \overrightarrow{GC} + \overrightarrow{CF}$ $= \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\overrightarrow{CB}$ $= \frac{1}{2}\left(\overrightarrow{DC} + \overrightarrow{CB}\right)$ $= \frac{1}{2}\overrightarrow{DB}$ $\overrightarrow{HE} = \overrightarrow{HA} + \overrightarrow{AE}$ $= \frac{1}{2}\overrightarrow{DA} + \frac{1}{2}\overrightarrow{AB}$ $= \frac{1}{2}\left(\overrightarrow{DA} + \overrightarrow{AB}\right)$ $= \frac{1}{2}\overrightarrow{DB}$ $= \boldsymbol{q}$
4 Interpret the answer.	Given that $\overrightarrow{GH} = \overrightarrow{FE} = \boldsymbol{p}$ and $\overrightarrow{GF} = \overrightarrow{HE} = \boldsymbol{q}$ then the midpoints *EFGH* of the quadrilateral *ABCD* join to form a parallelogram.

22 Diagonals of a parallelogram meet at right angles if and only if it is a rhombus

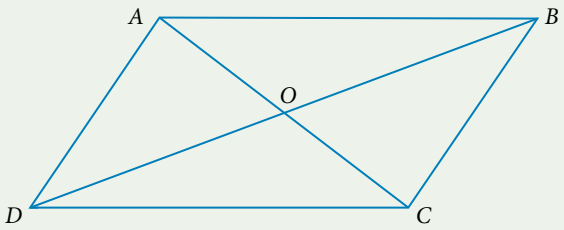
Consider the parallelogram $OABC$, where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$ as shown. Prove that the diagonals of a parallelogram meet at right angles if and only if it is a rhombus.



THINKING	WORKING
1 Define the diagonals in terms of the known vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.	The diagonals of the parallelogram are $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$, $\overrightarrow{CA} = \boldsymbol{a} + (-\boldsymbol{b})$ and if $OABC$ is a rhombus, then $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$.
2 If two vectors are parallel, then the dot product is equal to 0.	The dot product of the diagonals gives $(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = \lvert\boldsymbol{a}\rvert^2 - \lvert\boldsymbol{b}\rvert^2$. Given that $\lvert\boldsymbol{a}\rvert = \lvert\boldsymbol{b}\rvert$, then $(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = 0$.
3 Interpret the result.	If the parallelogram is a rhombus, then the diagonals meet at right angles.
4 Define vectors and establish relationships between vectors.	Let $\overrightarrow{OA} = \overrightarrow{CB} = \boldsymbol{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \boldsymbol{b}$. Then $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ and $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{CO}$.
5 The dot product can often help when establishing whether two vectors are perpendicular. Recall that the dot product of perpendicular vectors is equal to zero.	$\overrightarrow{OB}\cdot\overrightarrow{CA} = (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})$ $= \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{b}\cdot\boldsymbol{b}$ $= \lvert\boldsymbol{a}\rvert^2 - \lvert\boldsymbol{b}\rvert^2$ The diagonals $\overrightarrow{OB}$ and $\overrightarrow{CA}$ are perpendicular if and only if $\overrightarrow{OB}\cdot\overrightarrow{CA} = 0$.
6 Synthesise the expansion of the dot product into your proof to establish a relationship between the sides of the parallelogram.	$\overrightarrow{OB} \perp \overrightarrow{CA}$ $\Rightarrow \overrightarrow{OB}\cdot\overrightarrow{CA} = 0$ $\Rightarrow \lvert\boldsymbol{a}\rvert^2 - \lvert\boldsymbol{b}\rvert^2 = 0$ $\Rightarrow \lvert\boldsymbol{a}\rvert^2 = \lvert\boldsymbol{b}\rvert^2$
7 Relate your conclusion back to the question.	Therefore, the diagonals of the parallelogram will meet at 90° if and only if $\lvert\boldsymbol{a}\rvert^2 = \lvert\boldsymbol{b}\rvert^2$, i.e. only if the parallelogram is a rhombus.

23 Using the property of linear independence of vectors

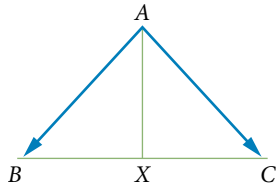
ABCD is a parallelogram. Use the property of linear independence to prove that the diagonals bisect each other at *O*.

THINKING	WORKING
1 Draw a parallelogram and label the vertices *A*, *B*, *C* and *D*. Label *O*, the point of intersection of the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$.	
2 Define vectors $\overrightarrow{AB}$ and $\overrightarrow{DA}$.	Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{DA} = \boldsymbol{b}$.
3 Define vectors $\overrightarrow{DC}$ and $\overrightarrow{CB}$.	Also $\overrightarrow{DC} = \boldsymbol{a}$ and $\overrightarrow{CB} = \boldsymbol{b}$.
4 Define vectors $\overrightarrow{AC}$ and $\overrightarrow{DB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.	$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ $= \overrightarrow{AB} - \overrightarrow{CB}$ $= \boldsymbol{a} - \boldsymbol{b}$ $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{CB}$ $= \boldsymbol{a} + \boldsymbol{b}$
5 Define vectors $\overrightarrow{AO}$ and $\overrightarrow{DO}$ in terms of vectors $\overrightarrow{AC}$ and $\overrightarrow{DB}$. Use m and n to indicate unknown scalar constants.	Let $\overrightarrow{AO} = m\overrightarrow{AC}$ and $\overrightarrow{DO} = n\overrightarrow{DB}$. $\overrightarrow{AO} = m\overrightarrow{AC}$ $= m(\boldsymbol{a} - \boldsymbol{b})$ $\overrightarrow{DO} = n\overrightarrow{DB}$ $= n(\boldsymbol{a} + \boldsymbol{b})$
6 If the diagonals bisect each other, then it needs to be shown that $m = n = \frac{1}{2}$. Develop an equation in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, m and n.	Consider ΔAOD. $\overrightarrow{DA} = \overrightarrow{DO} + \overrightarrow{OA}$ $\overrightarrow{DA} = \overrightarrow{DO} - \overrightarrow{AO}$ $\boldsymbol{b} = n(\boldsymbol{a} + \boldsymbol{b}) - m(\boldsymbol{a} - \boldsymbol{b})$ $\boldsymbol{b} = (n - m)\boldsymbol{a} + (n + m)\boldsymbol{b}$
7 Rearrange so that all vectors are on one side of the equation.	$(n - m)\boldsymbol{a} + (n + m - 1)\boldsymbol{b} = \boldsymbol{0}$
8 Use the property of linear independence of vectors to solve for m and n: $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ if and only if both $x = y = 0$.	Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent (because they are non-parallel vectors), $n - m = 0$ [1] $n + m - 1 = 0$ [2]
9 Solve equations [1] and [2] simultaneously.	$m = \frac{1}{2}$ and $n = \frac{1}{2}$
10 Relate your conclusion back to the question.	Thus, *O* is the midpoint of the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$.

EXERCISE 2.5 Vector proofs of geometric theorems

Worked Example 21

1 ABC is a triangle with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AC} = \boldsymbol{b}$. X is the midpoint of $\overrightarrow{BC}$ as shown.



(a) Determine an expression for $\overrightarrow{BX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

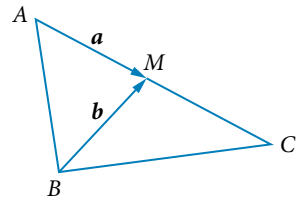
(b) Determine an expression for $\overrightarrow{AX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Determine $2\left(\overrightarrow{BX}\cdot\overrightarrow{BX}+\overrightarrow{AX}\cdot\overrightarrow{AX}\right)$ and hence prove that $\overrightarrow{AB}^2+\overrightarrow{AC}^2=2\left(\overrightarrow{AX}^2+\overrightarrow{BX}^2\right)$.

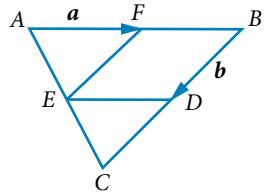
Worked Example 23

2 $ABCD$ is a rectangle. Use the property of linear independence to prove that the diagonals of a rectangle bisect each other.

3 ΔABC is such that M, the midpoint of AC, is equidistant from the three vertices, such that $\overrightarrow{AM}=\overrightarrow{BM}=\overrightarrow{CM}$. Let $\overrightarrow{AM} = \boldsymbol{a}$ and $\overrightarrow{BM} = \boldsymbol{b}$. Use a vector method to prove that $\angle ABC$ is a right angle.



4 $BDEF$ is a parallelogram contained within a triangle ABC as shown in the diagram. F is the midpoint of $\overrightarrow{AB}$.



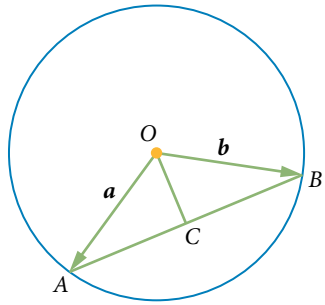
Let $\overrightarrow{AF} = \boldsymbol{a}$ and $\overrightarrow{BD} = \boldsymbol{b}$.

(a) Determine the vector $\overrightarrow{AE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

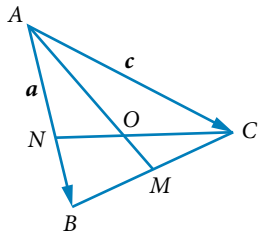
(b) Use vector methods to prove that $\overrightarrow{BD}=\frac{1}{2}\overrightarrow{BC}$.

5 Use vector methods to prove that the angle in a semicircle is a right angle.

6 A circle has its centre at O and a chord AB that is bisected by $\overrightarrow{OC}$. $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Use a vector method to prove that $\angle ACO$ is a right angle.



7 Consider ΔABC shown.



Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AC} = \boldsymbol{c}$, and let M be the midpoint of $\overrightarrow{BC}$. Another point N lies on $\overrightarrow{AB}$ such that $\overrightarrow{AN} = 2\overrightarrow{NB}$.

Let O be the point of intersection of the lines joining $\overrightarrow{CN}$ and $\overrightarrow{AM}$.

(a) Express vector $\overrightarrow{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(b) Demonstrate that $\overrightarrow{CN} = \frac{2}{3}\boldsymbol{a} - \boldsymbol{c}$.

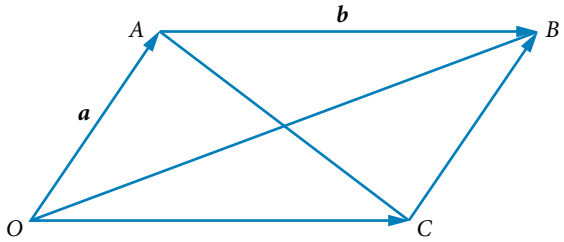
(c) If $\overrightarrow{AO} = x\overrightarrow{AM}$ and $\overrightarrow{ON} = y\overrightarrow{CN}$, determine expressions for $\overrightarrow{AO}$ in terms of x, $\boldsymbol{a}$ and $\boldsymbol{c}$, and $\overrightarrow{ON}$ in terms of y, $\boldsymbol{a}$ and $\boldsymbol{c}$.

(d) Use your results from part **(c)** to demonstrate that $\left(\frac{3x + 4y - 4}{6}\right)\boldsymbol{a} + \left(\frac{x - 2y}{2}\right)\boldsymbol{c} = \boldsymbol{0}$.

(e) Hence, demonstrate that $\left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{AM}\right|$ and $\left|\overrightarrow{CO}\right| = \frac{3}{5}\left|\overrightarrow{CN}\right|$.

Worked Example 22

8 Consider the parallelogram $OABC$, where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$.

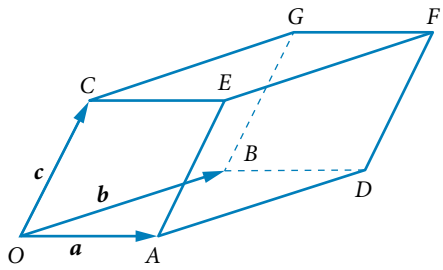


Prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

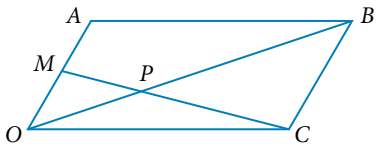
9 $OADBCEFG$ is a parallelepiped. The position vectors of A, B and C, with respect to O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

(a) Determine, in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, the position vectors of the points D, E, F and G with respect to O.

(b) Demonstrate that the diagonals of the parallelepiped bisect each other at a common point.



10 $OABC$ is a parallelogram and M is the midpoint of side $\overrightarrow{OA}$.



P is the point of intersection of $\overrightarrow{CM}$ and the diagonal $\overrightarrow{OB}$. Use the concept of linear independence of vectors to prove that P is a point of trisection of $\overrightarrow{OB}$.

2.6 The vector product

You have already encountered the scalar (or dot) product method of calculating the product between two vectors. There is a second way of calculating vector products, and this method yields a vector result.

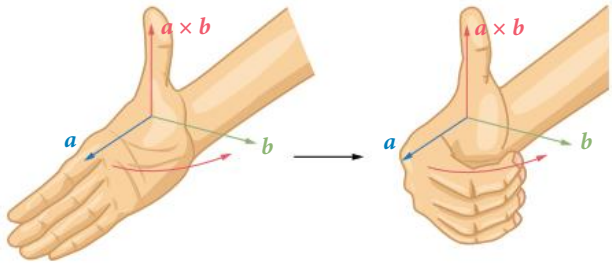
The *vector product* or *cross product* of two vectors results in a vector quantity. The symbol used to denote the cross product is '×' and the cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted $\boldsymbol{a}\times\boldsymbol{b}$.

The cross product has many applications in mathematics, such as to calculate the areas of triangles and parallelograms, and to calculate the volume of a parallelepiped. It can also be used to define a plane in three dimensions, and to calculate the distance between a point and a line in three-dimensional space.

Recall that the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$. The vector product has a similar definition, so it is important not to confuse the two formulas.

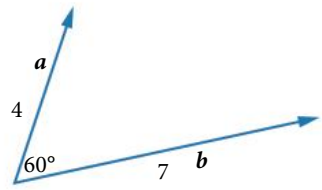
The *vector product* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\sin(\theta)\hat{\boldsymbol{n}}$, where θ is the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ in the right-hand sense.

To understand the phrase 'in the right-hand sense', consider a scenario where you extend your right hand and curve your fingers in a direction from vector $\boldsymbol{a}$ to vector $\boldsymbol{b}$. The direction of the vector product, $\boldsymbol{a}\times\boldsymbol{b}$, will be indicated by the direction of your right thumb.



24 The vector (cross) product of two vectors

Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ lie in a horizontal plane as shown. Calculate $\boldsymbol{a}\times\boldsymbol{b}$.

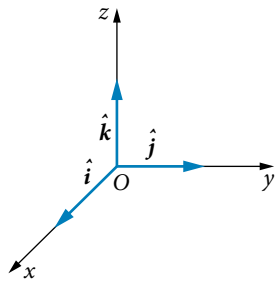


THINKING	WORKING
1 Recall the definition of the vector product.	$\boldsymbol{a}\times\boldsymbol{b}=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin(\theta)\hat{\boldsymbol{n}}$
2 Substitute given values to calculate the vector product.	$\boldsymbol{a}\times\boldsymbol{b}=4\times7\times\sin(60°)\hat{\boldsymbol{n}}$ $=\frac{21\sqrt{3}}{2}\hat{\boldsymbol{n}}$

3 Determine the direction of the resultant vector using the right-hand rule.	For $\boldsymbol{a}\times\boldsymbol{b}$ as shown, the direction of the vector product will be into the plane (or downwards).
4 Interpret your result.	$\boldsymbol{a}\times\boldsymbol{b}=\frac{21\sqrt{3}}{2}$ units into the plane.

The vector (cross) product in component form

Consider the unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$, located on the x-, y- and z-axes respectively.



These vectors all have a magnitude of 1, and the coordinate axes have a direction of 90° from one another. Using this information, it is possible to calculate the dot product of unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$.

Consider the case when these unit vectors are multiplied by themselves. From the definition of the vector product above,

$\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\sin(\theta)\hat{\boldsymbol{n}}$

$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{i}}=1\times1\times\sin(0°)\hat{\boldsymbol{n}}$

As $\sin(0°)=0$, $\hat{\boldsymbol{i}}\times\hat{\boldsymbol{i}}=0$. The same result occurs for $\hat{\boldsymbol{j}}\times\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}$.

Consider also when these unit vectors are multiplied by each other. For example,

$\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\sin(\theta)\hat{\boldsymbol{n}}$

$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}=1\times1\times\sin(90°)\hat{\boldsymbol{n}}$

$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}=1\times1\times1\times\hat{\boldsymbol{n}}$

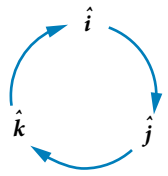
$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}=\hat{\boldsymbol{k}}$

This result is obtained because $\hat{\boldsymbol{k}}$ is perpendicular to $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ in the right-hand sense. This process can be continued for the other component vectors, with the following results:

$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}=\hat{\boldsymbol{k}}$ $\quad$ $\hat{\boldsymbol{i}}\times\hat{\boldsymbol{k}}=-\hat{\boldsymbol{j}}$ $\quad$ $\hat{\boldsymbol{j}}\times\hat{\boldsymbol{k}}=\hat{\boldsymbol{i}}$

$\hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}}=-\hat{\boldsymbol{k}}$ $\quad$ $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{i}}=\hat{\boldsymbol{j}}$ $\quad$ $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{j}}=-\hat{\boldsymbol{i}}$

To help remember these results, think of the component vectors in a clockwise circle:



To obtain the vector product using this diagram, follow the circle in a clockwise direction. For example, $\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}=\hat{\boldsymbol{k}}$. Following the circle in an anticlockwise direction yields a negative result, for example, $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{j}}=-\hat{\boldsymbol{i}}$.

Calculating the vector (cross) product for two vectors in component form

Making connections

The vector (cross) product in component form

Investigate the direction of the cross product of vectors by moving the sliders.

Consider the two three-dimensional vectors in component form, $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$. Using the distributive law of multiplication:

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \left(a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}\right) \times \left(b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}\right) \\
&= a_1b_1\left(\hat{\boldsymbol{i}} \times \hat{\boldsymbol{i}}\right) + a_1b_2\left(\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}\right) + a_1b_3\left(\hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}}\right) + a_2b_1\left(\hat{\boldsymbol{j}} \times \hat{\boldsymbol{i}}\right) + a_2b_2\left(\hat{\boldsymbol{j}} \times \hat{\boldsymbol{j}}\right) + a_2b_3\left(\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}}\right) \\
&\quad + a_3b_1\left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}}\right) + a_3b_2\left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{j}}\right) + a_3b_3\left(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}\right) \\
&= 0 + a_1b_2\hat{\boldsymbol{k}} - a_1b_3\hat{\boldsymbol{j}} - a_2b_1\hat{\boldsymbol{k}} + 0 + a_2b_3\hat{\boldsymbol{i}} + a_3b_1\hat{\boldsymbol{j}} - a_3b_2\hat{\boldsymbol{i}} + 0 \\
&= \left(a_2b_3 - a_3b_2\right)\hat{\boldsymbol{i}} - \left(a_1b_3 - a_3b_1\right)\hat{\boldsymbol{j}} + \left(a_1b_2 - a_2b_1\right)\hat{\boldsymbol{k}}
\end{aligned}$$

Conveniently, this expansion of the vector product takes the form of the determinant of a 3×3 matrix:

$$\begin{bmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

This information can be used when calculating the vector product of vectors in component form.

To remember the components of $\boldsymbol{a} \times \boldsymbol{b}$:

Form the 3×3 rectangular array $\begin{bmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$, where the first row consists of the three unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$, the second row consists of the components of vector $\boldsymbol{a}$ and the third row consists of the components of vector $\boldsymbol{b}$.

- To calculate the $\hat{\boldsymbol{i}}$ component of $\boldsymbol{a} \times \boldsymbol{b}$, delete the first column and calculate the determinant of the remaining 2×2 matrix below the $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ columns.
- To calculate the $\hat{\boldsymbol{j}}$ component of $\boldsymbol{a} \times \boldsymbol{b}$, delete the second column and calculate the negative of the determinant of the remaining 2×2 matrix below the $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{k}}$ columns.
- To calculate the $\hat{\boldsymbol{k}}$ component of $\boldsymbol{a} \times \boldsymbol{b}$, delete the third column and calculate the determinant of the remaining 2×2 matrix below the $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ columns.

Hence, $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\boldsymbol{k}}$

For two vectors in component form, $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$, the vector (cross) product can be calculated:

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}} \\ &= (a_2b_3 - a_3b_2)\hat{\boldsymbol{i}} - (a_1b_3 - a_3b_1)\hat{\boldsymbol{j}} + (a_1b_2 - a_2b_1)\hat{\boldsymbol{k}}\end{aligned}$$

25 Calculating the vector (cross) product of two vectors in component form

Calculate the cross product of vectors $\boldsymbol{a} = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$.

THINKING

1 Construct the matrix $\begin{bmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ by substituting the values.

2 Substitute values into the cross product formula:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}}$$

3 Calculate the cross product using the determinants of each 2×2 matrix and simplify.

WORKING

Since $\boldsymbol{a} = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, then

$$\begin{bmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} -2 & -3 \\ 3 & -4 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix}\hat{\boldsymbol{k}}$$

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= (1 \times (-4) - (-3) \times 2)\hat{\boldsymbol{i}} - ((-2) \times (-4) - (-3) \times 3)\hat{\boldsymbol{j}} \\ &\quad + ((-2) \times 2 - 1 \times 3)\hat{\boldsymbol{k}} \\ &= (-4 + 6)\hat{\boldsymbol{i}} - (8 + 9)\hat{\boldsymbol{j}} + (-4 - 3)\hat{\boldsymbol{k}} \\ &= 2\hat{\boldsymbol{i}} - 17\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}\end{aligned}$$

Calculating the vector (cross) product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ will result in a vector that is orthogonal (perpendicular to) both $\boldsymbol{a}$ and $\boldsymbol{b}$.

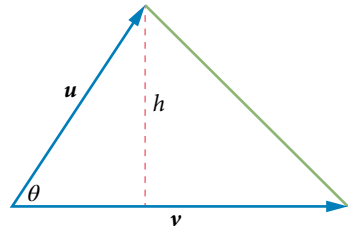
Technology worked example
Calculating the vector (cross) product

26 Calculating a vector orthogonal to two other vectors

Determine a vector orthogonal to both $\boldsymbol{a}=3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=-3\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ with a magnitude of 5.

THINKING	WORKING
1 The vector $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, so calculate $\boldsymbol{a}\times\boldsymbol{b}$: $\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{i}}-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{j}}+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}}$	$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix} 2 & -2 \\ 4 & 1 \end{vmatrix}\hat{\boldsymbol{i}}-\begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix}\hat{\boldsymbol{j}}+\begin{vmatrix} 3 & 2 \\ -3 & 4 \end{vmatrix}\hat{\boldsymbol{k}}$ $=(2\times1-(-2)\times4)\hat{\boldsymbol{i}}-(3\times1-(-2)\times(-3))\hat{\boldsymbol{j}}$ $+(3\times4-2\times(-3))\hat{\boldsymbol{k}}$ $=(2-(-8))\hat{\boldsymbol{i}}-(3-6)\hat{\boldsymbol{j}}+(12-(-6))\hat{\boldsymbol{k}}$ $=10\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+18\hat{\boldsymbol{k}}$
2 Calculate the unit vector in the direction of $\boldsymbol{a}\times\boldsymbol{b}$.	$\lvert\boldsymbol{a}\times\boldsymbol{b}\rvert=\sqrt{10^2+3^2+18^2}$ $=\sqrt{433}$ Therefore, the unit vector is $\frac{1}{\sqrt{433}}\left(10\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+18\hat{\boldsymbol{k}}\right)$.
3 Multiply the unit vector by the given magnitude to determine the required vector.	The required vector is $\frac{5\sqrt{433}}{433}\left(10\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+18\hat{\boldsymbol{k}}\right)$.

The area of a parallelogram and a triangle

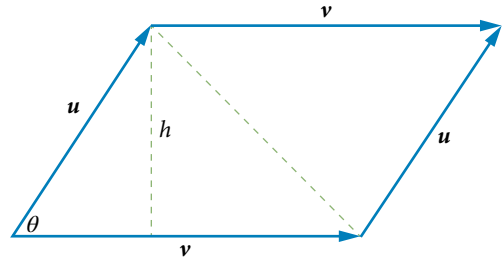


Consider a triangle defined by two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and their included angle θ:

The formula for the area of a triangle is $A=\frac{bh}{2}$, where b is the length of the base and h is perpendicular height. For the triangle shown:

$$A=\frac{1}{2}\times|\boldsymbol{v}|\times h$$
$$=\frac{1}{2}\times|\boldsymbol{v}|\times(|\boldsymbol{u}|\sin(\theta))$$
$$=\frac{1}{2}|\boldsymbol{u}||\boldsymbol{v}|\sin(\theta)$$
$$=\frac{1}{2}|\boldsymbol{u}\times\boldsymbol{v}|$$

Likewise, the area of a parallelogram formed by two vectors is simply double the area of the corresponding triangle.



For a triangle formed by two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$,
Area $=\frac{1}{2}|\boldsymbol{u}\times\boldsymbol{v}|$.
For a parallelogram with adjacent sides formed by two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, Area $=|\boldsymbol{u}\times\boldsymbol{v}|$.

27 Using the vector product to calculate the area of a triangle

Calculate the area of the triangle whose vertices are $A(2,3,1)$, $B(4,2,8)$ and $C(3,4,6)$.

THINKING	WORKING
1 Calculate vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ for two adjacent sides of the triangle.	Let $\boldsymbol{a} = \overrightarrow{AB}$ $= \left(4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}}\right) - \left(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$ $= 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$ Let $\boldsymbol{b} = \overrightarrow{AC}$ $= \left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}\right) - \left(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$ $= \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$
2 Substitute values into the cross product formula: $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\boldsymbol{k}}$	Then, $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} -1 & 7 \\ 1 & 5 \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} 2 & 7 \\ 1 & 5 \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \hat{\boldsymbol{k}}$ $= (-5-7)\hat{\boldsymbol{i}} - (10-7)\hat{\boldsymbol{j}} + (2+1)\hat{\boldsymbol{k}}$ $= -12\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$
3 The area of a triangle with adjacent sides $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by: Area $= \frac{1}{2}\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert$.	Area of triangle ABC: $A = \frac{1}{2}\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert$ $= \frac{1}{2}\sqrt{(-12)^2 + (-3)^2 + 3^2}$ $= \frac{1}{2}\sqrt{162}$ $= \frac{1}{2}\sqrt{81 \times 2}$ $= \frac{9}{2}\sqrt{2}$ square units

28 Using the cross product to calculate the area of a parallelogram

Calculate the area of the parallelogram having adjacent sides $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$.

THINKING

1 Substitute values into the cross product formula:

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}}$$

2 The area of a parallelogram with $\boldsymbol{a}$ and $\boldsymbol{b}$ adjacent sides of a parallelogram is given by: Area $= |\boldsymbol{a}\times\boldsymbol{b}|$

WORKING

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} 5 & 2 \\ 3 & 3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix}\hat{\boldsymbol{k}}$$

$$= (15-6)\hat{\boldsymbol{i}} - (12-4)\hat{\boldsymbol{j}} + (12-10)\hat{\boldsymbol{k}}$$

$$= 9\hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

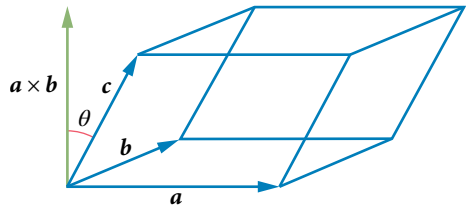
$$\text{Area} = |\boldsymbol{a}\times\boldsymbol{b}|$$

$$= \sqrt{9^2 + (-8)^2 + 2^2}$$

$$= \sqrt{149} \text{ square units}$$

The volume of a parallelepiped

A parallelepiped is a solid (3D) figure having six faces with opposite pairs of faces being congruent parallelograms. A parallelepiped can be defined by three vectors, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, that form the three edges coming from a common vertex.



The volume of the parallelepiped is the area of the base multiplied by the height. The area of the base is the area of the parallelogram with $\boldsymbol{a}$ and $\boldsymbol{b}$ as adjacent sides and is equal to $|\boldsymbol{a}\times\boldsymbol{b}|$. The height is the length of the projection of $\boldsymbol{c}$ onto $\boldsymbol{a}\times\boldsymbol{b}$ and is given by $|\boldsymbol{c}|\cos(\theta)$.

Therefore, the volume of the parallelepiped is given by

V = Area of base × height

$$= |\boldsymbol{a}\times\boldsymbol{b}||\boldsymbol{c}|\cos(\theta)$$

$$= |(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}|$$

The volume of a parallelepiped defined by vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is $V = |(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}|$. This is sometimes called the *scalar triple product* (as the result is a scalar quantity).

The volume can similarly be calculated:

$$V = |(\boldsymbol{a}\times\boldsymbol{c})\cdot\boldsymbol{b}|$$

$$V = |(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}|$$

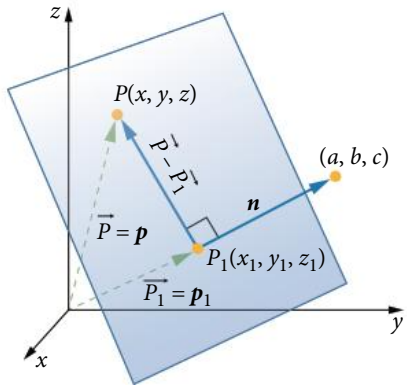
29 Using the vector product to calculate the volume of a parallelepiped

Calculate the volume of a parallelepiped having the following three vectors as edges: $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Substitute values into the cross product formula: $\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}}$	$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} 3 & -1 \\ -4 & 3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} 2 & 3 \\ 3 & -4 \end{vmatrix}\hat{\boldsymbol{k}}$ $= (9-4)\hat{\boldsymbol{i}} - (6+3)\hat{\boldsymbol{j}} + (-8-9)\hat{\boldsymbol{k}}$ $= 5\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}} - 17\hat{\boldsymbol{k}}$
2 Calculate the scalar product $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$.	$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \left(5\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}} - 17\hat{\boldsymbol{k}}\right)\cdot\left(4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}\right)$ $= 5\times 4 + (-9)\times 5 + (-17)\times 6$ $= 20 - 45 - 102$ $= -127$
3 Calculate the magnitude of the scalar triple product $\lvert(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\rvert$ to give the volume of the parallelepiped.	Volume: $V = \lvert(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\rvert$ $= \lvert-127\rvert$ $= 127$ cubic units

Vector and Cartesian equations of a plane in three dimensions

In three dimensions, a plane is defined by the equation $ax + by + cz + d = 0$. To determine the equation of a plane, it is necessary to know the coordinates of a point on the plane, $P_1(x_1, y_1, z_1)$, as well as a normal vector, $\boldsymbol{n} = \left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right)$, which is orthogonal (perpendicular) to the plane.



Given this information, a vector on the plane can be determined using the points $P_1(x_1, y_1, z_1)$ and a general point $P(x, y, z)$. Since the normal vector $\boldsymbol{n} = \left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right)$ is by definition orthogonal to the vector connecting points P and P_1, the dot product of these two vectors will be equal to zero:

$$\begin{aligned} \boldsymbol{n}\cdot(\boldsymbol{p} - \boldsymbol{p}_1) &= 0 \\ \boldsymbol{n}\cdot\boldsymbol{p} - \boldsymbol{n}\cdot\boldsymbol{p}_1 &= 0 \\ \boldsymbol{n}\cdot\boldsymbol{p} &= \boldsymbol{n}\cdot\boldsymbol{p}_1 \\ \boldsymbol{p}\cdot\boldsymbol{n} &= \boldsymbol{p}_1\cdot\boldsymbol{n} \end{aligned}$$

This is referred to as the vector form of the equation of a plane, where $\boldsymbol{p}$ is the general position vector $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, $\boldsymbol{p}_1$ is the position vector of a given point on the plane and $\boldsymbol{n}$ is a vector normal to the plane.

An alternative version of this formula is $\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$, where $\boldsymbol{r}$ is the general position vector $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ and $\boldsymbol{a}$ is the position vector of a given point on the plane.

The vector equation of a plane is $\boldsymbol{p} \cdot \boldsymbol{n} = \boldsymbol{p}_1 \cdot \boldsymbol{n}$, where $\boldsymbol{p}$ is the general position vector $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, $\boldsymbol{p}_1$ is the position vector of a given point on the plane and $\boldsymbol{n}$ is a vector normal to the plane.

An alternative version of this formula is $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$, where $\boldsymbol{r}$ is the general position vector $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, $\boldsymbol{a}$ is the position vector of a given point on the plane and $\boldsymbol{n}$ is a vector normal to the plane.

The Cartesian equation of a plane can be derived from the vector equation:

$$(\boldsymbol{p} - \boldsymbol{p}_1) = (x - x_1)\hat{\boldsymbol{i}} + (y - y_1)\hat{\boldsymbol{j}} + (z - z_1)\hat{\boldsymbol{k}}$$

$$\boldsymbol{n} \cdot (\boldsymbol{p} - \boldsymbol{p}_1) = \left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right) \cdot \left[(x - x_1)\hat{\boldsymbol{i}} + (y - y_1)\hat{\boldsymbol{j}} + (z - z_1)\hat{\boldsymbol{k}}\right]$$

$$0 = a(x - x_1) + b(y - y_1) + c(z - z_1)$$

WARNING

The vectors $\boldsymbol{n}$ and $(\boldsymbol{p} - \boldsymbol{p}_1)$ don't need to be drawn at the same point to be perpendicular.

The general Cartesian equation of a plane is $ax + by + cz + d = 0$. The Cartesian equation of a plane passing through the point $P_1(x_1, y_1, z_1)$ and perpendicular to the normal vector $\boldsymbol{n} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ is $0 = a(x - x_1) + b(y - y_1) + c(z - z_1)$.

30 Determining a vector equation of a plane in three dimensions

Determine the vector equation of a plane such that the point $A(5, -2, 3)$ lies on the plane and the vector $\boldsymbol{n} = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$ is normal to the plane.

THINKING	WORKING
1 Recall the general vector equation of a plane.	$\boldsymbol{p} \cdot \boldsymbol{n} = \boldsymbol{p}_1 \cdot \boldsymbol{n}$
2 Express the given point as a position vector and write the normal vector.	$\boldsymbol{p}_1 = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ $\boldsymbol{n} = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$
3 Substitute into the general equation and calculate the dot product.	$\boldsymbol{p} \cdot \boldsymbol{n} = \boldsymbol{p}_1 \cdot \boldsymbol{n}$ $\boldsymbol{p} \cdot \left(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}\right) = \left(5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right) \cdot \left(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}\right)$ $= -10 - 2 - 15$ $= -27$
4 Interpret your answer.	A vector equation of the plane is $\boldsymbol{p} \cdot \left(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}\right) = -27$.

31 Determining a Cartesian equation of a plane in three dimensions

Determine a Cartesian equation of the plane containing the points $A(1,0,2)$, $B(2,1,0)$ and $C(-1,2,1)$.

THINKING	WORKING
1 Determining the equation of a plane requires knowing a point on the plane and a vector perpendicular to the plane. The perpendicular vector can be determined by calculating the vector product of two vectors in the plane. Use the given points to determine these two vectors.	$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ $= \left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) - \left(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}\right)$ $= \left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right)$ $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$ $= \left(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right) - \left(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}\right)$ $= \left(-2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$
2 Calculate the vector product of the two vectors to determine the normal vector $\boldsymbol{n}$.	$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ $= \left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right) \times \left(-2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$ $= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 1 & -2 \\ -2 & 2 & -1 \end{vmatrix}$ $= \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} 1 & -2 \\ -2 & -1 \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} \hat{\boldsymbol{k}}$ $= (-1+4)\hat{\boldsymbol{i}} - (-1-4)\hat{\boldsymbol{j}} + (2+2)\hat{\boldsymbol{k}}$ $= 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$
3 Substitute the coefficients from the normal vector and the coordinates of one of the given points to determine the equation of the plane. Simplify your answer.	Use point $A(1,0,2)$ $0 = a(x - x_1) + b(y - y_1) + c(z - z_1)$ $= 3(x-1) + 5(y-0) + 4(z-2)$ $= 3x - 3 + 5y + 4z - 8$ Therefore $3x + 5y + 4z = 11$.

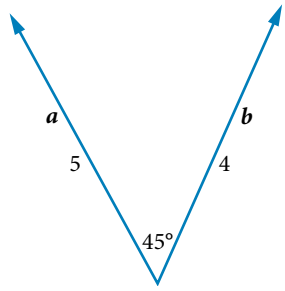
EXERCISE 2.6 The vector product

Worked Example

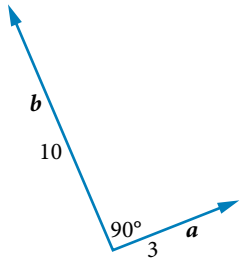
24

1 For each of the following pairs of vectors, use the definition of the cross product to calculate $\boldsymbol{a}\times\boldsymbol{b}$.

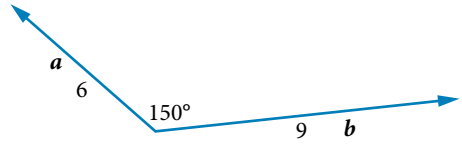
(a)



(b)



(c)



2 Let $\boldsymbol{a}=2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$, $\boldsymbol{b}=3\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$. Calculate each of the following:

(a) $\boldsymbol{a}\times\boldsymbol{b}$ (b) $\boldsymbol{a}\times\boldsymbol{c}$ (c) $\boldsymbol{b}\times\boldsymbol{c}$ (d) $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})$

25

3 Calculate the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ for each of the following:

(a) $\boldsymbol{a}=4\hat{\boldsymbol{i}}+5\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=-2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+5\hat{\boldsymbol{k}}$

(b) $\boldsymbol{a}=-\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=5\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}}$

4 Let $\boldsymbol{a}=\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}+5\hat{\boldsymbol{k}}$, $\boldsymbol{b}=2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=-2\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}+7\hat{\boldsymbol{k}}$. Determine each of the following:

(a) $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})$ (b) $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ (c) $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}$

29

5 Calculate the volume of the parallelepiped having the following three vectors as edges:

(a) $\boldsymbol{a}=3\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$, $\boldsymbol{b}=4\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=-2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}$

(b) $\boldsymbol{a}=4\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}$, $\boldsymbol{b}=-3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=5\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$

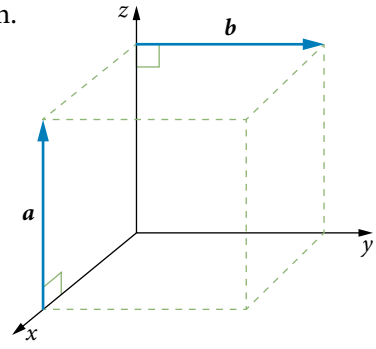
(c) $\boldsymbol{a}=5\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}$, $\boldsymbol{b}=-2\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=-3\hat{\boldsymbol{i}}+5\hat{\boldsymbol{k}}$

6 Calculate the volume of the parallelepiped having the vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{i}}+\hat{\boldsymbol{k}}$ as edges.

30

7 Determine the vector equation of a plane such that the point $A(10,3,-1)$ lies on the plane and the vector $\boldsymbol{n}=2\hat{\boldsymbol{i}}-5\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ is normal to the plane.

8 Consider the two vectors drawn in a three-dimensional space as shown.



(a) Identify which of the following statements is true.

A The dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ vectors is different to zero.

B The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ need to be drawn at the same point to be perpendicular.

C $\boldsymbol{b}$ is the only orthogonal vector to $\boldsymbol{a}$.

D The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not on the same plane.

(b) Explain why the incorrect statements in part (a) are false.

9 Let $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$, $\boldsymbol{b} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$. 26

Determine a vector $\boldsymbol{p}$ that is:

(a) orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, with a magnitude of 1.

(b) orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{c}$, with a magnitude of 10.

10 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. Calculate the area of the parallelogram whose adjacent sides are formed by the following vectors: 28

(a) $\boldsymbol{a}$ and $\boldsymbol{b}$ (b) $\boldsymbol{a}$ and $\boldsymbol{c}$ (c) $\boldsymbol{b}$ and $\boldsymbol{c}$

11 Calculate the area of the triangle whose vertices are given: 27

(a) $A(3,4,-2)$, $B(1,5,2)$, $C(-1,5,3)$ (b) $A(1,0,4)$, $B(-2,4,5)$, $C(2,-7,-3)$

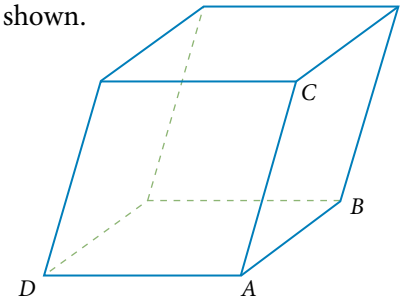
12 Consider the parallelepiped with adjacent edges $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ as shown.

Calculate the volume of this parallelepiped if:

(a) $A = (0,0,0), B = (1,2,1), C = (3,0,1), D = (-1,1,2)$

(b) $A = (1,1,1), B = (3,5,2), C = (5,7,5), D = (-3,-2,4)$

(c) $A = (3,4,5), B = (-3,7,0), C = (0,5,2), D = (9,11,23)$



13 Determine the equation of the plane containing each of the following sets of coordinate points: 31

(a) $A = (1,0,0), B = (2,-1,-1), C = (-1,0,1)$

(b) $A = (2,0,1), B = (3,3,1), C = (3,2,2)$

(c) $A = (-1,1,1), B = (0,2,3), C = (-3,1,2)$

14 Determine a vector perpendicular to the plane $2x - 3y + z = 6$.

15 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors in space.

(a) Prove that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors, then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$.

(b) Prove that if $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors.

16 The vectors $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ form a triangle ABC.

(a) Calculate the cross products $\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{a} \times \boldsymbol{c}$ and $\boldsymbol{b} \times \boldsymbol{c}$.

(b) Calculate the area of the triangle ABC.

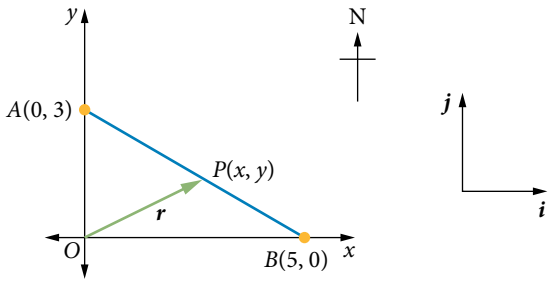
2.7 Vector equation of a straight line in the Cartesian plane

There are multiple ways of expressing a line on the Cartesian plane. You would be most familiar with the Cartesian form $y = mx + c$, which expresses a direct relationship between x and y values on the plane. However, when working with vectors it is often desirable to use a different approach.

The vector equation of a line in the plane

Consider the following scenario. Alexandra lives 3 km north of school and Bavneet lives 5 km east of school. There is a straight road that runs past both Alexandra's and Bavneet's houses. If Pieter walks along this road from Alexandra's house A to Bavneet's house B, can Pieter's position on the road be determined relative to the school, O?

Draw a diagram with Pieter at some point $P(x, y)$ along the road between A and B.



Let the position vector of point $P(x, y)$ be $\boldsymbol{r}$, where $\boldsymbol{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$.

The point $P(x, y)$ is on $\overline{AB}$ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, where $0 \leq \lambda \leq 1$, so that when $\lambda = 0$ Pieter is at A and when $\lambda = 1$ he is at B.

Now, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = 5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, so $\boldsymbol{r} = 3\hat{\boldsymbol{j}} + \lambda(5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}})$. This gives Pieter's position vector walking on the road between A and B.

The position vector of P can be written as $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ and so $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} = 3\hat{\boldsymbol{j}} + \lambda(5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}})$.

That is, $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} = 5\lambda\hat{\boldsymbol{i}} + (3 - 3\lambda)\hat{\boldsymbol{j}}$ and so $x = 5\lambda$ and $y = 3 - 3\lambda$.

The equations

$x = 5\lambda$ [1]

and $y = 3 - 3\lambda$ [2]

are known as the *parametric form* of the equations of a straight line.

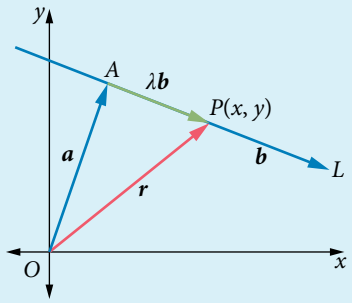
The parameter λ can be eliminated from these equations by rewriting equation [1] as $\lambda = \frac{x}{5}$ and equation [2] as $\lambda = \frac{3 - y}{3}$ and then equating, giving $\frac{x}{5} = \frac{3 - y}{3}$ or $3x + 5y = 15$, which is the Cartesian equation of the straight line passing through A and B.

This method of determining the equation to describe the position of a person can be applied to all objects travelling in a straight line. For example, when aircrafts are approaching landing, it is necessary to know their positions along their flight paths so that collisions can be avoided.

The vector equation of a line L passing through the point A with position vector $\boldsymbol{a}$ and in the direction of position vector $\boldsymbol{b}$ is given by $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where λ is a *scalar parameter*.

If point $P(x, y)$ is a point on the line L, then the vector $\overrightarrow{AP}$ is parallel to the vector $\boldsymbol{b}$, such that $\overrightarrow{AP} = \lambda\boldsymbol{b}, 0 \leq \lambda \leq 1$.

Now $\boldsymbol{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ and so $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ is the vector equation of line L.



The equation of a straight line in vector matrix form

Let the coordinates of points A and B be (a_1, a_2) and (b_1, b_2) respectively, and the coordinates of point P be (x, y).

The position vectors of point A and B are $\overrightarrow{OA} = \boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ respectively, and the position vector of point P is $\overrightarrow{OP} = \boldsymbol{r} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Since $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \end{bmatrix}$ in vector matrix form.

This gives the parametric form for the equation of a straight line:

$x = a_1 + \lambda b_1$ [1]

$y = a_2 + \lambda b_2$ [2]

From equation [1], $\lambda = \dfrac{x - a_1}{b_1}$ [3].

From equation [2], $\lambda = \dfrac{y - a_2}{b_2}$ [4].

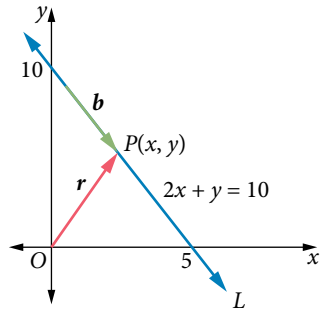
Eliminating λ by equating [3] and [4] gives the Cartesian form for the equation of a straight line $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2}$ or $b_2(x - a_1) = b_1(y - a_2)$.

A vector equation of a straight line in the Cartesian plane

Given the equation of a straight line, a possible vector equation can be determined.

32 Vector equation of a straight line in the Cartesian plane

Consider the straight line L in the Cartesian plane with equation $2x + y = 10$ as shown.



(a) Determine a vector equation of L using the point (5, 0).
Note: There are infinite possibilities, this example shows how to determine just two of them.

THINKING	WORKING
1 Determine the known position vector.	The position vector of the point $(5,0)$ is $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.
2 Determine a vector parallel to L.	The gradient of the line L is -2. Therefore a vector that runs parallel to L, for every unit to the right, runs 2 units down: $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$
3 Determine an equation of the line L, $\boldsymbol{r}_1 = \boldsymbol{a} + \lambda\boldsymbol{b}$.	Using the known position vectors, the position vector $P(x, y)$ on the line could be given as: $\boldsymbol{r} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \boldsymbol{r}_1 = 5\hat{\boldsymbol{i}} + \lambda\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

(b) Determine a vector equation of L using the point (0, 10).

1 Determine the known position vector.	The position vector for the point (0, 10) is $\begin{bmatrix} 0 \\ 10 \end{bmatrix}$.
2 Determine a vector parallel to L.	From part (a): $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ Note: As long as the ratio is the same i.e. $y = -2x$, then the values in this column vector can be anything.
3 Determine an equation of the line L, $\boldsymbol{r}_2 = \boldsymbol{a} + \lambda\boldsymbol{b}$.	Using the known position vectors, the position vector $P(x, y)$ on the line could be given as: $\boldsymbol{r}_2 = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \boldsymbol{r}_2 = 10\hat{\boldsymbol{j}} + \lambda\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

Note: In reality, your initial position vector may be any point along the line $2x + y = 10$.

(c) Determine the position $P(x, y)$ using each of your equations, when $\lambda = 1$.

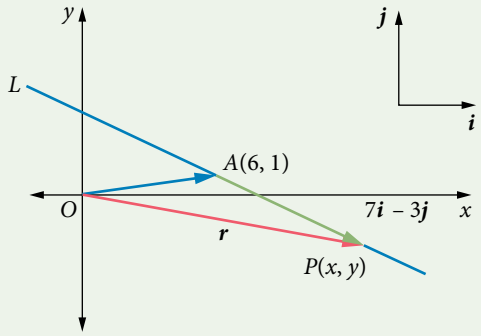
1 Substitute $\lambda = 1$ into each equation.	$\boldsymbol{r}_1 = 5\hat{\boldsymbol{i}} + 1 \times \left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ $= 6\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ $\quad \boldsymbol{r}_2(1) = 10\hat{\boldsymbol{j}} + 1 \times \left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ $= \hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$
2 Interpret the answer.	At $\lambda = 1$, the position $P(x, y)$ is 1 unit to the right and 2 units down from the chosen initial position vector. $P_1(6, -2)$ and $P_2(1, 8)$ Note: This interpretation is dependent on the direction of the vector parallel to L chosen in step 2.

A vector equation of a straight line passing through a given point

If a line passes through a given point and is parallel to a specified vector, a possible vector equation can be determined.

33 Determining a vector equation of a straight line passing through a given point

Determine a possible vector equation of the straight line L that passes through the point $A(6,1)$ and is parallel to the vector $7\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$.

THINKING	WORKING
1 Draw a diagram of the line L and the point A. Let the point P be any point on the line L with position vector $\boldsymbol{r}$, then $\boldsymbol{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$	
2 Determine the vectors $\overrightarrow{OA}$ and $\overrightarrow{AP}$.	$\overrightarrow{OA} = 6\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\overrightarrow{AP} = \lambda\left(7\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$
3 The equation of line L is $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ or $\boldsymbol{r} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.	Therefore, $\boldsymbol{r} = \left(6\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) + \lambda\left(7\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$ or $\boldsymbol{r} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ -3 \end{bmatrix}$.
4 Represent the vector equation of the line L in terms of the parameter λ, where $\lambda \in \mathbb{R}$. That is, $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ or $\boldsymbol{r} = \begin{bmatrix} a_1 + \lambda b_1 \\ a_2 + \lambda b_2 \end{bmatrix}$.	$\boldsymbol{r} = (6 + 7\lambda)\hat{\boldsymbol{i}} + (1 - 3\lambda)\hat{\boldsymbol{j}}$ or $\boldsymbol{r} = \begin{bmatrix} 6 + 7\lambda \\ 1 - 3\lambda \end{bmatrix}$

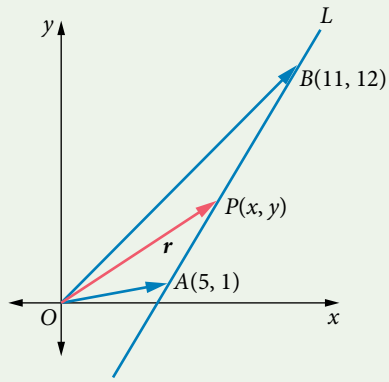
The vector equation of a straight line and straight line segment passing through two given points

If a line passes through two given points, its vector equation can be determined.

34 Determining the vector equation of a straight line passing through two given points

The straight line L passes through the points $A(5,1)$ and $B(11,12)$ in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$.

(a) Determine the vector equation of the straight line L.

THINKING	WORKING
1 Draw a diagram of the line L and the points A and B.	
2 Let the point P be any point on the line L with position vector $\boldsymbol{r}$, then $\boldsymbol{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$.	$\boldsymbol{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ and $\overrightarrow{AP} \parallel \overrightarrow{AB}$ so $\overrightarrow{AP} = \lambda \overrightarrow{AB}$, where $\lambda \in \mathbb{R}$.
3 Calculate the vector $\overrightarrow{AB}$ which will be the vector parallel to the line L.	$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ $= \overrightarrow{OB} - \overrightarrow{OA}$ $= \begin{bmatrix} 11 \\ 12 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ $= \begin{bmatrix} 6 \\ 11 \end{bmatrix}$
4 Determine the vector $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ in matrix form.	$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$, $\overrightarrow{OP} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \lambda \times \begin{bmatrix} 6 \\ 11 \end{bmatrix}$
5 Simplify the vector equation, where necessary.	$\boldsymbol{r} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ 11 \end{bmatrix}$ or $\boldsymbol{r} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \lambda\left(6\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}\right)$.

(b) Express the equation of L in both parametric and Cartesian form.

1 Determine the parametric form of L using with $P(x, y)$ being any point on the line L.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ 11 \end{bmatrix}$
2 Obtain the parametric equations.	The parametric equations are $x = 5 + 6\lambda$ and $y = 1 + 11\lambda$.
3 Make the parameter the subject of each of the parametric equations.	From $x = 5 + 6\lambda$, $\lambda = \frac{x-5}{6}$ [1]. From $y = 1 + 11\lambda$, $\lambda = \frac{y-1}{11}$ [2].
4 Equate these equations to eliminate the parameter.	$\frac{x-5}{6} = \frac{y-1}{11}$ $11x - 55 = 6y - 6$ $11x - 6y = 49$
5 Write the Cartesian form of L.	The Cartesian equation of L is $11x - 6y = 49$.

(c) Determine the vector equation of the line segment connecting points $A(5,1)$ and $B(11,12)$.

1 A line segment can be defined by restricting the domain of the parameter λ for the vector equation that has already been calculated. Write the vector equation and the position vectors of the two points.	$\boldsymbol{r} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \lambda\left(6\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}\right)$ $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ $\overrightarrow{OB} = 11\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$
2 Determine the value of λ corresponding to point A.	When $\lambda = 0$, $\boldsymbol{r} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} = \overrightarrow{OA}$.
3 Determine the value of λ corresponding to point B.	$5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \lambda\left(6\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}\right) = 11\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$ $(5 + 6\lambda)\hat{\boldsymbol{i}} + (1 + 11\lambda)\hat{\boldsymbol{j}} = 11\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$ $5 + 6\lambda = 11$ and $1 + 11\lambda = 12$ Therefore, $\lambda = 1$.
4 Interpret your answer.	The vector equation of the line segment connecting points $A(5,1)$ and $B(11,12)$ is $\boldsymbol{r} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \lambda\left(6\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}\right), 0 \leq \lambda \leq 1$.

35 Determining the Cartesian equation of a straight line specified in vector equation form

The vector equation of the line L is specified by $\boldsymbol{r} = \begin{bmatrix} 4-3\lambda \\ -3+8\lambda \end{bmatrix}$.

(a) Express the vector equation in the standard form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$.

THINKING	WORKING
1 The vector equation of the line L is $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ or $\boldsymbol{r} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.	$\boldsymbol{r} = \begin{bmatrix} 4-3\lambda \\ -3+8\lambda \end{bmatrix}$ $= \begin{bmatrix} 4 \\ -3 \end{bmatrix} + \begin{bmatrix} -3\lambda \\ 8\lambda \end{bmatrix}$ $= \begin{bmatrix} 4 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 8 \end{bmatrix}$
2 Express answer as the sum of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ components.	$\boldsymbol{r} = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \lambda\left(-3\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}\right)$

(b) Calculate a unit vector in the direction of L.

1 The direction of the line L is provided by the vector $\boldsymbol{b}$.	$\boldsymbol{b} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$
2 Since unit vector $\hat{\boldsymbol{b}} = \dfrac{\boldsymbol{b}}{\lvert\boldsymbol{b}\rvert}$, calculate $\lvert\boldsymbol{b}\rvert$.	$\lvert\boldsymbol{b}\rvert = \sqrt{(-3)^2 + 8^2}$ $= \sqrt{73}$
3 Write unit vector $\hat{\boldsymbol{b}}$.	Therefore, $\hat{\boldsymbol{b}} = \dfrac{1}{\sqrt{73}} \begin{bmatrix} -3 \\ 8 \end{bmatrix}$ $= \dfrac{\sqrt{73}}{73} \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

(c) Determine the Cartesian form of the line L.

1 If $P(x, y)$ is a point on the line L, then the position vector $\boldsymbol{r}$ is given by $\boldsymbol{r} = \begin{bmatrix} x \\ y \end{bmatrix}$.	$\boldsymbol{r} = \begin{bmatrix} 4-3\lambda \\ -3+8\lambda \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4-3\lambda \\ -3+8\lambda \end{bmatrix}$
2 Determine the parametric equations for L.	$x = 4 - 3\lambda$ or $\lambda = \dfrac{x-4}{-3}$ and $y = -3 + 8\lambda$ or $\lambda = \dfrac{y+3}{8}$.

3 Equate these equations to eliminate the parameter.	$\frac{x-4}{-3}=\frac{y+3}{8}$
4 Write the Cartesian form of L.	The Cartesian form of L is $8x-32=-3y-9$ or $8x+3y=23$

Calculating the angle between two straight lines

The angle between two straight lines can be determined by calculating the direction vectors corresponding to each of the straight lines. The scalar product of the direction vectors can then be used to calculate the angle between the two direction vectors.

36 Calculating the angle between two straight lines

Calculate the angle, correct to the nearest degree, between the straight lines L_1 with Cartesian equation $\frac{x+3}{5}=\frac{3-y}{2}$ and L_2 with Cartesian equation $\frac{x-4}{-7}=\frac{y+3}{5}$.

THINKING	WORKING
1 EExpress the line L_1 in vector form by introducing a parameter, λ.	Let $\frac{x+3}{5}=\frac{y-3}{-2}=\lambda$ giving the parametric equations $x=-3+5\lambda$ and $y=3-2\lambda$.
2 Express the parametric equations in vector form.	$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3+5\lambda \\ 3-2\lambda \end{bmatrix}$ $=\begin{bmatrix} -3 \\ 3 \end{bmatrix}+\lambda\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ or $\underset{\sim}{r}=-3\underset{\sim}{i}+3\underset{\sim}{j}+\lambda\left(5\underset{\sim}{i}-2\underset{\sim}{j}\right)$
3 Interpret the vector equation.	The line $\frac{x+3}{5}=\frac{y-3}{-2}$ is parallel to the vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$
4 Express the line L_2 in vector form by introducing a parameter.	Let $\frac{x-4}{-7}=\frac{y+3}{5}=\mu$ ggiving the parametric equations $x=4-7\mu$ and $y=-3+5\mu$.

5 Express the parametric equations in vector form.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4-7\mu \\ -3+5\mu \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -3 \end{bmatrix} + \mu \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$
$$\text{or} = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \mu\left(-7\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$$

6 Interpret the vector equation.

The line $\dfrac{x-4}{-7} = \dfrac{y+3}{5}$ is parallel to the vector $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$.

7 Calculate the magnitudes of the direction vectors.

$$\left\| \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right\| = \sqrt{5^2 + (-2)^2}$$
$$= \sqrt{29}$$
$$\left\| \begin{bmatrix} -7 \\ 5 \end{bmatrix} \right\| = \sqrt{7^2 + (-5)^2}$$
$$= \sqrt{74}$$

8 Calculate the scalar product of the direction vectors.

$$\left(5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right) \cdot \left(-7\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right) = 5 \times -7 + -2 \times 5$$
$$= -35 - 10$$
$$= -45$$

9 To calculate the angle between the two lines use their direction vectors along with the scalar product:
$\cos(\theta) = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right|}$, where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the direction vectors.

$$\theta = \cos^{-1}\left(\frac{\left(5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right) \cdot \left(-7\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)}{\left\| \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right\| \times \left\| \begin{bmatrix} -7 \\ 5 \end{bmatrix} \right\|} \right)$$
$$= \cos^{-1}\left(\frac{-45}{\sqrt{29} \times \sqrt{74}} \right)$$
$$\approx 166°$$

10 State the angle, correct to the nearest degree.

The angle between lines L_1 and L_2 is 166°.

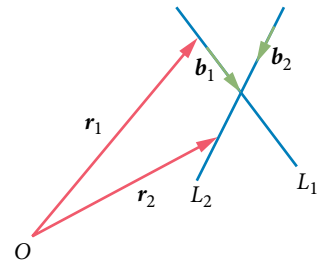
The relationship between two straight lines in a plane

There are three scenarios for two straight lines in the Cartesian plane.

1 Two lines are not parallel and therefore intersect at some point

Consider L_1, where $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda\boldsymbol{b}_1$ and L_2, where $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \mu\boldsymbol{b}_2$, which are not parallel.

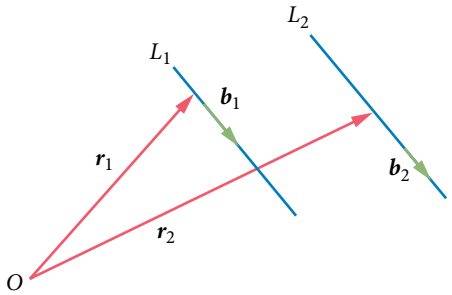
The lines have non-parallel direction vectors and a common point.



2 Two lines are parallel and therefore never meet

Consider L_1, where $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and L_2, where $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \mu \boldsymbol{b}_2$, and $\boldsymbol{b}_1 \parallel \boldsymbol{b}_2$.

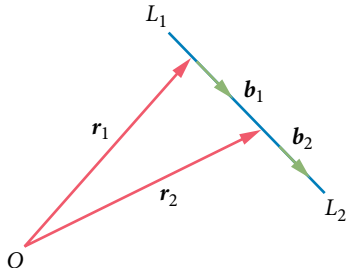
The lines have the same direction vectors and no common points.



3 Two lines are parallel and have a common point and therefore they are coincident

Consider L_1, where $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and L_2, where $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \mu \boldsymbol{b}_2$, and $\boldsymbol{b}_1 \parallel \boldsymbol{b}_2$.

The lines have the same direction vectors and all their points are common.



The point of intersection of two straight lines can be determined by equating their vector equations.

37 Calculating the point of intersection of two straight lines

Calculate the point of intersection of the lines L_1 and L_2 whose vector equations are $\boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \lambda\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \mu\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ respectively.

THINKING	WORKING
1 If the two lines intersect then there exists a point on L_1 and L_2 such that $\boldsymbol{r}_1 = \boldsymbol{r}_2$.	$3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \lambda\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right) = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \mu\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ $\therefore (3 + 2\lambda)\hat{\boldsymbol{i}} + (4 - 3\lambda)\hat{\boldsymbol{j}} = (4 + 3\mu)\hat{\boldsymbol{i}} + (-1 - 2\mu)\hat{\boldsymbol{j}}$
2 Equate the components on each side of the equation and simplify.	$3 + 2\lambda = 4 + 3\mu$ $2\lambda - 3\mu = 1 \quad [1]$ $4 - 3\lambda = 1 - 2\mu$ $-3\lambda + 2\mu = -5 \quad [2]$
3 Solve the simultaneous equations for λ and μ.	$2 \times [1] + 3 \times [2]$: $-5\lambda = -13$ $\lambda = \frac{13}{5}$ Substituting $\lambda = \frac{13}{5}$ into [1]: $2 \times \frac{13}{5} - 3\mu = 1$ $\mu = \frac{7}{5}$
4 Use vector equation $\boldsymbol{r}_1$ to calculate the position vector of the point of intersection.	$\boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \frac{13}{5}\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$ $= \frac{41}{5}\hat{\boldsymbol{i}} - \frac{19}{5}\hat{\boldsymbol{j}}$

5 Write the point of intersection of lines L_1 and L_2.

Lines L_1 and L_2 intersect at the point with coordinates $\left(\frac{41}{5}, -\frac{19}{5}\right)$.

6 Check the point of intersection using vector equation $\boldsymbol{r}_2$.

$\boldsymbol{r}_2 = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \frac{7}{5}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

$= \frac{41}{5}\hat{\boldsymbol{i}} - \frac{19}{5}\hat{\boldsymbol{j}}$

and the point of intersection is $\left(\frac{41}{5}, -\frac{19}{5}\right)$.

EXERCISE 2.7 Vector equation of a straight line in the Cartesian plane

Worked Example

33

1 Determine the vector equation of the line passing through the point A and parallel to the vector $\boldsymbol{b}$, where:

(a) A is $(-3, 1)$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

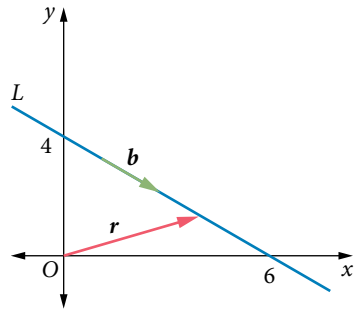
(b) A is $(7, -2)$, $\boldsymbol{b} = -4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$

(c) A is $(4, -8)$, $\boldsymbol{b} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$

(d) A is $(1, 0)$, $\boldsymbol{b} = \begin{bmatrix} -16 \\ 3 \end{bmatrix}$

32

2 Consider the straight line L in the Cartesian plane with equation $2x + 3y = 12$ as shown.



(a) Determine the vector equation of L in the form $a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + \lambda\left(c\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}}\right)$.

(b) Hence, determine the position vector when $\lambda = 2$.

(c) If the position vector $\boldsymbol{r} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, determine the value of λ.

3 Determine the vector equation of the straight line L that passes through the point $A(2, 4)$ and is parallel to the vector $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

34

4 The straight line L passes through the points $A(1, 3)$ and $B(8, 5)$.

(a) Determine the vector equation of the straight line L in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$.

(b) Express the equation of L in Cartesian form.

5 Determine a vector equation, $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, of the line passing through the given points:

(a) $A(1, 2)$ and $B(-2, 5)$

(b) $C(-3, 1)$ and $D(5, -6)$

(c) $E(0, -3)$ and $F(-6, -4)$

(d) $G(9, 0)$ and $H(-4, -9)$

6 Determine the vector equation of the line segment connecting points $A(1, 4)$ and $B(3, -2)$.

7 Determine the vector equation of the straight line, $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, defined by the following parametric equations:

(a) $x = 2 + 3\lambda, y = 4 - \lambda$ **(b)** $x = -5 + 3\lambda, y = 3 - 9\lambda$

(c) $x = -3 - 12\lambda, y = 1 + 11\lambda$ **(d)** $x = -2 - 6\lambda, y = -2 + 13\lambda$

8 Determine the parametric equations of the straight line having the given vector equation:

(a) $\boldsymbol{r} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \lambda\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ **(b)** $\boldsymbol{r} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} + \lambda\begin{bmatrix} -7 \\ -1 \end{bmatrix}$ **(c)** $\boldsymbol{r} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} 12 \\ -5 \end{bmatrix}$ **(d)** $\boldsymbol{r} = \begin{bmatrix} -4 - \frac{1}{2}\lambda \\ 5 + \frac{1}{4}\lambda \end{bmatrix}$

Worked Example 35

9 The vector equation of the line L is specified in vector column matrix form by $\boldsymbol{r} = \begin{bmatrix} -2 + 5\lambda \\ 1 - 4\lambda \end{bmatrix}$.

(a) Express the vector equation in the standard form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$.

(b) Determine a unit vector in the direction of L.

(c) Determine the Cartesian form of the line L.

10 Determine the Cartesian form of the straight line $ax + by = d$ having the following vector equations.

(a) $\boldsymbol{r} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \lambda\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ **(b)** $\boldsymbol{r} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \lambda\begin{bmatrix} -3 \\ -7 \end{bmatrix}$ **(c)** $\boldsymbol{r} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(d) $\boldsymbol{r} = \begin{bmatrix} \frac{1}{2} \\ \frac{9}{2} \end{bmatrix} - \lambda\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **(e)** $\boldsymbol{r} = \begin{bmatrix} 2 \\ -7 \end{bmatrix} + \lambda\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ **(f)** $\boldsymbol{r} = \begin{bmatrix} -5 \\ -9 \end{bmatrix} + \lambda\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

11 Calculate the angle θ, correct to the nearest degree, between the straight lines L_1 with Cartesian equation $\frac{x-2}{4} = \frac{y-4}{-3}$ and L_2 with Cartesian equation $\frac{x-5}{-2} = \frac{y+1}{6}$.

12 Calculate the angle θ, correct to the nearest degree, between the straight lines L_1 and L_2 with the following Cartesian equations: 36

(a) L_1: $\frac{x-1}{4} = \frac{y-2}{5}$ and L_2: $\frac{x-4}{1} = \frac{y-3}{2}$ **(b)** L_1: $\frac{x+3}{5} = \frac{y+4}{2}$ and L_2: $\frac{x-3}{4} = \frac{y-2}{-1}$

(c) L_1: $\frac{x-5}{-3} = \frac{y+4}{6}$ and L_2: $\frac{x+3}{-4} = \frac{y-2}{-3}$ **(d)** L_1: $\frac{x+1}{3} = y - 4$ and L_2: $x + 2 = \frac{y-3}{-6}$

13 Calculate the point of intersection of the lines L_1 and L_2 whose vector equations are $\boldsymbol{r}_1 = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \lambda\left(4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = -3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \mu\left(\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}\right)$ respectively.

14 Calculate the point of intersection of the lines L_1 and L_2 with the vector equations $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively. 37

(a) $\boldsymbol{r}_1 = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \lambda\left(2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = 2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} + \mu\left(4\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}}\right)$

(b) $\boldsymbol{r}_1 = -5\hat{\boldsymbol{j}} + \lambda\left(-\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = 5\hat{\boldsymbol{i}} + \mu\left(2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}\right)$

(c) $\boldsymbol{r}_1 = -\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} + \lambda\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \mu\left(-3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\right)$

(d) $\boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} + \lambda\left(4\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \mu\left(-3\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}\right)$

15 Determine the position vector of the point of intersection of each of the following pairs of lines L_1 and L_2 with the vector equations $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively:

(a) $\boldsymbol{r}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{r}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

(b) $\boldsymbol{r}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{r}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $\boldsymbol{r}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and $\boldsymbol{r}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

(d) $\boldsymbol{r}_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\boldsymbol{r}_2 = \begin{bmatrix} -1 \\ -4 \end{bmatrix} + \mu \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

16 Determine the equation of the line L that passes through the point $A(-2, 3)$ and is perpendicular to the line with equation $\boldsymbol{r} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \lambda\left(-5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right)$.

17 Determine the Cartesian equation of the line L that passes through the point $A(5, -4)$ and is perpendicular to the vector $2\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$.

18 Line L_1 is defined by the vector equation $\boldsymbol{r}_1 = 5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \lambda\left(-2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}\right)$ and line L_2 by the Cartesian equation $\frac{x+3}{-2} = \frac{y+1}{6}$.

(a) Determine the vector equation $\boldsymbol{r}_2$ of line L_2.

(b) Determine the Cartesian equation of line L_1.

(c) Calculate the point of intersection of lines L_1 and L_2.

19 Determine the vector equation $\boldsymbol{r}_3 = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ of the line that passes through the origin and through the point of intersection of the lines $\boldsymbol{r}_1 = -3\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}} + \lambda\left(\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = 2\hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}} + \mu\left(-12\hat{\boldsymbol{i}} + 17\hat{\boldsymbol{j}}\right)$.

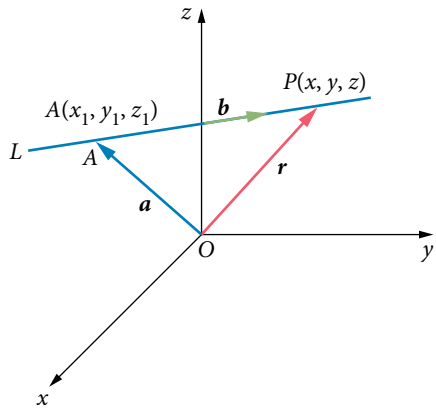
20 Consider the line L_1 with vector equation $\boldsymbol{r}_1 = \left(2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}\right) + \lambda\left(-5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$. Determine the point of intersection, if it exists, of L_1 with the line with the following vector equations:

(a) $\boldsymbol{r}_2 = \left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right) + \mu\left(-5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$ **(b)** $\boldsymbol{r}_3 = \left(4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}\right) + \rho\left(-5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$ **(c)** $\boldsymbol{r}_4 = \left(-3\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}\right) + \sigma\left(5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$

Straight lines in three-dimensional space 2.8

Straight lines in three dimensions are approached in the same way as lines in two dimensions.

For any point $P(x, y, z)$ on the line L with position vector $\boldsymbol{r}$ that passes through the point $A(x_1, y_1, z_1)$ and is parallel to a vector $\boldsymbol{b}$, in the direction of the line, the vector equation of the line can be written as $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$.



For example, the line L that passes through the point $(3, 4, -1)$ and having the direction vector $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ can be written as

$\boldsymbol{r} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + \lambda\left(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$ or as $\boldsymbol{r} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

The parametric and Cartesian forms of the equation are obtained by considering a point $P(x, y, z)$ on the line with position vector $\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

The parametric equations are $x = 3 + 2\lambda$, $y = 4 - \lambda$ and $z = -1 + 3\lambda$.

Solving each of these for λ gives $\lambda = \frac{x-3}{2}$, $\lambda = 4 - y$ and $\lambda = \frac{z+1}{3}$, therefore the Cartesian equation is $\frac{x-3}{2} = 4 - y = \frac{z+1}{3}$.

This means if you choose values of x, y and z that satisfy the Cartesian equation, then the point $P(x, y, z)$ will be on the line. To determine the coordinates of the point $P(x, y, z)$ on the line, the values of x, y and z must satisfy the Cartesian equation.

For example, if $\lambda = 2$, then the point $P(7, 2, 5)$ lies on the line because $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$.

To convert a Cartesian equation into parametric form, reverse the process and let the equation be equal to a parameter, say λ.

For example, if the Cartesian equation is $\frac{x+2}{5} = \frac{3-y}{2} = \frac{z+1}{6}$, then $\lambda = \frac{x+2}{5} = \frac{y-3}{-2} = \frac{z+1}{6}$.

So, $x = -2 + 5\lambda$, $y = 3 - 2\lambda$ and $z = -1 + 6\lambda$ and $\boldsymbol{r} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$.

In general, for a straight line L passing through the point $P(x_1, y_1, z_1)$ and in the direction of vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$:

- The Cartesian equation of the line is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
- The parametric form of the line is $\lambda = \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$, so $x = x_1 + \lambda a$, $y = y_1 + \lambda b$ and $z = z_1 + \lambda c$
- The vector equation of the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ or

$\boldsymbol{r} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}} + \lambda\left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right)$.

Occasionally these formulas may be expressed using notation for point $P(a_1, a_2, a_3)$ and direction vector $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$. When using this notation:

- The Cartesian equation of the line is $\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$, unless d_1 or d_2 or d_3 are zero.
- The parametric equation of the line is $\lambda = \frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$, so $x = a_1 + \lambda d_1$, $y = a_2 + \lambda d_2$ and $z = a_3 + \lambda d_3$.
- The vector equation of the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

38 Determining the Cartesian equation of a straight line in three-dimensional space

Determine the Cartesian equation of the straight line L passing through the point $(4, 6, 3)$ and having direction vector $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Identify the direction vector $\boldsymbol{b}$ of the line and the point the line goes through.	The direction vector of the line is $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and the line goes through the point $(x_1, y_1, z_1) = (4, 6, 3)$.
2 Determine the vector equation of the line L, $\boldsymbol{r} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}} + \lambda\left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right)$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$	$\boldsymbol{r} = \left(4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right) + \lambda\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

3 Determine the parametric equations.	$x = 4 + 3\lambda, y = 6 - 2\lambda$ and $z = 3 + \lambda$
4 Make the parameter λ the subject of each parametric equation.	$\lambda = \frac{x-4}{3}$, $\lambda = \frac{y-6}{-2}$ and $\lambda = \frac{z-3}{1}$
5 Write the Cartesian equation of the line L.	The Cartesian equation of the line L is $\frac{x-4}{3} = \frac{y-6}{-2} = \frac{z-3}{1}$ $\therefore \frac{x-4}{3} = \frac{y-6}{-2} = z - 3$

The vector equation of the line passing through two points in space

The vector equation of the line passing through two points in space can be determined.

39 The vector equation of the line passing through two points in space

Determine the matrix vector form of the equation of the line through the points $A(2, 1, -1)$ and $B(-4, 0, 3)$.

THINKING	WORKING
1 Determine the position vectors for the given points.	$\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = -4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}$
2 Determine the direction vector $\boldsymbol{b} = \overrightarrow{AB}$	$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ $= \overrightarrow{OB} - \overrightarrow{OA}$ $= \left(-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}\right) - \left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$ $= -6\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ or $\overrightarrow{AB} = \begin{bmatrix} -6 \\ -1 \\ 4 \end{bmatrix}$
3 Let the position vector of any point P on the line be $\boldsymbol{r}$. The vector equation of the line is $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$.	The vector equation of the line is $\boldsymbol{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ and so $\boldsymbol{r} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -6 \\ -1 \\ 4 \end{bmatrix}$.

Since the denominators of the Cartesian equation of a straight line provide the coefficients of the directional vector of the line, the angle between two lines specified in Cartesian form can be determined using the scalar product of these directional vectors. The angle between the two lines is the same as the angle between their direction vectors.

40 Determining the angle between two straight lines in space

Calculate the acute angle, correct to the nearest degree, between the directions of the straight lines L_1 and L_2 with Cartesian equations $\frac{x-1}{1}=\frac{y+4}{-1}=\frac{z}{4}$ and $\frac{x-2}{3}=\frac{y+2}{1}=\frac{z-3}{2}$ respectively.

THINKING	WORKING
1 Determine the direction vectors for each of the lines by examining the denominators of the Cartesian equations.	For L_1, the direction vector is $\boldsymbol{b}_1 = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$. For L_2, the direction vector is $\boldsymbol{b}_2 = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$.
2 Calculate the magnitude of each of the direction vectors.	$\lvert \boldsymbol{b}_1 \rvert = \sqrt{1^2 + (-1)^2 + 4^2}$ $= \sqrt{18}$ $= 3\sqrt{2}$ $\lvert \boldsymbol{b}_2 \rvert = \sqrt{3^2 + 1^2 + 2^2}$ $= \sqrt{14}$
3 Calculate the scalar product of the direction vectors.	$\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = (1 \times 3) + (-1 \times 1) + (4 \times 2)$ $= 10$
4 Calculate the angle between the two lines using the magnitudes of direction vectors and the scalar product: $\cos(\theta) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert \boldsymbol{a} \rvert \lvert \boldsymbol{b} \rvert}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the direction vectors.	$\theta = \cos^{-1}\left(\frac{\boldsymbol{b}_1 \cdot \boldsymbol{b}_2}{\lvert \boldsymbol{b}_1 \rvert \times \lvert \boldsymbol{b}_2 \rvert}\right)$ $= \cos^{-1}\left(\frac{10}{3\sqrt{2} \times \sqrt{14}}\right)$ $= \cos^{-1}\left(\frac{10}{3\sqrt{28}}\right)$ $= 50.95°$ (2 d.p.)
5 Write the acute angle between the two lines.	The angle between the lines is 51°.

Parallel and perpendicular lines in three dimensions

Two straight lines will be parallel if their direction vectors are parallel. Similarly, two straight lines will be perpendicular if their direction vectors are perpendicular.

For two straight lines L_1 and L_2 with vector equations $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_2$ respectively and with direction vectors $\boldsymbol{b}_1 = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b}_2 = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$ respectively:

- If the two lines are parallel, then $\boldsymbol{b}_1 = k\boldsymbol{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$.
  Elimination of the constant k results in $\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}$.
- If the two lines are perpendicular, then $\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = 0$, therefore $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

41 Parallel and perpendicular lines in three dimensions

Line L_1 passes through the points (1, -2, 4) and (5, 3, -2) while line L_2 passes through the points (3, 8, -2) and $(a, -2, 10)$, where $a \in \mathbb{R}$. Calculate the value(s) of a if:

(a) L_1 is parallel to L_2.

THINKING	WORKING
1 Determine the direction vectors for each of the lines.	Line L_1 passes through the points (1, -2, 4) and (5, 3, -2) and so the direction vector is $\boldsymbol{b}_1 = \begin{bmatrix} 5-1 \\ 3-(-2) \\ (-2)-4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$. Line L_2 passes through the points (3, 8, -2) and $(a, -2, 10)$ and so the direction vector is $\boldsymbol{b}_2 = \begin{bmatrix} a-3 \\ (-2)-8 \\ 10-(-2) \end{bmatrix} = \begin{bmatrix} a-3 \\ -10 \\ 12 \end{bmatrix}$
2 If the two lines are parallel, then $\boldsymbol{b}_1 = k\boldsymbol{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$.	If $L_1 \parallel L_2$, then $\boldsymbol{b}_1 = k\boldsymbol{b}_2$, $k \neq 0$. i.e., $\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} = k\begin{bmatrix} a-3 \\ -10 \\ 12 \end{bmatrix}$ and so $\frac{4}{a-3} = \frac{5}{-10} = \frac{-6}{12}$.
3 Solve for the unknown.	$\frac{4}{a-3} = -\frac{1}{2}$ $a-3 = -8$ $a = -5$

(b) L_1 is perpendicular to L_2.

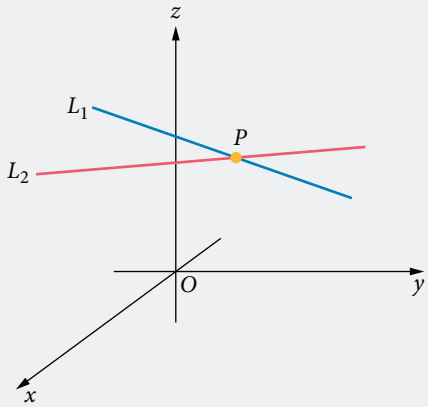
1 If the two lines are perpendicular, then $\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = 0$ and so $x_1x_2 + y_1y_2 + z_1z_2 = 0$.	If $L_1 \perp L_2$, then $\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = 0$ $\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} a-3 \\ -10 \\ 12 \end{bmatrix} = 0$ $4 \times (a-3) + 5 \times (-10) + (-6) \times 12 = 0$ $4a - 134 = 0$
2 Solve for the unknown.	$4a = 134$ $a = \frac{67}{2}$

The relationship between two straight lines in space

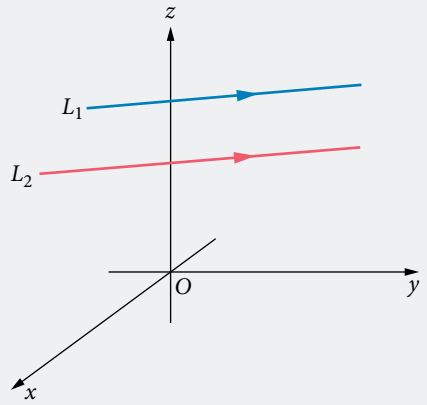
Technology worked example
Parallel and perpendicular lines in three dimensions

There are four possible relationships between two straight lines in space.

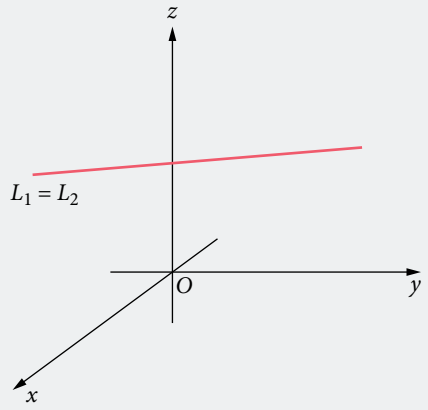
1 Two lines L_1 and L_2 may intersect at a point P



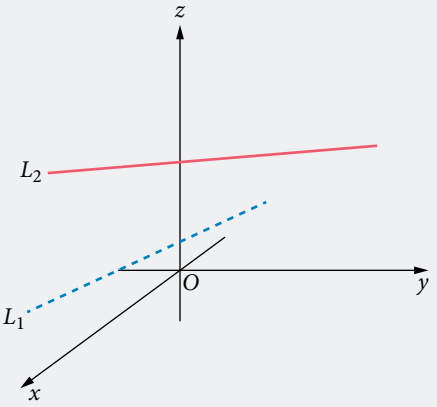
2 Two lines L_1 and L_2 may be parallel and never intersect



3 Two lines L_1 and L_2 may be parallel and coincident (i.e. they are the same line)



4 Two lines L_1 and L_2 may be neither parallel nor intersect



The first three cases are consistent with lines in a plane; however, the fourth situation is new for lines in three dimensions. Vectors that lie in the same plane are said to be *coplanar*.

Two lines that intersect at one or more points are coplanar (cases 1 and 3).

Two parallel lines are coplanar (case 2).

Two lines that are not parallel and do not intersect are said to be skew. Skew lines do not lie in the same plane, so they are not coplanar (case 4).

42 The relationship between two straight lines in space

Line L_1 passes through the points $A(3, -1, -1)$ and $B(5, 3, -2)$ while the line L_2 passes through the points $C(1, 2, -3)$ and $D(6, -2, 1)$. Demonstrate that the lines L_1 and L_2 are skew lines.

THINKING	WORKING
1 Determine the direction vectors of each of the lines.	For L_1, the direction vector is $\boldsymbol{b}_1 = (5-3)\hat{\boldsymbol{i}} + (3-(-1))\hat{\boldsymbol{j}} + ((-2)-(-1))\hat{\boldsymbol{k}}$ $= 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ For L_2, the direction vector is $\boldsymbol{b}_1 = (6-1)\hat{\boldsymbol{i}} + ((-2)-2)\hat{\boldsymbol{j}} + (1-(-3))\hat{\boldsymbol{k}}$ $= 5\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$
2 Determine the vector equations of each of the lines.	L_1 passes through $A(3, -1, -1)$, so $\boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + \lambda(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$. L_2 passes through $C(1, 2, -3)$, so $\boldsymbol{r}_2 = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + \mu(5\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}})$.
3 Determine the parametric equations for each of the lines.	For L_1: $x = 3 + 2\lambda$, $y = -1 + 4\lambda$ and $z = -1 - \lambda$ [1]. For L_2: $x = 1 + 5\mu$, $y = 2 - 4\mu$ and $z = -3 + 4\mu$ [2].
4 If lines L_1 and L_2 are parallel, then $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.	$\frac{2}{5} \neq \frac{4}{-4} \neq \frac{-1}{4}$ Thus, lines L_1 and L_2 are not parallel.
5 If lines L_1 and L_2 intersect, then the point (x_1, y_1, z_1) lies on both L_1 and L_2 for a value of λ and μ.	$3 + 2\lambda = 1 + 5\mu$ [3] $-1 + 4\lambda = 2 - 4\mu$ [4] $-1 - \lambda = -3 + 4\mu$ [5]
6 Use given information to determine the point (x_1, y_1, z_1), if it exists, by solving for λ and μ.	[4] + [5]: $-2 + 3\lambda = -1$ $\lambda = \frac{1}{3}$ Substituting $\lambda = \frac{1}{3}$ into [3]: $3 + 2 \times \frac{1}{3} = 1 + 5\mu$ $\mu = \frac{8}{15}$
7 Substitute the calculated values for λ and μ into another equation to check the validity of these values.	Substitute $\lambda = \frac{1}{3}$ into [4]: $-1 + 4 \times \frac{1}{3} = 2 - 4\mu$ $\mu = \frac{5}{12}$
8 Interpret the meaning of μ in terms of the two lines.	Since the equations result in different values of λ and μ, lines L_1 and L_2 do not intersect. Since they are not parallel, lines L_1 and L_2 are skew lines and not coplanar.

EXERCISE 2.8 Straight lines in three-dimensional space

1 Determine the vector equation of the line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ passing through the given point and parallel to the given vector.

(a) $A(1, 2, 3)$ that is also parallel to the vector $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$.

(b) $B(-1, -2, 3)$ that is also parallel to the vector $3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$.

2 Determine the parametric equations for each of the following lines:

(a) $\dfrac{x-5}{2} = \dfrac{y+1}{3} = \dfrac{z-3}{-2}$

(b) $\dfrac{x+4}{-5} = y+2 = \dfrac{z-4}{2}$

(c) $\dfrac{x-5}{-1} = \dfrac{5-y}{1} = \dfrac{2z-4}{3}$

(d) $\dfrac{4-y}{-3} = \dfrac{3z-6}{2}$

3 Determine the Cartesian equations for the following lines expressed as vector equations:

(a) $\boldsymbol{r}_1 = \left(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right) + \lambda\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$

(b) $\boldsymbol{r}_2 = \left(-3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}\right) + \mu\left(13\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}\right)$

Worked Example

(c) $\boldsymbol{r}_3 = \begin{bmatrix} -4 \\ -3 \\ -2 \end{bmatrix} + \rho\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

(d) $\boldsymbol{r}_4 = \begin{bmatrix} 0 \\ 2 \\ -7 \end{bmatrix} + \sigma\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

38 4 Determine the Cartesian equation of the straight line L passing through the point $(4, 3, 1)$ and having direction vector $2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

39 5 Determine the matrix vector equation of the line through the following pairs of points:

(a) $A(2, -3, -4)$ and $B(-1, 3, -2)$

(b) $C(8, 0, 2)$ and $D(4, -2, 2)$

(c) $E(0, -1, 12)$ and $F(7, 0, 9)$

(d) $G(-5, 0, -4)$ and $H(3, 0, 0)$

6 Determine the Cartesian form of the lines with the following vector equations:

(a) $\boldsymbol{r}_1 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

(b) $\boldsymbol{r}_2 = \begin{bmatrix} 3 \\ -5 \\ -4 \end{bmatrix} + \lambda\begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$

(c) $\boldsymbol{r}_3 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$

(d) $\boldsymbol{r}_4 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + \lambda\begin{bmatrix} 6 \\ 9 \\ 0 \end{bmatrix}$

7 Consider the two points $A(-3, 4, -7)$ and $B(-5, 0, 4)$.

(a) Determine the vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ of line L, passing through points A and B, in matrix vector form.

(b) Determine the parametric equations of line L.

(c) Determine the Cartesian equation of the line L passing through the points.

8 Demonstrate that the lines $\dfrac{x-3}{4} = \dfrac{y-6}{3} = z+2$ and $\dfrac{x+4}{8} = \dfrac{y-4}{6} = \dfrac{3-z}{2}$ are parallel.

9 The vector equation of the line passing through the point $A(5, -3, -1)$ and parallel to the vector $2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ is:

A $\boldsymbol{r} = \left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}\right) + \lambda\left(5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$

B $\boldsymbol{r} = \left(5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right) + \lambda\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}\right)$

C $\dfrac{x-2}{5} = \dfrac{y-3}{-3} = 4-z$

D $0 = 2(x-5) - 3(y+3) + 4(z+1)$

10 Determine the vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ of the line passing through the following pairs of points:

(a) $A(1, 3, 4)$ and $B(3, -2, 1)$ **(b)** $C(-3, -4, 2)$ and $D(1, 3, -4)$

(c) $E(-4, 0, -5)$ and $F(0, -4, -5)$ **(d)** $E(0, -7, 0)$ and $F(6, -7, -2)$

11 Determine the Cartesian equation of the line passing through the following pairs of points:

(a) $A(3, -2, 4)$ and $B(2, 1, -2)$ **(b)** $C(1, 2, 3)$ and $D(-3, -2, 3)$

Worked Example 40

12 Calculate the acute angle θ, correct to the nearest degree, between the directions of the straight lines L_1 and L_2 with Cartesian equations $\frac{x+3}{4} = \frac{y-3}{2} = \frac{z+2}{-3}$ and $\frac{x-1}{4} = \frac{y+4}{5} = \frac{z-1}{1}$ respectively.

13 Calculate the angle θ, correct to the nearest degree, between the directions of the following lines specified in Cartesian form:

(a) $\frac{x-2}{3} = \frac{y-3}{-2} = \frac{z-4}{5}$ and $x + 2 = \frac{y+4}{2} = \frac{z-2}{-4}$ **(b)** $\frac{x+5}{2} = \frac{2-y}{7} = z$ and $\frac{x-3}{5} = \frac{y-1}{-1} = z + 3$

14 Calculate the angle θ, correct to the nearest degree, between the directions of the following lines specified in vector form:

(a) $\boldsymbol{r}_1 = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{r}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$

(b) $\boldsymbol{r}_1 = \left(-\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right) + \lambda\left(3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = \left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right) + \mu\left(\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\right)$

15 Consider line L specified by the Cartesian equation $\frac{x+3}{-2} = \frac{2y-5}{5} = 4 - z$. Which of the following points lie on this line?

A $(-3, 2.5, 4)$ **B** $(-2, 5, 1)$ **C** $(3, -2.5, 4)$ **D** $(2, -5, 1)$

16 Line L_1 passes through the points $(2, 3, 1)$ and $(-2, 4, -2)$ while line L_2 passes through the points $(-4, 7, -1)$ and $(4, 5, a)$, where $a \in \mathbb{R}$.

Determine the value(s) of a if:

(a) L_1 is parallel to L_2. **(b)** L_1 is perpendicular to L_2.

17 Line L_1 passes through the points $A(2, -3, 4)$ and $B(5, 1, -3)$ while the line L_2 passes through the points $C(4, 1, -2)$ and $D(6, -2, 1)$. Demonstrate that the lines L_1 and L_2 are skew lines.

18 **(a)** Determine the coordinates of the point where the line $\boldsymbol{r}_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ intersects the xy-plane.

(b) Determine the coordinates of the point where the line $\boldsymbol{r}_2 = \left(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right) + \lambda\left(\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$ intersects the zy-plane.

(c) Determine the coordinates of the point where the line $\boldsymbol{r}_3 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$ intersects the xz-plane.

19 **(a)** The lines $\frac{x-2}{5} = \frac{y+2}{-3} = z + 4$ and $\frac{x+3}{2} = \frac{y-5}{7} = \frac{z+2}{3p+1}, p \neq -\frac{1}{3}$ meet at right angles. Determine the value of p.

(b) The lines $\frac{x+3}{2} = \frac{y-3}{q+2} = \frac{z-5}{4}$, $q \neq 2$ and $\frac{x-2}{5} = \frac{y-3}{3q+2} = \frac{z-5}{-4}$, $q \neq -\frac{2}{3}$ are perpendicular. Determine the value(s) of q.

2.9 Systems of linear equations

Review of inverse matrices

Previously you have learnt how to calculate the multiplicative inverse of a matrix, which is useful when solving matrix equations.

The inverse of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\mathbf{A}^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
The value $(ad-bc)$ is known as the *determinant* of matrix $\mathbf{A}$ and is written determinant $\mathbf{A}$ (often shortened to det $\mathbf{A}$ or $|\mathbf{A}|$). If determinant $\mathbf{A} \neq 0$, then matrix $\mathbf{A}^{-1}$ exists.

43 Calculating the inverse of a matrix

Let $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ -1 & 1 \end{bmatrix}$. Calculate $\mathbf{A}^{-1}$, if it exists.

THINKING	WORKING
1 Define the matrix.	$\mathbf{A} = \begin{bmatrix} 3 & -4 \\ -1 & 1 \end{bmatrix}$
2 Identify the values of a, b, c and d and calculate the determinant $(ad-bc)$	$a = 3$, $b = -4$, $c = -1$ and $d = 1$ det $\mathbf{A} = ad-bc$ $= (3\times1)-(-4\times-1)$ $= 3-(4) = -1$ So determinant $\mathbf{A} = -1$.
3 If determinant $\mathbf{A} \neq 0$, then the inverse exists.	$\mathbf{A}^{-1}$ exists.
4 Swap the elements on the main diagonal, change the sign of the elements on the secondary diagonal and divide by the determinant. $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$	$\mathbf{A}^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$ $\mathbf{A}^{-1} = \begin{bmatrix} -1 & -4 \\ -1 & -3 \end{bmatrix}$

Solving systems of linear equations using the inverse matrix

A system of linear equations can be set up and solved using matrix algebra. Consider the following pair of simultaneous linear equations with two unknowns:

$ax + by = u$ $\qquad$ $cx + dy = v$

Matrices can be defined as follows:

$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad$ $\mathbf{B} = \begin{bmatrix} u \\ v \end{bmatrix}$ $\qquad$ $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$

The matrix equation can then be created and solved by pre-multiplying both sides of the matrix equation by the inverse of the coefficient matrix $\mathbf{A}^{-1}$:

$$\mathbf{AX} = \mathbf{B}$$
$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

44 Solving simultaneous linear equations using inverse matrices

Solve the following pairs of simultaneous equations using inverse matrices.

$2x + 3y = -5$

$5x - 2y = 16$

THINKING	WORKING
1 Write the system of linear equations in the form $\mathbf{AX} = \mathbf{B}$.	$\mathbf{AX} = \mathbf{B}$ $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 16 \end{bmatrix}$
2 Determine whether $\mathbf{A}^{-1}$ exists.	$\det \mathbf{A} = (2 \times -2) - (3 \times 5)$ $= -4 - 15$ $= -19$ Therefore $\mathbf{A}^{-1}$ exists.
3 Pre-multiply both sides of $\mathbf{AX} = \mathbf{B}$ by $\mathbf{A}^{-1}$.	$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}^{-1}\begin{bmatrix} -5 \\ 16 \end{bmatrix}$
4 Determine $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.	$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} -5 \\ 16 \end{bmatrix}$ $= -\frac{1}{19}\begin{bmatrix} (-2 \times -5) + (-3 \times 16) \\ (-5 \times -5) + (2 \times 16) \end{bmatrix}$ $= -\frac{1}{19}\begin{bmatrix} 10 - 48 \\ 25 + 32 \end{bmatrix}$ $= -\frac{1}{19}\begin{bmatrix} -38 \\ 57 \end{bmatrix}$ $= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$
5 Interpret the result.	Therefore $x = 2$ and $y = -3$.

WARNING

When solving matrix equations, remember that matrix multiplication is not commutative.

To solve the equation $\mathbf{AX} = \mathbf{B}$ you need to *pre-multiply* by $\mathbf{A}^{-1}$ to get $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

To solve the equation $\mathbf{XA} = \mathbf{B}$ you need to *post-multiply* by $\mathbf{A}^{-1}$ to get $\mathbf{X} = \mathbf{BA}^{-1}$.

Solving simultaneous linear equations using Gaussian elimination

It is relatively straightforward to solve a system of two linear equations using inverse matrices. For systems of three or more linear equations, technology can be used to calculate the inverse coefficient matrix.

However, there is an additional manual technique for solving systems of equations using matrices. *Gaussian elimination* (named after German mathematician Carl Friedrich Gauss) involves performing elementary row operations to modify a matrix in a certain way. Gaussian elimination is also commonly known as *row reduction*.

Returning to the system of equations:

$ax+by=u$
$cx+dy=v$ where $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\mathbf{B}=\begin{bmatrix} u \\ v \end{bmatrix}$ $\mathbf{X}=\begin{bmatrix} x \\ y \end{bmatrix}$

A is known as the coefficient matrix and **B** is the constant matrix.

To solve a system of equations using Gaussian elimination, combine matrices **A** and **B** into a single *augmented matrix* $\left[\mathbf{A} \mid \mathbf{B}\right]$:

$$\left[\mathbf{A} \mid \mathbf{B}\right]=\left[\begin{array}{cc|c} a & b & u \\ c & d & v \end{array}\right]$$

The objective of this process is to form the identity matrix on the left side of the augmented matrix. When this is done, the solution will be on the right side of the matrix:

$$\left[\mathbf{A} \mid \mathbf{B}\right] \rightarrow \left[\mathbf{I} \mid \mathbf{X}\right]$$

45 Solving simultaneous linear equations using Gaussian elimination

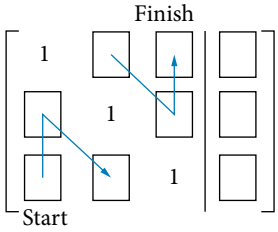
Solve the following pair of simultaneous equations using Gaussian elimination:

$x+2y=4$
$3x-y=5$

THINKING	WORKING
1 Define the coefficient and constant matrices.	$\mathbf{A}=\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ $\mathbf{B}=\begin{bmatrix} 4 \\ 5 \end{bmatrix}$
2 Convert to an augmented matrix.	$\left[\mathbf{A} \mid \mathbf{B}\right]=\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -1 & 5 \end{array}\right]$
3 Start at the bottom left corner. Eliminate the a_{21} element to make it zero. Communicate your steps.	In this case, work on row 2 and subtract 3 lots of row 1. $R_2-3R_1:\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -7 & -7 \end{array}\right]$
4 Eliminate the a_{12} element to make it zero. Communicate your steps.	Multiply row 1 by 7 and add 2 lots of row 2. $7R_1+2R_2:\left[\begin{array}{cc|c} 7 & 0 & 14 \\ 0 & -7 & -7 \end{array}\right]$

5 Once the necessary matrix elements are 'eliminated', multiply each row by an appropriate constant so that there is a 1 on the leading diagonal.	$R_1 \times \frac{1}{7}, R_2 \times -\frac{1}{7} : \left[\begin{array}{cc\|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$
6 Interpret your solution. The augmented matrix is now in the form $\left[\mathbf{I} \mid \mathbf{X}\right]$, so the solution $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be read from the right of the matrix.	$\left[\mathbf{I} \mid \mathbf{X}\right] = \left[\begin{array}{cc\|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$ so $x = 2$ and $y = 1$.
7 Verify your solutions by substituting values into original equation.	$(2)+2(1)=4$ $4=4$ $3(2)-(1)=5$ $5=5$ The solutions are verified.

Gaussian elimination can also be used to solve a system of three (or more) equations. Always begin at the bottom left, and try to eliminate the elements in column 1 before moving on. In general, you should work from the bottom left to the top right of the matrix as shown:



46 Solving a system of three linear equations using Gaussian elimination

Solve the following set of simultaneous equations using Gaussian elimination.

$2x + y - z = 2$

$x - 2y + 3z = 7$

$3x + 5y - z = 0$

THINKING	WORKING
1 Define the coefficient and constant matrices.	$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ 3 & 5 & -1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$
2 Convert to an augmented matrix.	$\left[\mathbf{A} \mid \mathbf{B}\right] = \left[\begin{array}{ccc\|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 7 \\ 3 & 5 & -1 & 0 \end{array}\right]$

3 Start at the bottom left corner. Eliminate the a_{31} element to make it zero. Communicate your steps.

$$R_3 - 3R_2 : \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 7 \\ 0 & 11 & -10 & -21 \end{array}\right]$$

4 Move up to element a_{21} and eliminate to make it zero. Communicate your steps.

$$2R_2 - R_1 : \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 7 & 12 \\ 0 & 11 & -10 & -21 \end{array}\right]$$

5 Eliminate element a_{32} to complete the elimination under the leading diagonal. At this stage it would also be wise to simplify the third row of the equation by multiplying by a constant.

$$5R_3 + 11R_2 : \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 7 & 12 \\ 0 & 0 & 27 & 27 \end{array}\right]$$

$$\frac{1}{27} \times R_3 : \left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -5 & 7 & 12 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

6 Eliminate element a_{12}. This will complete the elimination of columns 1 and 2.

$$5R_1 + R_2 : \left[\begin{array}{ccc|c} 10 & 0 & 2 & 22 \\ 0 & -5 & 7 & 12 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

7 Eliminate element a_{23}.

$$R_2 - 7R_3 : \left[\begin{array}{ccc|c} 10 & 0 & 2 & 22 \\ 0 & -5 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

8 Eliminate element a_{13}. This will complete the elimination of elements.

$$R_1 - 2R_3 : \left[\begin{array}{ccc|c} 10 & 0 & 0 & 20 \\ 0 & -5 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

9 Multiply each row by a constant so that there is a 1 in each element on the leading diagonal.

$$\frac{1}{10} \times R_1, \frac{-1}{5} \times R_2 : \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

10 Interpret your solution. The augmented matrix is now in the form $\left[\begin{array}{c|c} \mathbf{I} & \mathbf{X} \end{array}\right]$, so the solution $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be read from the right of the matrix.

$$\left[\begin{array}{c|c} \mathbf{I} & \mathbf{X} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

so $x = 2$, $y = -1$ and $z = 1$.

11 Verify your solution by substituting values into original equation.

$2(2)+(-1)-(1)=2$ (correct)

$(2)-2(-1)+3(1)=7$ (correct)

$3(2)+5(-1)-(1)=0$ (correct)

Solutions are verified.

Technology worked example

Solving a system of linear equations

Explore further

Gaussian elimination

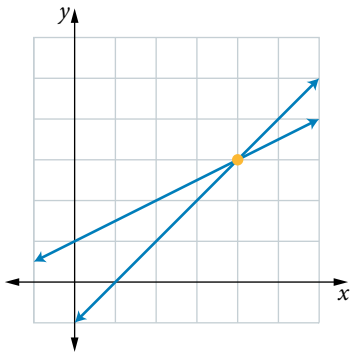
Use technology to perform row operations on a matrix to find its inverse.

Solution sets of linear equations

Recall that for a set of two linear equations on the Cartesian plane there are three possible sets of solutions:

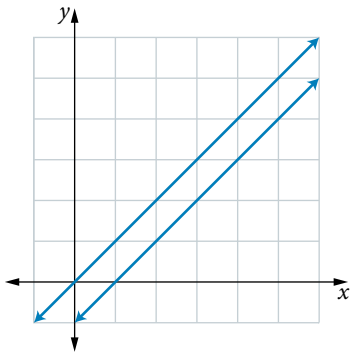
1 One unique solution

There will be one unique solution if the lines intersect at a point:



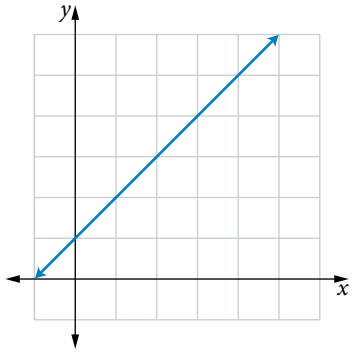
2 No solutions

There will be no solutions if the lines are parallel:



3 Infinitely many solutions

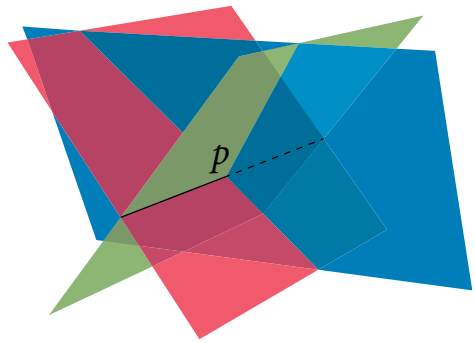
There will be infinitely many solutions if the lines are coincident (that is, if they are the same line):



For systems of three equations with three variables, each linear equation represents a plane in three dimensions. Again, there are three possible solution sets:

1 One unique solution

There will be one unique solution if the planes intersect at a single point:



2 No solutions

There will be no solutions if there is no single point common to all three planes. This could occur when at least two of the planes are parallel:

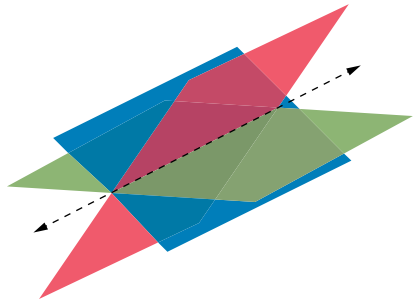


It could also occur for non-parallel planes with no common point:



3 Infinitely many solutions

There will be infinitely many solutions if the planes are coincident (that is, if they are the same plane). There will also be infinitely many solutions if the planes intersect at a single line:



47 Solution sets of linear equations

Use Gaussian elimination to demonstrate that there is no unique solution to the following set of linear equations:

$$x + 2y - z = 3$$
$$-3y - 3z = 3$$
$$y + z = 2$$

THINKING	WORKING
1 Define the coefficient and constant matrices.	$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & -3 \\ 0 & 1 & 1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$
2 Convert to an augmented matrix.	$\left[\begin{array}{c\|c} \mathbf{A} & \mathbf{B} \end{array}\right] = \left[\begin{array}{ccc\|c} 1 & 2 & -1 & 3 \\ 0 & -3 & -3 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$
3 The first column is already reduced. Start at element a_{32}. (Note that you are forced to work with row 2 here, because working with row 1 would introduce a non-zero element for a_{31}.)	$3R_3 + R_2 : \left[\begin{array}{ccc\|c} 1 & 2 & -1 & 3 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 9 \end{array}\right]$
4 Note that the third row has been reduced to zeros on the left and a non-zero on the right. This means that the system of equations is inconsistent. Interpret your result.	It is not possible to use Gaussian elimination to reduce the augmented matrix $\left[\begin{array}{c\|c} \mathbf{A} & \mathbf{B} \end{array}\right]$ to the form $\left[\begin{array}{c\|c} \mathbf{I} & \mathbf{X} \end{array}\right]$. Returning to algebraic form, the third row is $0x + 0y + 0z = 9$, hence $0 = 9$. Therefore, this system of equations is inconsistent and has no unique solution.

EXERCISE 2.9 Systems of linear equations

Worked Example 43

1 Calculate the inverse of the following matrices, if it exists.

(a) $\mathbf{A} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} -3 & 7 \\ 1 & 2 \end{bmatrix}$

2 Determine the value of p such that the following matrices have no inverse:

(a) $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ -3 & p \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} 1 & p^2 \\ 4 & p \end{bmatrix}$

3 Represent the following sets of linear equations as a matrix equation:

(a) $x - 2y = 5$ and $x - y = 1$ (b) $2x + 5y = 10$ and $x - 6y = 2$ (c) $7x - y = 4$ and $-6x - 3y = 0$

4 Represent the following sets of linear equations as a matrix equation:

(a) $3x-y+z=1$
$x-y-z=3$
$5x+3y+2z=-1$

(b) $x+z=5$
$-2y-z=-2$
$x+4y=-3$

5 When solving a system of linear equations using Gaussian elimination, a student has arrived at the intermediate step:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 1 & -1 & 1 & 9 \\ 0 & -1 & 2 & 4 \end{array}\right]$$

Which of the following row operations should be performed next?

A $R_2 - R_1$ **B** $R_3 - R_2$ **C** $R_1 - 2R_2$ **D** $R_2 - 1$

6 It is known that there is no unique solution to a set of three linear equations with three unknowns. Which of the following statements is valid?

A The three planes represented by the linear equations will meet at a single point.

B All three planes represented by the linear equations must be parallel.

C At least two planes represented by the linear equations must be parallel.

D There is no single point common to all three planes represented by the linear equations.

Worked Example

7 Solve the following pairs of simultaneous equations using inverse matrices.

(a) $x+y=4$
$2x-y=11$

(b) $4x-y=5$
$x-2y=-4$

(c) $-2x+y=-4$
$x+3y=-5$

8 Solve the following set of linear equations by using Gaussian elimination:

$2x+y=6$
$3x-y=-1$

9 Solve the following set of linear equations by using Gaussian elimination:

$x-y=-4$
$-2x-5y=1$

10 Solve the following set of linear equations by using Gaussian elimination:

$2x+3y=-2$
$x-y=4$

11 Solve the following set of linear equations by using Gaussian elimination:

$x+y+z=3$
$2x-y-z=0$
$x+2y-z=-1$

12 Solve the following set of linear equations by using Gaussian elimination:

$$x+2y-z=4$$
$$3x-y+z=5$$
$$2x+3y+2z=7$$

13 Solve the following set of linear equations by using Gaussian elimination:

$$3x-y+z=-1$$
$$2x+3y+z=4$$
$$5x+4y+2z=5$$

Worked Example 47

14 Use Gaussian elimination to demonstrate that there is no unique solution to the system of equations:

$$2x-y+3z=4$$
$$x+y-z=5$$
$$-6x+3y-9z=8$$

15 Consider the system of linear equations:

$$x-y+2z=-3$$
$$4x+4y-2z=1$$
$$-2x+2y-4z=6$$

(a) Use Gaussian elimination to demonstrate that there is an infinite number of solutions to this system of equations.

(b) Determine a solution for this system of equations for when $z = 1$.

2.10 Dominance and Leslie matrices

There are many real-life applications of matrices. In this section you will explore two of these applications:

- *Dominance matrices* are used for calculating ranking of sporting teams or individuals.
- *Leslie matrices* are used in population modelling.

Dominance matrices

Sporting competitions have different rules about how to determine a winning team or player in a round-robin competition. Sometimes it is obvious which are the dominant contestants. However, often the competition is very close and it may be necessary to determine a winner or a finalist from several competitors. Dominance matrices represent one method for evaluating the performance of teams and individuals in a sporting competition.

- If Team A play against Team B in a competition and Team A win, then Team A has *first-order dominance* over Team B.
- If Team A wins against Team B, and Team B wins against Team C, then Team A has *second-order dominance* over Team C.

This is significant, because in sport you often infer a transitive relationship between teams. If Team A is strong enough to defeat Team B, who have already defeated Team C, then you might reasonably expect Team A to defeat Team C. Of course, it doesn't always turn out that way.

Consider a hypothetical competition between four teams, A, B, C and D. After each team has played every other team once, the results are as follows:

- Team A defeats Teams B and C
- Team B defeats Teams C and D
- Team C defeats Team D
- Team D defeats Team A

These results can be shown on a win–loss table:

Team	Wins	Losses
A	2	1
B	2	1
C	1	2
D	1	2

Teams A and B have the strongest record, but which is the dominant team?

Constructing a dominance matrix

To construct a dominance matrix from the information given, create a 4×4 matrix where each team is allocated a row and a column. The team named in each row has the number 1 recorded in the column

relating to a team they defeated, and a 0 recorded in the column relating to a team that defeated them. For example, if Team *A* defeated Team *D*, place a 1 in the matrix element corresponding to Team *A*'s row and Team *D*'s column.

The dominance matrix for the example competition is shown:

$$\mathbf{M} = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 0 \\ B & 0 & 0 & 1 & 1 \\ C & 0 & 0 & 0 & 1 \\ D & 1 & 0 & 0 & 0 \end{array}$$

From this matrix, a *power vector* can be calculated by adding together all of the elements in each row:

$$\mathbf{P}_1 = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

In this case, Teams *A* and *B* have each won two games while Teams *C* and *D* have each won a single game. Now consider what happens when the dominance matrix is multiplied by itself (technology can be used to assist with this):

$$\mathbf{M}^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Consider carefully what this squared matrix represents. For example, note the element 2 in the first row and fourth column. This is the element relating to Team *A*'s record against Team *D*. From the initial list of results, you will note that *A* defeated *B*, who defeated *D*. Also, *A* defeated *C*, who defeated *D*. In other words, Team *A* has established *second-order dominance* over Team *D* on two separate occasions. This is what the element 2 in row 1, column 4 represents.

A dominance matrix **M** is a square matrix describing the results of the teams/players in each row against the teams/players in each column. The matrix $\mathbf{M}^2$ indicates second-order dominance. The matrix $\mathbf{M}^3$ indicates third-order dominance, etc.

It is now possible to calculate $\mathbf{P}_2$, the second-order power vector, by adding together each row in $\mathbf{M}^2$:

$$\mathbf{P}_2 = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

To determine the overall ranking for the teams, you can add $\mathbf{P}_1$ and $\mathbf{P}_2$ together. It may be appropriate to give greater weighting to first-order dominance, in which case the power vectors can be scaled by multiplying by a scalar. For example:

$$\mathbf{P} = \mathbf{P}_1 + 0.5\mathbf{P}_2$$

$$\mathbf{P} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 3.5 \\ 3 \\ 1.5 \\ 2 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

Therefore, Team *A* has the highest dominance ranking, followed by Team *B*, Team *D* and Team *C*.

- The first-order power vector $\mathbf{P}_1$ can be calculated by adding up the elements in each row of $\mathbf{M}$.
- The second-order power vector $\mathbf{P}_2$ can be calculated by adding the rows of $\mathbf{M}^2$, and so on.
- The overall power vector $\mathbf{P}$ can be calculated by adding each of $\mathbf{P}_1$, $\mathbf{P}_2$, etc. after applying a scalar multiple to give a weighting to each power vector if appropriate.
- The team or player with the highest value in $\mathbf{P}$ is said to be the dominant team or player.

48 Dominance matrices

Five teams compete in a local volleyball championship. After playing each other once, their results are summarised in the following table:

	Aspley	Birkdale	Capalaba	Dutton Park	Enoggera
Aspley		L	L	L	W
Birkdale			W	L	W
Capalaba				W	L
Dutton Park					W

The table is read from row to column, so Aspley lost to Birkdale, etc.

Use a matrix method to determine the most dominant team overall. Fully justify your solution and identify any assumptions you have made.

THINKING

1 Construct the dominance matrix from the given information.

WORKING

$$\mathbf{M} = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 0 & 0 & 0 & 1 \\ B & 1 & 0 & 1 & 0 & 1 \\ C & 1 & 0 & 0 & 1 & 0 \\ D & 1 & 1 & 0 & 0 & 1 \\ E & 0 & 0 & 1 & 0 & 0 \end{array}$$

2 Calculate the first-order power vector and determine whether it is sufficient to rank the teams.

$$\mathbf{P}_1 = \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Team *B* and Team *D* are tied on three wins each. Consider second-order dominance.

3 Use technology to calculate the second-order dominance matrix by squaring **M**.

$$\mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4 Calculate the second-order power vector.

$$\mathbf{P}_2 = \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \begin{bmatrix} 1 \\ 4 \\ 4 \\ 5 \\ 2 \end{bmatrix}$$

5 Calculate the overall power vector by adding $\mathbf{P}_1$ and $0.5\mathbf{P}_2$.

$$\mathbf{P}_1 + 0.5\mathbf{P}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 4 \\ 4 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 5 \\ 4 \\ 5.5 \\ 2 \end{bmatrix}$$

6 Rank the teams and identify any assumptions.

The most dominant team is Dutton Park, followed by Birkdale, Capalaba, Enoggera and Aspley.

This calculation takes into account second-order dominance, with a double-weighting towards first-order dominance.

Leslie matrices

Generally, it is not possible to predict the rate of growth of a certain population of animal based on the size of the population alone. There are several factors that will influence how quickly or how slowly a given population will grow. *Leslie matrices* (named after Oxford biomathematician Patrick H. Leslie) account for these factors.

Splitting a given population into smaller age categories allows a much more accurate population model to be developed. The two central factors in a Leslie model are the *survival rate* of a population at various ages and the *fecundity rate* (or the actual reproductive rate) at various ages.

Consider the following example. A particular species of hamster has a maximum life span of 4 years. On a certain island, the population of female hamsters is counted:

Age at the end of the year (x)	Number of female hamsters (N)
1	48
2	42
3	26
4	12

It is known that the survival rates of hamsters (expressed as a decimal) in various age categories can be summarised as:

Age at the end of the year (x)	Survival rate (S_x)
1	0.7
2	0.5
3	0.3

It is also known that the fecundity rates of hamsters (the number of female offspring per female hamster per year) can be summarised as:

Age at the end of the year (x)	Fecundity rate (F_x)
1	0.5
2	3.1
3	0.8
4	0.2

Knowing this information, it is possible to predict the population of hamsters the following year:

Age the following year (x)	Description	Number of female hamsters (N)
1	All newborn hamsters	$F_1 \times$ number of one-year-olds, plus $F_2 \times$ number of two-year-olds, plus $F_3 \times$ number of three-year-olds, plus $F_4 \times$ number of four-year-olds.
2	Surviving one-year-olds	$S_1 \times$ number of one-year-olds.
3	Surviving two-year-olds	$S_2 \times$ number of two-year-olds.
4	Surviving three-year-olds	$S_3 \times$ number of three-year-olds.

This calculation can be simplified using a Leslie matrix. The initial population of female hamsters can be expressed using the column matrix:

$$\mathbf{N}_0 = \begin{bmatrix} 48 \\ 42 \\ 26 \\ 12 \end{bmatrix}$$

The population of female hamsters after 1 year can therefore be calculated:

$$\mathbf{N}_1 = \begin{bmatrix} (48 \times 0.5) + (42 \times 3.1) + (26 \times 0.8) + (12 \times 0.2) \\ (48 \times 0.7) \\ (42 \times 0.5) \\ (26 \times 0.3) \end{bmatrix}$$

However, this is an unwieldy and time-consuming process. It is much simpler to construct a matrix equation that can be used to generate the following year's population:

$$\mathbf{N}_1 = \begin{bmatrix} 0.5 & 3.1 & 0.8 & 0.2 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \times \begin{bmatrix} 48 \\ 42 \\ 26 \\ 12 \end{bmatrix} \approx \begin{bmatrix} 177 \\ 34 \\ 21 \\ 8 \end{bmatrix}$$

(Note: as this is a model of an actual population, numbers in $\mathbf{N}_1$ have been rounded.)

As you can see, the following year's population can be calculated by multiplying a square matrix by a column matrix. The square matrix on the left is known as the Leslie matrix, and the column matrix on the right is simply the population matrix of the previous year.

The equation can be rewritten as follows:

$$\mathbf{N}_1 = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ S_1 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & S_3 & 0 \end{bmatrix} \times \mathbf{N}_0$$

The Leslie matrix $\mathbf{L}$ is constructed by assigning fecundity rates across the top row, survival rates along the sub-diagonal (beginning under F_1), and zeros in all other elements.

When calculating populations in successive years:

$\mathbf{N}_1 = \mathbf{L} \times \mathbf{N}_0$

$\mathbf{N}_2 = \mathbf{L} \times \mathbf{N}_1 = \mathbf{L} \times \mathbf{L} \times \mathbf{N}_0 = \mathbf{L}^2 \times \mathbf{N}_0$

$\mathbf{N}_3 = \mathbf{L} \times \mathbf{N}_2 = \mathbf{L} \times \mathbf{L}^2 \times \mathbf{N}_0 = \mathbf{L}^3 \times \mathbf{N}_0$

and so on.

In general, $\mathbf{N}_r = \mathbf{L} \times \mathbf{N}_{r-1}$ and $\mathbf{N}_r = \mathbf{L}^r \times \mathbf{N}_0$.

49 Leslie matrices

A particular species of mouse is known to have a maximum lifespan of 3 years. The initial female mouse population is given by the matrix $\mathbf{N}_0 = \begin{bmatrix} 16 \\ 25 \\ 14 \end{bmatrix}$ and the Leslie matrix associated with the population change of this mouse species is $\mathbf{L} = \begin{bmatrix} 0.5 & 2.1 & 0.3 \\ 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$.

(a) Determine the approximate female mouse population after 1 year.

THINKING	WORKING
1 Identify or construct the relevant matrices if they are not already given.	$\mathbf{L} = \begin{bmatrix} 0.5 & 2.1 & 0.3 \\ 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ and $\mathbf{N}_0 = \begin{bmatrix} 16 \\ 25 \\ 14 \end{bmatrix}$
2 Construct an equation for the Leslie model.	$\mathbf{N}_1 = \mathbf{L} \times \mathbf{N}_0$ $\mathbf{N}_1 = \begin{bmatrix} 0.5 & 2.1 & 0.3 \\ 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 16 \\ 25 \\ 14 \end{bmatrix}$
3 Calculate the matrix product to determine the population in the following year. Use technology to assist with this calculation where appropriate.	$\mathbf{N}_1 = \begin{bmatrix} 0.5 & 2.1 & 0.3 \\ 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 16 \\ 25 \\ 14 \end{bmatrix}$ $\mathbf{N}_1 \approx \begin{bmatrix} 65 \\ 10 \\ 10 \end{bmatrix}$
4 Add up the total female population and interpret your result.	The approximate number of female mice after 1 year is $65 + 10 + 10 = 85$ mice.

(b) Calculate the total population of mice after 5 years. Identify any assumptions you make and comment on the effects of these assumptions on your answer.

1 Set up the Leslie equation.	$\mathbf{N}_5 = \mathbf{L}^5 \times \mathbf{N}_0$ $\mathbf{N}_5 = \begin{bmatrix} 0.5 & 2.1 & 0.3 \\ 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}^5 \times \begin{bmatrix} 16 \\ 25 \\ 14 \end{bmatrix}$
2 Use technology to assist with this calculation.	$\mathbf{N}_5 \approx \begin{bmatrix} 208 \\ 78 \\ 27 \end{bmatrix}$

3 Interpret your result.	The approximate number of female mice after 5 years is $208 + 78 + 27 = 313$. Therefore, the total number of mice $= 313 \times 2 = 626$ mice.
4 Identify any assumptions and comment on their effects.	This assumes there are an equal number of male and female mice in the population. It also assumes the survival and fecundity rates remain constant over the five-year period. Given the rapid growth of younger mice over the five years, the survival rate of one-year-old mice may decrease if resources become scarce. If this occurs, the estimated population of 626 may be an overestimation.

The use of Leslie matrices is a form of population modelling. Like all population models, the Leslie method is based on certain assumptions. One such assumption is that survival and fecundity rates remain constant over time. In reality, these rates may fluctuate as a result of environmental factors, availability of resources, and the presence or absence of natural predators.

Explore further

Leslie matrices

Use technology to investigate the effects of changes in initial population, fecundity and survival rates on a population over time using Leslie matrices.

EXERCISE 2.10 Dominance and Leslie matrices

1 Consider the following first-order dominance matrix for five teams in a certain competition:

$$\mathbf{M} = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 1 & 1 & 1 & 1 \\ B & 0 & 0 & 0 & 0 & 1 \\ C & 0 & 1 & 0 & 1 & 1 \\ D & 0 & 1 & 0 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 0 \end{array}$$

(a) What was the result when Team D played Team E?

(b) How many victories did Team C have overall?

(c) Calculate the first-order power vector $\mathbf{P}_1$ for this competition.

(d) Is a second-order dominance calculation necessary to rank the teams in this competition? Justify your answer.

2 The second-order dominance matrix is calculated for a certain competition between four teams, *A*, *B*, *C* and *D*:

$$\mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

From this information, which of the following statements is true?

A Team *D* did not win a game.

B Team *B* won two games against Team *A*.

C Team *B* defeated two other teams who defeated Team *A*.

D Team *B* won three games overall.

3 Calculate the third-order dominance matrix for the following competition:

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4 Consider the following Leslie matrix equation:

$$\mathbf{N}_1 = \begin{bmatrix} 0.3 & 1.8 & 0.2 \\ 0.8 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 321 \\ 243 \\ 76 \end{bmatrix}$$

(a) Explain the meaning of the number 0.3 in the Leslie matrix.

(b) What percentage of two-year-old animals would be expected to survive to reach three years of age?

5 Which of the following is an example of a well-formed Leslie matrix?

A $\begin{bmatrix} 0.8 & 3.1 & 0.4 \\ 1.2 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$

B $\begin{bmatrix} 0 & 2.6 & 1.2 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$

C $\begin{bmatrix} 0.1 & 5.7 & 1.4 \\ 0.3 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0.2 & 1.9 & 0.3 \\ 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0 \end{bmatrix}$

Worked Example 48

6 Five players compete in a ten-pin bowling championship. After playing each other once, their results are summarised in the following table:

	Anuj	Bronte	Chara	Diego	Ezra
Anuj		Won	Lost	Won	Lost
Bronte			Won	Lost	Lost
Chara				Won	Won
Diego					Lost

The table is read from row to column, so Anuj defeated Bronte, etc.

Use a matrix method to determine the most dominant player overall. Fully justify your solution and identify any assumptions you have made.

Worked Example 49

7 A certain species of gerbil has a maximum life span of 4 years. The initial population of female gerbils in one particular environment, along with their survival and fecundity rates, is summarised in the following table:

Age at the end of the year (x)	Initial number of female gerbils (N)	Survival rate (S_x)	Fecundity rate (F_x)
1	170	0.4	0.1
2	85	0.8	0.7
3	40	0.5	2.4
4	15	0	0.5

(a) Construct a Leslie matrix to model this population of gerbils.

(b) Use the Leslie model to estimate the number of female gerbils after 1 year.

(c) Use the Leslie model to estimate the total number of gerbils after 5 years. Identify any assumptions you make and comment on the effects of your assumptions on your answer.

8 A certain species of gnat has three distinct life phases over a maximum lifespan of 3 days. On the first day of its life it is an egg, on the second day it hatches into larvae, and on the third day it is an adult. Only female adults can reproduce, and each female adult lays an average of 500 female eggs before it dies.

The survival rates of eggs and larvae are very low. 98% of eggs will never hatch and 90% of larvae will die before they reach adulthood.

Scientists release a number of these insects into a controlled environment. Use Leslie matrices to determine whether the population of gnats will increase, decline, or remain constant over the long term. Identify any assumptions you make and comment on the effects of these assumptions on your answer.

2.11 Further applications of matrices

Leontief matrices

Leontief matrices (named after Nobel Prize winning economist Wassily Leontief, 1905–1999) are used to analyse the interactions between various industries in an economic system. Central to Leontief's system is the calculation of inputs and outputs in each section of the economy. Using Leontief matrices, you can calculate the necessary production in an economic system to meet a particular consumer demand.

Although Leontief modelling can be applied in large economies involving hundreds of separate industries, it is helpful to start with a simplified example. Consider an economy that produces three different resources: electrical power P, natural resources N and manufactured goods M. These sections of the economy are all related in that the production of any one resource requires inputs from all three sections. For example, producing electrical power requires some natural resources and manufactured goods, and it even requires some electrical power too.

The Leontief method involves comparing inputs and outputs by calculating how much of each resource is consumed in the production process. To do this, you begin with a consumption matrix, $\mathbf{C}$. Consider the three-sector economy described above. Suppose you know the following information:

The production of 1 unit of P requires 0.1 units of P, 0.3 units of N and 0.2 units of M.

The production of 1 unit of N requires 0.2 units of P, 0.2 units of N and 0.3 units of M.

The production of 1 unit of M requires 0.3 units of P, 0.2 units of N and 0.1 units of M.

This information can be summarised into a consumption matrix:

$$\mathbf{C} = \begin{array}{c} \begin{matrix} P & N & M \end{matrix} \\ \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \end{array} \begin{matrix} P \\ N \\ M \end{matrix}$$

To read from this matrix, start with the column and then locate the row. For example, the 0.3 in the third column and first row means that to produce 1 unit of M requires 0.3 units of P.

Suppose you wanted to know the input of each resource, q, required to produce 12 units of P, 15 units of N and 20 units of M. This could be done by constructing a single-column production matrix, p, and multiplying it by the consumption matrix:

$$\mathbf{C} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \qquad p = \begin{bmatrix} 12 \\ 15 \\ 20 \end{bmatrix}$$

$$q = \mathbf{C}p$$

$$q = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 12 \\ 15 \\ 20 \end{bmatrix} = \begin{bmatrix} 10.2 \\ 10.6 \\ 8.9 \end{bmatrix}$$

This input matrix q is an indication of how many resources are 'used up' in the given production process. The portion remaining can be used to meet consumer demand d so that $p-q=d$:

$$p-q=d$$

$$d=\begin{bmatrix}12\\15\\20\end{bmatrix}-\begin{bmatrix}10.2\\10.6\\8.9\end{bmatrix}=\begin{bmatrix}1.8\\4.4\\11.1\end{bmatrix}$$

Suppose you know there is a certain demand for resources and you are trying to calculate the production required to meet that demand. In this case, the equations above can be combined:

$$p-\mathbf{C}p=d$$
$$(\mathbf{I}-\mathbf{C})p=d$$
$$p=(\mathbf{I}-\mathbf{C})^{-1}d$$

If the amount of resources consumed by an economy in the production process is too high, it may not be possible to meet a given demand. Generally, the lower the consumption of resources (and the smaller the numbers in the consumption matrix), the less production will be required to meet demand.

In a given economy:

- The *output* (or production) of the economy can be defined as p and the *input* of resources required to achieve a certain production can be defined as q.
- To calculate the input required to achieve a certain output, multiply by the consumption matrix: $q=\mathbf{C}p$.
- The difference between the output and the input can be used to meet consumer demand, d. That is, $p-q=d$.

Substituting and rearranging these equations, using the identity matrix gives:

$$p-\mathbf{C}p=d$$
$$(\mathbf{I}-\mathbf{C})p=d$$
$$p=(\mathbf{I}-\mathbf{C})^{-1}d$$

This allows you to calculate the production required to meet a given demand.

50 Calculating production using Leontief matrices

In a certain economy with three sectors A, B and C, the consumption matrix is defined as:

$$\mathbf{C}=\begin{array}{c}\begin{matrix}A & B & C\end{matrix}\\\begin{bmatrix}0.1 & 0.2 & 0.2\\0.1 & 0.1 & 0.2\\0.3 & 0.1 & 0.2\end{bmatrix}\end{array}\begin{matrix}A\\B\\C\end{matrix}$$

(a) Calculate the input required to produce 8 units of A, 6 units of B and 10 units of C.

THINKING	WORKING
1 Construct the production matrix from the given information.	$p = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix}$
2 Substitute the appropriate values for input and output, then perform the calculation.	$q = \mathbf{C}p$ $q = \begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix}$ $= \begin{bmatrix} 4 \\ 3.4 \\ 5 \end{bmatrix}$
3 Interpret your result.	The required input is 4 units of A, 3.4 units of B and 5 units of C.

(b) Determine the consumer demand that can be met by the production described in part **(a)**.

1 Write the relationship between input, output and consumer demand.	$p - q = d$
2 Set up the Leontief input–output equation to calculate the required production. Use technology to assist where appropriate.	$d = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix} - \begin{bmatrix} 4 \\ 3.4 \\ 5 \end{bmatrix}$ $= \begin{bmatrix} 4 \\ 2.6 \\ 5 \end{bmatrix}$
3 Interpret your result.	The demand that can be met is 4 units of A, 2.6 units of B and 5 units of C.

51 Calculating demand in an economy using Leontief matrices

In a certain economy with three sectors A, B and C, the consumption matrix is defined as:

$$\mathbf{C} = \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.2 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix}$$

Calculate the production required to meet a consumer demand of 20 units of A, 15 units of B and 18 units of C.

THINKING	WORKING
1 Write the relationship between demand, production and consumption.	$p = (\mathbf{I} - \mathbf{C})^{-1} d$
2 Substitute the appropriate values for consumption and demand, then perform the calculation. Use technology to assist with this calculation.	$p = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 20 \\ 15 \\ 18 \end{bmatrix}$ $p = \begin{bmatrix} 0.9 & -0.2 & -0.2 \\ -0.1 & 0.9 & -0.2 \\ -0.3 & -0.1 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 15 \\ 18 \end{bmatrix}$ $p = \begin{bmatrix} 37.84 \\ 29.85 \\ 40.42 \end{bmatrix}$ (2 d.p.)
3 Interpret your result.	The required production is 37.84 units of A, 29.85 units of B and 40.42 units of C.

Markov chains

A *Markov chain* (named after Russian mathematician Andrei Markov) is a mathematical model for calculating the probability associated with a sequence of events where the probability of a single event depends on a previous state. For example, there may be a higher probability that there will be sunny weather today given that it was sunny and not rainy yesterday.

Consider a hypothetical scenario where there are two major political parties in the town of Markoville: the Progressive Party and the Conservative Party. At one particular election, each of these parties achieved 50% of the vote.

The people of Markoville are fairly set in their ways when it comes to politics. A person who votes for the Progressive Party at one election has a 90% chance of voting for the Progressive Party again in the next election, whereas a Conservative Party supporter has an 85% chance of voting for the Conservative Party the next time around.

Given this information, what is likely to be the outcome of the next election, or the next election after that? What is likely to happen in terms of long-term support for either of the parties? These questions can be answered through the application of Markov chains.

To use this method, you must set up a *transition matrix*, **T**. This matrix contains the probabilities associated with the change in support from each event (in this case, an election) to the next event. In the Markoville election example there are only 2 political parties, so a 2×2 transition matrix is required. The general transition matrix can be constructed:

$$\mathbf{T} = \begin{array}{c} \\ \text{To} \\ \\ \end{array} \begin{array}{c} \text{From} \\ \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} \end{array} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

In this case, there is a probability of 0.9 that a Progressive Party voter will support the Progressives again in the next election, which means there is only a 0.1 probability that they will change their support to the Conservatives. There is a probability of 0.85 that a Conservative voter will remain loyal to that party at the next election, which means that there is a 0.15 probability that they will swap allegiances. This information can be summarised in the transition matrix:

$$\mathbf{T} = \begin{array}{c} \\ \text{To} \end{array} \begin{array}{c} \begin{array}{cc} \text{From} & \\ \text{Prog} & \text{Cons} \end{array} \\ \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Prog} \\ \text{Cons} \end{array}$$

The rule for complementary probabilities means the elements in each column of the transition matrix must add to give 1.

To calculate the likely state of affairs after the next election, multiply the transition matrix by the initial state matrix according to the rule:

$$\mathbf{S}_1 = \mathbf{T} \times \mathbf{S}_0$$

$$= \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$= \begin{array}{l} \begin{bmatrix} 0.525 \\ 0.475 \end{bmatrix} \\ \end{array} \begin{array}{l} \text{(Prog)} \\ \text{(Cons)} \end{array}$$

At the following election, this process can be repeated. To calculate the support after 2 elections, re-multiply $\mathbf{S}_1$ by the transition matrix.

$$\mathbf{S}_2 = \mathbf{T} \times \mathbf{S}_1$$

$$= \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix} \begin{bmatrix} 0.525 \\ 0.475 \end{bmatrix}$$

$$= \begin{array}{l} \begin{bmatrix} 0.54375 \\ 0.45625 \end{bmatrix} \\ \end{array} \begin{array}{l} \text{(Prog)} \\ \text{(Cons)} \end{array}$$

Under this model, it appears that the Progressive Party is gradually gaining long-term superiority over the Conservative Party in terms of electoral support.

A fundamental assumption behind Markov chains is that the system is closed, there are no outside influences, and the rules for transitional probabilities do not change over time. In reality, this is not likely to be the case.

In a Markov chain model with square transition matrix $\mathbf{T}$ and initial state matrix $\mathbf{S}_0$:

$\mathbf{S}_n = \mathbf{T} \times \mathbf{S}_{n-1}$ and $\mathbf{S}_n = \mathbf{T}^n \times \mathbf{S}_0$

The steady state

Over time, a Markov chain model will reach a *steady state*. This means that any further multiplication by the transition matrix will have no effect on each subsequent state as the system has reached equilibrium.

There are several methods for determining the steady state in a Markov chain model. The first is algebraic. As the steady state involves a scenario in which multiplying by the transition matrix $\mathbf{T}$ has no further effect, this can be represented by the following equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix}$$

Going back to the Markoville election example, the steady state can be calculated:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.9x + 0.15y \\ 0.1x + 0.85y \end{bmatrix}$$

Because the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ represents a probability event, $x + y = 1$. From the first row of the matrix equation above, $x = 0.9x + 0.15y$. By substituting $y = 1 - x$ into the equation:

$$\begin{aligned} x &= 0.9x + 0.15(1 - x) \\ 0.1x &= 0.15 - 0.15x \\ 0.25x &= 0.15 \\ x &= 0.6 \\ y &= 0.4 \end{aligned}$$

Therefore, over the long term, the Progressive Party can expect to reach a steady state of 60% electoral support, assuming there are no outside influences and that people continue to cast their vote along the lines of the stated probabilities.

52 Markov chains

In a certain town there are three major supermarkets—Cales, Wallies and Oldi. At the end of every month, a certain percentage of shoppers may choose to swap their preferred supermarket. Surveys of shoppers have uncovered the following information:

15% of Cales shoppers swap to Wallies, and 5% of Cales shoppers swap to Oldi
10% of Wallies shoppers swap to Cales, and 10% of Wallies shoppers swap to Oldi
5% of Oldi shoppers swap to Cales, and 5% of Oldi shoppers swap to Wallies

At the start of January, the market share of each supermarket is:

Cales: 45%
Wallies: 40%
Oldi: 15%

Use Markov chains to calculate the expected market share of the Oldi supermarket at the beginning of June. Identify any limitations of the model you have used.

THINKING	WORKING
1 Construct a transition matrix **T** based on the given information, and identify the initial state.	$\mathbf{T} = \begin{array}{c} \\ C \\ W \\ O \end{array} \begin{array}{c} \begin{array}{ccc} C & W & O \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.05 \\ 0.15 & 0.8 & 0.05 \\ 0.05 & 0.1 & 0.9 \end{bmatrix} \end{array}$ $\mathbf{S}_0 = \begin{bmatrix} 0.45 \\ 0.4 \\ 0.15 \end{bmatrix}$

2 From the beginning of January to the beginning of June is 5 months. Formulate the Markov equation.

$$\mathbf{S}_n = \mathbf{T}^n \times \mathbf{S}_0$$

$$\mathbf{S}_5 = \begin{bmatrix} 0.8 & 0.1 & 0.05 \\ 0.15 & 0.8 & 0.05 \\ 0.05 & 0.1 & 0.9 \end{bmatrix}^5 \begin{bmatrix} 0.45 \\ 0.4 \\ 0.15 \end{bmatrix}$$

$$= \begin{bmatrix} 0.317 \\ 0.360 \\ 0.323 \end{bmatrix} \text{ (3 d.p.)}$$

3 Interpret your result.

The market share of Oldi at the beginning of June is approximately 32.3%.

4 Discuss a limitation of your model.

This model is limited in that it does not take into account the changing motivation of customers in relation to factors such as sales and special promotions. It assumes that the transition probabilities for customers' shopping choices do not change over time.

Eigenvalues and eigenvectors

Technology worked example
Markov steady-state eigenvectors

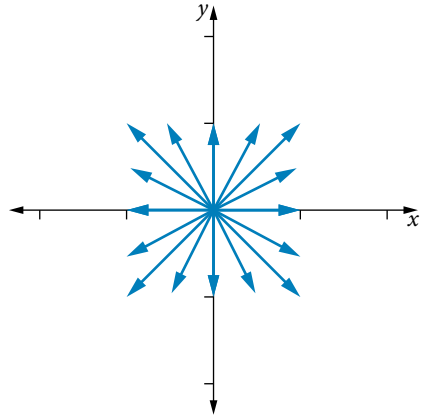
The steady state of a Markov chain model is an example of an *eigenvector*. Eigenvectors, and their associated eigenvalues, have application in diverse fields of mathematics and engineering such as image and sound processing, electrical engineering, structural stability, and even algorithms for the page rankings of popular search engines.

Eigenvectors represent vectors whose directions are unchanged after being multiplied by the matrix **A**. To understand eigenvectors, think about the effects of a matrix transformation. For example, recall that the transformation matrix $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has the effect of a vertical dilation of factor 2. If you consider the Cartesian plane before this transformation, there is an infinite number of vectors that could possibly be placed on the plane (only a small number are shown):

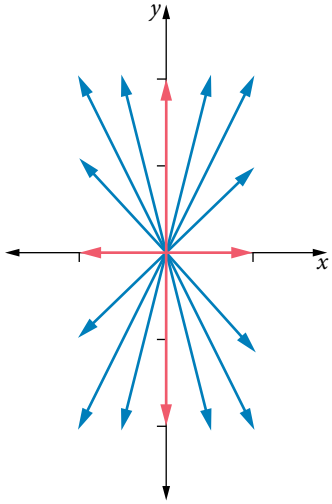
Eigenvectors and eigenvalues are related to a given matrix **A** by the equation:

$$\mathbf{AX} = \lambda\mathbf{X}$$

In this equation, **X** is a column matrix (vector) and λ is a scalar. **X** is called the *eigenvector* of matrix **A** and λ is the associated *eigenvalue* of matrix **A**.



When the transformation is applied, only very few of these vectors retain their original direction.



Once this dilation is applied, only the horizontal and vertical vectors retain their original direction. These vectors (coloured red in the diagram) are known as the eigenvectors of the transformation matrix $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

The vertical vectors have doubled in length—they correspond to an eigenvalue of 2.

The horizontal vectors remain the same length—they correspond to an eigenvalue of 1.

To calculate the eigenvectors and eigenvalues associated with a particular matrix, proceed as follows:

$\mathbf{AX} = \lambda\mathbf{X}$

$\mathbf{AX} - \lambda\mathbf{X} = 0$

Because λ is a scalar, multiply by the identity matrix $\mathbf{I}$:

$\mathbf{AX} - \lambda\mathbf{IX} = 0$

$(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = 0$

Therefore:

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

To calculate the eigenvalues associated with matrix $\mathbf{A}$, solve the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. To calculate the eigenvectors associated with matrix $\mathbf{A}$, substitute the eigenvalues back into the equation $\mathbf{AX} = \lambda\mathbf{X}$.

53 Eigenvalues and eigenvectors

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

(a) Calculate the eigenvalues associated with this matrix.

THINKING	WORKING
1 Write the formula for determining the eigenvalues of matrix $\mathbf{A}$.	$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
2 Substitute the relevant values.	$\det\left(\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$
	$\det\left(\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$
	$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{bmatrix}\right) = 0$

3 Calculate the determinant by solving the quadratic.	$(1-\lambda)(4-\lambda)-(-5\times-2)=0$ $4-5\lambda+\lambda^2-10=0$ $\lambda^2-5\lambda-6=0$ $(\lambda-6)(\lambda+1)=0$ $\lambda=6, \lambda=-1$
4 Interpret your result.	Therefore, the eigenvalues associated with $\mathbf{A}=\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ are $\lambda=6, \lambda=-1$.

(b) Calculate the eigenvectors associated with this matrix.

1 Substitute the first value for λ into the original equation. Use a general matrix $\begin{bmatrix} a \\ b \end{bmatrix}$ to represent the eigenvector.	$\mathbf{AX}=\lambda\mathbf{X}$ $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=6\begin{bmatrix} a \\ b \end{bmatrix}$
2 Expand the matrix equation for the first row. (There is no need to expand both rows.)	$a-2b=6a$ $-2b=5a$ $b=-\frac{5a}{2}$
3 Interpret your result.	Therefore, the eigenvector associated with the eigenvalue $\lambda=6$ is $\begin{bmatrix} a \\ -\frac{5a}{2} \end{bmatrix}$.
4 Substitute the second value for λ into the original equation.	$\mathbf{AX}=\lambda\mathbf{X}$ $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=-1\begin{bmatrix} a \\ b \end{bmatrix}$
5 Expand the matrix equation for the first row.	$a-2b=-a$ $-2b=-2a$ $b=a$
6 Interpret your result.	Therefore, the eigenvector associated with the eigenvalue $\lambda=-1$ is $\begin{bmatrix} a \\ a \end{bmatrix}$.

Explore further

Encryption and decryption with matrices

Explore the encryption of messages with matrices and decryption of the same message using the inverse of the matrix.

EXERCISE 2.11 Further applications of matrices

1 In a particular economy, the consumption matrix has been defined as follows:

$$\mathbf{C} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} \end{array}$$

(a) How many units of C are required to produce 1 unit of B?

(b) Determine the inputs required to produce 1 unit of A.

Worked Example 50

2 In a certain economy with three sectors A, B and C, the consumption matrix is defined as:

$$\mathbf{C} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \end{array}$$

(a) Calculate the input required to produce 10 units of A, 8 units of B and 12 units of C.

(b) Determine the consumer demand that can be met by the production described in part **(a)**.

3 Consider the following Markov transition matrix for a system with three options, A, B and C:

$$\mathbf{T} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.7 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.1 \\ 0.1 & x & 0.8 \end{bmatrix} \end{array}$$

(a) Determine the value of x in the second column of the matrix.

(b) What is the likelihood that outcome C occurs on one event, given that it occurred on the previous event?

(c) If outcome A occurs on a particular event, what is the least likely outcome at the next event?

4 In a certain economy with three sectors P, Q and R, the consumption matrix is defined as:

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$$\mathbf{C} = \begin{array}{c} \\ P \\ Q \\ R \end{array} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.2 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix} \end{array}$$

Calculate the production required to meet a consumer demand of 50 units of P, 60 units of Q and 40 units of R.

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5 In a certain country there are three major brands of soft drink: Ace, Bubbles and Crisp. Soft drink enthusiasts in this country have a preferred brand, and may change their preference every month according to the following rules:

20% of Ace drinkers swap to Bubbles, and 10% of Ace drinkers swap to Crisp
15% of Bubbles drinkers swap to Ace, and 10% of Bubbles drinkers swap to Crisp
10% of Crisp drinkers swap to Ace, and 10% of Crisp drinkers swap to Bubbles

At the start of January, the market share of each supermarket is:

Ace: 30%
Bubbles: 40%
Crisp: 30%

Use Markov chains to calculate the expected market share of each brand after 4 months. Identify any limitations of the model you have used.

6 On a certain remote island, the weather on any given day is either rainy or sunny. If it is a sunny day today then the probability that it will be sunny again tomorrow is 0.95, whereas if it is a rainy day today, there is a probability of 0.55 that it will rain again tomorrow. Using Markov chains, estimate the likely number of rainy days in a calendar year.

7 Consider the linear transformation defined by the matrix $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Which of the following is *not* an eigenvector of $\mathbf{T}$?

A $\begin{bmatrix} 12 \\ 0 \end{bmatrix}$ **B** $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ **C** $\begin{bmatrix} 0 \\ 8 \end{bmatrix}$ **D** $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

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8 Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$.

(a) Calculate the eigenvalues associated with this matrix.

(b) Calculate the eigenvectors associated with this matrix.

9 In a certain economy with three sectors *A*, *B* and *C*, the consumption matrix is defined as:

$$\mathbf{C} = \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0.2 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.3 \\ 0.2 & 0.1 & 0.2 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix}$$

(a) In this economy, there is a yearly consumer demand for 100 units each of resources *A*, *B* and *C*, but the factories have a maximum production limit of 300 units each of resources *A*, *B* and *C*. Demonstrate that this production limit will be insufficient to meet consumer demand.

(b) The factories responsible for producing resources *A* and *C* are able to make efficiencies in their production process. They each invest in machinery that allows them to cut their use of resource *B* by 20% in manufacturing their products. Determine whether this investment will now allow the economy to meet consumer demand of 100 units each of resources *A*, *B* and *C*.

Summary

2

Review of vectors

A unit vector is denoted using a 'hat' as $\hat{\boldsymbol{a}}$, where $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$ (and $|\boldsymbol{a}|$ is the magnitude of $\boldsymbol{a}$).

If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, then $|\boldsymbol{r}| = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \dfrac{y}{x}$, where θ is the angle made by the vector $\boldsymbol{r}$ with the positive x-axis.

If vector $\boldsymbol{r}$ has a magnitude of $|\boldsymbol{r}|$ and forms an angle of θ with the positive x-axis, where $0° \leq \theta \leq 360°$, then the vector can be written in polar form as $\boldsymbol{r} = (|\boldsymbol{r}|, \theta°)$.

If a vector $\boldsymbol{r}$ of magnitude $|\boldsymbol{r}|$ makes an angle of θ with the positive x-axis, then $\boldsymbol{r} = |\boldsymbol{r}|\cos(\theta)\hat{\boldsymbol{i}} + |\boldsymbol{r}|\sin(\theta)\hat{\boldsymbol{j}}$. In this case, $|\boldsymbol{r}|\cos(\theta)\hat{\boldsymbol{i}}$ is the *horizontal component* and $|\boldsymbol{r}|\sin(\theta)\hat{\boldsymbol{j}}$ is the *vertical component* of the vector $\boldsymbol{r}$.

If P is the point $P(x, y)$, then $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. The vector $\overrightarrow{OP}$ is the position vector of P relative to O.

If point A is defined by the position vector $\boldsymbol{a}$ and point B is defined by the position vector $\boldsymbol{b}$, then the position vector of B relative to A is $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \boldsymbol{b} - \boldsymbol{a}$. Likewise, the position vector of A relative to B is $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \boldsymbol{a} - \boldsymbol{b}$.

Vectors in three dimensions

In three-dimensional space, a third unit vector $\hat{\boldsymbol{k}}$ is parallel to the z-axis, and the position vector of a point with coordinates (x, y, z) will be $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$.

The magnitude of vector $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ is $|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$.

If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then $\hat{\boldsymbol{r}} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})$ or $\hat{\boldsymbol{r}} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|}$.

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then

$\boldsymbol{a} + \boldsymbol{b} = (x_1 + x_2)\hat{\boldsymbol{i}} + (y_1 + y_2)\hat{\boldsymbol{j}} + (z_1 + z_2)\hat{\boldsymbol{k}}$

$\boldsymbol{a} - \boldsymbol{b} = (x_1 - x_2)\hat{\boldsymbol{i}} + (y_1 - y_2)\hat{\boldsymbol{j}} + (z_1 - z_2)\hat{\boldsymbol{k}}$

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$, then $k\boldsymbol{a} = kx\hat{\boldsymbol{i}} + ky\hat{\boldsymbol{j}} + kz\hat{\boldsymbol{k}}$ for $k \in \mathbb{R}$.

Three-dimensional vectors in polar form

It is necessary to use two angles to define the direction of a vector in polar form. The azimuth angle θ is measured anticlockwise from the x-axis onto the projection of the vector to the x-y plane $(0° \leq \theta < 360°)$. The altitude angle ϕ is the angle from the x-y plane to the vector $(-90° \leq \theta < 90°)$.

The vector $\boldsymbol{r} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ with magnitude $|\boldsymbol{r}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}$ can be expressed in polar form as $\boldsymbol{r} = (|\boldsymbol{r}|, \theta, \phi)$, where θ is the azimuth angle of $\boldsymbol{r}$ $(0° \leq \theta < 360°)$ and ϕ is the altitude angle of $\boldsymbol{r}$ $(-90° \leq \phi < 90°)$.

The components of $\boldsymbol{r}$ can be determined:

$x_1 = |\boldsymbol{r}|\cos(\phi)\cos(\theta)$

$y_1 = |\boldsymbol{r}|\cos(\phi)\sin(\theta)$

$z_1 = |\boldsymbol{r}|\sin(\phi)$

The azimuth and altitude can be determined:

$$\theta = \tan^{-1}\left(\frac{y_1}{x_1}\right)$$

$$\phi = \sin^{-1}\left(\frac{z_1}{|\boldsymbol{r}|}\right)$$

Scalar product of vectors in three dimensions

If θ is the smaller angle between the positive directions of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the scalar (dot) product is $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\left(\frac{\pi}{2}\right) = 0$.

If $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = x_1x_2 + y_1y_2 + z_1z_2$.

If θ is the angle between vectors $\boldsymbol{a} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}$, then

$$\cos(\theta) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\boldsymbol{a}||\boldsymbol{b}|}.$$

The scalar projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

The vector projection or component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

The vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ (or the vector rejection of $\boldsymbol{a}$ from $\boldsymbol{b}$) is $\boldsymbol{a} - \left(\boldsymbol{a} \cdot \hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}}$.

Linearly dependent and independent vectors

Vectors in a set are *linearly dependent* if one vector in the set can be expressed as a linear combination of other vectors in the set. If no vector can be expressed in such a way, then the vectors in the set are *linearly independent*.

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be linearly dependent if and only if there are two scalars, x and y, such that $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ or $\boldsymbol{a} = -\frac{y}{x}\boldsymbol{b}$ (that is, one vector is a non-zero scalar multiple of the other). This implies that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors.

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be linearly independent if and only if the only solution to $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{0}$ is $x = y = 0$. This implies that $\boldsymbol{a}$ and $\boldsymbol{b}$ will be non-parallel vectors.

The vector product

The *vector product* or *cross product* of two vectors results in a vector quantity. The symbol used to denote the cross product is '×' and the cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted $\boldsymbol{a} \times \boldsymbol{b}$.

The vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin(\theta)\hat{\boldsymbol{n}}$, where θ is the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ in the right-hand sense.

For two vectors in component form, $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$, the vector (cross) product can be calculated:

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}}.$$

For a triangle formed by two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, area $= \frac{1}{2}|\boldsymbol{u}\times\boldsymbol{v}|$.

For a parallelogram with adjacent sides formed by two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, area $= |\boldsymbol{u}\times\boldsymbol{v}|$.

The volume of a parallelepiped defined by vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is $V = \left|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\right|$. This is sometimes called the *scalar triple product* (as the result is a scalar quantity).

In three dimensions, a plane is defined by the equation $ax + by + cz + d = 0$. To determine the equation of a plane, it is necessary to know the coordinates of a point on the plane, $P_1(x_1, y_1, z_1)$, as well as a normal vector, $\boldsymbol{n} = \left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right)$, which is orthogonal (perpendicular) to the plane.

The equation of a plane passing through point $P_1(x_1, y_1, z_1)$ and perpendicular to the vector $\boldsymbol{n} = \left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right)$ is given by $0 = a(x - x_1) + b(y - y_1) + c(z - z_1)$.

Vector equation of a straight line in the Cartesian plane

The vector equation of a line L passing through the point A with position vector $\boldsymbol{a}$ and in the direction of position vector $\boldsymbol{b}$, is given by $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$, where λ is a *scalar parameter*. If point $P(x, y)$ is a point on the line L, then the vector $\overrightarrow{AP}$ is parallel to the vector $\boldsymbol{b}$, such that $\overrightarrow{AP} = \lambda\boldsymbol{b}, 0 \leq \lambda \leq 1$. Now $\boldsymbol{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ and so $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ is the vector equation of line L.

There are three scenarios for two straight lines in the Cartesian plane:

- Two lines are not parallel and therefore intersect at some point.
- Two lines are parallel and therefore never meet.
- Two lines are parallel and have a common point and therefore they are coincident.

Straight lines in three-dimensional space

In general, for a straight line L passing through the point $P(x_1, y_1, z_1)$ and in the direction of vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$:

- The Cartesian equation of the line is $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.
- The parametric form of the line is $\lambda = \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$, so $x = x_1 + \lambda a$, $y = y_1 + \lambda b$ and $z = z_1 + \lambda c$.
- The vector equation of the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ or $\boldsymbol{r} = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}} + \lambda\left(a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}\right)$.

For two straight lines L_1 and L_2 with vector equations $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_2$ respectively and with direction vectors $\boldsymbol{b}_1 = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + z_1\hat{\boldsymbol{k}}$ and $\boldsymbol{b}_2 = x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}} + z_2\hat{\boldsymbol{k}}$ respectively:

- If the two lines are parallel, then $\boldsymbol{b}_1 = k\boldsymbol{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$. Elimination of the constant k results in $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.
- If the two lines are perpendicular, then $\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = 0$, therefore $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

There are four possible relationships between two straight lines in space:

- Two lines L_1 and L_2 may intersect at a point P (lines are coplanar).
- Two lines L_1 and L_2 may be parallel and never intersect (lines are coplanar).
- Two lines L_1 and L_2 may be parallel and coincident (i.e. they are the same line; lines are coplanar).
- Two lines L_1 and L_2 may be neither parallel nor intersect (lines are skew).

Systems of linear equations

Gaussian elimination represents the linear equations

$ax + by = u$

$cx + dy = v$

as a matrix equation where $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} u \\ v \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$. $\mathbf{A}$ is known as the coefficient matrix and $\mathbf{B}$ is the constant matrix.

To solve a system of equations using Gaussian elimination, combine matrices $\mathbf{A}$ and $\mathbf{B}$ into a single *augmented matrix* $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \end{array}\right]$: $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \end{array}\right] = \left[\begin{array}{cc|c} a & b & u \\ c & d & v \end{array}\right]$. The objective of this process is to form the identity matrix on the left side of the augmented matrix. When this is done, the solution will be on the right side of the matrix: $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \end{array}\right] \rightarrow \left[\begin{array}{c|c} \mathbf{I} & \mathbf{X} \end{array}\right]$.

For systems of three equations with three variables, each linear equation represents a plane in three dimensions. There are three possible solution sets:

- One unique solution (planes intersect at a single point).
- No solutions (either two or more planes are parallel, or three planes are non-parallel with no common point).
- Infinitely many solutions (either planes are coincident, or they intersect at a single line).

Dominance matrices

A dominance matrix $\mathbf{M}$ is a square matrix describing the results of the teams/players in each row against the teams/players in each column. The matrix $\mathbf{M}^2$ indicates second-order dominance. The matrix $\mathbf{M}^3$ indicates third-order dominance, etc.

- The first-order power vector $\mathbf{P}_1$ can be calculated by adding up the elements in each row of $\mathbf{M}$.
- The second-order power vector $\mathbf{P}_2$ can be calculated by adding the rows of $\mathbf{M}^2$, and so on.
- The overall power vector $\mathbf{P}$ can be calculated by adding each of $\mathbf{P}_1$, $\mathbf{P}_2$, etc. after applying a scalar multiple to give a weighting to each power vector if appropriate.
- The team or player with the highest value in $\mathbf{P}$ is said to be the dominant team or player.

Leslie matrices

The Leslie matrix $\mathbf{L}$ is constructed by assigning fecundity rates across the top row, survival rates along the sub-diagonal (beginning under F_1), and zeros in all other elements.

In general, $\mathbf{N}_r = \mathbf{L} \times \mathbf{N}_{r-1}$ and $\mathbf{N}_r = \mathbf{L}^r \times \mathbf{N}_0$.

Leontief matrices

The *output* (or production) of the economy can be defined as p and the *input* of resources required to achieve a certain production can be defined as q.

To calculate the input required to achieve a certain output, multiply by the consumption matrix: $q = \mathbf{C}p$.

The difference between the output and the input can be used to meet consumer demand, d. That is, $p - q = d$.

The production required to meet a particular demand is $p = (\mathbf{I} - \mathbf{C})^{-1} d$.

Markov chains

In a Markov chain model with square transition matrix $\mathbf{T}$ and initial state matrix $\mathbf{S}_0$, $\mathbf{S}_n = \mathbf{T} \times \mathbf{S}_{n-1}$ and $\mathbf{S}_n = \mathbf{T}^n \times \mathbf{S}_0$. The elements in each column of the transition matrix must add to give 1.

The steady state is reached when $\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{T}\begin{bmatrix} x \\ y \end{bmatrix}$.

Eigenvalues and eigenvectors

Eigenvectors and eigenvalues are related to a given matrix $\mathbf{A}$ by the equation $\mathbf{AX} = \lambda\mathbf{X}$.

In this equation, $\mathbf{X}$ is a column matrix (vector) and λ is a scalar. $\mathbf{X}$ is called the *eigenvector* of matrix $\mathbf{A}$ and λ is the associated *eigenvalue* of matrix A.

To calculate the eigenvalues associated with matrix $\mathbf{A}$, solve the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

To calculate the eigenvectors associated with matrix $\mathbf{A}$, substitute the eigenvalues back into the equation $\mathbf{AX} = \lambda\mathbf{X}$.

Chapter review 2

Exercise 2.2

1 Calculate the magnitude of each of the following vectors, and hence determine a unit vector parallel to the given vector:

(a) $\boldsymbol{a} = 7\hat{\boldsymbol{i}} - 24\hat{\boldsymbol{j}}$ **(b)** $\boldsymbol{b} = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$

Exercise 2.2

2 For each of the points P whose coordinates are given, determine the position vector $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ and hence determine a unit vector in the direction of P.

(a) $P(-2, 3, -3)$ **(b)** $P(3, -5, -2)$

Exercise 2.2

3 Given that $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, calculate:

(a) $\boldsymbol{a} + \boldsymbol{b}$ **(b)** $3\boldsymbol{a} - 2\boldsymbol{c}$ **(c)** $3\boldsymbol{c} - 5\boldsymbol{b} + 7\boldsymbol{a}$

Exercise 2.3

4 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$. Calculate the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

Exercise 2.6

5 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$. Calculate each of the following.

(a) $\boldsymbol{a} \times \boldsymbol{b}$ **(b)** $\boldsymbol{a} \times \boldsymbol{c}$ **(c)** $\boldsymbol{b} \times \boldsymbol{c}$

Exercise 2.7

6 A straight line L passes through the point $(4, -1)$ and is parallel to the vector $-2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$.

(a) Determine the vector equation of line L.

(b) Determine the parametric equations for line L.

(c) Determine the Cartesian equation for line L.

Exercise 2.8

7 Determine the vector matrix equation of the line L_1 that passes through the point $A(2, -1, 3)$ and is parallel to the line L_2, defined by the vector equation $\boldsymbol{r}_2 = \left(-2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}\right) + \lambda\left(3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$.

Exercise 2.8

8 Determine whether the straight lines with Cartesian equations $\frac{x-1}{2} = \frac{y+3}{-5} = \frac{z}{-3}$ and $\frac{x+1}{2} = \frac{y}{2} = z - 2$ are parallel.

Exercise 2.10

9 In a certain economy with three sectors A, B and C, the consumption matrix is defined as:

$$\mathbf{C} = \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \end{matrix}$$

(a) Calculate the input required to produce 20 units of A, 22 units of B and 18 units of C.

(b) Determine the consumer demand that can be met by the production described in part **(a)**.

10 Which of the following is a correctly constructed Markov chain transition matrix? Exercise 2.11

A $\mathbf{T} = \begin{bmatrix} 0.3 & 0.8 \\ 0.2 & 0.1 \\ 0.5 & 0.1 \end{bmatrix}$

B $\mathbf{T} = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix}$

C $\mathbf{T} = \begin{bmatrix} 1.5 & 0.4 \\ -0.5 & 0.6 \end{bmatrix}$

D $\mathbf{T} = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$

11 Rudi is orienteering in the forest. He heads from the starting point O for a distance of 3.5 km on a bearing of S15°E until he reaches checkpoint A. He then runs N82°E until he reaches checkpoint B, 4.2 km from A. Calculate Rudi's position relative to the starting point. Exercise 2.1

12 For the following pairs of vectors, determine the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, to the nearest degree. Exercise 2.3

(a) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}, \boldsymbol{b} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

(b) $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}, \boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

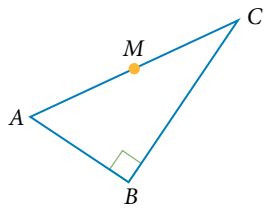
13 If $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, resolve $\boldsymbol{a}$ into a component parallel to $\boldsymbol{b}$ and a component perpendicular to $\boldsymbol{b}$. Exercise 2.3

14 If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent vectors, determine the values of x, y and z, where $x, y, z \in \mathbb{R}$, if: Exercise 2.4

(a) $(5x-7)\boldsymbol{a} + (y+7)\boldsymbol{b} + (3z+13)\boldsymbol{c} = \boldsymbol{0}$

(b) $(x^2 - 6x + 8)\boldsymbol{a} + (5y^2 - 10)\boldsymbol{b} + (z^3 - 8)\boldsymbol{c} = \boldsymbol{0}$

15 ABC is a right-angled triangle, as shown in the diagram. Let $\overrightarrow{AM} = \overrightarrow{MC} = \boldsymbol{a}$ and $\overrightarrow{BM} = \boldsymbol{b}$, where M is the midpoint of the hypotenuse $\overline{AC}$. Exercise 2.5



(a) Express $\overline{AB}$ and $\overline{BC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Prove that M is equidistant from the three vertices of ΔABC.

16 **(a)** Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}, \boldsymbol{b} = 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. Calculate the area of the parallelogram whose adjacent sides are formed by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Exercise 2.6

(b) Calculate the area of the triangle whose vertices are $A(3, 4, -2)$, $B(1, 5, 2)$, $C(-1, 5, 3)$.

(c) Calculate the volume of the parallelepiped defined by the following three vectors as edges: $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}, \boldsymbol{b} = 4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$.

17 Determine the equation of the plane containing the set of points $A = (-1, 1, 1)$, $B = (2, 0, -1)$, $C = (0, -1, 1)$. Exercise 2.6

18 Determine the coordinates of the point of intersection of the lines L_1 and L_2 with vector equations $\boldsymbol{r}_1 = \begin{bmatrix} -4 \\ 9 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -4 \end{bmatrix}$ and $\boldsymbol{r}_2 = \begin{bmatrix} -1 \\ 14 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ 5 \end{bmatrix}$. Exercise 2.7

Exercise 2.9

19 Solve the following set of simultaneous equations using Gaussian elimination.

$$x + y + z = -2$$
$$4x + y - z = -3$$
$$-x - 2y - 3z = 3$$

Exercise 2.10

20 Four teams compete in a local cricket competition. Their results can be summarised as follows:

- Team A defeats Team D
- Team B defeats Team A
- Team C defeats Team A and B
- Team D defeats Team B and C

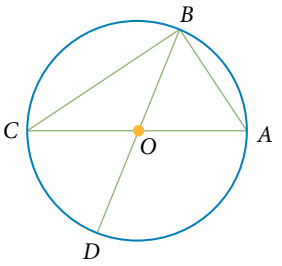
Determine the most dominant team in this competition.

Exercise 2.10

21 A particular species of mouse is known to have a maximum lifespan of 3 years. The initial female mouse population is given by the matrix $\mathbf{N}_0 = \begin{bmatrix} 80 \\ 35 \\ 20 \end{bmatrix}$ and the Leslie matrix associated with the population change of this mouse species is $\mathbf{L} = \begin{bmatrix} 0.2 & 2.2 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$. Calculate the total population of mice after 4 years. Identify any assumptions you make and comment on the effects of these assumptions on your answer.

Exercise 2.5

22 Consider the circle with centre O and radius $\overrightarrow{OA} = \boldsymbol{a}$. B and C are points on the circle and $\overrightarrow{OB} = \boldsymbol{b}$.



(a) Prove that the angle subtended by a diameter in a circle, $\angle ABC$, is a right angle.

(b) Prove that the lines joining the endpoints of the two non-parallel diameters AC and BD form a rectangle $ABCD$.

Exercise 2.6

23 Determine a vector perpendicular to the plane $x + 3y - 2z = 12$.

24 Determine whether there are no solutions, a unique solution, or an infinite number of solutions for the following system of linear equations:

$$2x + y - z = 5$$
$$x + y + 2z = 2$$
$$6x + 3y - 3z = 15$$

Exercise 2.11

25 Which of the following matrices will *not* have any real eigenvalues associated with them?

A $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

B $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

C $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

D $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

26 **(a)** Calculate the volume of the parallelepiped with adjacent edges $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ where $A=(-5,9,0)$, $B=(8,12,-15)$, $C=(2,0,20)$, $D=(5,-11,-12)$. Exercise 2.6

(b) Determine the value of v if the volume of the parallelepiped with adjacent edges $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$, where $A=(3,-7,5)$, $B=(2,-9,4)$, $C=(1,v,-15)$, $D=(2,-5,10)$, is equal to 670 cubic units.

27 The volume V of a tetrahedron can be calculated using the scalar triple product, where $V=\frac{1}{6}\left|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\right|$. Exercise 2.6

(a) Calculate the volume of the tetrahedron with vertices at the points $A=(5,7,2)$, $B=(6,15,-10)$, $C=(3,15,0)$, $D=(0,-1,22)$.

(b) Determine the value of w if the volume of the tetrahedron with adjacent edges $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$, where $A=(0,-9,3)$, $B=(-2,9,-5)$, $C=(3,15,0)$, $D=(4,w,-10)$, is equal to 243 cubic units.

28 **(a)** Determine the Cartesian form of the lines with parametric equation given by L_1: $x=1-3\lambda$, $y=-5+\lambda$, $z=3\lambda$ and L_2: $x=1-4\mu$, $y=-1$, $z=-6+2\mu$ Exercise 2.8

(b) Determine the point of intersection of these two lines.

(c) Calculate the acute angle between these two lines, correct to 2 decimal places.

(d) Calculate the coordinates of the point where L_1 cuts the xy-plane.

(e) Calculate the coordinates of the point where L_2 cuts the yz-plane.





3

Vector calculus

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3 Recall

Differentiate

1 Calculate $\frac{dx}{dt}$ for each of the following:

(a) $x = 1 - \cos(t)$ (b) $x = \cos^2(t)$ (c) $x = e^{2t} - 4t^3$ (d) $x = \sec(t)$

2 Calculate $\frac{dy}{dt}$ for each of the following:

(a) $y = 4\sin(t)$ (b) $y = \sin^2(t)$ (c) $y = \log_e(2t)$ (d) $y = \tan(t)$

Calculate the derivative for a given value

3 If $f(t) = 10t - 5t^2$, calculate $f'(2)$.

4 If $f(t) = \sin(2t)$, calculate without using technology $f'\left(\frac{\pi}{6}\right)$.

5 If $f(t) = 3\cos^2(\pi t)$, calculate without using technology $f'\left(\frac{1}{4}\right)$.

6 If $f(t) = \sqrt{2t+1}$, calculate $f'\left(\frac{1}{2}\right)$.

Anti-differentiate

7 Calculate the following:

(a) $\int \left(e^{2t} + e^{-t}\right) dt$ (b) $\int 3\sin(2t)\, dt$ (c) $\int (4t+1)^2\, dt$ (d) $\int \frac{1}{3t+2}\, dt$

Solve problems involving displacement and velocity

8 A particle moves in a straight line such that its position, x metres, relative to an origin O at time t seconds is given by $x(t) = -t^2 + 2t - 8$, $t \geq 0$.

(a) State the particle's initial position. (b) Determine the particle's velocity at $t = 4$.

(c) Determine when the particle's velocity is zero.

9 The velocity, v m/s, of a particle moving in a straight line at time t seconds is given by $v(t) = 6 - 2t$. At time $t = 0$, the particle is 3 metres to the right of the origin O.

(a) Determine $x(t)$. (b) Describe the particle's position relative to O after 4 seconds.

(c) Calculate the total distance travelled by the particle after 4 seconds.

Calculate the magnitude of a vector

10 Calculate the magnitude of the following vectors:

(a) $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ (b) $\boldsymbol{c} = -5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\sqrt{2}\hat{\boldsymbol{k}}$

Use the scalar product

11 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = -3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$. Calculate:

(a) $\boldsymbol{a} \cdot \boldsymbol{b}$ (b) $\boldsymbol{a} \cdot \boldsymbol{c}$ (c) $\boldsymbol{b} \cdot \boldsymbol{c}$

12 Calculate the angle, in radians, between each of the following pairs of vectors. State each angle correct to three significant figures.

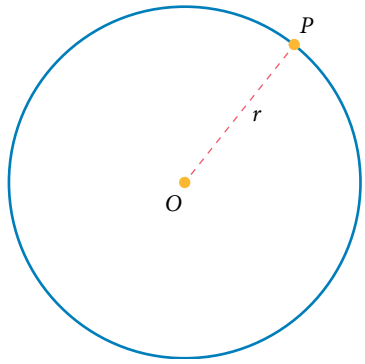
(a) $\boldsymbol{a} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ (b) $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

13 Let $\boldsymbol{a} = -\hat{\boldsymbol{i}} + m\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = m\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, $m \in \mathbb{R}$. If $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$, determine the value of m.

3.1 Position vectors as a function of time

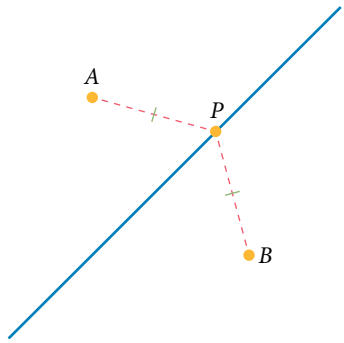
You have already seen how vectors can be used to represent the position, velocity and acceleration of a particle. You are also aware of the important relationship between equations of position, velocity and acceleration and the mathematical processes of differentiation and anti-differentiation. Combining these concepts, in this chapter you will utilise the technique of *vector calculus* to describe and calculate the position and movement of a particle with respect to time.

The locus of a point



In geometry, a *locus* (plural: *loci*) is a set of points satisfying one or more particular geometric conditions. For example, the circle can be defined as the locus of a point P on a plane such that P is a fixed distance (the radius r) from a single fixed point O.

You can think of the locus as the path traced out by a point P under certain conditions. A straight line can be defined as the locus of a point P on a plane such that P is equidistant from two fixed points, A and B.



The locus of a point can be described or represented by a Cartesian equation. The Cartesian equation describes the nature of the curve along which a particle travels. A Cartesian equation, however, does not provide any information about a particle's position at any given time. For this reason, it is better to express a particle's coordinates as a function of time. Hence if you express a particle's coordinates in terms of a third variable (parameter), say t, you can obtain a description of a particle's path and its location on this path at any time.

A particle's position at any time t is best described using a position vector as a function of time.

The concept of a position vector as a function of time will now be introduced.

Parametric and Cartesian equations

Remember what a position vector is:

> The position of any point P, relative to an origin O, is uniquely specified by the vector $\overrightarrow{OP}$, which is termed the *position vector* of P (relative to O).

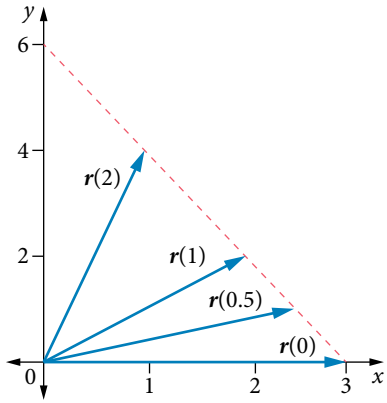
Consider a body moving in the Cartesian plane so that its position at time t is given by the position vector $\boldsymbol{r}(t) = (3 - t)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$, $t \geq 0$.

By substituting values of t into $\boldsymbol{r}(t)$, you can determine the position of the body on the Cartesian plane at these times.

For example:

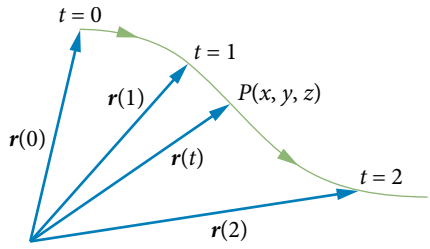
t	$\boldsymbol{r}(t)$	Position on Cartesian plane
0	$3\hat{\boldsymbol{i}}$	$(3,0)$
0.5	$2.5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$	$(2.5,1)$
1	$2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$	$(2,2)$
2	$\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$	$(1,4)$

The figure below shows these four position vectors and the path on the Cartesian plane that the particle appears to be taking.



Note that in the above example, the particle begins its journey (that is, when $t = 0$) at the point $(3,0)$ and continues along what appears to be a straight line. With an understanding of linear relations, you should recognise the equation of this straight line as $y = 6 - 2x$.

In general, when a particle moves in space, its position vector varies with time. In the following diagram, the position vector $\boldsymbol{r}(t)$ is dependent on time, t.



If the coordinates of the variable point $P(x, y, z)$ are expressed parametrically as $x = f(t)$, $y = g(t)$ and $z = h(t)$, then the *vector equation* of the path is given by the position vector $\boldsymbol{r}(t) = f(t)\hat{\boldsymbol{i}} + g(t)\hat{\boldsymbol{j}} + h(t)\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are unit vectors in the positive directions of the x, y and z axes respectively.

The equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ are called *parametric equations* and t is called the *parameter*.

In the introductory example, the parametric equations were $x = (3 - t)$ and $y = 2t$.

Rearranging the first equation to make the parameter t the subject gives $t = 3 - x$. Substituting $t = 3 - x$ into $y = 2t$ gives $y = 2(3 - x) = 6 - 2x$.

So, confirming the earlier assertion, $y = 6 - 2x$ is the Cartesian equation that describes the path of the body.

Technology worked example
Graphing parametric equations

1 Stating a vector equation from parametric equations

Determine the position vector equation of a curve which has parametric equations defined by $x = 2\cos(t)$, $y = 2\sin(t)$ and $z = 1$ for $t \geq 0$.

THINKING	WORKING
Synthesise the vector equation by substituting the given parametric equations into the general form $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$.	$x = 2\cos(t)$ $y = 2\sin(t)$ $z = 1$ Therefore, the vector equation of the curve is $\boldsymbol{r}(t) = 2\cos(t)\hat{\boldsymbol{i}} + 2\sin(t)\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $t \geq 0$.

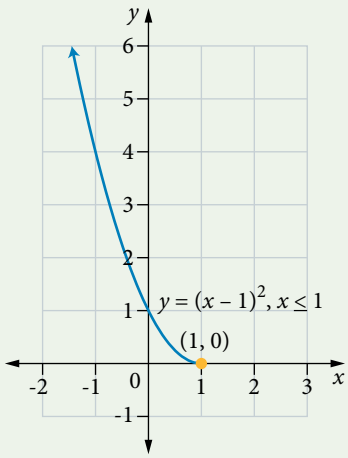
2 Determining and sketching a Cartesian equation from linear and quadratic parametric equations

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = (1-t)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine the Cartesian equation of the path of the particle and determine the domain.

THINKING	WORKING
1 Write and number the parametric equations.	$x = 1 - t$ [1] $y = t^2$ [2]
2 Express t in terms of x.	From [1] $t = 1 - x$.
3 Determine the allowed values of x.	If $t \geq 0$ then $x \leq 1$.
4 Substitute into the other parametric equation to eliminate t.	$y = (1-x)^2$
5 Write the Cartesian equation with domain.	The Cartesian equation is $y = (x-1)^2$, $x \leq 1$.

(b) Sketch the path of the body on the Cartesian plane.

Sketch the parabola. Ensure you use the correct domain.	The path is that of the parabola $y = (x-1)^2$, where $x \leq 1$. 

To summarise from Worked example 2:

- The parametric equations are $x = 1 - t$ and $y = t^2$, $t \geq 0$.
- The vector equation is $\boldsymbol{r}(t) = (1-t)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$, $t \geq 0$.
- The Cartesian equation is $y = (x-1)^2$, $x \leq 1$.

All three of these equations describe the position of the same particle, but note that the Cartesian equation describes the path or the *locus* of the particle. Knowing only the Cartesian equation is insufficient to determine the position of the particle at any time, t.

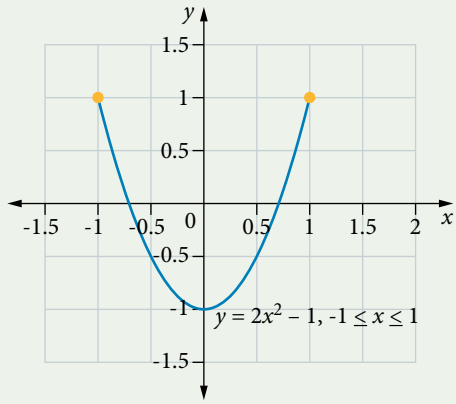
3 Determining and sketching a Cartesian equation from trigonometric parametric equations

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = \cos(t)\hat{\boldsymbol{i}} + \cos(2t)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine the Cartesian equation of the path of the particle, indicating the domain and range.

THINKING	WORKING
1 Write and number the parametric equations.	$x = \cos(t)$ [1] $y = \cos(2t)$ [2]
2 Determine the allowed values of x.	$-1 \leq \cos(t) \leq 1$, therefore $-1 \leq x \leq 1$.
3 Determine the allowed values of y.	$-1 \leq \cos(2t) \leq 1$, therefore $-1 \leq y \leq 1$.
4 Use the trigonometric identity $\cos(2t) = 2\cos^2(t) - 1$ to eliminate the parameter t.	$y = 2\cos^2(t) - 1$ $y = 2x^2 - 1$
5 Write the Cartesian equation with the domain and the range.	The Cartesian equation is $y = 2x^2 - 1$, $-1 \leq x \leq 1$. The range is $-1 \leq y \leq 1$.

(b) Sketch the path of the body on the Cartesian plane.

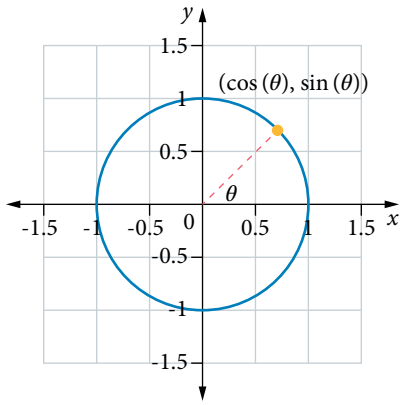
Sketch the parabola using appropriate technology. Ensure you use the correct domain.	The path is that of the parabola $y = 2x^2 - 1$, where $-1 \leq x \leq 1$. 

Circles, ellipses and hyperbolas

Particles sometimes move on paths described by circles, ellipses or hyperbolas.

Circles

You will recall from your study in Mathematical Methods that the unit circle, defined by the equation $x^2 + y^2 = 1$, consists of x- and y-coordinates that can be represented as trigonometric functions of the angle θ measured anticlockwise from the positive x-axis, such that $x = \cos(\theta)$ and $y = \sin(\theta)$. In this case, θ is the parameter.



If θ was replaced by another parameter, say t, the parametric equations of the unit circle become:

$x = \cos(t)$ and $y = \sin(t)$

If the centre of a circle of radius r is at the point (h, k), then the Cartesian equation is $(x-h)^2 + (y-k)^2 = r^2$.

The parametric equations of a circle of radius r and centre at (h, k) are $x = h + r\cos(t)$ and $y = k + r\sin(t)$, where $t \in [0, 2\pi]$.

The vector equation of a circle of radius r and centre at (h, k) is $\boldsymbol{r}(t) = (h + r\cos(t))\hat{\boldsymbol{i}} + (k + r\sin(t))\hat{\boldsymbol{j}}$, where $t \in [0, 2\pi]$.

Note that swapping cosine and sine in the parametric equations still produces equations that describe a circle.

4 Determining the Cartesian equation and sketching circular motion of a particle

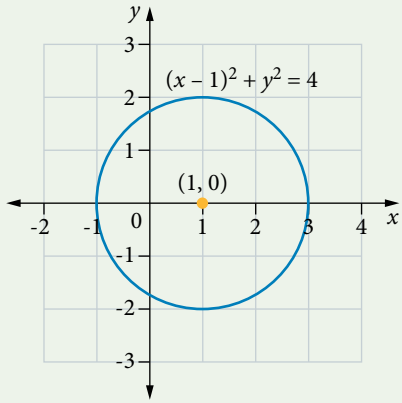
The position vector of a body at time t is given by $\boldsymbol{r}(t) = (1 + 2\cos(\pi t))\hat{\boldsymbol{i}} + 2\sin(\pi t)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine the Cartesian equation of the path of the particle, stating the domain and range.

THINKING	WORKING
1 Write and number the parametric equations.	$x = 1 + 2\cos(\pi t)$ [1] $y = 2\sin(\pi t)$ [2]
2 Express $\cos(\pi t)$ in terms of x.	From [1] you obtain $\cos(\pi t) = \frac{1}{2}(x-1)$.
3 Determine the possible values of x.	Since $-1 \leq \cos(\pi t) \leq 1$ you obtain $-1 \leq \frac{1}{2}(x-1) \leq 1$. Simplifying the inequality: $-1 \leq \frac{1}{2}(x-1) \leq 1$ $-2 \leq (x-1) \leq 2$ $-1 \leq x \leq 3$

4 Express $\sin(\pi t)$ in terms of y.	From [2] you obtain $\sin(\pi t) = \frac{1}{2}y$.
5 Determine the possible values of y.	Since $-1 \leq \sin(\pi t) \leq 1$ you obtain $-1 \leq \frac{1}{2}y \leq 1$. Simplifying the inequality: $-1 \leq \frac{1}{2}y \leq 1$ $-2 \leq y \leq 2$
6 Use $\cos^2(\pi t) + \sin^2(\pi t) = 1$ to eliminate t.	$\frac{1}{4}(x-1)^2 + \frac{1}{4}y^2 = 1$ $(x-1)^2 + y^2 = 4$
7 Write the Cartesian equation with the domain and the range.	The Cartesian equation is $(x-1)^2 + y^2 = 4$, $-1 \leq x \leq 3$, $-2 \leq y \leq 2$.

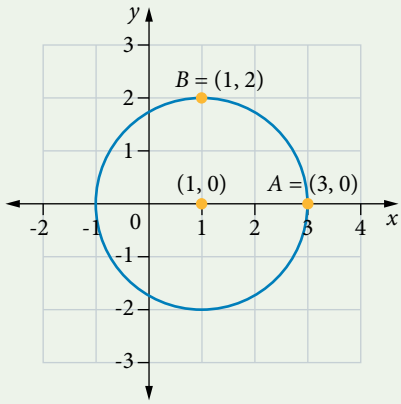
(b) Sketch the path of the body on the Cartesian plane.

Sketch the circle. Ensure you use the correct domain.	The path is that of a circle with centre $(1, 0)$ and radius 2. 

(c) Describe the motion of the body.

Substitute increasing values of t into $\boldsymbol{r}(t)$.	At $t = 0$: $x = 1 + 2\cos(0)$ $= 3$ $y = 2\sin(0)$ $= 0$ The body starts at $A(3, 0)$. At $t = \frac{1}{2}$: $x = 1 + 2\cos\left(\frac{\pi}{2}\right)$ $\quad$ $y = 2\sin\left(\frac{\pi}{2}\right)$ $= 1$ $\quad$ $= 2$ At $t = \frac{1}{2}$, the body is at $B(1, 2)$. The period of motion is $\frac{2\pi}{\pi} = 2$.

The body moves around the circle in an anticlockwise direction, returning to A at every 2 units of time.



Explore further

Graphing parametric equations

Create and explore the properties of circles and ellipses in parametric and Cartesian form.

Making connections

Explore the features of an ellipse

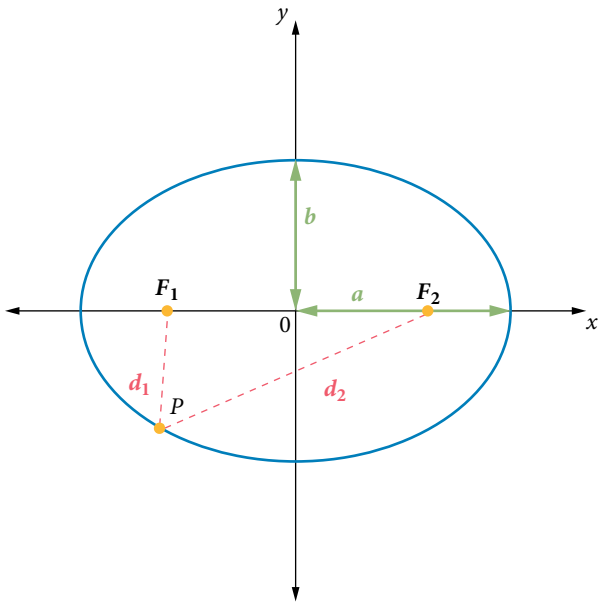
Explore the features of an ellipse by moving the sliders.

Ellipses

Ellipses have many practical applications in the real world. For example, the motions of comets and planets are defined by elliptical orbits. An ellipse can be considered as the locus of a point P on a plane such that the sum of its distances (d_1 and d_2) from two fixed points F_1 and F_2 is fixed. (The points F_1 and F_2 are known as the *foci* of the ellipse).

The standard Cartesian equation of an ellipse centred at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $a > b$, the ellipse has a horizontal *major axis* of $2a$ and a vertical *minor axis* of $2b$. If $b > a$ the ellipse has a vertical major axis, and if $a = b$ the ellipse is a circle.

If the centre of the ellipse is at the point (h, k), then the Cartesian equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.



The parametric equations of an ellipse centred at the origin are $x = a\cos(t)$ and $y = b\sin(t)$, where $t \in [0, 2\pi]$.

The parametric equations of an ellipse with its centre at (h, k) are $x = h + a\cos(t)$ and $y = k + b\sin(t)$.

The vector equation of an ellipse with its centre at (h, k) is $\boldsymbol{r}(t) = (h + a\cos(t))\hat{\boldsymbol{i}} + (k + b\sin(t))\hat{\boldsymbol{j}}$, where $t \in [0, 2\pi]$.

$x = h + a\cos(\omega t)$ and $y = k + b\sin(\omega t)$ are also parametric equations of an ellipse centred at (h, k). The variable ω determines the speed and direction of the particle.

Note that swapping cosine and sine in the parametric equations still produces equations that describe an ellipse.

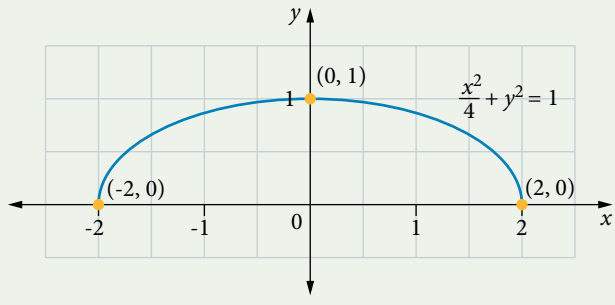
5 Determining the Cartesian equation and sketching elliptical motion of a particle

The position vector of a body at time t is given by $\boldsymbol{r}(t) = 2\cos(t)\hat{\boldsymbol{i}} + \sin(t)\hat{\boldsymbol{j}}$, $0 \le t \le \pi$.

(a) Determine the Cartesian equation of the path of the particle, indicating the domain and range.

THINKING	WORKING
1 Write and number the parametric equations.	$x = 2\cos(t)$ [1] $y = \sin(t)$ [2]
2 Express $\cos(t)$ in terms of x.	From [1] you obtain $\cos(t) = \frac{x}{2}$.
3 Determine the possible values of x.	Since $-1 \le \cos(t) \le 1$ for $0 \le t \le \pi$ you obtain $-1 \le \frac{x}{2} \le 1$. So $-2 \le x \le 2$.
4 Determine the possible values of y.	Since $0 \le \sin(t) \le 1$ for $0 \le t \le \pi$ you obtain $0 \le y \le 1$.
5 Use $\cos^2(t) + \sin^2(t) = 1$ to eliminate t.	$\frac{x^2}{4} + y^2 = 1$
6 Write the Cartesian equation with the domain and the range.	The Cartesian equation is $\frac{x^2}{4} + y^2 = 1$, $-2 \le x \le 2$. The range is $0 \le y \le 1$.

(b) Sketch the path of the body on the Cartesian plane.

Sketch the ellipse using appropriate technology. Ensure you use the correct domain and range.	The path is half (the upper branch) of an ellipse with centre at the origin. 

(c) Determine the body's initial position and its direction of motion.

1 Determine the position (x, y) at $t = 0$.	At $t = 0$: $x = 2\cos(0)$ $= 2$ $y = \sin(0)$ $= 0$ The body starts at $A(2, 0)$.
2 Substitute a larger value of t into $\boldsymbol{r}(t)$.	At $t = \frac{\pi}{2}$: $x = 2\cos\left(\frac{\pi}{2}\right)$ $= 0$ $y = \sin\left(\frac{\pi}{2}\right)$ $= 1$ At $t = \frac{\pi}{2}$, the body is at $B(0, 1)$. The body moves around the half-ellipse in an anticlockwise direction.
3 Interpret the answer.	Initially the body starts at $A(2, 0)$ and from there travels around the half-ellipse in an anticlockwise direction.

Explore further

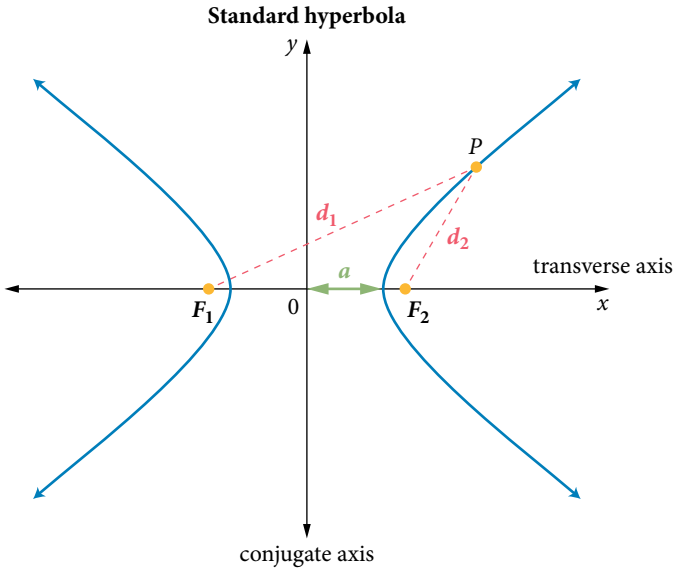
Hyperbolas in Cartesian and parametric form

Explore the properties of hyperbolas in parametric and Cartesian form.

Hyperbolas

A hyperbola can be described as the locus of a point P on a plane such that the absolute value of the difference of its distances from two given points F_1 and F_2 (the foci) is fixed.

The standard Cartesian equation of a hyperbola centred at the origin and with vertices $(\pm a, 0)$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The *transverse axis* passes through the centre and the two foci (F_1 and F_2), and the *conjugate axis* passes through the centre and is perpendicular to the transverse axis.



If the centre of the hyperbola is at the point (h, k), exactly half way between the two foci (F_1 and F_2), then the Cartesian equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

The parametric equations of a hyperbola centred at the origin and with vertices $(\pm a, 0)$ are $x = a\sec(t)$ and $y = b\tan(t)$.

The parametric equations of a hyperbola with its centre at (h, k) are $x = h + a\sec(t)$ and $y = k + b\tan(t)$.

The vector equation of a hyperbola with its centre at (h, k) is $\boldsymbol{r}(t) = (h + a\sec(t))\hat{\boldsymbol{i}} + (k + b\tan(t))\hat{\boldsymbol{j}}$.

Making connections

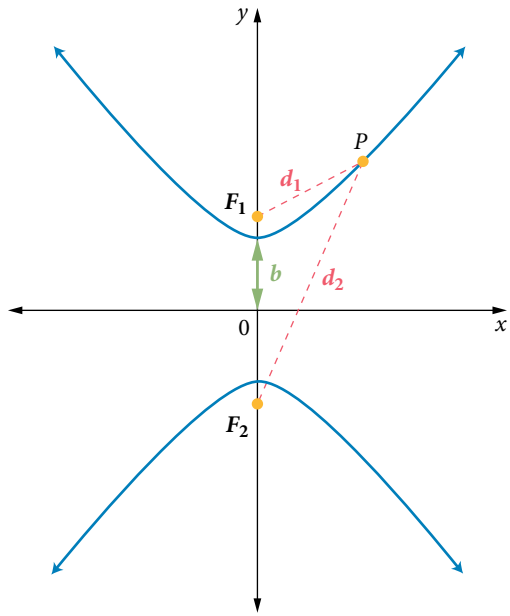
Explore the features of a standard hyperbola

Explore the features of a standard hyperbola by moving the sliders.

As with the ellipse, $x = h + a\sec(\omega t)$ and $y = k + b\tan(\omega t)$, are also parametric equations of a hyperbola centred at (h, k). The variable ω determines the speed and direction of the particle.

The standard Cartesian equation of the *conjugate hyperbola* centred at the origin and with vertices $(0, \pm b)$ is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

Conjugate hyperbola



The parametric equations for the *conjugate hyperbola* centred at the origin and with vertices $(0, \pm b)$ are $x = a\tan(t)$ and $y = b\sec(t)$.

The parametric equations for the *conjugate hyperbola* with its centre at (h, k) are $x = h + a\tan(t)$ and $y = k + b\sec(t)$.

The vector equation of the *conjugate hyperbola* with its centre at (h, k) is $\boldsymbol{r}(t) = (h + a\tan(t))\hat{\boldsymbol{i}} + (k + b\sec(t))\hat{\boldsymbol{j}}$.

Making connections

Explore the features of a conjugate hyperbola

Explore the features of a conjugate hyperbola by moving the sliders.

6 Determining the Cartesian equation and sketching hyperbolic motion of a particle

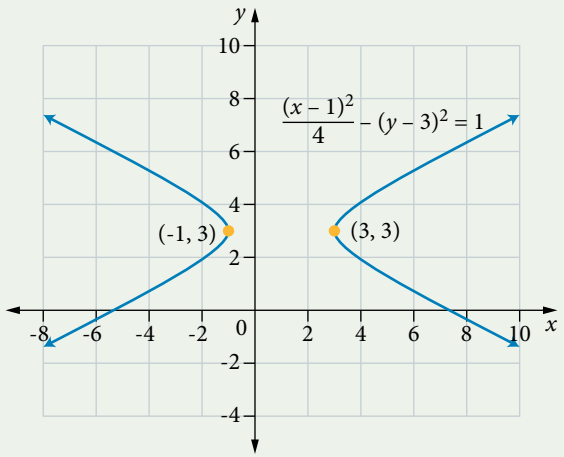
The position vector of a body at time t is given by $\boldsymbol{r}(t) = (1 + 2\sec(t))\hat{\boldsymbol{i}} + (3 + \tan(t))\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine the Cartesian equation of the path of the particle, determining the domain and range.

THINKING	WORKING
1 Write and number the parametric equations.	$x = (1 + 2\sec(t))$ [1] $y = (3 + \tan(t))$ [2]
2 Express $\sec(t)$ in terms of x.	From [1] you obtain $\sec(t) = \frac{x-1}{2}$.

3 Determine the possible values of x.	Now $\sec(t) \geq 1$ or $\sec(t) \leq -1$ for $t \geq 0$. Therefore $\frac{x-1}{2} > 1$ or $\frac{x-1}{2} \leq -1$ $x - 1 \geq 2$ or $x - 1 \leq -2$ $x \geq 3$ or $x \leq -1$
4 Express $\tan(t)$ in terms of y.	From [2] you obtain $\tan(t) = y - 3$.
5 Determine the possible values of y.	Since $-\infty < 3 + \tan(t) < \infty$ for $t \geq 0$, the range is $y \in (-\infty, \infty)$.
6 Use $\sec^2(t) - \tan^2(t) = 1$ to eliminate t.	$\frac{(x-1)^2}{4} - (y-3)^2 = 1$
7 Write the Cartesian equation with the domain and the range.	The Cartesian equation is $\frac{(x-1)^2}{4} - (y-3)^2 = 1, x \in (-\infty, -1] \cup [3, \infty)$. The range is $y \in (-\infty, \infty)$.

(b) Sketch the path of the body on the Cartesian plane.

Sketch the hyperbola using appropriate technology. Ensure you use the correct domain and range.	The path is a hyperbola with vertices $(-1, 3)$ and $(3, 3)$. 

Colliding particles

Two particles will collide if they share the same position at the same time.

Technology worked example
Colliding particles

7 Colliding particles

The paths of two particles with respect to time t are described by the vector equations $\boldsymbol{r}_1(t) = (2t-2)\hat{\boldsymbol{i}} + \left(t^2+6\right)\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_2(t) = t\hat{\boldsymbol{i}} + 5t\hat{\boldsymbol{j}}$.

(a) Determine the coordinates of the point at which the two particles collide.

THINKING	WORKING
1 Equate the $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ components.	$2t-2=t$ [1] $t^2+6=5t$ [2]
2 Solve the $\hat{\boldsymbol{i}}$ components for t.	$2t-2=t$ $t=2$
3 Solve the $\hat{\boldsymbol{j}}$ components for t.	$t^2+6=5t$ $t^2-5t+6=0$ $(t-2)(t-3)=0$ $t=2, t=3$
4 Interpret your result.	Therefore the particles share the same common point when $t=2$.
5 Determine when and where the particles collide.	When $t=2$, the particles have the position vector $2\hat{\boldsymbol{i}}+10\hat{\boldsymbol{j}}$; i.e. they collide at $(2,10)$.

(b) Determine the coordinates of the points at which the two paths cross. Verify your answer graphically.

1 Recall that the Cartesian equation of the position of each particle will define the locus, or the path traced out, by each particle. Begin by determining the Cartesian equation for the first particle.	$\boldsymbol{r}_1(t) = (2t-2)\hat{\boldsymbol{i}} + \left(t^2+6\right)\hat{\boldsymbol{j}}$ $x = 2t-2$ $\therefore t = \dfrac{x+2}{2}$ $y = t^2+6$ Therefore $y = \left(\dfrac{x+2}{2}\right)^2 + 6$ $y = \dfrac{(x+2)^2}{4} + 6$ [1]
2 Determine the Cartesian equation for the second particle.	$\boldsymbol{r}_2(t) = t\hat{\boldsymbol{i}} + 5t\hat{\boldsymbol{j}}$ $x = t$ $y = 5t$ Therefore $y = 5x$ [2]

3 Solve simultaneously to determine the points of intersection of the two loci.

$$[2] = [1]$$
$$5x = \frac{(x+2)^2}{4} + 6$$
$$20x = (x+2)^2 + 24$$
$$20x = x^2 + 4x + 4 + 24$$
$$0 = x^2 - 16x + 28$$
$$0 = (x-14)(x-2)$$

Therefore

$$x = 14,\ x = 2$$

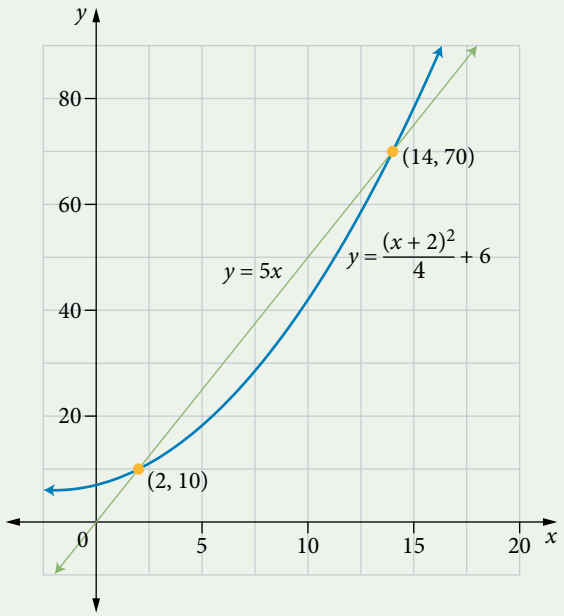
When $x = 14$, $y = 5(14) = 70$

When $x = 2$, $y = 5(2) = 10$

4 Interpret your result.

The paths of the two particles cross at the points $(14,70)$ and $(2,10)$.

5 Verify your solution graphically.



The points of intersection $(14,70)$ and $(2,10)$ have been verified graphically as required.

(c) Calculate the distance between the particles when $t = 1$.

1 Calculate the two position vectors when $t = 1$.

$$\boldsymbol{r}_1(1) = 7\hat{\boldsymbol{j}} \text{ and } \boldsymbol{r}_2(1) = \hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$$

2 Calculate the displacement between the two particles at $t = 1$.

$$\boldsymbol{r}_1(1) - \boldsymbol{r}_2(1) = 7\hat{\boldsymbol{j}} - \left(\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$$
$$= -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$$

3 Calculate the distance between the two particles at $t = 1$. Distance is the magnitude of displacement.

$$\left|\boldsymbol{r}_1(1) - \boldsymbol{r}_2(1)\right| = \sqrt{(-1)^2 + 2^2}$$
$$= \sqrt{5}$$

4 Interpret your result.

The distance between the two particles at $t = 1$ is $\sqrt{5}$ units.

EXERCISE 3.1 Position vectors as a function of time

Worked Example

1 Determine the vector equation of a curve which has parametric equations given by $x = a\cos(t)$, $y = a\sin(t)$ and $z = bt$ for $t \geq 0$.

2 Determine the Cartesian equation and sketch the path of a body whose position vector at time t, $t \geq 0$, is given by:

(a) $\boldsymbol{r}(t) = 2t\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}}$

(b) $\boldsymbol{r}(t) = (t+1)\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}}$

(c) $\boldsymbol{r}(t) = \left(t^2+1\right)\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}}$

(d) $\boldsymbol{r}(t) = 2t^2\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}$

3 The position vector of a particle at time t is given by $\boldsymbol{r}(t) = \left(\sqrt{t-3}\right)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$, $t \geq 3$.
Determine the Cartesian equation of the path of the particle and sketch the graph.

4 The position vector of a body at time t is given by $\boldsymbol{r}(t) = \cos^2(t)\hat{\boldsymbol{i}} + 2\sin^2(t)\hat{\boldsymbol{j}}$.

(a) Determine the Cartesian equation of the body's path.

(b) Describe the motion of the body.

5 The path of a particle satisfies the Cartesian equation $x^2 + (y+1)^2 = 1$. The position vector of the particle at time t, $t \geq 0$ could be given by which of the following equations?

A $\boldsymbol{r}(t) = (\cos(t)-1)\hat{\boldsymbol{i}} + \sin(t)\hat{\boldsymbol{j}}$

B $\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + (\sin(t)+1)\hat{\boldsymbol{j}}$

C $\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + (\cos(t)+1)\hat{\boldsymbol{j}}$

D $\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + (\cos(t)-1)\hat{\boldsymbol{j}}$

6 The position vector of a particle at time t is given by $\boldsymbol{r}(t) = \sin^2(t)\hat{\boldsymbol{i}} - 3\cos^2(t)\hat{\boldsymbol{j}}$, $0 \leq t \leq \dfrac{\pi}{2}$. The path along which the particle moves from $(0,-3)$ to $(1,0)$ is:

A straight

B circular

C elliptical

D hyperbolic

7 Two bodies, A and B, commence motion at time $t = 0$. At time t, their respective position vectors, $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$, are given by $\boldsymbol{r}_A = (6-t)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_B = (2t-5)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$. Determine whether the two bodies collide. If they do collide, calculate the time and location of the collision.

3 8 The position vector of a particle at time t is given by $\boldsymbol{r}(t) = \cos(\pi t)\hat{\boldsymbol{i}} + \cos(2\pi t)\hat{\boldsymbol{j}}$, $t \geq 0$. Determine the Cartesian equation of the particle's path and sketch the graph.

4 9 The position vector of a body at time t is given by $\boldsymbol{r}(t) = \left(2 + 2\cos(\pi t)\right)\hat{\boldsymbol{i}} + \left(1 + 2\sin(\pi t)\right)\hat{\boldsymbol{j}}$.

(a) Determine the Cartesian equation of the body's path and sketch the graph.

(b) Describe the motion of the body.

5 10 A particle moves so that its position vector at time t is given by $\boldsymbol{r}(t) = \cos(t)\hat{\boldsymbol{i}} + 2\sin(t)\hat{\boldsymbol{j}}$, $0 \leq t \leq \pi$.

(a) Determine the Cartesian equation of the body's path.

(b) Sketch the path and determine the particle's initial position and direction of motion.

11 The position vector of a particle at time t is given by $\boldsymbol{r}(t) = 6\sec(t)\hat{\boldsymbol{i}} + 4\tan(t)\hat{\boldsymbol{j}}$, $t \geq 0$. 6

(a) Determine the Cartesian equation of the particle's path.

(b) Sketch the path of the particle on the Cartesian plane.

12 The paths of two particles with respect to time t are described by the vector equations $\boldsymbol{r}_1(t) = (4t-5)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_2(t) = (t+1)\hat{\boldsymbol{i}} + (7t-10)\hat{\boldsymbol{j}}$. 7

(a) Determine the coordinates of the point at which the two particles collide.

(b) Calculate the coordinates of the points at which the two paths cross.

(c) Calculate the distance between the particles when $t = 1$.

13 The position vectors of two particles, A and B, are defined by the following equations:

$\boldsymbol{r}_A(t) = 5\hat{\boldsymbol{i}} + \left(t^2 - 6t + 10\right)\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, where $t \geq 0$

$\boldsymbol{r}_B(t) = 5\hat{\boldsymbol{i}} + \left(pt - 6\right)\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, where $t \geq 0$

It is known that the particles collide *only once* during their journey. Calculate the possible value(s) of p.

14 The position vectors of two particles, A and B, are defined by the following equations:

$\boldsymbol{r}_A(t) = \left(a\cos(t)\right)\hat{\boldsymbol{i}} + \left(b\sin(t)\right)\hat{\boldsymbol{j}}$, where $t \in [0, 2\pi]$

$\boldsymbol{r}_B(t) = \left(b\cos(t)\right)\hat{\boldsymbol{i}} + \left(a\sin(t)\right)\hat{\boldsymbol{j}}$, where $t \in [0, 2\pi]$

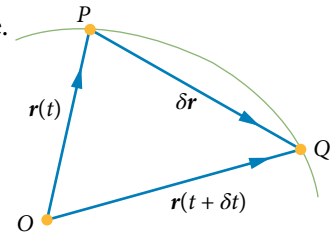
Determine the point(s) at which the paths of these particles intersect, in terms of a and b.

3.2 Differentiation of vector functions

Consider a function f that is differentiable throughout its domain.

The derivative of f is defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

By analogy, you can define the derivative of a vector function of a real variable.



Suppose $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+\delta t)$ are position vectors of neighbouring points P and Q on the smooth continuous curve defined by $\boldsymbol{r}(t)$.

By the triangle law of vector addition:

$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

$\overrightarrow{PQ} = \delta \boldsymbol{r}$

$$\begin{aligned}\delta \boldsymbol{r} &= \overrightarrow{PQ} \\ &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)\end{aligned}$$

So $\frac{\delta \boldsymbol{r}}{\delta t} = \frac{\boldsymbol{r}(t+\delta t) - r(t)}{\delta t}$, $\delta t \neq 0$.

Provided that the limit exists, the derivative $\frac{d\boldsymbol{r}}{dt}$ is defined as:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t}\end{aligned}$$

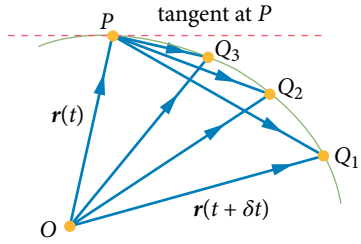
Re-expressing $\frac{\delta \boldsymbol{r}}{\delta t}$ as $\frac{1}{\delta t}(\delta \boldsymbol{r})$, see that $\frac{\delta \boldsymbol{r}}{\delta t}$ is a vector parallel to $\delta \boldsymbol{r}$. i.e. $\frac{\delta \boldsymbol{r}}{\delta t}$ is a vector whose direction is along $\overrightarrow{PQ}$.

By analogy with real (scalar) calculus, $\frac{d\boldsymbol{r}}{dt}$ is denoted as the rate of change of $\boldsymbol{r}$ with respect to t at the point P.

As $\delta t \to 0$, Q approaches P (shown by $Q_1, Q_2, Q_3 \ldots$).

As $\delta t \to 0$, $\boldsymbol{r}(t+\delta t) \to \boldsymbol{r}(t)$ and so the direction of $\frac{\delta \boldsymbol{r}}{\delta t}$ successively approaches the tangent at P.

Therefore, a geometric interpretation for $\frac{d\boldsymbol{r}}{dt}$ has been established.



Rules for differentiation of vector functions

Vector functions can be differentiated in a similar manner to real functions.

Derivative of a constant vector

This result can be verified as follows:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{c} - \boldsymbol{c}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{0}}{\delta t} \\ &= \boldsymbol{0}\end{aligned}$$

Note that $\frac{d\hat{\boldsymbol{i}}}{dt} = \frac{d\hat{\boldsymbol{j}}}{dt} = \frac{d\hat{\boldsymbol{k}}}{dt} = \boldsymbol{0}$.

If $\boldsymbol{r} = \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector, then $\frac{d\boldsymbol{r}}{dt} = 0$.

Derivative of a product of a scalar function and a constant vector *c*

This result can be verified as follows:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{s(t+\delta t)\boldsymbol{c} - s(t)\boldsymbol{c}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\left[s(t+\delta t) - s(t)\right]\boldsymbol{c}}{\delta t} \\ &= \frac{ds}{dt}\boldsymbol{c}\end{aligned}$$

If $\boldsymbol{r} = s\boldsymbol{c}$, where s is a function of t and $\boldsymbol{c}$ is a constant vector, then $\frac{d\boldsymbol{r}}{dt} = \frac{ds}{dt}\boldsymbol{c}$.

Derivative of a sum of vectors

This result can be verified as follows:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\boldsymbol{u}(t+\delta t) + \boldsymbol{v}(t+\delta t) - \left[\boldsymbol{u}(t) + \boldsymbol{v}(t)\right]}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\left[\boldsymbol{u}(t+\delta t) - \boldsymbol{u}(t)\right] + \left[\boldsymbol{v}(t+\delta t) - \boldsymbol{v}(t)\right]}{\delta t} \\ &= \frac{d\boldsymbol{u}}{dt} + \frac{d\boldsymbol{v}}{dt}\end{aligned}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are functions of t, and if $\boldsymbol{r} = \boldsymbol{u} + \boldsymbol{v}$, then $\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{u}}{dt} + \frac{d\boldsymbol{v}}{dt}$.

Using the last two rules and the fact that $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are constant vectors, gives:

If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then $\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\hat{\boldsymbol{i}} + \frac{dy}{dt}\hat{\boldsymbol{j}} + \frac{dz}{dt}\hat{\boldsymbol{k}}$ and $\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\boldsymbol{i}} + \frac{d^2y}{dt^2}\hat{\boldsymbol{j}} + \frac{d^2z}{dt^2}\hat{\boldsymbol{k}}$.
If $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$, then $\boldsymbol{r}'(t) = x'(t)\hat{\boldsymbol{i}} + y'(t)\hat{\boldsymbol{j}} + z'(t)\hat{\boldsymbol{k}}$ and $\boldsymbol{r}''(t) = x''(t)\hat{\boldsymbol{i}} + y''(t)\hat{\boldsymbol{j}} + z''(t)\hat{\boldsymbol{k}}$.

Hence, a vector expressed in component form can be differentiated term by term.

8 Differentiating a vector function

Determine $\frac{d\boldsymbol{r}}{dt}$ and $\frac{d^2\boldsymbol{r}}{dt^2}$ if $\boldsymbol{r} = 3t^2\hat{\boldsymbol{i}} + \left(4t - t^3\right)\hat{\boldsymbol{j}} + \sin(5t)\hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Differentiate each component of $\boldsymbol{r}$ to determine $\frac{d\boldsymbol{r}}{dt}$.	$\boldsymbol{r} = 3t^2\hat{\boldsymbol{i}} + \left(4t - t^3\right)\hat{\boldsymbol{j}} + \sin(5t)\hat{\boldsymbol{k}}$ $\frac{d\boldsymbol{r}}{dt} = 6t\hat{\boldsymbol{i}} + \left(4 - 3t^2\right)\hat{\boldsymbol{j}} + 5\cos(5t)\hat{\boldsymbol{k}}$

2 Differentiate each component of $\frac{d\boldsymbol{r}}{dt}$ to determine $\frac{d^2\boldsymbol{r}}{dt^2}$.	$\frac{d\boldsymbol{r}}{dt} = 6t\hat{\boldsymbol{i}} + \left(4 - 3t^2\right)\hat{\boldsymbol{j}} + 5\cos(5t)\hat{\boldsymbol{k}}$ $\frac{d^2\boldsymbol{r}}{dt^2} = 6\hat{\boldsymbol{i}} - 6t\hat{\boldsymbol{j}} - 25\sin(5t)\hat{\boldsymbol{k}}$

9 Differentiating and substituting a value into a vector function

Determine the value of $\boldsymbol{r}'(0)$ and $\boldsymbol{r}''(0)$ if $\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}} + e^{-2t}\hat{\boldsymbol{k}}$.

THINKING	WORKING
1 Differentiate each component of $\boldsymbol{r}(t)$ to determine $\boldsymbol{r}'(t)$.	$\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}} + e^{-2t}\hat{\boldsymbol{k}}$ $\boldsymbol{r}'(t) = \cos(t)\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}} - 2e^{-2t}\hat{\boldsymbol{k}}$
2 Substitute $t = 0$ into $\boldsymbol{r}'(t)$.	$\boldsymbol{r}'(0) = \cos(0)\hat{\boldsymbol{i}} + 4(0)\hat{\boldsymbol{j}} - 2e^{-2(0)}\hat{\boldsymbol{k}}$ $= \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$
3 Differentiate each component of $\boldsymbol{r}'(t)$ to determine $\boldsymbol{r}''(t)$.	$\boldsymbol{r}'(t) = \cos(t)\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}} - 2e^{-2t}\hat{\boldsymbol{k}}$ $\boldsymbol{r}''(t) = -\sin(t)\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4e^{-2t}\hat{\boldsymbol{k}}$
4 Substitute $t = 0$ into $\boldsymbol{r}''(t)$.	$\boldsymbol{r}''(0) = -\sin(0)\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4e^{-2(0)}\hat{\boldsymbol{k}}$ $= 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$
5 Write the answer.	$\boldsymbol{r}'(0) = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$ $\boldsymbol{r}''(0) = 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

Consider a curve described by the vector equation $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$. The gradient of the curve at the point (x, y) can be found using related rates, namely:

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$, where $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$.

Worked example 10 illustrates this.

10 Calculating the gradient at a point on a curve defined by a vector function

A curve is described by the vector equation $\boldsymbol{r}(t) = 2t\hat{\boldsymbol{i}} + (t+2)\hat{\boldsymbol{j}}, t \geq 0$.

(a) Calculate the gradient of the curve at the point when $t = 2$.

THINKING	WORKING
1 Write and number the parametric equations.	$x = 2t$ [1] $y = t + 2$ [2]

2 Calculate $\frac{dx}{dt}$ and $\frac{dy}{dt}$.	$\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 1$
3 Use the chain rule $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ to determine $\frac{dy}{dx}$ where $t = 2$.	$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ $= 1 \times \frac{1}{2}$ $= \frac{1}{2}$
4 Interpret the answer.	The gradient of the curve at $t = 2$ is $\frac{1}{2}$.

(b) Calculate $\boldsymbol{r}'(t)$.

Differentiate each component of $\boldsymbol{r}(t)$ to determine $\boldsymbol{r}'(t)$.	$\boldsymbol{r}(t) = 2t\hat{\boldsymbol{i}} + (t+2)\hat{\boldsymbol{j}}$ $\boldsymbol{r}'(t) = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}, t \geq 0$

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve at the point when $t = 2$.

1 Use $\hat{\boldsymbol{s}} = \frac{\boldsymbol{s}}{\lvert\boldsymbol{s}\rvert}$.	$\boldsymbol{s} = \boldsymbol{r}'(t)$ then $\lvert\boldsymbol{s}\rvert = \sqrt{2^2 + 1^2} = \sqrt{5}$
2 Write the answer.	$\hat{\boldsymbol{s}} = \frac{\boldsymbol{s}}{\lvert\boldsymbol{s}\rvert}$ $= \frac{\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)}{\sqrt{5}}$ $= \frac{1}{\sqrt{5}}\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)$

EXERCISE 3.2 Differentiation of vector functions

Worked Example

1 Calculate $\frac{d\boldsymbol{r}}{dt}$ and $\frac{d^2\boldsymbol{r}}{dt^2}$ for each of the following vector functions: 8

(a) $\boldsymbol{r} = 5t^2\hat{\boldsymbol{i}} - 2t^3\hat{\boldsymbol{j}} - t\hat{\boldsymbol{k}}$

(b) $\boldsymbol{r} = 4\sin(2t)\hat{\boldsymbol{i}} + 3\cos(2t)\hat{\boldsymbol{j}} + t^2\hat{\boldsymbol{k}}$

2 Calculate $\boldsymbol{r}'(\alpha)$ and $\boldsymbol{r}''(\alpha)$ for each of the following vector functions and values of α: 9

(a) $\boldsymbol{r}(t) = -\sin(2t)\hat{\boldsymbol{i}} + \cos(2t)\hat{\boldsymbol{j}} + e^{-t}\hat{\boldsymbol{k}}, \alpha = 0$

(b) $\boldsymbol{r}(t) = e^{-t}\hat{\boldsymbol{i}} + e^{t}\hat{\boldsymbol{j}} + \left(t^3 - 1\right)\hat{\boldsymbol{k}}, \alpha = 0$

(c) $\boldsymbol{r}(t) = 10t\hat{\boldsymbol{i}} + \left(10\sqrt{3} - 5t^2\right)\hat{\boldsymbol{j}}, \alpha = 2$

(d) $\boldsymbol{r}(t) = \sec(t)\hat{\boldsymbol{i}} + \tan(t)\hat{\boldsymbol{j}}, \alpha = \frac{\pi}{4}$

Worked Example

10

3 A curve is described by the vector equation $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + \frac{1}{t}\hat{\boldsymbol{j}}$, $t > 0$.

(a) Calculate the gradient of the curve at the point when $t = 1$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = 1$.

4 A curve is described by the vector equation $\boldsymbol{r}(t) = \left(t^2 + t\right)\hat{\boldsymbol{i}} + \left(t^2 - t\right)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Calculate the gradient of the curve at the point when $t = 0$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = 0$.

5 A curve is described by the vector equation $\boldsymbol{r}(t) = \cos(2t)\hat{\boldsymbol{i}} + \cos(t)\hat{\boldsymbol{j}}$, $0 \leq t \leq 2\pi$.

(a) Calculate the gradient of the curve at the point when $t = \frac{\pi}{3}$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = \frac{\pi}{3}$.

6 For each of the following vector functions, calculate $\frac{d\boldsymbol{r}}{dt}$ and determine any restrictions on t.

(a) $\boldsymbol{r} = t\left(\sin(t)\hat{\boldsymbol{i}} + \cos(t)\hat{\boldsymbol{j}}\right)$ (b) $\boldsymbol{r} = t\left(\log_e(t)\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}}\right)$

7 A curve is described by the vector equation $\boldsymbol{r}(t) = (5t + 2)\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}}$, $t \in \mathbb{R}$. The gradient of the curve at (x, y) is:

A 3 B 5 C $\frac{3}{5}$ D $\frac{5}{3}$

8 A curve is described by the vector equation $\boldsymbol{r}(t) = \cos^3(t)\hat{\boldsymbol{i}} + \sin^3(t)\hat{\boldsymbol{j}}$, $0 \leq t \leq 2\pi$.

(a) Calculate the gradient of the curve at the point when $t = \frac{2\pi}{3}$.

(b) Calculate $\boldsymbol{r}'(t)$.

(c) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = \frac{2\pi}{3}$.

9 A curve is described by the vector equation $\boldsymbol{r}(t) = e^{2t}\hat{\boldsymbol{i}} + \sin(2t)\hat{\boldsymbol{j}}$, $t \geq 0$. Calculate the corresponding time t at which the gradient of this curve is equal to $\frac{1}{2}$.

10 A curve C is defined by the parametric equations $x = 1 + t$ and $y = t^2$, $t \geq 0$.

(a) Determine the Cartesian equation of C and determine its domain.

(b) Sketch the graph of C.

(c) Determine the vector equation of C.

(d) Determine a unit vector, $\hat{\boldsymbol{s}}$, parallel to the tangent to the curve where $t = 1$.

11 The position vector of a particle at time t, is given by $\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}} + \cos(t)\hat{\boldsymbol{k}}$. Prove that $\boldsymbol{r}'(t)$ is perpendicular to $\boldsymbol{r}''(t)$.

12 The position vector of a particle at time t is given by $\boldsymbol{r}(t) = 4t\hat{\boldsymbol{i}} + 4t^2\hat{\boldsymbol{j}}$, $t \geq 0$. Determine when the magnitude of the angle between $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$ is $\frac{\pi}{4}$.

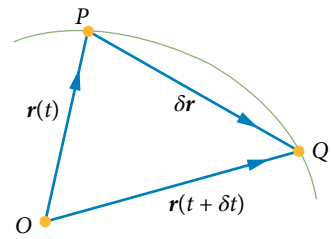
3.3 Velocity and acceleration vectors

You have learned how a moving particle can be defined by a position vector, in terms of time, t. You have also learned how it is possible to differentiate a vector function.

In your study of instantaneous rates of change, you will be aware of the relationship between the process of differentiation and the calculation of velocity and acceleration for rectilinear motion. Vectors offer a useful method for extending these calculations to objects moving in three dimensions.

Velocity vector

Consider again the vector $\boldsymbol{r}(t)$, which represents the position of a particle at point P at a certain time, t. After a short time, δt, the particle is at Q with position vector $\boldsymbol{r}(t+\delta t)$.



The displacement (or change in position) of the particle during the time interval δt is:

$$\delta \boldsymbol{r} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)$$

Since velocity is the time ratio of change of displacement, the *average velocity* of the particle during this time interval is:

$$\frac{\delta \boldsymbol{r}}{\delta t} = \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t},\ \delta t \neq 0$$

Re-expressing $\frac{\delta \boldsymbol{r}}{\delta t}$ as $\frac{1}{\delta t}(\delta \boldsymbol{r})$, you will see that $\frac{\delta \boldsymbol{r}}{\delta t}$ is a vector parallel to $\delta \boldsymbol{r}$. i.e. $\frac{\delta \boldsymbol{r}}{\delta t}$ is a vector whose direction is along $\overrightarrow{PQ}$.

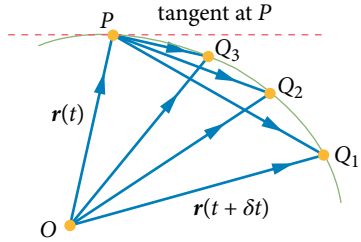
The instantaneous velocity of P at time t is therefore given by $\lim\limits_{\delta t \to 0} \dfrac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t}$.

By analogy with scalar calculus, this rate of change limit is denoted by $\frac{d\boldsymbol{r}}{dt}$ or $\dot{\boldsymbol{r}}$.

If $\boldsymbol{r}(t)$ defines the position vector of a particle P at time t, then the velocity at P is:

$$\boldsymbol{v} = \boldsymbol{r}'(t) = \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t}$$

The direction of $\boldsymbol{v}$ is the direction of $\overrightarrow{PQ}$ as $\delta t \to 0$. i.e. as $Q \to P$ (see following diagram).



Therefore, the instantaneous velocity of the particle, $\boldsymbol{v}$, is parallel to the path of the particle at P, in the direction of motion.

If $\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt}$, i.e. the velocity of the particle is the rate of change of position, then $v = |\boldsymbol{v}|$ is the speed of the particle.

11 Calculating the velocity vector

A particle moves so that its position vector at time t is given by $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (t-2)^2\,\hat{\boldsymbol{j}}$, $t \geq 0$.

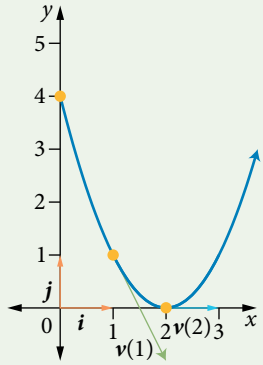
(a) Determine an expression for the velocity of the particle at time t.

THINKING	WORKING
Differentiate each component with respect to t to calculate $\boldsymbol{v}(t)$.	$\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (t-2)^2\,\hat{\boldsymbol{j}}$ $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ $\boldsymbol{v}(t) = \hat{\boldsymbol{i}} + (2t-4)\,\hat{\boldsymbol{j}}$

(b) Calculate the velocity of the particle at $t = 1$ and $t = 2$.

Substitute the given values of t into $\boldsymbol{v}(t)$.	$\boldsymbol{v}(t) = \hat{\boldsymbol{i}} + (2t-4)\,\hat{\boldsymbol{j}}$ $\boldsymbol{v}(1) = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ $\boldsymbol{v}(2) = \hat{\boldsymbol{i}}$

(c) Sketch the path of the particle and draw the velocity vectors at $t = 1$ and $t = 2$.

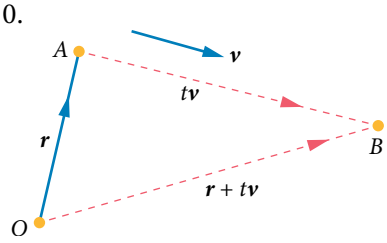
1 Write and number the parametric equations.	$x = t$ [1] $y = (t-2)^2$ [2]
2 Write the Cartesian equation with domain.	The Cartesian equation is $y = (x-2)^2$, $x \geq 0$.
3 Determine the location of the particle at $t = 1$ and $t = 2$. These will determine the location for the tails of the velocity vectors.	At $t = 1$: $\boldsymbol{r}(1) = 1\hat{\boldsymbol{i}} + (1-2)^2\,\hat{\boldsymbol{j}}$ $= 1\hat{\boldsymbol{i}} + 1\hat{\boldsymbol{j}}$ At $t = 2$: $\boldsymbol{r}(1) = 2\hat{\boldsymbol{i}} + (2-2)^2\,\hat{\boldsymbol{j}}$ $= 2\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$
4 Sketch the parabola with correct domain and velocity vectors at $t = 1$ and $t = 2$.	 The velocity vectors are tangential to the particle's path at the respective points.

Calculations involving position, velocity and acceleration vectors

In the following figure, a particle is at A with position vector $\boldsymbol{r}$ at time $t = 0$.

If the particle is moving with *constant* velocity $\boldsymbol{v}$, then it is moving in a straight line and in time interval t it undergoes a displacement $t\boldsymbol{v}$.

Hence at time t it has position vector $\overrightarrow{OB} = \boldsymbol{r} + t\boldsymbol{v}$.



Acceleration is the rate of change of velocity with respect to time. Calculation of acceleration from the velocity vector is somewhat similar to the calculation of velocity from the position vector, as previously described.

Unless the particle is moving in a straight line, $\boldsymbol{a}$ is *not* parallel to $\boldsymbol{v}$.

Making connections

Observe the vector direction and acceleration of a particle

Move the sliders to observe how the vector acceleration changes.

$$\boldsymbol{a} = \boldsymbol{v}'(t) = \dot{\boldsymbol{v}} = \frac{d\boldsymbol{v}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{v}(t+\delta t) - \boldsymbol{v}(t)}{\delta t} \text{ or, equivalently, } \boldsymbol{a} = \boldsymbol{r}''(t) = \ddot{\boldsymbol{r}} = \frac{d^2\boldsymbol{r}}{dt^2}$$

To calculate the position vector $\boldsymbol{r}$ given $\ddot{\boldsymbol{r}}$, anti-differentiate once to determine $\dot{\boldsymbol{r}}$ and then again to get $\boldsymbol{r}$.

Like real (scalar) functions, vector functions can be anti-differentiated.

For anti-differentiating vectors expressed in component form, use the following rule:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \frac{d}{dt}\left(\int x\,dt\right)\hat{\boldsymbol{i}} + \frac{d}{dt}\left(\int y\,dt\right)\hat{\boldsymbol{j}} + \frac{d}{dt}\left(\int z\,dt\right)\hat{\boldsymbol{k}} + \frac{d\boldsymbol{c}}{dt} \\ &= x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}} + \boldsymbol{0} \\ &= x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}\end{aligned}$$

If $\dfrac{d\boldsymbol{r}}{dt} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then:

$$\begin{aligned}\boldsymbol{r} &= \int\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}\right)dt \\ &= \left(\int x\,dt\right)\hat{\boldsymbol{i}} + \left(\int y\,dt\right)\hat{\boldsymbol{j}} + \left(\int z\,dt\right)\hat{\boldsymbol{k}} + \boldsymbol{c}\end{aligned}$$

Note that when anti-differentiating a vector function, a *vector* constant ($\boldsymbol{c}$) is introduced, the value of c can be determined with additional information.

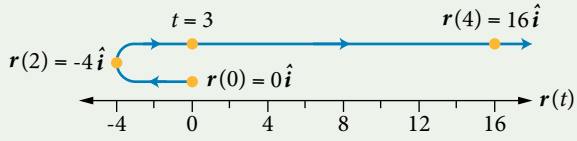
12 Determining a position vector from a velocity vector

The velocity of a particle at time t is given by $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} + (2t+2)\hat{\boldsymbol{j}}$, $t \geq 0$. Determine its position vector when $t = 1$, if $\boldsymbol{r} = \hat{\boldsymbol{i}}$ when $t = 0$.

THINKING	WORKING
1 Anti-differentiate each component with respect to t to calculate r.	$\dfrac{d\boldsymbol{r}}{dt} = \hat{\boldsymbol{i}} + (2t+2)\hat{\boldsymbol{j}}$ $\boldsymbol{r} = \int\left(\hat{\boldsymbol{i}} + (2t+2)\hat{\boldsymbol{j}}\right)dt$ $= t\hat{\boldsymbol{i}} + \left(t^2 + 2t\right)\hat{\boldsymbol{j}} + \boldsymbol{c}$
2 Apply the initial condition to determine $\boldsymbol{c}$.	$\boldsymbol{r}(0) = \hat{\boldsymbol{i}}$ and so $\boldsymbol{c} = \hat{\boldsymbol{i}}$.
3 Write the position vector $\boldsymbol{r}$.	$\boldsymbol{r} = (t+1)\hat{\boldsymbol{i}} + \left(t^2 + 2t\right)\hat{\boldsymbol{j}}$
4 Calculate $\boldsymbol{r}$ when $t = 1$.	$\boldsymbol{r}(1) = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

13 Motion in a straight line

(a) Sketch the position vector of a particle moving in a straight line is given as $\boldsymbol{r}(t) = \left(t^3 - 3t^2\right)\hat{\boldsymbol{i}}$ where t is the time in seconds for $0 \leq t \leq 4$ and $\boldsymbol{r}(t)$ represents the displacement in cm.

THINKING	WORKING
1 Determine the initial and final position of the particle.	$\boldsymbol{r}(0) = \left(0^3 - 3\times 0^2\right)\hat{\boldsymbol{i}} = 0\hat{\boldsymbol{i}}$ $\boldsymbol{r}(4) = \left(4^3 - 3\times 4^2\right)\hat{\boldsymbol{i}} = 16\hat{\boldsymbol{i}}$ Initially the particle has a displacement of 0 cm, after 4 seconds it is 16 cm to the right of where it began.
2 Differentiate the position vector to determine the velocity vector.	$\dot{\boldsymbol{r}}(t) = \left(3t^2 - 6t\right)\hat{\boldsymbol{i}}$
3 Determine the time when the particle is stationary.	The particle will be stationary when $\dot{\boldsymbol{r}}(t) = \boldsymbol{0}$: $0 = 3t^2 - 6t$ $0 = 3t(t-2)$ $t = 0$ or $t = 2$ The particle is stationary at $t = 0$ and again at $t = 2$, hence the particle changes directions 2 wseconds into the journey..
4 Determine the displacement of the particle when it is stationary.	From step 1, $\boldsymbol{r}(0) = 0\hat{\boldsymbol{i}}$ $\boldsymbol{r}(2) = \left(2^3 - 3\times 2^2\right)\hat{\boldsymbol{i}} = -4\hat{\boldsymbol{i}}$ The particle is 4 m to the left of the origin at $t = 2$.
5 Determine the time when the particle is at the origin for the second time.	Let $\boldsymbol{r}(t) = 0$ and solve for t: $0\hat{\boldsymbol{i}} = \left(t^3 - 3t^2\right)\hat{\boldsymbol{i}}$ $t^3 - 3t^2 = 0$ $t^2(t-3) = 0$ $t = 0$ or $t = 3$
6 Sketch the position of the particle.	
7 Describe the motion of the particle.	The particle travels with variable acceleration starting at the origin and travels 4 cm to the left over the first 2 seconds, then changes direction and travels 20 cm to the right in the next 2 seconds.

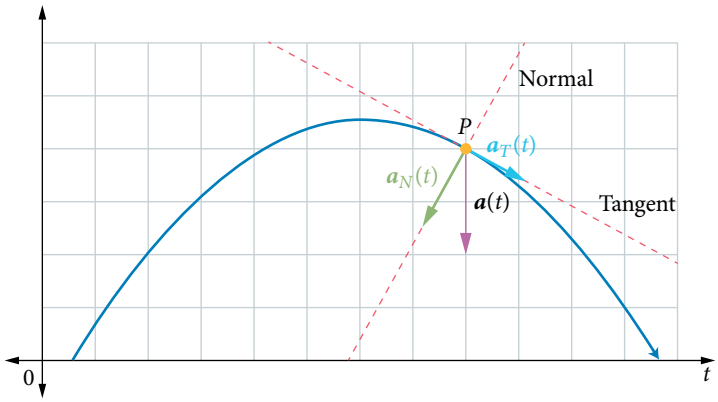
(b) A ball is dropped from a height of 2.8 m. Assuming the ball travels with a constant acceleration given by $\ddot{\boldsymbol{r}}(t) = -9.8\,\hat{\boldsymbol{j}}$ determine the time it takes for the ball to reach the ground correct to 2 decimal places.

1 Anti-differentiate the acceleration vector to determine the velocity vector.	$\dot{\boldsymbol{r}}(t) = \int \ddot{\boldsymbol{r}}(t)\,dt$ $= (-9.8t + c)\,\hat{\boldsymbol{j}}$ At $t = 0$, $\dot{\boldsymbol{r}}(t) = 0$ $\therefore c = 0$
2 Anti-differentiate the velocity vector to determine the position vector.	$\boldsymbol{r}(t) = \int \dot{\boldsymbol{r}}(t)\,dt$ $= \left(-4.9t^2 + d\right)\hat{\boldsymbol{j}}$ At $t = 0$, $\boldsymbol{r}(t) = 2.8$ $\therefore d = 2.8$
3 Determine the time taken to reach the ground by substituting the known value for $\boldsymbol{r}(t)$.	$\boldsymbol{r}(t) = \left(-4.9t^2 + 2.8\right)\hat{\boldsymbol{j}}$ Let $\boldsymbol{r}(t) = 0\,\hat{\boldsymbol{j}}$ and solve for t: $-4.9t^2 + 2.8 = 0$ $t^2 = \frac{2.8}{4.9}$ $t = \pm\sqrt{\frac{2.8}{4.9}}$ $t = \pm 0.75592...$
4 Interpret the answer.	Given that $t > 0$, then the ball takes approximately 0.76 seconds to reach the ground.

Technology worked example
Tangential and normal components of an acceleration vector

Resolving acceleration vectors into tangential and normal components

In general, $\boldsymbol{a}$ has a component *tangential* to the curve at P (the direction of $\boldsymbol{v}$) and a component *normal* to the curve at P. To understand how this works, consider the case of *projectile motion* (a form of motion you will study in greater depth later in this chapter). In simplified projectile motion, the only acceleration acting on a particle is due to gravity (and this acceleration acts in a downwards direction). The path traced out by the point is parabolic, and the acceleration vector $\boldsymbol{a}(t)$ can be split into the normal vector $\boldsymbol{a}_N(t)$, shown in green, and the tangential vector $\boldsymbol{a}_T(t)$, shown in blue:



Consider also a scenario where a car is travelling around a circular track at constant speed. Is the car accelerating?

By definition, the car is accelerating if its velocity is changing.

Although the magnitude of the car's velocity (its speed) is constant, the car is continually changing direction so that its velocity is changing. Hence, the car is accelerating.

14 Particles moving with a constant velocity vector

Two passenger ferries, *F* and *G*, travelling at constant velocities are observed from a clifftop at 11:00 am. Their position vectors, measured in kilometres and velocity vectors measured in km/h are:

$\boldsymbol{r}_F = 30\hat{\boldsymbol{i}} - 15\hat{\boldsymbol{j}}$ $\quad$ $\boldsymbol{v}_F = -10\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}}$

$\boldsymbol{r}_G = -10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ $\quad$ $\boldsymbol{v}_G = 10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

Determine whether the ferries will collide. Identify any assumptions you have made and the effects of these assumptions on your solution.

THINKING	WORKING
1 Use $\boldsymbol{r} + t\boldsymbol{v}$ to determine the position vectors of both ferries at time t hours after 11:00 am.	Ferry *F*: $\boldsymbol{r}_F = 30\hat{\boldsymbol{i}} - 15\hat{\boldsymbol{j}} + t\left(-10\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}}\right)$ $= (30 - 10t)\hat{\boldsymbol{i}} + (15t - 15)\hat{\boldsymbol{j}}$ Ferry *G*: $\boldsymbol{r}_G = -10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + t(10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}})$ $= (10t - 10)\hat{\boldsymbol{i}} + (5t + 5)\hat{\boldsymbol{j}}$
2 The two ferries will collide if there is a value of t (>0) such that $\boldsymbol{r}_F = \boldsymbol{r}_G$.	$(30 - 10t)\hat{\boldsymbol{i}} + (15t - 15)\hat{\boldsymbol{j}} = (10t - 10)\hat{\boldsymbol{i}} + (5t + 5)\hat{\boldsymbol{j}}$
3 Equate the components and solve for t.	$(30 - 10t)\hat{\boldsymbol{i}} = (10t - 10)\hat{\boldsymbol{i}}$ $\quad$ $(15t - 15)\hat{\boldsymbol{j}} = (5t + 5)\hat{\boldsymbol{j}}$ $20t = 40$ $\quad$ $10t = 20$ $t = 2$ $\quad$ $t = 2$
4 Write a conclusion.	$\boldsymbol{r}_F(2) = \boldsymbol{r}_G(2)$, hence the ferries will collide when $t = 2$. A collision will occur at 1:00 pm.
5 Identify any assumptions, and comment on the effects of these assumptions on your solution.	This model assumes there are no underlying currents that would affect the relative velocity of either ferry. It also assumes that the crew of both ferries are willing to continue travelling at a constant velocity despite the obvious dangers of an imminent collision. If these assumptions were incorrect, the ferries would not collide at 1:00 pm.

15 Calculating maximum and minimum speeds of a particle

At time t, a particle has position vector $\boldsymbol{r} = \big(\cos(t) + \cos(2t)\big)\hat{\boldsymbol{i}} + \big(\sin(t) + \sin(2t)\big)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine an expression for the speed of the particle.

THINKING	WORKING
1 Differentiate each component with respect to t to calculate $\dot{\boldsymbol{r}}$.	$\boldsymbol{r} = \big(\cos(t) + \cos(2t)\big)\hat{\boldsymbol{i}} + \big(\sin(t) + \sin(2t)\big)\hat{\boldsymbol{j}}$ $\dot{\boldsymbol{r}} = \big(-\sin(t) - 2\sin(2t)\big)\hat{\boldsymbol{i}} + \big(\cos(t) + 2\cos(2t)\big)\hat{\boldsymbol{j}}$
2 Recall that speed (a scalar) is the magnitude of the velocity vector. Use $v = \lvert\dot{\boldsymbol{r}}\rvert$ to determine the speed at time t.	$\dot{\boldsymbol{r}} = \big(-\sin(t) - 2\sin(2t)\big)\hat{\boldsymbol{i}} + \big(\cos(t) + 2\cos(2t)\big)\hat{\boldsymbol{j}}$ $v = \sqrt{\sin^2(t) + 4\sin(t)\sin(2t) + 4\sin^2(2t) + \cos^2(t) + 4\cos(t)\cos(2t) + 4\cos^2(2t)}$
3 Use the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to simplify v.	$v = \sqrt{\big(\sin^2(t) + \cos^2(t)\big) + 4\big(\sin^2(2t) + \cos^2(2t)\big) + 4\big(\sin(t)\sin(2t) + \cos(t)\cos(2t)\big)}$ $= \sqrt{1 + 4 + 4\big(\sin(t)\sin(2t) + \cos(t)\cos(2t)\big)}$
4 Use the trigonometric identity $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$ to simplify v.	$v = \sqrt{1 + 4 + 4\cos(2t - t)}$ $= \sqrt{5 + 4\cos(t)}$
5 Write v in terms of t.	$v = \sqrt{5 + 4\cos(t)}$

(b) Determine when the particle first attains its minimum and maximum speed.

1 The minimum speed occurs when $\cos(t) = -1$.	$v = \sqrt{5 + 4\cos(t)}$ The minimum value of v is 1 and occurs when $\cos(t) = -1$, i.e. when $t = \pi, 3\pi, \ldots$ It follows that the minimum speed is 1 and is first attained at $t = \pi$.
2 The maximum speed occurs when $\cos(t) = 1$.	The maximum value of v is $\sqrt{9} = 3$ and occurs when $\cos(t) = 1$, i.e. when $t = 0, 2\pi, \ldots$ It follows that the maximum speed is 3 and is first attained at $t = 0$.

16 Using vector calculus to describe the motion of a particle in two dimensions

The position vector of a particle at time t is $\boldsymbol{r} = \cos(2t)\hat{\boldsymbol{i}} + \sin(2t)\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $t \geq 0$.

(a) Calculate $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$.

THINKING	WORKING
1 Differentiate each component with respect to t to calculate $\dot{\boldsymbol{r}}$.	$\boldsymbol{r} = \cos(2t)\hat{\boldsymbol{i}} + \sin(2t)\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ $\dot{\boldsymbol{r}} = \left(\frac{d}{dt}\cos(2t)\right)\hat{\boldsymbol{i}} + \left(\frac{d}{dt}\sin(2t)\right)\hat{\boldsymbol{j}} + \left(\frac{d}{dt}1\right)\hat{\boldsymbol{k}}$ $= -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$
2 Differentiate each component with respect to t to calculate $\ddot{\boldsymbol{r}}$.	$\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$ $\ddot{\boldsymbol{r}} = -4\cos(2t)\hat{\boldsymbol{i}} - 4\sin(2t)\hat{\boldsymbol{j}}$

(b) Prove that the speed of the body is constant.

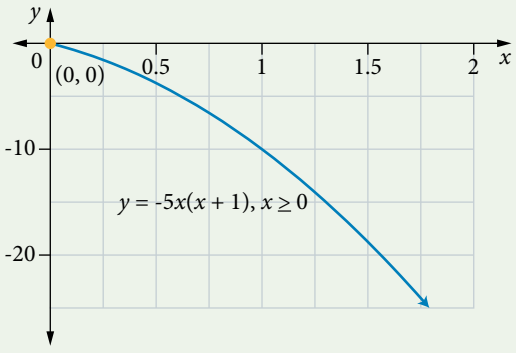
1 Calculate $\lvert\dot{\boldsymbol{r}}\rvert$.	$\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$ $\lvert\dot{\boldsymbol{r}}\rvert = \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2}$
2 Use the Pythagorean identity to simplify $\lvert\dot{\boldsymbol{r}}\rvert$.	$\lvert\dot{\boldsymbol{r}}\rvert = \sqrt{4\sin^2(2t) + 4\cos^2(2t)}$ $= \sqrt{4\left(\sin^2(2t) + \cos^2(2t)\right)}$ $= \sqrt{4}$ $= 2$
3 Write the conclusion.	As $\lvert\dot{\boldsymbol{r}}\rvert = 2$, the speed of the body is constant.

(c) Prove that the particle's acceleration is always perpendicular to its direction of motion.

1 Vectors are perpendicular if their dot product is equal to zero. Calculate $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}$.	$\ddot{\boldsymbol{r}} = -4\cos(2t)\hat{\boldsymbol{i}} - 4\sin(2t)\hat{\boldsymbol{j}}$ and $\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$ $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \left(-4\cos(2t)\hat{\boldsymbol{i}} - 4\sin(2t)\hat{\boldsymbol{j}}\right) \cdot \left(-2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}\right)$
2 Demonstrate that $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = 0$.	$\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \left(-4\cos(2t)\right)\left(-2\sin(2t)\right) + \left(-4\sin(2t)\right)\left(2\cos(2t)\right)$ $= 8\sin(2t)\cos(2t) - 8\sin(2t)\cos(2t)$ $= 0$
3 Write the conclusion.	So $\ddot{\boldsymbol{r}} \perp \dot{\boldsymbol{r}}$ since $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \boldsymbol{0}$ and $\ddot{\boldsymbol{r}}, \dot{\boldsymbol{r}} \neq \boldsymbol{0}$. Hence the particle's acceleration is always perpendicular to its direction of motion (the direction of $\dot{\boldsymbol{r}}$).

17 Sketching the path of a particle given its acceleration vector

The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}} = -10\hat{\boldsymbol{j}}$, $t \geq 0$. Sketch the path of the particle if $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{r} = 0$ when $t = 0$.

THINKING	WORKING
1 Anti-differentiate each component with respect to t to calculate $\dot{\boldsymbol{r}}$.	$\ddot{\boldsymbol{r}} = -10\hat{\boldsymbol{j}}$ $\dot{\boldsymbol{r}} = -10t\hat{\boldsymbol{j}} + \boldsymbol{c}$
2 Apply the initial condition to determine the constant vector $\boldsymbol{c}$.	At $t = 0$: $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ Hence $\boldsymbol{c} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$.
3 Anti-differentiate each component with respect to t to calculate $\boldsymbol{r}$.	$\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - (10t + 5)\hat{\boldsymbol{j}}$ $\boldsymbol{r} = t\hat{\boldsymbol{i}} - \left(5t^2 + 5t\right)\hat{\boldsymbol{j}} + \boldsymbol{d}$
4 Apply the initial condition to determine the constant vector $\boldsymbol{d}$.	At $t = 0$: $\boldsymbol{r} = \boldsymbol{0}$ Hence $\boldsymbol{d} = \boldsymbol{0}$.
5 Write the position vector $\boldsymbol{r}$.	$\boldsymbol{r} = t\hat{\boldsymbol{i}} - \left(5t^2 + 5t\right)\hat{\boldsymbol{j}}$
6 To sketch the path, convert the position vector into Cartesian form. Write and number the parametric equations.	$x = t$ [1] $y = -\left(5t^2 + 5t\right)$ [2]
7 Determine the possible values of x.	If $t \geq 0$ and $x = t$ then $x \geq 0$.
8 Substitute into the other parametric equation to eliminate t.	$y = -\left(5x^2 + 5x\right)$ So $y = -5x(x + 1)$.
9 Write the Cartesian equation with domain.	The Cartesian equation is $y = -5x(x + 1)$, $x \geq 0$.
10 Sketch the path of the particle with correct domain.	

Technology worked example
Sketching the path of a particle

EXERCISE 3.3 Velocity and acceleration vectors

Worked Example

11

1 Determine an expression for the velocity at time t for a particle that moves so that its position vector at time t is given by:

(a) $\boldsymbol{r} = (t+1)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ (b) $\boldsymbol{r} = t\hat{\boldsymbol{i}} + \left(t^2 - 1\right)\hat{\boldsymbol{j}}$

2 Determine an expression for the acceleration at time t for a particle that moves so that its velocity vector at time t is given by:

(a) $\boldsymbol{v} = \hat{\boldsymbol{i}} - 2t^2\hat{\boldsymbol{j}}$ (b) $\boldsymbol{v} = (15 - 10t)\hat{\boldsymbol{j}}$

3 Determine $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ for each of the following equations.

(a) $\boldsymbol{r} = 4t\hat{\boldsymbol{i}} - \left(t^2 - 2\right)\hat{\boldsymbol{j}} + 3t^2\hat{\boldsymbol{k}}$ (b) $\boldsymbol{r} = 26t\hat{\boldsymbol{i}} + 5\left(3t - t^2\right)\hat{\boldsymbol{j}}$

(c) $\boldsymbol{r} = 4\cos(2t)\hat{\boldsymbol{i}} + 3\sin(2t)\hat{\boldsymbol{j}}$ (d) $\boldsymbol{r} = e^t\hat{\boldsymbol{i}} + \left(e^{-2t} + 1\right)\hat{\boldsymbol{j}} - t^3\hat{\boldsymbol{k}}$

4 A particle moves so that its position vector at time t is given by $\boldsymbol{r} = 2\cos(t)\hat{\boldsymbol{i}} + 3\sin(t)\hat{\boldsymbol{j}} - 4t\hat{\boldsymbol{k}}$. Calculate the initial speed of the particle.

5 A particle moves so that at time $t \geq 0$, its position vector is given by $\boldsymbol{r} = \left(t^3 - t\right)\hat{\boldsymbol{i}} + (2t-1)^3\,\hat{\boldsymbol{j}}$. At $t = 1$, the magnitude of the particle's acceleration is:

A $2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ B $2\sqrt{10}$ C $6\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}}$ D $6\sqrt{17}$

12

6 Calculate the position vector $\boldsymbol{r}$ if:

(a) $\dot{\boldsymbol{r}} = 4t\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $\boldsymbol{r} = \hat{\boldsymbol{j}}$ at $t = 0$.

(b) $\dot{\boldsymbol{r}} = -2\sin(2t)\hat{\boldsymbol{i}} + 2\cos(2t)\hat{\boldsymbol{j}}$, $\boldsymbol{r} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ at $t = 0$.

(c) $\ddot{\boldsymbol{r}} = -10\hat{\boldsymbol{j}}$, $\boldsymbol{r} = \boldsymbol{0}$ and $\dot{\boldsymbol{r}} = 20\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$ at $t = 0$.

13

7 Sketch the graph of the following horizontal vectors using a number line.

(a) $\boldsymbol{r}(t) = \left(10t^3 - 5t - 4\right)\hat{\boldsymbol{i}}$ (b) $\boldsymbol{r}(t) = \left(4t^2 - 5t\right)\hat{\boldsymbol{i}}$ (c) $\boldsymbol{r}(t) = \left(-5t^3 + 3t^2\right)\hat{\boldsymbol{i}}$

8 An object is projected vertically upwards from ground level with an initial speed of 7 m/s, assuming the object experiences constant acceleration $\ddot{\boldsymbol{r}}(t) = -9.8\,\hat{\boldsymbol{j}}$ determine the maximum height reached by the object.

8 An object starting from rest, moves with constant acceleration of $3\hat{\boldsymbol{i}}$ m/s^2. If, on impact with another object, it is travelling at a velocity of $28\hat{\boldsymbol{i}}$ m/s, determine the displacement of the object when the collision occurred.

10 Cape gannets are often referred to as missile birds. They hunt for sardines with a diving motion with a maximum entry speed into the water of 115 km/h. If the birds plunge with $\ddot{\boldsymbol{r}}(t) = -32\,\hat{\boldsymbol{j}}$ m/s^2, determine the optimal height above the oceans surface that they should start their dive so that they enter the water with maximum speed.

11 Two snooker balls, the white ball and the black ball, simultaneously rebound slowly off adjacent cushions of a snooker table. Immediately after rebounding, their position and velocity vectors relative to a corner of the table are: 14

$\boldsymbol{r}_W = 0.5\hat{\boldsymbol{i}} \quad \boldsymbol{v}_W = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

$\boldsymbol{r}_B = 1.5\hat{\boldsymbol{j}} \quad \boldsymbol{v}_B = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

where distance is measured in metres and time in seconds.

(a) Prove that the paths of the two balls cross at right angles.

(b) Calculate how long after rebounding the balls collide.

(c) Identify any assumptions made in parts (a) and (b) and discuss the effects of these assumptions on your answer.

12 The position vector of a particle at time t is given by $\boldsymbol{r} = \sqrt{2}\cos(nt)\hat{\boldsymbol{i}} + \sin(nt)\hat{\boldsymbol{j}} + \sin(nt)\hat{\boldsymbol{k}}$, where n is a positive constant. 16

(a) Prove that the particle moves with constant speed.

(b) Prove that its acceleration is always perpendicular to its velocity.

13 A particle moves so that its position vector at time t is given by $\boldsymbol{r} = \left(e^{-t}\sin(t)\right)\hat{\boldsymbol{i}} + \left(e^{-t}\cos(t)\right)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine an expression for the particle's velocity.

(b) Determine an expression for the particle's speed.

(c) Determine an expression for the particle's acceleration.

14 At time t, a particle has position vector $\boldsymbol{r} = \left(3\sin(t) + \sin(2t)\right)\hat{\boldsymbol{i}} + \left(3\cos(t) - \cos(2t)\right)\hat{\boldsymbol{j}} + t\hat{\boldsymbol{k}}$, $t \geq 0$. 15

(a) Calculate the maximum and minimum speeds of the particle.

(b) Calculate the magnitude of its acceleration at $t = \dfrac{\pi}{2}$.

15 A particle moves so that at time $t > 0$, its position vector is given by $\boldsymbol{r} = \ln\left(t^2 + 2t\right)\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}}$. Calculate the particle's velocity vector at $t = 2$.

16 The acceleration of a particle moving in a plane is constant and is given by $\ddot{\boldsymbol{r}} = 18\hat{\boldsymbol{i}}$. At $t = 0$, $\dot{\boldsymbol{r}} = 6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{r} = \boldsymbol{0}$. Calculate the velocity and the position of the particle at $t = 1$. 17

17 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}} = e^{t}\hat{\boldsymbol{i}} + e^{-t}\hat{\boldsymbol{j}} + 6t\hat{\boldsymbol{k}}$. Calculate the particle's velocity and position at time t if $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{r} = 2\hat{\boldsymbol{j}}$ at $t = 0$.

18 The acceleration at time t seconds of a particle moving in a straight line is $\ddot{\boldsymbol{x}} = (4 - 6t)\hat{\boldsymbol{i}}$ m/s^2. The particle starts its motion 16 metres to the right of a fixed point O and moves away from O at a speed of 4 m/s.

(a) Determine the position where the particle is instantaneously at rest, relative to O.

(b) Determine when the particle passes through O.

19 A body moves from rest at the origin so that after t seconds its acceleration is given by $\ddot{\boldsymbol{r}} = 10\hat{\boldsymbol{i}} + e^{-0.1t}\hat{\boldsymbol{j}}$ m/s^2.

(a) Calculate its velocity after t seconds.

(b) Calculate its position vector at $t = 10$.

20 A particle moves so that its position vector at time t is $\boldsymbol{r} = t^2\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine an expression for the velocity at time t.

(b) Calculate the speed and direction of the particle at $t = 2$.

(c) Determine an expression for the acceleration at time t.

(d) Resolve the acceleration vector at $t = 1$ into tangential and normal components.

(e) At what angle to its direction of motion is the particle accelerating at $t = 0$ and $t = 1$?

21 A child is sitting still in some long grass watching a bee. The bee flies at constant speed in a straight line from its beehive to a flower and reaches the flower 3 seconds later. The position vector of the beehive relative to the child is $10\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ and the position vector of the flower relative to the child is $7\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$, where all distances are measured in metres.

(a) Calculate the speed of the bee.

(b) Calculate a unit vector in the direction of motion of the bee.

(c) Calculate the velocity of the bee.

(d) Calculate the closest distance that the bee comes to the child. Express your answer correct to one decimal place.

22 The velocity of a golf ball, t seconds, after being hit from a tee, is given by $\dot{\boldsymbol{r}} = 10\hat{\boldsymbol{i}} + (20 - 10t)\hat{\boldsymbol{j}}$ m/s where $\hat{\boldsymbol{i}}$ is a unit vector in the horizontal direction of the green and $\hat{\boldsymbol{j}}$ is a unit vector vertically upward.

(a) Calculate the initial speed and the angle of projection of the ball. Give your answers correct to one decimal place.

(b) Determine when the ball's velocity is horizontal.

(c) Taking the tee as the origin, determine an expression for the position vector $\boldsymbol{r}$ of the ball at time t seconds.

(d) Calculate the maximum height reached by the ball. Identify any assumptions and comment on the effect of these assumptions on your solution.

23 At time t, a particle has velocity $\boldsymbol{v} = 2\cos(t)\hat{\boldsymbol{i}} - 4\sin(t)\cos(t)\hat{\boldsymbol{j}}$, $t \geq 0$. At time $t = 0$, $\boldsymbol{r} = 3\hat{\boldsymbol{j}}$.

(a) Calculate the displacement of the particle at any time t.

(b) Calculate the displacement of the particle when it first comes to rest.

(c) Determine the Cartesian equation of the path and sketch the path of the particle.

(d) Determine the maximum speed attained by the particle.

Investigating curvilinear motion

3.4

Curvilinear motion refers to the motion of an object moving in a curved path (as opposed to straight-line, or *rectilinear* motion). Vector calculus is a handy tool for investigating such motion.

Circular motion

You were introduced to the concept of circular motion, as well as the parametric and vector equations describing such motion, earlier in the chapter.

- For a particle moving in a circle, with constant angular speed, the position vector can be defined by $\boldsymbol{r} = r\cos(nt)\hat{\boldsymbol{i}} + r\sin(nt)\hat{\boldsymbol{j}}$.
- The acceleration of a particle moving in a circle with constant angular speed is directed towards the centre of the circle.

18 Investigating circular motion

A particle's position vector at time t seconds is $\boldsymbol{r} = r\cos(nt)\hat{\boldsymbol{i}} + r\sin(nt)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Demonstrate that the particle moves with constant speed.

THINKING	WORKING
1 Differentiate each component with respect to t to determine $\dot{\boldsymbol{r}}$.	$\boldsymbol{r} = r\cos(nt)\hat{\boldsymbol{i}} + r\sin(nt)\hat{\boldsymbol{j}}$ $\dot{\boldsymbol{r}} = -rn\sin(nt)\hat{\boldsymbol{i}} + rn\cos(nt)\hat{\boldsymbol{j}}$
2 Use $v = \lvert\dot{\boldsymbol{r}}\rvert$ to determine the speed at time t.	$\dot{\boldsymbol{r}} = -rn\sin(nt)\hat{\boldsymbol{i}} + rn\cos(nt)\hat{\boldsymbol{j}}$ $v = \sqrt{(-rn\sin(nt))^2 + (rn\cos(nt))^2}$ $= \sqrt{(r^2n^2)} \times \sqrt{\sin^2(nt) + \cos^2(nt)}$ $= rn\sqrt{\sin^2(nt) + \cos^2(nt)}$ $= rn$ (which is constant as n is constant)

(b) Demonstrate that the acceleration vector is always perpendicular to the velocity vector.

1 Differentiate each component with respect to t to determine $\ddot{\boldsymbol{r}}$.	$\dot{\boldsymbol{r}} = -rn\sin(nt)\hat{\boldsymbol{i}} + rn\cos(nt)\hat{\boldsymbol{j}}$ $\ddot{\boldsymbol{r}} = -rn^2\cos(nt)\hat{\boldsymbol{i}} - rn^2\sin(nt)\hat{\boldsymbol{j}}$
2 Calculate $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}$.	$\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = r^2n^3\cos(nt)\sin(nt) - r^2n^3\sin(nt)\cos(nt)$ $= 0$
3 Write the conclusion.	The dot product is zero, therefore the acceleration vector is perpendicular to the velocity vector since $\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = 0$, and $\dot{\boldsymbol{r}}, \ddot{\boldsymbol{r}} \neq 0$.

(c) Demonstrate that the acceleration of the particle has constant magnitude and is directed towards the centre of the circle.

1 Express $\ddot{\boldsymbol{r}}$ in terms of $\boldsymbol{r}$.	$\ddot{\boldsymbol{r}} = -rn^2\cos(nt)\hat{\boldsymbol{i}} - rn^2\sin(nt)\hat{\boldsymbol{j}}$ $= -rn^2\left(\cos(nt)\hat{\boldsymbol{i}} + \sin(nt)\hat{\boldsymbol{j}}\right)$ $= -n^2\boldsymbol{r}$
2 Write the conclusion.	The acceleration is directed towards O, the centre of the circle, and has magnitude rn^2, which is constant.

Projectile motion

The motion of a body that is projected into the air is referred to as *projectile motion*. In a simplified projectile motion scenario, the only force acting upon the projected body is the force of gravity, which is a constant downwards force.

In two-dimensional simplified projectile motion, the acceleration vector of a particle is given by $\boldsymbol{a}(t) = -g\hat{\boldsymbol{j}}$, where g represents acceleration due to gravity, and where $\hat{\boldsymbol{j}}$ is a unit vector in the vertical direction. At sea level on Earth, $g \approx 9.8\,\text{m/s}^2$.

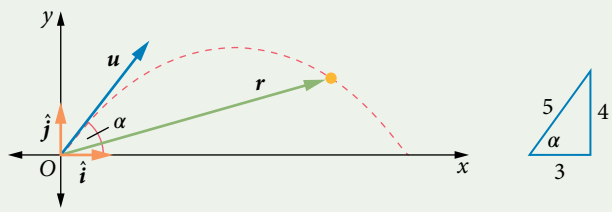
The acceleration vector can be anti-differentiated to determine vectors for velocity and position, using appropriate substitutions for initial velocity and position.

When investigating projectile motion, it is important to identify any assumptions you make and comment on the effects of those assumptions on your calculations. Typically, one major assumption you might make is to ignore the effects of air resistance or wind on the projected body. Air resistance is a retarding force, so ignoring this force may have the effect of over-estimating the horizontal range and maximum height reached by the projected body. Wind can of course have a significant effect on the motion of the projected body, as any golfer could attest.

19 Investigating projectile motion

A golf ball is struck down the middle of a fairway with an initial speed of 25 m/s at an angle of projection to the horizontal, α, where $\tan(\alpha) = \frac{4}{3}$.

(a) Taking unit vectors $\hat{\boldsymbol{i}}$ horizontally in the direction of motion and $\hat{\boldsymbol{j}}$ vertically upward, determine an expression for the initial velocity $\boldsymbol{u}$ m/s of the ball.

THINKING	WORKING
1 Draw a diagram.	

2 Express $\boldsymbol{u}$ in component form.	$\boldsymbol{u} = u\cos(\alpha)\hat{\boldsymbol{i}} + u\sin(\alpha)\hat{\boldsymbol{j}}$ $= 25\left(\frac{3}{5}\right)\hat{\boldsymbol{i}} + 25\left(\frac{4}{5}\right)\hat{\boldsymbol{j}}$ $= 15\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$
3 Write the answer.	$\boldsymbol{u} = 15\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$

(b) Taking the origin as the point of projection, determine an expression for the position vector $\boldsymbol{r}$ of the ball after t seconds.

1 The acceleration of the ball is g (gravity) and is acting in a downwards direction.	$\ddot{\boldsymbol{r}} = -g\hat{\boldsymbol{j}}$
2 Anti-differentiate each component with respect to t to calculate $\dot{\boldsymbol{r}}$.	$\dot{\boldsymbol{r}} = -gt\hat{\boldsymbol{j}} + \boldsymbol{c}$
3 Apply the initial condition to calculate $\boldsymbol{c}$.	At $t = 0$, $\dot{\boldsymbol{r}} = \boldsymbol{u} \Rightarrow \boldsymbol{c} = 15\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$.
4 Write the expression for $\dot{\boldsymbol{r}}$.	$\dot{\boldsymbol{r}} = 15\hat{\boldsymbol{i}} + (20 - gt)\hat{\boldsymbol{j}}$
5 Anti-differentiate each component with respect to t to calculate $\boldsymbol{r}$.	$\boldsymbol{r} = 15t\hat{\boldsymbol{i}} + \left(20t - \frac{1}{2}gt^2\right)\hat{\boldsymbol{j}} + \boldsymbol{d}$
6 Apply the initial condition to calculate $\boldsymbol{d}$.	At $t = 0$, $\boldsymbol{r} = \boldsymbol{0} \Rightarrow \boldsymbol{d} = \boldsymbol{0}$.
7 Write the expression for $\boldsymbol{r}$.	$\boldsymbol{r} = 15t\hat{\boldsymbol{i}} + \left(20t - \frac{1}{2}gt^2\right)\hat{\boldsymbol{j}}$

(c) Calculate the horizontal distance that the ball travels before hitting the ground. Give your answer correct to 1 decimal place and identify any assumptions you make (and comment on the effect of these assumptions on your answer).

1 The ball hits the ground when the $\hat{\boldsymbol{j}}$ component of the position vector, $\boldsymbol{r}$ is zero.	$20t - 4.9t^2 = 0$ $t(20 - 4.9t) = 0$ $t = 0$ or $t = \frac{20}{4.9}$ $= 4.08$ s (2 d.p.)
2 The horizontal distance that the ball has covered is the $\boldsymbol{i}$ component of $\boldsymbol{r}$ correct to 1 decimal place.	$x(\boldsymbol{r}) = 15(4.08)$ $= 61.2$ m (1 d.p.)
3 Interpret the answer.	The ball travels 61.2 m before hitting the ground.
4 Identify any assumptions and comment on the effect of those assumptions.	This answer ignores air resistance and assumes there is no wind. Air resistance would decrease the calculated value.

(d) Calculate the maximum height reached by the ball during its flight. Give your answer correct to 1 decimal place.

1 The ball reaches its maximum height when the $\hat{\boldsymbol{j}}$ component of $\dot{\boldsymbol{r}}$ is zero (i.e. the vertical component of velocity is zero).	$20 - 9.8t = 0$ $t = \dfrac{20}{9.8}$ $= 2.04\,\text{s}\,(2\text{ d.p.})$
2 Its height is given by the $\hat{\boldsymbol{j}}$ component of $\boldsymbol{r}$.	$h_{\max} = h(2.04)$ $= 2.04\left(20 - 4.9 \times \dfrac{20}{9.8}\right)$ $= 2.04 \times 10$ $= 20.4\text{ m}\,(1\text{ d.p.})$
3 Interpret the answer.	The ball reaches a maximum height of 20.4 m at 2.04 s.

e Explore further
Projectile motion
Explore the effects of initial velocity, both speed and angle, on a projectile.

c Making connections
Investigating projectile motion
Adjust time value to trace locus of projectile. Position, acceleration and velocity vectors have been drawn in.

t Technology worked example
Investigating projectile motion
Understand how the particle moves while changing the vector direction.

EXERCISE 3.4 Investigating curvilinear motion

Worked Example 18

1 A particle moves so that its position vector $\boldsymbol{r}$ at time t is given by $\boldsymbol{r} = 3\cos(2t)\hat{\boldsymbol{i}} + 3\sin(2t)\hat{\boldsymbol{j}}$, $t \geq 0$.
(a) Demonstrate that the particle moves in a circle and determine the Cartesian equation of its path.
(b) Demonstrate that the particle moves with constant speed.
(c) Demonstrate that the particle's acceleration has constant magnitude and is perpendicular to the direction of motion of the particle.

2 The position vector of a particle at time t seconds, $t \geq 0$, is $\boldsymbol{r} = \big(1 + \sin(4t)\big)\hat{\boldsymbol{i}} + \big(2 - \cos(4t)\big)\hat{\boldsymbol{j}}$ (units are metres).
(a) Determine the Cartesian equation of the path of the particle and sketch the result.
(b) Prove that its acceleration is always perpendicular to its velocity.

3 The position vector of a particle at time t is given by $\boldsymbol{r} = \sin(2t)\hat{\boldsymbol{i}} + \cos(2t)\hat{\boldsymbol{j}} + \sin(2t)\hat{\boldsymbol{k}}$.
(a) Which of the following will give the correct equation for the acceleration vector?
A $\boldsymbol{a} = -\boldsymbol{r}$ B $\boldsymbol{a} = -2\boldsymbol{r}$ C $\boldsymbol{a} = -4\boldsymbol{r}$ D $\boldsymbol{a} = 4\boldsymbol{v}$
(b) Explain the common error made by a student who selected the first incorrect option.

4 A particle is projected from the origin with an initial speed of V at an angle of α to the horizontal unit vector $\hat{\boldsymbol{i}}$. If $\hat{\boldsymbol{j}}$ is a unit vector in the vertical direction, then the vector equation for position at time t is:

A $\boldsymbol{r}(t) = V\cos(t)\hat{\boldsymbol{i}} + \left(V\sin(t) - \dfrac{gt^2}{2}\right)\hat{\boldsymbol{j}}$

B $\boldsymbol{r}(t) = Vt\cos(\alpha)\hat{\boldsymbol{i}} + \left(Vt\sin(\alpha) - \dfrac{gt^2}{2}\right)\hat{\boldsymbol{j}}$

C $\boldsymbol{r}(t) = V\sin(t)\hat{\boldsymbol{i}} + \left(V\cos(t) - \dfrac{gt^2}{2}\right)\hat{\boldsymbol{j}}$

D $\boldsymbol{r}(t) = V\cos(t)\hat{\boldsymbol{i}} + \left(V\sin(t) - 9.8t^2\right)\hat{\boldsymbol{j}}$

5 The position vector of a particle at time t seconds, $t \geq 0$, is given by $\boldsymbol{r} = 2\cos(3t)\hat{\boldsymbol{i}} + 2\sin(3t)\hat{\boldsymbol{j}} + 3t\hat{\boldsymbol{k}}$ (units are metres).

(a) Calculate the speed of the particle at time t.

(b) Calculate the magnitude of the acceleration at time t.

6 A particle moves so that its position vector at time t is given by $\boldsymbol{r} = 3\cos(t)\hat{\boldsymbol{i}} + 2\sin(t)\hat{\boldsymbol{j}}$, $0 \leq t \leq 2\pi$.

(a) Determine the Cartesian equation of the path of the particle and sketch the path.

(b) Determine when the velocity of the particle is perpendicular to its position vector and hence determine the position vectors at these times.

(c) Use technology to calculate the maximum and minimum speeds of the particle.

(d) Demonstrate that the particle's acceleration is directed towards the origin and is equal in magnitude to the particle's distance from the origin.

(e) Determine when the acceleration is perpendicular to the velocity.

Worked Example 19

7 A body is projected from level ground with a velocity of $7\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}}$ m/s, where $\hat{\boldsymbol{i}}$ is horizontal and $\hat{\boldsymbol{j}}$ is vertically up.

(a) Calculate the initial speed and angle of projection of the body. Give the angle of projection correct to the nearest tenth of a degree.

(b) Calculate the time of flight of the body. Give your answer correct to 2 decimal places.

(c) Calculate the horizontal distance travelled by the body. Give your answer correct to 1 decimal place. Identify any assumptions you have made and comment on the effect of these assumptions on your answer.

(d) Calculate the maximum height reached by the particle. Give your answer correct to 1 decimal place.

8 The velocity, in m/s, at time t seconds, of a ball hit from a height of 2 metres above ground level is given by $\dot{\boldsymbol{r}} = 12\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}} + (30 - 9.8t)\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are unit vectors in the east, north and vertically up directions respectively and the origin is at ground level. Calculate the height, above ground level in metres, after 2 seconds (to 1 decimal place).

9 A cricket ball is thrown from a height of 1 metre with a speed of 30 m/s and at an angle of 60° to the horizontal. Calculate the horizontal distance travelled by the ball before it lands. Identify any assumptions you have made and comment on the effect of the assumptions on your answer.

10 A baseball is hit horizontally with a speed of 200 km/h from a height of 1.225 m. The ball is hit directly towards a fielder who is standing 32 metres away in the outfield, and the fielder is able to move at 6 m/s along the ground. The fielder can catch the ball if it reaches them 'on the full' (without bouncing). Will the fielder take the catch in this case? Justify your response, identify any assumptions you have made, and comment on the effect of these assumptions on your answer.

11 A projectile is launched at an angle of 45° to the horizontal. Determine the launching speed, correct to 1 decimal place, necessary to achieve (relative to a horizontal plane through the point of projection):

(a) a horizontal range of 100 m. **(b)** a height of 30 m.

12 A particle is projected from level ground with a speed of 19.6 m/s at an angle of projection of α. (Ignore air resistance and assume there are no other forces).

(a) Prove that the particle reaches its maximum height halfway through its flight.

(b) Determine the angle α for which the range of the particle is a maximum, and determine this maximum range. Express the maximum range correct to 1 decimal place.

13 A particle is projected from the origin with initial speed u and angle of projection α.

(a) Demonstrate that the Cartesian equation of its path is $y = x\tan(\alpha) - \frac{gx^2}{2u^2}\sec^2(\alpha)$.

(b) Hence determine an expression for the projectile's range, R.

(c) Hence determine an expression for the projectile's maximum height, $y_{\max}$.

(d) Prove that for a given initial speed, the maximum range of the projectile occurs when $\alpha = 45°$ and is given by $R = \frac{u^2}{g}$.

14 A golf ball is hit from the ground with a velocity $20\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 15\hat{\boldsymbol{k}}$ m/s, where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are unit vectors horizontally forward, horizontally to the left, and vertically upwards, respectively. After being hit, the ball is under a gravitational acceleration of $-10\hat{\boldsymbol{k}}$ m/s^2, and also a 'hook' (i.e. a horizontal acceleration to the left) of $4\hat{\boldsymbol{j}}$ m/s^2.

(a) Determine the expression for $\boldsymbol{r}$, the position vector of the ball at time t.

(b) When the ball strikes the ground, calculate how far it will be to the left of the line along the horizontally forward direction.

(c) Determine when the speed of the ball is a minimum. Give your answer correct to 2 decimal places.

15 A golf ball is hit from a point O with initial velocity V at an angle θ to the horizontal ground. The ball first hits the ground at a point P which is at the same horizontal level as O.

(a) Given that $V\cos(\theta) = 6u$ and $V\sin(\theta) = \frac{5u}{2}$, determine, in terms of u and g, the time taken to travel from O to P.

(b) Determine the range, R, of the golf ball in terms of u and g.

(c) Express V in terms of u.

(d) Calculate in terms of u the golf ball's minimum speed during its flight.

Summary

The locus of a point

In geometry, a *locus* (plural: *loci*) is defined as a set of points satisfying one or more particular geometric conditions.

A circle can be defined as the locus of a point P on a plane such that P is a fixed distance (the radius) from a single fixed point O.

A straight line can be defined as the locus of a point P on a plane such that P is equidistant from two fixed points, A and B.

An ellipse can be defined as the locus of a point P on a plane such that the sum of its distances from two fixed points F_1 and F_2 is fixed. (The points F_1 and F_2 are known as the *foci* of the ellipse).

A hyperbola can be defined as the locus of a point P on a plane such that the absolute value of the difference of its distances from two given points F_1 and F_2 (the foci) is fixed.

Parametric and Cartesian equations

A particle's position at any time t can be described using a position vector as a function of time.

The position of any point P, relative to an origin O, is uniquely specified by the vector $\overrightarrow{OP}$, which is termed the *position vector* of P (relative to O).

If the coordinates of the variable point $P(x, y, z)$ are expressed parametrically as $x = f(t)$, $y = g(t)$ and $z = h(t)$, then the *vector equation* of the path is given by the position vector $\boldsymbol{r}(t) = f(t)\hat{\boldsymbol{i}} + g(t)\hat{\boldsymbol{j}} + h(t)\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are unit vectors in the positive directions of the x, y and z axes respectively.

The equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ are called *parametric equations* and t is called the *parameter*.

Circles

The Cartesian equation of a circle with radius r and centre (h, k) is $(x-h)^2 + (y-k)^2 = r^2$.

The parametric equations of a circle of radius r and centre at (h, k) are $x = h + r\cos(t)$ and $y = k + r\sin(t)$, where $t \in [0, 2\pi]$.

The vector equation of a circle of radius r and centre at (h, k) is $\boldsymbol{r}(t) = (h + r\cos(t))\hat{\boldsymbol{i}} + (k + r\sin(t))\hat{\boldsymbol{j}}$, where $t \in [0, 2\pi]$.

Ellipses

The Cartesian equation of an ellipse with centre at the point (h, k), horizontal major axis of $2a$ and vertical minor axis of $2b$ is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

The parametric equations of an ellipse centred at the origin are $x = a\cos(t)$ and $y = b\sin(t)$, where $t \in [0, 2\pi]$.

The parametric equations of an ellipse with its centre at (h, k) are $x = h + a\cos(t)$ and $y = k + b\sin(t)$.

The vector equation of an ellipse with its centre at (h, k) is $\boldsymbol{r}(t) = (h + a\cos(t))\hat{\boldsymbol{i}} + (k + b\sin(t))\hat{\boldsymbol{j}}$, where $t \in [0, 2\pi]$.

Hyperbolas

The Cartesian equation of a hyperbola with horizontal transverse axis of $2a$, vertical conjugate axis of $2b$ and centre (h, k), is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

The parametric equations of a hyperbola centred at the origin and with vertices $(\pm a, 0)$ are $x = a\sec(t)$ and $y = b\tan(t)$.

The parametric equations of a hyperbola with its centre at (h, k) are $x = h + a\sec(t)$ and $y = k + b\tan(t)$.

The vector equation of a hyperbola with its centre at (h, k) is $\boldsymbol{r}(t) = (h + a\sec(t))\hat{\boldsymbol{i}} + (k + b\tan(t))\hat{\boldsymbol{j}}$.

The parametric equations for the *conjugate hyperbola* centred at the origin and with vertices $(0, \pm b)$ are $x = a\tan(t)$ and $y = b\sec(t)$.

The parametric equations for the *conjugate hyperbola* with its centre at (h, k) are $x = h + a\tan(t)$ and $y = k + b\sec(t)$.

The vector equation of the *conjugate hyperbola* with its centre at (h, k) is $\boldsymbol{r}(t) = (h + a\cos(t))\hat{\boldsymbol{i}} + (k + b\sec(t))\hat{\boldsymbol{j}}$.

Colliding particles

Two particles will collide if they share the same position at the same time.

Differentiation of vector functions

Provided that the limit exists, the derivative $\dfrac{d\boldsymbol{r}}{dt}$ is defined as:

$$\frac{d\boldsymbol{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \boldsymbol{r}}{\delta t}$$
$$= \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t + \delta t) - \boldsymbol{r}(t)}{\delta t}$$

If $\boldsymbol{r} = \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector, then $\dfrac{d\boldsymbol{r}}{dt} = 0$.

If $\boldsymbol{r} = s\boldsymbol{c}$, where s is a function of t and $\boldsymbol{c}$ is a constant vector, then $\dfrac{d\boldsymbol{r}}{dt} = \dfrac{ds}{dt}\boldsymbol{c}$.

If $\boldsymbol{r} = \boldsymbol{u} + \boldsymbol{v}$, then $\dfrac{d\boldsymbol{r}}{dt} = \dfrac{d\boldsymbol{u}}{dt} + \dfrac{d\boldsymbol{v}}{dt}$.

If $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then $\dfrac{d\boldsymbol{r}}{dt} = \dfrac{dx}{dt}\hat{\boldsymbol{i}} + \dfrac{dy}{dt}\hat{\boldsymbol{j}} + \dfrac{dz}{dt}\hat{\boldsymbol{k}}$ and $\dfrac{d^2\boldsymbol{r}}{dt^2} = \dfrac{d^2x}{dt^2}\hat{\boldsymbol{i}} + \dfrac{d^2y}{dt^2}\hat{\boldsymbol{j}} + \dfrac{d^2z}{dt^2}\hat{\boldsymbol{k}}$.

If $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$, then $\boldsymbol{r}'(t) = x'(t)\hat{\boldsymbol{i}} + y'(t)\hat{\boldsymbol{j}} + z'(t)\hat{\boldsymbol{k}}$ and $\boldsymbol{r}''(t) = x''(t)\hat{\boldsymbol{i}} + y''(t)\hat{\boldsymbol{j}} + z''(t)\hat{\boldsymbol{k}}$. Hence, a vector expressed in component form can be differentiated term by term.

Velocity vector

If $\boldsymbol{r}(t)$ defines the position vector of a particle P at time t, then the velocity at P is:

$$\boldsymbol{v} = \boldsymbol{r}'(t) = \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t}$$

If $\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt}$, i.e. the velocity of the particle is the rate of change of position, then $v = |\boldsymbol{v}|$ is the speed of the particle.

Acceleration vector

$$\boldsymbol{a} = \boldsymbol{v}'(t) = \dot{\boldsymbol{v}} = \frac{d\boldsymbol{v}}{dt} = \lim_{\delta t \to 0} \frac{\boldsymbol{v}(t+\delta t) - \boldsymbol{v}(t)}{\delta t} \text{ or, equivalently, } \boldsymbol{a} = \boldsymbol{r}''(t) = \ddot{\boldsymbol{r}} = \frac{d^2\boldsymbol{r}}{dt^2}$$

Anti-differentiation of vector functions

If $\dfrac{d\boldsymbol{r}}{dt} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then:

$$\begin{aligned}\boldsymbol{r} &= \int \left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}\right) dt \\ &= \left(\int x\,dt\right)\hat{\boldsymbol{i}} + \left(\int y\,dt\right)\hat{\boldsymbol{j}} + \left(\int z\,dt\right)\hat{\boldsymbol{k}} + \boldsymbol{c}\end{aligned}$$

Circular motion

For a particle moving in a circle with constant angular speed, the position vector can be defined by $\boldsymbol{r} = r\cos(nt)\hat{\boldsymbol{i}} + r\sin(nt)\hat{\boldsymbol{j}}$.

The acceleration of a particle moving in a circle with constant angular speed is directed towards the centre of the circle.

Projectile motion

In two-dimensional simplified projectile motion, the acceleration vector of a particle is given by $\boldsymbol{a}(t) = -g\hat{\boldsymbol{j}}$, where g represents acceleration due to gravity, and where $\hat{\boldsymbol{j}}$ is a unit vector in the vertical direction. At sea level on Earth, $g \approx 9.8\,\text{m/s}^2$.

The acceleration vector can be anti-differentiated to determine vectors for velocity and position, using appropriate substitutions for initial velocity and position.

Chapter review 3

Exercise 3.1

1 The position of a body at time t is given by $\boldsymbol{r}(t) = (t-2)\hat{\boldsymbol{i}} + 4(t-2)^2\,\hat{\boldsymbol{j}}$, $t \geq 0$. Determine the Cartesian equation of the body's path.

Exercise 3.2

2 If $\boldsymbol{r}(t) = \hat{\boldsymbol{i}} - 3t^2\,\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$, determine an expression for $\ddot{\boldsymbol{r}}(t)$.

Exercise 3.2

3 A particle moves so that its position vector at time t is $\boldsymbol{r}(t) = 4\sin(2t)\hat{\boldsymbol{i}} - 3\cos(2t)\,\hat{\boldsymbol{j}}$, $t \geq 0$. Calculate the initial speed of the particle.

Exercise 3.3

4 A particle moves so that its position vector at time t is $\boldsymbol{r}(t) = \left(t^2+1\right)\hat{\boldsymbol{i}} + \ln(2t+3)\,\hat{\boldsymbol{j}}$, $t \geq 0$. Determine the particle's acceleration vector.

Exercise 3.3

5 A particle starts at position $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and moves so that its velocity at time t is $2t\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$. The position vector of the particle at time t is given by:

A $2\hat{\boldsymbol{i}}$ **B** $t^2\hat{\boldsymbol{i}} - t\hat{\boldsymbol{j}}$ **C** $\left(t^2-1\right)\hat{\boldsymbol{i}} + (1-t)\,\hat{\boldsymbol{j}}$ **D** $\left(t^2+1\right)\hat{\boldsymbol{i}} - (t+1)\,\hat{\boldsymbol{j}}$

Exercise 3.3

6 A particle moves so that its position vector at time t is $\boldsymbol{r}(t) = \left(t^2-1\right)\hat{\boldsymbol{i}} + \left(t^4 - 2t^3\right)\hat{\boldsymbol{j}}$, $t \geq 0$. Calculate the particle's acceleration vector at $t = 1$.

Exercise 3.3

7 The velocity and acceleration of a particle are parallel for:

A all motion **B** straight-line motion only

C circular motion only **D** projectile motion only

Exercise 3.4

8 The position vector of a particle moving in a circle is given by $\boldsymbol{r}(t) = \cos(3t)\hat{\boldsymbol{i}} + \sin(3t)\,\hat{\boldsymbol{j}}$. Determine the acceleration of the particle in terms of $\boldsymbol{r}$.

Exercise 3.4

9 A projectile is launched with a speed of $10\sqrt{3}$ m/s at an angle of 60° to the horizontal. If $\hat{\boldsymbol{i}}$ is horizontal and $\hat{\boldsymbol{j}}$ is vertically upwards in the plane of motion of the projectile, determine an expression for the initial position vector $\boldsymbol{v}$ of the ball.

Exercise 3.4

10 The position vector of a projectile at time t seconds, relative to a point on level ground, is $\boldsymbol{r}(t) = 10t\hat{\boldsymbol{i}} + \left(19.6t - 4.9t^2\right)\hat{\boldsymbol{j}}$ m, where $\hat{\boldsymbol{i}}$ is horizontal and $\hat{\boldsymbol{j}}$ is vertically upwards. Calculate the maximum height reached by the projectile.

Exercise 3.1

11 At time t, a particle has position vector $\boldsymbol{r}(t) = 2\sin(t)\hat{\boldsymbol{i}} + \left(2\cos(2t)+2\right)\hat{\boldsymbol{j}}$, $t \geq 0$.

(a) Determine the Cartesian equation of the path of the particle.

(b) Sketch the path.

(c) Describe the motion of the particle.

12 A curve C is defined by the parametric equations $x = 3 + t$ and $y = t^2 - 5$, $t \geq 0$. Exercise 3.1

(a) Determine the Cartesian equation of C and determine its domain.

(b) Sketch the graph of C.

(c) Determine the vector equation of C.

13 A particle moves so that its position vector at time t is $\boldsymbol{r}(t) = t^2\hat{\boldsymbol{i}} - 2t\hat{\boldsymbol{j}}$, $t \geq 0$. Exercise 3.3

(a) Determine an expression for the velocity of the particle at time t.

(b) Calculate its velocity at $t = 0$, $t = 1$ and $t = 2$.

14 A particle moves so that its position vector at time t is $\boldsymbol{r}(t) = \left(t^2 + 1\right)\hat{\boldsymbol{i}} - 2t\hat{\boldsymbol{j}}$, $t \geq 0$. Exercise 3.3

(a) Calculate the average velocity between times t and $t + \delta t$.

(b) Hence determine an expression for the velocity of the particle at time t.

15 Two ships are observed from a station O. Ship A is observed at 1:00 pm and ship B at 2:00 pm. Exercise 3.3
Relative to O, the respective position and velocity vectors of the ships at these times are given by

$\boldsymbol{r}_A = 2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ and $\boldsymbol{v}_A = 5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ $\quad$ $\boldsymbol{r}_B = 11\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ and $\boldsymbol{v}_B = 2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$,

where time is measured in hours and velocity in km/h.

Determine the time at which the two ships collide. Identify any assumptions you have made and comment on the effect of these assumptions on your answer.

16 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = -4\cos(2t)\hat{\boldsymbol{j}} - 4\sin(2t)\hat{\boldsymbol{k}}$, $t \geq 0$. Exercise 3.3

Determine an expression for the position vector at time t if initially $\dot{\boldsymbol{r}}(t) = 2\left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}\right)$ and $\boldsymbol{r} = \hat{\boldsymbol{j}}$.

17 The position vector of a particle at time t is given by $\boldsymbol{r}(t) = \cos(t)\hat{\boldsymbol{i}} + \sqrt{3}\sin(t)\hat{\boldsymbol{j}}$, $0 \leq t \leq 2\pi$. Exercise 3.3

(a) Determine when the velocity is perpendicular to the position vector.

(b) Determine when the acceleration is perpendicular to the velocity.

(c) Calculate the particle's displacement from $t = 0$ to $t = \dfrac{\pi}{2}$.

18 The position vector, $\boldsymbol{r}(t)$, of a particle at time t is given by $\boldsymbol{r}(t) = \left(e^{-t}\sin(t)\right)\hat{\boldsymbol{i}} + \left(e^{-t}\cos(t)\right)\hat{\boldsymbol{j}}$. Exercise 3.3
Calculate the speed of the particle at time t. Give your answer in the form ce^{-t}, where c is a constant.

19 The initial velocity of a projectile launched from ground level is $12\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ m/s, where $\hat{\boldsymbol{i}}$ is horizontal and $\hat{\boldsymbol{j}}$ is vertically up. Exercise 3.3

(a) Determine the initial speed and angle of projection.

(b) Determine the maximum height reached by the projectile.

20 A rock is thrown from a height of 6 metres above ground level with a speed of $4\sqrt{g}$ m/s and at an angle of 45° to the horizontal. Exercise 3.4

Taking the origin as ground level directly below the point of projection, determine the Cartesian equation of the trajectory of the stone.

Exercise 3.4

21 Initially a motor boat is at a point J at the end of a jetty and a police boat is at a point P. The position vector of P relative to J is $400\hat{\boldsymbol{i}} - 600\hat{\boldsymbol{j}}$. The motor boat leaves point J and travels with constant velocity $6\hat{\boldsymbol{i}}$. At the same time, the police boat leaves its position at P and travels with constant velocity $v\left(8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}\right)$, where v is a real number. All distances are measured in metres and all times are measured in seconds. The police boat meets the motor boat after t seconds.

(a) Calculate the value of t.

(b) Determine the value of v.

(c) Determine the speed of the police boat.

(d) Determine the closest distance that the police boat gets to J.

Exercise 3.4

22 The position vector, $\boldsymbol{r}(t)$, of a particle at time t is given by $\boldsymbol{r}(t) = t^2\hat{\boldsymbol{i}} + \left(e^{-t} + 1\right)\hat{\boldsymbol{j}} - t\hat{\boldsymbol{k}}$.

(a) Determine an expression for the particle's velocity at time t.

(b) Calculate the exact initial speed of the particle.

(c) Resolve the acceleration vector at $t = 1$ into its exact tangential component.

The tangential component of acceleration is parallel to the velocity unit vector.

(d) Resolve the acceleration vector at $t = 1$ into its exact normal component.

Exercise 3.3

23 A radar facility tracks a plane flying with constant speed in a straight line. Relative to the radar facility, the plane's initial and final vectors are $2\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $8\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 13\hat{\boldsymbol{k}}$ respectively. All distances are in kilometres.

(a) Determine a unit vector in the direction of motion in the plane.

(b) Determine the plane's position when it is closest to the radar facility.

(c) Calculate the distance from the facility at this point. Give your answer correct to 2 decimal places.

Exercise 3.4

24 A missile is launched with speed of 1000 m/s from a launching pad at an origin O in a direction due east, at an angle of elevation α to the horizontal.

(a) If $\tan(\alpha) = \frac{4}{3}$, verify that the initial velocity $\boldsymbol{u}$ of the missile can be written as $\boldsymbol{u} = 200\left(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}\right)$, where $\hat{\boldsymbol{i}}$ is a unit horizontal vector due east and $\hat{\boldsymbol{j}}$ is a unit vector in the vertically upwards direction.

(b) Assuming that the only force acting on the missile is its weight, determine the position vector $\boldsymbol{r}$ relative to O of the missile t seconds after launching. Assume the acceleration due to gravity to be $10\ \text{m/s}^2$.

(c) Simultaneously with the launching of the missile, a pilotless target aircraft is launched from a point which is 44 080 m due east of the missile site and at the same height above sea level. This aircraft is guided to travel with constant speed along a straight-line path.

If the aircraft's velocity at any time is given by $\boldsymbol{v} = a\left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)$, where a is constant, determine its position vector $\boldsymbol{v}$ relative to O, when t seconds have elapsed after launching.

(d) Given that the missile hits the target aircraft, calculate how long after launching this occurs. Give your answer correct to the nearest second.

(e) Calculate the speed at which the target aircraft is travelling, correct to 2 significant figures.

25 Particles A and B move in a plane so that at any time $t \geq 0$, their position vectors are $\boldsymbol{r}_A = 2t\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_B = 4\sin(\alpha t)\hat{\boldsymbol{i}} + 4\cos(\alpha t)\hat{\boldsymbol{j}}$, where α is a positive constant. Exercise 3.4

(a) Determine the speed of B in terms of α.

(b) Determine the Cartesian equations of the paths of A and B.

(c) On the same set of axes, sketch the paths of A and B and indicate the directions of travel.

(d) Determine the point(s) where the paths of A and B cross.

(e) Calculate the least value of α, correct to 2 decimal places, for which A and B will collide.

(f) Assuming that the collision in part **(e)** occurs, calculate the distance that has been travelled by B when it collides with A, giving your answer correct to 1 decimal place.





4

Complex numbers

Recall 196

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4 Recall

Classifying numbers

1 Determine the sets from the list $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{N}$ and $\mathbb{C}$ to which each of the following belong.

(a) -3 **(b)** $\sqrt{\frac{4}{9}}$ **(c)** $4+i$

Identify union and intersection of two sets

2 Consider the two sets $A = \{2, 3, 4, 5, 6, 7, 8\}$ and $B = \{2, 4, 6, 8, 10\}$, which are subsets of the universal set $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(a) List the elements in the following sets:

(i) $A \cup B$ **(ii)** $A \cap B$ **(iii)** $(A \cup B)'$

(b) Illustrate the sets on a Venn diagram.

Express numbers in terms of *i*

3 Express each of these numbers in terms of i, where $i = \sqrt{-1}$.

(a) $\sqrt{-100}$ **(b)** $\sqrt{-8}$

Multiply a complex number by a real number

4 Multiply these complex numbers by the real numbers and simplify the answer.

(a) $2(8-12i)$ **(b)** $3(4-i)-4(3-5i)$

Multiply complex numbers in Cartesian form

5 Simplify the following, expressing your answer in the form $x + yi$.

(a) $(2+3i)(7-2i)$ **(b)** $(9+2i)(i-3)$

(c) $\left(\sqrt{3}+i\right)^2$ **(d)** $\left(4+\sqrt{5}i\right)\left(4-\sqrt{5}i\right)$

Divide by complex numbers in Cartesian form

6 Simplify the following, expressing your answer in the form $x + yi$.

(a) $\frac{1}{4-i}$ **(b)** $\frac{2+i}{1+3i}$

Simplify expressions containing complex numbers

7 If $z = -2 + 3i$, determine an expression for the following, leaving your answer in simplified form.

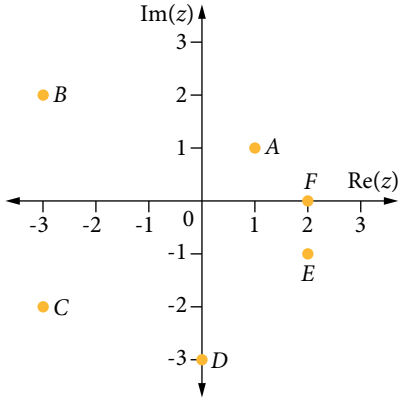
(a) $z\overline{z}$ **(b)** $\left(z-\overline{z}\right)^2$ **(c)** $\frac{z}{\overline{z}}$

Use the definition of equality of complex numbers to solve equations

8 Determine the values of x and y so that $(x+yi)(1+4i) = 6+7i$.

Recognise complex numbers in the complex plane

9 Express the complex numbers shown on the complex plane in the form $x + yi$.



Use trigonometric identities to simplify expressions

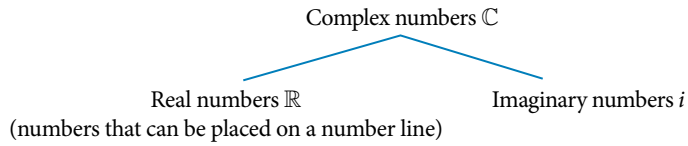
10 Simplify these expressions using trigonometric identities:

(a) $\dfrac{\cos(2\theta)}{\cos(\theta)+\sin(\theta)}$

(b) $\dfrac{\cos(A)\cos(B)-\sin(A)\sin(B)}{\sin(A+B)}$

(c) $\dfrac{(\cos(\theta)+\sin(\theta))^2}{1+\sin(2\theta)}$

4.1 Complex number review



You have progressively widened your understanding of number systems, starting with the set of natural numbers $\mathbb{N}$, and extended that to the set of integers $\mathbb{Z}$, then to rational numbers or numbers that can be expressed as fractions $\mathbb{Q}$, and then to irrational numbers $\mathbb{Q}'$. These are all included in the set of real numbers $\mathbb{R}$. More recently, you have met the set of complex numbers $\mathbb{C}$, which contains all the real numbers and the imaginary number i.

A complex number consists of two parts, a *real part* and an *imaginary part*, and is defined as:

$\mathbb{C} = \{z : z = x + yi, \text{ where } x, y \in \mathbb{R}, \text{ and } i^2 = \sqrt{-1}\}$

The real component of z is $\text{Re}(z) = x$ and the imaginary component is $\text{Im}(z) = y$.

The complex conjugate of z is represented by $\overline{z} = x - yi$.

WARNING

The imaginary part of a complex number is the coefficient of i, and should not include i.

Two complex numbers are considered to be equal if and only if their real components are equal and their imaginary components are equal.

Complex numbers can be used to find solutions to equations and can be used in other areas of mathematics and their applications such as electrical engineering.

You have already established that the laws of arithmetic hold for complex numbers, and these can easily be verified or proved.

For addition		For multiplication	
$z_1 + z_2 = z_2 + z_1$	Commutative law	$z_1 z_2 = z_2 z_1$	Commutative law
$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$	Associative law	$(z_1 z_2) z_3 = z_1 (z_2 z_3)$	Associative law
$z_1 + (-z_1) = 0 + 0i$	Additive inverse	$z_1 \times \frac{1}{z_1} = 1 + 0i$	Multiplicative inverse
$0 + 0i$	Identity element	$1 + 0i$	Identity element

1 Real and imaginary components of a complex number

Determine the real and imaginary components of these complex numbers.

(a) $z = 3(4 - 2i)$

THINKING	WORKING
1 Express the complex number in the form $x + yi$.	$3(4 - 2i) = 12 - 6i$
2 Identify and write the real and imaginary components.	$\text{Re}(z) = 12$ $\text{Im}(z) = -6$

(b) $z = \dfrac{1}{4 - i}$

1 Express the complex number in the form $x + yi$.	$\dfrac{1}{4-i} = \dfrac{1}{4-i} \times \dfrac{4+i}{4+i}$ $= \dfrac{4+i}{16-i^2}$ $= \dfrac{4+i}{16-(-1)}$ $= \dfrac{4+i}{17}$ $= \dfrac{4}{17} + \dfrac{i}{17}$
2 Identify and write the real and imaginary components.	$\text{Re}(z) = \dfrac{4}{17}$ $\text{Im}(z) = \dfrac{1}{17}$

(c) $\left(\sqrt{3} - 4i\right)^2$

1 Express the complex number in the form $x + yi$.	$\left(\sqrt{3} - 4i\right)^2 = 3 - 8\sqrt{3}\,i + 16i^2$ $= -13 - 8\sqrt{3}\,i$
2 Identify and write the real and imaginary components.	$\text{Re}(z) = -13$ $\text{Im}(z) = -8\sqrt{3}$

2 Expressing complex numbers in Cartesian form

If $z_1 = -2 + 3i$ and $z_2 = 1 + i$, express the following in the form $x + yi$.

(a) $z_1 \times \overline{z}_2$

THINKING	WORKING
1 Substitute the given numbers and simplify the expression.	$\overline{z}_2 = 1 - i$ $z_1 \times \overline{z}_2 = (-2 + 3i)(1 - i)$ $= -2 + 5i - 3i^2$ $= -2 + 3 + 5i$ $= 1 + 5i$
2 Express in the required form.	$z_1 \times \overline{z}_2 = 1 + 5i$

(b) $\dfrac{z_1 - z_2}{z_1 + z_2}$

1 Substitute the given numbers and simplify the expression.	$\dfrac{z_1 - z_2}{z_1 + z_2} = \dfrac{-2 + 3i - 1 - i}{-2 + 3i + 1 + i}$ $= \dfrac{-3 + 2i}{-1 + 4i} \times \dfrac{-1 - 4i}{-1 - 4i}$ $= \dfrac{3 + 10i - 8i^2}{1 + 16}$ $= \dfrac{11 + 10i}{17}$
2 Express in the required form.	$\dfrac{z_1 - z_2}{z_1 + z_2} = \dfrac{11}{17} + \dfrac{10i}{17}$

Technology worked example
Complex number review

3 Using powers of *i* to simplify expressions

Simplify the following complex numbers.

(a) i^{55}

THINKING	WORKING
1 Group the factors of i.	$i^{55} = \left(i^4\right)^{13} \times i^2 \times i$
2 Simplify the answer.	$\left(i^4\right)^{13} \times i^2 \times i = 1 \times -1 \times i$ (since $i^4 = -1 \times -1$) $= -i$

(b) $3i^7 \times 4i^2$

1 Multiply the given terms.	$3i^7 \times 4i^2 = 12i^9$
2 Group the factors of i and simplify the answer.	$12i^9 = 12\left(i^4\right)^2 \times i$ $= 12i$ (since $i^4 = i^2 \times i^2 = -1 \times -1$)

4 Raising complex numbers to a power

Expand and simplify $(2-i)^3$.

METHOD 1

THINKING	WORKING
1 Express in factors and expand the first two factors.	$(2-i)^3 = (2-i)(2-i)(2-i)$ $= \left(4-4i+i^2\right)(2-i)$ $= (3-4i)(2-i)$
2 Complete the expansion, collecting real and imaginary components to simplify.	$(3-4i)(2-i) = 6-11i+4i^2$ $= 2-11i$

METHOD 2

Use the cubic expansion where $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ to write the function in expanded form. Simplify.	$(2-i)^3 = 2^3 - 3 \times 2^2 \times i + 3 \times 2 \times i^2 - i^3$ $= 8 - 12i + 6 \times -1 - i^2 \times i$ $= 8 - 12i - 6 + 1i$ $= 2 - 11i$

EXERCISE 4.1 Complex number review

1 Determine the sets from the list $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-, \mathbb{N}$ to which each of the following belong.

(a) $(3+i)^2$ **(b)** i^6 **(c)** $\sqrt{\frac{23}{2}}$ **(d)** $\sqrt{-4i^4}$

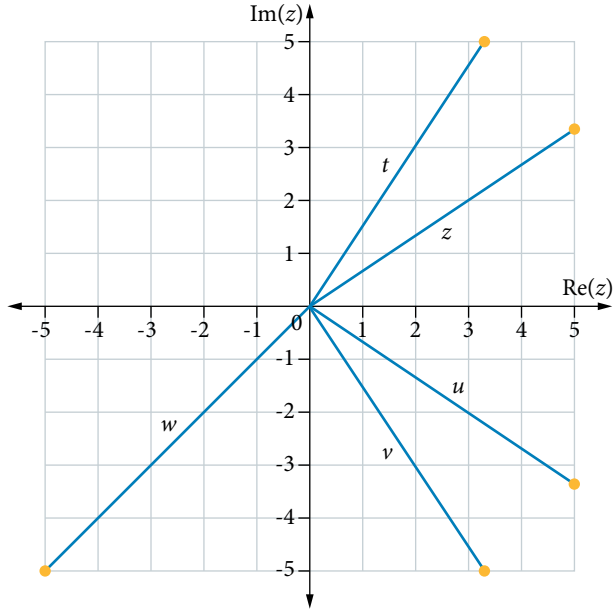
2 Express each of the following in terms of the imaginary number i.

(a) $\sqrt{-16}$ **(b)** $\sqrt{-11}$ **(c)** $\sqrt{-28}$ **(d)** $\sqrt{-2} \times \sqrt{-3} \times \sqrt{-12}$

3 Determine the complex conjugate of these complex numbers.

(a) $5 + 27i$ **(b)** $(3-4i)^2$ **(c)** $2(3+2i) - 4(1-3i)$ **(d)** $\frac{1}{6-i}$

4 Determine the values of the complex numbers shown in the graph.



Worked Example

1

5 Determine the real and imaginary parts of these complex numbers.

(a) $z = (3-2i)(4+7i)$ (b) $z = 4i(-3i)$ (c) $z = (2+i)(2-i)$

(d) $4+2i-(3-6i)$ (e) $z = \left(1+\sqrt{7}\,i\right)^2$ (f) $z = (u-2vi)+(v-2ui)$

3

6 Simplify each of the following.

(a) i^{31} (b) $\left(3i^5\right)^2\left(2i^3\right)^3$ (c) $2-i^5+i^{10}-i^{15}$

2

7 Express each of the following in the form $x + yi$.

(a) $3(5-4i)+2(-1-i)$ (b) $(4-i)(2+i)-(3-i)(2+i)$ (c) $(2+3i)^2(2-3i)^2$

(d) $\dfrac{1}{(1+i)^2}$ (e) $3+4i+\dfrac{25}{3+4i}$ (f) $\dfrac{2(5-i)}{3+i}$

(g) $z = \left(a^2-bi\right)-\left(b^2-ai\right)$

4

8 Expand and simplify $(1+i)^3$.

9 Express each of the following in the form $x + yi$.

(a) $(0.6-3i)(1.5+2.1i)$ (b) $4(23+i)-8(5+16i)$

10 Express each of the following in the form $x + yi$.

(a) $\dfrac{3}{12-5i}$ (b) $\dfrac{5+i}{7-2i}$

11 Express each of the following in the form $x + yi$.

(a) $(1+i)^4$ (b) $(2+i)^3(1-2i)^2$

12 Determine the values of a and b, given that:

(a) $a+3i = 2-bi$ (b) $a+bi = b+3i$

(c) $(a+b)+2i = 1+(a-b)i$ (d) $(a+bi)(2-i) = a+3i$

13 If $u = 3+5i$, $v = -1+i$, and $w = -4-2i$, determine the following:

(a) u^2 (b) vw (c) $\text{Re}\left(u^2-vw\right)$ (d) $\dfrac{u}{\overline{v}}$

14 Determine the values of the real numbers p and q, given that $\dfrac{\sqrt{15}-i}{\sqrt{15}+i} = p+qi$.

15 Determine the values of a and b, given that $(a+i)(1+bi) = 3b+ai$.

16 Express these complex numbers in the form $x + yi$.

(a) $\dfrac{8+5i}{4-3i} + \dfrac{2+i}{4+3i}$ **(b)** $(4+i)(3+2i)(1-i)$ **(c)** $\dfrac{(2+i)(3-2i)(1+2i)}{(1-i)^2}$

17 The imaginary component of $-3i + 2$ is equal to:

A 2 **B** $-3i$ **C** $2+3i$ **D** -3

18 **(a)** Which of the following statements is false?

A A real number is a complex number.

B The difference between a complex number and its conjugate is imaginary for $y \in \mathbb{R}$.

C The product of a complex number and its conjugate is always real.

D A rational number is a real number.

(b) Explain why the other three alternative statements in part **(a)** are true.

19 Express the following complex numbers in the form $x + yi$.

(a) $7i^{-5}$ **(b)** $(1-i)^{-4}$

20 Simplify the product $\dfrac{i}{10} \times \dfrac{2i^2}{9} \times \dfrac{3i^3}{8} \times \ldots \times \dfrac{8i^8}{3} \times \dfrac{9i^9}{2} \times 10i^{10}$.

21 Evaluate the sum $i^0 + i^1 + i^2 + i^3 + i^4 + \ldots + i^{97} + i^{98} + i^{99} + i^{100}$.

22 Determine the complex number $z = x + yi$ that satisfies the equation $z - 3iz = 20$.

23 Use the substitution $z = 2 - 3i$ to verify the rule $\left(z^{-1}\right)^2 = \left(z^2\right)^{-1}$.

24 **(a)** Express $i^{4n}, i^{4n+1}, i^{4n+2}, i^{4n+3}$ in simplest form for $n \in \mathbb{N}$.

(b) Use the results found in part **(a)** to simplify i^{98} and i^{103}.

25 If $u = a + bi$ and $v = c + di$,

(a) show that $\overline{u+v} = \overline{u} + \overline{v}$.

(b) verify the result in part **(a)** for $u = 2 + 3i$ and $v = 1 - 4i$.

26 If $u = a + bi$ and $v = c + di$, verify that $uv = vu$.

27 Given that $c - di$ is equal to the square of its conjugate, determine all possible values of c and d.

4.2 Modulus and argument form of a complex number

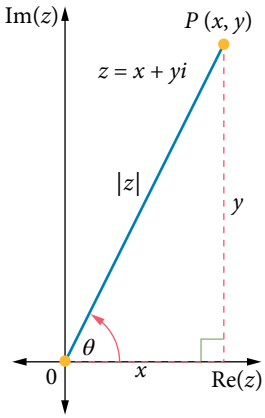
Modulus and argument

The *modulus* of a complex number z is written as $|z|$, and when

$z = x + yi$

$|z| = \sqrt{x^2 + y^2}$

The reason for this definition is clear when a point is seen in the complex plane (also known as the Argand plane). The modulus is the distance of the point P from the origin as given by Pythagoras' theorem.



WARNING

The modulus of z is the distance of z from the origin in the Argand plane, hence the modulus does not contain i and is always positive.

The *argument* of a complex number is the angle measured in radians, in an anti-clockwise direction from the positive real axis. It is the angle θ in the diagram above, where $\tan(\theta) = \frac{y}{x}$. The domain of the principal argument, denoted by $\text{Arg}(z)$, is defined as the coterminal angle in the domain $-\pi < \theta \leq \pi$.

5 Determining the modulus of a complex number

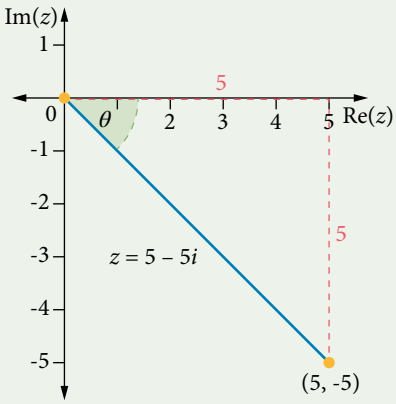
If $z = 7 - \sqrt{2}i$, determine the value of $|z|$.

THINKING	WORKING
1 Write the formula for the modulus.	$\|z\| = \sqrt{x^2 + y^2}$
2 Substitute the given values.	$\|z\| = \sqrt{7^2 + \left(\sqrt{2}\right)^2}$
3 Simplify and write the answer.	$\sqrt{7^2 + \left(\sqrt{2}\right)^2} = \sqrt{49 + 2}$ $\|z\| = \sqrt{51}$
4 Interpret the answer.	The modulus of $z = 7 - \sqrt{(2)}i$ is $\sqrt{51}$.

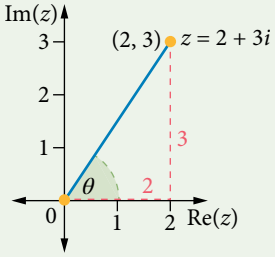
6 Determining the principal argument of a complex number

Determine the principal argument of the following.

(a) $5 - 5i$

THINKING	WORKING
1 Sketch a diagram to determine the quadrant for the complex number.	 Fourth quadrant
2 Determine the acute angle θ made with the real axis.	$\tan(\theta) = \frac{5}{-5}$ $\tan(\theta) = -1$ $\theta = \tan^{-1}(-1)$ $\theta = -\frac{\pi}{4}$
3 Determine the argument and write the answer.	$\text{Arg}(5 - 5i) = -\frac{\pi}{4}$

(b) $2 + 3i$, leaving your answer correct to 3 decimal places.

1 Sketch a diagram to determine the quadrant of the argument.	 First quadrant
2 Determine the acute angle θ made with the real axis.	$\tan(\theta) = \frac{3}{2}$ $\theta = 0.98279\ldots^{c}$ $\text{Arg}(z) = 0.983^{c}$ (3 d.p.)
3 Determine the angle and write the answer.	$\text{Arg}(2 + 3i) = 0.983$

WARNING

It is not sufficient to just use $\text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right)$ to determine the required angle, since $-\pi < \text{Arg}(z) \leq \pi$. A diagram should be sketched to help determine the quadrant in which the angle lies.

C Making connections

Modulus and argument form of complex numbers

For the complex number $a + bi$, move sliders m (modulus) and c to show the position vector for the complex number.

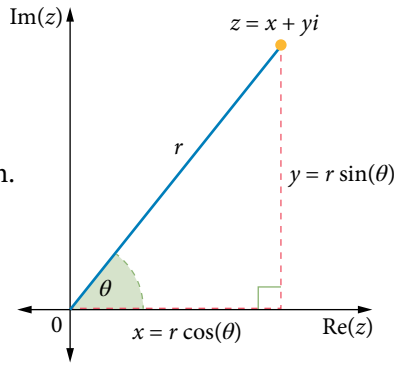
t Technology worked example

The modulus and argument of a complex number

Polar form of a complex number

Once the modulus and argument of a complex number $x + yi$ are known, you can express the number in terms of the polar coordinates r and θ, where

$$r = |z| = \sqrt{x^2 + y^2}$$
$$\theta = \text{Arg}(z)$$



This is also known as modulus argument or Mod-Arg form.

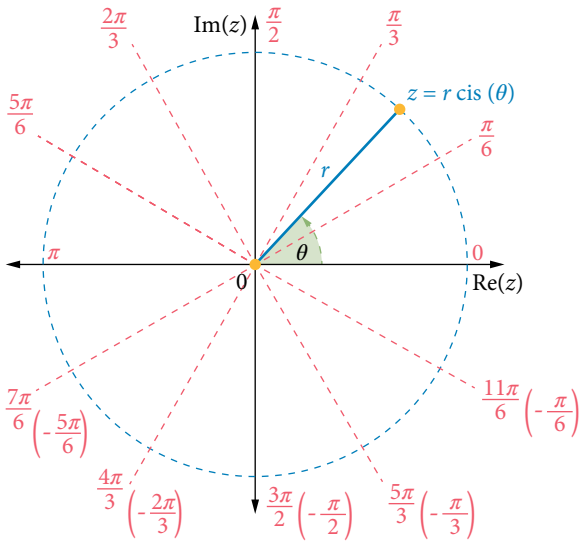
Note that using right-angled triangle trigonometry:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Thus $z = x + yi$ can be written in polar form as

$$z = r\cos(\theta) + r\sin(\theta)i = r(\cos(\theta) + i\sin(\theta))$$

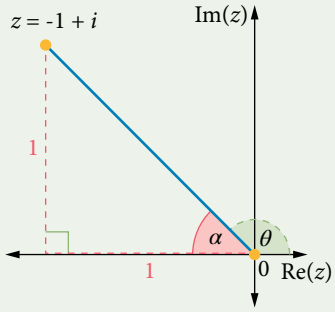
This is often abbreviated to $z = r\text{cis}(\theta)$, where $\text{cis}(\theta)$ is defined as $\cos(\theta) + i\sin(\theta)$.



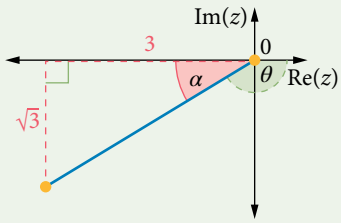
7 Converting Cartesian coordinates to polar coordinates

Convert the following complex numbers to polar form.

(a) $z = -1 + i$

THINKING	WORKING
1 Determine the value of r, where $r = \sqrt{x^2 + y^2}$.	$r = \sqrt{(-1)^2 + 1^2}$ $= \sqrt{2}$
2 Sketch a graph to identify the quadrant for the argument.	
3 Calculate the acute angle α made with the real axis and hence calculate θ, the angle made with the positive real axis.	$\tan(\alpha) = \frac{1}{1}$ $\tan(\alpha) = 1$ $\alpha = \frac{\pi}{4}$ Therefore $\theta = \frac{3\pi}{4}$
4 Convert the number to polar form $z = r\text{cis}(\theta)$.	$z = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

(b) $-3 - \sqrt{3}i$

1 Determine the value of r, where $r = \sqrt{x^2 + y^2}$.	$r = \sqrt{(-3)^2 + \left(-\sqrt{3}\right)^2}$ $= \sqrt{12}$ $= 2\sqrt{3}$
2 Sketch a graph to identify the quadrant for the argument.	
3 Determine the value of the acute angle α made with the real axis and hence θ, the angle made with the positive real axis.	$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ $\alpha = \frac{\pi}{6}$ Therefore $\theta = -\frac{5\pi}{6}$
4 Convert the number to polar form $z = r\text{cis}(\theta)$.	$z = 2\sqrt{3}\,\text{cis}\left(-\frac{5\pi}{6}\right)$

8 Converting polar coordinates to Cartesian coordinates

Express $z = 3\text{cis}\left(\frac{2\pi}{3}\right)$ in Cartesian coordinates.

THINKING	WORKING
1 Substitute the given values for r and θ into the rule $z = r\cos(\theta) + r\sin(\theta)i$.	$z = 3\cos\left(\frac{2\pi}{3}\right) + 3\sin\left(\frac{2\pi}{3}\right)i$
2 Calculate the real and imaginary components.	$z = 3 \times -\frac{1}{2} + 3 \times \frac{\sqrt{3}}{2}i$
3 Write the complex number in Cartesian form.	$z = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

When calculating angles, it is often more straightforward to use positive values for the x and y components and use the inverse tangent ratio to calculate the acute angle (to the nearest Re(z) axis). From here, the argument can be determined as follows:

- For acute angle α in quadrant 1, argument $\theta = \alpha$
- For acute angle α in quadrant 2, argument $\theta = \pi - \alpha$
- For acute angle α in quadrant 3, argument $\theta = -\pi + \alpha$
- For acute angle α in quadrant 4, argument $\theta = -\alpha$

Multiplication and division of complex numbers in polar form

You have already met the rules for multiplication and division of complex numbers in polar form.

For multiplication, the moduli are multiplied and the arguments are added.

For division, the moduli are divided and the arguments are subtracted.

If $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$, then

$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

9 Multiplication and division of complex numbers in polar form

If $z_1 = 6\text{cis}\left(\frac{\pi}{4}\right)$ and $z_2 = 2\text{cis}\left(\frac{7\pi}{6}\right)$, determine the following in polar form.

(a) $z_1 z_2$

THINKING	WORKING
1 Substitute the given values into the rule $z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$.	$z_1 z_2 = 6 \times 2\text{cis}\left(\frac{\pi}{4} + \frac{7\pi}{6}\right)$
2 Simplify the expression, checking that the argument is in the allowed domain.	$= 12\text{cis}\left(\frac{17\pi}{12}\right)$
3 Write the answer.	$= 12\text{cis}\left(-\frac{7\pi}{12}\right)$

(b) $\frac{z_1}{z_2}$

1 Substitute the given values into the rule $\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$.	$\frac{z_1}{z_2} = \frac{6}{2}\text{cis}\left(\frac{\pi}{4} - \frac{7\pi}{6}\right)$
2 Simplify the expression, checking that the argument is in the allowed domain.	$= 3\text{cis}\left(-\frac{11\pi}{12}\right)$
3 Write the answer.	$= 3\text{cis}\left(-\frac{11\pi}{12}\right)$

Powers of complex numbers in polar form

It is quite a lengthy process to raise complex numbers in Cartesian form to a power without the use of technology. In polar form, however, de Moivre's theorem provides a relatively easy way to raise complex numbers to any integer power.

Explore further

Complex numbers in polar form

Explore conversion between standard and polar form of a complex number.

De Moivre's theorem states that if $z = r\,\text{cis}(\theta)$, then $\left(r\,\text{cis}(\theta)\right)^n = r^n\text{cis}(n\theta)$.

This is a special case of the rule for multiplication and you will be asked to prove the theorem later in this section.

10 Raising complex numbers in polar form to a power

For $z = 2\text{cis}\left(\frac{\pi}{3}\right)$ calculate z^4 in the form $z = r\,\text{cis}(\theta)$.

THINKING	WORKING
1 Substitute the given values into de Moivre's formula and simplify. $\left(r\,\text{cis}(\theta)\right)^n = r^n\text{cis}(n\theta)$	$\left(2\text{cis}\left(\frac{\pi}{3}\right)\right)^4 = 2^4\text{cis}\left(4 \times \frac{\pi}{3}\right)$ $= 16\text{cis}\left(\frac{4\pi}{3}\right)$
2 Verify that the argument is within the allowed domain and adjust it if necessary.	Given that $\frac{4\pi}{3} > \pi$, then $\theta = -\frac{2\pi}{3}$
3 Write the answer.	$z = 16\text{cis}\left(-\frac{2\pi}{3}\right)$

11 Proving identities involving complex numbers in polar form

Prove that $\dfrac{1}{\cos(\theta)+i\sin(\theta)} = \cos(\theta)-i\sin(\theta)$

THINKING	WORKING
1 Multiply the numerator and denominator by the conjugate of the denominator.	$\dfrac{1}{\cos(\theta)+i\sin(\theta)} \times \dfrac{\cos(\theta)-i\sin(\theta)}{\cos(\theta)-i\sin(\theta)}$
2 Simplify the expression using the Pythagorean identity.	$= \dfrac{\cos(\theta)-i\sin(\theta)}{\cos^2(\theta)+\sin^2(\theta)}$ $= \dfrac{\cos(\theta)-i\sin(\theta)}{1}$
3 Write the answer.	Therefore $\dfrac{1}{\cos(\theta)+i\sin(\theta)} = \cos(\theta)-i\sin(\theta)$ as required.

Technology worked example
Raising complex numbers in polar form to a power

EXERCISE 4.2 Modulus and argument form of a complex number

Worked Example

1 Determine the modulus of the following complex numbers.

(a) $-\sqrt{6}-\sqrt{3}i$ (b) $a+bi+2$ (c) $a+bi+3-4i$ (d) $\dfrac{(2-i)(2+i)}{i}$

2 Determine the principal argument, in radians, of the following complex numbers.

(a) $12+12i$ (b) $2-2\sqrt{3}i$ (c) $-1-5i$ (d) $-1+2i$

3 Determine the principal argument, in degrees, of the following complex numbers. Leave your answers correct to 1 decimal place.

(a) $-8-24i$ (b) $4-3i$

4 Express these complex numbers in Cartesian form.

(a) $2\,\text{cis}\left(\dfrac{\pi}{4}\right)$ (b) $\text{cis}\left(-\dfrac{2\pi}{3}\right)$ (c) $\text{cis}\left(\dfrac{\pi}{2}\right)$

(d) $6\,\text{cis}(\pi)$ (e) $8\,\text{cis}\left(\dfrac{3\pi}{4}\right)$ (f) $\sqrt{7}\,\text{cis}(0)$

5 Convert these complex numbers to Cartesian form.

(a) $10\,\text{cis}(60°)$ (b) $2\,\text{cis}(-90°)$ (c) $\text{cis}(150°)$ (d) $\text{cis}(-120°)$

6 Determine the polar form of these complex numbers, expressing the argument in radians. 7

(a) $4\sqrt{3}-4i$ (b) $9+9i$ (c) $0+5i$

(d) $(2i+3)(3-2i)$ (e) $(0+i)\left(\sqrt{3}+i\right)$

7 Express these complex numbers in polar form, with the argument in degrees and answers correct to 1 decimal place.

(a) $2+7i$ (b) $4-i$ (c) $-3+2i$ (d) $(-6-2i)^2$

8 Simplify these expressions, leaving your answer in the form $r\,\text{cis}(\theta)$. 9

(a) $5\,\text{cis}(45°)\times 2\,\text{cis}(45°)$ (b) $2\,\text{cis}(150°)\times\sqrt{3}\,\text{cis}(170°)$ (c) $4\,\text{cis}\left(\frac{\pi}{4}\right)\times 3\,\text{cis}\left(\frac{4\pi}{3}\right)$

(d) $\dfrac{12\,\text{cis}\left(\frac{\pi}{2}\right)}{3\,\text{cis}\left(-\frac{\pi}{4}\right)}$ (e) $\dfrac{3\,\text{cis}\left(\frac{\pi}{6}\right)}{\text{cis}(2\pi)}$ (f) $\dfrac{\left(4\,\text{cis}\left(\frac{\pi}{4}\right)\right)\left(3\,\text{cis}\left(\frac{\pi}{6}\right)\right)}{6\,\text{cis}\left(\frac{\pi}{12}\right)}$

(g) $\dfrac{6\,\text{cis}\left(\frac{\pi}{4}\right)}{\left(3\,\text{cis}\left(\frac{\pi}{3}\right)\right)\left(2\,\text{cis}\left(\frac{\pi}{6}\right)\right)}$ (h) $\dfrac{a^4\,\text{cis}(A+B)}{a\,\text{cis}(A-B)}$

9 Use de Moivre's theorem to express the following in the form $r\text{cis}(\theta)$. 10

(a) $\left(2\,\text{cis}\left(\frac{\pi}{4}\right)\right)^6$ (b) $\left(3\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^4$ (c) $\left(6\,\text{cis}\left(\frac{\pi}{6}\right)\right)^3$

(d) $\left(\sqrt{3}-i\right)^5$ (e) $(1+i)^8$ (f) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^4$

(g) $\left(3\,\text{cis}\left(\frac{\pi}{6}\right)\right)^3\left(2\,\text{cis}\left(-\frac{\pi}{2}\right)\right)^2$ (h) $\dfrac{\left(\text{cis}(120°)\right)^4}{\left(3\,\text{cis}(-30°)\right)^3}$

10 Given that $w=5\,\text{cis}\left(\frac{7\pi}{8}\right)$, $z=2\,\text{cis}\left(-\frac{\pi}{6}\right)$ determine the principal argument of w^2z.

11 Prove that $\text{cis}(n\theta)\,\text{cis}(-n\theta)=1$

12 If $u=2-2i$ and $v=\sqrt{3}+i$, determine the following:

(a) The polar form of u and v. (b) $\left|\dfrac{u^4}{v^3}\right|$ (c) $\text{Arg}\left(\dfrac{u^4}{v^3}\right)$

13 If a complex number lies on the negative imaginary axis in the complex plane, which one of the following is true?

A The modulus is negative B The argument is $\frac{\pi}{2}$

C The argument is $-\frac{\pi}{2}$ D It is a multiple of i^2

14 Choose the statement that is true about a complex number $z=x+yi$ and its conjugate $\overline{z}=x-yi$ sketched on the Argand plane.

A $\overline{z}$ is a reflection of z in the line $y=x$. B $\overline{z}$ is a reflection of z in the imaginary axis.

C $\overline{z}$ is a reflection of z in the real axis. D The difference in their arguments is always π.

15 Prove that if $z=\text{cis}(\theta)$, then $\overline{z}=\dfrac{1}{z}$.

Worked Example 11

16 If $4 - 3i = 5\,\text{cis}(\theta)$, determine the following without calculating the value of θ. Express your answer in the form $x + yi$.

(a) $5\big(\cos(\theta) - i\sin(\theta)\big)$ **(b)** $\cos(2\theta) + i\sin(2\theta)$

17 Consider the complex numbers $u = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ and $v = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$.

(a) Determine $\frac{u}{v}$ in polar and Cartesian form.

(b) Use your answers to part **(a)** to determine an exact expression for $\sin\left(\frac{\pi}{12}\right)$.

18 Prove that $r_1\,\text{cis}(\theta_1) \times r_2\,\text{cis}(\theta_2) = r_1 r_2\,\text{cis}\big(\theta_1 + \theta_2\big)$.

19 Use the method of mathematical induction to prove de Moivre's theorem for positive integer powers, i.e. prove $\big(r\,\text{cis}(\theta)\big)^n = r^n\text{cis}(n\theta),\ n \in \mathbb{Z}^+$.

20 By letting $n = -m$, where $n < 0$ and therefore $m \in \mathbb{Z}^+$, prove that de Moivre's theorem is valid for negative integer powers, i.e. prove $\big(r\,\text{cis}(\theta)\big)^n = r^n\text{cis}(n\theta),\ n \in \mathbb{Z}^-$.

21 Prove that $\frac{1}{2i}\big(\text{cis}(\theta) - \text{cis}(-\theta)\big) = \sin(\theta)$.

22 Use de Moivre's theorem to prove that $\overline{z^n} = (\overline{z})^n$.

23 The series expansions for e^x, $\cos(x)$ and $\sin(x)$ are as follows:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Use these expressions to prove that $\text{cis}(\theta) = e^{i\theta}$.

24 Given that $\text{cis}(\theta) = e^{i\theta}$, express these complex numbers in Cartesian form.

(a) $6e^{i\pi}$ **(b)** $2e^{-\frac{2\pi}{3}i}$ **(c)** $3e^{i\frac{\pi}{2}} \times 2e^{\frac{\pi}{4}}$

Roots of complex numbers

4.3

Determining square roots of a complex number

You know that $i^2 = -1$ and so i is a square root of -1. But, just as -2 (as well as +2) is a square root of 4, $-i$ is also a square root of -1 as:

$$\begin{aligned}(-i)^2 &= (-i)(-i)\\ &= i^2\\ &= -1\end{aligned}$$

In general, if z is a square root of a complex number, then so is $-z$. The simplest way to find a square root of a complex number is to use polar form and de Moivre's theorem.

12 Determining the square roots of a complex number

Determine the square roots of $3 + 4i$.

THINKING	WORKING
1 Let $z = x + iy$ and z^2 be equal to the given number.	$z^2 = 3 + 4i$ $x^2 + 2ixy - y^2 = 3 + 4i$
2 Equate real and imaginary parts.	$x^2 - y^2 = 3 \quad [1]$ $2xy = 4$ $xy = 2 \quad [2]$
3 Solve the simultaneous equations.	$y = \frac{2}{x}$ Substitute the expression for y into [1] $x^2 - \frac{4}{x^2} = 3$ $x^4 - 3x^2 - 4 = 0$ $(x^2 - 4)(x^2 + 1) = 0$ $x = \pm 2$ or $\pm i$ (discard unreal value of x) $y = \pm 1$
4 Write the answers.	The square roots of $3 + 4i$ are $2 + i$ and $-2 - i$

This method is only useful for determining square roots and can get quite complicated.

A more general method of determining the square roots of a complex number is demonstrated in the next worked example. This method can be used more generally to determine cube roots, fourth roots, etc. It uses the fact that $\text{cis}(\theta) = \text{cis}(\theta + 2k\pi)$, $k \in \mathbb{Z}$. This approach requires the use of de Moivre's theorem for rational powers. You have already proved the theorem for integers but proving it for non-integers is beyond the scope of this course.

To determine roots of complex numbers, de Moivre's theorem can be written as

$$\left(r\,\text{cis}(\theta)\right)^{\frac{1}{n}} = r^{\frac{1}{n}}\text{cis}\left(\frac{\theta}{n}\right)$$

13 Using a general method to determine the square roots of a complex number

Determine the roots of the following complex numbers.

(a) The cube roots of $2\,\text{cis}\left(\frac{\pi}{4}\right)$.

THINKING	WORKING
1 Express the complex number in the form of $\text{cis}(\theta + 2k\pi)$, $k \in \mathbb{Z}$.	$2\,\text{cis}\left(\frac{\pi}{4}\right) = 2\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right),\ k \in \mathbb{Z}$
2 Set up the required equation.	$\left(2\,\text{cis}\left(\frac{\pi}{4}\right)\right)^{\frac{1}{3}} = \left(2\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)\right)^{\frac{1}{3}}$ $= 2^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)$
3 Identify the three values of k required and substitute.	$\frac{\pi}{12} + \frac{2k\pi}{3} = \frac{\pi}{12}, \frac{\pi}{12} + \frac{2\pi}{3}, \frac{\pi}{12} + \frac{4\pi}{3}$ $= \frac{\pi}{12}, \frac{9\pi}{12}, \frac{17\pi}{12}$ $= \frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}$ $= \frac{\pi}{12}, \frac{3\pi}{4}, \frac{-5\pi}{12}$
4 Write the answers.	$\left(2\,\text{cis}\left(\frac{\pi}{4}\right)\right)^{\frac{1}{3}} = 2^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{12}\right),\ 2^{\frac{1}{3}}\text{cis}\left(\frac{3\pi}{4}\right),\ 2^{\frac{1}{3}}\text{cis}\left(\frac{-5\pi}{12}\right)$

(b) The square roots of $1 - \sqrt{3}i$.

1 Express the complex number in polar form.	Modulus $= \sqrt{1+3}$ $= 2$ Argument $= \tan^{-1}\left(-\sqrt{3}\right)$ (fourth quadrant) $= -\frac{\pi}{3}$ $1 - \sqrt{3}i = 2\text{cis}\left(-\frac{\pi}{3}\right)$

2 Express the complex number in the form of $\text{cis}(\theta + 2k\pi),\ k \in \mathbb{Z}$.	$1 - \sqrt{3}i = 2\,\text{cis}\left(-\frac{\pi}{3} + 2k\pi\right),\ k \in \mathbb{Z}$
3 Set up the required equation.	$\left(1 - \sqrt{3}\right)^{\frac{1}{2}} = \left(2\,\text{cis}\left(-\frac{\pi}{3} + 2k\pi\right)\right)^{\frac{1}{2}}$ $= \sqrt{2}\,\text{cis}\left(-\frac{\pi}{6} + k\pi\right)$
4 Identify the two values of k required and substitute, in this case, $k = 0$ and $k = 1$.	$-\frac{\pi}{6} + k\pi = -\frac{\pi}{6}, \frac{5\pi}{6}$
5 Write the answers.	$\sqrt{2}\,\text{cis}\left(-\frac{\pi}{6}\right), \sqrt{2}\,\text{cis}\left(\frac{5\pi}{6}\right)$ or $\frac{\sqrt{2}}{2}\left(\sqrt{3} - i\right), \frac{\sqrt{2}}{2}\left(-\sqrt{3} + i\right)$

Determining the *n*th root of a complex number

Determining the nth root of a complex number z is equivalent to solving the equation $z^n = a$ or $z = a^{\frac{1}{n}}$. To solve for the required number of solutions (roots) the fundamental theorem of algebra is used.

The fundamental theorem of algebra states that a polynomial of degree n has n linear factors in $\mathbb{C}$ (the set of complex numbers). Hence, a polynomial equation of degree n has n roots in $\mathbb{C}$ (provided that repeated roots are counted the appropriate number of times).

As a result of this theorem, you would expect to find two square roots, three cube roots, four fourth roots etc. of any complex number.

14 Determining the *n*th root of a complex number

Determine the cube roots of -8, leaving your answer in polar form.

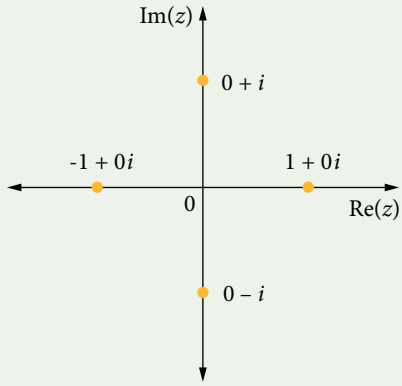
THINKING	WORKING
1 Convert the complex number to polar form.	$-8 = 8\,\text{cis}(\pi)$
2 Express the complex number in the form $\text{cis}(\theta) = \text{cis}(\theta + 2k\pi), k \in \mathbb{Z}$.	$8\,\text{cis}(\pi) = 8\,\text{cis}(\pi + 2k\pi), k \in \mathbb{Z}$
3 Set up the required equation.	$\left(8\,\text{cis}(\pi)\right)^{\frac{1}{3}} = \left(8\,\text{cis}(\pi + 2k\pi)\right)^{\frac{1}{3}}$ $= 2\,\text{cis}\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)$

4 Choose the three values $k = 0,\ 1,\ 2$ and substitute, checking that the answers are within the given domain.	$\theta = \frac{\pi}{3}, \frac{\pi}{3} + \frac{2\pi}{3}, \frac{\pi}{3} + \frac{4\pi}{3}$ $= \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$ Note that $\frac{5\pi}{3}$ is outside the domain and needs to be rewritten as $-\frac{\pi}{3}$.
5 Write the answers.	$2\text{cis}\left(\frac{\pi}{3}\right),\ 2\text{cis}(\pi),\ 2\text{cis}\left(-\frac{\pi}{3}\right)$

You could have chosen any three consecutive integer values of k, but that could have led to arguments outside the domain $-\pi < \theta \leq \pi$. It is often easier to start with $k = 0,\ 1, \ldots$, and then make any adjustments required.

15 Plotting roots of complex numbers on the complex plane

Determine the fourth roots of unity (1), and plot them on the complex plane in Cartesian form.

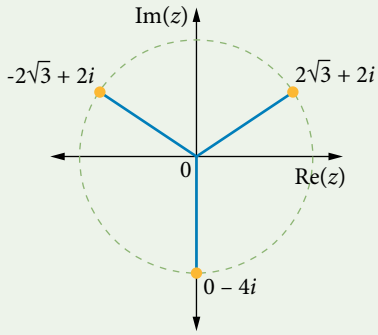
THINKING	WORKING
1 Convert the complex number to polar form.	$1 = 1\text{cis}(0)$
2 Express the complex number in the form $\text{cis}(\theta) = \text{cis}(\theta + 2k\pi),\ k \in \mathbb{Z}$.	$1\text{cis}(0) = 1\text{cis}(0 + 2k\pi),\ k \in \mathbb{Z}$
3 Set up the required equation.	$1\text{cis}(0)^{\frac{1}{4}} = \left(1\text{cis}(0 + 2k\pi)\right)^{\frac{1}{4}}$ $= \text{cis}\left(0 + \frac{k\pi}{2}\right)$
4 Identify the four values $k = 0,\ 1,\ 2,\ 3$ and substitute to determine the arguments.	$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
5 Write the answers in polar and Cartesian form.	$1^{\frac{1}{4}} = \text{cis}(0), \text{cis}\left(\frac{\pi}{2}\right), \text{cis}(\pi), \text{cis}\left(-\frac{\pi}{2}\right)$ $= 1 + 0i, 0 + i, -1 + 0i, 0 - i$
6 Sketch the answers on the complex plane.	

Symmetry of roots of complex numbers

In Worked example 15, the roots were symmetrically placed about the origin at equal angles of $\frac{\pi}{2}$. This will always occur with solutions to equations of the type $z^n = a$, with the roots separated by angles of $\frac{2\pi}{n}$ about the origin. They will all have the same modulus. It is therefore possible to just find the first value of the root and use this property to determine the remaining ones.

16 Using the rotational symmetry of roots of complex numbers

Determine the cube roots of $64i$ in both polar and Cartesian form and sketch them on the complex plane.

THINKING	WORKING
1 Convert the complex number to polar form.	$64i = 64\,\text{cis}\left(\frac{\pi}{2}\right)$
2 Express the complex number in the form $\text{cis}(\theta) = \text{cis}(\theta + 2k\pi), k \in \mathbb{Z}$.	$64\,\text{cis}\left(\frac{\pi}{2}\right) = 64\,\text{cis}\left(\frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$
3 Set up the required equation.	$\left(64\,\text{cis}\left(\frac{\pi}{2}\right)\right)^{\frac{1}{3}} = \left(64\,\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)\right)^{\frac{1}{3}}$ $= 4\,\text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$
4 Let $k = 0$ and substitute to determine one argument.	$\theta = \frac{\pi}{6}$
5 Deduce the other arguments using the rotational symmetry of the roots.	$\frac{2\pi}{n} = \frac{2\pi}{3}$ $\theta = \frac{\pi}{6} + \frac{2\pi}{3}, \frac{\pi}{6} + \frac{4\pi}{3}$ $\theta = \frac{5\pi}{6}, \frac{9\pi}{6}$
6 Write the answers in polar and Cartesian form.	$(64i)^{\frac{1}{3}} = 4\,\text{cis}\left(\frac{\pi}{6}\right), 4\,\text{cis}\left(\frac{5\pi}{6}\right), 4\,\text{cis}\left(-\frac{\pi}{2}\right)$ $= 2\sqrt{3} + 2i, -2\sqrt{3} + 2i, 0 - 4i$
7 Plot the answers on the complex plane.	

Technology worked example
Determine the roots of complex numbers

EXERCISE 4.3 Roots of complex numbers

Worked Example

12 13 14 15 16

1 Determine the square roots of $-5 - 12i$.

2 Determine the square roots of these complex numbers in polar form.

(a) $-25i$ **(b)** $\sqrt{3} + i$

3 Determine the required roots of these complex numbers.

(a) The cube roots of -1. **(b)** The cube roots of $-1 + \sqrt{3}i$.

(c) The fourth roots of i. **(d)** The fourth roots of $8 + 8i$.

4 Determine the cube roots of -1 and use your answers to deduce the cube roots of -27.

5 Determine the roots of the following complex numbers in Cartesian form and plot them on the complex plane.

(a) The square roots of -49. **(b)** The fourth roots of 81.

(c) The cube roots of 1. **(d)** The fourth roots of -16.

6 Consider the equation $z^5 = 32$.

(a) To solve this equation, you need to find the fifth roots of 32. Write one of these roots in polar form.

(b) Use the rotational symmetry of roots of this type of equation to determine the angle between each of the five roots.

(c) Given that the modulus of the five roots is the same, determine the remaining four roots in polar form.

7 For each of these equations, determine the number of roots and the spacing between consecutive arguments of the roots.

(a) $z^2 = -7$ **(b)** $z^5 + 2 = 0$ **(c)** $z^3 = 4i$ **(d)** $z^9 = -i$

8 Consider the equation $z^6 = 1$. To solve this equation, you need to find the sixth roots of 1.

(a) Write one of these roots in polar form.

(b) Use the rotational symmetry of roots of this type of equation to determine the angle between each of the six roots.

(c) Given that the modulus of the six roots is the same, determine the remaining five roots in polar form.

(d) Plot the six solutions on the complex plane.

9 Determine the fourth roots of 625 in polar and Cartesian form.

10 Solve the equation $z^6 = -i$.

11 If 1, u and v are the cube roots of unity, prove the following:

(a) $u + v = -1$ **(b)** $uv = 1$ **(c)** $u = v^2$

12 **(a)** Determine the square roots of $2 + 2\sqrt{3}i$ and $-2 - 2\sqrt{3}i$.

(b) Demonstrate that if z is a square root of w, then iz is a square root of $-w$.

(c) Verify that this property holds for the complex numbers in part **(a)**.

13 Which of the following statements about the sixth roots of 64 is false?

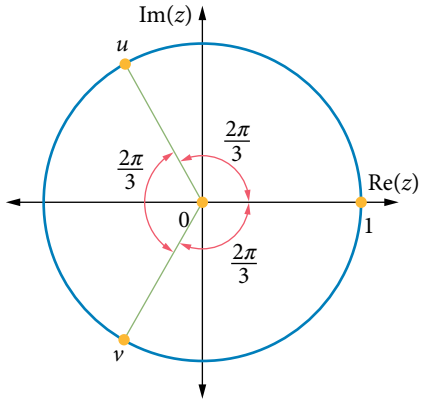
A There are six of them.

B Their arguments are separated by angles of size $\frac{\pi}{3}$.

C $4\,\mathrm{cis}(\pi)$ is one of the roots.

D There are conjugate pairs among the roots.

14 The diagram shows three complex numbers, u, v and 1.



(a) Which one of the following statements is correct?

A $\mathrm{Arg}(v) = \frac{4\pi}{3}$ **B** $u^3 = 1$

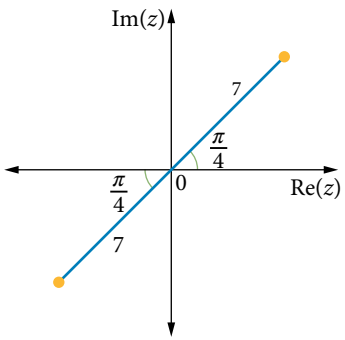
C $u = -v$ **D** u and v are cube roots of 8

(b) Explain the common errors that lead to the each of the incorrect options in part **(a)**.

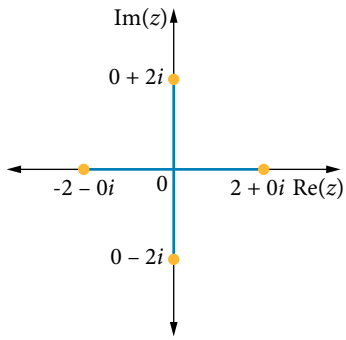
15 Without converting to polar coordinates, determine the two square roots of $5 + 12i$ in Cartesian form.

16 Determine the equation that has its roots represented in the following graphs. Express your answer in the form $z^n = x + yi$.

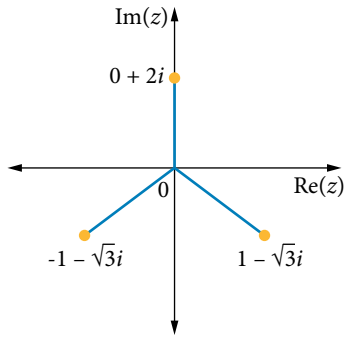
(a)



(b)



(c)



17 Consider the complex equation $z^4 + ni = 0$, where n is a positive real number.
Determine all solutions of the equation in terms of n.

18 Prove that the product of any sixth root of 1 and any twelfth root of 1 is also a twelfth root of 1.

4.4 The complex plane

Subsets of the complex plane

Just as relations such as $x^2 + y^2 = 4$ for $x, y \in \mathbb{R}$ define subsets of the real Cartesian plane and can be represented as graphs, relations such as $|z| = 2$ for $z \in \mathbb{C}$ define subsets of the complex plane (Argand plane).

When graphing these relations, it is useful to express them in terms of x and y, the real and imaginary components of the complex number. The resulting graphs are often called *loci*, which is the plural of *locus*, roughly meaning 'path'.

Remember that if $z = x + iy$, then

$|z| = \sqrt{x^2 + y^2}$

$\text{Re}(z) = x$

$\text{Im}(z) = y$

Loci involving Re(z) and/or Im(z)

17 Sketching a vertical line

Sketch the locus satisfying the condition $\text{Re}(z) = 4$.

THINKING

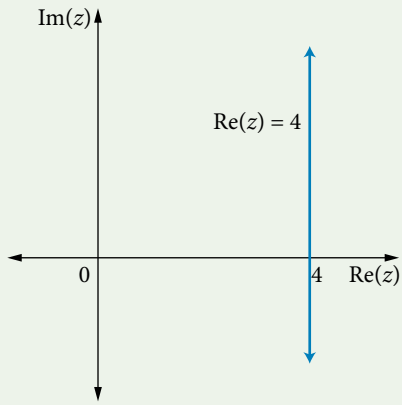
1 Replace $\text{Re}(z)$ with x and $\text{Im}(z)$ with y (if required).

2 Sketch the locus.

WORKING

$\text{Re}(z) = 4$

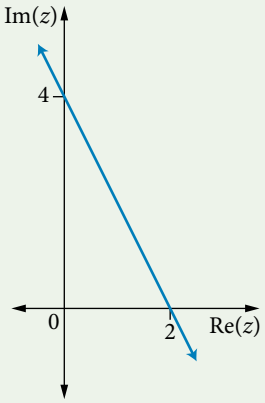
$x = 4$



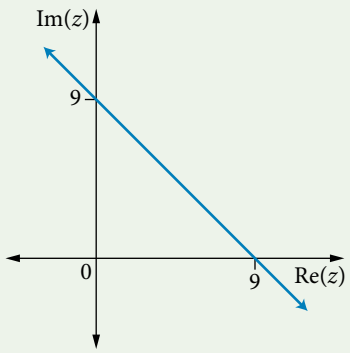
18 Sketching an oblique straight line

Sketch the locus satisfying the given condition.

(a) $2\text{Re}(z) + \text{Im}(z) = 4$.

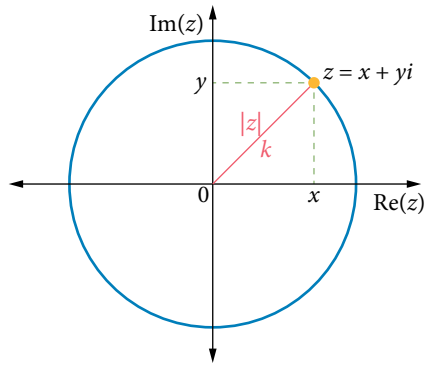
THINKING	WORKING
1 Replace $\text{Re}(z)$ with x, and replace $\text{Im}(z)$ with y.	$2x + y = 4$ $y = -2x + 4$
2 Sketch the locus.	

(b) $\text{Re}(z - 2 + i) + \text{Im}(z - 2 - 5i) = 2$

1 Replace $\text{Re}(z)$ with x and $\text{Im}(z)$ with y, collecting real and imaginary components.	$\text{Re}(z - 2 + i) = \text{Re}(z) + \text{Re}(-2) + \text{Re}(i)$ $= \text{Re}(x + yi) + \text{Re}(-2) + \text{Re}(i)$ $= x - 2$ $\text{Im}(z - 2 - 5i) = \text{Im}(z) + \text{Im}(-2) + \text{Im}(-5i)$ $= \text{Im}(x + yi) + \text{Im}(-2) + \text{Im}(-5i)$ $= y - 5$ $(x - 2) + (y - 5) = 2$
2 Organise the linear equation.	$y = -x + 9$
3 Sketch the locus.	

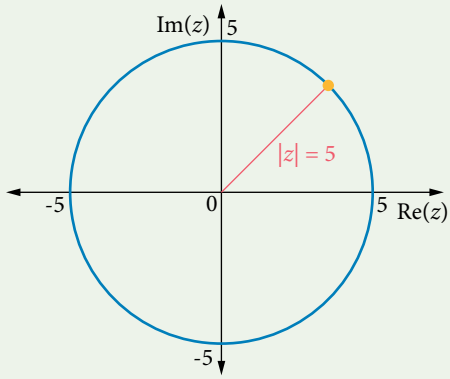
Loci involving modulus

$|z|$ or r represent the modulus of a complex number, which is the equivalent to the distance from the origin. Thus, the locus $|z| = k$ can be thought of as the set of all complex numbers that are a fixed distance, k, from the origin. This yields a circle, with centre at the origin and radius k.



19 Sketching a circle with centre at the origin

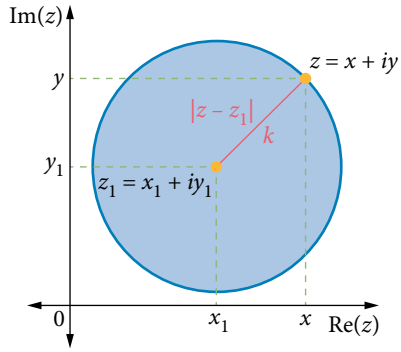
Sketch the locus defined by $|z| = 5$.

THINKING	WORKING
1 Express the relation in terms of x and y.	$\sqrt{x^2 + y^2} = 5$ $x^2 + y^2 = 25$
2 Identify the locus.	All points are 5 units from the origin. This is a circle with centre at the origin and radius 5 units.
3 Sketch the graph.	

If z_1 is a fixed complex number, the expression $|z - z_1|$ can be interpreted as the distance between the general point $z = x + iy$ and the fixed point $z_1 = x_1 + iy_1$.

Consider the relation $|z - z_1| = k$:

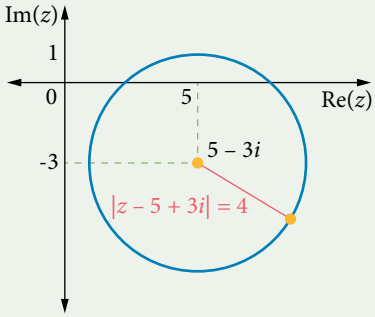
$$|z - z_1| = k$$
$$|x + iy - (x_1 + iy_1)| = k$$
$$|(x - x_1) + i(y - y_1)| = k$$
$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = k$$
$$(x - x_1)^2 + (y - y_1)^2 = k^2$$



This locus is a circle with centre $z_1 = x_1 + iy_1$ and radius k. It can be seen as a circle with radius k that has been translated x_1 units parallel to the x-axis and y_1 units parallel to the y-axis.

20 Sketching any circle

Sketch the locus satisfying the condition $|z - 5 + 3i| = 4$.

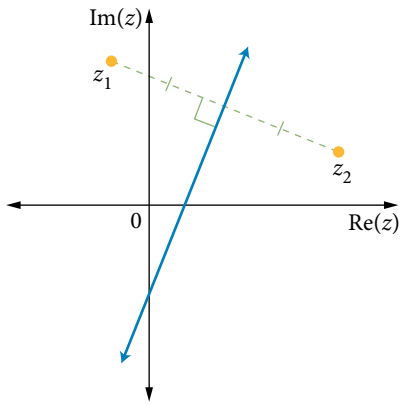
THINKING	WORKING
1 Substitute with z with $x + iy$.	$\lvert x + iy - 5 + 3i \rvert = 4$
2 Identify the locus.	Note that $\lvert z - 5 + 3i \rvert = 4$ can be rewritten as $\lvert z - (5 - 3i) \rvert = 4$. This represents a circle of radius 4 that has been translated 5 units to the right and 3 units down. It has centre $5 - 3i$. (This can be verified by squaring both sides and determining the Cartesian equation.) $\sqrt{(x-5)^2 + (y+3)^2} = 4$ $(x-5)^2 + (y+3)^2 = 4^2$
3 Sketch the graph.	

It is possible to approach the above example, and many others, in three different ways:

The Cartesian equation	Recognising the graph as a circle with centre $(5, -3)$ and radius 4.
The transformation	Regarding the graph as a circle centred on the origin with radius 4 that has been translated 5 units in the positive real direction and 3 units in the negative imaginary direction.
The definition	Considering the graph to be the locus of all points which are a distance of 4 units from the point $5 - 3i$.

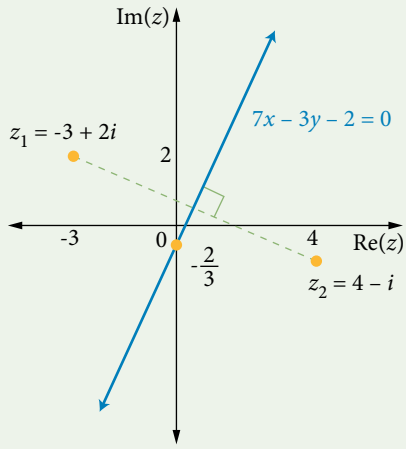
Some examples might be better suited to a particular approach, but often any of the three methods can be used.

The locus represented by a relation of the type $|z - z_1| = |z - z_2|$ can be interpreted as the path of a general point that moves equidistant from *two* fixed points. This will then yield the line that is the perpendicular bisector of the line segment joining z_1 and z_2.



21 Sketching the locus of a point which is equidistant from two fixed points

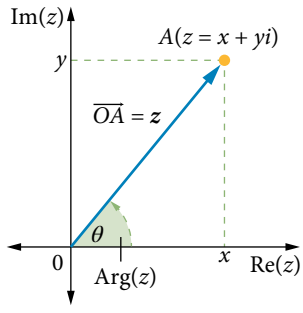
Sketch the locus satisfying the condition $|z + 3 - 2i| = |z - 4 + i|$ and describe it geometrically.

THINKING	WORKING
1 Replace z with $x + iy$.	$\|z + 3 - 2i\| = \|z - 4 + i\|$ $\|(x + iy) + (3 - 2i)\| = \|(x + iy) + (-4 + i)\|$
2 Collect real and imaginary components.	$\|(x + 3) + i(y - 2)\| = \|(x - 4) + i(y + 1)\|$
3 Express the modulus and square both sides.	$\sqrt{(x + 3)^2 + (y - 2)^2} = \sqrt{(x - 4)^2 + (y + 1)^2}$ $(x + 3)^2 + (y - 2)^2 = (x - 4)^2 + (y + 1)^2$
4 Expand the brackets and simplify the equation.	$x^2 + 6x + 9 + y^2 - 4y + 4 = x^2 - 8x + 16 + y^2 + 2y + 1$ $6x - 4y + 13 = -8x + 2y + 17$ $14x - 6y - 4 = 0$ $7x - 3y - 2 = 0$
5 Sketch the graph and interpret the result.	

The linear equation is the perpendicular bisector of the line segment joining the points $-3 + 2i$ and $4 - i$.

Loci involving Arg(z)

A complex number z can be represented by a vector for which the modulus gives the magnitude and the argument gives the direction. The locus defined by the relation $\text{Arg}(z) = \theta$ is a ray from the origin making an angle of θ with the positive direction of the real axis.



22 Sketching the locus defined by $\text{Arg}(z) = \theta$

Sketch the locus that satisfies the condition $\text{Arg}(z) = \frac{\pi}{6}$.

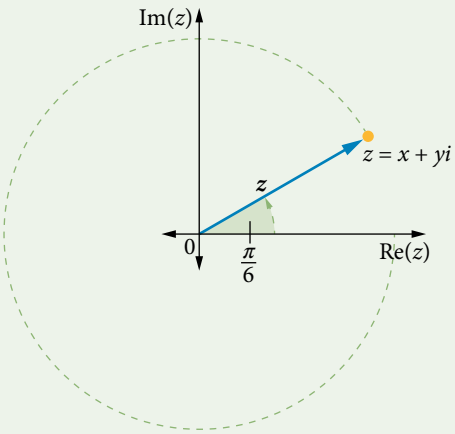
THINKING

1 Identify the type of locus. $\text{Arg}(z) = \theta$ represents a ray from the origin making an angle θ with the positive real axis.

2 Sketch the graph.

WORKING

$\text{Arg}(z) = \frac{\pi}{6}$ represents a ray from the origin making an angle $\frac{\pi}{6}$ with the positive real axis.



Consider the complex numbers $z = x + iy$ and $z_1 = x_1 + iy_1$:

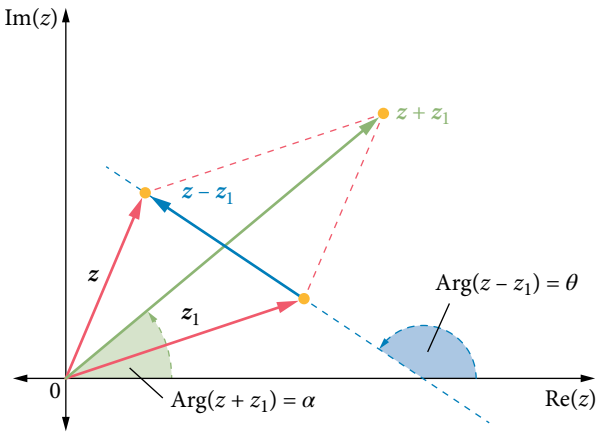
- $z = x + iy$ corresponds to the vector $\boldsymbol{z} = \begin{bmatrix} x \\ y \end{bmatrix}$
- $z_1 = x_1 + iy_1$ corresponds to the vector $\boldsymbol{z}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$
- The complex number $z - z_1$ and $z + z_1$ corresponds to the vectors $\boldsymbol{z} - \boldsymbol{z}_1$ and $\boldsymbol{z} + \boldsymbol{z}_1$ respectively.
- $\text{Arg}(z - z_1) = \theta$ and $\text{Arg}(z + z_1) = \alpha$ defines the angles at which the vectors are inclined to the positive real axis.

Note that the above can be shown by considering how complex numbers can be represented by vectors on the complex plane.

Making connections

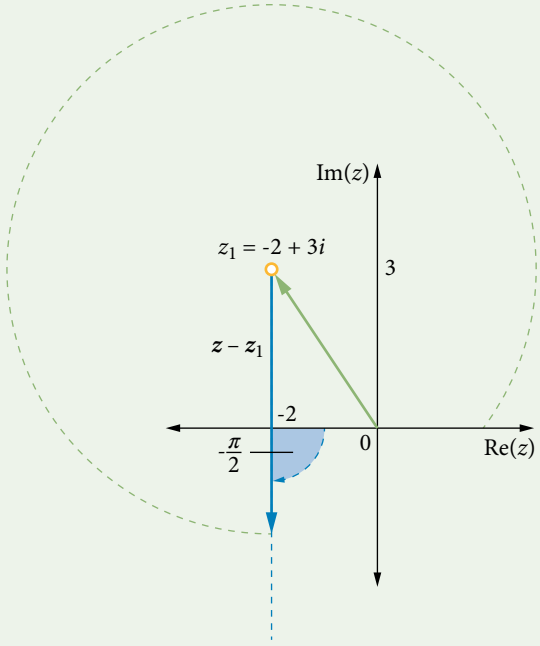
Addition of complex numbers

For the complex number $a + bi$ and $c + di$, move sliders for a, b, c and d and observe how $(a + c) + (b + d)i$ moves around the complex plane.



23 Sketching the locus defined by $\text{Arg}(z - z_1) = \theta$

Sketch the locus that satisfies the condition $\text{Arg}(z + 2 - 3i) = -\frac{\pi}{2}$.

THINKING	WORKING
1 Express in the form $\text{Arg}(z - z_1)$.	$\text{Arg}(z - (-2 + 3i)) = -\frac{\pi}{2}$
2 Identify the starting point for the ray and the angle made with the positive real axis.	$z_1 = -2 + 3i$
3 Note that the starting point is *not* included in the locus, as the argument is undefined at that point.	$\theta = -\frac{\pi}{2}$ This represents all complex values of z such that $z + 2 - 3i$ lies on the negative imaginary axis.
4 Sketch the graph.	 Note that the vector z lies on a vertical line with starting point $-2 + 3i$.

e Explore further

Drawing regions on the plane

Draw regions on the plane that involve Cartesian or polar conditions, or both.

t Technology worked example

Loci of points on the complex plane

WARNING

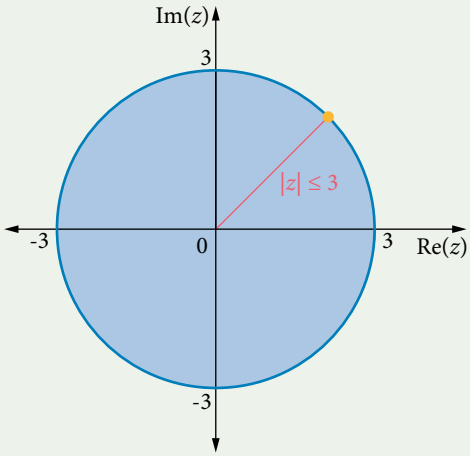
For loci containing $\text{Arg}(z)$, the starting point for the ray is represented by an open circle. This shows that the point is not included in the locus. An argument is only meaningful when a direction is defined, and a single point has no direction.

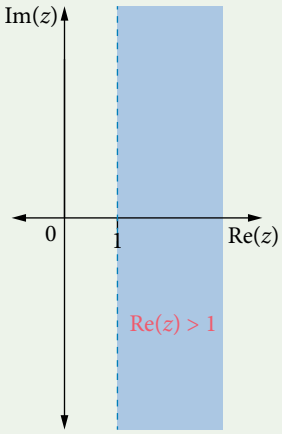
Inequalities can be represented in the complex plane and used to solve problems such as the following worked example.

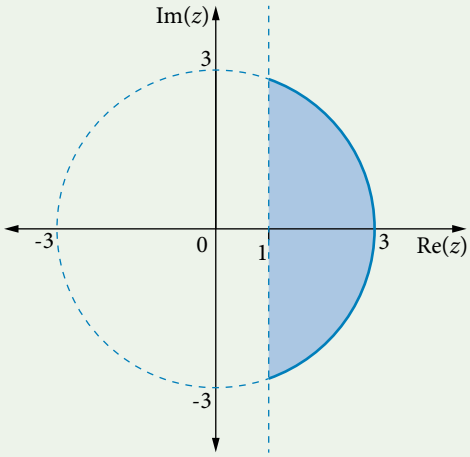
24 Sketching regions of the complex plane

Sketch the region defined by $\{z : |z| \leq 3\} \cap \{z : \text{Re}(z) > 1\}$.

THINKING	WORKING
1 Identify each of the two regions.	$\|z\| \leq 3$ represents the points inside and on a circle with centre at the origin and radius 3. $\text{Re}(z) > 1$ represents the points for which $x > 1$.
2 Sketch the regions. Because both conditions hold, the relevant locus is given by the intersection of the two regions (shaded).	Note that there is a solid boundary for the circle and a dotted one for the line, as points on the line are not included.







EXERCISE 4.4 The complex plane

Worked Example 17

1 Sketch the following loci.

(a) $\text{Re}(z) = 2$

(b) $\text{Im}(z) = -3$

Worked Example

18

2 Sketch the loci defined by the following:

(a) $\text{Re}(z) = \text{Im}(z)$ (b) $\text{Re}(z) = \text{Im}(z) + 2$

(c) $4\,\text{Re}(z) - 3\,\text{Im}(z) = 12$ (d) $2\,\text{Re}(z + 2 + i) + \text{Im}(z - 3 + 2i) = 1$

19

3 Sketch the following subsets of the complex plane.

(a) $|z| = 4$ (b) $|z| = \sqrt{7}$

20

4 Sketch the following subsets of the complex plane.

(a) $|z - 2| = 1$ (b) $|z + 4 - 3i| = 5$ (c) $|z - 1 - i| = \sqrt{2}$ (d) $|z + 3 - 2i| = 6$

21

5 Sketch the locus satisfying the following conditions.

(a) $|z + 2 - i| = |z - 3|$ (b) $|z - 4| = |z - 6|$ (c) $|z + 2 + i| = |z - 2 - i|$

22

6 Sketch the following subsets of the complex plane.

(a) $\text{Arg}(z) = \dfrac{2\pi}{3}$ (b) $\text{Arg}(z) = -40°$ (c) $\text{Arg}(z) = -\dfrac{\pi}{2}$ (d) $\text{Arg}(\overline{z}) = -\dfrac{\pi}{6}$

23

7 Sketch the following subsets of the complex plane.

(a) $\text{Arg}(z - 1) = \dfrac{\pi}{2}$ (b) $\text{Arg}(z + i) = \dfrac{\pi}{4}$ (c) $\text{Arg}(z + 1 - i) = -\dfrac{3\pi}{4}$ (d) $\text{Arg}(z - 2 + 4i) = 150°$

8 Sketch the locus of a point defined by $\left\{z : \text{Arg}(z) = \dfrac{\pi}{3}\right\} \cup \left\{z : \text{Arg}(z) = -\dfrac{5\pi}{6}\right\}$.

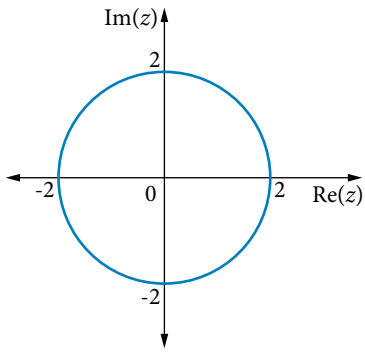
9 Use technology to sketch these subsets of the complex plane.

(a) $\text{Im}(z) - 5\,\text{Re}(z) = 10$ (b) $|z| = 9$ (c) $|z + 1 - 3i| = 4$

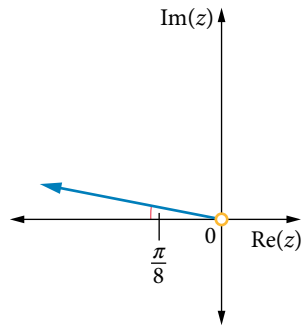
(d) $\text{Arg}(z) = -\dfrac{\pi}{4}$ (e) $\text{Arg}(z + 4 - 2i) = \dfrac{2\pi}{3}$

10 Determine the subsets of the complex plane that have the following graphs.

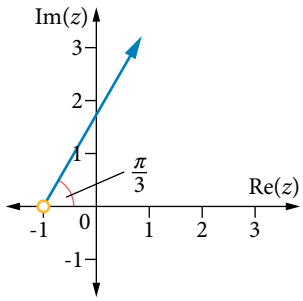
(a)



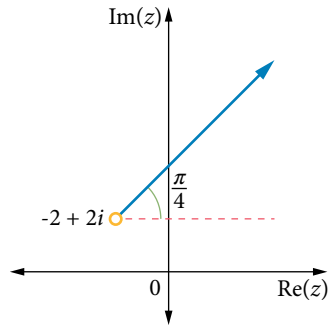
(b)



(c)

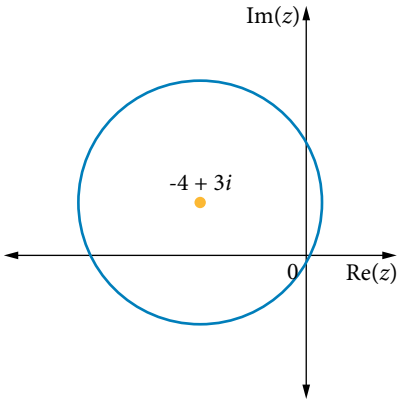


(d)

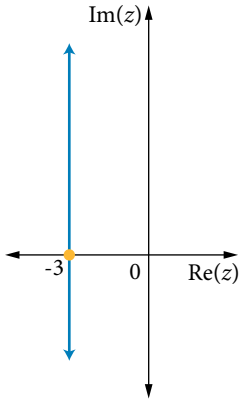


11 Determine the subsets of the complex plane that have the following graphs.

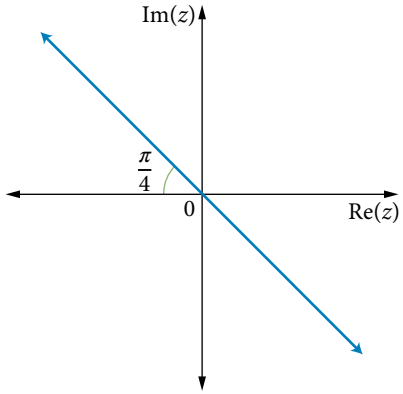
(a)



(b)



12 Which of the following statements in *not* a description of the graph?



A $\text{Re}(z) + \text{Im}(z) = 0$

B $\left\{z : \text{Arg}(z) = \frac{3\pi}{4}\right\} \cup \left\{z : \text{Arg}(z) = -\frac{\pi}{4}\right\} \cup \{0\}$

C $\text{Arg}(z) = \frac{3\pi}{4}$

D $y = -x$

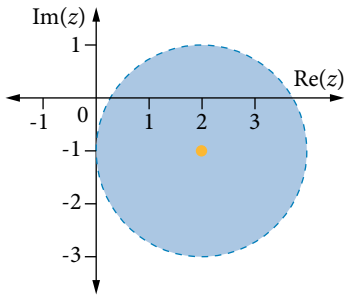
Worked Example 24

13 Sketch the regions in the complex plane defined as follows:

(a) $\text{Re}(z) \leq 5$ **(b)** $\text{Im}(z) > -1$ **(c)** $\text{Re}(z) + \text{Im}(z) < 2$ **(d)** $|z| < 6$

(e) $|z + 1 + i| \leq 4$ **(f)** $0° < \text{Arg}(z) < 60°$ **(g)** $\{z : |z| > 3\} \cap \{z : |z - 3| < 3\}$

14 Which of the following is the correct relation for the graph?



A $|z + 2 - i| \leq 2$

B $|z + 2 - i| < 2$

C $|z - 2 + i| \leq 2$

D $|z - 2 + i| < 2$

15 Sketch the region represented by $-\frac{\pi}{4} \leq \text{Arg}(z + 2 - i) \leq \frac{3\pi}{4}$.

16 Determine the point of intersection of the regions defined by $\text{Arg}(z) = \frac{3\pi}{4}$ and $\text{Im}(z) - 2\text{Re}(z) = 2$.

17 Prove that if $|z| = 2$ and $|z - 2\sqrt{2} - 2\sqrt{2}i| = 2$, then $\text{Re}(z) = \text{Im}(z)$.

18 If $|z + 1| = 3$, determine the greatest and least possible values for $|z|$.

19 Given $|z| \leq 2$ and $|z - 2 - 2i| \leq 2$, explain why $|z| \geq 2\left(\sqrt{2} - 1\right)$.

4.5 The remainder and factor theorem

Polynomials

A polynomial is an expression consisting of a sum of terms of the form ax^n, where n is a natural number or zero.

WARNING

An expression containing negative, fractional or irrational powers is not a polynomial.

Any polynomial can be expressed in the form:
$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x$.

The numbers represented by a_n, $a_{n-1}, \ldots a_2$, a_1, and a_0 are called coefficients.

The degree of the polynomial is the highest power of x present.

In this section, only real factors and real solutions are considered. In the next section, you will use the techniques covered here to determine complex factors of polynomials complex solutions to polynomial equations.

Division of polynomials

The four terms in any division have specialised names:

If b is a factor of a then:

$$\frac{a}{b} = \frac{\text{dividend}}{\text{divisor}} = \text{quotient} = c$$

If b is not a factor of a then:

$$\frac{a}{b} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}} = c + \frac{r}{b}$$

In general:

$\text{dividend} = \text{quotient} \times \text{divisor} + \text{remainder}$

For example:

$$\frac{16}{5} = 3 + \frac{1}{5} \rightarrow 16 = 3 \times 5 + 1$$

The remainder theorem

Before calculators were in common use, division was done using a multi-stage algorithm called long division. This algorithm is sometimes referred to as synthetic division.

Key points for synthetic division:

STEP 1: Both the numerator and denominator should be arranged so that each is in descending powers of x.

STEP 2: If any terms with consecutive powers of x are missing, these should be included by using 0 as their coefficient.

STEP 3: Each term of the quotient is obtained by dividing the first term of the divisor into the first term of the new dividend.

STEP 4: The result is written above the dividend.

STEP 5: After each division, the next step is to subtract and then bring down the next term.

25 Using synthetic division

Determine the quotient and remainder when $x^4 - 3x^3 + 2x^2 + x - 4$ is divided by $x + 3$.

THINKING	WORKING
1 Set up the divisor and the dividend for long division.	$x+3\overline{)x^4-3x^3+2x^2+x-4}$
2 Follow the steps in the algorithm:	$\begin{array}{r} x^3 \\ x+3\overline{)x^4-3x^3+2x^2+x-4} \\ \underline{-\left(x^4+3x^3\right)} \\ -6x^3+2x^2 \end{array}$
	$\begin{array}{r} x^3-6x^2 \\ x+3\overline{)x^4-3x^3+2x^2+x-4} \\ \underline{-\left(x^4+3x^3\right)} \\ -6x^3+2x^2 \\ \underline{-\left(-6x^3-18x^2\right)} \\ 20x^2+x \end{array}$
	$\begin{array}{r} x^3-6x^2+20x \\ x+3\overline{)x^4-3x^3+2x^2+x-4} \\ \underline{-\left(x^4+3x^3\right)} \\ -6x^3+2x^2 \\ \underline{-\left(-6x^3-18x^2\right)} \\ 20x^2+x \\ \underline{-\left(20x^2+60x\right)} \\ -59x-4 \end{array}$
	$\begin{array}{r} x^3-6x^2+20x-59 \\ x+3\overline{)x^4-3x^3+2x^2+x-4} \\ \underline{-\left(x^4+3x^3\right)} \\ -6x^3+2x^2 \\ \underline{-\left(-6x^3-18x^2\right)} \\ 20x^2+x \\ \underline{-\left(20x^2+60x\right)} \\ -59x-4 \\ \underline{-(-59x-177)} \\ 173 \end{array}$
3 Identify the quotient and remainder.	The quotient is $x^3 - 6x^2 + 20x - 59$. The remainder is 173.

The remainder theorem provides a quick and convenient method for finding the remainder when a polynomial is divided by a linear expression.

The remainder theorem states, in its simplest form, that when a polynomial, $p(x)$, is divided by $x - a$, the remainder is $p(a)$.

Consider the function from the previous worked example, $p(x) = x^4 - 3x^3 + 2x^2 + x - 4$ and $p(-3) = 173$.

Proof:

Let $p(x)$ be a polynomial in x, with quotient, Q, and remainder, R, when $p(x)$ is divided by $x - a$.

Then, in all cases, $p(x) \equiv (x - a)Q + R$. (Note that $\equiv$ means identically equal.)

Since this is true for all values of x, then it is true for $x = a$.

Substituting $x = a$ gives:

$$\begin{aligned} p(a) &= (a - a)Q + R \\ &= 0 \times Q + R \\ &= R \end{aligned}$$

26 Determining a remainder using the simpler form of the remainder theorem

Determine the remainder when $p(x) = x^4 - 2x^3 + x - 5$ is divided by $x + 1$.

THINKING	WORKING
1 Express the divisor in the form $x - a$.	$x + 1 = x - (-1)$
2 Substitute for a in the polynomial to calculate the value of $p(a)$.	$\begin{aligned} p(-1) &= (-1)^4 - 2(-1)^3 + (-1) - 5 \\ &= 1 + 2 - 1 - 5 \\ &= -3 \end{aligned}$
3 Interpret the answer.	The remainder is -3.

More generally, the remainder theorem states that when any polynomial, $p(x)$, is divided by a linear expression $ax - b$, then the remainder is $p\left(\frac{b}{a}\right)$.

27 Determining a remainder using the general remainder theorem

Determine the remainder when $p(x) = x^3 - 2x + 1$ is divided by $3x - 2$.

THINKING	WORKING
1 Identify the values of a and b in the divisor.	$ax - b = 3x - 2$ $\frac{b}{a} = \frac{2}{3}$
2 Substitute for $\frac{b}{a}$ into the polynomial to calculate the value of $p\left(\frac{b}{a}\right)$.	$p(x) = x^3 - 2x + 1$ $p\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right) + 1$ $= \frac{8}{27} - \frac{4}{3} + 1$ $= -\frac{1}{27}$
3 Interpret the answer.	The remainder is $\frac{-1}{27}$.

Technology can be used to divide polynomials to determine the quotient and remainder.

Technology worked example
Factor theorem and solving polynomial equations

The factor theorem

The factor theorem is a convenient method to use when attempting to factorise a polynomial.

The factor theorem, in its simplest form, states that $(x - a)$ will be a factor of a polynomial, $p(x)$, if and only if $p(a) = 0$.

Remember that 'if and only if' is a two-way implication. It signifies that, in this case:

$(x - a)$ is a factor of $p(x)$ implies that $p(a) = 0$,

and also

$p(a) = 0$ implies that $(x - a)$ is a factor of $p(x)$.

For example, the parabola with equation $y = x^2 - 3x - 4$ has a factor of $x - 4$ if and only if the graph has an x-intercept at $x = 4$, i.e. one of the roots of the equation $x^2 - 3x - 4 = 0$ is 4.

Proof:

Let $p(x)$ be a polynomial in x, and let Q be the quotient when $p(x)$ is divided by $x - a$.

If $(x - a)$ is a factor of $p(x)$, then it divides exactly with zero remainder.

That is, $p(x) \equiv (x - a) \times Q + 0$

This is true for all values of x. Therefore, it is true for $x = a$.

Substituting $x = a$ gives:

$$\begin{aligned} p(a) &= (a-a)\times Q \\ &= 0\times Q \\ &= 0 \end{aligned}$$

The proof of the implication the other way is as follows. If $p(a) = 0$, then, by the remainder theorem, when $p(x)$ is divided by $(x - a)$, the remainder is zero. But if there is zero remainder, then $(x - a)$ must divide exactly into $p(x)$. Thus, $(x - a)$ is a factor of $p(x)$.

Using this theorem for factorisation relies on substituting carefully chosen numbers into the polynomial until the value 0 is obtained, thus indicating a zero remainder and hence a factor.

First try integer factors of the constant term, but note that the factors of some polynomials may involve fractions. For example, if $p\left(\frac{1}{2}\right) = 0$, then $(2x - 1)$ is a factor.

28 Factorising a polynomial

Factorise the following polynomials.

(a) $p(x) = x^3 - 3x - 2$

THINKING	WORKING
1 Determine the possible values of x that may give zero. This can be done by identifying the factors of the constant term.	The factors of -2 are ± 1 and ± 2.
2 Substitute factors into $p(x)$ until a solution is found.	$p(1) = 1^3 - 3(1) - 2$ $= 1 - 3 - 2$ $\neq 0$ Given that $x = 1$ is not a solution to $p(x) = 0$, then $(x - 1)$ is not a factor. $p(-1) = (-1)^3 - 3(-1) - 2$ $= -1 + 3 - 2$ $= 0$ Given that $x = -1$ is a solution to $p(x) = 0$, then $(x + 1)$ is a factor.
3 Use long division or an inspection method to determine the remaining factors.	$x^3 - 3x - 2 = (x + 1)\left(x^2 + bx - 2\right)$ Equate coefficients: $-3x = 1\times bx + x\times -2$ $-3x = bx - 2x$ $\therefore -3 = b - 2$ $b = -1$ Therefore: $x^3 - 3x - 2 = (x + 1)\left(x^2 - x - 2\right)$ $= (x + 1)(x - 2)(x + 1)$ $= (x + 1)^2(x - 2)$

(b) $p(x) = 2x^3 + 5x^2 - x - 6$

1 Determine the possible values of x that may give zero. This can be done by identifying the factors of the constant term.	The factors of -6 are ± 1, ± 2, ± 3 and ± 6.
2 Substitute factors into $p(x)$ until a solution is found.	$p(1) = 2\times(1)^3 + 5\times(1)^2 - (1) - (6)$ $= 2 + 5 - 1 - 6$ $= 0$ Given that $x = 1$ is a solution to $p(x) = 0$, then $(x-1)$ is a factor.
3 Use long division or an inspection method to determine the remaining factors.	$2x^3 + 5x^2 - x - 6 = (x-1)\left(2x^2 + bx + 6\right)$ Equating coefficients: $5x^2 = -1\times 2x^2 + x\times bx$ $5x^2 = -2x^2 + bx^2$ $\therefore 5 = -2 + b$ $b = 7$ Therefore: $2x^3 + 5x^2 - x - 6 = (x-1)\left(2x^2 + 7x + 6\right)$
4 Simplify the quadratic factor.	$\left(2x^2 + 7x + 6\right) = 2x^2 + 3x + 4x + 6$ $= x(2x+3) + 2(2x+3)$ $= (x+2)(2x+3)$
5 Write the complete factorisation.	$2x^3 + 5x^2 - x - 6 = (x-1)\left(2x^2 + 7x + 6\right)$ $= (x-1)(x+2)(2x+3)$

Alternatively, once one factor has been determined, then others can be obtained by long division. Thus, having found that $(x-1)$ is a factor of $2x^3 + 5x^2 - x - 6$, divide by $x - 1$:

$$
\begin{array}{r}
2x^2 + 7x + 6 \\
x - 1 \overline{)\,2x^3 + 5x^2 - x - 6} \\
\underline{-\left(2x^3 - 2x^2\right)} \\
7x^2 - x \\
\underline{-\left(7x^2 - 7x\right)} \\
6x - 6 \\
\underline{-(6x - 6)} \\
0
\end{array}
$$

Explore further

Roots of polynomials

Use graphing software to find the roots of polynomials.

Then factorise the quotient $2x^2 + 7x + 6$:

$2x^2 + 7x + 6 = (2x + 3)(x + 2)$

The complete factorisation is:

$2x^3 + 5x^2 - x - 6 = (x - 1)(2x + 3)(x + 2)$

Any quadratic expression may be factorised using the quadratic formula, so finding one factor of a cubic polynomial guarantees that the expression can be fully factorised.

This division method may be used in all cases. It is particularly useful when the coefficient of the highest power of x is not 1.

Technology worked example
Linear divisors and synthetic division

The factor theorem can also be used to solve equations. The solutions or roots can be determined directly from the factors.

WARNING

Be careful to distinguish between factors of polynomials and solutions (or roots) of equations:

- Determine factors of polynomials.
  e.g. $(x - 2)$ and $(x + 1)$ are the factors of the polynomial $x^2 - x - 2$.
- Determine solutions (or roots) of equations.
  e.g. 2 and -1 are the roots of the equation $x^2 - x - 2 = 0$ because $x^2 - x - 2 = (x + 1)(x - 2)$, and when this is set to zero, the null factor law gives these solutions.

29 Solving an equation using the factor theorem

Solve the equation $2x^3 + 5x^2 - x - 6 = 0$.

THINKING	WORKING
1 Determine the factors of the expression on the left-hand side and write the equation in factorised form.	The factors for the polynomial have already been found in Worked example 28. $(2x + 3)(x - 1)(x + 2) = 0$
2 Use the null factor law.	$2x + 3 = 0$ $x - 1 = 0$ $x + 2 = 0$
3 Write the solutions.	$x = -\frac{3}{2}, 1$ or -2

EXERCISE 4.5 The remainder and factor theorem

Worked Example

1 Divide the first expression by the second to determine the quotient and remainder. 25

(a) $x^3 + 3x^2 - 2x + 1; x + 1$ (b) $x^4 + 2x^3 - x^2 + x - 5; x - 1$ (c) $x^4 + 3x^3 - x + 4; x + 2$

(d) $x^4 + x^2 + 1; x + 1$ (e) $x^3 - 5x^2 + 2x + 4; x - 3$ (f) $2x^3 - 5x^2 + x + 14; 2x - 3$

(g) $6x^4 - 20x^2 - 3x + 8; 3x + 3$

2 $y = \dfrac{3x-5}{x-4}$ can be expressed in the form $y = P + \dfrac{Q}{x-4}$. Determine the values of P and Q.

3 Determine the values of P and Q if $x^3 + 27 = (x + 3)(x^2 + Px + Q)$.

4 Determine the remainder when the following divisions are carried out. 26

(a) $p(x) = x^2 - 4x + 1$ divided by $x - 1$ (b) $p(x) = x^2 - 4x + 1$ divided by $x + 2$

(c) $p(x) = 3x^2 + 4x + 1$ divided by $x + 1$ (d) $p(x) = 3x^3 + 5x^2 - x + 6$ divided by $x - 2$

(e) $p(x) = x^3 + 2x^2 - 3x + 1$ divided by $x + 3$ (f) $p(x) = 2x^3 + 5x - 6$ divided by $x + 4$

5 Determine the remainder when each of these divisions is carried out. 27

(a) $p(x) = x^2 - 4x + 1$ divided by $2x + 1$ (b) $p(x) = x^3 + 2x^2 - 3x + 1$ divided by $2x - 3$

(c) $p(x) = x^3 + 5x^2 - x + 6$ divided by $3x + 1$ (d) $p(x) = 3x^2 + 4x - 1$ divided by $2x - 1$

6 Determine the remainder when $p(x) = 7x^5 - x^4 + 15x^2 - 6x - 3$ is divided by x.

7 Verify that $x - 2$ is a factor of $p(x) = x^3 - 11x + 14$. 28

8 Verify that $x + 1$ is a factor of $p(x) = x^4 - x^3 + x^2 - 2x - 5$.

9 Use the factor theorem to express the following as the product of linear factors.

(a) $p(x) = x^3 - 4x^2 + x + 6$ (b) $p(x) = x^3 - 4x^2 - x + 4$

(c) $p(x) = x^3 + 3x^2 - x - 3$ (d) $p(x) = x^3 + 6x^2 + 3x - 10$

(e) $p(x) = x^3 + 3x^2 - 4x - 12$ (f) $p(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$

10 Use the factor theorem to express the following as the product of linear factors.

(a) $p(x) = 2x^3 + 3x^2 - 18x + 8$ (b) $p(x) = 2x^3 + x^2 - 2x - 1$

(c) $p(x) = 3x^3 + 8x^2 - 7x - 12$ (d) $p(x) = 4x^3 + 13x^2 - 32x + 15$

11 The cubic polynomials in this question each have three factors, two of which are identical. Factorise each polynomial fully.

(a) $p(x) = x^3 - 3x + 2$ (b) $p(x) = x^3 + 2x^2 - 4x - 8$ (c) $p(x) = 4x^3 - 11x^2 + 10x - 3$

12 Solve the following equations. 29

(a) $3x^3 - 7x^2 - 7x + 3 = 0$ (b) $x^3 - 4x^2 + x + 6 = 0$

(c) $x^3 + x^2 - 8x - 12 = 0$ (d) $4x^3 - 5x^2 - 7x + 2 = 0$

13 Consider the polynomial $p(x) = 5x^3 - 8x^2 - 19x - 6 = 0$.

(a) Determine the remainder when the polynomial is divided by $x - 7$.

(b) Factorise the polynomial into three linear factors.

14 Given that $p(x) = x^3 - 3ax^2 - 8x - 8$ has remainder 8 when divided by $x - 4$, determine the value of a.

15 Given that the remainder is 10 when $p(x) = 2x^3 + 5x^2 - 5mx + 16$ is divided by $x + 2$, determine the value of m.

16 The polynomial $p(x) = x^3 + 2x^2 + qx - 8$ has the same remainder when divided either by $x + 1$ or $x - 2$. Determine the value of q.

17 The two polynomials $p(x) = x^2 - 2ax - 6$ and $q(x) = x^3 - 8x + 9a$ both have the same remainder when divided by $x + 3$. Determine the value of a.

18 Solve the equation $x^5 + 4x^4 - 10x^2 - x + 6 = 0$.

19 Prove the general form of the remainder theorem.

20 If $\dfrac{1 - x^4}{1 - x} = Ax^3 + Bx^2 + Cx + D$, determine the values of A, B, C and D.

21 The polynomial $p(x) = x^3 + ax^2 - 3x + 2b$ has a remainder of 4 when divided by $x - 3$, and $q(x) = x^3 + ax^2 - x - 3b$ has a remainder of -8 when divided by $x - 1$. Determine the values of a and b.

22 If $x - q$ is a factor of $p(x) = x^3 + 2x^2 - 9x - 18$, determine the possible values of q.

23 Determine the quotient and remainder for $\dfrac{4x^3 - 5x^2 + 12}{x^2 + x - 1}$.

24 What is the remainder when $4x^2 - x + 11$ is divided by $2x^2 + 3x - 7$?

25 The equation $x^4 + 3x^3 - 5x^2 - 6x - 8 = 0$ has only two real roots. Determine these roots and explain why there are only two of them.

Complex factors of polynomials

4.6

In the previous section you considered polynomials with real factors, which were factorised over $\mathbb{R}$, and led to equations that had real roots. In this section, factorisation and solutions of equations will be extended to include factors and roots that are complex.

The fundamental theorem of algebra plays an important role throughout this section. As a reminder:

The fundamental theorem of algebra states that a polynomial of degree n has n linear factors in $\mathbb{C}$ (the set of complex numbers). Hence a polynomial equation of degree n has n roots in $\mathbb{C}$ (provided that repeated roots are counted the appropriate number of times).

Quadratic polynomials

The same techniques you have used for factorisation can still be effective; completing the square, difference of squares, the factor theorem etc. When solving quadratic equations, the quadratic formula can be used if factors are not obvious, or for any roots that are irrational or non-real.

Using complex numbers allows you to factorise the sum of two squares.

$$\begin{aligned} a^2 + b^2 &= a^2 - b^2 i^2 \\ &= (a + ib)(a - ib) \end{aligned}$$

30 Factorising the sum of two squares over $\mathbb{C}$

Factorise the following expressions into linear factors.

(a) $5z^2 + 20$

THINKING	WORKING
1 Extract the common factor first.	$5z^2 + 20 = 5(z^2 + 4)$
2 Apply $i^2 = -1$ to generate the difference of two squares.	$\begin{aligned} 5(z^2 + 4) &= 5(z^2 - 4i^2) \\ &= 5(z^2 - (2i)^2) \end{aligned}$
3 Factorise using difference of squares.	$5z^2 + 20 = 5(z + 2i)(z - 2i)$

(b) $16z^4 - 625$

1 Factorise into two quadratic factors using difference of squares.	$16z^4 - 625 = \left(4z^2 - 25\right)\left(4z^2 + 25\right)$
2 Apply $i^2 = -1$ to generate the difference of two squares.	$\begin{aligned} \left(4z^2 - 25\right)\left(4z^2 + 25\right) &= \left(4z^2 - 25\right)\left(4z^2 - 25i^2\right) \\ &= \left((2z)^2 - 5^2\right)\left((2z)^2 - (5i)^2\right) \end{aligned}$
3 Factorise again, using difference of squares.	$16z^4 - 625 = (2z - 5)(2z + 5)(2z - 5i)(2z + 5i)$

31 Completing the square to factorise over $\mathbb{C}$

Factorise $z^2 + 6z + 13 = 0$.

THINKING	WORKING
1 Complete the square to express the LHS as the sum of a perfect square and a number.	$z^2 + 6z + 13 = z^2 + 6z + 9 - 9 + 13$ $= (z + 3)^2 + 4$
2 Express as the difference of squares.	$= (z + 3)^2 - 4i^2$ $= (z + 3)^2 - (2i)^2$
3 Factorise.	$z^2 + 6z + 13 = (z + 3 - 2i)(z + 3 + 2i)$

To solve the quadratic equation $az^2 + bz + c = 0$, the quadratic formula, $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ may be used.

When there are unreal roots, the discriminant will be negative. Either factorising or the formula may be used to find the two roots of a quadratic equation, which will always be complex conjugates.

32 Solving a quadratic equation over $\mathbb{C}$

(a) Solve the equation $z^2 - 2z + 4 = 0$ using the quadratic formula.

THINKING	WORKING
1 Identify the coefficients a, b and c.	$a = 1, b = -2, c = 4$
2 Write the quadratic formula	$z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3 Substitute the values of a, b and c.	$z = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 4}}{2 \times 1}$
4 Simplify.	$= \dfrac{2 \pm \sqrt{4 - 16}}{2}$ $= \dfrac{2 \pm \sqrt{-12}}{2}$
5 Substitute $i^2 = -1$.	$= \dfrac{2 \pm \sqrt{12i^2}}{2}$
6 Simplify.	$= \dfrac{2 \pm 2i\sqrt{3}}{2}$ $= 1 \pm i\sqrt{3}$
7 Write the solutions.	$z = 1 \pm i\sqrt{3}$

(b) Solve the equation $z^2 - 2z + 4 = 0$ using completing the square.

1 Complete the square.	$z^2 - 2z + 4 = 0$ $z^2 - 2z + 1 - 1 + 4 = 0$ $(z-1)^2 + 3 = 0$
2 Substitute $i^2 = -1$.	$(z-1)^2 - 3i^2 = 0$
3 Rearrange the equation and take the square root of both sides.	$(z-1)^2 = 3i^2$ $z - 1 = \pm\sqrt{3}i$
4 Solve for z to determine the solution.	$z = 1 \pm \sqrt{3}i$

When you know the roots of an equation, you can write the equation in factor form and then expand to obtain the polynomial that forms one side of the equation.

For example, the equation with real roots 1 and -6 must be of the form $a(z-1)(z+6) = 0, \ a \neq 0$. So, in its expanded form, the equation is $a\left(z^2 + 5z - 6\right) = 0$.

A similar method can be used to obtain an equation when you are given complex roots.

33 Determining an equation from its roots

Determine, in expanded form, the cubic equation with the roots 1 and i (repeated). Assume that the coefficient of z^3 is 1.

THINKING	WORKING
1 Write the equation in factorised form.	$(z-1)(z-i)^2 = 0$
2 Expand and simplify the repeated factors.	$(z-1)(z^2 - 2iz - 1) = 0$
3 Expand in full, in decreasing powers of z.	$z^3 - (1+2i)z^2 + (-1+2i)z + 1 = 0$

Recall that the factor theorem states:

- if a polynomial $p(z)$ has a factor $az - b$, then $p\left(\frac{b}{a}\right) = 0$
- if $p\left(\frac{b}{a}\right) = 0$, then $az - b$ is a factor.

You can use this result to check for or determine complex factors.

34 Verifying factors of a polynomial

Verify that $z - i$ and $z + 2i$ are factors of $p(z) = 3z^3 - iz^2 + 10z - 8i$ and determine the third factor.

THINKING	WORKING
1 Use the factor theorem to test for the first given factor.	$p(i) = 3i^3 - i \times i^2 + 10i - 8i$ $= -3i + i + 10i - 8i$ $= 0$ $z - i$ is a factor.
2 Use the factor theorem to test for the second given factor.	$p(-2i) = 3 \times (-2i)^3 - i \times (-2i)^2 + 10 \times -2i - 8i$ $= 3 \times -8i^3 - i \times 4i^2 - 20i - 8i$ $= 24i + 4i - 20i - 8i$ $= 0$ $z + 2i$ is a factor.
3 Call the third factor $az + b$. Equate coefficients of corresponding powers of z.	$3z^3 - iz^2 + 10z - 8i = (z - i)(z + 2i)(az + b)$ Terms in z^3: $a = 3$ Constant terms: $-8i = -i \times 2i \times b$ $-8i = 2b$ $b = -4i$
4 Interpret the answer.	Given that $p(z - i) = 0$ and $p(z + 2i) = 0$ then $z - i$ and $z + 2i$ are factors of $p(z)$. The third factor is $3z - 4i$.

The conjugate root theorem

Recall that if $z = a + bi$, then the complex conjugate is $\overline{z} = a - bi$.

It is important to note that a pair of factors containing conjugates, $\big(z - (a + ib)\big)\big(z - (a - ib)\big)$, will expand to produce real coefficients.

$$\begin{aligned}\big(z - (a + ib)\big)\big(z - (a - ib)\big) &= (z - a - ib)(z - a + ib)\\ &= \left((z - a)^2 - (ib)^2\right)\\ &= (z - a)^2 + b^2\\ &= z^2 - 2az + a^2 + b^2\end{aligned}$$

If the conjugates are the roots of an equation $\big(z - (a + ib)\big)\big(z - (a - ib)\big) = 0$, then the equation will have real coefficients. Such roots are called complex conjugate pairs.

The conjugate root theorem states:

For polynomials $p(z)$ with real coefficients, the complex roots of $P(z) = 0$ occur in conjugate pairs.

The equation might also have real roots as well as complex conjugate pairs, or only real roots.

This means that if $p(z) = 0$ is an equation with real coefficients and a complex root α, then $\overline{\alpha}$ is also a root.

If $p(\alpha) = 0$, then $p(\overline{\alpha}) = 0$.

35 Determining the roots of an equation with real coefficients

$-3 + 2i$ is a root of the cubic equation $2z^3 + 9z^2 + 8z - 39 = 0$. Determine the other two roots by expressing the equation in factorised form.

THINKING	WORKING
1 Use the conjugate root theorem to identify another root.	$-3 - 2i$ is a root (conjugate of $-3 + 2i$).
2 Since the expression is cubic, it has three roots. You have found two, so there is a linear factor remaining corresponding to the third root.	The third root corresponds to a factor of the form $az + b$.
3 Write the polynomial as a product of factors.	$2z^3 + 9z^2 + 8z - 39$ $= (z - (-3 + 2i))(z - (-3 - 2i))(az + b)$
4 Equate the coefficients of z^3.	By inspection, $a = 2$.
5 Equate the constant terms.	$-(-3 + 2i) \times -(-3 - 2i) \times b = -39$
6 Simplify and solve for b.	$(-3 + 2i)(-3 - 2i)b = -39$ $(9 + 4)b = -39$ $13b = -39$ $b = -3$
7 Write the third factor and the third root of the equation.	The third factor is $2z - 3$ and, therefore, the third root is $\frac{3}{2}$.
8 Write the three roots.	$-3 + 2i, -3 - 2i, \frac{3}{2}$

Technology worked example

Determine the roots of an equation with real coefficients

36 Determining factors of a polynomial with real coefficients

Determine the equation of a quartic polynomial, with real coefficients, in expanded form that have $z - 2i$ and $z + 1 - i$ as two of their factors.

THINKING	WORKING
1 Classify the other two factors that make up the conjugate pairs.	$z + 2i,\ z + 1 + i$
2 Write the polynomial in factor form.	$p(z) = a(z - 2i)(z + 2i)(z + 1 - i)(z + 1 + i)\ \ a \neq 0$
3 Expand in pairs.	$p(z) = a\left(z^2 - 4i^2\right)\left((z + 1)^2 - i^2\right)$ $= a\left(z^2 + 4\right)\left((z + 1)^2 + 1\right)$ $= a\left(z^2 + 4\right)\left(z^2 + 2z + 2\right)$
4 Complete the expansion.	$p(z) = a\left(z^4 + 2z^3 + 6z^2 + 8z + 8\right)$

EXERCISE 4.6 Complex factors of polynomials

Worked Example

30 1 Factorise to linear factors over $\mathbb{C}$.

(a) $z^2 + 9$ (b) $4z^2 + 1$ (c) $16z^2 + 81$

(d) $3z^2 + 15$ (e) $16z^2 + 100$ (f) $z^4 - 81$

32 2 Solve the following equations using the quadratic formula.

(a) $z^2 + 2z + 5 = 0$ (b) $2z^2 - z + 3 = 0$ (c) $z^2 - 4z + 6 = 0$ (d) $z^2 - 4z + 3 = 0$

31 3 Factorise these quadratic polynomials by completing the square.

(a) $z^2 + 4z + 6$ (b) $z^2 - 8z + 21$

4 Solve these quadratic equations by completing the square.

(a) $z^2 + 2z + 2 = 0$ (b) $z^2 - 6z + 18 = 0$ (c) $z^2 - 10z + 29 = 0$ (d) $z^2 + 12z + 100 = 0$

5 Consider the polynomial $p(z) = z^3 - 4z^2 - 2z + 20$.

(a) Express $p(z)$ as the product of three factors. (b) Solve the equation $p(z) = 0$.

33 6 Determine all equations in terms of z for each given set of roots.

(a) Quadratic equations with roots $z = 1, z = i$.

(b) Cubic equations with roots $z = 1, z = i$, and real coefficients.

(c) Cubic equations with roots $z = 2,\ z = 3i,\ z = -i$ and real coefficients.

(d) Quartic equations with roots $z = -i, z = 2i$ and real coefficients.

7 Determine the values of real numbers p and q, given that one root of the equation $z^2 + pz + q = 0$ is $-1 + 3i$.

8 Given that one root of an equation is $z_1 = 3(1+i)$, determine a quadratic equation that has the roots z_1 and $\overline{z_1}$. Worked Example

9 **(a)** Verify that $z+i$ is a factor of $3z^2 + (-2+3i)z - 2i$ and determine the other factor. 34

(b) Verify that $z+1$ is a factor of $2z^2 + (2-i)z - i$ and determine the other factor.

(c) Verify that $z-2i$ is a factor of $z^3 + (4-2i)z^2 + (3-8i)z - 6i$ and determine the other factor.

(d) Verify that $z-i$ and $z+2i$ are factors of $4z^3 + iz^2 + 11z - 6i$ and determine the other factor.

10 Given that $z = 3+i$ is a root of $z^3 - 7z^2 + 16z - 10 = 0$, determine the other two roots. 35

11 Given that $z = 3i$ is a root of the equation $2z^3 - z^2 + 18z - 9 = 0$, determine the other two roots.

12 Determine the linear factors of the following equations over $\mathbb{C}$.

(a) $z^3 - iz^2 - z + i = 0$ **(b)** $z^3 + 8 = 0$

13 Which of the following is a true statement?

A The equation $(z-i)^2(z+3) = 0$ has only two solutions, i and 3.

B The roots $z = \pm 2$ are a complex conjugate pair.

C The polynomial $z^3 + 1$ has three real linear factors.

D A polynomial of degree 5 has five linear factors.

14 What is the lowest possible degree of an equation with real coefficients that has $2+i$ and -3 among its roots?

A 3 **B** 2 **C** 4 **D** 1

15 Determine a quartic polynomial with real coefficients in expanded form that has amongst its factors $z-3-i$ and $z+i$. 36

16 Given that $z = (a+i)^4$ where a is real, determine values of a such that z is real.

17 If $z = 1+i$ is one root of the equation $z^2 - z + 1 - i = 0$, determine the other root.

18 $\sqrt{3}i$ is a root of the equation $2z^3 - z^2 + mz - 3 = 0$. Use the factor theorem to determine the value of m.

19 Factorise the following polynomials.

(a) $z^3 + (-1+3i)z^2 - (2+3i)z + 2$ **(b)** $z^3 - 2(1+i)z^2 + 3iz + (1-i)$

(c) $z^3 + (6-i)z^2 + (8-6i)z - 8i$

20 One of the factors of $z^4 + 6z^2 + 25$ is $z^2 - 2z + 5$.

(a) Determine the other factor.

(b) Determine the remaining quadratic factor and hence determine all four solutions to the equation $z^4 + 6z^2 + 25 = 0$.

(c) Verify your answers to parts **(a)** and **(b)** using technology.

21 Given that $z = 2+3i$ is a solution of $z^4 - 5z^3 + 18z^2 - 17z + 13 = 0$, determine the other three solutions.

22 Determine all the solutions of the equation $\left(\frac{1+z}{z}\right)^4 - 1 = 0$.

4

Summary

Complex numbers review

Complex numbers $\mathbb{C}$

Real numbers $\mathbb{R}$ (numbers that can be placed on a number line)

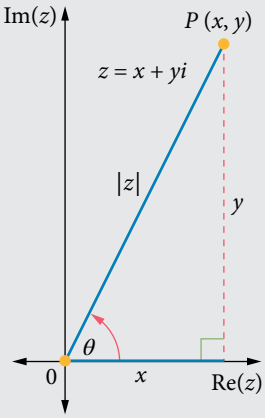
Imaginary numbers i

The laws of arithmetic for complex numbers

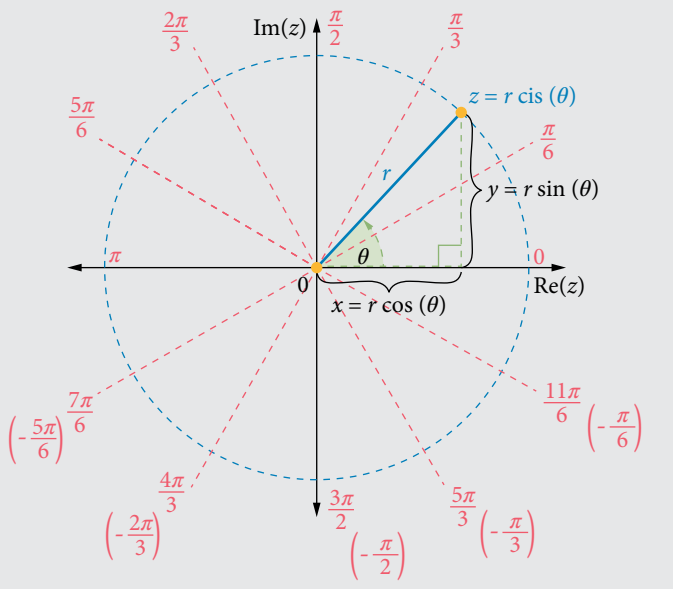
For addition		For multiplication	
$z_1 + z_2 = z_2 + z_1$	Commutative law	$z_1 z_2 = z_2 z_1$	Commutative law
$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$	Associative law	$(z_1 z_2) z_3 = z_1 (z_2 z_3)$	Associative law
$z_1 + (-z_1) = 0 + 0i$	Additive inverse	$z_1 \times \frac{1}{z_1} = 1 + 0i$	Multiplicative inverse
$0 + 0i$	Identity element	$1 \times 0i$	Identity element

Modulus and argument form

Modulus and argument of a complex number



Polar form of a complex number



De Moivre's theorem:

$z = r\,\mathrm{cis}(\theta)$ then $(r\,\mathrm{cis}(\theta))^n = r^n \mathrm{cis}(n\theta)$

Roots of complex numbers

Square roots of complex numbers:

$i^2 = -1$ and so i is a square root of -1. However, just as -2 (as well as +2) is a square root of 4, $-i$ is also a square root of -1 as:

$$(-i)^2 = (-i)(-i)$$
$$= i^2$$
$$= -1$$

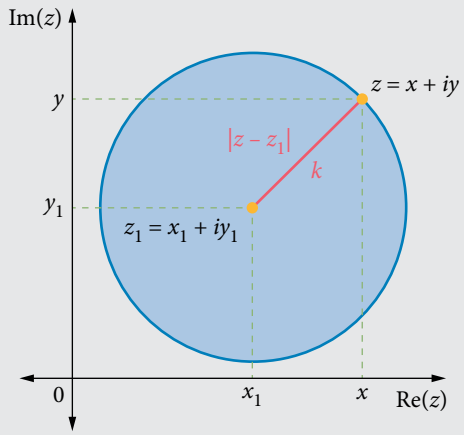
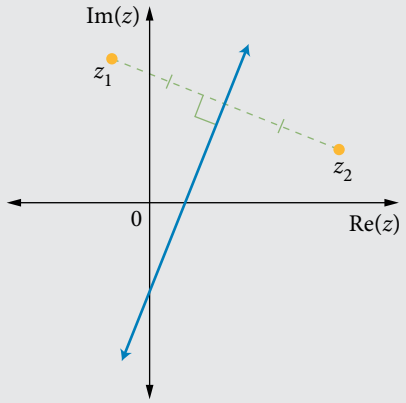
*n*th roots of complex numbers:

Determining the *n*th root of a complex number z is equivalent to solving the equation

$z^n = a$ or $z = a^{\frac{1}{n}}$.

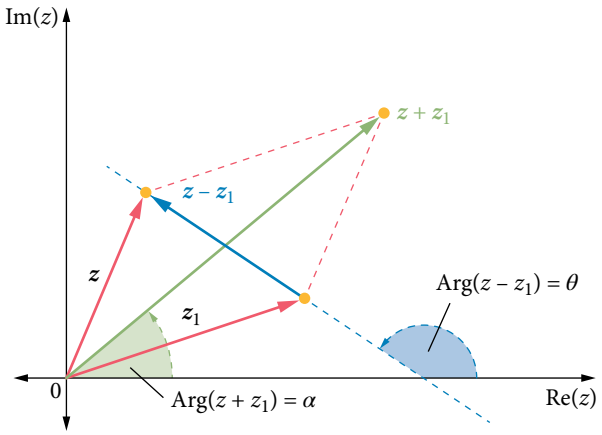
The complex plane

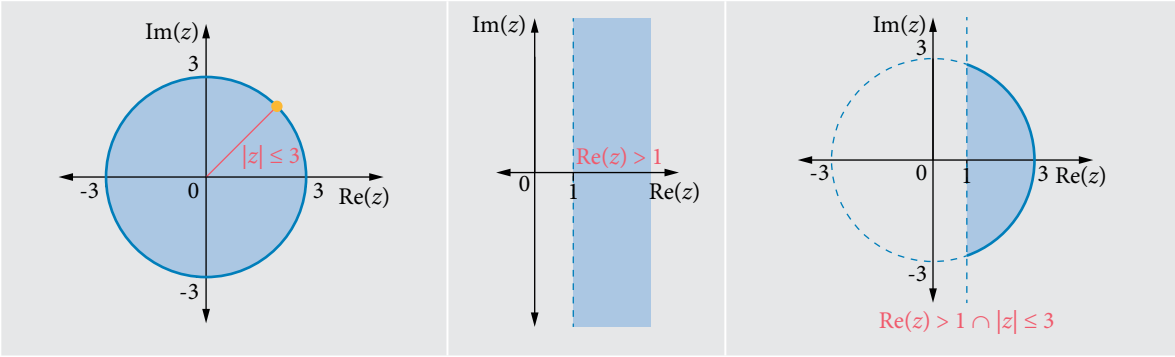
The locus of points representing straight lines and circles:

$\lvert z - z_1 \rvert = k$	$\lvert z - z_1 \rvert = \lvert z - z_2 \rvert$
	

Consider the complex numbers $z = x + iy$ and $z_1 = x_1 + iy_1$, where $z = x + iy$ corresponds to the vector $z = \begin{bmatrix} x \\ y \end{bmatrix}$ and $z_1 = x_1 + iy_1$ corresponds to the vector $z_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

- The complex number $z - z_1$ and $z + z_1$ corresponds to the vectors $\boldsymbol{z} - \boldsymbol{z}_1$ and $\boldsymbol{z} + \boldsymbol{z}_1$ respectively.
- $\text{Arg}(z - z_1) = \theta$ and $\text{Arg}(z + z_1) = \alpha$ defines the angles at which the vectors are inclined to the positive real axis.





Polynomials

Synthetic division:

Key points for synthetic division:

STEP 1: Both the numerator and denominator should be arranged so that each is in descending powers of x.

STEP 2: If any terms with consecutive powers of x are missing, these should be included by using 0 as their coefficient.

STEP 3: Each term of the quotient is obtained by dividing the first term of the divisor into the first term of the new dividend.

STEP 4: Write the term of the quotient found above the dividend.

STEP 5: After each division, the next step is to subtract and then bring down the next term.

The remainder theorem:

When a polynomial $p(x)$ is divided by $x - a$, the remainder is $p(a)$.

The factor theorem:

$(x - a)$ will be a factor of a polynomial $p(x)$ if and only if $p(a) = 0$.

Complex factors of polynomials

Factors of quadratic polynomials using completing the square and the quadratic formula:

Using complex numbers allows you to factorise the sum of two squares.

$$\begin{aligned} a^2 + b^2 &= a^2 - b^2 i^2 \\ &= (a + ib)(a - ib) \end{aligned}$$

The conjugate root theorem for equations with real coefficients:

Given $\big(z - (a + ib)\big)\big(z - (a - ib)\big)$, these will expand to produce real coefficients.

$$\begin{aligned} \big(z - (a + ib)\big)\big(z - (a - ib)\big) &= (z - a - ib)(z - a + ib) \\ &= \Big((z - a)^2 - (ib)^2\Big) \\ &= (z - a)^2 + b^2 \\ &= z^2 - 2az + a^2 + b^2 \end{aligned}$$

If the conjugates are the roots of an equation $\big(z - (a + ib)\big)\big(z - (a - ib)\big) = 0$, then the equation will have real coefficients. Such roots are called complex conjugate pairs.

Chapter review 4

1 Express each of the following in the form $x + yi$. Exercise 4.1

(a) $(5-2i)(3+i)$ (b) $\left(\sqrt{7}-3i\right)^2$ (c) $\dfrac{3i}{4-2i}$ (d) $\left(\dfrac{1+i}{1-i}\right)^2$

(e) $z^{-1}+\overline{z}$, where $z = 3+2i$ (f) $\overline{z_1 \times z_2}$, where $z_1 = 5+i,\ z_2 = 1-2i$

2 Determine real numbers x and y such that: Exercise 4.1

(a) $2(x+yi)-1=(4-i)^2$ (b) $(x+yi)(2-4i)=15i$

3 Express each of the following in modulus–argument (polar) form. Exercise 4.2

(a) $-5+5i$ (b) $4\sqrt{3}+4i$

4 Express each of the following in Cartesian form. Exercise 4.2

(a) $6\text{cis}\left(-\dfrac{\pi}{3}\right)$ (b) $7\text{cis}\left(-\dfrac{5\pi}{6}\right)$

5 Use de Moivre's theorem to determine the following, expressing your answer in Cartesian form. Exercise 4.2

(a) $\text{cis}\left(\dfrac{\pi}{4}\right)^8$ (b) $\left(3\text{cis}\left(\dfrac{\pi}{3}\right)\right)^4$ (c) $\left(\sqrt{2}\text{cis}\left(\dfrac{5\pi}{6}\right)\right)^6$ (d) $\dfrac{-2+2i}{-1-i}$

6 Express the following in the form $x + yi$, leaving your answers correct to 2 decimal places. Exercise 4.2

(a) $\left(2\text{cis}\left(\dfrac{\pi}{5}\right)\right)^6$ (b) $(0.03+0.7i)^{-3}$

7 Solve the following equations over $\mathbb{C}$, leaving your answers in the form $x + yi$ where exact values can be used. Exercise 4.3

(a) $z^2 = -81$ (b) $z^2 = 4i$

(c) $z^3 = \sqrt{2}+\sqrt{2}i$ (leave in polar form) (d) $z^3 + 27i = 0$

8 Sketch the solutions found in question **7** on the complex plane. Exercise 4.3

(a) $z = \pm 9i$ (b) $z = \sqrt{2}+\sqrt{2}i,\ z = -\sqrt{2}-\sqrt{2}i$

(c) $z = 2^{\frac{1}{3}}\text{cis}\left(\dfrac{\pi}{12}\right),\ 2^{\frac{1}{3}}\text{cis}\left(\dfrac{3\pi}{4}\right),\ 2^{\frac{1}{3}}\text{cis}\left(-\dfrac{7\pi}{12}\right)$ (d) $z = \dfrac{3\sqrt{3}}{2}-\dfrac{3}{2}i,\ 0+3i,\ -\dfrac{3\sqrt{3}}{2}-\dfrac{3}{2}i$

9 Sketch the following loci on the complex plane. Exercise 4.4

(a) $|z| = 11$ (b) $|z+2i| = 3$

(c) $\text{Arg}(z) = \dfrac{2\pi}{3}$ (d) $5\,\text{Re}(z)+\text{Im}(z)+7=0$

(e) The perpendicular bisector of the line segment joining the points $1+i$ and $2+0i$.

10 Determine, in polar form, the following roots: Exercise 4.3

(a) $(125)^{\frac{1}{3}}$ (b) $(-i)^{\frac{1}{4}}$

Exercise 4.3 **11** **(a)** Determine, in polar form, the seventh roots of unity.

(b) Determine the angle separating consecutive roots of $(1)^{\frac{1}{7}}$.

Exercise 4.5 **12** Determine the remainder in each of the following divisions.

(a) $p(x) = x^3 - 3x^2 + 8x - 9 \div (x - 2)$ **(b)** $p(x) = x^3 + x^2 + x + 12 \div (x + 1)$

(c) $p(x) = x^3 + 2x^2 - 7 \div (2x - 1)$

Exercise 4.5 **13** Use synthetic division to determine the remainder when $p(x) = x^3 - 2x + 5$ is divided by $x + 1$.

Exercise 4.5 **14** Demonstrate that $x - 2$ is a factor of the following polynomials.

(a) $p(x) = x^3 + x^2 - 6x$ **(b)** $p(x) = x^4 - 3x^3 + 2x^2 - x + 2$

Exercise 4.5 **15** Determine the linear factors of these polynomials:

(a) $p(x) = x^3 + x^2 - 10x + 8$ **(b)** $p(x) = 2x^3 + 5x^2 + x - 2$

Exercise 4.5 **16** Factorise the following:

(a) $2x^3 - 3x^2 - 8x + 12$ **(b)** $x^4 - 3x^3 - 3x^2 + 7x + 6$

Exercise 4.6 **17** Solve the following equations over $\mathbb{C}$, leaving your answers in the form $x + yi$.

(a) $z^2 = -32$ **(b)** $z^2 + 8z + 20 = 0$

(c) $z^3 - z^2 + 9z - 9 = 0$, given that one of the roots is $z = 3i$.

(d) $z^3 + 3z^2 + 4z + 2 = 0$, given that one of the roots is $z = -1 + i$.

Exercise 4.6 **18** Determine the roots of the equation $2z^3 - 7z^2 + 10z - 6 = 0$ over $\mathbb{C}$.

Exercise 4.3 **19** **(a)** Write the fifth roots of unity.

(b) Sketch the fifth roots of unity on the complex plane.

Exercise 4.2 **20** Simplify the following, leaving your answers in polar form.

(a) $\dfrac{(1+i)^3}{\left(\sqrt{3}+i\right)^4}$ **(b)** $\dfrac{(1-i)^2}{(1+i)^3}$

Exercise 4.1 **21** If $z_1 = 4 - i$, $z_2 = 2 + i$, $z_3 = i$, prove that $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (the associative law for multiplication).

Exercise 4.4 **22** Sketch the following subsets of the complex plane.

(a) The locus of the points described by $\text{Arg}(z - 3i) = \dfrac{\pi}{4}$. **(b)** $|z - 1 - i| < 1$

(c) $-2\,\text{Re}(z) + \text{Im}(z) \leq -4$

Exercise 4.5 **23** Determine the values of a and b if the polynomial $p(x) = 6x^3 + 7x^2 + ax + b$ is divisible by $2x - 1$ and $x + 1$.

Exercise 4.1 **24** Determine the value of p and q such that $\dfrac{3 + 2i}{1 - i} = 5p + qi$.

Exercise 4.3 **25** Express $(8 - 8i)^{-7}$ in the form $x + yi$.

26 Which one of the following is defined on the complex plane by $|z - 2i| > 9$? Exercise 4.4

- **A** A circle with centre $0 + 2i$ and radius 3
- **B** A circle with centre $0 + 2i$ and radius 9
- **C** The region outside the circle with centre $0 - 2i$ and radius 9
- **D** The region outside the circle with centre $0 + 2i$ and radius 9

27 Sketch the locus of a point defined by $\left\{z : \text{Arg}(z + 1 - i) = \frac{\pi}{6}\right\} \cup \left\{z : \text{Arg}(z + 1 - i) = -\frac{5\pi}{6}\right\}$. Exercise 4.4

28 **(a)** By factorising $z^2 + 4$, demonstrate that $z^2 + 4$ is a factor of $p(z) = z^3 + 8z^2 + 4z + 32$. Exercise 4.6

(b) Determine the remaining factor.

29 **(a)** If $(x + yi)^2 = 8 + 6i$, determine the values of x and y. Exercise 4.1

(b) Hence determine the square roots of $8 + 6i$.

30 Prove that $z\overline{u} + \overline{z}u$ is real for all values of z and u. Exercise 4.1

31 **(a)** Use de Moivre's theorem to write an expression for $(\text{cis}(\theta))^3$. Write your answer in the form $x + iy$. Exercise 4.2

(b) Expand $(\cos(\theta) + i\sin(\theta))^3$.

(c) Hence express $\sin(3\theta)$ in terms of $\sin(\theta)$.

32 Given that ω is a complex cube root of unity, determine the value of $\omega + \omega^3 + \omega^5$. Exercise 4.3

33 **(a)** If $z^n = \cos(n\theta) + i\sin(n\theta)$, demonstrate that $\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$. Exercise 4.2

(b) Write an expression for $z^n + \frac{1}{z^n}$.

(c) Expand $\left(z + \frac{1}{z}\right)^3$.

(d) Hence express $\cos^3(\theta)$ in terms of cosines of multiple angles.

Exam review UNIT 3

Paper 1: Technology-free

Exercise 2.2

1 If $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, calculate:

(a) $|\boldsymbol{a} + \boldsymbol{b}|$ (b) $|3\boldsymbol{a} - 2\boldsymbol{b}|$

Exercise 2.2

2 Points A and B have position vectors $-2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ respectively. Calculate the position vector of B relative to A.

Exercise 3.2

3 A curve is described by the vector equation $\boldsymbol{r}(t) = (3 - 4t)\hat{\boldsymbol{i}} - 5t\hat{\boldsymbol{j}}$, $t \in \mathbb{R}$. Calculate the gradient of the curve at (x, y).

Exercise 4.1

4 Express each of these complex numbers in the form $x + yi$:

(a) $(3+6i)(2i-7)$ (b) $4i^{15} - 12i^{101}$ (c) $\dfrac{3}{5-i}$

(d) $\dfrac{2+i}{7+2i}$ (e) $(1-i)^3$ (f) $\dfrac{}{(3-4i)(2+i)^2}$

Exercise 4.2

5 Simplify the following, expressing your answer in the form $x + yi$:

(a) $3\text{cis}\left(\dfrac{\pi}{4}\right) \times -4\text{cis}\left(\dfrac{3\pi}{2}\right)$ (b) $\dfrac{11\text{cis}\left(\frac{\pi}{3}\right)}{\text{cis}\left(\frac{5\pi}{6}\right)}$ (c) $\dfrac{\left(2\text{cis}\left(\frac{\pi}{3}\right)\right)^4}{\left(\text{cis}\left(\frac{\pi}{6}\right)\right)^2}$ (d) $\left(-2 + 2\sqrt{3}i\right)^6$

Exercise 1.2

6 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \ldots + n \times 2^{n-1} = (n-1)2^n + 1$$

Exercise 1.2

7 Use mathematical induction to prove the following result for all $n \in \mathbb{Z}^+$:

$$\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n} = \frac{2^{2n} - 1}{3 \times 2^{2n}}$$

Exercise 1.3

8 Use mathematical induction to prove that $8^n - 8$ is divisible by 7 for all integers $n \geq 2$.

Exercise 4.1

9 Determine the cube roots of $27i$ in Cartesian form.

Exercise 3.4

10 At time t, a particle has position vector $\boldsymbol{r} = \left(\sin(t) + \sin(2t)\right)\hat{\boldsymbol{i}} + \left(\cos(t) - \cos(2t)\right)\hat{\boldsymbol{j}}$, $t \geq 0$. Calculate the particle's minimum and maximum speeds.

Exercise 1.3

11 Prove that if a is an integer, then $1 - (-a)^n$ is divisible by $1 + a$ for all $n \in \mathbb{Z}^+$.

Paper 2: Technology-active

1 Calculate $\frac{d\boldsymbol{r}}{dt}$ and $\frac{d^2\boldsymbol{r}}{dt^2}$ for $\boldsymbol{r} = 3\tan(2t)\hat{\boldsymbol{i}} + 3\sec(2t)\hat{\boldsymbol{j}}$. Exercise 3.3

2 Determine the modulus and argument of the following complex numbers. Leave the argument correct to the nearest whole degree. Exercise 4.2

(a) $10+10i$ **(b)** $2-5i$ **(c)** $-2-8i$

3 If vectors $\boldsymbol{a} = -5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, then calculate the vector projection of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$. Exercise 2.3

4 If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent vectors, determine the values of x, y and z, where $x, y, z \in \mathbb{R}$, if: Exercise 2.4

(a) $(x+5)\boldsymbol{a} + (y-1)\boldsymbol{b} + z\boldsymbol{c} = \boldsymbol{0}$ **(b)** $(2x^2 - x - 6)\boldsymbol{a} + (y^3 - 27)\boldsymbol{b} + (z^2 - 4)\boldsymbol{c} = \boldsymbol{0}$

5 Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 2 & 5 \end{bmatrix}$. Exercise 2.11

(a) Calculate the eigenvalues associated with this matrix.

(b) Calculate the eigenvectors associated with this matrix.

6 Calculate the angle, correct to the nearest degree, between the pair of vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$. Exercise 3.2

7 Sketch the following subsets of the complex plane. Exercise 4.4

(a) $\text{Im}(z) = 3$ **(b)** $3\text{Re}(z) - \text{Im}(z) = 2$ **(c)** $|z| = 3$

(d) $|z-1+3i| = |z+2+i|$ **(e)** $\text{Arg}(z-1+2i) = \frac{\pi}{3}$

8 Factorise the following polynomials over $\mathbb{C}$. Exercise 4.6

(a) $81z^4 - 256$

(b) $z^2 - 4z + 10$

(c) $2z^3 - 11z^2 + 38z - 39$, given that $z - (2-3i)$ is one of the factors.

9 A football is kicked towards the goal posts with an initial speed of 29 m/s at an angle of projection to the horizontal, α, where $\tan(\alpha) = \frac{21}{20}$. The goalposts are 70 metres horizontally from the position the ball is kicked, and the crossbar of the goalposts is 3 metres from the ground. The ball needs to travel over the crossbar for the kick to be successful. Will the kick be successful in this case? Justify your answer, identify any assumptions you have made and comment on the effect of these assumptions on your answer. Exercise 3.4





5

Integration techniques

Recall 256

5.1 Inverse trigonometric functions 257

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Summary 314

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5 Recall

Factorise quadratic expressions

1 Factorise each of the following quadratic expressions:

(a) $4x^2 - 9$ (b) $x^2 - 2x - 15$ (c) $3x^2 - 14x - 5$

Use the technique of completing the square

2 (a) Express $x^2 + 10x + 31$ in the form $(x + b)^2 + c$.

(b) Express $4x^2 + 12x + 10$ in the form $(ax + b)^2 + c$.

Use polynomial long division

3 Express each of the following in quotient–remainder form:

(a) $\dfrac{3x^2 + 4x - 5}{x^2 - 1}$ (b) $\dfrac{2x^3 - 3x^2 + x + 5}{x^2 - 3}$

Simplify expressions involving logarithms

4 Express each of the following as single logarithms:

(a) $\ln(6) - 2\ln(3)$ (b) $3\ln(a) + \ln(b) - \ln(c)$

Use the chain, product, and quotient rules to differentiate functions

5 Differentiate each of the following:

(a) $y = \left(3x^2 + 5x + 2\right)^7$ (b) $y = x^3 \sin(5x)$ (c) $y = \dfrac{\ln(x)}{x}$ (d) $y = \tan(x)$

Integrate using standard integral rules

6 Determine the following indefinite integrals:

(a) $\int x^2 + 6x + 7\, dx$ (b) $\int \sqrt{3x + 4}\, dx$

(c) $\int 3\cos(7x)\, dx$ (d) $\int e^{4x}\, dx$

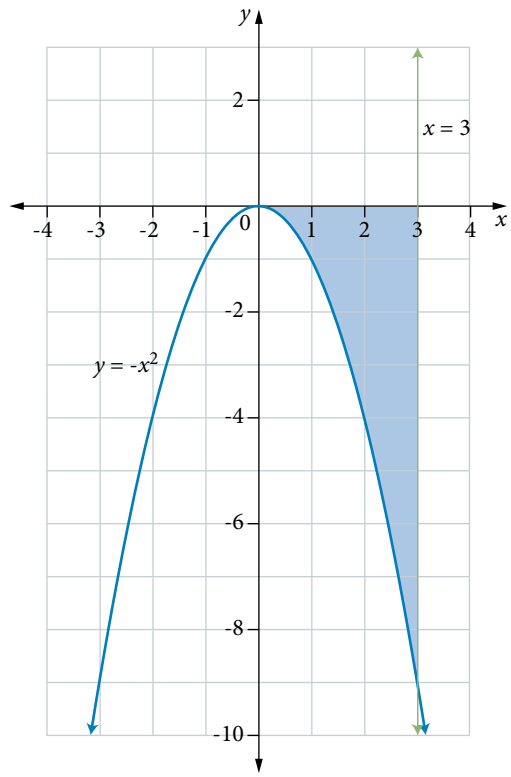
Calculate definite integrals

7 Calculate the exact value of the following:

(a) $\int_1^2 \dfrac{1}{\sqrt{x}}\, dx$ (b) $\int_0^1 e^{3x}\, dx$

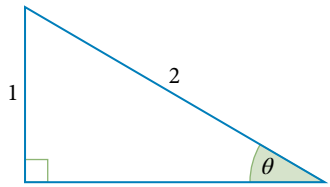
Calculate areas using definite integrals

8 Calculate the exact area bounded by the function $y = -x^2$, the x-axis and the line $x = 3$ as shown in the diagram.



5.1 Inverse trigonometric functions

Consider the following right-angled triangle.



When first learning about trigonometry, you would have learnt that the value of θ could be determined with the aid of the $\sin^{-1}$ command on your calculator.

$$\sin(\theta) = \frac{1}{2}$$
$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$
$$\theta = 30°$$

WARNING

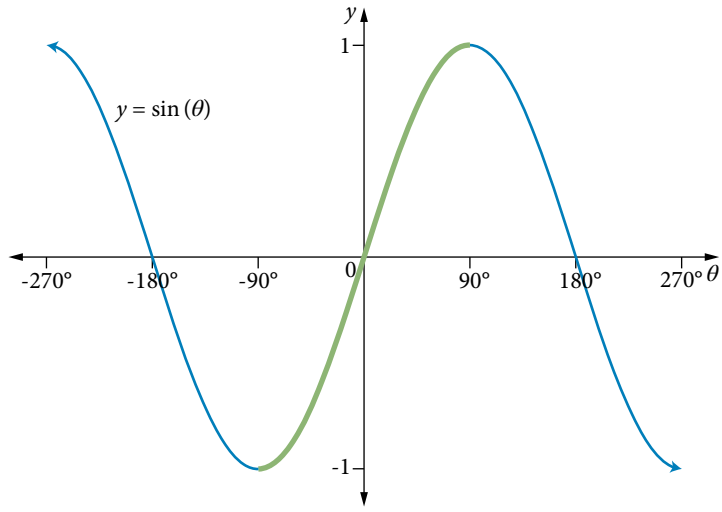
Some sources use the inverse sine notation $\text{Sin}^{-1}(x)$ or $\arcsin(x)$ instead of $\sin^{-1}(x)$.

The $\sin^{-1}$ command on your calculator gives an angle θ that has a particular sine ratio.

However, if angles greater than 90° and negative angles are allowed, there are infinitely many angles with a given sine ratio. For example, all of the angles 30°, 150°, 390°, -210°, etc. have a sine of $\frac{1}{2}$. All of the angles -45°, -135°, 225°, 315°, etc. have a sine of $-\frac{\sqrt{2}}{2}$. How does your calculator know which angle to give you when determining $\sin^{-1}\left(\frac{1}{2}\right)$ or $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$?

As a convention, $\sin^{-1}(x)$ refers to the angle θ *in the interval* $[-90°, 90°]$ for which $\sin(\theta) = x$. This guarantees that $\sin^{-1}(x)$ can only equal a single value, (which means $y = \sin^{-1}(x)$ is actually a function: the *inverse sine function*).

For example, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$. The fact that every number from -1 to 1 has exactly one angle whose sine is equal to this number can be readily seen in the graph of $y = \sin(\theta)$ shown. Given a particular y-value (a value of $\sin(\theta)$), there is exactly one possible x-value (a value of θ) within $[-90°, 90°]$ (shown in green).



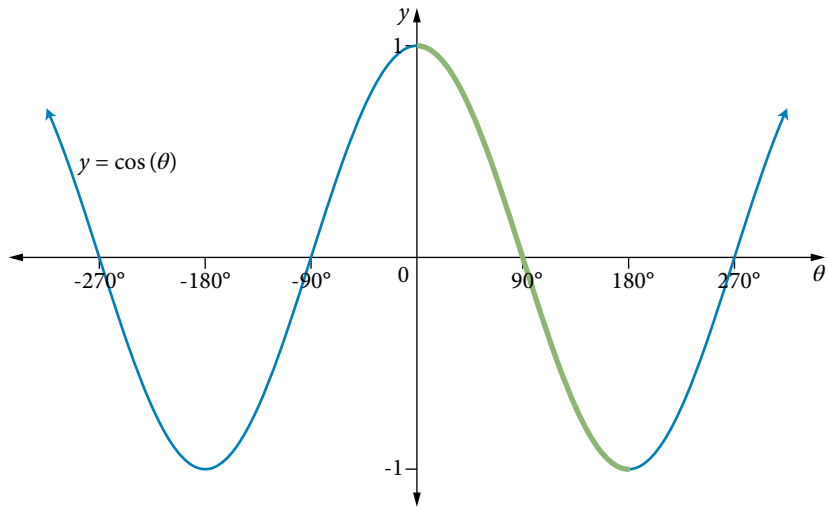
WARNING

Note that $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$.

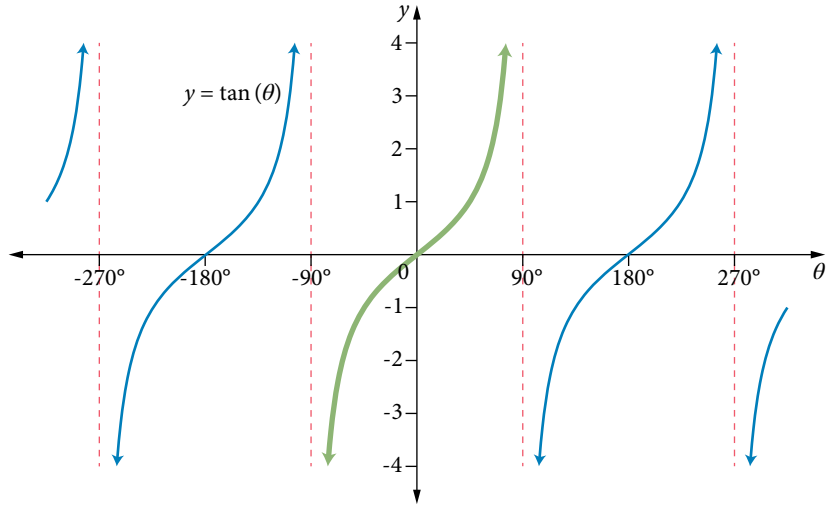
The expression $\sin^{-1}(x)$ refers to the *inverse sine* of x, while the expression $\frac{1}{\sin(x)}$ refers to the *reciprocal sine* of x, which is very different.

Inverse cosine and inverse tangent functions are defined similarly to the inverse sine function.

The inverse cosine of a number, $\cos^{-1}(x)$, is the angle θ in the interval $[0,180°]$ for which $\cos(\theta) = x$. The interval $[-90°, 90°]$ cannot be used for $\cos^{-1}(x)$ as there are always two angles in this interval with a given cosine. For example, $\cos(60°)$ and $\cos(-60°)$ are both equal to $\frac{1}{2}$. However, as indicated in the following graph, given a particular y-value (a value of $\cos(\theta)$) there is exactly one possible value of θ within $[0,180°]$.



The inverse tangent of a number, $\tan^{-1}(x)$ is the angle θ in the interval $(-90°, 90°)$ for which $\tan(\theta) = x$. Note that -90° and 90° are not included in the interval as the tangent of these angles is undefined. The graph of $y = \tan(\theta)$ showing this interval is shown.



While degrees may be used in the context of inverse trigonometric functions, it is more common to use radians. Given the calculus focus of this chapter, many of the results in subsequent sections are only valid when working in radians.

$\arcsin(x) = \sin^{-1}(x) = \theta$ such that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\sin(\theta) = x$.
$\arccos(x) = \cos^{-1}(x) = \theta$ such that $0 \leq \theta \leq \pi$ and $\cos(\theta) = x$.
$\arctan(x) = \tan^{-1}(x) = \theta$ such that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan(\theta) = x$.

WARNING

$\sin^{-1}(x) \neq \frac{1}{\sin(x)}$, $\cos^{-1}(x) \neq \frac{1}{\cos(x)}$, $\tan^{-1}(x) \neq \frac{1}{\tan(x)}$

1 Determining exact values of inverse trigonometric functions

Determine the exact value of each of the following:

(a) $\arcsin(-1)$

THINKING	WORKING
$\arcsin(x)$ is equal to the angle from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ whose sine is equal to x.	Let $y = \arcsin(-1)$. So $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\sin(y) = -1$. Thus, $\arcsin(-1) = -\frac{\pi}{2}$.

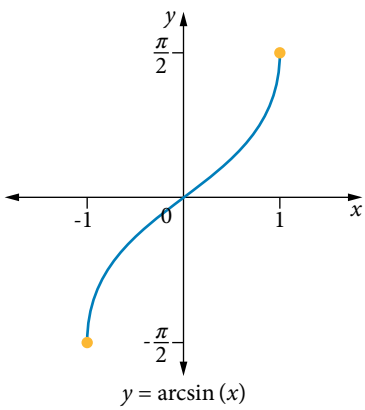
(b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\arccos(x)$ is equal to the angle from 0 to π whose cosine is equal to x.	Let $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. So $0 \leq y \leq \pi$ and $\cos(y) = \frac{\sqrt{3}}{2}$. Since $\cos(y)$ is positive and $0 \leq y \leq \pi$, y must be in the first quadrant. Thus, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

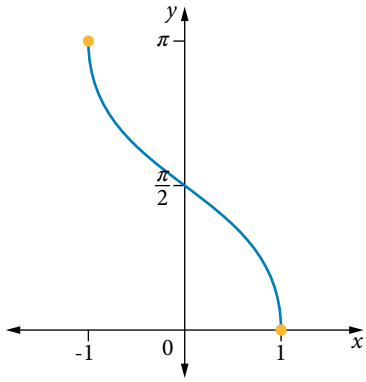
(c) $\arctan\left(-\sqrt{3}\right)$

$\arctan(x)$ is equal to the angle from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ whose tangent is equal to x.	Let $y = \arctan\left(-\sqrt{3}\right)$. So $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan(y) = -\sqrt{3}$. Since $\tan(y)$ is negative, y must be in the fourth quadrant. Thus, $\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$.

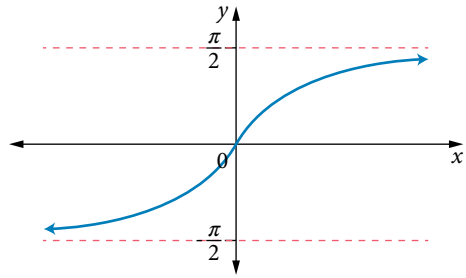
The graphs of the functions $y = \arcsin(x)$, $y = \arccos(x)$ and $y = \arctan(x)$, known collectively as the standard inverse trigonometric functions, are shown.



$y = \arcsin(x)$



$y = \arccos(x)$



$y = \arctan(x)$

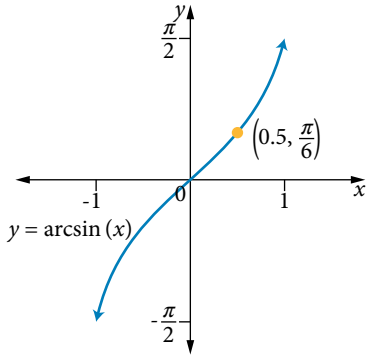
From the graphs, the following should be clear:

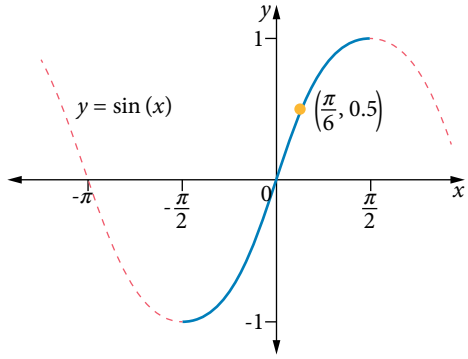
The domain of $y = \arcsin(x)$ is $[-1,1]$ and range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

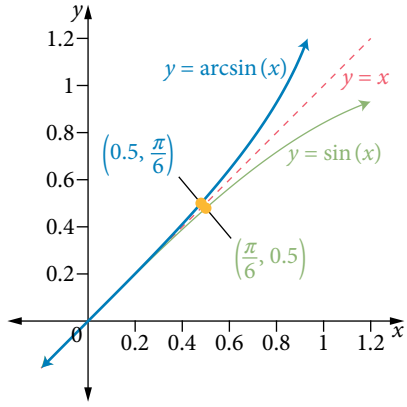
The domain of $y = \arccos(x)$ is $[-1,1]$ and range is $[0, \pi]$.

The domain of $y = \arctan(x)$ is $\mathbb{R}$ and range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

If the point (p,q) is on the graph of $y = \arcsin(x)$, then the point (q,p) will be on the graph of $y = \sin(x)$ (since $\arcsin(p) = q$ means that $\sin(q) = p$).





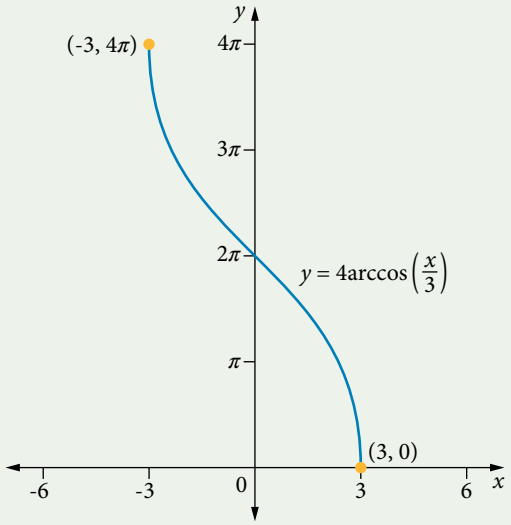


Graphs of functions related to the standard inverse trigonometric functions can readily be sketched by transforming the graphs of these standard functions.

2 Sketching graphs of simple transformations of inverse trigonometric functions

Determine the domain and range of the following functions, and sketch the graph of each:

(a) $y = 4\arccos\left(\frac{x}{3}\right)$

THINKING	WORKING
1 The domain is the set of possible values of x. The standard arccos function is defined provided the argument is from -1 to 1.	Domain: $-1 \leq \frac{x}{3} \leq 1$ $-3 \leq x \leq 3$ Domain is $[-3, 3]$.
2 The range is the set of possible values of y. The standard arccos function produces y-values from 0 to π.	Range: $0 \leq \arccos\left(\frac{x}{3}\right) \leq \pi$ $0 \leq 4\arccos\left(\frac{x}{3}\right) \leq 4\pi$ Range is $[0, 4\pi]$.
3 The shape of the graph will be the same as that of the graph of $y = \arccos(x)$.	

(b) $y = -3\tan^{-1}(x-1)$

1 $\arctan(x)$ is defined for all real numbers x. Thus, the domain of this function is also $\mathbb{R}$.

Domain is $\mathbb{R}$.

2 The range is the set of possible values of y. The standard arctan function produces y-values strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Range:

$$-\frac{\pi}{2} < \tan^{-1}(x-1) < \frac{\pi}{2}$$

$$\frac{3\pi}{2} > -3\tan^{-1}(x-1) > -\frac{3\pi}{2}$$

Range is $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

3 The shape of the graph will be the same as that of the graph of $y = \arctan(x)$, but reflected in the x-axis, and translated horizontally. Intercepts can be determined by setting x or y to 0 in the equation.

y-intercept:

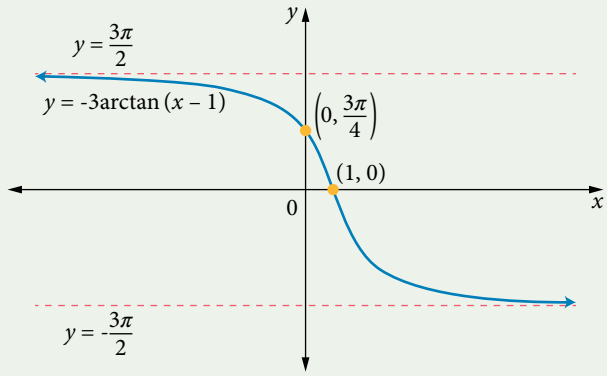
$$\begin{aligned} y &= -3\tan^{-1}(0-1) \\ &= -3\tan^{-1}(-1) \\ &= -3 \times -\frac{\pi}{4} \\ &= \frac{3\pi}{4} \end{aligned}$$

The coordinates of the y-intercept are $\left(0, \frac{3\pi}{4}\right)$.

x-intercept:

$$\begin{aligned} 0 &= -3\tan^{-1}(x-1) \\ 0 &= \tan^{-1}(x-1) \\ \tan(0) &= x-1 \\ 0 &= x-1 \\ 1 &= x \end{aligned}$$

The coordinates of the x-intercept are $(1, 0)$.



WARNING

When using your calculator for trigonometric or inverse trigonometric functions (either graphing or calculating values), ensure the mode of your calculator is set to the correct option: degrees or, more commonly, radians.

Combining trigonometric and inverse trigonometric functions

Consider the expression $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$. This could be calculated by determining the value of $\arcsin\left(\frac{1}{2}\right)$, and then finding the sine of this value:

$$\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

In general, if x is any real number with $-1 \le x \le 1$, then $\sin(\arcsin(x)) = x$. $\arcsin(x)$ is equal to the angle from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ whose sine is equal to x, so taking the sine of this angle must necessarily give x.

$\sin(\arcsin(x)) = x$ for all real numbers x with $-1 \le x \le 1$.
$\cos(\arccos(x)) = x$ for all real numbers x with $-1 \le x \le 1$.
$\tan(\arctan(x)) = x$ for all real numbers x.

However, the situation is not quite so simple with expressions of the form $\arcsin(\sin(x))$ as opposed to $\sin(\arcsin(x))$.

For example, although it is true that $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$, consider the expression $\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right)$.

Carefully working from the inside set of brackets outwards:

$$\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \neq \frac{5\pi}{6}$$

$\arcsin(\sin(x))$ will not be equal to x if the angle x is outside of the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Instead, $\arcsin(\sin(x))$ is equal to the coterminal angle within the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; that is, the angle that has the same sine as x. In the previous example, $\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right)$ was determined to be $\frac{\pi}{6}$ since $\frac{\pi}{6}$ is the angle within the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ that has the same sine as $\frac{5\pi}{6}$.

$\arcsin(\sin(x)) = \theta$ such that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\sin(\theta) = \sin(x)$.
$\arccos(\cos(x)) = \theta$ such that $0 \le \theta \le \pi$ and $\cos(\theta) = \cos(x)$.
$\arctan(\tan(x)) = \theta$ such that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan(\theta) = \tan(x)$.

WARNING
If the angle domain is not defined, then multiple solutions exist.

3 Combining trigonometric and inverse trigonometric functions 1

Determine the exact value of each of the following:

(a) $\cos\left(\arccos\left(-0.3\right)\right)$

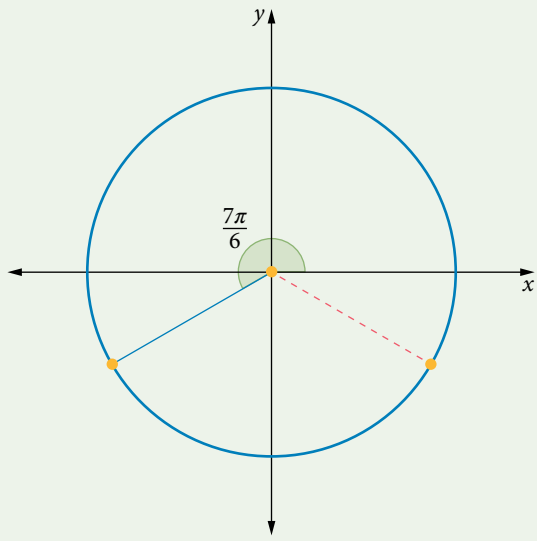
THINKING	WORKING
$\cos\left(\arccos\left(x\right)\right)$ is always equal to x (provided x is in the domain of $\arccos\left(x\right)$).	$-1 \leq -0.3 \leq 1$ Therefore $\cos\left(\arccos\left(-0.3\right)\right) = -0.3$

(b) $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$

1 $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$ is equal to the angle from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (first or fourth quadrants) whose sine is equal to the sine of $\frac{7\pi}{6}$.

Given that $\frac{7\pi}{6}$ does not lie within the given domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ then $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$ is equal to the angle with the equivalent y-coordinate in either the first or fourth quadrant.

2 The sine of an angle is equal to the y-coordinate when drawn on the unit circle.



3 Identify the angle in the first or fourth quadrant with the same y-coordinate.

$\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right) = -\frac{\pi}{6}$

Problems involving the composition of trigonometric and inverse trigonometric functions can become more challenging when the functions involved are not inverses of each other, as the following Worked examples illustrate.

4 Combining trigonometric and inverse trigonometric functions 2

Determine the exact value of each of the following:

(a) $\cos\left(\arcsin\left(\frac{\sqrt{3}}{2}\right)\right)$

THINKING	WORKING
Calculate the value of the inner function and substitute the result into the outer function.	$\cos\left(\arcsin\left(\frac{\sqrt{3}}{2}\right)\right) = \cos\left(\frac{\pi}{3}\right)$ $= \frac{1}{2}$

(b) $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$

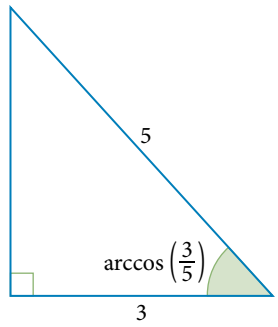
1 Define y as the inner function and determine the quadrant in which y exists.	Let $y = \arccos\left(\frac{3}{5}\right)$. So $0 \leq y \leq \pi$ and $\cos(y) = \frac{3}{5}$. Since $\cos(y)$ is positive, y must be in the first quadrant.
2 Use the property $\sin^2(\theta) + \cos^2(\theta) = 1$ to solve for y. Alternatively, recognise the Pythagorean triad 3,4,5 in order to create the sine ratio.	$\sin^2(y) = 1 - \cos^2(y)$ $= 1 - \left(\frac{3}{5}\right)^2$ $= 1 - \frac{9}{25}$ $= \frac{16}{25}$ $\sin(y) = \pm\sqrt{\frac{16}{25}}$ $= \pm\frac{4}{5}$
3 Interpret the answer.	Since y is in the first quadrant, $\sin(y) = \frac{4}{5}$. Thus, $\sin\left(\arccos\left(\frac{3}{5}\right)\right) = \frac{4}{5}$.

(c) $\sin\left(\arctan\left(-\frac{12}{5}\right)\right)$

1 Define y as the inner function and determine the quadrant in which y exists.	Let $y = \arctan\left(-\frac{12}{5}\right)$. So $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan(y) = -\frac{12}{5}$. Since $\tan(y)$ is negative, y must be in the fourth quadrant.
2 Use the property $\text{cosec}^2(y) = 1 + \cot^2(y)$ to solve for y. Alternatively, recognise the Pythagorean triad 5,12,13 to create the sine ratio.	$\text{cosec}^2(y) = 1 + \cot^2(y)$ $= 1 + \frac{25}{144}$ $= \frac{169}{144}$ $\frac{1}{\sin^2(y)} = \frac{169}{144}$ $\sin^2(y) = \frac{144}{169}$ $\sin(y) = \pm\sqrt{\frac{144}{169}}$ $= \pm\frac{12}{13}$
3 Interpret the answer.	Since y is in the fourth quadrant, $\sin(y) = -\frac{12}{13}$. Thus, $\sin\left(\arctan\left(-\frac{12}{5}\right)\right) = -\frac{12}{13}$.

As an alternative to using trigonometric identities to solve problems such as those in the previous examples, it may be possible to use a simple right-angled triangle. Consider part **(b)** of Worked example 4: $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$. A right-angled triangle can be constructed with an angle of $\arccos\left(\frac{3}{5}\right)$ as shown.

Using Pythagoras' theorem, the remaining side of this triangle can be determined to be 4 units. Then $\sin\left(\arccos\left(\frac{3}{5}\right)\right) = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$.



Although this approach is simpler for problems such as this one, it becomes more complicated when the argument to the inverse trigonometric function is negative, or for general proofs.

EXERCISE 5.1 Inverse trigonometric functions

Worked Example

1 Determine the exact values of the following: 1

(a) $\arcsin(0)$ (b) $\arctan(0)$ (c) $\cos^{-1}\left(\frac{1}{2}\right)$

(d) $\tan^{-1}\left(\sqrt{3}\right)$ (e) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ (f) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(g) $\arctan(1)$ (h) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ (i) $\arcsin\left(-\frac{1}{2}\right)$

2 The value of $\arcsin(0.3)$ is closest to:

A 0.3047 B 17.458 C 0.2956 D 0.0052

3 The value of $\arccos(0.9)$ is closest to:

A 64.16° B 25.84° C 1.12° D 0.45°

4 Determine the domain and range of the following functions, and sketch the graph of each. 2

(a) $f(x) = 2\arcsin\left(\frac{x}{3}\right)$ (b) $y = -3\cos^{-1}(2x)$ (c) $y = 2\arctan(5x)$

(d) $y = \arcsin(x-1)$ (e) $y = 2\arccos(x+0.5)$ (f) $y = -2\arctan\left(x+\sqrt{3}\right)$

5 Determine the exact value of each of the following: 3

(a) $\sin\left(\arcsin\left(\frac{1}{2}\right)\right)$ (b) $\cos\left(\arccos(0.7)\right)$

(c) $\tan\left(\arctan(-2.9)\right)$ (d) $\sin\left(\arcsin(-2.9)\right)$

6 Determine the exact value of each of the following:

(a) $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$ (b) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ (c) $\arccos\left(\cos\left(\frac{4\pi}{3}\right)\right)$

(d) $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$ (e) $\arctan\left(\tan\left(-\frac{\pi}{7}\right)\right)$ (f) $\arctan\left(\tan\left(-\frac{8\pi}{7}\right)\right)$

(g) $\arcsin\left(\sin\left(-\frac{\pi}{5}\right)\right)$ (h) $\arcsin\left(\sin\left(-\frac{6\pi}{5}\right)\right)$ (i) $\cos^{-1}\left(\cos\left(-\frac{6\pi}{5}\right)\right)$

(j) $\arctan\left(\tan(-\pi)\right)$

Worked Example

4

7 Determine the exact value of each of the following:

(a) $\sin\left(\arccos(-1)\right)$ **(b)** $\tan\left(\arcsin\left(-\frac{1}{2}\right)\right)$ **(c)** $\cos\left(\arcsin\left(\frac{3}{5}\right)\right)$

(d) $\sin\left(\arccos\left(-\frac{1}{3}\right)\right)$ **(e)** $\tan\left(\arccos\left(-\frac{2}{3}\right)\right)$ **(f)** $\cos\left(\tan^{-1}(3)\right)$

(g) $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$

8 Which of the following statements is true for all real values of x, where $-1 \leq x \leq 1$?

A $\sin^{-1}(x) = \frac{1}{\sin(x)}$ **B** $\tan^{-1}(x) = \frac{\sin^{-1}(x)}{\cos^{-1}(x)}$

C $\left(\sin^{-1}(x)\right)^2 + \left(\cos^{-1}(x)\right)^2 = 1$ **D** none of the above

9 **(a)** Which of the following statements is true?

A $\arccos\left(\cos(4)\right) = 4$ **B** $\cos\left(\arccos(4)\right) = 4$

C $\arccos\left(\cos(4)\right) = 2\pi - 4$ **D** $\arccos\left(\cos(4)\right) =$ undefined

(b) Explain the possible errors or misconceptions from selecting the incorrect options in part **(a)**.

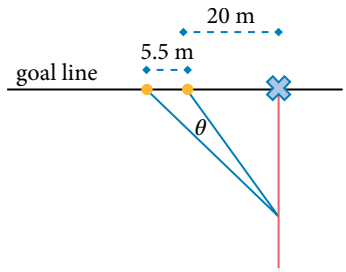
10 Prove that if $-1 \leq x \leq 1$, $\sin\left(\arccos(x)\right) = \sqrt{1-x^2}$.

11 Prove that if $-1 \leq x \leq 1$, $\cos\left(\arctan(x)\right) = \frac{1}{\sqrt{1+x^2}}$.

12 **(a)** Use the fact that $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$ to prove that $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$.

(b) Evaluate the reasonableness of this result with reference to the angles in a right-angled triangle, and the graphs of $y = \sin^{-1}(x)$ and $y = \cos^{-1}(x)$.

13 A rugby league player is about to attempt to kick a goal to convert a try. The try was scored 20 metres from the nearest goal post, as shown, and so the player can kick from anywhere along the red line (which is perpendicular to the goal line). Given that the distance between the two goal posts is 5.5 metres, how far along the red line from the goal line should the player attempt the kick from, in order to maximise the kicking angle, θ? Express your answer correct to 2 decimal places.

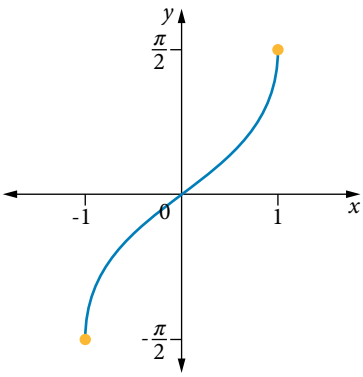


14 Solve the equation $\arccos(x) - \arccos(3x) = \frac{\pi}{3}$ algebraically, given that $0 < x < 1$. Evaluate the reasonableness of your solution(s).

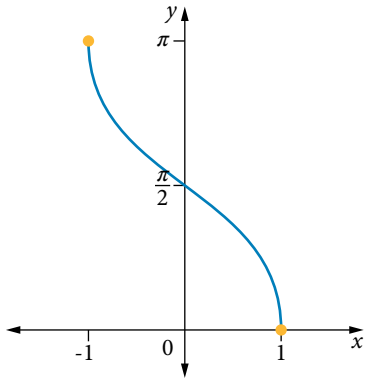
Derivatives of inverse trigonometric functions

5.2

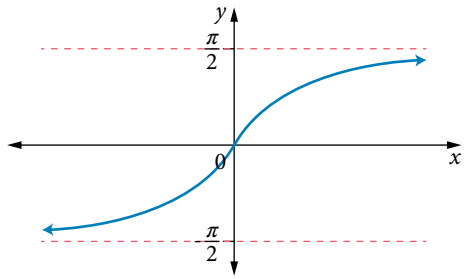
The graphs of the inverse trigonometric functions introduced in the previous section are shown.



$y = \arcsin(x)$



$y = \arccos(x)$



$y = \arctan(x)$

To determine the derivatives of each of these functions, the following general result will be used:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \text{ if } \frac{dx}{dy} \neq 0$$

This result may seem strange at first. However, consider the simple function $y = \frac{1}{3}x - 2$. This can be rearranged to give $x = 3y - 6$. From the original function, it is clear that $\frac{dy}{dx} = \frac{1}{3}$. From the rearranged version, it is clear that $\frac{dx}{dy} = 3$.

Thus, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Now consider the problem of determining the derivative of the arcsine function.

Let $y = \arcsin(x)$. So $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $x = \sin(y)$.

$\frac{dx}{dy} = \cos(y)$, so $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

Since $\sin^2(y) + \cos^2(y) = 1$,

$$\begin{aligned}\cos(y) &= \pm\sqrt{1 - \sin^2(y)} \\ &= \pm\sqrt{1 - x^2}\end{aligned}$$

Also, since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos(y) \geq 0$ then $\cos(y) = \sqrt{1 - x^2}$.

Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\cos(y)} \\ &= \frac{1}{\sqrt{1 - x^2}}\end{aligned}$$

So $\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1 - x^2}}$.

Similar results for the derivatives of the arccosine and arctangent functions can be established using a similar method (the proofs of these are left as problems in Exercise 5.2).

In summary:

The derivative of a function is a formula for the gradient of the function at any value of x.

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

From the derivative expressions above:

- The expression for the derivative of $y = \arcsin(x)$ is positive for all permissible values of x. This makes sense as the graph of $y = \arcsin(x)$ is always increasing (so the gradient is always positive).
- The expression for the derivative of $y = \arccos(x)$ is negative for all permissible values of x. This makes sense as the graph of $y = \arccos(x)$ is always decreasing (so the gradient is always negative).
- The expression for the derivative of $y = \arctan(x)$ is positive for all permissible values of x. This makes sense as the graph of $y = \arctan(x)$ is always increasing (so the gradient is always positive).
- The expressions for the derivatives of both the arcsine and arccosine functions are only defined for $-1 < x < 1$, even though the original functions are defined for $-1 \leq x \leq 1$. This makes sense as the graphs of the arcsine and arccosine functions become vertical at $x = \pm 1$, meaning that the gradient at these points is undefined.

These derivative rules can be used in conjunction with the chain, product and quotient rules to differentiate more complicated functions involving the inverse trigonometric functions.

5 Using the basic inverse trigonometric derivative rules

Determine the derivative of each of the following:

(a) $y = \arccos(3x + 2)$

METHOD 1

THINKING	WORKING
1 Define the expression within the parentheses.	$y = \arccos(u)$, where $u = 3x + 2$ $\therefore \frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = 3$
2 Use the rule for the derivative of $\arccos(x)$ in combination with the chain rule to determine $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= \frac{-1}{\sqrt{1-u^2}} \times 3$ $= \frac{-1}{\sqrt{1-(3x+2)^2}} \times 3$ $= \frac{-3}{\sqrt{1-(3x+2)^2}}$

METHOD 2

1 Use the single variable x and determine $\frac{dy}{dx}$. The chain rule can also be used without formally introducing a new variable.

$$y = \arccos(3x+2)$$
$$\frac{d(\arccos(x))}{dx} = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d(3x+2)}{dx} = 3$$

2 Determine the derivative of the main function and multiply by its derivative.

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(3x+2)^2}} \times 3 \text{ (using the chain rule)}$$
$$= \frac{-3}{\sqrt{1-(3x+2)^2}}$$

(b) $y = \arcsin\left(\frac{x}{5}\right)$

1 Define the expression within the parentheses.

$$y = \arcsin(u), \text{ where } u = \frac{x}{5}$$
$$\therefore \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \text{ and } \frac{du}{dx} = \frac{1}{5}$$

2 Use the chain rule to determine the derivative.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{\sqrt{1-u^2}} \times \frac{1}{5}$$
$$= \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} \times \frac{1}{5}$$
$$= \frac{1}{\sqrt{1-\frac{x^2}{25}}} \times \frac{1}{\sqrt{25}}$$
$$= \frac{1}{\sqrt{25\left(1-\frac{x^2}{25}\right)}}$$
$$= \frac{1}{\sqrt{25-x^2}}$$

(c) $y = 2\sqrt{x}\arctan(x)$

1 Recognise the function as the product of two sub-functions.	Let $y = uv$, where $u = 2\sqrt{x} = 2x^{\frac{1}{2}}$ and $v = \arctan(x)$ Then $\frac{du}{dx} = x^{-\frac{1}{2}}$ and $\frac{dv}{dx} = \frac{1}{1+x^2}$
2 Use the product rule to differentiate the function.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= 2x^{\frac{1}{2}} \times \frac{1}{1+x^2} + \arctan(x) \times x^{-\frac{1}{2}}$ $= \frac{2\sqrt{x}}{1+x^2} + \frac{\arctan(x)}{\sqrt{x}}$

Part **(b)** in Worked example 5 demonstrated that $\frac{d}{dx}\left(\arcsin\left(\frac{x}{5}\right)\right) = \frac{1}{\sqrt{25-x^2}}$.

In general, if $a > 0$, $\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2-x^2}}$. This result can be formally proved:

Let $y = \arcsin\left(\frac{x}{a}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \times \frac{1}{a} \text{ (using the chain rule)}$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \times \frac{1}{\sqrt{a^2}} \text{ (provided } a > 0)$$

$$= \frac{1}{\sqrt{a^2-x^2}}$$

If $a > 0$, $\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2-x^2}}$.

If $a > 0$, $\frac{d}{dx}\left(\arccos\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2-x^2}}$.

If $a \neq 0$, $\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2}$.

Similar results for the arccosine and arctangent functions can be established using a similar method (the proofs of these are left as problems in Exercise 5.2).

These derivative results may seem unnecessary since a function such as $y = \arcsin\left(\frac{x}{5}\right)$ could readily be differentiated in the manner previously demonstrated. However, these derivative results are useful since these types of functions occur so frequently, and it is useful being able to quickly determine the derivative when they do.

6 Using the more general inverse trigonometric derivative rules

Use the rules for the derivatives of $\arcsin\left(\frac{x}{a}\right)$, $\arccos\left(\frac{x}{a}\right)$ and $\arctan\left(\frac{x}{a}\right)$ to determine the derivative of each of the following:

(a) $y = 5\arccos\left(\frac{x}{6}\right)$

THINKING	WORKING
Recall the general rule to differentiate $\arccos\left(\frac{x}{a}\right)$ to determine $\frac{dy}{dx}$.	In the form $\arccos\left(\frac{x}{a}\right)$, where $a = 6$: $\frac{dy}{dx} = 5 \times \frac{-1}{\sqrt{a^2 - x^2}}$ $= 5 \times \frac{-1}{\sqrt{6^2 - x^2}} = \frac{-5}{\sqrt{36 - x^2}}$

(b) $y = 3\arctan\left(\frac{x}{7}\right)$

Recall the general rule to differentiate $\arctan\left(\frac{x}{a}\right)$ to determine $\frac{dy}{dx}$.	In the form $\arctan\left(\frac{x}{a}\right)$, where $a = 7$: $\frac{dy}{dx} = 3 \times \frac{a}{a^2 + x^2}$ $= 3 \times \frac{7}{7^2 + x^2} = \frac{21}{49 + x^2}$

Remember that the derivative of a function is a formula for the gradient of the function at any value of x. You can then determine the gradient at specific values of x.

7 Calculating gradients at particular values of x

Determine the exact gradient of the function $y = 7\sin^{-1}(5x)$ at the point where $x = \frac{1}{10}$.

THINKING	WORKING
1 Recall the general rule to differentiate $\arcsin\left(\frac{x}{a}\right)$ to determine $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{7}{\sqrt{1 - (5x)^2}} \times 5$ (using the chain rule) $= \frac{35}{\sqrt{1 - 25x^2}}$
2 Substitute the value of x to calculate the instantaneous gradient or rate of change at that point.	$\frac{dy}{dx}_{x=\frac{1}{10}} = \frac{35}{\sqrt{1 - 25\left(\frac{1}{10}\right)^2}}$ $= \frac{35}{\sqrt{1 - \frac{25}{100}}}$ $= \frac{35}{\sqrt{\frac{75}{100}}}$ $= \frac{35}{\sqrt{\frac{3}{4}}}$ $= \frac{35}{\frac{\sqrt{3}}{2}}$ $= \frac{70}{\sqrt{3}}$ $= \frac{70\sqrt{3}}{3}$

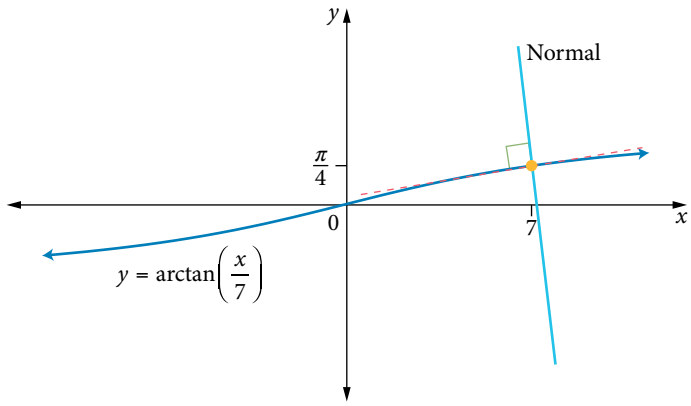
Graphics calculators typically have a command that allows the gradient of any function at a particular x-value to be calculated. If you use such a command with trigonometric or inverse trigonometric functions, your calculator must be in radian mode.

Technology worked example
Calculating the instantaneous rate of change

In Mathematical Methods, you learnt how to determine the equations of tangents and normals to simple functions. These ideas can be applied to inverse trigonometric functions (and related functions), as demonstrated in Worked example 8.

8 Determining the equation of a tangent and the equation of the normal to a function

The diagram shows the normal to the function $y = \arctan\left(\frac{x}{7}\right)$ at the point where $x = 7$. Determine the equation of this normal in the form $y = mx + c$ and evaluate the reasonableness of your solution with reference to the diagram.



THINKING	WORKING
1 Determine the derivative function where $\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}$.	$y = \tan^{-1}\left(\frac{x}{7}\right)$ $\frac{dy}{dx} = \frac{7}{49 + x^2}$
2 Substitute the x-coordinate of the relevant point into the derivative to obtain the gradient of the tangent at this point. The gradient of the normal at this point is the negative reciprocal of the gradient of the tangent.	$\frac{dy}{dx}_{x=7} = \frac{7}{49 + 7^2}$ $= \frac{7}{98}$ $= \frac{1}{14}$ $\therefore m_T = \frac{1}{14}$ and $\therefore m_N = -14$

3 Determine the y-coordinate of the point on the function.

$$y(7) = \tan^{-1}\left(\frac{7}{7}\right)$$
$$= \tan^{-1}(1)$$
$$= \frac{\pi}{4}$$

The coordinates of the point on the curve are $\left(7, \frac{\pi}{4}\right)$.

4 Use the rule $y - y_1 = m(x - x_1)$ to determine the equation of the normal.

Equation of normal:

$$y - \frac{\pi}{4} = -14(x - 7)$$
$$y = -14x + 98 + \frac{\pi}{4}$$

5 Evaluate the reasonableness of your solution with reference to the diagram.

From the equation obtained, the gradient is -14 and the y-intercept is $98 + \frac{\pi}{4}$. Both values are reasonable as the diagram indicates that the normal has a steep negative gradient and a large positive y-intercept.

EXERCISE 5.2 Derivatives of inverse trigonometric functions

Worked Example

1 Determine the derivative of each of the following: 5

(a) $y = \arcsin(5x)$ (b) $y = \arccos\left(\frac{x}{3}\right)$ (c) $y = 3\arctan(-2x)$

(d) $y = 3\arcsin(5 - 3x)$ (e) $y = 7\arccos\left(\sqrt{x}\right)$ (f) $y = 7\arcsin\left(\frac{1}{x}\right)$

(g) $y = x\arctan(5x)$ (h) $y = \sqrt{x}\arcsin\left(\frac{x}{2}\right)$ (i) $y = \frac{\arccos(x)}{\sqrt{x}}$

2 Use the rules for the derivatives of $\arcsin\left(\frac{x}{a}\right)$, $\arccos\left(\frac{x}{a}\right)$ and $\arctan\left(\frac{x}{a}\right)$ to determine the derivative of each of the following: 6

(a) $y = \arcsin\left(\frac{x}{7}\right)$ (b) $y = 4\arccos\left(\frac{x}{3}\right)$ (c) $y = 2\arctan\left(\frac{x}{5}\right)$

(d) $y = \arccos(6x)$ (e) $y = 2\arcsin(3x)$ (f) $y = 3\arctan(5x)$

3 Which of the following statements is true for the function $y = \arccos\left(\frac{x}{3}\right)$?

A The domain of the function is $[-1,1]$ and the domain of its derivative is $[-1,1]$.

B The domain of the function is $[-3,3]$ and the domain of its derivative is $[-3,3]$.

C The domain of the function is $[-3,3]$ and the domain of its derivative is $(-3,3)$.

D The domain of the function is $[-1,1]$ and the domain of its derivative is $(-1,1)$.

Worked Example

4 Determine the domain and range of the derivative of $y = \arctan\left(\frac{x}{5}\right)$.

7

5 Determine the gradient of the function $y = \tan^{-1}(6x)$ at the point where $x = \frac{1}{2}$.

6 Determine the gradient of the function $y = \arcsin\left(\frac{1}{\sqrt{x}}\right)$ at the point where $x = 5$.

7 Determine the gradient of the function $y = \arctan\left(\frac{x}{5}\right)$ at the point where $x = 0.99$. Express your answer correct to 2 decimal places.

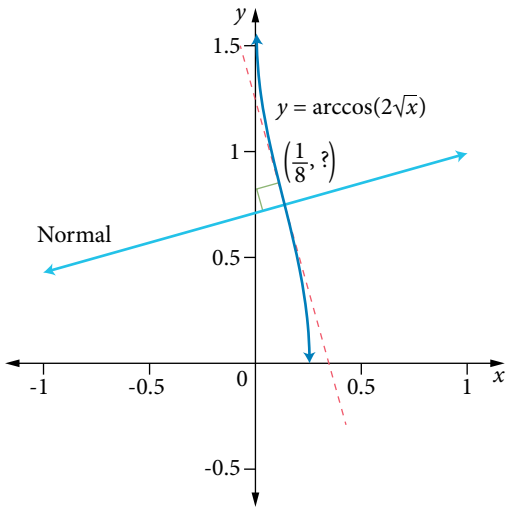
8 Determine the gradient of the function $y = \sin^{-1}(x^2)$ at the point where $x = 0.52$. Express your answer correct to 2 decimal places.

8

9 Determine the equation of the tangent to the function $y = \arccos\left(\frac{x}{6}\right)$ at the point where $x = 3$.

10 Determine the equation of the tangent to the function $y = \tan^{-1}(3x)$ at the point where $x = 0$.

11 The diagram shows the normal to the function $y = \arccos\left(2\sqrt{x}\right)$ at the point where $x = \frac{1}{8}$. Determine the equation of this normal in the form $y = mx + c$ and evaluate the reasonableness of your solution with reference to the diagram.



12 (a) Which one of the following is the derivative of $y = \cos^{-1}\left(-\frac{x}{3}\right)$?

A $\frac{1}{\sqrt{9-x^2}}$ B $\frac{-1}{\sqrt{9-x^2}}$ C $\frac{1}{\sqrt{9+x^2}}$ D $\frac{-1}{\sqrt{9+x^2}}$

(b) Explain a possible misconception if a student selected the first incorrect option in part (a).

13 Determine the gradient of the function $y = \cos^{-1}(2x-1)$ at the point with y-coordinate equal to $\frac{\pi}{2}$.

14 The volume in cm^3 of water in a tank after t hours is modelled by $V(t) = \tan^{-1}(2.7t)$. How many minutes (to the nearest minute) pass before the rate at which the volume is increasing reaches $1.197\,cm^3$/hour?

15 **(a)** Prove that $\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$.

(b) **(i)** Prove that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$.

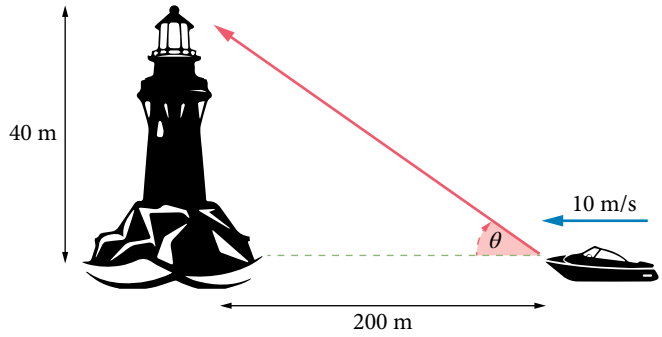
(ii) Hence, prove that $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$.

16 Using the result from question **15** part **(a)**, in conjunction with the chain rule, prove that if $a > 0$, $\frac{d}{dx}\left(\arccos\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2-x^2}}$.

17 Using the result from question **15** part **(b)**, in conjunction with the chain rule, prove that if a is any real number, $\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x}$.

18 Determine the coordinates in exact form of the point(s) on the function $y = 4\arcsin(7-3x)$ at which the gradient of the normal is equal to $\frac{1}{24}$.

19 A boat is moving at 10 m/s towards a 40 m high lighthouse. Determine how fast (in radians per second) the angle of elevation from the boat to the top of the lighthouse is increasing by when the boat is 200 m from the base of the lighthouse. Express your answer correct to 4 decimal places.



5.3 Anti-derivatives in logarithmic and inverse trigonometric form

In this section, several additional integral rules will be introduced to add to other standard rules you have already learnt in Mathematical Methods. The first rule relates to the logarithmic function, while the remaining rules relate to the inverse trigonometric functions studied in the previous two sections.

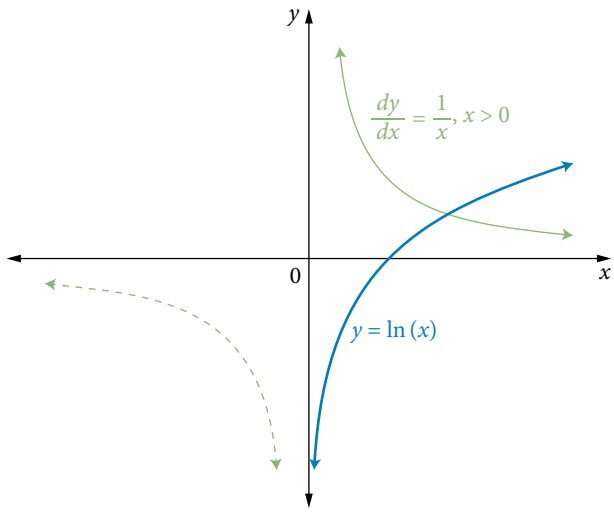
Integrals of the form $\int \frac{1}{ax+b}\,dx$

In Mathematical Methods, you learnt that the derivative of the function $y = \ln(x)$ is $\frac{dy}{dx} = \frac{1}{x}$, $x > 0$. This rule may be demonstrated using a similar method that was used to establish the derivative of the inverse sine function in the previous section:

If $y = \ln(x)$, then $x = e^y$ and so

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}}\\ &= \frac{1}{e^y}\\ &= \frac{1}{x}\end{aligned}$$

Even though $\frac{1}{x}$ is defined for negative values of x, since $y = \ln(x)$ is only defined for $x > 0$, its derivative cannot possibly be defined for values of x less than 0. Hence, it is more accurate to say that the derivative of the function $y = \ln(x)$ is equal to the part of the function $y = \frac{1}{x}$ for which $x > 0$ (see diagram).



Similarly, the corresponding anti-derivative rule $\int \frac{1}{x}dx = \ln(x) + c$ is only valid for $x > 0$.

Ideally, you would like to determine the anti-derivative of $\frac{1}{x}$ when $x < 0$. Consider the function $y = \ln(-x)$. This function is defined for all $x < 0$. To determine its derivative, the chain rule may be used by letting $y = \ln(u)$, where $u = -x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times -1 \\ &= \frac{-1}{-x} \\ &= \frac{1}{x}\end{aligned}$$

Therefore, if $x < 0$, $\int \frac{1}{x} dx = \ln(-x) + c$.

In summary:

$$\int \frac{1}{x} dx = \begin{cases} \ln(x) + c & x > 0 \\ \ln(-x) + c & x < 0 \end{cases}$$

This can be written more succinctly using the absolute value function. Recall that $|x|$ is equal to x if $x \geq 0$ and $-x$ if $x < 0$.

$$\int \frac{1}{x} dx = \ln|x| + c \quad \text{for } x \neq 0$$

To generalise this rule slightly, the chain rule may be used to establish that if a and b are real numbers with $a \neq 0$, then both $\frac{d}{dx}\left(\frac{1}{a}\ln(ax+b)\right) = \frac{1}{ax+b}$, for $ax + b > 0$ and $\frac{d}{dx}\left(\frac{1}{a}\ln(-(ax+b))\right) = \frac{1}{ax+b}$, for $ax + b < 0$. Thus:

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + c \quad \text{for } x \neq -\frac{b}{a}$$

9 Determining indefinite integrals of the form $\int \frac{1}{ax+b} dx$

Determine the following indefinite integrals:

(a) $\int \frac{1}{3x} dx$

METHOD 1

THINKING	WORKING
Take out common factors to express the integral in the form $\int \frac{1}{x} dx$. Then apply the rule for $\int \frac{1}{x} dx$.	$\int \frac{1}{3x} dx = \frac{1}{3}\int \frac{1}{x} dx$ $= \frac{1}{3}\ln\|x\| + c$

METHOD 2

Apply the rule for $\int \frac{1}{ax+b} dx$ with $a = 3$ and $b = 0$.	$\int \frac{1}{3x} dx = \frac{1}{3}\ln\|3x\| + c$

(b) $\int \frac{7}{-3x+4}dx$

1 Take out common factors to express the integral in the form $\int \frac{1}{ax+b}dx$.	$\int \frac{7}{-3x+4}dx = 7 \times -\frac{1}{3}\int \frac{-3}{4-3x}dx$
2 Integrate the function.	$= -\frac{7}{3}\ln\lvert 4-3x\rvert + c$

(c) $\int \frac{(x+3)^2}{9x}dx$

1 Expand the numerator and separate the integrand into multiple terms.	$\int \frac{(x+3)^2}{9x}dx = \int \frac{x^2+6x+9}{9x}dx$ $= \int \frac{x^2}{9x} + \frac{6x}{9x} + \frac{9}{9x}dx$ $= \int \frac{1}{9}x + \frac{2}{3} + \frac{1}{x}dx$
2 Integrate each term.	$= \frac{1}{18}x^2 + \frac{2}{3}x + \ln\lvert x\rvert + c$

In part **(a)** of Worked example 9, the two different methods produced answers that look slightly different: $\frac{1}{3}\ln\lvert x\rvert + c$ and $\frac{1}{3}\ln\lvert 3x\rvert + c$. To see why these expressions are equivalent, it is important to understand that the 'c' when integrating is used to represent a constant that can take on any real value.

Thus, even though $\int 2x\ dx = x^2 + c$, it would also be valid to say that $\int 2x\ dx = x^2 + 7 + c$ or even $\int 2x\ dx = x^2 + 41 - 13c$ as '$7 + c$' or '$41 - 13c$' are both constant expressions that can take on any real values. For simplicity, if an indefinite integral involves constant terms in addition to the c that is obtained, usually you combine them into a single constant term with a different name (often c_1). For example, you could write

$$\begin{aligned}\int 2x\ dx &= x^2 + 41 - 13c \\ &= x^2 + c_1, \text{ where } c_1 = 41 - 13c\end{aligned}$$

Returning to the case of $\int \frac{1}{3x}dx$, using logarithm rules, the result $\frac{1}{3}\ln\lvert 3x\rvert + c$ may be simplified as follows to obtain the result $\frac{1}{3}\ln\lvert x\rvert + c$:

$$\begin{aligned}\frac{1}{3}\ln\lvert 3x\rvert + c &= \frac{1}{3}\left(\ln(3) + \ln\lvert x\rvert\right) + c \\ &= \frac{1}{3}\ln\lvert x\rvert + \frac{1}{3}\ln(3) + c \\ &= \frac{1}{3}\ln\lvert x\rvert + c_1, \text{ where } c_1 = \frac{1}{3}\ln(3) + c\end{aligned}$$

Integrals of the form $\int \frac{\pm 1}{\sqrt{a^2 - x^2}}\,dx$ and $\int \frac{a}{a^2 + x^2}\,dx$

The previous section established the following derivative rules for inverse trigonometric functions:

If $a > 0$,
$$\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}.$$

If $a > 0$,
$$\frac{d}{dx}\left(\arccos\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - x^2}}.$$

If $a \neq 0$,
$$\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}.$$

These give rise to the integral rules in the highlighted box.

If $a > 0$, $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c.$

If $a > 0$, $\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + c.$

If $a \neq 0$, $\int \frac{a}{a^2 + x^2}\,dx = \arctan\left(\frac{x}{a}\right) + c.$

The second of these rules is actually unnecessary since any integral of the form $\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx$ may be rewritten as $-\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$, and then the first rule applied.

For example, consider the integral $\int \frac{-1}{\sqrt{9 - x^2}}\,dx$. Using the second rule with $a = 3$, you would obtain $\int \frac{-1}{\sqrt{9 - x^2}}\,dx = \arccos\left(\frac{x}{3}\right) + c$.

Alternatively, this integral could be rewritten as $-\int \frac{1}{\sqrt{9 - x^2}}\,dx$. Then applying the first rule with $a = 3$ gives

$$\int \frac{-1}{\sqrt{9 - x^2}}\,dx = -\int \frac{1}{\sqrt{9 - x^2}}\,dx = -\arcsin\left(\frac{x}{3}\right) + c.$$

Like the example of $\int \frac{1}{3x}\,dx$ previously discussed, it may seem strange that two seemingly different answers may be obtained for the same indefinite integral. However, they are both valid. From the previous exercise you will have proved that $\arccos(x) + \arcsin(x) = \frac{\pi}{2}$, and so

$$\begin{aligned}\arccos\left(\frac{x}{3}\right) + c &= -\arcsin(x) + \frac{\pi}{2} + c \\ &= -\arcsin(x) + c_1, \text{ where } c_1 = c + \frac{\pi}{2}\end{aligned}$$

The following Worked examples demonstrate how to apply these new integral rules.

10 Determining indefinite integrals of the form $\int \frac{\pm 1}{\sqrt{a^2-x^2}}\,dx$ and $\int \frac{a}{a^2+x^2}\,dx$

Determine the following indefinite integrals:

(a) $\int \frac{2}{\sqrt{3-x^2}}\,dx$

THINKING	WORKING
1 Simplify the integral so it has the form $\frac{1}{\sqrt{a^2-x^2}}$.	$\int \frac{2}{\sqrt{3-x^2}}\,dx = 2\int \frac{1}{\sqrt{3-x^2}}\,dx$ $= 2\int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-x^2}}\,dx$
2 Use the appropriate rule to integrate.	$2\int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-x^2}}\,dx = 2\arcsin\left(\frac{x}{\sqrt{3}}\right)+c$

(b) $\int \frac{-5}{\sqrt{1-9x^2}}\,dx$

METHOD 1

1 Simplify the integral so it has the form $\frac{1}{\sqrt{a^2-x^2}}$.	$\int \frac{-5}{\sqrt{1-9x^2}}\,dx = -5\int \frac{1}{\sqrt{1-9x^2}}\,dx$ $= -5\int \frac{1}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}\,dx$ $= -5\int \frac{1}{3\sqrt{\frac{1}{9}-x^2}}\,dx$ $= -\frac{5}{3}\int \frac{1}{\sqrt{\frac{1}{9}-x^2}}\,dx$ $= -\frac{5}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2-x^2}}\,dx$
2 Use the appropriate rule to integrate.	$-\frac{5}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2-x^2}}\,dx = -\frac{5}{3}\arcsin\left(\frac{x}{\frac{1}{3}}\right)+c$ $= -\frac{5}{3}\arcsin(3x)+c$

METHOD 2

1 Simplify the integral so it has the form $\frac{-1}{\sqrt{a^2 - x^2}}$.

$$\begin{aligned}\int \frac{-5}{\sqrt{1-9x^2}}\,dx &= 5\int \frac{-1}{\sqrt{1-9x^2}}\,dx \\ &= 5\int \frac{-1}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}\,dx \\ &= 5\int \frac{-1}{3\sqrt{\frac{1}{9}-x^2}}\,dx \\ &= \frac{5}{3}\int \frac{-1}{\sqrt{\frac{1}{9}-x^2}}\,dx \\ &= \frac{5}{3}\int \frac{-1}{\sqrt{\left(\frac{1}{3}\right)^2-x^2}}\,dx\end{aligned}$$

2 Use the appropriate rule to integrate.

$$\begin{aligned}\frac{5}{3}\int \frac{-1}{\sqrt{\left(\frac{1}{3}\right)^2-x^2}}\,dx &= \frac{5}{3}\arccos\left(\frac{x}{\frac{1}{3}}\right)+c \\ &= \frac{5}{3}\arccos(3x)+c\end{aligned}$$

(c) $\int \frac{1}{9+x^2}\,dx$

1 Simplify the integral so it has the form $\frac{a}{a^2 + x^2}$.

$$\begin{aligned}\int \frac{1}{9+x^2}\,dx &= \int \frac{1}{3^2+x^2}\,dx \\ &= \int \frac{1}{3}\times\frac{3}{3^2+x^2}\,dx \\ &= \frac{1}{3}\int \frac{3}{3^2+x^2}\,dx\end{aligned}$$

2 Use the appropriate rule to integrate.

$$\frac{1}{3}\int \frac{3}{3^2+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+c$$

(d) $\int \frac{-5}{7+2x^2}\,dx$

1 Simplify the integral so it has the form $\frac{a}{a^2 + x^2}$.

$$\begin{aligned}\int \frac{-5}{7+2x^2}\,dx &= -5\int \frac{1}{2\left(\frac{7}{2}+x^2\right)}\,dx \\ &= -\frac{5}{2}\int \frac{1}{\left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2+x^2}\,dx \\ &= -\frac{5}{2}\int \frac{\sqrt{2}}{\sqrt{7}}\times\frac{\frac{\sqrt{7}}{\sqrt{2}}}{\left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2+x^2}\,dx \\ &= -\frac{5\sqrt{2}}{2\sqrt{7}}\int \frac{\frac{\sqrt{7}}{\sqrt{2}}}{\left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2+x^2}\,dx\end{aligned}$$

2 Use the appropriate rule to integrate.

$$-\frac{5\sqrt{2}}{2\sqrt{7}}\int\frac{\frac{\sqrt{7}}{\sqrt{2}}}{\left(\frac{\sqrt{7}}{\sqrt{2}}\right)^2+x^2}\,dx=-\frac{5\sqrt{2}}{2\sqrt{7}}\arctan\left(\frac{x}{\frac{\sqrt{7}}{\sqrt{2}}}\right)+c$$

$$=-\frac{5\sqrt{2}}{2\sqrt{7}}\arctan\left(\frac{\sqrt{2}x}{\sqrt{7}}\right)+c$$

Definite integrals

The new integral rules introduced in this section may be applied to definite integrals using the techniques learnt in Mathematical Methods.

11 Calculating definite integrals

Calculate the following definite integrals:

(a) $\int_{-7}^{-e}\frac{1}{3x}dx$

THINKING

1 Use the appropriate rule to integrate, placing the result in square brackets with the upper and lower integration limits shown on the right.

2 Substitute the limits and simplify the result.

WORKING

$$\int_{-7}^{-e}\frac{1}{3x}dx=\frac{1}{3}\int_{-7}^{-e}\frac{1}{x}dx$$

$$=\frac{1}{3}\left[\ln|x|\right]_{-7}^{-e}$$

$$=\frac{1}{3}\left(\ln|-e|-\ln|-7|\right)$$

$$=\frac{1}{3}\left(1-\ln(7)\right)$$

(b) $\int_0^{\sqrt{3}}\frac{3}{\sqrt{4-x^2}}dx$

1 Simplify the integral so it has the form $\frac{1}{\sqrt{a^2-x^2}}$.

2 Use the appropriate rule to integrate, placing the result in square brackets with the upper and lower integration limits shown on the right.

3 Substitute the limits and simplify the result.

$$\int_0^{\sqrt{3}}\frac{3}{\sqrt{4-x^2}}dx=3\int_0^{\sqrt{3}}\frac{1}{\sqrt{2^2-x^2}}dx$$

$$=3\left[\arcsin\left(\frac{x}{2}\right)\right]_0^{\sqrt{3}}$$

$$=3\left(\arcsin\left(\frac{\sqrt{3}}{2}\right)-\arcsin\left(\frac{0}{2}\right)\right)$$

$$=3\left(\frac{\pi}{3}-0\right)$$

$$=\pi$$

WARNING

Graphics calculators typically have a command that calculates definite integrals. If you use such a command with trigonometric or inverse trigonometric functions, your calculator must be in radian mode.

Areas bounded by functions

Technology worked example
Calculating definite integrals

In Mathematical Methods, you learnt that the area bounded by a function $f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is equal to:

$\int_a^b f(x)\,dx$ if the function lies above the x-axis from a to b, and

$-\int_a^b f(x)\,dx$ if the function lies below the x-axis from a to b.

The new integral rules introduced in this section mean that problems relating to areas bounded by functions of the form $y = \frac{1}{ax+b}$, $y = \frac{1}{\sqrt{a^2 - x^2}}$, $y = \frac{-1}{\sqrt{a^2 - x^2}}$ or $y = \frac{a}{a^2 + x^2}$ may now be readily solved.

12 Solving area problems

Determine the exact value of k, given that the area bounded by the function $y = \frac{1}{\sqrt{16 - x^2}}$, the y-axis, the x-axis and the line $x = k$, where $k > 0$, is equal to $\frac{\pi}{3}$ units^2.

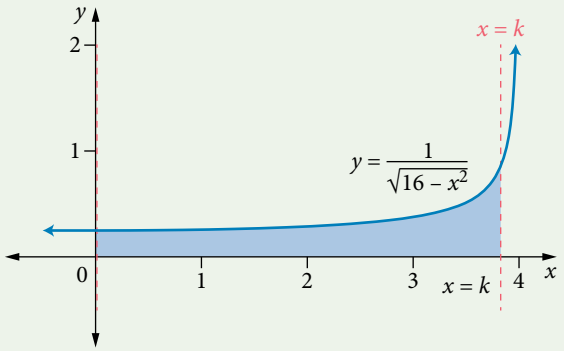
THINKING

1 Express the area described as a definite integral.

WORKING

Given that the function $y = \frac{1}{\sqrt{16 - x^2}}$ lies entirely above the x-axis, the area described is given by:

$$\text{Area} = \int_0^k \frac{1}{\sqrt{16 - x^2}}\,dx$$



2 Substitute the value for the area and obtain an expression for the definite integral in terms of k.

$$\frac{\pi}{3} = \left[\arcsin\left(\frac{x}{4}\right)\right]_0^k$$
$$= \arcsin\left(\frac{k}{4}\right) - \arcsin(0)$$
$$= \arcsin\left(\frac{k}{4}\right) - 0$$

3 Solve the equation to determine the value of k.

$$\sin\left(\frac{\pi}{3}\right) = \frac{k}{4}$$
$$\frac{\sqrt{3}}{2} = \frac{k}{4}$$
$$k = 2\sqrt{3}$$

EXERCISE 5.3 Anti-derivatives in logarithmic and inverse trigonometric form

Worked Example

9

1 Determine the following anti-derivatives.

(a) $\int \frac{7}{2x-5}\,dx$ (b) $\int \frac{-3}{9-x}\,dx$ (c) $\int \frac{5}{4x}\,dx$

(d) $\int \frac{5}{-7x}\,dx$ (e) $\int \frac{x^5+4x-7}{4x}\,dx$ (f) $\int \frac{(x-1)^2}{2x}\,dx$

10

2 Determine the following anti-derivatives.

(a) $\int \frac{7}{\sqrt{9-x^2}}\,dx$ (b) $\int \frac{-5}{\sqrt{16-x^2}}\,dx$ (c) $\int \frac{3}{9+x^2}\,dx$

(d) $\int \frac{3}{\sqrt{2-x^2}}\,dx$ (e) $\int \frac{-2}{\sqrt{1-25x^2}}\,dx$ (f) $\int \frac{7}{\sqrt{9-4x^2}}\,dx$

(g) $\int \frac{-3}{\sqrt{5-2x^2}}\,dx$ (h) $\int \frac{4}{4+x^2}\,dx$ (i) $\int \frac{\sqrt{2}}{\sqrt{8-x^2}}\,dx$

(j) $\int \frac{1}{100+x^2}\,dx$ (k) $\int \frac{2}{\sqrt{4-3x^2}}\,dx$ (l) $\int \frac{2}{5+x^2}\,dx$

(m) $\int \frac{1}{1+7x^2}\,dx$ (n) $\int \frac{\sqrt{2}}{\sqrt{8-x^2}}\,dx$ (o) $\int \frac{3}{5+2x^2}\,dx$

11

3 Calculate the following definite integrals, expressing your answers in exact form.

(a) $\int_{-5}^{-e} \frac{1}{4x}\,dx$ (b) $\int_{2}^{3} \frac{4}{2x-7}\,dx$ (c) $\int_{1}^{2} \frac{-1}{\sqrt{4-x^2}}\,dx$

(d) $\int_{0}^{\frac{1}{3}} \frac{1}{\sqrt{1-9x^2}}\,dx$ (e) $\int_{0}^{1} \frac{1}{1+3x^2}\,dx$ (f) $\int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{2-4x^2}}\,dx$

4 Demonstrate that $\int_{4}^{5} \frac{3}{x+1} - \frac{3}{x-1}\,dx = 3\ln\left(\frac{9}{10}\right)$.

5 The value of $\int_{1}^{3.2} \frac{19}{7x^2+4}\,dx$ is closest to:

A 1.49 B 2.58 C 3.72 D 4.81

6 The value of $\int_{0}^{1.5} \frac{-5}{\sqrt{9-2x^2}}\,dx$ is closest to:

A 2.78 B -2.78 C 1.35 D -1.35

7 Demonstrate that $\int \frac{3}{2x+3} - \frac{1}{2x-5}\,dx = \frac{1}{2}\ln\left|\frac{(2x+3)^3}{2x-5}\right| + c$.

8 Determine the exact area enclosed by the function $y = \frac{7}{\sqrt{4-x^2}}$, the x-axis and the lines $x = -1$ and $x = \sqrt{3}$.

9 Determine the exact area enclosed by the function $y = \frac{-1}{\sqrt{1-9x^2}}$, the x-axis, the y-axis and the line $x = \frac{\sqrt{3}}{6}$.

Worked Example 12

10 Determine the exact value of k, given that the area bounded by the function $y = \frac{3}{\sqrt{4-x^2}}$, the x-axis the line $x = 2$, and the line $x = k$, where $k < 2$, is equal to $\frac{\pi}{2}$ units^2.

11 Determine the exact value of k, given that the area bounded by the function $y = \frac{4}{4+x^2}$, the x-axis, the line $x = k$, where $k < 2$, and the line $x = 2$ is equal to π units^2.

12 Determine the exact value of k, given that the area bounded by the function $y = \frac{1}{3x}$, the x-axis, the line $x = k$, where $k < 0$, and the line $x = -1$ is equal to $\frac{1}{3}$ units^2.

13 Which of the following is *not* equal to $\int \frac{1}{-2x}\,dx$? Explain your reasoning.

A $-\frac{1}{2}\ln|2x| + c$ **B** $\frac{1}{2}\ln\left|\frac{1}{x}\right| + c$ **C** $-\ln\left(\sqrt{|x|}\right) + c$ **D** $\frac{1}{2}\ln|-2x| + c$

14 Which of the following statements is not true?

A $\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}\,dx$ **B** $\arccos(x) = -\arcsin(x)$

C $[\arccos(x)]_a^b = -[\arcsin(x)]_a^b$ **D** $\frac{d}{dx}(\arccos(x)) = -\frac{d}{dx}(\arcsin(x))$

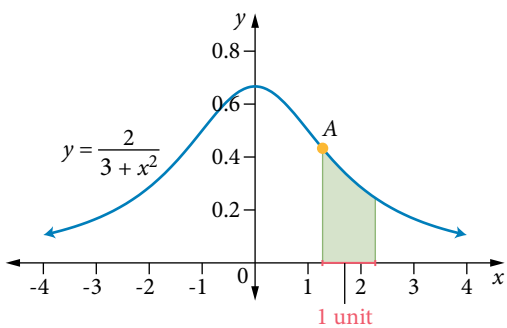
15 **(a)** Differentiate $y = \arccos(x+b)$ and hence determine $\int \frac{-1}{\sqrt{1-(x+b)^2}}\,dx$.

(b) Use the integral from part **(a)** to determine $\int \frac{-1}{\sqrt{-x^2-10x-24}}\,dx$.

16 **(a)** Differentiate $y = \arctan(ax+b)$ and hence determine $\int \frac{a}{1+(ax+b)^2}\,dx$.

(b) Use the integral from part **(a)** to determine $\int \frac{12}{9x^2+12x+5}\,dx$.

17 Determine the coordinates (correct to 2 decimal places) of the point A in the diagram, given that the shaded area is 0.331 units^2.



5.4 Integration by substitution

The chain rule allows you to differentiate a complicated function by transforming it into a simpler function in terms of a different variable u. For example, to use the chain rule to differentiate $y = \left(x^2 + 3\right)^7$ with respect to x, firstly make the substitution $u = x^2 + 3$ so the function becomes $y = u^7$. The chain rule then says that the derivative with respect to x is equal to the derivative of this new simpler function with respect to u, multiplied by $\frac{du}{dx}$. In symbols:

Let $u = x^2 + 3$ and $y = \left(x^2 + 3\right)^7 = u^7$ then:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 7u^6 \times \frac{du}{dx} \\ &= 7\left(x^2 + 3\right)^6 \times 2x\end{aligned}$$

The *substitution rule* is similar to the chain rule in that it sometimes allows a complicated integral to be transformed into a simpler one in terms of a different variable u:

The substitution rule:

$$\int f(u) \times \frac{du}{dx}\, dx = \int f(u)\, du$$

For example, $\int 7\left(x^2 + 3\right)^6 \times 2x\ dx$. The function $7\left(x^2 + 3\right)^6 \times 2x$ can be expressed as $7u^6 \times \frac{du}{dx}$ by making the substitution $u = x^2 + 3$ and $\frac{du}{dx} = 2x$. Then,

$$\begin{aligned}\int 7\left(x^2 + 3\right)^6 \times 2x\ dx &= \int 7u^6 \times \frac{du}{dx}\, dx \\ &= \int 7u^6\ du \\ &= u^7 + c \\ &= \left(x^2 + 3\right)^7 + c\end{aligned}$$

The substitution rule is essentially a restatement of the chain rule. The chain rule says the derivative of $g(u)$ with respect to x is equal to $g'(u) \times \frac{du}{dx}$. This is equivalent to saying that an anti-derivative of $g'(u) \times \frac{du}{dx}$ is $g(u)$, which is the same as saying an anti-derivative of an anti-derivative of $f(u) \times \frac{du}{dx}$ is $\int f(u)\ du$ (letting $f(u) = g'(u)$).

Using the substitution rule to integrate is also known as using a *change of variables*. While it is sometimes difficult to know what expression to set the variable u to represent, a general rule is to let u be equal to the largest expression whose derivative is equal to, or only differs by a constant from, a factor of the integral. For example, in the integral $\int 3x^4 \sqrt{x^5 + 7}\ dx$ it would be a good idea to set $u = x^5 + 7$ since its derivative $(5x^4)$ only differs by a multiplying constant from $3x^4$, which is a factor of the integrand.

WARNING

For harder problems, sometimes you need to experiment with multiple substitutions before you determine substitutions that work.

13 Integration by substitution

Determine each of the following indefinite integrals.

(a) $\int x^2\left(x^3+7\right)^9\,dx$

METHOD 1

THINKING	WORKING
1 Define the inner function as u and determine $\frac{du}{dx}$.	Let $u = x^3 + 7$. Then $\frac{du}{dx} = 3x^2 \quad \therefore \frac{1}{3}\frac{du}{dx} = x^2$
2 Rewrite the integral in terms of u and $\frac{du}{dx}$. It will almost certainly be necessary to multiply $\frac{du}{dx}$ by a constant for the rewritten integral to be equivalent to the original.	$\int x^2\left(x^3+7\right)^9\,dx = \int\left(x^3+7\right)^9 x^2\,dx$ $= \int u^9\,\frac{1}{3}\frac{du}{dx}\,dx$ $= \frac{1}{3}\int u^9\,\frac{du}{dx}\,dx$
3 Use $\int f(u)\times\frac{du}{dx}\,dx = \int f(u)\,du$ to form a simple integral in terms of u, then determine this integral and change back to the original variable x.	$= \frac{1}{3}\int u^9\,du$ $= \frac{1}{30}u^{10} + c$ $= \frac{1}{30}\left(x^3+7\right)^{10} + c$

METHOD 2

1 Define the inner function as u and determine $\frac{du}{dx}$.	Let $u = \left(x^3+7\right)^9, \quad (u)^{\frac{1}{9}} = \left(x^3+7\right)$. Then $\frac{du}{dx} = 27\left(x^3+7\right)^8 x^2$ $\therefore \frac{1}{27\,(u)^{\frac{8}{9}}}\frac{du}{dx} = x^2$
2 Rewrite the integral in terms of u and $\frac{du}{dx}$.	$\int x^2\left(x^3+7\right)^9\,dx = \int\left(x^3+7\right)^9 x^2\,dx$ $= \int u\,\frac{1}{27\,(u)^{\frac{8}{9}}}\frac{du}{dx}\,dx$ $= \frac{1}{27}\int u^{\frac{1}{9}}\,\frac{du}{dx}\,dx$

3 Use $\int f(u) \times \frac{du}{dx}\, dx = \int f(u)\, du$ to form a simple integral in terms of u, then determine this integral and change back to the original variable x.

$$\frac{1}{27}\int u^{\frac{1}{9}} \frac{du}{dx}\, dx = \frac{1}{27}\int u^{\frac{1}{9}}\, du$$
$$= \frac{1}{27} \times \frac{9}{10} u^{\frac{10}{9}} + c$$
$$= \frac{1}{30}\left(x^3 + 7\right)^{10} + c$$

(b) $\int \frac{7x}{\sqrt{x^2 - 5}}\, dx$

1 Define an appropriate expression as u and determine $\frac{du}{dx}$.

Let $u = x^2 - 5$. Then $\frac{du}{dx} = 2x \quad \therefore \frac{1}{2}\frac{du}{dx} = x$

2 Rewrite the integral in terms of u and $\frac{du}{dx}$. It will almost certainly be necessary to multiply $\frac{du}{dx}$ by a constant for the rewritten integral to be equivalent to the original.

$$\int \frac{7x}{\sqrt{x^2 - 5}}\, dx = 7\int \frac{1}{\sqrt{u}} \times \frac{1}{2}\frac{du}{dx}\, dx$$
$$= \frac{7}{2}\int \frac{1}{\sqrt{u}}\, du$$
$$= \frac{7}{2}\int \frac{1}{(u)^{\frac{1}{2}}}\, du$$

3 Use $\int f(u) \times \frac{du}{dx}\, dx = \int f(u)\, du$ to form a simple integral in terms of u, then determine this integral and change back to the original variable x.

$$= \frac{7}{2}\int u^{-\frac{1}{2}}\, du$$
$$= 7u^{\frac{1}{2}} + c$$
$$= 7\sqrt{x^2 - 5} + c$$

(c) $\int \frac{\cos\left(x^{\frac{1}{3}} - 6\right)}{x^{\frac{2}{3}}}\, dx$

1 Define the expression in the parenthesis as u and determine $\frac{du}{dx}$.

Let $u = x^{\frac{1}{3}} - 6$. Then

$$\frac{du}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$
$$3\frac{du}{dx} = \frac{1}{x^{\frac{2}{3}}}$$

2 Rewrite the integral in terms of u and $\frac{du}{dx}$.

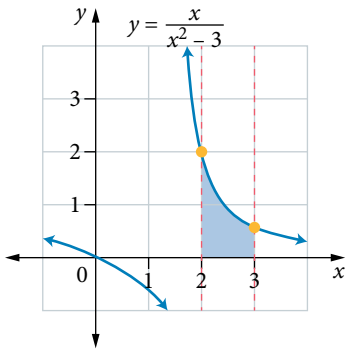
$$\int \frac{\cos\left(x^{\frac{1}{3}} - 6\right)}{x^{\frac{2}{3}}}\, dx = \int \cos(u)\; 3\frac{du}{dx}\, dx$$

3 Use $\int f(u) \times \frac{du}{dx}\, dx = \int f(u)\, du$ to form a simple integral in terms of u, then determine this integral and change back to the original variable, x.	$\int \cos(u)\ 3\frac{du}{dx}\, dx = 3\int \cos(u)\, du$ $= 3\sin(u) + c$ $= 3\sin\left(x^{\frac{1}{3}} - 6\right) + c$

When determining definite integrals using a change of variable, it is not necessary to express the anti-derivative in terms of the original variable provided you change the integration limits to values corresponding to the new variable. Worked example 14 illustrates this.

14 Calculating definite integrals using substitution

Calculate the exact value of the integral $\int_2^3 \frac{x}{x^2 - 3} dx$ and interpret the result.



THINKING	WORKING
1 Define an appropriate expression as u and determine $\frac{du}{dx}$.	Let $u = x^2 - 3$. Then $\frac{du}{dx} = 2x$.
2 Change the integration limits to values corresponding to the new variable.	$x = 2 \Rightarrow u = 2^2 - 3$ $= 1$ $x = 3 \Rightarrow u = 3^2 - 3$ $= 6$
3 Use $\int f(u) \times \frac{du}{dx}\, dx = \int f(u)\, du$ to form a simple integral in terms of u, then integrate and substitute the new integration limits.	$\int_2^3 \frac{x}{x^2 - 3} dx = \int_{u=1}^{u=6} \frac{1}{u} \frac{1}{2} \frac{du}{dx}\, dx$ $= \frac{1}{2}\int_{u=1}^{u=6} \frac{1}{u}\, du$ $= \frac{1}{2}\left[\ln\lvert u\rvert\right]_{u=1}^{u=6}$ $= \frac{1}{2}\left(\ln(6) - \ln(1)\right)$ $= \frac{1}{2}\left(\ln(6) - 0\right)$ $= \frac{1}{2}\ln(6)$

4 Interpret the answer.	The area bound by the curve, the x-axis and the lines $x = 2$ and $x = 3$ is $\frac{1}{2}\ln(6)$ units^2.

C Making connections

Area under the curve

Move the sliders a and b to change the limits of integration in the function 'y' and observe how the shape of the curve and the limits change after applying the substitution $u = \sin(x)$, but the total area does not change.

The substitution method for integration can sometimes be used for integrals when you may not initially be sure if it can be expressed in the form $f(u) \times \frac{du}{dx}$.

Consider the integral $\int 2x^3\left(x^2 - 10\right)^7\, dx$. Let $u = x^2 - 10$ so $\frac{du}{dx} = 2x$. Then,

$$\begin{aligned}\int 2x^3\left(x^2 - 10\right)^7\, dx &= \int x^2\left(x^2 - 10\right)^7\, 2x\, dx \\ &= \int x^2\, u^7\, \frac{du}{dx}\, dx\end{aligned}$$

Although at this stage you have incorporated $\frac{du}{dx}$ as a factor, and the variable x is still present. Since $u = x^2 - 10$, the x^2 in the integral can be replaced with $u + 10$. This gives:

$$\begin{aligned}\int 2x^3\left(x^2 - 10\right)^7\, dx &= \int x^2\left(x^2 - 10\right)^7\, 2x\, dx \\ &= \int x^2\, u^7\, \frac{du}{dx}\, dx \\ &= \int (u + 10)u^7\, \frac{du}{dx}\, dx \\ &= \int (u + 10)u^7\, du \\ &= \int u^8 + 10u^7\, du \\ &= \frac{1}{9}u^9 + \frac{10}{8}u^8 + c \\ &= \frac{1}{9}\left(x^2 - 10\right)^9 + \frac{5}{4}\left(x^2 - 10\right)^8 + c\end{aligned}$$

In summary, in a situation where $\frac{du}{dx}$ has been incorporated as a factor but the variable x is still present, rearrange the expression for u as needed so you can express the integral solely in terms of u.

15 Determine more integrals in which substitution may be used

Determine the following indefinite integrals using the substitution provided.

(a) $\int x(3x-5)^4\,dx$ (Let $u = 3x - 5$)

THINKING	WORKING
1 Use the definition of u to find dx in terms of du and x in terms of u.	Let $u = 3x - 5$ then $\frac{du}{dx} = 3$ and therefore $x = \frac{1}{3}(u+5)$ and $dx = \frac{1}{3}du$.
2 Rewrite the integrand in terms of u and du, that is $\int f(u) \times \frac{du}{dx}dx = \int f(u)\,du$.	$\int x(3x-5)^4 dx = \int \frac{1}{3}(u+5) \times u^4 \times \frac{1}{3}du$ $= \frac{1}{9}\int \left(u^5 + 5u^4\right)du$
3 Determine the resulting integral.	$= \frac{1}{9}\left(\frac{1}{6}u^6 + u^5 + c_1\right)$ $= \frac{1}{54}u^6 + \frac{1}{9}u^5 + c$
4 Rewrite the expression in terms of x by substituting for u.	$= \frac{1}{54}(3x-5)^6 + \frac{1}{9}(3x-5)^5 + c$

(b) $\int \frac{x^5}{x^2+1}dx$ (Let $u = x^2 + 1$)

1 Use the definition of u to find dx in terms of du and x in terms of u.	Let $u = x^2 + 1$ then $\frac{du}{dx} = 2x$, $x^2 = u - 1$ and $x = \pm\sqrt{u-1}$. Which gives $x^5 = \pm(u-1)^2 \times \sqrt{u-1}$ and $dx = \frac{1}{2x}du$ $= \frac{du}{\pm 2\sqrt{u-1}}$
2 Rewrite the integrand in terms of u and du, that is $\int f(u) \times \frac{du}{dx}dx = \int f(u)\,du$.	$\int \frac{x^5}{x^2+1}dx = \int \frac{\pm(u-1)^2\sqrt{u-1}}{u} \times \frac{du}{\pm 2\sqrt{u-1}}$ $= \int \frac{(u-1)^2}{2u}du$ $= \frac{1}{2}\int \frac{u^2 - 2u + 1}{u}du$ $= \frac{1}{2}\int \left(u - 2 + \frac{1}{u}\right)du$
3 Determine the resulting integral.	$= \frac{1}{4}u^2 - u + \frac{1}{2}\ln\lvert u\rvert + c$
4 Rewrite the expression in terms of x by substituting for u.	$= \frac{1}{4}\left(x^2+1\right) - \left(x^2+1\right) + \frac{1}{2}\ln\left(x^2+1\right) + c$

EXERCISE 5.4 Integration by substitution

Worked Example 13

1 Determine the following anti-derivatives.

(a) $\int x\left(3x^2+4\right)^5\,dx$

(b) $\int (2x+3)\sqrt{2x^2+6x+5}\,dx$

(c) $\int 2x^2\left(x^3-3\right)^{10}\,dx$

(d) $\int xe^{-x^2}\,dx$

(e) $\int (9x+6)\sin\left(3x^2+4x\right)dx$

(f) $\int \sin^3(x)\cos(x)\,dx$

(g) $\int (1+x)e^{3x^2+6x+1}\,dx$

(h) $\int \sec^2(x)\left(1+\tan(x)\right)^4\,dx$

2 Determine the following anti-derivatives.

(a) $\int \frac{5x}{\left(x^2-3\right)^3}\,dx$

(b) $\int \frac{5x}{\sqrt{x^2-3}}\,dx$

(c) $\int \frac{5x}{x^2+3}\,dx$

(d) $\int \frac{2x^3}{\sqrt{3-x^4}}\,dx$

(e) $\int \frac{2x^3}{3-x^4}\,dx$

(f) $\int \frac{\sin(2x)}{\cos^3(2x)}\,dx$

(g) $\int \frac{\sin(x)}{\cos(x)}\,dx$

(h) $\int \frac{3x+3}{x^2+2x+3}\,dx$

(i) $\int \frac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)}\,dx$

(j) $\int \frac{e^x-e^{-x}}{\left(e^x+e^{-x}\right)^2}\,dx$

3 Determine each of the following indefinite integrals:

(a) $\int \frac{\left(3+\sqrt{x}\right)^8}{\sqrt{x}}\,dx$

(b) $\int \frac{e^{\tan(x)}}{\cos^2(x)}\,dx$

(c) $\int \frac{\ln(x+2)}{x+2}\,dx$

(d) $\int \frac{5x\ln\left(x^2+1\right)}{x^2+1}\,dx$

(e) $\int \frac{x}{\sqrt{1-\left(2x^2+3\right)^2}}\,dx$

(f) $\int \frac{x+2}{9+\left(x^2+4x\right)^2}\,dx$

(g) $\int \frac{5}{x\ln(x)}\,dx$

(h) $\int \sin(x)\sec^2(x)\,dx$

(i) $\int \frac{1}{\left(9+x^2\right)\sqrt{\arctan\left(\frac{x}{3}\right)}}\,dx$

(j) $\int \frac{1}{\cos^2(x)\left(7-\tan(x)\right)^5}\,dx$

14

4 Calculate the exact value of each of the following:

(a) $\int_0^1 x\sqrt{x^2+3}\,dx$

(b) $\int_0^1 \frac{x^2}{3x^3+2}\,dx$

(c) $\int_0^{\frac{\pi}{6}} \cos(x)e^{\sin(x)}\,dx$

(d) $\int_1^e \frac{\ln(x)}{x}\,dx$

5 $\int_1^2 x^2\left(x^3+5\right)^7\,dx$ is equivalent to which of the following?

A $\int_1^2 u^7\,du$

B $\frac{1}{3}\int_1^2 u^7\,du$

C $\int_6^{13} u^7\,du$

D $\frac{1}{3}\int_6^{13} u^7\,du$

15

6 Determine the following anti-derivatives using the substitution provided.

(a) $\int x(2x+1)^7\,dx$ (Let $u = 2x+1$)

(b) $\int x^2\sqrt{x-3}\,dx$ (Let $u = x-3$)

(c) $\int \frac{x}{(1-x)^{10}}\,dx$ (Let $u = 1-x$)

(d) $\int \frac{x^2}{x+5}\,dx$ (Let $u = x+5$)

(e) $\int \frac{x^2}{(x-3)^2}\,dx$

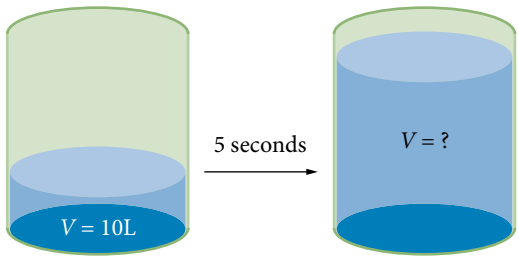
(f) $\int x^5\sqrt{1-x^2}\,dx$

(g) $\int \frac{x^3}{9-x^2}\,dx$

(h) $\int \frac{e^{3x}}{e^x-1}\,dx$

7 By using the substitution $u = x+1$, calculate the exact value of $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$.

8 The rate at which the volume of water in a tank is increasing is given by $\frac{dV}{dt} = 1.44t\,e^{0.1t^2}$, where V represents the volume in litres and t the time in seconds. Initially the tank contains 10 litres of water. Determine its volume, to the nearest litre, after 5 seconds.



9 **(a)** Which of the following statements is true regarding $\int \cos\left(x^2+3\right)dx$?

A $\int \cos\left(x^2+3\right)dx = \sin\left(x^2+3\right)$

B $\int \cos\left(x^2+3\right)dx = \frac{1}{2}\sin\left(x^2+3\right)$

C $\int \cos\left(x^2+3\right)dx = \dfrac{1}{2x}\sin\left(x^2+3\right)$

D None of the previous alternatives are true.

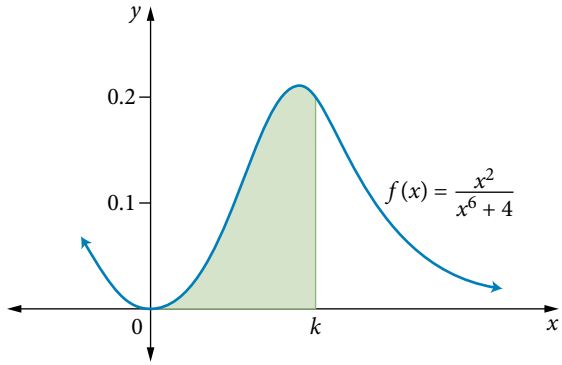
(b) Explain why each of the other alternatives in part **(a)** are incorrect.

10 Express $x^2+6x+18$ in the form $s+(x+t)^2$. Hence, determine $\int \dfrac{1}{x^2+6x+18}dx$.

11 Determine $\int \dfrac{3+x}{4+x^2}dx$.

12 Differentiate $y = x\arcsin(x)$ and hence determine $\int \arcsin(x)dx$.

13 Calculate the exact value of k, given that the shaded area in the diagram is equal to $\dfrac{\pi}{24}$ units^2. Evaluate the reasonableness of your solution with the aid of technology.



Integration with trigonometric identities 5.5

The focus of this chapter has been to develop techniques to enable the integration of a wider range of functions. This section will continue that process by using trigonometric identities to simplify or rearrange expressions to make them easier to integrate.

The following three identities are particularly important:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$\tan^2(x) = \sec^2(x) - 1$ (more commonly written in the form $\sec^2(x) = 1 + \tan^2(x)$

The following Worked examples illustrate how these identities may be used to transform a difficult integral into one that may be readily determined using the standard rules for anti-derivatives of $\sin(x)$, $\cos(x)$, and $\sec^2(x)$.

16 Integration with trigonometric identities

Determine the following indefinite integrals:

(a) $\int \sin^2(3x)\,dx$

THINKING	WORKING
1 Use the appropriate identity to express the integrand in a form that may be readily integrated.	$\sin^2(a) = \frac{1 - \cos(2a)}{2}$, hence $\sin^2(3x) = \frac{1}{2}(1 - \cos(6x))$
2 Integrate.	$\int \sin^2(3x)\,dx = \int \frac{1}{2}(1 - \cos(6x))\,dx$ $= \frac{1}{2}\left[\int dx - \int \cos(6x)\,dx\right]$ $= \frac{1}{2}\left(x - \frac{1}{6}\sin(6x)\right) + c$ $= \frac{1}{2}x - \frac{1}{12}\sin(6x) + c$

(b) $\int 5 + 3\tan^2(2x)\,dx$

1 Use the appropriate identity to express the integrand in a form that may be readily integrated.	$\tan^2(2x) = \sec^2(2x) - 1$

2 Integrate.

$$\begin{aligned}\int 5+3\tan^2(2x)\,dx &= \int 5+3\left(\sec^2(2x)-1\right)dx\\ &= \int 2+3\sec^2(2x)\,dx\\ &= 2x+\frac{3}{2}\tan(2x)+c\end{aligned}$$

The method of integrating by substitution may also be used in combination with trigonometric identities. The well-known identity $\sin^2(x)+\cos^2(x)=1$ is often used to rewrite integrals in a form that then enables particular substitutions.

17 Integration with trigonometric identities and substitution

Determine the following indefinite integrals:

(a) $\int \sin^3(x)\,dx$

THINKING

1 Use the identity $\sin^2(x)+\cos^2(x)=1$ to replace $\sin^2(x)$ with $1-\cos^2(x)$.

2 Observe that the derivative of $\cos(x)$ is a factor of the integrand, then let u be equal to $\cos(x)$.

3 Apply the substitution and integrate.

WORKING

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \sin(x)\sin^2(x)\,dx\\ &= \int \sin(x)\left(1-\cos^2(x)\right)dx\end{aligned}$$

Let $u=\cos(x)$.

Then $\frac{du}{dx}=-\sin(x)$.

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \sin(x)\left(1-\cos^2(x)\right)dx\\ &= -\int \left(1-u^2\right)\frac{du}{dx}\,dx\\ &= \int \left(u^2-1\right)du\\ &= \frac{1}{3}u^3-u+c\\ &= \frac{1}{3}\cos^3(x)-\cos(x)+c\end{aligned}$$

(b) $\int \cos^3(x)\sin^{10}(x)\,dx$

1 Recognise that the derivative of $\sin(x)$ is a factor of the integrand and define u as $\sin(x)$.

Let $u=\sin(x)$.

Then $\frac{du}{dx}=\cos(x)$.

2 Apply the substitution and integrate.

$$\begin{aligned}\int \cos^3(x)\sin^{10}(x)\,dx &= \int \cos^2(x)u^{10} \times \frac{du}{dx}\,dx \\ &= \int \left(1-\sin^2(x)\right)u^{10} \times \frac{du}{dx}\,dx \\ &= \int \left(1-u^2\right)u^{10}\,du \\ &= \int u^{10}-u^{12}\,du \\ &= \frac{1}{11}u^{11}-\frac{1}{13}u^{13}+c \\ &= \frac{1}{11}\sin^{11}(x)-\frac{1}{13}\sin^{13}(x)+c\end{aligned}$$

The method used in part (b) of Worked example 17 may be used to determine any integral of the form $\int \cos^m(x)\sin^n(x)\,dx$, where m and n are positive integers with at least one of m or n being odd. Once a function with an odd power is identified, define the variable u as the other function. (In Worked example 17, $\cos(x)$ had an odd power so the variable u was assigned to $\sin(x)$.)

You should attempt such an integral using this method when both powers are even, so you can understand the desirability of an odd power.

In the case where both powers are even (and not identical), the identities introduced at the beginning of this section may be used to reduce the powers by 1. This will result in a simpler expression to be integrated.

In the special case in which both powers are even and are identical, the integral may be determined with the aid of the following additional identity:

$$\sin(2x) = 2\sin(x)\cos(x)$$

This identity is useful in several other situations, so it is worthwhile keeping it in mind.

EXERCISE 5.5 Integration with trigonometric identities

Worked Example 16

1 Determine the following anti-derivatives.

(a) $\int \cos^2\left(\frac{x}{2}\right)dx$

(b) $\int \sin^2(4x)\,dx$

(c) $\int \tan^2(3x)\,dx$

(d) $\int \left(x+4\sin^2(x)\right)dx$

(e) $\int \left(\tan^2(3\theta)-1\right)d\theta$

(f) $\int 2\sin^2(5\theta)\,d\theta$

(g) $\int \left(5-2\tan^2(x)\right)dx$

(h) $\int \tan^2\left(\frac{x}{2}\right)dx$

2 Calculate the exact value of each of the following definite integrals:

(a) $\int_0^{\frac{\pi}{2}} 2\sin^2(x)\,dx$ **(b)** $\int_{-\frac{\pi}{4}}^{0} 1+\tan^2(x)\,dx$ **(c)** $\int_0^{\frac{\pi}{2}} \cos(\theta)-\cos^2(\theta)\,d\theta$

(d) $\int_0^{\pi} \sin^2\left(\frac{\theta}{2}\right)d\theta$ **(e)** $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \tan^2(2x)\,dx$ **(f)** $\int_0^{\frac{\pi}{4}} \frac{1+\sin^2(x)}{\cos^2(x)}\,dx$

3 Prove that $\int_0^{\frac{1}{m}} \cos^2\left(\frac{\pi m x}{2}\right) = \frac{1}{2m}$.

Worked Example

4 Determine $\int \cos^4(x)\,dx$ by first expressing it as $\int \left(\cos^2(x)\right)^2\,dx$.

17

5 Determine the following indefinite integrals:

(a) $\int \cos^3(x)\,dx$ **(b)** $\int \sin^5(x)\,dx$ **(c)** $\int \sin^3(x)\cos^7(x)\,dx$

(d) $\int \sin^4(x)\cos^3(x)\,dx$ **(e)** $\int \cos^5(x)\sin^4(x)\,dx$ **(f)** $\int \cos^2(x)\sin^7(x)\,dx$

(g) $\int \sin^3(x)\cos^3(x)\,dx$

6 By making use of the identity $\sin(2A) = 2\sin(A)\cos(A)$, determine $\int \sin^2(x)\cos^2(x)\,dx$.

7 By making use of the identity $\sin(2A) = 2\sin(A)\cos(A)$, together with other relevant identities, determine $\int \sin^4(x)\cos^4(x)\,dx$.

8 **(a)** Use an appropriate identity to express $\cos(x)$ in terms of $\cos^2\left(\frac{x}{2}\right)$.

(b) Hence, determine $\int \frac{1}{1+\cos(x)}\,dx$.

9 Calculate the exact value of $\int_0^{\frac{\pi}{2}} 2\cos^2(x) - \cos(x)\sin^2(x)\,dx$.

10 **(a)** Determine:

(i) $\int \cos^2(2x)\,dx$ **(ii)** $\int \cos^3(2x)\,dx$.

(b) Using the results in part **(a)**, determine $\int \cos^2(x)\sin^4(x)\,dx$.

11 **(a)** By demonstrating that $\frac{d}{dx}\left(e^x f(x)\right) = e^x\left(f(x)+f'(x)\right)$, deduce that $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x) + c$.

(b) Use the identities $\sin(2A) = 2\sin(A)\cos(A)$ and $\cos(2A) = 2\cos^2(A) - 1$ to demonstrate that $g(x) = \frac{2+\sin(2x)}{1+\cos(2x)}$ can be expressed in the form $f(x)+f'(x)$ for a suitable function $f(x)$.

(c) Use the result of parts **(a)** and **(b)** to calculate the exact value of $\int_0^{\frac{\pi}{3}} e^x\left(\frac{2+2\sin(2x)}{1+\cos(2x)}\right)dx$.

5.6 Partial fractions

A rational function can be integrated by expressing it as a sum of simpler fractions, called partial fractions, each of which can be readily integrated.

Consider the integral $\int \frac{28}{(x+2)(x-5)}dx$. It can be readily verified that $\frac{28}{(x+2)(x-5)} = \frac{-4}{x+2} + \frac{4}{x-5}$.

Thus,

$$\begin{aligned}\int \frac{28}{(x+2)(x-5)}dx &= \int \frac{-4}{x+2} + \frac{4}{x-5}dx \\ &= -4\ln|x+2| + 4\ln|x-5| + c\end{aligned}$$

Normally you will not be told the partial fractions corresponding to a rational expression as you were in this case. The following theorem provides a process for determining the partial fractions for a rational expression when the denominator is a factorable quadratic and the numerator is linear or constant.

If no pair of $ax+b$, $cx+d$, and $ex+f$ have any common factors, then non-zero real numbers A and B exist such that

$$\frac{ax+b}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}, \text{ and } \frac{ax+b}{(cx+d)^2} = \frac{A}{cx+d} + \frac{B}{(cx+d)^2}$$

The first situation in this theorem is applicable to the expression $\frac{28}{(x+2)(x-5)}$ as the denominator consists of two distinct linear factors. Thus, this theorem guarantees you will be able to find non-zero real numbers A and B such that $\frac{28}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5}$. To determine the values of A and B:

$$\begin{aligned}\frac{28}{(x+2)(x-5)} &= \frac{A}{x+2} + \frac{B}{x-5} \\ &= \frac{A(x-5)+B(x+2)}{(x+2)(x-5)}\end{aligned}$$

Since the denominators on the left- and right-hand side are equal:

$$\begin{aligned}28 &= A(x-5)+B(x+2) \\ 28 &= Ax-5A+Bx+2B \\ 0x+28 &= (A+B)x+(2B-5A)\end{aligned}$$

Equating the coefficients of x and the constant terms on both sides gives
$0 = A+B$ and $28 = 2B-5A$.

$$\begin{cases} 28 = -5A+2B \\ 0 = A+B \end{cases}$$

Solving these two equations simultaneously then gives $A = -4$ and $B = 4$. Thus,
$\frac{28}{(x+2)(x-5)} = \frac{-4}{x+2} + \frac{4}{x-5}$, as expected.

In the following Worked examples, an alternative method for determining the values of A and B is demonstrated. This method avoids the need to solve simultaneous equations by substituting certain values of x that eliminate one of the variables in equation.

18 Expressing rational expressions in partial fraction form

Express the following rational expressions in partial fraction form.

(a) $\dfrac{28}{(x+2)(x-5)}$

THINKING	WORKING
1 Express the fraction in the appropriate form.	$\dfrac{28}{(x+2)(x-5)} = \dfrac{A}{x+2} + \dfrac{B}{x-5}$
2 Express the partial fraction expression with the same denominator as the original expression, then equate the numerators.	$\dfrac{A}{x+2} + \dfrac{B}{x-5} = \dfrac{A(x-5)+B(x+2)}{(x+2)(x-5)}$ $\therefore\ 28 = A(x-5)+B(x+2) \quad [1]$
3 Set the terms $B(x+2)$ and $A(x-5)$ equal to zero, where A and B are non-negative real numbers, solving for x.	$B(x+2) = 0$ $x = -2$ and $A(x-5) = 0$ $x = 5$
4 Substitute the values of x into [1] to determine the values of A and B.	Setting $x = -2$: $28 = A(-2-5)+B(-2+2)$ $28 = -7A$ $A = -4$ Setting $x = 5$: $28 = A(5-5)+B(5+2)$ $28 = 7B$ $B = 4$
5 Substitute the values for A and B and write the result.	$\therefore\ \dfrac{28}{(x+2)(x-5)} = \dfrac{-4}{x+2} + \dfrac{4}{x-5}$

(b) $\dfrac{x-2}{x(x-3)}$

1 Express the fraction in the appropriate form.	$\dfrac{x-2}{x(x-3)} = \dfrac{A}{x} + \dfrac{B}{x-3}$
2 Express the partial fraction expression with the same denominator as the original expression, then equate the numerators.	$\dfrac{A}{x} + \dfrac{B}{x-3} = \dfrac{A(x-3)+Bx}{x(x-3)}$ $\therefore\ x-2 = A(x-3)+Bx \quad [1]$

3 Substitute suitable values of x into [1] to determine the values of A and B.

Setting $x = 0$:

$$-2 = A(0-3) + B \times 0$$
$$-2 = -3A$$
$$A = \frac{2}{3}$$

Setting $x = 3$:

$$1 = A(3-3) + B \times 3$$
$$1 = 3B$$
$$B = \frac{1}{3}$$

4 Write the result.

$$\therefore \frac{x-2}{x(x-3)} = \frac{\frac{2}{3}}{x} + \frac{\frac{1}{3}}{x-3}$$
$$= \frac{2}{3x} + \frac{1}{3(x-3)}$$

(c) $\dfrac{3x-4}{(3x+1)^2}$

1 Where there are n repeated linear factors, the partial fraction form will contain n fractions each with the factor raised to increasing integer powers up to $(ax+b)^n$.

$$\frac{3x-4}{(3x+1)^2} = \frac{A}{3x+1} + \frac{B}{(3x+1)^2}$$

2 Express the partial fraction expression with the same denominator as the original expression, then equate the numerators.

$$\frac{A}{3x+1} + \frac{B}{(3x+1)^2} = \frac{A(3x+1)+B}{(3x+1)^2}$$
$$\therefore 3x - 4 = A(3x+1) + B \quad [1]$$

3 Substitute suitable values of x into [1] to determine the values of A and B.

Setting $x = -\frac{1}{3}$:

$$3 \times -\frac{1}{3} - 4 = A\left(3 \times -\frac{1}{3} + 1\right) + B$$
$$-5 = 0A + B$$
$$B = -5$$

Setting $x = 0$:

$$-4 = A(3 \times 0 + 1) + B$$
$$-4 = A + B$$
$$-4 = A + -5$$
$$A = 1$$

4 Write the result.

$$\therefore \frac{3x-4}{(3x+1)^2} = \frac{1}{3x+1} - \frac{5}{(3x+1)^2}$$

It may seem strange that values of x may be substituted into equation [1] that cause the original rational expression to be undefined. (For example, in part (a) the original rational function is undefined when $x = -2$ or $x = 5$.) However, it can be shown that equation [1], which represents a different form of the expression in the numerator, is valid for all values of x. Thus, there is no problem.

The following Worked examples illustrate the process of integrating a rational function by first expressing it in partial fraction form. It is important for the denominator in a rational function to be factorised to do this.

19 Integrating using partial fractions

Determine the following idefinite integrals by first expressing the integrand in partial fraction form.

(a) $\int \frac{9x+8}{2x^2-5x-3}\,dx$

THINKING	WORKING
1 Factorise the denominator of the integrand.	$\frac{9x+8}{2x^2-5x-3} = \frac{9x+8}{(2x+1)(x-3)}$
2 Express the integrand in partial fraction form.	$\frac{9x+8}{(2x+1)(x-3)} = \frac{A}{2x+1} + \frac{B}{x-3}$ $= \frac{A(x-3)+B(2x+1)}{(2x+1)(x-3)}$ $\therefore\ 9x+8 = A(x-3)+B(2x+1)$ Setting $x = -\frac{1}{2}$: $9\times -\frac{1}{2}+8 = A\left(-\frac{1}{2}-3\right)$ $\frac{7}{2} = -\frac{7}{2}A$ $A = -1$ Setting $x = 3$: $9\times 3+8 = (2\times 3+1)B$ $35 = 7B$ $B = 5$ $\therefore\ \frac{9x+8}{2x^2-5x-3} = \frac{-1}{2x+1} + \frac{5}{x-3}$
3 Use partial fractions to determine the integral.	$\int \frac{9x+8}{2x^2-5x-3}\,dx = \int \frac{-1}{2x+1} + \frac{5}{x-3}\,dx$ $= \int \frac{-1}{2x+1}\,dx + \int \frac{5}{x-3}\,dx$ $= -\frac{1}{2}\ln\lvert 2x+1\rvert + 5\ln\lvert x-3\rvert + c$

(b) $\int \frac{2x+13}{x^2+10x+25}\,dx$

1 Factorise the denominator of the integrand.	$\frac{2x+13}{x^2+10x+25} = \frac{2x+13}{(x+5)^2}$
2 Express the integrand in partial fraction form.	$\frac{2x+13}{(x+5)^2} = \frac{A}{x+5} + \frac{B}{(x+5)^2}$ $= \frac{A(x+5)+B}{(x+5)^2}$ $\therefore\ 2x+13 = A(x+5)+B$ Setting $x = -5$: $2\times -5+13 = B$ $B = 3$ Setting $x = 0$: $13 = 5A+B$ $13 = 5A+3$ $10 = 5A$ $A = 2$ $\therefore\ \frac{2x+13}{x^2+10x+25} = \frac{2}{x+5} + \frac{3}{(x+5)^2}$
3 Use partial fractions to determine the integral.	$\int \frac{2x+13}{x^2+10x+25}\,dx = \int \frac{2}{x+5} + \frac{3}{(x+5)^2}\,dx$ $= \int \frac{2}{x+5}\,dx + \int 3(x+5)^{-2}\,dx$ $= 2\ln\lvert x+5\rvert - \frac{3}{x+5} + c$

Suppose the denominator of a rational expression is a factorable quadratic (such as the examples described so far), but with a numerator that is quadratic or even cubic (as opposed to linear or constant). Worked example 20 illustrates how the techniques previously described may still be used provided the expression is first rewritten as the sum of a polynomial and a fraction with the same denominator as the original expression, but with a linear or constant numerator. Polynomial long division can always be used to accomplish this.

20 Integrating rational expressions with different degrees

Demonstrate that $\frac{2x^3+x^2-9x+6}{x^2-4} = 2x+1+\frac{-x+10}{x^2-4}$. Hence, determine $\int \frac{2x^3+x^2-9x+6}{x^2-4}\,dx$.

THINKING	WORKING
1 The initial equality can be shown by starting with the right-hand side and demonstrating that it is equal to the left-hand side. Alternatively, polynomial long division may be used to directly obtain the given statement.	**METHOD 1** $\text{RHS} = 2x+1+\dfrac{-x+10}{x^2-4}$ $= \dfrac{(2x+1)\left(x^2-4\right)}{x^2-4}+\dfrac{-x+10}{x^2-4}$ $= \dfrac{2x^3-8x+x^2-4-x+10}{x^2-4}$ $= \dfrac{2x^3+x^2-9x+6}{x^2-4}$ $= \text{LHS}$ **METHOD 2** $x^2+0x-4 \overline{)\, 2x^3+x^2-9x+6}$ has quotient $2x+1$ $-(2x^3+0x^2-8x)$ x^2-x+6 $-(x^2+0x-4)$ $-x+10$ Thus, $\dfrac{2x^3+x^2-9x+6}{x^2-4} = 2x+1+\dfrac{-x+10}{x^2-4}$
2 Express the fractional part of the expression in partial fraction form.	$\dfrac{-x+10}{x^2-4} = \dfrac{-x+10}{(x-2)(x+2)}$ $= \dfrac{A}{x-2}+\dfrac{B}{x+2}$ $= \dfrac{A(x+2)+B(x-2)}{(x-2)(x+2)}$ $\therefore\ -x+10 = A(x+2)+B(x-2)$ Let $x = 2$: $8 = 4A$ $A = 2$ Let $x = -2$: $12 = -4B$ $B = -3$ $\therefore\ \dfrac{-x+10}{(x-2)(x+2)} = \dfrac{2}{x-2}-\dfrac{3}{x+2}$
3 Determine the integral.	$\displaystyle\int \frac{2x^3+x^2-9x+6}{x^2-4}\,dx = \int 2x+1+\frac{2}{x-2}-\frac{3}{x+2}\,dx$ $= x^2+x+2\ln\lvert x-2\rvert-3\ln\lvert x+2\rvert+c$

Partial fraction decompositions exist for a much wider class of rational expressions than discussed here. Some examples are:

- $\dfrac{px^2+qx+r}{(ax+b)(cx+d)(ex+f)} = \dfrac{A}{ax+b}+\dfrac{B}{cx+d}+\dfrac{C}{ex+f}$
- $\dfrac{px^2+qx+r}{(ax+b)(cx+d)^2} = \dfrac{A}{ax+b}+\dfrac{B}{cx+d}+\dfrac{C}{(cx+d)^2}$
- $\dfrac{px^2+qx+r}{(ax+b)(cx^2+dx+e)} = \dfrac{A}{ax+b}+\dfrac{Bx+C}{cx^2+dx+e}$ (where cx^2+dx+e cannot be factored into linear factors).

EXERCISE 5.6 Partial fractions

Worked Example 18

1 Express the following rational expressions in partial fraction form:

(a) $\dfrac{8}{(x-1)(x+3)}$ (b) $\dfrac{2x+20}{(x-5)(x+5)}$ (c) $\dfrac{10x+1}{(2x-1)(2x+1)}$

(d) $\dfrac{10}{x(x-5)}$ (e) $\dfrac{2x+3}{(x+1)(x+3)}$ (f) $\dfrac{4x+1}{2x(3x+1)}$

(g) $\dfrac{3x}{(x-1)^2}$ (h) $\dfrac{x+5}{(x+2)^2}$ (i) $\dfrac{x}{(2x+3)^2}$

2 Anti-differentiate each of the expressions in question 1.

Worked Example 19

3 Determine the following indefinite integrals by first expressing the integrand in partial fraction form.

(a) $\displaystyle\int \frac{12}{x^2-6x+8}\,dx$ (b) $\displaystyle\int \frac{3}{x^2+3x}\,dx$ (c) $\displaystyle\int \frac{3x+8}{3x^2-x}\,dx$

(d) $\displaystyle\int \frac{x}{x^2-6x+9}\,dx$ (e) $\displaystyle\int \frac{5x+3}{4x^2+4x+1}\,dx$

4 Calculate the exact value of each of the following definite integrals:

(a) $\displaystyle\int_1^2 \frac{6}{x^2+2x}\,dx$ (b) $\displaystyle\int_3^4 \frac{3x-13}{3x^2-5x-2}\,dx$ (c) $\displaystyle\int_2^3 \frac{3}{4x^2-1}\,dx$ (d) $\displaystyle\int_0^1 \frac{3x-4}{9x^2+6x+1}\,dx$

5 (a) Which of the following statements is true?

A $\dfrac{x^2+3x+5}{(x+3)(x+2)}$ can be expressed in the form $\dfrac{A}{x+3}+\dfrac{B}{x+2}$, where A and B are non-zero real numbers.

B $\dfrac{2x+5}{(x+3)^2}$ can be expressed in the form $\dfrac{A}{x+3}+\dfrac{B}{x+3}$, where A and B are non-zero real numbers.

C $\dfrac{2x+5}{x^2-1}$ can be expressed in the form $\dfrac{A}{x+1}+\dfrac{B}{x-1}$, where A and B are non-zero real numbers.

D $\dfrac{2x+5}{x^2+1}$ can be expressed in the form $\dfrac{A}{x+1}+\dfrac{B}{x-1}$, where A and B are non-zero real numbers.

(b) Explain some possible common errors for the incorrect options.

Worked Example

6 Demonstrate that $\frac{3x^2+4x-5}{x^2-1} = 3+\frac{4x-2}{x^2-1}$. Hence, determine $\int \frac{3x^2+4x-5}{x^2-1}dx$.

20

7 Demonstrate that $\frac{x^3+3x^2-4}{x^2+x-6} = x+2+\frac{4x+8}{x^2+x-6}$. Hence, determine $\int \frac{x^3+3x^2-4}{x^2+x-6}dx$.

8 Demonstrate that $\frac{x^3+3}{x^2-x} = x+1+\frac{x+3}{x^2-x}$. Hence, determine $\int \frac{x^3+3}{x^2-x}dx$.

9 Determine the values of A, B and C, given that $\frac{2x-3}{x(x+1)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$. Hence, determine $\int \frac{2x-3}{x(x+1)^2}dx$.

10 Determine the values of A, B and C, given that $\frac{x-5}{x^2(x-2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$. Hence, determine $\int \frac{x-5}{x^3-2x^2}dx$.

11 Determine the values of A, B and C, given that $\frac{x+9}{2x(2x+3)^2} = \frac{A}{2x}+\frac{B}{2x+3}+\frac{C}{(2x+3)^2}$. Hence, determine $\int \frac{x+9}{2x(2x+3)^2}dx$.

12 Determine each of the following anti-derivatives:

(a) $\int \frac{1}{4+x}dx$ (b) $\int \frac{1}{4+x^2}dx$ (c) $\int \frac{1}{4-x^2}dx$ (d) $\int \frac{x}{4-x^2}dx$

13 The time taken for a particular online video to be viewed by n individuals is given by $t = \int \frac{3000010}{(n+1)(300000-n)}dn$, where t is measured in days since the moment the video is uploaded. Determine how many individuals will have viewed the video after 5 days have passed.

14 Determine $\int \frac{x^2}{4-x^2}dx$.

15 Prove that $\int_0^{\frac{\pi}{3}} \frac{\cos(x)}{1-\sin^2(x)}dx = \frac{1}{2}\ln\left(7+4\sqrt{3}\right)$.

16 Determine the values of A, B, C and D, given that $\frac{5x^3+6x^2+15x+4}{x(x+1)\left(x^2+4\right)} = \frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2+4}$. Hence, determine $\int \frac{5x^3+6x^2+15x+4}{x(x+1)\left(x^2+4\right)}dx$.

5.7 Integration by parts

The product rule for differentiation states that if a function can be expressed as the product of two functions, u and v, then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

If u' and v' are used to denote $\frac{du}{dx}$ and $\frac{dv}{dx}$ respectively, then $\frac{d}{dx}(uv) = u\,v' + v\,u'$.

If the derivative of uv is equal to $u\,v' + v\,u'$, it follows that an anti-derivative of $u\,v' + v\,u'$ is uv. In symbols, $\int (u\,v' + v\,u')\,dx = uv$. Therefore,

$$\int (u\,v' + v\,u')\,dx = uv$$
$$\int u\,v'\,dx + \int v\,u'\,dx = uv$$
$$\int u\,v'\,dx = uv - \int v\,u'\,dx$$

Which gives the following important rule:

$$\int u\,v'\,dx = uv - \int v\,u'\,dx$$

This rule is useful when you are integrating the product of two functions. Generally, start by defining one of the functions as variable u and the other as variable v'. Next, determine the derivative of u (u') and an anti-derivative of v' (v) in order to apply the rule. The right-hand side of the rule contains another integral; however, this new integral will often be one that can be readily determined.

Using this rule is known as *integrating by parts*.

21 Using integration by parts

Use the method of integration by parts to determine the following indefinite integrals:

(a) $\int xe^{2x}\,dx$

THINKING	WORKING
1 Identify the functions and break up the integrand into the product of two expressions, letting u be equal to one and $\frac{dv}{dx}$ (or simply v') be equal to the other.	Let $u = x$ and $v' = e^{2x}$.
2 Determine the derivative of u and an anti-derivative of v'.	Then $u' = 1$ and $v = \frac{1}{2}e^{2x}$.
3 Apply the rule for integration by parts and determine the new simpler integral that arises.	$\int xe^{2x}\,dx = \int uv'\,dx$ $= uv - \int u'v\,dx$ $= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$ $= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

(b) $\int \ln(x)\,dx$

1 Identify the functions and break up the integrand into the product of two expressions, letting u be equal to one expression and $\frac{dv}{dx}$ (or simply v') be equal to the other.	Let $u = \ln(x)$ and $v' = 1$.
2 Determine the derivative of u and an anti-derivative of v'.	Then $u' = \frac{1}{x}$ and $v = x$.
3 Apply the rule for integration by parts and determine the new simpler integral that arises.	$\int \ln(x)dx = \int uv'\,dx$ $= uv - \int u'v\,dx$ $= x\ln(x) - \int \frac{1}{x}x\,dx$ $= x\ln(x) - \int 1\,dx$ $= x\ln(x) - x + c$

When using integration by parts, the function defined as v' must be one that can be readily integrated (in order to determine an anti-derivative, v). For this reason, logarithmic and inverse trigonometric functions are nearly always assigned to the variable u as they do not have simple integrals.

Aside from this, the choice about which function to assign to u and which to assign to v' should be made based on which choice gives rise to a simpler integral on the right-hand side of the rule. You are encouraged to try both possible choices in each of the previous examples in order to understand the reasons for the choices made.

WARNING

Although the method of integration by parts frequently produces a simpler integral, it is occasionally necessary to use the technique again (or some other technique) if this new integral produced cannot be readily determined.

22 Using integration by parts more than once

Use the method of integration by parts to determine the following indefinite integrals:

(a) $\int x^2 \cos(x)\,dx$

THINKING	WORKING
1 Proceed using integration by parts.	Let $u = x^2$ and $v' = \cos(x)$. Then $u' = 2x$ and $v = \sin(x)$. $\int x^2\cos(x)dx = \int uv'\,dx$ $= uv - \int u'v\,dx$ $= x^2\sin(x) - \int 2x\sin(x)dx$ $= x^2\sin(x) - 2\int x\sin(x)dx$ [1]

2 If the new integral cannot readily be determined, use integration by parts (or some other technique) to determine it.

For $\int x\sin(x)\,dx$:

Let $u = x$ and $v' = \sin(x)$.

Then $u' = 1$ and $v = -\cos(x)$.

$$\begin{aligned}\int x\sin(x)\,dx &= \int uv'\,dx\\ &= uv - \int u'v\,dx\\ &= -x\cos(x) - \int -\cos(x)\,dx\\ &= -x\cos(x) + \int \cos(x)\,dx\\ &= -x\cos(x) + \sin(x) + c\end{aligned}$$

3 Substitute this integral into [1].

$$\begin{aligned}\int x^2\cos(x)\,dx &= x^2\sin(x) - 2\int x\sin(x)\,dx\\ &= x^2\sin(x) - 2\left(-x\cos(x) + \sin(x) + c\right)\\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) - 2c\\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + c_1,\\ &\quad \text{where } c_1 = -2c\end{aligned}$$

(b) $\int e^x\sin(x)\,dx$

1 Proceed using integration by parts.

Let $u = e^x$ and $v' = \sin(x)$.

Then $u' = e^x$ and $v = -\cos(x)$.

$$\begin{aligned}\int e^x\sin(x)\,dx &= \int uv'\,dx\\ &= uv - \int u'v\,dx\\ &= -e^x\cos(x) - \int -e^x\cos(x)\,dx\\ &= -e^x\cos(x) + \int e^x\cos(x)\,dx \quad [1]\end{aligned}$$

2 If the new integral cannot readily be determined, use integration by parts (or some other technique) to determine it. In this case, using integration by parts again does not result in a simple integral; however, it does result in an expression involving the original integral that may be desirable.

For $\int e^x\cos(x)\,dx$:

Let $u = e^x$ and $v' = \cos(x)$.

Then $u' = e^x$ and $v = \sin(x)$.

$$\begin{aligned}\int e^x\cos(x)\,dx &= \int uv'\,dx\\ &= uv - \int u'v\,dx\\ &= e^x\sin(x) - \int e^x\sin(x)\,dx\end{aligned}$$

3 Substitute this integral into [1].

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx$$
$$= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx$$

4 Rearrange the equation to solve for the original integral.

$$2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x) + c$$
$$\int e^x \sin(x)\,dx = \frac{1}{2}\left(-e^x \cos(x) + e^x \sin(x) + c\right)$$
$$\int e^x \sin(x)\,dx = -\frac{e^x}{2}(\cos(x) - \sin(x)) + c_1, \text{ where } c_1 = \frac{1}{2}c$$

EXERCISE 5.7 Integration by parts

Worked Example

21

1 Use the method of integration by parts to determine the following indefinite integrals:

(a) $\int x \sin(x)\,dx$ (b) $\int e^{4x} x\,dx$ (c) $\int x \ln(x)\,dx$

(d) $\int e^{2x}(3x-2)\,dx$ (e) $\int x^3 \ln(x)\,dx$ (f) $\int \ln(3x)\,dx$

2 Calculate the exact value of each of the following:

(a) $\int_0^{\frac{\pi}{3}} x \sin(x)\,dx$ (b) $\int_1^9 \sqrt{x} \ln(x)\,dx$

3 Prove that $\int_1^e x^n \ln(x)\,dx = \dfrac{ne^{n+1}+1}{(n+1)^2}$.

22

4 Use the method of integration by parts to determine the following indefinite integrals:

(a) $\int x^2 e^{2x}\,dx$ (b) $\int x^2 \sin(3x)\,dx$ (c) $\int (x+3)^2 e^{-x}\,dx$

(d) $\int e^x \cos(x)\,dx$ (e) $\int e^{2x} \sin\left(\dfrac{x}{2}\right)dx$

5 Use the method of integration by parts to determine the following indefinite integrals:

(a) $\int \arcsin(x)\,dx$

(b) $\int \arccos\left(\dfrac{x}{a}\right)dx$

(c) $\int \arctan\left(\dfrac{x}{a}\right)dx$

6 Let $I_n = \int x^n e^x \, dx$. Demonstrate that $I_n = x^n e^x - nI_{n-1}$, and then use this result to determine $\int x^3 e^x \, dx$.

7 Determine $\int x^2 \arcsin(x) \, dx$.

8 Determine the value of k, correct to 2 decimal places, given that $0 < k < \frac{\pi}{2}$ and $\int_0^k x \sec^2(x) = 10$.

Using technology or otherwise, use your rounded value for k to calculate $\int_0^k x \sec^2(x)$ and clarify why a discrepancy is reasonable.

5 Summary

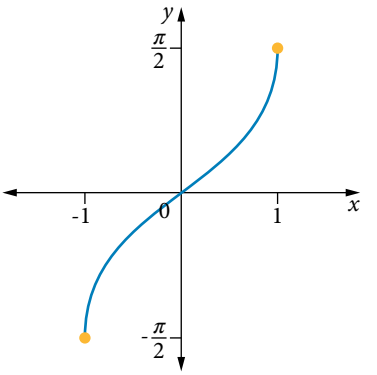
Inverse trigonometric functions

$\arcsin(x) = \sin^{-1}(x) = \theta$ such that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\sin(\theta) = x$.

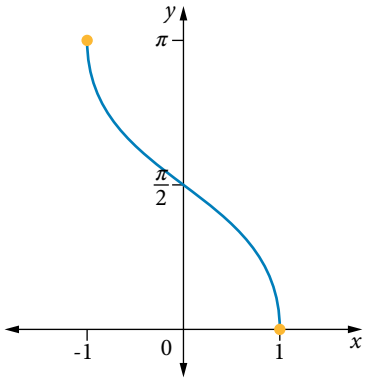
$\arccos(x) = \cos^{-1}(x) = \theta$ such that $0 \leq \theta \leq \pi$ and $\cos(\theta) = x$.

$\arctan(x) = \tan^{-1}(x) = \theta$ such that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan(\theta) = x$.

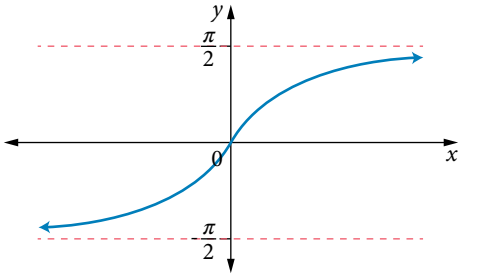
The graphs of the three inverse trigonometric functions:



$y = \arcsin(x)$



$y = \arccos(x)$



$y = \arctan(x)$

$\sin^{-1}(x) \neq \frac{1}{\sin(x)}$, $\cos^{-1}(x) \neq \frac{1}{\cos(x)}$, $\tan^{-1}(x) \neq \frac{1}{\tan(x)}$.

Derivatives of inverse trigonometric functions

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arccos(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{1+x^2}$$

If $a > 0$, $\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}$.

If $a > 0$, $\frac{d}{dx}\left(\arccos\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - x^2}}$.

If $a \neq 0$, $\frac{d}{dx}\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}$.

Anti-derivatives relating to logarithmic and inverse trigonometric functions

$$\int \frac{1}{x}\,dx = \ln|x| + c \text{ for } x \neq 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c \text{ for } x \neq -\frac{b}{a}$$

If $a > 0$, $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$.

If $a > 0$, $\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + c$.

If $a \neq 0$, $\int \frac{a}{a^2 + x^2}\,dx = \arctan\left(\frac{x}{a}\right) + c$.

Integration by substitution

The substitution rule (change of variable rule):

$$\int f(u) \times \frac{du}{dx}\,dx = \int f(u)\,du$$

When determining definite integrals using a change of variable, it is not necessary to express the anti-derivative in terms of the original variable provided you change the integration limits to values corresponding to the new variable.

Integration with trigonometric identities

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$\tan^2(x) = \sec^2(x) - 1$ (more commonly written in the form $\sec^2(x) = 1 + \tan^2(x)$)

$$\sin(2x) = 2\sin(x)\cos(x)$$

Partial fractions

The following rule allows certain rational expressions to be expressed as the sum of simpler fractions (known as partial fractions), each of which can be readily integrated:

If no pair of linear factors, $ax + b$, $cx + d$, and $ex + f$ have any common factors, then non-zero real numbers A and B exist such that

$$\frac{ax + b}{(cx + d)(ex + f)} = \frac{A}{cx + d} + \frac{B}{ex + f}, \text{ and}$$

$$\frac{ax + b}{(cx + d)^2} = \frac{A}{cx + d} + \frac{B}{(cx + d)^2}$$

for suitable non-zero real numbers A and B.

Integration by parts

$$\int u\,v'\,dx = uv - \int u'v\,dx$$

When using integration by parts, the function assigned to v' must be one that can be readily integrated (in order to determine an anti-derivative, v). For this reason, logarithmic and inverse trigonometric functions are nearly always assigned to the variable u as they do not have simple integrals.

Chapter review 5

Exercise 5.1 **1** Determine the exact values of the following:

(a) $\arcsin(1)$ **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$

Exercise 5.1 **2** Determine the domain and range of the function $y = 2\sin^{-1}\left(x - \frac{1}{2}\right)$, and sketch its graph.

Exercise 5.2 **3** Determine the derivative of each of the following:

(a) $y = 7\arccos(3x)$ **(b)** $y = (2x+3)\arctan(x)$

Exercise 5.3 **4** Determine the following indefinite integrals:

(a) $\int \frac{-3}{\sqrt{5-x^2}}\,dx$ **(b)** $\int \frac{2}{1+3x^2}\,dx$

Exercise 5.3 **5** Determine the exact value of $\int_0^{\frac{1}{4}} \frac{3}{\sqrt{1-4x^2}}\,dx$.

Exercise 5.4 **6** Determine the following anti-derivatives:

(a) $\int x^3\left(x^4+3\right)^6\,dx$ **(b)** $\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$ **(c)** $\int \frac{\ln(3x-1)}{3x-1}\,dx$

Exercise 5.4 **7** Determine the exact value of $\int_0^2 \frac{x}{1+x^2}\,dx$.

Exercise 5.5 **8** Determine the following indefinite integrals:

(a) $\int \cos^2\left(\frac{x}{6}\right)dx$ **(b)** $\int \tan^2(7x)\,dx$

Exercise 5.5 **9** Determine the exact value of $\int_0^{\frac{\pi}{6}} \sin(x) - \sin^2(x)\,dx$.

Exercise 5.6 **10** Determine the following anti-derivatives by first expressing the integrand in partial fraction form:

(a) $\int \frac{5x-1}{x^2-2x-15}\,dx$ **(b)** $\int \frac{12x-5}{9x^2-6x+1}\,dx$

Exercise 5.6 **11** Demonstrate that $\int_2^3 \frac{2x-9}{(x+6)(x-1)}\,dx = 2\ln\left(\frac{27}{32}\right)$.

Exercise 5.7 **12** Use the method of integration by parts to determine the following indefinite integrals:

(a) $\int x\ln(5x)\,dx$ **(b)** $\int x\sqrt{x+1}\,dx$

Exercise 5.1 **13** Determine the exact value of $\sin\left(\arccos\left(-\frac{12}{13}\right)\right)$.

Exercise 5.1 **14** Prove that if $-1 \le x \le 1$, $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$.

Exercise 5.3 **15** Determine the exact value of k, given that $k < -1$ and $\int_k^{-1} \frac{5}{2x+1}\,dx = -1$.

16 Let p and q be positive real numbers. Determine $\int \frac{1}{p+qx^2}\,dx$. Exercise 5.3

17 By considering the substitution $u = x^2+5$, determine $\int x^3\left(x^2+5\right)^7\,dx$. Exercise 5.4

18 Determine $\int \sin^4(x)\,dx$ by first expressing it as $\int\left(\sin^2(x)\right)^2\,dx$. Exercise 5.5

19 Demonstrate that $\frac{4x^3+8x^2+10x+4}{4x^2+4x+1} = x+1+\frac{5x+3}{4x^2+4x+1}$. Exercise 5.6

Hence, determine $\int \frac{4x^3+8x^2+10x+4}{4x^2+4x+1}\,dx$.

20 Determine the values of A, B and C, given that $\frac{8x+25}{x(x+5)^2} = \frac{A}{x}+\frac{B}{x+5}+\frac{C}{(x+5)^2}$. Exercise 5.6

Hence, determine $\int \frac{8x+25}{x(x+5)^2}\,dx$.

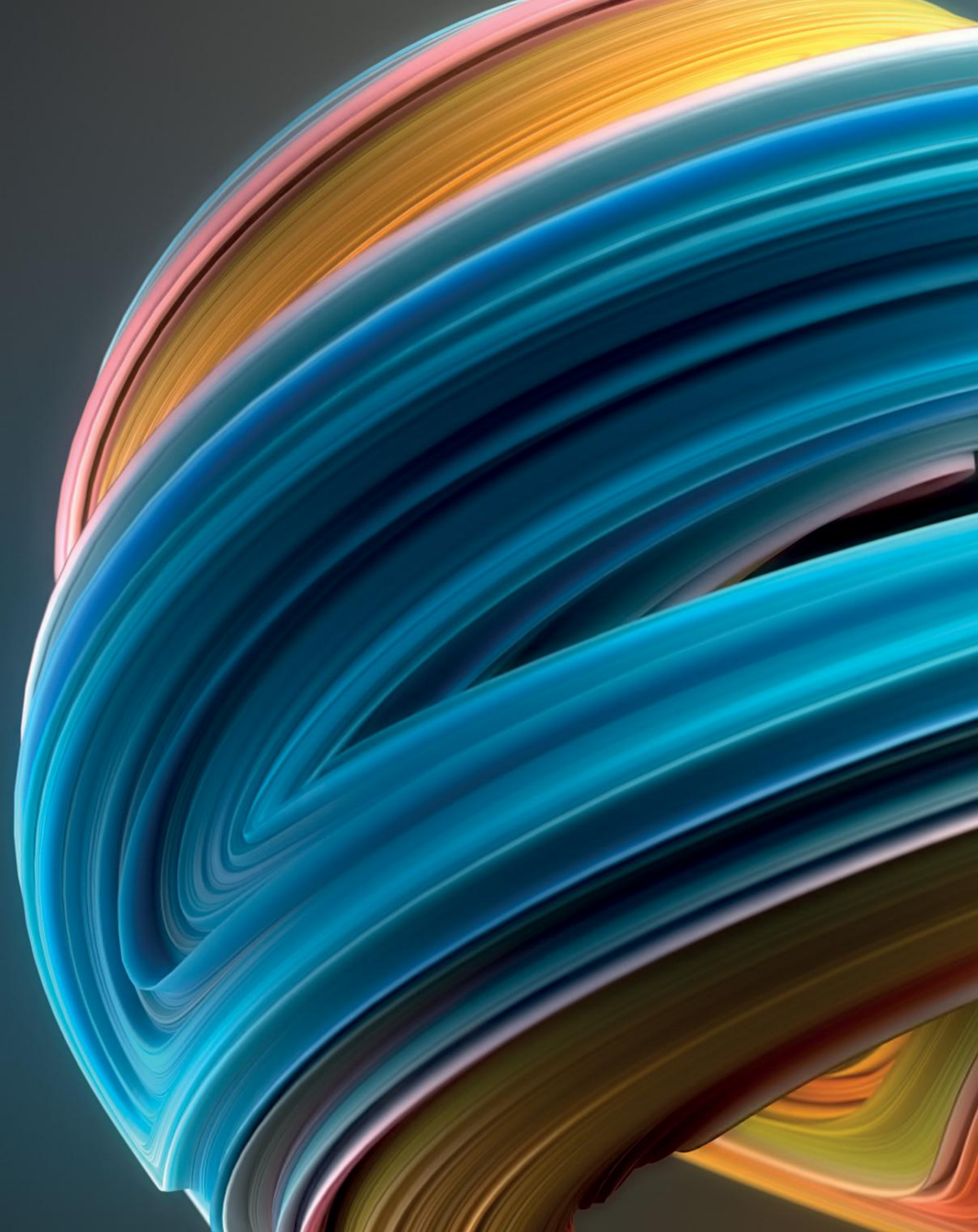
21 Prove that $\sin\left(\arccos(x)\right) = \cos\left(\arcsin(x)\right)$ for all real values of x with $-1 \le x \le 1$. Exercise 5.2

22 **(a)** Explain why $\arcsin(-k) = -\arcsin(k)$. Exercise 5.3

(b) Determine the exact value of k, given that $\int_{-k}^{k} \frac{3}{\sqrt{1-x^2}} = \pi$.

23 The number of fish in a lake is increasing at the rate $\frac{dP}{dt} = \frac{30t}{t^2+26}$, where P is the number of fish and t is the time in days. Initially there are 50 fish in the lake. Determine how fast the population is increasing when there are 100 fish in the lake. Express your answer correct to 1 decimal place. Exercise 5.4

24 Let $a > 0$. Determine the exact value of a if $\int_0^{\frac{a}{2}} \arcsin\left(\frac{x}{a}\right)dx = 1$. Exercise 5.7





6

Applications of integration

Recall 320

6.1 Areas between curves 321

6.2 Volumes of solids of revolution 334

6.3 Numerical integration using Simpson's rule 343

6.4 The exponential distribution 348

Summary 359

Chapter review 361

6

Recall

Determine indefinite integrals

1 Determine the following:

(a) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\int \cos^2(x)\,dx$ (c) $\int \frac{2x}{x^2-5}\,dx$

(d) $\int xe^x\,dx$ (e) $\int \frac{1}{x^2-9}\,dx$

Calculate definite integrals without technology

2 Determine the exact value of the following:

(a) $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx$

(b) $\int_0^1 \frac{e^x}{1+e^x}\,dx$

Calculate definite integrals with technology

3 Determine the value of $\int_1^2 x^x\,dx$ correct to 3 decimal places.

Sketch the graphs of simple functions without technology

4 Sketch each of the following functions, showing all turning points, intercepts and asymptotes.

(a) $f(x) = \sqrt{x-3}$ (b) $f(x) = 9 - x^2$ (c) $f(x) = \frac{2}{x-6}$

Use the complement rule for probabilities

5 When tossing a fair coin 5 times, the probability of obtaining at least two tails is 0.8125. Determine the probability of obtaining fewer than two tails.

Calculate conditional probabilities

6 Suppose that a weather forecaster predicts that the probability of it raining on Wednesday is 0.5, and that the probability of it raining on both Wednesday and Thursday is 0.4. Determine the probability that it rains on Thursday, given that it does rain on Wednesday.

Areas between curves

6.1

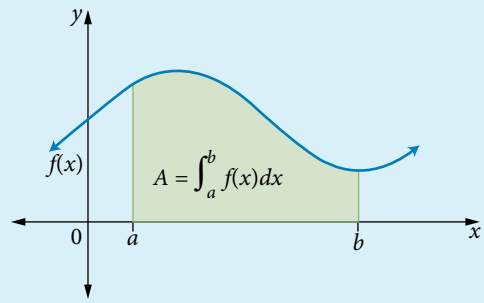
In Mathematical Methods, you will have learnt how to use definite integrals to calculate areas under curves and areas between two curves. In this section, these ideas are applied to more complex functions and situations where the integration techniques of Chapter 5 will often be needed.

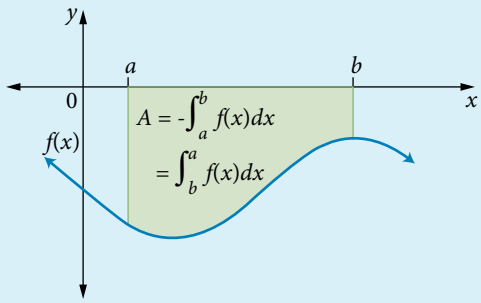
The area bound by a function and the *x*-axis

Recall the following important rule regarding areas between functions and the x-axis:

The area bounded by a function $f(x)$, the x-axis, and the lines $x = a$ and $x = b$ (where $a < b$) is equal to:

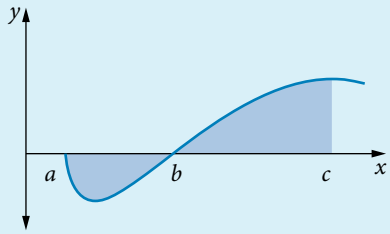
$\int_a^b f(x)\,dx$ if the function lies above the x-axis from a to b, and $-\int_a^b f(x)\,dx$ if the function lies below the x-axis from a to b.





The area bound by the curve and the x-axis over the interval shown is

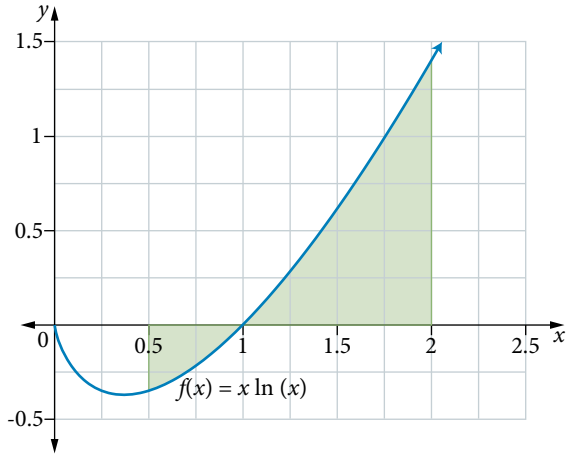
$-\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ or $\int_b^a f(x)\,dx + \int_b^c f(x)\,dx$.



If part of the function is above the x-axis and part below over the interval $[a, b]$, then the total area may be calculated by partitioning the area into sections with each section entirely above or entirely below the x-axis. Each of these areas may be calculated using a definite integral (which may have to be made positive by multiplying by -1 or taking the absolute value), and then added together to give the total area. The following example illustrates this.

1 Determining the area between a function and the *x*-axis

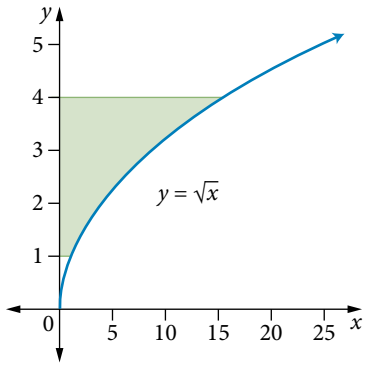
Determine the exact area of the shaded region in the diagram.



THINKING	WORKING
1 Split the function into two parts.	$A = -\int_{0.5}^{1} x\ln(x)\,dx + \int_{1}^{2} x\ln(x)\,dx$
2 Determine any difficult integrals separately.	To determine $\int x\ln(x)\,dx$, use integration by parts: Let $u = \ln(x)$ and $v' = x$. Then $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$. $\int x\ln(x)\,dx = \int uv'\,dx$ $= uv - \int u'v\,dx$ $= \frac{1}{2}x^2\ln(x) - \int \frac{1}{x}\frac{1}{2}x^2\,dx$ $= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx$ $= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + c$
3 Determine and evaluate the expression for the area.	$A = -\int_{0.5}^{1} x\ln(x)\,dx + \int_{1}^{2} x\ln(x)\,dx$ $= -\left[\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right]_{0.5}^{1} + \left[\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right]_{1}^{2}$ $= -\left(-\frac{1}{4} - \left(\frac{1}{2}\times\frac{1}{4}\ln\left(\frac{1}{2}\right) - \frac{1}{4}\times\frac{1}{4}\right)\right) + \frac{1}{2}\times 4\ln(2) - 1 - \left(-\frac{1}{4}\right)$ $= -\left(-\frac{1}{4} - \frac{1}{8}\ln\left(\frac{1}{2}\right) + \frac{1}{16}\right) + 2\ln(2) - \frac{3}{4}$ $= -\frac{9}{16} + \frac{1}{8}\ln\left(2^{-1}\right) + 2\ln(2)$ $= -\frac{9}{16} - \frac{1}{8}\ln(2) + 2\ln(2)$ $= \frac{15\ln(2)}{8} - \frac{9}{16}$ units^2

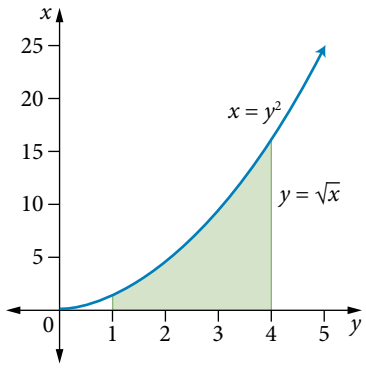
The area bound by a function and the y-axis

Consider the area shaded in the diagram.



In this case, the required area is between the function and y-axis from $y = 1$ to $y = 4$. While it is possible to determine this area by finding an area between the function and the x-axis, and then subtracting the area of a particular rectangle, it is also possible to express such an area as a single definite integral.

To see how this is done, imagine rotating the diagram so that the axes are swapped, as shown.



Now that the x-axis is the vertical axis, it makes sense to make x the subject of rule for the function.

Now consider,

$$y = \sqrt{x}$$
$$y^2 = x$$
$$x = y^2$$

It can be seen that the required area is simply equal to $\int_1^4 y^2\,dy$.

$$\begin{aligned} A &= \int_1^4 y^2\,dy \\ &= \left[\frac{1}{3}y^3\right]_1^4 \\ &= \frac{1}{3}4^3 - \frac{1}{3} \\ &= 21 \text{ units}^2 \end{aligned}$$

If a function $y = f(x)$ lies entirely to the right of the y-axis for the interval $a \leq y \leq b$, the function can be rearranged to obtain $x = g(y)$ for $a \leq y \leq b$. Now, the area bounded by the part of the function to the right of the y-axis, the y-axis, and the lines $y = a$ and $y = b$ is equal to $\int_{y=a}^{y=b} g(y)\,dy$.

Making connections

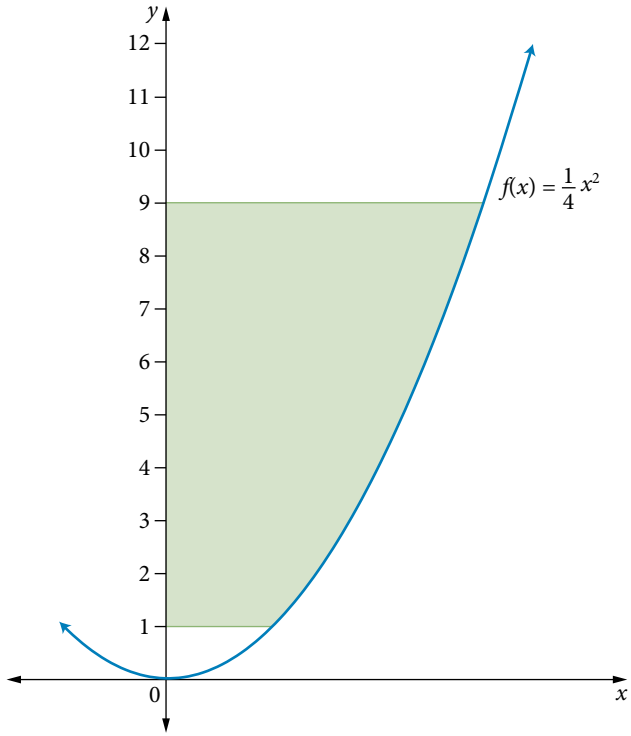
Areas between curves

Move the sliders to see what happens to the area between the curves.

While this idea will be used sparingly in the subsequent exercise, it is useful to keep in mind as it can dramatically simplify certain area calculations. It is also important from a conceptual understanding standpoint in order to make sense of volumes of revolution about the y-axis (Section 6.3). A further example of using this technique is illustrated in Worked example 2.

2 Determining the area between a function and the *y*-axis

Determine the exact area of the shaded region in the diagram.



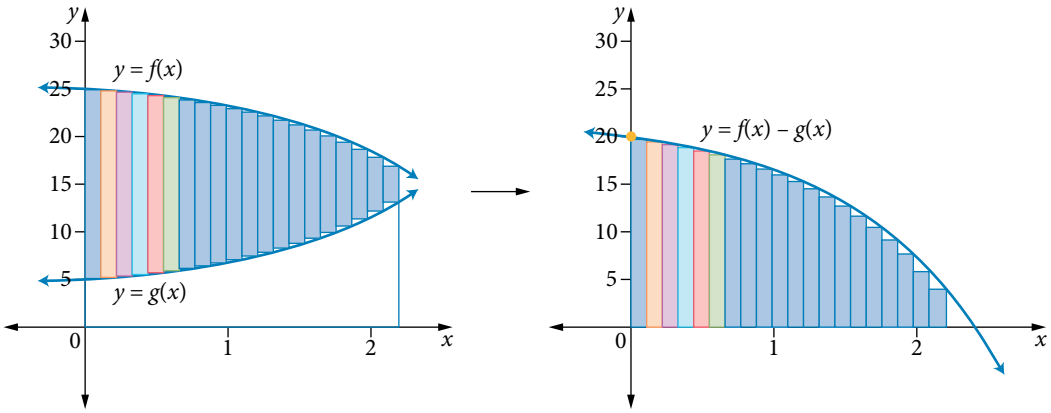
THINKING	WORKING
1 Express x in terms of y for the function given.	$y = \frac{1}{4}x^2$ $4y = x^2$ $x = \pm\sqrt{4y}$ $x = 2\sqrt{y}$ (As you are only interested in the part of the curve where $x \geq 0$, the negative expression for x may be discarded.)
2 Express the area as a definite integral, with y as the variable of integration. Finally, calculate the definite integral.	$A = \int_1^9 2y^{\frac{1}{2}}dy$ $= \left[\frac{4}{3}y^{\frac{3}{2}}\right]_1^9$ $= \frac{4}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$ $= \frac{4}{3}(27 - 1)$ $= \frac{104}{3}$ units^2

Areas between functions

Recall the following important rule regarding areas between functions:

If $f(x) \geq g(x)$ over the interval $[a,b]$, then the area between these functions over this interval is equal to $\int_a^b f(x) - g(x)\,dx$.

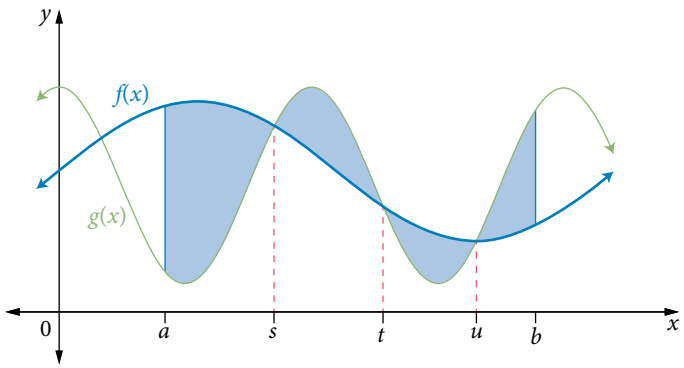
The following diagram illustrates why this makes sense.



Imagine breaking up the area between the functions into a series of thin vertical strips. The length of a strip at a particular x-value, say $x = p$, is equal to $f(p) - g(p)$. (For example, the first strip in the diagram has length approximately equal to 20.)

By graphing $y = f(x) - g(x)$, the height of the curve above the x-axis is always equal to the length of the strip. Thus, the area between the individual functions is the same as the area between the difference function and the x-axis.

Notice that if $f(x) \geq g(x)$ over $[a,b]$, then $f(x) - g(x)$ can never be negative over this interval. Thus, the area between the functions will be equal to $\int_a^b f(x) - g(x)\,dx$ regardless of whether the individual functions $f(x)$ and $g(x)$ are above or below the x-axis. The individual functions may even cross the x-axis in the interval.



If the two functions intersect over $[a,b]$, to obtain the area between the two functions you must add together the separate areas between the points of intersection. Each such partial area may be obtained by subtracting the lower function from the upper function, and integrating this over the required subinterval, as shown.

Given that $\int_p^q f(x) - g(x)\,dx = -\int_p^q g(x) - f(x)\,dx$, the shaded area in the diagram is also given by the following expression.

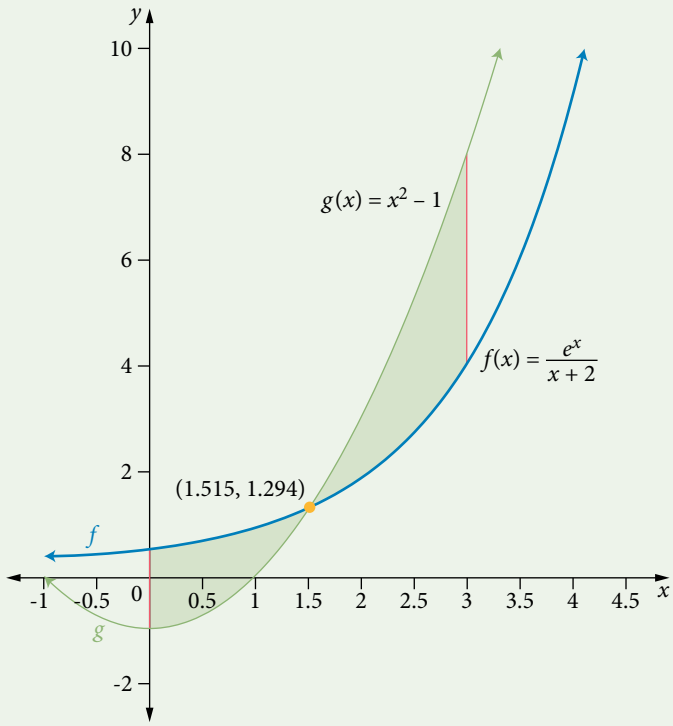
$$\text{Shaded area} = \int_a^s f(x) - g(x)\,dx + -\int_s^t f(x) - g(x)\,dx + \int_t^u f(x) - g(x)\,dx + -\int_u^b f(x) - g(x)\,dx$$

Note that in this expression, the function being integrated is the same in each term.

The following worked examples involve calculating areas between two functions. In Worked example 3, it is assumed that technology is available to assist with calculating points of intersection and definite integrals. In Worked example 4, it is assumed that no technology is available. (Section 5.3 contains an example of how to calculate a definite integral with technology.)

3 Determining the area between two functions using technology

Determine the area between the functions $f(x) = \dfrac{e^x}{x+2}$ and $g(x) = x^2 - 1$ for $0 \leq x \leq 3$, correct to 2 decimal places.

THINKING	WORKING
1 Use technology to sketch both functions on the same axes and shade the required area.	
2 Determine the coordinates of any points of intersection.	The coordinates of the point of intersection, using technology, are (1.515, 1.294).
3 Write an expression for the shaded area using definite integrals, and calculate the integral using technology.	$A = \int_0^{1.515} \dfrac{e^x}{x+2} - \left(x^2 - 1\right) dx + \int_{1.515}^{3} \left(x^2 - 1\right) - \dfrac{e^x}{x+2} dx$ $= 1.589 + 2.821$ (using technology) $= 4.41 \text{ units}^2$

Explore further

Area between curves

Use graphing software to calculate and display the area between two curves.

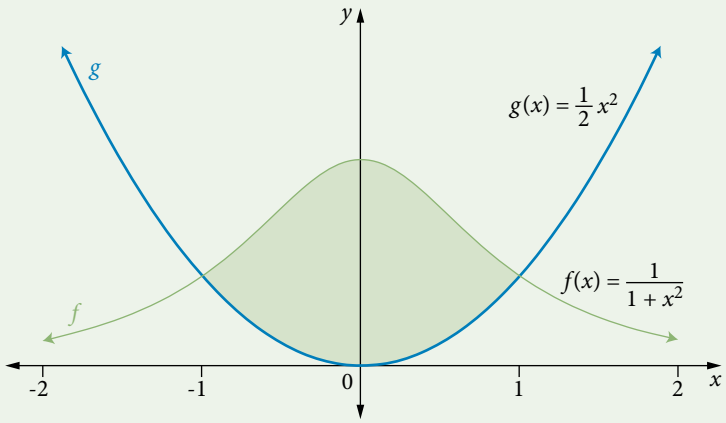
4 Determining the area between two functions

Determine the exact area bounded by the functions $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = \dfrac{1}{2}x^2$.

THINKING

1 Sketch the functions on the same set of axes and shade the required area.

WORKING



2 Determine the x-coordinates of any points of intersection and mark these on your sketch.

Points of intersection:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$
$$x^2 + x^4 = 2$$
$$\left(x^2\right)^2 + x^2 - 2 = 0$$
$$\left(x^2 - 1\right)\left(x^2 + 2\right) = 0$$
$$x = \pm 1$$

3 Write an expression for the shaded area using definite integrals, and evaluate. Often, the area calculation may be simplified by noting any symmetry.

As both curves are symmetric about the y-axis, the required area is equal to double the area to the right of the y-axis:

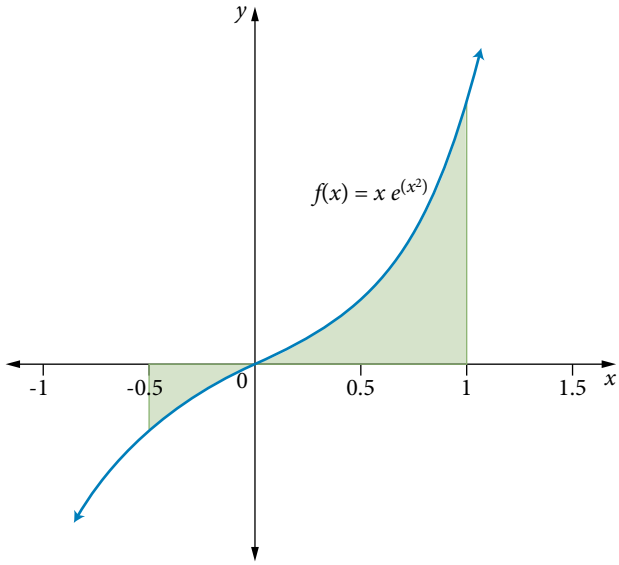
$$A = 2\int_0^1 \frac{1}{x^2+1} - \frac{1}{2}x^2\,dx$$
$$= 2\left[\arctan(x) - \frac{1}{6}x^3\right]_0^1$$
$$= 2\left(\left(\arctan(1) - \frac{1}{6}(1)^3\right) - \left(\arctan(0) - \frac{1}{6}(0)^3\right)\right)$$
$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right)$$
$$= \frac{\pi}{2} - \frac{1}{3} \text{ units}^2$$

EXERCISE 6.1 Areas between curves

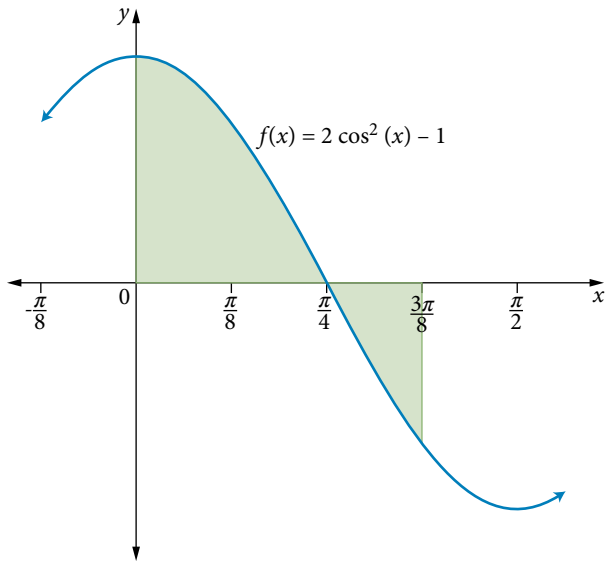
Worked Example

1 For each of the following, determine the exact area of the shaded region.

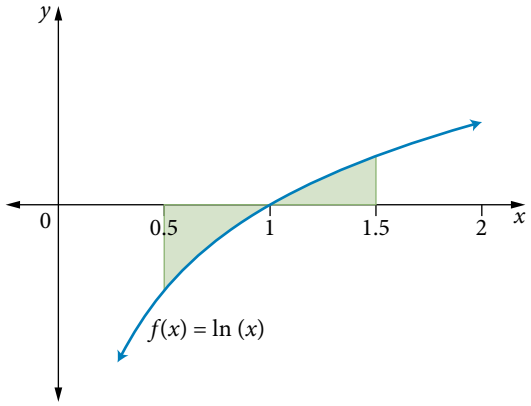
(a)



(b)

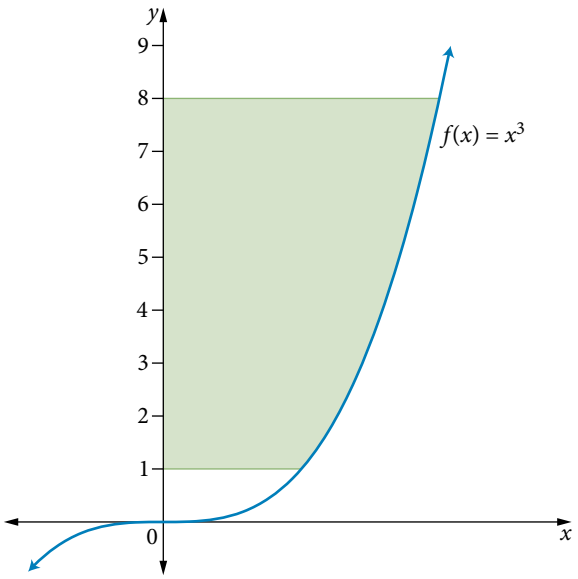


(c)

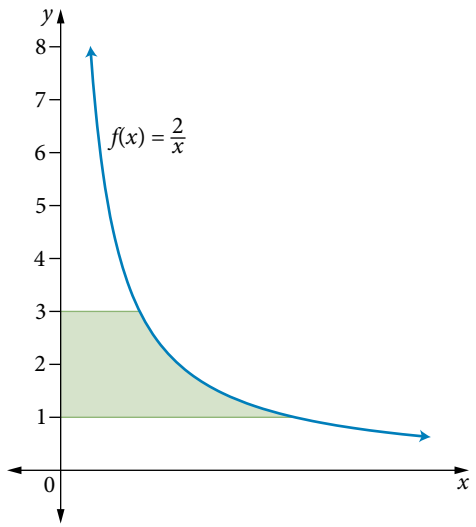


2 For each of the following, determine the exact area of the shaded region.

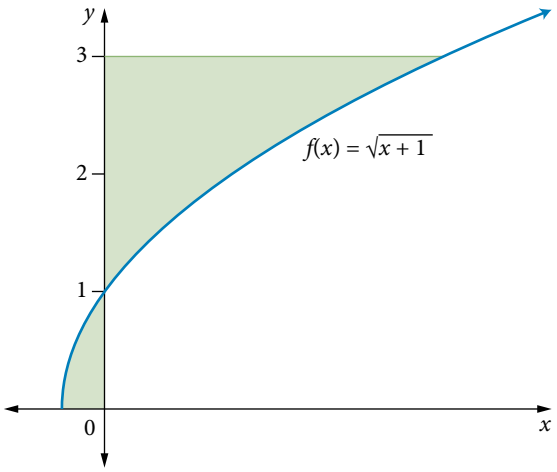
(a)



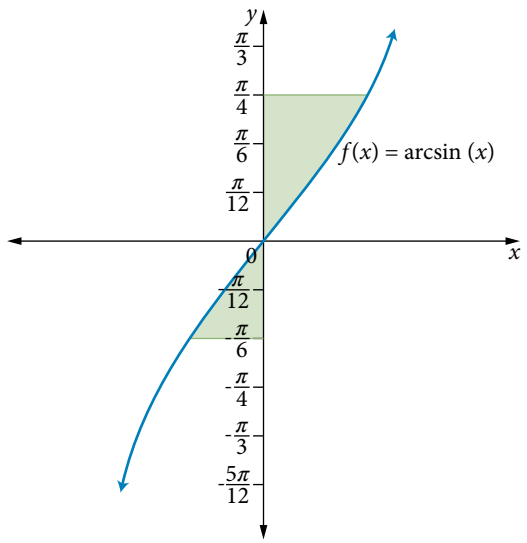
(b)



(c)



(d)



3 For each of the following, determine the area between the functions for the interval given. Give your answer correct to 2 decimal places.

(a) $f(x) = \sqrt{x}e^x$ and $g(x) = 6 - x^2$ for $0 \leq x \leq 2$

(b) $f(x) = \dfrac{1}{x}$ and $g(x) = x + 1$ for $-3 \leq x \leq -1$

4 For each of the following, determine the area bounded by the pair of functions.

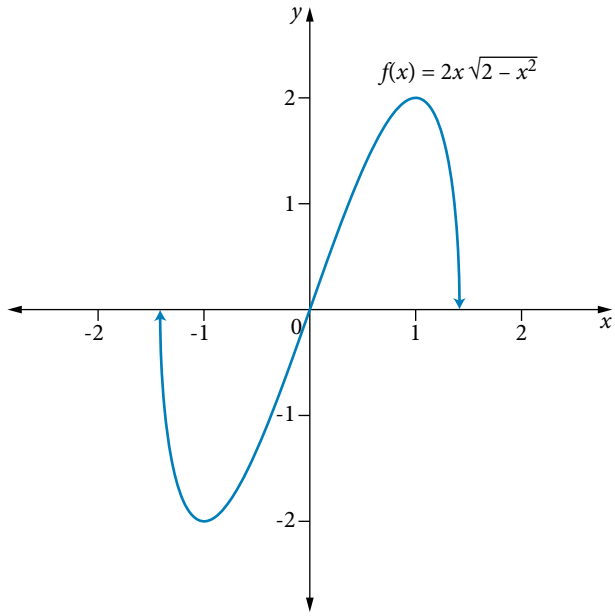
(a) $f(x) = \ln(x)$ and $g(x) = x^2 - 2$

(b) $f(x) = x^2 - 2$ and $g(x) = 1 - 2x^2$

Worked Example

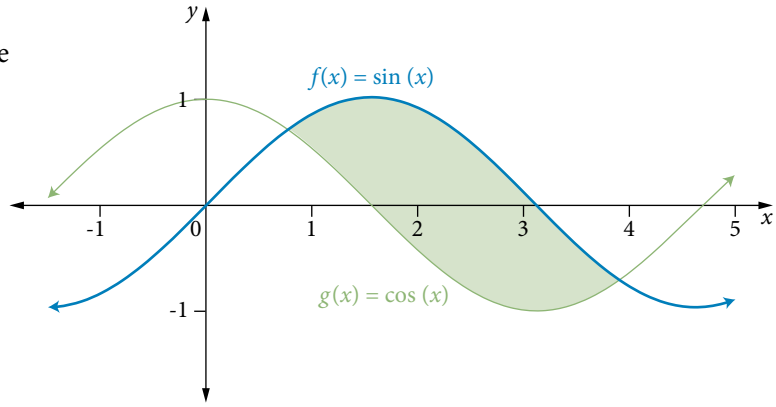
4

5 Determine the exact area bounded by $f(x) = 2x\sqrt{2 - x^2}$ and $g(x) = 2x$. The graph of $f(x)$ is shown to assist you.

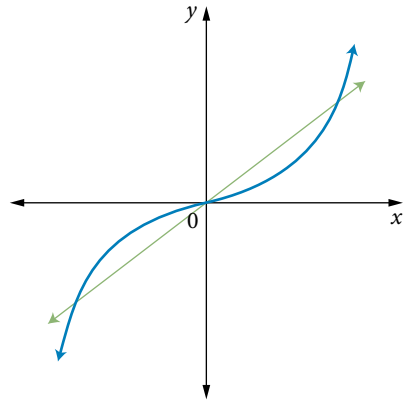


6 Determine the exact area bounded by $f(x) = e^x$, $g(x) = 3e^{-x}$ and the y-axis.

7 The graphs of $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are shown. Determine the exact area of the shaded region.



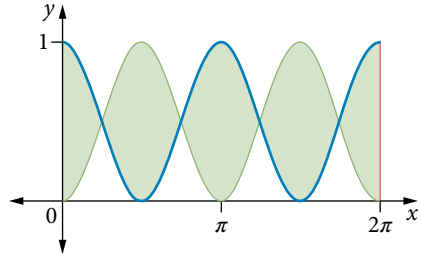
8 The diagram shows the graphs of $y = xe^{x^2}$ and $y = ex$. Determine the area between these functions for $1 \leq x \leq 2$.



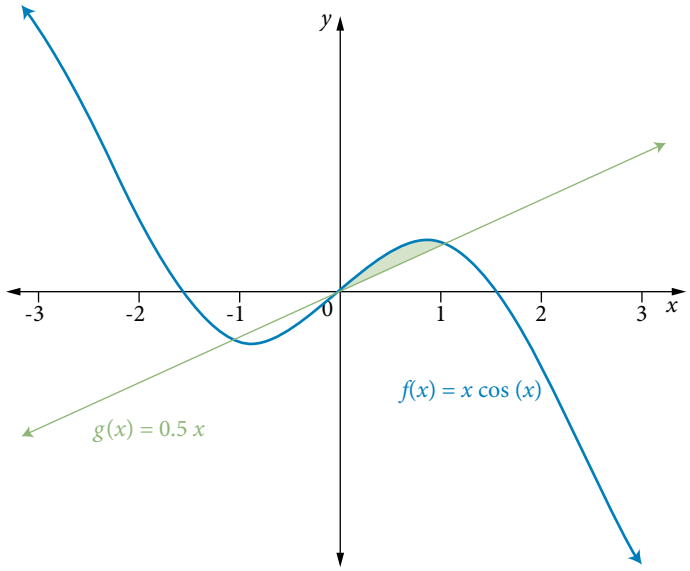
9 The graphs of $y = \cos^2(x)$ and $y = \sin^2(x)$ are shown for $0 \leq x \leq 2\pi$.

(a) Determine the exact area of the shaded region.

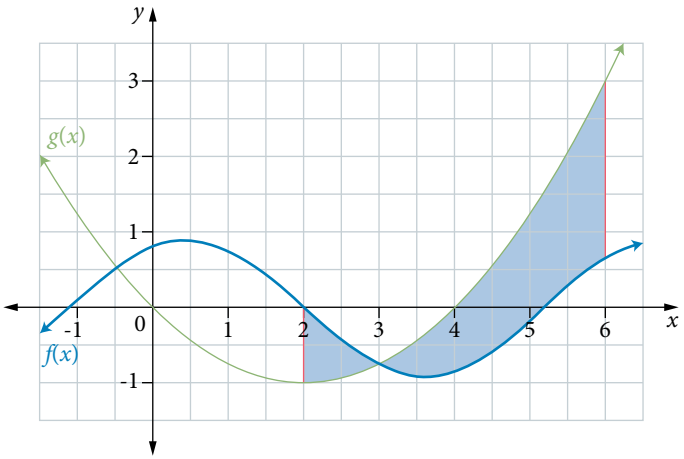
(b) Deduce an expression for the area enclosed by these two functions for $0 \leq x \leq 2\pi n$, where n is an integer.



10 The graphs of $f(x) = x\cos(x)$ and $g(x) = \frac{1}{2}x$ are shown. Determine the exact area of the shaded region.



11 Which of the following is a valid expression for the shaded area in the diagram?



A $\int_2^6 g(x) - f(x)\,dx$

B $\left|\int_2^6 g(x) - f(x)\,dx\right|$

C $\int_2^3 g(x) - f(x)\,dx + \int_3^6 f(x) - g(x)\,dx$

D $\int_2^3 f(x) - g(x)\,dx + \int_3^6 g(x) - f(x)\,dx$

12 Functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, where $a < b$ (and there are no other points of intersection). $f(x)$ lies above the x-axis and $g(x)$ lies below the x-axis for $a < x < b$.

(a) Which of the following is equal to the area bounded by the functions?

A $\int_a^b f(x) - g(x)\,dx$

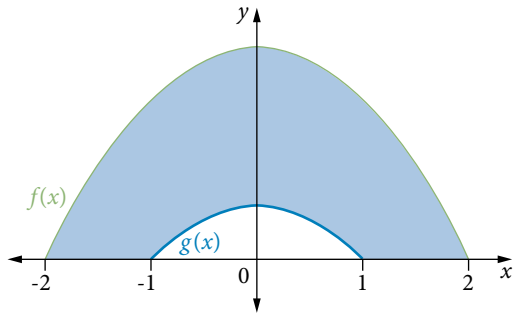
B $\int_a^b f(x) + g(x)\,dx$

C $-\int_a^b f(x) + g(x)\,dx$

D $\int_a^b g(x) - f(x)\,dx$

(b) Explain the common misconception made by a student who chose the first of the incorrect options in part **(a)**.

13 The shaded area in the diagram is given by:



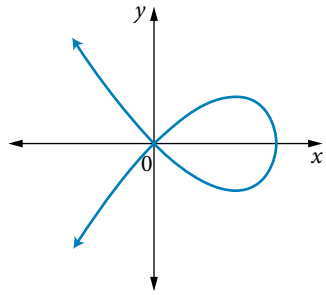
A $\int_{-2}^{2} f(x) - g(x)\,dx$

B $\int_{-1}^{1} f(x) - g(x)\,dx$

C $\int_{-1}^{1} g(x) - f(x)\,dx$

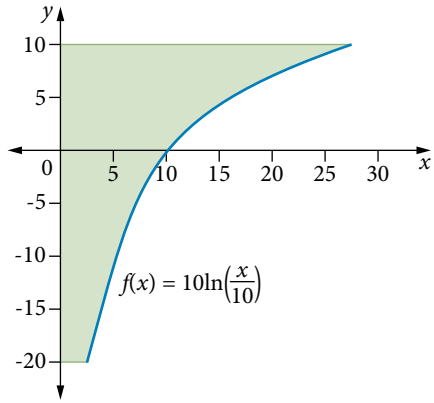
D $\int_{-2}^{2} f(x)\,dx - \int_{-1}^{1} g(x)\,dx$

14 The curve shown has equation $y^2 = x^2(1 - x)$. Determine the exact area of the enclosed region.

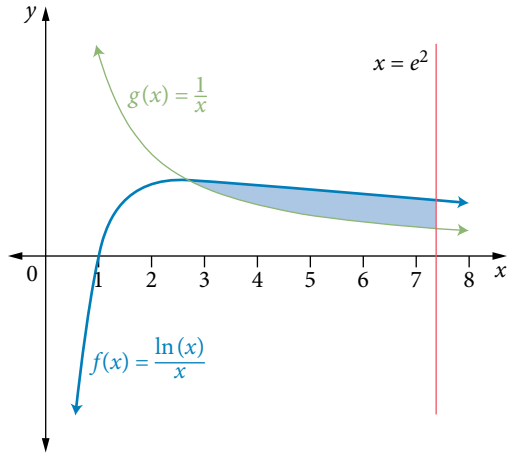


15 The diagram shows the cross-section of a wooden shelf attached to a wall. Units are centimetres.

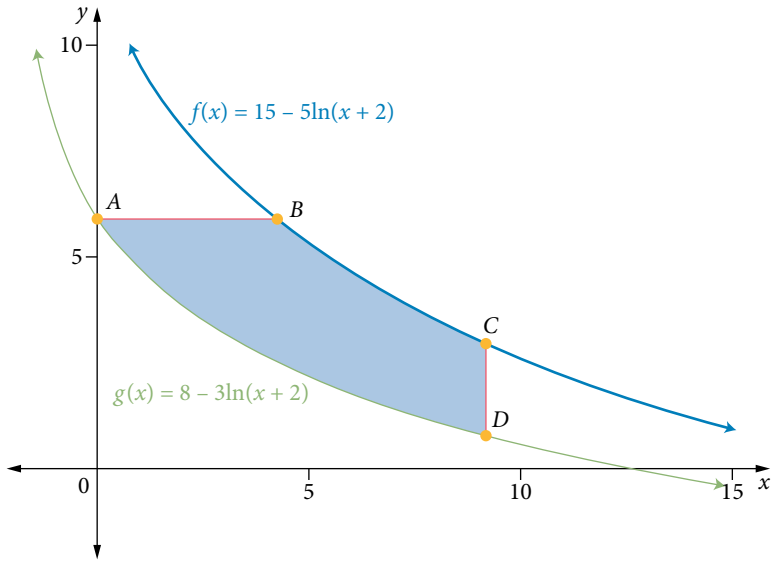
The length of the shelf is 80 cm. If the density of the wood is 0.5 grams per cubic centimetre, determine an exact expression for the mass of the shelf.



16 Determine the exact area of the shaded region in the diagram.



17 The shaded region in the diagram represents a section of a path that must be covered with pavers. (Units are metres.) The edges of the path are formed by the functions $f(x) = 15 - 5\ln(x+2)$ and $g(x) = 8 - 3\ln(x+2)$. Lines AB and CD are parallel to the x-axis and y-axis respectively, and point A is the intersection of the function $g(x)$ with the y-axis. If the length of line CD is 2.17 metres, determine the area that must be paved.



18 Let $a > 0$ and consider the functions $f(x) = xe^{ax}$ and $g(x) = x$.

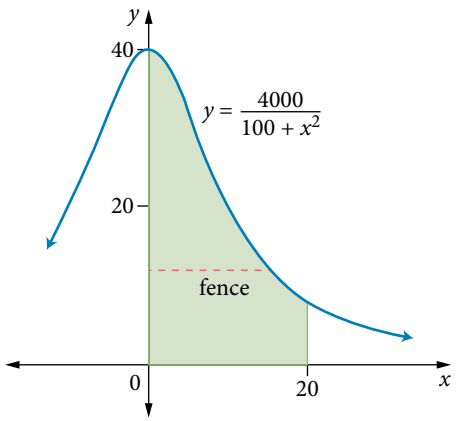
(a) Demonstrate that the only location where these functions intersect is the origin.

(b) Demonstrate that for $x > 0$, $f(x) > g(x)$.

(c) Determine the value of a, given that the area between these functions for $0 \leq x \leq 1$ is equal to 1 square unit.

19 The diagram shows a region of land bounded by the coordinate axes, the line $x = 20$ and the function $y = \dfrac{4000}{100 + x^2}$. The region is divided into two smaller regions of equal area by a fence.

Determine the length of the fence correct to 2 decimal places. Units are metres. Also evaluate the reasonableness of your solution by comparing your result to what you would expect the approximate length of the fence to be based on the diagram.



20 Consider the line $y = mx$ and the curve $y = \dfrac{x}{x^2 + 1}$, and suppose that the line and curve enclose a region in the first quadrant.

(a) Demonstrate that $0 < m < 1$.

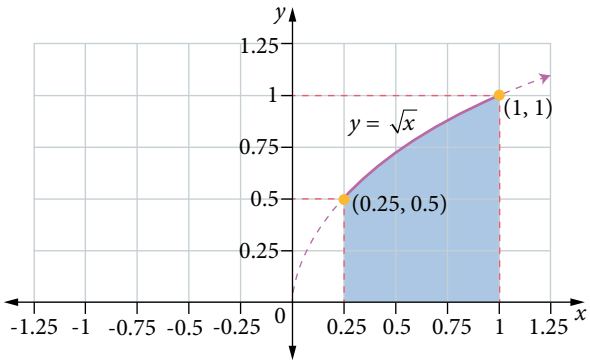
(b) Demonstrate that the y-coordinate of the curve is always greater than or equal to the y-coordinate of the line for any x-value corresponding to the enclosed region.

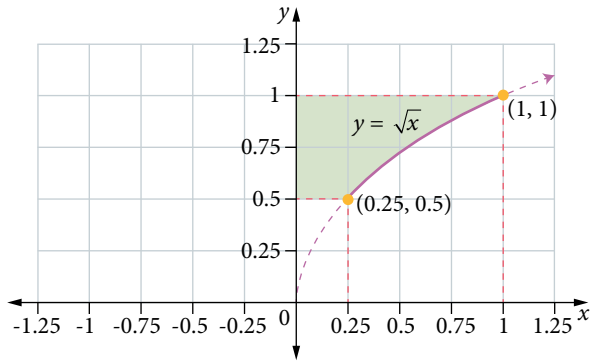
(c) Demonstrate that the area of the region is equal to $\dfrac{1}{2}\ln\left(\dfrac{1}{m}\right) - \dfrac{(1-m)^2}{2m}$.

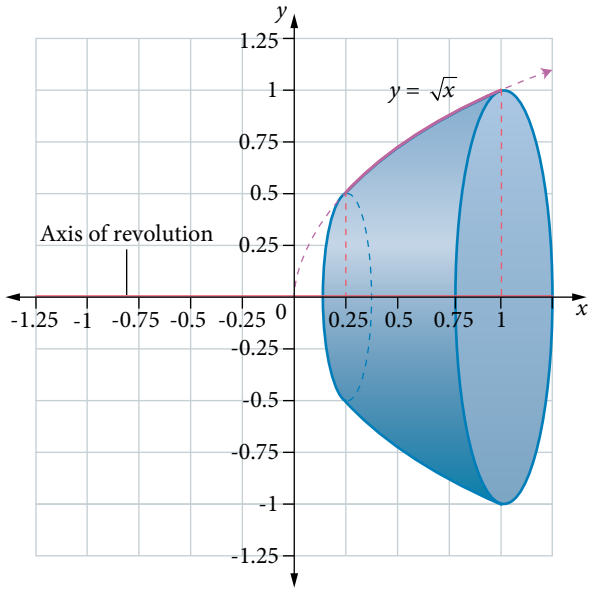
6.2 Volumes of solids of revolution

A solid of revolution is the solid formed by revolving a plane region (area) about a horizontal or vertical line called the *axis of revolution* that does not pass through the plane. In most cases this axis is one of the coordinate axes.

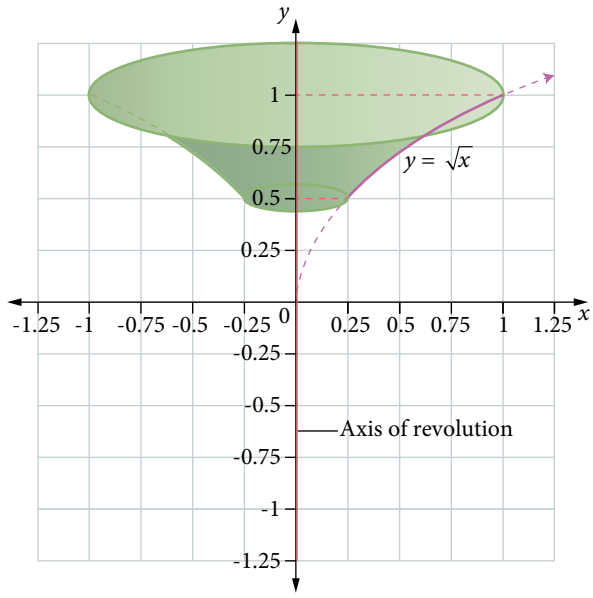
For example, the figures below illustrate the solids formed when the part of the function $y = \sqrt{x}$ between $(0.25, 0.5)$ and $(1,1)$ is rotated about each axis.







Solid of revolution formed when $y = \sqrt{x}$ is rotated about x-axis



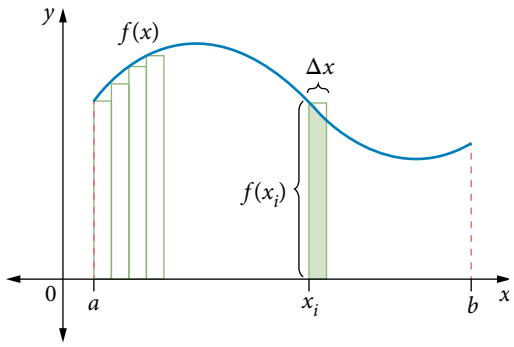
Solid of revolution formed when $y = \sqrt{x}$ is rotated about y-axis

The volume of a solid of revolution can be determined using integration. The two cases of rotating a function about the x-axis and y-axis will be considered separately.

Volumes of solids of revolution about the *x*-axis

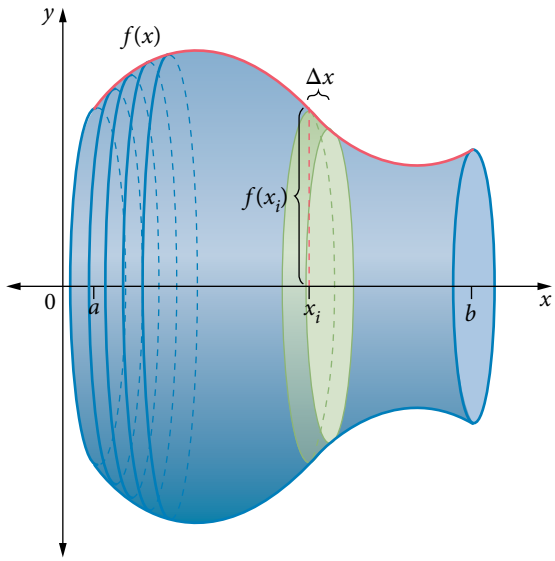
The problem of determining the volume of the solid generated by rotating the section of the curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is closely related to the problem of determining the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$. Recall from Mathematical Methods that the area problem may be solved as follows:

First imagine dividing the interval $[a, b]$ into a long sequence of equally spaced values, $x_1, x_2, x_3, \ldots x_n$ with $x_1 = a$ and $x_n = b$, and where the distance between consecutive values is Δx. For each x_i, create a thin rectangle of thickness Δx and height $f(x_i)$. The area of such a rectangle is given by the product $f(x_i) \times \Delta x$.



By adding all these products together and making Δx infinitesimal (as $\Delta x \to 0$), you obtain the required area. This special sum of products of the form $f(x_i) \times \Delta x$ is equal to the definite integral $\int_a^b f(x)\,dx$.

Now consider the problem of determining the volume of revolution. Once, again, imagine dividing the interval $[a, b]$ into a long sequence of equally spaced values, $x_1, x_2, x_3, \ldots x_n$ with $x_1 = a$ and $x_n = b$, and where the distance between consecutive values is Δx. This time, for each x_i, create a thin cylinder or disk of radius $f(x_i)$, and thickness Δx. The volume of such a cylinder is given by the product $\pi\left(f(x_i)\right)^2 \times \Delta x$. (Recall that the volume of a cylinder is $\pi r^2 h$.)



By adding all these products and making Δx infinitesimal (as $\Delta x \to 0$), you obtain the required volume of revolution. Just as the sum of the products of the form $f(x_i) \times \Delta x$ is equal to $\int_a^b f(x)\,dx$, the sum of products of the form $\pi\left(f(x_i)\right)^2 \times \Delta x$ is equal to $\int_a^b \pi\left(f(x)\right)^2 dx$, or equivalently, $\pi\int_a^b y^2\,dx$.

The volume of the solid generated by rotating the section of the curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is given by $V = \pi\int_a^b y^2\,dx$.

While it is common to speak of 'rotating a curve or function' about an axis, it is actually the region between the curve and the axis that you should imagine is being rotated to produce the solid. For this reason, when applying the previous rule, it is helpful to shade the area being rotated in order to better visualise the resulting solid of revolution. This is demonstrated in Worked example 5.

Making connections

Volumes of solids of revolution about the *x*-axis

Move the sliders to see how a solid of revolution is generated.

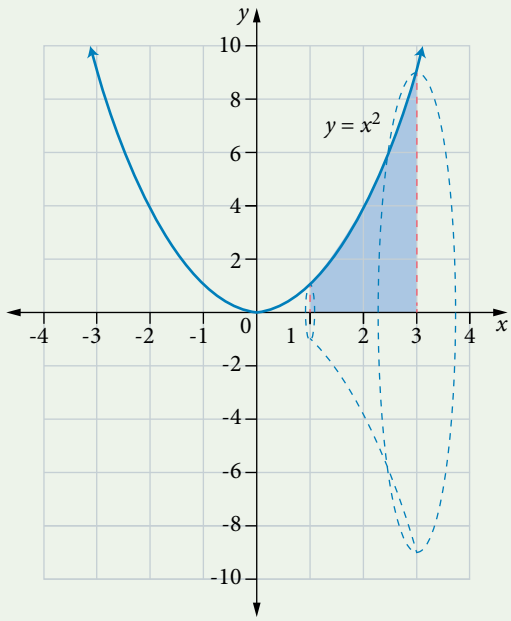
5 Determining the volume of a solid of revolution about the *x*-axis

Determine the volume of the solid of revolution when the curve $y = x^2, 1 \leq x \leq 3$ is rotated about the x-axis.

THINKING

1 Sketch the curve and shade the region to be rotated.

WORKING



2 Use the formula for the volume of a solid of revolution about the x-axis to determine the required volume.

$$\begin{aligned} V &= \pi \int_1^3 y^2 \, dx \\ &= \pi \int_1^3 \left(x^2\right)^2 dx \\ &= \pi \int_1^3 x^4 \, dx \\ &= \pi \left[\frac{1}{5}x^5\right]_1^3 \\ &= \frac{\pi}{5}\left(3^5 - 1^5\right) \\ &= \frac{242\pi}{5} \text{ units}^3 \end{aligned}$$

Volumes of solids of revolution about the *y*-axis

Similar reasoning to that used to establish the formula for the volume of a solid of revolution about the x-axis may be used to establish the result for the volume of a solid of revolution about the y-axis.

The volume of the solid generated by rotating the section of the curve $x = g(y)$ from $y = a$ to $y = b$ about the y-axis is given by $V = \pi \int_a^b x^2 \, dy$.

Note that applying this rule requires the equation of the curve to be in the form $x = g(y)$, as Worked example 6 illustrates.

6 Determining the volume of a solid of revolution about the *y*-axis

Determine the volume of the solid of revolution when the curve $y = x^2 + 2$, $2 \leq y \leq 4$, $x \geq 0$, is rotated about the *y*-axis.

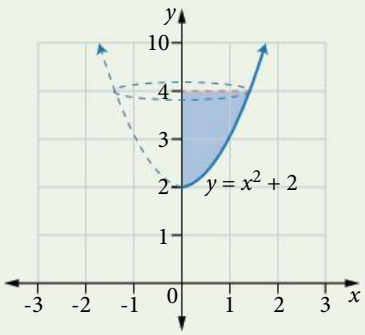
THINKING

1 Sketch the curve and shade the region to be rotated.

2 Express x as a function of y for the curve.

3 Use the formula for the volume of a solid of revolution about the *y*-axis to determine the required volume.

WORKING



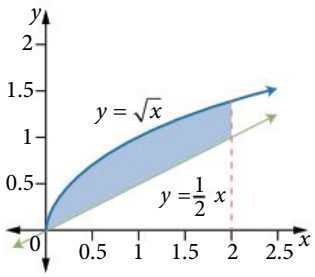
$$y = x^2 + 2$$
$$y - 2 = x^2$$
$$x = \sqrt{y-2} \quad (\text{for } x \geq 0)$$

$$V = \pi \int_2^4 x^2 \, dy$$
$$= \pi \int_2^4 \left(\sqrt{y-2}\right)^2 dy$$
$$= \pi \int_2^4 y - 2 \, dy$$
$$= \pi \left[\frac{1}{2}y^2 - 2y\right]_2^4$$
$$= \pi\left(8 - 8 - (2 - 4)\right)$$
$$= 2\pi \text{ units}^3$$

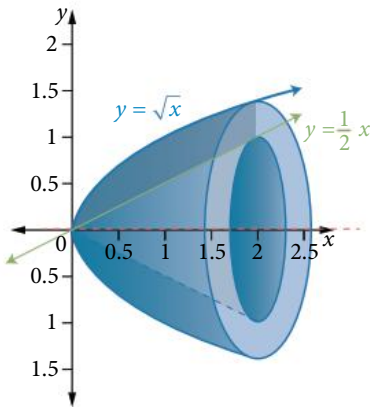
Volumes formed by rotating a region between two curves

A region between two curves may be rotated about one of the coordinate axes to produce a solid with a hollowed-out portion as shown.

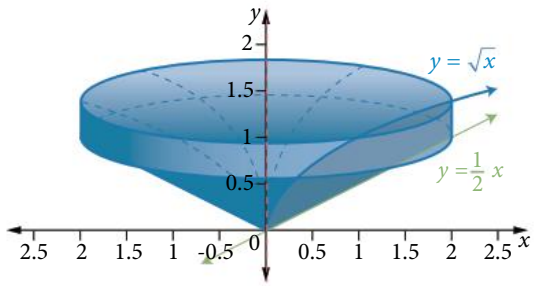
Shaded region between two curves



Rotate shaded region about *x*-axis



Rotate shaded region about *y*-axis

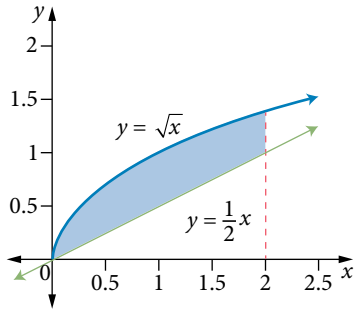


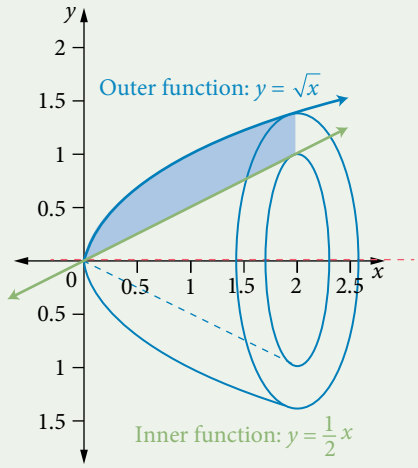
Notice that the hollowed-out portion is actually the volume obtained by rotating the 'inner curve' (the one closer to the axis of rotation) about the x-axis or y-axis, as shown separately in the previous graphs. Thus, the required volume may be obtained by subtracting this hollowed-out volume from the volume obtained by rotating the 'outer curve' (the one furthest from the axis of rotation) about the axis of rotation.

The volume of a solid formed by rotating a *region between two curves* about an axis can be obtained by subtracting the volume of revolution of the inner curve (the one closest to the axis of rotation) from the volume of revolution of the outer curve.

7 Determining the volume of a solid of revolution generated by a region between curves

Determine the volume of the solid generated when the region between the curves $y = \sqrt{x}$ and $y = \frac{1}{2}x$ from $x = 0$ to $x = 2$ (shown in the diagram) is rotated about the x-axis.



THINKING	WORKING
1 Determine which is the 'outer' function and which is the 'inner' function relative to the axis of rotation.	
2 Subtract the volume of revolution of the inner function from that of the outer function.	$V = \pi\int_0^2 \left(\sqrt{x}\right)^2 dx - \pi\int_0^2 \left(\frac{1}{2}x\right)^2 dx$

3 Combine into a single definite integral and integrate.

$$\begin{aligned} V &= \pi\int_0^2 \left(\sqrt{x}\right)^2 - \left(\frac{1}{2}x\right)^2 dx \\ &= \pi\int_0^2 x - \frac{1}{4}x^2\, dx \\ &= \pi\left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^2 \\ &= \pi\left(\frac{1}{2}\times 2^2 - \frac{1}{12}\times 2^3 - \left(\frac{1}{2}\times 0^2 - \frac{1}{12}\times 0^3\right)\right) \\ &= \pi\left(2 - \frac{2}{3}\right) \\ &= \frac{4\pi}{3}\text{ units}^3 \end{aligned}$$

4 Interpret the answer.

The volume of the solid generated when the region between the curves $y = \sqrt{x}$ and $y = \frac{1}{2}x$ from $x = 0$ to $x = 2$ is rotated about the x-axis is $\frac{4\pi}{3}$ units^3.

Notice in Worked example 7 that the required volume, $\pi\int_0^2 \left(\sqrt{x}\right)^2 dx - \pi\int_0^2 \left(\frac{1}{2}x\right)^2 dx$, was written as a single integral, $\pi\int_0^2 \left(\sqrt{x}\right)^2 - \left(\frac{1}{2}x\right)^2 dx$. This can always be done to simplify calculations.

WARNING

It is a common misconception that an expression such as $\pi\int_0^2 \left(\sqrt{x}\right)^2 dx - \pi\int_0^2 \left(\frac{1}{2}x\right)^2 dx$ is equivalent to $\pi\int_0^2 \left(\sqrt{x} - \frac{1}{2}x\right)^2 dx$. It is not, as $a^2 - b^2 \neq (a-b)^2$.

EXERCISE 6.2 Volumes of solids of revolution

Worked Example

1 Determine the volume of the solid of revolution when the following curves are rotated about the x-axis. 5

(a) $y = 2x^3,\ 0 \leq x \leq 1$

(b) $y = -e^{3x},\ 0 \leq x \leq \frac{1}{2}$

(c) $y = \sqrt{x},\ 1 \leq x \leq 3$

(d) $y = \cos(x),\ 0 \leq x \leq \frac{\pi}{3}$.

2 Determine the volume of the solid of revolution when the following curves are rotated about the y-axis. 6

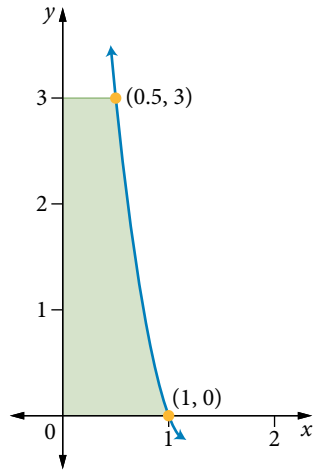
(a) $y = x^2 - 1,\ -1 \leq y \leq 8,\ x \geq 0$

(b) $y = 3 - \frac{1}{2}x,\ 0 \leq y \leq 2$

(c) $y = \frac{3}{x+1},\ 1 \leq y \leq 2$

(d) $y = \sqrt{2x} + 1,\ 1 \leq y \leq 2$

3 The section of the curve $y = \frac{1}{x^2} - 1$ between the points $(0.5, 3)$ and $(1, 0)$ is rotated about the y-axis. Determine the exact volume generated.



4 Determine the volume of the solid generated when the section of the curve $y = x^2$ from $x = 2$ to $x = 3$ is rotated about the y-axis.

5 Determine the volume of the solid generated when the section of the curve $y = \arcsin(x)$ from $x = 0$ to $x = 1$ is rotated about the y-axis.

6 Determine the volume of the solid generated when the region bounded by the curve $y = x(x - 2)$ and the x-axis is rotated about the x-axis.

7 The shape and size of an egg laid by a particular species of bird can be modelled approximately by rotating the region bounded by the graph of $y = \frac{3}{5}\sqrt{25 - x^2}$ and the x-axis about the x-axis. (Units are cm.) Determine the volume of the egg in cubic centimetres correct to 1 decimal place.

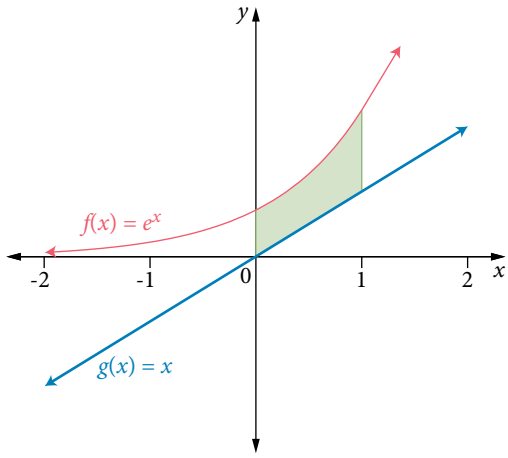
8 Determine the volume generated when the region bounded by the curve $y = e^x$, the y-axis and the line $y = 5$ is rotated about the y-axis. Express your answer correct to 2 decimal places.

9 Determine the volume of the solid generated when the region in the first quadrant bounded by the curve $y = 4 - x^2$ and the coordinate axes is rotated about:

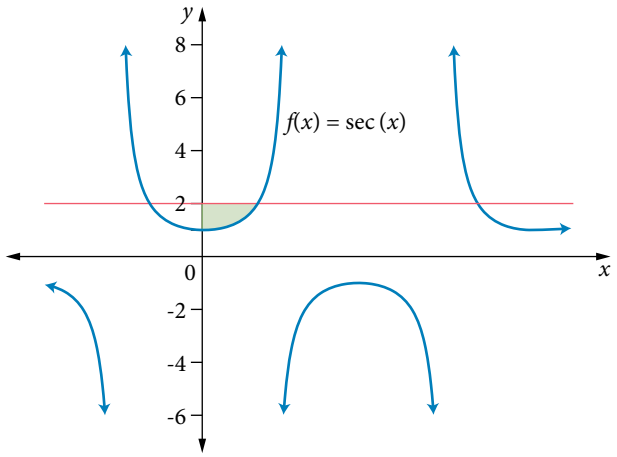
(a) the x-axis. **(b)** the y-axis.

Worked Example

7 **10** Determine the volume of the solid generated when the region between the curves $y = e^x$ and $y = x$ from $x = 0$ to $x = 1$ (shown in diagram) is rotated about the x-axis.



11 Determine the volume of the solid generated when the region enclosed by the curve $f(x) = \sec(x)$, the line $y = 2$ and the y-axis (shown in the diagram) is rotated about the x-axis.



12 Determine the volume of the solid generated when the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ is rotated about the x-axis.

13 Suppose $f(x) \geq g(x)$ for $a \leq x \leq b$.

(a) The volume generated by rotating the region between $f(x)$ and $g(x)$ from $x = a$ to $x = b$ about the x-axis is given by:

A $\pi \int_a^b f(x) - g(x)\,dx$

B $\pi \int_a^b \left(f(x) - g(x)\right)^2 dx$

C $\pi \int_a^b \left(f(x)\right)^2 - \left(g(x)\right)^2 dx$

D $\pi \int_a^b \left(g(x)\right)^2 - \left(f(x)\right)^2 dx$

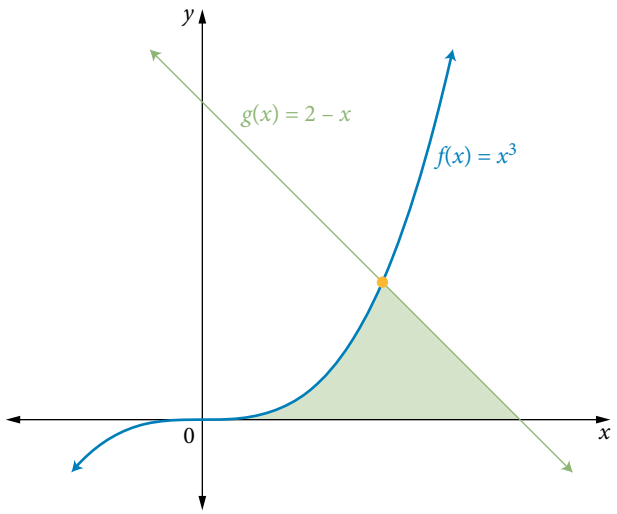
(b) Explain the common misconception related to the second incorrect option in part **(a)**.

14 Determine the volume, correct to 3 decimal places, generated by rotating the region bounded by $y = e^{0.5x}$ and $y = 2x$ about:

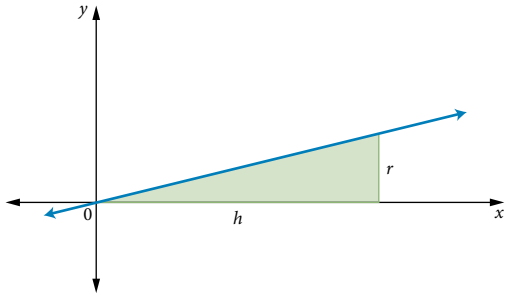
(a) the x-axis.

(b) the y-axis.

15 Determine the volume, correct to 3 decimal places, generated by rotating the shaded area in the diagram about the x-axis.



16 Determine the volume, correct to 3 decimal places, generated by rotating the shaded area in the diagram about the x-axis.



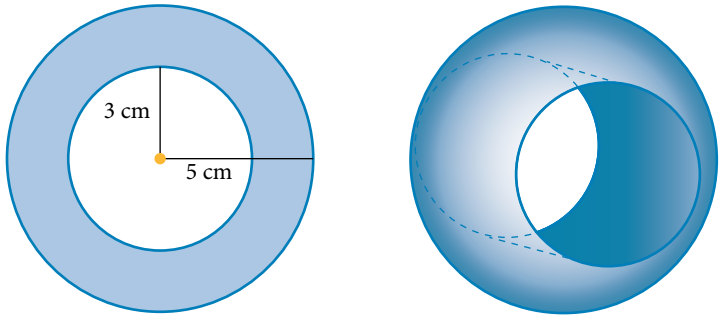
17 By determining a suitable volume of a solid of revolution, prove that the volume of a cone with radius r and height h is equal to $V = \frac{1}{3}\pi r^2 H$.

18 By determining a suitable volume of a solid of revolution, prove that the volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

19 A fruit bowl of volume 12π cubic units is designed by rotating the part of the graph of $y = \frac{1}{4}x^2$ in the first quadrant from $y = 0$ to $y = a$, where $a > 0$, about the y-axis. Determine the value of a.

20 Let $c > 0$ and consider the part of the curve $y = x^3$ from the point with x-coordinate 0 to the point with x-coordinate c. Determine the value of c if the volume of the solid formed by rotating this section of the curve about the x-axis is equal to the volume of the solid formed by rotating this section of the curve about the y-axis.

21 A solid sphere of radius 5 cm has a circular hole of radius 3 cm drilled through the centre along a diameter of the sphere, as shown. Determine the volume remaining.



22 The curve with equation $y = \sqrt{4 - 3x^2}$ is cut by a straight line of gradient -1. The curve and the straight line have the same y-intercept. Calculate the volume generated when the region bounded by the curve and the straight line is rotated about the y-axis.

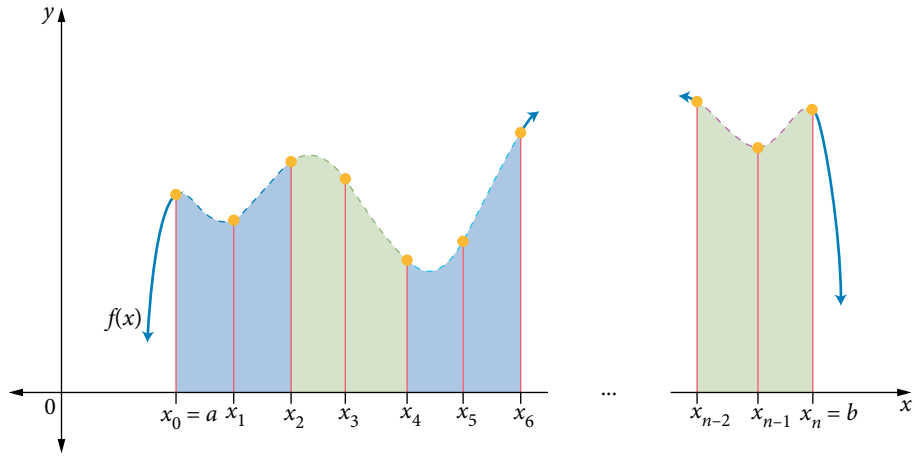
23 Determine the volume obtained when the region bound by the curve $y = x(6 - x)$ and the x-axis is rotated about the y-axis. Express your answer correct to 1 decimal place.

Numerical integration using Simpson's rule 6.3

In Mathematical Methods, you learnt how to use the trapezoidal rule to approximate the values of definite integrals. The trapezoidal rule is an example of a numerical integration technique. Such techniques are useful in determining approximations of definite integrals to specified levels of accuracy when either the indefinite integral cannot be readily determined or when the function to be integrated is only known at certain values.

In this section you will learn a technique for numerical integration known as Simpson's rule. Simpson's rule is known to be considerably more accurate than the trapezoidal rule, and is often the technique used by calculators to calculate definite integrals.

Simpson's rule for approximating a definite integral of the form $\int_a^b f(x)\,dx$ involves approximating the function $f(x)$ by a series of parabolas, each of which passes through three points of the original function, as shown below.

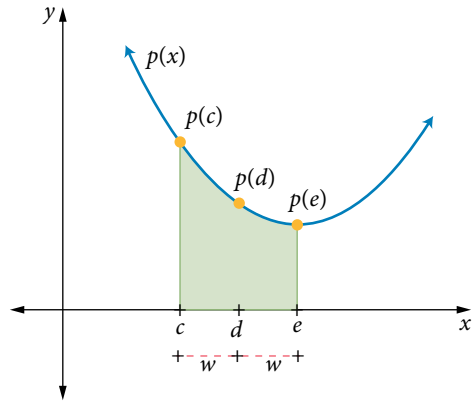


The values $x_0, x_1, x_2, \ldots x_n$ are chosen to divide the main interval $[a, b]$ into n equally spaced sub-intervals. Notice that since each parabola spans two sub-intervals, the value of n must be chosen to be an even number.

If $p(x)$ is a parabola and c, d, and e are equally spaced x-values with w representing the width between consecutive values (as shown), then:

$$\int_c^e p(x) = \frac{w}{3}(p(c) + 4p(d) + p(e))$$

(A proof of this result is a challenging problem in Exercise 6.3.)



Applying this result to each parabola in the series allows you to obtain an approximation for the value of $\int_a^b f(x)dx$:

$$\begin{aligned}\int_a^b f(x)dx &= \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \int_{x_4}^{x_6} f(x)dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)dx \\ &\approx \frac{w}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) \\ &+ \frac{w}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right) \\ &+ \frac{w}{3}\left(f(x_4) + 4f(x_5) + f(x_6)\right) \\ &+ \ldots \\ &+ \frac{w}{3}\left(f(x_{n-3}) + 4f(x_{n-2}) + f(x_{n-1})\right) \\ &+ \frac{w}{3}\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right) \\ &= \frac{w}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \ldots + 4f(x_{n-1}) + f(x_n)\right)\end{aligned}$$

This final expression may be written more succinctly as follows:

$$\int_a^b f(x) \approx \frac{w}{3}\sum_{i=1}^{n} f(x_i) \times M_i \quad \text{where } M_i = \begin{cases} 1 & i = 0 \text{ or } i = n \\ 4 & i = 1,3,5,\ldots, n-1 \\ 2 & i = 2,4,6,\ldots, n-2 \end{cases}$$

The values of M_i may be thought of as *multipliers*. The first and last multipliers are 1, while the inner multipliers alternate between 4 and 2.

In summary:

Simpson's rule:

When n is an even positive integer and $x_0, x_1, \ldots, x_n$ partition the interval $[a, b]$ into n equally spaced subintervals with the width of each subinterval being w, an approximation for the definite integral $\int_a^b f(x)dx$ is:

$$\int_a^b f(x)dx \approx \frac{w}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \ldots + 4f(x_{n-1}) + f(x_n)\right)$$

Equivalently,

$$\int_a^b f(x) \approx \frac{w}{3}\sum_{i=1}^{n} f(x_i) \times M_i \quad \text{where } M_i = \begin{cases} 1 & i = 0 \text{ or } i = n \\ 4 & i = 1, 3, 5,\ldots, n-1 \\ 2 & i = 2, 4, 6,\ldots, n-2 \end{cases}$$

Simpson's rule is most conveniently implemented by constructing a table containing the values of x_i, $f(x_i)$, and M_i as Worked example 8 illustrates. Note that the expression $\frac{w}{3}\sum_{i=1}^{n} f(x_i) \times M_i$ will often be written as simply $\frac{w}{3}\sum f(x) \times M$ for convenience.

8 Using Simpson's rule to approximate a definite integral

Use Simpson's rule with $n = 6$ subintervals to approximate $\int_1^{2.2} \frac{1}{x}\,dx$. Express your answer correct to 3 decimal places.

THINKING

1 Define the function being integrated.

2 Determine the subinterval width using the formula $w = \frac{b-a}{n}$.

3 Construct a table with x-values from a to b, increasing by w. Determine the values of $f(x)$ for each x-value. Determine the appropriate multiplier, M corresponding to each x-value. Remember that the multipliers follow the pattern: 1, 4, 2, 4, 2, 4, …, 2, 4, 1

4 Apply Simpson's rule using the previously calculated values to determine the approximation of the definite integral.

WORKING

Let $f(x) = \frac{1}{x}$.

$$w = \frac{2.2 - 1}{6}$$
$$= 0.2$$

x	$f(x)$	M
1	1	1
1.2	0.8333	4
1.4	0.7143	2
1.6	0.625	4
1.8	0.5556	2
2	0.5	4
2.2	0.4545	1

$$\int_1^{2.2} f(x)\,dx \approx \frac{0.2}{3}\sum f(x) \times M$$
$$= \frac{0.2}{3}\left(1 \times 1 + 0.8333 \times 4 + 0.7143 \times 2 + \ldots + 0.4545 \times 1\right)$$
$$= \frac{0.2}{3}(11.8275\ldots)$$
$$= 0.789 \ \ (3\text{d.p.})$$

The approximation for $\int_1^{2.2} \frac{1}{x}\,dx$ calculated in Worked example 8 differs from the actual value of this integral by less than 0.000 05. Simpson's rule, in general, gives much better approximations than using rectangles or trapeziums. It can be shown that Simpson's rule produces the exact result for definite integrals in which the integrand is a polynomial of degree at most 3.

EXERCISE 6.3 Numerical integration using Simpson's rule

Worked Example 8

1 Use Simpson's rule to approximate each of the following definite integrals using the specified number of subintervals. Express each answer correct to 3 decimal places.

(a) $\int_2^5 2^x \, dx, n = 6$

(b) $\int_1^3 \ln(x) dx, n = 10$

(c) $\int_2^3 -3x \, dx, \ n = 8$

(d) $\int_0^{\pi} \sin(x) dx, n = 4$

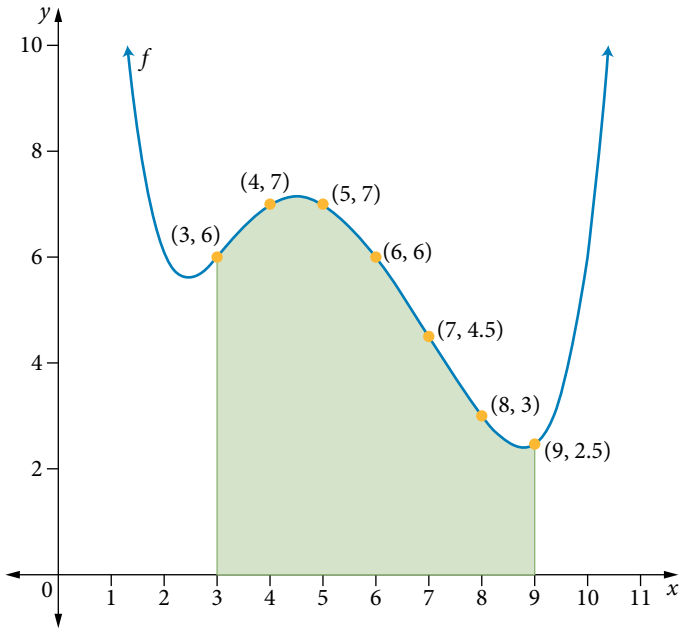
(e) $\int_1^3 \sqrt{x} \, dx, n = 2$

2 Determine the Simpson's rule approximation of $\int_0^2 x^4 dx$ with subintervals of width 0.5, expressing your answer correct to 3 decimal places. Then prove that your result is within 0.02 of the actual value of this integral.

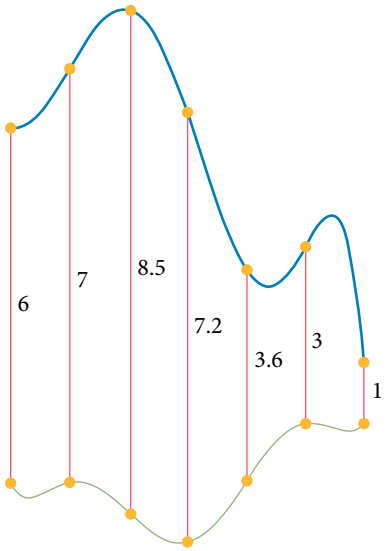
3 Determine the Simpson's rule approximation of $\int_0^{0.6} \frac{1}{\sqrt{1-x^2}} dx$ with $n = 6$ subintervals, expressing your answer correct to 6 decimal places. Then prove that your result is within 0.000 02 of the actual value of this integral.

4 Use Simpson's rule with $n = 4$ to determine an approximation for the area between the x-axis and the function $f(x) = x^x$ from $x = 1$ to $x = 2$. Evaluate the reasonableness of your approximation.

5 Use Simpson's rule to determine an approximation of the shaded area shown.



6 A landscaper wishes to cover an irregularly shaped section of land in turf and has taken measurements (in metres) of the length of the land at regular intervals 2 metres apart:

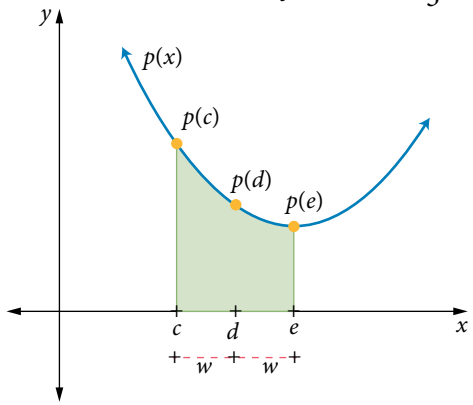


If turf costs \$7 per square metre, use Simpson's rule to determine the approximate total cost.

7 Let I_n be the Simpson's rule approximation for $\int_1^2 \frac{1}{x}\,dx$ using n intervals. It can be shown that $\left|I_n - \int_1^2 \frac{1}{x}\,dx\right| \leq \frac{2}{15n^4}$. Determine the number of intervals required in order for the Simpson's rule approximation to be within 0.0001 of the true value of this integral. Evaluate the reasonableness of your result.

8 Prove that using Simpson's rule with $n = 2$ produces the exact value of $\int_a^b x^3\,dx$.

9 Prove that if $p(x)$ is a parabola and c, d and e are equally spaced x-values, with w representing the width between consecutive values (as shown), then $\int_c^e p(x)\,dx = \frac{w}{3}(p(c) + 4p(d) + p(e))$.



6.4 The exponential distribution

In Mathematical Methods, you were introduced to the concept of the random variable—a variable whose values are the possible outcomes of a random process or phenomenon. Random variables may be discrete (where there is only a finite number of possible values) or continuous (where the values comprise all infinitely many real numbers in some interval). Examples of discrete random variables include:

- the number of tails obtained when a coin is tossed 10 times
- the number of times a darts player hits the bullseye in a game
- the number obtained when a die is rolled.

Examples of continuous random variables include:

- the time spent waiting for a bus at a bus stop
- the height of a student at a school
- the distance travelled by a car before it needs repair.

The subsequent discussion focuses on a particular type of continuous random variable known as an *exponentially distributed continuous random variable*.

An *exponentially distributed random variable* is a variable representing the time or distance before the next occurrence of a random event, where such random events occur independently at a constant average rate, λ.

Consider a call centre that receives a call on average every 5 seconds, which can also be written as $\frac{1}{5} = 0.2$ calls per second on average. Let X represent the time before the next call is received. Notice that X represents the time before the next occurrence of a random event (a call being received). Also, note that calls being received occur independently (one call being received does not influence the likelihood of another being received), and that these calls occur at a constant average rate ($\lambda = 0.2$ calls per second). Thus, X is an exponentially distributed random variable.

Closely related to the value of λ for an exponentially distributed random variable is the mean μ.

The mean μ (also known as the expected value) of an exponentially distributed random variable is the average time or distance before the next occurrence of the event.
It is always the case that $\lambda = \frac{1}{\mu}$ (or equivalently $\mu = \frac{1}{\lambda}$).

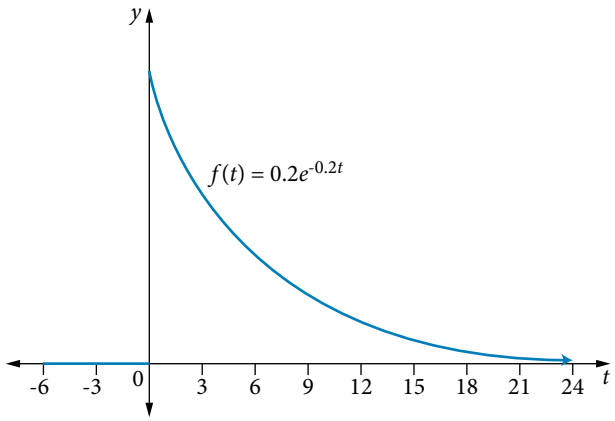
For the call centre example described, $\mu = 5$ seconds and $\lambda = \frac{1}{5} = 0.2$ calls per second.

Any continuous random variable has a special function associated with it known as a *probability density function* (sometimes abbreviated to pdf). Such a function can be used to determine probabilities associated with the variable. For a continuous variable, the probability density function is also known as the *probability distribution*, as it defines how probabilities are distributed across the possible values of the variable.

For an exponentially distributed random variable, the probability density function is given by:

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

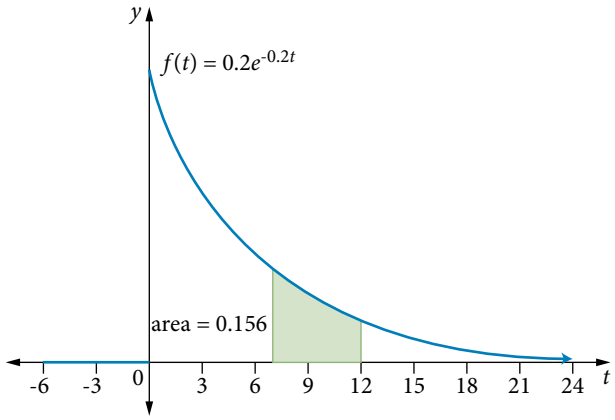
The probability density function for the call centre example is shown:



Probability density functions always have the property that $P(a < X < b) = \int_a^b f(t)\,dt$. In other words, the probability of the variable taking on a value between a and b is equal to the area between the function and the x-axis from $x = a$ to $x = b$.

For example, the probability that the time before the next phone call is received is between 7 and 12 minutes is given by:

$$\begin{aligned} P(7 < X < 12) &= \int_7^{12} 0.2e^{-0.2t}\,dt \\ &\approx 0.156 \end{aligned}$$



A counter-intuitive feature of any continuous random variable is that the probability of it taking on any *particular* value is always zero. For example, in the call centre scenario, the probability that the time before the next call is exactly 7 minutes is zero (since such a probability would correspond to an area under the curve of width zero). Thus, for continuous random variables, it only makes sense to ask about the probability of the variable taking on the values within some interval.

For this reason, it is not necessary to distinguish between the phrases 'less than' and 'less than or equal to' (and likewise between 'greater than' and 'greater than or equal to') when working with continuous random variables. That is, probabilities such as $P(X < 5)$ and $P(X \leq 5)$ are equal since the probability that X is equal to 5 is zero.

WARNING

Although probabilities such as $P(X < 5)$ and $P(X \leq 5)$ are equal when X is a continuous random variable, they are *not* equal when X is a discrete variable.

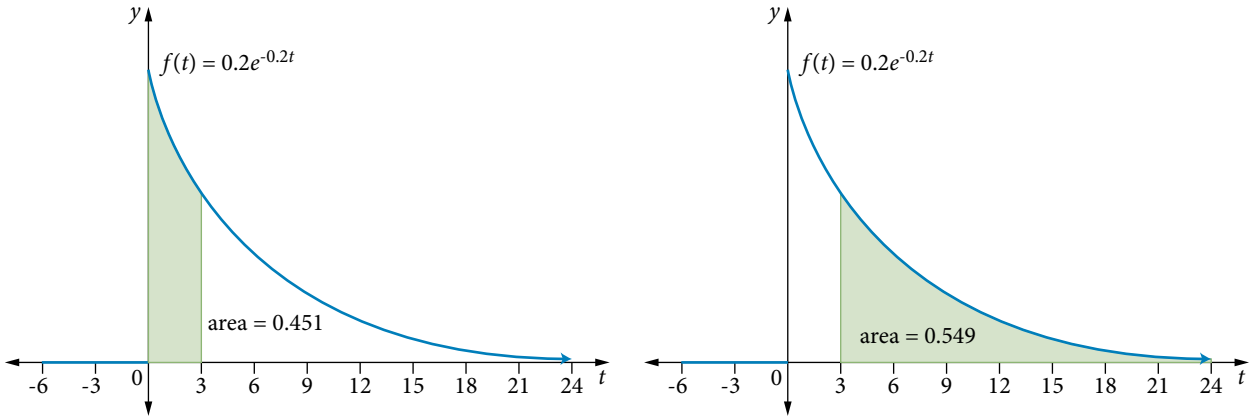
When studying probability and statistics, always pay attention to whether you are working with continuous or discrete variables as different rules apply in these situations.

Returning to the call centre example, consider the probability that the time before the next call is less than 3 minutes. This probability corresponds to the area under the probability density function from $x = 0$ to $x = 3$ and is thus given by:

$$P(X < 3) = \int_0^3 0.2e^{-0.2t}\,dt$$
$$\approx 0.451$$

Similarly, the probability that the time before the next call is greater than or equal to 3 minutes corresponds to the area under the probability density function starting at $x = 3$ and continuing indefinitely to the right. This area can be denoted as $\int_3^\infty 0.2e^{-0.2t}\,dt$, and can be evaluated using technology by replacing the upper limit of ∞ with an arbitrarily chosen large number (such as 1×10^{99}). This is only reasonable as the probability density function approaches the x-axis as x approaches infinity, meaning that as x becomes large, the additional area under the function is negligible. Thus,

$$P(X \geq 3) = \int_0^\infty 0.2e^{-0.2t}\,dt$$
$$\approx 0.549$$



Notice that $P(X \geq 3) + P(X < 3) = 1$. This is to be expected as these are complementary events. This also demonstrates that the total area under the probability density function is equal to 1. This can be shown to be true for any probability density function of the form $f(t) = \lambda e^{-\lambda t}$ as follows:

Let $b > 0$. Then the area under the curve $f(t) = \lambda e^{-\lambda t}$ from $t = 0$ to $t = b$ is given by:

$$\int_0^b \lambda e^{-\lambda t}\,dt$$
$$= \left[-e^{-\lambda t}\right]_0^b$$
$$= -e^{-\lambda b} - -e^0$$
$$= -\frac{1}{e^{\lambda b}} + 1$$

Notice that as t approaches ∞, the first term in the previous expression approaches 0, and so the value of the definite integral approaches 1.

In summary:

For an exponentially distributed random variable X with rate λ, the probability density function is given by:

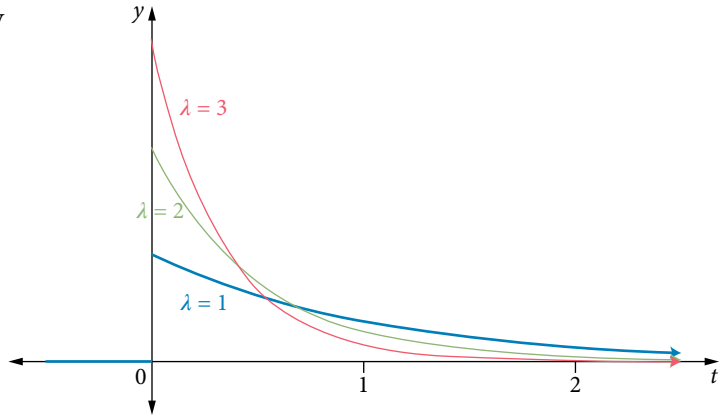
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Probability density functions always have the property that $P(a < X < b) = \int_a^b f(t)\,dt$. In other words, the probability of the variable taking on a value between a and b is equal to the area between the function and the x-axis from $x = a$ to $x = b$. Consequently, the probability of X taking on any particular value is zero (since this would correspond to an area of width zero).

It is not necessary to distinguish between the phrases 'less than' and 'less than or equal to' (and likewise between 'greater than' and 'greater than or equal to') when working with continuous random variables.

The total area under the curve of any probability density function is always 1.

The following diagram shows the probability density function corresponding to random exponentially distributed variables with $\lambda = 1$, 2 and 3.



Notice that an area close to $\lambda = 0$ will be larger for a higher value of λ. A higher value of λ (the rate of events per unit of time) means that the random events occur more frequently, and so it is more likely that the next occurrence of the event will occur sooner.

9 Determining probabilities associated with exponentially distributed variables

Let X represent the time before the next bus arrives at a bus stop. Suppose that X is an exponentially distributed random variable, and that on average, a bus arrives every 8 minutes.

(a) Determine the mean μ, the rate λ, and the probability density function for X.

THINKING	WORKING
1 Determine the mean.	$\mu = 8$ minutes
2 Use the formula $\lambda = \frac{1}{\mu}$ to determine the rate λ.	$\lambda = \frac{1}{\mu}$ $= \frac{1}{8}$ $= 0.125$ buses per minute
3 The pdf is given by $f(t) = \lambda e^{-\lambda t}$.	$f(t) = 0.125e^{-0.125t}$

(b) Determine the probability, correct to 3 decimal places, that the next bus will arrive within the next 3 minutes.

Use the rule $P(a < X < b) = \int_a^b f(t)\,dt$ to express the required probability as a definite integral, then use technology to calculate to 3 decimal places. Note that a probability of the form $P(X < b)$ is given by $\int_0^b f(t)\,dt$.

$$P(X < 3) = \int_0^3 0.125e^{-0.125t}\,dt$$
$$\approx 0.313 \quad \text{(using technology)}$$

(c) Determine the probability, correct to 3 decimal places, that the next bus doesn't arrive within the next 15 minutes.

Use the rule $P(a < X < b) = \int_a^b f(t)\,dt$ to express the required probability as a definite integral, then use technology to calculate to 3 decimal places. Note that a probability of the form $P(X > a)$ is given by $\int_a^\infty f(t)\,dt$, which is equal to $1 - \int_0^a f(t)\,dt$ since the total area under the pdf is 1.

$$P(X > 15) = \int_{15}^\infty 0.125e^{-0.125t}\,dt$$
$$= 1 - \int_0^{15} 0.125e^{-0.125t}\,dt$$
$$\approx 0.153 \quad \text{(using technology)}$$

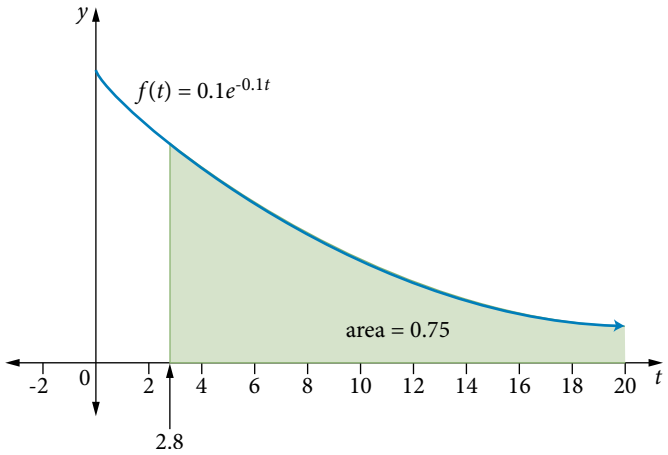
Note that in Worked example 9 part **(c)**, the integral $\int_{15}^\infty 0.125e^{-0.125t}\,dt$ could have been calculated directly using technology as described earlier. However, $\int_{15}^\infty 0.125e^{-0.125t}\,dt$ was rewritten as $1 - \int_0^{15} 0.125e^{-0.125t}\,dt$ to avoid dealing with the ∞ integration limit. This is essentially applying the complement rule for probabilities: $P(A') = 1 - P(A)$. In this case, the complement of the event $X > 15$ is $X \leq 15$, which corresponds to the area under the probability density function from 0 to 15.

Quantiles

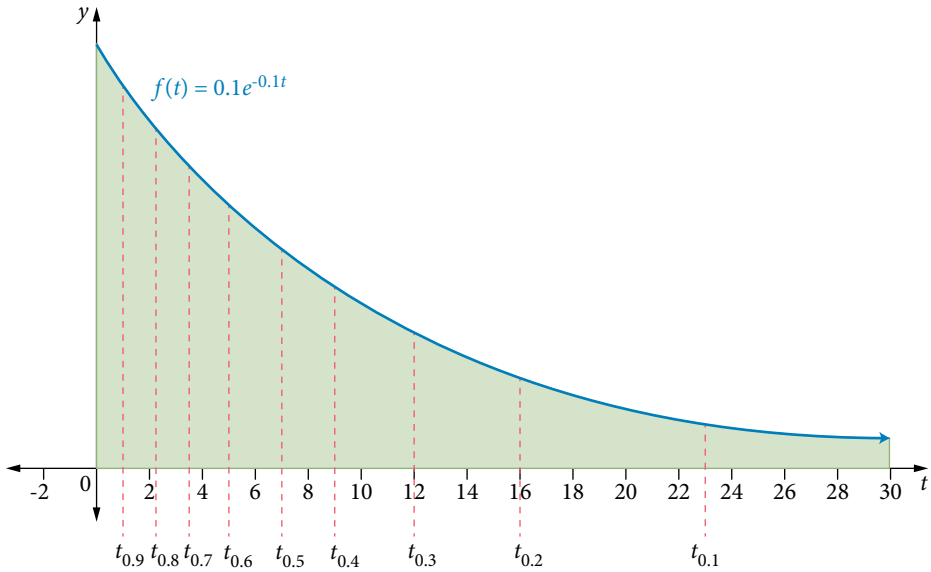
When working with continuous random variables, as well as being able to calculate probabilities (such as $P(X > 2.5)$), it is useful to be able to reverse this process – in other words, to have a particular probability in mind (such as 0.1) and determine the value of x for which $P(X > x) = 0.1$. Such a value of x is called a quantile. More formally:

Given a particular probability, α, the quantile for the continuous random variable X corresponding to α is defined to be the value t_α such that $P(X > t_\alpha) = \alpha$.

To illustrate, consider the diagram showing the probability density function $f(t) = 0.1e^{-0.1t}$. It can be seen in the diagram that the probability that X is greater than 2.8 is 0.75. Therefore $t_{0.75} = 2.8$.



The following diagram shows the quantiles $t_{0.1}, t_{0.2}, \ldots t_{0.9}$ for the probability density function $f(t) = 0.1e^{-0.1t}$. These quantiles are known as *deciles* as they partition the possible values of the random variable into 10 intervals, each having equal probability. Other common types of quantiles are *percentiles* (100 intervals) and *quartiles* (4 intervals).



The most important quantile is $t_{0.5}$ as it represents the value above (and below) that there is a 50% probability. In other words, $t_{0.5}$ represents the median value of the variable. For the probability density function $f(t) = 0.1e^{-0.1t}$, the median is approximately 6.9.

It is interesting to compare the median to the mean for an exponentially distributed random variable. For the example shown, the rate is $\lambda = 0.1$ and so the mean is $\mu = \frac{1}{0.1} = 10$. Thus, the median (6.9) is less than the mean. This is true in general for an exponentially distributed random variables as the probability density function is positively skewed.

Imagine a situation in which a car drives past, on average, every 10 seconds (so $\mu = 10$ seconds). Now, imagine simulating this situation many times, each time recording the time you must wait until the next car drives past. There will be many times recorded that are between 0 and 10 seconds, and many times between 10 and 20 seconds. But there will also occasionally be some large waiting times such as 30 seconds or 40 seconds. These larger times dramatically affect the mean, which means that in order

for the mean waiting time to be 10 seconds, there must be more waiting times less than 10 seconds than greater than 10 seconds. In other words, the median waiting time will be less than the mean.

Explore further

Exponential probability density function

Use technology to explore the effect of the λ on the exponential probability density function.

For any exponentially distributed random variable, the median will be less than the mean.

Worked example 10 illustrates how to calculate quantiles for an exponentially distributed variable.

10 Determining quantiles for exponentially distributed variables

Let X be an exponentially distributed random variable with $\lambda = 0.2$. Determine each of the following quantiles:

(a) $t_{0.5}$ (the median)

THINKING	WORKING
1 Assign a variable to the quantile being determined and use the definite integral of t_α to write an appropriate probability statement.	Let $t_{0.5} = x$. $P(X \geq x) = 0.5$
2 Rewrite the probability statement in the form $P(X \leq x)$ using the complement rule.	$P(X \leq x) = 1 - 0.5$ $P(X \leq x) = 0.5$
3 Express the probability as a definite integral, then solve the resulting equation.	$\int_0^x 0.2e^{-0.2t}\,dt = 0.5$ $\left[-e^{-0.2t}\right]_0^x = 0.5$ $\left[e^{-0.2t}\right]_0^x = -0.5$ $e^{-0.2x} - e^0 = -0.5$ $e^{-0.2x} = 0.5$ $-0.2x = \ln(0.5)$ $x = \frac{\ln(0.5)}{-0.2}$ ≈ 3.466
4 Write the answer.	$t_{0.5} = 3.466$ (3 d.p.)

(b) $t_{0.9}$

1 Assign a variable to the quantile being determined and use the definite integral of t_α to write an appropriate probability statement.	Let $t_{0.9} = x$. $P(X \geq x) = 0.9$
2 Rewrite the probability statement in the form $P(X \leq x)$ using the complement rule.	$P(X \leq x) = 1 - 0.9$ $P(X \leq x) = 0.1$

3 Express the probability as a definite integral, then solve the resulting equation.	$\int_0^x 0.2e^{-0.2t}\,dt = 0.1$ $\left[-e^{-0.2t}\right]_0^x = 0.1$ $\left[e^{-0.2t}\right]_0^x = -0.1$ $e^{-0.2x} - e^0 = -0.1$ $e^{-0.2x} = 0.9$ $-0.2x = \ln(0.9)$ $x = \dfrac{\ln(0.9)}{-0.2}$ ≈ 0.527
4 Write the answer.	$t_{0.9} = 0.527$ (3 d.p.)

It is important to remember that quantiles are not probabilities, even though they may occasionally have values between 0 and 1 (such as in part (b) of Worked example 10).

WARNING

Quantiles are not probabilities. They are values of the random variable.

In some real-life situations it will be necessary to make the assumption that a random variable is exponentially distributed. This is a valid assumption provided the variable represents the time or distance before the next occurrence of some event that is reasonable to assume occurs randomly and independently.

Worked example 11 requires such an assumption to be made initially to then apply the techniques for determining probabilities and quantiles discussed previously.

11 Modelling situations with an exponentially distributed random variable

A particular brand of battery lasts 8 years on average.

(a) Determine the probability, correct to 3 decimal places, that a battery will last longer than 8 years. Evaluate the reasonableness of your result.

THINKING	WORKING
1 Identify the random variable that appears to be exponentially distributed and determine the value of λ.	Let X be the time before a battery fails. Then, assuming X is exponentially distributed, $\mu = 8$ years, and so $\lambda = \dfrac{1}{8}$ batteries per year.
2 Determine the required probability.	$P(X > 8) = \int_8^\infty \dfrac{1}{8} e^{-\frac{1}{8}t}\,dt$ ≈ 0.368 (using technology)
3 Evaluate the reasonableness of your result.	The result is reasonable since the median is less than the mean lifespan in years.

(b) Determine the smallest value of x such that 70% of batteries last longer than x years. Express your answer correct to 1 decimal place.

1 Write an equation involving the probability given. If necessary, use the complement rule to rewrite the equation to involve a probability of the form $P(X \leq x)$.	$P(X \geq x) = 0.7$ $P(X \leq x) = 0.3$
2 Express the probability as a definite integral, then solve the resulting equation.	$\int_0^x \frac{1}{8} e^{-\frac{1}{8}t}\, dt = 0.3$ $\left[-e^{-\frac{1}{8}t}\right]_0^x = 0.3$ $\left[e^{-\frac{1}{8}t}\right]_0^x = -0.3$ $e^{-\frac{x}{8}} - e^0 = -0.3$ $e^{-\frac{x}{8}} = 0.7$ $-\frac{x}{8} = \ln(0.7)$ $x = -8\ln(0.7)$ $x \approx 2.9$
3 Interpret your answer.	Therefore, 70% of batteries will last at least 2.9 years.

EXERCISE 6.4 The exponential distribution

Worked Example 9

1 Let X represent the time before the next earthquake in a particular region. X is an exponentially distributed random variable and the region experiences an earthquake, on average, every 5 years.

(a) Determine the mean μ, the rate λ and the probability density function for X.

(b) Determine the probability, correct to 3 decimal places, that there will be an earthquake within the next 2 years.

(c) Determine the probability, correct to 3 decimal places, that the next earthquake will take longer than 10 years to occur.

2 Let X represent the time between arrivals of planes at a particular airport. X is an exponentially distributed random variable and a plane arrives, on average, every 15 minutes.

(a) Determine the mean μ (in hours), the rate λ (in arrivals per hour), and the probability density function for X.

(b) Determine the probability, correct to 3 decimal places, that it will be longer than an hour before the next plane arrives.

(c) Determine the probability, correct to 3 decimal places, that the next plane will arrive within 9 minutes.

3 Let X represent the distance between faults in a long coil of copper wire. X is an exponentially distributed random variable and there are, on average, 10 faults per metre of wire.

(a) Determine the mean μ, the rate λ and the probability density function for X.

(b) Determine the probability, correct to 3 decimal places, that the next 20 cm of wire will have no faults in it.

(c) Determine the probability, correct to 3 decimal places, that there will be at least one fault in the next 50 cm of wire.

4 **(a)** Which one of the following variables could be exponentially distributed?

A The number of times a coin shows tails from 10 tosses.

B The running times of a large group of athletes in a 100 metre sprint.

C The number of cracks per metre along a footpath.

D The distance between consecutive cracks along a footpath.

(b) Explain the common error made by the student who selected one of the incorrect options in part **(a)**.

5 A student completes, on average, 15 homework problems per hour. Assuming that the time taken to complete the next problem is exponentially distributed, the probability that the student will complete their next problem within 30 minutes is given by which of the following?

A $\int_0^{30} 15e^{-15t}\,dt$ **B** $\int_0^{0.5} 15e^{-15t}\,dt$ **C** $\int_0^{30} 4e^{-4t}\,dt$ **D** $\int_0^{0.5} 4e^{-4t}\,dt$

6 Let X be an exponentially distributed random variable with probability density function $f(t) = 3e^{-3t}$.

(a) Determine the rate λ and the mean μ.

(b) Determine the exact value of:

(i) $P(X < 7)$ **(ii)** $P(X \leq 7)$ **(iii)** $P(1 < X < 2)$ **(iv)** $P(X > 5)$

7 Let X be an exponentially distributed random variable with probability density function $f(t) = 0.2e^{-0.2t}$.

(a) Determine the rate λ and the mean μ.

(b) Determine the exact value of

(i) $P(X < 1)$ **(ii)** $P(X \geq 10)$

8 Let X be an exponentially distributed random variable with $\lambda = 4$. Determine each of the following quantiles:

Worked Example 10

(a) $t_{0.5}$ (the median) **(b)** $t_{0.95}$ **(c)** $t_{0.25}$

9 X is a random variable. The quantile $t_{0.3}$ represents which of the following?

A the probability that X is less than 0.3

B the probability that X is greater than 0.3

C the value of x for which the probability that X is less than x is 0.3

D the value of x for which the probability that X is greater than x is 0.3

10 The waiting time for customers at a bank is exponentially distributed. On average, customers have to wait 1 minute to be served. The probability that a customer is served within 1 minute must be which of the following?

A 1 **B** 0.5 **C** less than 0.5 **D** greater than 0.5

11 The distance that a particular car can run before its battery wears out is exponentially distributed with a mean of 20 000 km. The owner of the car intends to take a 1000 km trip. Determine the probability that the trip will be completed without having to replace the battery. Express your answer correct to 3 decimal places. Evaluate the reasonableness of your result.

Worked Example 11

12 The waiting time to be served at a fast food restaurant is 3 minutes on average.

(a) Determine the probability, correct to 3 decimal places, that you have to wait longer than 3 minutes to be served. Evaluate the reasonableness of your result.

(b) Determine the least value of x for which the store can truthfully claim that 99% of customers will be served within x minutes. Express your answer correct to 1 decimal place.

13 A particular model of mobile phone lasts an average of 5 years before a malfunction occurs.

(a) What percentage of customers who purchase such a phone will experience a malfunction within a year?

(b) If the phone manufacturer wishes to advertise a length of time within which 90% of their phones will not malfunction, what is the greatest whole number of months for which this is true?

14 If a store wishes to be sure that 85% of customers will be served within 5 minutes, determine the mean waiting time needed to achieve this. Express your answer in minutes, correct to 2 decimal places.

15 An aircraft engine has a 90% chance of lasting at least 6 years without needing to be repaired. What is the longest period of time for which there is a 95% chance that the engine will last without needing to be repaired? Express your answer in hours, correct to 2 decimal places.

16 Prove that the median of an exponentially distributed random variable with rate λ is equal to $\frac{1}{\lambda}\ln(2)$.

17 One of the most important properties of an exponential distribution is the 'memoryless' property. Informally, this means that the probability that the next occurrence of a random event occurs after an interval of length x is independent of how much time has already elapsed (a) without the event happening. Symbolically, this can be written as:

$P(X > a + x \mid X > a) = P(X > x)$.

Prove that this property holds for any exponential distribution.

Summary

The area bound by a function and the *x*-axis

The area bounded by a function $f(x)$, the x-axis, and the lines $x = a$ and $x = b$ (where $a < b$) is equal to

$\int_a^b f(x)\,dx$ if the function lies above the x-axis from a to b, and

$-\int_a^b f(x)\,dx$ if the function lies below the x-axis from a to b.

The area between a function and the *y*-axis

If a function $y = f(x)$ lies entirely to the right of the y-axis for the interval $a \leq y \leq b$, the function can be rearranged to obtain $x = g(y)$ for $a \leq y \leq b$. Now, the area bounded by the part of the function to the right of the y-axis, the y-axis, and the lines $y = a$ and $y = b$ is equal to $\int_{y=a}^{y=b} g(y)\,dy$.

The area between two functions

If $f(x) \geq g(x)$ over the interval $[a,b]$, then the area between these functions over this interval is equal to $\int_a^b f(x) - g(x)\,dx$.

If the two functions intersect over $[a,b]$, then in order to obtain the area between the two functions, you must add together the separate areas between the points of intersection. Each such partial area may be obtained by subtracting the lower function from the upper function, and integrating this over the required subinterval.

Volumes of solids of revolution

A solid of revolution is the solid formed when a curve is rotated about one of the coordinate axes.

The volume of the solid generated by rotating the section of the curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is given by $V = \pi\int_a^b y^2\,dx$.

The volume of the solid generated by rotating the section of the curve $x = g(y)$ from $y = a$ to $y = b$ about the y-axis is given by $V = \pi\int_a^b x^2\,dy$.

The volume of a solid formed by rotating a *region between two curves* about an axis can be obtained by subtracting the volume of revolution of the inner curve (the one closest to the axis of rotation) from the volume of revolution of the outer curve.

Numerical integration using Simpson's rule

Simpson's rule:

When n is an even positive integer and $x_0, x_1, \ldots x_n$ partition the interval $[a, b]$ into n equally spaced subintervals with the width of each subinterval being w, an approximation for the definite integral $\int_a^b f(x)\,dx$ is:

$$\int_a^b f(x)\,dx \approx \frac{w}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+\ldots+4f(x_{n-1})+f(x_n)\right)$$

Equivalently,

$$\int_a^b f(x) \approx \frac{w}{3}\sum_{i=1}^{n} f(x_i)\times M_i \quad \text{where} \quad M_i = \begin{cases} 1 & i = 0 \text{ or } i = n \\ 4 & i = 1,3,5,\ldots,n-1 \\ 2 & i = 2,4,6,\ldots,n-2 \end{cases}$$

The exponential distribution

An *exponentially distributed random variable* is a variable representing the time or distance before the next occurrence of a random event, where such random events occur independently at a constant average rate, λ.

The mean μ (also known as the expected value) of an exponentially distributed random variable is the average time or distance before the next occurrence of the event.

It is always the case that $\lambda = \frac{1}{\mu}$ (or equivalently $\mu = \frac{1}{\lambda}$).

For an exponentially distributed random variable X with rate λ, the probability density function is given by:

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Probability density functions always have the property that $P(a < X < b) = \int_a^b f(t)\,dt$. In other words, the probability of the variable taking on a value between a and b is equal to the area between the function and the x-axis from $x = a$ to $x = b$. Consequently, the probability of X taking on any particular value is zero (since this would correspond to an area of width zero).

It is not necessary to distinguish between the phrases 'less than' and 'less than or equal to' (and likewise between 'greater than' and 'greater than or equal to') when working with continuous random variables.

The total area under the curve of any probability density function is always 1.

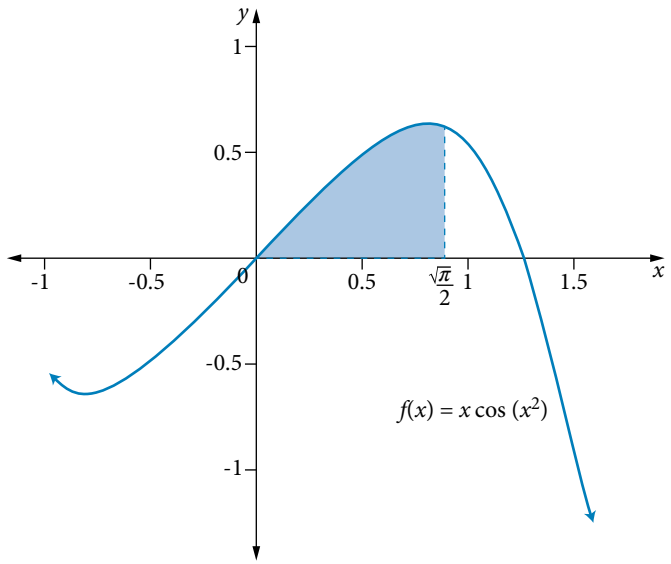
Given a particular probability, α, the quantile for the continuous random variable X corresponding to α is defined to be the value t_α such that $P(X > t_\alpha) = \alpha$.

For any exponentially distributed random variable, the median will be less than the mean.

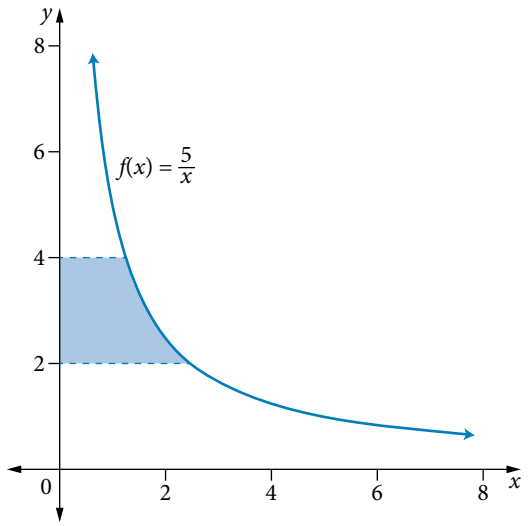
Chapter review 6

1 For each of the following, determine the exact area of the shaded region. Exercise 6.1

(a)



(b)



2 Determine the area between the functions $f(x) = x^3$ and $g(x) = x + 2$ for $-2 \leq x \leq 3$. Give your answer correct to 2 decimal places. Exercise 6.1

3 Determine the exact volume of the solid of revolution when the part of the curve $y = e^{2x}$ with $0 \leq x \leq 1$ is rotated about the x-axis. Exercise 6.2

4 Determine the exact volume of the solid of revolution when the part of the line $y = 2x + 4$ with $0 \leq y \leq 4$ is rotated about the y-axis. Exercise 6.2

5 Determine the volume of the solid generated when the curve $y = \arctan(x)$ from the point where $x = 0$ to the point where $x = 1$ is rotated about the y-axis. Exercise 6.2

Exercise 6.3

6 Use Simpson's rule to approximate $\int_0^4 \sqrt{x}\,dx$ using $n = 8$ subintervals. Express your answer correct to 3 decimal places.

Exercise 6.4

7 Let X represent the time before the next car drives by a particular location. X is an exponentially distributed random variable and a car drives by, on average, every 3 minutes.

(a) Determine the mean μ, the rate λ and the probability density function for X.

(b) Determine the probability, correct to 3 decimal places, that a car will drive by in the next 15 seconds.

(c) Determine the probability, correct to 3 decimal places, that no cars will drive by in the next 5 minutes.

Exercise 6.4

8 Let X be an exponentially distributed random variable with probability density function $f(t) = 2e^{-2t}$.

(a) Determine the rate λ and the mean μ.

(b) Determine the exact value of:

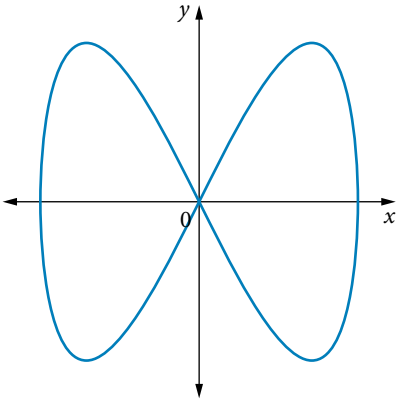
(i) $P(X < 5)$ **(ii)** $P(X \leq 5)$ **(iii)** $P(X > 1)$

Exercise 6.4

9 Let X be an exponentially distributed random variable with $\mu = 5$. Determine the value of x for which $P(X > x) = 0.3$. Express your answer correct to 3 decimal places.

Exercise 6.1

10 The curve shown has equation $y^2 = 4x^2\left(1 - x^2\right)$. Determine the exact area of the enclosed region.



Exercise 6.2

11 A hemispherical bowl of radius 10 cm is filled with rice to a depth of 5 cm. Determine the volume of rice in the bowl.

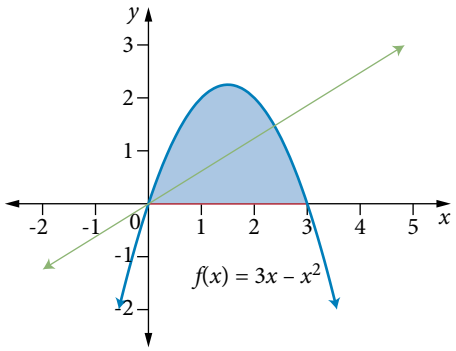
Exercise 6.3

12 Demonstrate that using Simpson's rule with $n = 2$ subintervals produces the approximation $\frac{2\pi}{3}$ for $\int_0^{\pi} \sin(x)\,dx$. Demonstrate that this approximation is within 0.1 of the actual value of this integral.

Exercise 6.4

13 A particular website receives 18 hits on average per hour. Determine the maximum length of time for which there is at least a 50% chance that the website receives no hits. Express your answer as a number of minutes, correct to 2 decimal places.

14 In the diagram, the straight line divides the shaded region into two regions of equal area. Determine the gradient of this line. Exercise 6.1



15 A vase is constructed by rotating a section of the function $f(x) = 20\ln(x)$ about the y-axis. The base of the vase has a radius of 2 cm and the vase is wider at the top than the base. Determine the height of the vase in cm, correct to 1 decimal place, if it holds 480 mL of water when full. Exercise 6.2

16 Prove that using Simpson's rule with $n = 2$ produces the exact value of $\int_a^b x^2\,dx$. Exercise 6.3

17 A manufacturing company is concerned about the number of accidents occurring in the workplace. At present, there is a 50% chance that the next accident will occur within 10 days. If the company wishes there to be only a 20% chance that the next accident will occur within 10 days, by what percentage will the company need to decrease the average number of accidents per year? Express your answer correct to the nearest per cent. Exercise 6.4





7

Rates of change and differential equations

Recall 366

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7 Recall

Solve an equation involving an absolute value

1 Solve each of the following equations.

(a) $|x| = 2$ (b) $|x-1| = 2$ (c) $\{x : |x-1| = |a-b|\}$

Use the product rule and quotient rule to differentiate functions

2 Determine the derivative of the following functions:

(a) $x^2 \log_e(x)$ (b) $\dfrac{\log_e\left(x^2+1\right)}{2x}$ (c) $\cot(x)$

(d) $\cos(x)\cot(x)$ (e) $\sin^{-1}(x)\operatorname{cosec}(x)$

Use the chain rule to determine a derivative of a composite function

3 Determine the derivative of the following:

(a) $\log_e\left(g(x)\right),\ g(x) > 0$ (b) $e^{g(x)}$ (c) $\left(g(x)\right)^n$ (d) $\tan^{-1}\left(x^2\right)$

Recognise you can calculate the value of $\int_a^b \frac{1}{x}dx$ where $x \neq 0$ only when a and b have the same sign i.e. when $0 < ab < \infty$

4 Calculate the value of the following definite integrals if they exist.

(a) $\int_{-3}^{-1} \frac{1}{x}dx$ (b) $\int_1^2 \frac{1}{x}dx$ (c) $\int_{-2}^{-1} \frac{1}{x}dx$ (d) $\int_{-1}^2 \frac{1}{x}dx$

Use an appropriate substitution to determine an anti-derivative

5 Use an appropriate substitution to determine an anti-derivative of the following.

(a) $\dfrac{\cos(x)}{\sqrt{1-\sin(x)}}$ (b) $\dfrac{2x}{x^2+1}$ (c) $\dfrac{2x}{x^4+1}$ (d) $2x\cos\left(x^2\right)$

(e) $3\sin^2(x)\cos(x)$ (f) $e^{\tan(x)}\sec^2(x)$ (g) $x^2\sin(x)$

Use an appropriate substitution to determine a definite integral

6 Use the method of substitution, if necessary, to evaluate the definite integral in each of the following:

(a) $\int_0^1 \frac{2}{2x+1}dx$ (b) $\int_0^1 \frac{x}{\sqrt{x^2+1}}dx$ (c) $3\int_0^{\frac{\pi}{2}} \sin(x)\cos^2(x)dx$

Use a derivative to determine an anti-derivative

7 (a) Determine the derivative of $\log_e\left(x+\sqrt{x^2+1}\right)$ and hence, determine an anti-derivative of $\dfrac{2}{\sqrt{x^2+1}}$.

(b) Determine the derivative of $\sqrt{x^2-1}$ and hence, determine an anti-derivative of $\dfrac{2x}{\sqrt{x^2-1}}$.

Use partial fractions to determine an anti-derivative

8 Use partial fractions to determine an anti-derivative of each of the following:

(a) $\dfrac{1}{x(x-2)}$ (b) $\dfrac{1}{(x-2)(x-1)}$

Use the trigonometric double-angle formulas to determine an anti-derivative of an expression involving even powers of cos(*x*) or sin(*x*)

9 Use an appropriate double-angle formula to determine the following:

(a) $\int \cos^2(x)\,dx$ **(b)** $\int \sin^2(x)\,dx$ **(c)** $\int 4\sin^2(x)\cos^2(x)\,dx$

Anti-differentiate the derivative of an inverse trigonometric function

10 Determine an anti-derivative of the following:

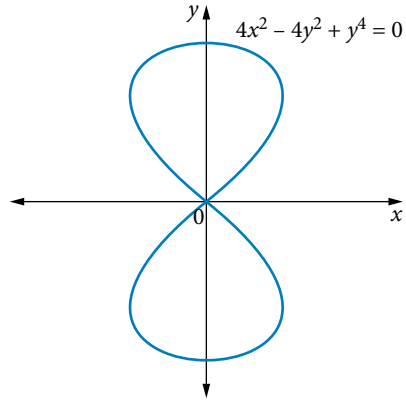
(a) $\dfrac{2}{4x^2+1}$ **(b)** $\dfrac{1}{\sqrt{4-x^2}}$ **(c)** $\dfrac{1}{x^2+2x+2}$

7.1 Implicit differentiation

The relationship between a pair of variables (for example, x and y) can be expressed explicitly (for example, $y = x + 1$) or implicitly (for example, $-x + y = 1$). The *explicit form* (or functional form) of a rule is written as $y = f(x)$ to distinguish between the roles of the dependent (y) and independent (x) variables. The dependent variable changes in response to changes in the independent variable.

By contrast, the relationship between a pair of variables defining an implicit relationship (for example, $y^3 + y - x = 1$) is more symmetric, with neither variable assuming the role of a dependent or independent variable, but both variables cooperating to maintain the equality in response to a change in the other variable. An example of such a relationship is the circle, where either of the variables, x or y, can assume the role of the dependent variable or the independent variable.

The graph of a relation is often called a level curve (similar to a contour line on a topographical map). The level curve of the relation $4x^2 - 4y^2 + y^4 = 0$ is shown.



$4x^2 - 4y^2 + y^4 = 0$ is not a function because it fails the vertical line test; that is, there is not a unique y-value for each x-value. However, its graph is a smooth curve so each point on the curve has a unique tangent. Consequently, it should be possible to calculate a derivative of this implicitly defined relationship.

The level curve of the relation $y^4 - 4\left(y^2 - x^2\right) = 0$ indicates that there must exist one or more functions $y = f(x)$ such that $(f(x))^4 - 4\left((f(x))^2 - x^2\right) = 0$ for every value of x in some interval of the number line. It can be verified by substitution that any one of four possibilities,

$f_1(x) = \sqrt{2}\sqrt{1 + \sqrt{1 - x^2}}$, $f_2(x) = \sqrt{2}\sqrt{1 - \sqrt{1 - x^2}}$, $f_3(x) = -\sqrt{2}\sqrt{1 + \sqrt{1 - x}}$ or $f_4(x) = -\sqrt{2}\sqrt{1 - \sqrt{1 - x^2}}$ satisfies $(f(x))^4 - 4\left((f(x))^2 - x^2\right) = 0$. Each of these functions is called a 'branch' of the implicit relation.

The relation $g(x, y) = c$ defines y as an implicit function of x on an interval $I = \{x : a \leq x \leq b\}$, if there exists one or more functions $f(x)$ defined on I, such that $g(x, f(x)) = c$ for every x in I.

Technology worked example
Drawing the level curves of an implicitly defined relation

For example, $x^2 + y^2 = 1$ does define y as an implicit function of x with two separate branches, each with an explicit rule of $f(x) = \sqrt{1 - x^2}$ or $f(x) = -\sqrt{1 - x^2}$ for every value of x on the interval $\{x : -1 \leq x \leq 1\}$. By contrast, the seemingly similar equation $x^2 + y^2 = -1$ does not define y as an implicit function of x because $x^2 + y^2 = -1$ represents a relation between x and y that is not a function. The domain of any corresponding explicit rule $y = f(x)$ would be the empty set ($x \in \varnothing$).

WARNING

It is possible to determine the derivative of any one of these four branches of $y^4 - 4\left(y^2 - x^2\right) = 0$. However, it is not always possible to write an implicitly defined relation such as $g(x, y) = c$ in a usable explicit function form $y = f(x)$. Therefore, you should be able to compute the derivative $\frac{dy}{dx}$ of the relation in implicit form.

In this section you will learn how to arrive at such a derivative by using the chain rule to determine $\frac{dy}{dx}$ from an implicitly defined relation such as $g(x, y) = c$ directly, without first converting $g(x, y) = c$ into an explicit function of the form $y = f(x)$.

The derivative $\frac{dy}{dx}$ of an implicitly defined function $g(x, y) = c$ can be calculated using the standard chain rule procedure:

- Differentiate both sides of the given equation with respect to x, differentiating any expressions of the form $h(y)$ with the chain rule $\frac{dh}{dx} = \frac{dh}{dy}\frac{dy}{dx}$.
- Collect all terms involving $\frac{dy}{dx}$ onto the left-hand side of the equation, and all terms independent of $\frac{dy}{dx}$ onto the right-hand side.
- Solve for $\frac{dy}{dx}$ by dividing both sides of the derived equation by the coefficient of $\frac{dy}{dx}$.

1 Calculating $\frac{dy}{dx}$ and $\frac{dx}{dy}$ from an implicitly defined relation

Given the implicitly defined relation $x^2y + y^2 = 10$:

(a) determine an expression for $\frac{dy}{dx}$.

THINKING	WORKING
1 Differentiate both sides with respect to x, differentiating any expressions of the form $h(y)$ by the chain rule $\frac{dh}{dx} = \frac{dh}{dy}\frac{dy}{dx}$.	To differentiate x^2y, use the product rule: Let $u = x^2$ and $v = y$ then: $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{dv}{dy} \times \frac{dy}{dx} = 1 \times \frac{dy}{dx}$ $\frac{d}{dx}\left(x^2y\right) = u\frac{dv}{dx} + v\frac{du}{dx}$ $= x^2 \times 1\frac{dy}{dx} + y \times 2x$ $= x^2\frac{dy}{dx} + 2xy$ Use the chain rule to differentiate y^2 with respect to x $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$ Therefore: $\underbrace{\left(2xy + x^2\frac{dy}{dx}\right)}_{\frac{d}{dx}\left(x^2y\right)} + \underbrace{2y\frac{dy}{dx}}_{\frac{d}{dx}\left(y^2\right)} = \underbrace{0}_{\frac{d}{dx}(10)}$
2 Collect all terms with $\frac{dy}{dx}$ on the left-hand side.	$\left(x^2 + 2y\right)\frac{dy}{dx} = -2xy$

3 Solve for $\frac{dy}{dx}$.	$\frac{dy}{dx} = -\frac{2xy}{x^2 + 2y}$

(b) determine an expression for $\frac{dx}{dy}$.

1 Differentiate each term of the equation separately with respect to y.	Note: x^2y is a product between x^2 and y. Hence the product rule will be used to differentiate this term. $\underbrace{\left(x^2 + 2xy\frac{dx}{dy}\right)}_{\frac{d}{dx}(x^2y)} + \underbrace{2y}_{\frac{d}{dx}(y^2)} = \underbrace{0}_{\frac{d}{dy}(10)}$
2 Solve for $\frac{dx}{dy}$.	$2xy\frac{dx}{dy} = -x^2 - 2y$ $\frac{dx}{dy} = \frac{-x^2 - 2y}{2xy}$ $\frac{dx}{dy} = \frac{-\left(x^2 + 2y\right)}{2xy}$

(c) multiply $\frac{dy}{dx}$ by $\frac{dx}{dy}$ and express $\frac{dx}{dy}$ in terms of $\frac{dy}{dx}$.

1 Multiply the two answers and simplify.	$\frac{dy}{dx} \times \frac{dx}{dy} = \frac{-2xy}{x^2 + 2y} \times \frac{-(x^2 + 2y)}{2xy}$ $= 1$
2 Write the relationship.	$\therefore \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$

For any function $f(x)$, the first derivative of the function, $f'(x)$, is the gradient of the tangent to the curve of $f(x)$ at point x. In other words, the first derivative gives an indication of whether the function $f(x)$ is increasing or decreasing and by how much.

The second derivative of $f(x)$ is $f''(x)$ or the derivative of $f'(x)$. The second derivative indicates the concavity of the curve of $f(x)$ and the value of x for which $f(x)$ has a point of inflection.

- When $f''(x) < 0$, $f(x)$ opens upwards (concave up).
- When $f''(x) > 0$, $f(x)$ opens downwards (concave down).
- When $f''(x) = 0$, $f(x)$ may have a point of inflection.

Technology worked example
Determine the derivative of an implicitly defined relation

WARNING

When calculating a derivative $\frac{dy}{dx}$ at a specific point (a, b), be sure to substitute for the coordinates of the point only after the derivative $\frac{dy}{dx}$ has been calculated. This order of operations is made clear by writing $\left.\frac{dy}{dx}\right|_{\substack{x=a \\ y=b}}$.

2 Determining the derivative of an implicitly defined relation at a point

Consider the curve $ye^x + x^2y^2 = x + 3$. Determine the slope of the curve at the point $(0, 3)$.

THINKING	WORKING
1 Differentiate each side of the equation with respect to x and collect terms with $\frac{dy}{dx}$.	$\underbrace{\left(ye^x + e^x\frac{dy}{dx}\right)}_{\frac{d}{dx}(ye^x)} + \underbrace{\left(2xy^2 + 2x^2y\frac{dy}{dx}\right)}_{\frac{d}{dx}(x^2y^2)} = \underbrace{1}_{\downarrow\ \frac{d}{dx}(x+3)}$ $e^x\frac{dy}{dx} + 2x^2y\frac{dy}{dx} = 1 - ye^x - 2xy^2$ $\left(e^x + 2x^2y\right)\frac{dy}{dx} = 1 - ye^x - 2xy^2$
2 Solve for $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{1 - ye^x - 2xy^2}{\left(e^x + 2x^2y\right)}$
3 Calculate the derivative at the required point by substituting for x and y.	$\left.\frac{dy}{dx}\right\|_{\substack{x=0\\y=3}} = \frac{1 - 3e^0 - 2\times 0\times 3^2}{(e^0 + 2\times 0^2\times 3)} = -2$

3 Determining the second derivative $\frac{d^2y}{dx^2}$ of an implicitly defined relation

The variables x and y are implicitly related by $x^2 - y^2 = 1$.

(a) Determine $\frac{dy}{dx}$.

THINKING	WORKING
1 Differentiate each side of the equation with respect to x.	$\underbrace{(2x)}_{\frac{d}{dx}(x^2)} - \underbrace{2y\frac{dy}{dx}}_{\frac{d}{dx}(y^2)} = \underbrace{0}_{\downarrow\ \frac{d}{dx}(1)}$
2 Solve for $\frac{dy}{dx}$.	$2y\frac{dy}{dx} = 2x$ $\frac{dy}{dx} = \frac{x}{y}$

(b) Hence, verify that $\frac{d^2y}{dx^2} = \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}$.

1 To determine $\frac{d^2y}{dx^2}$, differentiate the expression for $\frac{dy}{dx}$ with respect to x.	Using the product rule considering $u = \frac{1}{y}$ and $v = x$: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x}{y}\right)$ $= \frac{1}{y}\underbrace{(1)}_{\frac{d}{dx}(x)} + x\underbrace{\left(\frac{-1}{y^2}\frac{dy}{dx}\right)}_{\frac{d}{dx}\left(\frac{1}{y}\right)}$

2 Write the answer.	$\frac{d^2y}{dx^2} = \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}$

(c) Determine an expression for $\frac{d^2y}{dx^2}$ in terms of variables x and y only.

1 Use the answer to part (a) to eliminate $\frac{dy}{dx}$ in part (b).	$\frac{d^2y}{dx^2} = \left(\frac{1}{y} - \frac{x}{y^2} \times \frac{x}{y}\right)$
2 Simplify.	$\frac{d^2y}{dx^2} = -\frac{1}{y^3}\left(x^2 - y^2\right)$
3 Write the answer.	Given that $x^2 - y^2 = 1$, then: $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$

EXERCISE 7.1 Implicit differentiation

1 Plot each of the following implicitly defined relations.

Worked Example

(a) $x^3 + y^3 = 3xy$ (b) $\left(x^2 + y^2\right)^2 = 4xy^2$ (c) $y^2\left(x^2 + y^2\right) = 4x^2$

1

2 Determine the derivative $\frac{dy}{dx}$ for each of the following implicitly defined relations.

(a) $x^4 = x^2 - y^2$ (b) $x^3 + y^3 = 3xy$ (c) $x^4 + y^4 = 1$

(d) $y^2\left(x^2 + y^2\right) = x^2$ (e) $\left(x^2 + y^2\right)^2 = 4xy^2$

3 Use technology to determine the derivative $\frac{dy}{dx}$ in each of the following implicitly defined relations.

(a) $y^2 = x^3(1 - x)$ (b) $(x - 1)\left(x^2 + y^2\right) = x^2$

(c) $\left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) = 1$ (d) $\left(x^2 + y^2 - 2x\right)^2 = 4\left(x^2 + y^2\right)$

2

4 Consider the family of curves defined by the relation $3x^2 + kx - xy + y^2 + y = 6$, where $k \in \mathbb{R}$.

(a) Verify that every curve in the family passes through the point $(0, 2)$, and determine the other point of intersection with the y-axis.

(b) Determine an expression for $\frac{dy}{dx}$ in terms of x, y and k.

(c) Hence, calculate the gradient of the curve at the point $(1,1)$.

5 Consider the implicitly defined relation $y^2 = x^3 + 3x^2$.

(a) Determine an expression for $\frac{dy}{dx}$.

(b) Hence, determine the equation of the tangent(s) to the curve when $x = 1$.

6 If $x^3 + 2xy + y^3 = 8$, determine an expression for $\frac{dy}{dx}$.

7 If $y^2 - 2xy = 8$, determine an expression for $\frac{dy}{dx}$.

8 If $\cos(x) = e^y, -\frac{\pi}{2} < x < \frac{\pi}{2}$, determine an expression for $\frac{dy}{dx}$.

9 If $\tan\left(xy^2\right) = x$, determine an expression for $\frac{dy}{dx}$.

Worked Example 3

10 Consider the relation $x^2 + y^2 = 1$. Determine an expression for $\frac{d^2y}{dx^2}$ in terms of y.

11 If $y^3 = x^4$, determine expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in simplest form.

12 Consider the curve defined by $5x^2 + 6xy + 5y^2 = 8$.

(a) Verify that $\frac{dy}{dx} = -\frac{5x + 3y}{3x + 5y}$.

(b) Point P is a point on the curve with a horizontal tangent. Verify that the x-coordinate and y-coordinate at point P are related by $y = -\frac{5}{3}x$.

(c) If the y-coordinate at point P is $y = -\sqrt{\frac{5}{2}}$, determine the x-coordinate at point P.

(d) Determine the value of $\frac{d^2y}{dx^2}$ at point P.

(e) Is point P a local maximum, a local minimum or neither? Justify your answer.

13 Consider the derivative of $\cos(x) - xy = 6$ with respect to x.

(a) Which one of the following statements is incorrect?

A The equation cannot be differentiated with respect to x without using implicit differentiation.

B There will only be one term in $\frac{dy}{dx}$.

C There will be three terms on the left-hand side.

D The given relation is a function.

(b) Explain why the other statements in part **(a)** are correct.

14 **(a)** Which one of the following statements is true for the curve with equation $x^2 + \frac{y^2}{4} = 1$?

A It has a defined derivative with respect to x at the point $\left(\frac{1}{4}, 1\right)$.

B It has a defined derivative with respect to x at the point $\left(\frac{1}{2}, -\sqrt{3}\right)$.

C The derivative at the point $(1, 0)$ is defined.

D The derivative is not equal to zero anywhere on the curve.

(b) Explain why the other statements in part **(a)** are false.

15 Consider the curve $3y^2 = 4x^2 - 4xy$.

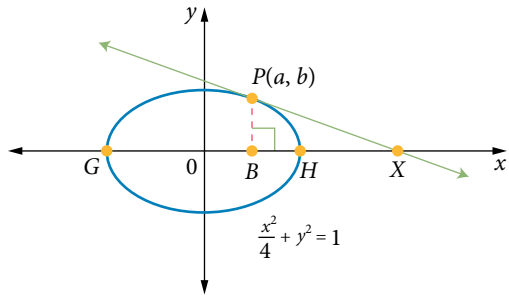
(a) Determine the values of m for which points on the line $y = mx$ are also points on the curve $3y^2 = 4x^2 - 4xy$.

(b) If $3y^2 = 4x^2 - 4xy$, determine an expression for $\frac{dy}{dx}$.

(c) Hence, verify your answer to part **(a)**.

16 Consider the implicitly defined relation $\frac{x^2}{4} + y^2 = 1$.

Point $P(a,b)$ is a point on the graph of $\frac{x^2}{4} + y^2 = 1$ in the first quadrant.



(a) Determine the ratio of the lengths $\frac{BH}{BG}$ in terms of a.

(b) Determine an expression for b in terms of a.

(c) Determine an expression for $\frac{dy}{dx}$ at the point $P(a,b)$ in terms of a only.

(d) Hence, determine the equation of the tangent to the curve $\frac{x^2}{4} + y^2 = 1$ at the point $P(a,b)$.

(e) Determine the coordinates of the point X, the intercept of the tangent with the x-axis.

(f) Determine the ratio of the lengths $\frac{XH}{XG}$ in terms of a and compare this result with the ratio $\frac{BH}{BG}$ found in part **(a)**.

17 Consider the implicitly defined curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are real constants.

(a) Determine $\left.\frac{dy}{dx}\right|_{(x_0,\, y_0)}$

(b) The equation of the tangent to the curve at the point (x_0, y_0) can be written in the form $Ax + By = 1$, where A and B are real constants. Determine expressions for A and B in terms of x_0, y_0, a and b.

18 Consider the equation $x^2 + 6x + y^2 + 2y + 10 = 0$.

(a) Determine an expression for $\frac{dy}{dx}$.

(b) Calculate the value of the gradient at the point (1,1) and the point (-3,-1).

(c) Comment on your answer to part **(b)**. (Hint: Complete the square on the x-terms and y-terms of $x^2 + 6x + y^2 + 2y + 10 = 0$.)

19 According to the Cobb–Douglas production model, the output Q (in units per week) of a factory can be modelled by $Q = Kx^a y^{1-a}$, where x (in working hours per week) is the size of the work force, and y (in dollars per week) is the operating budget. The numbers K and a are constants that depend on the particular situation studied, with $0 < a < 1$.

At a certain factory, output is modelled by $Q = 100x^{\frac{3}{4}} y^{\frac{1}{4}}$. The factory currently has on operating budget of \$50 000 per week, and a work force of 400 working hours per week.

(a) If factory output per week (Q) is to remain constant, calculate $\left.\frac{dy}{dx}\right|_{\substack{x=400 \\ y=50000}}$ with the appropriate units and interpret the result.

(b) Hence, determine the rate of change in the operating budget per week to offset a reduction of 10 hours per week in working hours.

Related rates

7.2

A pair of variables (for example, x and y) can be related by either an explicit function in the form $y = f(x)$, or by an implicit relation in the form $g(x, y) = c$. In both cases, one of the variables must change in response to a change in the other variable to maintain the equality embodied within the relationship.

However, many problems encountered in the practice of mathematics in the real world are defined parametrically, where each of the variables depends on a third variable called a *parameter* (often a time or an angle variable). You often encounter problems in either the explicit parametric form $y(t) = f(x(t))$ or the implicit parametric form $g(x(t), y(t)) = c$, where c is a real constant, and t is the parameter.

In this section you will investigate the related rates, which model the relationship between the rates of change in a pair of variables as a parameter (usually a time variable) changes. Consequently, the phrase 'rate of change' without further qualification will often mean a rate of change with respect to this time parameter.

For example, consider a car traveling at 100 km/hour at a particular time when fuel is being consumed at the rate of 10 litres per 100 km. To determine how many litres of fuel are being consumed per hour, you must multiply the following rates:

$$\begin{aligned}\text{Rate of fuel consumption} &= \frac{\text{distance (km)}}{\text{time (hour)}} \times \frac{\text{consumption (L)}}{\text{distance (km)}} \\ &= \frac{100 \text{ km}}{1 \text{ hour}} \times \frac{10 \text{ L}}{100 \text{ km}} \\ &= \frac{10 \text{ L}}{1 \text{ hour}} \\ &= 10 \text{ L/hour}\end{aligned}$$

Note that the units of the right-hand side simplify to the required units of the answer on the left-hand side.

All related rates problems considered in this chapter can be classified into one of two cases.

Case 1: variables related by an explicit parametric equation

If the variables in the model (for example, x and y) are related by an explicit parametric equation such as $y(t) = f(x(t))$, then the two rates $\frac{dy}{dt}$ and $\frac{dx}{dt}$ can be related by the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, where $\frac{dy}{dx} = \frac{d\left(f(x)\right)}{dx}$.

For example, consider the relationship $y = \frac{1}{x^2}$, which expresses y as an explicit function of x. The time rate of change in the dependent variable $\frac{dy}{dt}$ is now related to the time rate of change in the independent variable $\frac{dx}{dt}$ by the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$
$$\frac{dy}{dt} = \frac{-2}{x^3}\frac{dx}{dt}$$

Case 2: variables related by an implicit parametric equation

If the variables in the model (for example, x and y) are related by an implicit parametric equation such as $g(x(t), y(t)) = c$), then the two rates $\frac{dy}{dt}$ and $\frac{dx}{dt}$ can be related by the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, where $\frac{dy}{dx}$ is an implicit derivative.

For example, consider the implicit relationship $g(x, y) \equiv x^2 + xy + y^2 = 1$.

To determine an expression for $\frac{dy}{dt}$ in terms of $\frac{dx}{dt}$, an implicit derivative of both sides with respect to time gives:

$$\begin{aligned} 1 &= x^2 + xy + y^2 \\ \frac{d}{dt}(1) &= \frac{d}{dt}(x^2 + xy + y^2) \\ 0 &= 2x\frac{dx}{dt} + y\frac{dx}{dt} + x\frac{dy}{dt} + 2y\frac{dy}{dt} \\ 0 &= (2x + y)\frac{dx}{dt} + (x + 2y)\frac{dy}{dt} \\ -(x + 2y)\frac{dy}{dt} &= (2x + y)\frac{dx}{dt} \\ \frac{dy}{dt} &= -\left(\frac{2x + y}{x + 2y}\right)\frac{dx}{dt} \end{aligned}$$

Alternatively, you can achieve the same result with the following procedure:

$$1 = x^2 + xy + y^2$$

$$\frac{d}{dt}(1) = \frac{d}{dt}(x^2 + xy + y^2)$$

$$0 = \frac{d}{dx}(x^2 + xy + y^2)\Big|_{y \text{ constant}} \times \frac{dx}{dt} + \frac{d}{dy}(x^2 + xy + y^2)\Big|_{x \text{ constant}} \times \frac{dy}{dt}.$$

$$0 = (2x + y)\frac{dx}{dt} + (x + 2y)\frac{dy}{dt} = 0$$

$\frac{dy}{dt} = -\frac{(2x + y)}{(x + 2y)}\frac{dx}{dt}$ as determined above.

Related rates problems can be broken down into a set of steps:

STEP 1: Draw a diagram if appropriate, labelling constants with their numerical values and variables with a pronumeral.

STEP 2: Write the given information and the required rate.

STEP 3: Write an equation, using the chain rule, with the required rate as the subject.

STEP 4: Substitute the derivatives needed in the chain rule. (If the variables are related implicitly, an implicit differentiation will be required to calculate a derivative.)

STEP 5: If necessary, solve algebraically for the required rate.

STEP 6: Substitute known values for variables and rates.

STEP 7: Interpret the solution in the context of the original question and express the answer with the appropriate units.

WARNING

When writing the given rates of change, make them positive if the variable is increasing and negative if the variable is decreasing.

4 Using the chain rule to solve a related rates problem

A spherical hot air balloon develops a leak so that t hours later the radius r metres is given by $r(t) = 24 - 3t^2$.

Assuming the balloon remains spherical (that is, $V(r) = \frac{4\pi}{3}r^3$), determine the rate at which the balloon is losing air 2 minutes after the leak commenced.

THINKING	WORKING
1 Identify the key elements in the problem then write the given information and the required rate.	Given: $V(r) = \frac{4\pi}{3}r^3$ $r(t) = 24 - 3t^2$ Required to determine: $\left.\frac{dV}{dt}\right\|_{t=\frac{2}{60}}$
2 Apply the chain rule to write the required rate of change in terms of other related rates.	Volume is an explicit parametric function of time, so $V(t) \equiv V(r(t))$ $\left.\frac{dV}{dt}\right\| = \frac{dV}{dr} \times \frac{dr}{dt}$
3 Substitute the derivatives needed into the related rates equation.	$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4\pi}{3}r^3\right) \times \frac{d}{dt}\left(24 - 3t^2\right)$ $= 4\pi\left(r^2\right) \times (-6t)$ $= 4\pi\left(24 - 3t^2\right)^2 \times (-6t)$
4 Calculate the required rate at the given time.	$\left.\frac{dV}{dt}\right\|_{t=\frac{2}{60}} = 4\pi\left(24 - 3 \times \left(\frac{2}{60}\right)^2\right)^2 \times -6 \times \left(\frac{2}{60}\right)$ $= -1447.24$ (2 d.p.)
5 Interpret the solution in the context of the question and write the answer.	After 2 minutes, the balloon is losing air at a rate of approximately 1447 cubic metres per hour.

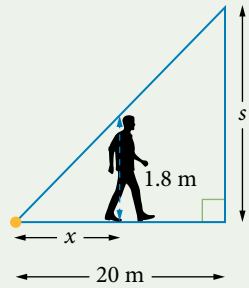
WARNING

Never substitute values for changing quantities into a related rates problem until after all derivatives have been taken. Substituting too soon makes changeable variables behave like constants, with zero derivatives.

5 Using related rates to determine shadow lengths

A man 1.8 m tall walks away from a spotlight positioned on the floor and angled towards a wall 20 m away. The man walks away from the spotlight towards the wall at 2 m/s.

Calculate the rate of change in the height of the man's shadow when he is 10 m from the light.

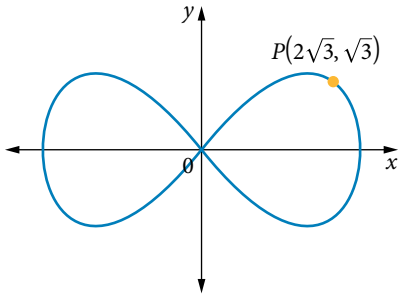
THINKING	WORKING
1 Draw a diagram and define the variables.	Let s be the height of the man's shadow, x be the distance between the man and the light and t be the time after he starts walking, in seconds. 
2 Write the given information and the required rate.	Given: $\frac{dx}{dt} = 2$ m/s; The man is 1.8 m tall. The distance between the light and the wall is 20 m. Required to determine: $\frac{ds}{dt}$ when $x = 10$ m.
3 Use the chain rule to write the required rate of change in terms of other related rates.	$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$
4 Substitute the derivatives needed into the related rates equation.	Using similar triangles, $\frac{1.8}{s} = \frac{x}{20}$ $s = \frac{36}{x}$ $\therefore \frac{ds}{dx} = -\frac{36}{x^2}$ Therefore $\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$ $= \frac{36}{x^2} \times 2$ $= -\frac{72}{x^2}$
5 Calculate the required rate at the given value.	$\left.\frac{ds}{dt}\right\|_{x=10} = -\frac{72}{(10)^2}$ $= -0.72$

6 Interpret the solution with appropriate units.	The height of the man's shadow on the wall is decreasing at a rate of 0.72 m/s when he is 10 metres from the wall.

6 Using implicit differentiation to solve a related rates problem

A toy car travels in an around a toy racetrack in the shape of a figure eight as shown.

The equation of the toy racetrack is given by $x^4 - 16\left(x^2 - y^2\right) = 0$, where x and y are measured in metres. At the instant the toy car is at the point $P(2\sqrt{3}, \sqrt{3})$, $\frac{dx}{dt} = -5$ m/s. Determine $\frac{dy}{dt}$ at this same instant.

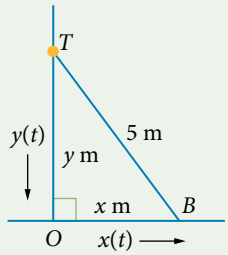


THINKING	WORKING
1 Identify the given information and the required rate.	Given: $x^4 - 16\left(x^2 - y^2\right) = 0$ and $\frac{dx}{dt} = -5$ Required to determine: $\frac{dy}{dt}$ at the point $P(2\sqrt{3}, \sqrt{3})$
2 Use the chain rule to write the required rate of change in terms of other related rates.	$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$
3 Differentiate implicitly to determine $\frac{dy}{dx}$.	$\frac{d}{dx}\left(x^4 - 16\left(x^2 - y^2\right)\right) = \frac{d}{dx}(0)$ $4x^3 - 16\left(2x - 2y\frac{dy}{dx}\right) = 0$ $2x - 2y\frac{dy}{dx} = \frac{x^3}{4}$ $\frac{dy}{dx} = \frac{8x - x^3}{8y}$
4 Calculate the required rate at the given value.	$\frac{dy}{dt} = \left.\frac{dy}{dx}\right\|_{\substack{x=2\sqrt{3} \\ y=\sqrt{3}}} \times \frac{dx}{dt}$ $= \frac{16\sqrt{3} - 8\left(\sqrt{3}\right)^3}{8\sqrt{3}} \times -5$ $= -1 \times -5$
5 Interpret the solution in the context of the question.	When the toy car is at point $P(2\sqrt{3}, \sqrt{3})$, the speed along the y-axis is 5 m/s.

7 Using implicit differentiation to solve the sliding ladder problem

A 5 m long ladder rests against a wall at point T that is y m above the ground, with the foot of the ladder B at x m from the wall. At time $t = 0$ seconds, $x = 3$ m and the foot of the ladder begins sliding along the ground away from the wall at a rate of 1 m/s.

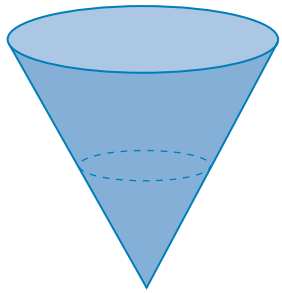
Calculate the rate at which the top end of the ladder at point T starts sliding down the wall.

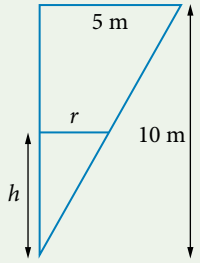
THINKING	WORKING
1 Draw and label a diagram.	
2 Identify the given information and the required rates of change.	Given: $\left.\frac{dx}{dt}\right\|_{t=0} = 1\text{ m/s}$ Required to determine: $\left.\frac{dy}{dt}\right\|_{t=0}$
3 Use the chain rule to write the required rate of change in terms of other related rates.	$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$
4 The two variables are related by an implicit relation. Differentiate both sides with respect to x.	$x^2 + y^2 = 25$ $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$ $2x + 2y \times \frac{dy}{dx} = 0$ $2y \times \frac{dy}{dx} = -2x$ $\therefore \frac{dy}{dx} = -\frac{x}{y}$
5 Solve for any other needed information.	Using Pythagoras theorem when $x = 3$, $y = 4$.
6 Calculate the required rate at the given value.	$\left.\frac{dy}{dt}\right\|_{t=0} = -\left.\frac{x}{y}\frac{dx}{dt}\right\|_{t=0}$ $= -\frac{3}{4} \times 1\text{ m/s}$ $= -\frac{3}{4}\text{ m/s}$
7 Interpret the solution in the context of the question.	When the ladder first begins to slip, the top moves downward at $\frac{3}{4}$ m/s.

8 Using related rates to solve a problem involving a cone

A water tank is in the shape of an inverted cone as shown.

The height of the cone is 10 m and the radius of the base is 5 m. Water flows into the empty tank at the rate of 2 m^3/h. How fast is the water level rising in the cone when the water is 5 m deep?



THINKING	WORKING
1 Draw and label a diagram with the appropriate variables and units.	Let V be the volume of water in the tank; h be the height of the water level after t hours; r be the radius of surface of the water after t hours. A cross-section of half of the cone is shown. 
2 Identify the given information and the required rate.	Given: $\frac{dV}{dt} = 2\ m^3/h$ Height of tank: 10 m Radius of tank base: 5 m Required: $\frac{dh}{dt}$ when $h = 5$
3 Use the chain rule to write the required rate of change in terms of other related rates.	In this case it is more convenient to write the chain rule in a slightly different form as you know the rule for volume in terms of the radius and height of the cone. $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ Volume of a cone: $V = \frac{1}{3}\pi r^2 h$
4 Additional information must be used to eliminate one of the variables in the rule for the volume.	Using similar triangles to express r in terms of h, $\frac{r}{h} = \frac{5}{10}$ $r = \frac{1}{2}h$ Substitute for r: $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ $= \frac{1}{12}\pi h^3$

5 Substitute the known derivatives into the chain rule.	$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $2 = \frac{d}{dh}\left(\frac{1}{12}\pi h^3\right) \times \frac{dh}{dt}$ $2 = \frac{\pi h^2}{4}\frac{dh}{dt}$
6 Calculate the required rate at the given value.	$\frac{dh}{dt} = \frac{4}{\pi \times 5^2} \times 2$ $= \frac{8}{25\pi}$ m/h $= 0.10$ m/h (2 d.p.)
7 Interpret the solution in the context of the question.	The water level is rising at approximately 0.10 m/h when the water is 5 m high.

EXERCISE 7.2 Related rates

1 Use the chain rule to determine each of the following.

(a) The radius r of a circle is related to its area A by the equation $A = \pi r^2$. Write an equation that relates $\frac{dA}{dt}$ to $\frac{dr}{dt}$.

(b) The volume V of a cube is related to the side length a by the equation $V = a^3$. Write an equation that relates $\frac{dV}{dt}$ to $\frac{da}{dt}$.

(c) The power P in a circuit is related to the current I by the equation $P = RI^2$, where the resistance R is assumed to remain constant. Write an equation for $\frac{dP}{dt}$ in terms of $\frac{dI}{dt}$.

(d) A particle travels along a path modelled by the curve with equation $y = 3x^2$. Write an equation for $\frac{dy}{dt}$ in terms of $\frac{dx}{dt}$.

Worked Example

(e) The light intensity I units at depth x metres in ocean water is given by the equation $I = e^{-0.005x}$. Write an equation for $\frac{dI}{dt}$ in terms of $\frac{dx}{dt}$.

4

2 Use the chain rule for each of the following problems.

(a) A circular puddle of water is increasing in size during a rain storm with its radius r cm increasing at a rate of 3 cm/minute. Determine the rate at which its area A cm^2 is increasing when the radius is 20 cm.

(b) An ice cube is melting, and the length of its side L cm is decreasing by 0.5 cm/hour. Determine the rate at which its volume V cm^3 is decreasing when the side length is 3 cm.

5

3 A light is 4.5 metres directly above a path. A woman 1.5 metres tall is walking along the path away from the light at a constant speed of 0.8 metres/second. At what speed is the length of her shadow increasing?

4 Let $y = \sin(x^2)$. Given that $\frac{dx}{dt} = 2$, determine $\frac{dy}{dt}$ when $x = \sqrt{\pi}$.

5 The area A of a triangle is related to the side lengths a, b and included angle θ by the formula $A = \frac{1}{2}ab\sin(\theta)$. Assume that the angle θ is a constant but the side lengths a and b are each differentiable functions of t. Determine an equation that relates $\frac{dA}{dt}$, $\frac{da}{dt}$ and $\frac{db}{dt}$.

Worked Example 6

6 A toy train runs clockwise around an elliptical track, modelled by the equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$. At the instant when the train is at the point $\left(1, \frac{4\sqrt{2}}{3}\right)$, $\frac{dx}{dt} = 16$ cm/second. Determine the value of $\frac{dy}{dt}$ at that instant.

7 Use the chain rule and implicit differentiation to determine the following.

(a) If $\frac{x}{y} + x = 2$, write an equation for $\frac{dy}{dt}$ in terms of x, y and $\frac{dx}{dt}$.

(b) If $xy^2 + y = 6$ and $\frac{dx}{dw} = 26$ at the point $(7, -1)$, determine the value of $\frac{dy}{dw}$ at that point.

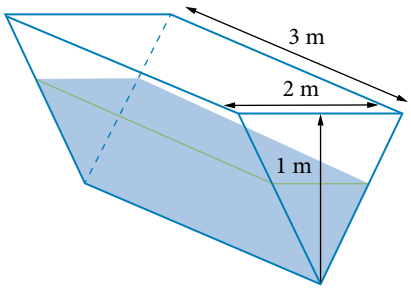
8 A ladder 6 metres long has its upper end against a vertical wall and its lower end on a horizontal floor. The lower end moves away from the wall at a speed of 1.5 metres/second. Determine the speed at which the upper end moves down the wall 2 seconds after it begins to move.

9 A right circular cylinder of volume V cubic units has height h and base radius r.

(a) Determine the rate of change of the volume with respect to the radius if the height stays constant.

(b) Determine the rate of change of the volume with respect to the height if the radius stays constant.

10 A trough is being filled with water. The trough is a triangular prism 3 metres long, 1 metre high, and 2 metres wide at the top. The trough is being filled at a rate of $\frac{1}{100}$ cubic metres per minute.

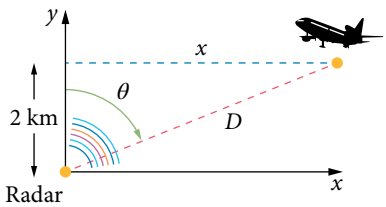


(a) If the depth of water is h metres, demonstrate that the volume is $V = 3h^2$ cubic metres.

(b) Determine an expression for $\frac{dh}{dt}$ in terms of h and $\frac{dV}{dt}$.

(c) At what rate is the depth of the water changing when the water is 20 centimetres deep?

11 An aircraft flying at a fixed altitude of 2 km and a constant velocity passes directly over a radar antenna. When the aircraft is 5 km past the antenna, the radar detects that the distance D is changing at the rate of 460 km/h.



(a) Determine the speed of the aircraft in the form $m\sqrt{n}$, at that instant, where m and n are natural numbers.

(b) Determine an expression for $\frac{d\theta}{dt}$ after t hours.

(c) Hence, demonstrate that $\frac{d\theta}{dt} = \frac{184\sqrt{29}}{D^2}$.

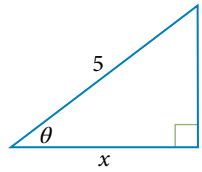
(d) Determine the rate of change of θ with respect to time when the aircraft is 5 km past the antenna. Express your answer correct to 4 decimal places.

12 According to the theory of relativity, the mass M kg of an object travelling at a speed v km/s is given by $M = m\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, where m kg is the mass of the object when at rest and c km/s is the speed of light.

Captain Kirk, whose mass at rest is 80 kg, is captain of the *Starship Enterprise*. If at a particular instant the *Starship's* speed is $0.8c$ km/s (that is, 0.8 times the speed of light) and increasing at a rate of $0.025c$ km/s^2, at what rate is Captain Kirk's mass increasing?

13 In the triangle shown, θ decreases at a rate of 3 radians per second.

At what rate is x increasing at the instant when $x = 3$?



14 The volume (V, measured in cubic units) of an expanding cube is increasing three times as fast as its side length (x, measured in linear units).

(a) Which of the following is *not* a correct statement and why?

A $\frac{dV}{dt} = 3\frac{dx}{dt}$

B $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

C $\frac{dV}{dt} = x^3\frac{dx}{dt}$

D $x = 1$

(b) Explain why the other statements in part **(a)** are correct.

15 **(a)** Which one of the following statements is correct if $x^3 + y^2 - y = 7$?

A $3x^2 + 2y - 1 = 7$

B $3x^2\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 7$

C $3x^2\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 0$

D $3x^2 + 2y\frac{dy}{dx} - \frac{dy}{dx} = 0$

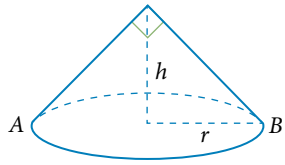
(b) Explain why the other statements in part **(a)** are incorrect.

16 The volume V of a cone with a base of area A square metres and a height h metres is given by $V = \frac{1}{3}Ah$. Provided $A > 0$, A is decreasing at 2 square metres per minute, and h is increasing at 2 metres per minute. Determine a relationship between A and h for which the volume is decreasing.

Worked Example

8

17 A conical heap of sand has a vertical angle of 90° as shown. Sand is being poured onto the heap from above. If the height $(h\,\text{cm})$ is increasing at a rate of 0.6 cm/min, determine the rate at which the volume $(V\,\text{cm}^3)$ is increasing when the height is 10 cm.



18 A particle travels anticlockwise around a vertical circle of radius 2 metres at a constant speed of 12 revolutions per minute. The equation of the circle is given by $x^2 + y^2 = 4$, where the x- and y-axes are horizontal and vertical, respectively. When the particle is at the point $P(x, y)$, the radius OP makes an angle θ with the x-axis.

(a) Express x and y in terms of θ.

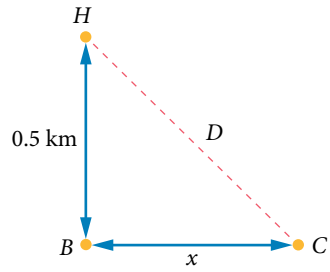
(b) Determine the value of $\frac{d\theta}{dt}$, expressing your answer in radians per second.

(c) Express $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of θ.

(d) Determine the horizontal velocity of the particle when $\theta = \frac{\pi}{6}$.

(e) Determine the coordinates of the particle when it is falling most rapidly.

19 A police helicopter at H hovers 0.5 kilometres above point B along a highway with a speed limit of 100 km/h. A car is observed at point C on the highway below, moving away from the helicopter. The radar measures the distance D as 1 km and the car travelling at a rate of 95 km/h.



(a) Demonstrate that $D^2 - x^2 = 0.25$.

(b) Hence, determine an expression for $\frac{dx}{dD}$.

(c) Is the car travelling above the speed limit?

20 Consider the functions $f(x) = e^x$ and $g(x) = \log_e(x)$.

(a) Determine the derivatives of $f(x)$ when $x = 1$ with the derivative of $g(x)$ when $x = f(1)$, and write a relationship connecting them.

(b) If $y = f(x)$ is a function of x with an inverse function $x = f^{-1}(y)$, demonstrate that $\frac{df}{dx} = \frac{1}{\left(\frac{df^{-1}}{dy}\right)}$.

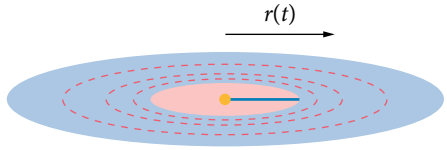
7.3 Introduction to differential equations

The study of differential equations began with work carried out by Sir Isaac Newton (1642–1727) to investigate the orbits of the planets about the Sun. Newton referred to these equations as *fluxional equations*. The term 'differential equation' was suggested by Newton's contemporary, Gottfried Leibnitz.

Differential equations are a powerful way to represent and predict the behaviour of systems that are changing with time. Applications include growth rates, situations involving velocity and acceleration, and modelling problems in a variety of physical contexts.

Writing equations to represent the information in the following situations results in a differential equation in each case:

- A circular oil slick is expanding so that its radius is increasing at 0.03 m/s. The variables here are the radius r metres and the time t seconds. The increase of 0.03 m/s represents a rate of change of r with respect to t, hence $\frac{dr}{dt} = 0.03$.



- A curve has the property that, at the point (x, y) on the curve, the sum of the squares of the y-coordinate and the gradient total one unit. In this case $y^2 + \left(\frac{dy}{dx}\right)^2 = 1$.

These are both first order differential equations, as the highest derivative involved is the first derivative. This section is almost entirely concerned with first order equations.

Whereas the solution of an algebraic equation such as $x + 1 = 0$ is a number, the solution of a differential equation is a function, often with an arbitrary constant as part of the solution.

9 Setting up differential equations

Write a differential equation that represents each of the following situations.

(a) In a country town during summer, the rate of increase of the number of mosquitoes is proportional to the number of mosquitoes present.

THINKING	WORKING
1 Define variables.	Let M be the number of mosquitoes present and t be the time in days.
2 Write the differential equation.	$\frac{dM}{dt} = kM$, where k is a constant.

(b) The gradient of a curve at a point on the curve is twice the product of its x and y coordinates at that point.

1 Define variables.	The coordinates of a point are (x, y).
2 Write the differential equation.	$\frac{dy}{dx} = 2xy$

A differential equation of the general form $\frac{dy}{dx} = ky$, is one in which the rate of change of a variable y with respect to a second variable x, is directly proportional to y.

10 Verification of a solution to a differential equation

Verify by differentiation that $y = 3e^{-x}$ is a solution of the differential equation $\frac{dy}{dx} = -y$.

THINKING	WORKING
1 Determine the left-hand side of the differential equation.	$\text{LHS} = \frac{dy}{dx}$ $= \frac{d}{dx}\left(3e^{-x}\right)$ $= -3e^{-x}$
2 Determine the right-hand side of the differential equation.	$\text{RHS} = -y$ $= -3e^{-x}$
3 Verify that the two sides are equal.	$\text{LHS} = -3e^{-x}$ $= \text{RHS}$

Because a constant has a derivative of zero, it is possible for a differential equation to have multiple solutions. For example, if $\frac{dy}{dx} = 2x$, then the solution is $y = x^2 + c$, where the constant of integration represents any real number; that is, $c \in \mathbb{R}$. This is called the general solution. If further information is given, then you can determine the value of the constant of integration, c, and you have a particular solution.

Determining solutions to a differential equation of the form $\frac{dy}{dx} = f(x)$

If $\frac{dy}{dx} = f(x)$, then $y = \int f(x)\,dx$

11 Determining solutions to a differential equation of the form $\frac{dy}{dx} = f(x)$

Consider the differential equation $\frac{dy}{dx} = 2\cos(2x)$.

(a) Determine the general solution.

THINKING	WORKING
1 Anti-differentiate to determine an expression for y in terms of x, including the arbitrary constant.	$y = \int 2\cos(2x)\,dx$
2 Write the general solution.	$y = \sin(2x) + c$

(b) If $y = 0$ when $x = \frac{\pi}{4}$, determine the particular solution.

1 Substitute the given values into the general solution to determine the value of the constant.	$0 = \sin\left(\frac{\pi}{2}\right) + c$ $0 = 1 + c$ $c = -1$
2 Write the particular solution.	$y = \sin(2x) - 1$

Differential equations of the form $\frac{dy}{dx} = f(y)$

If a differential equation is of the form $\frac{dy}{dx} = f(y)$, then a different technique is needed. Recall that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$. This is used in solving this type of differential equation.

If $\frac{dy}{dx} = f(y)$

$$\frac{dx}{dy} = \frac{1}{f(y)}$$

$$x = \int \frac{1}{f(y)}\, dy$$

12 Determining a general solution of a differential equation of the form $\frac{dy}{dx} = f(y)$

Solve the differential equation $\frac{dy}{dx} = y^2 + 1$.

THINKING	WORKING
1 Use $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ to rewrite the equation.	$\frac{dx}{dy} = \frac{1}{y^2 + 1}$
2 Anti-differentiate to determine an expression for x in terms of y.	$x = \int \frac{1}{y^2 + 1}\, dy$ $= \tan^{-1}(y) + c$
3 Rearrange to make y the subject for the general solution.	$y = \tan(x - c)$

13 Determining a particular solution to a differential equation of the form $\frac{dy}{dx} = f(y)$

If $y = 0$ when $x = 2$, solve the differential equation $\frac{dy}{dx} = y + 1$.

THINKING	WORKING
1 Use $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ to rewrite the equation.	$\frac{dx}{dy} = \frac{1}{y+1}$
2 Anti-differentiate to determine an expression for x in terms of y.	$x = \int \frac{1}{y+1}\,dy$ $= \log_e(y+1) + c$
3 Rearrange the equation to make y the subject.	$y = e^{x-c} - 1$ $= Ae^x - 1$ (where $A = e^{-c}$)
4 Substitute the given values into the general solution to determine the value of the constant.	$0 = Ae^2 - 1$ $A = e^{-2}$
5 Write the particular solution.	$y = e^{-2}e^x - 1$ $= e^{x-2} - 1$

Applying the solution technique of Worked example 13 shows that the general solution to a differential equation of this form is $y = Ae^{kx}$. By substitution, you can see that when $x = 0$, $y = A$. So, if you are given an initial condition, you can deduce the value of A. Since the solution function is exponential, you have exponential growth if $k > 0$ and exponential decay if $k < 0$.

14 Solving a growth or decay problem

The number of bacteria in a sample of milk increases at a rate proportional to the number present. If the number of bacteria doubles in half an hour, calculate how long will elapse before an initial number of 1000 will become 1 000 000 (one million).

THINKING	WORKING
1 Define the variables and create the differential equation.	Let N be the number of bacteria present after t hours. $\frac{dN}{dt} = kN$
2 Use $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ to rewrite the equation.	$\frac{dt}{dN} = \frac{1}{kN}$
3 Anti-differentiate.	$t = \frac{1}{k}\int \frac{1}{N}\,dN$ $= \frac{1}{k}\log_e(N) + c$

4 Rearrange the equation.	$k(t-c) = \log_e(N)$ $N = e^{kt-kc}$ $N = e^{-kc} \cdot e^{kt}$ $N = N_0 e^{kt}$ Where N_0 is the initial value of $N(e^{-kc})$ when $t = 0$.
5 Use the given information to substitute into the equation.	When $t = \frac{1}{2}$, $N = 2N_0$ $2N_0 = N_0 e^{\frac{k}{2}}$ $\log_e(2) = \frac{k}{2}$ $k = 2\log_e(2)$ $N = N_0 e^{2\log_e(2)t}$
6 Use the initial and final conditions to solve the problem.	$1\,000\,000 = 1000e^{2\log_e(2)\,t}$ $1000 = e^{1.386\,29...}$ $t = \frac{\log_e(1000)}{1.386\,29...}$ $t = 4.98$ (2 d.p.)
7 Interpret the answer.	It takes 4.98 hours for the number of bacteria to grow to one million.

Explore further

Differential equations

Use technology to explore the relationship between a first order differential equation and its solution.

15 Verifying a solution to a differential equation using implicit differentiation

Verify by implicit differentiation that $3x^2 + 2xy + 3y^2 - 4 = 0$ is a solution of the differential equation $\frac{dy}{dx} = -\frac{3x+y}{x+3y}$.

THINKING	WORKING
1 Differentiate the solution equation implicitly.	$6x + 2x\frac{dy}{dx} + 2y + 6y\frac{dy}{dx} = 0$
2 Solve for $\frac{dy}{dx}$.	$\frac{dy}{dx}(2x+6y) = -6x - 2y$ $\frac{dy}{dx} = \frac{-6x-2y}{2x+6y}$ $\frac{dy}{dx} = -\frac{3x+y}{x+3y}$ as required

Differential equations of the form $\frac{dy}{dx} = f(x)g(y)$

In this case, the separation of variables technique is used.

$$\text{If } \frac{dy}{dx} = f(x)g(y)$$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

16 Using separation of variables to solve a differential equation

Determine the particular solution of the differential equation $\frac{dy}{dx} = \frac{\sin(x)}{3y^2}$ if $y = 0$ when $x = \frac{\pi}{3}$.

THINKING	WORKING
1 Separate the variables onto each side.	$\int 3y^2\,dy = \int \sin(x)\,dx$
2 Anti-differentiate both sides.	$y^3 = -\cos(x) + c$
3 Substitute the given conditions to determine the value of the constant of integration.	$0 = -\frac{1}{2} + c$ $c = \frac{1}{2}$
4 Write the answer.	$y^3 = -\cos(x) + \frac{1}{2}$ or $y = \left(-\cos(x) + \frac{1}{2}\right)^{\frac{1}{3}}$

EXERCISE 7.3 Introduction to differential equations

Worked Example 9

1 Create a differential equation to represent each of these situations.

(a) A hot thin metal bar is cooled so that its length, L cm, decreases at 0.0003 cm/min.

(b) A curve has the property that, at the point (x, y) on the curve, the sum of the square of the gradient and the x-coordinate is equal to zero.

(c) The volume of a balloon is increasing at a rate that is proportional to its radius.

(d) Newton's law of cooling states that the rate of change of cooling of a body is proportional to the difference in temperature between the body and its surroundings.

10

2 Verify by differentiation that the given function is a solution of the corresponding differential equation.

(a) $y = \frac{x^3}{3} - \frac{1}{x} + 7; \quad \frac{dy}{dx} - x^2 - \frac{1}{x^2} = 0$

(b) $y = xe^{-x^2}; \quad \frac{dy}{dx} + 2xy = e^{-x^2}$

(c) $y = -\frac{2}{3} + e^{-3x}; \quad \frac{dy}{dx} + 3y + 2 = 0$

(d) $y = x^n; \quad \frac{dy}{dx} - \frac{n}{x}y = 0$

11

3 Determine a general solution for each of the following differential equations.

(a) $\frac{dy}{dx} - \frac{2}{\sqrt{1-x^2}} = 0$

(b) $\frac{dz}{dt} = \frac{t}{1+t^2}$

(c) $\frac{dy}{dx} = x^2\sqrt{x}$

(d) $\frac{dy}{dx} = 2\cos^2(x)$

4 Determine the particular solution for the differential equations in question **3** under the following conditions.

(a) $y = 5$ when $x = 1$

(b) $z = 0$ when $t = 0$

(c) $y = 1$ when $x = 1$

(d) $y = 0$ when $x = \pi$

12

5 Determine the general solution for each of the following differential equations.

(a) $\frac{dy}{dx} = 2y - 1$ **(b)** $2\frac{dy}{dx} - 3y = 0$ **(c)** $y^2\frac{dy}{dx} = 1$ **(d)** $\frac{dh}{dt} = \frac{3}{h}$

13

6 Determine the particular solution for each of the following differential equations under the given conditions.

(a) $\frac{dy}{dx} = -y$, if $y = 10$ when $x = 0$.

(b) $\frac{dy}{dx} = 1 + y^2$, if $y = \sqrt{3}$ when $x = 0$.

(c) $\frac{dy}{dx} = 3(1+y)^2$, if $y = 2$ when $x = 3$.

(d) $\frac{dy}{dx} = y - y^2$, if $y = \frac{1}{2}$ when $x = 0$.

14

7 A radioactive isotope decays at a rate proportional to the mass M milligrams present at time t years.

(a) Write the differential equation involving M and t and hence express M in terms of t.

(b) If 5 milligrams of the isotope decays to 4 milligrams over 1 year, determine

(i) the amount remaining after a second year (leave your answer correct to 1 decimal place).

(ii) the half-life of this isotope (leave your answer correct to 1 decimal place).

8 Verify that $y = e^{x^n}$ is a solution of the differential equation $\frac{dy}{dx} - nx^{n-1}y = 0$.

15

9 Use implicit differentiation to demonstrate that the given equation is a solution to the corresponding differential equation.

(a) $x^2 + 2y^2 = c; \; 2y\frac{dy}{dx} + x = 0$

(b) $2x^2 - y^2 = c; \; y\frac{dy}{dx} = 2x$

10 Sketch the suggested particular solutions of these differential equations.

(a) $x\dfrac{dy}{dx} - 3y = 0$; general solution $y = cx^3$; for $c = \pm 1, \pm 2$.

(b) $y\dfrac{dy}{dx} = x$; general solution $x^2 - y^2 = c$; for $c = \pm 1, \pm 2$.

(c) $4y\dfrac{dy}{dx} + x = 0$; general solution $x^2 + 4y^2 = c$; for $c = 1, 2, 3, 4$.

(d) $x\dfrac{dy}{dx} + y = 0$; general solution $xy = c$; for $c = \pm 1, \pm 2$.

Worked Example 16

11 Use separation of variables to determine solutions to the following differential equations.

(a) $\dfrac{dy}{dx} = 4y^2x$ **(b)** $\dfrac{dy}{dx} = \dfrac{\cos(2x)}{\sqrt{y}}$ **(c)** $\dfrac{dy}{dx} = \dfrac{e^x}{\left(1 + y^2\right)}$

(d) $\dfrac{dy}{dx} = \dfrac{x}{y}$, if the solution curve passes through the point $(4, 6)$.

(e) $x\cos\left(y\right)\dfrac{dy}{dx} = 1 + \sin\left(y\right)$, if the solution curve passes through the point $(1, 0)$.

(f) $\dfrac{dy}{dx} = xe^{-2y}$, if the solution curve passes through the point $(0, 0)$.

12 If $\left(x^2 + 4\right)\dfrac{dy}{dx} = 4$ and when $x = 2$, $y = 2$, determine the value of y when $x = 2\sqrt{3}$.

13 Verify that the given general solution is a solution of the differential equation, and determine the particular solution.

(a) $\dfrac{dy}{dx} = y(1 - y)$; general solution $y = \dfrac{e^x}{A + e^x}$ and $y(0) = 2$.

(b) $x^2\dfrac{dy}{dx} = -y^2$; general solution $y = \dfrac{-x}{Ax + 1}$ and $y(1) = -\dfrac{1}{3}$.

14 Verify that the given general solution is a solution of the corresponding differential equation.

(a) $y^3 - 3xy = c$; $(y^2 - x)\dfrac{dy}{dx} = y$ **(b)** $x^2 + xy^2 = c$; $2xy\dfrac{dy}{dx} + y^2 + 2x = 0$

(c) $\dfrac{1}{x^2} - \dfrac{y^2}{x^4} = c$; $\dfrac{dy}{dx} = -\dfrac{x}{y} + \dfrac{2y}{x}$

15 Which of the following statements is correct and why?

A The solution of a differential equation is always a number.

B The graphs of several particular solutions of a differential equation are the same shape but translated up or down.

C The solution of a differential equation is always a function.

D A differential equation can have many solutions.

16 Consider the differential equation $\frac{dy}{dx} = (y-1)(y+2)$.

(a) Determine an expression for x in terms of y.

(b) Rearrange the equation to make y the subject to determine the general solution.

(c) Determine the solution if the curve passes through the point $(0, 6)$.

17 The amount $m(t)$ of medication in mg/L remaining in the bloodstream t hours after ingesting it can be modelled by the differential equation

$$\frac{dm}{dt} = -3m + 4e^{-2t}$$

The first term on the right-hand side represents the rate at which the medication is absorbed from the blood into the body and the second term represents the rate at which the medicine enters the bloodstream (at first rapidly as most of the pill dissolves, and later more slowly when only a small amount of the pill remains)

(a) Verify by differentiation that $m(t) = 4(e^{-2t} - e^{-3t})$ is a solution of $\frac{dm}{dt} = -3m + 4e^{-2t}$.

(b) What is the initial amount of medication in the bloodstream?

(c) When is the amount of medication in the bloodstream at its greatest?

(d) What is the long-term amount of medication in the bloodstream?

18 If a differential equation contains a second derivative, it is called a second order differential equation. For example, consider $\frac{d^2y}{dx^2} + 4y = 0$.

(a) Verify that $y = a\sin(2x) + b\cos(2x)$ is a solution.

(b) Determine the particular solution under the conditions $y(0) = 1$ and $\left.\frac{dy}{dx}\right|_{x=0} = 2$.

19 The population of fish in a lake is initially 10 000. The population would increase at a rate of 20% per year, except that there is a fishing quota of k fish per year taken from the lake. The population P after t years is modelled by the solution of the differential equation:

$$\frac{dP}{dt} = \frac{1}{5}P - k \text{ with } P = 10\,000 \text{ when } t = 0.$$

In the equation above, k is the constant number of fish removed from the lake each year due to fishing (the fishing quota).

(a) If the fishing quota is set at $k = 1000$ per year, verify by differentiation that the number of fish in the lake t years later would be $P(t) = 5000\left(1 + e^{\frac{t}{5}}\right)$.

(b) Alternatively, if the fishing quota is set at $k = 3000$ per year, verify by differentiation that the number of fish in the lake t years later would be $P = 5000\left(3 - e^{\frac{t}{5}}\right)$.

(c) More generally, assume an arbitrary but fixed fishing quota of k fish per year. Verify by differentiation that the number of fish in the lake t years later would be $P(t) = 5\left(k + (2000 - k)e^{\frac{t}{5}}\right)$.

(d) Hence, choose a fishing quota k fish per year to maintain the fish population at 10 000.

Qualitative and numerical solutions of differential equations

7.4

Slope (direction or gradient) fields

Recall that $\frac{dy}{dx}$ represents the slope of a curve at any point on it. A differential equation, such as $\frac{dy}{dx} = f(x, y)$, can be considered a description of the slope of the tangent to a curve. It provides enough information for you to sketch a graphical representation of the solution curve without actually solving the differential equation. The following methods use a slope field, which is sometimes referred to as a direction or gradient field. In this section you will consider two methods of constructing a slope field: the Cartesian grid method and the isocline method.

Method 1 Slope field construction on a Cartesian plane

This method involves the following steps:

STEP 1 Evaluate $f(x, y) = \frac{dy}{dx}$ on a predetermined set of points (x, y) in the Cartesian plane (grid).

STEP 2 At each given point (x, y) on the grid, draw a short line segment with the same slope as calculated for $\frac{dy}{dx}$. (Note that the angle made by each line segment with the positive x-axis is θ, where $\tan(\theta) = \frac{dy}{dx}$.)

Although this method is labour intensive, it is relatively easy to implement using technology.

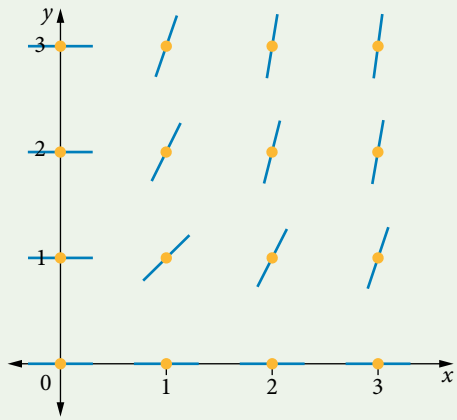
17 Using the grid method to construct the slope field of a first order differential equation

Construct the slope field for $\frac{dy}{dx} = xy$ on the following points.

$(0,0)$ $(0,1)$ $(0,2)$ $(0,3)$
$(1,0)$ $(1,1)$ $(1,2)$ $(1,3)$
$(2,0)$ $(2,1)$ $(2,2)$ $(2,3)$
$(3,0)$ $(3,1)$ $(3,2)$ $(3,3)$

THINKING	WORKING
1 Calculate the value of the gradient $f(x, y) = xy$ on the given grid.	$f(0,0) = 0$ $f(0,1) = 0$ $f(0,2) = 0$ $f(0,3) = 0$ $f(1,0) = 0$ $f(1,1) = 1$ $f(1,2) = 2$ $f(1,3) = 3$ $f(2,0) = 0$ $f(2,1) = 2$ $f(2,2) = 4$ $f(2,3) = 6$ $f(3,0) = 0$ $f(3,1) = 3$ $f(3,2) = 6$ $f(3,3) = 9$

2 At each such point (x, y) on the grid, draw line segments of slope $f(x, y)$.



Method 2 Slope field construction on the isoclines

This method is almost the reverse of the previous method, in that you choose a value for the gradient and then determine a set of points that have that gradient.

STEP 1 Choose values of c where $f(x, y) = c$, and c is a real constant and rearrange the resulting equation.

STEP 2 Draw the curve containing points with the chosen value of the gradient. If $c = 0$ the curve is called a *null-cline* and otherwise it is called an *isocline*.

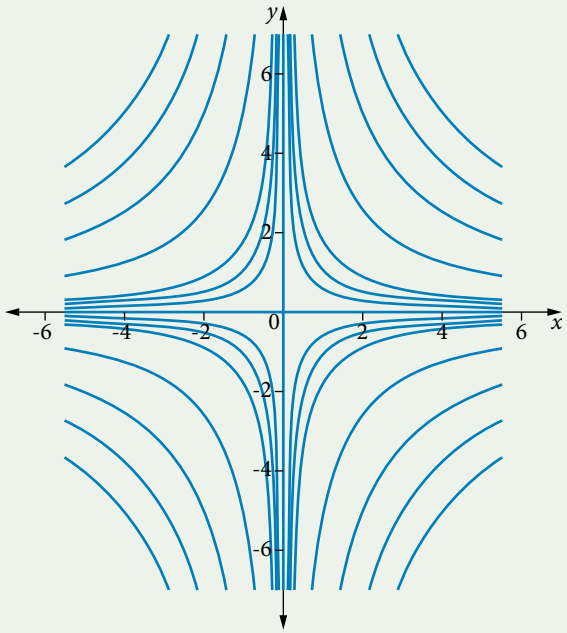
STEP 3 Along each null-cline or isocline $f(x, y) = c$, draw a number of short parallel line segments, each of slope c, to construct the slope field for the differential equation.

18 Using the method of isoclines to construct the slope field of a first order differential equation

Construct the slope field of $\frac{dy}{dx} = xy$ on the null-cline $c = 0$ and the isoclines, $c = \pm\frac{1}{\sqrt{3}}$, $c = \pm 1$, and $c = \pm\sqrt{3}$.

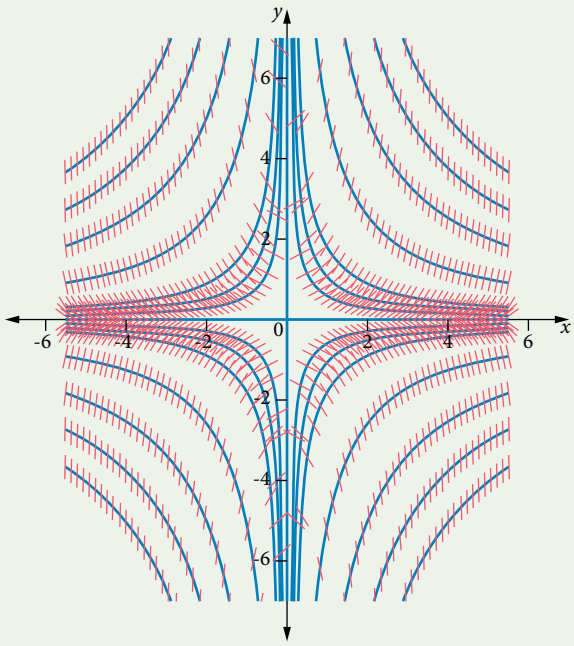
THINKING	WORKING
1 Substitute the given values of c into the differential equation.	$xy = 0, xy = \pm\frac{1}{\sqrt{3}}, xy = \pm 1, xy = \pm\sqrt{3}$
2 Rearrange the resulting equations if required.	$y = 0, y = \pm\frac{1}{\sqrt{3}\,x}, y = \pm\frac{1}{x}, y = \pm\frac{\sqrt{3}}{x}$

3 Draw the null-cline and isocline curves.



4 On each isocline, $f(x, y) = c$, draw all the parallel line segments of slope c.

The isoclines of slope c with $c = \pm\frac{1}{\sqrt{3}}, 0, \pm 1$ are used to construct the slope field.



WARNING

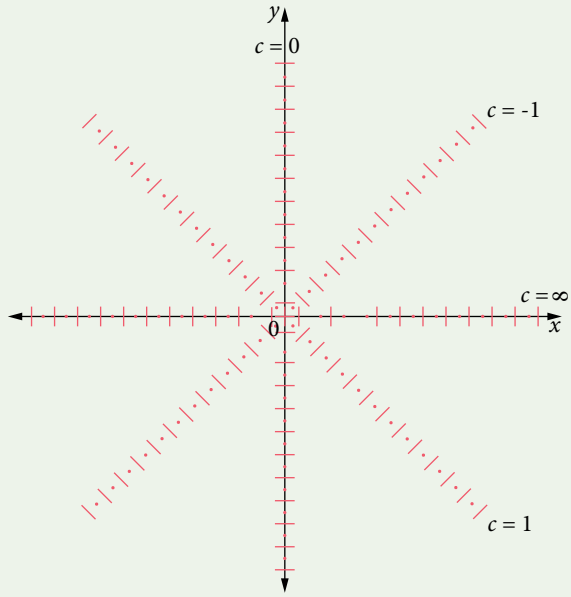
The isocline curves are not solution curves for the differential equation. They are just a series of points for which the gradient has a particular value.

Once the slope field has been constructed, it is possible to get a qualitative picture of the general solution of the differential equation. The resulting curves are called integral curves. If a point on the solution curve is known, then you can sketch an approximation of the particular solution.

19 Using a slope field to sketch solutions of a differential equation

Consider the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

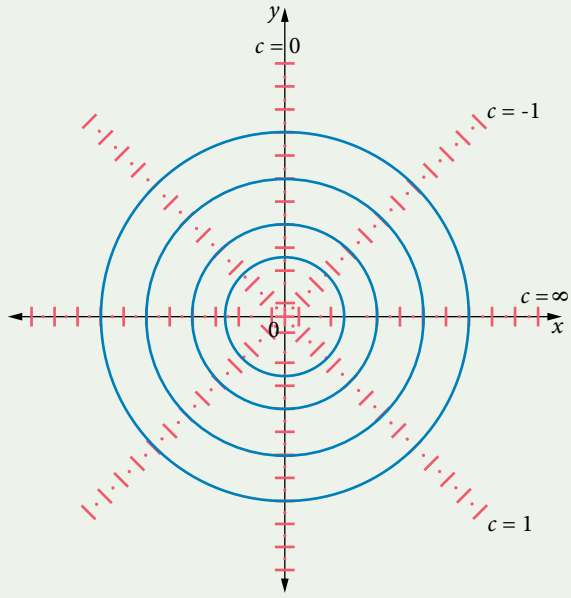
(a) Use the isocline method to construct a slope field with $\frac{dy}{dx} = c$, where $c = \pm 1, 0, \infty$.

THINKING	WORKING
1 Substitute the given values of c into the differential equation.	$-\frac{x}{y} = \pm 1, 0, \infty$
2 Rearrange the resulting equations if required.	$y = -x, y = x, x = 0, y = 0$
3 Draw the null-cline and isocline curves and mark line segments with the corresponding gradients.	

(b) Sketch four possible integral curves to illustrate the general solution.

Follow the gradients of the slope field to deduce the approximate shape of the solution (integral) curves.

The curves appear to be circles centred at the origin.



In Worked example 19, using the technique of separation of variables, you can confirm that the particular solution you have determined is correct. This will not always be the case.

$$\frac{dy}{dx} = \frac{y^2}{2} = -\frac{x^2}{2} + c_1$$
$$\int y\,dy = x^2 + y^2 = 2c_1$$
$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$
$$x^2 + y^2 = 2c_1$$

t Technology worked example
Constructing a slope field

Given that $2c_1$ is a constant, then the equation can be written in the form $x^2 + y^2 = c^2$.

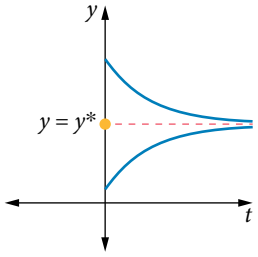
Steady state solutions of differential equations of the form $\frac{dy}{dx} = f(y)$

For a steady state solution, it is necessary for the solution equation to remain constant. This means that $\left.\frac{dy}{dx}\right|_{y=y*} = 0$, where $y*$ is the steady state solution. All steady state solutions $y = y*$ of the differential equation $\frac{dy}{dx} = f(y)$ are the null-clines, where $f(y*) = 0$ because they are places where the gradient is zero. Consequently, they appear as horizontal line segments on the slope field.

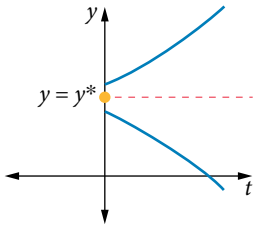
In the long-term, as $x \to \infty$, every solution y of $\frac{dy}{dx} = f(y)$ either moves towards or moves away from the null-clines of the slope field. An equilibrium solution is said to be stable if all nearby solutions converge towards the equilibrium solution in the long-term, as $x \to \infty$. Otherwise, the nearby solutions diverge from the equilibrium solution for large values of the independent variable. In this latter case, the solution $y = y*$ is classified as an unstable equilibrium solution.

The equilibrium (or steady state) solutions $y = y*$ of a differential equation of the form $\frac{dy}{dx} = f(y)$ occur at the roots $f(y*) = 0$.

Nearby solutions converge toward the stable equilibrium solution $y = y*$



Nearby solutions diverge away from the unstable equilibrium solution $y = y*$



e Explore further
Slope fields and differential equations
Use technology to explore slope fields of differential fields and solutions when given an initial condition.

One way to test the stability of a steady state solution is to calculate the value of $f'(y)$ (the derivative of the slope function with respect to y) at the values of $y*$ determined. If $f'(y) < 0$, then the solution is stable. If $f'(y) > 0$, then the solution is unstable.

c Making connections
Slope fields
Move the sliders to explore a slope field for the differential equation $\frac{dy}{dx} = f(x, y)$.

20 Determining a steady state solution for a differential equation of the form $\frac{dy}{dt} = f(y)$

Consider the differential equation $\frac{dy}{dx} = -2(y - 5)$.

(a) Construct the slope field of $\frac{dy}{dx} = -2(y - 5)$ on a suitable grid for $0 \leq x \leq 10$ and $0 \leq y \leq 10$.

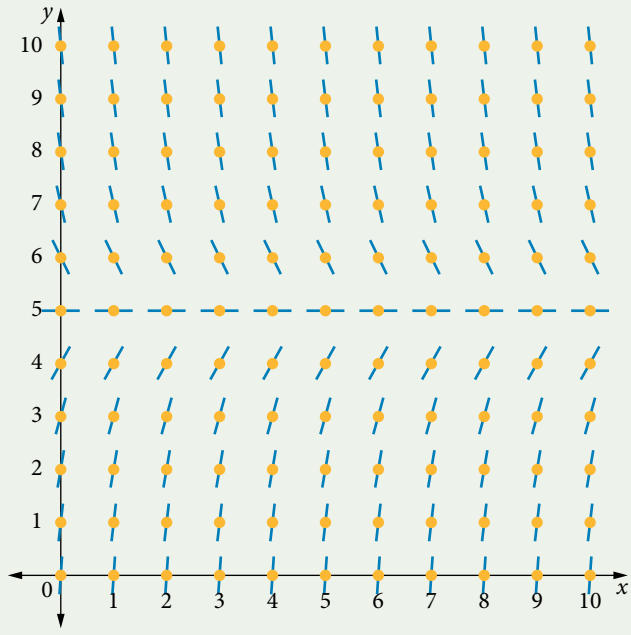
THINKING

1 Construct a table of values demonstrating the slope from $0 \leq y \leq 10$.

WORKING

y	Slope $\frac{dy}{dx}$
0	$-2(0 - 5) = 10$
1	$-2(1 - 5) = 8$
2	$-2(2 - 5) = 6$
3	$-2(3 - 5) = 4$
4	$-2(4 - 5) = 2$
5	$-2(5 - 5) = 0$
6	$-2(6 - 5) = -2$
7	$-2(7 - 5) = -4$
8	$-2(8 - 5) = -6$
9	$-2(9 - 5) = -8$
10	$-2(10 - 5) = -10$

2 Construct a grid and at each point (x, y), draw line segments of the slope function.



(b) Determine and classify any steady state solutions of $\frac{dy}{dx} = -2(y-5)$.

1 Steady state solutions of $\frac{dy}{dx} = f(y)$ are null-clines $\{y : f(y) = 0\}$.

$-2(y-5) = 0$

$y = 5$

2 Determine the value of $f'(y)$.

For $f(y) = \frac{dy}{dx} = -2y + 10$, $f'(y) = -2$

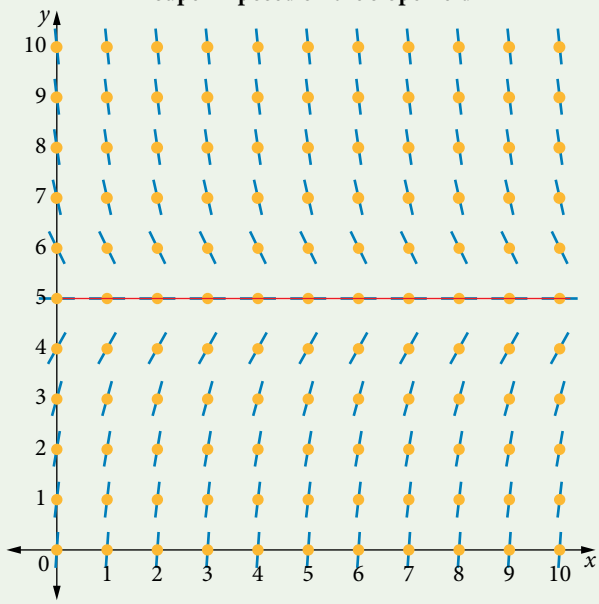
Given that $f'(y) < 0$, the solution $y = 5$ is stable.

Note that the slopes converge towards the line $y = 5$ as x increases, consistent with the stability.

(c) Draw and label the equilibrium solution(s) of $\frac{dy}{dx} = -2(y-5)$ on the slope field.

Draw and label the equilibrium solution(s) required.

The equilibrium solution superimposed on the slope field



Notice that the line segments on the slope field above $y = 5$ slope downwards, whereas those below slope upwards, appearing to converge towards $y = 5$.

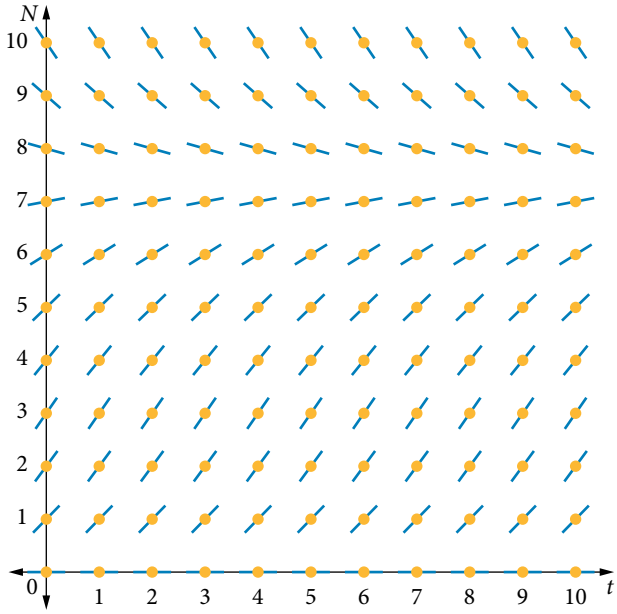
The slope field and the classification of its equilibrium solution(s) gives useful qualitative information about the general solution of $\frac{dy}{dx} = f(y)$. This qualitative information can be augmented with additional information about the concavity of the solution curves.

The concavity $\frac{d^2y}{dx^2}$ of a differential equation of the form $\frac{dy}{dx} = f(y)$ is given by $\frac{d^2y}{dx^2} = \frac{df(y)}{dy} \times \frac{dy}{dx} = f'(y)f(y)$. Therefore, y is concave up when $f'(y)$ and $f(y)$ have the same sign, and concave down when they differ in sign.

An understanding of the qualitative features of the general solution of a differential equation of the form $\frac{dy}{dx} = f(y)$ assists in determining its particular solutions, as shown in Worked example 21.

21 Analysing a first order differential equation of the form $\frac{dy}{dt} = f(y)$ qualitatively

The slope field of the differential equation $\frac{dN}{dt} = \frac{1}{2}N\left(2 - \log_e(N)\right)$ is shown.



This differential equation has null-clines at $N = 0$ and $N = s$, where s is a positive real number.

(a) Determine the value of s and use a sign diagram to classify the stability of the solution $N = s$.

THINKING	WORKING
1 Null-clines occur where the gradient function is equal to zero.	$\frac{dN}{dt} = \frac{1}{2}N\left(2 - \log_e(N)\right)$ $\frac{1}{2}N\left(2 - \log_e(N)\right) = 0$ $N = 0 \text{ or } N = e^2$ $\therefore s = e^2$
2 Determine the derivative of the gradient function at the required value.	$\frac{d}{dN}\left(\frac{1}{2}N\left(2 - \log_e(N)\right)\right) = \frac{d}{dN}\left(N - \frac{1}{2}N - \log_e(N)\right)$ $= 1 - \frac{N}{2} \cdot \frac{1}{N} - \frac{1}{2}\log_e(N)$ $= \frac{1}{2}\left(1 - \log_e(N)\right)$ $= \frac{1}{2}(1 - \log_e(e^2))$ $= \frac{1}{2}(1 - 2)$ $= -\frac{1}{2}$
3 Classify the stability of the solution.	Given that $\frac{d}{dN} < 0$ then the null-cline $N = e^2$ is a stable equilibrium solution.

(b) Hence, write the long-term value of N (that is, write the value of $\lim_{t\to\infty} N$).

In the long-term, all solutions move towards the stable equilibrium solution.

$\lim_{t\to\infty} N = e^2$.

(c) Verify by substitution that $\log_e(N) = 2 - 3e^{\frac{-t}{2}}$, $t \geq 0$ satisfies the differential equation $\frac{dN}{dt} = \frac{1}{2}N\left(2 - \log_e(N)\right), t \geq 0$.

1 Differentiate the given solution function.

Using implicit differentiation:

$$\frac{1}{N}\cdot\frac{dN}{dt} = \frac{3}{2}e^{-\frac{t}{2}}$$
$$\therefore \frac{dN}{dt} = N\cdot\frac{3}{2}e^{-\frac{t}{2}}$$

2 Determine whether the left-hand side of the equation is equal to the right hand side for the given solution.

For $\frac{dN}{dt} = \frac{1}{2}N(2 - \log_e(N))$ then:

$$\text{LHS} = \frac{dN}{dt} = N\cdot\frac{3}{2}e^{-\frac{t}{2}}$$

$$\text{RHS} = \frac{1}{2}N(2 - \log_e(N)) = \frac{1}{2}N\left(2 - \left(2 - 3e^{-\frac{t}{2}}\right)\right) = \frac{1}{2}N\cdot 3e^{-\frac{t}{2}} = N\cdot\frac{3}{2}e^{-\frac{t}{2}} = \text{LHS}$$

Therefore $\log_e(N) = 2 - 3e^{-\frac{t}{2}}$ satisfies the differential equation.

(d) If $N = a$ when $t = 0$, determine the exact value of a.

Substitute $t = 0$ and $N = a$ into the solution equation.

$$\log_e(N) = 2 - 3e^{\frac{-t}{2}}$$
$$\log_e(a) = -1$$
$$a = e^{-1}$$
$$a = \frac{1}{e}$$

(e) Determine an expression for $\frac{d^2N}{dt^2}$ in terms of N and $\log_e(N)$.

If $\frac{dN}{dt} = f(N)$ then:

$$\frac{d^2N}{dt^2} = \frac{df}{dN}\times\frac{dN}{dt} = f(N)f'(N)$$

$$\frac{d^2N}{dt^2} = \underbrace{\left(\frac{1}{2}N\left(2 - \log_e(N)\right)\right)}_{f(N)}\times\underbrace{\left(\frac{1}{2}\left(1 - \log_e(N)\right)\right)}_{f'(N)} = \frac{1}{4}N\left(1 - \log_e(N)\right)\left(2 - \log_e(N)\right)$$

(f) The graph of N as a function of t has a point of inflection with coordinates $(t, N) = (b\log_e(c), e^d)$, where b, c and d are positive integers. Determine the coordinates of this point of inflection.

1 The concavity of the solution must change sign either side of a point of inflection, therefore solve for t when $\frac{d^2N}{dt^2} = 0$.

$$\frac{d^2N}{dt^2} = 0$$

$$\frac{1}{4}N\left(1-\log_e(N)\right)\left(2-\log_e(N)\right) = 0$$

Therefore, $N = 0$, $N = e$ and $N = e^2$.

The function is undefined for $N = 0$ and $N = e^2$ is a steady state solution, so the only point of inflection occurs when $N = e$.

2 Substitute in the solution equation.

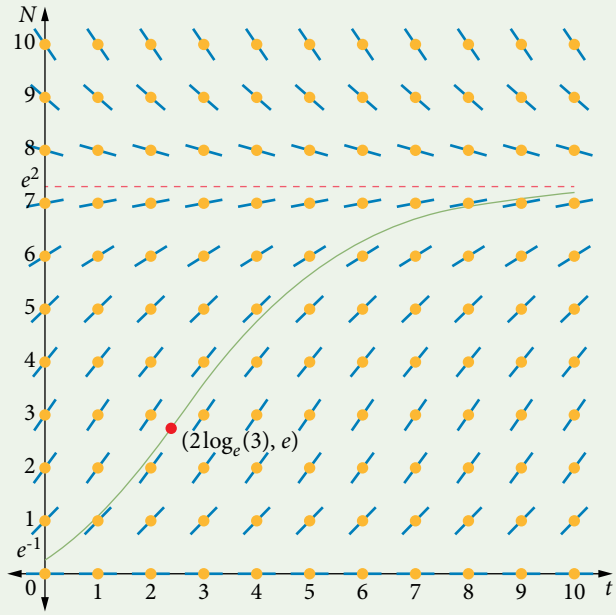
$$\log_e(e) = 2 - 3e^{\frac{-t}{2}}$$
$$1 = 2 - 3e^{\frac{-t}{2}}$$
$$-1 = -3e^{\frac{-t}{2}}$$
$$\frac{1}{3} = e^{\frac{-t}{2}}$$
$$\log_e\left(\frac{1}{3}\right) = -\frac{t}{2}$$
$$t = 2\log_e(3)$$

3 Interpret the solution.

The point of inflection occurs at $(2\log_e(3), e)$.

(g) Sketch the solution to $\frac{dN}{dt} = \frac{1}{2}N(2-\log(N))$, $N(0) = a$ on the slope field showing the coordinates of the point of inflection and the equilibrium solution $N = e^2$.

Sketch the particular solution, indicating the point of inflection and the steady state (equilibrium) solution.



The solution curve follows the directions of the local line segments in the slope field, converging towards the line $y = e^2$, as expected.

To assist with identification of the solution curve in a differential equation $\frac{dy}{dx} = f(x, y)$ associated with the given slope field, you should consider the answers to the following questions:

- What is the equation of the null-clines at $f(x, y) = 0$ and/or isoclines at $f(x, y) = c$?
- Is the slope constant along the x-axis?
- Are the slopes constant in rows (so the slopes depend on y only)?
- Is the slope constant along the y-axis?
- Are the slopes constant in columns (so the slopes depend on x only)?
- Is there even or odd symmetry of the slopes about either of the axes?

EXERCISE 7.4 Qualitative and numerical solutions of differential equations

Worked Example 17

1 Use the grid method to construct a slope field for each of the following differential equations at the points $(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)$.

(a) $\frac{dy}{dx} = x + y$ **(b)** $\frac{dy}{dx} = -xy$

2 Consider the differential equation $\frac{dy}{dx} = 1 + y^2$.

(a) Determine the angle the solution function makes with the positive direction of the x-axis for each of the given values of y. Leave your answers correct to the nearest whole degree. (Hint: The angle of inclination θ of the line segments of the slope field is given by $\tan(\theta) = 1 + y^2$, which is independent of x.)

(i) $y = 0$ **(ii)** $y = \pm 1$ **(iii)** $y = \pm 2$ **(iv)** $y = \pm 3$

(b) Sketch the slope field using the values determined in part **(a)**.

(c) Verify by substitution that $y = \tan(x)$ is a solution of $\frac{dy}{dx} = 1 + y^2$ and, on the slope field, sketch the particular solution of the curve that passes through the origin.

3 Consider the differential equation $\frac{dy}{dx} = x - y + 1$. 18

(a) Use the isocline method to sketch the slope field, where $\frac{dy}{dx} = c$ and $c = -2, -1, 0, 1, 2, 3$.

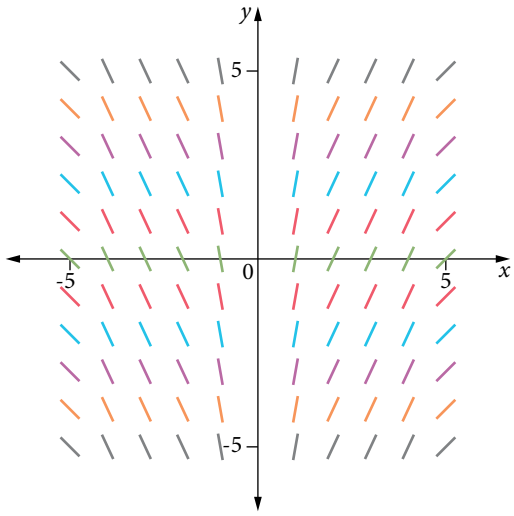
(b) Write the equation of the null-cline.

(c) The function $y = x$ appears to be a solution of $\frac{dy}{dx} = x - y + 1$. Verify that it is a solution.

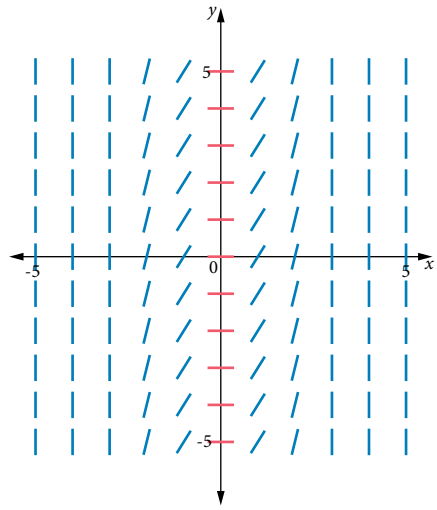
4 On the slope diagram drawn in question **3**, sketch solution curves (integral curves) passing through $(0,1)$ and $(0,-1)$. 19

5 For each of the following slope fields, sketch a solution curve through the given point.

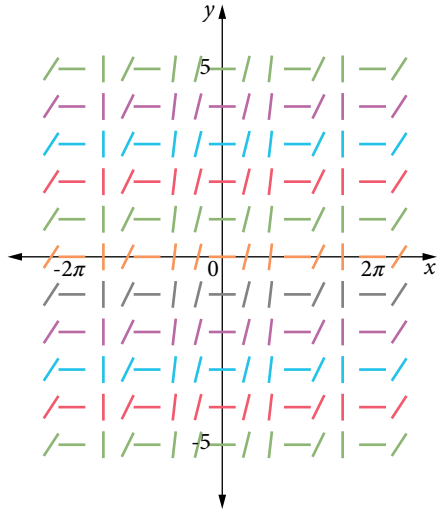
(a) Passing through $(2, 2)$.



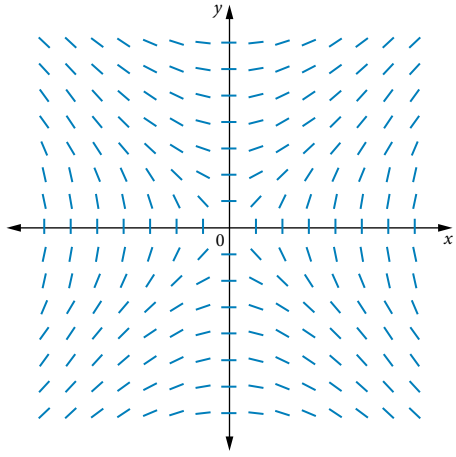
(b) Passing through $(1, 2)$.



(c) Passing through $(1, 3)$.



6 The graph shown is the slope field of a first order differential equation.



Suggest a differential equation for which it is the slope field.

7 Sketch the slope field for the differential equation $x\dfrac{dy}{dx} - y = 0$.

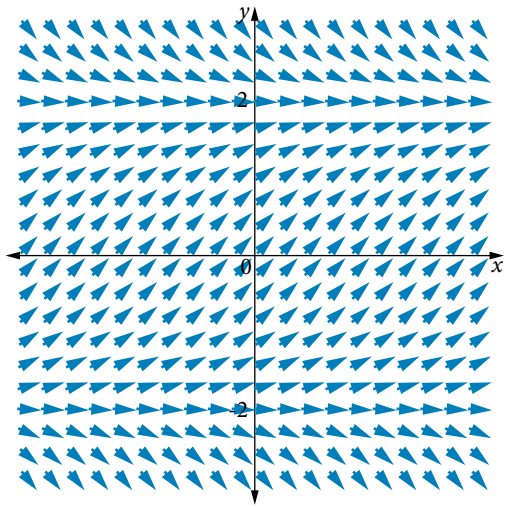
20

8 Consider the differential equation $\frac{dy}{dx} = y - 3$.

(a) Construct a slope field in the first quadrant for $x = 0$ to 7, $y = 0$ to 7.

(b) Determine and classify the steady state (equilibrium) solution.

9 The slope field of $\frac{dy}{dx} = f(y)$ is shown. Determine and classify any steady state solutions.



10 Consider the differential equation $\frac{dy}{dx} = 2y(2 - y)$.

(a) Which of the following statements is false and why?

A There are steady state solutions $y = 0$, $y = 2$.

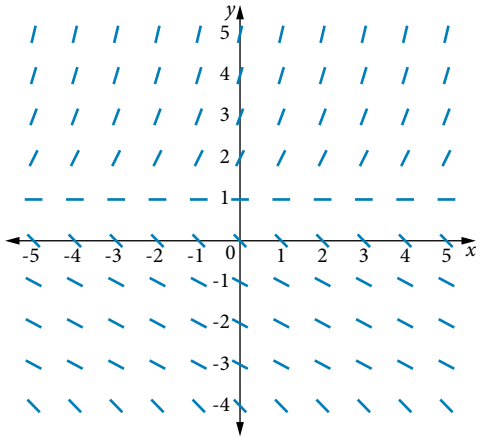
B $y = 0$ is a stable solution and $y = 2$ is an unstable solution.

C $y = 2$ is a stable solution and $y = 0$ is an unstable solution.

D The slope field line segments along the line $y = 2$ all have the same gradient.

(b) Explain why the other statements in part **(a)** are true.

11 Consider the slope field shown.



(a) Which one of the following differential equations could be represented by the slope field shown and why?

A $\frac{dy}{dx} = \frac{x-1}{2}$ **B** $\frac{dy}{dx} = \frac{y}{2}$ **C** $\frac{dy}{dx} = \frac{xy}{2}$ **D** $\frac{dy}{dx} = \frac{y-1}{2}$

(b) Explain why the other options in part **(a)** are incorrect.

21

12 The population P (measured in hundreds) of kangaroos in a national park grows according to the differential equation $\frac{dP}{dt} = P\left(1 - \frac{P}{20}\right)$, where t is measured in months.

(a) Classify the steady state solutions of $\frac{dP}{dt} = P\left(1 - \frac{P}{20}\right)$.

(b) Determine an expression for $\frac{d^2P}{dt^2}$ in factored form. (Hint: Use implicit differentiation.)

(c) Hence, determine the population corresponding to the greatest growth rate of kangaroos.

(d) What is the long-term population of kangaroos in the park?

(e) Verify that $P = \frac{8000e^t}{1 + 400e^t}$ is a solution of the differential equation.

(f) Determine the number of kangaroos present at the point of inflection of the solution curve.

13 Consider the differential equation $\frac{dy}{dx} = y - x$.

(a) Sketch the slope field of the isoclines $y - x = C,\ C \in \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\}$.

(b) Sketch the particular solution $\frac{dy}{dx} = y - x,\ y(0) = 2$ on your answer to part **(a)**. Record the coordinates of the particular solution at $x = 1.5$ on your sketch.

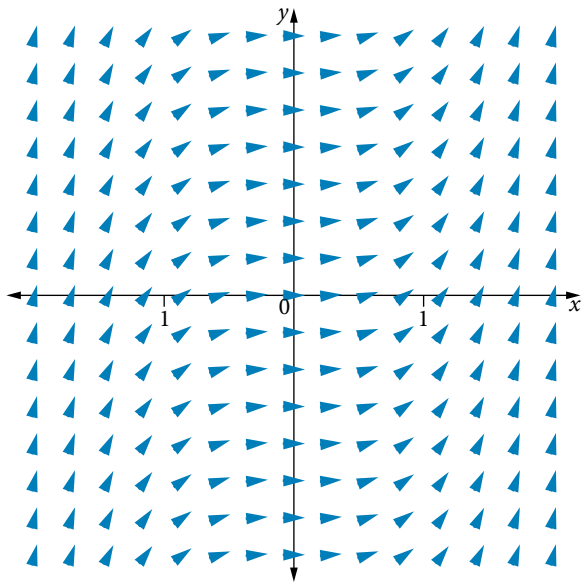
(c) Verify that $y = x + e^x + 1$ is a solution of $\frac{dy}{dx} = y - x$.

14 Consider the differential equation $\frac{dy}{dx} = 2y + 2x + 3$.

(a) Determine $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Determine the values of the constants m, c and r for which $y = e^{rx} + mx + c$ is a solution of the differential equation.

15 A first order differential equation has a slope field shown in the following diagram.



(a) Suggest, with reasons, the simplest first order differential equation consistent with the slope field shown.

(b) Suggest a possible general solution for your differential equation.

Analytical solutions of differential equations 7.5

In the previous sections you solved differential equations and analysed them qualitatively using the graphical method of slope or direction fields. In this section you will set up and solve differential equations that model practical situations.

22 Setting up and solving differential equations

(a) A company discharges toxic waste into a storage tank at a rate of $60t(2-t)$ L/min over a period of t minutes. Construct an appropriate differential equation and calculate the total volume of toxic waste discharged after 2 minutes.

THINKING	WORKING
1 Define the variables required.	Let W litres be the volume of waste at time t minutes.
2 Write the differential equation.	$\frac{dW}{dt} = 60t(2-t)$
3 Determine the general solution.	$W = \int (120t - 60t^2)\, dt$ $= 60t^2 - 20t^3 + c$
4 Use additional information to determine a particular solution.	$W(0) = 0$ $c = 0$ $W = 60t^2 - 20t^3$
5 Calculate and interpret the answer.	If $t = 2$ $W = 240 - 160$ $= 80$ 80 litres have been discharged.

(b) The gradient at any point (x, y) on a curve is equal to the product of the x-coordinate and the cube root of the y-coordinate. If the curve passes through the point $(0, 8)$, determine its equation.

1 Define the variables required.	Variables are given.
2 Write the differential equation.	$\frac{dy}{dx} = y^{\frac{1}{3}}x$
3 Determine the general solution.	Using separation of variables, $\int y^{-\frac{1}{3}}\, dy = \int x\, dx$ $\frac{3}{2}y^{\frac{2}{3}} = \frac{x^2}{2} + c$ $3y^{\frac{2}{3}} = x^2 + 2c$ $y = \left(\frac{x^2 + 2c}{3}\right)^{\frac{3}{2}}$

4 Use additional information to determine a particular solution.	Passes through $(0, 8)$ $3(8)^{\frac{2}{3}} = 0 + 2c$ $2c = 12$ $y = \left(\frac{x^2 + 12}{3}\right)^{\frac{3}{2}}$
5 Write the answer.	$y = \left(\frac{x^2 + 12}{3}\right)^{\frac{3}{2}}$

The three types of differential equations you have solved in this chapter are as follows:

Type 1	Type 2	Type 3 Separation of variables
$\frac{dy}{dx} = f(x)$ $y = \int f(x)\,dx$	$\frac{dy}{dx} = f(y)$ $\frac{dx}{dy} = \frac{1}{f(y)}$ $x = \int f(y)\,dy$	$\frac{dy}{dx} = f(x)g(y)$ $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

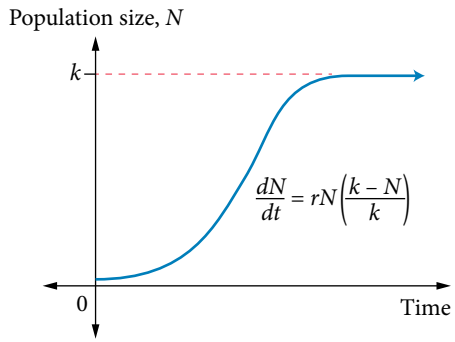
The logistic equation

A special case of the type 2 differential equation is the logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right)$$

where N is the number present in the population at time t, r is the growth rate and k is the maximum capacity in the given situation.

The logistic equation is used to model population growth and in economics, chemistry, statistics and many other fields. Without the second term, the equation is simply the exponential growth or decay model that you have met before as a type 2 differential equation. The second term introduces factors impeding growth as N increases, such as scarcity of resources or space.



The graph of the solution curve looks similar to the one shown. The population approaches the maximum capacity over a long period of time.

The general solution is $N(t) = \frac{kN_0}{(k - N_0)e^{-rt} + N_0}$, where N_0 is the starting value of the population.

23 Solving the logistic equation

The number of people N who hear some gossip after t days satisfies the equation $\frac{dN}{dt} = N\left(1 - \frac{N}{500}\right)$. Write the growth rate and the maximum number of people available, and determine the number of people who have heard the gossip after 7 days if one person starts spreading the gossip.

THINKING	WORKING
1 Write the logistic equation in standard form.	$\frac{dN}{dt} = N\left(1 - \frac{N}{500}\right)$
2 Write the growth rate and the maximum capacity.	Growth rate: 1 person per day The rate of change cannot be negative, as this would imply that people have unheard the gossip, hence, the maximum capacity is 500.
3 Determine the general solution.	$500\frac{dN}{dt} = N(500 - N)$ $500\int \frac{1}{N(500-N)}\,dN = \int 1\,dt$ using separation of variables Using partial fractions, $\frac{1}{N(500-N)} = \frac{A}{N} + \frac{B}{500-N}$ $\frac{1}{N(500-N)} = \frac{A(500-N)+BN}{N(500-N)}$ $\therefore 1 = A(500-N)+BN$ Let $N = 0$ Let $N = 500$ $1 = 500A$ $1 = 500B$ $A = \frac{1}{500}$ $B = \frac{1}{500}$ Therefore $500\int \frac{1}{500N} + \frac{1}{500(500-N)}\,dN = \int 1\,dt$ $\int \frac{1}{N} + \frac{1}{(500-N)}\,dN = \int 1\,dt$ $\log_e(N) - \log_e(500-N) = t + c$ $\log_e\left(\frac{N}{500-N}\right) = t + c$ $\frac{N}{500-N} = e^{t+c}$ Rearranging gives $N = \frac{500}{1+e^{-(t+c)}}$

4 Substitute the initial value to determine a particular solution.	$N = 1$ when $t = 0$ $1 = \frac{500}{1 + e^{-c}}$ $1 + e^{-c} = 500$ $e^{-c} = 499$ $N = \frac{500}{1 + e^{-t}e^{-c}}$ $= \frac{500}{1 + 499e^{-t}}$
5 Answer the question and write the answer.	When $t = 7$, $N(7) = \frac{500}{1 + 499e^{-7}}$ $= 343.63\ldots$ ≈ 343 Therefore, 343 people have heard the gossip after 7 days.

24 Constructing a more complex differential equation

A tank is initially filled with 50 L of a salt solution containing 25 kg of salt. Another salt solution containing 3 kg of salt per litre runs into the tank at a constant rate of 6 L/min. In addition to this, a salt solution of uniform concentration runs out of the tank at a rate of 4 L/min. Assuming that the concentration is kept uniform by stirring, set up a differential equation indicating the rate of change of the mass of salt in the tank at any time.

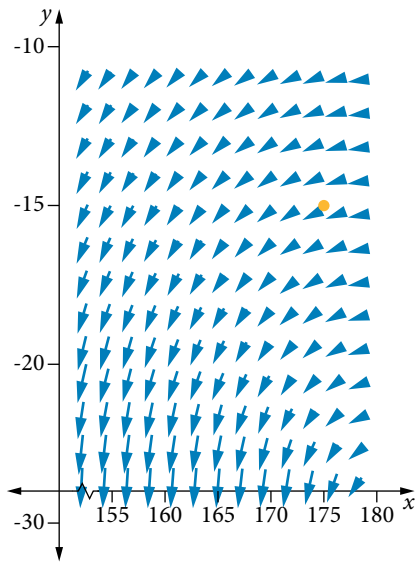
THINKING	WORKING
1 Define the variables required.	Let M kg be the mass of salt in the container after t minutes.
2 Set up the differential equation.	$\frac{dM}{dt} = \left.\frac{dM}{dt}\right\|_{in} - \left.\frac{dM}{dt}\right\|_{out}$ $= 3\left(\text{kg/L}\right) \times 6\left(\text{L/ min}\right) - \frac{M}{50 + (6-4)t}\left(\text{kg/L}\right) \times 4\left(\text{L/ min}\right)$ $= 18 - \frac{4M}{50 + 2t}$
3 Write the answer.	$\frac{dM}{dt} = 18 - \frac{4M}{50 + 2t}$

EXERCISE 7.5 Analytical solutions of differential equations

Worked Example 22

1 Effluent is dumped into the ocean and dissipates due to currents at a rate given by $\frac{dV}{dt} = \frac{k}{(1+t)^{\frac{3}{2}}}$, where V litres is the volume of effluent at the site of the dumping, t hours is the time since the dumping occurred and k is a constant. If 10 000 litres are dumped initially and the effluent totally disperses eventually (so that after a long time the volume remaining is negligible), determine how long it takes for half the effluent to disperse.

2 On a particular day, the synoptic scale airflow over the Coral Sea is modelled by the differential equation $\frac{dy}{dx} = \frac{180 - x}{4(25 + y)}$ with the slope field as shown.



In this figure, x is the longitude and y is the latitude (both are measured in degrees, with southern latitudes negative).

Tropical Cyclone Wendy is shown at the point longitude 175°E and latitude 15°S.

(a) Determine the particular solution of $\frac{dy}{dx} = \frac{180 - x}{4(25 + y)}$, $y(175) = -15$.

(b) Hence, plot the path of Cyclone Wendy on the slope field.

3 An insect population P experiences a seasonal growth rate given by $\frac{dP}{dt} = \frac{\pi}{12}\sin\left(\frac{\pi}{6}t\right)P$, $P(0) = 1$, where P is measured in millions, and t is the number of months since the beginning of spring. Express the time variation of the insect population P, as a function of t, and sketch this variation over the course of 1 year.

4 The fuel-air pressure P (Pa) inside the cylinder of a combustion engine with a 10 : 1 compression ratio is related to the volume V cm^3 of the cylinder by the differential equation: $\frac{dP}{dV} = -1.400\frac{P}{V}$.

(a) Determine the particular solution of this equation if the volume of the fuel-air in the uncompressed cylinder is 600 cm^3 when the fuel-air pressure is 100 000 Pa. (Express your answer to 4 significant figures.)

(b) Hence, calculate the air pressure P when the cylinder is fully compressed to 60 cm^3. (Express your answer to 2 significant figures.)

5 Water is poured at a rate of $0.5\,\text{m}^3/\text{h}$ into a tank whose shape is a vertical prism with a cross-sectional area of $14\,\text{m}^2$. A hole at the base of the tank allows the water to leak out at a rate of $0.1\sqrt{x}\,\text{m}^3/\text{h}$, where x m is the height of the water at time t hours.

(a) If $V\,\text{m}^3$ is the volume of water in the tank at time t hours, set up a differential equation to model the situation.

(b) Using the chain rule, obtain a differential equation relating x to t.

6 The number of bacteria in a sample of milk increases at a rate proportional to the number present. If the number of bacteria doubles in half an hour, calculate how long will elapse before an initial number of one thousand will become one million. Leave your answer correct to the nearest whole hour.

7 A particle moving in a straight line has a velocity at time t seconds given by the differential equation $\frac{dx}{dt} = \frac{1}{\sqrt{16 - t^2}}$, $0 \leq t < 4$, where x m is the displacement from an origin.

(a) Determine the displacement as a function of time, given that when $t = 0$, $x = \pi$.

(b) After how many seconds is the displacement of the particle $\frac{7\pi}{6}$ metres to the right of the origin?

8 Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference between the temperature of the surroundings and that of the object. A packet of meat at a temperature of 20°C is placed in a freezer whose temperature is kept at a constant -18°C.

(a) Set up a differential equation to model the situation.

(b) If it takes 15 minutes for the temperature of the meat to drop to 10°C, how much longer to the nearest minute does it take for the meat's temperature to fall another 10°C and reach freezing point?

9 Determine the equation of each of the following graphs.

(a) The graph passes through $(1, 2)$ and has a slope $\frac{3y}{x^2}$ at each point (x, y).

(b) The gradient of the tangent to the graph is given by $\frac{-2y}{x}$, and the graph passes through the point $(1, 2)$.

10 Lupita's lawn is left unattended so that an infestation of weeds begins to take over. The rate of growth of the weeds is proportional to the area of lawn not yet invaded by weeds.

(a) If $W\,\text{m}^2$ is the area of lawn taken over by weeds after t weeks, and the lawn has a total area of $A\,\text{m}^2$, write the differential equation that models the situation.

(b) The area taken over by weeds grows from one quarter to one half the total area of the lawn in T weeks. If $t = 0$ when $W = \frac{1}{4}A$, solve the differential equation in part **(a)**.

11 The half-life of the radioactive isotope radium-226 is 1600 years to the nearest 100 years. In other words, it takes about 1600 years for half of the unstable radioactive atoms in radium-226 to decay. Given that the rate of decay is proportional to the number of atoms present, determine the following:

(a) The time it takes for a quantity of radium-226 to decay to 20% of its initial amount, leaving your answer to the nearest 100 years.

(b) The percentage of radium-226 that decays in any 100-year period. Leave your answer correct to 1 decimal place.

12 The acceleration of an object of mass M kg is partially impeded by air resistance proportional to the velocity v m/s of the object. The motion is modelled by the differential equation $\frac{dv}{dt} = g - \frac{k}{M}v$, where M, g and k are constants.

(a) Determine an expression for the velocity in terms of time, t seconds.

(b) If an object of mass 4 kg falls from rest with air resistance that is 2% of its velocity, determine the time when its velocity is 25 m/s. Assume that $g = 9.8$ and leave your answer correct to 2 decimal places.

13 With respect to time (t), the rate of change of the volume (V) of a cube of side length x is proportional to its surface area (A). Choose the differential equation that correctly describes the situation and describe why the other options are incorrect.

A $\frac{dV}{dx} = 3x^2$ **B** $\frac{dV}{dA} = k$, a constant **C** $\frac{dV}{dt} = A$ **D** $\frac{dV}{dt} = kx^2$

14 Select the true statement from the following.

A $\frac{dy}{dx} = x + y$ can be solved by separation of variables.

B $\frac{dy}{dx} = y^2 + 1$ can be solved by direct integration with respect to y.

C $\frac{dy}{dx} = \frac{2}{x-1}$ can be solved by direct integration with respect to x.

D $\frac{dy}{dx} + \frac{d^2y}{dx^2} - x = 0$ is a first order differential equation.

Worked Example 23

15 Suppose that 50 rabbits are inadvertently introduced to a small island off the Victorian coast that was previously rabbit-free. The rabbits have no natural enemies on the island but, due to food constraints, the island can only support a population of 5000. The rate of growth of the rabbit population at time t months (after their initial introduction) satisfies the differential equation $\frac{dN}{dt} = kN(5000 - N)$, where N is the number of rabbits present after t months.

(a) If the initial rate of increase of the population is 45 rabbits per month, determine the value of k.

(b) Solve the differential equation and hence determine the number of rabbits present 6 months after their introduction.

(c) How many rabbits are present when the population is growing most rapidly? (Hint: Use the chain rule to determine when $\frac{d^2N}{dt^2} = 0$.)

(d) Determine and classify, as stable or unstable, any steady state solutions of the differential equation.

16 Set up (but do not attempt to solve) differential equations that model the following situations. 24

(a) The rate of change of the displacement of an object is made up of two parts: a fixed part and a part that is proportional to the square root of the displacement.

(b) A family of curves that have a gradient equal to the product of the reciprocal of the x-coordinate and the square of the y-coordinate.

(c) A vat contains 100 litres of paint obtained by mixing red and blue paint in the ratio 3 : 1. The paint manufacturer wishes to make paint in various shades of purple and achieves this by drawing off the mixture in the vat at 5 litres per minute while simultaneously adding more red and blue paint into the vat at rates of 2 and 3 litres per minute respectively. If the paint mixture in the vat contains x litres of red paint at time t minutes, set up a differential equation involving x and t, assuming that the paint in the tank is kept well mixed.

(d) A tank is in the shape of a right cylinder, with radius r. The volume of liquid V decreases as it flows out of the tank at a rate proportional to the square root of the depth of liquid h in the tank. Set up a differential equation describing the rate of change of the depth with respect to time.

(e) A container is initially filled with 100 litres of water and has 20 kg of copper sulphate dissolved in it. A solution containing 2 kg of copper sulphate per litre is poured into the container at a rate of 5 L/min and the stirred solution runs out at a rate of 3 L/min. Set up a differential equation involving the amount of copper sulphate Q kg at time t minutes.

17 Mikail is having a picnic and takes some soft cheese out of an esky where the temperature is 8°C. He then leaves it standing in an area where the constant temperature is 23°C, and the soft cheese temperature reaches 12°C after 5 minutes. Mikail prefers to eat soft cheese when its temperature is at least 15°C but no more than 20°C. Assume that Newton's law (in this situation, of warming) applies: the rate of change of temperature of a body is proportional to the difference between the temperature of the surroundings and that of the body.

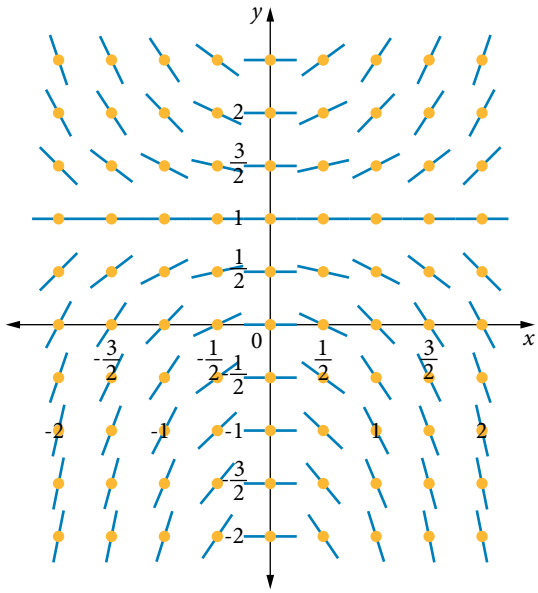
(a) Determine, to the nearest minute, the earliest time at which Mikail should take his first bite.

(b) Determine the length of time Mikail has in which to eat the soft cheese.

18 Consider the differential equation $\frac{dy}{dx} = x(y-1)$.

(a) Determine the general solution.

(b) The slope field of $\frac{dy}{dx} = x(y-1)$ is shown below.



This slope field indicates that $y = 1$ should be a solution curve for $\frac{dy}{dx} = x(y-1)$. Explain why this is not a possible particular solution.

19 Determine the general solution of the differential equation $\frac{dy}{dx} = \sqrt{1 - x^2}$ using the substitution $x = \sin(\theta)$.

20 The rate at which a disease spreads during an epidemic is believed to be proportional to the product of the fraction of the population already infected and the fraction not yet infected.

(a) If y is the fraction of the population infected after t months, set up a differential equation to represent this situation.

(b) If the fraction infected increases from one quarter to one half of the population in 1 month, solve the differential equation giving your answer in the form $y = \frac{1}{1 + ab^t}$, where a and b are constants to be specified.

21 Consider the differential equation $\frac{dy}{dx} = -\frac{2xy}{\log_e(y)}$, $y > 0$.

(a) Determine the general solution $g(x, y) = c$ of this differential equation as an implicit relation between x and y.

(b) Determine the particular solution passing through the point $(0, e)$.

(c) Explain why $x = 1$ cannot be the solution to part **(b)**.

22 Consider the differential equation $\frac{dy}{dx} = \frac{5 - x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the differential equation for $0 < x < 1$ such that the line $y = 5$ is a tangent to the graph of $y = f(x)$. Determine the x-coordinate of the point P of tangency.

(b) Determine an expression for $\frac{d^2y}{dx^2}$.

(c) Hence, determine the nature of the solution of the differential equation at the point P.

(d) Let $y = g(x)$ be the particular solution to the given differential equation for $0 < x < 10$ with the initial condition $g(5) = -5$. Determine the rule $y = g(x)$.

23 A space probe is fired vertically upwards from the surface of a planet with a radius R. If the atmospheric drag is ignored, the outward velocity (v m/s) of the probe at height h metres above the surface of the planet is modelled by the differential equation

$\frac{dv}{dh} = -\frac{gR^2}{v(R + h)^2}$, where $v = u$ when $h = 0$ and g is the gravitational acceleration on the surface of the planet.

(a) Verify that $v^2 = u^2 - \frac{2gR}{1 + \frac{R}{h}}$.

(b) Hence, determine the minimum launch velocity u for the probe to escape the planet's gravity.

7

Summary

Implicit differentiation

Given an implicit relationship $g(x, y) = c$, the derivative $\frac{dy}{dx}$ can be determined using the chain rule.

If $y = f(x(t))$, then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$.

A first order differential equation $\frac{dy}{dx} = f(x, y)$ specifies a relationship between a dependent variable y and its derivative $\frac{dy}{dx}$.

Related rates

The general solution of an ordinary differential equation is a rule that specifies an analytical relationship $g(x, y) = c$ between y and x. Where possible, this relationship should be rearranged to express $y = F(x, c)$, so that y is an explicit function of x.

All related rates problems considered in this chapter can be classified into one of two cases:

Case 1	If the variables in the model (for example, x and y) are related by an explicit parametric equation such as $y(t) = f(x(t))$, then the two rates $\frac{dy}{dt}$ and $\frac{dx}{dt}$ can be related by the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, where $\frac{dy}{dx} = \frac{d\left(f(x)\right)}{dx}$.
Case 2	If the variables in the model (for example, x and y) are related by an implicit parametric equation such as $g(x(t), y(t)) = c$, then the two rates $\frac{dy}{dt}$ and $\frac{dx}{dt}$ can be related by the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, where $\frac{dy}{dx}$ is an implicit derivative.

Differential equations

A particular solution of a differential equation can be found with additional information:

- You have learned techniques of solving differential equations of the form $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = f(y)$ and, using separation of variables, $\frac{dy}{dx} = f(x)f(y)$.
- Qualitative (or graphical) methods are based upon the slope field of $\frac{dy}{dx} = f(x, y)$. Each point in the x, y-plane corresponds to a unique value of the slope, $f(x, y)$, and consequently a unique value of $\frac{dy}{dx}$, and is represented by a small line (tangent) segment with the appropriate slope.
- Steady state solutions may be found and classified as stable or unstable.

Qualitative and numerical solutions of differential equations

Qualitative methods provide a valuable understanding of the information by sketching a graphical representation of the solution curve. These methods are:

Method 1 – Slope field construction on a Cartesian plane. This method involves the following steps:

STEP 1 Evaluate $f(x, y) = \frac{dy}{dx}$ on a predetermined set of points (x, y) in the Cartesian plane (grid).

STEP 2 At each given point (x, y) on the grid, draw a short line segment with the same slope as calculated for $\frac{dy}{dx}$. (Note that the angle made by each line segment with the positive x-axis is θ, where $\tan(\theta) = \frac{dy}{dx}$.)

Method 2 – Slope field construction on the isoclines. This method is almost the reverse of the previous method in that you choose a value for the gradient and then determine a set of points that have that gradient.

STEP 1 Choose values of c where $f(x, y) = c$ and c is a real constant, and rearrange the resulting equation.

STEP 2 Draw the curve containing points with the chosen value of the gradient. If $c = 0$ the curve is called a null-cline and otherwise it is called an isocline.

STEP 3 Along each null-cline or isocline $f(x, y) = c$, draw a number of short parallel line segments, each of slope c, to construct the slope field for the differential equation.

Analytical solutions of differential equations

Many natural phenomena are modelled by differential equations, many of which may be solved analytically. Some examples include growth and decay, the logistic equation for inhibited growth, chemical solutions and motion of particles.

Chapter review 7

Exercise 7.1

1 Use implicit differentiation to determine an expression for $\frac{dy}{dx}$ in each of the following.

(a) $3xy + y^2 = 7$

(b) $2x^2 - \sin(2y) + 1 = 0$

(c) $\frac{1}{y} + e^{x+y} = 4$

(d) $\log_e(x) + 2\log_e(y) + \frac{x}{y} = 9$

Exercise 7.1

2 Consider the implicitly defined curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a and b are real constants.

(a) Determine $\left.\frac{dy}{dx}\right|_{(x_0,\, y_0)}$

(b) The equation of the tangent to the curve at the point (x_0, y_0) can be written in the form $Ax + By = 1$, where A and B are real constants.

Determine expressions for A and B in terms of x_0, y_0, a and b.

Exercise 7.1

3 Given the relation $2x^2 + 3xy + y^2 = 12$, determine the gradient of the tangent to the curve at the point in the first quadrant where $x = 1$.

Exercise 7.2

4 Given the relation $xy^2 - 2y + \log_e(x - 2) = 1$, determine the gradient of the normal to the curve of the relation at the point in the first quadrant where $x = 3$.

Exercise 7.2

5 If $r = \frac{1 + x}{1 + x^2}$, determine $\frac{dx}{dt}$ when $x = 2$, given that $\frac{dr}{dt} = 14$.

Exercise 7.2

6 A camera at ground level is 400 m away from a hot air balloon just prior to the balloon lifting off. The balloon lifts off and the camera records the balloon rising into the sky at a constant rate of 10 m/s.

(a) If θ is the angle of elevation of the balloon, express the height h of the balloon in terms of this angle.

(b) At what rate is the camera's angle of elevation, θ radians, changing when the balloon is 300 m above the ground?

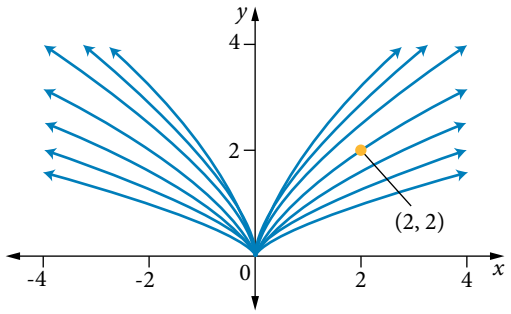
Exercise 7.2

7 There is a time when the rate of increase of the volume (V cubic units) of an expanding sphere is equal to the rate of increase of the surface area (S square units). Determine the radius (r) of the sphere at that time.

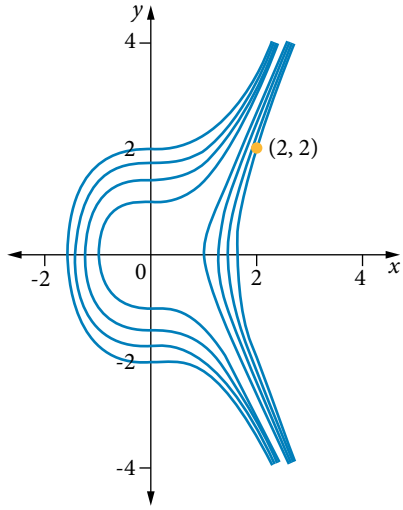
Exercise 7.3

8 The graphs on the following page show some possible solutions for the given differential equation. Determine the particular solution by determining C for the graph passing through the indicated point.

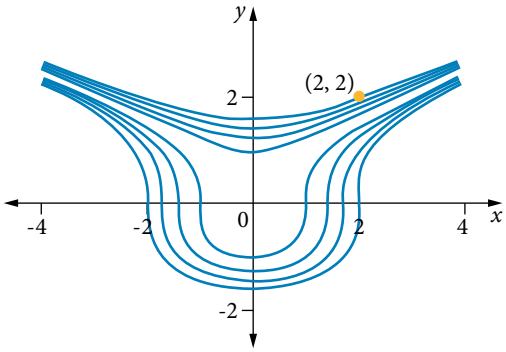
(a) $y^3 = Cx^2;\ 3x\dfrac{dy}{dx} - 2y = 0$



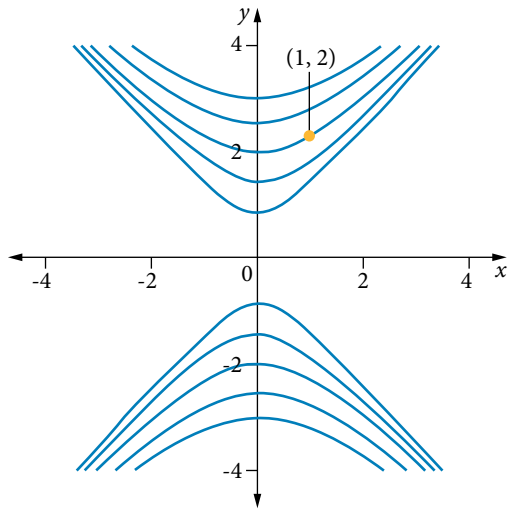
(b) $y^2 - x^3 = C;\ 2y\dfrac{dy}{dx} - 3x^2 = 0$



(c) $y^3 - x^2 = C;\ 3y^2\dfrac{dy}{dx} - 2x = 0$



(d) $y^2 - x^2 = C;\ y\dfrac{dy}{dx} - x = 0$



9 For each of the particular solutions you found in question **8**, verify by differentiation that each one is a solution of the given differential equation. Exercise 7.3

(a) $y^3 = 2x^2$ **(b)** $y^2 - x^3 + 4 = 0$ **(c)** $y^3 - x^2 = 4$ **(d)** $y^2 - x^2 = 0$

10 The gradient of the tangent to a curve at any point (x, y) is $\dfrac{x}{x+1}$, $x > -1$. If the curve passes through the point (1,1), determine the equation of the curve. Exercise 7.3

11 Solve the differential equation $\dfrac{dy}{dx} = 1 - 2y$, given that $y = -1$ when $x = 0$. Exercise 7.3

12 Determine y as a function of x if y satisfies the differential equation $\dfrac{dy}{dx} = y^2 - 4$ and $y = 0$ when $x = 0$. Exercise 7.3

Exercise 7.3 **13** A species of tuna is declining so that T, the number of tuna after time t years from now, satisfies the differential equation $\frac{dT}{dt} = -0.1T$.

(a) Determine the general solution to this differential equation, where $T(0) = A$ is the initial population.

(b) Determine the time it will take for the number of tuna to fall to one quarter of the present value. Leave your answer correct to 1 decimal place.

Exercise 7.3 **14** Consider the differential equation $\frac{dy}{dx} = 2x\left(1 + y^2\right)$, $y(0) = 1$.

Determine the particular solution of the differential equation.

Exercise 7.3 **15** Verify that the functions given are solutions of the corresponding differential equations.

(a) $y = \log_e\left(\sin(x)\right); \frac{dy}{dx} = \cot(x)$

(b) $y = \sqrt{x^2 + 3}; \frac{dy}{dx} - \frac{x}{y} = 0$

(c) $y = Ax^b; x\frac{dy}{dx} = by$

(d) $y = \log_e\left(\frac{2-x}{1-x}\right); \frac{dy}{dx} = \frac{1}{(2-x)(1-x)}$

Exercise 7.1 **16** Consider the curve with equation $2xy - 3y^2 = 1$.

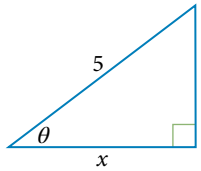
(a) Verify that the equation is a solution, in implicit form, of the differential equation $\frac{dy}{dx} = \frac{y}{3y - x}$.

(b) Determine all the points on the curve with a tangent of slope 1.

(c) Demonstrate that there are no points on the curve with a horizontal tangent.

(d) The coordinates (x, y) on the curve $2xy - 3y^2 = 1$ are functions of time. At time $t = 2$, $y = 1$ and $\frac{dy}{dt} = 3$. Determine the value of $\frac{dx}{dt}$ when $t = 2$.

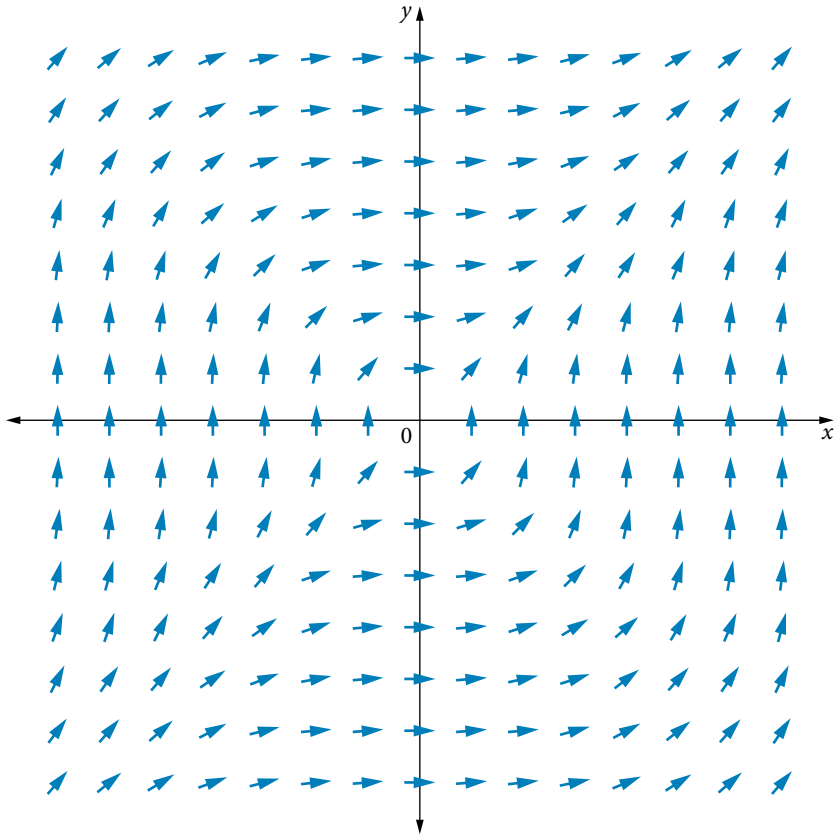
Exercise 7.2 **17** In the triangle shown, θ increases at a rate of 3 radians per second.



At what rate is x decreasing at the instant when $x = 3$ units, assuming the triangle remains right-angled?

18 The slope field of $\frac{dy}{dx} = \frac{x^2}{y^2}$ is shown.

Exercise 7.4



(a) Explain why the line segments are vertical along the x-axis.

(b) Explain why the line segments are horizontal along the y-axis.

(c) Write the equation of a null-cline on the slope field.

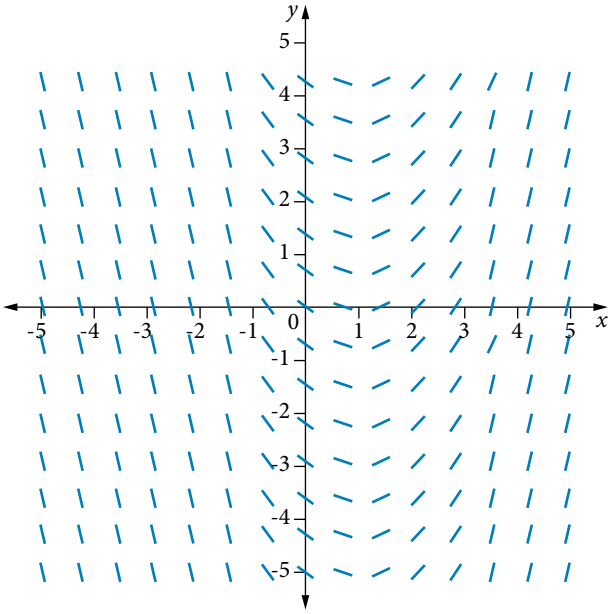
(d) The line with equation $y = x$ appears to be a solution of the differential equation. Verify that it is a solution by determining the general solution.

(e) Sketch the particular solutions for $A = -2$, $A = 0$, $A = 2$.

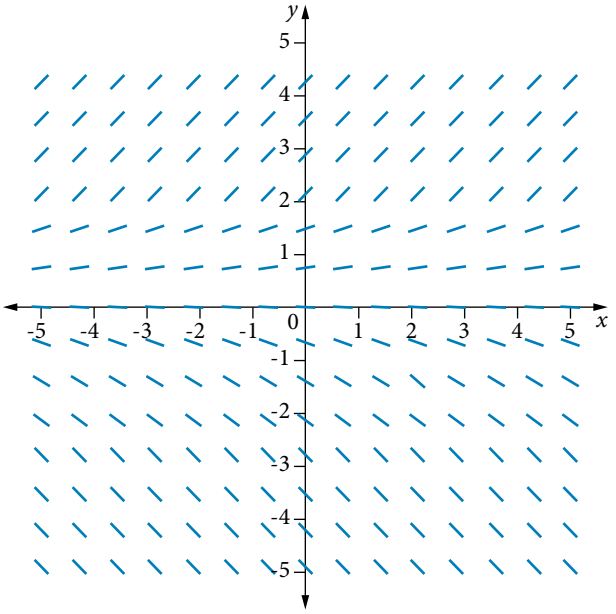
Exercise 7.4

19 Four slope fields are shown below. Match each of the following differential equations with one of the slope fields.

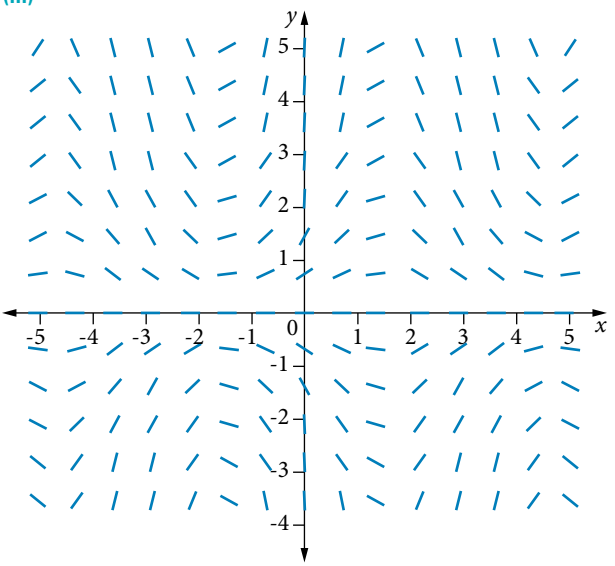
(I)



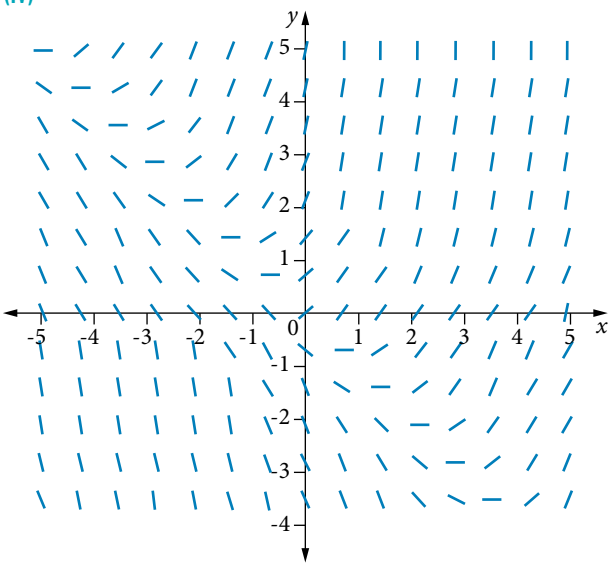
(II)



(III)



(IV)



(a) $\frac{dy}{dx} = x - 1$ **(b)** $\frac{dy}{dx} = x + y$ **(c)** $\frac{dy}{dx} = \frac{y}{3}$ **(d)** $\frac{dy}{dx} = y\cos(x)$

Exercise 7.5

20 Consider the family of curves $\frac{y^2}{x^3} = c$ as a general solution.

(a) Write a differential equation that has this family of curves as its general solution.

(b) Orthogonal trajectories of a curve intersect it everywhere at right angles. Determine the equation of the orthogonal trajectories of $\frac{y^2}{x^3} = c$.

21 If $x^2 + \log(xy) + y^2 = 4$, determine an expression for $\frac{dy}{dx}$. Exercise 7.1

22 If $y = x^2 - \frac{4}{x}$, determine the value of $\frac{dy}{dt}$ when $x = 2$ and $\frac{dx}{dt} = 3$. Exercise 7.2

23 Consider the slope field of the differential equation $\frac{dy}{dx} = y^2 - 1$. Exercise 7.4

(a) Describe the slope field in each of the intervals $-\infty < y < -1$, $-1 < y < 1$ and $1 < y < \infty$, and sketch it.

(b) The sketch suggests two steady state solutions. Write these two solutions and determine the stability of each one.

24 Water is brought to the boil in an electric kettle and some of it is used to make coffee. The unused water in the kettle is then left in the kitchen whose constant temperature is 15°C. If the water temperature drops from 100°C to 80°C in 10 minutes, how much longer will it take to drop a further 20°C? Assume Newton's law of cooling and leave your answer correct to the nearest minute. Exercise 7.5

25 Consider the differential equation $\frac{dy}{dx} = x - 2y$, for which the solution is $y = g(x)$. Exercise 7.5
Determine the concavity of the particular solution curve that contains the point (0, -1) at that point.

26 When added to water, 5 grams of salt dissolves at a rate equal to 10% of the amount of undissolved salt per hour. Exercise 7.3

(a) If x is the number of grams of undissolved salt after t hours, set up a differential equation describing the rate of change of the amount of undissolved salt. Leave your answers correct to 1 decimal place.

(b) Determine the time taken when the amount of undissolved salt reaches 2 grams.

(c) Conditions are changed so that the rate of change of the amount of undissolved salt now satisfies the differential equation $\frac{dx}{dt} = -0.1(x - 1)$. Determine the difference this makes to the amount of salt remaining after a long time.

27 The population $P(t)$ of a certain species satisfies the differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{10\,000}\right)$, with an initial population $P(0) = 4000$, where t is the time in years. Exercise 7.5

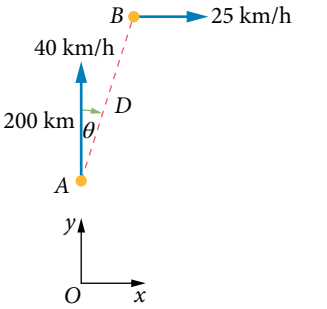
(a) Without solving the equation, determine the population as t approaches infinity.

(b) Solve the differential equation to determine the size of the population after 1 year. Leave your answer correct to the nearest whole number.

28 A quantity of sugar has been dissolved in a tank containing 200 litres of pure water. At time $t = 0$ minutes, pure water is poured into the tank at a rate of 5 litres per minute. The tank is kept well stirred. At the same time, the sugar solution is drawn from a tap at the bottom of the tank at a rate of 7 litres per minute. Set up a differential equation for the mass m grams of sugar in the tank after t minutes. Exercise 7.5

Exercise 7.1

29 Initially, ship A is at point O, 200 km due south of ship B. Ship A sails northward (in the direction of increasing y) at $\frac{dy}{dt} = 40$ km/h and ship B sails eastward (in the direction of increasing x) at $\frac{dx}{dt} = 25$ km/h.



Let D be the distance between the ships.

(a) Verify that $D^2 = x^2 + (200 - y)^2$.

(b) Determine an expression for the rate of change of the distance between the ships.

(c) Determine the rate of change of the distance between the ships after 2 hours.

Exercise 7.2

30 Fourier's law of heat conduction states that the rate of heat transfer $\frac{dQ}{dt}$ through an ice sheet in Commonwealth Bay, Antarctica, is given by the differential equation

$$\frac{dQ}{dt} = \frac{k(T_w - T_a)}{h}$$

where k is the thermal conductivity of the ice, h is the thickness of the ice sheet, and T_w and T_a are the temperatures at the ice/water interface and the ice/air interface respectively.

As the seawater loses Q joules in heat through the ice layer, the rate of increase in ice thickness h is given by

$$\frac{dh}{dQ} = \frac{1}{L\rho}$$

where L is the latent heat of seawater (the amount of heat loss required to freeze 1 kilogram of it) and ρ is the density of the ice.

(a) Determine the rate of increase in the ice sheet thickness $\frac{dh}{dt}$.

(b) If $h(0) = h_0$, determine $h(t)$. Assume that $\frac{k(T_w - T_a)}{L\rho}$ is a positive constant.

Exercise 7.3

31 The relationship between air temperature T K and wind speed V m/s is modelled by the differential equation $\frac{dT}{dV} = -\frac{V}{C_P}$, where C_P is a constant known as the specific heat of the air mass.

(a) Determine the general solution for the wind speed $V(T)$ as a function of temperature.

(b) Outside a particular tropical cyclone, the wind speed is zero and the temperature is 308 K. Determine the particular solution for the wind speed $V(T)$ as a function of temperature.

(c) If $C_P = 1004.6$, determine the wind speed near the centre of a tropical cyclone if the temperature there drops to 307 K. Express your answer correct to the nearest metre per second.

Exercise 7.2

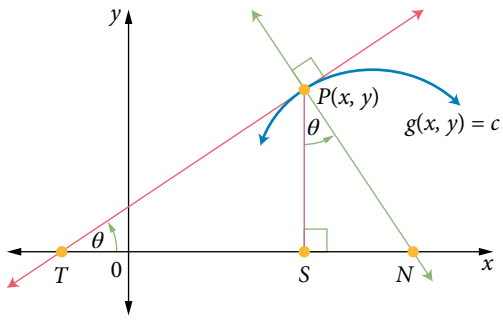
32 The relationship between x, the number of predators in hundreds, and y, the number of prey in thousands, of a particular environment at time t years is modelled by the pair of differential equations $\frac{dy}{dt} = -\frac{1}{400}(x - 2)$ and $\frac{dx}{dt} = \frac{1}{100}(y - 1)$.

(a) Determine a differential equation involving $\frac{dy}{dx}$, x and y only.

(b) At some point in time there were 100 predators (i.e. $x = 1$) and 3000 prey (i.e. $y = 3$). Verify that $a^2(x - h)^2 + b^2(y - k)^2 = 1$ for a suitable choice of the real numbers a, b, h and k.

(c) The graph of the solution in part (b) is an ellipse with a centre at (h, k). Sketch your solution from part (b) and hence write the maximum number of predators and maximum number of prey the environment can support. Express your answer correct to the nearest 10 predators or prey.

33 A portion of the implicit relation $g(x, y) = c$ is shown in the following figure. Exercise 7.3



Each point P on this curve has a subnormal SN of one half of a unit in length (that is, $SN = \frac{1}{2}$). You may also assume that the curve $g(x, y) = c$ passes through the origin.

(a) Determine an equation for the slope $\frac{dy}{dx}$ of the tangent to this curve.

(b) Hence, determine an implicit relation of the form $g(x, y) = c$ for the curve.

34 While on an unauthorised trip to the local fast food outlet during their study period, a pair of Year 12 students are convinced they have just seen Elvis buying a hamburger. Exercise 7.5

When returning to the school, rumour of their amazing sighting spreads throughout the school community at the rate $\frac{dp}{dt} = \frac{1}{10}p(1 - p)$, where p is the proportion of the school community that has already heard the rumour t minutes after their return to school.

(a) What proportion of the school community has heard the rumour when it is spreading most rapidly?

(b) By the beginning of the afternoon period, 20% of the school community had already heard the rumour. Determine $p(t)$ at time t minutes since the beginning of the afternoon period.

(c) At what time (correct to the nearest minute) is the rumour spreading most rapidly?

35 The height $h(x)$ m of a tsunami, as a function of the coordinate x m along a direction at right angles to the wavefront, is the solution of the differential equation: Exercise 7.5

$$\begin{cases} \frac{dh}{dx} = 2h\sqrt{2-h}, \quad x \geq 1. \\ h(1) = 2 \end{cases}$$

(a) Verify that $h(x) = 2$ is a solution of $h'(x) = 2h\sqrt{2-h}$.

(b) Construct a differential equation for the new dependent variable $u = \sqrt{2-h}$.

(c) Hence, solve for $u(x)$.

(d) Determine the height $h(x)$ of the tsunami for $x \geq 1$.

(e) Determine the height $h(x)$ of the tsunami for $x \in [0, 3]$ and sketch its graph.





8

Modelling motion

Recall 430

8.1 Linear motion with constant acceleration 432

8.2 Linear motion with non-constant acceleration 449

8.3 Newton's laws of motion 460

8.4 Simple harmonic motion 478

Summary 487

Chapter review 489

8 Recall

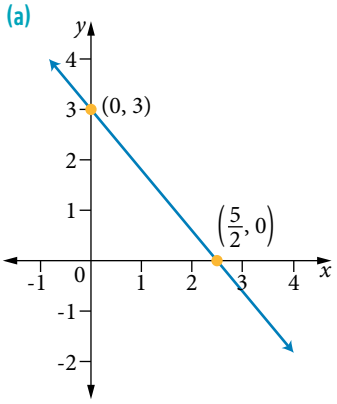
Determine the gradient of a straight line passing through two points

1 Determine the gradient of the straight line passing through the given points.

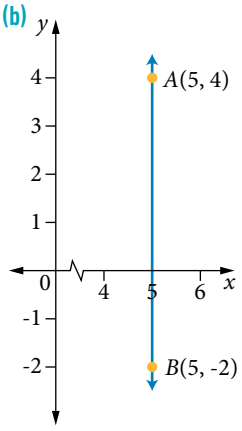
(a) $M(6, 3)$ and $N(-1, 10)$ (b) $A(-3, 5)$ and $B(0, -1)$ (c) $P(12, 1)$ and $Q(4, -11)$

2 Determine the gradient of the straight line passing through the given points in the graphs below.

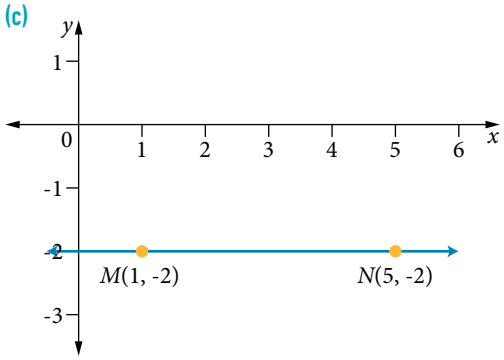
(a)



(b)



(c)



Convert units of time, distance and speed

3 Calculate the converted quantity in each of the following.

(a) Convert 60 km/h into m/s, correct to 1 decimal place.

(b) Convert 40 m/s into km/h.

(c) Convert 5.4 m/s into km/min, correct to 3 decimal places.

(d) Convert 48 m/min into km/s, correct to 4 decimal places.

(e) Convert 2.85 km/min into m/s, correct to 1 decimal place.

(f) Convert 0.013 m/s into cm/min.

Use the constant speed formula to determine speed, distance or time

4 Determine the required quantity for each of the following, using v, d and t to represent speed, distance and time respectively.

(a) Determine the walking speed of a person who travels 1 km in 10 min.

(b) Determine the distance travelled by a car travelling at an average speed of 80 km/h for 3.5 hours.

(c) Determine the time taken, in minutes, correct to the nearest minute, to travel 97 km at a constant speed of 110 km/h.

(d) Determine the speed of light in m/s if light travels 1080 million km in 1 hour.

(e) Determine the time taken, in seconds correct to 1 decimal place, to travel 65 m at 28 cm/s.

Differentiate functions and determine the gradient at a given point

5 Differentiate the following functions and calculate the gradient at the given point.

(a) $y = 3x^2 - 7x + 11$ at $x = -1$ (b) $f'(3)$ if $f(t) = -6t^2 + 9t + 5$

(c) $x = 2\cos(\pi t) + 1$ at $t = \frac{4}{3}$

(d) $y = 3e^{2x+1} - 5$ at $x = 0$

(e) $v = (2+t)^3$ if $v = 27$

Integrate functions and determine the value of definite integrals

6 (a) Given that $\frac{dh}{dt} = 5t - 3$, determine the expression for $h(t)$.

(b) Given that $f'(x) = \frac{7}{(1+2x)^2}$, determine the expression for $f(x)$ if $f(x) = 3$ when $x = 0$.

(c) Determine the value of $\int_0^{\frac{\pi}{4}} 1 + \sec^2(x)\, dx$.

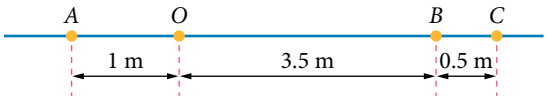
(d) Determine the value of $\int_0^1 \left(2x^2 - 3x + 1\right) dx$.

(e) Determine the value of $\int_1^2 \frac{2}{3-t}\, dt$.

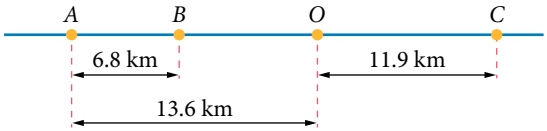
Calculate distance and displacement

7 Determine each of the following displacements or distances.

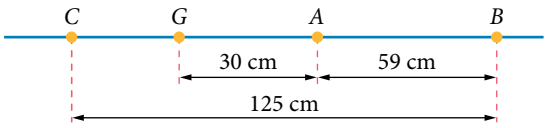
(a) The displacements of particles A, B and C with the positions relative to a reference point O as indicated in the diagram.



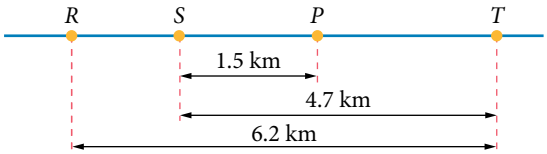
(b) The displacements of particles A, B and C with the positions relative to a reference point O as indicated in the diagram.



(c) The displacements of points A, B and C relative to the reference point G as indicated in the diagram.



(d) The displacements of points S, T and R relative to the reference point P as indicated in the diagram.



(e) The total distance and the final displacement of a particle that travels along the path $PTRS$, using the diagram in part (d).

8.1 Linear motion with constant acceleration

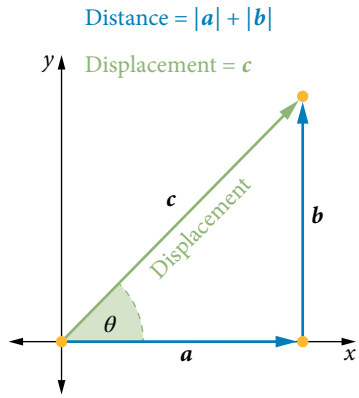
Kinematics is a branch of physics that describes the motion of particles, bodies and systems of bodies without reference to the forces that produce the motion. The word 'kinematics' has its roots in the Greek word *kinema* meaning movement or motion.

In this chapter objects will be treated as particles or points, as the motion of the object is the subject of interest and not its shape or any other physical characteristics. Motion of a particle in a straight line is called rectilinear or linear motion.



Distance and displacement

The displacement of a particle describes the position of a particle in relation to a reference or starting position. As a vector quantity this requires both magnitude and direction. The distance travelled by a particle is a scalar quantity so has only magnitude. The two values can differ. For example, if a particle moves 2 m to the right and then 1 m to the left its displacement is 1 m to the right of its starting position but the distance travelled is 3 m.



When a particle undergoes rectilinear motion there are only two possible directions for its motion. These are typically referred to as *positive* and *negative* as this allows visualisation of the motion.

Velocity and speed both describe how fast the particle is moving at a particular point in time.

Velocity (v) is a vector quantity so requires both magnitude and direction.

Speed is a scalar quantity and is the absolute value of the velocity. Hence, it can be represented by $|v|$. It has only magnitude.

Units

The most frequently used units describing linear motion are units in the SI system, that is, metres and seconds, although kilometres and hours are also sometimes used.

SI or the International System of Units is the metric system used around the world today. The abbreviation SI comes from the French Système Internationale d'Unités.

Average and instantaneous speed and velocity

Average velocity represents the rate of change of displacement with respect to some stated time interval. When the time interval approaches zero, the average velocity approaches the instantaneous velocity.

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time interval}}$$

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}}$$

$$v_{\text{instantaneous}} = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right), \text{ where } \Delta x \text{ is the displacement in time interval } \Delta t$$
$$= \frac{dx}{dt}$$

The same expression can be used for instantaneous speed, where Δx represents the distance travelled.

Notation

Distance and displacement are usually represented by x, s or $\int v(t)\,dt$.

Speed or velocity can be represented by v, $\frac{dx}{dt}$, $x'(t)$ or $\dot{x}$.

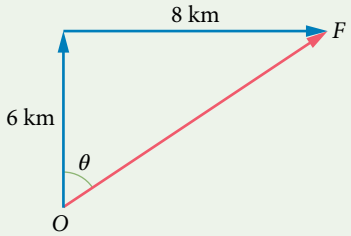
1 Determining average speed and velocity for motion in one direction

Determine the average speed and the average velocity of a car travelling 150 km on a straight road in 3 hours from east to west. Give your answers in km/h.

THINKING	WORKING
1 Recall the formula for average speed.	$\text{Average speed} = \frac{\text{distance travelled}}{\text{time interval}}$
2 Determine the average speed.	$\text{Average speed} = \frac{150\,\text{km}}{3\,\text{h}}$ $= 50\,\text{km/h}$
3 Given that displacement and distance are equal in this example, include a direction to give the average velocity.	$\text{Average velocity} = \frac{\text{displacement}}{\text{time interval}}$ $= \frac{150\,\text{km west}}{3\text{h}}$ $= 50\,\text{km/h west}$ This direction is sometimes considered to be negative, so the velocity could then be written as -50 km/h, where the negative describes the direction.

2 Determining average speed and velocity for motion in more than one direction

Calculate the average speed and the average velocity of a cyclist travelling 6 km north for 20 minutes and then turning east and travelling 8 km for another 30 minutes. Give your answers in km/h.

THINKING	WORKING
1 Draw a diagram to represent the motion.	

2 Determine the distance travelled and the displacement.

(Recognising this as the 6, 8, 10 Pythagorean triad could save some work.)

$$\begin{aligned}\text{Distance} &= 6 + 8\\ &= 14\,\text{km}\\ \text{Displacement} &= |OF|\\ &= \sqrt{6^2 + 8^2}\\ &= \sqrt{36 + 64}\\ &= \sqrt{100}\\ &= 10\,\text{km}\end{aligned}$$

$$\theta = \tan^{-1}\left(\frac{8}{6}\right)$$

$$\approx 53^\circ$$

$$\text{Displacement} = 10\,\text{km N53°E}$$

3 Convert the time taken to the required units.

$$\text{Total time} = \frac{(20 + 30)}{60}$$

$$= \frac{5}{6}\ \text{hours}$$

4 Determine the average speed and velocity.

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time interval}}$$

$$= \frac{14\,\text{km}}{\frac{5}{6}\text{h}}$$

$$= 16.8\,\text{km/h}$$

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time interval}}$$

$$= \frac{10\,\text{km N53°E}}{\frac{5}{6}\text{h}}$$

$$= 12\,\text{km/h N53°E}$$

Acceleration

Acceleration is the rate of change of the velocity of a particle with respect to time. As such acceleration is a vector quantity. The most frequently used units for acceleration are m/s^2, but other units might sometimes be used.

Sign convention for acceleration:

- If both the acceleration and the velocity of a particle are in the same direction (both positive or both negative), the velocity of the object is increasing.
- If the acceleration and the velocity of a particle are in opposite directions (one positive and one negative), the velocity of the object is decreasing.

Since acceleration represents the rate of change of displacement with respect to time, when the time interval approaches zero, average acceleration approaches the instantaneous acceleration, hence:

$$a_{\text{instantaneous}} = \lim_{\Delta t \to 0}\left(\frac{\Delta v}{\Delta t}\right)$$
$$= \frac{dv}{dt}$$

Notation

Acceleration can be represented as a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$ or $\ddot{x}$.

This leads to another representation for velocity, as $\int a(t)\,dt$.

In this section, you will only be considering motion under constant acceleration.

3 Determining acceleration from a given displacement

Calculate the acceleration of a particle at time t seconds with displacement x metres, where $x = 2t^2 - 3t + 1$.

THINKING	WORKING
1 Determine an expression for the velocity of the particle at any time t.	$v = \frac{dx}{dt}$ $= (4t - 3)\,\text{m/s}$
2 Determine an expression for the acceleration of the particle at any time t.	$a = \frac{dv}{dt}$ $= 4\,\text{m/s}^2$
3 Write the answer.	$a = 4\,\text{m/s}^2$

4 Determining velocity and acceleration at a given time

A particle is moving in a straight line with displacement x metres from a fixed reference point O at time t seconds. The displacement of the particle is given by $x(t) = 4t^2 + 2t - 2$, where $t \geq 0$.

(a) Determine the initial displacement of the particle.

THINKING	WORKING
1 Substitute $t = 0$ into the displacement equation.	$x(t) = 4t^2 + 2t - 2$ $x(0) = 4 \times 0^2 + 2 \times 0 - 2$ $= -2\,\text{m}$
2 Interpret the answer.	The particle's initial position is 2 m left of the reference point O. or $x(0) = -2\,\text{m}$ or 2 m to the left of the reference point

(b) Determine the displacement, velocity and acceleration of the particle, 2 seconds after the beginning of its motion.

1 Substitute the value of t into the displacement equation.	$x(t) = 4t^2 + 2t - 2$ $x(2) = 4 \times 2^2 + 2 \times 2 - 2$ $= 18\text{ m}$
2 Determine an expression for the velocity of the particle's motion and substitute the given value of t.	$v = \frac{dx}{dt}$ $= (8t + 2)\text{ m/s}$ $v(2) = 8 \times 2 + 2$ $= 18\text{ m/s}$
3 Determine an expression for the acceleration of the particle's motion and substitute the given value of t.	$a = \frac{dv}{dt}$ $= 8\text{ m/s}^2$
4 Interpret the answer.	2 seconds after the beginning of the motion the particle is 18 m right of O, moving at 18 m/s right, constantly accelerating by 8 m/s^2.

Making connections

Linear motion with constant acceleration

Move the slider to observe linear motion with constant acceleration.

Technology worked example

Calculating displacement, velocity and acceleration

5 Determining velocity and displacement from a given acceleration

A body moving in a straight line is slowing down at a constant rate of 2 m/s^2.

(a) If the initial velocity is 12 m/s, determine an expression for the velocity of the body at time t seconds.

THINKING	WORKING
1 Write the formula for velocity when acceleration is given.	$v = \int a(t)\,dt$
2 Determine the expression for the velocity of the body at any time t.	Note: Slowing down requires a negative sign. $a = -2\text{ m/s}^2$ $v = \int (-2)\,dt$ $= -2t + c$
3 Determine the value of the constant, c.	$v(0) = 12\text{ m/s}$ $-2 \times 0 + c = 12$ $c = 12$
4 Interpret the answer.	$v(t) = (-2t + 12)\text{ m/s}$

(b) If the body was 3 m to the left of a reference point O at $t = 1$ s, determine the expression for its displacement x in terms of time t.

1 Write the formula required to obtain the displacement expression from a given velocity.	$x = \int v(t)\,dt$
2 Determine the expression for the displacement of the body at any time t.	$x(t) = -t^2 + 12t + c$
3 Determine the value of the constant, c.	$x(1) = -3$ $-3 = -1^2 + 12 \times 1 + c$ $c = -3 + 1 - 12$ $c = -14$
4 Write the answer.	$x(t) = -t^2 + 12t - 14$

Equations of motion for constant acceleration

Motion in a straight line of a particle with constant acceleration can be described by the equations of motion using the following notation:

- u represents the initial velocity of the particle
- v represents the final velocity of the particle
- t represents the time interval of the motion
- a represents the constant acceleration of the particle
- s represents the displacement of the particle , from its initial position

The equations can be derived as follows, when $t = 0$ and $s = 0$:

$v = \int a\,dt \quad v(0) = u$
$= at + c \quad\quad = c$
$\quad\quad\quad\quad v = u + at$

$s = \int v\,dt \quad\quad s(0) = 0$
$= \int (u + at)\,dt \quad\quad = c$
$= ut + \frac{1}{2}at^2 + c \quad\quad s = ut + \frac{1}{2}at^2$

$$v = u + at \quad [1]$$

$$s = ut + \frac{1}{2}at^2 \quad [2]$$

By rearranging [1] $\quad a = \frac{v - u}{t}$

From [2] $\quad s = ut + \frac{1}{2}at^2$

substituting a in [2] $\quad s = ut + \frac{1}{2}\left(\frac{v - u}{t}\right)t^2$

$2s = 2ut + vt - ut$

$s = \frac{(u + v)t}{2}$

By rearranging [3] $\quad t = \frac{2s}{u + v}$

From [1] $\quad v = u + at$

substituting t in [1] $\quad v = u + a\left(\frac{2s}{u + v}\right)$

$(v - u)(v + u) = 2as$

$v^2 = u^2 + 2as$

$$s = \frac{1}{2}(u + v)t \quad [3]$$

$$v^2 = u^2 + 2as \quad [4]$$

In this course you will consider objects moving with both constant and non-constant acceleration, so to avoid confusion, calculus will be used to solve problems.

6 Using the equations for constant acceleration

An object, starting at the origin, moves in an easterly direction with an initial velocity of 2.4 m/s. At time $t = 4$ s the object is at rest.

(a) If the object moves with a constant acceleration find this acceleration.

THINKING	WORKING
1 Identify the given information.	$v(0) = 2.4\,\text{m/s}$, $v(4) = 0\,\text{m/s}$
2 Recall an appropriate rule to calculate acceleration, substitute values and evaluate.	$a = \dfrac{\Delta v}{\Delta t}$ $= \dfrac{0 - 2.4}{4 - 0}$ $= -0.6$
3 Interpret the answer.	The object is accelerating at $-0.6\,\text{m/s}^2$ in an easterly direction.

(b) Determine the position of the particle when it becomes stationary.

1 Use $v(t) = \int a(t)dt$ to determine the velocity equation. Substitute known values for calculate the value of the constant.	$v(t) = \int -0.6\,dt$ $= -0.6t + c$ $v(0) = 2.4 \quad \therefore c = 0$ and $v(t) = -0.6t + 2.4$
2 Use $x(t) = \int v(t)dt$ to determine the displacement. Substitute known values and integrate to find the displacement equation.	$x(t) = \int (-0.6t + 2.4)\,dt$ $= -0.3t^2 + 2.4t + d$ $x(0) = 0 \quad \therefore d = 0$ and $x(t) = -0.3t^2 + 2.4t$
3 Substitute the given time when stationary to find the object's displacement.	$x(4) = -0.3 \times 4^2 + 2.4 \times 4$ $= 4.8$
4 Interpret the answer.	The object comes to rest 4.8 m East of the starting position.

(c) At what time was the object 3 m to the East of the starting position for the first time?

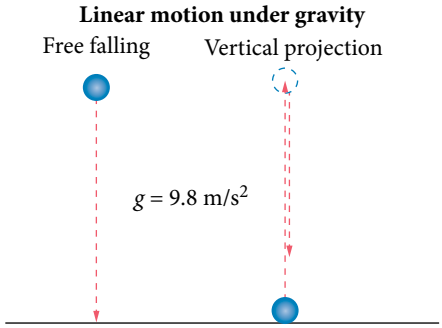
1 Substitute the known values and evaluate.	$3 = -0.3t^2 + 2.4t$ $0.3t^2 - 2.4t + 3 = 0$ $t^2 - 8t + 10 = 0$ where $a = 1$, $b = -8$, $c = 10$ $t = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 1 \times 10}}{2 \times 1}$ $= \dfrac{8 \pm \sqrt{24}}{2}$ $= t = 6.449\ldots$ or $1.550\ldots$
2 Interpret the answer.	The first time the object is 3 m East of the starting position is after 1.55 s.

Vertical motion under gravity

Linear vertical motion occurs when an object is either falling towards Earth's surface or is projected vertically upwards or downwards. A free falling object is an object falling with a constant acceleration due to gravity, where air resistance is negligible. Near Earth's surface, the acceleration (also called a) due to gravity (g) has a value of approximately $9.8\,\text{m/s}^2$, or $g = 9.8\,\text{m/s}^2$.

When solving problems involving vertical motion under gravity it is possible to assign upwards as positive or downwards as positive. To assist your understanding of the question use a diagram to indicate the direction of positive force.

It is generally the case that upwards is considered positive so assume this unless you are specifically told otherwise.



7 Considering vertical motion under gravity

A ball is dropped from the top of a building and reaches the ground in 4.2 seconds.

Determine the height the ball was dropped from and the speed of the ball when it hits the ground.

THINKING	WORKING
1 Recall an equation to calculate the velocity of the ball and use the velocity equation to calculate the position equation.	$v(t) = \int a(t)\,dt$ $= \int 9.8\,dt$ $= 9.8t + c$ $v(0) = 0 \quad \therefore c = 0$ $x(t) = \int v(t)\,dt$ $= \int 9.8\,dt$ $= 4.9t^2 + d$
2 Substitute the known time and position (assuming that x represents the distance from the ground).	$x(4.2) = 0$ $0 = 4.9 \times 4.2^2 + d$ $d = 86.4$ (1 d.p.) So $x = 4.9t^2 + 86.4$
3 Interpret the result.	At $t = 0$, $x = 86.4$ The ball is dropped from a height of 86.4 m.
4 Substitute the time when the ball hits the ground into the velocity equation.	$v(4.2) = 9.8 \times 4.2$ ≈ 41
5 Interpret the answer.	The ball hits the ground at a speed of 41 m/s.

Explore further

Vertical projectile motion

Use technology to explore the concept of projectile motion.

8 Considering vertical projectile motion

An object is projected vertically upward from a platform 10 m above the ground with a speed of 25 m/s.

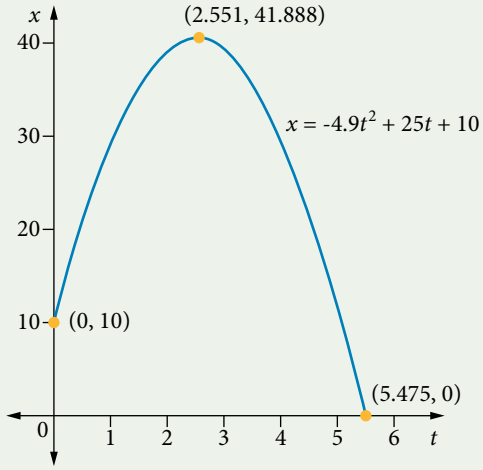
(a) Determine the maximum height reached by the object and the time it occurs. Give your answer correct to 1 decimal place.

THINKING	WORKING
1 Identify the given information.	At $t = 0$, the velocity is 25 m/s with gravity acting in the opposite direction, so $a = -9.8\ \text{m/s}^2$.
2 Recall an appropriate rule to calculate acceleration.	$v(t) = \int a(t)\,dt$ $= \int -9.8\,dt$ $= -9.8t + c$
3 Substitute the given information to determine the value of the constant of integration.	$v(0) = 25$ $25 = -9.8 \times 0 + c$ $\therefore c = 25$ So, $v(t) = -9.98t + 25$
4 The maximum height will be reached when the velocity is 0 m/s when the object momentarily stops and then begins its descent.	When $v = 0$: $0 = -9.8t + 25$ $9.8t = 25$ $t = 2.55\ldots$ s
5 Determine the position equation, where x represents the height above the ground and at $t = 0$, $x = 10$.	$x(t) = \int v(t)\,dt$ $= \int -9.8t + 25\,dt$ $= -4.9t^2 + 25t + c$ At $t = 0$, $x = 10$ $\therefore c = 10$ So, $x(t) = -4.9t^2 + 25t + 10$
6 Substitute the time when the object reaches the maximum height and interpret the answer.	$x(2.55\ldots) = -4.9 \times (2.55\ldots)^2 + 25 \times 2.55\ldots + 10$ $= 41.88\ldots$ The object reaches a maximum height of 41.9 m approximately 2.6 s after it is projected.

(b) Determine the total time taken for the object to fall to the ground, correct to 1 decimal place.

1 Let $x = 0$ and solve for t.

$0 = -4.9t^2 + 25t + 10$



3 Interpret the answer.

The object hits the ground approximately 5.5 s after it is launched.

Velocity–time graphs

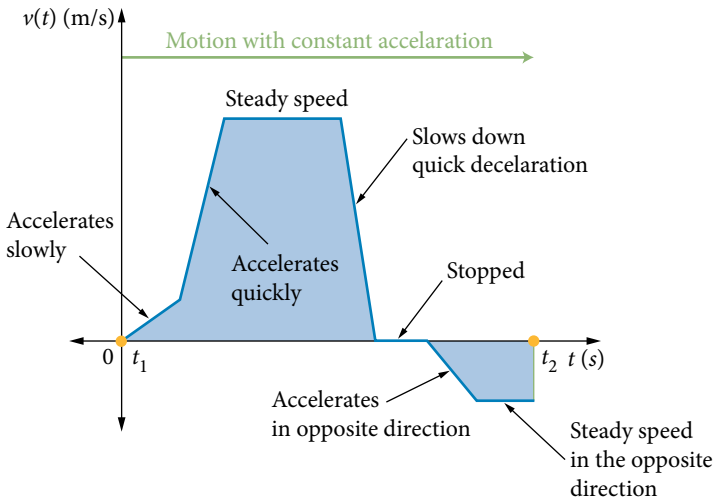
The motion of an object can be represented graphically using velocity–time graphs. The variables are velocity (the dependent variable on the vertical axis) and time (the independent variable on the horizontal axis).

Acceleration is the rate of change of velocity, which means the gradient of the velocity–time graph represents the acceleration.

Motion with constant acceleration has a constant gradient and is represented by a straight line segment.

Displacement is given by

$$x = \int_{t_1}^{t_2} v(t)\, dt$$
$$= x\Big|_{t_1}^{t_2}$$
$$= x_2 - x_1$$



This means the area under the graph represents the displacement of the object between t_1 and t_2.

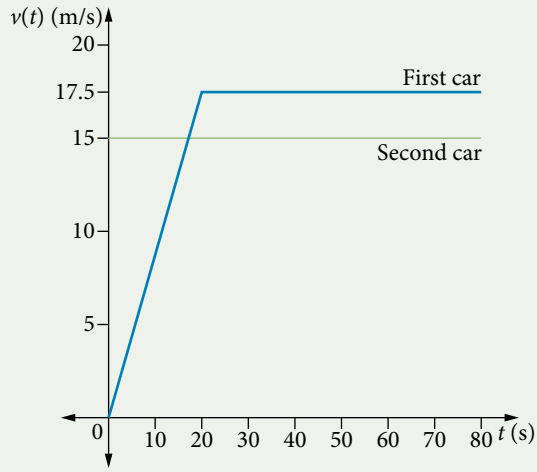
Hence, the distance travelled will be given by $\left|\int_{t_1}^{t_2} v(t)\, dt\right| = |x_2 - x_1|$.

The area under the velocity–time curve in a time interval represents the change in displacement for the time interval specified.

9 Using velocity–time graphs

A car starts from rest accelerating at a constant rate and travels for 20 seconds until it reaches a speed of 63 km/h and maintains this speed afterwards. At the moment the car is starting, a second car is overtaking by travelling in the same direction at a speed of 54 km/h.

(a) Sketch a velocity–time graph to illustrate the velocity of the two cars.

THINKING	WORKING
1 Convert measurements to the same units.	Car 1: After 20 seconds, constant velocity is $v = 63$ km/h $= 17.5$ m/s Car 2: Constant velocity is $v = 54$ km/h $= 15$ m/s
2 Sketch the graph.	

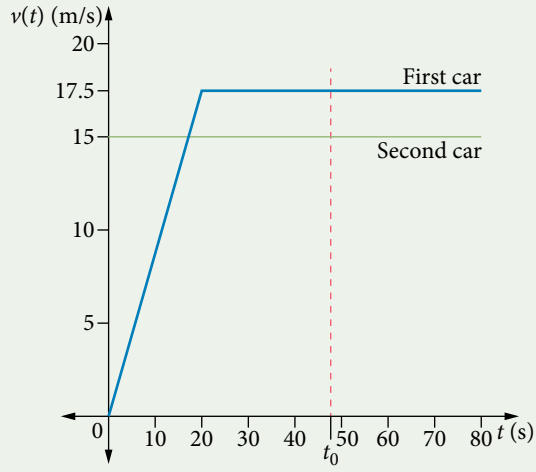
(b) Determine the time taken for the first car take to overtake the second car.

1 Determine the distance travelled by the two cars until they are at the same position. This is found from the area under the velocity–time graph.

Let t_0 be the time taken.

$$\text{Distance}_1 = \frac{1}{2} \times 17.5 \times 20 + 17.5(t_0 - 20)$$
$$= 175 + 17.5t_0 - 350$$
$$= (17.5t_0 - 175)\,\text{m}$$

$$\text{Distance}_2 = 15t_0\,\text{m}$$



2 Equate the distances and solve to determine the time taken.	$15t_0 = 17.5t_0 - 175$ $t_0 = 70$
3 Interpret the answer.	The first car will overtake the second car after 70 seconds from the starting point.

(c) Determine the distance travelled by each car until the overtaking point.

1 Determine the area under the graph for each car.	$\text{Distance}_1 = 175 + 17.5 \times (70 - 20)$ $= 175 + 17.5 \times 50$ $= 175 + 875$ $= 1050\,\text{m}$ Verify the distance by checking that car 2 has travelled the same distance. $\text{Distance}_2 = 15 \times 2$ $= 15 \times 70$ $= 1050\,\text{m}$
2 Interpret the answer.	The overtaking point will occur 1050 m after the starting point.

10 Further applications of velocity-time graphs

A bus travels 3.6 km in a straight line between two bus stops A and B. When leaving stop A, the bus starts accelerating at a rate of $0.4\,\text{m/s}^2$ until it reaches a maximum speed of 57.6 km/h. The bus maintains this maximum speed until it approaches stop B, slowing down at a rate of $0.5\,\text{m/s}^2$ so that it comes to a stop at B.

(a) Determine the time and distance travelled by the bus between leaving stop A and reaching its maximum speed.

THINKING	WORKING
1 Convert the speed to metres per second.	$57.6\,\text{km/h} \div 3.6 = 16\,\text{m/s}$
2 Use the formula $a = \frac{\Delta v}{\Delta t}$ to calculate the time interval required for the bus to accelerate to maximum speed.	$0.4 = \frac{16 - 0}{\Delta t}$ $\Delta t = \frac{16}{0.4}$ $\Delta t = 40\ \text{s}$
3 Interpret the value.	The bus takes 40 seconds to reach its maximum speed.
4 Use $v(t) = \int a(t)\,dt$ to calculate the velocity equation.	$v(t) = \int 0.4\,dt$ $= 0.4t + c$ Given that the bus started from rest then at $t = 0$, $v = 0$. $\therefore\ v(t) = 0.4t$

5 Use $x(t) = \int v(t)\,dt$ to calculate the displacement equation, where x represents the distance from A.	$x(t) = \int 0.4\,dt$ $= 0.2t^2 + c_1$ Given that the bus starts at A, then $c_1 = 0$ and: $x(t) = 0.2t^2$
6 Substitute the time taken to reach the maximum speed, into the displacement equation to calculate the distance travelled during the initial period of acceleration.	$x(40) = 0.2 \times 40^2$ $= 320\text{ m}$
7 Write the answer.	During the initial period of acceleration, the bus takes 40 s to reach maximum speed in which time it travels a total distance of 320 m.

(b) Determine the time the bus takes to come to rest from the moment it begins to slow down and the distance covered during this time.

1 Identify the known information.	The initial velocity $v_1 = 16\text{ m/s}$ and the final velocity is $v_2 = 0\text{ m/s}$.
2 Use the formula $a = \frac{\Delta v}{\Delta t}$ to calculate the time interval required for the bus to slow to a stop.	$-0.5 = \frac{0 - 16}{\Delta t}$ $\Delta t = \frac{-16}{0.5}$ $\Delta t = 32\text{ s}$
3 Interpret the value.	The bus takes 32 seconds to slow to a stop.
4 Use $v(t) = \int a(t)\,dt$ to calculate the velocity equation, taking t to represent the time when the bus starts to slow.	$v(t) = \int -0.5\,dt$ $= -0.5t + c_2$ Given that the initial velocity of the bus during this part of the journey is 16 m/s then: $16 = -0.5 \times 0 + c_2$ $c_2 = 16,\ \therefore v = -0.5t + 16$
5 Use $x(t) = \int v(t)\,dt$ to calculate the displacement equation, where x represents the displacement from the point where the bus starts to slow.	$x(t) = \int -0.5t + 16\,dt$ $= -0.25t^2 + 16t + c_3$ At $t = 0$, $x = 0$ and therefore $c_3 = 0$
6 Substitute the time taken for the bus to stop into the displacement equation to calculate the distance travelled during the initial period of acceleration.	$x(32) = -0.25 \times 32^2 + 16 \times 32$ $= 256\text{ m}$
7 Write the answer.	During the approach to B, the bus takes 32 seconds to slow to a stop over a distance of 256 m.

(c) Sketch a velocity–time graph to illustrate the motion of the bus.

1 Determine the total distance travelled by the bus when it is travelling at a constant velocity.

The distance between the stops A and B is 3.6 km = 3600 m and travels 320 m accelerating to reach maximum speed and 256 m slowing to a stop.

$3600 - 320 - 256 = 3024$ m

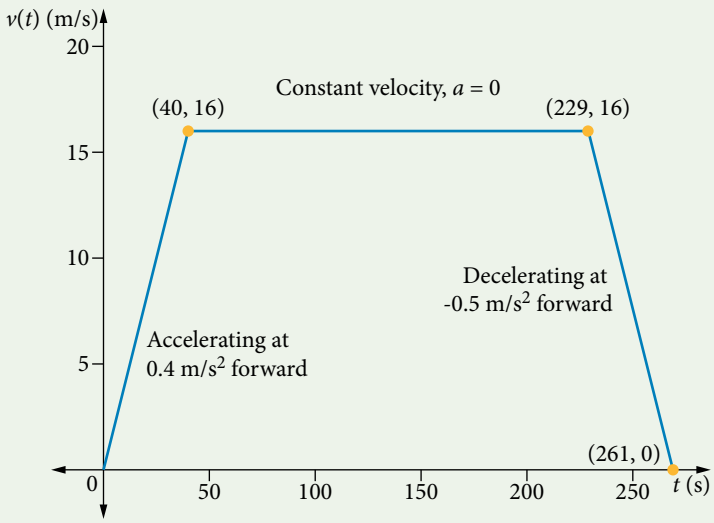
2 Use $\frac{\Delta x}{\Delta t}$ to determine the time spent travelling at a constant velocity.

$\Delta t = \frac{3024}{16}$

$\Delta t = 189$ s

The bus spends 189 seconds travelling at a constant velocity.

3 Sketch the graph.



EXERCISE 8.1 Linear motion with constant acceleration

Worked Example 1

1 Determine the average velocity for the following situations:

(a) A student walks down the hallway from the art room to the science room, a distance of 55 m, in 8 seconds.

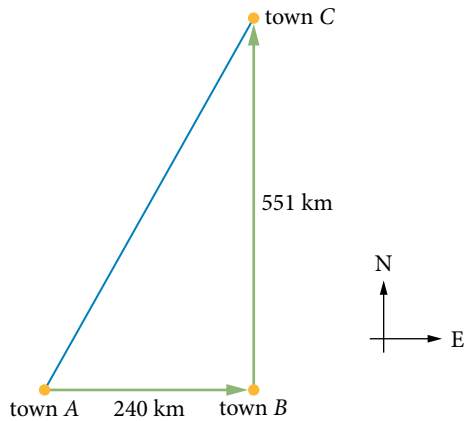
(b) A car travels a distance of 15 km due west in a time of 10 minutes.

Worked Example 2

2 Determine the average speed and the average velocity for the following situations:

(a) A particle moves 28 m to the left of an origin in 56 seconds.

(b) A car travels 240 km east for 3 hours and then 551 km north for 4.5 hours, as shown in the diagram.



3 Determine an expression for the velocity v and acceleration a for each of the following displacement functions $x(t)$.

(a) $x = -2t^2 - 5t + 9$ (b) $x = (3t + 4)^2$ (c) $x = -4 - 6t$

4 A particle is moving in a straight line with displacement x metres from a fixed reference point O at any time t seconds. The displacement of the particle is given by the equation $x(t) = 10t^2 + 19t - 56$, where $t \geq 0$.

Determine the velocity and acceleration of the particle 2 seconds after the beginning of its motion.

5 The position x km of a taxi is given by $x(t) = (1 - 2t)^2$ at time t hours.

(a) Determine the velocity and acceleration of the taxi after 45 minutes.

(b) Is the taxi speeding up or slowing down at this time?

6 The acceleration of a moving object is 5 m/s^2 in a forward direction. If the object's initial velocity is -5 m/s and its initial displacement $x(0) = 0$, determine its displacement relative to its starting point at $t = 6$ s.

7 A toy car has an acceleration a of 2 m/s^2 and starts with an initial velocity $v = 0$ m/s. After 3 seconds, its position is 10 m from its starting point. Determine its displacement relative to its starting point after 2 seconds.

8 A particle moving in a straight line has a displacement x m, given by $x = -5t^2 + 6t - 11$.

(a) Determine the velocity v and acceleration a at time t seconds.

(b) Determine the velocity of the particle at $t = 0$ and $t = 1$ s.

(c) Determine the displacement when the particle comes to rest.

9 A stone rolls down a hill with acceleration 2 m/s^2. It is initially at rest. Determine each of the following:

(a) Its displacement after 12 seconds

(b) The time it takes to reach a speed of 25 m/s

(c) Its speed after 15 seconds

(d) The time it takes to travel 20 metres

(e) The speed after 12 seconds

10 Consider the following objects falling freely with acceleration due to gravity of 9.8 m/s^2 downwards. Give your answers correct to 1 decimal place.

(a) An object falls freely from a height of 65 m. Determine the speed of the object and the time taken when it reaches the ground.

(b) An object is dropped and reaches the ground after 7 seconds. Calculate the speed when reaching the ground and the height from which the object was dropped.

(c) If an object dropped from the top of a cliff reaches sea level with a speed of 117.6 m/s, determine the time taken to reach sea level and the height of the cliff.

11 Consider the following objects, which are projected upwards, reach their zenith and fall freely experiencing an acceleration due to gravity of 9.8 m/s^2 downwards.

(a) An object is projected vertically upward from a platform 23 m above the ground with a speed of 20 m/s.

(i) Assuming the object does not hit the platform, determine the time it takes the object to reach the ground, correct to 2 decimal places.

(ii) Determine the total distance travelled, correct to 2 decimal places.

(b) A particle is projected vertically upward from the top of a 58 m high building with a speed of 32 m/s. Determine the maximum height from the ground that the particle reaches, correct to 1 decimal place.

(c) What is the time taken by an object to reach the ground if the object is projected vertically upward from ground level with a speed of 14 m/s?

Worked Example 9

12 A particle is moving in a positive direction, in a straight line, with an initial velocity of 1.8 m/s. At time $t = 3$ s, the particle is at rest.

(a) If the particle is moving with a constant acceleration, determine the acceleration.

(b) Use a velocity–time graph to determine the position of the particle when it becomes stationary.

(c) At what time is the particle 12 m on the negative side of the starting point?

13 An object moves with velocity in a positive direction and acceleration in a negative direction.

(a) Which one of the following statements is correct?

A The object is travelling faster in a negative direction.

B The object is travelling in a negative direction and slowing down.

C The object is travelling faster in a positive direction.

D The object is travelling in a positive direction and slowing down.

(b) Explain why the other statements in part (a) are false.

14 A cyclist travels 3 km due north and then 4 km due east. Determine which of the following statements is incorrect and explain why.

A The total distance travelled is 7 km.

B Both the distance travelled and the size of the displacement from the original position are the same.

C The displacement from the original position is 5 km at an angle of $\tan^{-1}\left(\frac{4}{3}\right)$ with due north.

D The distance travelled is greater than or equal to the size of the displacement.

15 A body is moving in a straight line with a velocity of 14 m/s E. It moves with constant acceleration and its velocity after 3 seconds is 7 m/s W. 10

(a) Determine the acceleration.

(b) Determine the time when the particle comes to rest and changes direction.

(c) Determine the displacement of the body after 6 seconds relative to the starting point.

(d) Use a velocity–time graph to determine the distance covered by the particle over the 6 second time interval.

16 A car starts from rest with a constant acceleration of 0.1 m/s^2 until it reaches a speed of 24 m/s and maintains this speed for 30 minutes. Then the car slows down at a constant rate of -0.1 m/s^2 until it stops. Sketch a velocity–time graph representing this motion and determine the time taken for the whole journey. Use km/h for velocity and hours for time.

17 A tram moved from rest with constant acceleration, and after 10 s has travelled 400 m forward.

(a) Determine the tram's acceleration in m/s^2.

(b) Determine the velocity, in m/s, of the tram at the end of the 10 s.

(c) Determine the tram's velocity, in m/s, when it has travelled 100 m.

(d) Determine the tram's velocity when it has travelled 200 m. Give your answer in exact form.

(e) Determine the tram's velocity when it has travelled 300 m. Give your answer in exact form.

(f) If the distance travelled by the tram from rest is s and the time taken to travel this distance is t, determine an expression for the velocity of the tram in terms of x and t.

18 The velocity of a car, v m/s, is given by the function (where t is measured in seconds)

$$v(t) = \begin{cases} 5t, & t \in [0,6) \\ 30, & t \in [6,9) \\ -5t + 75, & t \in [9,15] \end{cases}$$

(a) Determine the velocity at $t = 0,\ t = 4,\ t = 8,\ t = 12$.

(b) Describe the motion forward.

(c) Use a velocity–time graph to determine the total distance travelled by the car.

(d) At $t = 15$ s the car changes direction, momentarily stopping and then accelerating at $4\ m/s^2$ towards the starting point for 3 seconds and then maintains this speed until it reaches the starting point.

What is the car's velocity when it stops accelerating?

(e) How long does it take the car to reach the starting point from the moment it changed direction?

(f) Sketch the graph of the return journey on the velocity–time graph from part **(c)**.

19 A car is driving at a constant speed of 70 km/h in a 60 km/h zone and passes a stationary police car at time $t = 0$. The police car starts to follow the speeding car, accelerating for 25 seconds from rest until it reaches a speed of 95 km/h. The police car maintains this speed until it catches up with the speeding car. Give your answers correct to 2 decimal points, where necessary.

(a) Determine the time it takes the police car to catch up with the speeding car.

(b) Determine the total distance travelled by each car, from the start of pursuit, until they are in the same position.

(c) The speeding car slows down until it comes to rest after 500 m. Determine the constant rate at which the car slows down.

(d) How long does it take the speeding car to come to rest after starting to slow down?

(e) Determine expressions for $v(t)$ for both cars during the motion.

(f) Sketch a velocity–time graph to represent the motion of the two cars.

Linear motion with non-constant acceleration

8.2

In this section rectilinear motion will be considered further, with non-constant acceleration. The techniques used in the previous section will be applied to linear motion with variable acceleration.

11 Determining velocity and acceleration from a displacement function

The displacement of an object travelling in a straight line x metres from a fixed origin O at time t seconds is given by $x(t) = -2t^3 + 3t^2 + 5t - 6$, where the right of the origin is considered positive.

(a) Determine the displacement, velocity and acceleration of the object at $t = 1$ s.

THINKING	WORKING
1 Determine the displacement of the object at the required time.	$x(1) = -2 \times 1^3 + 3 \times 1^2 + 5 \times 1 - 6$ $= -2 + 3 + 5 - 6$ $= 0$ m
2 Determine an expression for the velocity of the object.	$v(t) = \frac{dx}{dt}$ $= \frac{d}{dt}\left(-2t^3 + 3t^2 + 5t - 6\right)$ $= -6t^2 + 6t + 5$
3 Determine the velocity of the object at the required time.	$v(1) = -6 \times 1^2 + 6 \times 1 + 5$ $= -6 + 6 + 5$ $= 5$ m/s
4 Determine an expression for the acceleration of the object.	$a(t) = \frac{dv}{dt}$ $= \frac{d}{dt}\left(-6t^2 + 6t + 5\right)$ $= -12t + 6$
5 Determine the acceleration of the object at the required time.	$a(1) = -12 \times 1 + 6$ $= -12 + 6$ $= -6$ m/s^2
6 Interpret the answer.	At $t = 1$ s the object is at the origin moving at 5 m/s to the right and slowing at a rate of 6 m/s^2 to the right.

(b) Determine the other time(s) when the object is at the origin.

1 Let the displacement of the object be zero and solve for the time.	$-2t^3 + 3t^2 + 5t - 6 = 0$ $(t-1)(-2t^2 + t + 6) = 0$ $(t-1)(-2t^2 - 3t + 4t + 6) = 0$ $(t-1)[-t(2t+3) + 2(2t+3)] = 0$ $(t-1)(2-t)(2t+3) = 0$ $t = 1$, $t = 2$ or $t = -1.5$ Discard $t = -1.5$ as $t \geq 0$.
2 Interpret the answer.	The object is also at the origin at time $t = 2$ s.

(c) Determine the time when the body is moving at maximum velocity and the value of the maximum velocity.

1 A local maximum (or a local minimum) velocity occurs when its derivative, the acceleration of the body, is zero. Given that the equation for the velocity represents a convex parabola it has only one stationary point, which must be a maximum.	$a(t) = 0$ $-12t + 6 = 0$ $t = 0.5$ s
2 Determine the value of the maximum velocity.	$v(0.5) = -6 \times 0.5^2 + 6 \times 0.5 + 5$ $= -1.5 + 3 + 5$ $= 6.5$ m/s
3 Interpret the answer.	The maximum velocity of the object is 6.5 m/s to the right and it occurs at $t = 0.5$ s.

In Worked example 11 it was assumed that the displacement equation may not apply before $t = 0$. This means that the 'start' time, $t = 0$, is not necessarily the start of the *motion*.

12 Determining velocity from an acceleration function

The acceleration of a toy car is given by $3\sec^2(t)\,\text{m/s}^2$, $0 \leq t < \frac{\pi}{3}$ at time t seconds for any time t within the specified domain. If it starts from rest, determine the velocity of the racing car after $\frac{\pi}{3}$ seconds.

THINKING	WORKING
1 Use integration to write an expression for the velocity.	$v = \int \sec^2(t)\,dt$ $= \tan(t) + c$ $v(0) = 0$ $c = 0$ $v = \tan(t)$
2 Determine the velocity at the required time.	$v\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right)$ $= \sqrt{3}$

3 Interpret the answer.	The velocity of the toy car after $\frac{\pi}{3}$ seconds is $\sqrt{3}$ m/s in a positive direction.

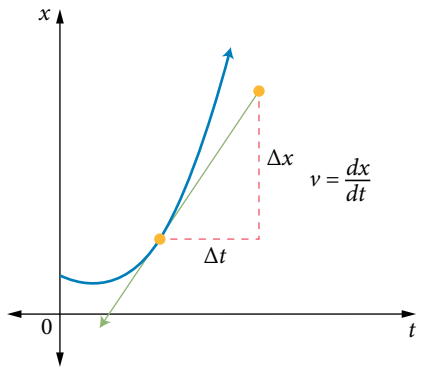
13 Determining displacement from an acceleration function

The acceleration of a particle is given by $a = 2e^{0.1t}$ m/s^2 at time t seconds for a finite domain applied to t. Determine the displacement x after 10 seconds if the initial velocity is 6 m/s and the initial displacement is zero. Leave your answer correct to 1 decimal place.

THINKING	WORKING
1 Determine an expression for the velocity.	$v = \int 2e^{0.1t}\,dt$ $= 20e^{0.1t} + c$ $v(0) = 6$ $6 = 20e^0 + c$ $c = -14$ $\therefore v = 20e^{0.1t} - 14$
2 Determine an expression for the displacement.	$x = \int v\,dt$ $= \int \left(20e^{0.1t} - 14\right)dt$ $= 200e^{0.1t} - 14t + d$ $x(0) = 0$ $0 = 200 + d$ $d = -200$ $\therefore x = 200e^{0.1t} - 14t - 200$
3 Determine the displacement at the required time and interpret the answer.	$x(10) = 200e^1 - 140 - 200$ $= 203.7\ (1\text{ d.p.})$ The displacement of the object after 10 seconds is 203.7 meters in the positive direction.

Displacement–time graphs

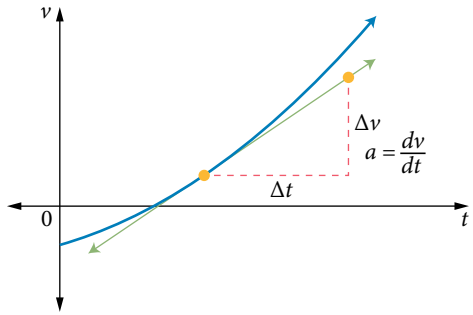
The gradient of the tangent to the displacement–time curve at any time t represents the instantaneous velocity of the particle at time t.



Velocity–time graphs

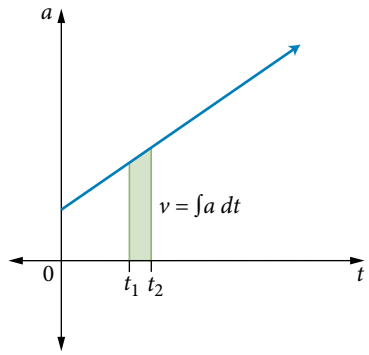
The gradient of the tangent to the velocity–time curve at any time t represents the instantaneous acceleration of the particle at time t.

The area under the velocity–time curve in a time interval represents the change in displacement for the time interval specified.



Acceleration–time graphs

The area under the acceleration–time curve in a time interval represents the change in velocity for the time interval specified.

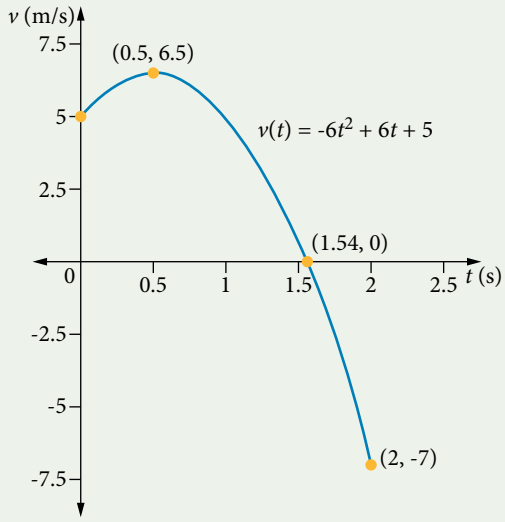


14 Velocity–time graphs with non-constant acceleration

The velocity of an object travelling in a straight line at time t seconds is given by $v(t) = -6t^2 + 6t + 5$. Sketch a velocity–time graph and describe the motion of the object for $t \in [0, 2]$. Use technology to determine and label key features of the graph.

THINKING	WORKING
1 Determine key features of the graph; axes intercepts, turning points and endpoints of the domain.	Domain: $t \in [0, 2]$ t-intercept occurs when $v(t) = 0$ $-6t^2 + 6t + 5 = 0$ $t = -0.54$ or $t = 1.54$ s Discard $t = -0.54$ as $t \in [0, 2]$ When $t = 0$, $v(0) = -6(0)^2 + 6(0) + 5$ v-intercept occurs when $t = 0$, $v(0) = 5$. Maximum point has coordinates (0.5, 6.5). Endpoints of the domain: $v(2) = -6 \times 2^2 + 6 \times 2 + 5$ $= -24 + 12 + 5$ $= -7$ The endpoints of the domain have coordinates (0, 5) and (2, -7).

2 Sketch a velocity–time graph for the required time interval and label all key features.



3 Describe the motion of the object.

The object is moving to the right away from its starting point from $t = 0$ to $t = 0.5$ s accelerating from $v = 5$ m/s to a maximum velocity of 6.5 m/s.

The object continues moving to the right from $t = 0.5$ s, slowing down until it becomes stationary at $t = 1.54$ s.

At $t = 1.54$ s the object changes direction and accelerates to 7 m/s at $t = 2$ s back towards its starting point.

Technology worked example
Velocity–time graphs

Technology worked example
Displacement and acceleration from a velocity equation

Different notation for acceleration

There are situations when the velocity of a particle is given in terms of its displacement, $v(x)$. In this case, the acceleration can only be found as a function of x, not of t, and is given by:

Using the chain rule	Substituting $\frac{dx}{dt} = v$	Using the chain rule in [2]
$a = \frac{dv}{dt}$ $= \frac{dv}{dx}\frac{dx}{dt}$	into equation [1] $a = v\frac{dv}{dx}$	$a = v\frac{dv}{dx}$ $= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx}$ $= \frac{dv}{dx}\frac{d}{dv}\left(\frac{1}{2}v^2\right)$ $= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
$a = \frac{dv}{dx}\frac{dx}{dt}$ [1]	$a = v\frac{dv}{dx}$ [2]	$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ [3]

Alternative forms for acceleration are:

$$a = \frac{dv}{dt}; \quad a = \frac{d}{dx}\left(\frac{1}{2}v^2\right); \quad a = \frac{d^2x}{dt^2}; \quad a = v\frac{dv}{dx};$$

15 Determining acceleration in terms of displacement

A particle is moving in a straight line with a velocity $v(x) = 3x - 10$, where x is the displacement of the particle at any time t relative to an origin O. Determine an expression for the particle's acceleration, $a(x)$.

THINKING	WORKING
1 Recall an appropriate formula for the acceleration of the particle.	The velocity is given in terms of displacement, so use $a(x) = v\frac{dv}{dx}$
2 Substitute the velocity expression into the formula.	$a(x) = (3x - 10)\frac{d(3x - 10)}{dx}$
3 Differentiate to determine $\frac{dv}{dx}$ and simplify.	$a(x) = (3x - 10) \times 3$ $= 9x - 30$

WARNING

When the velocity is given as a function of the displacement, it is not possible to differentiate with respect to time.

16 Determining velocity and acceleration in terms of displacement

The velocity of a particle is $v = 2$ m/s when it passes through the reference point O. If the acceleration of the particle is $a(x) = 49x - 14$, determine the particle's velocity and acceleration when the position of the particle is $x = 1$ m.

THINKING	WORKING
1 Recall an appropriate formula to determine the velocity of the particle.	$a(x) = 49x - 14$ The acceleration is given in terms of displacement, so use $a(x) = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
2 Substitute the equation for the acceleration into the formula.	$49x - 14 = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
3 Integrate with respect to x.	$\frac{1}{2}v^2 = \int (49x - 14)\,dx$ $\frac{1}{2}v^2 = \frac{49}{2}x^2 - 14x + c$

4 Substitute the given conditions to determine the value of the constant of integration c.	At $x = 0$, $v = 2$ m/s $\frac{1}{2} \times (2)^2 = \frac{49}{2} \times 0^2 - 14 \times 0 + c$ $2 = c$
5 Determine the equation for the velocity of the particle.	$\frac{1}{2}v^2 = \frac{49}{2}x^2 - 14x + 2$ $v^2 = 49x^2 - 28x + 4$ $v = \pm\sqrt{49x^2 - 28x + 4}$ $v = \pm\sqrt{(7x - 2)^2}$ $= \pm(7x - 2)$
6 Determine the acceleration and the velocity of the particle for the required displacement.	$a(1) = 49 \times 1 - 14$ $= 35$ m/s^2 $v(1) = \pm(7 \times 1 - 2)$ $= \pm 5$ m/s At $x = 0$ the particle is moving to the right at 2 m/s but its acceleration is to the left i.e. it is slowing down. However it does not come to a rest before arriving at a displacement of 0.29 m to the right, at which point it begins speeding up and continues to do so until reaching a displacement of 1 m to the right at which the velocity is 5 m/s to the right. Hence, discard $v = -5$ m/s
7 Interpret the answer.	When the particle is 1 m to the right of the origin, it is accelerating at 35 m/s^2 to the right and has an instantaneous velocity of 5 m/s to the right.

EXERCISE 8.2 Linear motion with non-constant acceleration

Worked Example 11

1 The displacement of a particle travelling in a straight line x metres from a fixed reference point O at time t seconds ($t \geq 0$) is given by $x(t) = -t^3 - 2t^2 + 3t$.

(a) Determine the displacement of the particle at $t = 2$ s.

(b) Determine the velocity of the particle at $t = 2$ s.

(c) Determine the acceleration of the particle at $t = 2$ s.

2 For each of the following displacement functions, determine an expression for the acceleration.

(a) $x(t) = 3\cos(2t)$

(b) $x(t) = \sqrt{4-2t}$

(c) $x(t) = t + e^{10t}$

(d) $x(t) = 0.5\log_e(6t-1)$

3 The displacement of a particle x km from a reference point O at any time t hours is modelled by the equation $x(t) = 4t^4 + 8t^3 - 31t^2 + 7t + 12$. Determine the times at which the particle is at the reference point.

4 The displacement of a particle is given by $x(t) = 2t^3 - 5t + 9\,\text{m}$, $0 \le t \le 3\,\text{s}$.

Worked Example

(a) Determine the time when the velocity of the particle is a minimum.

(b) Determine the minimum velocity of the particle.

14

5 The displacement of a particle is given by $x(t) = -8t^4 + 3t^2 - 11\,\text{m}$, $0 \le t < 1\,\text{s}$.

(a) Determine the time when the acceleration of the particle is equal to zero.

(b) Determine the velocity of the particle at this time.

(c) Sketch a velocity–time graph to determine if the velocity you found in part (b) is a maximum or a minimum value.

6 The displacement of a particle is modelled by the equation $x(t) = 2\sin(t) - t\,\text{m}$, $0 \le t \le \frac{2\pi}{3}\,\text{s}$.

(a) Determine an expression for the velocity of the object.

(b) Sketch a velocity–time graph labelling key features.

12

7 The acceleration of a particle is modelled by the equation $a(t) = -3\cos(t)\,\text{m/s}^2$, $0 \le t \le \frac{3\pi}{4}\,\text{s}$ and at $t = 0$, $v(t) = 1$. Determine an expression for the velocity of the object.

13

8 Determine an expression for the displacement x for each of the following, given acceleration a and velocity v:

(a) $a(t) = \cos\left(\frac{1}{2}t\right)$, $v(0) = 5$ and $x(0) = 0$

(b) $a(t) = \frac{1}{(t-10)^2}\,\text{km/h}^2$, $0 \le t \le 9\,\text{h}$ and $v(t) = 2\,\text{km/h}$ at $t = 9\,\text{h}$ and the initial displacement is at the origin.

9 If the velocity of an object is described by the equation $v(t) = 3e^{-t}\,\text{m/s}$, and $x = 5\,\text{m}$ at $t = 0\,\text{s}$, determine expressions for its displacement and acceleration at time t.

10 A car is driven with an initial velocity u from a reference point O in a straight line.

(a) Determine expressions for the car's displacement and velocity at time t if $a(t) = 3t - 16\,\text{m/s}^2$ and $u = \frac{17}{2}\,\text{m/s}$.

(b) Determine expressions for the car's displacement and velocity at time t if $a(t) = -2.4t + 21.8\,\text{m/s}^2$ and $u = 3.6\,\text{m/s}$.

(c) Determine the expressions for the car's displacement and velocity at time t if $a(t) = (\sin(2\pi t) + 3)\,\text{m/s}^2$ and $u = -\frac{1}{2\pi}\,\text{m/s}$.

15

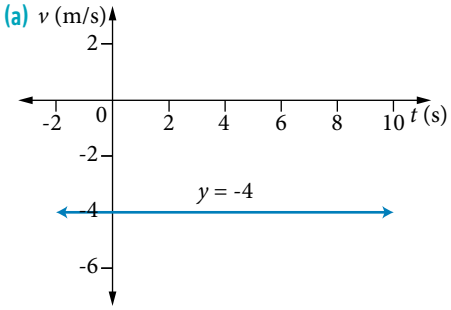
11 Determine the acceleration for each of the following:

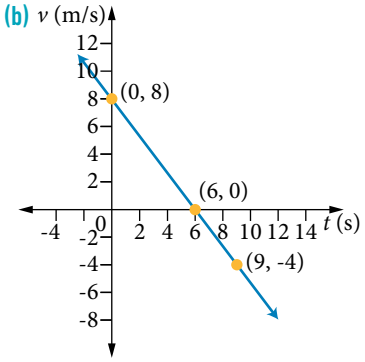
(a) A particle is moving in a straight line with a velocity $v(x) = -6x + 11$, where x is the displacement of the particle at time t for $0 \le t \le 20$, relative to a reference point O.

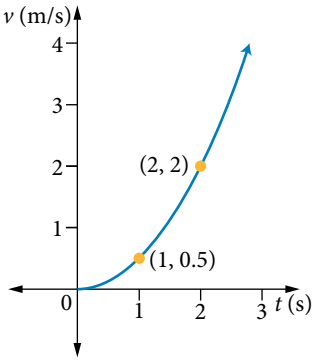
(b) A particle is moving in a straight line with a velocity $v(x) = 2x^2 + 9x - 1$, where x is the displacement of the particle at time t relative to a reference point, O.

(c) A particle is moving in a straight line with a velocity $v(x) = e^{2x^2}$, where x is the displacement of the particle at time t relative to a reference point O.

12 For each of the following velocity–time graphs (either straight lines or parabolas), determine an expression for velocity v and acceleration a in terms of time t.

(a) 

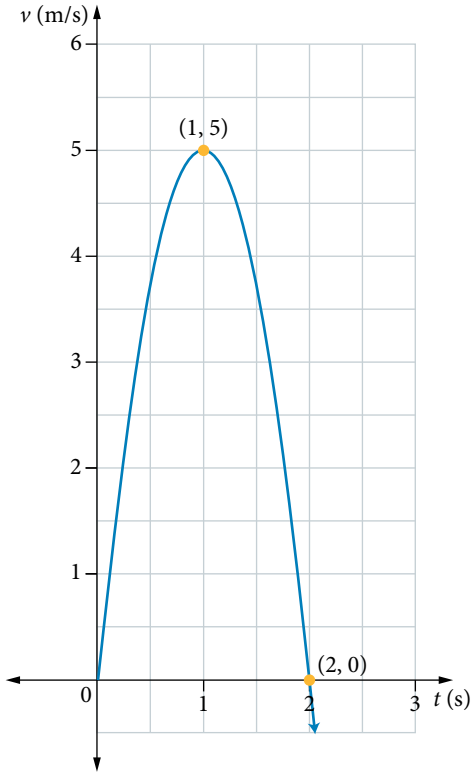
(b) 

(c) 

13 Consider the velocity–time graph shown.

(a) Determine an expression for velocity v and acceleration a in terms of time t.

(b) Determine the displacement and the distance travelled from the initial position in 9 seconds.



Worked Example 16

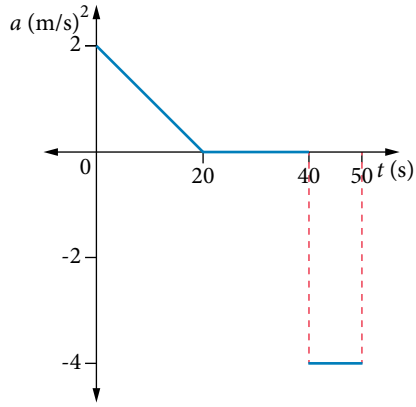
14 The velocity of a particle is $v = 7$ m/s when it passes through the reference point O, and the acceleration of the particle is given by $a(x) = (9x - 21)\text{m/s}^2$.

(a) Determine an expression for the velocity of the particle.

(b) Determine the particle's velocity and acceleration when the particle is 1 m to the right of the reference point.

(c) Determine the acceleration and velocity of the particle when its position is 2 m to the left of the reference point.

15 The acceleration–time graph for the motion of a particle that starts with a positive velocity is shown below.



(a) Which one of the following is a false statement and why?

A The particle speeds up in the first 20 seconds.

B The particle slows down in the final 10 seconds.

C The particle is stationary from 20 seconds to 40 seconds.

D The particle is travelling at constant velocity for some part of its journey.

(b) Explain why the other options in part **(a)** are true.

16 The initial velocity of the particle described in question **15** is 3 m/s.

(a) Which one of the following statements is true and why?

A The velocity at $t = 20$ is given by $v = -\frac{t}{10} + 2$.

B The velocity in the middle section of the journey is 23 m/s.

C The velocity is positive throughout the journey.

D The displacement in the first 20 seconds is $\frac{1}{2} \times 20 \times 2$ m.

(b) Explain why the other options in part **(a)** are false.

17 Consider the acceleration of an object given by the equation $a(t) = mt + d$, where m and d are real constants. The object starts from rest, at a fixed point O at time $t = 0$.

(a) Determine the equation for the velocity of the object at any time t.

(b) Determine the equation for the displacement of the object at any time t.

(c) Determine expressions for the velocity and displacement of the object any time t if the initial velocity of the object is u.

(d) Determine the equations for the velocity and displacement of the object at any time t if the initial velocity of the object is u and the starting position is x_0.

18 An object is moving in a straight line with velocity v. The object is decelerating at a rate of $\frac{1}{3}v$.

(a) Determine the velocity equation of the object at any distance x.

(b) Determine the exact values of acceleration and velocity of the moving object at $x = 3$ if $v(0) = 10$ m/s.

19 The acceleration a of an object is given by $a = x + 1$, where x is the displacement from a point 2 m to the right of the origin after t seconds.

(a) Determine an expression for the velocity in terms of x if the initial velocity is 1 m/s.

(b) Determine, in exact form, the displacement of the object after 3 seconds.

20 The displacement of a particle moving in a straight line is given by $x = (7t - 1)e^{-\frac{2}{3}t}$ m, where the time is measured in seconds.

- **(a)** Determine the position of the particle at time $t = 0$ and $t = 2$ s, correct to 1 decimal place.
- **(b)** Sketch a displacement–time graph and describe the motion of the particle in terms of its displacement for the first 8 seconds of its motion. Label key features of the graph.
- **(c)** Use calculus to determine the maximum displacement of the particle and the time at which this occurs, correct to 2 decimal places. Check your answers with the value sketch in part **(b)**.
- **(d)** Determine an expression for the velocity of the particle in terms of t.
- **(e)** Determine the acceleration of the particle at any time t.
- **(f)** Determine the particle's maximum velocity in both the positive and negative directions of the motion during the first 8 seconds of its motion, correct to 2 decimal places.
- **(g)** Sketch a velocity–time graph for the motion of the particle for the first 8 seconds and describe the motion. Label key features of the graph.

21 The acceleration of a body (m/s^2) is directly proportional to the square of its velocity (m/s), $a = k\sqrt{v}$.

- **(a)** Determine the velocity of the body in terms of time, t, u and k.
- **(b)** Determine the value of k if the body was initially at rest and its velocity was 16 m/s after 12 seconds.
- **(c)** Determine the equation for the acceleration of the body in terms of time t.
- **(d)** Determine the times at which both the velocity and the acceleration of the body have the same magnitude.

22 A stone is dropped from a height and falls under the influence of gravity and air resistance. The gravitational acceleration is g and the retardation due to air resistance is proportional to the speed v, and equal in magnitude to $\frac{v}{20}$. The distance travelled in time t is x.

- **(a)** Using downwards as the positive direction, set up a differential equation for the acceleration of the stone.
- **(b)** Determine an expression for the velocity at time t.
- **(c)** Determine the terminal velocity, that is, the constant velocity that the stone reaches after falling for a long time.
- **(d)** Determine the distance fallen in time t.

8.3 Newton's laws of motion

Sir Isaac Newton (1642–1727) was one of the greatest mathematicians and scientists of all time. Newton developed the principles of modern physics, published in 1687 in his book *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*). He is well known for formulating the three laws of motion known as Newton's laws of motion.

Newton's first law of motion

The focus of the first part of this chapter has been the mathematical description of motion. However, the factors that produce and affect the motion of particles or bodies have not been discussed.

In this section motion will be further discussed from the point of view of *dynamics*, the branch of physics that studies forces and their effect on motion. Kinematics and dynamics are part of a larger branch of physics called mechanics.

The quantity that affects motion is called a *force*. Force is a vector quantity and is informally defined as 'a push or a pull acted upon an object by another object'. In other words, a force is an interaction between two objects where the action of one object will change the motion of the other object. There are many types of forces, such as weight, friction, magnetic forces and electric forces.

Newton's first law of motion

An object either remains at rest or continues to move at a constant velocity unless acted upon by an external force.

1 newton of force is the force required to accelerate an object with a mass of 1 kilogram by 1 m/s^2. You can also interpret this as kg wt.

Observe a few examples of Newton forces:

				
1 N	800 N	4000 N	20 000 N	100 000 N
The weight of an apple	Biting force of adult human	The force required to break an adult femur	The biting force of adult saltwater crocodile	The force applied by a seatbelt and airbag to a restrained passenger in a collision at 100 km/h

The first law of motion is also known as the law of inertia, where inertia is the resistance of the body to a change in its motion. For example, when you are standing in a moving vehicle and it begins to decelerate, if you are not holding on you often 'fall' forward (in the same way you can feel your body push back into your aeroplane seat during take off). In mechanics this means the resistance of the body to a change in velocity.

If more than one force acts on a body to produce the motion, then the sum of all forces is required. This is called the resultant force or net force, $F_{\text{net}} = \sum F$.

Forces are vectors, so the net force acting on a body is the vector sum of all forces (vectors) acting on the body.

In the situation below, the forces act in the same direction and are therefore added.

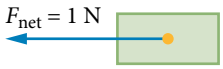




In the situation below, the forces are also added, regardless of their direction. In this case their sign simply states the direction in which they are acting.

-1 + 2 = 1 N, given that it is positive, the resultant or net force is to the right.



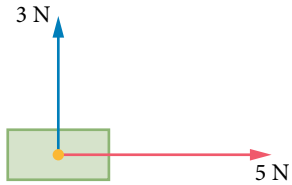


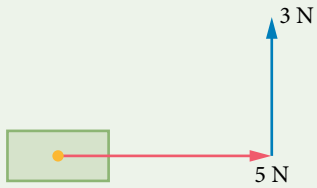
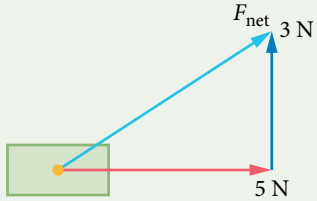
If forces are not acting in the same line, the magnitude and direction of the net force can be calculated by applying Pythagoras' theorem (only after decomposition into orthogonal components), sine or cosine rules or trigonometric ratios.

17 Determining resultant force

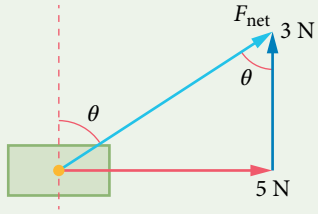
Determine the net force acting on each of the bodies shown in the diagrams. Give your answers correct to 1 decimal place.

(a)



THINKING	WORKING
1 Move one of the forces such that its tail is at the head of the other force.	
2 Draw the vector connecting the tail of one force with the head of the other force.	
3 Calculate the magnitude of the net force.	Using Pythagoras' theorem, $F_{net} = \sqrt{5^2 + 3^2}$ $= \sqrt{25 + 9}$ $= \sqrt{34}$ $= 5.8\,\text{N}$

4 Determine the angle required for the direction of the net force.



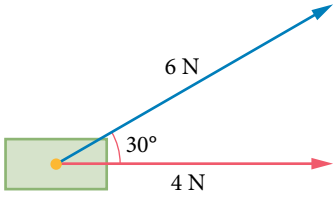
5 Calculate the value of the angle required.

$$\theta = \tan^{-1}\left(\frac{5}{3}\right)$$
$$= 59.0°$$

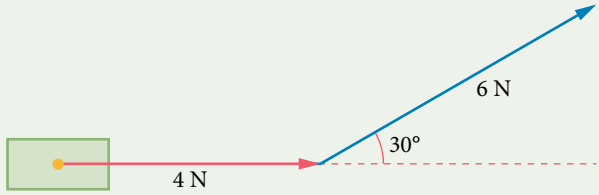
6 Interpret the answer.

The resultant or net force acting on the body is 5.8 N at an angle of 59.0° clockwise from vertical upwards.

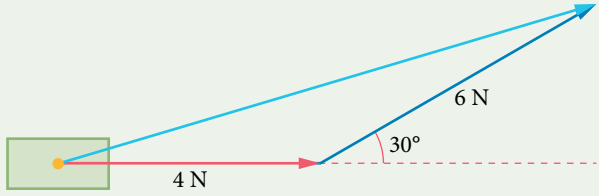
(b)



1 Move one of the forces such that its tail is at the head of the other force.

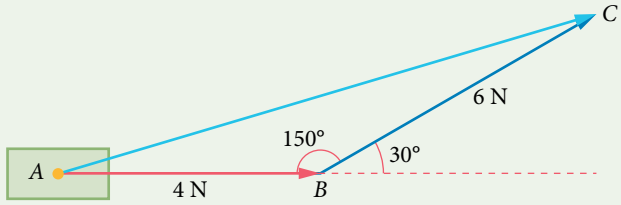


2 Draw the addition of the vectors by connecting the tail of one force with the head of the other force.



3 Label the triangle formed and determine the magnitude of the supplementary angle.

$$\angle ABC = 180° - 30°$$
$$= 150°$$



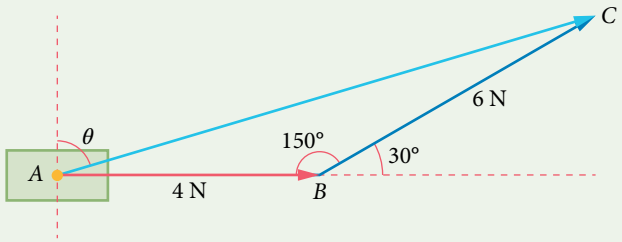
4 Calculate the magnitude of the net force.

Use the cosine rule in ΔABC.

$$\begin{aligned}(AC)^2 &= AB^2 + BC^2 - 2AB \times BC\cos(\angle ABC)\\ &= 4^2 + 6^2 - 2 \times 4 \times 6\cos(150°)\\ &= 52 - 48 \times (-0.866)\\ &= 93.57\ldots\end{aligned}$$

$$\begin{aligned}AC &= \sqrt{93.57\ldots}\\ &= 9.7 \text{ (1d.p.)}\end{aligned}$$

5 Determine the angle required for the direction of the net force.



Using the sine rule,

$$\frac{6}{\sin(\angle BAC)} = \frac{\sqrt{93.57\ldots}}{\sin(150°)}$$

$$\sin(\angle BAC) = \frac{6 \times \sin(150°)}{\sqrt{93.57\ldots}}$$

$$\angle BAC = \sin^{-1}(0.31\ldots)$$

$$\angle BAC = 18.1° \text{ (1 d.p.)}$$

$$\theta = 90 - 18.1$$

$$\theta = 71.9°$$

6 Interpret the answer.

The resultant or net force acting on the particle is 9.7 N acting at an angle 71.9° clockwise from vertical upwards.

Newton's second law of motion

To introduce this law of motion, a new concept needs to be defined: *momentum*, which is mass in motion. Given that all objects have mass, then if an object is in motion it has momentum. Momentum is directly proportional to the mass of an object and directly proportional to the velocity of an object.

Consider, for example, the force exerted on your hands when catching a beach ball compared to a bowling ball. A beach ball with a mass of approximately 100 g exerts very little force. Regardless of how fast it is thrown it is very unlikely that it will have enough momentum to knock you down. A bowling ball, however, with a mass of approximately 5 kg (that is, 50 times the mass of a beach ball) will have enough momentum to knock you down.

The momentum (p) of a body is the product of the mass of the body (m) and its velocity (v). The product of scalar and vector quantities produces a vector quantity to describe the magnitude and direction of the 'mass in motion'.

$$p = mv$$

Momentum is a vector quantity and possesses both a magnitude and direction. A scalar quantity possesses only a magnitude and no direction. Mass is an example of a scalar quantity.

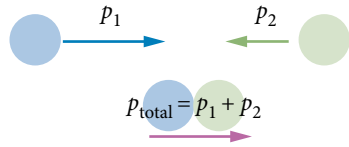
The SI unit for momentum is kg m/s.

Quite often the idea of momentum is used when two objects interact, predominantly when they collide. The objects may start moving together or may come together, and after the collision they end up together or move apart. No matter what, the forces between those objects will not change the total momentum.

WARNING

A mass is a scalar quantity (that is, it has no direction), whereas velocity is a vector quantity.

A momentum diagram consists of a labelled vector representing the momentum of the two objects before the interaction or collision, a labelled vector representing the total momentum $p_1 + p_2$ and then vectors representing the respective momentums of the objects afterwards. For example:



18 Determining momentum

Determine the momentum of a 70 kg body travelling at 72 km/h.

THINKING	WORKING
1 Identify the information given and recall the formula required.	$m = 70\,\text{kg}$ $v = 72\,\text{km/h}$ $p = mv$
2 Convert any units if necessary.	Convert velocity to m/s: $v = 72\,\text{km/h}$ $= \dfrac{72 \times 1000\,\text{m}}{3600\,\text{s}}$ $= 20\,\text{m/s}$
3 Substitute the known values into the formula.	$p = 70 \times 20$ $= 1400\,\text{kg m/s}$
4 Interpret the answer.	A 70 kg body traveling at 72 km/h has a momentum of 1400 kg m/s in the direction of travel.

Given that the mass of a body is constant, then a change in momentum is determined by a change in the velocity of the body.

$$\begin{aligned}\text{Impulse} &= \text{change in momentum} \\ &= \Delta p \\ &= m\Delta v\end{aligned}$$

Note: A positive impulse Δp corresponds to an increase in velocity of the body, a body gaining momentum. While a negative impulse, Δp corresponds to a decrease in velocity of the body, decreasing momentum.

The SI unit for momentum is kg m/s or N s.

19 Determining a change in momentum

Calculate the change in momentum of a 2 tonne truck slowing from 80 km/h to 50 km/h, correct to 1 decimal place.

THINKING	WORKING
1 Write the information given and recall the formula required.	$\Delta p = m\Delta v$ $m = 2$ tonne $\Delta v = 50 - 80 = -30$ km/h
2 Convert units if necessary.	The change in momentum is measured in kg ms. $m = 2$ tonne $= 2000 \times 1000$ kg $= 2000$ kg $\Delta v = -30$ km/h $= \frac{-30 \times 1000\text{ m}}{3600\text{ s}}$ $= \frac{25}{3}$ m/s
3 Substitute the known values into the formula.	$\Delta p = m\Delta v$ $= 2000 \times -\frac{25}{3}$ $= -16\,666.7$ kg m/s (1 d.p.)
4 Interpret the answer.	Without considering the forces applied, this value could be interpreted as braking force over time. When a 2 tonne truck slows from 80 km/h to 50 km/h there is a decrease in momentum of 16666.7 kg m/s.

Newton's second law of motion in terms of momentum:

The net force of an object is equal to the rate of change of its momentum Δp over a time interval Δt.

$F = \frac{\Delta p}{\Delta t}$; $F_{\text{net}} = \frac{dp}{dt}$

Differentiating the momentum with respect to time,

$$\begin{aligned} F_{\text{net}} &= \frac{dp}{dt} \\ &= \frac{d(mv)}{dt}, \text{ where } m \text{ is a constant} \\ &= m\frac{dv}{dt} \\ &= ma \end{aligned}$$

Newton's second law of motion in terms of force:

The net force required to produce an acceleration (a) on a body of mass (m) is given by ma, as follows:

$F_{\text{net}} = ma$

The SI unit for force is the newton; $1\,\text{N} = 1\,\text{kg m/s}^2$.

20 Change in momentum and average net force applied forward, in a straight line, to a body

Andre is pushing a 120 g toy trolley so that over 3 seconds, its velocity increases from 0.6 m/s to 1.2 m/s. Calculate the change in momentum of the trolley and the average net force applied to it. Give your answers correct to 3 decimal places.

THINKING	WORKING
1 Identify the information given.	$m = 0.120\,\text{kg}$ $\Delta v = 1.2 - 0.6 = 0.6\,\text{m/s}$ $\Delta t = 3\,\text{s}$
2 Calculate the change in momentum.	$\Delta p = m\Delta v$ $= 0.12 \times 0.6$ $= 0.072\,\text{kg m/s}$ $= 0.072\,\text{N s}$
3 Determine the net force applied to achieve the change in momentum.	$F = \frac{\Delta p}{\Delta t}$ $= \frac{0.072}{3}$ $= 0.024\,\text{N}$
4 Interpret the answer.	In order for Andre's trolley to increase in velocity from 0.6 m/s to 1.2 m/s over a 3 second interval, he needed to apply a pushing force of 0.024 N.

21 Determining acceleration and net force

A car uniformly accelerates from rest to 10 m/s in 5 seconds. Calculate the car's acceleration and the net force on it, if its mass is 1140 kg.

THINKING	WORKING
1 Identify the information given and the formula required.	$v_1 = 0$ $v_2 = 10\,\text{m/s}$ $\Delta t = 5\,\text{s}$ $a = \frac{\Delta v}{\Delta t}$
2 Determine the acceleration.	$a = \frac{v_2 - v_1}{\Delta t}$ $= \frac{10 - 0}{5}$ $= 2\,\text{m/s}^2$
3 Write the information given and recall the formula required.	$m = 1140\,\text{kg}$ $a = 2\,\text{m/s}^2$ $F = ma$
4 Calculate the net force.	$F = 1140 \times 2$ $= 2280\,\text{N}$
5 Interpret the answer.	For the car to accelerate from rest to 10 m/s (36 km/h) in 5 seconds, a force of 2280 N must be applied parallel to the car's velocity.

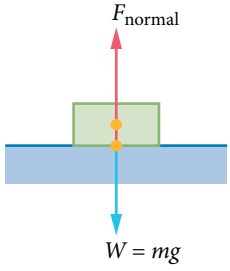
Technology worked example
Newton's second law of motion

Weight and the normal force

A force that acts on all objects is the *gravitational force*. The effect of this force is called 'weight' and is defined as the inverse of the upward supporting force; $W = mg$, where m is the mass of the body (kg) and g is the acceleration due to gravity (m/s^2).

Weight is directed downward in the direction of the total gravitational field at any point. In the case of Earth, the weight is not directed toward the centre because Earth is not an exact sphere and does not have uniform density. The direction is approximately, but not exactly, toward the centre and is neither time- nor position-constant.

Although an object sitting on the ground acts on Earth with a weight W, the object is not moving into the ground. The ground acts on the object with an equal but opposite force on the object. In the diagram, the upwards force is called the normal force F_{normal} acting on the object from the ground up.



Magnitude of F_{normal} = magnitude of W; however, the two forces are directly opposite.

WARNING

Newton's third law is commonly known as: Every action has an equal and opposite reaction. This means that when an object, A, exerts a force on a *different* object, B, then object B automatically and always exerts a force of equal magnitude but opposite in direction back on object A unless one or both of the objects breaks.

It is a common misconception that the weight force and the normal force are third law pairs, but this is *not* the case. This is because they are acting on the same object.

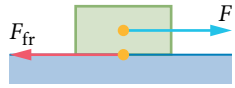
22 Determining weight and reaction in the absence of acceleration

Determine the weight of a 65 kg person sitting on a chair and the normal force from the chair onto the body.

THINKING	WORKING
1 Write the information given and the formula required to determine the weight of the person.	$m = 65\text{ kg}$ $W = mg$
2 Substitute the known values.	$F_{\text{normal}} = W$ $F_{\text{normal}} = 637\text{ N up}$
3 Determine the reaction force.	$W = 65 \times 9.8$ $= 637\text{ N down}$
4 Interpret the answer.	A person sitting on a chair exerts a 637 N force down. The reaction force from the chair onto the body is opposite but equal in magnitude; that is, 637 N up.

Friction

Friction, F_{fr} occurs at the point(s) of contact between an object and a surface as the object is moving (or about to move) across the surface. It is a force that opposes the direction of motion.



If $F > F_{\text{fr}}$, the object is accelerating with a net force $F_{\text{net}} = F - F_{\text{fr}}$ in the direction of F.

If $F < F_{\text{fr}}$, assume that the object is stationary as the pulling force F is not large enough to overcome the maximum friction force between the object and the surface. This maximum friction force is called the *limiting friction force*.

When $F =$ limiting F_{fr}, the object is stationary and the object is said to be in *limiting equilibrium* and about to move.

WARNING

The static coefficient of friction is not the same (in general) as the coefficient of kinetic (or rolling) friction. If you only consider objects at rest, you can ignore this. You cannot ignore this if you consider objects both at rest and in motion under the effect of friction.

Friction depends on the nature of the two surfaces in contact. All materials have a coefficient of friction, μ, whose value depends on the structural properties of the material. Usually $\mu < 1$; however, there are instances when μ takes larger values. A coefficient of friction equal to 1 means that the frictional force is equal to the normal force. For example, the more slippery the surface is, the smaller the force required to move an object, e.g. $0.02 < \mu_{ice} < 0.09$. On the other hand, the less slippery the surface is, the greater the force required to move the object., e.g. $\mu_{rubber} > 1$.

There is a relationship between the friction force on an object and the normal reaction to it when an object is in limiting equilibrium.

In limiting equilibrium, $F_{fr} = \mu F_{normal}$.

23 Moving an object on a rough horizontal surface

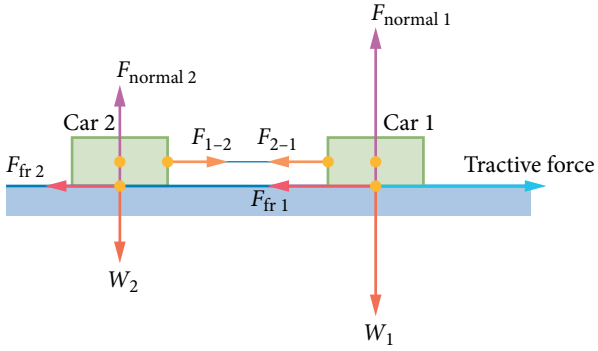
Determine the minimum force required to move a 6 kg object along a horizontal surface with a coefficient of friction $\mu = 0.7$.

THINKING	WORKING
1 Identify the information given and the formula required to determine the normal reaction.	$m = 6\,\text{kg}$ $\mu = 0.7$ $F_{normal} = mg$
2 Calculate the normal force.	$F_{normal} = 6 \times 9.8$ $= 58.8\,\text{N}$
3 Substitute the known values into the formula for the friction force.	$F_{fr} = \mu F_{normal}$ $F_{fr} = 0.7 \times 58.8$ $= 41.16\,\text{N}$
4 Interpret the answer.	The object is in limiting equilibrium and about to move when $F = F_{fr}$ $= 41.16\,\text{N}$

Systems of two connected objects

There are situations when two connected objects are in a rectilinear motion. For example, a car pulling a trailer, a person pushing a trolley in a supermarket or a person in an elevator going up or down.

The following diagram displays all forces acting on two connected cars in motion.



$W_1 = F_{\text{normal 1}}$ and $W_2 = F_{\text{normal 2}}$

$F_{\text{fr1}} = \mu F_{\text{normal 1}}$ and $F_{\text{fr2}} = \mu F_{\text{normal 2}}$

F_{1-2} is the force exerted by car 1 on car 2 and F_{2-1} is the force exerted by car 2 on car 1.

$F_{1-2} = F_{2-1}$

$F_{\text{net}} = \text{Driving force} - \sum F_{\text{fr}}$

24 Determining forces on two connected objects

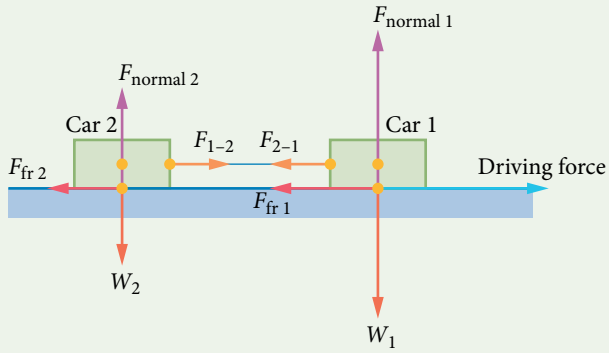
A four-wheel drive vehicle of mass 2100 kg is towing a car of mass 1500 kg. The two vehicles are accelerating east on a road with a coefficient of kinetic friction between the tyres and the road of $\mu = 0.75$.

(a) Determine the friction forces on the two cars.

THINKING

1 Draw a diagram showing all forces involved.

WORKING



2 Recall the formula required and substitute known values.

The cars are in motion, so the coefficient of friction is for kinetic friction.

$$\begin{aligned} F_{\text{fr1}} &= \mu F_{\text{normal 1}} \\ &= \mu W_1 \\ &= \mu m_1 g \\ &= 0.75 \times 2100 \times 9.8 \\ &= 15\,435\,\text{N} \end{aligned}$$

$$\begin{aligned} F_{\text{fr2}} &= \mu F_{\text{normal 2}} \\ &= \mu W_2 \\ &= \mu m_2 g \\ &= 0.75 \times 1500 \times 9.8 \\ &= 11\,025\,\text{N} \end{aligned}$$

3 Write the answer.

$F_{\text{fr1}} = 15\,435\,\text{N west}$

$F_{\text{fr2}} = 11\,025\,\text{N west}$

(b) Determine the net force on the system if the driving force $D = 32\,000\,\text{N}$ due east when the cars are moving.

1 Write an equation of motion involving all horizontal forces.

$$\begin{aligned} F_{\text{net}} &= D - F_{\text{fr1}} - F_{\text{fr2}} \\ &= 32\,000 - 15\,435 - 11\,025 \\ &= 5540\,\text{N} \end{aligned}$$

2 Interpret the answer.

The net force exerted on the system is 5540 N east.

(c) Determine the acceleration of the system of two vehicles.

1 Write the second law of motion for this system.

$$F_{\text{net}} = m_{\text{system}} \times a$$

2 Determine the total mass of the system and the acceleration.

$$\begin{aligned} m &= m_1 + m_2 \\ &= 2100 + 1500 \\ &= 3600\,\text{kg} \end{aligned}$$

$$\begin{aligned} 5540 &= 3600 \times a \\ a &= \frac{5540}{3600} \\ a &= 1.5\,\text{m/s}^2 \ (1\,\text{d.p.}) \end{aligned}$$

3 Interpret the answer.

The acceleration of the system is $1.5\,\text{m/s}^2$ east.

(d) Determine the tension in the rope between the vehicles.

1 Determine the net force on car 2.

$$\begin{aligned} F_{\text{net2}} &= m_2 \times a \\ &= 1500 \times 1.53\ldots \\ &= 2308\,\text{N west} \end{aligned}$$

2 Create an equation of motion involving all horizontal forces applying to car 2.

$$F_{\text{net2}} = F_{1-2} + \left(-F_{\text{fr2}}\right)$$

3 Substitute known values and determine the required force.

$$\begin{aligned} 2308 &= F_{1-2} - 11025 \\ F_{1-2} &= 2308 + 11025 \\ &= 13\,333\,\text{N west} \end{aligned}$$

4 Interpret the answer.

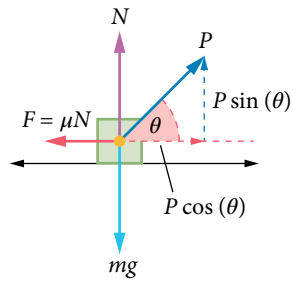
The pulling force exerted on Car 2 is 13 333 N to the west.

(e) Verify that the magnitude of the force exerted by car 1 on the rope is the same as the tension of the rope calculated in part **(d)**.

1 Determine the net force on car 1.	$F_{net1} = m_1 \times a$ $= 2100 \times 1.53\ldots$ $= 3232\,\text{N}$
2 Write an equation of motion involving all horizontal forces applying to car 1.	$F_{net1} = D - F_{2-1} + \left(-F_{fr1}\right)$
3 Substitute known values in the equation and determine the required force.	$3232 = 32\,000 - F_{1-2} - 15\,435$ $F_{2-1} = 32\,000 - 3232 - 15\,435$ $= 13\,333\,\text{N}$
4 Write the answer.	$F_{2-1} = F_{1-2} = 13\,333\,\text{N}$ as required

Resolution of forces

When a force is not acting horizontally or vertically, it is often useful to resolve it into two components: one vertical and one horizontal. For example, in the following diagram, the force P N acts on a mass m kg in a vertical plane at an angle of θ to the horizontal, producing a horizontal acceleration a m/s^2. The coefficient of friction is μ and there is a normal reaction N on the mass by the surface.



Use Newton's second law $F_{resultant} = ma$ in each of the two directions.

Resolving horizontally:

$P\cos(\theta) - \mu N = ma$

Resolving vertically:

$N + P\sin(\theta) - mg = 0$

Rectilinear motion on an inclined plane

An inclined plane is a surface that makes an angle $\theta > 0$ with the horizontal plane.

Note: Forces act on objects at different points. For example, the gravitational force acts at the centre of gravity of the object and a frictional force acts at the point of contact between two surfaces. For the purpose of this chapter, all forces will be drawn with their tails in the gravitational centre of the object.

Consider an object of mass m kg on an inclined plane at an angle θ to the horizontal surface. The forces acting on the object are its weight due to gravity, the frictional force on the surface of the plane and the normal reaction of the plane.

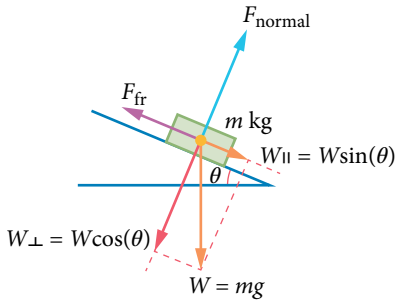
In this case, it is more convenient to resolve the weight force into a component parallel to the inclined plane $\left(W_{\parallel}\right)$ and a component perpendicular to the inclined plane $\left(W_{\perp}\right)$. Using trigonometric ratios, the two components can be written in terms of W:

$W_{\parallel} = W\sin(\theta)$

$W_{\perp} = W\cos(\theta)$

The perpendicular component is equal and opposite to the normal reaction force. Their magnitudes are equal. $\left|F_{\text{normal}}\right| = \left|W_{\perp}\right|$.

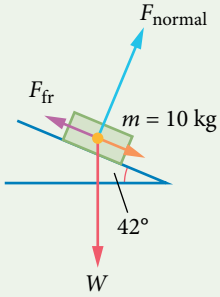
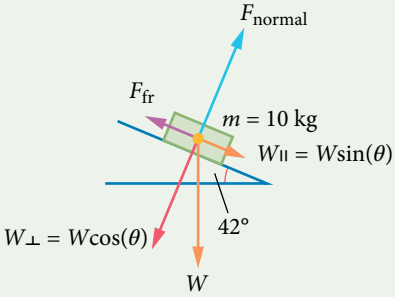
The net force downwards on the object parallel to the ramp is $F_{\text{net}} = W_{\parallel} - F_{\text{fr}}$.



25 Determining forces on an inclined plane

A 10 kg object is sitting at limiting equilibrium on a plane inclined at 42° to the horizontal surface. For the following questions, give your answers correct to 1 decimal place.

(a) Determine the value of the normal reaction force.

THINKING	WORKING
1 Draw a diagram showing all forces involved.	
2 Draw the parallel and perpendicular components of the weight.	
3 Determine the normal reaction force.	$\left\|F_{\text{normal}}\right\| = \left\|W_{\perp}\right\|$ $= mg\cos(\theta)$ $= 10 \times 9.8 \times \cos(42°)$ $= 72.8\,\text{N}$
4 Write the answer.	$F_{\text{normal}} = 72.8\,\text{N}$ perpendicularly upwards from the plane

(b) Determine the value of the maximum frictional force.

1 Write an equation involving the maximum frictional force.	At limiting equilibrium, $F_{fr} = W_{\|\|}$.
2 Determine the maximum frictional force.	$\begin{aligned}\|F_{fr}\| &= \|W\sin(\theta)\| \\ &= mg\sin(\theta) \\ &= 10 \times 9.8 \times \sin(42°) \\ &= 65.6\,\text{N}\end{aligned}$
3 Write the answer.	$F_{fr} = 65.6\,\text{N}$ up the plane

(c) Determine the coefficient of friction between the object and the inclined plane.

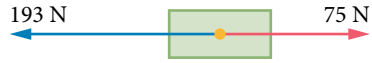
1 Write an equation involving the coefficient of friction.	$\|F_{fr}\| = \|\mu F_{normal}\|$ for limiting equilibrium.
2 Substitute and determine the value of μ.	$\begin{aligned}\mu &= \left\|\frac{F_{fr}}{F_{normal}}\right\| \\ &= \frac{65.6}{72.8} \\ &= 0.9\end{aligned}$
3 Write the answer.	$\mu = 0.9$

EXERCISE 8.3 Newton's laws of motion

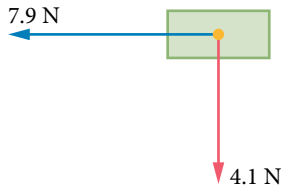
Worked Example 17

1 Determine the net force acting on the body shown in each of the following diagrams. Give your answers correct to 1 decimal place.

(a)

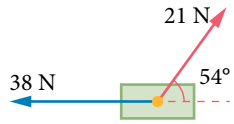


(b)

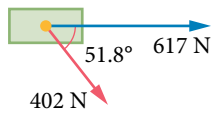


2 Determine the magnitude and the direction of the net force acting on the body in each of the following cases, correct to 1 decimal place. Use north as the direction of a force acting vertically upwards.

(a)



(b)



3 A particle of mass 10 kg is being pulled along a smooth horizontal surface with an acceleration of $2\,m/s^2$ by a force P acting in a vertical plane at an angle of 30° to the horizontal. 22

(a) Determine, in exact form, the size of the force P.

(b) Determine the normal reaction, in terms of g, of the plane on the particle.

4 Determine the momentum p of each of the following. 18

(a) A 25 kg child walking at 1 m/s.

(b) A 2.31 tonne car travelling at 100 km/h.

(c) A 500 g disc thrown at 9.4 m/s (the initial momentum in this case).

(d) An 83 kg cyclist travelling at 27.6 km/h.

5 Determine the change in momentum Δp of the following, leaving your answer correct to 1 decimal place: 19

(a) A 1.6 tonne car travelling in a straight line, slowing down from 95 km/h to 43 km/h.

(b) A 129 kg horse accelerating in a straight line from 16 km/h to 46 km/h.

6 A 600 g ball is thrown vertically upwards with a speed of 5 m/s and reaches a speed of 3.7 m/s in 2.8 seconds. Give your answers to the following questions correct to 2 decimal places. 20

(a) Determine the change in momentum of the ball.

(b) Determine the net force due to gravity applied to the ball.

7 A car with a mass of 1.35 tonnes is uniformly accelerating from rest to 60 km/h in 5 seconds. Give your answers to the following questions correct to 2 decimal places. 21

(a) Determine the car's acceleration over the given time interval in m/s^2.

(b) What is the net force required to achieve this acceleration for the car?

(c) Determine the change in momentum Δp of the car.

8 A particle of mass 6 kg is moving at a rate of 2 m/s. What constant force will bring it to rest in 5 seconds?

9 A 12.3 kg dog is sitting on a boy's lap. Calculate the weight of the dog and the reaction force from the boy onto the dog. Give your answers correct to 2 decimal places.

10 Calculate the maximum frictional force when an 18.3 kg object is in limiting equilibrium on a surface with a coefficient of friction $\mu = 0.55$. Give your answer correct to 1 decimal place. 23

11 Two trailers of mass 1367 kg and 694 kg are connected and moving together. They are moving east, from rest, on a surface with a coefficient of friction of $\mu = 0.26$. Give your answers to the following correct to 2 decimal places. 24

(a) Determine the friction forces acting on each of the two trailers.

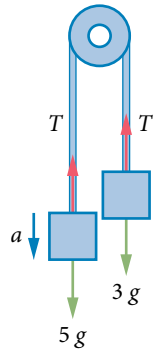
(b) Determine the net force of the system of two objects if the driving force $D = 18\,251$ N due east.

(c) Determine the acceleration of the system of two cars.

(d) Determine the magnitude of the force exerted by the first object onto the second object.

(e) Verify that the force exerted by the second object onto the first object has the same magnitude as the force exerted by the first object onto the second object.

12 A mass of 5 kg and a mass of 3 kg are suspended by a string over a smooth pulley as shown in the diagram. The tension in the string T acting on each of the masses is of same magnitude throughout, and the the two masses accelerate in opposite directions at a m/s^2. Determine the acceleration and the tension in the string in terms of the acceleration due to gravity g.



13 A trolley of mass 1 kg is being pulled along a rough horizontal table, the coefficient of friction between the two objects is 0.4, with an acceleration of 3 m/s^2 to the right, by a rope at an angle of $\tan^{-1}\left(\frac{3}{4}\right)$ degrees to the horizontal, in a vertical plane. Determine the size of the tension T in the rope and the normal reaction N of the table on the trolley. Give your answer correct to 1 decimal place.

14 A ball of mass 50 g is let fall from rest from a height of 3 metres above the ground. It bounces back up to a height of 1 metre. Determine the impulse exerted by the ground on the ball, correct to 1 decimal place.

Worked Example 25

15 A 694 kg object is sitting at limiting equilibrium on a plane inclined at 46° to the horizontal. Give your answers to the following correct to 1 decimal place.

(a) Determine the normal reaction force of the plane on the object.

(b) Determine the maximum frictional force.

(c) Determine the coefficient of friction between the object and the inclined plane.

(d) Determine the normal reaction force if the angle of inclination is halved.

(e) Determine the new maximum frictional force.

(f) Determine the coefficient of friction between the object and the inclined plane.

16 A box sits on a table at rest and is not about to move.

(a) Which one of the following is a correct statement?

A The normal reaction of the table on the box is equal to the mass of the box.

B The friction force on the box is equal to the coefficient of friction multiplied by the normal reaction.

C The friction force on the box is positive.

D The normal reaction of the table on the box is equal to the weight of the box.

(b) Explain why the other statements in part **(a)** are incorrect.

17 A mass of 100 g has an initial velocity of 2 m/s and a final velocity of -5 m/s. Which one of the following is the change in momentum that it undergoes?

A 700 kg m/s **B** -0.7 kg m/s **C** 0.7 kg m/s **D** -700 kg m/s

18 A sled of mass m kg is sliding down an icy slope that makes an angle of 28° to the horizontal. The coefficient of friction between the slope and the sled is 0.1.

(a) Draw a diagram showing all the forces on the sled and determine its acceleration correct to 1 decimal place.

(b) If the sled started from rest, assuming that the change in the coefficient of friction when stationary and sliding is negligible, determine the time it takes to travel a distance of 30 m down the slope.

19 A particle of mass 2 kg moves in a straight line such that at time t seconds its displacement from the origin of motion is x metres and its speed is v m/s.

(a) If the resultant force on the particle is $(1 + 2t)$ N, determine its velocity in terms of t if $v = 3$ when $t = 0$.

(b) If the resultant force on the particle is $(4 - 2x)$ N, determine its velocity in terms of x if $v = 3$ when $x = -2$.

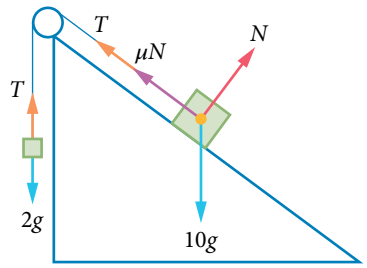
(c) If the resultant force on the particle is $(1 - v)$ N, determine the velocity in terms of t if $v(0) = 2$.

20 A girl of mass 50 kg is in a lift that is accelerating upwards. She stands on a set of bathroom scales inside the lift and the reading on the scales is 60 kg.

(a) Determine the acceleration of the lift in terms of g, the acceleration due to gravity.

(b) What would the reading on the scales be if the acceleration was $\frac{g}{4}$ downwards?

21 A smooth pulley is fixed to the top of a slope that makes an angle of 30° with the horizontal. A mass of 10 kg is on the slope and is connected, by a light rope, to a 2 kg block hanging over the other side of the pulley. The coefficient of friction for the mass on the plane is 0.2. The block is accelerating down the plane. Determine the acceleration a m/s^2 and the tension in the rope T N correct to 2 decimal places.



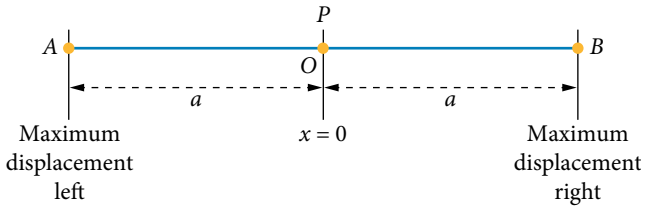
22 An object of mass m kg is projected up an inclined plane that makes an angle of θ with the horizontal. The coefficient of friction between the mass and the plane is μ. Determine the acceleration of the object a m/s^2 and how far up the plane it travels before coming to rest.

8.4 Simple harmonic motion

The motion of an object hanging from a stretched spring is an example of a *simple harmonic motion* (often SHM is used as shorthand). The discovery of simple harmonic motion is attributed to one of the most prominent scientists of the Renaissance, Galileo Galilei (1564–1642), also sometimes called 'the father of modern physics'.

Periodic oscillatory motion is back-and-forth motion that repeats itself over equal periods of time. Simple harmonic motion is a special type of periodic oscillatory motion where the acceleration is directly proportional, and opposite in direction, to the displacement.

Consider the particle P of negligible mass moving back and forth between points A and B as shown in the diagram. At the reference point (origin) O, the displacement of the particle is zero. The reference point is halfway between the endpoints A and B. The particle momentarily stops at points A and B before changing direction; the particle *oscillates* between points A and B.



Making connections

Harmonic motion in a straight line

Observe how the periodic motion of an object repeats itself over equal intervals of time.

If the particle moves from O to A and momentarily stops at point A, this means that the particle P is slowing down between O and A. The direction of the acceleration is towards the reference point O. Similarly, when the particle moves from O to B the acceleration is also pointing towards the reference point. When the particle moves from A to O or from B to O, the particle is increasing speed, with acceleration pointing towards O.

The acceleration in simple harmonic motion is always directed towards the reference point, at the centre of the motion (equilibrium position). Acceleration is always opposed to the velocity when the object moves away from the origin and parallel to velocity when the object moves toward the origin.

The acceleration is proportional to the displacement of the object from the centre of the motion, and is always in the opposite direction to that displacement. This leads to the differential equation $\frac{d^2x}{dt^2} = -\omega^2 x$.

The squaring of the n ensures that the constant is always negative. You can verify that a general solution of this differential equation is $x = A\sin(\omega t + \alpha)$ or $x = A\cos(\omega t + \beta)$, where A is the amplitude in metres, ω is the angular frequency in radians per second, T is the period measured in seconds and f is the frequency or the number of oscillations per second.

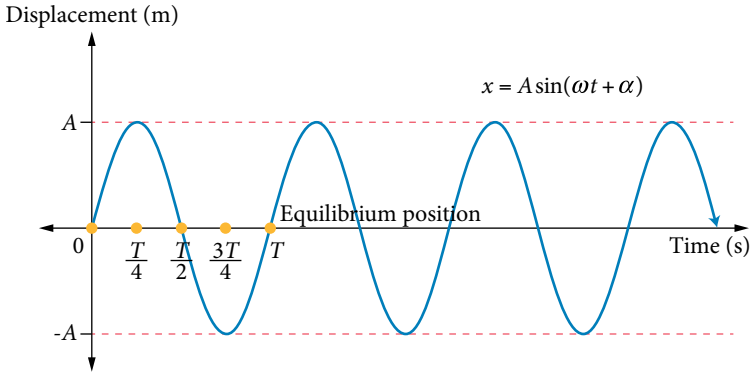
$$x = A\sin(\omega t + \alpha) \qquad x = A\cos(\omega t + \beta)$$
$$\dot{x} = A\omega\cos(\omega t + \alpha) \qquad \dot{x} = A\omega\sin(\omega t + \beta)$$
$$\ddot{x} = -A\omega^2\sin(\omega t + \alpha) \qquad \ddot{x} = -A\omega^2\cos(\omega t + \beta)$$
$$= -\omega^2 x \qquad = -\omega^2 x$$

$x = A\sin(\omega t + \alpha)$ or $x = A\cos(\omega t + \beta)$
$\dot{x} = A\omega\cos(\omega t + \alpha)$ or $\dot{x} = -A\omega\sin(\omega t + \beta)$
$\ddot{x} = -\omega^2 x$

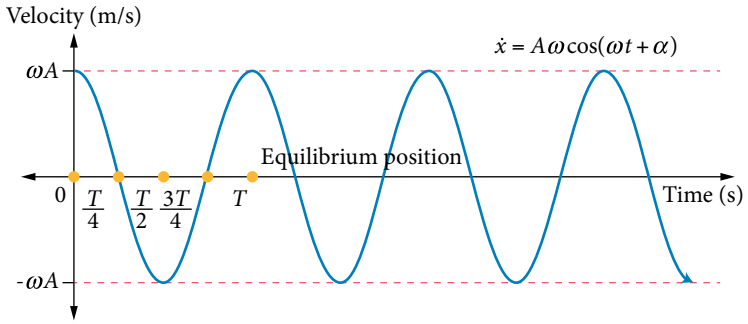
Graphs of simple harmonic motion

The following displacement–time graph shows the motion of a particle that has its initial position at the centre of motion.

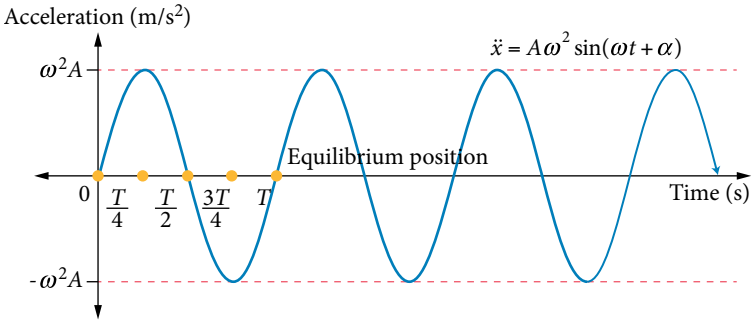
The amplitude A is the maximum displacement from the equilibrium position.



The following velocity–time graph shows the motion of a particle that has its maximum velocity at $t = 0$, indicating that its initial position is at the centre of motion where the maximum speed occurs.



The following acceleration–time graph shows the motion of a particle with initial displacement at the centre of motion where the acceleration is zero. The maximum acceleration occurs at the extreme ends of the motion.



The period of simple harmonic motion is the time taken to complete one cycle and is given by $T = \frac{2\pi}{\omega}$.

Rearranged this gives $\omega = \frac{2\pi}{T}$.

The frequency (number of cycles per second) is the reciprocal of the time taken, $f = \frac{1}{T}$.

The phase α depends on the starting position of the motion.

26 Determining time taken in simple harmonic motion

A particle is oscillating with simple harmonic motion of period 5 s. Determine the time taken for the particle to move from the equilibrium position to half of its amplitude for the first time, correct to 1 decimal place.

THINKING	WORKING
1 Identify the known information and recall an appropriate formula.	At $t = 0$, $x = 0$ and therefore assume $\alpha = 0$ $T = 5$

2 Use the given displacement and the formula using the period.

$x = \frac{A}{2}$ [1]

$\omega = \frac{2\pi}{T}$

$= \frac{2\pi}{5}$

$x = A\sin(\omega t)$

$x = A\sin\left(\frac{2\pi}{5}t\right)$ [2]

$[1] = [2]$

$\frac{A}{2} = A\sin\left(\frac{2\pi}{5}t\right)$

$\frac{2\pi}{5}t = \sin^{-1}\left(\frac{1}{2}\right)$

$\frac{2\pi}{5}t = \frac{\pi}{6}$

$t = \frac{5}{12} \approx 0.4$ s

3 Interpret the answer.

The particle takes approximately 0.4 seconds to move from the equilibrium position to one-half of its amplitude.

27 Using the equations of simple harmonic motion

An object is oscillating in simple harmonic motion modelled by the equation $x = 3\sin\left(2\pi t - \frac{\pi}{4}\right)$ m.

Determine the object's initial displacement, velocity and acceleration, correct to 2 decimal places.

THINKING	WORKING
1 Determine the initial displacement.	$x = 3\sin\left(2\pi t - \frac{\pi}{4}\right)$ $x(0) = 3\sin\left(-\frac{\pi}{4}\right)$ $= -\frac{3}{\sqrt{2}}$ $= -2.12$ (2 d.p.)
2 Determine the equation for the velocity of the body.	$\dot{x} = 6\pi\cos\left(2\pi t - \frac{\pi}{4}\right)$
3 Determine the initial velocity.	$\dot{x}(0) = 6\pi\cos\left(-\frac{\pi}{4}\right)$ $= \frac{6\pi}{\sqrt{2}}$ $= 13.33$ (2 d.p.)
4 Determine the equation for the acceleration of the body.	$\ddot{x} = -12\pi^2\sin\left(2\pi t - \frac{\pi}{4}\right)$

5 Determine the initial acceleration.

$$\ddot{x}(0) = -12\pi^2 \sin\left(-\frac{\pi}{4}\right)$$
$$= \frac{12\pi^2}{\sqrt{2}}$$
$$= 83.75 \text{ (2 d.p.)}$$

6 Write the answer.

$x = -2.12$ m
$\dot{x} = 13.33$ m/s
$\ddot{x} = 83.75$ m/s^2

Oscillating springs

The motion of a mass m attached to a light elastic spring is described by Hooke's law, which states that the force F exerted by the spring on the mass is proportional to the extension or compression x of the spring from its natural length, $F = -kx$

The force exerted by the spring is always towards its natural length position, opposing the direction of motion.

While the mass is in equilibrium, using Newton's second law:

$$m\ddot{x} + kx = 0$$
$$m\ddot{x} = -kx$$
$$\ddot{x} = -\frac{k}{m}x$$

This is the same rule you have been using for simple harmonic motion, where $\frac{k}{m} = \omega^2$.

The constant k is known as the stiffness of the spring, and typically is expressed in units N/m or kg/s^2. The oscillatory motion can be started when either the object is pulled down (away from the fixed point) or pushed up (toward the fixed point) and then released.

The displacement from the equilibrium position is therefore modelled by $x = A\sin(\omega t + \alpha)$.

The period is

$$T = \frac{2\pi}{n}$$

Given that $\omega = \sqrt{\frac{k}{m}}$, then

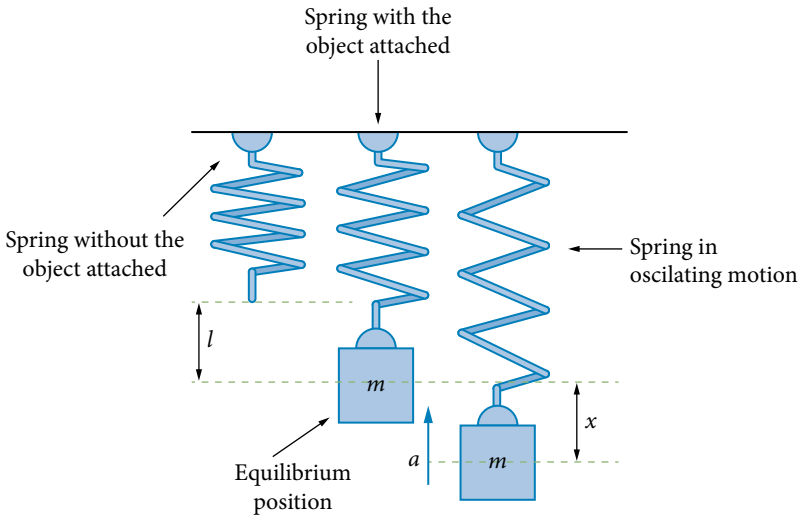
$$T = 2\pi\sqrt{\frac{m}{k}}$$

The spring in the following diagram is hanging from a fixed point. When an object of mass m is attached to the spring, the spring will extend by a length l. This position is considered to be the equilibrium position. The spring and the object are at rest.

The period of simple harmonic motion for a mass m oscillating with displacement x from the equilibrium position is

$$T = \frac{2\pi}{\omega}$$
$$= 2\pi\sqrt{\frac{m}{k}}$$

where k and ω are constants.



28 Vertical oscillation of a light spring

An object of mass 2 kg is attached to a vertical spring and it is in equilibrium at $t = 0$. The motion has an amplitude of 0.05 m and the period is 1 second.

(a) Determine an expression for the displacement of the mass from the centre of motion.

THINKING	WORKING
Recall the formula for displacement and substitute the given values.	$x = A\sin(\omega t + \alpha)$, $x = 0$, $A = 0.5$ and $\alpha = 0$ $\therefore x = 0.5\sin(\omega t)$

(b) Determine the stiffness constant of the spring in terms of π.

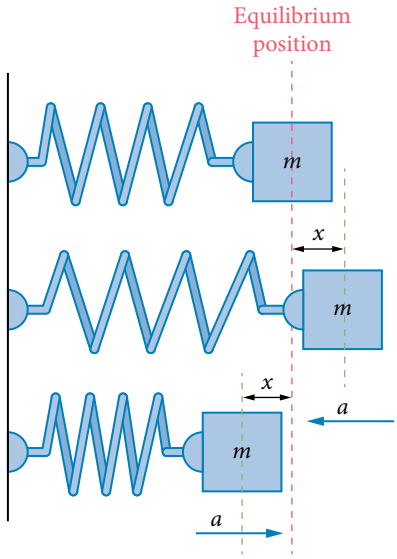
1 Substitute the given values into the equation, $T = 2\pi\sqrt{\frac{m}{k}}$ and solve for k.	$T^2 = 4\pi^2\frac{m}{k}$ $k = \frac{4\pi^2 m}{T^2}$ $= 8\pi^2$
2 Interpret the answer.	The stiffness constant is $8\pi^2$ N/m.

(c) Determine the velocity after $\frac{1}{6}$ seconds in terms of π.

1 Write an expression for the velocity.	$x = A\sin(\omega t) \quad \therefore \dot{x} = A\omega\cos(\omega t)$
2 Determine the value of the constant ω, using the relationship $T = \frac{2\pi}{\omega}$.	$T = \frac{2\pi}{\omega}$ Here it is given that $T = 1$, hence: $1 = \frac{2\pi}{\omega}$ $\omega = 2\pi$
3 Substitute for the constant into the velocity equation.	For $A = 0.05$ and $\omega = 2\pi$. $\dot{x} = A\omega\cos(\omega t)$ $= 0.05 \times 2\pi \times \cos(2\pi t)$ $= \frac{\pi}{10}\cos(2\pi t)$
4 Determine the required value.	$\dot{x}\left(\frac{1}{6}\right) = \frac{\pi}{10}\cos\left(\frac{2\pi}{6}\right)$ $= \frac{\pi}{10} \times \frac{1}{2}$ $= \frac{\pi}{20}$
5 Interpret the answer.	The object is travelling at a velocity of $\frac{\pi}{20}$ m/s towards the attachment point.

The spring in the following diagram is fixed to the left hand edge of an almost frictionless plane and has an object of mass m attached to the right. The spring and the object are at rest. Consider that the object is free to move and there are no other forces such as friction on it. This position is considered the equilibrium position (the spring is neither extended nor compressed).

The oscillatory motion can be started when either the object is pulled to the right (away from the fixed point) or pushed to the left (toward the fixed point) and then released.



29 Horizontal oscillation of a light spring

An object of mass 2 kg is attached to a horizontal spring whose stiffness constant k is 1000 N/m.

(a) Determine the period of the oscillation correct to 2 decimal places.

THINKING	WORKING
1 Write the information given and identify the formula required.	$m = 2$ kg $T = 2\pi\sqrt{\frac{m}{k}}$
2 Substitute the given values into the formula and calculate the period of the oscillation.	$T = 2\pi\sqrt{\frac{2}{1000}}$ $= 0.28$ s (2 d.p.)
3 Interpret the answer.	The period of the oscillation is 0.28 seconds.

(b) Determine the acceleration when the mass is 20 cm in the positive direction from its equilibrium position correct to 1 decimal place.

1 Write the formula for acceleration and determine the constant n.	$\ddot{x} = -\omega^2 x$ $T = \frac{2\pi}{\omega}$ $\omega = \frac{2\pi}{0.28}$ $= 22.43\ldots$
2 Determine the acceleration.	$\ddot{x} = -(22.43\ldots)^2 \times 0.2$ $= -100.7$ (1 d.p.)
3 Interpret the answer.	Acceleration is 100.7 m/s^2 in the negative direction.

30 Determining velocity in terms of displacement for simple harmonic motion

A particle moves in simple harmonic motion and comes to rest when the displacement is $\frac{10}{\pi}$ m.

The period of the motion is 3 seconds and the amplitude is A m. Determine the velocity when the particle is in the equilibrium position, $x = 0$.

THINKING	WORKING
1 Use an appropriate rule for acceleration to determine an expression for velocity in terms of displacement.	$\ddot{x} = -\omega^2 x$ [1] $\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$ [2] [1] = [2] $\frac{1}{2}v^2 = \int -\omega^2 x \, dx$ $\frac{1}{2}v^2 = -\frac{\omega^2 x^2}{2} + c_1$ $v^2 = -\omega^2 x^2 + 2c_1$ When $v = 0$, $x = A$: $2c_1 = \omega^2 A^2$ Therefore $v^2 = \omega^2(A^2 - x^2)$
2 Determine the amplitude and the value of the constant ω.	Amplitude is $\frac{10}{\pi}$ where the particle is at rest. The period of motion is 3 seconds. $\text{Period} = \frac{2\pi}{\omega}$ $\omega = \frac{2\pi}{3}$
3 Determine the required velocity.	$v = \pm\sqrt{\omega^2(A^2 - x^2)}$ $= \pm\omega\sqrt{A^2 - x^2}$ $\therefore v(0) = \pm\frac{2\pi}{3} \times \sqrt{\left(\frac{10}{\pi}\right)^2 - 0^2}$ $= \pm\frac{20}{3}$
4 Interpret the answer.	The velocity of the particle is $\frac{20}{3}$ m/s in either direction.

The relationship found in Worked example 30 also defines simple harmonic motion, giving the velocity in terms of the displacement and amplitude.

$v^2 = \omega^2(A^2 - x^2)$

EXERCISE 8.4 Simple harmonic motion

Worked Example

1 A particle is oscillating with simple harmonic motion of period 2 s. Determine the time taken for the particle to move from the equilibrium position to half of its amplitude for the first time. 26

2 A particle is oscillating with simple harmonic motion modelled by the function $x = 6\sin\left(\pi t + \frac{\pi}{6}\right)$. 27

(a) Determine the initial displacement and the displacement after 1 second.

(b) At what time and displacement is the particle stationary for the first time?

(c) At what time and displacement is the speed of the particle at its maximum value for the first time?

(d) Determine the initial acceleration.

(e) Determine the acceleration at the origin and at the each end of the motion.

(f) Determine the time taken for the particle to move from one extreme position to the other 12 times.

3 The displacement of a moving particle is modelled by the function $x = 4\cos\left(2t + \frac{\pi}{4}\right)$. Demonstrate that the motion is simple harmonic.

4 A particle moves so that its velocity is modelled by the function $\dot{x} = 6\cos\left(2t - \frac{\pi}{6}\right)$.

(a) Determine the period and amplitude of the motion.

(b) Determine the position after 5 seconds correct to 1 decimal place if the initial displacement was $x = -3$.

5 A particle moves with simple harmonic motion of amplitude 6 m. Determine, correct to 2 decimal places, how long it takes for the particle to travel 4 m from its central position if the period is 10 s.

6 A mass of 3 kg is attached to a light spring hanging from a fixed point. It is oscillating with simple harmonic motion with a period of 5 s and amplitude 0.2 m. 28

(a) Determine the stiffness constant of the spring k in terms of π.

(b) If the mass is at its lowest point when $t = 0$, determine an expression for its displacement at any time.

7 An object of mass 4 kg is attached to a horizontal spring whose constant k is 750 N/m. It oscillates with simple harmonic motion on a smooth surface. 29

(a) Determine the period of the motion.

(b) Determine the acceleration when the object is 10 cm away from its central position.

8 A child is standing on a moving sprung platform whose spring has a constant k of 3430 N/m. The period of the simple harmonic motion is 0.64 s. Determine the mass of the child correct to the nearest whole kilogram.

9 An air-track glider is attached to a horizontal spring pulled 16 cm to the left. The glider is released at $t = 0$ and makes 6 complete oscillations in 3 seconds. Give your answers to correct to 2 decimal places.

(a) Determine the period of oscillation for this simple harmonic motion.

(b) Write the amplitude of the oscillation.

(c) Determine the maximum speed of the object attached to the spring.

(d) Determine the position of the object 1.2 s after it is released.

(e) Determine the velocity of the object at $t = 1.2$ s.

10 A bungee cord has a spring constant of 320 N/m. Roshan, weighing 80 kg, falls down from the bungee jumping platform and his motion is recorded from the moment he reaches the lowest point after the jump, $t = 0$. The amplitude of his oscillation is 2.05 m.

(a) Write the equations modelling Roshan's position and velocity as functions of time.

(b) Determine Roshan's position and velocity at $t = 0$ and $t = 1$ s.

(c) Sketch both the displacement–time and the velocity–time graphs for the first full oscillation, on the same set of axes.

Worked Example 30

11 A particle moves with simple harmonic motion with a period of 5 s and amplitude $\frac{20}{\pi}$ m.

(a) Determine the velocity when the particle is in the central position.

(b) Determine the velocity when the particle is one quarter of the way to its maximum displacement.

12 A particle is moving with simple harmonic motion in a straight line. The velocity of the particle is $\sqrt{7}$ m/s at $x = -2$ m and 4 m/s at $x = 0$ m.

(a) Determine the amplitude of the simple harmonic motion.

(b) Determine the period of the motion.

13 The displacement of a particle of m kg from a central position is given by $x = 4\sin\left(\frac{\pi t}{6}\right) + 2$.

(a) Determine an expression for the acceleration of the particle.

(b) Write your acceleration expression in terms of $x - 2$.

(c) Determine the displacement correct to 1 decimal place when the acceleration is $-\pi$ m/s^2.

14 **(a)** Which of the following relations does not represent simple harmonic motion?

A $\ddot{x} = -23x$ **B** $\ddot{x} = 3\cos(\pi t)$ **C** $v^2 = 4\left(x^2 - 3^2\right)$ **D** $\ddot{x} = -\sin\left(2t - \frac{\pi}{3}\right)$

(b) Explain why the other options in part **(a)** are examples of simple harmonic motion.

15 The velocity of a particle is modelled by the equation $v^2 = n^2\left(A^2 - (x-1)^2\right)$.

(a) Determine an expression for the acceleration of the particle.

(b) By setting $x - 1 = y$, verify that the motion is simple harmonic.

16 The velocity of a moving particle is given by $v^2 = 32 - 12x - 2x^2$.

(a) Verify that the motion is simple harmonic.

(b) Determine the amplitude and the central position of the motion.

17 A particle's position is given by $x = a\sin(nt) + b\cos(nt)$.

(a) Verify that the particle is moving with simple harmonic motion.

(b) Determine an expression for the amplitude of the motion.

18 A mass of 3 kg is hanging from a spring of natural length 50 cm. The mass is at rest and has extended the spring by 10 cm. It is pulled down by a further 4 cm and let go. Determine the period (correct to 1 decimal place) and amplitude of the resulting simple harmonic motion. (Hint: Hooke's law tells you that the force the spring exerts on the mass is $F = -kx$.)

Summary

Linear motion with constant acceleration

Average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

Average velocity $= \dfrac{\text{displacement}}{\text{time interval}}$

Instantaneous velocity, $v_{\text{instantaneous}} = \dfrac{dx}{dt}$

Average acceleration $= \dfrac{\text{change in velocity}}{\text{time interval}}$

Instantaneous acceleration, $a_{\text{instantaneous}} = \dfrac{dv}{dt}$

Displacement, $x = \int v\,dt$

Velocity, $v = \int a(t)\,dt$

The gradient of a line in a velocity–time graph represents the acceleration of the body over the given time interval. The algebraic sum of the area between the velocity–time graph and the time axis represents the displacement of the body from the reference point.

When the acceleration is constant, the following formulas can be used:

$v = u + at$

$v^2 = u^2 + 2ax$

$x = ut + \dfrac{1}{2}at^2$

$x = \dfrac{1}{2}(u + v)t$

u represents the initial velocity of the particle

v represents the final velocity of the particle

t represents the time interval of the motion

a represents the constant acceleration of the particle

x represents the displacement of the particle, from its initial position

Different formulas for acceleration are used, depending on the given or required information.

$$a = \ddot{x}$$
$$= \frac{d^2x}{dt^2}$$
$$= v\frac{dv}{dx}$$
$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Newton's first law of motion

An object either remains at rest or continues to move at a constant velocity unless acted upon by an external force. Forces can be resolved into mutually perpendicular components to determine resultant or net forces in two directions.

Newton's second law of motion

The net force on an object is equal to the rate of change of its momentum over time.

Momentum is a vector quantity, $p = mv$

Change in momentum (impulse), $\Delta p = m\Delta v$

$$F_{\text{net}} = \frac{dp}{dt}$$
$$= ma$$

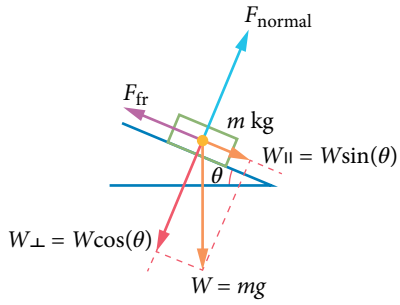
Weight and the normal force

A weight force due to gravity $W = mg$, acting on an object is equal and opposite to the normal reaction force.

Friction

Friction occurs at the point(s) of contact between an object and a surface as a force is applied to the object. The friction is known as either static friction (when the object is stationary) or kinetic friction (when the object is moving). The friction is calculated as $F_{\text{fr}} = \mu F_{\text{normal}}$, where μ is the coefficient of friction.

Forces acting on a body on an inclined plane are as shown.



Simple harmonic motion

Simple harmonic motion is a periodic motion modelled by the following:

$$\ddot{x} = -\omega^2 x$$
$$x = A\sin(\omega t + \alpha)$$
$$\dot{x} = \omega A\cos(\omega t + \alpha)$$
$$v^2 = \omega^2\left(A^2 - x^2\right)$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$

for displacement x m, velocity v or $\dot{x}$ m/s, acceleration $\ddot{x}$ m/s^2, amplitude A m, time t s, period T s, mass m kg, stiffness constant for a spring k N/m.

Chapter review 8

1 The Shanghai Maglev train is the fastest commercial train in the world. The train can travel a distance of 30 km in a minimum time of 4.186 minutes. Determine the maximum possible speed of the train. Exercise 8.1

2 The tallest building in the world is the Burj Khalifa skyscraper in Dubai, United Arab Emirates, with a height of 829.8 m. Determine the time it would take for a ball dropped from the top of the building to reach the ground, assuming the acceleration is constant at $g = 9.8\,\text{m/s}^2$. State your answer to the nearest second. Exercise 8.1

3 A car accelerates with an average acceleration of $3.5\,\text{m/s}^2$ and reaches a speed of 60 km/h in 5 seconds. Determine the initial velocity of the car. Exercise 8.1

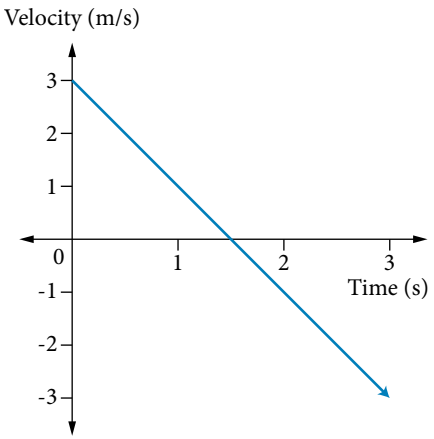
4 The velocity v m/s of a particle at any time t is modelled by the equation $v = -t^2 - 7t + 8$. Exercise 8.2

(a) Determine the acceleration of the particle at time $t = 3$ s.

(b) Determine the acceleration when the velocity is zero.

5 A particle is moving in a straight line with displacement x metres from a fixed reference point O at any time t seconds. The displacement of the particle is modelled by the equation $x = 2t^2 - 8t - 3$, where $t \geq 0$. Determine the initial displacement, velocity and acceleration of the particle. Exercise 8.1

6 A velocity–time graph for an object is shown. Exercise 8.1



(a) Determine the displacement of the object after 2 seconds if it starts at the origin.

(b) Determine the acceleration of the object.

7 An object travelling at 48.6 km/h in a forward direction slows down at a uniform rate and comes to rest after 8 seconds. Determine the distance travelled by the object while slowing to a rest. Exercise 8.1

Exercise 8.1

8 Two identical balls are thrown from the same height, ball P upwards and ball Q downwards, with the same initial velocity. Assuming there is no air resistance, which of the following statements is correct?

- **A** Ball P reaches the maximum height at the same time as ball Q reaches the ground.
- **B** The acceleration of ball P and ball Q is equal.
- **C** The maximum height reached by ball P is the same as the distance travelled by ball Q when it reaches the ground.
- **D** Ball P moves at the same speed as ball Q during the first few seconds of motion.

Exercise 8.3

9 A net force of 28 N is applied to a particle and makes the particle accelerate at 0.7 m/s^2. Determine the mass of the particle.

Exercise 8.4

10 A horizontal force of 300 N and a vertical force of 400 N are applied to a mass of 20 kg. Determine the magnitude and direction of acceleration of the mass.

Exercise 8.3

11 A 1 tonne body is being pulled vertically up by a crane with a force of 10 kN. Determine the acceleration of the body.

Exercise 8.3

12 A car is travelling east on a straight road. The mass of the car is 1.26 tonnes and it is acted upon by a net force of 2540 N east. Determine the acceleration of the car, correct to 2 decimal places.

Exercise 8.4

13 An object of mass 4 tonnes is attached to a crane by an elastic cable with a stiffness constant k of 3270 N/m and is oscillating with simple harmonic motion. Determine the period of the oscillation correct to 2 decimal places.

Exercise 8.4

14 A particle is moving so that its position x m at time t s is modelled by the function $x = -3\sin\left(\pi t - \frac{\pi}{2}\right)$. Determine the following:

- **(a)** The initial position and velocity of the particle.
- **(b)** The maximum speed of the particle and the time at which it first occurs.
- **(c)** The period and amplitude of the motion.
- **(d)** The two extreme positions of the particle.
- **(e)** The acceleration when the particle is at $x = -3$ m.

Exercise 8.2

15 The velocity of a car is described by the equation $v(t) = 12t^2 + 6t - 7$ m/s, and $x = 15$ m at $t = 0$. Determine expressions for the displacement and acceleration of the car at time t s.

Exercise 8.2

16 An object is projected with an initial velocity of 3 m/s from a reference point O in a straight line. If the acceleration of the object at any time t is given by $a(t) = -8t + 1$, determine the expressions for its displacement and velocity at any time t s.

Exercise 8.3

17 A mass of 11 kg is being pulled to the right along a rough horizontal surface with an acceleration of 2 m/s^2 by a horizontal force. The coefficient of friction between the mass and the surface is 0.6. Determine the magnitude of the force and the normal reaction of the surface correct to 1 decimal place.

Exercise 8.3

18 Determine the magnitude of the force and the normal reaction on the mass in question **17** if the pulling force is acting at an angle of 18° to the horizontal and the acceleration is still 2 m/s^2.

19 A bullet of mass 80 g travelling at 500 m/s is fired into a practice target wall and comes to rest in 0.001 seconds, embedded in the wall. Exercise 8.3

(a) Determine the magnitude of the impulse exerted on the bullet by the target.

(b) Determine the distance the bullet travels after entering the target. Assume the deceleration of the bullet is constant.

20 A particle is attached to a horizontal spring and its position is modelled by the equation $x = -5\sin(1.2t - 4)$. Exercise 8.4

(a) Determine expressions for the velocity and acceleration of the particle at time t.

(b) Determine the velocity and acceleration when the particle has reached half of its amplitude in the positive direction.

21 Determine an expression for the velocity v and position x for the following, given the acceleration and initial conditions. Exercise 8.2

(a) $a = \dfrac{2}{(1+t)^2}$ and $v(0) = 4$, $x(0) = 0$

(b) $a = 3e^{-2t+1}$ and $v(0) = \dfrac{1}{2}e$, $x(0) = 0$

(c) $a = \sec^2(\pi t)$ for $t < 0.4$ and $v(0) = 0$, $x(0) = 0$

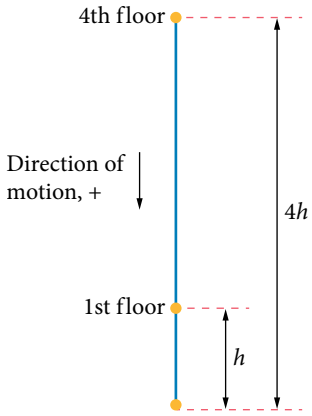
22 A mass of $m = 10$ kg is moving with simple harmonic motion attached to a horizontal spring with stiffness constant $k = 25$ N/m. The velocity $\dot{x}$ m/s is zero when the displacement from the centre of motion is $x = 3$ m. Give your answers correct to 2 decimal places. Exercise 8.4

(a) The displacement x m is modelled by the equation $x = A\cos(nt)$. Determine the value of A and n.

(b) Between which two times is the velocity negative and the acceleration positive for the first time?

(c) Determine the acceleration at $x = -2$.

23 An object is dropped from the first floor of a building and then from the fourth floor of the building as shown. The height of each floor is h and the air resistance is negligible. Taking downwards as the positive direction, compare the time taken to reach the ground for the two cases. Exercise 8.1



Exercise 8.2

24 The displacement of a particle is modelled by the equation $x(t) = 3\cos(t) - 2t$ metres, $0 \leq t \leq \pi$ s.

(a) Sketch the graph of the velocity of the particle for the time interval given.

(b) Determine the exact value of the total displacement for the whole time interval shown.

Exercise 8.3

25 A net force of 768 N is applied to a body at an angle of 30° to the horizontal. Determine the following, giving your answers correct to 1 decimal place.

(a) The magnitudes of the vertical and horizontal components of the net force.

(b) The magnitude and the direction of the acceleration of the body if its mass is 60 kg.

(c) The normal reaction of the surface on the mass of 60 kg if it can only move horizontally.

Exercise 8.1

26 The acceleration due to gravity on Mars is 3.7 m/s^2, on Jupiter is 24.8 m/s^2 and on the Moon is 1.6 m/s^2. Determine how long it would take objects, correct to 1 decimal place, to fall 250 m on Mars, Jupiter, the Moon and Earth. Assume atmospheric resistance is negligible in all places.

Exercise 8.3

27 A car of mass 1.5 tonnes is travelling in a straight line from a stationary position due to a constant driving force of 1600 N. The car accelerates for 28 seconds at 0.7 m/s^2.

(a) Determine the magnitude of the normal reaction force of the road on the car.

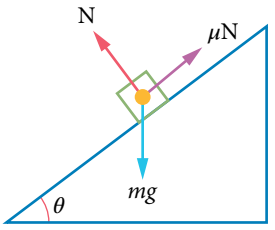
(b) Determine the resultant force acting on the car.

(c) Determine the speed of the car at the end of the 28 seconds.

(d) If the driving force is doubled after the first 28 seconds, determine the new acceleration, and the speed of the car after another 28 seconds.

Exercise 8.3

28 A mass m is sliding down an inclined plane for which the coefficient of friction is μ. The plane makes an angle θ with the horizontal. The forces on the mass are shown in the diagram.



(a) Determine an expression for the acceleration of the mass.

(b) Determine a relationship between θ and μ such that the mass moves down the plane with a constant velocity.

Exercise 8.1

29 Chloe is a sprinter and she starts a race from rest, accelerating with a constant acceleration for 12.5 s while covering 100 m. At the end of the sprint Chloe slows down at a constant rate until she stops after another 100 m over the next 20 seconds.

(a) Determine Chloe's acceleration for the first 100 m of the motion.

(b) What was Chloe's velocity at the end of the first 100 m of motion?

(c) Determine the rate at which Chloe slowed down.

(d) Determine the average velocity over the 200 m distance.

(e) Sketch a position–time graph for the whole motion.

Exercise 8.2

30 An object is moving in a straight line with velocity v. The object is decelerating at a rate of half of its velocity. Determine the velocity and acceleration of the moving object at $x = 2$ m if $v = 13$ m/s at $x = 0$ s.

31 When a mass of 5 kg is attached to a vertical spring and is resting in equilibrium, the spring is extended beyond its natural length by 0.2 m. Give your answers correct to 2 decimal places. Exercise 8.4

(a) Determine the stiffness constant of the spring, k.

(b) The mass is then pulled down a further 0.1 m, and let go. Determine the period T and frequency f of the resulting simple harmonic motion.

(c) Write the maximum and minimum extension positions of the mass during its motion, and hence determine the amplitude.

(d) Determine expressions for the position and velocity of the mass in terms of time.

32 The velocity $\dot{x}$ m/s of a particle at displacement x m is modelled by the equation $\dot{x} = \frac{\pi}{3}\sqrt{20 - x^2}$. Exercise 8.4

(a) Verify that the particle moves with simple harmonic motion.

(b) Determine the period and amplitude of the motion.

(c) Determine the first two times that the speed is zero if $x(0) = 0$.

33 A mass of 6 kg hanging vertically is connected by a string passing over a smooth pulley to a mass of 3 kg on an inclined plane that makes an angle of 10° with the horizontal. The coefficient of friction between the 3 kg mass and the plane is 0.2. The acceleration of the masses are equal in magnitude, a, with the 3 kg mass moving up the plane. Exercise 8.3

(a) Draw a diagram showing the forces on each mass.

(b) Determine the magnitude of the tension T correct to 1 decimal place.

34 An object is moving so that its velocity v m/s at position x m is modelled by the equation $v^2 = -25\left(3 + x^2 - 4x\right)$. Verify that the motion is simple harmonic. Exercise 8.4

35 A particle is projected vertically upwards with an initial velocity of 100 m/s. The retardation due to air resistance is $0.002v^2$ m/s^2. Determine the height reached by the particle before it comes to rest and begins to descend, correct to the nearest whole number. Exercise 8.2

36 A particle moves in a straight line with acceleration $a = \frac{g}{2}e^{-0.1t}$. Exercise 8.2

(a) Determine an expression for the velocity of the particle if it starts from rest.

(b) Determine the velocity of the particle after a long period of time.

(c) Determine an expression for the acceleration in terms of the velocity.

(d) Determine the acceleration when the particle has reached its terminal (long-term) velocity.





9

Statistical inference

Recall 496

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9.2 Sampling distributions 504

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9 Recall

Distinguish between continuous and discrete scales

1 Determine if the following variables are associated with continuous or discrete scales:

(a) The number of emails received in one day

(b) The temperature of a child at any one time

(c) The number of lollies eaten at a party

(d) The amount of water consumed in one day

Identify key statistical information

2 Identify the population, the sample, the parameter and the statistic for each of the following cases:

(a) A researcher wants to estimate the average (mean) weight of boys between 10 and 12 years old in Australia. From a random sample of 344 such boys, the average weight is 36.7 kg.

(b) A survey was conducted on 1378 randomly selected teachers in the Queensland public system. Of the 1378 teachers surveyed, 923 said that they would not leave the public system.

(c) An Australian study stated that the average amount of sleep that is required by a 15-year-old is approximately 8.5 hours with a standard deviation of 40 minutes. A random sample of 1783 15-year-olds found that the average amount of sleep obtained was 7.8 hours.

(d) A standard unbiased die is rolled 50 times and the number 6 is recorded nine times.

Calculate the mean and standard deviation from a set of data

3 Calculate the mean (μ) and standard deviation (σ) from the following sets of data, rounded to the nearest whole.

(a) {7, 9, 10, 11, 13}

(b)

x	f
3500	5
4000	8
4200	5
4300	2

(c)

Class	f
$120 < 130$	2
$130 < 140$	5
$140 < 150$	25
$150 < 160$	10
$160 < 170$	8

Identify independent events

4 Determine if the following events are independent.

(a) The outcome of two rolls of a fair die.

(b) From 10 students, one is chosen as the captain and then from the remaining students a vice captain is chosen.

(c) Drawing a 3 from a deck of cards, replacing it and then drawing an ace.

Calculate the expected value, variance and standard deviation of a discrete probability distribution

5 For the following discrete probability distributions, calculate the expected value, $E(X)$, the variance, $\text{Var}(X)$, and the standard deviation, $\sigma(X)$, correct to 2 decimal places.

(a)

x	1	2	3	5
$P(X = x)$	0.02	0.51	0.17	0.3

(b)

x	2	4	8	16	32
$P(X = x)$	0.25	0.3	0.2	0.15	0.1

(c)

x	3	6	9	12	15
$P(X = x)$	0.01	0.35	0.4	0.2	0.04

Identify the common features of the normal distribution and apply them

6 What percentage is captured by ±1, ±2 and ±3 standard deviations from the mean?

7 The average IQ is 100 with a standard deviation of 15 and is normally distributed. What range of scores would fall into one, two and three standard deviations from the mean?

9.1 Sample means

Inferential statistics

When you were first introduced to the topic of statistics, you saw how various statistical calculations could be used to describe features of a particular data set. For example, the *mean* is a useful indication of the central tendency of the data set in the absence of outliers or significant skewness in the distribution, and the *standard deviation* is a measure of spread. These basic statistics that describe the features of a data set are known as *descriptive statistics*.

While descriptive statistics are useful for summarising information about a specific population or data set, *inferential statistics* allow you to infer features of a population by analysing the characteristics of samples that are drawn from that population. In real life, inferential statistics are used routinely by government agencies, researchers, scientists and social scientists as they can give a very accurate generalisation of a population based on a relatively small sample from that population. You can also use inferential statistics to indicate how confident you are that the characteristics of your sample will match the characteristics of the population from which the sample was taken.

A review of statistical terminology and the specific notation used with statistical analysis:

Term	Definition
Population	The *population* (or parent population) is the total of all the units that make up a particular group.
Unit	A *unit* is a single member of a finite population. It may be a person, animal, company or object.
Census	A *census* is data collected from the entire population.
Sample	A *sample* is a portion of units taken from the entire population.
Parameter	A *parameter* is a numerical descriptive measure, such as mean or standard deviation, that describes an unchanging and generally unknown characteristic of a population.
Statistic	A *statistic* is a calculated numerical measure, such as mean or standard deviation, that describes a characteristic of a sample.

Statistical notation is as follows:

	Sample statistic	Population parameter
Number of members	n	N
Mean	$\overline{X}$ or $\overline{x}$	μ
Standard deviation	s	σ

Sample means

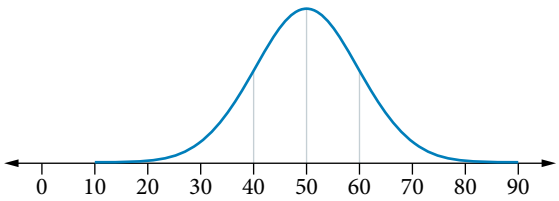
Consider a certain population that has a particular measurable aspect (for example, the height of female adults). In a given population, this measurable aspect takes the form of a random variable, X. The population mean for X is denoted by μ (which in practical terms is generally unknown).

A random sample on X of size n is defined as n random variables, $X_1, X_2, \ldots, X_n$. The random variable that describes the set of all possible values of the mean from sample size n is defined as the *sample mean* $\overline{X} = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$.
Suppose an actual random sample of size n is taken from variable X. This takes the form $x_1, x_2, \ldots, x_n$. Calculating the mean of this sample gives $\overline{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$. Please note that in the case of an actual and calculated sample mean, $\overline{x}$ is used.

This observed value $\overline{x}$ is referred to as a *point estimate* of the population mean μ. A point estimate is a statistic which is used to estimate a population parameter. In many cases, this is the best that can be done. It is often impractical or even impossible to calculate the actual population mean (or other parameters of interest). After all, the point of statistics is to estimate difficult-to-measure values using practical approximations.

A more accurate estimation of the population mean can be obtained by drawing multiple samples (rather than a single sample) and then calculating the mean of the multiple sample means. Since all sample statistics are known to be only approximations of the population parameters they estimate, and their value will differ between samples, drawing multiple samples allow these errors to be refined. The distribution of all possible sample means of size n has a mean of μ, the same as the population mean for X.

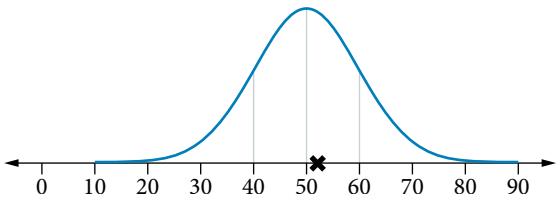
Consider a particular aptitude test that is given to a large number of students. Suppose that it is known that the population of student results for this test has a mean μ of 50 and a standard deviation σ of 10.



Now suppose that a sample of $n = 15$ student results is drawn at random from this population:

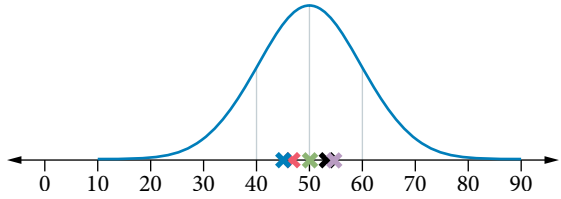
54.9	47.9	42.6
55.8	52.6	57
51.3	38.8	42.9
59.9	42.9	68.2
49	53.7	65.5

The mean of this first sample is $\overline{x}_1 = 52.2$ and is marked on the diagram as follows:



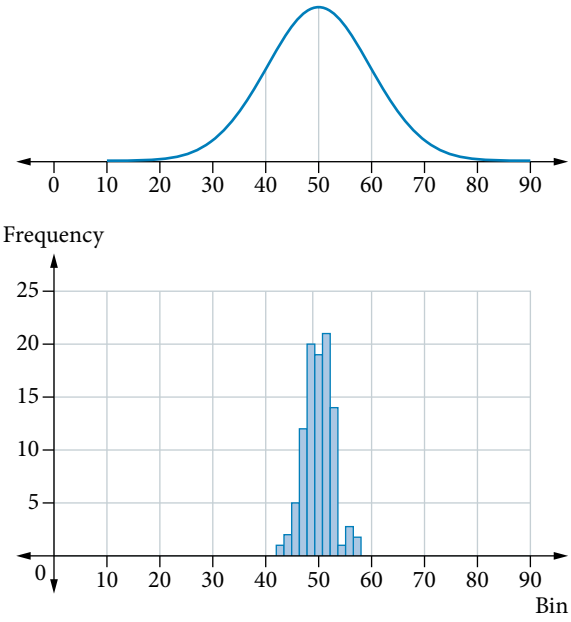
This process is repeated another four times, and the means of each of these five samples are summarised in the table and on the following graph:

Sample	x_i	Mean
1	$\overline{x}_1$	52.2
2	$\overline{x}_2$	47.5
3	$\overline{x}_3$	45.9
4	$\overline{x}_4$	53.2
5	$\overline{x}_5$	50.2



The value of any calculated statistic for a random sample taken from a population will be just one of a whole range of possible values. This is called *sampling variability* and is demonstrated in the summary of different values for the sample means above. Sampling variability indicates that statistics (sample mean, sample standard deviation etc.) are random variables themselves and have a distribution known as the *sampling distribution*.

Consider what occurs when the sampling process is repeated a large number of times (say, 100 samples overall). The means of these samples can be plotted as a histogram and compared with the original population distribution as follows:



As you would expect, the sampling distribution appears to share the same mean as the population. In other words, $\dfrac{\overline{x}_1 + \overline{x}_2 + \overline{x}_3 + \ldots + \overline{x}_{100}}{100} \approx \mu$.

However, notice that the variation or spread of the sampling distribution is much less than that of the original population. The reason for this will become clear later in this chapter.

t Technology worked example

Generating and graphing a simple random sample

e Explore further

Calculate summary statistics

Use technology to calculate summary statistics.

Mean and variance of the sample mean

To calculate the mean and variance of sample means you will need to draw upon your knowledge of the binomial distribution, namely:

For n random variables $X_1, X_2, \cdots X_n$:
$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$
and $E(aX) = aE(X)$.

Mean of the sample mean

Each random variable X_i in the random sample has the same distribution as X, therefore $E(X_i) = \mu$. Using this result, the expected value of the sample mean can be calculated as follows:

$$\begin{aligned}
E(\bar{X}) &= E\left(\frac{X_1 + \ldots + X_n}{n}\right) \\
&= E\left(\frac{X_1}{n}\right) + E\left(\frac{X_2}{n}\right) + \ldots + E\left(\frac{X_n}{n}\right) \\
&= \frac{1}{n}E(X_1) + \frac{1}{n}E(X_2) + \ldots + \frac{1}{n}E(X_n) \\
&= \frac{1}{n}\mu + \frac{1}{n}\mu + \ldots + \frac{1}{n}\mu \\
&= \frac{1}{n}(\mu + \mu + \ldots + \mu) \\
&= \frac{1}{n}(n\mu) \\
&= \mu
\end{aligned}$$

If a random sample size n on X has an expected value $E(X) = \mu$, then the expected value of the sample mean is equal to the expected value of the population mean, that is, $E(\bar{X}) = \mu$.

This result is significant – it means that the expected value of the sample mean is actually the population mean, μ. Although it may seem obvious, this is a powerful result – the sample mean can be used to provide an unbiased estimate of the population mean.

Variance of the sample mean

Note that $\text{var}(X_i) = \sigma^2$ and in a random sample, $X_1, X_2, \ldots X_n$ are mutually independent.

Recall that if $Y = aX + b$, then $\text{var}(Y) = a^2\,\text{var}(X)$. Using these results, the variance can be calculated as follows:

$$\begin{aligned}
\text{var}(\bar{X}) &= \text{var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) \\
&= \left(\frac{1}{n}\right)^2 \text{var}(X_1 + X_2 + \ldots + X_n) \\
&= \left(\frac{1}{n}\right)^2 \text{var}(X_1) + \left(\frac{1}{n}\right)^2 \text{var}(X_2) + \ldots + \left(\frac{1}{n}\right)^2 \text{var}(X_n) \\
&= \left(\frac{1}{n}\right)^2 \sigma + \left(\frac{1}{n}\right)^2 \sigma + \ldots + \left(\frac{1}{n}\right)^2 \sigma \\
&= \left(\frac{1}{n}\right)^2 (\sigma^2 + \sigma^2 + \ldots + \sigma^2) \\
&= \left(\frac{1}{n}\right)^2 n\sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

A random sample of size n on X where $\text{var}(X) = \sigma^2$ has $\text{var}(\bar{X}) = \dfrac{\sigma^2}{n}$ and standard deviation $\text{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

From this, you obtain the standard deviation of the sample mean, $\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$.

Because $\sqrt{n}$ is in the denominator of $\text{sd}(\bar{X})$, the variance of the distribution of sample means becomes smaller as n becomes larger. For very large samples, this means it is very likely that $\bar{X}$ will be close to the population mean μ.

Throughout the rest of this chapter, the standard deviation will be used rather than the variance, as the standard deviation of $\bar{X}$ is given in the same units as X and μ. Further calculations of the expected value $E(\bar{X})$ will be correct to 1 decimal place and all other calculations correct to 2 decimal places.

Technology worked example
Calculating the mean, variance and standard deviation

1 Calculating the standard deviation of the sample means, σ known

The air in the Sydney Harbour tunnel is tested every 30 minutes to monitor levels of carbon monoxide. Tests show that the carbon monoxide levels have a mean, $\mu = 0.21$ ppm (parts per million) and a standard deviation, $\sigma = 1.45$ ppm. Calculate the standard deviation of the mean values that you would expect over an 8 hour period of testing.

THINKING	WORKING
1 Determine the value of n.	$n = 2 \times 8$ $= 16$
2 Substitute the values into the formula for the standard deviation of sample means.	$\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$ $= \frac{1.45}{\sqrt{16}}$ $= 0.3625$
3 Write your answer.	$\text{sd}(\bar{X}) = 0.36$ ppm to 2 decimal places.

EXERCISE 9.1 Sample means

1 Calculate the mean (to 1 decimal place where necessary), the variance and standard deviation (to 2 decimal places where necessary) for the following sets of sample data.

Worked Example 1

(a) 7, 39, 13, 9, 45, 47, 22 (b) 0.25, 0.01, 0.658, 0.00148, 0.27, 4.2, 1.002

2 Calculate the standard deviation of the sample means to 2 decimal places, given the following values:

(a) $\sigma = 1.36$, $n = 10$ (b) $\sigma = 32.89$, $n = 380$ (c) $\sigma = 0.047$, $n = 5$

3 Identify the population, the sample, the parameter and the statistic for the following scenarios

(a) A researcher wants to estimate the average height of girls between the age of 10 and 15 years. For a random sample of 247 such girls, the average height is 136.9 cm.

(b) The Dairy Farm Holders Association of Australia (DFHAA) wants to estimate the average (mean) daily consumption of calcium by male teenagers in Australia. From a random sample of 1342 male teenagers, the DFHAA calculates a sample mean of 1121 mg of calcium consumed.

(c) An education group wants to estimate the mean score on the reading comprehension section of the Year 9 NAPLAN test for students whose first language is not English. A simple random sample of 423 Year 9 students whose first language is not English is chosen.

4 A population has a mean of 56.3 and a standard deviation of 8. A random sample of 800 is selected. What is the expected mean of the sample?

A 19.9 **B** 0.283 **C** 8 **D** 56.3

5 Which of the following offers the best description of the sample mean?

A Numerical values calculated from the population used to estimate the population parameter

B Numerical value calculated from the sample used to estimate the population mean

C Numerical value calculated from the sample used to estimate a sample statistic

D Numerical values calculated from the sample used to estimate the spread of data

6 A machine grinds discs to a target diameter $\mu = 40.125$ mm and $\sigma = 0.02$ mm. A disc is checked every 12 hours over the period of one working week (5 days). Calculate the expected mean diameter (to 1 decimal place) and standard deviation of the sample means (to 2 decimal places) after a week's worth of testing.

7 A simple random sample of 10 apprentices was obtained and their hourly rate of pay recorded in the table.

If the apprentices all work a 40-hour week, estimate:

(a) the mean weekly income for apprentices.

(b) the standard deviation of the sample means of weekly incomes.

Apprentice	Hourly pay rate
1	\$17.67
2	\$18.24
3	\$17.20
4	\$16.80
5	\$17.50
6	\$16.23
7	\$17.00
8	\$18.90
9	\$19.10
10	\$16.66

8 During a stock market crash, stocks lost nearly 20% of their value in a single day. Over a year, the mean return of all stocks was $\mu = -3.5\%$ (that is, there was an overall loss). The standard deviation of the returns was $\sigma = 26\%$. A financer studies five stocks and records each return. What is the expected mean of the five returns and standard deviation of sample means?

9 For many years, the most widely-used measure of human intelligence has been the IQ test. IQ test results are standardised with a population mean $\mu = 100$ and a population standard deviation $\sigma = 15$. A researcher initially collects IQ data from groups with a sample size of $n = 10$ people. She later adjusts her experimental technique such that the sample size is increased to $n = 20$ people. Determine the effect this change would have on the mean of the sample means in the study and the standard deviation of the samples means in the study.

9.2 Sampling distributions

Population parameters and sample estimates

The aim of inferential statistics is to infer characteristics about a particular population by examining the mathematical features of samples that are drawn from that population. For example, a researcher may be interested in investigating the height of Year 12 students in Queensland. The mean height of this group of students is referred to as a *population parameter*, μ.

Unless a census is taken whereby the height of every member of this population is measured, the true mean remains unknown. However, sampling techniques can be used to *estimate* the value of μ. As with any form of estimation, there is some uncertainty involved in drawing conclusions about a population from samples of that population, but that degree of uncertainty can itself be calculated and quantified.

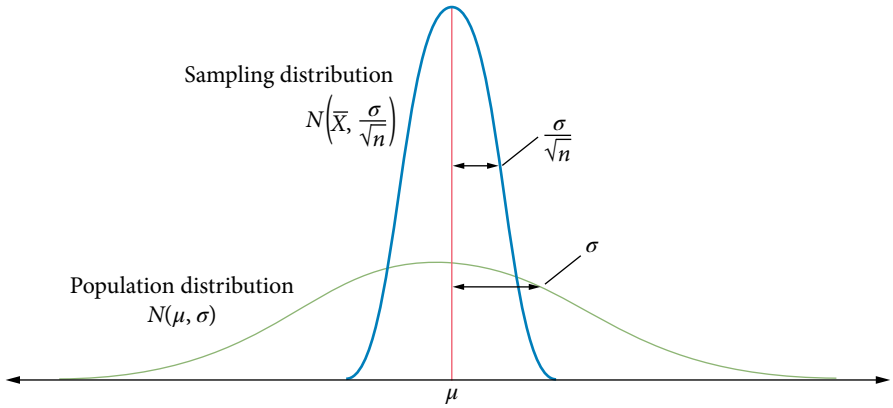
In the previous section you were introduced to the concept of sampling distributions of the sample mean. You will now investigate these distributions in greater detail and observe how the shape of these distributions relate to the shape of the population distribution.

Mean and standard deviation of a sampling distribution

The sample mean $\bar{X}$ will be at, above or below the population mean μ. The point estimate $\bar{x}$, from a single sample, is considered to be an *unbiased estimate* of μ. In other words, over many samples, it will *on average* approximate the population mean.

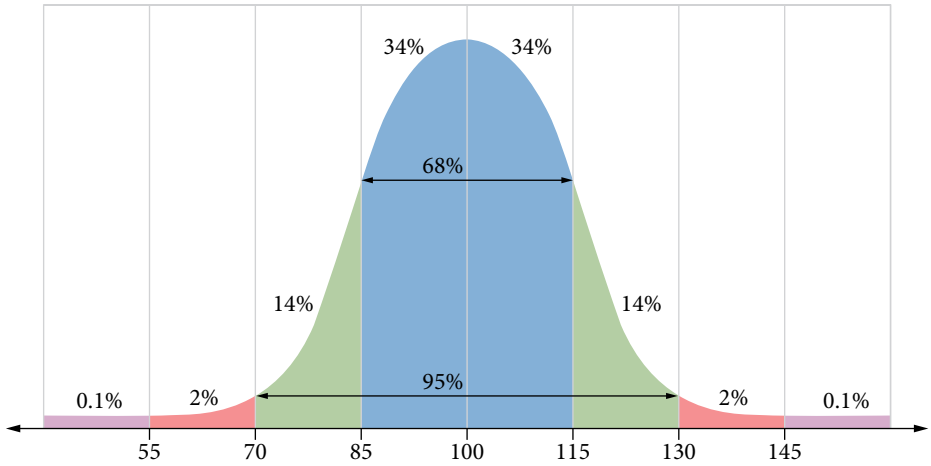
One type of sampling variability, given as a single value, is the standard deviation. This statistic is a numerical value that measures how much the sample statistic varies from sample to sample. The standard deviation of $\bar{X}$ is $\frac{\sigma}{\sqrt{n}}$, where σ is the population standard deviation. Note that the standard deviation of the sample means is smaller than the population standard deviation by a factor of $\sqrt{n}$.

The following graph demonstrates the difference between population standard deviation and the standard deviation of the sample means.

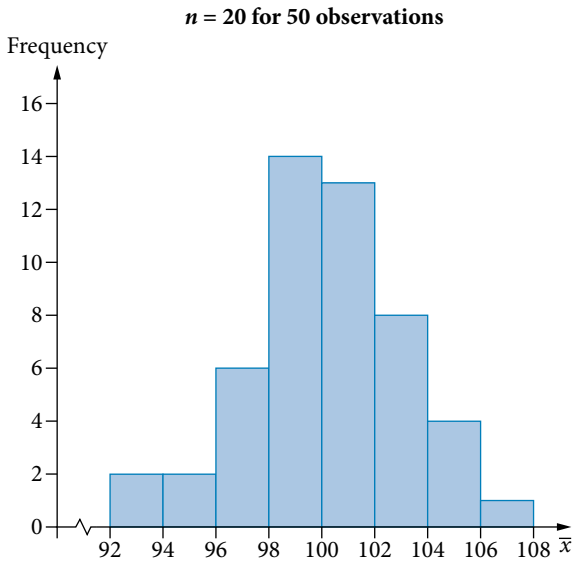


Sampling distribution of a normal population

Intelligence quotient (IQ) scores are normally distributed with a mean $\mu = 100$ and $\sigma = 15$. The distribution of these scores is graphed below.



A random sample of 20 people was taken and the point estimate of the mean IQ, $\bar{x}$, was calculated. This process was then repeated 50 times to produce a sample distribution of the sample means:

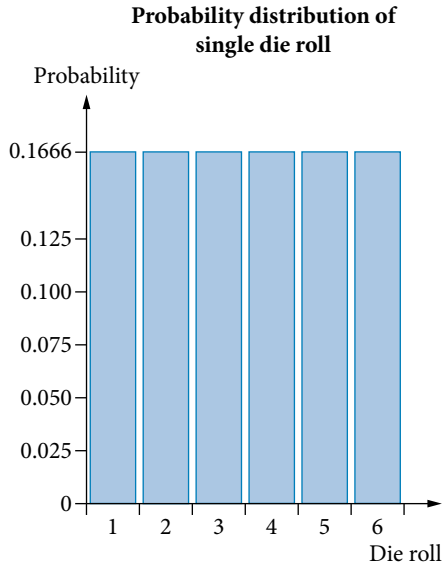


From this graph it is apparent that the distribution of sample means is beginning to take the shape of a normal distribution, with the same mean as the IQ distribution (100) but with considerably less variation than the population distribution.

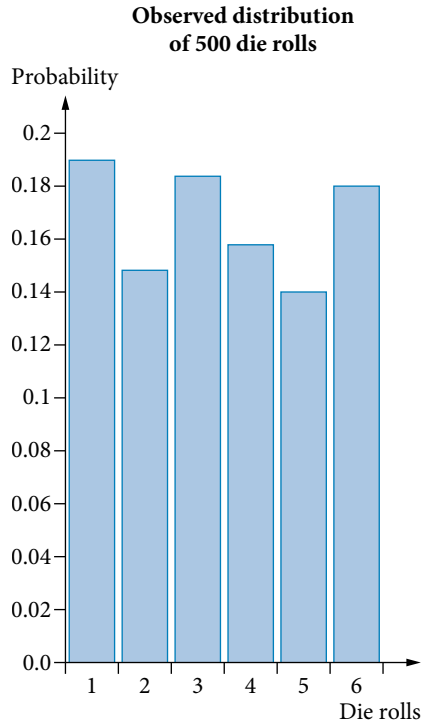
However, what happens when the population is not normally distributed?

Sampling distribution of a uniform population

Consider the theoretical probability distribution of rolling a single die. The expected distribution of the population is uniform.



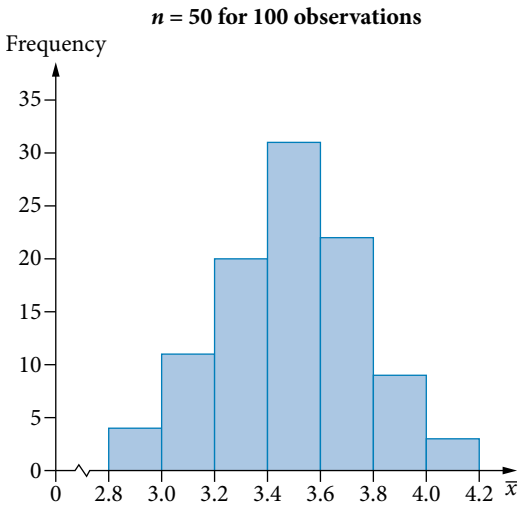
If a single sample of 500 observations of a die roll is taken, you would expect to see some random variation from the theoretical uniform distribution.

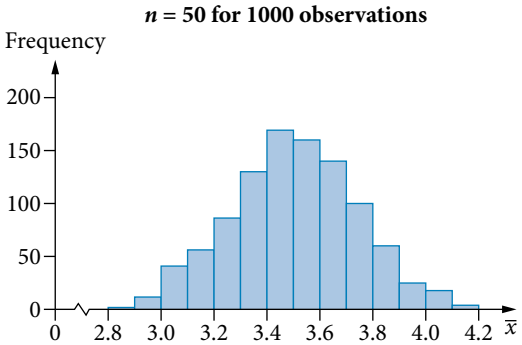


Now consider what happens if several samples are taken and the distribution of sample means is graphed. It is known that the theoretical (population) mean of rolling a die is 3.5. A random sample of 50 rolls of a die has a calculated mean of 3.57. This figure alone does not give you an indication about the distribution, therefore you repeat this process and collate the results. The graph shows the result of repeating the above process 100 times.

As expected, the sample means of the 100 experiments vary, but the most obvious feature is that the distribution no longer resembles a uniform distribution. Instead, the distribution appears bell-shaped and symmetrical. Repeating the same process 1000 times gives a better indication of the shape of the distribution of the sample means.

As more observations are made, the distribution of sample means begins to resemble more closely a normal distribution centred around a mean of approximately 3.5. Note that the column width and vertical axis have been modified to accommodate the larger number of experiments.





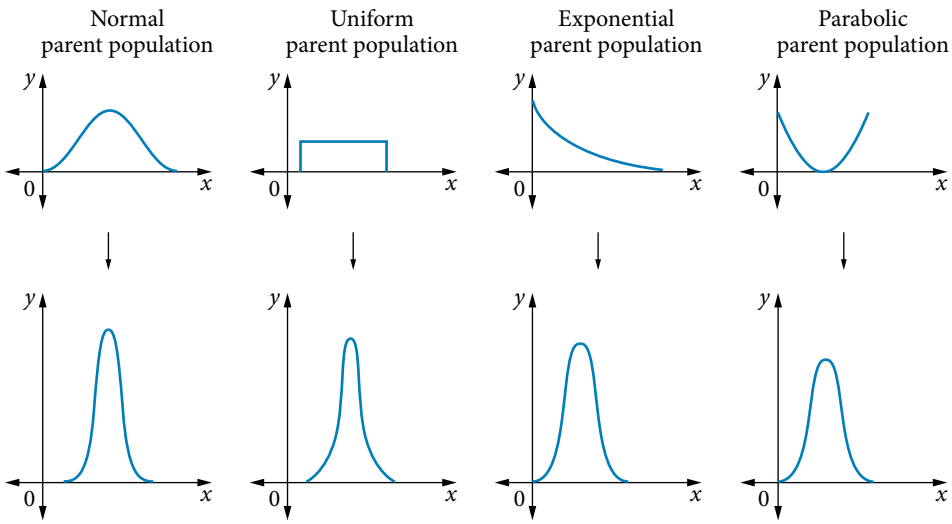
Explore further

Sampling distribution of a uniform population

Use technology to graph an increasing number of sample means and observe trends as the sample size increases.

The central limit theorem

In the previous examples you observed how the distribution of sample means resembled a normal distribution when samples were taken from a normal and a uniform parent population. This pattern holds true for all parent populations, regardless of their shape, provided that the sample size n is sufficiently large.



Sampling distributions of $\bar{x}$ for $n = 30$

This result, known as the *central limit theorem*, is important in inferential statistics, as it makes it possible to draw inferences about an unknown population mean, regardless of the shape of the population.

The *central limit theorem* (CLT) states that when the sample size n is sufficiently large, $n \geq 30$, the distribution of the sample mean is approximately normal. For any parent population with mean μ and standard deviation σ, the distribution of the sample mean $\bar{X}$ tends to a normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.

WARNING

If the parent population is normally distributed, then the sampling distribution of the sample mean $\bar{X}$ will also be normally distributed for any sample size n. However, for parent populations with other shapes (such as uniform, exponential or parabolic), the distribution of the sample mean $\bar{X}$ will tend towards a normal shape only as n becomes sufficiently large.

Both the mean μ and the standard deviation $\frac{\sigma}{\sqrt{n}}$ of the sample means are defined using parameters that may be unknown. When this is the case, the exact mean and standard deviation cannot be calculated. However, using the statistics from our sample we can replace μ with $\bar{x}$ and $\frac{\sigma}{\sqrt{n}}$ with $\frac{s}{\sqrt{n}}$.

The distribution of $\bar{X}$ for sample means, when μ and σ are unknown, is defined as: $\bar{X} \sim N\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$, where the expected value $E(\bar{X}) \approx \mu$ and the standard error of the sample means $se(\bar{X}) \approx \frac{s}{\sqrt{n}}$.

Note that the sample standard deviation can be calculated manually as follows: $s = \sqrt{\left[\frac{\sum x^2}{n} - (\bar{x})^2\right] \times \frac{n}{n-1}}$

Technology worked example
Point estimate and standard error of the sample mean

2 Calculating the standard deviation of the sample mean, σ known

A brand of rope is known to have a mean breaking strength of 10 kg with a standard deviation of 0.2 kg. A random sample of 50 ropes are tested for breaking strength. Determine the standard deviation of the sample mean of breaking strength of the rope to 2 decimal places.

THINKING	WORKING
1 Use the formula for the standard deviation of sample means.	$\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$ $= \frac{0.2}{\sqrt{50}}$ $= 0.028\,28\ldots$
2 Write your answer.	$\text{sd}(\bar{X}) = 0.03$ to 2 decimal places.

Applying the central limit theorem requires the value of n to be sufficiently large.

For most distributions, a sample size of $n \geq 30$ will be sufficient for the CLT to apply.

3 Estimating the population mean from the sample mean

A sample of 1000 students was drawn from a population of 2000 students. The students sampled were found to spend, on average, \$13.50 a day on lunch. The standard deviation from this amount is \$3.

(a) Estimate the mean of the population.

THINKING	WORKING
1 Recall the formula for the expected value.	$E(\bar{X}) = \mu$
2 Substitute the given value.	$\bar{X} = \$13.50$
3 Write your answer.	$\mu \approx \$13.50$

(b) Calculate the standard error of the sample means (to 2 decimal places).

1 Recall the formula for the standard error.	$se(\bar{X}) = \frac{s}{\sqrt{n}}$
2 Substitute given values.	$s = 3, n = 1000$ $se(\bar{X}) = \frac{3}{\sqrt{1000}}$ $= 0.0948\ldots$
3 Write your answer.	$se(\bar{X}) = \$0.09$ to 2 decimal places.

(c) Determine the shape of the sampling distribution of the sample mean.

1 Observe the sample size.	$n = 1000$
2 Assess normality.	Is $n \geq 30$?
3 Write your answer.	As $n \geq 30$, the sampling distribution of the sample mean will approximate a normal distribution centred about the mean of \$13.50.

Standardising the sample mean

Any normal distribution can be standardised to have a mean of 0 and a variance of 1. If X is normally distributed with population parameters μ and σ^2, then $Z = \frac{X - \mu}{\sigma}$ is the standardised form of X.

This information can be used to standardise the sampling distribution of the sample mean in cases where the central limit theorem applies. To do this, subtract the mean of $\bar{X}$ (which is μ) and divide by the standard deviation of $\bar{X}$ (which is $\frac{\sigma}{\sqrt{n}}$).

For a population with a mean of μ and standard deviation of σ, the sampling distribution of the sample mean can be standardised to $\frac{\bar{X} - \mu}{(\sigma/\sqrt{n})} \sim N(0,1)$. This standardisation is only appropriate for normally distributed parent populations, and all other parent populations where the sample size is sufficiently large ($n \geq 30$).

EXERCISE 9.2 Sampling distributions

1 Which of the following distributions of sample means is *not* likely to approximate a normal distribution?

A A distribution from a uniform parent population using sample size $n = 5$.

B A distribution from a normal parent population using sample size $n = 5$.

C A distribution from an exponential parent population using sample size $n = 30$.

D A distribution from a logarithmic parent population using sample size $n = 30$.

2 The age of each person from a simple random sample of 8 employees was obtained. The ages were: 25, 32, 26, 40, 50, 54, 22, 23. Calculate the point estimate and standard error of the sample means, correct to 2 decimal places.

Worked Example 2

3 The mean of a population of scores is 130 with a standard deviation of 20 points. A researcher takes a sample of 500 people's scores to estimate the mean score of students in the state. What is the standard deviation of the sample means, correct to 2 decimal places?

Worked Example 3

4 XYZ Corporation runs a general knowledge competition for all of its employees. An analysis of the scores shows they are normally distributed with a mean of 75 and standard deviation of 15. A simple random sample of 25 is taken from the population of 500.

(a) Estimate the mean of the sample.

(b) Calculate the standard deviation of the sample means.

(c) Determine the shape of the sampling distribution of the sample mean.

5 A simple random sample of 100 observations was taken from a population. The sample mean is 80 and the standard deviation 12. What is the value of the standard deviation of the distribution of means?

A 0.12 B 1.2 C 8.00 D 12.0

6 Which of the following selections is a property of the sampling distribution of $\bar{X}$?

A The standard deviation of the sample means is the same as the standard deviation of the parent population.

B The mean of the sampling distribution is the same as the mean of the population.

C $\bar{X}$ always has a normal distribution.

D The sampling distribution of $\bar{X}$ has the same distribution as the parent population.

7 A researcher is collecting data about the weight of a certain population of approximately 50 000 mice in an ecosystem. What sample size should the researcher use to ensure that the standard error of the sampling distribution is no greater than $\frac{1}{10}$ of the value of the population standard deviation?

8 Over 10 000 students took a statistics exam. A free-response section of the exam consisted of a marking scale of 0 to 4. A random sample of 25 student papers provided the following scores:

1 0 1 0 0 0 3 1 1 1 2 0 0 2 1 1 0 2 4 1 0 2 0 3 4.

(a) Calculate the point estimate.

(b) Calculate the standard error of the sample means, to 3 decimal places.

(c) Determine whether the central limit theorem would apply to the sampling distribution of this particular sample.

9 A super welterweight boxing team has four boxers who weigh 68.9 kg, 70.7 kg, 72.5 kg and 74.3 kg. Determine all possible random samples (with replacement) of $n = 2$ and calculate the mean for each of these samples. Using this information, calculate the following:

(a) The mean of the weight of the four boxers.

(b) The standard deviation of the four boxers' weights.

(c) The mean of the sample means.

(d) The standard error of the sample means.

10 According to an ABC Public Service survey of 500 people, it was found that the average yearly salary in Australia is $79 767 with a standard deviation of $9100. This survey was repeated many times with random samples being taken from the same population and the point estimate of each sample recorded. Use this information to answer the following questions.

(a) Is it likely that the point estimate from each survey will always be $79 767?

(b) Could the value of the point estimate be lower than $45 000? What could this be caused by?

(c) Is the point estimate of $79 767 close to the true population mean? Justify your answer.

(d) In what scenario could the point estimate not be representative of the true population mean?

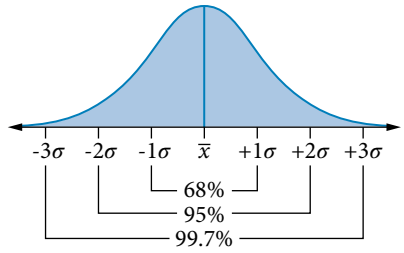
9.3 Confidence intervals

Using the central limit theorem, the distribution of $\overline{X}$ can be approximated by a normal distribution regardless of the type of population distribution. This allows you to estimate the mean and standard deviation of the sample means from the point estimate and standard deviation of the sample even when the population parameters are not known. In this section, the degree of confidence you can have in this estimation will be discussed.

Confidence intervals

A *confidence interval* is a specific interval based around the point estimate $\overline{x}$. When the population parameters are unknown, the true mean *may* be located within this interval. The interval acts as an indicator of the accuracy for the estimate of the population mean and allows you to make a statement of the probability that this interval contains the population mean.

The construction of a confidence interval of a sample mean is similar to that of a sample proportion, and uses probability ranges and their corresponding standard deviations from the mean such as the ones in the following graph.



From the diagram, a 68% range occurs within approximately one standard deviation of the mean, a 95% range occurs within approximately two standard deviations of the mean and a 99.7% range occurs within approximately three standard deviations of the mean. This is known as the 68–95–99.7 rule. Initially, you assess normality by ensuring $n \geq 30$ for the sample. Once this is confirmed, you may calculate the confidence interval.

Consider a medical researcher who is interested in determining the average height of adult females in Australia. The researcher chooses a random sample of 400 women and determines that the mean height from this sample is 166.5 centimetres. The true population mean μ could lie above this sample mean or it could lie below it.

Suppose that the standard deviation of this population, σ, is known to be 10 centimetres. Given that $n \geq 30$, then the distribution of $\overline{X}$ will be normally distributed with a mean of μ and a standard deviation of $\frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{400}} = 0.5$ cm.

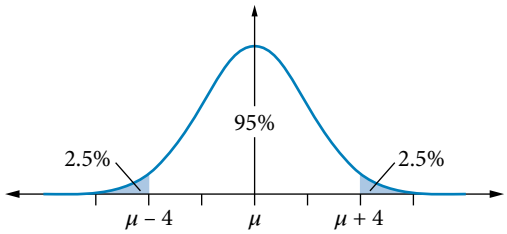
Therefore the approximate 95% confidence interval for the sample is $166.5 \pm (2 \times 0.5)$ centimetres, that is, with repeated samples you would expect the true population parameter to fall between 165.5 – 167.5 centimetres 95% of the time.

The confidence interval can be calculated for every point estimate. If the researcher in this particular study chooses another random sample of 400 women and calculates their mean height to be 164 centimetres, then the confidence interval for this sample would be $164 \pm (2 \times 0.5)$ centimetres, giving a 95% confidence interval of (163, 165) cm.

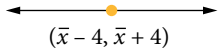
WARNING

In this scenario, the standard deviation of the parent population, σ, is normally an unknown parameter. For the purposes of our approximate methods, the population standard deviation can be replaced by the sample standard deviation, s. For large values of n, the sample standard deviation s is a good estimate of σ.

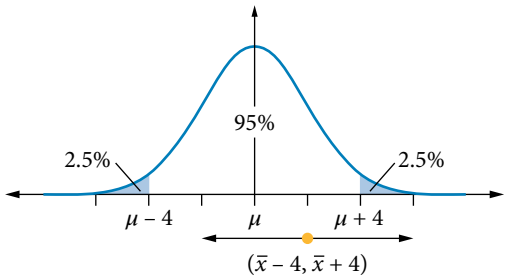
To understand the meaning of this confidence interval, consider once again the study of average adult female height. The distribution of sample means for this study can be shown in the following graph:



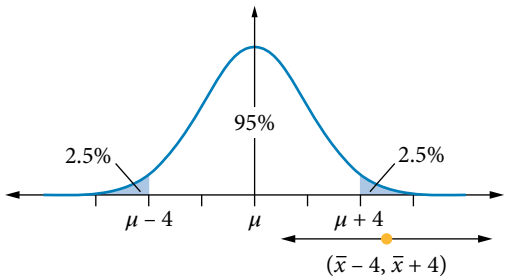
In this diagram, the actual population mean, μ, is unknown. However, 95% of sample means should lie 4 centimetres (two standard deviations) either side of μ. Now consider a point estimate that was calculated in this study. This point estimate can be represented by plotting a dot with arrows on either side to show the confidence interval:



Because it is known that approximately 95% of sample means will lie within 2 standard deviations of the population mean μ, there is a 95% probability that any particular sample mean will lie in this region:



Importantly, note that the confidence interval for the point estimate will *include* the population mean, μ, and that this will occur 95% of the time. However, by chance alone, the point estimate may lie outside the unshaded region of the graph. In other words, the point estimate may lie further than 2 standard deviations from the mean:

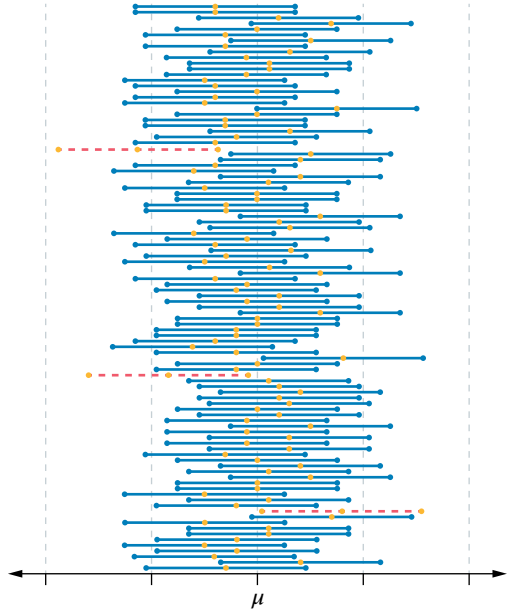


In this case, the confidence interval does *not* include the population mean, μ. While this may be unfortunate for a researcher who is relying on a single sample to estimate the population mean, it is purely a reflection of the inherent uncertainty involved in inferential statistics. If 100 such random samples are taken, with a confidence interval of 95%, you would expect 5 such samples to occur by chance alone.

Confidence levels

A confidence interval requires you to define the *level* or *degree* to which you wish to express the confidence about μ. Selecting the level of confidence is a decision made by the researcher in relation to the context. In application, confidence intervals are most commonly calculated at a 95% level. However, various levels such as 90%, 98% and 99% are also used.

To illustrate what it means to have a confidence level of 95%, consider the following graph of 100 point estimates around a known population mean, μ. Most of the time the confidence intervals for each sample include (overlap) the population mean. However, in this particular group of samples, there are three (indicated as dashed red lines) whose confidence intervals do *not* include the population mean. It is not possible to know with absolute certainty whether the confidence interval for any single point estimate will include the mean, but it is possible to express a certain level of confidence that it will (in this case, the level of confidence is 95%).



A 95% confidence level means that the probability that the confidence interval will capture the true population mean μ in repeated samples is 0.95 or 95%.

4 Calculating an approximate confidence interval using the empirical method

The AAA corporation gives each of its employees an aptitude test. The current year's scores are normally distributed with a mean of 60 and standard deviation of 13. A random sample of 36 employee's results are collected. Calculate a 95% confidence interval, correct to 2 decimal places, for the average range of scores of the aptitude test since testing began.

THINKING	WORKING
1 Define the required statistics.	$\mu = E(\bar{X}) = 60$ $\text{sd}(\bar{X}) = \frac{13}{\sqrt{36}}$ $= 2.17$
2 Define the required interval where 95% lies approximately 2 standard deviations either side of the mean..	$\left(\mu \pm \left(2 \times \text{sd}(\bar{X})\right)\right)$
3 Write your answer.	The confidence interval is (55.66, 64.34).
4 Interpret your answer.	There is a 95% probability that the confidence interval (55.66, 64.34) will capture the true population proportion of the average range of scores.

Confidence intervals and *z* scores

Using the empirical method to calculate confidence intervals works well when you want to determine 68%, 95% and 99.7% intervals. However, should you wish to determine other intervals, you would be required to work with other calculations. To overcome this, you can standardise a distribution and work with *z*-scores rather than standard deviations. Taking this approach will also require the sample mean to be standardised.

For a population with a mean of μ and standard deviation of σ, the sampling distribution of the sample mean can be standardised to $z = \dfrac{\bar{X} - \mu}{\left(\sigma/\sqrt{n}\right)} \sim \text{N}(0,1)$. The sample standard deviation s can be used as an approximation to the population standard deviation σ if the true value of the population standard deviation is not available.

Utilising the standard normal distribution $Z \sim \text{N}(0,1)$, the sample mean has an approximate standardised value of $z = \dfrac{\bar{X} - \mu}{\left(s/\sqrt{n}\right)}$.

Technology can be readily used to both calculate *z*-scores and illustrate corresponding regions under graphs of probability density functions.

Calculating *z*-scores using technology

Technology worked example
Calculating *z*-scores for confidence intervals

To calculate *z*-scores the inverse normal function or equivalent is used.

For a given probability, mean and standard deviation, the inverse normal function calculates the *z*-score for which the area to the left of this score is equal to the probability.

The above calculations give an interval centred around μ, which leads to redefining the 95% confidence interval.

$$P\left(-1.96 < \frac{\bar{X} - \mu}{\left(s/\sqrt{n}\right)} < 1.96\right) \approx 0.95$$

You can rearrange this expression by multiplying all sides of the inequality by the denominator to obtain the required confidence interval.

$$P\left(-1.96\frac{s}{\sqrt{n}} < \bar{X} - \mu < 1.96\frac{s}{\sqrt{n}}\right) \approx 0.95$$

Then

$$P\left(\bar{X} - 1.96\frac{s}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{s}{\sqrt{n}}\right) \approx 0.95$$

The above expression gives an interval centred around μ.

For any given probability, the confidence interval can be calculated using the formulas shown below, where the appropriate *z*-score is calculated from the standard normal distribution.

The general formula for a confidence interval of a sample mean is:

Sample mean $\pm$ (*z*-score) $\times$ (standard deviation of the sample means)

$\bar{x} \pm z\dfrac{\sigma}{\sqrt{n}}$ when σ is known or

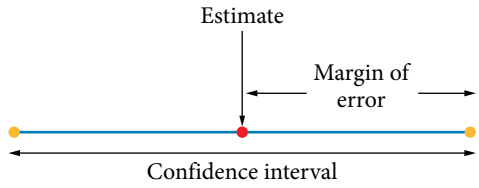
$\bar{x} \pm z\dfrac{s}{\sqrt{n}}$ when σ is unknown.

$\left(\bar{x} + z\dfrac{s}{\sqrt{n}}, \bar{x} - z\dfrac{s}{\sqrt{n}}\right)$

Technology worked example
Calculating confidence intervals

The population standard deviation is normally unknown. However, the sample standard deviation can be used as a reasonable estimate of this parameter.

Margin of error and confidence intervals



For any given study, the statistical researcher will typically decide on a level of confidence. This will then determine the value of the z-score, which in turn varies the length of the interval. Understanding the effect of varying the level of confidence is essential. Recall that a confidence interval is built around the sample proportion. It does not reflect the variability of the unknown population mean, rather, it provides a range of values that may include the unknown population mean. Other measures such as the level of confidence, the standard error and the margin of error all affect the precision of the results.

Begin by looking at the margin of error. The margin of error is defined as the distance between the sample mean and the endpoint of the interval. It is sometimes referred to as the *radius* of the confidence interval.

When the confidence level varies, the confidence interval will also vary, which, in turn, increases or decreases the margin of error. The larger the margin of error, the wider the interval.

The margin of error is defined as:
$E = z\dfrac{\sigma}{\sqrt{n}}$ when σ is known, or
$E = z\dfrac{s}{\sqrt{n}}$ when σ is unknown,
where the z-score indicates the level of confidence.

Technology worked example
Margin of error

The margin of error is linked to the size of the chosen confidence interval. It is important to remember that it is a measure of precision. The margin of error indicates how much the sample results potentially differ from those of the actual population within a specific level of confidence.

Recall the previous definition of the confidence interval: $\bar{x} \pm z\dfrac{\sigma}{\sqrt{n}}$ or $\bar{x} \pm z\dfrac{s}{\sqrt{n}}$

This can now be redefined using the margin of error.

A confidence interval can be expressed as $\mu \pm E$,
where $E = z\dfrac{\sigma}{\sqrt{n}}$ when σ is known, or $E = z\dfrac{s}{\sqrt{n}}$
when σ is unknown.

Margin of error and sample size

Understanding the relationship between the level of confidence, the interval and the margin of error is essential. In statistical inference, the idea of precision is linked to the margin of error in the confidence interval.

Selecting a lower confidence interval is one method by which the margin of error can be reduced. However, a smaller confidence interval also means you will be making an incorrect estimate of the population mean more often, which is not ideal. Fortunately, there is another method for reducing the margin of error in the sampling process.

Consider the formula for margin of error: $E = z\frac{\sigma}{\sqrt{n}}$ when σ is known, or $E = z\frac{s}{\sqrt{n}}$ when σ is unknown. By increasing the value of n, the sample size, the denominator in this formula increases and therefore the margin of error becomes smaller. It is therefore possible to choose an appropriate sample size that will allow you to obtain a confidence interval with a specified margin of error. This can be achieved by rearranging the margin of error formulas:

$$E = z\frac{\sigma}{\sqrt{n}}$$
$$\sqrt{n} = z\frac{\sigma}{E}$$
$$n = \left(\frac{z\sigma}{E}\right)^2$$

The sample size required, when defined by a specific confidence level z and margin of error E, is given by:

$n = \left(\frac{z\sigma}{E}\right)^2$ when σ is known

$n = \left(\frac{zs}{E}\right)^2$ when σ is unknown

where E is the required margin of error and z is the z-score relating to the required level of confidence.

5 Using z-scores to calculate a sample size for a given margin of error

A demographer wants to estimate the mean salary for full-time working adults in a particular town. It is known that the population standard deviation of full-time salaries is $22 500. How large a sample should be selected so that the estimate with a 95% confidence level is within $1000 of the population mean?

THINKING	WORKING
1 Summarise the known information.	$\sigma = \$22500$, $E = \$1000$
2 Use technology (invnorm command) to determine the z-score that corresponds to a 95% confidence interval.	The z-score ≈ 1.96.
3 Determine the required sample size for a known population standard deviation.	Required sample size: $n = \left(\frac{z\sigma}{E}\right)^2$ $= \left(\frac{1.96 \times 22500}{1000}\right)^2$ ≈ 1945
4 Interpret your answer.	Therefore, a sample of approximately 1945 will be required.

EXERCISE 9.3 Confidence intervals

Worked Example 4

1 A publishing company wants to sell a new textbook. Research of 25 comparable textbooks shows that the average price is $145. The standard deviation is $35 and the population of textbook prices is normally distributed. Using the empirical method, calculate the following confidence intervals to the nearest whole number.

(a) 68% confidence interval (b) 95% confidence interval (c) 99.7% confidence interval

2 A Government welfare agency wishes to estimate the time it takes to fill out various forms. 200 randomly selected adults were timed as they filled out a particular form. The calculated mean was 12.8 minutes with a standard deviation of 1.7 minutes. Calculate the margin of error, correct to 3 decimal places, for the following confidence intervals.

(a) 90% confidence interval
(b) 95% confidence interval
(c) 98% confidence interval
(d) 99% confidence interval

3 Which of the following does *not* affect the margin of error in a statistical study?

A sample size
B population mean
C confidence interval
D population standard deviation

4 Calculate the margin of error, correct to 3 decimal places, and the point estimate, correct to 2 decimal places, for the following confidence intervals:

(a) (0.554, 0.647)
(b) (0.405, 0.495)
(c) (0.748, 0.792)

5 Holding all other factors constant, what effect does doubling the sample size have on the margin of error of a study?

A halves the margin of error
B doubles the margin of error
C decreases the margin of error by approximately 30%
D increases the margin of error by approximately 30%

Worked Example 5

6 The finance department of Pacific University wants to estimate the mean debt of this year's graduating students. It is known that the population standard deviation of the debt is $11 800. How large a sample should be selected so that the estimate with a 99% confidence level is within $800 of the population mean?

7 A company manufactures batteries with a lifespan that is normally distributed with a standard deviation of 10.3 hours. If a sample of 40 batteries has an average life of 247 hours, calculate the following confidence intervals using z-score values, correct to 2 decimal places, for the population mean of all batteries produced by the company.

(a) 90% confidence interval
(b) 95% confidence interval
(c) 98% confidence interval
(d) 99% confidence interval

8 The following data gives the speeds (in km/h as measured by radar) of 30 cars travelling on the Pacific Motorway.

98	110	115	102	116
124	112	109	101	100
99	104	115	117	107
97	103	113	105	106
104	112	105	100	99
110	110	119	98	109

Assuming that the speeds of all cars travelling on the Pacific Motorway have a normal distribution when there is no undue congestion, construct the following confidence intervals, correct to 2 decimal places, for the mean speed of all cars travelling on the Pacific Motorway.

(a) 90% confidence interval
(b) 95% confidence interval
(c) 98% confidence interval
(d) 99% confidence interval

9 A random sample of 35 observations was collected and is shown in the table.

42	51	43	31	28	36	49
29	46	37	32	27	33	41
47	41	28	46	34	39	48
26	35	37	38	46	48	39
29	31	44	41	37	38	46

The population data is normally distributed. Calculate the margin of error, correct to 3 decimal places, for the following confidence intervals.

(a) 90% confidence interval

(b) 95% confidence interval

(c) 98% confidence interval

(d) 99% confidence interval

10 A survey of 500 Brisbane secondary students reports that the average mark in their Specialist Mathematics exam was 58% with a standard deviation of 8%.

(a) Calculate, correct to 2 decimal places, the standard deviation of the sample means.

(b) Assess normality and determine if a confidence interval can be constructed.

(c) Calculate the margin of error at the 98% confidence level.

(d) If possible, construct a 98% confidence interval, correct to 2 decimal places, on the true mean of Specialist Mathematics results.

11 A certain study is conducted using a confidence interval of 99%. Assuming the sample size is kept constant and the same parent population is being drawn from, calculate the confidence interval required in order to halve the margin of error (correct to 2 decimal places).

12 A navy would like to record the average time a diver needs to spend underwater fixing the hull of a ship. Real-life dives are difficult to assess so divers use a tank and a simulated repair exercise. The population standard deviation is 20 minutes. A margin of error of 8 minutes on a 95% confidence interval is satisfactory; however, a margin of error of 5 minutes is more desirable. If the navy were to use a margin of error of 5 minutes rather than 8 minutes, how many more dives would be required?

13 At a bottling plant, a machine is set to fill bottles with a capacity of 955 mL. The amount varies from bottle to bottle but, when the machine is working correctly, the average amount of liquid is approximately 946.35 mL. The standard deviation of the amount of liquid in all bottles is 4.4 mL. The quality control department takes a sample of 25 bottles each week, calculates the mean amount of liquid per bottle and produces a 99% confidence interval for the population mean. If the upper limit of this confidence interval is greater than 950.79 mL or the lower limit of the confidence interval is less than 941.92 mL, the machine is stopped and adjusted. A recent sample of 25 bottles produced a mean amount of liquid of 944.58 mL. Based on this sample, decide if the machine needs to be stopped and adjusted. Assume that the amounts of liquid are normally distributed.

14 Consider again the study conducted in question **10**—that is, a survey of 500 Brisbane secondary students reporting that the average mark in their Specialist Mathematics exam was 58% with a standard deviation of 8%. Suppose you wish to refine the estimate of the population mean and decide to cut the width of the interval in half. From this new information, answer the following questions.

(a) Will doubling the sample size cut the width of the interval in half? Justify your answer.

(b) What size sample is needed to cut the interval width in half?

9

Summary

Inferential statistics

Inferential statistics allow you to infer features of a population by analysing the characteristics of samples statistics, when one or more samples are drawn from that population.

Mean and variance of the sample mean

If a random sample size n on X has an expected value $E(X) = \mu$, then the expected value of the sample mean is equal to the expected value of the population mean. In other words, $E(\bar{X}) = \mu$.

A random sample of size n on X where $\text{var}(X) = \sigma^2$ has $\text{var}(\bar{X}) = \dfrac{\sigma^2}{n}$ and standard deviation $\text{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

Sampling distributions

The *central limit theorem* (CLT) states that when the sample size n is sufficiently large, the distribution of the sample means is approximately normal. For any parent population with mean μ and standard deviation σ, the distribution of the sample mean $\bar{X}$ tends to a normal distribution with mean μ and standard deviation $\dfrac{\sigma}{\sqrt{n}}$.

If the parent population is normally distributed, then the distribution of the sample mean $\bar{X}$ will also be normally distributed for any sample size n. However, even for parent populations with other shapes (such as uniform, exponential or parabolic), then the distribution of the sample mean $\bar{X}$ will still tend towards a normal shape provide that only as n becomes sufficiently large. (For most distributions, a sample size of $n \geq 30$ will be sufficient for the CLT to apply.)

The distribution of $\bar{X}$ for sample means when μ and σ is unknown is $\bar{X} \sim N\left(\bar{x}, \dfrac{s}{\sqrt{n}}\right)$, where the expected value $E(\bar{X}) \approx \mu$ and the standard error of the sample means $se(\bar{X}) \approx \dfrac{s}{\sqrt{n}}$.

Confidence intervals

A *confidence interval* is a specific interval based around the point estimate $\bar{x}$. When the population parameters are unknown, the true mean *may* be located within this interval. The interval acts as an indicator of the accuracy for the estimate of the population mean and allows you to make a statement of the probability that this interval contains the population mean.

A confidence interval requires you to define the *level* or *degree* to which you wish to express your confidence about μ. Selecting the level of confidence is a decision made by the researcher in relation to the context. In application, confidence intervals are most commonly calculated at a 95% level. However, various levels such as 90%, 98% and 99% are also used.

A 95% confidence level means that the probability that the confidence interval will capture the true population mean μ in repeated samples is 0.95 or 95%.

Utilising the standard normal distribution $Z \sim \mathrm{N}(0,1)$, the sample mean has an approximate standardised value of $\frac{\overline{X} - \mu}{\left(s/\sqrt{n}\right)}$.

The general formula for a confidence interval of a sample mean:

Sample mean ± (z-score) × (standard deviation of the sample means)

$\overline{x} \pm z\frac{\sigma}{\sqrt{n}}$ when σ is known, or

$\overline{x} \pm z\frac{s}{\sqrt{n}}$ when σ is unknown.

$\left(\overline{x} + z\frac{s}{\sqrt{n}}, \overline{x} - z\frac{s}{\sqrt{n}}\right)$

Margin of error:

$E = z\frac{\sigma}{\sqrt{n}}$ when σ is known, or

$E = z\frac{s}{\sqrt{n}}$ when σ is unknown

where the z-score indicates the level of confidence.

Confidence interval:

$\mu \pm E$, where $E = z\frac{\sigma}{\sqrt{n}}$ when σ is known, or $E = z\frac{s}{\sqrt{n}}$ when σ is unknown.

The sample size required when defined by a specific confidence level z and margin of error E:

$n = \left(\frac{z\sigma}{E}\right)^2$ when σ is known

$n = \left(\frac{zs}{E}\right)^2$ when σ is unknown

where E is the required margin of error and z is the z-score relating to the required level of confidence.

Chapter review 9

Exercise 9.1

1 Determine whether the following statements are true or false and explain your reasoning.

(a) A statistic obtained from a simple random sample is a random variable.

(b) The sample mean of a single sample is known as the point estimate.

(c) A parameter will vary from sample to sample.

(d) The standard deviation of sample means is the same calculation as the standard deviation for a sample.

Exercise 9.1

2 Calculate the standard deviation of the sample means for the following information. Answer correct to 3 decimal places.

(a) $\sigma = 2.34$ and $n = 28$ (b) $\sigma = 54.98$ and $n = 54$ (c) $\sigma = 0.02$ and $n = 357$

Exercise 9.1

3 Calculate the point estimate to 2 decimal places and standard deviation correct to 3 decimal places of the following sets of data.

(a) 423.7, 211.45, 554.03, 400.01, 333.33 (b) $\ln(2)$, e^2, $2^{-3}\frac{\pi}{2}$, 1

Exercise 9.2

4 A sample of 95 observations is taken from a population. Determine the likely shape of the sampling distribution of $\overline{X}$.

Exercise 9.2

5 The standard blood sugar levels in an adult should read between 4 and 8. A random sample of 1345 adults showed an average reading of 6.3 with a standard deviation of 1.4. Calculate the standard error of the sample means, correct to 3 decimal places.

Exercise 9.3

6 A random sample of 100 visitors to a theme park in Queensland spent, on average, \$142 on the day with a standard deviation of \$47.50. Using the empirical method, calculate a 98% confidence interval for the mean amount of money spent by all visitors.

Exercise 9.3

7 A researcher surveys 250 people who have been arrested for driving under the influence of alcohol. The average blood alcohol reading is 0.073% with a standard deviation of 0.13%. Calculate the margin of error, as a percentage, for a 90% confidence interval, correct to 3 decimal places.

Exercise 9.3

8 A technology developer is preparing her results for the board. She has produced a new mobile phone battery that, at a 95% confidence level, has a life of (4.3 ± 1.2) days. Which of the following conditions could she use to adjust her findings so that she may obtain a smaller margin of error?

A Increase the confidence level to 100% as this will produce no margin of error

B Decrease the confidence level to 90%

C Increase the confidence level to 99%

D Take a new sample from the population using the exact same size

Exercise 9.3

9 A survey was conducted and the results were given as an average of 75% with a margin of error of 4%. Calculate the confidence interval for this survey.

10 An orthopaedic ward at a children's hospital recorded the ages (in years) of the children on their ward and obtained the results shown. Exercise 9.2

6	1	8	16	12	4
2	17	1	2	8	5
4	12	1	13	7	9
4	16	9	15	2	15

Calculate the point estimate, correct to 1 decimal place, and the standard error of the sample means, correct to 1 decimal place. (Since ages are always rounded down, you need to add 0.5 years to the point estimate to get the true mean age).

11 A local gym wishes to estimate the average amount of time (in hours) that members spend each week in the centre. A random sample of 53 members shows an average of 2.5 hours with a standard deviation of 1.3 hours. Calculate, correct to 3 decimal places, the following confidence intervals for the mean amount of time that all members spend in this particular gym. Exercise 9.3

(a) 90% confidence interval
(b) 95% confidence interval
(c) 98% confidence interval
(d) 99% confidence interval

12 Calculate the sample size required if a 90% confidence interval is to be created for the average diameter of washers produced by a company with a margin of error of no more than 0.02 mm. Assume that the standard deviation is 0.035 mm. Exercise 9.3

13 A random sample has a mean of 93.24 and a standard deviation of 13.4. Assuming the population is normally distributed, calculate a 99% confidence interval and margin of error for the following sample sizes: Exercise 9.3

(a) $n = 25$ **(b)** $n = 1000$ **(c)** $n = 10\,000$

14 A survey was conducted to research the link between students who are studying and working at the same time. To begin with, researchers interviewed 200 students and found that the average time spent at work (per week) is 7.1 hours. From previous studies, a standard deviation of 5 hours has been obtained. Exercise 9.3

(a) Calculate a 95% confidence interval based on the sample of 200.
(b) Explain how to interpret this confidence interval.
(c) Why isn't a 100% confidence interval used?
(d) Researchers are not satisfied with their confidence interval and would like to conduct another survey to determine a shorter confidence interval. What should be changed in order for them to be successful?

15 Harrison works for a political polling company that wishes to conduct a survey on voting intention in a particular electorate. His initial survey involves 40 participants and produces a margin of error of $E = 2.6\%$ for the primary survey question. The company policy is that all such polls must have a maximum margin of error of $E = 2\%$, and Harrison is instructed to adjust the sample size to conform to the company policy. Determine what sample size Harrison must use for this poll. Exercise 9.3

Exam review

UNIT 4

Paper 1: Technology-free

Exercise 5.1

1 Determine the domain and range of the function $y = \arctan(x+1)$ and sketch its graph.

Exercise 5.2

2 Determine the derivative of each of the following:

(a) $y = \arctan(5x)$

(b) $y = x\arccos\left(\frac{1}{x}\right)$

Exercise 5.3

3 Determine the following indefinite integrals:

(a) $\int \frac{3}{\sqrt{2-9x^2}}\,dx$

(b) $\int \frac{1}{10+x^2}\,dx$

Exercise 5.4

4 Determine the following:

(a) $\int \frac{x^2}{5+\left(x^3+4\right)^2}\,dx$

(b) $\int_3^4 x\sqrt{x^2-8}\,dx$

Exercise 6.3

5 Use Simpson's rule to approximate $\int_1^2 \ln(x)\,dx$ using $n = 4$ subintervals. Express your answer correct to 3 decimal places. Evaluate the reasonableness of your result.

Exercise 7.1

6 Verify that the given relation is a solution of the corresponding differential equations.

(a) $y = \sin^{-1}(x) + 7;\ \sqrt{1-x^2}\,\frac{dy}{dx} = 1$

(b) $y = bx^m;\ x\frac{dy}{dx} = my$

(c) $y^2 - 4x^2 = 3;\ \frac{dy}{dx} = \frac{4x}{y}$

(d) $y = \ln\left|\tan(x)\right|;\ \frac{dy}{dx} - 2\text{cosec}(2x) = 0$

Exercise 7.3

7 Determine the solution of the following differential equations.

(a) $\frac{dy}{dt} = -y$

(b) $\frac{dy}{dx} = \cos^2(x)$ and when $x = 0$, $y = 5$

(c) $x^2\frac{dy}{dx} = y^2 + 1$

(d) $\frac{dy}{dx} = xe^{-y}$ and when $x = 0$, $y = 1$

Exercise 8.2

8 A cyclist accelerates from rest for 8 seconds to a speed of 10 m/s, then maintains that speed. A second cyclist is travelling at a constant speed of 7.5 m/s when she passes the first cyclist who is just starting.

(a) Draw a velocity–time graph to represent the motion of the two cyclists.

(b) Which of the two cyclists is ahead after 30 seconds?

Exercise 8.4

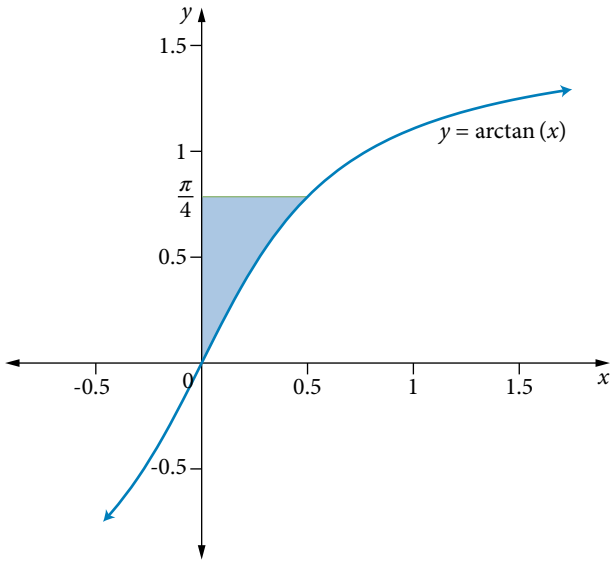
9 The displacement x of a particle moving with simple harmonic motion at time t is modelled by the equation $x = 2\sin\left(\frac{3t}{\pi}\right)$.

(a) Determine the expression for the acceleration $\ddot{x}$ of the particle, in terms of time and in terms of position.

(b) Determine the period of the motion.

10 Prove that $\int_0^a \arctan\left(\frac{x}{a}\right) dx = \frac{a}{2}\left(\frac{\pi}{2} - \ln(2)\right)$. Exercise 5.7

11 Determine the exact area of the shaded region shown. Exercise 6.1



12 A particular brand of battery lasts 2 years on average before failing. Exercise 6.4

(a) Determine the exact probability that such a battery will last at least 3 years.

(b) Determine the greatest value of d for which the manufacturer can truthfully claim that 90% of batteries will last beyond d days. Express your answer as a whole number of days.

13 Determine an expression in terms of x and y only for $\frac{d^2y}{dx^2}$ if $x^3 - y^2 = 7$. Exercise 7.3

14 A body of mass 3 kg moves in a straight line with velocity v m/s given by $v = \frac{20}{(3+t)^2}$ at time t seconds. Determine the following: Exercise 8.4

(a) the initial acceleration of the body

(b) the resultant force acting on the body when its velocity is $\frac{5}{4}$ m/s

(c) the position of the body after 3 seconds given that the body was initially located at the origin

15 By making use of the identity $\sin(2A) = 2\sin(A)\cos(A)$ and other relevant identities, determine $\int \sin(x)\sin^2(2x)\cos^3(x)\,dx$. Exercise 5.5

16 The velocity v of a particle at position x is modelled by $v = \sqrt{18x - 2x^2}$. Determine the acceleration a when $x = 8$. Exercise 8.2

17 The acceleration a of a particle with velocity v at position x is modelled by $a = 2\sqrt{1+v^2}$. Determine an expression for velocity in terms of position if $v = 0$ when $x = 0$. Exercise 8.3

Paper 2: Technology-active

Exercise 6.2

1 Determine the exact volume of the solid of revolution when the region of the curve $y = x^3$ from $x = 1$ to $x = 2$ is rotated about the x-axis.

Exercise 7.2

2 Determine an expression for $\frac{dy}{dx}$ for the following relations:

(a) $\sin(2x) + xy^3 = 14$ **(b)** $(x - y)^2 = 3, \ y \neq x$ **(c)** $3x + \log_e(y^2) = 0, \ y \neq 0$

Exercise 8.1

3 A hiker walks 5 km due north in 30 minutes, then walks due east for 12 km in an hour. Determine the average speed and the average velocity of the hiker in km/h, correct to 1 decimal place.

Exercise 9.1

4 The mean of a population of scores is 450 with a standard deviation of 40 points. A researcher takes a sample of 200 people's scores to estimate the mean score of students in the state. What is the standard deviation of the sample means, correct to 2 decimal places?

Exercise 9.3

5 Calculate the margin of error, correct to 2 decimal places, and the point estimate, correct to 2 decimal places, for the following confidence intervals:

(a) $(1.78, 2.42)$ **(b)** $(10.58, 14.12)$

Exercise 6.2

6 Determine the volume, correct to 2 decimal places, generated by rotating the region bounded by $y = e^x$ and $y = \ln(x) + 3$ about the y-axis.

Exercise 7.2

7 A spherical hot air balloon develops a leak so that t minutes later the radius r metres of the balloon is given by $r = 8 - 0.03t^2$. Assuming that the balloon remains spherical, determine the rate at which the balloon is losing volume 3 minutes after the leak started.

Exercise 7.4

8 Sketch the slope field of the differential equation $\frac{dy}{dx} = y - 3$.

Exercise 8.4

9 A box of mass 12 kg can be pulled at constant speed up a slope angled at 30° to the horizontal by a force of 100 N acting parallel to the slope.

(a) Determine the coefficient of friction between the box and the slope correct to 3 decimal places.

(b) If the force pulling on the box is removed, determine the acceleration with which the box slides down the slope.

Exercise 9.1

10 A machine fills potato chip packets to a target mass of $\mu = 250$ g and $\sigma = 4$ g. The machine is checked every 8 hours over the period of one working week (5 days). Calculate the expected mean mass (to 1 decimal place) and standard deviation of the sample means (to 2 decimal places) after a week's worth of testing.

11 A researcher is investigating starting salaries of law graduates. It is known that the population standard deviation of starting law salaries is $14 000. How large a sample should be selected so that the estimate with a 99% confidence level is within $500 of the population mean? Exercise 9.3

12 A tank is filled with 80 litres of a salt solution containing 80 kg of salt. A fresh salt solution containing 4 kg of salt per litre is poured into the tank at a rate of 6 litres per minute and the mixture runs out of the tank at the same rate. The solution is stirred continuously. Exercise 7.5

Construct a differential equation describing the rate of change of the quantity of salt Q kg in the tank with respect to time t minutes.

Exam review

UNITS 3 & 4

Paper 1: Technology-free

Exercise 1.1

1 Prove that $n\left(n^2+5\right)$ is divisible by 6 for $n=5$, $n=2$ and $n=1$.

Exercise 2.2

2 Given that $\boldsymbol{a}=\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$, $\boldsymbol{b}=2\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=5\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$, calculate:

(a) $\boldsymbol{a}+\boldsymbol{b}$ (b) $2\boldsymbol{a}+3\boldsymbol{c}$

Exercise 2.2

3 If $\boldsymbol{a}=3\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=5\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}$, calculate $\left|-3\boldsymbol{a}+4\boldsymbol{b}\right|$

Exercise 2.2

4 Points A and B have position vectors $-2\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}$ and $3\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$ respectively. Calculate the position vector of A relative to B.

Exercise 2.6

5 Let $\boldsymbol{a}=2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}$, $\boldsymbol{b}=\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$ and $\boldsymbol{c}=-2\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$. Calculate each of the following:

(a) $\boldsymbol{a}\times\boldsymbol{b}$ (b) $\boldsymbol{a}\times\boldsymbol{c}$

Exercise 3.1

6 The position of a body at time t is given by $\boldsymbol{r}(t)=(t+1)^2\,\hat{\boldsymbol{i}}+(t-3)\,\hat{\boldsymbol{j}}$, $t\geq 0$. Determine the Cartesian equation of the body's path.

Exercise 5.1

7 Determine the exact value of $\sin\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)$.

Exercise 5.1

8 Determine the domain and range of the function $y=-3\arcsin\left(\frac{x}{2}\right)$ and sketch its graph.

Exercise 5.3

9 Determine the following indefinite integrals:

(a) $\int\frac{-1}{\sqrt{5-x^2}}\,dx$ (b) $\int\frac{1}{2+5x^2}\,dx$

Exercise 5.4

10 Determine the following anti-derivatives:

(a) $\int\frac{\cos(x)}{\sin(x)}\,dx$ (b) $\int\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Exercise 5.4

11 Determine the following:

(a) $\int\frac{1}{\sqrt{x}\sqrt{4-\left(\sqrt{x}+1\right)^2}}\,dx$ (b) $\int_0^3 xe^{x^2}\,dx$ (c) $\int_0^{\frac{\pi}{3}}\frac{\sin(x)-\cos(x)}{\left(\sin(x)+\cos(x)\right)^2}\,dx$

Exercise 5.5

12 Determine the following indefinite integrals:

(a) $\int\cos(x)+\sin^2(2x)\,dx$ (b) $\int 2-\tan^2(3x)\,dx$

Exercise 5.6

13 Demonstrate that $\int_3^4\frac{3x-1}{(x-2)(x+3)}\,dx=\ln\left(\frac{7}{3}\right)$.

Exercise 5.6

14 Demonstrate that $\int_0^1\frac{x+5}{(x+4)^2}\,dx=\ln\left(\frac{5}{4}\right)+\frac{1}{20}$.

Exercise 2.7

15 Determine the Cartesian equation and sketch the path of a body whose position vector at time t, $t\geq 0$, is given by $\boldsymbol{r}(t)=(t+2)\hat{\boldsymbol{i}}+\left(t^2-2t\right)\hat{\boldsymbol{j}}$.

Exercise 8.1

16 Determine the vector equation of the velocity of a particle that has parametric position equations given by $\boldsymbol{x}=m\sin(t)$, $\boldsymbol{y}=m\cos(t)$ and $\boldsymbol{z}=nt^2$ for $t\geq 0$.

17 Simplify the following expressions. Exercise 4.2

(a) $\dfrac{\text{cis}(12\theta)^2}{\text{cis}(\theta)^5\,\text{cis}(4\theta)^3}$ **(b)** $\dfrac{(1+8i)^2}{2+i}$

18 Determine the three cube roots of 64, leaving your answers in polar form. Exercise 4.3

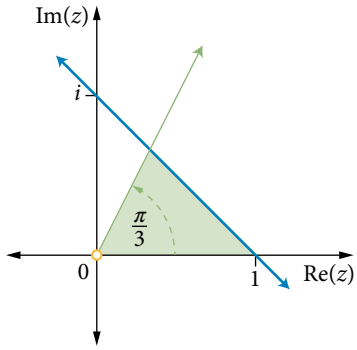
19 Factorise the following polynomials: Exercise 4.6

(a) z^2+27 over $\mathbb{C}$ **(b)** $z^2+16z+100$ **(c)** $p(x)=x^3-21x-20$ over $\mathbb{R}$

(d) $z^3-2z^2+16z-32$ over $\mathbb{C}$, given that $z-4i$ is a factor

20 Solve the equation $(z+2i)^3=1$. Exercise 4.6

21 Construct an expression in terms of z that defines the subset of the complex plane shown. Exercise 4.4



22 If $(1+i)^n=x+yi$, prove that, where $n\in\mathbb{Z}$ and x, y are real, $x^2+y^2=2^n$. Exercise 4.1

23 Determine the value of the gradient at the point $(0,2)$ for the relation $e^x y+3xy^2+8=5y$. Exercise 7.5

24 **(a)** Verify by differentiation that $y=2e^{-x}$ is a solution of the differential equation $\dfrac{dy}{dx}=-y$. Exercise 7.3

(b) Determine the second derivative of the relation $x^2+y^2=-1$ in terms of x and y only.

25 Determine the velocity v and acceleration a for the particles with position x modelled by the following functions: Exercise 8.1

(a) $x=e^{-2t}$ **(b)** $x=\sqrt{3-t}$ **(c)** $x=4\sin(\pi t)+3\cos(\pi t)$

26 Determine the final velocity v of these particles moving with uniform acceleration a at time t with position x and initial velocity u: Exercise 8.1

(a) $u=3,\ a=-4,\ t=9$ **(b)** $x=3,\ u=1,\ t=5$

27 Use mathematical induction to prove that $3^{2n-1}+2^{n+1}$ is divisible by 7 for all $n\in\mathbb{Z}^+$. Exercise 1.3

28 Solve the following set of linear equations by using Gaussian elimination: Exercise 2.9

$3x-y=10$

$x+4y=-1$

29 By making use of the identity $\sin(2A)=2\sin(A)\cos(A)$ and other relevant identities, determine $\int_0^{\frac{\pi}{12}} 32\sin^2(x)\cos^2(x)\,dx$. Exercise 5.5

30 Consider an exponentially distributed random variable with rate λ. Prove that $t_{0.25}=\dfrac{2\ln(2)}{\lambda}$. Exercise 6.4

31 Sketch and describe the region of the complex plane defined by $|z-2|=|z+i|$. Exercise 4.4

32 Determine the equation of the normal to the function $y=\sin^{-1}\left(-\dfrac{\sqrt{x}}{2}\right)$ at $x=3$. Exercise 7.5

Exercise 7.5

33 Determine the solution of each of these differential equations:

(a) $\frac{dy}{dx} = 2x(y-11),\ y > 11$

(b) $\frac{dy}{dx} = -y\sin(x),\ y\left(\frac{\pi}{2}\right) = 3$

(c) $\frac{dy}{dx} = (3-y)(3+y)$, given that $y(0) = y_0$

Exercise 5.7

34 Prove that $\int \sin^n(x)dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$.

Exercise 5.7

35 Determine $\int 2x\ln(x+1)dx$.

Exercise 6.1

36 Determine the value of the positive real number a if the area bounded by the functions $y = ax^2$ and $y = a\sqrt{x}$ is equal to 1 square unit.

Paper 2: Technology-active

Exercise 6.3

1 Use Simpson's rule with $n = 4$ subintervals to determine an approximation for the area between the x-axis and the function $f(x) = \arcsin(x)$ from $x = 0$ to $x = 1$. Evaluate the reasonableness of your approximation.

Exercise 6.3

2 Use Simpson's rule to approximate $\int_0^3 3^x\,dx$ using $n = 6$ subintervals. Express your answer correct to 3 decimal places.

Exercise 6.4

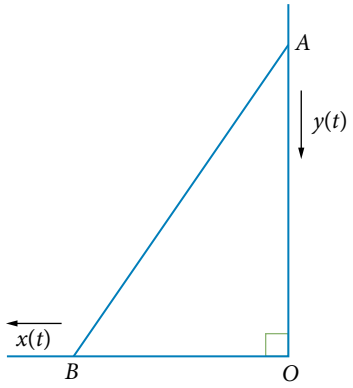
3 Let X represent the time before the next error that a typist makes. On average, the typist makes an error every 20 seconds.

(a) Determine the mean μ, the rate λ and the probability density function for X.

(b) Determine the probability, correct to 3 decimal places, that the typist will not make a mistake in the next minute.

Exercise 7.3

4 A 4 m long ladder rests against a wall at point A that is y m above the ground, with the foot of the ladder B at x m from the wall. At time $t = 0$ seconds, $x = 1$ m and the foot of the ladder begins sliding along the ground away from the wall at a constant rate of 1.2 m/s. Determine the rate at which the top end of the ladder at point A starts sliding down the wall. Give your answer correct to 2 decimal places.



Exercise 8.3

5 A box of mass 12 kg is on a horizontal table and is being pulled along by a force of 4 N acting at an angle of 6° to the table. Determine the normal reaction of the table on the box correct to 1 decimal place.

Exercise 9.3

6 A random sample of 64 visitors to a certain restaurant spent, on average, \$65 per person for their meal, with a standard deviation of \$15. Using the empirical method, calculate a 98% confidence interval for the mean amount of money spent by all visitors.

7 Given vectors $\boldsymbol{a} = -5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$: Exercise 2.3

(a) determine the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$. **(b)** determine the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

8 Calculate $\dfrac{d\boldsymbol{r}}{dt}$ and $\dfrac{d^2\boldsymbol{r}}{dt^2}$ for $\boldsymbol{r} = -5t^2\hat{\boldsymbol{i}} + 50t\hat{\boldsymbol{j}} - \sqrt{t}\hat{\boldsymbol{k}}$. Exercise 3.2

9 Determine the equation of the plane containing the set of points Exercise 2.6
$A = (1,0,3), B = (0,1,-1), C = (-2,1,1)$.

10 Calculate the point of intersection of the lines L_1 and L_2 whose vector equations are Exercise 2.7
$\boldsymbol{r}_1 = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \lambda\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$ and $\boldsymbol{r}_2 = -6\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \mu\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)$ respectively.

11 Solve the following set of simultaneous equations using Gaussian elimination: Exercise 2.9

$$\begin{aligned} x + 2z &= 4 \\ 2x - y - z &= -8 \\ x + 3y + z &= 4 \end{aligned}$$

12 Calculate the angle, correct to the nearest degree, between the following pair of vectors: Exercise 3.2
$\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$.

13 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = 8\sin(2t)\hat{\boldsymbol{i}} + 8\cos(2t)\hat{\boldsymbol{k}}$, $t \geq 0$. Exercise 3.3
Determine an expression for the position vector at time t if initially $\dot{\boldsymbol{r}}(0) = 4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{r} = -2\hat{\boldsymbol{k}}$.

14 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}} = e^t\hat{\boldsymbol{i}} + 4e^{-2t}\hat{\boldsymbol{j}}$, $t \geq 0$. Sketch the path of the Exercise 3.3
particle if $\dot{\boldsymbol{r}} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\boldsymbol{r} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ when $t = 0$.

15 Consider a slope field that has all its line segments with a slope of -2 all over the Cartesian plane. Exercise 7.4

(a) Write the differential equation that this slope field represents.

(b) Write the equation of the family of curves that is the general solution of this differential equation.

(c) Construct a set of isoclines for the differential equation

$$\begin{aligned} \frac{dy}{dx} &= -\frac{2x}{y} \\ &= c \end{aligned}$$

for the following values: $c = 0, \pm 1, \pm 2$, undefined.

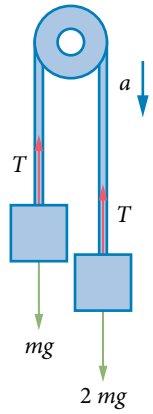
(d) Describe a curve that could be a solution to the differential equation in part **(c)**.

16 A ball of mass 0.5 kg is dropped from a height of 10 m and when it hits the ground it bounces back up with a velocity of 7 m/s. Exercise 3.1

(a) Determine the change in momentum of the ball as it bounces.

(b) Determine the maximum height the ball reaches after it has bounced.

17 Two masses of m kg and $2m$ kg are joined by a light string over a smooth pulley. Exercise 8.3
The system is accelerating in a clockwise direction as shown in the diagram.
Determine the acceleration and the tension in the string in terms of g and m.



Exercise 9.3

18 Calculate the sample size required if a 95% confidence interval is to be created for the average weight of baked beans in a tin with a margin of error of no more than 2 g. Assume that the standard deviation is 3 g.

Exercise 2.6

19 Determine a vector perpendicular to the plane $4x + y - 2z = 4$.

Exercise 2.6

20 Determine a vector perpendicular to the plane $2x + y + 3z = 18$.

Exercise 4.3

21 The position vector, $\boldsymbol{r}(t)$, of a particle at time t is given by $\dot{\boldsymbol{r}}(t) = 4t\hat{\boldsymbol{i}} + 2t^3\hat{\boldsymbol{j}} + \left(e^t - 1\right)\hat{\boldsymbol{k}}$.

(a) Determine an expression for the particle's velocity at time t.

(b) Calculate the exact initial speed of the particle.

(c) Resolve the acceleration vector at $t = 1$ into its exact tangential component.

Exercise 6.2

22 A vase is to be constructed by rotating the section of the function of the form $f(x) = a\sqrt{x}$ from $x = 0$ to $x = h$ about the x-axis. The vase must have the property that the sum of the height and the radius of the circular opening is equal to 30 cm. Determine the maximum possible volume of such a vase correct to the nearest cubic centimetre.

Exercise 2.1

23 A radar facility tracks an aircraft flying with constant speed in a straight line. Relative to the radar facility, the aircraft's initial and final position vectors are $5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$ and $8\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 13\hat{\boldsymbol{k}}$ respectively. Units are in kilometres.

(a) Determine the aircraft's position when it is closest to the radar facility.

(b) Calculate the distance from the facility at this point. Give your answer correct to 2 decimal places.

Exercise 7.5

24 A holding tank at a desalination plant contains 400 cubic metres of brine solution, 30 tonnes of which is salt. At time $t = 0$ minutes, the plant engineer turns on an inflow of sea water flowing at 10 cubic metres per minute, with a salt concentration of 40 kilogram per cubic metre. The well-mixed solution also flows out of the holding tank to a storage pond at the same rate of 10 cubic metres per minute.

(a) Construct a differential equation to model the rate of change of mass m kg of salt in the tank after t minutes.

(b) Determine an expression for m in terms of t.

(c) Determine the amount of salt in the tank after a very long time.

Exercise 8.1

25 A particle moves so that its acceleration a is modelled by $a = -10v$, where v is its velocity at time t. It starts from the origin with an initial velocity of 2 m/s.

(a) Determine an expression for the velocity in terms of the position x.

(b) Determine an expression for the velocity and position in terms of time.

Answers

Chapter 1: Induction and proof

RECALL

1 (a) $2(k+1)(2k-3)^2$ (b) $(k+5)(2k+1)$
(c) $2k^2(3k-1)(k+2)$

2 (a) $\frac{7k+2}{6}$ (b) $\frac{3}{k+1}$ (c) $\frac{5}{k+2}$

3 (a) a^{k+1} (b) 17×3^k (c) 48×7^k

4 (a) $(k+2)(k+1)$ (b) $k!k$ (c) $\frac{1}{(k+1)!}$

5 $$\begin{aligned}\text{LHS} &= \frac{1}{k}+\frac{1}{k+1}\\ &= \frac{k+1}{k(k+1)}+\frac{k}{k(k+1)}\\ &= \frac{k+1+k}{k(k+1)}\\ &= \frac{2k+1}{k(k+1)}\\ &= \frac{2k+2-1}{k(k+1)}\\ &= \frac{2(k+1)-1}{k(k+1)}\\ &= \text{RHS}\end{aligned}$$

6 Let o be an odd integer, and e be an even integer.
Then $o = 2s+1$ and $e = 2t$ where $s, t \in \mathbb{Z}$.
$$\begin{aligned}o+e &= 2s+1+2t\\ &= 2(s+t)+1\end{aligned}$$
Since $s+t \in \mathbb{Z}$, it follows that $o+e$ is odd.

EXERCISE 1.1

1 (a) If $n=5$: $1^3+2^3+3^3+4^3+5^3 = 225$
If $n=6$: $1^3+2^3+3^3+4^3+5^3+6^3 = 441$
If $n=2$: $1^3+2^3 = 9$
If $n=1$: $1^3 = 1$

(b) If $n=5$: $4+7+10+13+16 = 50$
If $n=6$: $4+7+10+13+16+19 = 69$
If $n=2$: $4+7 = 11$
If $n=1$: 4

(c) If $n=5$: $1+3+5+7+9+11+13 = 49$
If $n=6$: $1+3+5+7+9+11+13+15 = 64$
If $n=2$: $1+3+5+7 = 16$
If $n=1$: $1+3+5 = 9$

(d) If $n=5$: $-2+4-6+8-10 = -6$
If $n=6$: $-2+4-6+8-10+12 = 6$
If $n=2$: $-2+4 = 2$
If $n=1$: -2

(e) If $n=5$: $2^0+2^1+2^2+2^3+2^4+2^5+2^6+2^7 = 255$
If $n=6$: $2^0+2^1+2^2+2^3+2^4+2^5+2^6+2^7+2^8 = 511$
If $n=2$: $2^0+2^1+2^2+2^3+2^4 = 31$
If $n=1$: $2^0+2^1+2^2+2^3 = 15$

2 (a) $1^3+2^3+3^3+\cdots(n-1)^3+n^3$
(b) $4+7+10+\cdots+(3n-2)+(3n+1)$
(c) $1+3+5+\cdots+(2n+1)+(2n+3)$
(d) $-2+4-6+\cdots+(-1)^{n-1}2(n-1)+(-1)^n 2n$
(e) $2^0+2^1+\cdots+2^{n+1}+2^{n+2}$

3 (a) $\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$ (b) $2+4+6+\cdots+2n$
(c) $4+6+\cdots+2(n+1)$ (d) $3+9+27+\cdots+3^n$
(e) $1+3+9+27+\cdots+3^{n-1}$

4 (a) $1+8+27+64+125$ (b) $1+3+5+7+9$
(c) $-1+4-9+16$ (d) $1+2+6+\cdots+n!$
(e) $1+9+\cdots+(2n-1)^2$ (f) $9+16+\cdots+n^2$
(g) $\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}$
(h) $-1+8-27+\cdots+(-1)^n n^3$
(i) $1-8+27+\cdots+(-1)^{n+1}n^3$
(j) $a+a^2+a^3+\cdots+a^n$
(k) $1+2^a+3^a+\cdots+n^a$

5 (a) $$\begin{aligned}2\times 3^{k+1}-3^{k+1}+21 &= 2\times 3\times 3^k-3\times 3^k+21\\ &= 6\times 3^k-3\times 3^k+21\\ &= 3\times 3^k+21\\ &= 3(3^k+7)\end{aligned}$$
Since k is a non-negative integer, 3^k+7 is an integer.
Thus, $2\times 3^{k+1}-3^{k+1}+21$ is divisible by 3.

(b) $3 \times 4^k - 6 \times 4^{k+1} + 28$
$= 3 \times 4^k - 6 \times 4 \times 4^k + 28$
$= 3 \times 4^k - 24 \times 4^k + 28$
$= -21 \times 4^k + 28$
$= 7\left(-3 \times 4^k + 4\right)$

Since k is a non-negative integer, $-3 \times 4^k + 4$ is an integer. Thus, $2 \times 3^{k+1} - 3^{k+1} + 20$ is divisible by 7.

(c) $5^{k+2} - 7 \times 5^k + 12$
$= 5^2 \times 5^k - 7 \times 5^k + 12$
$= 25 \times 5^k - 7 \times 5^k + 12$
$= 18 \times 5^k + 12$
$= 6\left(3 \times 5^k + 2\right)$

Since k is a non-negative integer, $3 \times 5^k + 2$ is an integer. Thus, $5^{k+2} - 7 \times 5^k + 12$ is divisible by 6.

6 (a) When $n = 5$:

LHS $= 2 + 4 + 6 + 8 + 10$
$= 30$

RHS $= 5 \times (5 + 1)$
$= 30$
$=$ LHS

So the statement is true for $n = 5$.

When $n = 2$:

LHS $= 2 + 4$
$= 6$

RHS $= 2 \times 3$
$= 6$
$=$ LHS

So the statement is true for $n = 2$.

When $n = 1$:

LHS $= 2$

RHS $= 1 \times 2$
$= 2$
$=$ LHS

So the statement is true for $n = 1$.

(b) When $n = 5$:

LHS $= 1 + 2 + 2^2 + 2^3 + 2^4$
$= 31$

RHS $= 2^5 - 1$
$= 31$
$=$ LHS

So the statement is true for $n = 5$.

When $n = 2$:

LHS $= 1 + 2$
$= 3$

RHS $= 2^2 - 1$
$= 3$
$=$ LHS

So the statement is true for $n = 2$.

When $n = 1$:

LHS $= 1$

RHS $= 2^1 - 1$
$= 1$
$=$ LHS

So the statement is true for $n = 1$.

(c) When $n = 5$:

LHS $= 1 \times 2 + 2 \times 5 + 3 \times 8 + 4 \times 11 + 5 \times 14$
$= 150$

RHS $= 5^2(5 + 1)$
$= 25 \times 6$
$= 150$
$=$ LHS

So the statement is true for $n = 5$.

When $n = 2$:

LHS $= 1 + 2 + 2 \times 5$
$= 12$

RHS $= 2^2(2 + 1)$
$= 12$
$=$ LHS

So the statement is true for $n = 2$.

When $n = 1$:

LHS $= 1 \times 2$
$= 2$

RHS $= 1^2(1 + 1)$
$= 2$
$=$ LHS

So the statement is true for $n = 1$.

(d) When $n = 5$:

$5^3 + 2 \times 5 = 135$
$= 3 \times 45$

When $n = 5$, the given expression can be written as an integer multiple of 3; therefore, it is divisible by 3.

When $n = 2$:

$2^3 + 2 \times 2 = 12$
$= 3 \times 4$

When $n = 2$, the given expression can be written as an integer multiple of 3; therefore, it is divisible by 3.

When $n = 1$:

$1^3 + 2 \times 1 = 3$
$= 3 \times 1$

When $n = 1$, the given expression can be written as an integer multiple of 3; therefore, it is divisible by 3.

(e) When $n = 5$:

$3^4 + 1 = 82$
$= 2 \times 41$

When $n = 5$, the given expression can be written as an integer multiple of 2; therefore, it is divisible by 2.

When $n = 2$:

$3^2 + 1 = 10$
$= 2 \times 5$

When $n = 2$, the given expression can be written as an integer multiple of 2; therefore, it is divisible by 2.

When $n = 1$:

$3^0 + 1 = 2$
$= 2 \times 1$

When $n = 1$, the given expression can be written as an integer multiple of 2; therefore, it is divisible by 2.

(f) When $n = 5$:

$3^4 - 2 \times 5 + 5 = 76$
$= 4 \times 19$

When $n = 5$, the given expression can be written as an integer multiple of 4; therefore, it is divisible by 4.

When $n = 2$:

$3^1 - 2 \times 2 + 5 = 4$
$= 4 \times 1$

When $n = 2$, the given expression can be written as an integer multiple of 4; therefore, it is divisible by 4.

When $n = 1$:

$3^0 - 2 \times 1 + 5 = 4$
$= 4 \times 1$

When $n = 1$, the given expression can be written as an integer multiple of 4; therefore, it is divisible by 4.

7 (a) C

(b) A possible misconception could be that when writing expressions for sums the variable n must represent the number of terms being added. While this is a common convention, it need not be true.

8 (a) C

(b) For option A, a common misconception is that any number of the form $6A$ must be divisible by 6. However, this is only true if you can be certain that A is an integer. In this alternative, the expression within the parentheses is not an integer when $k = 1$, for example.

For option B, notice that $24 + 6^{2k-3} = 6\left(4 + 6^{2k-4}\right)$. This is an integer divisible by 6 provided the expression within the parentheses is an integer. However, if $k = 1$, for example, $6 \times \left(4 + 6^{-2}\right) = 24\frac{1}{6}$ is obtained.

For option D, if the expression in the parentheses was always a multiple of 3, then this alternative would always be divisible by 6. However, this is not the case, as when $k = 3$, for example, the expression in the parentheses is equal to 19.

9 (a) $2 + 4 + 6 + \cdots + 2n$ **(b)** $\sum_{r=1}^{n} 2r$

(c) For $n = 1$, the sum is equal to 2 and $2 = 1 \times 2$, hence $2 = 1 \times (1 + 1)$.

For $n = 2$, the sum is equal to $2 + 4 = 6$ and $6 = 2 \times 3$, hence $6 = 2 \times (2 + 1)$.

For $n = 3$, the sum is equal to $2 + 4 + 6 = 12$ and $12 = 3 \times 4$, hence $12 = 3 \times (3 + 1)$.

For $n = 4$, the sum is equal to $2 + 4 + 6 + 8 = 20$ and $20 = 4 \times 5$, hence $20 = 4 \times (4 + 1)$.

In general, it appears that $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

(d) When $n = 10$:

LHS $= 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$
$= 110$

RHS $= 10 \times 11$
$= 110$

Since LHS = RHS, the conjecture is true for $n = 10$.

10 Since $5^k - 1$ is divisible by 4, it must be the case that $5^k - 1 = 4A$ where $A \in \mathbb{Z}$.

To show that $5^{k+1} - 1$ is divisible by 4, it is necessary to show that it can be expressed as 4 multiplied by an integer:

$5^{k+1} - 1$
$= 5 \times 5^k - 1$
$= 5 \times 5^k - 5 + 5 - 1$
$= 5\left(5^k - 1\right) + 5 - 1$
$= 5 \times 4A + 5 - 1$
$= 20A + 4$
$= 4(5A + 1)$

Since A is an integer, the number $5A + 1$ must also be an integer. Thus, if $5^k - 1$ is divisible by 4, $5^{k+1} - 1$ must also be divisible by 4.

EXERCISE 1.2

1 When $n = 1$:

RHS $= 1(1 + 1)$
$= 2$
$=$ LHS

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$2 + 4 + 6 + \cdots + 2k = k(k + 1)$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$2 + 4 + 6 + \cdots + 2(k + 1) = (k + 1)(k + 2)$

LHS $= 2 + 4 + 6 + \cdots + 2k + 2(k + 1)$
$= k(k + 1) + 2(k + 1)$ using [1]
$= (k + 1)(k + 2)$
$=$ RHS

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

2 When $n = 1$:

$\text{LHS} = 1^2$
$= 1$

$\text{RHS} = \dfrac{1(2\times 1+1)(1+1)}{6}$
$= \dfrac{6}{6}$
$= 1$
$= \text{LHS}$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(2k+1)(k+1)}{6}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \dfrac{(k+1)\big(2(k+1)+1\big)(k+1+1)}{6}$

$\text{RHS} = \dfrac{(k+1)\big(2(k+1)+1\big)(k+1+1)}{6}$
$= \dfrac{(k+1)(2k+2+1)(k+2)}{6}$
$= \dfrac{(k+1)(2k+3)(k+2)}{6}$

$\text{LHS} = 1^2 + 2^2 + 3^2 + \cdots + (k+1)^2$
$= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$
$= \dfrac{k(2k+1)(k+1)}{6} + (k+1)^2$ using [1]
$= \dfrac{k(2k+1)(k+1)}{6} + \dfrac{6(k+1)^2}{6}$
$= \dfrac{k(2k+1)(k+1) + 6(k+1)^2}{6}$
$= \dfrac{(k+1)\big(k(2k+1) + 6(k+1)\big)}{6}$
$= \dfrac{(k+1)\big(2k^2 + k + 6k + 6\big)}{6}$
$= \dfrac{(k+1)\big(2k^2 + 7k + 6\big)}{6}$
$= \dfrac{(k+1)(2k+3)(k+2)}{6}$
$= \text{RHS}$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

3 When $n = 1$:

$\text{LHS} = 1^3$
$= 1$

$\text{RHS} = \dfrac{1}{3}(2\times 1-1)(2\times 1+1)$
$= 1$
$= \text{LHS}$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \dfrac{1}{3}k(2k-1)(2k+1)$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$1^2 + 3^2 + 5^2 + \cdots + \big(2(k+1)-1\big)^2$
$= \dfrac{1}{3}(k+1)\big(2(k+1)-1\big)\big(2(k+1)+1\big)$

$\text{RHS} = \dfrac{1}{3}(k+1)\big(2(k+1)-1\big)\big(2(k+1)+1\big)$
$= \dfrac{1}{3}(k+1)(2k+1)(2k+3)$

$\text{LHS} = 1^2 + 3^2 + 5^2 + \cdots + \big(2(k+1)-1\big)^2$
$= 1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + \big(2(k+1)-1\big)^2$
$= \dfrac{1}{3}k(2k+1)(2k-1) + \big(2(k+1)-1\big)^2$ (using [1])
$= \dfrac{1}{3}k(2k+1)(2k-1) + (2k+1)^2$
$= \dfrac{1}{3}(2k+1)\big(k(2k-1) + 3(2k+1)\big)$
$= \dfrac{1}{3}(2k+1)\big(2k^2 - k + 6k + 3\big)$
$= \dfrac{1}{3}(2k+1)\big(2k^2 + 5k + 3\big)$
$= \dfrac{1}{3}(2k+1)(2k+3)(k+1)$
$= \text{RHS}$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

4 When $n = 1$:

$\text{LHS} = 1\times 2$
$= 2$

$\text{RHS} = 1^2(1+1)$
$= 2$
$= \text{LHS}$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$1\times 2 + 2\times 5 + \cdots + k(3k-1) = k^2(k+1)$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$1\times 2 + 2\times 5 + \cdots + (k+1)\big(3(k+1)-1\big) = (k+1)^2(k+2)$

$\text{LHS} = 1\times 2 + 2\times 5 + \cdots + k(3k-1) + (k+1)\big(3(k+1)-1\big)$
$= k^2(k+1) + (k+1)\big(3(k+1)-1\big)$ using [1]
$= k^2(k+1) + (k+1)(3k+2)$
$= (k+1)\big(k^2 + 3k + 2\big)$
$= (k+1)(k+1)(k+2)$
$= (k+1)^2(k+2)$
$= \text{RHS}$

Hence, by induction, the original statement must be true for all $n \in \mathbb{Z}^+$.

5 When $n = 3$:

LHS $= 7$

RHS $= 3^2 + 2 \times 3 - 8$
$= 9 + 6 - 8$
$= 7$
$=$ LHS

So the statement is true for $n = 3$.

Let $k \geq 3$ and assume statement is true for $n = k$.
That is, assume

$7 + 9 + 11 + \cdots + (2k + 1) = k^2 + 2k - 8$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$7 + 9 + 11 + \cdots + \big(2(k + 1) + 1\big) = (k + 1)^2 + 2(k + 1) - 8$

RHS $= (k + 1)^2 + 2(k + 1) - 8$
$= k^2 + 2k + 1 + 2k + 2 - 8$
$= k^2 + 4k - 5$

LHS $= 7 + 9 + 11 + \cdots + \big(2(k + 1) + 1\big)$
$= 7 + 9 + 11 + \cdots + (2k + 1) + \big(2(k + 1) + 1\big)$
$= k^2 + 2k - 8 + \big(2(k + 1) + 1\big)$ using [1]
$= k^2 + 2k - 8 + 2k + 2 + 1$
$= k^2 + 4k - 5$
$=$ RHS

Hence, by induction, the original statement must be true for all integers $n \geq 3$.

6 The statement can be rewritten as:

$3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n + 1)}{2}$ where $n \in \mathbb{Z}^+$.

When $n = 1$:

LHS $= 3$

RHS $= \dfrac{3 \times 1(1 + 1)}{2}$
$= 3$
$=$ LHS

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$3 + 6 + 9 + \cdots + 3k = \dfrac{3k(k + 1)}{2}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$3 + 6 + 9 + \cdots + 3(k + 1) = \dfrac{3(k + 1)(k + 2)}{2}$

LHS $= 3 + 6 + 9 + \cdots + 3(k + 1)$
$= 3 + 6 + 9 + \cdots + 3k + 3(k + 1)$
$= \dfrac{3k(k + 1)}{2} + 3(k + 1)$ (using [1])
$= \dfrac{3k(k + 1)}{2} + \dfrac{6(k + 1)}{2}$
$= \dfrac{3k(k + 1) + 6(k + 1)}{2}$
$= \dfrac{3(k + 1)(k + 2)}{2}$
$=$ RHS

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

7 The statement can be rewritten as:

$1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$ for $n \in \mathbb{Z}^+$.

When $n = 1$:

LHS $= 1^3$
$= 1$

RHS $= \dfrac{1^2(1 + 1)^2}{4}$
$= 1$
$=$ LHS

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$1^3 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{k^2(k + 1)^2}{4}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$1^3 + 2^3 + 3^3 + \cdots + (k + 1)^3 = \dfrac{(k + 1)^2(k + 2)^2}{4}$

LHS $= 1^3 + 2^3 + 3^3 + \cdots + (k + 1)^3$
$= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k + 1)^3$
$= \dfrac{k^2(k + 1)^2}{4} + (k + 1)^3$ using [1]
$= \dfrac{k^2(k + 1)^2 + 4(k + 1)^3}{4}$
$= \dfrac{(k + 1)^2\big(k^2 + 4(k + 1)\big)}{4}$
$= \dfrac{(k + 1)^2(k^2 + 4k + 4)}{4}$
$= \dfrac{(k + 1)^2(k + 2)^2}{4}$
$=$ RHS

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

8 The statement given can be rewritten as:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

When $n = 1$:

$$\text{LHS} = \frac{1}{1\times 2} = \frac{1}{2}$$

$$\text{RHS} = \frac{1}{1+1} = \frac{1}{2} = \text{LHS}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad [1]$$

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

$$\begin{aligned}\text{LHS} &= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{(k+1)(k+2)}\\ &= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{k(k+1)}\frac{1}{(k+1)(k+2)}\\ &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \text{using [1]}\\ &= \frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}\\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

9 When $n = 1$:

$$\text{LHS} = \frac{1}{1\times 3} = \frac{1}{3}$$

$$\text{RHS} = \frac{1}{2\times 1+1} = \frac{1}{3} = \text{LHS}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \quad [1]$$

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)} = \frac{k+1}{2(k+1)+1}$$

$$\begin{aligned}\text{LHS} &= \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)}\\ &= \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}\\ &\quad+\frac{1}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)}\\ &= \frac{k}{2k+1}+\frac{1}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)} \quad \text{using [1]}\\ &= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\\ &= \frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}\\ &= \frac{k(2k+3)+1}{(2k+1)(2k+3)}\\ &= \frac{2k^2+3k+1}{(2k+1)(2k+3)}\\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}\\ &= \frac{k+1}{2(k+1)+1}\\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

10 The statement can be rewritten as:

$$3+3^2+3^3+\cdots+3^n=\frac{3^{n+1}-3}{2}.$$

When $n = 1$:

$$\text{LHS} = 3^1 = 3$$

$$\text{RHS} = \frac{3^{1+1}-3}{2} = 3 = \text{LHS}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$$3 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1} - 3}{2} \quad [1]$$

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$$3 + 3^2 + 3^3 + \cdots + 3^{k+1} = \frac{3^{k+2} - 3}{2}$$

$$\begin{aligned}
\text{LHS} &= 3 + 3^2 + 3^3 + \cdots + 3^{k+1} \\
&= 3 + 3^2 + 3^3 + \cdots + 3^k + 3^{k+1} \\
&= \frac{3^{k+1} - 3}{2} + 3^{k+1} \quad \text{using [1]} \\
&= \frac{3^{k+1} - 3}{2} + \frac{2 \times 3^{k+1}}{2} \\
&= \frac{3^{k+1} - 3 + 2 \times 3^{k+1}}{2} \\
&= \frac{3 \times 3^{k+1} - 3}{2} \\
&= \frac{3^{k+2} - 3}{2} \\
&= \text{RHS}
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

11 The statement can be rewritten as:

$$1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \cdots + n \times 3^n = \frac{3}{4}\left((2n-1)3^n + 1\right).$$

When $n = 1$:

$$\begin{aligned}
\text{LHS} &= 1 \times 3 \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{3}{4}\left((2 \times 1 - 1)3^1 + 1\right) \\
&= \frac{3}{4}(4) \\
&= 3 \\
&= \text{LHS}
\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$$1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \cdots + k \times 3^k = \frac{3}{4}\left((2k-1)3^k + 1\right) \quad [1]$$

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$$1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \cdots + (k+1) \times 3^{k+1} = \frac{3}{4}\left(\left(2(k+1)-1\right)3^{k+1} + 1\right)$$

$$\begin{aligned}
\text{RHS} &= \frac{3}{4}\left((2k + 2 - 1)3^{k+1} + 1\right) \\
&= \frac{3}{4}\left((2k+1)3^{k+1} + 1\right)
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= 1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \cdots + (k+1) \times 3^{k+1} \\
&= 1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \cdots + k \times 3^k + (k+1) \times 3^{k+1} \\
&= \frac{3}{4}\left((2k-1)3^k + 1\right) + (k+1) \times 3^{k+1} \quad \text{using [1]} \\
&= \frac{3}{4}\left(2k \times 3^k - 3^k + 1\right) + (k+1) \times 3^{k+1} \\
&= \frac{3}{4}2k \times 3^k - \frac{3}{4}3^k + \frac{3}{4} + k3^{k+1} + 3^{k+1} \\
&= \frac{1}{4}2k \times 3^{k+1} - \frac{1}{4}3^{k+1} + k3^{k+1} + 3^{k+1} + \frac{3}{4} \\
&= \frac{3}{2}2k \times 3^k + \frac{3}{4}3^k + \frac{3}{4} \\
&= \frac{3}{4}\left(2k \times 3^{k+1} + 3^{k+1} + 1\right) \\
&= \frac{3}{4}\left((2k+1)3^{k+1} + 1\right) \\
&= \text{RHS}
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

12 When $n = 1$:

$$\begin{aligned}
\text{LHS} &= a \\
\text{RHS} &= \frac{a\left(r^1 - 1\right)}{r - 1} \\
&= a \\
&= \text{LHS}
\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$$a + ar + ar^2 + \cdots + ar^{k-1} = \frac{a\left(r^k - 1\right)}{r - 1} \quad [1]$$

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$$a + ar + ar^2 + \cdots + ar^k = \frac{a\left(r^{k+1} - 1\right)}{r - 1}$$

$$\begin{aligned}
\text{LHS} &= a + ar + ar^2 + \cdots + ar^k \\
&= a + ar + ar^2 + \cdots + ar^{k-1} + ar^k \\
&= \frac{a\left(r^k - 1\right)}{r - 1} + ar^k \quad \text{(using [1])} \\
&= \frac{a\left(r^k - 1\right)}{r - 1} + \frac{ar^k(r-1)}{r-1} \\
&= \frac{a\left(r^k - 1 + r^k(r-1)\right)}{r-1} \\
&= \frac{a\left(r^k - 1 + r^{k+1} - r^k\right)}{r-1} \\
&= \frac{a\left(r^{k+1} - 1\right)}{r-1} \\
&= \text{RHS}
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

13 When $n = 1$:

$$\text{LHS} = \frac{1}{2!} = \frac{1}{2}$$

$$\text{RHS} = \frac{(1+1)!-1}{(1+1)!} = \frac{2!-1}{2!} = \frac{1}{2} = \text{LHS}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = \frac{(k+1)!-1}{(k+1)!} \quad [1]$$

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k+1}{(k+2)!} = \frac{(k+2)!-1}{(k+2)!}$$

$$\begin{aligned}
\text{LHS} &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k+1}{(k+2)!} \\
&= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= \frac{(k+1)!-1}{(k+1)!} + \frac{(k+1)}{(k+2)!} \quad \text{(using [1])} \\
&= \frac{\big((k+1)!-1\big)(k+2)}{(k+1)!(k+2)} + \frac{(k+1)}{(k+2)!} \\
&= \frac{\big((k+1)!-1\big)(k+2)}{(k+2)!} + \frac{(k+1)}{(k+2)!} \\
&= \frac{(k+1)!(k+2)-(k+2)+k+1}{(k+2)!} \\
&= \frac{(k+2)!-k-2+k+1}{(k+2)!} \\
&= \frac{(k+2)!-1}{(k+2)!} \\
&= \text{RHS}
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

14 When $n = 1$:

$$\text{LHS} = -2$$

$$\text{RHS} = \frac{2}{3}\left((-2)^1 - 1\right) = \frac{2}{3}(-3) = -2 = \text{LHS}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$$-2 + 4 - 8 + \cdots + (-2)^k = \frac{2}{3}\left((-2)^k - 1\right) \quad [1]$$

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$$-2 + 4 - 8 + \cdots + (-2)^{k+1} = \frac{2}{3}\left((-2)^{k+1} - 1\right)$$

$$\begin{aligned}
\text{LHS} &= -2 + 4 - 8 + \cdots + (-2)^{k+1} \\
&= -2 + 4 - 8 + \cdots + (-2)^k + (-2)^{k+1} \\
&= \frac{2}{3}\left((-2)^k - 1\right) + (-2)^{k+1} \quad \text{using [1]} \\
&= \frac{2}{3}(-2)^k - \frac{2}{3} + (-2)^{k+1} \\
&= -\frac{1}{3} \times -2 \times (-2)^k + (-2)^k - \frac{2}{3} \\
&= -\frac{1}{3}(-2)^{k+1} + (-2)^{k+1} - \frac{2}{3} \\
&= \frac{2}{3}(-2)^{k+1} - \frac{2}{3} \\
&= \frac{2}{3}\left((-2)^{k+1} - 1\right) \\
&= \text{RHS}
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

15 (a) C

(b) Option A would be a valid way to prove that the statement is true for all positive integers. But as there exists odd positive integers for which the statement is untrue, this alternative is not valid.

Option B would be a valid way to prove that the statement is true for all positive integers greater than or equal to 4. But it does not prove that the statement is true for $n = 2$.

Option D would be a valid way to prove the statement is true for all positive odd integers. This is clearly not valid.

16 (a) D

(b) With most sums, the expression obtained by substituting $n = k+1$ can be obtained by simply adding on one extra term to the expression obtained by substituting $n = k$. Because this sum is unusual in that every term depends on the value of n, changing the value of n from k to $k+1$ actually changes every term in the sum. Not doing this for all the terms is a common error, leading us to suspect that alternatives A, B and C must be incorrect.

17 Proceeding by induction on n:

When $n = 1$:

$$\text{LHS} = F_1^2 = 1$$

$$\text{RHS} = F_1 \times F_2 = 1 \times 1 = 1 = \text{LHS}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 = F_k F_{k+1} \quad [1]$$

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$$F_1^2 + F_2^2 + F_3^2 + \cdots + F_{k+1}^2 = F_{k+1} F_{k+2}$$

$$\begin{aligned}
\text{LHS} &= F_1^2 + F_2^2 + F_3^2 + \cdots + F_{k+1}^2 \\
&= F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 + F_{k+1}^2 \\
&= F_k F_{k+1} + F_{k+1}^2 \quad \text{using [1]} \\
&= F_{k+1}\left(F_k + F_{k+1}\right) \\
&= F_{k+1} F_{k+2} \\
&= \text{RHS}
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

18 Proceeding by induction on n:

When $n = 0$:

$$\begin{aligned}
\text{RHS} &= \frac{\sin\left(\left(0 + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \frac{\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1 \\
&= \text{LHS}
\end{aligned}$$

So the statement is true for $n = 0$.

Let $k \geq 0$ and assume statement is true for $n = k$.
That is, assume

$$1 + \cos(x) + \cos(2x) \cdots + \cos(kx) = \frac{\sin\left(\left(k + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \quad [1]$$

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$$1 + \cos(x) + \cos(2x) \cdots + \cos(kx) + \cos\left((k+1)x\right) = \frac{\sin\left(\left(k + 1 + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2}$$

$$\begin{aligned}
\text{LHS} &= 1 + \cos(x) + \cos(2x) \cdots + \cos(kx) + \cos\left((k+1)x\right) \\
&= \frac{\sin\left(\left(k + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} + \cos\left((k+1)x\right) \quad \text{using [1]} \\
&= \frac{\sin\left(\left(k + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{2\cos\left((k+1)x\right)\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \frac{\sin\left(\left(k + \frac{1}{2}\right)x\right) + 2\cos\left((k+1)x\right)\sin\left(\frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \frac{\sin\left(\left(k + \frac{3}{2}\right)x\right) + \sin\left((k+1)x + \frac{1}{2}x\right) - \sin\left((k+1)x - \frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} \\
&\quad + \frac{1}{2} \quad \text{(using the identity given)} \\
&= \frac{\sin\left(kx + \frac{1}{2}x\right) + \sin\left(kx + x + \frac{1}{2}x\right) - \sin\left(kx + x - \frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \frac{\sin\left(kx + \frac{1}{2}x\right) + \sin\left(kx + \frac{3}{2}x\right) - \sin\left(kx + \frac{1}{2}x\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \frac{\sin\left(k\left(x + \frac{3}{2}\right)\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \frac{\sin\left(k\left(x + 1 + \frac{1}{2}\right)\right)}{2\sin\left(\frac{1}{2}x\right)} + \frac{1}{2} \\
&= \text{RHS}
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 0$.

19 The statement can be rewritten as:

$$\begin{aligned}
&1^2 2^{n-1} + 2^2 2^{n-2} + 3^2 2^{n-3} + \cdots + (n-1)^2 2^1 + n^2 2^0 \\
&= 2^{n+3} - 2^{n+1} - n^2 - 4n - 6
\end{aligned}$$

Proceeding by induction on n:

When $n = 1$:

$$\begin{aligned}
\text{LHS} &= 1^2 \times 2^0 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= 2^{1+3} - 2^{1+1} - 1^2 - 4 \times 1 - 6 \\
&= 16 - 4 - 1 - 4 - 6 \\
&= 1 \\
&= \text{LHS}
\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$$\begin{aligned}
&1^2 2^{k-1} + 2^2 2^{k-2} + 3^2 2^{k-3} + \cdots + (k-1)^2 2^1 + k^2 2^0 \\
&= 2^{k+3} - 2^{k+1} - k^2 - 4k - 6 \quad [1]
\end{aligned}$$

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$$\begin{aligned}
&1^2 2^k + 2^2 2^{k-1} + 3^2 2^{k-2} + \cdots + k^2 2^1 + (k+1)^2 2^0 \\
&= 2^{k+4} - 2^{k+2} - (k+1)^2 - 4(k+1) - 6
\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 2^{k+4} - 2^{k+2} - (k+1)^2 - 4(k+1) - 6\\ &= 2^{k+4} - 2^{k+2} - \left(k^2 + 2k + 1\right) - 4(k+1) - 6\\ &= 2^{k+4} - 2^{k+2} - k^2 - 2k - 1 - 4k - 4 - 6\\ &= 2^{k+4} - 2^{k+2} - k^2 - 6k - 11\end{aligned}$$

When working with the left-hand side, it is necessary to somehow incorporate equation 1. How to do this is not obvious for this problem. One solution is to make part of this left-hand side resemble the left-hand side of equation 1 by taking out a factor of 2 from all but the final term, as demonstrated in the second line below.

$$\begin{aligned}\text{LHS} &= 1^2 2^k + 2^2 2^{k-1} + 3^2 2^{k-2} + \cdots + k^2 2^1 + (k+1)^2 2^0\\ &= 2\left(1^2 2^{k-1} + 2^2 2^{k-2} + 3^2 2^{k-3} + \cdots + k^2 2^0\right) + (k+1)^2 2^0\\ &= 2\left(2^{k+3} - 2^{k+1} - k^2 - 4k - 6\right) + (k+1)^2 2^0 \quad \text{using [1]}\\ &= 2^{k+4} - 2^{k+2} - 2k^2 - 8k - 12 + k^2 + 2k + 1\\ &= 2^{k+4} - 2^{k+2} - k^2 - 6k - 11\\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

EXERCISE 1.3

1 (a) The statement given can be rewritten as: $11^n - 1 = 10A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 11^1 - 1\\ &= 10\\ &= 10 \times 1\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$11^k - 1 = 10A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$11^{k+1} - 1 = 10B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 11^{k+1} - 1\\ &= 11 \times 11^k - 1\\ &= 11(10A + 1) - 1 \quad \text{using [1]}\\ &= 110A + 11 - 1\\ &= 110A + 10\\ &= 10(11A + 1)\\ &= \text{RHS} \ (\text{since } A \in \mathbb{Z} \Rightarrow 11A + 1 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

(b) The statement given can be rewritten as: $5^n + 3 = 4A$ where $A \in \mathbb{Z}$.

When $n = 0$:

$$\begin{aligned}\text{LHS} &= 5^0 + 3\\ &= 4\\ &= 4 \times 1\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 0$.

Let $k \geq 0$ and assume statement is true for $n = k$.
That is, assume

$5^k + 3 = 4A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$5^{k+1} + 3 = 4B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 5^{k+1} + 3\\ &= 5 \times 5^k + 3\\ &= 5(4A - 3) + 3 \quad \text{using [1]}\\ &= 20A - 15 + 3\\ &= 20A - 12\\ &= 4(5A - 3)\\ &= \text{RHS} \ (\text{since } A \in \mathbb{Z} \Rightarrow 5A - 3 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 0$.

(c) The statement given can be rewritten as: $4^n - 7 = 3A$ where $A \in \mathbb{Z}$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= 4^2 - 7\\ &= 9\\ &= 3 \times 3\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$.
That is, assume

$4^k - 7 = 3A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$4^{k+1} - 7 = 3B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 4^{k+1} - 7\\ &= 4 \times 4^k - 7\\ &= 4(3A + 7) - 7 \quad \text{using [1]}\\ &= 12A + 28 - 7\\ &= 12A + 21\\ &= 3(4A + 7)\\ &= \text{RHS} \ (\text{since } A \in \mathbb{Z} \Rightarrow 4A + 7 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

(d) The statement given can be rewritten as: $3^{3n} + 2^{n+2} = 5A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 3^3 + 2^{1+2}\\ &= 27 + 8\\ &= 5 \times 7\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$3^{3k} + 2^{k+2} = 5A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$3^{3(k+1)} + 2^{k+1+2} = 5B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}
\text{LHS} &= 3^{3(k+1)} + 2^{k+1+2} \\
&= 3^{3k+3} + 2^{k+3} \\
&= 3^3 \times 3^{3k} + 2^{k+3} \\
&= 27 \times 3^{3k} + 2^{k+3} \\
&= 27\left(5A - 2^{k+2}\right) + 2^{k+3} \quad \text{using [1]} \\
&= 27 \times 5A - 27 \times 2^{k+2} + 2^{k+3} \\
&= 27 \times 5A - 27 \times 2^{k+2} + 2 \times 2^{k+2} \\
&= 27 \times 5A - 25 \times 2^{k+2} \\
&= 5\left(27A - 5 \times 2^{k+2}\right) \\
&= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow 27A - 5 \times 2^{k+2} \in \mathbb{Z})
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

(e) The statement given can be rewritten as:

$11^{n+2} + 12^{2n+1} = 133A$ where $A \in \mathbb{Z}$.

When $n = 0$:

$$\begin{aligned}
\text{LHS} &= 11^{0+2} + 12^{0+1} \\
&= 121 + 12 \\
&= 133 \\
&= 133 \times 1 \\
&= \text{RHS}
\end{aligned}$$

So the statement is true for $n = 0$.

Let $k \geq 0$ and assume statement is true for $n = k$. That is, assume

$11^{k+2} + 12^{2k+1} = 133A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$11^{k+1+2} + 12^{2(k+1)+1} = 133B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}
\text{LHS} &= 11^{k+1+2} + 12^{2(k+1)+1} \\
&= 11^{k+3} + 12^{2k+3} \\
&= 11 \times 11^{k+2} + 12^{2k+3} \\
&= 11\left(133A - 12^{2k+1}\right) + 12^{2k+3} \quad \text{using [1]} \\
&= 11 \times 133A - 11 \times 12^{2k+1} + 12^2 \times 12^{2k+1} \\
&= 11 \times 133A - 11 \times 12^{2k+1} + 144 \times 12^{2k+1} \\
&= 11 \times 133A + 133 \times 12^{2k+1} \\
&= 133\left(11A + 12^{2k+1}\right) \\
&= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow 11A + 12^{2k+1} \in \mathbb{Z})
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 0$.

(f) The statement given can be rewritten as:

$5^{2n+1} + 2^{2n+1} = 7A$ where $A \in \mathbb{Z}$.

When $n = 0$:

$$\begin{aligned}
\text{LHS} &= 5^{0+1} + 2^{0+1} \\
&= 7 \\
&= 7 \times 1 \\
&= \text{RHS}
\end{aligned}$$

So the statement is true for $n = 0$.

Let $k \geq 0$ and assume statement is true for $n = k$. That is, assume

$5^{2k+1} + 2^{2k+1} = 7A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$5^{2(k+1)+1} + 2^{2(k+1)+1} = 7B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}
\text{LHS} &= 5^{2k+3} + 2^{2k+3} \\
&= 5^2 \times 5^{2k+1} + 2^{2k+3} \\
&= 25\left(7A - 2^{2k+1}\right) + 2^{2k+3} \quad \text{using [1]} \\
&= 25 \times 7A - 25 \times 2^{2k+1} + 2^2 \times 2^{2k+1} \\
&= 25 \times 7A - 25 \times 2^{2k+1} + 4 \times 2^{2k+1} \\
&= 25 \times 7A - 21 \times 2^{2k+1} \\
&= 7\left(25A - 3 \times 2^{2k+1}\right) \\
&= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow 25A - 3 \times 2^{2k+1} \in \mathbb{Z})
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 0$.

(g) The statement given can be rewritten as:

$3 \times 5^{2n+1} + 2^{3n+1} = 17A$ where $A \in \mathbb{Z}$.

When $n = 0$:

$$\begin{aligned}
\text{LHS} &= 3 \times 5^{0+1} + 2^{0+1} \\
&= 17 \\
&= 17 \times 1 \\
&= \text{RHS}
\end{aligned}$$

So the statement is true for $n = 0$.

Let $k \geq 0$ and assume statement is true for $n = k$. That is, assume

$3 \times 5^{2k+1} + 2^{3k+1} = 17A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$3 \times 5^{2(k+1)+1} + 2^{3(k+1)+1} = 17B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}
\text{LHS} &= 3 \times 5^{2(k+1)+1} + 2^{3(k+1)+1} \\
&= 3 \times 5^{2k+3} + 2^{3k+4} \\
&= 5^2 \times 3 \times 5^{2k+1} + 2^{3k+4} \\
&= 25\left(17A - 2^{3k+1}\right) + 2^{3k+4} \quad \text{using [1]} \\
&= 25 \times 17A - 25 \times 2^{3k+1} + 2^3 \times 2^{3k+1} \\
&= 25 \times 17A - 25 \times 2^{3k+1} + 8 \times 2^{3k+1} \\
&= 25 \times 17A - 17 \times 2^{3k+1} \\
&= 17\left(25A - 2^{3k+1}\right) \\
&= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow 25A - 2^{3k+1} \in \mathbb{Z})
\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 0$.

(h) The statement given can be rewritten as: $n^3 - n = 3A$ where $A \in \mathbb{Z}$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= 2^3 - 2\\ &= 6\\ &= 3 \times 2\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$. That is, assume

$k^3 - k = 3A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$(k+1)^3 - (k+1) = 3B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= (k+1)^3 - (k+1)\\ &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= k^3 + 3k^2 + 2k\\ &= 3A + k + 3k^2 + 2k \quad \text{using [1]}\\ &= 3A + 3k^2 + 3k\\ &= 3\left(A + k^2 + k\right)\\ &= \text{RHS} \ (\text{since } A, k \in \mathbb{Z} \Rightarrow A + k^2 + k \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

(i) Let n be a positive integer. The statement given can be rewritten as: $n^3 + (n+1)^3 + (n+2)^3 = 9A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 1^3 + 2^3 + 3^3\\ &= 36\\ &= 9 \times 4\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$k^3 + (k+1)^3 + (k+2)^3 = 9A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$(k+1)^3 + (k+2)^3 + (k+3)^3 = 9B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= (k+1)^3 + (k+2)^3 + (k+3)^3\\ &\quad k^3 + (k+1)^3 + (k+2)^3 - k^3 + (k+3)^3\\ &= 9A - k^3 + (k+3)^3 \quad \text{using [1]}\\ &= 9A - k^3 + k^3 + 9k^2 + 27k + 27\\ &= 9A + 9k^2 + 27k + 27\\ &= 9\left(A + k^2 + 3k + 3\right)\\ &= \text{RHS} \ (\text{since } A, k \in \mathbb{Z} \Rightarrow A + k^2 + 3k + 3 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all positive integers.

(j) The statement given can be rewritten as: $4^n - 3n - 1 = 9A$ where $A \in \mathbb{Z}$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= 4^2 - 3 \times 2 - 1\\ &= 9\\ &= 9 \times 1\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$. That is, assume

$4^k - 3k - 1 = 9A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$4^{k+1} - 3(k+1) - 1 = 9B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 4^{k+1} - 3(k+1) - 1\\ &= 4 \times 4^k - 3k - 4\\ &= 4(9A + 3k + 1) - 3k - 4 \quad \text{using [1]}\\ &= 36A + 12k + 4 - 3k - 4\\ &= 36A + 9k\\ &= 9(4A + k)\\ &= \text{RHS} \ (\text{since } A, k \in \mathbb{Z} \Rightarrow 4A + k \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

(k) The statement given can be rewritten as:

$3^{2n+2} - 8n - 9 = 64A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 3^{2\times 1+2} - 8 \times 1 - 9\\ &= 81 - 8 - 9\\ &= 64\\ &= 64 \times 1\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$3^{2k+2} - 8k - 9 = 64A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$3^{2(k+1)+2} - 8(k+1) - 9 = 64B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 3^{2(k+1)+2} - 8(k+1) - 9\\ &= 3^{2k+4} - 8k - 17\\ &= 3^2 \times 3^{2k+2} - 8k - 17\\ &= 9(64A + 8k + 9) - 8k - 17 \quad \text{using [1]}\\ &= 9 \times 64A + 72k + 81 - 8k - 17\\ &= 3 \times 64A + 64k + 64\\ &= 64(3A + k + 1)\\ &= \text{RHS} \ (\text{since } A, k \in \mathbb{Z} \Rightarrow 3A + k + 1 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

2 The statement given can be rewritten as:

$7^{2n} + 2^{3n-3}3^{n-1} = 25A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 7^{2\times1} + 2^{3\times1-3} \times 3^{1-1} \\ &= 50 \\ &= 25 \times 2 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$7^{2k} + 2^{3k-3}3^{k-1} = 25A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$7^{2(k+1)} + 2^{3(k+1)-3}3^{k+1-1} = 25B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 7^{2(k+1)} + 2^{3(k+1)-3}3^{k+1-1} \\ &= 7^{2k+2} + 2^{3k}3^k \\ &= 7^2 \times 7^{2k} + 2^{3k}3^k \\ &= 49\left(25A - 2^{3k-3}3^{k-1}\right) + 2^{3k}3^k \quad \text{using [1]} \\ &= 49 \times 25A - 49 \times 2^{3k-3}3^{k-1} + 2^{3k}3^k \\ &= 49 \times 25A - 49 \times 2^{3k-3}3^{k-1} + 2^3 \times 2^{3k-3} \times 3 \times 3^{k-1} \\ &= 49 \times 25A - 49 \times 2^{3k-3}3^{k-1} + 24 \times 2^{3k-3}3^{k-1} \\ &= 49 \times 25A - 25 \times 2^{3k-3}3^{k-1} \\ &= 25\left(49A - 2^{3k-3}3^{k-1}\right) \\ &= \text{RHS} \ (\text{since } A, k \in \mathbb{Z} \Rightarrow 49A - 2^{3k-3}3^{k-1} \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

3 The statement given can be rewritten as:
$5^{n+1} + 2 \times 3^n + 1 = 8A$ where $A \in \mathbb{Z}$.

When $n = 0$:

$$\begin{aligned}\text{LHS} &= 5^{0+1} + 2 \times 3^0 + 1 \\ &= 8 \\ &= 8 \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 0$.

Let $k \geq 0$ and assume statement is true for $n = k$.
That is, assume

$5^{k+1} + 2 \times 3^k + 1 = 8A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$5^{k+1+1} + 2 \times 3^{k+1} + 1 = 8B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 5^{k+1+1} + 2 \times 3^{k+1} + 1 \\ &= 5 \times 5^{k+1} + 2 \times 3^{k+1} + 1 \\ &= 5\left(8A - 2 \times 3^k - 1\right) + 2 \times 3^{k+1} + 1 \quad \text{using [1]} \\ &= 40A - 10 \times 3^k - 5 + 2 \times 3^{k+1} + 1 \\ &= 40A - 10 \times 3^k + 2 \times 3 \times 3^k - 4 \\ &= 40A - 10 \times 3^k + 6 \times 3^k - 4 \\ &= 40A - 4 \times 3^k - 4 \\ &= 40A - 4\left(3^k + 1\right)\end{aligned}$$

Note that as k is a non-negative integer, 3^k must be odd (since multiplying an odd number by itself any number of times can never introduce a factor of 2). It follows that $3^k + 1$ must be even, and hence, this expression can be expressed as $2C$ where $C \in \mathbb{Z}$. Thus,

$$\begin{aligned}&40A - 4\left(3^k + 1\right) \\ &= 40A - 4(2C) \\ &= 40A - 8C \\ &= 8(5A - C) \\ &= 8B \text{ where } B \in \mathbb{Z} \ (\text{since } A, C \in \mathbb{Z} \Rightarrow 5A - C \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 0$.

4 The statement given can be rewritten as: $x^n - 1 = (x - 1)A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= x^1 - 1 \\ &= (x - 1) \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$x^k - 1 = (x - 1)A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$x^{k+1} - 1 = (x - 1)B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= x^{k+1} - 1 \\ &= x \times x^k - 1 \\ &= x\left((x - 1)A + 1\right) - 1 \quad \text{using [1]} \\ &= x(x - 1)A + x - 1 \\ &= (x - 1)(xA + 1) \\ &= \text{RHS} \ (\text{since } A, x \in \mathbb{Z} \Rightarrow xA + 1 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

5 The statement given can be rewritten as:

$(x + 1)^n - nx - 1 = x^2A$ where $A \in \mathbb{Z}$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= (x + 1)^2 - 2x - 1 \\ &= x^2 + 2x + 1 - 2x - 1 \\ &= x^2 \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$.
That is, assume

$(x + 1)^k - kx - 1 = x^2A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$(x+1)^{k+1} - (k+1)x - 1 = x^2 B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= (x+1)^{k+1} - (k+1)x - 1 = x^2 B \\ &= (x+1)(x+1)^k - kx - x - 1 \\ &= (x+1)\left(x^2 A + kx + 1\right) - kx - x - 1 \quad \text{using [1]} \\ &= x^3 A + kx^2 + x + x^2 A + kx + 1 - kx - x - 1 \\ &= x^3 A + kx^2 + x^2 A \\ &= x^2 (xA + k + A) \\ &= \text{RHS} \ (\text{since } A, x \in \mathbb{Z} \Rightarrow xA + k + A \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

6 The statement given can be rewritten as:

$(n+1)(n+2)\cdots(2n) = 2^n A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= (1+1) \\ &= 2 \\ &= 2^1 \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$(k+1)(k+2)\cdots(2k) = 2^k A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$(k+1+1)(k+1+2)\cdots(2(k+1)) = 2^{k+1} B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= (k+1+1)(k+1+2)\cdots(2(k+1)) = 2^{k+1} B \\ &= (k+2)(k+3)\cdots(2k+2) \\ &= (k+2)(k+3)\cdots(2k)(2k+1)(2k+2) \\ &= (k+2)(k+3)\cdots(2k)(2k+1) \times 2(k+1) \\ &= 2(k+1)(k+2)(k+3)\cdots(2k)(2k+1) \\ &= 2 \times 2^k A(2k+1) \ \text{using [1]} \\ &= 2^{k+1} A(2k+1) \\ &= \text{RHS} \ (\text{since } A, k \in \mathbb{Z} \Rightarrow A(2k+1) \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

CHAPTER REVIEW 1

1 When $n = 5$:

$$\begin{aligned}\text{LHS} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \\ &= \frac{16}{32} + \frac{8}{32} + \frac{4}{32} + \frac{2}{32} + \frac{1}{32} \\ &= \frac{31}{32}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 1 - \frac{1}{2^5} \\ &= 1 - \frac{1}{32} \\ &= \frac{31}{32} \\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 5$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= \frac{1}{2} + \frac{1}{4} \\ &= \frac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 1 - \frac{1}{2^2} \\ &= \frac{3}{4} \\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 2$.

When $n = 1$:

$$\text{LHS} = \frac{1}{2}$$

$$\begin{aligned}\text{RHS} &= 1 - \frac{1}{2^1} \\ &= \frac{1}{2} \\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

2 (a) $\sqrt[3]{2} + \sqrt[3]{3} + \sqrt[3]{4} + \sqrt[3]{5}$

(b) $1 - 1! + 2! - 3! + \cdots + (-1)^n n!$

3 (a) $7 + 12 + 17 + \cdots + (5n+2)$

(b) $7 + 12 + 17 + \cdots + (5n+2) = \sum_{r=1}^{n} 5r + 2$

4 C

5 When $n = 1$:

$$\begin{aligned}\text{RHS} &= \frac{1}{2}(1+1) \\ &= 1 \\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$1 + 2 + 3 + \cdots + k = \frac{k}{2}(k+1)$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$1 + 2 + 3 + \cdots + (k+1) = \frac{k+1}{2}(k+2)$

$$\begin{aligned}\text{LHS} &= 1+2+3+\cdots+(k+1)\\ &= 1+2+3+\cdots+k+(k+1)\\ &= \frac{k}{2}(k+1)+(k+1) \quad \text{using [1]}\\ &= \left(\frac{k}{2}+1\right)(k+1)\\ &= \frac{1}{2}(k+1)(k+2)\\ &= \frac{k+1}{2}(k+2)\\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

6 The statement can be rewritten as:

$$\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\frac{3^2}{5\times 7}+\cdots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$$

When $n = 1$:

$$\begin{aligned}\text{LHS} &= \frac{1^2}{1\times 3}\\ &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1(1+1)}{2(2\times 1+1)}\\ &= \frac{2}{6}\\ &= \frac{1}{3}\\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$$\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\frac{3^2}{5\times 7}+\cdots+\frac{k^2}{(2k-1)(2k+1)}$$
$$=\frac{k(k+1)}{2(2k+1)} \qquad [1]$$

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$$\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\frac{3^2}{5\times 7}+\cdots+\frac{(k+1)^2}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)}$$
$$=\frac{(k+1)(k+2)}{2\big(2(k+1)+1\big)}$$

$$\begin{aligned}\text{RHS} &= \frac{(k+1)(k+2)}{2\big(2(k+1)+1\big)}\\ &= \frac{(k+1)(k+2)}{2(2k+3)}\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\frac{3^2}{5\times 7}+\cdots+\frac{k^2}{(2k-1)(2k+1)}\\ &\quad+\frac{(k+1)^2}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)}\\ &= \frac{k(k+1)}{2(2k+1)}+\frac{(k+1)^2}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)} \quad \text{using [1]}\\ &= \frac{k(k+1)}{2(2k+1)}+\frac{(k+1)^2}{(2k+1)(2k+3)}\\ &= \frac{k(k+1)(2k+3)}{2(2k+1)(2k+3)}+\frac{2(k+1)^2}{2(2k+1)(2k+3)}\\ &= \frac{k(k+1)(2k+3)+2(k+1)^2}{2(2k+1)(2k+3)}\\ &= \frac{(k+1)\big(k(2k+3)+2(k+1)\big)}{2(2k+1)(2k+3)}\\ &= \frac{(k+1)\big(2k^2+3k+2k+2\big)}{2(2k+1)(2k+3)}\\ &= \frac{(k+1)\big(2k^2+5k+2\big)}{2(2k+1)(2k+3)}\\ &= \frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)}\\ &= \frac{(k+1)(k+2)}{2(2k+3)}\\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

7 The statement can be rewritten as:

$5+5^2+5^3+\cdots+5^n=\frac{5}{4}\left(5^n-1\right)$ for $n \in \mathbb{Z}^+$.

When $n = 1$:

$$\begin{aligned}\text{RHS} &= \frac{5}{4}\left(5^1-1\right)\\ &= 5\\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$5+5^2+5^3+\cdots+5^k=\frac{5}{4}\left(5^k-1\right)$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$5+5^2+5^3+\cdots+5^{k+1}=\frac{5}{4}\left(5^{k+1}-1\right)$

$$\begin{aligned}\text{LHS} &= 5 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1}\\ &= \frac{5}{4}\left(5^k - 1\right) + 5^{k+1} \quad \text{using [1]}\\ &= \frac{5}{4} \times 5^k - \frac{5}{4} + 5^{k+1}\\ &= \frac{1}{4} \times 5^{k+1} - \frac{5}{4} + 5^{k+1}\\ &= \frac{5}{4} \times 5^{k+1} - \frac{5}{4}\\ &= \frac{5}{4}\left(5^{k+1} - 1\right)\\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

8 When $n = 1$:

$$\begin{aligned}\text{RHS} &= \frac{1}{2}\left(2a + (1-1)d\right)\\ &= \frac{1}{2}(2a)\\ &= a\\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$$\begin{aligned}&a + (a+d) + (a+2d) + \cdots + \left(a + (k-1)d\right)\\ &= \frac{k}{2}\left(2a + (k-1)d\right) \qquad [1]\end{aligned}$$

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$$a + (a+d) + (a+2d) + \cdots + (a+kd) = \frac{k+1}{2}(2a + kd)$$

$$\begin{aligned}\text{LHS} &= a + (a+d) + (a+2d) + \cdots + \left(a + (k-1)d\right) + (a+kd)\\ &= \frac{k}{2}\left(2a + (k-1)d\right) + (a+kd) \quad \text{using [1]}\\ &= \frac{k}{2}(2a + kd - d) + a + kd\\ &= \frac{1}{2}\left(2ak + k^2d - kd\right) + a + kd\\ &= \frac{1}{2}\left(2ak + k^2d - kd + 2a + 2kd\right)\\ &= \frac{1}{2}\left(2ak + k^2d + kd + 2a\right)\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{2}(k+1)(2a + kd)\\ &= \frac{1}{2}\left(2ak + k^2d + 2a + kd\right)\\ &= \text{LHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

9 When $n = 1$:

$$\begin{aligned}\text{LHS} &= 3 \times 2^0\\ &= 3\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 1 + (2 \times 1 - 1)2^1\\ &= 3\\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$$\begin{aligned}&3 \times 2^0 + 5 \times 2^1 + 7 \times 2^2 + 9 \times 2^3 + \cdots + (2k+1)2^{k-1}\\ &= 1 + (2k-1)2^k \qquad [1]\end{aligned}$$

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$$\begin{aligned}&3 \times 2^0 + 5 \times 2^1 + 7 \times 2^2 + 9 \times 2^3 + \cdots + \left(2(k+1)+1\right)2^k\\ &= 1 + \left(2(k+1)-1\right)2^{k+1}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 1 + \left(2(k+1)-1\right)2^{k+1}\\ &= 1 + (2k+1)2^{k+1}\end{aligned}$$

$$\begin{aligned}\text{LHS} &= 3 \times 2^0 + 5 \times 2^1 + 7 \times 2^2 + 9 \times 2^3 + \cdots + \left(2(k+1)+1\right)2^k\\ &= 1 + (2k-1)2^k + \left(2(k+1)+1\right)2^k \quad \text{using [1]}\\ &= 1 + (2k-1)2^k + (2k+3)2^k\\ &= 1 + 2^k\left(2k - 1 + 2k + 3\right)\\ &= 1 + 2^k\left(4k + 2\right)\\ &= 1 + 2^k\left(2(k+1)\right)\\ &= 1 + 2^{k+1}(2k+1)\\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

10 The statement given can be rewritten as: $9^n - 1 = 8A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 9^1 - 1\\ &= 8\\ &= 8 \times 1\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$9^k - 1 = 8A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$9^{k+1} - 1 = 8B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 9^{k+1} - 1\\ &= 9 \times 9^k - 1\\ &= 9(8A + 1) - 1 \quad \text{using [1]}\\ &= 9 \times 8A + 9 - 1\\ &= 9 \times 8A + 8\\ &= 8(9A + 1)\\ &= \text{RHS} \ (\text{since } A \in \mathbb{Z} \Rightarrow 9A + 1 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

11 The statement given can be rewritten as: $5^n + 2 \times 11^n = 3A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 5^1 + 2 \times 11^1 \\ &= 27 \\ &= 3 \times 9 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$5^k + 2 \times 11^k = 3A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$5^{k+1} + 2 \times 11^{k+1} = 3B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 5^{k+1} + 2 \times 11^{k+1} \\ &= 5 \times 5^k + 2 \times 11^{k+1} \\ &= 5 \times \left(3A - 2 \times 11^k\right) + 2 \times 11^{k+1} \quad \text{using [1]} \\ &= 15A - 10 \times 11^k + 2 \times 11 \times 11^k \\ &= 15A - 10 \times 11^k + 22 \times 11^k \\ &= 15A + 12 \times 11^k \\ &= 3\left(5A + 4 \times 11^k\right) \\ &= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow 5A + 4 \times 11^k \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

12 The statement given can be rewritten as: $n^3 + 2n = 3A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 1^3 + 2 \times 1 \\ &= 3 \\ &= 3 \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$.
That is, assume

$k^3 + 2k = 3A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$(k+1)^3 + 2(k+1) = 3B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= (k+1)^3 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 2k + 3k^2 + 3k + 3 \\ &= 3A + 3k^2 + 3k + 3 \quad \text{using [1]} \\ &= 3\left(A + k^2 + k + 1\right) \\ &= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow A + k^2 + k + 1 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

13 Let n be a positive integer. The statement given can be rewritten as: $n(n+1) = 2A$ where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 1(1+1) \\ &= 2 \\ &= 2 \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$k(k+1) = 2A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$. That is, must demonstrate

$(k+1)(k+1+1) = 2B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= (k+1)(k+2) \\ &= k(k+1) + 2(k+1) \\ &= 2A + 2(k+1) \quad \text{using [1]} \\ &= 2(A + k + 1) \\ &= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow A + k + 1 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

14 The statement given can be rewritten as: $3^n - 1 - 2n = 4A$ where $A \in \mathbb{Z}$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= 3^2 - 1 - 2 \times 2 \\ &= 4 \\ &= 4 \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$.
That is, assume

$3^k - 1 - 2k = 4A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k + 1$.
That is, must demonstrate

$3^{k+1} - 1 - 2(k+1) = 4B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 3^{k+1} - 1 - 2(k+1) \\ &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3 \times 3^k - 2k - 3 \\ &= 3(4A + 1 + 2k) - 2k - 3 \quad \text{using [1]} \\ &= 12A + 3 + 6k - 2k - 3 \\ &= 12A + 4k \\ &= 4(3A + k) \\ &= \text{RHS} \quad (\text{since } A, k \in \mathbb{Z} \Rightarrow 3A + k \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

15 Proceeding by induction on n:

When $n = 2$, the statement reduces to Ramanujan's result:

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \ldots}}}}} = 3.$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$.
That is, assume

$$\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+\ldots}}}}} = k+1 \quad [1]$$

Must demonstrate statement is true when $n = k+1$.
That is, must demonstrate

$$\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+(k+4)\sqrt{1+\ldots}}}}} = k+2 \quad [2]$$

As it is not obvious how one could simplify the left-hand side of this equation and incorporate equation [1], attempting to rearrange equation [1] would be a sensible place to start:

$$\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+\ldots}}}}} = k+1$$

$$1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+\ldots}}}} = (k+1)^2 \quad \text{(squaring both sides)}$$

$$k\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+\ldots}}}} = (k+1)^2 - 1$$

$$\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+\ldots}}}} = \frac{(k+1)^2-1}{k} \quad [1']$$

Returning to equation [2]:

$$\begin{aligned}\text{LHS} &= \sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+(k+4)\sqrt{1+\ldots}}}}} \\ &= \frac{(k+1)^2-1}{k} \quad \text{using [1']} \\ &= \frac{k^2+2k+1-1}{k} \\ &= \frac{k^2+2k}{k} \\ &= \frac{\cancel{k}(k+2)}{\cancel{k}} \\ &= k+2 \\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

16 The statement given can be rewritten as: $7^n - 4^n - 3^n = 12A$ where $A \in \mathbb{Z}$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= 7^2 - 4^2 - 3^2 \\ &= 49 - 16 - 9 \\ &= 36 \\ &= 12 \times 3 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$. That is, assume

$7^k - 4^k - 3^k = 12A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$7^{k+1} - 4^{k+1} - 3^{k+1} = 12B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 7^{k+1} - 4^{k+1} - 3^{k+1} \\ &= 7 \times 7^k - 4^{k+1} - 3^{k+1} \\ &= 7\left(12A + 4^k + 3^k\right) - 4^{k+1} - 3^{k+1} \quad \text{using [1]} \\ &= 7 \times 12A + 7 \times 4^k + 7 \times 3^k - 4 \times 4^k - 3 \times 3^k \\ &= 7 \times 12A + 3 \times 4^k + 4 \times 3^k \\ &= 7 \times 12A + 3 \times 4 \times 4^{k-1} + 4 \times 3 \times 3^{k-1} \\ &= 12\left(7A + 4^{k-1} + 3^{k-1}\right) \\ &= \text{RHS} \ \left(\text{since } A, k \in \mathbb{Z} \Rightarrow 7A + 4^{k-1} + 3^{k-1} \in \mathbb{Z}\right)\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

17 The statement given can be rewritten as: $2^n - (-1)^n - 3 = 6A$ where $A \in \mathbb{Z}$.

When $n = 3$:

$$\begin{aligned}\text{LHS} &= 2^3 - (-1)^3 - 3 \\ &= 8 - (-1) - 3 \\ &= 6 \\ &= 6 \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 3$.

Let $k \geq 3$ and assume statement is true for $n = k$.
That is, assume

$2^k - (-1)^k - 3 = 6A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k+1$.
That is, must demonstrate

$2^{k+1} - (-1)^{k+1} - 3 = 6B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 2^{k+1} - (-1)^{k+1} - 3 \\ &= 2 \times 2^k - (-1)^{k+1} - 3 \\ &= 2\left(6A + (-1)^k + 3\right) - (-1)^{k+1} - 3 \quad \text{using [1]} \\ &= 12A + 2 \times (-1)^k + 6 - (-1)^{k+1} - 3 \\ &= 12A + 2 \times (-1)^k - (-1)(-1)^k + 3 \\ &= 12A + 2 \times (-1)^k + (-1)^k + 3 \\ &= 12A + 3(-1)^k + 3 \\ &= 12A + 3\left((-1)^k + 1\right)\end{aligned}$$

Since $(-1)^k = \pm 1$, the expression $12A + 3\left((-1)^k + 1\right)$ is equal to either $12A$ or $12A + 6$. These two expressions are equivalent to $6 \times 2A$ and $6(2A+1)$ respectively, implying that $12A + 3\left((-1)^k + 1\right)$ must be divisible by 6.

Hence, by induction, the original statement must be true for all integers $n \geq 3$.

Chapter 2: Vectors and matrices

RECALL

1 (a) $3\sqrt{7}$ (b) $3\sqrt{6}$ (c) $9\sqrt{2}$ (d) $11\sqrt{3}$

2 (a) $2\sqrt{10}$ (b) $\sqrt{145}$ (c) $\sqrt{170}$ (d) $\sqrt{149}$

3 (a) 0.4226 (b) 0.4067 (c) -0.8192
(d) -0.6157 (e) 0.7002 (f) -1.8807

4 (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) -1
(d) $-\frac{\sqrt{3}}{2}$ (e) $-\frac{\sqrt{3}}{3}$ (f) $\frac{\sqrt{2}}{2}$

5 (a) 21.56° (Note that for $\sin^{-1}$, calculator gives only one 'principal' answer from -90° to 90°.)
(b) 169.14° (c) -26.03° (d) 82.78°
(e) 55.52° (Note that for $\tan^{-1}$, calculator gives only one 'principal' answer from -90° to 90°.)
(f) -34.95°

6 (a) 60°,120° (b) 45°,315° (c) 60°,240°
(d) 120°,240° (e) 135°,315° (f) 225°,315°

7 (a) $c = 34.41,\ A = 54.46°,\ B = 35.54°$
(b) $c = 159.67,\ A = 49.83°,\ B = 40.17°$
(c) $a = 13.51,\ b = 63.58,\ A = 12°$

8 (a) $c = 47.03,\ B = 35.03°,\ C = 88.97°$
(b) $b = 47.77,\ A = 55.40°,\ B = 44.60°$

9 (a) $c = 18.97,\ A = 49.84°,\ B = 65.16°$
(b) $a = 90.34,\ B = 47.53°,\ C = 36.47°$

10 (a) $4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ (b) $12\hat{\boldsymbol{i}} + 35\hat{\boldsymbol{j}}$

11 (a) 11 (b) -6

EXERCISE 2.1

1 Displacement is 500 m N37°E of O.

2 (a) Roslyn: 100 m, west of O
Francois: 1000 m, S53.13°W from O
Gabrielle: 774.47 m, N18.83°E from O
(b) Roslyn: 900 m, Francois: 1400 m, Gabrielle: 800 m

3 11.98 km on a bearing of N61.50°E relative to O.

4 (a) $3\boldsymbol{a}$ (b) $2\boldsymbol{b}$ (c) $2\boldsymbol{a}$ (d) $2\boldsymbol{a} + \boldsymbol{b}$
(e) $3\boldsymbol{b} - \boldsymbol{a}$ (f) $-3\boldsymbol{a} - 3\boldsymbol{b}$ (g) $3\boldsymbol{a} - 2\boldsymbol{b}$ (h) $\boldsymbol{b} - 3\boldsymbol{a}$

5 (a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $\boldsymbol{a} - \boldsymbol{b}$ (c) $-\boldsymbol{a} - \boldsymbol{b}$ (d) $\boldsymbol{b} - \boldsymbol{a}$

6 (a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ (c) $-(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$
(d) $-(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$

7 $\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}),\ \overrightarrow{AN} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b},\ \overrightarrow{MN} = -\frac{1}{2}\boldsymbol{a}$

8 (a) $\boldsymbol{a} + \boldsymbol{b}$ (b) $-\boldsymbol{c}$ (c) $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$
(d) $\boldsymbol{b} + \boldsymbol{c}$ (e) $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

9 $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b},\ \overrightarrow{AC} = \boldsymbol{b} - \boldsymbol{a}$

10 $-\boldsymbol{b} - \boldsymbol{a}$

11 (a) $\hat{\boldsymbol{a}} = \frac{1}{5}\left(4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$ (b) $\hat{\boldsymbol{b}} = \frac{1}{25}\left(24\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}\right)$

12 $\boldsymbol{c} = (13, 112.62°)$

13 (a) $\boldsymbol{r} = \frac{5}{2}\hat{\boldsymbol{i}} + \frac{5\sqrt{3}}{2}\hat{\boldsymbol{j}}$ (b) $\boldsymbol{r} = \frac{25\sqrt{2}}{2}\hat{\boldsymbol{i}} + \frac{25\sqrt{2}}{2}\hat{\boldsymbol{j}}$
(c) $\boldsymbol{r} = -17.10\hat{\boldsymbol{i}} + 28.47\hat{\boldsymbol{j}}$

14 (a) $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}},\ \frac{\sqrt{13}}{13}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ (b) $12\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}},\ \frac{1}{13}\left(12\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$
(c) $-2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}},\ \frac{\sqrt{2}}{2}\left(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)$ (d) $-3\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}},\ \frac{\sqrt{58}}{58}\left(-3\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}\right)$

15 (a) $5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ (b) $2\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$ (c) $-18\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$
(d) $-32\hat{\boldsymbol{i}} - 34\hat{\boldsymbol{j}}$

16 (a) $|\boldsymbol{a}| = \sqrt{10}, |\boldsymbol{b}| = \sqrt{5}, |\boldsymbol{c}| = \sqrt{34}$
(b) $\hat{\boldsymbol{a}} = \frac{\sqrt{10}}{10}\left(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right),\ \hat{\boldsymbol{b}} = \frac{\sqrt{5}}{5}\left(-2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right),\ \hat{\boldsymbol{c}} = \frac{\sqrt{34}}{34}\left(-5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right)$

17 (a) C
(b) For option A: the magnitude is 1, but vector is not parallel to original.
For option B: the magnitude is 1, but vector is not parallel to original.
For option D: the vector is not a unit vector as magnitude does not equal 1.

18 $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

19 (a) (i) $\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$ (ii) $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$ (iii) $\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$
(iv) $\boldsymbol{a} - \boldsymbol{b}$
(b) $\overrightarrow{CN} = \frac{1}{3}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b},\ \overrightarrow{CA} = \boldsymbol{a} - \boldsymbol{b}$. So $\overrightarrow{CN} = \frac{1}{3}\overrightarrow{CA}$, hence points A, N, C are collinear as the two vectors $\overrightarrow{CN}$ and $\overrightarrow{CA}$ have the same direction and point C in common.

20 (a) (i) $\boldsymbol{a} - \boldsymbol{b}$ (ii) $\frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}$ (iii) $\frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$
(iv) $\frac{1}{3}\boldsymbol{a} + \frac{1}{6}\boldsymbol{b}$ (v) $\frac{1}{2}\boldsymbol{b}$
(b) As $\overrightarrow{AC} = \frac{1}{2}\boldsymbol{b}$ then $\overrightarrow{AC} = \frac{1}{2}\overrightarrow{OB}$, meaning $\overrightarrow{AC}$ is parallel to $\overrightarrow{OB}$ and is half the length of $\overrightarrow{OB}$.

EXERCISE 2.2

1 (a) $\overrightarrow{CB} = \boldsymbol{a}$ (b) $\overrightarrow{EF} = \boldsymbol{c}$ (c) $\overrightarrow{FG} = -\boldsymbol{a}$
(d) $\overrightarrow{EA} = -\boldsymbol{d}$ (e) $\overrightarrow{OG} = \boldsymbol{c} + \boldsymbol{d}$ (f) $\overrightarrow{AC} = -\boldsymbol{a} + \boldsymbol{c}$
(g) $\overrightarrow{GA} = \boldsymbol{a} - \boldsymbol{c} - \boldsymbol{d}$ (h) $\overrightarrow{BD} = \boldsymbol{d} - \boldsymbol{a} - \boldsymbol{c}$

2 (a) $|\boldsymbol{a}| = 3, \hat{\boldsymbol{a}} = \frac{1}{3}\left(2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$

(b) $|\boldsymbol{b}| = \sqrt{22}, \hat{\boldsymbol{b}} = \frac{\sqrt{22}}{22}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$

(c) $|\boldsymbol{c}| = \sqrt{30}, \hat{\boldsymbol{c}} = \frac{\sqrt{30}}{30}\left(-5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$

(d) $|\boldsymbol{d}| = \sqrt{38}, \hat{\boldsymbol{d}} = \frac{\sqrt{38}}{38}\left(-2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$

3 (a) $|\boldsymbol{a}| = \sqrt{11}, |\boldsymbol{b}| = \sqrt{38}$

(b) $\hat{\boldsymbol{a}} = \frac{\sqrt{11}}{11}\left(3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right), \hat{\boldsymbol{b}} = \frac{\sqrt{38}}{38}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$

(c) $\boldsymbol{a} + \boldsymbol{b} = 6\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ (d) $4\boldsymbol{a} + 2\boldsymbol{b} = 18\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{k}}$

(e) $2\boldsymbol{c} - 2\boldsymbol{b} + 3\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 9\hat{\boldsymbol{k}}$

(f) $4(\boldsymbol{a} + \boldsymbol{c}) - 3\boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 11\hat{\boldsymbol{k}}$

4 (a) $\overrightarrow{OA} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}, \overrightarrow{OB} = -4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

(b) $\overrightarrow{AB} = -5\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

5 (a) C

(b) The common error is to think the magnitude of a vector is negative when it's always positive or equal to zero.

6 (a) $\overrightarrow{OP} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}, \hat{\overrightarrow{OP}} = \frac{\sqrt{17}}{17}\left(2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$

(b) $\overrightarrow{OP} = -3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}, \hat{\overrightarrow{OP}} = \frac{\sqrt{38}}{38}\left(-3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)$

(c) $\overrightarrow{OP} = -2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}, \hat{\overrightarrow{OP}} = \frac{\sqrt{30}}{30}\left(-2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$

(d) $\overrightarrow{OP} = -3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}, \hat{\overrightarrow{OP}} = \frac{\sqrt{70}}{70}\left(-3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}\right)$

7 (a) $|\boldsymbol{a}| = \sqrt{17}, |\boldsymbol{b}| = \sqrt{3}, |\boldsymbol{c}| = \sqrt{38}$

(b) $\hat{\boldsymbol{a}} = \frac{\sqrt{17}}{17}\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right), \hat{\boldsymbol{b}} = \frac{\sqrt{3}}{3}\left(-\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right),$

$\hat{\boldsymbol{c}} = \frac{\sqrt{38}}{38}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}\right)$

8 $\overrightarrow{AB} = -2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 10\hat{\boldsymbol{k}}, \left|\overrightarrow{AB}\right| = \sqrt{108} = 6\sqrt{3}$

$\hat{\overrightarrow{AB}} = -\frac{\sqrt{3}}{9}\left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$

9 (a) $\overrightarrow{AB} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}, \overrightarrow{AC} = 3\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}} + 9\hat{\boldsymbol{k}}$

(b) $\overrightarrow{AC} = 3\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}} + 9\hat{\boldsymbol{k}}$

$= 3\left(\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$

$= 3\overrightarrow{AB}$

$\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{AB}$, hence the vectors are parallel.

10 $5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

11 $\overrightarrow{AB} = (6-5)\hat{\boldsymbol{i}} + (1-2)\hat{\boldsymbol{j}} + (4-(-2))\hat{\boldsymbol{k}}$

$= \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$

$\overrightarrow{CD} = (2-3)\hat{\boldsymbol{i}} + (-2-(-3))\hat{\boldsymbol{j}} + (-4-2)\hat{\boldsymbol{k}}$

$= -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$

$\overrightarrow{AB} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$

$= -\overrightarrow{CD}$

Hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

12 (a) $\boldsymbol{u} = \frac{25\sqrt{2}}{2}\hat{\boldsymbol{i}} - \frac{25\sqrt{6}}{2}\hat{\boldsymbol{j}} - 25\sqrt{2}\hat{\boldsymbol{k}}$

(b) $\boldsymbol{v} = \left(5\sqrt{6}, 135°, 54.74°\right)$ (2 d.p.)

13 (a) $(x-4)^2 + y^2 + (z-1)^2 = 36$

(b) $(x+3)^2 + (y+1)^2 + (z-5)^2 = 3$

14 (a) Centre is $(1, 2, 0)$ and radius is 2.

(b) Centre is $(-5, 1, -2)$ and radius is $\sqrt{11}$.

15 (a) $\overrightarrow{AB} = -3\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$

(b) $\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 3^2} = \sqrt{18}$ or $3\sqrt{2}$

$\left|\overrightarrow{OA}\right| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$\left|\overrightarrow{OB}\right| = \sqrt{(-1)^2 + (-2)^2 + 2^2} = 3$

Hence ΔOAB is isosceles as $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = 3$.

(c) $M = \left(\frac{1}{2}, -\frac{1}{2}, 2\right)$ (d) $\overrightarrow{OM} = \frac{1}{2}\hat{\boldsymbol{i}} - \frac{1}{2}\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

(e) $\left|\overrightarrow{OM}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 2^2} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$ or $\frac{3\sqrt{2}}{2}$

As $\left|\overrightarrow{AB}\right| = 3\sqrt{2}$ and M is the midpoint of $\overrightarrow{AB}$ then $\overrightarrow{AM} = \overrightarrow{BM} = \frac{3\sqrt{2}}{2}$. Hence M is equidistant from vertices O, A, and B, the distance being $\frac{3\sqrt{2}}{2}$.

16 $x = 5, y = 12, z = -3$

17 $y = -\frac{3}{5}, z = -\frac{1}{10}$

18 The area of the circle is 16π square units.

19 5.22 light years (2 d.p.).

EXERCISE 2.3

1 (a) $\boldsymbol{a} \cdot \boldsymbol{b} = 8$ (b) $\boldsymbol{a} \cdot \boldsymbol{b} = 20$ (c) $\boldsymbol{a} \cdot \boldsymbol{b} = -6$

(d) $\boldsymbol{a} \cdot \boldsymbol{b} = -37$ (e) $\boldsymbol{a} \cdot \boldsymbol{b} = 2$ (f) $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

2 $z = \frac{3}{2}$

3 If two vectors are parallel, one will be a (scalar) multiple of the other.

$\boldsymbol{c} = 4\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}} - 14\hat{\boldsymbol{k}}$

$\boldsymbol{c} = 2\left(2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}\right) = 2\boldsymbol{a}$

Hence, $\boldsymbol{c}$ is parallel to $\boldsymbol{a}$.

If two vectors are perpendicular, their dot product will equal zero.

$\boldsymbol{a} \cdot \boldsymbol{b} = 2 \times 3 + 5 \times 3 - 7 \times 3$

$\boldsymbol{a} \cdot \boldsymbol{b} = 0$

Hence, $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$.

4 (a) $3\sqrt{2}$ (b) $-3\sqrt{2}$ (c) $-3\sqrt{2}$ (d) 0

5 60°

6 A

7 B

8 (a) $\theta = 54°$ (b) $\theta = 70°$ (c) $\theta = 74°$

9 If two vectors are at right angles, their scalar product will be zero.

$\overrightarrow{AC} = (5-2)\hat{\boldsymbol{i}} + (-2-3)\hat{\boldsymbol{j}} + (4-2)\hat{\boldsymbol{k}}$
$= 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

$\overrightarrow{BD} = (1-(-2))\hat{\boldsymbol{i}} + (5-2)\hat{\boldsymbol{j}} + (7-4)\hat{\boldsymbol{k}}$
$= 3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

$\overrightarrow{AC} \cdot \overrightarrow{BD} = 3 \times 3 - 5 \times 3 + 2 \times 3 = 0$

Hence, $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$.

10 If two vectors are at right angles, their scalar product will be zero.

$\overrightarrow{AB} = (2-3)\hat{\boldsymbol{i}} + (-3-1)\hat{\boldsymbol{j}} + (5-1)\hat{\boldsymbol{k}}$
$= -\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

$\overrightarrow{BC} = (0-2)\hat{\boldsymbol{i}} + (-2-(-3))\hat{\boldsymbol{j}} + (2-5)\hat{\boldsymbol{k}}$
$= -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

$\overrightarrow{AC} = (0-3)\hat{\boldsymbol{i}} + (-2-1)\hat{\boldsymbol{j}} + (2-1)\hat{\boldsymbol{k}}$
$= -3\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

Checking scalar products, $\overrightarrow{AB} \cdot \overrightarrow{BC} = -14$, $\overrightarrow{AB} \cdot \overrightarrow{AC} = 19$, $\overrightarrow{BC} \cdot \overrightarrow{AC} = -2 \times -3 - 1 \times 3 - 3 \times 1 = 0$.

Hence, the triangle is right-angled with the right angle at vertex C (that is, $\angle BCA = 90°$).

11 (a) $-2p^2 - 20p$ (b) $p = 0$ and $p = -10$

12 (a) $\dfrac{-\sqrt{30}}{2}$ (b) $-\dfrac{1}{2}\left(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$

(c) $\dfrac{5}{2}\hat{\boldsymbol{i}} - \dfrac{1}{2}\hat{\boldsymbol{k}}$

13 (a) $\dfrac{-15\sqrt{14}}{14}$ (b) $-\dfrac{15}{14}\left(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\right)$

(c) $\dfrac{44}{14}\hat{\boldsymbol{i}} + \dfrac{13}{14}\hat{\boldsymbol{j}} + \dfrac{25}{14}\hat{\boldsymbol{k}}$

14 $\dfrac{13}{21}\left(-\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}\right), \dfrac{34}{21}\underset{\sim}{i} + \dfrac{37}{21}\underset{\sim}{j} + \dfrac{10}{21}\underset{\sim}{k}$

15 $\boldsymbol{u} = -\dfrac{19}{14}\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\right), \boldsymbol{v} = \dfrac{1}{14}\left(10\hat{\boldsymbol{i}} - 23\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$

16 Internal angles are 31.73°, 88.07° and 60.20° (to 2 d.p.). All angles are within the specified range, so the shade sail meets the specifications.

EXERCISE 2.4

1 (a) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.

(b) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.

(c) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.

(d) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.

(e) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.

(f) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.

(g) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent.

(h) $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.

2 B

3 (a) $x\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right) + y\left(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right) + z\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right) = 0$

$(2x + y + z)\hat{\boldsymbol{i}} + (-3x + y - 2z)\hat{\boldsymbol{j}} = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$

$2x + y + z = 0 \quad [1]$
$-3x + y - 2z = 0 \quad [2]$

From [2], $y = 3x + 2z \quad [3]$

Substituting [3] into [1] gives:

$2x + (3x + 2z) + z = 0$
$5x + 3z = 0$
Thus $z = -\dfrac{5}{3}x$ and $y = -\dfrac{x}{3}$

Thus, since y and z are functions of x, the vectors are linearly dependent.

(b) $x\left(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right) + y\left(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right) + z\left(-\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right) = \boldsymbol{0}$

$(3x + 2y - z)\hat{\boldsymbol{i}} + (2x - 3y + 3z)\hat{\boldsymbol{j}} = 0i + 0\hat{\boldsymbol{j}}$

$3x + 2y - z = 0 \quad [1]$
$2x - 3y + 3z = 0 \quad [2]$

From [1], $z = 3x + 2y \quad [3]$

Substituting [3] into [2] gives:

$2x - 3y + 3(3x + 2y) = 0$
$11x + 3y = 0$
Thus $y = -\dfrac{11}{3}x$ and $z = \dfrac{-13x}{3}$

Thus, since y and z are functions of x, the vectors are linearly dependent.

4 (a) $x = \dfrac{3}{5}$, $y = -\dfrac{1}{2}$

(b) $x = \pm\sqrt{\dfrac{5}{3}} = \pm\dfrac{\sqrt{15}}{3}$, $y = 2$ or $y = \dfrac{1}{2}$

5 $x = 2$ or $\frac{1}{2}$, $y = \frac{5}{3}$

6 **(a)** $x = 3, y = -1, z = 0$

(b) $x = -\frac{1}{2}, y = 0, z = -3$

(c) $x = 2, -\frac{1}{3}, y = \pm 3, z = -2$

(d) $x = \pm 4,\ y = \pm 2\sqrt{3}, z = -2, -\frac{4}{3}$

7 **(a)** D

(b) The two given vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, are non-parallel vectors defined in the $\hat{\boldsymbol{j}}$–$\hat{\boldsymbol{k}}$ plane. Any additional vector with an $\hat{\boldsymbol{i}}$ component therefore creates a set of vectors that are necessarily linearly independent.

Option A: vector $\boldsymbol{c} = 5\hat{\boldsymbol{i}}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$, creating a set of linearly independent vectors so it is incorrect.

Option B: vector $\boldsymbol{c} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ has an $\hat{\boldsymbol{i}}$ component and is not coplanar with $\boldsymbol{a}$ and $\boldsymbol{b}$ so it is incorrect.

Option C: vector $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{k}}$ has an $\hat{\boldsymbol{i}}$ component and is not coplanar with $\boldsymbol{a}$ and $\boldsymbol{b}$ so it is incorrect.

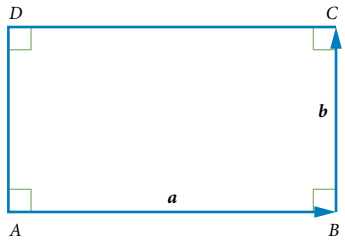
EXERCISE 2.5

1 **(a)** $\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$ **(b)** $\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(c) $2\left(\overrightarrow{BX} \cdot \overrightarrow{BX} + \overrightarrow{AX} \cdot \overrightarrow{AX}\right) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$

$$2\left(\overrightarrow{AX}^2 + \overrightarrow{BX}^2\right) = 2\left(\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\right)^2 + 2\left(\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\right)^2$$
$$= 2\left(\frac{1}{4}\left(|\boldsymbol{b}|^2 + |\boldsymbol{a}|^2\right) + \frac{1}{4}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)\right)$$
$$= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$$

2



Let $\overrightarrow{AB} = \boldsymbol{a}$.

$\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore \overrightarrow{DC} = \boldsymbol{a}$

Let $\overrightarrow{BC} = \boldsymbol{b}$.

$\overrightarrow{BC} = \overrightarrow{AD}$

$\therefore \overrightarrow{AD} = \boldsymbol{b}$

Also $\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$ and $\overrightarrow{BD} = \boldsymbol{b} - \boldsymbol{a}$.

Let M be the point of intersection of $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

$\overrightarrow{AD} = \boldsymbol{b}$

$\overrightarrow{AD} = \overrightarrow{AM} + \overrightarrow{MD}$
$= \overrightarrow{AM} - \overrightarrow{DM}$
$= m\left(\overrightarrow{AC}\right) + n\left(\overrightarrow{BD}\right)$

$\boldsymbol{b} = m(\boldsymbol{a} + \boldsymbol{b}) + n(\boldsymbol{b} - \boldsymbol{a})$

$m - n = 0 \quad [1]$
$m + n - 1 = 0 \quad [2]$

$m = \frac{1}{2}$ and $n = \frac{1}{2}$

Thus, M is the midpoint of the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

3 Show that $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

Let $\overrightarrow{AM} = \boldsymbol{a}$ and $\overrightarrow{BM} = \boldsymbol{b}$

$\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MB}$
$= \overrightarrow{AM} - \overrightarrow{BM}$
$= \boldsymbol{a} - \boldsymbol{b}$

$\overrightarrow{BC} = \overrightarrow{BM} + \overrightarrow{MC}$
$= \boldsymbol{b} + \boldsymbol{a}$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{b} + \boldsymbol{a})$
$= |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2$

Since $AM = BM, |\boldsymbol{a}| = |\boldsymbol{b}|$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = |\boldsymbol{a}|^2 - |\boldsymbol{a}|^2$
$= 0$

Hence $\angle ABC$ is a right angle.

4 **(a)** $\overrightarrow{AE} = \boldsymbol{a} + \boldsymbol{b}$

(b) $\overrightarrow{AC} = m\overrightarrow{AE}$
$= m(\boldsymbol{a} + \boldsymbol{b})$

$\overrightarrow{BC} = n\overrightarrow{BD}$
$= n\boldsymbol{b}$

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$
$= \overrightarrow{AC} - \overrightarrow{BC}$
$= m(\boldsymbol{a} + \boldsymbol{b}) - n\boldsymbol{b}$

Since $\overrightarrow{AB} = 2\boldsymbol{a}$

$2\boldsymbol{a} = m(\boldsymbol{a} + \boldsymbol{b}) - n\boldsymbol{b}$

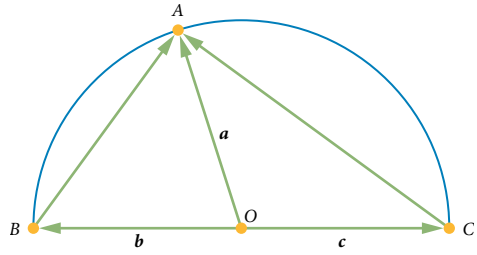
so $m = 2$ and $m - n = 0$

$\therefore n = 2$

$\overrightarrow{BC} = 2\overrightarrow{BD}$

$\therefore \overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$

5 Consider ΔABC:



Let $\overrightarrow{OA} = \boldsymbol{a}$, let $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

The angle $\angle BAC$ will be a right angle if $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are orthogonal, in other words, if $\overrightarrow{BA} \cdot \overrightarrow{CA} = 0$.

Because $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ all represent the radius of a circle, it follows that $|\boldsymbol{a}| = \boldsymbol{b} = |\boldsymbol{c}|$. Also, because they lie on the same diameter, $\boldsymbol{c} = -\boldsymbol{b}$ (they are in opposite directions).

Now $\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b}$ and $\overrightarrow{CA} = \boldsymbol{a} - \boldsymbol{c}$.

$$\begin{aligned}\overrightarrow{BA} \cdot \overrightarrow{CA} &= (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{c}) \\ &= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c}\end{aligned}$$

Because $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$ and $\boldsymbol{c} = -\boldsymbol{b}$,

$$\begin{aligned}\overrightarrow{BA} \cdot \overrightarrow{CA} &= |\boldsymbol{a}|^2 + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{a} - |\boldsymbol{b}|^2 \\ &= |\boldsymbol{a}|^2 + \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b} - |\boldsymbol{b}|^2 \\ &= |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2\end{aligned}$$

And because $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$,

$$\begin{aligned}\overrightarrow{BA} \cdot \overrightarrow{CA} &= |\boldsymbol{a}|^2 - |\boldsymbol{a}|^2 \\ &= 0\end{aligned}$$

Because $\overrightarrow{BA} \cdot \overrightarrow{CA} = 0$, then $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are perpendicular.

6 Show that $\overrightarrow{OC} \cdot \overrightarrow{AC} = 0$.

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

$$\overrightarrow{AC} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) \\ &= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} \\ &= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\end{aligned}$$

$$\begin{aligned}\overrightarrow{OC} \cdot \overrightarrow{AC} &= \left(\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\right) \cdot \left(\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\right) \\ &= \frac{1}{4}\left(\boldsymbol{b}^2 - \boldsymbol{a}^2\right)\end{aligned}$$

Since $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ and both vectors are radii,

$$|\boldsymbol{b}| = |\boldsymbol{a}|$$

$$\begin{aligned}\overrightarrow{OC} \cdot \overrightarrow{AC} &= \frac{1}{4}\left(\boldsymbol{a}^2 - \boldsymbol{a}^2\right) \\ &= 0\end{aligned}$$

Hence $\angle AOC$ is a right angle.

7 (a) $\overrightarrow{AM} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$

(b)

$$\begin{aligned}\overrightarrow{CN} &= -\overrightarrow{AC} + \frac{2}{3}\overrightarrow{AB} \\ &= -\boldsymbol{c} + \frac{2}{3}\boldsymbol{a} \\ &= \frac{2}{3}\boldsymbol{a} - \boldsymbol{c}\end{aligned}$$

(c) $\overrightarrow{AO} = \frac{x}{2}(\boldsymbol{a} + \boldsymbol{c}), \overrightarrow{ON} = y\left(\frac{2}{3}\boldsymbol{a} - \boldsymbol{c}\right)$

(d)

$$\overrightarrow{AO} + \overrightarrow{ON} + \overrightarrow{NA} = \boldsymbol{0}$$

$$\frac{x}{2}(\boldsymbol{a} + \boldsymbol{c}) + y\left(\frac{2}{3}\boldsymbol{a} - \boldsymbol{c}\right) - \frac{2}{3}\boldsymbol{a} = \boldsymbol{0}$$

$$\frac{x}{2}\boldsymbol{a} + \frac{x}{2}\boldsymbol{c} + \frac{2y}{3}\boldsymbol{a} - y\boldsymbol{c} - \frac{2}{3}\boldsymbol{a} = \boldsymbol{0}$$

$$\left(\frac{x}{2} + \frac{2y}{3} - \frac{2}{3}\right)\boldsymbol{a} + \left(\frac{x}{2} - y\right)\boldsymbol{c} = \boldsymbol{0}$$

$$\left(\frac{3x + 4y - 4}{6}\right)\boldsymbol{a} + \left(\frac{x - 2y}{2}\right)\boldsymbol{c} = \boldsymbol{0}$$

(e) $\frac{3x + 4y - 4}{6} = 0$ and $\frac{x - 2y}{2} = 0$

$3x + 4y = 4$ [1]

$x - 2y = 0$ [2]

Adding [1] and 2 × [2] gives: $5x = 4$

$\therefore\ x = \frac{4}{5}$

Substituting $x = \frac{4}{5}$ in [2] gives: $\frac{4}{5} - 2y = 0$

$\therefore\ y = \frac{2}{5}$

Now, if $\overrightarrow{AO} = x\overrightarrow{AM}$ and $\overrightarrow{ON} = y\overrightarrow{CN}$, then

$\overrightarrow{AO} = \frac{4}{5}\overrightarrow{AM}$ and $\overrightarrow{ON} = \frac{2}{5}\overrightarrow{CN}$

$$\begin{aligned}\overrightarrow{AM} &= \overrightarrow{AO} + \overrightarrow{OM} \\ \overrightarrow{OM}l &= \overrightarrow{AM} - \overrightarrow{AO} \\ &= \overrightarrow{AM} - \frac{4}{5}\overrightarrow{AM} \\ &= \frac{1}{5}\overrightarrow{AM}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{CN} &= \overrightarrow{CO} + \overrightarrow{ON} \\ \overrightarrow{CO}l &= \overrightarrow{CN} - \overrightarrow{ON} \\ &= \overrightarrow{CN} - \frac{2}{5}\overrightarrow{CN} \\ &= \frac{3}{5}\overrightarrow{CN}\end{aligned}$$

$\therefore \left|\overrightarrow{OM}\right| = \frac{1}{5}\left|\overrightarrow{AM}\right|$ and $\left|\overrightarrow{CO}\right| = \frac{3}{5}\left|\overrightarrow{CN}\right|$

8 Start by determining the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\overrightarrow{OA} = \overrightarrow{CB} = \boldsymbol{a}$$

$$OC = \overrightarrow{AB} = \boldsymbol{b}$$

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \boldsymbol{a} + \boldsymbol{b}$$

$$\overrightarrow{AC} = \overrightarrow{AB} - \overrightarrow{BC} = \boldsymbol{b} - \boldsymbol{a}$$

Then calculate the sum of the squares of the lengths of the sides of the parallelogram:

$$\begin{aligned}&\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{OC}\right|^2 + \left|\overrightarrow{CB}\right|^2 \\ &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + |\boldsymbol{b}|^2 + |\boldsymbol{a}|^2 \\ &= 2\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)\end{aligned}$$

Determine the sum of the squares of the lengths of the diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\left|\overrightarrow{OB}\right|^2 + \left|\overrightarrow{AC}\right|^2$$
$$= |\boldsymbol{a}+\boldsymbol{b}|^2 + |\boldsymbol{b}-\boldsymbol{a}|^2$$
$$= (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b}) + (\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a})$$
$$= \boldsymbol{a}\cdot\boldsymbol{a} + 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{b} - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{a}$$
$$= 2\boldsymbol{a}\cdot\boldsymbol{a} + 2\boldsymbol{b}\cdot\boldsymbol{b}$$
$$= 2|\boldsymbol{a}|^2 + 2|\boldsymbol{b}|^2$$
$$= 2\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)$$

Now $\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{OC}\right|^2 + \left|\overrightarrow{CB}\right|^2 = 2\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)$

And $\left|\overrightarrow{OB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 2\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)$

Therefore $\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{OC}\right|^2 + \left|\overrightarrow{CB}\right|^2 = \left|\overrightarrow{OB}\right|^2 + \left|\overrightarrow{AC}\right|^2$

9 (a) $\overrightarrow{OD} = \boldsymbol{a}+\boldsymbol{b}$, $\overrightarrow{OE} = \boldsymbol{a}+\boldsymbol{c}$, $\overrightarrow{OF} = \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$, $\overrightarrow{OG} = \boldsymbol{b}+\boldsymbol{c}$

(b) $\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{EC}$

$\overrightarrow{AG} = \overrightarrow{AE} + \overrightarrow{AB}$

$\overrightarrow{AG} = \boldsymbol{c}+\boldsymbol{b}-\boldsymbol{a}$

Let M be the midpoint of $\overline{\text{AG}}$.

$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$\overrightarrow{OM} = \boldsymbol{a} + \frac{1}{2}(\boldsymbol{c}+\boldsymbol{b}-\boldsymbol{a})$

$\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$

Let P be the midpoint of $\overline{OF}$.

$\overrightarrow{OP} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$

$\overrightarrow{OM} = \overrightarrow{OP}$, hence

$M = P$ and the diagonals $\overrightarrow{OF}$ and $\overrightarrow{AG}$ meet at a common point and bisect each other.

$\overrightarrow{CD} = \boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}$

Let Q be the midpoint of $\overline{CD}$.

$\overrightarrow{CQ} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c})$

$\overrightarrow{OQ} = \boldsymbol{c} + \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c})$

$\overrightarrow{OQ} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$

but $\overrightarrow{OM} = \overrightarrow{OP} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$

Hence $M = P = Q$ and the diagonals $\overrightarrow{OF}$, $\overrightarrow{CD}$ and $\overrightarrow{AG}$ meet at a common point and bisect each other.

$\overrightarrow{EB} = \boldsymbol{b}-\boldsymbol{a}-\boldsymbol{c}$

Let R be the midpoint of $\overline{EB}$, then

$\overrightarrow{OR} = \boldsymbol{a}+\boldsymbol{c} + \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a}-\boldsymbol{c})$

$\overrightarrow{OR} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$

Hence $\overrightarrow{OM} = \overrightarrow{OP} = \overrightarrow{OQ} = \overrightarrow{OR}$ and $M = P = Q = R$ and all four diagonals meet at a common point and bisect each other.

10 Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$.

Let P be the point of intersection of $\overrightarrow{OB}$ and $\overrightarrow{MC}$.

$\overrightarrow{OB} = \boldsymbol{a}+\boldsymbol{b}$

Let m be a scalar factor such that

$\overrightarrow{OP} = m\overrightarrow{OB}$

$= m(\boldsymbol{a}+\boldsymbol{b})$

$\overrightarrow{MC} = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$

Let n be a scalar factor such that

$\overrightarrow{MP} = n\overrightarrow{MC}$

$= n\left(-\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}\right)$

$\frac{1}{2}\boldsymbol{a} = m(\boldsymbol{a}+\boldsymbol{b}) + n\left(-\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}\right)$

$\frac{1}{2}\boldsymbol{a} = \left(m + \frac{1}{2}n\right)\boldsymbol{a} + (m-n)\boldsymbol{b}$

$m = n$ and $\frac{1}{2} = m + \frac{1}{2}n$

$\frac{1}{2} = m + \frac{1}{2}m$

$\frac{1}{2} = \frac{3m}{2}$

$m = \frac{1}{3}$

Hence P is a point of trisection of $\overrightarrow{OB}$.

EXERCISE 2.6

1 (a) $\boldsymbol{a}\times\boldsymbol{b}$ is a vector of magnitude $10\sqrt{2}$ with direction into the plane.

(b) $\boldsymbol{a}\times\boldsymbol{b}$ is a vector of magnitude 30 with direction out of the plane.

(c) $\boldsymbol{a}\times\boldsymbol{b}$ is a vector of magnitude 27 with direction into the plane.

2 (a) $10\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} - 11\hat{\boldsymbol{k}}$ **(b)** $-22\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

(c) $-6\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} + 13\hat{\boldsymbol{k}}$ **(d)** $-12\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$

3 (a) $28\hat{\boldsymbol{i}} - 18\hat{\boldsymbol{j}} + 22\hat{\boldsymbol{k}}$ **(b)** $-\hat{\boldsymbol{i}} - 22\hat{\boldsymbol{j}} - 13\hat{\boldsymbol{k}}$

4 (a) $-53\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$ **(b)** $41\hat{\boldsymbol{i}} + 66\hat{\boldsymbol{j}} - 26\hat{\boldsymbol{k}}$

(c) 0

5 (a) 7 cubic units **(b)** 77 cubic units **(c)** 54 cubic units

6 1 cubic unit

7 A vector equation of the plane is $\boldsymbol{p}\cdot\left(2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right) = 4$.

8 (a) D

(b) The dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ vectors is equal to zero because these are orthogonal, therefore option **A** is false.

The direction of a vector is independent of its drawn position. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ don't need to be drawn at the same point to be perpendicular, therefore option **B** is false.

Any vector in the *ZOY* plane is orthogonal to $\boldsymbol{a}$. Vector $\boldsymbol{b}$ is not the only orthogonal vector to $\boldsymbol{a}$.

9 (a) $\frac{\sqrt{2}}{18}\left(-11\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}-5\hat{\boldsymbol{k}}\right)$ (b) $\frac{10\sqrt{78}}{234}\left(-23\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}-13\hat{\boldsymbol{k}}\right)$

10 (a) Area $=\sqrt{465}$ square units (b) Area $=3\sqrt{42}$ square units

(c) Area $=\sqrt{429}$ square units

11 (a) Area $=\frac{\sqrt{41}}{2}$ square units

(b) Area $=\frac{\sqrt{1130}}{2}$ square units

12 (a) 12 cubic units (b) 40 cubic units (c) 9 cubic units

13 (a) $x-y+2z=1$ (b) $3x-y-z=5$ (c) $x-5y+2z=-4$

14 A vector perpendicular to the plane $2x-3y+z=6$ will be $n\left(2\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}\right)$, where $n\in\mathbb{R}$.

15 (a) Let $\boldsymbol{a}=a_1\hat{\boldsymbol{i}}+a_2\hat{\boldsymbol{j}}+a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=b_1\hat{\boldsymbol{i}}+b_2\hat{\boldsymbol{j}}+b_3\hat{\boldsymbol{k}}$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors, then $\boldsymbol{a}=m\boldsymbol{b}$, $m\in\mathbb{R}$.

Thus, $\boldsymbol{a}=m\left(b_1\hat{\boldsymbol{i}}+b_2\hat{\boldsymbol{j}}+b_3\hat{\boldsymbol{k}}\right)=mb_1\hat{\boldsymbol{i}}+mb_2\hat{\boldsymbol{j}}+mb_3\hat{\boldsymbol{k}}$.

Now,

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{b}&=\begin{vmatrix}mb_2 & mb_3\\ b_2 & b_3\end{vmatrix}\hat{\boldsymbol{i}}-\begin{vmatrix}mb_1 & mb_3\\ b_1 & b_3\end{vmatrix}\hat{\boldsymbol{j}}+\begin{vmatrix}mb_1 & mb_2\\ b_1 & b_2\end{vmatrix}\hat{\boldsymbol{k}}\\&=(mb_2b_3-mb_3b_2)\hat{\boldsymbol{i}}-(mb_1b_3-mb_3b_1)\hat{\boldsymbol{j}}\\&\quad+(mb_1b_2-mb_2b_1)\hat{\boldsymbol{k}}\\&=0\hat{\boldsymbol{i}}+0\hat{\boldsymbol{j}}+0\hat{\boldsymbol{k}}\\&=\boldsymbol{0}\end{aligned}$$

(b) For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin(\theta)$, where θ is the angle between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

If $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$, then $|\boldsymbol{a}\times\boldsymbol{b}|=\boldsymbol{0}$ and so $|\boldsymbol{a}|=0$, $|\boldsymbol{b}|=0$ or $\sin(\theta)=0$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors, if $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$, then $\sin(\theta)=0$.

If $\sin(\theta)=0$, then $\theta=0°$ or $180°$ and so $\boldsymbol{a}$ and $\boldsymbol{b}$ must be parallel vectors.

16 (a) $\boldsymbol{a}\times\boldsymbol{b}=-7\hat{\boldsymbol{i}}+10\hat{\boldsymbol{j}}+39\hat{\boldsymbol{k}}$, $\boldsymbol{a}\times\boldsymbol{c}=-7\hat{\boldsymbol{i}}+10\hat{\boldsymbol{j}}+39\hat{\boldsymbol{k}}$, $\boldsymbol{b}\times\boldsymbol{c}=-7\hat{\boldsymbol{i}}+10\hat{\boldsymbol{j}}+39\hat{\boldsymbol{k}}$

(b) $\text{Area}=\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$

$=\frac{1}{2}\sqrt{(-7)^2+10^2+39^2}$

$=\frac{1}{2}\sqrt{1670}$ square units

EXERCISE 2.7

1 (a) $\boldsymbol{r}=-3\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+\lambda(2\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}})$ (b) $\boldsymbol{r}=7\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+\lambda(-4\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}})$

(c) $\boldsymbol{r}=\begin{bmatrix}4\\-8\end{bmatrix}+\lambda\begin{bmatrix}5\\-9\end{bmatrix}$ (d) $\boldsymbol{r}=\begin{bmatrix}1\\0\end{bmatrix}+\lambda\begin{bmatrix}-16\\3\end{bmatrix}$

2 (a) $\boldsymbol{r}=\begin{bmatrix}6\\0\end{bmatrix}+\lambda\begin{bmatrix}3\\-2\end{bmatrix}$ or $\boldsymbol{r}=6\hat{\boldsymbol{i}}+\lambda\left(3\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}\right)$

(b) $\boldsymbol{r}=\begin{bmatrix}12\\-4\end{bmatrix}$ or $\boldsymbol{r}=12\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}$ (c) $\lambda=-1$

3 $\boldsymbol{r}=(2+3\lambda)\hat{\boldsymbol{i}}+(4-2\lambda)\hat{\boldsymbol{j}}$ or $\boldsymbol{r}=\begin{bmatrix}2+3\lambda\\4-2\lambda\end{bmatrix}$

4 (a) $\underset{\sim}{r}=\begin{bmatrix}1\\3\end{bmatrix}+\lambda\begin{bmatrix}7\\2\end{bmatrix}$ or $\underset{\sim}{r}=(1+7\lambda)\underset{\sim}{i}+(3+2\lambda)\underset{\sim}{j}$

(b) $2x-7y=-19$

5 (a) $\boldsymbol{r}=\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+\lambda\left(-3\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}\right)$ (b) $\boldsymbol{r}=-3\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+\lambda\left(8\hat{\boldsymbol{i}}-7\hat{\boldsymbol{j}}\right)$

(c) $\boldsymbol{r}=-3\hat{\boldsymbol{j}}+\lambda\left(-6\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}\right)$ (d) $\boldsymbol{r}=9\hat{\boldsymbol{i}}+\lambda\left(-13\hat{\boldsymbol{i}}-9\hat{\boldsymbol{j}}\right)$

6 The vector equation of the line segment is $\boldsymbol{r}=\left(\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}\right)+\lambda\left(2\hat{\boldsymbol{i}}-6\hat{\boldsymbol{j}}\right)$, $0\le\lambda\le1$.

7 (a) $\boldsymbol{r}=2\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}+\lambda\left(3\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}\right)$ (b) $\boldsymbol{r}=-5\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+\lambda\left(3\hat{\boldsymbol{i}}-9\hat{\boldsymbol{j}}\right)$

(c) $\boldsymbol{r}=-3\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+\lambda\left(-12\hat{\boldsymbol{i}}+11\hat{\boldsymbol{j}}\right)$

(d) $\boldsymbol{r}=-2\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+\lambda\left(-6\hat{\boldsymbol{i}}+13\hat{\boldsymbol{j}}\right)$

8 (a) $x=3+4\lambda$, $y=5+7\lambda$ (b) $x=0-7\lambda$, $y=-3-\lambda$

(c) $x=4+12\lambda$, $y=-5\lambda$

(d) $\boldsymbol{x}=-4-\frac{1}{2}\lambda$, $\boldsymbol{y}=5+\frac{1}{4}\lambda$

9 (a) $\boldsymbol{r}=\begin{bmatrix}-2\\1\end{bmatrix}+\lambda\begin{bmatrix}5\\-4\end{bmatrix}$ or $\boldsymbol{r}=(-2+5\lambda)\hat{\boldsymbol{i}}+(1-4\lambda)\hat{\boldsymbol{j}}$

(b) $\hat{\boldsymbol{b}}=\frac{1}{\sqrt{41}}\left(5\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}\right)$

(c) $4x+5y=-3$

10 (a) $3x+y=11$ (b) $7x-3y=-13$ (c) $2x+y=-7$

(d) $4x+8y=38$ or $2x+4y=19$ (e) $x=2$ (f) $y=-9$

11 $\theta=145°$

12 (a) $\theta=12°$ (b) $\theta=36°$ (c) $\theta=100°$ (d) $\theta=99°$

13 $\left(\frac{-46}{19},\frac{-36}{19}\right)$

14 (a) $\left(0,\frac{23}{2}\right)$ (b) $(1,-14)$

(c) $(-1,-6)$ (d) $\left(-\frac{47}{19},\frac{182}{19}\right)$

15 (a) $12\hat{i} - 15\hat{j}$ (b) $\frac{9}{2}\hat{i} + \frac{7}{4}\hat{j}$ (c) $-\frac{7}{2}\hat{i} + \frac{33}{2}\hat{j}$

(d) $\frac{19}{8}\hat{i} - \frac{7}{4}\hat{j}$

16 $\boldsymbol{r} = -2\hat{i} + 3\hat{j} + \mu\left(2\hat{i} + 5\hat{j}\right)$

17 $2x - 7y = 38$

18 (a) $\boldsymbol{r}_2 = -3\hat{i} - \hat{j} + \mu\left(-2\hat{i} + 6\hat{j}\right)$

(b) $\frac{x-5}{-2} = \frac{y+3}{7}$ (c) $(49, -157)$

19 The position vector is $\boldsymbol{r}_3 = \frac{442}{149}\hat{i} - \frac{1396}{149}\hat{j}$.

20 (a) $\left(\frac{20}{3}, \frac{21}{5}\right)$

(b) The lines are parallel and do not intersect

(c) No point of intersection

EXERCISE 2.8

1 (a) $\boldsymbol{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} + 2\hat{j} + 4\hat{k}\right)$

(b) $\boldsymbol{r} = \left(-\hat{i} - 2\hat{j} + 3\hat{k}\right) + \lambda(3\hat{i} + 7\hat{j} - 5\hat{k})$

2 (a) $x = 5 + 2\lambda$, $y = -1 + 3\lambda$ and $z = 3 - 2\lambda$

(b) $x = -4 - 5\lambda$, $y = -2 + \lambda$ and $z = 4 + 2\lambda$

(c) $x = 5 - \lambda$, $y = 5 - \lambda$ and $z = 2 + \frac{3}{2}\lambda$

(d) $y = 4 + 3\lambda$ and $z = 2 + \frac{2}{3}\lambda$

3 (a) $x - 1 = \frac{y-3}{-2} = z + 2$ (b) $x = -3, \frac{y}{13} = \frac{z-2}{-5}$

(c) $x + 4 = \frac{z+2}{-3}$, $y = -3$ (d) $x = z + 7$, $y = 2$

4 $\frac{x-4}{2} = \frac{y-3}{-5} = \frac{z-1}{-1}$

5 (a) $\boldsymbol{r} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix}$ (b) $\boldsymbol{r} = \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$

(c) $\boldsymbol{r} = \begin{bmatrix} 0 \\ -1 \\ 12 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}$ (d) $\boldsymbol{r} = \begin{bmatrix} -5 \\ 0 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

6 (a) $\frac{x-5}{2} = \frac{y-3}{4} = \frac{z-2}{6}$ (b) $\frac{x-3}{-2} = y + 5 = \frac{z+4}{-5}$

(c) $x - 2 = \frac{y+2}{-4} = \frac{z}{-1}$ or $x - 2 = \frac{y+2}{-4} = -z$

(d) $\frac{x}{6} = \frac{y-2}{9}$, $z = -3$

7 (a) $\boldsymbol{r} = \begin{bmatrix} -3 \\ 4 \\ -7 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ -4 \\ 11 \end{bmatrix}$

(b) $x = -3 - 2\lambda, y = 4 - 4\lambda, z = -7 + 11\lambda$

(c) $\frac{x+3}{-2} = \frac{y-4}{-4} = \frac{z+7}{11}$

8 The line $\frac{x-3}{4} = \frac{y-6}{3} = z + 2$ is in the direction of vector $\boldsymbol{b}_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$.

The line $\frac{x+4}{8} = \frac{y-4}{6} = \frac{3-z}{-2}$ is in the direction of vector $\boldsymbol{b}_2 = \begin{bmatrix} 8 \\ 6 \\ 2 \end{bmatrix}$. $\boldsymbol{b}_2 = 2\boldsymbol{b}_1$, therefore the lines are parallel.

9 B

10 (a) $\boldsymbol{r} = \left(\hat{i} + 3\hat{j} + 4\hat{k}\right) + \lambda\left(2\hat{i} - 5\hat{j} - 3\hat{k}\right)$

(b) $\boldsymbol{r} = \left(-3\hat{i} - 4\hat{j} + 2\hat{k}\right) + \lambda\left(4\hat{i} + 7\hat{j} - 6\hat{k}\right)$

(c) $\boldsymbol{r} = \left(-4\hat{i} - 5\hat{k}\right) + \lambda\left(4\hat{i} - 4\hat{j}\right)$

(d) $\boldsymbol{r} = \left(-7\hat{j}\right) + \lambda\left(6\hat{i} - 2\hat{k}\right)$

11 (a) $\frac{x-3}{-1} = \frac{y+2}{3} = \frac{z-4}{-6}$ (b) $\frac{x-1}{-4} = \frac{y-2}{-4}, z = 3$

12 49°

13 (a) 138° (b) 62°

14 (a) 88° (b) 26°

15 A

16 (a) $a = 5$ (b) $a = \frac{37}{-3}$

17 Line L_1 passes through the points $A(2, -3, 4)$ and $B(5, 1, -3)$ and so the direction vector is $\boldsymbol{b}_1 = \begin{bmatrix} 5-2 \\ 1-(-3) \\ -3-4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$.

Line L_2 passes through the points $C(4, 1, -2)$ and $D(6, -2, 1)$ and so the direction vector is $\boldsymbol{b}_2 = \begin{bmatrix} 6-4 \\ -2-1 \\ 1-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$.

Vector equations are:

$\boldsymbol{r}_1 = \left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) + \lambda\left(3\hat{i} + 4\hat{j} - 7\hat{k}\right)$ and

$\boldsymbol{r}_2 = \left(4\hat{i} + \hat{j} - 2\hat{k}\right) + \mu\left(2\hat{i} - 3\hat{j} + 3\hat{k}\right)$.

Direction vectors are:

$\boldsymbol{b}_1 = 3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\boldsymbol{b}_2 = 2\hat{i} - 3\hat{j} + 3\hat{k}$, and $\boldsymbol{b}_1 \neq k\boldsymbol{b}_2$ for $k \in \mathbb{R}$, so L_1 and L_2 are not parallel.

For L_1: $x = 2 + 3\lambda$, $y = -3 + 4\lambda$ and $z = 4 - 7\lambda$

For $k \in \mathbb{R}$: $x = 4 + 2\mu$, $y = 1 - 3\mu$ and $z = -2 + 3\mu$

If lines L_1 and L_2 intersect, then a common point (x_1, y_1, z_1) lies on both L_1 and L_2 for a value of λ and μ.

Use given information to determine this point (x_1, y_1, z_1), if it exists, by solving for λ and μ:

$2 + 3\lambda = 4 + 2\mu \Rightarrow 3\lambda - 2\mu = 2$

$-3 + 4\lambda = 1 - 3\mu \Rightarrow 4\lambda + 3\mu = 4$

$4 - 7\lambda = -2 + 3\mu \Rightarrow 7\lambda + 3\mu = 6$

This system of three equations has no solution. Taking the equations in pairs gives a different relation between λ and μ for each different pair of equations.

Since different values of λ and μ are needed to equate parametric equations, L_1 and L_2 do not intersect. As they are not parallel and do not intersect, they are skew lines.

18 **(a)** The coordinates of the point are $(1, -5, 0)$.

(b) The coordinates of the point are $(0, -7, 5)$.

(c) The coordinates of the point are $(-2, 0, 9)$.

19 **(a)** $p = \frac{10}{3}$ **(b)** $q = \frac{-4 \pm \sqrt{22}}{3}$

EXERCISE 2.9

1 **(a)** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ **(b)** $\mathbf{B}^{-1} = -\frac{1}{13}\begin{bmatrix} 2 & -7 \\ -1 & -3 \end{bmatrix}$

2 **(a)** $p = \frac{-15}{2}$ **(b)** $p = 0$ and $p = \frac{1}{4}$

3 **(a)** $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 5 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -1 \\ -6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

4 **(a)** $\begin{bmatrix} 3 & -1 & 1 \\ 1 & -1 & -1 \\ 5 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$

5 A

6 D

7 **(a)** $x = 5$ and $y = -1$ **(b)** $x = 2$ and $y = 3$

(c) $x = 1$ and $y = -2$

8 $x = 1$ and $y = 4$

9 $x = -3$ and $y = 1$

10 $x = 2$ and $y = -2$

11 $x = 1$, $y = 0$ and $z = 2$

12 $x = 2$, $y = 1$ and $z = 0$

13 $x = -1$, $y = 1$ and $z = 3$

14 $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -1 \\ -6 & 3 & -9 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 1 & 1 & -1 & 5 \\ -6 & 3 & -9 & 8 \end{array}\right]$$

$$R_3 + 3R_1 : \left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 1 & 1 & -1 & 5 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

It is not possible to use Gaussian elimination to reduce the augmented matrix $[\,\mathbf{A}|\mathbf{B}\,]$ to the form $[\,\mathbf{I}|\mathbf{X}\,]$. Returning to algebraic form, the third row is $0x + 0y + 0z = 20$, hence $0 = 20$. Therefore, this system of equations is inconsistent and has no unique solution.

15 **(a)** $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 4 & -2 \\ -2 & 2 & -4 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix}$

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 4 & 4 & -2 & 1 \\ -2 & 2 & -4 & 6 \end{array}\right]$$

$$R_3 + 2R_1 : \left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 4 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 - 4R_1 : \left[\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 8 & -10 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It is not possible to use Gaussian elimination to reduce the augmented matrix $[\,\mathbf{A}|\mathbf{B}\,]$ to the form $[\,\mathbf{I}|\mathbf{X}\,]$, so there is no unique solution. However, returning to expanded linear form:

$x - y + 2z = -3 \quad [1]$

$8y - 10z = 13 \quad [2]$

Rearranging equation $[2]$ gives:

$y = \frac{5}{4}z + \frac{13}{8}$

Substituting into equation $[1]$:

$x - \left(\frac{5}{4}z + \frac{13}{8}\right) + 2z = -3$

$x - \frac{5}{4}z - \frac{13}{8} + 2z = -3$

$x = -\frac{3}{4}z - \frac{11}{8}$

Therefore, values of x and y will vary with z to form an infinite set of solutions in the form $\left(\left(-\frac{3}{4}z - \frac{11}{8}\right), \left(\frac{5}{4}z + \frac{13}{8}\right), z\right)$

(b) $x = -\frac{17}{8}$, $y = \frac{23}{8}$, $z = 1$

EXERCISE 2.10

1 **(a)** Team *D* defeated Team *E*. **(b)** 3 victories

(c) $\mathbf{P}_1 = \begin{bmatrix} 1+1+1+1 \\ 1 \\ 1+1+1 \\ 1+1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}$

(d) No, the first-order power vector is sufficient to rank the teams. Team *A* is first, followed by Team *C*, then Team *D*, Team *B* and Team *E*.

2 C

3 $\mathbf{M}^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4 **(a)** The element corresponding to 0.3 is the fecundity rate for one-year-old animals.

(b) 40% of two-year-old animals would be expected to survive to reach three years of age.

5 C

6 The first-order power matrix shows Chara and Ezra tied on three wins, Anuj placing third on 2 wins and Bronte and Diego tied with 1 win. By considering the sum of the first and second-order matrix, the most dominant player is Chara, followed by Ezra, Anuj, Bronte and Diego.

7 **(a)** $\mathbf{L} = \begin{bmatrix} 0.1 & 0.7 & 2.4 & 0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$

(b) 336 female gerbils

(c) The estimated total number of gerbils after 5 years is 978.

This assumes there are equal number of male and female gerbils in the population. It also assumes the survival and fecundity rates remain constant over the five-year period and there are no significant external or environmental effects. If this is not the case, then the estimated population of 978 may be incorrect.

8 The Leslie matrix for this population is:

$\mathbf{L} = \begin{bmatrix} 0 & 0 & 500 \\ 0.02 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$

The population is stable over time. This assumes the survival and fecundity rates remain constant over time, and there are no external influences on the population. Even a slight change to these variables will cause the population to grow or decline instead of remaining stable.

EXERCISE 2.11

1 **(a)** 0.1 units of *C*

(b) 0.2 units of *A*, 0.2 units of *B* and 0.4 units of *C*.

2 **(a)** Required input is 5.6 units of *A*, 5.2 units of *B* and 6.2 units of *C*.

(b) The demand that can be met is 4.4 units of *A*, 2.8 units of *B* and 5.8 units of *C*.

3 **(a)** $x = 0.1$ **(b)** 0.8 **(c)** Outcome $C\,(p = 0.1)$

4 The required production is 110.13 units of *P*, 113.67 units of *Q* and 89.11 units of *R*.

5 The market share after 4 months (rounded to 1 d.p.) is:

Ace: 29.8%
Bubbles: 37.6%
Crisp: 32.5%

Some limitations of this model include that it does not take into account any external factors such as advertising campaigns, and it does not factor in new customers or those who may have ceased drinking soft drinks altogether in an effort to lead a healthier lifestyle.

6 37 days

7 B

8 **(a)** $\lambda = 7, \lambda = -2$

(b) The eigenvector associated with the eigenvalue $\lambda = 7$ is $\begin{bmatrix} a \\ \frac{5a}{4} \end{bmatrix}$.

The eigenvector associated with the eigenvalue $\lambda = -2$ is $\begin{bmatrix} a \\ -a \end{bmatrix}$.

9 **(a)** To meet a consumer demand of 100 units each of resources *A*, *B* and *C*:

$$p = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.3 \\ 0.2 & 0.1 & 0.2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$$

$$p = \begin{bmatrix} 0.8 & -0.2 & -0.3 \\ -0.3 & 0.8 & -0.3 \\ -0.2 & -0.1 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$$

$$p = \begin{bmatrix} 296.50 \\ 326.15 \\ 239.89 \end{bmatrix} \text{ (2 d.p.)}$$

The required production of 326.15 units of resource *B* is larger than the production capacity of 300 units of resource *B*. Therefore, the required demand cannot be met owing to the production limit of resource *B* being reached.

(b) The new production requirement is $p = \begin{bmatrix} 280.80 \\ 278.23 \\ 229.98 \end{bmatrix}$ (2 d.p.). Therefore, consumer demand can be met.

CHAPTER REVIEW 2

1 (a) $|\boldsymbol{a}| = 25, \hat{\boldsymbol{a}} = \frac{1}{25}\left(7\hat{\boldsymbol{i}} - 24\hat{\boldsymbol{j}}\right)$

(b) $|\boldsymbol{b}| = \sqrt{74}, \hat{\boldsymbol{b}} = \frac{1}{\sqrt{74}}\left(4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}\right) = \frac{\sqrt{74}}{74}\left(4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}\right)$

2 (a) $\overrightarrow{OP} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$,
$\hat{\overrightarrow{OP}} = \frac{1}{\sqrt{22}}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\right) = \frac{\sqrt{22}}{22}\left(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\right)$

(b) $\overrightarrow{OP} = 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$,
$\hat{\overrightarrow{OP}} = \frac{1}{\sqrt{38}}\left(3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right) = \frac{\sqrt{38}}{38}\left(3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right)$

3 (a) $6\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ (b) $11\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$ (c) $3\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}} - 26\hat{\boldsymbol{k}}$

4 $\frac{8}{\sqrt{29}} = \frac{8\sqrt{29}}{29}$

5 (a) $-11\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ (b) $9\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ (c) $\boldsymbol{b} \times \boldsymbol{c} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$

6 (a) $\boldsymbol{r} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \lambda\left(-2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}\right)$ (b) $x = 4 - 2\lambda,\ y = -1 + 5\lambda$

(c) $\frac{x-4}{-2} = \frac{y+1}{5}$

7 $\boldsymbol{r}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$

8 The lines are not parallel.

9 (a) The required input is 11.6 units of A, 8.2 units of B and 14 units of C.

(b) The demand that can be met is 8.4 units of A, 11.8 units of B and 4 units of C.

10 B

11 5.79 km S61.10°E (2 d.p.)

12 (a) $\boldsymbol{a} \cdot \boldsymbol{b} = 0, \theta = 90°$ (b) $\boldsymbol{a} \cdot \boldsymbol{b} = 10, \theta = 50°$

13 $\frac{2}{9}\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right),\ \frac{1}{9}\left(-31\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}} - 23\hat{\boldsymbol{k}}\right)$

14 (a) $x = \frac{7}{5},\ y = -7,\ z = -\frac{13}{3}$

(b) $x = 2$ or $4,\ y = \pm\sqrt{2},\ z = 2$

15 (a) $\overrightarrow{AB} = \boldsymbol{a} - \boldsymbol{b},\ \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$

(b) $\overrightarrow{AM} = \overrightarrow{MC}$
$= \overrightarrow{BM}$
$= |\boldsymbol{a}|$

Since $|\boldsymbol{a}| = |\boldsymbol{b}|$ because

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = \boldsymbol{a}^2 - \boldsymbol{b}^2 = 0$

16 (a) Area $= \sqrt{465}$ (b) Area $= \frac{\sqrt{41}}{2}$ (c) 7 cubic units

17 $4x + 2y + 5z = 3$

18 $(11, -1)$

19 $x = 1,\ y = -5$ and $z = 2$.

20 Team D has the highest dominance, followed by Team C, Team A and Team B.

21 The approximate number of female mice after 5 years is $161 + 71 + 17 = 249$. Therefore, the total number of mice $= 249 \times 2 = 498$ mice.

This assumes there are equal numbers of male and female mice in the population. It also assumes the survival and fecundity rates remain constant over the five-year period and that there is no unexpected immigration or emigration among the mouse population. A reduction in fecundity or survival rates will cause the population to grow slower or recede.

22 (a) Show that $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are both radii,

$|\boldsymbol{a}| = |\boldsymbol{b}|$
$\overrightarrow{CO} = \overrightarrow{OA} = \boldsymbol{a}$
$\overrightarrow{CB} = \boldsymbol{a} + \boldsymbol{b}$
$\therefore \overrightarrow{BC} = -\boldsymbol{a} - \boldsymbol{b}$
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= -\boldsymbol{a} + \boldsymbol{b}$
$\overrightarrow{AB} \cdot \overrightarrow{BC} = (-\boldsymbol{a} + \boldsymbol{b}) \cdot (-\boldsymbol{a} - \boldsymbol{b})$
$= |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2$
$= |\boldsymbol{a}|^2 - |\boldsymbol{a}|^2$
$= 0$

Hence $\angle ABC$ is a right angle.

(b) Show that $\overrightarrow{DC} = \overrightarrow{AB},\ \overrightarrow{AD} = \overrightarrow{BC}$ and $\overrightarrow{CD} \cdot \overrightarrow{AD} = 0$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$
$\overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC}$
$= \boldsymbol{b} - \boldsymbol{a}$
$\therefore \overrightarrow{DC} = \overrightarrow{AB}$
$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$
$= -\boldsymbol{a} - \boldsymbol{b}$
$\therefore \overrightarrow{AD} = \overrightarrow{BC}$
$\overrightarrow{CD} \cdot \overrightarrow{AD} = \overrightarrow{AB} \cdot \overrightarrow{BC}$
$= 0$

23 A vector perpendicular to the plane $x + 3y - 2z = 12$ will be $n\left(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right)$, where $n \in \mathbb{R}$.

24 There is an infinite number of solutions in the form $(3z + 3, -5z - 1, z)$.

25 D

26 (a) Volume = 8206 cubic units (b) $\frac{266}{3}$

27 (a) $21\frac{1}{3}$ cubic units (b) $w = 5$ or $w = 102.2$

28 (a) $L_1 : \frac{x-1}{-3} = y+5 = \frac{z}{3}$ and $L_2 : \frac{x-1}{-4} = \frac{z+6}{2}$, $y = -1$

(b) No point of intersection (c) 22.57°

(d) (1, -5, 0) (e) $\left(0, -1, -\frac{11}{2}\right)$

Chapter 3: Vector calculus

RECALL

1 (a) $\frac{dx}{dt} = \sin(t)$

(b) $\frac{dx}{dt} = -2\sin(t)\cos(t)$ or $\frac{dx}{dt} = -\sin(2t)$

(c) $\frac{dx}{dt} = 2e^{2t} - 12t^2$

(d) $\frac{dx}{dt} = \frac{\sin(t)}{\cos^2(t)}$ or $\frac{dx}{dt} = \tan(t)\sec(t)$

2 (a) $\frac{dy}{dt} = 4\cos(t)$

(b) $\frac{dy}{dt} = 2\sin(t)\cos(t)$ or $\frac{dy}{dt} = \sin(2t)$

(c) $\frac{dy}{dt} = \frac{1}{t}$

(d) $\frac{dy}{dt} = \sec^2(t)$ or $\frac{dy}{dt} = \frac{1}{\cos^2(t)}$

3 $f'(2) = -10$

4 $f'\left(\frac{\pi}{6}\right) = 1$

5 $f'\left(\frac{1}{4}\right) = -3\pi$

6 $f'\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2}$

7 (a) $\frac{1}{2}e^{2t} - e^{-t} + c$ (b) $-\frac{3}{2}\cos(2t) + c$

(c) $\frac{1}{12}(4t+1)^3 + c$ (d) $\frac{1}{3}\log_e(|3t+2|) + c$

8 (a) 8 metres to the left of O (b) -6 m/s (c) 1 s

9 (a) $x(t) = -t^2 + 6t + 3$

(b) The particle is 11 metres to the right of O after 4 seconds.

(c) The distance travelled after 4 seconds is 10 m.

10 (a) $|\boldsymbol{a}| = \sqrt{14}$ (b) $|\boldsymbol{c}| = 7$

11 (a) $\boldsymbol{a} \cdot \boldsymbol{b} = 14$ (b) $\boldsymbol{a} \cdot \boldsymbol{c} = -3$ (c) $\boldsymbol{b} \cdot \boldsymbol{c} = -5$

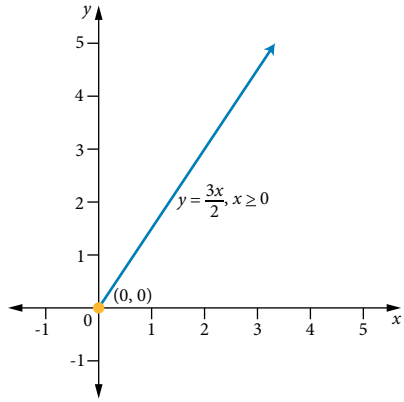
12 (a) 1.57 (b) 2.43

13 $m = -12$

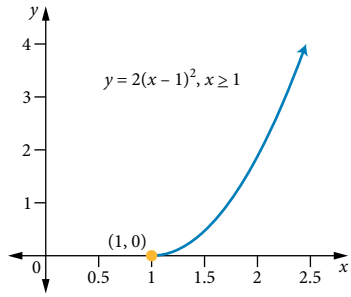
EXERCISE 3.1

1 $\boldsymbol{r}(t) = a\cos(t)\hat{\boldsymbol{i}} + a\sin(t)\hat{\boldsymbol{j}} + bt\hat{\boldsymbol{k}}$

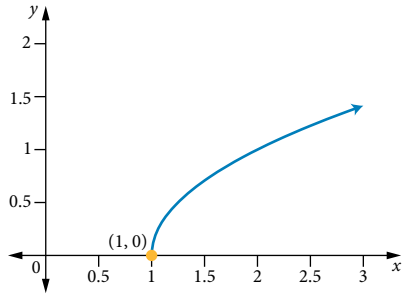
2 (a) $y = \frac{3x}{2}, x \geq 0$



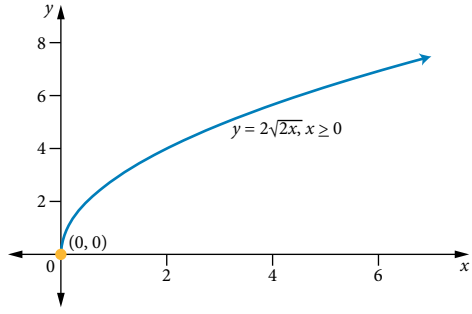
(b) $y = 2(x-1)^2, x \geq 1$



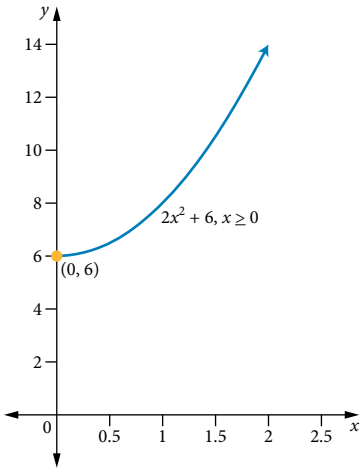
(c) $y = \sqrt{x-1}, x \geq 1$



(d) $y = 2\sqrt{2x}, x \geq 0$



3 $y = 2x^2 + 6,\ x \geq 0$



4 (a) $y = 2 - 2x,\ 0 \leq x \leq 1$

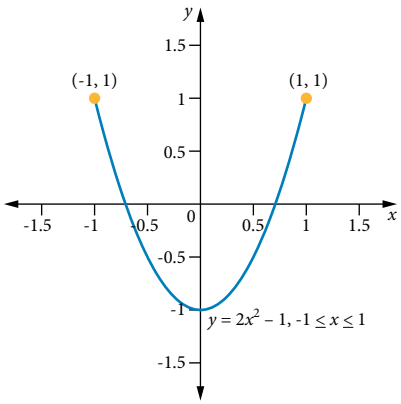
(b) The body starts at $(1,0)$ and moves back and forth along a straight line between $(1,0)$ and $(0,2)$.

5 D

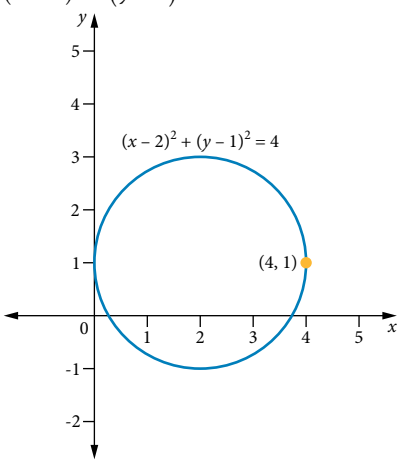
6 A

7 The bodies will collide when $t = \frac{11}{3}$. This occurs at the point $\left(\frac{7}{3}, \frac{22}{3}\right)$.

8 $y = 2x^2 - 1, -1 \leq x \leq 1$



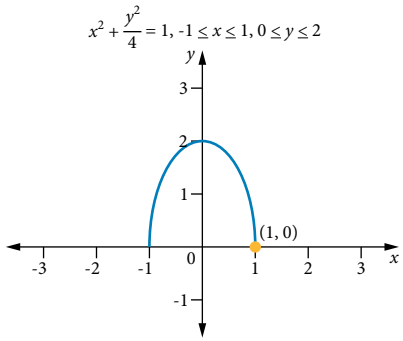
9 (a) $(x-2)^2 + (y-1)^2 = 4$



(b) The body starts at $(4,1)$ and moves in an anticlockwise direction along a circular path.

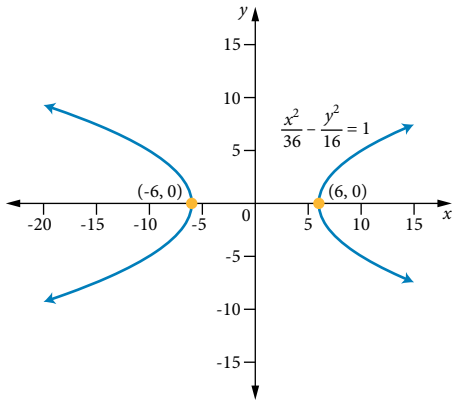
10 (a) $x^2 + \frac{y^2}{4} = 1,\ -1 \leq x \leq 1$

(b) The particle's initial position is $(1,0)$ and it travels in an anticlockwise direction in the path of a semi-ellipse.



11 (a) $\frac{x^2}{36} - \frac{y^2}{16} = 1,\ x \in (-\infty, -6] \cup [6, \infty)$

(b)



12 (a) The two particles collide at $(3,4)$.

(b) The paths cross at $(3,4)$ and $(99,676)$.

(c) 5 units

13 $p = 2$

14 The four points of intersection are:

$$\left(\frac{ab}{\sqrt{a^2+b^2}}, \frac{ab}{\sqrt{a^2+b^2}}\right)$$

$$\left(\frac{-ab}{\sqrt{a^2+b^2}}, \frac{ab}{\sqrt{a^2+b^2}}\right)$$

$$\left(\frac{ab}{\sqrt{a^2+b^2}}, \frac{-ab}{\sqrt{a^2+b^2}}\right)$$

$$\left(\frac{-ab}{\sqrt{a^2+b^2}}, \frac{-ab}{\sqrt{a^2+b^2}}\right)$$

EXERCISE 3.2

1 (a) $\frac{d\boldsymbol{r}}{dt} = 10t\hat{\boldsymbol{i}} - 6t^2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\frac{d^2\boldsymbol{r}}{dt^2} = 10\hat{\boldsymbol{i}} - 12t\hat{\boldsymbol{j}}$

(b) $\frac{d\boldsymbol{r}}{dt} = 8\cos(2t)\hat{\boldsymbol{i}} - 6\sin(2t)\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$ and

$\frac{d^2\boldsymbol{r}}{dt^2} = -16\sin(2t)\hat{\boldsymbol{i}} - 12\cos(2t)\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

2 (a) $\boldsymbol{r}'(0) = -2\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{r}''(0) = -4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

(b) $\boldsymbol{r}'(0) = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{r}''(0) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

(c) $\boldsymbol{r}'(2) = 10\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}}$ and $\boldsymbol{r}''(2) = -10\hat{\boldsymbol{j}}$

(d) $\boldsymbol{r}'\left(\frac{\pi}{4}\right) = \sqrt{2}\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{r}''\left(\frac{\pi}{4}\right) = 3\sqrt{2}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

3 (a) $\frac{dy}{dx} = -1$ (b) $\boldsymbol{r}'(t) = \hat{\boldsymbol{i}} - \frac{1}{t^2}\hat{\boldsymbol{j}}$ (c) $\hat{\boldsymbol{s}} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)$

4 (a) $\frac{dy}{dx} = -1$ (b) $\boldsymbol{r}'(t) = (2t+1)\hat{\boldsymbol{i}} + (2t-1)\hat{\boldsymbol{j}}$

(c) $\hat{\boldsymbol{s}} = \frac{1}{\sqrt{2}}\left(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)$

5 (a) $\frac{dy}{dx} = \frac{1}{2}$ (b) $\boldsymbol{r}'(t) = -2\sin(2t)\hat{\boldsymbol{i}} - \sin(t)\hat{\boldsymbol{j}}$

(c) $\hat{\boldsymbol{s}} = -\frac{1}{\sqrt{5}}\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right)$

6 (a) $\frac{d\boldsymbol{r}}{dt} = \left(\sin(t) + t\cos(t)\right)\hat{\boldsymbol{i}} + \left(\cos(t) - t\sin(t)\right)\hat{\boldsymbol{j}}$

(b) $\frac{d\boldsymbol{r}}{dt} = \left(1 + \log_e(t)\right)\hat{\boldsymbol{i}} + \left(te^t + e^t\right)\hat{\boldsymbol{j}}$; Restriction: $t \geq 0$

7 C

8 (a) $\frac{dy}{dx} = \sqrt{3}$

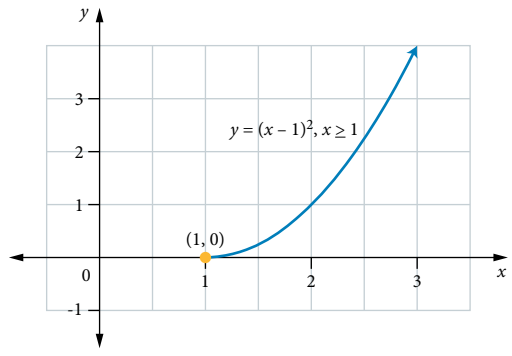
(b) $\boldsymbol{r}'(t) = -3\cos^2(t)\sin(t)\hat{\boldsymbol{i}} + 3\sin^2(t)\cos(t)\hat{\boldsymbol{j}}$

(c) $\hat{\boldsymbol{s}} = -\frac{1}{2}\left(\hat{\boldsymbol{i}} + \sqrt{3}\hat{\boldsymbol{j}}\right)$

9 $t = 0.27$ (2 d.p.)

10 (a) $y = (x-1)^2$, $x \geq 1$

(b)



(c) $\boldsymbol{r}(t) = (1+t)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$, $t \geq 0$ (d) $\hat{\boldsymbol{s}} = \frac{1}{\sqrt{5}}\left(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right)$

11 Begin by calculating $\boldsymbol{r}'(t)$:

$\boldsymbol{r}(t) = \sin(t)\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}} + \cos(t)\hat{\boldsymbol{k}}$

$\boldsymbol{r}'(t) = \cos(t)\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \sin(t)\hat{\boldsymbol{k}}$

Then calculate $\boldsymbol{r}''(t)$:

$\boldsymbol{r}'(t) = \cos(t)\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \sin(t)\hat{\boldsymbol{k}}$

$\boldsymbol{r}''(t) = -\sin(t)\hat{\boldsymbol{i}} - \cos(t)\hat{\boldsymbol{k}}$

Two vectors are perpendicular if their dot product is zero.

$\boldsymbol{r}'(t) = \cos(t)\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \sin(t)\hat{\boldsymbol{k}}$

$\boldsymbol{r}''(t) = -\sin(t)\hat{\boldsymbol{i}} - \cos(t)\hat{\boldsymbol{k}}$

$\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t) = \left(\cos(t) \times -\sin(t)\right) + (1 \times 0) + \left(-\sin(t) \times -\cos(t)\right)$

$= -\cos(t)\sin(t) + \sin(t)\cos(t)$

$= 0$

Hence, $\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t) = 0$; $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$ are always perpendicular.

12 $t = \frac{1}{2}$

EXERCISE 3.3

1 (a) $\boldsymbol{v} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ (b) $\boldsymbol{v} = \hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$

2 (a) $\boldsymbol{a} = -4t\hat{\boldsymbol{j}}$ (b) $\boldsymbol{a} = -10\hat{\boldsymbol{j}}$

3 (a) $\dot{\boldsymbol{r}} = 4\hat{\boldsymbol{i}} - 2t\hat{\boldsymbol{j}} + 6t\hat{\boldsymbol{k}}$ and $\ddot{\boldsymbol{r}} = -2\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$

(b) $\dot{\boldsymbol{r}} = 26\hat{\boldsymbol{i}} + 5(3-2t)\hat{\boldsymbol{j}}$ and $\ddot{\boldsymbol{r}} = -10\hat{\boldsymbol{j}}$

(c) $\dot{\boldsymbol{r}} = -8\sin(2t)\hat{\boldsymbol{i}} + 6\cos(2t)\hat{\boldsymbol{j}}$ and

$\ddot{\boldsymbol{r}} = -16\cos(2t)\hat{\boldsymbol{i}} - 12\sin(2t)\hat{\boldsymbol{j}}$

(d) $\dot{\boldsymbol{r}} = e^t\hat{\boldsymbol{i}} - 2e^{-2t}\hat{\boldsymbol{j}} - 3t^2\hat{\boldsymbol{k}}$ and $\ddot{\boldsymbol{r}} = e^t\hat{\boldsymbol{i}} + 4e^{-2t}\hat{\boldsymbol{j}} - 6t\hat{\boldsymbol{k}}$

4 Initial speed is 5.

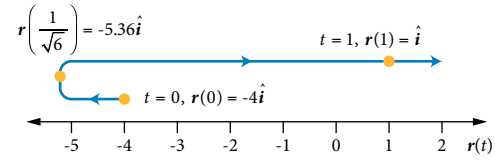
5 D

6 (a) $\boldsymbol{r} = 2t^2\hat{\boldsymbol{i}} + (1-t)\hat{\boldsymbol{j}}$

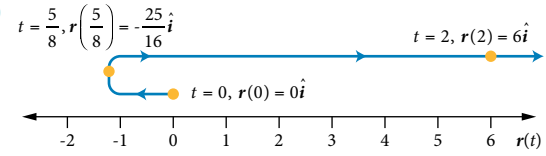
(b) $\boldsymbol{r} = \cos(2t)\hat{\boldsymbol{i}} + \sin(2t)\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

(c) $\boldsymbol{r} = 20t\hat{\boldsymbol{i}} + \left(20t - 5t^2\right)\hat{\boldsymbol{j}}$

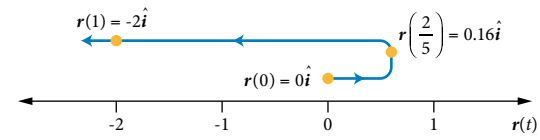
7 (a)



(b)



(c)



8 2.5 m

9 130.67 m in a positive direction

10 16 m above the ocean's surface

11 (a) Vectors at right angle to each other have a zero dot product.

$$\begin{aligned}\boldsymbol{v}_W \cdot \boldsymbol{v}_B &= 0\\ \boldsymbol{v}_W &= \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\\ \boldsymbol{v}_B &= 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\\ \boldsymbol{v}_W \cdot \boldsymbol{v}_B &= \left(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\right)\cdot\left(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)\\ &= 1\times 2 + 2\times -1\\ &= 0\\ &\therefore \boldsymbol{v}_W \perp \boldsymbol{v}_B\end{aligned}$$

(b) The balls will collide after 0.5 seconds.

(c) These calculations assume that the balls continue travelling with constant velocity. Given the friction of the table surface, this assumption is not likely to be correct. If the assumption did not hold, the balls either would not collide or would collide at a later time.

12 (a) $\boldsymbol{r} = \sqrt{2}\cos(nt)\hat{\boldsymbol{i}} + \sin(nt)\hat{\boldsymbol{j}} + \sin(nt)\hat{\boldsymbol{k}}$

$$\begin{aligned}v &= |\boldsymbol{v}| = |\dot{\boldsymbol{r}}|\\ \dot{\boldsymbol{r}} &= -\sqrt{2}n\sin(nt)\hat{\boldsymbol{i}} + n\cos(nt)\hat{\boldsymbol{j}} + n\cos(nt)\hat{\boldsymbol{k}}\\ v &= \sqrt{\left(-\sqrt{2}n\sin(nt)\right)^2 + \left(n\cos(nt)\right)^2 + \left(n\cos(nt)\right)^2}\\ &= \sqrt{2n^2\sin^2(nt) + 2n^2\cos^2(nt)}\\ &= \sqrt{2n^2\left(\sin^2(nt) + \cos^2(nt)\right)}\\ &= \sqrt{2n^2(1)}\\ &= \sqrt{2}n\end{aligned}$$

(b) Perpendicular vectors have a zero dot product.

$$\begin{aligned}\boldsymbol{r} &= \sqrt{2}\cos(nt)\hat{\boldsymbol{i}} + \sin(nt)\hat{\boldsymbol{j}} + \sin(nt)\hat{\boldsymbol{k}}\\ \dot{\boldsymbol{r}} &= -\sqrt{2}n\sin(nt)\hat{\boldsymbol{i}} + n\cos(nt)\hat{\boldsymbol{j}} + n\cos(nt)\hat{\boldsymbol{k}}\\ \ddot{\boldsymbol{r}} &= -\sqrt{2}n^2\cos(nt)\hat{\boldsymbol{i}} - n^2\sin(nt)\hat{\boldsymbol{j}} - n^2\sin(nt)\hat{\boldsymbol{k}}\\ \dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}} &= 2n^3\cos(nt)\sin(nt) - 2n^3\sin(nt)\cos(nt)\\ &= 0\\ &\therefore \dot{\boldsymbol{r}} \perp \ddot{\boldsymbol{r}}\end{aligned}$$

13 (a) $\dot{\boldsymbol{r}} = e^{-t}\left(\cos(t) - \sin(t)\right)\hat{\boldsymbol{i}} - e^{-t}\left(\sin(t) + \cos(t)\right)\hat{\boldsymbol{j}}$

(b) $\sqrt{2}e^{-t}$ **(c)** $\ddot{\boldsymbol{r}} = -2e^{-t}\cos(t)\hat{\boldsymbol{i}} + 2e^{-t}\sin(t)\hat{\boldsymbol{j}}$

14 (a) $v_{\max} = \sqrt{26}$, $v_{\min} = \sqrt{2}$ **(b)** 5

15 $\dot{\boldsymbol{r}} = \frac{3}{4}\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$

16 $\dot{\boldsymbol{r}} = 24\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$; $\boldsymbol{r} = 15\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

17 $\dot{\boldsymbol{r}} = e^{t}\hat{\boldsymbol{i}} - e^{-t}\hat{\boldsymbol{j}} + \left(3t^2 + 1\right)\hat{\boldsymbol{k}}$;
$\boldsymbol{r} = \left(e^{t} - 1\right)\hat{\boldsymbol{i}} + \left(e^{-t} + 1\right)\hat{\boldsymbol{j}} + \left(t^3 + t\right)\hat{\boldsymbol{k}}$

18 (a) 24 units to the right of O.

(b) $t = 4$, 4 seconds into the journey, the particle passes back through the origin

19 (a) $\dot{\boldsymbol{r}} = 10t\hat{\boldsymbol{i}} + 10\left(1 - e^{-0.1t}\right)\hat{\boldsymbol{j}}$ m/s

(b) $\boldsymbol{r} = 500\hat{\boldsymbol{i}} + 100e^{-1}\hat{\boldsymbol{j}}$

20 (a) $\boldsymbol{v} = 2t\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

(b) Particle's speed is $\sqrt{17}$ and direction is 14.0° to the $\hat{\boldsymbol{i}}$ direction.

(c) $\boldsymbol{a} = 2\hat{\boldsymbol{i}}$ **(d)** $\frac{4}{5}\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\right), \frac{2}{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ **(e)** 90°, 26.6°

21 (a) 3 m/s **(b)** $\frac{1}{3}\left(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right)$

(c) $-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ m/s **(d)** 10.2 m

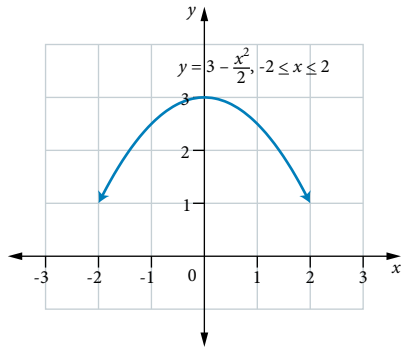
22 (a) Initial speed is 22.4 m/s and the angle projection is 63.4° to the horizontal.

(b) After 2 seconds. **(c)** $\boldsymbol{r} = 10t\hat{\boldsymbol{i}} + \left(20t - 5t^2\right)\hat{\boldsymbol{j}}$

(d) 20 m. This assumes that the fairway is level and horizontal, and ignores other factors such as wind resistance. If these assumptions are not correct, the ball's maximum height will differ from this solution.

23 (a) $\boldsymbol{r} = 2\sin(t)\hat{\boldsymbol{i}} + \left(2 + \cos(2t)\right)\hat{\boldsymbol{j}}$ **(b)** $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

(c) $y = 3 - \frac{1}{2}x^2, -2 \le x \le 2$



(d) The maximum speed is 2.5.

EXERCISE 3.4

1 (a) $\boldsymbol{r} = 3\cos(2t)\hat{\boldsymbol{i}} + 3\sin(2t)\hat{\boldsymbol{j}}$

$$x = 3\cos(2t) \Rightarrow \cos(2t) = \frac{x}{3}, \cos^2(2t) = \frac{x^2}{9}$$

$$y = 3\sin(2t) \Rightarrow \sin(2t) = \frac{y}{3}, \sin^2(2t) = \frac{y^2}{9}$$

$$\sin^2(2t) + \cos^2(2t) = \frac{x^2}{9} + \frac{y^2}{9} = 1$$

$$\therefore x^2 + y^2 = 9$$

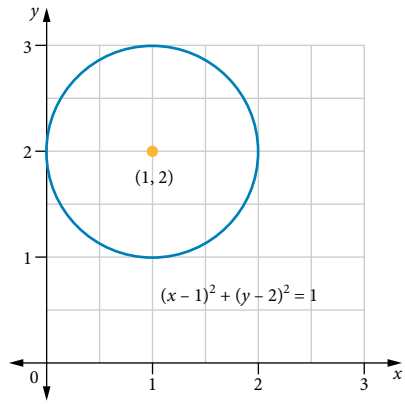
(b) $\boldsymbol{r} = 3\cos(2t)\hat{\boldsymbol{i}} + 3\sin(2t)\hat{\boldsymbol{j}}$

$$\begin{aligned}\dot{\boldsymbol{r}} &= -6\sin(2t)\hat{\boldsymbol{i}} + 6\cos(2t)\hat{\boldsymbol{j}}\\ v &= |\dot{\boldsymbol{r}}| = \left|-6\sin(2t)\hat{\boldsymbol{i}} + 6\cos(2t)\hat{\boldsymbol{j}}\right|\\ &= \sqrt{\left(-6\sin(2t)\right)^2 + \left(6\cos(2t)\right)^2}\\ &= \sqrt{36\left(\sin^2(2t) + \cos^2(2t)\right)}\\ &= 6\end{aligned}$$

(c) The dot product is equal to zero for perpendicular vectors.

$$\dot{\boldsymbol{r}} = -6\sin(2t)\hat{\boldsymbol{i}} + 6\cos(2t)\hat{\boldsymbol{j}}$$
$$\ddot{\boldsymbol{r}} = -12\cos(2t)\hat{\boldsymbol{i}} - 12\sin(2t)\hat{\boldsymbol{j}}$$
$$\boldsymbol{a} = |\ddot{\boldsymbol{r}}| = \sqrt{\left(-12\cos(2t)\right)^2 + \left(-12\sin(2t)\right)^2}$$
$$= 12$$
$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = \left(-6\sin(2t)\hat{\boldsymbol{i}} + 6\cos(2t)\hat{\boldsymbol{j}}\right) \cdot \left(-12\cos(2t)\hat{\boldsymbol{i}} - 12\sin(2t)\hat{\boldsymbol{j}}\right)$$
$$= 72\sin(2t)\cos(2t) - 72\sin(2t)\cos(2t) = 0$$
$$\therefore \dot{\boldsymbol{r}} \perp \ddot{\boldsymbol{r}}$$

2 (a) $(x-1)^2 + (y-2)^2 = 1$



(b) The dot product of perpendicular vectors equals zero.

$$\boldsymbol{r} = \left(1 + \sin(4t)\right)\hat{\boldsymbol{i}} + \left(2 - \cos(4t)\right)\hat{\boldsymbol{j}}$$
$$\dot{\boldsymbol{r}} = 4\cos(4t)\hat{\boldsymbol{i}} + 4\sin(4t)\hat{\boldsymbol{j}}$$
$$\ddot{\boldsymbol{r}} = -16\sin(4t)\hat{\boldsymbol{i}} + 16\cos(4t)\hat{\boldsymbol{j}}$$
$$\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = \left(4\cos(4t)\hat{\boldsymbol{i}} + 4\sin(4t)\hat{\boldsymbol{j}}\right) \cdot \left(-16\sin(4t)\hat{\boldsymbol{i}} + 16\cos(4t)\hat{\boldsymbol{j}}\right)$$
$$= -64\sin(4t)\cos(4t) + 64\sin(4t)\cos(4t)$$
$$= 0$$
$$\therefore \dot{\boldsymbol{r}} \perp \ddot{\boldsymbol{r}}$$

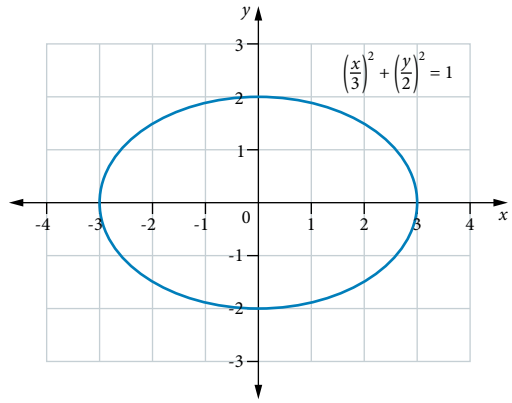
3 (a) C

(b) Option A fails to take into account multiplication by scalar.

4 B

5 (a) $3\sqrt{5}\ \text{ms}^{-1}$ (b) $18\ \text{ms}^{-2}$

6 (a) $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$



(b) $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi;\ 3\hat{\boldsymbol{i}}, 2\hat{\boldsymbol{j}}, -3\hat{\boldsymbol{j}}, -2\hat{\boldsymbol{j}}, 3\hat{\boldsymbol{i}}$

(c) Maximum speed is 3; minimum speed is 2.

(d) Acceleration should be opposite and parallel to position; i.e. a negative scalar multiple of the position vector.

$$\boldsymbol{r} = 3\cos(t)\hat{\boldsymbol{i}} + 2\sin(t)\hat{\boldsymbol{j}}$$
$$\dot{\boldsymbol{r}} = -3\sin(t)\hat{\boldsymbol{i}} + 2\cos(t)\hat{\boldsymbol{j}}$$
$$\ddot{\boldsymbol{r}} = -3\cos(t)\hat{\boldsymbol{i}} - 2\sin(t)\hat{\boldsymbol{j}}$$
$$= -\left(3\cos(t)\hat{\boldsymbol{i}} + 2\sin(t)\hat{\boldsymbol{j}}\right)$$
$$= -\boldsymbol{r}$$
$$\therefore |\ddot{\boldsymbol{r}}| = |-\boldsymbol{r}| = |\boldsymbol{r}|$$

(e) $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

7 (a) $25\ \text{ms}^{-1}$; 73.7° (b) 4.90 seconds

(c) 34.3 metres. This assumes no air resistance or wind. Air resistance may decrease the range of the body.

(d) 29.4 metres

8 42.4 metres

9 80.1 metres. This assumes no air resistance and also a level, horizontal playing field. Air resistance would decrease the range of the cricket ball. A playing surface with an uphill slope would reduce the range, and a surface with a downhill slope would increase the range.

10 The ball will travel 27.8 metres horizontally, at which point the fielder is still 29 metres horizontally from the point of impact. Therefore, the fielder will not catch the ball.

This calculation involves several assumptions. It assumes no air resistance, which if factored into the calculation would reduce the range of the ball. It assumes no wind is acting on the ball. It also assumes a level, horizontal playing surface. The fielder is assumed to run with constant velocity, and it is assumed that the fielder begins to move the instant the ball is hit towards them. The only way that the fielder could take the catch is if they started moving towards the ball before it was hit.

11 (a) 31.3 m/s (b) 34.3 m/s

12 (a) Begin by determining the time at which the projectile reaches its maximum height—this will occur when vertical velocity is equal to zero.

$$\boldsymbol{a} = -g\hat{\boldsymbol{j}}$$
$$\boldsymbol{v}(t) = -gt\hat{\boldsymbol{j}} + \boldsymbol{c}$$
$$\boldsymbol{v}(0) = 19.6\cos(\alpha)\hat{\boldsymbol{i}} + 19.6\sin(\alpha)\hat{\boldsymbol{j}} = \boldsymbol{c}$$
$$\boldsymbol{v}(t) = -gt\hat{\boldsymbol{j}} + 19.6\cos(\alpha)\hat{\boldsymbol{i}} + 19.6\sin(\alpha)\hat{\boldsymbol{j}}$$
$$= 19.6\cos(\alpha)\hat{\boldsymbol{i}} + \left(19.6\sin(\alpha) - gt\right)\hat{\boldsymbol{j}}$$
$$\boldsymbol{v}_{\text{vertical max}}: \left(19.6\sin(\alpha) - gt\right) = 0 \Rightarrow$$
$$t = \frac{19.6\sin(\alpha)}{9.8}$$
$$= 2\sin(\alpha)$$

Now determine an expression for the total time of flight—this will occur when the vertical component of displacement is equal to zero.

$$\boldsymbol{r}(t) = 19.6\cos(\alpha)t\hat{\boldsymbol{i}} + \left(19.6\sin(\alpha)t - \frac{gt^2}{2}\right)\hat{\boldsymbol{j}} + \boldsymbol{d}$$

$$\boldsymbol{r}(t) = 0 = \boldsymbol{d}$$

$$\boldsymbol{r}(t) = 19.6\cos(\alpha)t\hat{\boldsymbol{i}} + \left(19.6\sin(\alpha)t - \frac{gt^2}{2}\right)\hat{\boldsymbol{j}}$$

$$\boldsymbol{r}_{vertzero}\left(19.6\sin(\alpha)t - \frac{gt^2}{2}\right) = 0 \Rightarrow$$

$$t\left(19.6\sin(\alpha) - \frac{gt}{2}\right) = 0$$

$$t = 0, 19.6\sin(\alpha) - \frac{gt}{2} = 0$$

$$t = 0, t = \frac{2 \times 19.6\sin(\alpha)}{9.8}$$

$$t = 0, 4\sin(\alpha)$$

$t = 0$ corresponds to the initial position of the particle, while $t = 4\sin(\alpha)$ is total time of flight. $4\sin(\alpha) = 2\left(2\sin(\alpha)\right)$ hence the particle reaches its maximum height halfway through its flight.

(b) 45° and 39.2 metres

13 (a) Begin with acceleration. Anti-differentiate twice to get the position vector which contains the x and y parametric equations.

$$\boldsymbol{a} = -g\hat{\boldsymbol{j}}$$

$$\boldsymbol{v}(t) = -gt\hat{\boldsymbol{j}} + c$$

$$\boldsymbol{v}(0) = u\cos(\alpha)\hat{\boldsymbol{i}} + u\sin(\alpha)\hat{\boldsymbol{j}} = \boldsymbol{c}$$

$$\boldsymbol{v}(t) = u\cos(\alpha)\hat{\boldsymbol{i}} + \left(u\sin(\alpha) - gt\right)\hat{\boldsymbol{j}}$$

$$\boldsymbol{r}(t) = u\cos(\alpha)t\hat{\boldsymbol{i}} + \left(u\sin(\alpha)t - \frac{gt^2}{2}\right)\hat{\boldsymbol{j}} + \boldsymbol{d}$$

$$\boldsymbol{r}(t) = 0 = \boldsymbol{d} \therefore$$

$$\boldsymbol{r}(t) = u\cos(\alpha)t\hat{\boldsymbol{i}} + \left(u\sin(\alpha)t - \frac{gt^2}{2}\right)\hat{\boldsymbol{j}}$$

$$x = u\cos(\alpha)t \Leftrightarrow t = \frac{x}{u\cos(\alpha)}$$

$$y = u\sin(\alpha)t - \frac{gt^2}{2}$$

$$y = u\sin(\alpha) \times \frac{x}{u\cos(\alpha)} - \frac{g\left(\frac{x}{u\cos(\alpha)}\right)^2}{2}$$

$$y = x\tan(\alpha) - \frac{gx^2}{2u^2\cos^2(\alpha)}$$

(b) $R = \dfrac{u^2\sin(2\alpha)}{g}$ **(c)** $y_{\max} = \dfrac{u^2\sin^2(\alpha)}{2g}$

(d)

$$R = \frac{u^2\sin(2\alpha)}{g}$$

$$\frac{dR}{d\alpha} = \frac{2u^2\cos(2\alpha)}{g}$$

$$\frac{dR}{d\alpha} = 0 \Rightarrow$$

$$\frac{2u^2\cos(2\alpha)}{g} = 0 \Rightarrow$$

$$\cos(2\alpha) = 0$$

$$2\alpha = 90°$$

$$\alpha = 45°$$

$$R_{\max} = \frac{u^2\sin(2\alpha)}{g}$$

$$= \frac{u^2\sin(90°)}{g}$$

$$= \frac{u^2}{g}$$

14 (a) $\boldsymbol{r} = 20t\hat{\boldsymbol{i}} + 2t^2\hat{\boldsymbol{j}} + \left(15t - 5t^2\right)\hat{\boldsymbol{k}}$ **(b)** 18 metres

(c) 1.29 seconds

15 (a) $t = \dfrac{5u}{g}$ **(b)** $R = \dfrac{30u^2}{g}$ **(c)** $V^2 = (6u)^2 + \left(\dfrac{5u}{2}\right)^2$

(d) $6u$

CHAPTER REVIEW 3

1 $y = 4x^2, x \geq -2$

2 $\ddot{\boldsymbol{r}}(t) = -6\boldsymbol{j}$

3 $v(0) = 8$

4 $\ddot{\boldsymbol{r}}(t) = 2\hat{\boldsymbol{i}} - \dfrac{4}{(2t+3)^2}\hat{\boldsymbol{j}}$

5 C

6 $\ddot{\boldsymbol{r}}(1) = 2\hat{\boldsymbol{i}}$

7 B

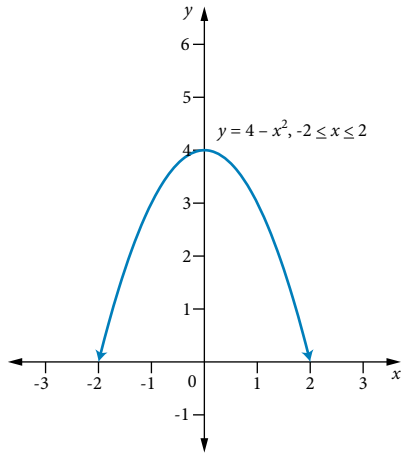
8 $\ddot{\boldsymbol{r}}(t) = -9\boldsymbol{r}$

9 $\boldsymbol{v}(0) = 5\sqrt{3}\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}}$

10 19.6 m

11 (a) $y = 4 - x^2, -2 \leq x \leq 2$

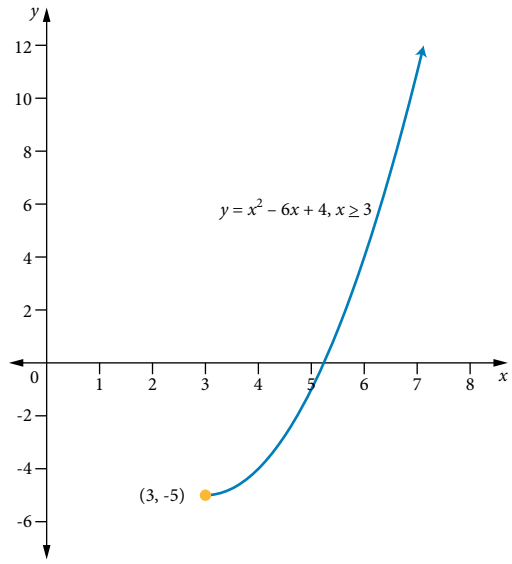
(b)



(c) Particle starts at $(0,4)$ and moves along the parabola towards $(2,0)$. The particle then moves along this parabola between the points $(2,0)$ and $(-2,0)$.

12 (a) $y = x^2 - 6x + 4,\ x \geq 3$

(b)



(c) $\boldsymbol{r}(t) = (t+3)\hat{\boldsymbol{i}} + \left(t^2 - 5\right)\hat{\boldsymbol{j}}$

13 (a) $\boldsymbol{v}(t) = 2t\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

(b) $\boldsymbol{v}(t) = 2t\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

$\boldsymbol{v}(0) = 2\hat{\boldsymbol{j}}$

$\boldsymbol{v}(1) = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

$\boldsymbol{v}(2) = 4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

14 (a) $\boldsymbol{v}_{av}(t) = (2t + \delta t)\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ (b) $\boldsymbol{v}(t) = 2t\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

15 3:20 pm. This answer assumes that there are no currents or prevailing winds affecting the velocity of the ships—in other words, it assumes that the velocity of the two ships remains constant. If this was not the case, then the ships may not collide after the specified time or at all.

16 $\boldsymbol{r}(t) = 2t\hat{\boldsymbol{i}} + \cos(2t)\hat{\boldsymbol{j}} + \sin(2t)\hat{\boldsymbol{k}}$

17 (a) $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (b) $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(c) $-\hat{\boldsymbol{i}} + \sqrt{3}\hat{\boldsymbol{j}}$

18 $\sqrt{2}e^{-t}$

19 (a) 13 m/s and 22.6° (b) 1.28 m

20 $y = 6 + x - \frac{1}{16}x^2$

21 (a) $t = 200$ s (b) $v = 0.5$ (c) 5 m/s

(d) Closest distance is 720 m.

22 (a) $\dot{\boldsymbol{r}}(t) = 2t\hat{\boldsymbol{i}} - e^{-t}\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ (b) Initial speed is $\sqrt{2}$.

(c) $\boldsymbol{a}_t(1) = \frac{4 - e^{-2}}{5 + e^{-2}}\left(2\hat{\boldsymbol{i}} - e^{-1}\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$

(d) $\boldsymbol{a}_n(1) = \frac{1}{5 + e^{-2}}\left(\left(2 + 4e^{-2}\right)\hat{\boldsymbol{i}} + 9e^{-1}\hat{\boldsymbol{j}} + \left(4 - e^{-2}\right)\hat{\boldsymbol{k}}\right)$

23 (a) $\frac{1}{3}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)$ (b) $\frac{1}{3}\left(10\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}} + 11\hat{\boldsymbol{k}}\right)$

(c) 7.28 km

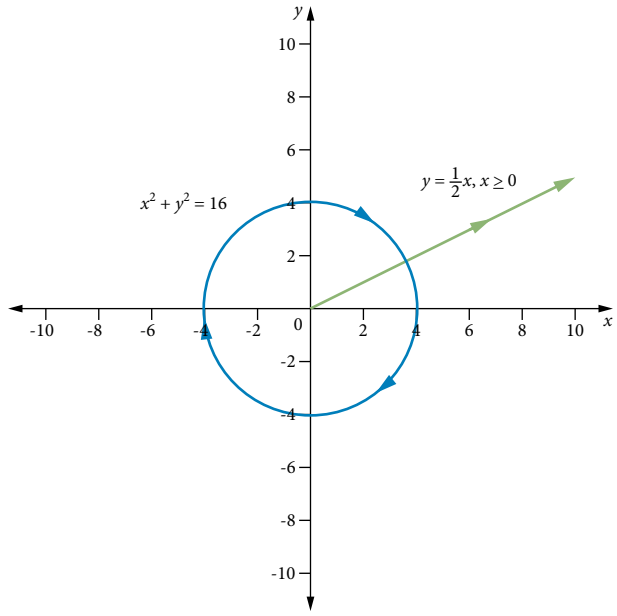
24 (a) $\tan(\alpha) = \frac{4}{3} \Rightarrow \boldsymbol{u} = 600\hat{\boldsymbol{i}} + 800\hat{\boldsymbol{j}}$

(b) $\boldsymbol{r} = 600t\hat{\boldsymbol{i}} + \left(800t - 5t^2\right)\hat{\boldsymbol{j}}$

(c) $\boldsymbol{v} = \left(at + 44\,080\right)\hat{\boldsymbol{i}} + at\hat{\boldsymbol{j}}$ (d) 116 seconds (e) 310 m/s

25 (a) $v = 4\alpha$ (b) $y = \frac{1}{2}x, x \geq 0$ and $x^2 + y^2 = 16$

(c)



(d) $\left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ (e) 0.62 (f) 4.4

Chapter 4: Complex numbers

RECALL

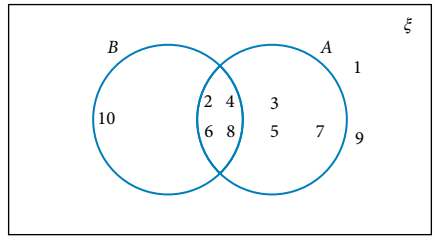
1 (a) -3 is an element of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}^-$.

(b) $\sqrt{\frac{4}{9}}$ is an element of $\mathbb{R}$ and $\mathbb{Q}$.

(c) $4 + i$ is an element of $\mathbb{C}$.

2 (a) (i) $\{2,3,4,5,6,7,8,10\}$ (ii) $\{2,4,6,8\}$ (iii) $\{1,9\}$

(b)



3 (a) $10i$ (b) $2\sqrt{2}i$

4 (a) $16 - 24i$ (b) $17i$

5 (a) $20 + 17i$ (b) $-29 + 3i$ (c) $2 + 2\sqrt{3}i$ (d) 21

6 (a) $\frac{4}{17} + \frac{1}{17}i$ (b) $\frac{1}{2} - \frac{1}{2}i$

7 (a) 13 (b) -36 (c) $\frac{-5}{13} - \frac{12}{13}i$

8 $x = 2,\ y = -1$

9 A: $1 + i$, B: $-3 + 2i$, C: $-3 - 2i$, D: $0 - 3i$, E: $2 - i$, F: $2 + 0i$

10 (a) $\cos(\theta) - \sin(\theta)$ (b) $\cot(A + B)$ (c) 1

EXERCISE 4.1

1 (a) $\mathbb{C}$ (b) $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}^-$ (c) $\mathbb{C}, \mathbb{R}$ (d) $\mathbb{C}$

2 (a) $4i$ (b) $\sqrt{11}\,i$ (c) $2\sqrt{7}\,i$ (d) $6\sqrt{2}\,i$

3 (a) $5 - 27i$ (b) $-7 + 24i$ (c) $2 - 16i$ (d) $\frac{6}{37} - \frac{1}{37}i$

4 $t = 3.2 + 5i$
$z = 5 + 3.2i$
$w = -5 - 5i$
$v = 3.2 - 5i$
$u = 5 - 3.2i$

5 (a) $\text{Re}(z) = 26,\ \text{Im}(z) = 13$ (b) $\text{Re}(z) = 12,\ \text{Im}(z) = 0$
(c) $\text{Re}(z) = 5,\ \text{Im}(z) = 0$ (d) $\text{Re}(z) = 1,\ \text{Im}(z) = 8$
(e) $\text{Re}(z) = -6,\ \text{Im}(z) = 2\sqrt{7}$
(f) $\text{Re}(z) = u + v,\ \text{Im}(z) = -(2v + 2u)$

6 (a) $-i$ (b) $-72i$ (c) 1

7 (a) $13 - 14i$ (b) $2 + i$ (c) $169 + 0i$
(d) $0 - \frac{1}{2}i$ (e) $6 + 0i$ (f) $\frac{14}{5} - \frac{8i}{5}$
(g) $\text{Re}(z) = a^2 - b^2,\ \text{Im}(z) = a - b$

8 $-2 + 2i$

9 (a) $7.2 - 3.24i$ (b) $52 - 124\,i$

10 (a) $\frac{36}{169} + \frac{15}{169}i$ (b) $\frac{33}{53} + \frac{17}{53}i$

11 (a) -4 (b) $38 - 41\,i$

12 (a) $a = 2,\ b = -3$ (b) $a = 3,\ b = 3$
(c) $a = \frac{3}{2},\ b = -\frac{1}{2}$ (d) $a = -1,\ b = 1$

13 (a) $-16 + 30i$ (b) $6 - 2i$
(c) -22 (d) $-4 - i$

14 $p = \frac{7}{8},\ q = \frac{-\sqrt{15}}{8}$

15 $a = 2,\ b = \frac{1}{2}$

16 (a) $\frac{28}{25} + \frac{42}{25}i$ (b) $21 + i$ (c) $-\frac{15}{2} + 5i$

17 D

18 (a) B
(b) A: The set of reals is a subset of the set of complex numbers, so the statement is true.
C: $(x + yi)(x - yi) = x^2 + y^2$, which is real, hence the statement is true.
D: True because $\mathbb{Q} \subset \mathbb{R}$.

19 (a) $0 - 7i$ (b) $-\frac{1}{4} + 0i$

20 $-i$

21 1

22 $z = 2 + 6i$

23 Consider the left-hand side

$$(2 - 3i)^{-1} = \frac{1}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$$
$$= \frac{2 + 3i}{13}$$
$$\left(\frac{2 + 3i}{13}\right)^2 = \frac{4 + 12i - 9}{169}$$
$$= -\frac{5}{169} + \frac{12}{169}i$$

Consider the right-hand side

$$(2 - 3i)^2 = 4 - 12i - 9$$
$$= -5 - 12i$$
$$\frac{1}{-5 - 12i} = \frac{1}{-5 - 12i} \times \frac{-5 + 12i}{-5 + 12i}$$
$$= \frac{-5 + 12i}{25 + 144}$$
$$= \frac{-5}{169} + \frac{12}{169}i$$
$$= \text{LHS}$$

24 (a) $1,\ i,\ -1,\ -i$ (b) $-1,\ -i$

25 (a)
$$u + v = a + bi + c + di$$
$$= a + c + (b + d)i$$
$$\overline{u + v} = a + c - (b + d)i$$
$$\bar{u} + \bar{v} = a - bi + c - di$$
$$= a + c - (b + d)i$$
$$= \overline{u + v}$$

(b)
$$u + v = 2 + 3i + 1 - 4i$$
$$= 3 - i$$
$$\overline{u + v} = 3 + i$$
$$\bar{u} + \bar{v} = 2 - 3i + 1 + 4i$$
$$= 3 + i$$

26
$$uv = (a + bi)(c + di)$$
$$= ac + (bc + ad)i - bd$$
$$vu = (c + di)(a + bi)$$
$$= ca + (da + cb)i - db$$
$$= ac + (bc + ad)i - bd$$
$$= vu$$

27

$c: 0$	1	$-\frac{1}{2}$	$-\frac{1}{2}$
$d: 0$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$

EXERCISE 4.2

1 (a) 3 (b) $\sqrt{(a + 2)^2 + b^2}$
(c) $\sqrt{(a + 3)^2 + (b - 4)^2}$ (d) 5

2 (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) -1.768 (d) 2.035

3 (a) $-108.4°$ (b) $-36.9°$

4 (a) $\sqrt{2} + \sqrt{2}\,i$ (b) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
(c) $0 + i$ (d) $-6 + 0i$
(e) $-4\sqrt{2} + 4\sqrt{2}\,i$ (f) $\sqrt{7} + 0\,i$

5 (a) $5 + 5\sqrt{3}\,i$ (b) $0 - 2i$
(c) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ (d) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

6 **(a)** $8\text{cis}\left(-\frac{\pi}{6}\right)$ **(b)** $9\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ **(c)** $5\text{cis}\left(\frac{\pi}{2}\right)$

(d) $13\text{cis}(0)$ **(e)** $2\text{cis}\left(\frac{2\pi}{3}\right)$

7 **(a)** $7.3\text{cis}(74.1°)$ **(b)** $4.1\,\text{cis}(-14.0°)$

(c) $3.6\text{cis}(146.3°)$ **(d)** $6.3\text{cis}(-161.6°)$

8 **(a)** $10\text{cis}(90°)$ **(b)** $2\sqrt{3}\text{cis}(-40°)$

(c) $12\text{cis}\left(-\frac{5\pi}{12}\right)$ **(d)** $4\text{cis}\left(\frac{3\pi}{4}\right)$

(e) $3\text{cis}\left(\frac{\pi}{6}\right)$ **(f)** $2\text{cis}\left(\frac{\pi}{3}\right)$

(g) $\text{cis}\left(-\frac{\pi}{4}\right)$ **(h)** $a^3\text{cis}(2B)$

9 **(a)** $64\text{cis}\left(-\frac{\pi}{2}\right)$ **(b)** $81\text{cis}\left(\frac{2\pi}{3}\right)$

(c) $216\text{cis}\left(\frac{\pi}{2}\right)$ **(d)** $32\text{cis}\left(-\frac{5\pi}{6}\right)$

(e) $16\text{cis}(0)$ **(f)** $\text{cis}\left(-\frac{2\pi}{3}\right)$

(g) $108\text{cis}\left(-\frac{\pi}{2}\right)$ **(h)** $\frac{1}{27}\text{cis}(-150°)$

10 $-\frac{5\pi}{12}$

11 $\begin{aligned}\text{cis}(n\theta)\text{cis}(-n\theta) &= \text{cis}(n\theta - n\theta)\\ &= \text{cis}(0)\\ &= \cos(0) + i\sin(0)\\ &= 1 + 0i\\ &= 1\end{aligned}$

12 **(a)** $u = 2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$, $v = 2\text{cis}\left(\frac{\pi}{6}\right)$ **(b)** 8

(c) $\frac{\pi}{2}$

13 C

14 C

15 $\begin{aligned}z &= \cos(\theta) + i\sin(\theta)\\ \overline{z} &= \cos(\theta) - i\sin(\theta)\\ \frac{1}{z} &= \frac{1}{\cos(\theta) + i\sin(\theta)} \times \frac{\cos(\theta) + i\sin(\theta)}{\cos(\theta) - i\sin(\theta)}\\ &= \frac{\cos(\theta) + i\sin(\theta)}{\cos^2(\theta) + \sin^2(\theta)}\\ &= \frac{\cos(\theta) - i\sin(\theta)}{1}\\ &= \cos(\theta) - i\sin(\theta)\\ &= \overline{z}\\ \overline{z} &= \frac{1}{z}\end{aligned}$

16 **(a)** $4 + 3i$ **(b)** $\frac{7}{25} - \frac{24}{25}i$

17 **(a)** $\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right), \frac{\left(1+\sqrt{3}\right)}{2} + \frac{i\left(\sqrt{3}-1\right)}{2}$

(b) $\frac{\left(\sqrt{3}-1\right)}{2\sqrt{2}}$

---

18 $\begin{aligned}r_1\text{cis}(\theta_1) \times r_2\text{cis}(\theta_2) &= \left(r_1\cos(\theta_1) + r_1\sin(\theta_1)i\right) \times \left(r_2\cos(\theta_2) + r_2\sin(\theta_2)i\right)\\ &= r_1\cos(\theta_1)r_2\cos(\theta_2) + r_1r_2i\left(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)\right) + r_1\sin(\theta_1)r_2\sin(\theta_2)i^2\\ &= r_1r_2\left(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right) + r_1r_2i\left(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)\right)\\ &= r_1r_2\cos\left(\theta_1 + \theta_2\right) + r_1r_2i\sin\left(\theta_1 + \theta_2\right)\\ &= r_1r_2\text{cis}\left(\theta_1 + \theta_2\right)\end{aligned}$

19 If $n = 1$, $\left(r\text{cis}(\theta)\right)^1 = r^1\text{cis}(1 \times \theta)$ so the theorem is true for $n = 1$.

Assume it is true for $n = k$,

so $\left(r\text{cis}(\theta)\right)^k = r^k\text{cis}(k\theta)$.

Now, for $n = k + 1$,

$\begin{aligned}\left(r\text{cis}(\theta)\right)^{k+1} &= \left(r\text{cis}(\theta)\right)^k r\text{cis}(\theta)\\ &= r^k\text{cis}(k\theta)r\text{cis}(\theta)\\ &= r^{k+1}\left(\cos(k\theta) + i\sin(k\theta)\right)\left(\cos(\theta) + i\sin(\theta)\right)\\ &= r^{k+1}\left(\cos(k\theta)\cos(\theta) + i^2\sin(k\theta)\sin(\theta) + i\sin(k\theta)\cos(\theta) + i\cos(k\theta)\sin(\theta)\right)\\ &= r^{k+1}\left(\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) + i\left(\sin(k\theta)\cos(\theta) + \cos(k\theta)\sin(\theta)\right)\right)\\ &= r^{k+1}\left(\cos(k\theta + \theta) + i\sin(k\theta + \theta)\right)\\ &= r^{k+1}\text{cis}\left((k+1)\theta\right)\end{aligned}$

Therefore, if it is true for $n = k$ then it is true for $n = k + 1$. It is shown to be true for $n = 1$ and is therefore true for $n = 2, 3, \ldots$ and for all $n \in \mathbb{Z}^+$.

20 $\left(r\text{cis}(\theta)\right)^n = \left(r\text{cis}(\theta)\right)^{-m}$

$$= \frac{1}{\left(r\text{cis}(\theta)\right)^m}$$

$$= \frac{1}{r^m \text{cis}(m\theta)} \quad \text{(already proved as } m > 0\text{)}$$

$$= \frac{1}{r^m}\left(\frac{1}{\cos(m\theta) + i\sin(m\theta)} \times \frac{\cos(m\theta) - i\sin(m\theta)}{\cos(m\theta) - i\sin(m\theta)}\right)$$

$$= \frac{1}{r^m}\left(\frac{\cos(m\theta) - i\sin(m\theta)}{\cos^2(m\theta) + \sin^2(m\theta)}\right)$$

$$= \frac{1}{r^m}\left(\cos(m\theta) - i\sin(m\theta)\right)$$

Now, you can substitute $-n$ for m

$$\left(r\text{cis}(\theta)\right)^n = \frac{1}{r^{-n}}\left(\cos(-n\theta) + i\sin(-n\theta)\right)$$

$$= r^n\left(\cos(n\theta) - i\sin(n\theta)\right)$$

$$= r^n \text{cis}(n\theta)$$

Thus, the theorem holds for negative integer powers. If you include the trivial case for $n = 0$, then the theorem holds for all $n \in \mathbb{Z}$.

21 $\frac{1}{2i}\left(\text{cis}(\theta) - \text{cis}(-\theta)\right) = \frac{1}{2i}\left(\cos(\theta) + i\sin(\theta) - \cos(-\theta) - i\sin(-\theta)\right)$

$$= \frac{1}{2i}\left(\cos(\theta) - \cos(\theta) + i\sin(\theta) + i\sin(\theta)\right)$$

$$= \frac{1}{2i}\left(2i\sin(\theta)\right)$$

$$= \sin(\theta)$$

22 Let

$z = r\text{cis}(\theta)$

$\overline{z} = r\cos(\theta) - ri\sin(\theta)$

$= r\cos(-\theta) + ri\sin(-\theta)$

$= r\text{cis}(-\theta)$

$z^n = r^n \text{cis}(n\theta)$

$\overline{z^n} = r^n \text{cis}(-n\theta)$

$\left(\overline{z}\right)^n = r^n\left(\text{cis}(-\theta)\right)^n$

$= r^n \text{cis}(-n\theta)$

$= \overline{z^n}$

23 $\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} \ldots$$

$$= 1 + \frac{i\theta}{1!} - \frac{(\theta)^2}{2!} - \frac{i(\theta)^3}{3!} + \frac{(\theta)^4}{4!} + \frac{i(\theta)^5}{5!} - \frac{(\theta)^6}{6!} - \ldots$$

$$= 1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} - \frac{(\theta)^6}{6!} + \ldots + i\left(\frac{i\theta}{1!} - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} - \ldots\right)$$

$\therefore \text{cis}(\theta) = e^{i\theta}$

24 (a) $-6 + 0i$ (b) $-1 - i\sqrt{3}$ (c) $-3\sqrt{2} + 3\sqrt{2}i$

EXERCISE 4.3

1 $2 - 3i,\ -2 + 3i$

2 (a) $5\text{cis}\left(-\frac{\pi}{4}\right),\ 5\text{cis}\left(\frac{3\pi}{4}\right)$ (b) $\sqrt{2}\text{cis}\left(\frac{\pi}{12}\right),\ \sqrt{2}\text{cis}\left(\frac{-11\pi}{12}\right)$

3 (a) $\text{cis}\left(\frac{\pi}{3}\right),\ \text{cis}(\pi),\ \text{cis}\left(\frac{-\pi}{3}\right)$

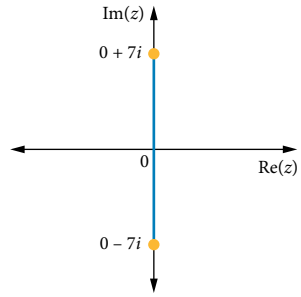
(b) $2^{\frac{1}{3}}\text{cis}\left(\frac{2\pi}{9}\right),\ 2^{\frac{1}{3}}\text{cis}\left(\frac{8\pi}{9}\right),\ 2^{\frac{1}{3}}\text{cis}\left(-\frac{4\pi}{9}\right)$

(c) $\text{cis}\left(\frac{\pi}{8}\right),\ \text{cis}\left(\frac{5\pi}{8}\right),\ \text{cis}\left(-\frac{7\pi}{8}\right),\ \text{cis}\left(-\frac{3\pi}{8}\right)$

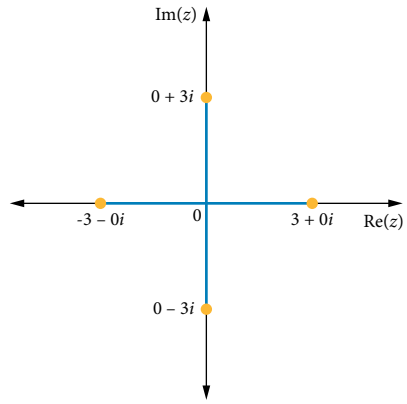
(d) $2^{\frac{7}{8}}\text{cis}\left(\frac{\pi}{16}\right),\ 2^{\frac{7}{8}}\text{cis}\left(\frac{9\pi}{16}\right),\ 2^{\frac{7}{8}}\text{cis}\left(-\frac{15\pi}{16}\right),\ 2^{\frac{7}{8}}\text{cis}\left(-\frac{7\pi}{16}\right)$

4 $\text{cis}\left(\frac{\pi}{3}\right), \text{cis}(\pi), \text{cis}\left(-\frac{\pi}{3}\right); 3\text{cis}\left(\frac{\pi}{3}\right), 3\text{cis}(\pi), 3\text{cis}\left(-\frac{\pi}{3}\right)$

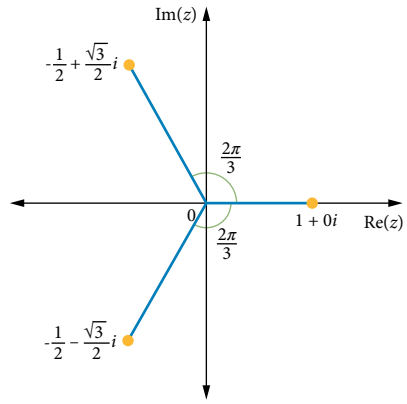
5 (a)



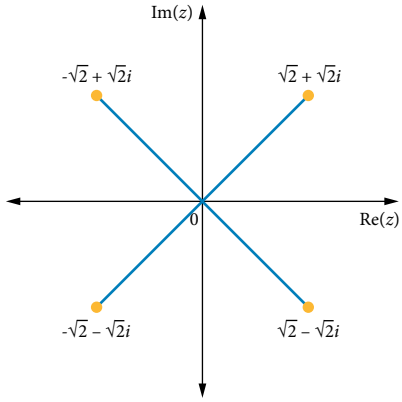
(b)



(c)



(d)



6 **(a)** $2\text{cis}(0)$ **(b)** $\frac{2\pi}{5}$

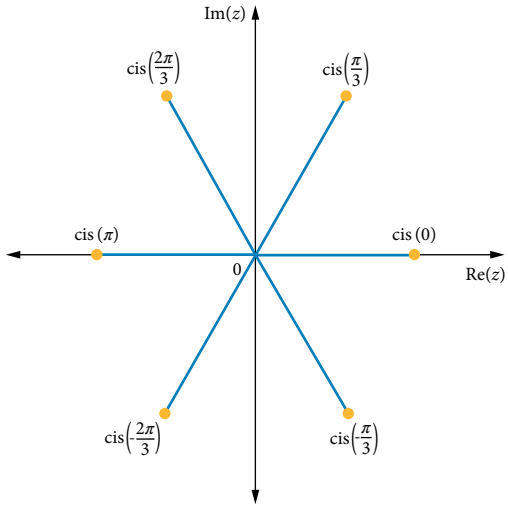
(c) $2\text{cis}\left(\frac{2\pi}{5}\right), 2\text{cis}\left(\frac{4\pi}{5}\right), 2\text{cis}\left(-\frac{4\pi}{5}\right), 2\text{cis}\left(-\frac{2\pi}{5}\right)$

7 **(a)** $2, \pi$ **(b)** $5, \frac{2\pi}{5}$ **(c)** $3, \frac{2\pi}{3}$ **(d)** $9, \frac{2\pi}{9}$

8 **(a)** $\text{cis}(0)$ **(b)** $\frac{\pi}{3}$

(c) $\text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}(\pi), \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\left(-\frac{\pi}{3}\right)$

(d) $(1)^{\frac{1}{6}} = \text{cis}(0), \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}(\pi),$
$\text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\left(-\frac{\pi}{3}\right)$



9 $5\text{cis}(0), 5\text{cis}\left(\frac{\pi}{2}\right), 5\text{cis}(\pi), 5\text{cis}\left(-\frac{\pi}{2}\right)$
$= 5 + 0i, 0 + 5i, -5 + 0i, 0 - 5i$

10 $\text{cis}\left(-\frac{\pi}{12}\right), \text{cis}\left(\frac{\pi}{4}\right), \text{cis}\left(\frac{7\pi}{12}\right), \text{cis}\left(\frac{11\pi}{12}\right), \text{cis}\left(-\frac{3\pi}{4}\right), \text{cis}\left(-\frac{5\pi}{12}\right)$

11 **(a)** $u + v = -\frac{1}{2} + \frac{\sqrt{3}}{2}i + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= -1$
Hence, $u + v = -1$

(b) Let $u = \text{cis}\left(\frac{2\pi}{3}\right), \quad v = \text{cis}\left(-\frac{2\pi}{3}\right)$

$uv = \text{cis}\left(\frac{2\pi}{3} - \frac{2\pi}{3}\right)$
$= \text{cis}(0)$
$= 1 + 0i$
$= 1$
Hence, $uv = 1$

(c) $v^2 = \text{cis}\left(-\frac{4\pi}{3}\right)$
$= \text{cis}\left(\frac{2\pi}{3}\right)$
$= u$
Hence, $u = v^2$

12 **(a)** $\pm 2\text{cis}\left(\frac{\pi}{6}\right), \pm 2\text{cis}\left(-\frac{\pi}{3}\right)$

(b) $z^2 = w$
$(iz)^2 = i^2 z^2$
$= -z^2$
$= -w$
Hence, $(iz)^2 = -w$

(c) Choosing one of the square roots, let $z = 2\text{cis}\left(\frac{\pi}{6}\right)$ and therefore $w = 2 + 2\sqrt{3}i$.

$z = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$
$= \sqrt{3} + i$
$iz = -1 + \sqrt{3}i$
$= 2\text{cis}\left(\frac{2\pi}{3}\right)$
$(iz)^2 = 4\text{cis}\left(\frac{4\pi}{3}\right)$
$= 4\text{cis}\left(-\frac{2\pi}{3}\right)$
$= 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$= -2 - 2\sqrt{3}i$
$= -w$
Hence, $(iz)^2 = -w$

13 C

14 **(a)** B

(b) Option A: The principle argument of v is $-\frac{2\pi}{3}$.

Option C: The complex numbers are conjugates, so only the imaginary parts will be the opposite sign; the real parts are equal.

Option D: The cube root of 8 is 2.

15 $3 + 2i, \ -3 - 2i$

16 **(a)** $z^2 = 0 + 49i$ **(b)** $z^4 = 16 + 0i$ **(c)** $z^3 = 0 - 8i$

17 $n^{\frac{1}{4}}\text{cis}\left(-\frac{\pi}{8}\right), n^{\frac{1}{4}}\text{cis}\left(\frac{3\pi}{8}\right), n^{\frac{1}{4}}\text{cis}\left(\frac{7\pi}{8}\right), n^{\frac{1}{4}}\text{cis}\left(-\frac{5\pi}{8}\right)$

18 Let u be the sixth root and v be the twelfth root.

$$1 = 1\text{cis}(0)$$
$$= \text{cis}(0 + 2k_1\pi), \quad k_1 \in \mathbb{Z}$$
$$u = \left(\text{cis}(0 + 2k_1\pi)\right)^{\frac{1}{6}}$$
$$= \text{cis}\left(\frac{k_1\pi}{3}\right)$$
$$v = \left(\text{cis}(0 + 2k_2\pi)\right)^{\frac{1}{12}}, \quad k_2 \in \mathbb{Z}$$
$$= \text{cis}\left(\frac{k_2\pi}{6}\right)$$

The product of these:

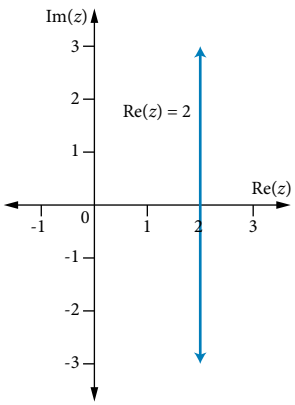
$$uv = \text{cis}\left(\frac{k_1\pi}{3}\right)\text{cis}\left(\frac{k_2\pi}{6}\right)$$
$$= \text{cis}\left(\frac{k_1\pi}{3} + \frac{k_2\pi}{6}\right)$$
$$= \text{cis}\left(\frac{(2k_1 + k_2)\pi}{6}\right)$$

Now, as k_1 and $k_2$are integers, $2k_1 + k_2$ is also an integer. The product is of the same form as v, and is therefore a twelfth root of 1.

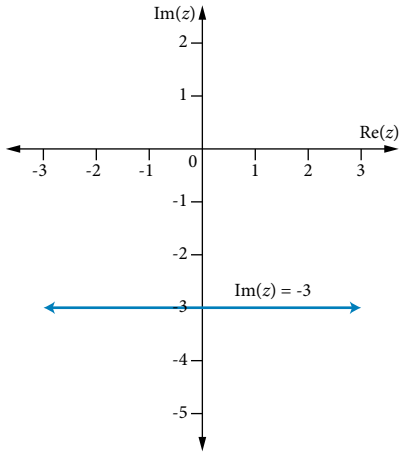
Hence, the product of the sixth and twelfth root is a twelfth root.

EXERCISE 4.4

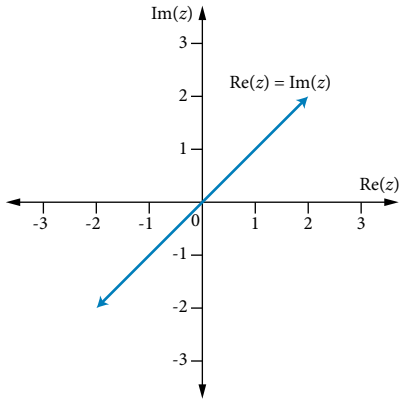
1 **(a)**



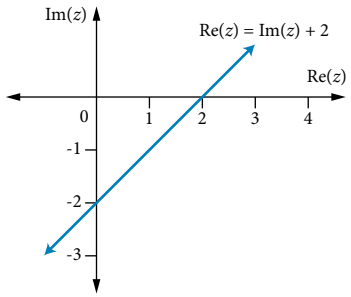
(b)



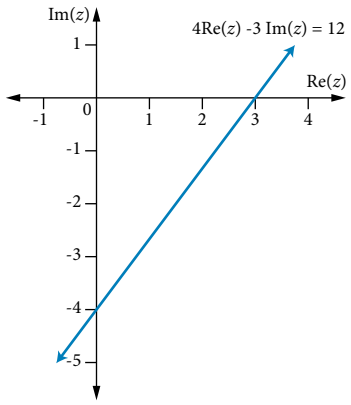
2 **(a)**



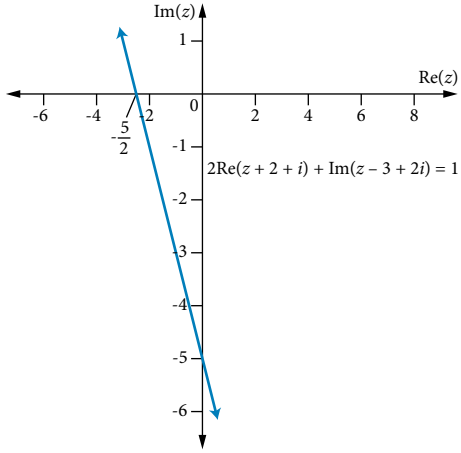
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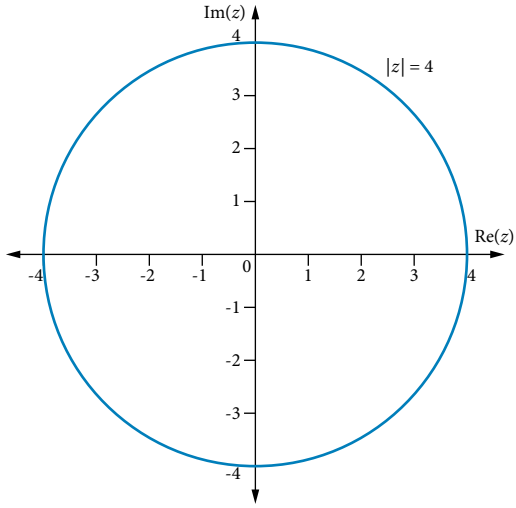
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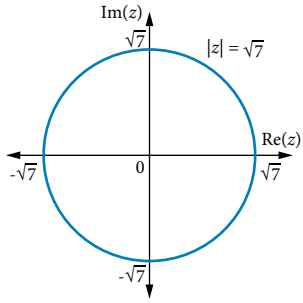
(d)



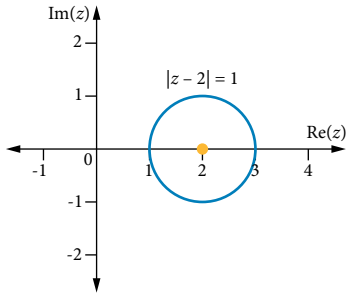
3 (a)



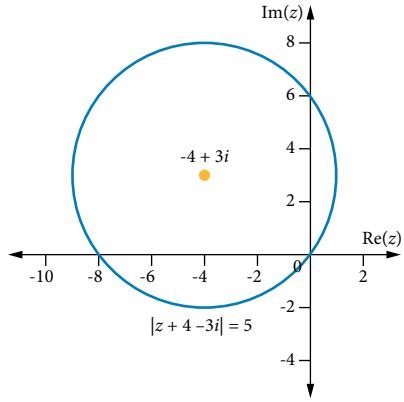
(b)



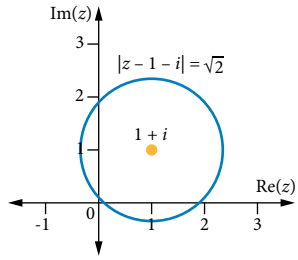
4 (a)



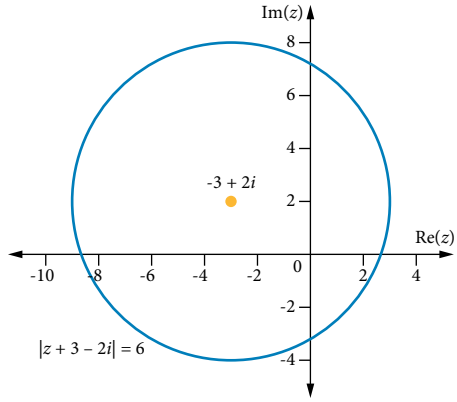
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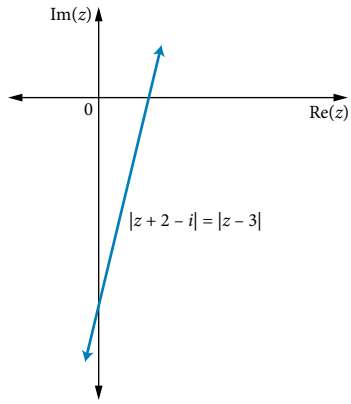
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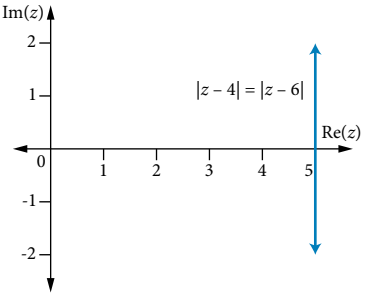
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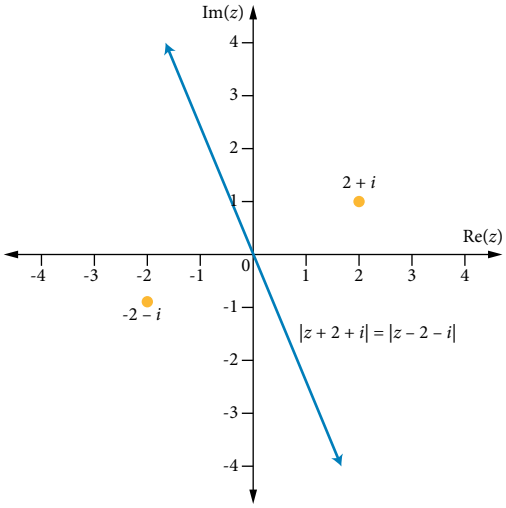
5 (a)



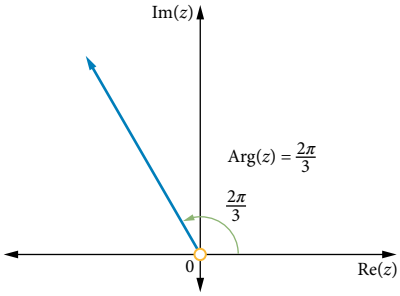
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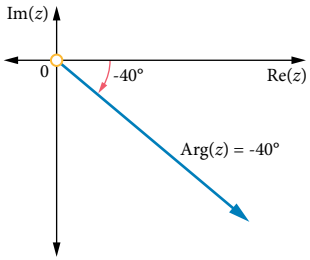
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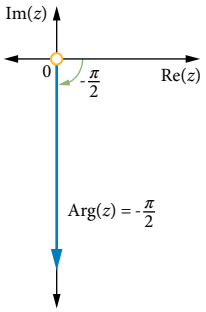
6 (a)



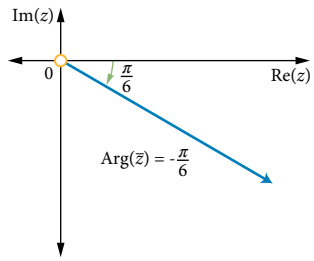
(b)



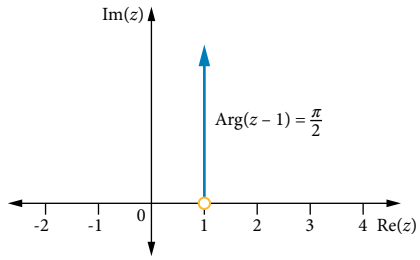
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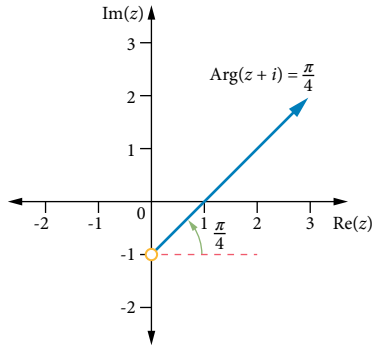
(d)



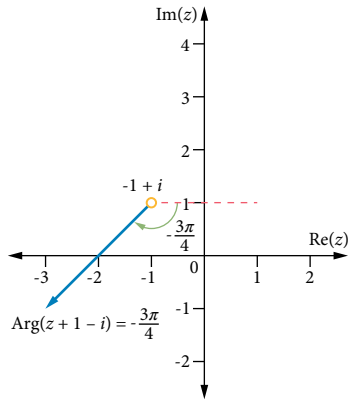
7 (a)



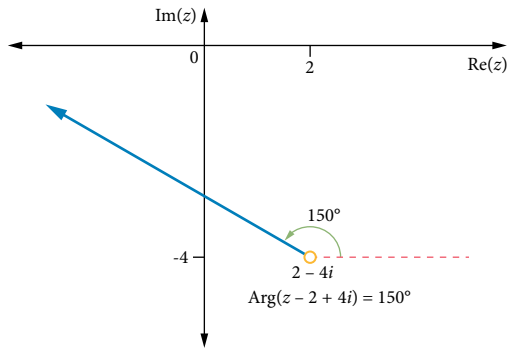
(b)



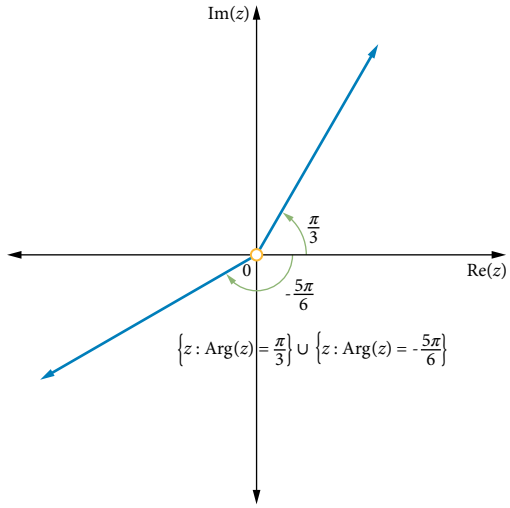
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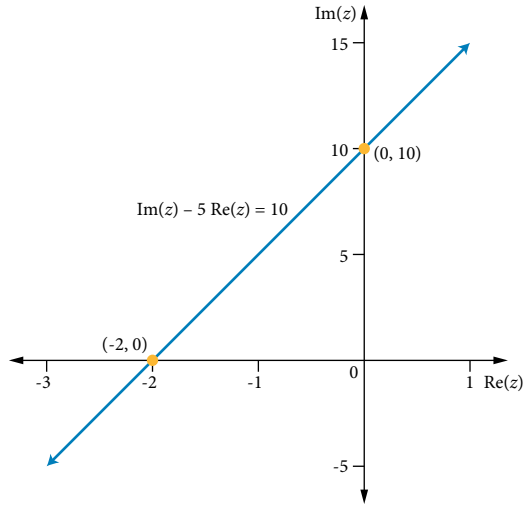
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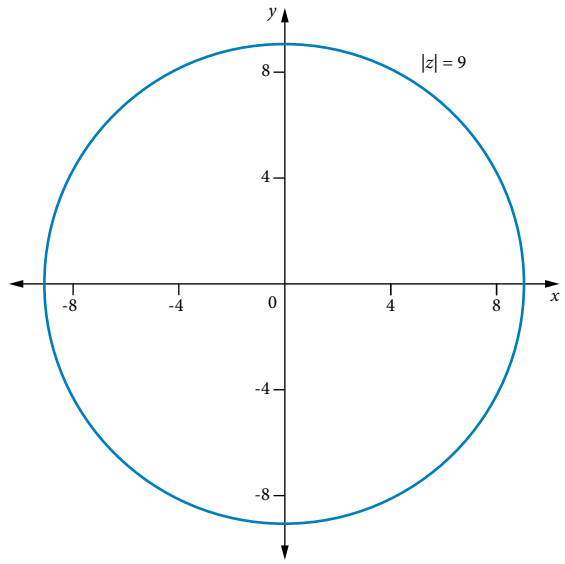
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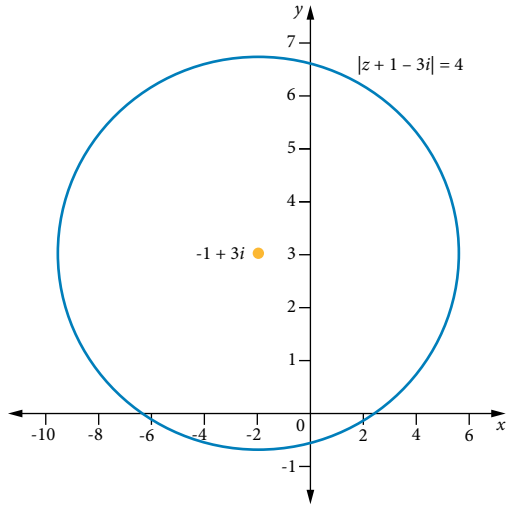
9 (a)



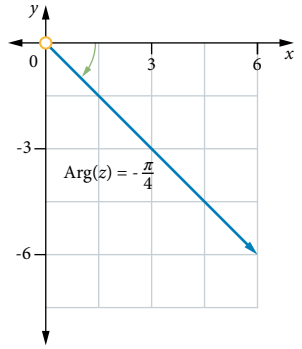
(b)



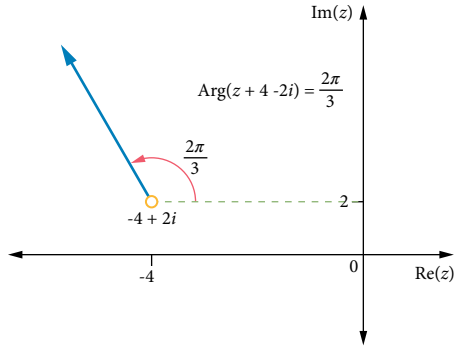
(c)



(d)



(e)



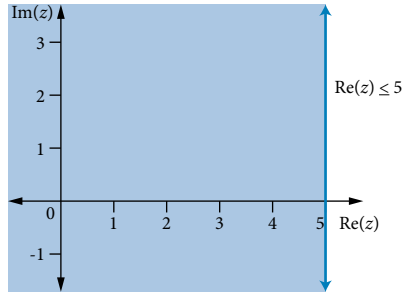
10 (a) $|z| = 2$ **(b)** $\text{Arg}(z) = \frac{7\pi}{8}$

(c) $\text{Arg}(z+1) = \frac{\pi}{3}$ **(d)** $\text{Arg}(z+2-2i) = \frac{\pi}{4}$

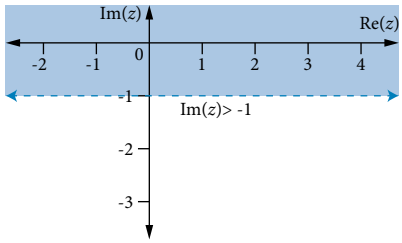
11 (a) $|z+4-3i| = 5$ **(b)** $|z+4| = |z+2|$

12 C

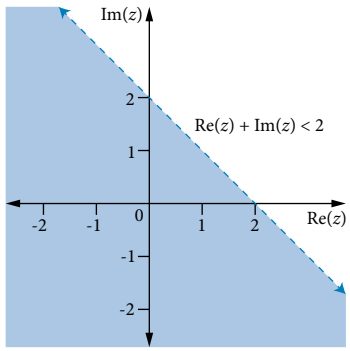
13 (a)



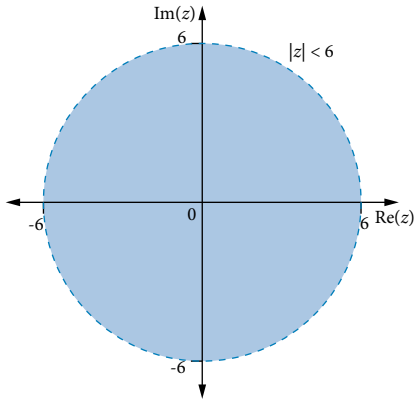
(b)



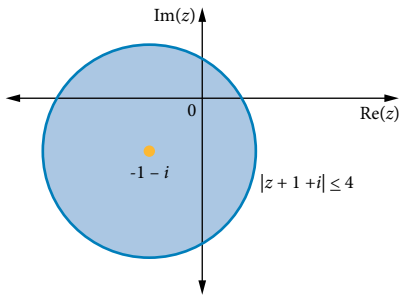
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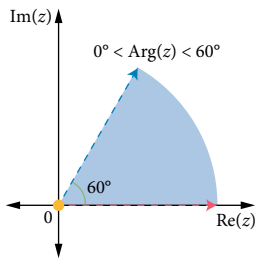
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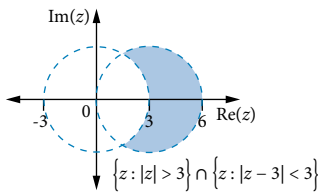
(e)



(f)

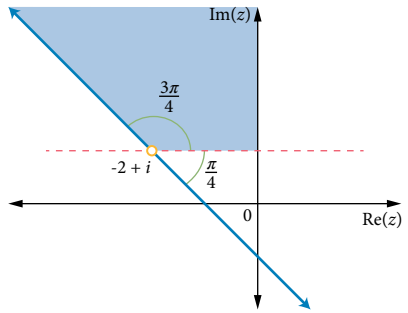


(g)



14 D

15



16 $-\frac{2}{3}+\frac{2}{3}i$

17

$$|z| = 2$$
$$\sqrt{x^2+y^2} = 2$$
$$x^2+y^2 = 4 \quad [1]$$
$$|z-2\sqrt{2}-2\sqrt{2}i| = 2$$
$$\sqrt{\left(x-2\sqrt{2}\right)^2+\left(y-2\sqrt{2}\right)^2} = 2$$
$$\left(x-2\sqrt{2}\right)^2+\left(y-2\sqrt{2}\right)^2 = 4$$
$$x^2-4\sqrt{2}x+8+y^2-4\sqrt{2}y+8 = 4 \quad [2]$$

Solving [1] and [2] simultaneously using technology or by hand gives $x = \sqrt{2}, y = \sqrt{2}$, therefore $\text{Re}(z) = \text{Im}(z)$ as required.

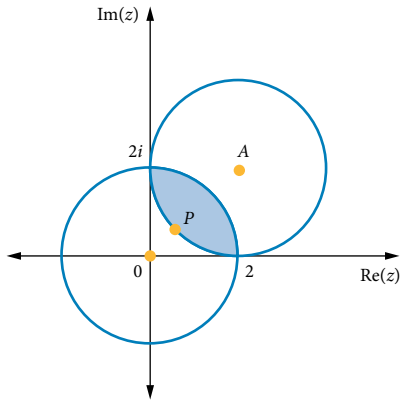
Note that with two overlapping circles with centres on the line $y = x$, the intersections will be two points on that line, and hence $\text{Re}(z) = \text{Im}(z)$.

In general, intersection points will lie on the perpendicular bisector of the line joining the centres of the circles.

18 Greatest value 2, least value -4.

19 $|z| \leq 2$ represents the points inside and on a circle with centre at the origin and radius 2.

$|z-2-2i| \leq 2$ represents the points inside and on a circle with centre at the point $2+2i$ and radius 2. The intersection of the two regions is shaded.



P is the point in this locus with the smallest value of $|z|$, because it is closest to the origin.

O, P and A are collinear, so the distance $OP = OA - AP$

$OA = |2 + 2i|$

$= \sqrt{8}$

$= 2\sqrt{2}$

$AP = 2$ (the radius of the circle)

$OP = 2\sqrt{2} - 2$

$= 2\left(\sqrt{2} - 1\right)$

Because P is the closest point in the locus to the origin, $|z|$, which is the distance from the origin, it must be greater than or equal to OP.

$|z| \geq 2\left(\sqrt{2} - 1\right)$

EXERCISE 4.5

1 (a) The quotient is $x^2 + 2x - 4$; the remainder is 5.

(b) The quotient is $x^3 + 3x^2 + 2x - 4$; the remainder is -2.

(c) The quotient is $x^3 + x^2 - 2x + 3$; the remainder is -2.

(d) The quotient is $x^3 - x^2 + 2x - 2$; the remainder is 3.

(e) The quotient is $x^2 - 2x - 4$; the remainder is -8.

(f) The quotient is $x^2 - x - 1$; the remainder is 11.

(g) The quotient is $2x^3 - 2x^2 - \frac{14}{3}x + \frac{11}{3}$; the remainder is -3.

2 By inspection, $P = 3$ and $Q = 7$.

3 $P = -3, Q = 9$

4 (a) -2 (b) 13 (c) 0

(d) 48 (e) 1 (f) -154

5 (a) $3\frac{1}{4}$ (b) $4\frac{3}{8}$ (c) $6\frac{23}{27}$ (d) $\frac{7}{4}$

6 -3

7 $p(2) = 0$, so $x - 2$ is a factor of $x^3 - 11x + 14$.

8 $p(-1) = 0$, so $x + 1$ is a factor of $x^4 - x^3 + x^2 - 2x - 5$.

9 (a) $x^3 - 4x^2 + x + 6 = (x + 1)(x - 2)(x - 3)$

(b) $x^3 - 4x^2 - x + 4 = (x - 1)(x + 1)(x - 4)$

(c) $x^3 + 3x^2 - x - 3 = (x - 1)(x + 1)(x + 3)$

(d) $x^3 + 6x^2 + 3x - 10 = (x - 1)(x + 2)(x + 5)$

(e) $x^3 + 3x^2 - 4x - 12 = (x - 2)(x + 2)(x + 3)$

(f) $x^4 + 3x^3 - 7x^2 - 27x - 18 = (x + 1)(x + 2)(x + 3)(x - 3)$

10 (a) $2x^3 + 3x^2 - 18x + 8 = (x - 2)(2x - 1)(x + 4)$

(b) $2x^3 + x^2 - 2x - 1 = (x - 1)(x + 1)(2x + 1)$

(c) $3x^3 + 8x^2 - 7x - 12 = (x + 1)(x + 3)(3x - 4)$

(d) $4x^3 + 13x^2 - 32x + 15 = (x - 1)(x + 5)(4x - 3)$

11 (a) $(x - 1)^2(x + 2)$ (b) $(x + 2)^2(x - 2)$

(c) $(x - 1)^2(4x - 3)$

12 (a) $x = -1, x = \frac{1}{3}, x = 3$ (b) $x = -1, x = 2, x = 3$

(c) $x = -2, x = -2, x = 3$ (d) $x = -1, x = \frac{1}{4}, x = 2$

13 (a) 1184 (b) $(5x + 2)(x - 3)(x + 1)$

14 $\frac{1}{3}$

15 -1

16 -5

17 2

18 $x = -3, x = -2, x = -1, x = 1, x = 1$

19 Let $p(x)$ be a polynomial in x, with quotient Q, and remainder R, when $p(x)$ is divided by $ax - b$.

Then, in all cases,

$p(x) \equiv Q(ax - b) + R$

$\equiv aQ\left(x - \frac{b}{a}\right) + R$

(Note that $\equiv$ means identically equal.)

Since this is true for all values of x, then it is true for $x = \frac{b}{a}$.

Substituting $x = \frac{b}{a}$ gives

$p\left(\frac{b}{a}\right) = aQ\left(\frac{b}{a} - \frac{b}{a}\right) + R$

$= aQ(0) + R$

$= R$

20 $A = 1, B = 1, C = 1, D = 1$

21 $a = -2, b = 2$

22 $q = -3, -2, 3$

23 $4x - 9 + \frac{13x + 3}{x^2 + x - 1}$

24 $-7x + 25$

25 2, -4. The remaining quadratic equation has a negative discriminant.

EXERCISE 4.6

1 (a) $(z - 3i)(z + 3i)$ (b) $(2z - i)(2z + i)$

(c) $(4z - 9i)(4z + 9i)$ (d) $3(z + \sqrt{5}i)(z - \sqrt{5}i)$

(e) $4(2z - 5i)(2z + 5i)$

(f) $(z - 3)(z + 3)(z - 3i)(z + 3i)$

2 (a) $z = -1 + 2i, z = -1 - 2i$

(b) $z = \frac{1}{4} + \frac{\sqrt{23}}{4}i, z = \frac{1}{4} - \frac{\sqrt{23}}{4}i$

(c) $z = 2 + \sqrt{2}i, z = 2 - \sqrt{2}i$ (d) $z = 1 + 0i, z = 3 + 0i$

3 (a) $\left(z + 2 + \sqrt{2}i\right)\left(z + 2 - \sqrt{2}i\right)$ (b) $\left(z - 4 + \sqrt{5}i\right)\left(z - 4 - \sqrt{5}i\right)$

4 (a) $z = -1 + i, z = -1 - i$ (b) $z = 3 + 3i, z = 3 - 3i$

(c) $z = 5 + 2i, z = 5 - 2i$ (d) $z = -6 + 8i, z = -6 - 8i$

5 **(a)** $(z+2)(z-3-i)(z-3+i)$

(b) $z = -2, 3+i, 3-i$

6 **(a)** $a\left(z^2-(1+i)z+i\right)=0$ **(b)** $a\left(z^3-z^2+z-1\right)=0$

(c) $a\left(z^3-2(i+1)z^2+(3+4i)z-6\right)=0$

(d) $a\left(z^4+5z^2+4\right)=0$

7 $p=2, q=10$

8 $z^2-6z+18=0$

9 **(a)** $3z-2$ **(b)** $2z-i$ **(c)** $z+3$ **(d)** $4z-3i$

10 $z=1, 3+i, 3-i$

11 $z = 3i, -3i, \frac{1}{2}$

12 **(a)** $(z-1)(z-i)(z+1)$

(b) $(z+2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)$

13 D

14 A

15 $p(z)=z^4-6z^3+11z^2-6z+10$

16 $0, \pm 1$

17 $-i$

18 $m=6$

19 **(a)** $(z-1)(z+i)(z+2i)$ **(b)** $(z-1)(z-i)(z-1-i)$

(c) $(z-i)(z+2)(z+4)$

20 **(a)** z^2+2z+5 **(b)** $1+2i, 1-2i, -1+2i, -1-2i$

(c) $z^2+2z+5; 1+2i, 1-2i, -1+2i, -1-2i$

21 $2-3i, \frac{1}{2}+\frac{\sqrt{3}i}{2}, \frac{1}{2}-\frac{\sqrt{3}i}{2}$

22 $-\frac{1}{2}, -\frac{1}{2}+\frac{1}{2}i$ and $-\frac{1}{2}-\frac{1}{2}i$

CHAPTER REVIEW 4

1 **(a)** $17-i$ **(b)** $-2-6\sqrt{7}i$ **(c)** $\frac{-3}{10}+\frac{3i}{5}$

(d) -1 **(e)** $\frac{42}{13}-\frac{28}{13}i$ **(f)** $7+9i$

2 **(a)** $x=8, y=-4$ **(b)** $x=-3, y=\frac{3}{2}$

3 **(a)** $5\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$ **(b)** $8\text{cis}\left(\frac{\pi}{6}\right)$

4 **(a)** $3-3\sqrt{3}i$ **(b)** $-\frac{7\sqrt{3}}{2}-\frac{7}{2}i$

5 **(a)** $1+0i$ **(b)** $-\frac{81}{2}-\frac{81\sqrt{3}}{2}i$

(c) $-8+0i$ **(d)** $0-2i$

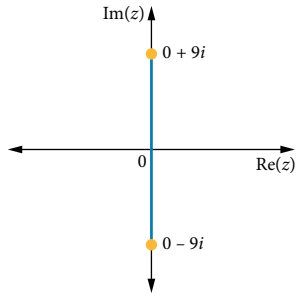
6 **(a)** $-51.78-37.62i$ **(b)** $-0.37+2.88i$

7 **(a)** $0 \pm 9i$ **(b)** $\sqrt{2}+\sqrt{2}i, -\sqrt{2}-\sqrt{2}i$

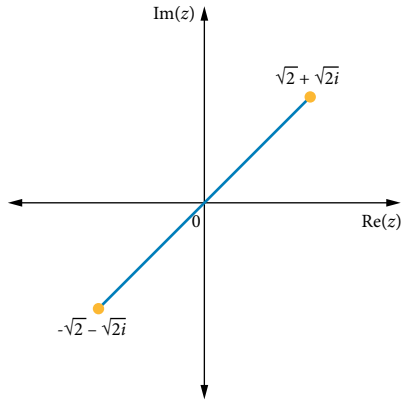
(c) $2^{\frac{1}{3}}\text{cis}\left(\frac{\pi}{12}\right), 2^{\frac{1}{3}}\text{cis}\left(\frac{3\pi}{4}\right), 2^{\frac{1}{3}}\text{cis}\left(-\frac{7\pi}{12}\right)$

(d) $\frac{3\sqrt{3}}{2}-\frac{3}{2}i, 0+3i, -\frac{3\sqrt{3}}{2}-\frac{3}{2}i$

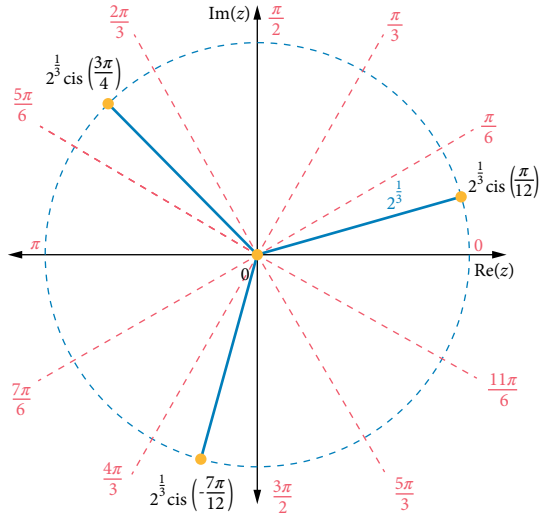
8 **(a)**



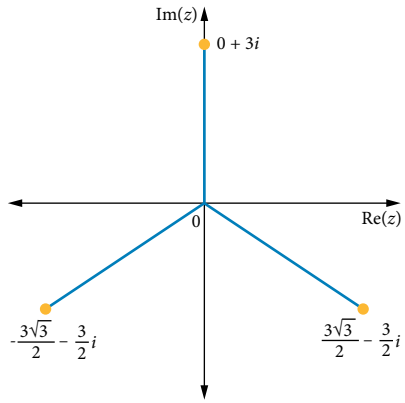
(b)



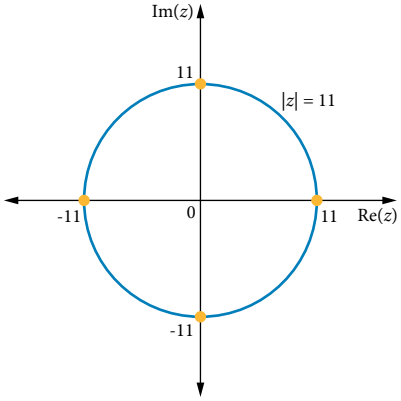
(c)



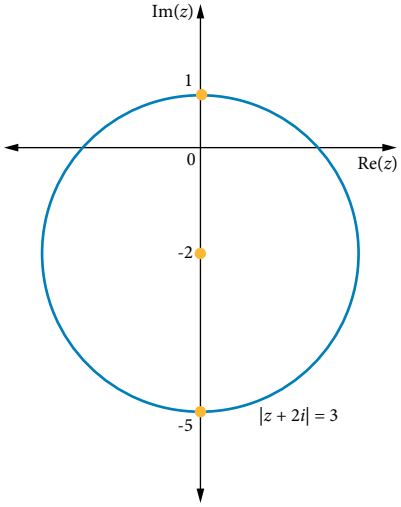
(d)



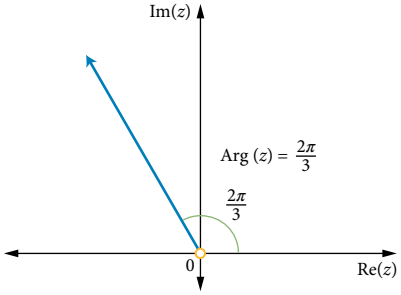
9 (a)



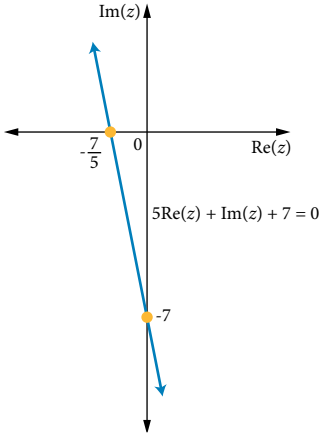
(b)



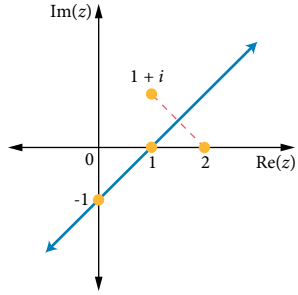
(c)



(d)



(e)



10 (a) $5\text{cis}(0), 5\text{cis}\left(\frac{2\pi}{3}\right), 5\text{cis}\left(-\frac{2\pi}{3}\right)$

(b) $\text{cis}\left(-\frac{\pi}{8}\right), \text{cis}\left(\frac{3\pi}{8}\right), \text{cis}\left(\frac{7\pi}{8}\right), \text{cis}\left(-\frac{5\pi}{8}\right)$

11 (a) $\text{cis}(0), \text{cis}\left(\frac{2\pi}{7}\right), \text{cis}\left(\frac{4\pi}{7}\right), \text{cis}\left(\frac{6\pi}{7}\right), \text{cis}\left(-\frac{6\pi}{7}\right), \text{cis}\left(-\frac{4\pi}{7}\right), \text{cis}\left(-\frac{2\pi}{7}\right)$

(b) $\frac{2\pi}{7}$

12 (a) 3 **(b)** 11 **(c)** $-\frac{51}{8}$

13 6

14 (a) $p(2) = 2^3 + 2^2 - 6(2)$
$= 0$
The remainder is zero, so $x - 2$ is a factor.

(b) $p(2) = 2^4 - 3(2)^3 + 2(2)^2 - 2 + 2$
$= 0$
The remainder is zero, so $x - 2$ is a factor.

15 (a) $(x-1)(x-2)(x+4)$ **(b)** $(x+1)(x+2)(2x-1)$

16 (a) $(x+2)(x-2)(2x-3)$ **(b)** $(x+1)^2(x-2)(x-3)$

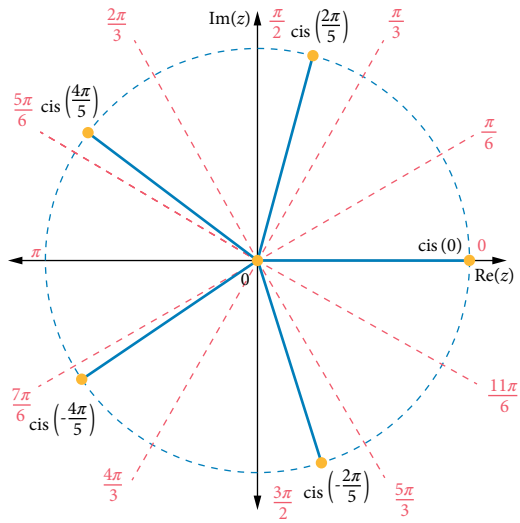
17 (a) $0 + 4\sqrt{2}i, 0 - 4\sqrt{2}i$ **(b)** $-4 + 2i, -4 - 2i$

(c) $z = 3i, -3i, 1$ **(d)** $z = -1, -1 + i, -1 - i$

18 $z = 1 + i,\ 1 - i,\ \frac{3}{2}$

19 (a) $\text{cis}(0), \text{cis}\left(\frac{2\pi}{5}\right), \text{cis}\left(\frac{4\pi}{5}\right), \text{cis}\left(-\frac{4\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right)$

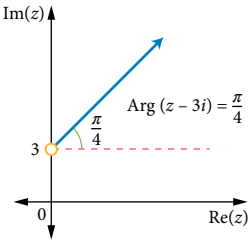
(b)



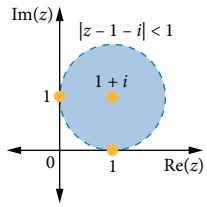
20 **(a)** $2^{-\frac{5}{2}}\operatorname{cis}\left(\frac{\pi}{12}\right)$ **(b)** $(2)^{-\frac{1}{2}}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

21
$$\begin{aligned} z_1z_2 &= (4-i)(2+i)\\ &= 8+2i+1\\ &= 9+2i\\ (z_1z_2)z_3 &= (9+2i)(i)\\ &= -2+9i\\ z_2z_3 &= (2+i)(i)\\ &= -1+2i\\ z_1(z_2z_3) &= (4-i)(-1+2i)\\ &= -4+9i+2\\ &= -2+9i\\ (z_1z_2)z_3 &= z_1(z_2z_3) \end{aligned}$$

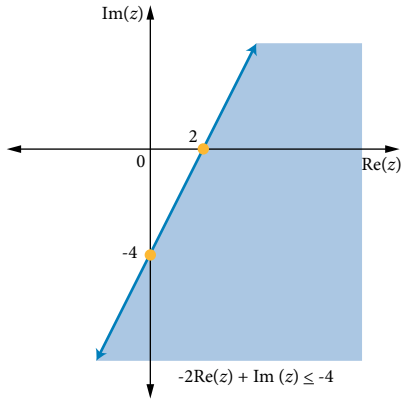
22 **(a)**



(b)



(c)



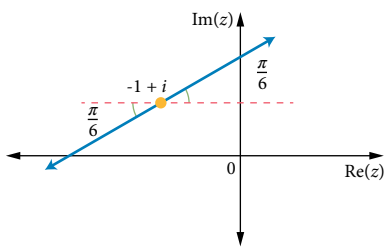
23 $a = -1,\ b = -2$

24 $p = \frac{1}{10},\ q = \frac{5}{2}$

25 $\frac{1}{2^{25}} - i\frac{1}{2^{25}}$

26 D

27



28 **(a)**
$$\begin{aligned} z^2+4 &= z^2-4i^2\\ &= (z-2i)(z+2i) \end{aligned}$$
$$\begin{aligned} p(2i) &= (2i)^3+8(2i)^2+4(2i)+32\\ &= -8i+32+8i+32\\ &= 0 \end{aligned}$$
$z - 2i$ is a factor of $p(z)$. Another factor is $z + 2i$, since coefficients are real so there is a conjugate pair.

$z^2 + 4$ is a factor of $p(z)$.

(b) $z + 8$

29 **(a)** $x = 3,\ y = 1$ or $x = -3,\ y = -1$

(b) $\sqrt{8+6i} = 3+i,\ -3-i$

30
$$\begin{aligned} z\overline{u}+\overline{z}u &= (x+yi)(a-ib)+(x-yi)(a+ib)\\ &= xa+yai-xbi+yb+(xa-yai+xbi+yb)\\ &= 2xa+2yb \end{aligned}$$
This is a real number as required.

31 **(a)** $\cos(3\theta)+i\sin(3\theta)$

(b) $\cos^3(\theta)+3\cos^2(\theta)i\sin(\theta)-3\cos(\theta)\sin^2(\theta)-i\sin^3(\theta)$

(c) $\sin(3\theta) = 3\sin(\theta)-4\sin^3(\theta)$

32 0

33 **(a)**
$$\begin{aligned} z^{-n} &= \operatorname{cis}(-n\theta)\\ &= \cos(-n\theta)+i\sin(-n\theta) \quad \text{(using de Moivre's theorem)}\\ &= \cos(n\theta)-i\sin(n\theta) \end{aligned}$$
$$\frac{1}{z^n} = \cos(n\theta)-i\sin(n\theta)$$

(b) $2\cos(n\theta)$ **(c)** $z^3+3z+\frac{3}{z}+\frac{1}{z^3}$

(d) $\cos^3(\theta) = \frac{1}{4}\cos(3\theta)+\frac{3}{4}\cos(\theta)$

Exam review: Unit 3

PAPER 1: TECHNOLOGY-FREE

1 **(a)** $|\boldsymbol{a}+\boldsymbol{b}| = \sqrt{74}$ **(b)** $|3\boldsymbol{a}-2\boldsymbol{b}| = \sqrt{266}$

2 $\overrightarrow{AB} = 5\hat{\boldsymbol{i}}+5\hat{\boldsymbol{j}}+5\hat{\boldsymbol{k}}$

3 $\frac{dy}{dx} = \frac{5}{4}$

4 **(a)** $-33-36i$ **(b)** $0-16i$ **(c)** $\frac{15}{26}+\frac{3}{26}i$

(d) $\frac{16}{53}+\frac{3}{53}i$ **(e)** $-2-2i$ **(f)** $-7+24i$

5 **(a)** $-6\sqrt{2}+6\sqrt{2}\,i$ **(b)** $0-11i$

(c) $-16+0i$ **(d)** 4^6+0i

6 When $n = 1$:

$\text{RHS} = (1-1)2^1+1$

$= 1$

$= \text{LHS}$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + k \times 2^{k-1} = (k-1)2^k + 1$ [1]

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + (k+1) \times 2^k = k2^{k+1} + 1$

$$\begin{aligned}\text{LHS} &= 1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + k \times 2^{k-1} + (k+1) \times 2^k \\ &= (k-1)2^k + 1 + (k+1)2^k \quad \text{using [1]} \\ &= 2^k(k - 1 + k + 1) + 1 \\ &= 2^k \times 2 + 1 \\ &= 2^{k+1} + 1 \\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

7 When $n = 1$:

$$\begin{aligned}\text{RHS} &= \frac{2^{2\times 1} - 1}{3 \times 2^{2\times 1}} \\ &= \frac{3}{12} \\ &= \frac{1}{4} \\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^k} = \frac{2^{2k} - 1}{3 \times 2^{2k}}$ [1]

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{k+1}} = \frac{2^{2(k+1)} - 1}{3 \times 2^{2(k+1)}}$

$$\begin{aligned}\text{LHS} &= \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^k} + \frac{1}{4^{k+1}} \\ &= \frac{2^{2k} - 1}{3 \times 2^{2k}} + \frac{1}{4^{k+1}} \quad \text{using [1]} \\ &= \frac{2^{2k} - 1}{3 \times 2^{2k}} + \frac{1}{\left(2^2\right)^{k+1}} \\ &= \frac{2^{2k} - 1}{3 \times 2^{2k}} + \frac{1}{2^{2k+2}} \\ &= \frac{2^2\left(2^{2k} - 1\right)}{3 \times 2^2 \times 2^{2k}} + \frac{3}{3 \times 2^{2k+2}} \\ &= \frac{2^2\left(2^{2k} - 1\right) + 3}{3 \times 2^{2k+2}} \\ &= \frac{2^{2k+2} - 2^2 + 3}{3 \times 2^{2k+2}} \\ &= \frac{2^{2k+2} - 1}{3 \times 2^{2k+2}} \\ &= \frac{2^{2(k+1)} - 1}{3 \times 2^{2(k+1)}} \\ &= \text{RHS}\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

8 The statement given can be rewritten as: $8^n - 8 = 7A$ where $A \in \mathbb{Z}$.

When $n = 2$:

$$\begin{aligned}\text{LHS} &= 8^2 - 8 \\ &= 56 \\ &= 7 \times 8 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 2$.

Let $k \geq 2$ and assume statement is true for $n = k$. That is, assume

$8^k - 8 = 7A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$8^{k+1} - 8 = 7B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 8^{k+1} - 8 \\ &= 8 \times 8^k - 8 \\ &= 8(7A + 8) - 8 \quad \text{using [1]} \\ &= 56A + 64 - 8 \\ &= 56A + 56 \\ &= 7(8A + 8) \\ &= \text{RHS} \quad (\text{since } A \in \mathbb{Z} \Rightarrow 8A + 8 \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 2$.

9 $\frac{3\sqrt{3}}{2} + \frac{3}{2}i, \ -\frac{3\sqrt{3}}{2} + \frac{3}{2}i, \ -3i$

10 $v_{\max} = 3, v_{\min} = 1$

11 The statement given can be rewritten as: $1 - (-a)^n = (1 + a)A$, where $A \in \mathbb{Z}$.

When $n = 1$:

$$\begin{aligned}\text{LHS} &= 1 - (-a)^1 \\ &= 1 + a \\ &= (1 + a) \times 1 \\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$1 - (-a)^k = (1 + a)A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$1-(-a)^{k+1}=(1+a)B$ where $B\in\mathbb{Z}$:

$$\begin{aligned}\text{LHS}&=1-(-a)^{k+1}\\&=1-(-a)(-a)^k\\&=1-(-a)\left(1-(1+a)A\right)\ \text{ using [1]}\\&=1+a\left(1-(1+a)A\right)\\&=1+a(1-A-aA)\\&=1+a-aA-a^2A\\&=1+a-aA(1+a)\\&=(1+a)(1-aA)\\&=\text{RHS}\ \ (\text{since } A,a\in\mathbb{Z}\Rightarrow 1-aA\in\mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n\geq 1$.

PAPER 2: TECHNOLOGY-ACTIVE

1 $\dfrac{d\boldsymbol{r}}{dt}=\dfrac{6}{\cos^2(2t)}\hat{\boldsymbol{i}}+\dfrac{6\sin(2t)}{\cos^2(2t)}\hat{\boldsymbol{j}}$ and

$$\frac{d^2\boldsymbol{r}}{dt^2}=\frac{24\sin(2t)}{\cos^3(2t)}\hat{\boldsymbol{i}}+\frac{12\left(\sin^2(2t)+1\right)}{\cos^3(2t)}\hat{\boldsymbol{j}}$$

2 **(a)** $10\sqrt{2}$; $45°$ **(b)** $\sqrt{29}$; $292°$ **(c)** $2\sqrt{17}$; $256°$

3 $\dfrac{1}{11}\left(-51\hat{\boldsymbol{i}}+23\hat{\boldsymbol{j}}+18\hat{\boldsymbol{k}}\right)$

4 **(a)** $x=-5, y=1, z=0$

(b) $x=2,-\dfrac{3}{2}$, $y=3$, $z=\pm 2$

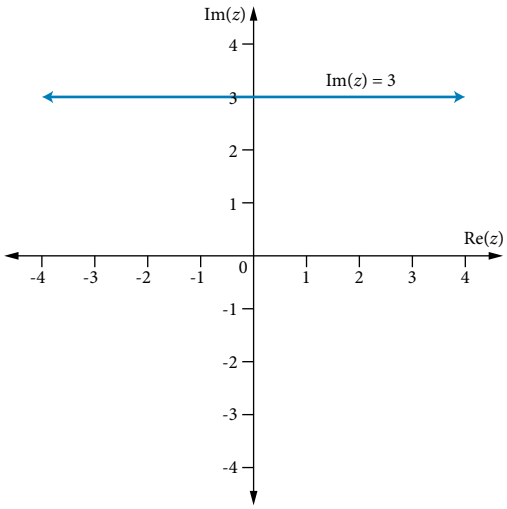
5 **(a)** $\lambda=7, \lambda=-1$

(b) The eigenvector associated with the eigenvalue $\lambda=7$ is $\begin{bmatrix}a\\a\end{bmatrix}$.

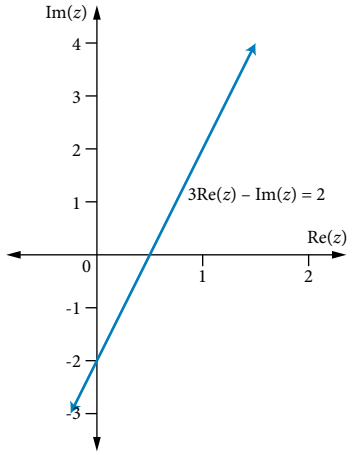
The eigenvector associated with the eigenvalue $\lambda=-1$ is $\begin{bmatrix}a\\-\frac{a}{3}\end{bmatrix}$.

6 $\theta=54°$

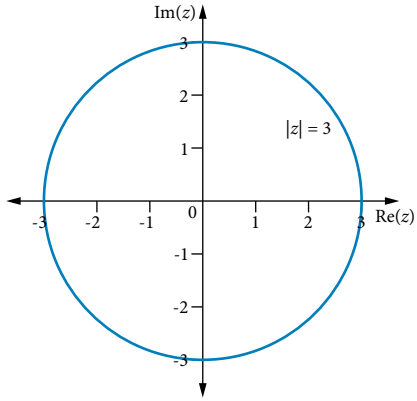
7 **(a)**



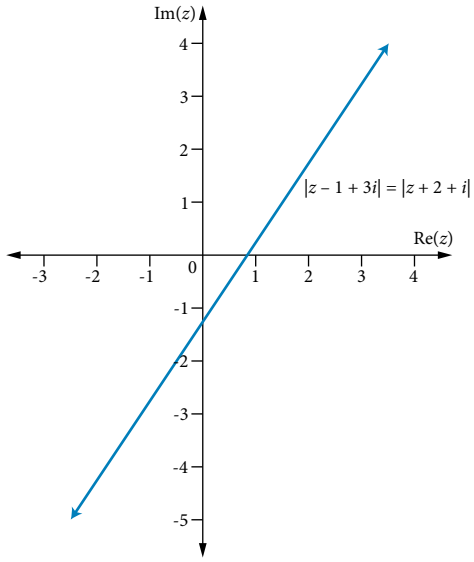
(b)



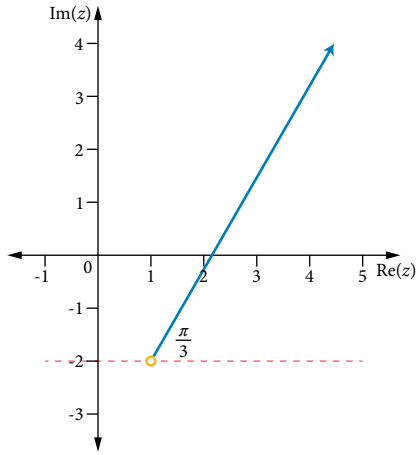
(c)



(d)



(e)



8 (a) $(3z-4)(3z+4)(3z-4i)(3z+4i)$

(b) $\left(z-2-\sqrt{6}\,i\right)\left(z-2+\sqrt{6}\,i\right)$

(c) $(z-2-3i)(z-2+3i)(2z-3)$

9 The ball will pass over the crossbar with a height of 13.475 metres. The kick will therefore be successful.

This assumes there was no wind or air resistance to affect the path of the ball, and that the only force acting on the ball was gravity. The consideration of air resistance or wind may affect the path of the ball, causing it to miss to the side or fall short of the goalposts.

Chapter 5: Integration techniques

RECALL

1 (a) $(2x-3)(2x+3)$ (b) $(x-5)(x+3)$ (c) $(3x+1)(x-5)$

2 (a) $(x+5)^2+6$ (b) $(2x+3)^2+1$

3 (a) $3+\dfrac{4x-2}{(x-1)(x+1)}$ (b) $2x-3+\dfrac{7x-4}{x^2-3}$

4 (a) $\ln\left(\dfrac{2}{3}\right)$ (b) $\ln\left(\dfrac{a^3b}{c}\right)$

5 (a) $7\left(3x^2+5x+2\right)^6(6x+5)$

(b) $3x^2\sin(5x)+5x^3\cos(5x)$

(c) $\dfrac{1-\ln(x)}{x^2}$ (d) $\sec^2(x)$

6 (a) $\dfrac{1}{3}x^3+3x^2+7x+c$ (b) $\dfrac{2}{9}(3x+4)^{\frac{3}{2}}+c$

(c) $\dfrac{3}{7}\sin(7x)+c$ (d) $\dfrac{1}{4}e^{4x}+c$

7 (a) $2\sqrt{2}-2$ (b) $\dfrac{1}{3}\left(e^3-1\right)$

8 9 units^2

EXERCISE 5.1

1 (a) 0 (b) 0 (c) $\dfrac{\pi}{3}$

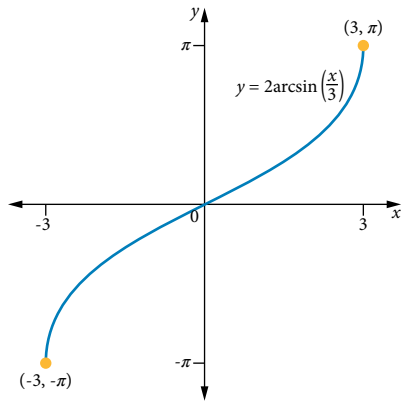
(d) $\dfrac{\pi}{3}$ (e) $-\dfrac{\pi}{4}$ (f) $\dfrac{5\pi}{6}$

(g) $\dfrac{\pi}{4}$ (h) $-\dfrac{\pi}{6}$ (i) $-\dfrac{\pi}{6}$

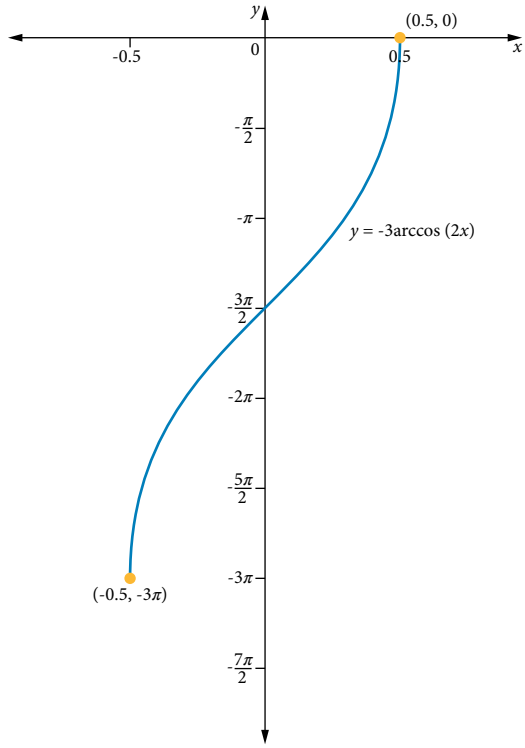
2 A

3 B

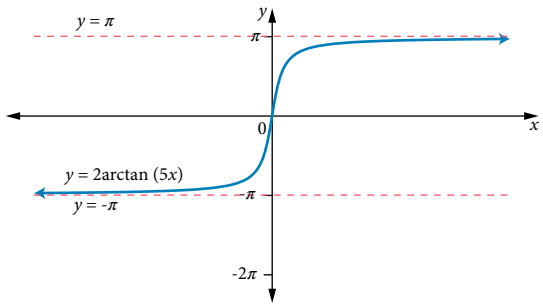
4 (a) domain: $[-3,3]$, range: $[-\pi,\pi]$



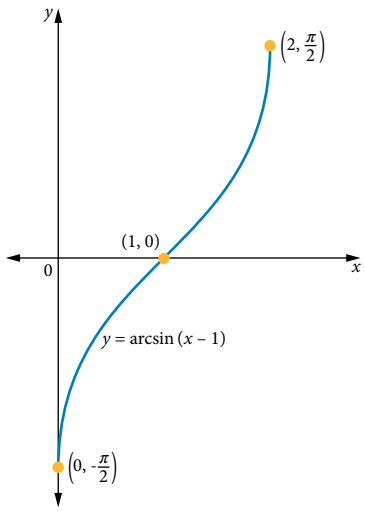
(b) domain: $\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$, range: $[-3\pi,0]$



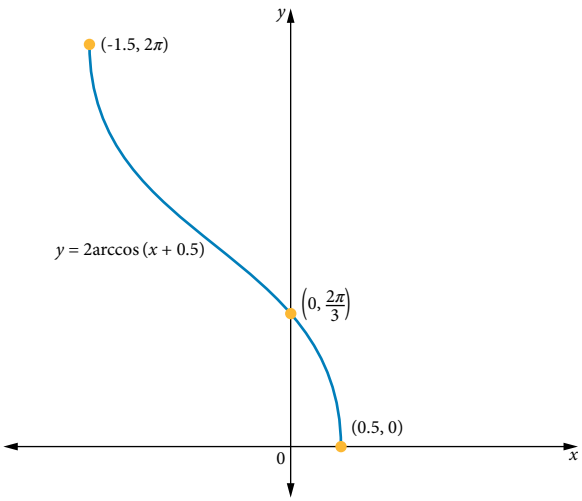
(c) domain: $\mathbb{R}$, range: $(-\pi,\pi)$



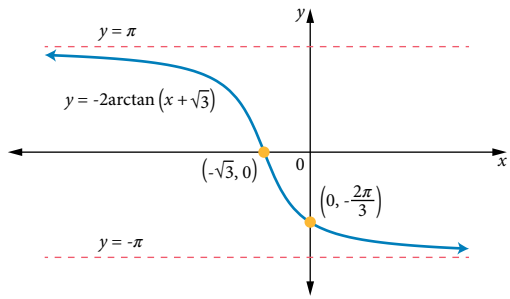
(d) domain: $[0, 2]$, range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(e) domain: $[-1.5, 0.5]$, range: $[0, 2\pi]$



(f) domain: $\mathbb{R}$, range: $(-\pi, \pi)$



5 (a) $\frac{1}{2}$ (b) 0.7 (c) -2.9

(d) $\sin(\arcsin(-2.9))$ is undefined

6 (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $-\frac{\pi}{3}$

(e) $-\frac{\pi}{7}$ (f) $-\frac{\pi}{7}$ (g) $-\frac{\pi}{5}$ (h) $\frac{\pi}{5}$

(i) $\frac{4\pi}{5}$ (j) 0

7 (a) 0 (b) $-\frac{\sqrt{3}}{3}$ (c) $\frac{4}{5}$ (d) $\frac{2\sqrt{2}}{3}$

(e) $-\frac{\sqrt{5}}{2}$ (f) $\frac{1}{\sqrt{10}}$ (g) $-\frac{3}{5}$

8 D

9 (a) C

(b) A common misconception is that cos(arccos(x)) is always equal to x. Indeed, this is the case provided that arccos(x) is actually defined. But in order for this to be the case, $-1 \leq x \leq 1$. Thus, arccos(4) is undefined, and therefore, so is cos(arccos(4)).

10 Let $y = \arccos(x)$. So $0 \leq y \leq \pi$ and $\cos(y) = x$.

$$\sin^2(y) = 1 - \cos^2(y)$$
$$= 1 - x^2$$
$$\sin(y) = \pm\sqrt{1 - x^2}$$

Since $0 \leq y \leq \pi$, $\sin(y) \geq 0$ so $\sin(y) = \sqrt{1 - x^2}$.

Thus, $\sin(\arccos(x)) = \sqrt{1 - x^2}$.

11 Let $y = \arctan(x)$. So $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan(y) = x$.

$$\sec^2(y) = 1 + \tan^2(y)$$
$$= 1 + x^2$$
$$\frac{1}{\cos^2(y)} = 1 + x^2$$
$$\cos^2(y) = \frac{1}{1 + x^2}$$
$$\cos(y) = \pm\frac{1}{\sqrt{1 + x^2}}$$

Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos(y) \geq 0$ so $\cos(y) = \frac{1}{\sqrt{1 + x^2}}$.

Thus, $\cos(\arctan(x)) = \frac{1}{\sqrt{1 + x^2}}$.

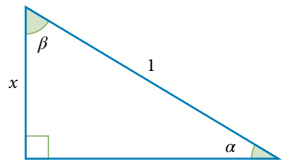
12 (a) Let $u = \sin^{-1}(x)$ and $v = \cos^{-1}(x)$. So $x = \sin(u)$, $x = \cos(v)$, $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, and $0 \leq v \leq \pi$. Then

$$\sin(u) = \cos(v)$$
$$\sin(u) = \sin\left(\frac{\pi}{2} - v\right)$$

Since $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq \frac{\pi}{2} - v \leq \frac{\pi}{2}$,

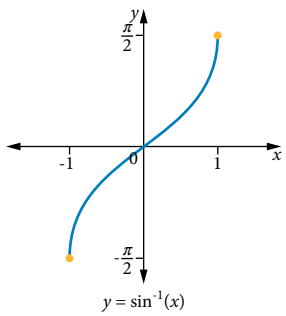
$$u = \frac{\pi}{2} - v$$
$$u + v = \frac{\pi}{2}$$
$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

(b) Consider a right-angled triangle with hypotenuse of length 1 and one of the shorter two sides having length x.

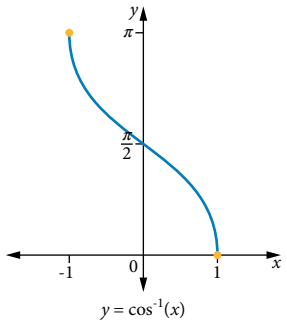


Angles α and β are equal to $\sin^{-1}(x)$ and $\cos^{-1}(x)$ respectively. And since $\alpha + \beta = \frac{\pi}{2}$, the result follows.

Now consider the graphs of $y = \sin^{-1}(x)$ and $y = \cos^{-1}(x)$.



$y = \sin^{-1}(x)$



$y = \cos^{-1}(x)$

The graph of $y = \cos^{-1}(x)$ can be obtained from the graph of $y = \sin^{-1}(x)$ by performing a reflection in the x-axis, followed by a vertical translation of $\frac{\pi}{2}$ units. The function $y = -\sin^{-1}(x)$ is obtained by performing the reflection, and then adding $\frac{\pi}{2}$ incorporates the translation. Thus the functions $y = -\sin^{-1}(x) + \frac{\pi}{2}$ and $y = \cos^{-1}(x)$ are equivalent. The result follows.

13 22.58 m

14 $x = \sqrt{\frac{3}{28}}$

EXERCISE 5.2

1 (a) $\frac{5}{\sqrt{1-25x^2}}$ **(b)** $\frac{-1}{\sqrt{9-x^2}}$ **(c)** $\frac{-6}{1+4x^2}$

(d) $\frac{-9}{\sqrt{1-(5-3x)^2}}$ **(e)** $\frac{-7}{2\sqrt{x}\sqrt{1-x}}$ **(f)** $\frac{-7}{x\sqrt{x^2-1}}$

(g) $\arctan(5x) + \frac{5x}{1+25x^2}$

(h) $\frac{\arcsin\left(\frac{x}{2}\right)}{2\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{4-x^2}}$

(i) $-\frac{\sqrt{x}}{x\sqrt{1-x^2}} - \frac{\arccos(x)}{2x\sqrt{x}}$

2 (a) $\frac{1}{\sqrt{49-x^2}}$ **(b)** $\frac{-4}{\sqrt{9-x^2}}$ **(c)** $\frac{10}{25+x^2}$

(d) $\frac{-6}{\sqrt{1-36x^2}}$ **(e)** $\frac{6}{\sqrt{1-9x^2}}$ **(f)** $\frac{15}{1+25x^2}$

3 C

4 domain: $\mathbb{R}$, range: $\left(0, \frac{1}{5}\right]$

5 $\frac{3}{5}$

6 $-\frac{1}{20}$

7 0.19

8 1.08

9 $y - \frac{\pi}{3} = \frac{-1}{\sqrt{27}}(x-3)$

10 $y = 3x$

11 $y = \frac{1}{4}x - \frac{1}{32} + \frac{\pi}{4}$

From this equation, the gradient is $\frac{1}{4}$ and the y-intercept is $-\frac{1}{32} + \frac{\pi}{4}$. Note that since π is slightly greater than 3, $\frac{\pi}{4}$ is slightly greater than $\frac{3}{4}$. The number $\frac{1}{32}$ is certainly less than $\frac{1}{4}$, so the y-intercept is certainly a number between 0.5 and 1. Both of these values are reasonable as the diagram indicates that the normal has a shallow positive gradient (so between 0 and 1) and a y-intercept between 0.5 and 1.

12 (a) A

(b) You may recognise that the function $y = \cos^{-1}\left(-\frac{x}{3}\right)$ has the form $y = \cos^{-1}\left(\frac{x}{a}\right)$ with $a = -3$, and hence be tempted to apply the rule $\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2-x^2}}$, leading to an answer of $\frac{-1}{\sqrt{(-3)^2-x^2}} = \frac{-1}{\sqrt{9-x^2}}$ (alternative B) for the derivative.

13 -2

14 25 minutes

15 **(a)** Let $y = \arccos(x)$. So $0 \le y \le \pi$ and $x = \cos(y)$.

$\frac{dx}{dy} = -\sin(y)$, so $\frac{dy}{dx} = \frac{-1}{\sin(y)}$.

Since $\sin^2(y) + \cos^2(y) = 1$,

$\sin(y) = \pm\sqrt{1 - \cos^2(y)}$

$= \pm\sqrt{1 - x^2}$

Also, since $0 \le y \le \pi$, $\sin(y) \ge 0$ and so $\sin(y) = \sqrt{1 - x^2}$.

Thus,

$\frac{dy}{dx} = \frac{-1}{\sin(y)}$

$= \frac{-1}{\sqrt{1 - x^2}}$

(b) **(i)** Let $y = \tan(x) = \frac{\sin(x)}{\cos(x)}$. Let $u = \sin(x)$ and $v = \cos(x)$. Using the quotient rule,

$\frac{dy}{dx} = \frac{u'v - v'u}{v^2}$

$= \frac{\cos(x)\cos(x) - -\sin(x)\sin(x)}{\cos^2(x)}$

$= \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$

$= \frac{1}{\cos^2(x)}$

$= \sec^2(x)$

(ii) Let $y = \arctan(x)$. So $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $x = \tan(y)$.

$\frac{dx}{dy} = \sec^2(y)$, so $\frac{dy}{dx} = \frac{1}{\sec^2(y)}$.

Since $\sec^2(y) = 1 + \tan^2(y)$,

$\frac{dy}{dx} = \frac{1}{1 + \tan^2(y)}$

$= \frac{1}{1 + x^2}$

16 Let $y = \arccos\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \times \frac{1}{a}$ (using the chain rule)

$= \frac{-1}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{1}{\sqrt{a^2}}$ (provided $a > 0$)

$= \frac{-1}{\sqrt{a^2 - x^2}}$

17 Let $y = \arctan\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x}{a}\right)^2} \times \frac{1}{a}$ (using the chain rule)

$= \frac{1}{1 + \frac{x^2}{a^2}} \times \frac{a}{a^2}$

$= \frac{a}{a^2 + x^2}$

18 $\left(\frac{\sqrt{3}}{6} + \frac{7}{3}, -\frac{4\pi}{3}\right)$ and $\left(-\frac{\sqrt{3}}{6} + \frac{7}{3}, \frac{4\pi}{3}\right)$

19 0.0096 radians/second

EXERCISE 5.3

1 **(a)** $\frac{7}{2}\ln|2x - 5| + c$ **(b)** $3\ln|9 - x| + c$

(c) $\frac{5}{4}\ln|x| + c$ **(d)** $-\frac{5}{7}\ln|x| + c$

(e) $\frac{1}{20}x^5 + x - \frac{7}{4}\ln|x| + c$

(f) $\frac{1}{4}x^2 - x + \frac{1}{2}\ln|x| + c$

2 **(a)** $7\arcsin\left(\frac{x}{3}\right) + c$

(b) $-5\arcsin\left(\frac{x}{4}\right) + c$ or $5\arccos\left(\frac{x}{4}\right) + c$

(c) $\arctan\left(\frac{x}{3}\right) + c$ **(d)** $3\arcsin\left(\frac{x}{\sqrt{2}}\right) + c$

(e) $-\frac{2}{5}\arcsin(5x) + c$ or $\frac{2}{5}\arccos(5x) + c$

(f) $\frac{7}{2}\arcsin\left(\frac{2x}{3}\right) + c$

(g) $-\frac{3}{\sqrt{2}}\arcsin\left(\frac{\sqrt{2}x}{\sqrt{5}}\right) + c$ or $-\frac{3\sqrt{2}}{2}\arcsin\left(\frac{\sqrt{10}x}{5}\right) + c$

(h) $2\arctan\left(\frac{x}{2}\right) + c$

(i) $\sqrt{2}\arcsin\left(\frac{x}{2\sqrt{2}}\right) + c$ or $\sqrt{2}\arcsin\left(\frac{\sqrt{2}x}{4}\right) + c$

(j) $\frac{1}{10}\arctan\left(\frac{x}{10}\right) + c$

(k) $\frac{2}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}x}{2}\right) + c$ or $\frac{2\sqrt{3}}{3}\arcsin\left(\frac{\sqrt{3}x}{2}\right) + c$

(l) $\frac{2}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + c$ or $\frac{2\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}x}{5}\right) + c$

(m) $\frac{\sqrt{7}}{7}\arctan\left(\sqrt{7}x\right) + c$

(n) $\sqrt{2}\arcsin\left(\frac{x}{2\sqrt{2}}\right) + c$ or $\sqrt{2}\arcsin\left(\frac{\sqrt{2}x}{4}\right) + c$

(o) $\frac{3\sqrt{2}}{2\sqrt{5}}\arctan\left(\frac{\sqrt{2}x}{\sqrt{5}}\right)+c$ or $\frac{3\sqrt{10}}{10}\arctan\left(\frac{\sqrt{10}x}{5}\right)+c$

3 (a) $\frac{1}{4}\left(1-\ln(5)\right)$ (b) $-2\ln(3)$ (c) $-\frac{\pi}{3}$

(d) $\frac{\pi}{6}$ (e) $\frac{\pi\sqrt{3}}{9}$ (f) $\frac{\pi}{8}$

4 $\int_4^5 \frac{3}{x+1}-\frac{3}{x-1}dx = 3\left[\ln|x+1|-\ln|x-1|\right]_4^5$

$= 3\left[\ln\left|\frac{x+1}{x-1}\right|\right]_4^5$

$= 3\left(\ln\left(\frac{6}{4}\right)-\ln\left(\frac{5}{3}\right)\right)$

$= 3\ln\left(\frac{\frac{6}{4}}{\frac{5}{3}}\right)$

$= 3\ln\left(\frac{18}{20}\right)$

$= 3\ln\left(\frac{9}{10}\right)$

5 A

6 B

7 $\int \frac{3}{2x+3}-\frac{1}{2x-5}dx = \int \frac{3}{2x+3}dx-\int \frac{1}{2x-5}dx$

$= \frac{3}{2}\ln|2x+3|-\frac{1}{2}\ln|2x-5|+c$

$= \frac{1}{2}\left(3\ln|2x+3|-\ln|2x-5|\right)$

$= \frac{1}{2}\left(\ln|2x+3|^3-\ln|2x-5|\right)$

$= \frac{1}{2}\ln\left(\frac{|2x+3|^3}{|2x-5|}\right)+c$

$= \frac{1}{2}\ln\left|\frac{(2x+3)^3}{2x-5}\right|+c$

8 $\frac{7\pi}{2}$ units^2

9 $\frac{\pi}{9}$ units^2

10 $\sqrt{3}$

11 -2

12 $-e$

13 The integration can be determined as either

$\int \frac{1}{-2x}dx = -\frac{1}{2}\ln|x|+c$ or equivalently, $-\frac{1}{2}\ln|-2x|+c$.

For option A: $-\frac{1}{2}\ln|2x|+c = -\frac{1}{2}\ln|-2x|+c$ since $|-2x| = |2x|$.

Thus, A is a valid expression for the integral.

For option B:

$\frac{1}{2}\ln\left|\frac{1}{x}\right|+c = \frac{1}{2}\ln\left(\frac{1}{|x|}\right)+c = \frac{1}{2}\ln|x|^{-1}+c = -\frac{1}{2}\ln|x|+c$.

Thus, B is a valid expression for the integral.

For option C: $-\ln\left(\sqrt{|x|}\right)+c = -\ln\left(|x|^{\frac{1}{2}}\right)+c = -\frac{1}{2}\ln|x|+c$.

Thus, C is a valid expression for the integral.

This leaves option D as the only alternative that is not a valid expression for the integral.

14 B

15 (a) $\frac{d}{dx}\left(\arccos(x+b)\right) = \frac{-1}{\sqrt{1-(x+b)^2}}$

$\int \frac{-1}{\sqrt{1-(x+b)^2}}dx = \arccos(x+b)+c$

(b) $\arccos(x+5)+c$

16 (a) $\frac{d}{dx}\left(\arctan(ax+b)\right) = \frac{a}{1+(ax+b)^2}$

$\int \frac{a}{1+(ax+b)^2}dx = \arctan(ax+b)+c$

(b) $4\arctan(3x+2)+c$

17 (1.27,0.43)

EXERCISE 5.4

1 (a) $\frac{1}{36}\left(3x^2+4\right)^6+c$ (b) $\frac{1}{3}\left(2x^2+6x+5\right)^{\frac{3}{2}}+c$

(c) $\frac{2}{33}\left(x^3-3\right)^{11}+c$ (d) $-\frac{1}{2}e^{-x^2}+c$

(e) $-\frac{3}{2}\cos\left(3x^2+4x\right)+c$ (f) $\frac{1}{4}\sin^4(x)+c$

(g) $\frac{1}{6}e^{3x^2+6x+1}+c$ (h) $\frac{1}{5}\left(1+\tan(x)\right)^5+c$

2 (a) $-\frac{5}{4\left(x^2-3\right)^2}+c$ (b) $5\sqrt{x^2-3}+c$

(c) $\frac{5}{2}\ln\left|x^2+3\right|+c$ (d) $-\sqrt{3-x^4}+c$

(e) $-\frac{1}{2}\ln\left|3-x^4\right|+c$ (f) $\frac{1}{4\cos^2(2x)}+c$

(g) $-\ln\left|\cos(x)\right|+c$ (h) $\frac{3}{2}\ln\left|x^2+2x+3\right|+c$

(i) $-\ln\left|\sin(x)+\cos(x)\right|+c$ (j) $-\frac{1}{e^x+e^{-x}}+c$

3 (a) $\frac{2}{9}\left(3+\sqrt{x}\right)^9+c$ (b) $e^{\tan(x)}+c$

(c) $\frac{1}{2}\left(\ln(x+2)\right)^2+c$ (d) $\frac{5}{4}\left(\ln\left(x^2+1\right)\right)^2+c$

(e) $\frac{1}{4}\arcsin\left(2x^2+3\right)+c$ (f) $\frac{1}{6}\arctan\left(\frac{x^2+4x}{3}\right)+c$

(g) $5\ln\left|\ln(x)\right|+c$ (h) $\frac{1}{\cos(x)}+c$ or $\sec(x)+c$

(i) $\frac{2}{3}\sqrt{\arctan\left(\frac{x}{3}\right)}+c$ (j) $\frac{1}{4\left(7-\tan(x)\right)^4}+c$

4 (a) $\frac{8}{3}-\sqrt{3}$ (b) $\frac{1}{9}\ln\left(\frac{5}{2}\right)$ (c) $e^{\frac{1}{2}}-1$ (d) $\frac{1}{2}$

5 D

6 (a) $\frac{1}{36}(2x+1)^9 - \frac{1}{32}(2x+1)^8 + c$

(b) $\frac{2}{7}(x-3)^{\frac{7}{2}} + \frac{12}{5}(x-3)^{\frac{5}{2}} + 6(x-3)^{\frac{3}{2}} + c$

(c) $\frac{1}{9(1-x)^9} - \frac{1}{8(1-x)^8} + c$

(d) $\frac{1}{2}(x+5)^2 - 10(x+5) + 25\ln|x+5| + c$

(e) $x - 3 + 6\ln|x-3| - \frac{9}{x-3} + c$

(f) $-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + \frac{2}{5}\left(1-x^2\right)^{\frac{5}{2}} - \frac{1}{7}\left(1-x^2\right)^{\frac{7}{2}} + c$

(g) $-\frac{1}{2}\left(9\ln\left|9-x^2\right| - 9 + x^2\right) + c$

(h) $\frac{1}{2}\left(e^x-1\right)^2 + 2\left(e^x-1\right) + \ln\left|e^x-1\right| + c$

7 $\frac{8}{3}$

8 91L

9 (a) D

(b) Given that it is true that, for example, $\int \cos(7x+3)dx = \frac{1}{7}\sin(7x+3)$, many students (noting that 7 was the derivative of $7x+3$ in this case) incorrectly conclude that since $2x$ is the derivative of x^2+3, it must be the case that $\int \cos\left(x^2+3\right)dx = \frac{1}{2x}\sin\left(x^2+3\right)$. The difference in these two integrals can be seen by attempting to solve each using a substitution:

For $\int \cos(7x+3)dx$, let $u = 7x+3$. Then $\frac{du}{dx} = 7$. Thus,

$$\begin{aligned}\int \cos(7x+3)dx &= \int \cos(u)\frac{1}{7}\frac{du}{dx}dx \\ &= \frac{1}{7}\int \cos(u)du \\ &= \frac{1}{7}\sin(u) + c \\ &= \frac{1}{7}\sin(7x+3) + c\end{aligned}$$

For $\int \cos(7x+3)dx$, let $u = x^2+3$. Then $\frac{du}{dx} = 2x$. Thus,

$$\int \cos\left(x^2+3\right)dx = \int \cos(u)\frac{1}{2x}\frac{du}{dx}dx$$

However, this expression is not equal to $\frac{1}{2x}\int \cos(u)du$ as the expression $\frac{1}{2x}$ contains a variable and thus cannot be taken out in front of the integral in the manner that the constant $\frac{1}{7}$ could.

It can also be seen by differentiating the expression $\frac{1}{2x}\sin\left(x^2+3\right)$ using the product rule that the derivative is not equal to $\cos\left(x^2+3\right)$. Likewise for options A and B, these are also incorrect.

10 $\frac{1}{3}\arctan\left(\frac{x+3}{3}\right) + c$

11 $\frac{3}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln\left(4+x^2\right) + c$

12 $\frac{d}{dx}x\arcsin(x) = \arcsin(x) + \frac{x}{\sqrt{1-x^2}}$

$\int \arcsin(x)dx = x\arcsin(x) + \sqrt{1-x^2} + c$

13 $k = \sqrt[3]{2}$

EXERCISE 5.5

1 (a) $\frac{1}{2}\left(x + \sin(x)\right) + c$ (b) $\frac{1}{2}\left(x - \frac{1}{8}\sin(8x)\right) + c$

(c) $\frac{1}{3}\tan(3x) - x + c$ (d) $\frac{1}{2}x^2 + 2x - \sin(2x) + c$

(e) $\frac{1}{3}\tan(3\theta) - 2\theta + c$ (f) $\theta - \frac{1}{10}\sin(10\theta) + c$

(g) $7x - 2\tan(x) + c$ (h) $2\tan\left(\frac{x}{2}\right) - x + c$

2 (a) $\frac{\pi}{2}$ (b) 1 (c) $1 - \frac{\pi}{4}$

(d) $\frac{\pi}{2}$ (e) $\frac{4-\pi}{4}$ (f) $2 - \frac{\pi}{4}$

3 $$\begin{aligned}\int_0^{\frac{1}{m}} \cos^2\left(\frac{\pi m x}{2}\right) &= \frac{1}{2}\int_0^{\frac{1}{m}} 1 + \cos(\pi m x)\,dx \\ &= \frac{1}{2}\left[x + \frac{1}{\pi m}\sin(\pi m x)\right]_0^{\frac{1}{m}} \\ &= \frac{1}{2}\left(\left(\frac{1}{m} + \frac{1}{\pi m}\sin(\pi)\right) - \left(0 + \frac{1}{\pi m}\sin(0)\right)\right) \\ &= \frac{1}{2}\left(\left(\frac{1}{m} + 0\right) - (0+0)\right) \\ &= \frac{1}{2m}\end{aligned}$$

4 $\frac{3x}{8} + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$

5 (a) $\sin(x) - \frac{1}{3}\sin^3(x) + c$

(b) $-\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + c$

(c) $\frac{1}{10}\cos^{10}(x) - \frac{1}{8}\cos^8(x) + c$

(d) $\frac{1}{5}\sin^5(x) - \frac{1}{7}\sin^7(x) + c$

(e) $\frac{1}{5}\sin^5(x) - \frac{2}{7}\sin^7(x) + \frac{1}{9}\sin^9(x) + c$

(f) $-\frac{1}{3}\cos^3(x) + \frac{3}{5}\cos^5(x) - \frac{3}{7}\cos^7(x) + \frac{1}{9}\cos^9(x) + c$

(g) $\frac{1}{16}\left(\frac{1}{3}\cos^3(2x) - \cos(2x)\right) + c$

6 $\frac{1}{8}x-\frac{1}{32}\sin(4x)+c$

7 $\frac{1}{128}\left(3x-\sin(4x)+\frac{1}{8}\sin(8x)\right)+c$

8 **(a)** $\cos(x)=2\cos^2\left(\frac{x}{2}\right)-1$ **(b)** $\tan\left(\frac{x}{2}\right)+c$

9 $\frac{\pi}{2}-\frac{1}{3}$

10 **(a)** **(i)** $\frac{1}{2}x+\frac{1}{8}\sin(4x)+c$ **(ii)** $\frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)+c$

(b) $\frac{1}{8}\left(\frac{1}{2}x-\frac{1}{2}\sin(2x)-\frac{1}{8}\sin(4x)+\frac{1}{2}\sin(2x)-\frac{1}{6}\sin^3(2x)\right)+c$

11 **(a)** Let $u=e^x$ and $v=f(x)$. Then using the product rule,

$$\begin{aligned}\frac{d}{dx}\left(e^x f(x)\right)&=u'v+v'u\\&=e^x f(x)+f'(x)e^x\\&=e^x\left(f(x)+f'(x)\right)\end{aligned}$$

Hence, $\int e^x\left(f(x)+f'(x)\right)dx=e^x f(x)+c$.

(b) $$\begin{aligned}g(x)&=\frac{2+\sin(2x)}{1+\cos(2x)}\\&=\frac{2+2\sin(x)\cos(x)}{1+2\cos^2(x)-1}\\&=\frac{1+\sin(x)\cos(x)}{\cos^2(x)}\\&=\frac{1}{\cos^2(x)}+\frac{\sin(x)}{\cos(x)}\\&=\sec^2(x)+\tan(x)\\&=f(x)+f'(x)\ \text{with}\ f(x)=\tan(x), f'(x)=\sec^2(x)\end{aligned}$$

(c) $\sqrt{3}e^{\frac{\pi}{3}}$

EXERCISE 5.6

1 **(a)** $\frac{2}{x-1}-\frac{2}{x+3}$ **(b)** $\frac{3}{x-5}-\frac{1}{x+5}$

(c) $\frac{3}{2x-1}+\frac{2}{2x+1}$

(d) $\frac{-2}{x}+\frac{2}{x-5}$ **(e)** $\frac{\frac{1}{2}}{x+1}+\frac{\frac{3}{2}}{x+3}$ **(f)** $\frac{1}{2x}+\frac{\frac{1}{2}}{3x+1}$

(g) $\frac{3}{x-1}+\frac{3}{(x-1)^2}$ **(h)** $\frac{1}{x+2}+\frac{3}{(x+2)^2}$

(i) $\frac{\frac{1}{2}}{2x+3}-\frac{\frac{3}{2}}{(2x+3)^2}$

2 **(a)** $2\ln|x-1|-2\ln|x+3|+c$ or $2\ln\left|\frac{x-1}{x+3}\right|+c$

(b) $3\ln|x-5|-\ln|x+5|+c$

(c) $\frac{3}{2}\ln|2x-1|+\ln|2x+1|+c$

(d) $-2\ln|x|+2\ln|x-5|+c$ or $-2\ln\left|\frac{x}{x-5}\right|+c$

(e) $\frac{1}{2}\ln|x+1|+\frac{3}{2}\ln|x+3|+c$

(f) $\frac{1}{2}\ln|x|+\frac{1}{6}\ln|3x+1|+c$

(g) $3\ln|x-1|-\frac{3}{x-1}+c$

(h) $\ln|x+2|-\frac{3}{x+2}+c$

(i) $\frac{1}{4}\ln|2x+3|+\frac{3}{4(2x+3)}+c$

3 **(a)** $6\ln|x-4|-6\ln|x-2|+c$ **(b)** $\ln|x|-\ln|x+3|+c$

(c) $-8\ln|x|+9\ln|3x-1|+c$ **(d)** $\ln|x-3|-\frac{3}{x-3}+c$

(e) $\frac{5}{4}\ln|2x+1|-\frac{1}{4(2x+1)}+c$

4 **(a)** $3\ln\left(\frac{3}{2}\right)$ **(b)** $\ln\left(\frac{169}{200}\right)$

(c) $-\frac{3}{4}\ln\left(\frac{21}{25}\right)$ **(d)** $\frac{1}{3}\ln(4)-\frac{5}{4}$

5 **(a)** C

(b) For option A, since the numerator is a quadratic function (like the denominator), this expression cannot be expressed in the form given.

For option B, since the denominator is a perfect square, it cannot be expressed in the form given—rather, it could be expressed in the form $\frac{A}{x+3}+\frac{B}{(x+3)^2}$.

For option D, the denominator in this case cannot be factorised into linear factors.

6 $3x+\ln|x-1|+3\ln|x+1|+c$

7 $\frac{1}{2}x^2+2x+\frac{16}{5}\ln|x-2|+\frac{4}{5}\ln|x+3|+c$

8 $\frac{1}{2}x^2+x-3\ln|x|+4\ln|x-1|+c$

9 $A=-3, B=3, C=5$

$$\int\frac{2x-3}{x(x+1)^2}dx=-3\ln|x|+3\ln|x+1|-\frac{5}{x+1}+c$$

10 $A=\frac{3}{4}, B=\frac{5}{2}, C=-\frac{3}{4}$

$$\int\frac{x-5}{x^2(x-2)}dx=\frac{3}{4}\ln|x|-\frac{5}{2x}+\frac{3}{4}\ln|x-2|+c$$

11 $A=1, B=-1, C=-\frac{5}{2}$

$$\int\frac{x+9}{2x(2x+3)^2}dx=\frac{1}{2}\ln|x|-\frac{1}{2}\ln|2x+3|+\frac{5}{4(2x+3)}+c$$

12 (a) $\ln|4+x|+c$

(b) $\frac{1}{2}\arctan\left(\frac{x}{2}\right)+c$

(c) $-\frac{1}{4}\ln|2-x|+\frac{1}{4}\ln|2+x|+c$

(d) $-\frac{1}{2}\ln|4-x^2|+c$

13 105 512

14 $-x-\ln|2-x|+\ln|2+x|+c$

15 Let $u=\sin(x)$. Then $\frac{du}{dx}=\cos(x)$.

$x=0\Rightarrow u=\sin(0)=0$

$x=\frac{\pi}{3}\Rightarrow u=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$

$$\int_0^{\frac{\pi}{3}}\frac{\cos(x)}{1-\sin^2(x)}dx=\int_{u=0}^{u=\frac{\sqrt{3}}{2}}\frac{1}{1-u^2}\frac{du}{dx}dx$$

$$=\int_{u=0}^{u=\frac{\sqrt{3}}{2}}\frac{1}{1-u^2}du$$

$$\frac{1}{1-u^2}=\frac{1}{(1-u)(1+u)}$$

$$=\frac{A}{1-u}+\frac{B}{1+u}$$

$$=\frac{A(1+u)+B(1-u)}{(1-u)(1+u)}$$

$\therefore 1=A(1+u)+B(1-u)$

Setting $u=1$:

$1=2A$

$A=\frac{1}{2}$

Setting $u=-1$:

$1=2B$

$B=\frac{1}{2}$

$$\therefore\frac{1}{1-u^2}=\frac{\frac{1}{2}}{1-u}+\frac{\frac{1}{2}}{1+u}$$

$$\int_0^{\frac{\pi}{3}}\frac{\cos(x)}{1-\sin^2(x)}dx=\int_{u=0}^{u=\frac{\sqrt{3}}{2}}\frac{1}{1-u^2}du$$

$$=\int_{u=0}^{u=\frac{\sqrt{3}}{2}}\frac{\frac{1}{2}}{1-u}+\frac{\frac{1}{2}}{1+u}du$$

$$=\frac{1}{2}\left[-\ln|1-u|+\ln|1+u|\right]_{u=0}^{u=\frac{\sqrt{3}}{2}}$$

$$=\frac{1}{2}\left[\ln\left(\frac{|1+u|}{|1-u|}\right)\right]_{u=0}^{u=\frac{\sqrt{3}}{2}}$$

$$=\frac{1}{2}\left(\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right)-\ln(1)\right)$$

$$=\frac{1}{2}\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\times\frac{1+\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}\right)$$

$$=\frac{1}{2}\ln\left(\frac{\frac{7}{4}+\sqrt{3}}{\frac{1}{4}}\right)$$

$$=\frac{1}{2}\ln\left(7+4\sqrt{3}\right)$$

16 $A=1,B=2,C=2,D=3$

$$\int\frac{5x^3+6x^2+15x+4}{x(x+1)(x^2+4)}dx=\ln|x|+2\ln|x+1|+\ln|x^2+4|$$

$$+\frac{3}{2}\arctan\left(\frac{x}{2}\right)+c$$

EXERCISE 5.7

1 (a) $-x\cos(x)+\sin(x)+c$ (b) $\frac{1}{4}xe^{4x}-\frac{1}{16}e^{4x}+c$

(c) $\frac{1}{2}x^2\ln(x)-\frac{1}{4}x^2+c$ (d) $(3x-2)\frac{1}{2}e^{2x}-\frac{3}{4}e^{2x}+c$

(e) $\frac{1}{4}x^4\ln(x)-\frac{1}{16}x^4+c$ (f) $x\ln(3x)-x+c$

2 (a) $-\frac{\pi}{6}+\frac{\sqrt{3}}{2}$ (b) $18\ln(9)-\frac{104}{9}$

3 Let $u=\ln(x)$ and $v'=x^n$.

Then $u'=\frac{1}{x}$ and $v=\frac{1}{n+1}x^{n+1}$.

$$\int x^n\ln(x)dx=\int uv'dx$$

$$=uv-\int u'v\,dx$$

$$=\frac{x^{n+1}}{n+1}\ln(x)-\int\frac{1}{x}\frac{1}{n+1}x^{n+1}dx$$

$$=\frac{x^{n+1}}{n+1}\ln(x)-\frac{1}{n+1}\int x^n dx$$

$$=\frac{x^{n+1}}{n+1}\ln(x)-\frac{1}{(n+1)^2}x^{n+1}+c$$

$$\int_1^e x^n\ln(x)dx=\left[\frac{x^{n+1}}{n+1}\ln(x)-\frac{1}{(n+1)^2}x^{n+1}\right]_1^e$$

$$=\left[\frac{(n+1)x^{n+1}\ln(x)-x^{n+1}}{(n+1)^2}\right]_1^e$$

$$=\frac{(n+1)e^{n+1}\ln(e)-e^{n+1}}{(n+1)^2}-\left(\frac{(n+1)1^{n+1}\ln(1)-1^{n+1}}{(n+1)^2}\right)$$

$$=\frac{(n+1)e^{n+1}-e^{n+1}}{(n+1)^2}-\left(-\frac{1}{(n+1)^2}\right)$$

$$=\frac{ne^{n+1}+e^{n+1}-e^{n+1}+1}{(n+1)^2}$$

$$=\frac{ne^{n+1}+1}{(n+1)^2}$$

4 (a) $\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}+c$

(b) $-\frac{1}{3}x^2\cos(3x)+\frac{2}{9}x\sin(3x)+\frac{2}{27}\cos(3x)+c$

(c) $-(x+3)^2e^{-x}-2(x+3)e^{-x}-2e^{-x}+c$

(d) $\frac{1}{2}e^x\left(\sin(x)+\cos(x)\right)+c$

(e) $\frac{2}{17}e^{2x}\left(-\cos\left(\frac{x}{2}\right)+4\sin\left(\frac{x}{2}\right)\right)+c$

5 **(a)** $x\arcsin(x)+\sqrt{1-x^2}+c$

(b) $x\arccos\left(\frac{x}{a}\right)-\sqrt{a^2-x^2}+c$

(c) $x\arctan\left(\frac{x}{a}\right)-\frac{a}{2}\ln\left(a^2+x^2\right)+c$

6 $x^3e^x-3x^2e^x+6xe^x-6e^x+c$

7 $\frac{1}{3}x^3\arcsin(x)+\frac{1}{3}\left(1-x^2\right)^{\frac{1}{2}}-\frac{1}{9}\left(1-x^2\right)^{\frac{3}{2}}+c$

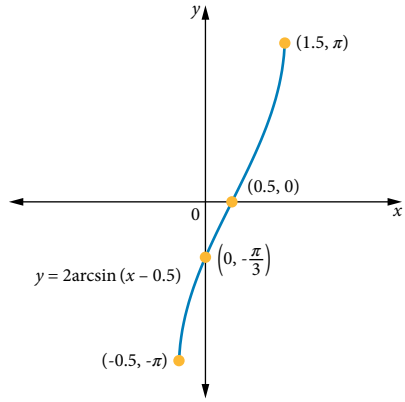
8 $k=1.45$ (2 d.p.)

Using technology to calculate $\int_0^{1.45}x\sec^2(x)$ gives a result of 9.83. The discrepancy between this value and the expected value of 10 is reasonable, as the graph of $y=x\sec^2(x)$ has an asymptote at $x=\frac{\pi}{2}\approx 1.57$. Thus, as k approaches this value, $\int_0^k x\sec^2(x)$ (which represents the area bounded by the function from $x=0$ to $x=k$), it increases in magnitude incredibly fast, meaning that a small change in k caused by rounding has a large impact on the value of $\int_0^k x\sec^2(x)$.

CHAPTER REVIEW 5

1 **(a)** $\frac{\pi}{2}$ **(b)** $\frac{2\pi}{3}$

2 domain: $[-0.5,1.5]$, range: $[-\pi,\pi]$



3 **(a)** $\frac{-21}{\sqrt{1-9x^2}}$ **(b)** $2\arctan(x)+\frac{2x+3}{1+x^2}$

4 **(a)** $-3\arcsin\left(\frac{x}{\sqrt{5}}\right)+c$ or $3\arccos\left(\frac{x}{\sqrt{5}}\right)+c$

(b) $\frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right)+c$

5 $\frac{\pi}{4}$

6 **(a)** $\frac{1}{28}\left(x^4+3\right)^7+c$ **(b)** $\sqrt{x^2-2x}+c$

(c) $\frac{1}{6}\left(\ln(3x-1)\right)^2+c$

7 $\frac{1}{2}\ln(5)$

8 **(a)** $\frac{1}{2}\left(x+3\sin\left(\frac{x}{3}\right)\right)+c$ **(b)** $\frac{1}{7}\tan(7x)-x+c$

9 $1-\frac{3\sqrt{3}}{8}-\frac{\pi}{12}$

10 **(a)** $2\ln|x+3|+3\ln|x-5|+c$

(b) $\frac{4}{3}\ln|3x-1|+\frac{1}{3(3x-1)}+c$

11 $\frac{2x-9}{(x+6)(x-1)}=\frac{A}{x+6}+\frac{B}{x-1}$

$=\frac{A(x-1)+B(x+6)}{(x+6)(x-1)}$

$\therefore 2x-9=A(x-1)+B(x+6)$

Setting $x=-6$:

$-21=-7A$

$A=3$

Setting $x=1$:

$-7=7B$

$B=-1$

$\therefore \frac{2x-9}{(x+6)(x-1)}=\frac{3}{x+6}-\frac{1}{x-1}$

$\int_2^3\frac{2x-9}{(x+6)(x-1)}\,dx=\int_2^3\frac{3}{x+6}-\frac{1}{x-1}\,dx$

$=\left[3\ln|x+6|-\ln|x-1|\right]_2^3$

$=3\ln(9)-\ln(2)-\left(3\ln(8)-\ln(1)\right)$

$=3\ln(9)-\ln(2)-3\ln(8)$

$=\ln\left(9^3\right)-\ln(2)-\ln\left(8^3\right)$

$=\ln\left(\frac{3^6}{2\times 2^9}\right)$

$=\ln\left(\frac{3^6}{2^{10}}\right)$

$=\ln\left(\left(\frac{3^3}{2^5}\right)^2\right)$

$=2\ln\left(\frac{27}{32}\right)$

12 **(a)** $\frac{1}{2}x^2\ln(5x)-\frac{1}{4}x^2+c$

(b) $\frac{2}{3}x(x+1)^{\frac{3}{2}}-\frac{4}{15}(x+1)^{\frac{5}{2}}+c$

13 $\frac{5}{13}$

14 Let $y = \arctan(x)$. So $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan(y) = x$.

$$\csc^2(y) = 1 + \cot^2(y)$$
$$\frac{1}{\sin^2(y)} = 1 + \frac{1}{\tan^2(y)}$$
$$\frac{1}{\sin^2(y)} = 1 + \frac{1}{x^2}$$
$$\frac{1}{\sin^2(y)} = \frac{x^2+1}{x^2}$$
$$\sin^2(y) = \frac{x^2}{x^2+1}$$
$$\sin(y) = \pm\frac{x}{\sqrt{x^2+1}}$$

Note that since $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $\sin(y)$ is positive if and only if $\tan(y) = x$ is positive.

Therefore, $\sin(y) = \frac{x}{\sqrt{1+x^2}}$.

That is, $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$.

15 $-\frac{1}{2}e^{\frac{2}{5}} - \frac{1}{2}$

16 $\frac{\sqrt{q}}{q\sqrt{p}}\arctan\left(\frac{\sqrt{q}x}{\sqrt{p}}\right) + c$

17 $\frac{1}{18}\left(x^2+5\right)^9 - \frac{5}{16}\left(x^2+5\right)^8 + c$

18 $\frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$

19 $\frac{1}{2}x^2 + x + \frac{5}{4}\ln|2x+1| - \frac{1}{4(2x+1)} + c$

20 $A = 1, B = -1, C = 3$

$$\int \frac{8x+25}{x(x+5)^2}dx = \ln|x| - \ln|x+5| - \frac{3}{(x+5)} + c$$

21 Let $u = \arccos(x)$ and $v = \arcsin(x)$. Then $0 \le u \le \pi$, $\cos(u) = x$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$, and $\sin(v) = x$.

Must demonstrate that $\sin(u) = \cos(v)$.

$$\sin^2(u) = 1 - \cos^2(u)$$
$$\sin^2(u) = 1 - x^2$$
$$\sin(u) = \pm\sqrt{1-x^2}$$

Since $0 \le u \le \pi$, $\sin(u) = \sqrt{1-x^2}$.

$$\cos^2(v) = 1 - \sin^2(v)$$
$$\cos^2(v) = 1 - x^2$$
$$\cos(v) = \pm\sqrt{1-x^2}$$

Since $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$, $\cos(v) = \sqrt{1-x^2}$.

Thus, $\sin(u) = \cos(v)$. That is, $\sin(\arccos(x)) = \cos(\arcsin(x))$.

22 (a) Let $p = \arcsin(-k)$. Then $\sin(p) = -k$. Given any angle p drawn on a unit circle diagram, the angle $-p$ will be vertically opposite, and thus the y-coordinate will have the opposite sign. (Recall that the sine of an angle is equal to the y-coordinate when the angle is drawn on a unit circle diagram.) Therefore, $\sin(-p) = -\sin(p) = -(-k) = k$. Therefore, $-p = \arcsin(k)$ and so $p = -\arcsin(k)$.

(b) $k = \frac{1}{2}$

23 1.1 fish per day

24 $\frac{12}{\pi + 6\sqrt{3} - 12}$

Chapter 6: Applications of integration

RECALL

1 (a) $\arcsin(x) + c$

(b) $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$

(c) $\ln|x^2 - 5| + c$

(d) $xe^x - e^x + c$

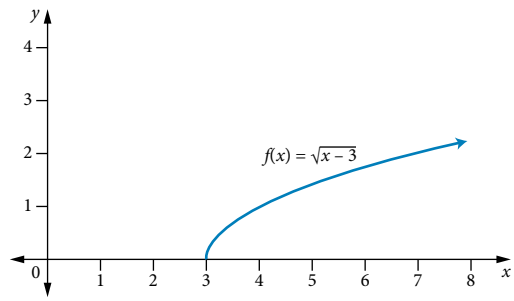
(e) $\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + c$

2 (a) $\frac{\pi}{3}$

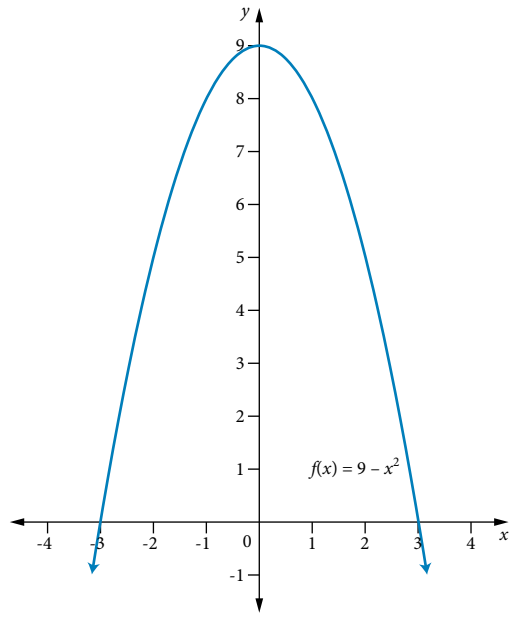
(b) $\ln\left(\frac{1+e}{2}\right)$

3 2.050

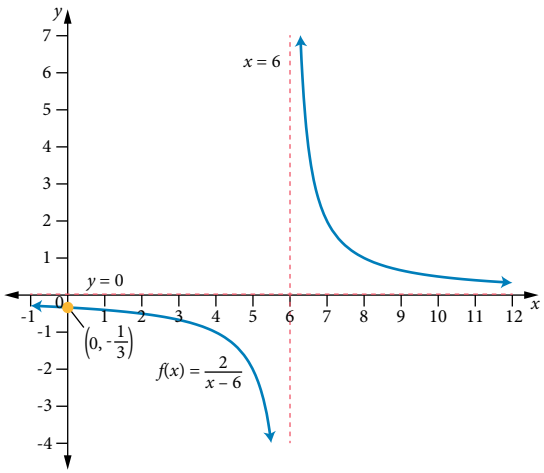
4 (a)



(b)



(c)



5 0.1875

6 0.8

EXERCISE 6.1

1 (a) $\frac{1}{2}e^{0.25} + \frac{1}{2}e - 1$ units^2 (b) $1 - \frac{\sqrt{2}}{4}$ units^2

(c) $0.5\ln(0.5) + 1.5\ln(1.5)$ units^2

2 (a) $11\frac{1}{4}$ units^2 (b) $2\ln(3)$ units^2

(c) $7\frac{1}{3}$ units^2 (d) $2 - \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}$ units^2

3 (a) 7.35 units^2 (b) 1.48 units^2

4 (a) 1.12 units^2 (b) 4 units^2

5 $\frac{2^{\frac{7}{2}} - 10}{3}$ units^2

6 $4 - 2\sqrt{3}$ units^2

7 $2\sqrt{2}$ units^2

8 $\frac{1}{2}e^4 - 2e$ units^2

9 (a) 4 units^2 (b) $4n$ units^2

10 $\frac{6\sqrt{3}\pi - \pi^2 - 18}{36}$ units^2

11 D

12 (a) A

(b) One common misconception could be thinking that because one curve is above the x-axis and the other below, then the area between them must involve adding areas (and consequently choosing option B incorrectly). While the required area can be interpreted as adding the area above to the area below, the integral corresponding to the area below is negative, meaning it must be multiplied by -1 to make it positive. Thus, option A is correct in this situation (as it would be even if you were not given information about whether the functions were above or below the x-axis).

13 D

14 $\frac{8}{15}$ units^2

15 $4000\left(e^1 - e^{-2}\right)$ g

16 $\frac{1}{2}$ units^2

17 21.93 m^2

18 (a) To determine the x-coordinates of the points of intersection:

$$xe^{ax} = x$$
$$xe^{ax} - x = 0$$
$$x\left(e^{ax} - 1\right) = 0$$
$$x = 0 \text{ or } e^{ax} = 1$$

If $e^{ax} = 1$, then $ax = \ln(1) = 0$, so $x = 0$. Since $g(0) = f(0) = 0$, the only place where the functions intersect is at the origin.

(b) Since $x > 0$ and $a > 0$, $e^{ax} > 1$ and so $xe^{ax} > x$. The result follows.

(c) $a = 1.55$

19 15.77 m

20 (a) The line and curve will enclose a region if there are two or more points of intersection. Points of intersection occur at points where

$$mx = \frac{x}{x^2 + 1}$$
$$mx\left(x^2 + 1\right) = x$$
$$mx\left(x^2 + 1\right) - x = 0$$
$$x\left(m\left(x^2 + 1\right) - 1\right) = 0$$
$$x = 0 \text{ or } m\left(x^2 + 1\right) - 1 = 0$$

$m\left(x^2 + 1\right) - 1 = 0$ implies that

$$m\left(x^2 + 1\right) = 1$$
$$x^2 + 1 = \frac{1}{m}$$
$$x^2 = \frac{1}{m} - 1$$
$$x = \pm\sqrt{\frac{1}{m} - 1}$$

Discard the negative solution as we are only concerned with points of intersection for $x \geq 0$. Thus, the line and the curve intersect at $x = 0$ and $x = \sqrt{\frac{1}{m} - 1}$. There will be two points of intersection provided $\frac{1}{m} - 1 > 0$, that is, when $\frac{1}{m} > 1$. This will be true provided $0 < m < 1$.

(b) Note that if $0 < x < \sqrt{\frac{1}{m} - 1}$, then $x^2 < \frac{1}{m} - 1$ and so $x^2 + 1 < \frac{1}{m}$. This implies that $\frac{1}{x^2 + 1} > m$ and so

$\frac{x}{x^2+1} > mx$ (meaning that the curve is above the line in this interval).

(c) The required area is given by the following:

$$\begin{aligned}
A &= \int_0^p \frac{x}{x^2+1} - mx\,dx \quad \text{where } p = \sqrt{\frac{1}{m}-1} \\
&= \left[\frac{1}{2}\ln\left(x^2+1\right) - \frac{1}{2}mx^2\right]_0^p \\
&= \frac{1}{2}\ln\left(p^2+1\right) - \frac{1}{2}mp^2 - \left(\frac{1}{2}\ln\left(0^2+1\right) - \frac{1}{2}m0^2\right) \\
&= \frac{1}{2}\ln\left(p^2+1\right) - \frac{1}{2}mp^2 \\
&= \frac{1}{2}\ln\left(\left(\sqrt{\frac{1}{m}-1}\right)^2+1\right) - \frac{1}{2}m\left(\sqrt{\frac{1}{m}-1}\right)^2 \\
&= \frac{1}{2}\ln\left(\frac{1}{m}-1+1\right) - \frac{1}{2}m\left(\frac{1}{m}-1\right) \\
&= \frac{1}{2}\ln\left(\frac{1}{m}\right) - \frac{1}{2}m\left(\frac{1-m}{m}\right) \\
&= \frac{1}{2}\ln\left(\frac{1}{m}\right) - \frac{(1-m)}{2}
\end{aligned}$$

EXERCISE 6.2

1 **(a)** $\frac{4\pi}{7}$ units^3 **(b)** $\frac{1}{6}\pi\left(e^3-1\right)$ units^3

(c) 4π units^3 **(d)** $\frac{\pi}{2}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)$ units^3

2 **(a)** $\frac{81\pi}{2}$ units^3 **(b)** $\frac{104\pi}{3}$ units^3

(c) $\pi\left(\frac{11}{4}-6\ln(2)\right)$ units^3 **(d)** $\frac{\pi}{20}$ units^3

3 $\pi\ln(4)$ units^3

4 $\frac{65\pi}{2}$ units^3

5 $\frac{\pi^2}{4}$ units^3

6 $\frac{16\pi}{15}$ units^3

7 188.5 cm^3

8 15.26 units^3 (2 d.p.)

9 **(a)** $\frac{256\pi}{15}$ units^3 **(b)** 8π units^3

10 $\pi\left(\frac{1}{2}e^2-\frac{5}{6}\right)$ units^3

11 $\pi\left(\frac{4\pi}{3}-\sqrt{3}\right)$ units^3

12 $\frac{3\pi}{10}$ units^3

13 **(a)** C

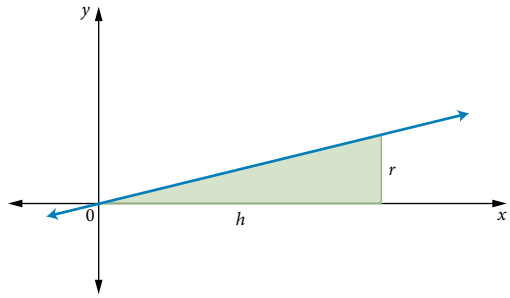
(b) A common misconception is that the required volume is equal to $\pi\int_a^b \left(f(x)-g(x)\right)^2 dx$. However, this expression gives a different volume. Specifically, it gives the resulting volume if the region between the functions were to be 'moved down' to the x-axis and then rotated about the x-axis. This expression is different to option C, as $(a-b)^2 \neq a^2-b^2$.

14 **(a)** 106.397 units^3 (3 d.p.) **(b)** 60.296 units^3 (3 d.p.)

15 1.496 units^3

16 $\frac{1}{3}\pi r^2 h$

17



A cone of radius r and height h is obtained by rotating the line $y = \frac{r}{h}x$ from $x = 0$ to $x = h$ about the x-axis. The resulting volume is:

$$\begin{aligned}
V &= \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx \\
&= \pi\int_0^h \frac{r^2}{h^2}x^2\,dx \\
&= \pi\left[\frac{r^2}{3h^2}x^3\right]_0^h \\
&= \pi\left(\frac{r^2}{3h^2}h^3\right) \\
&= \frac{1}{3}\pi r^2 h
\end{aligned}$$

18 The equation of a circle of radius r centred at the origin is $x^2 + y^2 = 1$. Solving for y gives $y = \pm\sqrt{r^2-x^2}$. The function $y = \sqrt{r^2-x^2}$ describes a semicircle with radius r. Rotating this semicircle about the x-axis produces a sphere of radius r. The resulting volume is

$$\begin{aligned}
V &= \pi\int_{-r}^{r}\left(\sqrt{r^2-x^2}\right)^2 dx \\
&= 2\times\pi\int_0^r\left(\sqrt{r^2-x^2}\right)^2 dx \text{ (due to the symmetry of the function)} \\
&= 2\pi\int_0^r r^2-x^2\,dx \\
&= 2\pi\left[r^2x-\frac{1}{3}x^3\right]_0^r \\
&= 2\pi\left(r^2r-\frac{1}{3}r^3\right) \\
&= 2\pi\times\frac{2}{3}r^3 \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

19 $a = \sqrt{6}$

20 $c = \sqrt{\frac{21}{5}} = \frac{\sqrt{105}}{5}$

21 $\frac{256\pi}{3}$ cm^3

22 $\frac{2\pi}{9}$ units^3

23 678.6 units^3

EXERCISE 6.3

1 **(a)** 40.399 **(b)** 1.295 (3 d.p.) **(c)** -7.5
(d) 2.005 **(e)** 2.796

2 6.417

The actual value of this definite integral is given by:

$$\begin{aligned}\int_0^2 x^4\,dx &= \frac{1}{5}\left[x^5\right]_0^2\\ &= \frac{1}{5}\left(2^5 - 0^5\right)\\ &= 6.4\end{aligned}$$

The difference between these results is

$$\begin{aligned}|6.417 - 6.4| &= 0.017\\ &< 0.02\end{aligned}$$

3 0.643517

The actual value of this definite integral is given by:

$$\begin{aligned}\int_0^{0.6} \frac{1}{\sqrt{1-x^2}}\,dx &= \left[\arcsin(x)\right]_0^{0.6}\\ &= \arcsin(0.6) - \arcsin(0)\\ &= 0.643501 \text{ (6 d.p.)}\end{aligned}$$

The difference between these results is

$$\begin{aligned}|0.643517 - 0.643501| &= 0.000016\\ &< 0.00002\end{aligned}$$

4 2.051 (3 d.p.) Reasonable as the actual value of the integral is 2.0504 (4 d.p.).

5 31.833 units^2

6 $466.67

7 $n = 8$. The number of intervals must be even and greater than or equal to 6.04.

8 The exact value of $\int_a^b x^3\,dx$ is:

$$\begin{aligned}\int_a^b x^3\,dx &= \left[\frac{1}{4}x^4\right]_a^b\\ &= \frac{1}{4}\left(b^4 - a^4\right)\\ &= \frac{b^4 - a^4}{4}\end{aligned}$$

Let $f(x) = x^3$. $w = \frac{b-a}{2}$.

x	$f(x)$	M
a	a^3	1
$\frac{a+b}{2}$	$\left(\frac{a+b}{2}\right)^3$	4
b	b^3	1

Note that

$$\begin{aligned}a + w &= a + \frac{b-a}{2}\\ &= \frac{2a}{2} + \frac{b-a}{2}\\ &= \frac{a+b}{2}\end{aligned}$$

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{w}{3}\sum f(x)\times M\\ &= \frac{\frac{b-a}{2}}{3}\left(f(a)\times 1 + f\left(\frac{a+b}{2}\right)\times 4 + f(b)\times 1\right)\\ &= \frac{b-a}{6}\left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right)\\ &= \frac{b-a}{6}\left(a^3 + 4\frac{(a+b)^3}{8} + b^3\right)\\ &= \frac{b-a}{6}\left(a^3 + \frac{(a+b)^3}{2} + b^3\right)\\ &= \frac{b-a}{6}\frac{1}{2}\left(2a^3 + (a+b)^3 + 2b^3\right)\\ &= \frac{b-a}{12}\left(2a^3 + a^3 + 3a^2b + 3ab^2 + b^3 + 2b^3\right)\\ &= \frac{(b-a)\left(3a^3 + 3a^2b + 3ab^2 + 3b^3\right)}{12}\\ &= \frac{(b-a)\left(a^3 + a^2b + ab^2 + b^3\right)}{4}\\ &= \frac{a^3b + a^2b^2 + ab^3 + b^4 - a^4 - a^3b - a^2b^2 - ab^3}{4}\\ &= \frac{b^4 - a^4}{4}\end{aligned}$$

9 First note that if c, d, and e are equally spaced, then d must be the mean of c and e. Thus, $d = \frac{c+e}{2}$.

Let $p(x) = Ax^2 + Bx + C$.

$$\int_c^e p(x)dx = \int_c^e Ax^2 + Bx + C\,dx$$
$$= \left[\frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx\right]_c^e$$
$$= \frac{1}{6}\left[2Ax^3 + 3Bx^2 + 6Cx\right]_c^e$$
$$= \frac{1}{6}\left(2Ae^3 + 3Be^2 + 6Ce - \left(2Ac^3 + 3Bc^2 + 6Cc\right)\right)$$
$$= \frac{1}{6}\left(2Ae^3 - 2Ac^3 + 3Be^2 - 3Bc^2 + 6Ce - 6Cc\right)$$
$$= \frac{1}{6}\left(2A\left(e^3 - c^3\right) + 3B\left(e^2 - c^2\right) + 6C(e - c)\right)$$
$$= \frac{1}{6}\left(2A(e - c)\left(e^2 + ce + e^2\right) + 3B(e - c)(e + c) + 6C(e - c)\right)$$
$$= \frac{1}{6}(e - c)\left(2A\left(e^2 + ce + e^2\right) + 3B(e + c) + 6C\right)$$
$$= \frac{1}{6}2w\left(2Ae^2 + 2Ace + 2Ac^2 + 3Be + 3Bc + 6C\right)$$
$$= \frac{w}{3}\left(\begin{matrix} Ac^2 + Bc + C + Ae^2 + Be + C + Ac^2 + 2Ace + Ae^2 \\ + 2Bc + 2Be + 4C \end{matrix}\right)$$
$$= \frac{w}{3}\left(p(c) + p(e) + A\left(c^2 + 2ce + e^2\right) + 2B(c + e) + 4C\right)$$
$$= \frac{w}{3}\left(p(c) + p(e) + A(c + e)^2 + 2B(c + e) + 4C\right)$$
$$= \frac{w}{3}\left(p(c) + p(e) + 4A\left(\frac{c + e}{2}\right)^2 + 4B\left(\frac{c + e}{2}\right) + 4C\right)$$
$$= \frac{w}{3}\left(p(c) + p(e) + 4A(d)^2 + 4B(d) + 4C\right)$$
$$= \frac{w}{3}\left(p(c) + p(e) + 4p(d)\right)$$
$$= \frac{w}{3}\left(p(c) + 4p(d) + p(e)\right)$$

EXERCISE 6.4

1 (a) $\mu = 5$ years

$\lambda = 0.2$ earthquakes per year

$f(t) = 0.2e^{-0.2t}$

(b) 0.330 (c) 0.135

2 (a) $\lambda = 4$ arrivals per hour

$f(t) = 4e^{-4t}$

(b) 0.018 (c) 0.451

3 (a) $\mu = 0.1$ metres

$\lambda = 10$ faults per metre

$f(t) = 10e^{-10t}$

(b) 0.135 (c) 0.993

4 (a) D

(b) Options A and C are both discrete variables, and therefore cannot be exponentially distributed.

Option B is continuous, but it is not exponentially distributed as it does not represent the time or distance before the next occurrence of a random event that occurs continuously and independently at a constant average rate.

A common misconception is that exponentially distributed variables must always be times. This is not the case.

5 B

6 (a) $\lambda = 3,\ \mu = \frac{1}{3}$

(b) (i) $1 - \frac{1}{e^{21}}$ (ii) $1 - \frac{1}{e^{21}}$

(iii) $\frac{1}{e^3} - \frac{1}{e^6}$ (iv) $\frac{1}{e^{15}}$

7 (a) $\lambda = 0.2,\ \mu = 5$

(b) (i) $1 - \frac{1}{e^{0.2}}$ (ii) $\frac{1}{e^2}$

8 (a) 0.173 (b) 0.013 (c) 0.347

9 C

10 D

11 0.951

Since it takes 20 000 km, on average, for a battery to wear out, the probability that it wears out within only 1000 km should be quite low, leading to a high probability that the trip will be able to be completed without having to replace the battery. As the result obtained (0.951) is quite high, this result is reasonable.

12 (a) 0.367, given that the median waiting time is less than the mean, then it is less likely to wait longer than the mean waiting time

(b) $x = 13.8$ minutes

13 (a) 18.1% (b) 6 months

14 2.64 minutes

15 2.92 years

16 Let X be exponentially distributed with rate λ. The probability density function is therefore $f(t) = \lambda e^{-\lambda t}$.

Let M be the median. Then

$$P(X < M) = \frac{1}{2}$$
$$\int_0^M \lambda e^{-\lambda t}\,dt = \frac{1}{2}$$
$$\left[-e^{-\lambda t}\right]_0^M = \frac{1}{2}$$
$$-e^{-\lambda M} - -e^0 = \frac{1}{2}$$
$$-e^{-\lambda M} + 1 = \frac{1}{2}$$
$$e^{-\lambda M} = \frac{1}{2}$$
$$-\lambda M = \ln\left(\frac{1}{2}\right)$$
$$-\lambda M = -\ln(2)$$
$$M = \frac{1}{\lambda}\ln(2)$$

17 LHS $= \dfrac{P(X > a+x \cap X > a)}{P(X > a)}$

$= \dfrac{P(X > a+x)}{P(X > a)}$

$= \dfrac{1 - P(X < a+x)}{1 - P(X < a)}$

$= \dfrac{1 - \int_0^{a+x} \lambda e^{-\lambda t}\, dt}{1 - \int_0^{a} \lambda e^{-\lambda t}\, dt}$

$= \dfrac{1 - \left[-e^{-\lambda t}\right]_0^{a+x}}{1 - \left[-e^{-\lambda t}\right]_0^{a}}$

$= \dfrac{1 + \left[e^{-\lambda t}\right]_0^{a+x}}{1 + \left[e^{-\lambda t}\right]_0^{a}}$

$= \dfrac{1 + e^{-\lambda(a+x)} - e^0}{1 + e^{-\lambda a} - e^0}$

$= \dfrac{e^{-\lambda a - \lambda x}}{e^{-\lambda a}}$

$= e^{-\lambda a - \lambda x - (-\lambda a)}$

$= e^{-\lambda x}$

RHS $= P(X > x)$

$= 1 - P(X < x)$

$= 1 - \int_0^{x} \lambda e^{-\lambda t}\, dt$

$= 1 - \left[-e^{-\lambda t}\right]_0^{x}$

$= 1 + \left[e^{-\lambda t}\right]_0^{x}$

$= 1 + e^{-\lambda x} - e^0$

$= e^{-\lambda x}$

$=$ LHS

CHAPTER REVIEW 6

1 **(a)** $\dfrac{\sqrt{2}}{4}$ units^2 **(b)** $5\ln(2)$ units^2

2 21.47 units^2

3 $\dfrac{\pi}{4}\left(e^4 - 1\right)$ units^3

4 $\dfrac{16\pi}{3}$ units^3

5 $\pi\left(1 - \dfrac{\pi}{4}\right)$ units^3

6 5.305

7 **(a)** $\mu = 3$ minutes

$\lambda = \dfrac{1}{3}$ cars per minute

$f(t) = \dfrac{1}{3}e^{-\frac{1}{3}t}$

(b) 0.080 **(c)** 0.189

8 **(a)** $\lambda = 2,\ \mu = \dfrac{1}{2}$

(b) **(i)** $1 - \dfrac{1}{e^{10}}$ **(ii)** $1 - \dfrac{1}{e^{10}}$ **(iii)** $\dfrac{1}{e^2}$

9 6.020

10 $\dfrac{8}{3}$ units^2

11 654.50 units^3

12 Let $f(x) = \sin(x)$.

$w = \dfrac{\pi - 0}{2}$

$= \dfrac{\pi}{2}$

x	$f(x)$	M
0	0	1
$\frac{\pi}{2}$	1	4
π	0	1

Using Simpson's rule,

$\int_0^{\pi} f(x)dx \approx \frac{\frac{\pi}{2}}{3}\sum f(x) \times M$

$= \dfrac{\pi}{6}(0 \times 1 + 1 \times 4 + 0 \times 1)$

$= \dfrac{2\pi}{3}$

The actual value of $\int_0^{\pi} \sin(x)dx$ is given by:

$\int_0^{\pi} \sin(x)dx = \left[-\cos(x)\right]_0^{\pi}$

$= -\cos(\pi) + \cos(0)$

$= 1 + 1$

$= 2$

The difference between the Simpson's rule approximation and this is $\dfrac{2\pi}{3} - 2 \approx 0.094$. Therefore, the approximation is within 0.1 of the actual value of the integral.

13 2.31 minutes

14 0.619

15 15.727 cm

16 The exact value of $\int_a^b x^2\, dx$ is:

$\int_a^b x^2\, dx = \left[\dfrac{1}{3}x^3\right]_a^b$

$= \dfrac{1}{3}\left(b^3 - a^3\right)$

$= \dfrac{b^3 - a^3}{3}$

Let $f(x) = x^2$. $w = \dfrac{b-a}{2}$.

x	$f(x)$	M
a	a^2	1
$\frac{a+b}{2}$	$\left(\frac{a+b}{2}\right)^2$	4
b	b^2	1

Note that

$$\begin{aligned} a+w &= a+\frac{b-a}{2} \\ &= \frac{2a}{2}+\frac{b-a}{2} \\ &= \frac{a+b}{2} \end{aligned}$$

$$\begin{aligned} \int_a^b f(x)dx &\approx \frac{w}{3}\sum f(x)\times M \\ &= \frac{\frac{b-a}{2}}{3}\left(f(a)\times 1+f\left(\frac{a+b}{2}\right)\times 4+f(b)\times 1\right) \\ &= \frac{b-a}{6}\left(a^2+4\left(\frac{a+b}{2}\right)^2+b^2\right) \\ &= \frac{b-a}{6}\left(a^2+4\frac{(a+b)^3}{4}+b^2\right) \\ &= \frac{b-a}{6}\left(a^2+(a+b)^2+b^2\right) \\ &= \frac{b-a}{6}\left(2a^2+2ab+2b^2\right) \\ &= \frac{b-a}{3}\left(a^2+ab+b^2\right) \\ &= \frac{(b-a)\left(a^2+ab+b^2\right)}{3} \\ &= \frac{a^2b+ab^2+b^3-a^3-a^2b-ab^2}{3} \\ &= \frac{b^3-a^3}{3} \end{aligned}$$

17 68%

Chapter 7: Rates of change and differential equations

RECALL

1 **(a)** -2, 2 **(b)** -1, 3 **(c)** $1-|a-b|, 1+|a-b|$

2 **(a)** $2x\log_e(x)+x$ **(b)** $\frac{1}{x^2+1}-\frac{\log_e\left(x^2+1\right)}{2x^2}$

(c) $-\text{cosec}^2(x)$ **(d)** $-\cos(x)-\text{cosec}(x)\cot(x)$

(e) $\frac{\text{cosec}(x)}{\sqrt{1-x^2}}-\sin^{-1}(x)\times\left(\cot(x)\text{cosec}(x)\right)$

3 **(a)** $\frac{g'(x)}{g(x)}$ **(b)** $e^{g(x)}\times g'(x)$

(c) $n\left(g(x)\right)^{n-1}\times g'(x)$ **(d)** $\frac{2x}{1+x^4}$

4 **(a)** $-\log_e(3)$ **(b)** $\log_e(2)$

(c) $-\log_e(2)$ **(d)** The integral does not exist.

5 **(a)** $-2\sqrt{1-\sin(x)}+c$ **(b)** $\log_e\left(x^2+1\right)+c$

(c) $\tan^{-1}\left(x^2\right)+c$ **(d)** $\sin\left(x^2\right)+c$

(e) $\sin^3(x)+c$ **(f)** $e^{\tan(x)}+c$

(g) $2x\sin(x)-\left(x^2-2\right)\cos(x)+c$

6 **(a)** $\log_e(3)$ **(b)** $\sqrt{2}-1$ **(c)** 1

7 **(a)** $2\log_e\left(x+\sqrt{x^2+1}\right)+c$ **(b)** $2\sqrt{x^2-1}$

8 **(a)** $\frac{1}{2}\log_e\left|\frac{x-2}{x}\right|+c$ **(b)** $\log_e\frac{|x-2|}{|x-1|}+c$

9 **(a)** $\frac{x}{2}+\frac{1}{4}\sin(2x)+c$ **(b)** $\frac{x}{2}-\frac{1}{4}\sin(2x)+c$

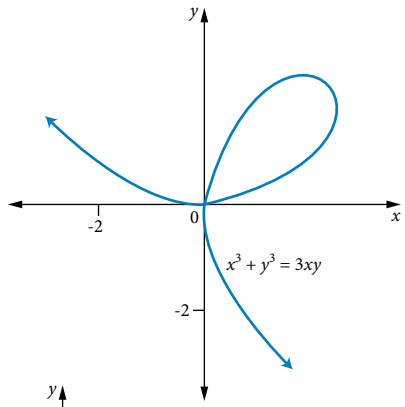
(c) $\frac{x}{2}-\frac{1}{8}\sin(4x)+c$

10 **(a)** $\tan^{-1}(2x)+c$ **(b)** $\sin^{-1}\left(\frac{x}{2}\right)+c$

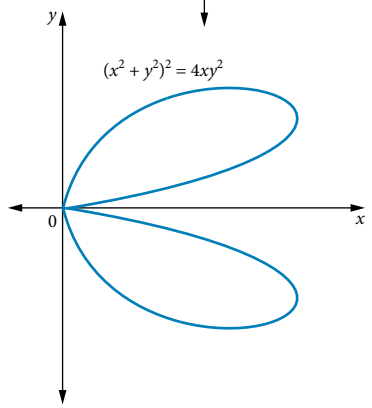
(c) $\tan^{-1}(x+1)+c$

EXERCISE 7.1

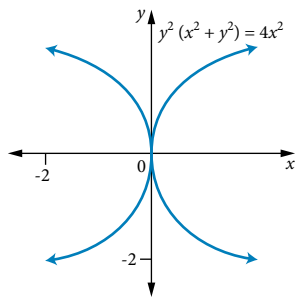
1 **(a)**



(b)



(c)



2 (a) $\frac{dy}{dx} = \frac{x - 2x^3}{y}$ (b) $\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$ (c) $-\frac{x^3}{y^3}$

(d) $\frac{x - xy^2}{y\left(x^2 + 2y^2\right)}$ (e) $\frac{y^2 - x^3 - xy^2}{y\left(x^2 + y^2 - 2x\right)}$

3 (a) $\frac{-x^2(4x - 3)}{2y}$ (b) $\frac{-\left(3x^2 - 4x + y^2\right)}{2(x - 1)y}$

(c) $\frac{-x(x^2 + y^2 - 1)}{\left(x^2 + y^2 + 1\right)y}$ (d) $\frac{-\left(x^3 - 3x^2 + xy^2 - y^2\right)}{\left(x^2 - 2x + y^2 - 2\right)y}$

4 (a) If $x = 0$

$y^2 + y - 6 = 0$

$(y - 2)(y + 3) = 0$

$y = 2, -3$

$\therefore$ the y-intercepts are $(0, 2)$ and $(0, -3)$

(b) $\frac{y - 6x - k}{2y - x + 1}$ (c) $-\frac{7}{2}$

5 (a) $\frac{3x(x + 2)}{2y}$ (b) $y = \frac{1}{4}(9x - 1),\ y = \frac{1}{4}(-9x + 1)$

6 $-\frac{3x^2 + 2y}{2x + 3y^2}$

7 $\frac{y}{y - x}$

8 $-\tan(x)$

9 $\frac{\cos^2\left(xy^2\right) - y^2}{2xy}$

10 $-\frac{1}{y^3}$

11 $\frac{dy}{dx} = \frac{4x^3}{3y^2}$ and $\frac{d^2y}{dx^2} = \frac{4x^2}{9y^2}$

12 (a) $\underbrace{(10x)}_{\frac{d}{dx}\left(5x^2\right)} + \underbrace{\left(6y + 6x\frac{dy}{dx}\right)}_{\frac{d}{dx}(6xy)} + \underbrace{\left(10y\frac{dy}{dx}\right)}_{\frac{d}{dx}\left(5y^2\right)} = \underbrace{0}_{\frac{d}{dx}(8)}$

$(6x + 10y)\frac{dy}{dx} = -(10x + 6y)$

$\frac{dy}{dx} = -\frac{5x + 3y}{3x + 5y}$

(b) For a horizontal tangent at P, the gradient is zero.

$\left.\frac{dy}{dx}\right|_P = \left.-\frac{5x + 3y}{3x + 5y}\right|_P$

$= 0$

$5x + 3y = 0$

$y = -\frac{5x}{3}$

(c) $\frac{3\sqrt{10}}{10}$

(d) $\frac{5\sqrt{10}}{16}$ (e) $P\left(\frac{3\sqrt{10}}{10}, -\sqrt{\frac{5}{2}}\right)$ is a local minimum.

13 (a) A

(b) Option B is true because it will be contained in the derivative of the product xy.

Option C is true as the product rule is needed for the second term.

Option D is true as the graph of the relation passes the vertical line test.

14 (a) B

(b) Option A is false, the point is not on the curve.

Option C is false, the derivative at that point can be found but the function is undefined.

Option D is false, the derivative is zero at the points $(0, -2)$ and $(0, 2)$.

15 (a) $m = -2$ or $m = \frac{2}{3}$ (b) $\frac{dy}{dx} = \frac{4x - 2y}{2x + 3y}$

(c) On the line $y = mx$,

$\left.\frac{dy}{dx} = \frac{4x - 2y}{2x + 3y}\right|_{y=mx}$

$= m$

$\frac{4x - 2mx}{2x + 3mx} = m$

$\frac{4 - 2m}{2 + 3m} = m,\ x \neq 0$

$4 - 2m = 2m + 3m^2$

$3m^2 + 4m - 4 = 0$

$m = -2$ or $m = \frac{2}{3}$

16 (a) $\frac{BH}{BG} = \frac{2 - a}{2 + a}$ (b) $b = \frac{1}{2}\sqrt{4 - a^2}$ (c) $-\frac{a}{2\sqrt{4 - a^2}}$

(d) $y = -\frac{a}{2\sqrt{4 - a^2}}x + \frac{2}{\sqrt{4 - a^2}}$ (e) X is at $\left(\frac{4}{a}, 0\right)$.

(f) $\frac{XH}{XG} = \frac{2 - a}{2 + a}$

$= \frac{BH}{BG}$

17 (a) $\left.\frac{dy}{dx}\right|_{(x_0, y_0)} = -\frac{b^2x_0}{a^2y_0}$ (b) $A = \frac{x_0}{a^2}$ and $B = \frac{y_0}{b^2}$

18 (a) $\frac{dy}{dx} = -\frac{x + 3}{y + 1}$

(b) $\left.\frac{dy}{dx}\right|_{(1,1)} = -2$ and $\left.\frac{dy}{dx}\right|_{(-3,-1)}$ undefined

(c) $\left.\frac{dy}{dx}\right|_{(1,1)} = -2$ is meaningless.

$\left.\frac{dy}{dx}\right|_{(-3,-1)}$ undefined, consistent with the nature of the relation.

19 (a) $\left.\frac{dy}{dx}\right|_{\substack{x=400 \\ y=50\,000}} = -375$ (dollars/working hour)

In words, every hour increase in working hours per week has a corresponding decrease of \$375 in capital investment per week, maintaining a constant weekly output Q.

(b) An extra \$3750 in capital must be allocated to offset the reduction of 10 hours per week in working hours if output is to be maintained.

EXERCISE 7.2

1 (a) $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ (b) $\frac{dV}{dt} = 3a^2 \frac{da}{dt}$ (c) $\frac{dP}{dt} = 2RI \frac{dI}{dt}$

(d) $\frac{dy}{dt} = 6x \frac{dx}{dt}$ (e) $\frac{dI}{dt} = -0.005e^{-0.005x} \frac{dx}{dt}$

2 (a) The area is increasing at a rate of 120π cm^2/minute.

(b) The volume is decreasing at a rate of $\frac{27}{2}$ cm^3/hour.

3 The length of the shadow is increasing at a rate of 0.4 metres/second.

4 $-4\sqrt{\pi}$

5 $\frac{dA}{dt} = \frac{1}{2}b\sin(\theta)\frac{da}{dt} + \frac{1}{2}a\sin(\theta)\frac{db}{dt}$

6 $-\frac{8\sqrt{2}}{3}$ cm/second

7 (a) $\frac{dy}{dt} = \left(\frac{y + y^2}{x}\right) \times \frac{dx}{dt}$ (b) 2

8 The upper end of the ladder is sliding down at $\frac{\sqrt{3}}{2}$ metres/second.

9 (a) The rate of change is $2\pi rh$ cubic units.

(b) πr^2

10 (a) $\frac{\text{width}}{2} = \frac{h}{1}$ (using similar triangles)

width $= 2h$

The area A of the water at the end of the trough is

$A = \frac{1}{2}\text{width} \times h$

$= \frac{1}{2}2h \times h$

$= h^2$

The volume of water V is then given by

$V = 3A$

$= 3h^2$

(b) $\frac{1}{6h}\frac{dV}{dt}$

(c) $\frac{1}{120}$ metres/minute

11 (a) $92\sqrt{29}$ (b) $\frac{d\theta}{dt} = \frac{46\sqrt{29}}{\sec^2(\theta)}$

(c) $\sec^2(\theta) = \frac{D^2}{4}$ (d) 34.1679 radians/hour

$\frac{d\theta}{dt} = \frac{46\sqrt{29}}{\sec^2(\theta)}$

$= \frac{46\sqrt{29}}{\left(\frac{D^2}{4}\right)}$

$= \frac{184\sqrt{29}}{D^2}$

12 $\frac{200}{27}$ kg/s

13 12 units/second

14 (a) Option C is a false statement because the volume expression has not been differentiated.

(b) Option A is a true statement because of the given information.

Option B is a true statement because $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$ is equal to $\frac{dV}{dx} \times \frac{dx}{dt}$.

Option D is a true statement because the expressions in A and B are equal.

15 (a) D

(b) Option A is false because the chain rule including $\frac{dy}{dx}$ has been ignored.

Option B is false because an incorrect $\frac{dy}{dx}$ is included in the first term and the right side is incorrect.

Option C is false because an incorrect $\frac{dy}{dx}$ is included in the first term.

16 V is decreasing only when $A < h$.

17 The volume is increasing at a rate of 60π cm^3/min.

18 (a) $x = 2\cos(\theta)$ and $y = 2\sin(\theta)$ (b) $\frac{2\pi}{5}$ radians/second

(c) $\frac{dx}{dt} = -\frac{4\pi}{5}\sin(\theta)$ m/s and $\frac{dy}{dt} = \frac{4\pi}{5}\cos(\theta)$ m/s

(d) $\left.\frac{dx}{dt}\right|_{\theta=\frac{\pi}{6}} = -\frac{2\pi}{5}$ m/s (e) (-2, 0)

19 (a) $0.25 + x^2 = D^2 \Rightarrow D^2 - x^2 = 0.25$ (b) $\frac{dx}{dD} = \frac{D}{x}$

(c) Yes, the car is travelling above the speed limit.

20 (a) e, $\frac{1}{e}$, $g'(f(1)) = \frac{1}{f'(1)}$

(b) $y = f(x)$

$= f\left(f^{-1}(y)\right)$

Differentiating both sides with respect to y:

$1 = \frac{dy}{dy}$

$= \frac{df\left(f^{-1}(y)\right)}{dy}$

$= \left.\frac{df(x)}{dx}\right|_{x=f^{-1}(y)} \times \frac{df^{-1}(y)}{dy}$

$\frac{df^{-1}(y)}{dy} = \frac{1}{\left(\left.\frac{df(x)}{dx}\right|_{x=f^{-1}(y)}\right)}$

But $y = f(x) \Leftrightarrow x = f^{-1}(y)$, therefore

$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$ whenever $y = f(x)$ has a derivative f' and an inverse f^{-1}.

EXERCISE 7.3

1 (a) $\frac{dL}{dt} = -0.0003$ (b) $\left(\frac{dy}{dx}\right)^2 + x = 0$

(c) $\frac{dV}{dt} = kr$ (d) $\frac{dT}{dt} = k(T - S)$

2 (a) $\dfrac{dy}{dx} = x^2 + \dfrac{1}{x^2}$

$\dfrac{dy}{dx} - x^2 - \dfrac{1}{x^2} = 0$

$y = x^2 + \dfrac{1}{x^2}$ is a solution of $\dfrac{dy}{dx} - \dfrac{x^3}{3} + \dfrac{1}{x} = 0$.

(b) $\dfrac{dy}{dx} = e^{-x^2} - 2x\left(xe^{-x^2}\right)$

$= e^{-x^2} - 2xy$

$\dfrac{dy}{dx} + 2xy = e^{-x^2}$

$y = xe^{-x^2}$ is a solution of $\dfrac{dy}{dx} + 2xy = e^{-x^2}$.

(c) $\dfrac{dy}{dx} + 3y + 2 = -3e^{-3x} + 3\left(-\dfrac{2}{3} + e^{-3x}\right) + 2$

$= -3e^{-3x} - 2 + 3e^{-3x} + 2$

$= 0$

$y = -\dfrac{2}{3} + e^{-3x}$ is a solution of $\dfrac{dy}{dx} + 3y + 2 = 0$.

(d) $\dfrac{d\left(x^n\right)}{dx} = nx^{n-1}$

$\dfrac{dy}{dx} - \dfrac{n}{x}y = nx^{n-1} - \dfrac{n}{x}x^n$

$= nx^{n-1} - nx^{n-1}$

$= 0$

$y = x^n$ is a solution of $\dfrac{dy}{dx} - \dfrac{n}{x}y = 0$.

3 (a) $y = 2\sin^{-1}(x) + c$ **(b)** $z = \dfrac{1}{2}\log_e\left(1 + t^2\right) + c$

(c) $y = \dfrac{2}{7}x^{\frac{7}{2}} + c$ **(d)** $y = x + \dfrac{1}{2}\sin(2x) + c$

4 (a) $y = 2\sin^{-1}(x) + 5 - \pi$ **(b)** $z = \dfrac{1}{2}\log_e\left(1 + t^2\right)$

(c) $y = \dfrac{2}{7}x^{\frac{7}{2}} + \dfrac{5}{7}$ **(d)** $y = x + \dfrac{1}{2}\sin(2x) - \pi$

5 (a) $y = \dfrac{1}{2}\left(Ae^{2x} + 1\right)$ **(b)** $y = Ae^{\frac{3x}{2}}$

(c) $y = (3x + 3c)^{\frac{1}{3}}$ **(d)** $h = \pm\sqrt{2(3t + c)}$

6 (a) $y = 10e^{-x}$ **(b)** $y = \tan\left(x + \dfrac{\pi}{3}\right)$

(c) $y = \dfrac{25 - 9x}{9x - 28}$ **(d)** $y = \dfrac{e^x}{1 + e^x}$

7 (a) $M = Ae^{-kt}$

(b) (i) There are 3.2 milligrams remaining after two years.

(ii) The half-life is 3.1 years.

8 $\dfrac{dy}{dx} = nx^{n-1}e^{x^n}$

$\dfrac{dy}{dx} - nx^{n-1}y = nx^{n-1}e^{x^n} - nx^{n-1}e^{x^n}$

$= 0$

$y = e^{x^n}$ is a solution of $\dfrac{dy}{dx} - nx^{n-1}y = 0$.

9 (a) $x^2 + 2y^2 = c$

$2x + 4y\dfrac{dy}{dx} = 0$

$x + 2y\dfrac{dy}{dx} = 0$

which is the same as the differential equation.

$x^2 + 2y^2 = c$ is a solution of $2y\dfrac{dy}{dx} + x = 0$.

(b) $2x^2 - y^2 = c$

$4x - 2y\dfrac{dy}{dx} = 0$

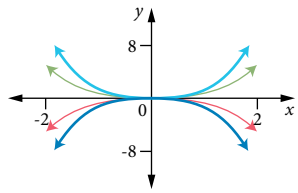
$2x - y\dfrac{dy}{dx} = 0$

$y\dfrac{dy}{dx} = 2x$

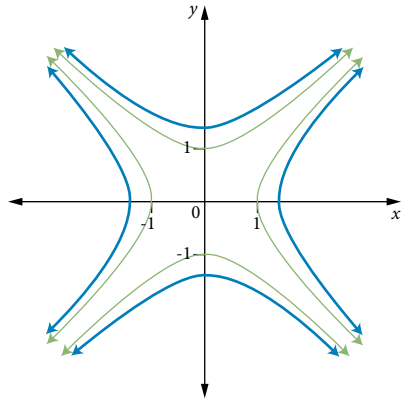
which is the same as the differential equation.

$2x^2 - y^2 = c$ is a solution of $y\dfrac{dy}{dx} = 2x$.

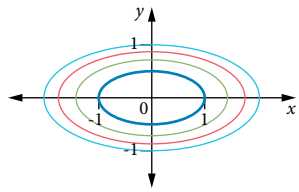
10 (a)



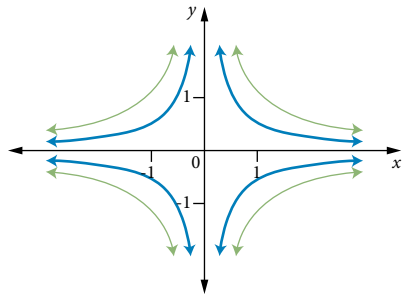
(b)



(c)



(d)



11 **(a)** $y = -\dfrac{1}{2x^2 + c}$ **(b)** $y = \left(\dfrac{3}{4}\sin(2x) + D\right)^{\frac{2}{3}}$

(c) $3y + y^3 - 3e^x = c$ **(d)** $y^2 = x^2 + 20$

(e) $y = \sin^{-1}(x - 1)$ **(f)** $y = \dfrac{1}{2}\log_e\left(x^2 + 1\right)$

12 $\dfrac{\pi}{6} + 2$

13 **(a)** $y = \dfrac{e^x}{A + e^x}$

$$\frac{dy}{dx} = \frac{\left(A + e^x\right)e^x - e^x e^x}{\left(A + e^x\right)^2}$$

$$= \frac{Ae^x}{\left(A + e^x\right)^2}$$

$$y(1 - y) = \frac{e^x}{A + e^x}\left(1 - \frac{e^x}{A + e^x}\right)$$

$$= \frac{e^x A + e^x e^x - e^x e^x}{\left(A + e^x\right)^2}$$

$$= \frac{Ae^x}{\left(A + e^x\right)^2}$$

$$= \frac{dy}{dx}$$

$y(0) = 2$

$$2 = \frac{1}{A + 1}$$

$$A = -\frac{1}{2}$$

$$y = \frac{e^x}{-\frac{1}{2} + e^x}$$

$$= \frac{2e^x}{2e^x - 1}$$

(b) $y = \dfrac{-x}{Ax + 1}$

$$\frac{dy}{dx} = \frac{-(Ax + 1) + xA}{(Ax + 1)^2}$$

$$x^2\frac{dy}{dx} = \frac{-x^2(Ax + 1) + x^3 A}{(Ax + 1)^2}$$

$$= \frac{-x^2(Ax + 1 - xA)}{(Ax + 1)^2}$$

$$= -\frac{x^2}{(Ax + 1)^2}$$

$$-y^2 = -\left(\frac{-x}{Ax + 1}\right)^2$$

$$= x^2\frac{dy}{dx}$$

If $y(1) = -\dfrac{1}{3}$,

$$-\frac{1}{3} = \frac{-1}{A + 1}$$

$$A = 2$$

$$y = \frac{-x}{2x + 1}$$

14 **(a)** $y^3 - 3xy = c$

$$3y^2\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$$

$$(3y^2 - 3x)\frac{dy}{dx} - 3y = 0$$

$$(y^2 - x)\frac{dy}{dx} = y$$

$y^3 - 3xy = c$ is a solution of $(y^2 - x)\dfrac{dy}{dx} = y$.

(b) $x^2 + xy^2 = c$

$$2x + y^2 + 2xy\frac{dy}{dx} = 0$$

$$2xy\frac{dy}{dx} + y^2 + 2x = 0$$

$x^2 + xy^2 = c$ is a solution of $2xy\dfrac{dy}{dx} + y^2 + 2x = 0$.

(c) $-2x^{-3} - \dfrac{d}{dx}\left(x^{-4}\right)y^2 - x^{-4}\dfrac{d}{dx}\left(y^2\right) = 0$

$$-2x^{-3} + 4x^{-5}y^2 - x^{-4}2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x^{-3} - 4x^{-5}y^2}{-2x^{-4}y}$$

$$\frac{dy}{dx} = -\frac{2x^{-3}}{2x^{-4}y} + \frac{4x^{-5}y^2}{2x^{-4}y}$$

$$\frac{dy}{dx} = -\frac{x}{y} + \frac{2y}{x}$$

$\dfrac{1}{x^2} - \dfrac{y^2}{x^4} = c$ is a solution of $\dfrac{dy}{dx} = -\dfrac{x}{y} + \dfrac{2y}{x}$.

15 Option D is correct, because the solution to a differential equation contains an arbitrary constant that can take many values.

16 **(a)** $x = \dfrac{1}{3}\log_e\left(\dfrac{y - 1}{y + 2}\right) - c$

(b) The general solution is $y = \dfrac{1 + 2Ae^{3x}}{1 - Ae^{3x}}$.

(c) $y = \dfrac{8 + 10e^{3x}}{8 - 5e^{3x}}$

17 (a) LHS $= \dfrac{dm}{dt}$

$= \dfrac{d}{dt}4(e^{-2t} - e^{-3t})$

$= 4(-2e^{-2t} + 3e^{-3t})$

$= -8e^{-2t} + 12e^{-3t}$

RHS $= -3m + 4e^{-2t}$

$= -3 \times 4(e^{-2t} - e^{-3t}) + 4e^{-2t}$

$= -8e^{-2t} + 12e^{-3t}$

LHS = RHS

$m(t) = 4(e^{-2t} - e^{-3t})$ is a solution of $\dfrac{dm}{dt} = -3m + 4e^{-2t}$.

(b) There is no medication in the bloodstream initially.

(c) $t = 0.41$ (2 d.p.)

(d) The long-term amount of medication in the bloodstream is zero.

18 (a) $\dfrac{d^2y}{dx^2} + 4y = 0$

$y = a\sin(2x) + b\cos(2x)$

$\dfrac{dy}{dx} = 2a\cos(2x) - 2b\sin(2x)$

$\dfrac{d^2y}{dx^2} = -4a\sin(2x) - 4b\cos(2x)$

$= -4\left(a\sin(2x) + b\cos(2x)\right)$

$= -4y$

$y = a\sin(2x) + b\cos(2x)$ is a solution of $\dfrac{d^2y}{dx^2} + 4y = 0$.

(b) The particular solution is $y = \sin(2x) + \cos(2x)$.

19 (a) $\dfrac{dP}{dt} = 5000\dfrac{d}{dt}\left(1 + e^{\frac{t}{5}}\right)$

$= 1000e^{\frac{t}{5}}$

Substituting the given expression for P, with $k = 1000$

$\dfrac{1}{5}P - 1000 = \dfrac{1}{5} \times 5000\left(1 + e^{\frac{t}{5}}\right) - 1000$

$= 1000 + 1000e^{\frac{t}{5}} - 1000$

$= 1000e^{\frac{t}{5}}$

$= \dfrac{dP}{dT}$

The number of fish in the lake would be $P(t) = 5000\left(1 + e^{\frac{t}{5}}\right)$

Initially,

$P(0) = 5000\left(1 + e^{\frac{1}{5} \times 0}\right)$

$= 10\,000$

which is consistent with the given information.

The particular solution is $P(t) = 5000\left(1 + e^{\frac{t}{5}}\right)$.

(b) $\dfrac{dP}{dt} = 5000\dfrac{d}{dt}\left(3 - e^{\frac{t}{5}}\right)$

$= -1000e^{\frac{t}{5}}$

Substituting the given expression for P, with $k = 3000$,

$\dfrac{1}{5}P - 3000 = \dfrac{1}{5} \times 5000\left(3 - e^{\frac{t}{5}}\right) - 3000$

$= 3000 - 1000e^{\frac{t}{5}} - 3000$

$= -1000e^{\frac{t}{5}}$

$= \dfrac{dP}{dt}$

Initially,

$P(0) = 5000\left(3 - e^{\frac{1}{5} \times 0}\right)$

$= 10\,000$

which is consistent with the given information.

Therefore, the particular solution is

$P(t) = 5000\left(3 - e^{\frac{t}{5}}\right)$

(c) $\dfrac{dP}{dt} = 5\dfrac{d}{dt}\left(k + (2000 - k)e^{\frac{t}{5}}\right)$

$= (2000 - k)e^{\frac{t}{5}}$

Substituting the given expression for P,

$\dfrac{1}{5}P - k = \dfrac{1}{5} \times 5\left(k + (2000 - k)e^{\frac{t}{5}}\right) - k$

$= k + (2000 - k)e^{\frac{t}{5}} - k$

$= (2000 - k)e^{\frac{t}{5}}$

$= \dfrac{dP}{dt}$

Initially,

$P(0) = 5\left(k + (2000 - k)e^{1/5 \times}\right)$

$= 10\,000$

which is consistent with the given information.

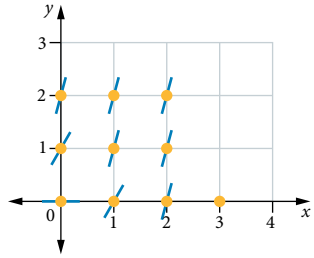
Therefore, the particular solution is

$P(t) = 5\left(k + (2000 - k)e^{t/5}\right)$

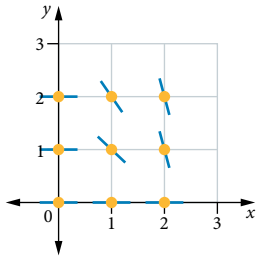
(d) $k = 2000$

EXERCISE 7.4

1 (a)

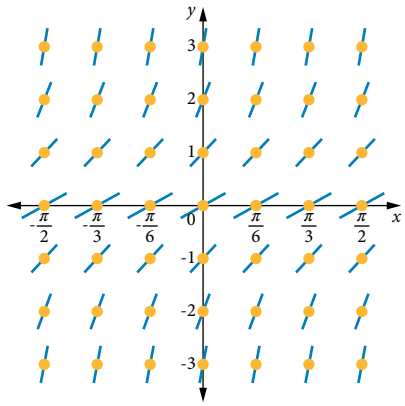


(b)



2 (a) (i) 45° **(ii)** 63° **(iii)** 79° **(iv)** 84°

(b) Note that the slope field is symmetrical about the y-axis.



(c) LHS $= \dfrac{dy}{dx}$

$= \dfrac{d}{dx}\tan(x)$

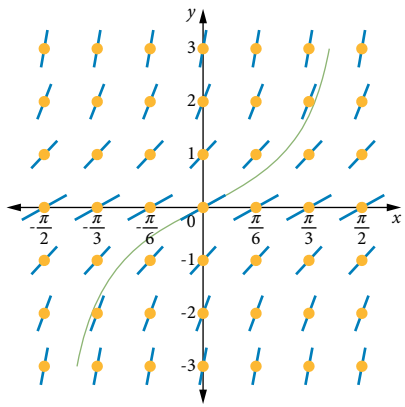
$= \sec^2(x)$

$= 1 + \tan^2(x)$

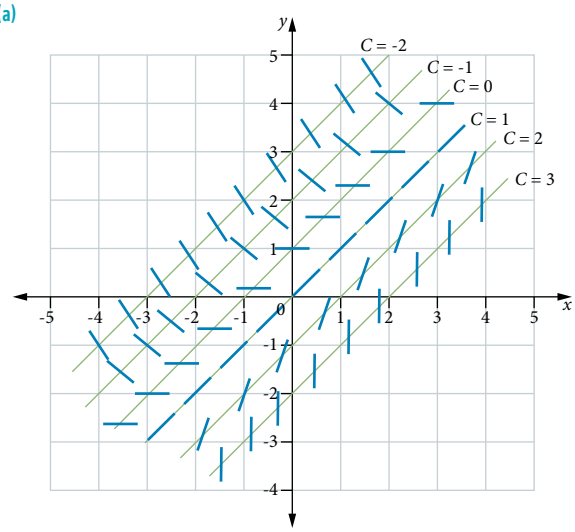
$= 1 + y^2$

$=$ RHS

The function $y = \tan(x)$ is a solution of $\dfrac{dy}{dx} = 1 + y^2$.



3 (a)



(b) $y = x + 1$

(c) $\dfrac{dy}{dx} = x - y + 1$

If $y = x$, $\dfrac{dy}{dx} = 1$

LHS $= 1$

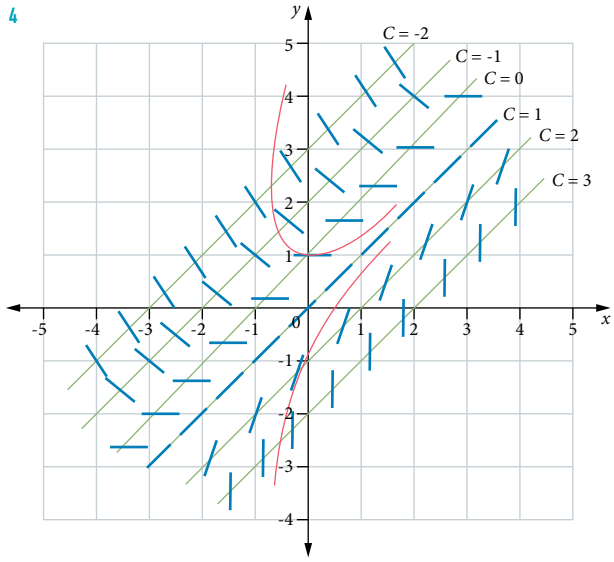
RHS $= x - x + 1$

$= 1$

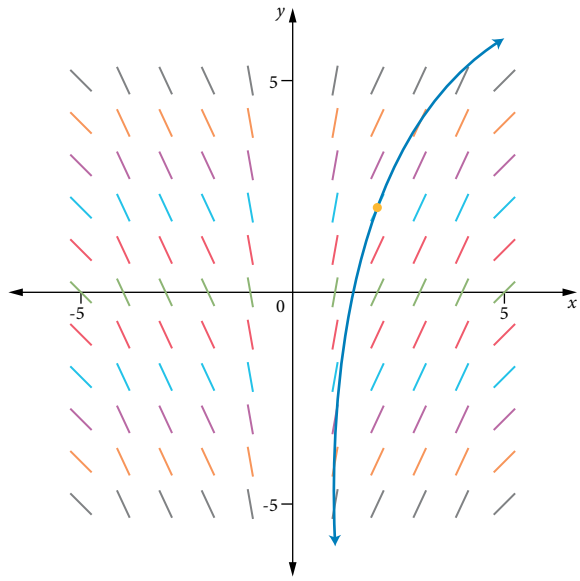
$=$ LHS

$y = x$ is a solution of $\dfrac{dy}{dx} = x - y + 1$.

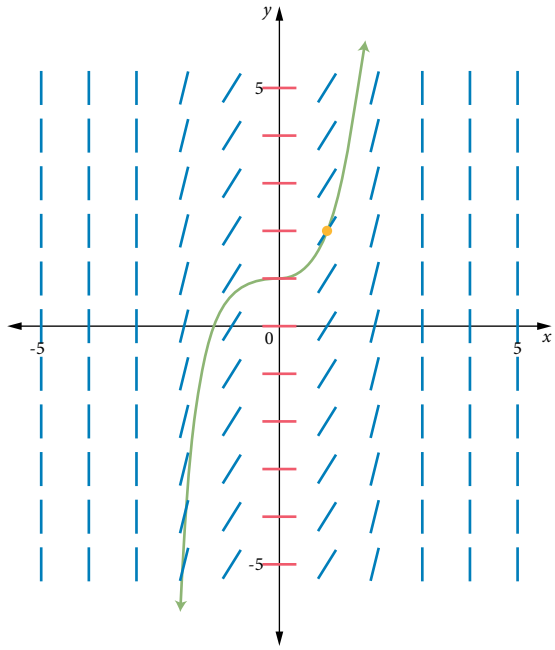
4



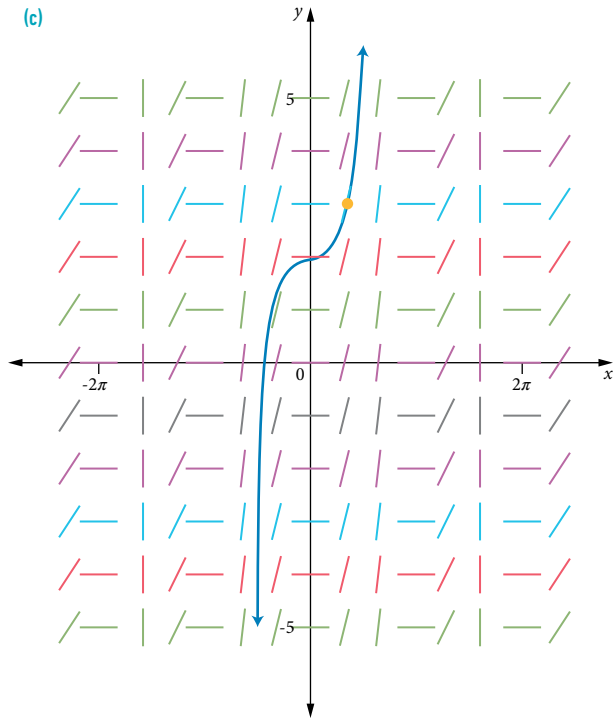
5 **(a)**



(b)

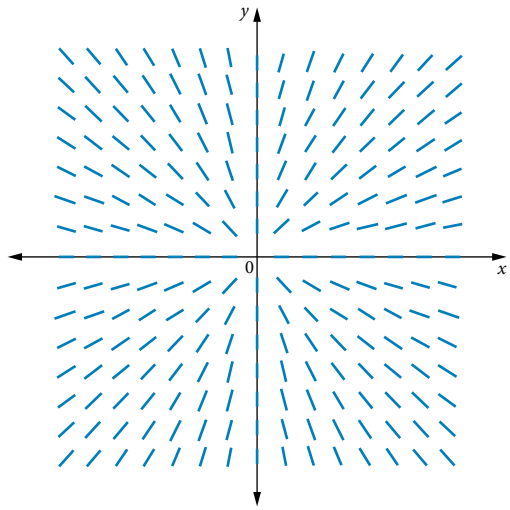


(c)

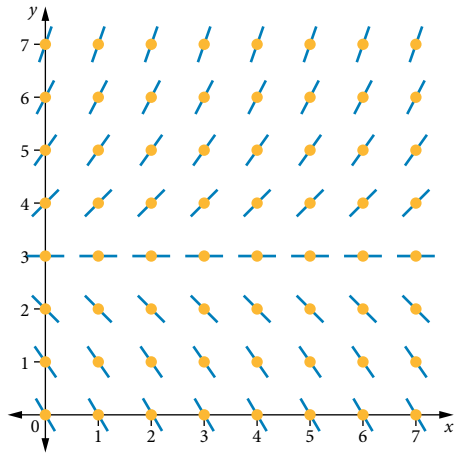


6 $\dfrac{dy}{dx} = \dfrac{x}{y}$

7



8 **(a)**



(b) $y = 3$, unstable

9 $y = 2$; stable equilibrium

$y = -2$; unstable equilibrium

10 (a) Option B is false, because at $y = 0$,

$$\frac{d\left(2y(2-y)\right)}{dy} = 4 - 0$$
$$> 0$$

Therefore, it is unstable.

(b) Option A is true, because these are values for which the gradient is zero.

Option C is true, because at $y = 2$,

$$\frac{d\left(2y(2-y)\right)}{dy} = 4 - 8$$
$$< 0$$

Option D is true, because the slopes are independent of x.

11 (a) Option D, because there is a null-cline where $\frac{dy}{dx} = 0$ at $y = 1$. Also, the slopes increase above this line and become more negative below it.

(b) Option A is false, because this equation would have a null-cline along the line $x = 1$.

Option B is false, because this equation would have a null-cline along the line $y = 0$.

Option C is false, because this equation would have null-clines along the lines $y = 0$, $x = 0$.

12 (a) $P = 0$ is an unstable steady state solution and $P = 20$ is a stable steady state solution.

(b) $\frac{d^2P}{dt^2} = \frac{1}{200}P(P-10)(P-20)$

(c) $P = 10$

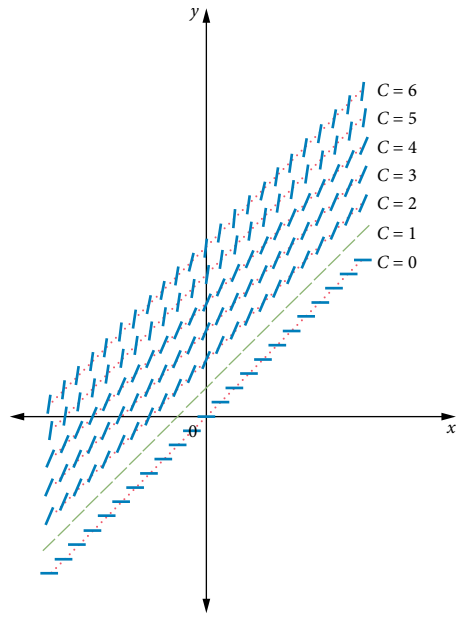
(d) 2000 kangaroos

(e) $$\frac{dP}{dt} = \frac{\left(1+400e^t\right)8000e^t - 8000e^t \times 400e^t}{\left(1+400e^t\right)^2}$$
$$= P - P \times \frac{400e^t}{\left(1+400e^t\right)}$$
$$= P - P \times \frac{P}{20}$$

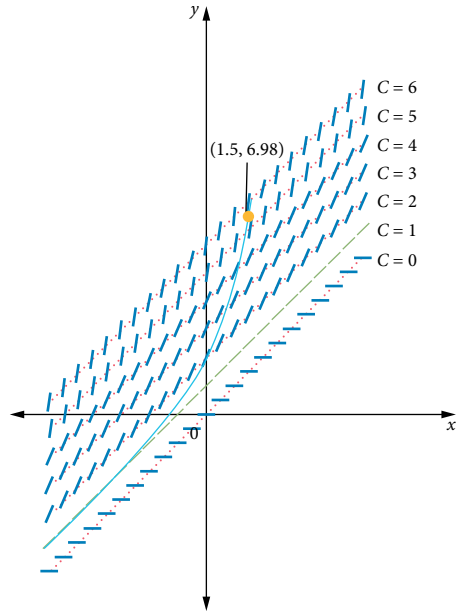
$P = \frac{8000e^t}{1+400e^t}$ is a solution.

(f) There are 1000 kangaroos present.

13 (a)



(b)



(c) LHS $= \frac{dy}{dx}$
$= \frac{d}{dx}\left(x + e^x + 1\right)$
$= 1 + e^x$
$= (x + e^x + 1) - x$
$= y - x$
$=$ RHS

Finally, $y(0) = 1 + e^0 + 0 = 2$ so the initial condition is satisfied also.

$y = x + e^x + 1$ is a solution.

14 (a) $\frac{d^2y}{dx^2} = 4y + 4x + 8$ **(b)** $r = 2$ and $m = -1$, $c = -2$

15 (a) $\frac{dy}{dx} = x^2$

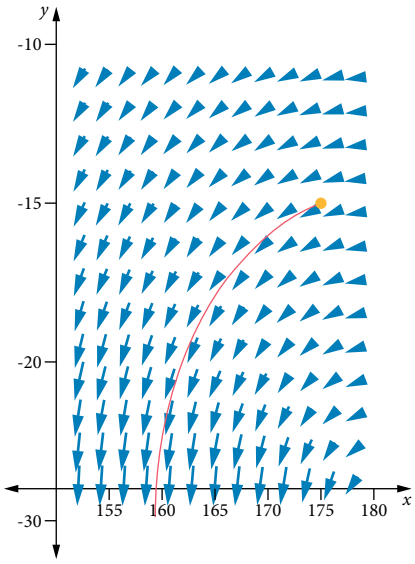
(b) $y = Ax^3 + c$ for a suitable value of A and an arbitrary value of c.

EXERCISE 7.5

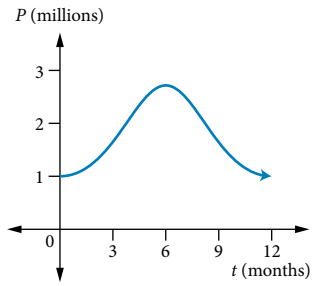
1 3 hours

2 (a) $y = \frac{1}{2}\left(-50 + \sqrt{425 - (x-180)^2}\right)$

(b)



3 $P(t) = e^{\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi}{6}t\right)}$



4 (a) $P = \frac{7.752 \times 10^8}{V^{1.4}}$ (b) 2.5×10^6 Pa

5 (a) $\frac{dV}{dt} = 0.5 - 0.1\sqrt{x}$ (b) $\frac{dx}{dt} = \frac{1}{140}\left(5 - \sqrt{x}\right)$

6 5 hours

7 (a) $x = \sin^{-1}\left(\frac{t}{4}\right) + \pi$ (b) 2 seconds

8 (a) $\frac{dT}{dt} = k(T + 18)$ (b) 22 minutes more are needed.

9 (a) $y = 2e^{3 - \frac{3}{x}}$ (b) $y = \frac{2}{x^2}$

10 (a) $\frac{dW}{dt} = k(A - W)$

(b) $W = A - \frac{3}{4}Ae^{\log_e\left(\frac{2}{3}\right)t}$ or $W = A - A\left(\frac{1}{2}\right)^t$

11 (a) 3700 years (b) 4.2%

12 (a) $v = \frac{Mg}{k} - Ae^{-\frac{kt}{M}}$ (b) 8.12 seconds

13 Option A is a true statement, but not what is asked for.

Option B is a false statement, because this rate depends on the side length.

Option C is a false statement, because this rate is missing a constant.

Option D is a true statement, because this has the surface area as $6x^2$.

14 C

15 (a) $\frac{1}{5500}$ (b) 3513 (c) 2500

(d) $N = 0$; unstable

$N = 5000$; stable

16 (a) $\frac{dx}{dt} = k_1 + k_2\sqrt{x}$ (b) $\frac{dy}{dx} = \frac{y^2}{x}$

(c) $\frac{dx}{dt} = -\left(\frac{x}{20} - 2\right)$ (d) $\frac{dh}{dt} = \frac{kh^{\frac{1}{2}}}{\pi r^2}$

(e) $\frac{dQ}{dt} = 10 - \frac{3Q}{100 + 2t}$

17 (a) 10 minutes (b) 16 minutes

18 (a) $y = 1 \pm Ae^{\frac{1}{2}x^2}$ (b) $y \neq 1$ because $A \neq 0$

19 $y = \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1 - x^2} + c$

20 (a) $\frac{dy}{dt} = ky(1 - y)$

(b) $y = \frac{1}{1 + \frac{3^t}{3}}$

$a = \frac{1}{3}, b = 3$

21 (a) $\frac{1}{2}\left(\log_e(y)\right)^2 + x^2 = c$ (b) $\left(\log_e(y)\right)^2 + 2x^2 = 1$

(c) If $x = 1$ then $\left(\log_e(y)\right)^2 = -1$, which has no solution when y is a real number.

22 (a) $x = 5$ (b) $\frac{d^2y}{dx^2} = -\frac{1}{y} - \frac{(5-x)^2}{y^3}$

(c) Local maximum (d) $y = -\sqrt{25 - (x-5)^2}$.

23 (a) $\int v\frac{dv}{dh}dh = -gR^2\int\frac{1}{(R+h)^2}dh$

$\frac{1}{2}v^2 = \frac{gR^2}{R+h} + c$

Using the initial condition $v = u$ when $h = 0$

$\frac{1}{2}u^2 = gR + c$

$c = \frac{1}{2}u^2 - gR$

$\frac{1}{2}v^2 = \frac{1}{2}u^2 + \frac{gR^2}{R+h} - gR$

$= \frac{1}{2}u^2 - \frac{gRh}{R+h}$

$= \frac{1}{2}u^2 - \frac{gR}{1 + \frac{R}{h}}$

Multiplying both sides by 2 given the required result.

$$v^2 = u^2 - \frac{2gR}{1+\frac{R}{h}}$$

(b) $u = \sqrt{2gR}$

CHAPTER REVIEW 7

1 (a) $-\dfrac{3y}{3x+2y}$ (b) $\dfrac{2x}{\cos(2y)}$

(c) $\dfrac{y^2e^{x+y}}{1-y^2e^{x+y}}$ (d) $\dfrac{y^2+xy}{x^2-2xy}$

2 (a) $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = \dfrac{b^2x_0}{a^2y_0}$ (b) $A = \dfrac{x_0}{a^2}$ and $B = -\dfrac{y_0}{b^2}$

3 $-\dfrac{10}{7}$

4 2

5 -50

6 (a) $h = 400\tan(\theta)$ (b) $\dfrac{2}{125}$ radian/s

7 $r = 2$

8 (a) $y^3 = 2x^2$ (b) $y^2 - x^3 + 4 = 0$

(c) $y^3 - x^2 = 4$ (d) $y^2 - x^2 = 3$

9 (a) $3y^2\dfrac{dy}{dx} - 4x = 0$

$3x\dfrac{dy}{dx} - \dfrac{4x^2}{y^2} = 0$ $\left(\text{multiplying both sides by } \dfrac{x}{y^2}\right)$

$3x\dfrac{dy}{dx} - \dfrac{2y^3}{y^2} = 0$

$3x\dfrac{dy}{dx} - 2y = 0$

$y^3 = 2x^2$ is a solution.

(b) $y^2 - x^3 + 4 = 0$

$2y\dfrac{dy}{dx} - 3x^2 = 0$

$y^2 - x^3 + 4 = 0$ is a solution.

(c) $y^3 - x^2 = 4$

$3y^2\dfrac{dy}{dx} - 2x = 0$

$y^3 - x^2 = 4$ is a solution.

(d) $y^2 - x^2 = 0$

$2y\dfrac{dy}{dx} - 2x = 0$

$y\dfrac{dy}{dx} - x = 0$

$y^2 - x^2 = 0$ is a solution.

10 $x + \log\left(\dfrac{2}{x+1}\right)$

11 $-\dfrac{3e^{-2x}}{2} + \dfrac{1}{2}$

12 $y = \dfrac{2\left(1-e^{4x}\right)}{1+e^{4x}}$

13 (a) $T(t) = Ae^{\frac{-t}{10}}$ (b) 13.9 years

14 $y = \tan\left(x^2 + \dfrac{\pi}{4}\right)$

15 (a) $y = \log_e\left(\sin(x)\right)$

$\dfrac{dy}{dx} = \dfrac{1}{\sin(x)} \times \cos(x)$

$= \cot(x)$

$y = \log_e\left(\sin(x)\right)$ is a solution.

(b) $y = \sqrt{x^2+3}$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left(x^2+3\right)^{-\frac{1}{2}} \times 2x$

$= \dfrac{x}{\sqrt{x^2+3}}$

$= \dfrac{x}{y}$

$\dfrac{dy}{dx} - \dfrac{x}{y} = 0$

$y = \sqrt{x^2+3}$ is a solution.

(c) $y = Ax^b$

$\dfrac{dy}{dx} = Abx^{b-1}$

$x\dfrac{dy}{dx} = Abx^b$

$= by$

$y = Ax^b$ is a solution.

(d) $y = \log_e\left(\dfrac{2-x}{1-x}\right)$

$y = \log_e(2-x) - \log_e(1-x)$

$\dfrac{dy}{dx} = -\dfrac{1}{2-x} + \dfrac{1}{1-x}$

$= \dfrac{-1+x+2-x}{(2-x)(1-x)}$

$= \dfrac{1}{(2-x)(1-x)}$

$y = \log_e\left(\dfrac{2-x}{1-x}\right)$ is a solution.

16 (a) $2xy - 3y^2 = 1$

$$2y + 2x\frac{dy}{dx} - 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2y}{2x - 6y}$$

$$= \frac{y}{3y - x}$$

$2xy - 3y^2 = 1$ is a solution.

(b) $(2, 1)$ or $(-2, -1)$

(c) For a horizontal tangent:

$$\frac{dy}{dx} = 0$$

$$\frac{y}{3y - x} = 0$$

$$y = 0$$

However, $y = 0$ substituted in the equation of the curve implies that $0 = 1$, which is impossible.

There are no points on the curve with a horizontal tangent.

(d) 3

17 $\frac{dx}{dt}$ is decreasing at 12 units/second.

18 (a) $\frac{dy}{dx}$ is undefined **(b)** $\frac{dy}{dx} = 0$.

(c) $x = 0$

(d) $\int y^2\, dy = \int x^2\, dx$

$$\frac{y^3}{3} = \frac{x^3}{3} + c$$

$$y = \left(x^3 + A\right)^{\frac{1}{3}}, \quad A = 3c$$

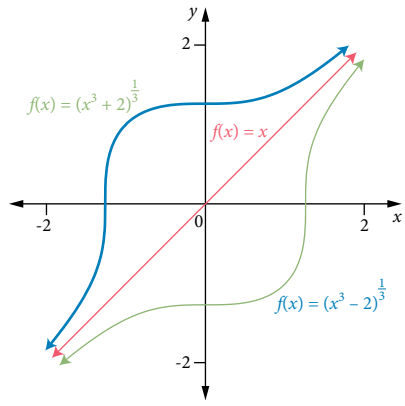
Let $A = 0$,

$$y = \left(x^3\right)^{\frac{1}{3}}$$

$$= x$$

$y = x$ is a solution.

(e)



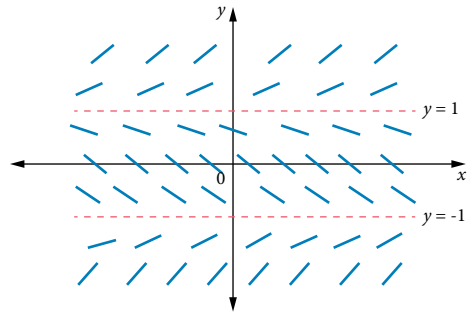
19 (a) I **(b)** IV **(c)** II **(d)** III

20 (a) $\frac{dy}{dx} = \frac{3y}{2x}$ **(b)** $3y^2 + 2x^2 = c$

21 $\frac{dy}{dx} = -\frac{y\left(1 + 2x^2\right)}{x\left(1 + 2y^2\right)}$

22 15

23 (a)



(b) $y = -1$ is a stable solution. $y = 1$ is an unstable solution.

24 14 minutes

25 The graph is sloping upwards to the right and concave down.

26 (a) $\frac{dx}{dt} = -\frac{1}{10}x$ **(b)** 9.2 hours

(c) The amount of undissolved salt approaches 1 gram instead of 0 grams.

27 (a) 20000 **(b)** 12976

28 $\frac{dm}{dt} = -\frac{7m}{200 - 2t}$

29 (a) The coordinates of ships A and B are $A(0, y)$ and $B(x, 200)$ respectively.

$$D = AB$$

$$= \sqrt{(x - 0)^2 + (200 - y)^2} \text{ (Pythagoras' theorem)}$$

$$D^2 = x^2 + (200 - y)^2$$

(b) $\frac{dD}{dt} = \frac{x}{D}\frac{dx}{dt} - \frac{(200 - y)}{D}\frac{dy}{dt}$

(c) $\left.\frac{dD}{dt}\right|_{t=2} = -\frac{355}{13}\ \frac{\text{km}}{\text{h}}$

30 (a) $\frac{dh}{dt} = \frac{k(T_w - T_a)}{L\rho}\frac{1}{h}$

(b) $h(t) = \sqrt{h_0^2 + \frac{2k(T_w - T_a)}{L\rho}t}$

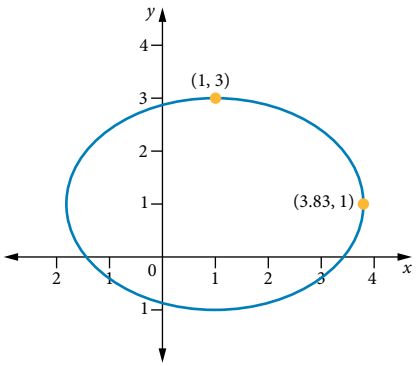
31 (a) $V = \sqrt{2(c - C_P T)}$ **(b)** $V(T) = \sqrt{2C_P(308 - T)}$

(c) 45 m s^{-1}

32 (a) $\frac{dy}{dx} = -\frac{x - 2}{4(y - 1)}$

(b) $\left(\frac{1}{\sqrt{17}}\right)^2 (x - 1)^2 + \left(\frac{2}{\sqrt{17}}\right)^2 (y - 1)^2 = 1$

(c)



Maximum predators: 383

Maximum prey: 3000

33 (a) $\frac{dy}{dx} = \frac{1}{2y}$ (b) $y^2 - x = 0.$

34 (a) $p = \frac{1}{2}$ (b) $p(t) = \frac{1}{1 + 4e^{\frac{-t}{10}}}$

(c) 14 minutes after the beginning of the afternoon period.

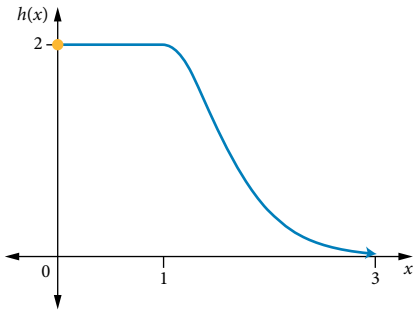
35 (a) LHS $= \frac{d}{dx}(2)$
$= 0$
RHS $= 2 \times 2\sqrt{2-2}$
$= 0$
Also, the initial condition $h(1) = 2$ is satisfied.
$\therefore h(x) = 2$ is a solution.
$h(x) = 2$ is a solution.

(b) $\frac{du}{dx} = u^2 - 2$
$u(1) = 0$

(c) $u(x) = -\sqrt{2}\left(\frac{1 - e^{-2\sqrt{2}(x-1)}}{1+e^{-2\sqrt{2}(x-1)}}\right)$

(d) $h(x) = 2\left(1 - \frac{\left(1 - e^{-2\sqrt{2}(x-1)}\right)^2}{\left(1 + e^{-2\sqrt{2}(x-1)}\right)^2}\right), \quad x \geq 1$

(e) $h(x) = \begin{cases} 2 & x < 1 \\ 2\left(1 - \frac{\left(1 - e^{-2\sqrt{2}(x-1)}\right)^2}{\left(1 + e^{-2\sqrt{2}(x-1)}\right)^2}\right) & x \geq 1 \end{cases}$



Chapter 8: Modelling motion

RECALL

1 (a) -1 (b) -2 (c) $\frac{3}{2}$

2 (a) $-\frac{6}{5}$ (b) Undefined (c) 0

3 (a) 16.7 m/s (b) 144 km/h (c) 0.324
(d) 0.0008 (e) 47.5 m/s (f) 78 cm/min

4 (a) 0.1 km/min (b) 280 km (c) 53 min
(d) 300000000 m/s (e) 232.1 s

5 (a) $6x - 7$; -13 (b) $-12t + 9$; -27 (c) $\pi\sqrt{3}$
(d) $6e$ (e) 27

6 (a) $h(t) = \frac{5}{2}t^2 - 3t + c$ (b) $f(x) = -\frac{7}{2(1+2x)} + \frac{13}{2}$
(c) $\frac{\pi}{4} + 1$ (d) $\frac{1}{6}$ (e) $2\ln(2)$

7 (a) -1 m; 3.5 m; 4 m (b) -13.6 km; -6.8 km, 11.9 km
(c) 30 cm, 89 cm, -36 cm
(d) -1.5 km, 3.2 km, -3 km
(e) 10.9 km; -1.5 km

EXERCISE 8.1

1 (a) Average speed: 6.875 m/s (b) 90 km/h west

2 (a) Average speed: $\frac{1}{2}$ m/s; Average velocity: $-\frac{1}{2}$ m/s
(b) Average speed: 105.5 km/h;
Average velocity: 80.1 km/h N23.5°E

3 (a) $v = -4t - 5$, $a = -4$ (b) $v = 6(3t + 4)$, $a = 18$ (c) $v = -6$, $a = 0$

4 velocity = 59 m/s; acceleration = 20 m/s^2

5 (a) 2 km/h; 8 km/h^2 (b) accelerating

6 The displacement of the object is 60 m to the right of its starting point

7 5 m from the starting point

8 (a) $-10t + 6$ m/s; -10 m/s^2 (b) $\dot{x}(0)$6 m/s; $\dot{x}(1)$-4 m/s
(c) -9.2 m

9 (a) 144 m downhill (b) 12.5 s (c) 30 m/s
(d) $2\sqrt{5}$ s (e) 24 m/s

10 (a) 35.7 m/s; 3.6 s
(b) 68.6 m/s; 240.1 m
(c) 12 s; -705.6 m (downwards) The rock ends up below where it is dropped, at the bottom of the cliff.

11 (a) (i) 5.02 s (ii) 63.82 m
(b) 110.2 m (c) 2.9 s

12 (a) -0.6 m/s^2 (slowing down)
(b) 2.7 m (c) 10 s

13 **(a)** D

(b) Option A is false, because the velocity is positive

Option B is false, because the velocity is positive

Option C is false, because the acceleration opposes the velocity so the object slows down

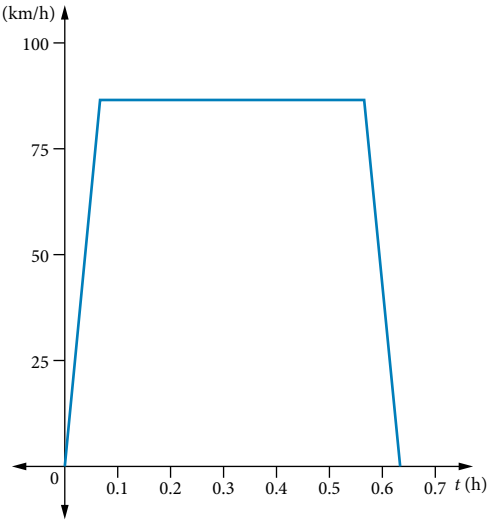
14 Option B is incorrect because the distance travelled is greater than the size of the displacement.

15 **(a)** 7 m/s^2 W **(b)** 2 s

(c) 42 m west of the starting point.

(d) 70 m

16



Time taken is 0.63 hours.

17 **(a)** 8 m/s^2 **(b)** 80 m/s **(c)** 40 m/s

(d) $40\sqrt{2}$ m/s **(e)** $40\sqrt{3}$ m/s

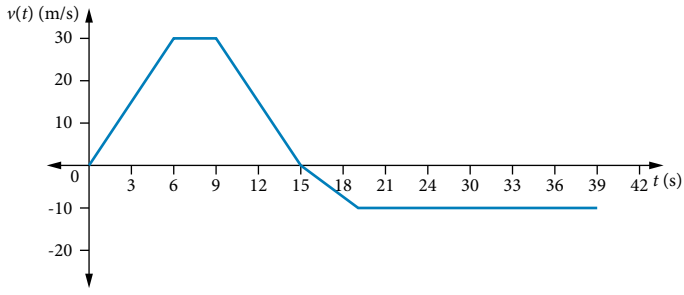
(f) **(i)** $v = \frac{2s}{t}$ **(ii)** $a = \frac{2s}{t^2}$

18 **(a)** 0 m/s, 20 m/s, 30 m/s, 15 m/s

(b) The car accelerates for 6 s from rest to 30 m/s forward. It travels at a constant velocity for 3 s and then slows down for 6 s until it stops.

(c) 270 m **(d)** 12 m/s backward **(e)** 24 s

(f)



19 **(a)** 47.5 s **(b)** 923.61 m

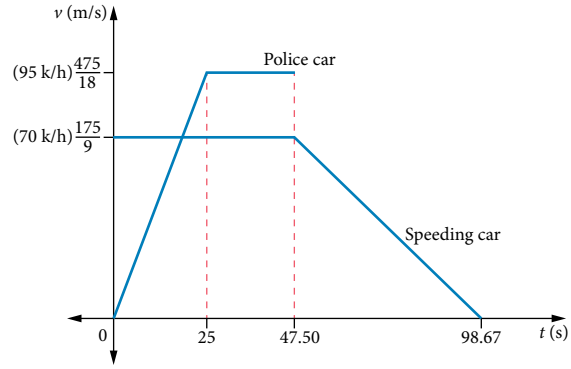
(c) 0.38 m/s^2 **(d)** 51.43 s

(e) Speeding car:

$$v(t) = \begin{cases} 19.44, & t \in [0, 47.5) \\ -0.38t + 19.44, & t \in [47.5, 98.67] \end{cases}$$

Police car: $v(t) = \begin{cases} 1.06t, & t \in [0, 25) \\ \frac{475}{18}, & t \in [25, 47.5] \end{cases}$

(f)



EXERCISE 8.2

1 **(a)** -10 m **(b)** -17 m/s **(c)** -16 m/s^2

2 **(a)** $-12\cos(2t)$ **(b)** $-(4-2t)^{-\frac{3}{2}}$

(c) $100e^{10t}$ **(d)** $\frac{-18}{(6t-1)^2}$

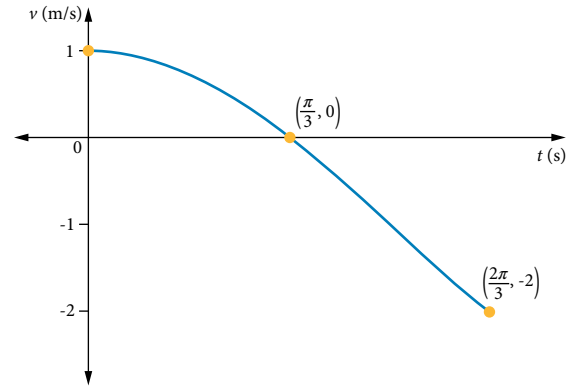
3 1, $\frac{3}{2}$ hours

4 **(a)** 0 s **(b)** -5 m/s

5 **(a)** $\frac{1}{4}$ s **(b)** 1 m/s **(c)** Maximum

6 **(a)** $v(t) = 2\cos(t) - 1$ m/s, $0 \le t \le \frac{2\pi}{3}$ s

(b)



7 $1 - 3\sin(t)$ m/s

8 **(a)** $-4\cos\left(\frac{1}{2}t\right) + 5t + 4$

(b) $\log_e\left(\frac{10}{t-10}\right) + t, \ 0 \le t < 10$

9 $-3e^{-t} + 8$ m; $-3e^{-t}$ m/s^2

10 **(a)** $x(t) = \frac{1}{2}t^3 - 8t^2 + \frac{17}{2}t$ m; $v(t) = \frac{3}{2}t^2 - 16t + \frac{17}{2}$ m/s

(b) $x(t) = -0.4t^3 + 10.9t^2 + 3.6t$ m;

$v(t) = -1.2t^2 + 21.8t + 3.6$ m/s

(c) $x(t) = \frac{3}{2}t^2 - \frac{\sin(2\pi t)}{4\pi^2}$ m; $v(t) = -\frac{1}{2\pi}\cos(2\pi t) + 3t$

11 **(a)** $a(x) = 36x - 66$

(b) $a(x) = 8x^3 + 54x^2 + 77x - 9$

(c) $4xe^{4x^2}$

12 **(a)** -4 m/s; 0 m/s^2 **(b)** $-\frac{4}{3}t + 8$ m/s; $-\frac{4}{3}$ m/s^2

(c) $\frac{1}{2}t^2$ m/s; t m/s^2

13 **(a)** $-5t^2 + 10t$ m/s; $-10t + 10$ m/s^2

(b) 18 m; 30 m

14 **(a)** $v(x) = 7 - 3x$ m/s

(b) $a(1) = -12$ m/s^2; $v(1) = 4$ m/s

(c) -39 m/s^2; 13 m/s

15 **(a)** Option C is false, because the particle is travelling at constant velocity at this time.

(b) Option A is true, because the acceleration is decreasing but still positive.

Option B is true, because the acceleration is negative and the velocity is positive.

Option D is true, because the velocity is constant for the middle part of the journey.

16 **(a)** Option B is true as:

$$a = -\frac{t}{10} + 2$$

$$v = -\frac{t^2}{20} + 2t + 3$$

$$v(20) = -\frac{400}{20} + 40 + 3$$

$$= 23$$

Hence, the velocity in the middle section of the journey is 23 m/s.

(b) Option A is false, because this is the expression for the acceleration.

Option C is false, because

$u = 23$

$a = -4$

$t = 10$

$v = u + at$

$v(10) = 23 - 4 \times 10$

$= -17$

Hence, the velocity changes to negative in the final 10 seconds.

Option D is false, because that is the area under an acceleration–time graph and is the change in velocity.

17 **(a)** $v(t) = \frac{mt^2}{2} + dt$ **(b)** $x(t) = \frac{mt^3}{6} + \frac{dt^2}{2}$

(c) $v(t) = \frac{mt^2}{2} + dt + u$; $x(t) = \frac{mt^3}{6} + \frac{dt^2}{2} + ut$

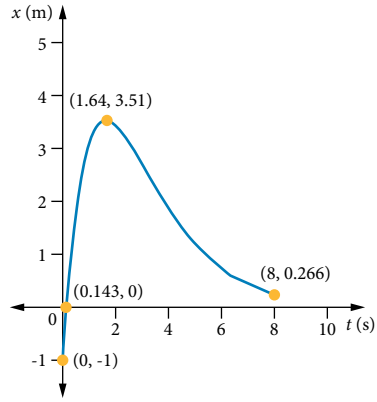
(d) $v(t) = \frac{mt^2}{2} + dt + u$; $x(t) = \frac{mt^3}{6} + \frac{dt^2}{2} + ut + x_0$

18 **(a)** $v(x) = -\frac{1}{3}x + c$ **(b)** 9, -3

19 **(a)** $v = x + 1$ **(b)** $\frac{e^3}{3} - 1$ m

20 **(a)** -1.0 m; 3.4 m

(b)



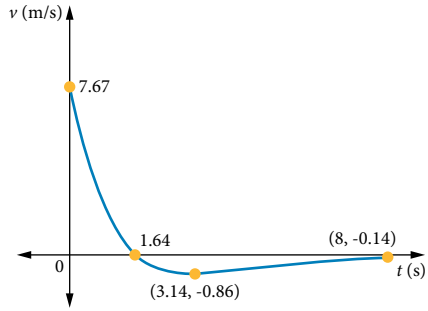
The initial velocity of the particle is 7.67 m/s right. The particle's velocity decreases until it stops at $t = 1.67$ s, when it changes direction and accelerates to reach 0.86 m/s at $t = 3.14$ s, when it starts decelerating to reach 0.14 m/s at $t = 8$ s.

(c) 3.51 m, 1.64 s **(d)** $v(t) = \frac{1}{3}(-14t + 23)e^{-\frac{2}{3}t}$

(e) $a(t) = \frac{4}{9}(7t - 22)e^{-\frac{2}{3}t}$

(f) $v_{\max(positive)} = 7.67$ m/s ; $v_{\max(negative)} = -0.86$ m/s

(g)



The initial velocity of the particle is 7.67 m/s right. The particle's velocity decreases until it stops at $t = 1.67$ s when it changes direction and accelerates to 0.86 m/s at $t = 3.14$ s when it start decelerating to 0.14 m/s at $t = 8$ s.

21 **(a)** $v = \left(\frac{kt + 2\sqrt{u}}{2}\right)^2$ **(b)** $k = \pm\frac{2}{3}$

(c) $\frac{2t}{9}$ **(d)** $t = 0, 2$ s

22 **(a)** $\frac{dv}{dt} = \frac{20g - v}{20}$ **(b)** $v = 20g\left(1 - e^{-\frac{t}{20}}\right)$

(c) $20g$

(d) $x = 20gt + 400g\left(e^{-\frac{t}{20}} - 1\right)$

EXERCISE 8.3

1 **(a)** 118.0 N to the left

(b) 8.9 N, 62.6° clockwise from vertically down

2 **(a)** $F_{net} = 30.8$ N, N56.5°W

(b) $F_{net} = 921.4$ N, S69.9°E

3 (a) $\frac{40\sqrt{3}}{3}$ N (b) $10g - \frac{20\sqrt{3}}{3}$

4 (a) 25 kg m/s (b) $p = 64166.7$ kg m/s

(c) 4.7 kg m/s (d) 636.3 kg m/s

5 (a) $\Delta p = -23111.1$ kg m/s (b) $\Delta p = 1075$ kg m/s

6 (a) -0.78 kg m/s (b) -0.28 N

7 (a) $a = 3.33$ m/s^2 (b) $F_{\text{net}} = 4500$ N

(c) $p = 22500$ kg m/s

8 A retarding force of 2.4 N.

9 $W = 120.54$ N down

$F_{\text{normal}} = 120.54$ N up

10 98.6 N

11 (a) 3483.12 N; 1768.31 N (b) $F_{\text{net}} = 12999.6$ N

(c) $a = 6.31$ m/s^2 (d) 12105.3 N

(e)
$$\begin{aligned} m_1 &= 1367 \\ \Sigma F_1 &= m_1 a = 1367 \times 6.307\,42 \\ &= 8622.25 \\ &= F_{2,1} - F_{\mu 1} \\ F_{2,1} - F_{\mu 1} &= 8622.25 \\ F_{2,1} &= 8622.25 + F_{\mu 1} \\ &= 8622.25 + 3483.12 \\ &= 12105.4 \text{ N} \end{aligned}$$

12 $\frac{g}{4}$; $\frac{15g}{4}$

13 5.4 N; 130.8 N

14 0.6 kg m/s

15 (a) 4724.5 N (b) 4892.4 N (c) 1.0

(d) 6260.5 (e) 2657.4 (f) 0.4

16 (a) D

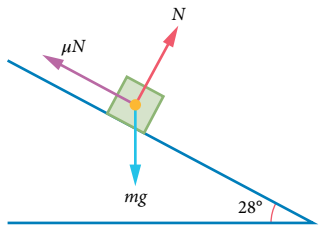
(b) Option A is incorrect because the normal reaction is equal to the weight, not the mass of the box.

Option B is incorrect because the box is not about to move so there is no friction force on it.

Option C is incorrect because the box is not about to move so there is no friction force on it.

17 B

18 (a)



3.7 m/s^2

(b) $t = 4.0$

19 (a) $v(t) = \frac{t}{2} + \frac{t^2}{2} + 3$ (b) $v(x) = +\sqrt{4x - x^2 + 21}$

(c) $v = e^{\frac{t}{2}} + 1$

20 (a) $\frac{g}{5}$ (b) 37.5 kg wt

21 1.04 m/s^2; 21.68 N

22 $a = -g\left(\sin(\theta) + \mu\cos(\theta)\right)$; $s = \frac{u^2}{2g\left(\sin(\theta) + \mu\cos(\theta)\right)}$

EXERCISE 8.4

1 $\frac{1}{6}$ s

2 (a) 3 m; -3 m (b) $\frac{1}{3}$ s; 6 m (c) $\frac{5}{6}$ s; 0 m

(d) $3\pi^2$ m/s^2 in a negative direction

(e) 0 m/s^2; $\mp 6\pi^2$ m/s^2 (f) 12 s

3 $\dot{x} = -8\sin\left(2t + \frac{\pi}{4}\right)$

$\ddot{x} = -16\cos\left(2t + \frac{\pi}{4}\right)$

$= -4x$

This is of the form $\ddot{x} = -n^2 x$, which defines simple harmonic motion.

4 (a) π s; 3 m

(b) 1.7 m in a negative direction

5 1.16 s

6 (a) $\frac{12\pi^2}{25}$ N/m (b) $x = 0.2\sin\left(\frac{2\pi}{5}t + \frac{3\pi}{2}\right)$

7 (a) 0.46 s

(b) 18.75 m/s^2 in a direction opposing the displacement

8 36 kg

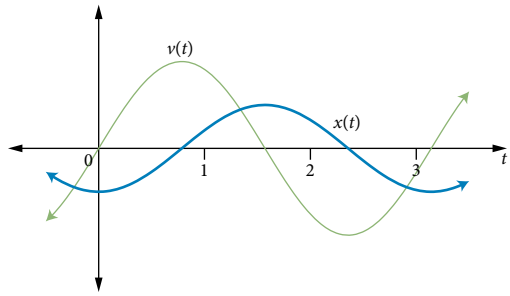
9 (a) 0.50 s (b) 0.16 m (c) 2.01 m/s

(d) 0.13 m (e) 1.18 m/s

10 (a) $x(t) = 2.05\sin\left(2t - \frac{\pi}{2}\right)$; $\dot{x}(t) = 4.10\cos\left(2t - \frac{\pi}{2}\right)$

(b) $x(0) = -2.05$ m; $\dot{x}(0) = 0$ m/s; $x(1) = 0.85$ m; $\dot{x}(1) = 3.73$ m/s

(c)



11 (a) ± 8 m/s (b) $2\sqrt{15}$ m/s

12 (a) $\frac{8}{3}$ m (b) 3π s

13 (a) $\ddot{x} = -\frac{\pi^2}{9}\sin\left(\frac{\pi t}{6}\right)$ (b) $\ddot{x} = -\frac{\pi^2}{36}(x - 2)$

(c) 13.5 m

14 (a) C

(b) Option A is correct because it represents the basic simple harmonic motion equation.

Option B is correct because it can be differentiated to show that it is of the form $\ddot{x} = -n^2 x$.

Option D is correct because it can be differentiated to show that it is of the form $\ddot{x} = -n^2 x$.

15 **(a)** $\ddot{x} = -2n^2(x-1)$

(b) $y = x - 1$
$\dot{y} = \dot{x}$
$\ddot{y} = \ddot{x}$
$\ddot{y} = -2n^2 y$

which represents SHM, hence the motion is simple harmonic.

16 **(a)** $v^2 = 32 - 12x - 2x^2$

$\frac{1}{2}v^2 = 16 - 6x - x^2$

$\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$
$= -6 - 2x$
$= -2(3 + x)$

Now let $y = 3 + x$

$\dot{y} = \dot{x}$
$\ddot{y} = \ddot{x}$
$\ddot{y} = -2y$

which represents SHM as $\ddot{y} = -n^2 y$, $n = \sqrt{2}$, hence the motion is simple harmonic.

(b) 5 m; $x = 3$

17 **(a)** $\dot{x} = na\cos(nt) - nb\sin(nt)$
$\ddot{x} = -n^2 a\sin(nt) - n^2 b\cos(nt)$
$= -n^2 x$

which describes SHM, hence the particle moves with simple harmonic motion.

(b) $\sqrt{a+b}$

18 0.6 s; 0.04 m

CHAPTER REVIEW 8

1 430 km/h

2 13 s

3 -0.8 m/s or -2.88 km/h

4 **(a)** -13 m/s^2 **(b)** -9 m/s^2

5 -3 m; -8 m/s; 4 m/s^2

6 **(a)** $\frac{7}{4}$ m **(b)** -2 m/s^2

7 54 m

8 B

9 40 kg

10 500 N at an angle of $\tan^{-1}\left(\frac{4}{3}\right)$ with the horizontal.

11 0.2 m/s^2

12 2.02 m/s^2 east

13 6.95 s

14 **(a)** 3 m; 0 m/s **(b)** 3π m/s; $\frac{1}{2}$ s **(c)** 2 s; 3 m

(d) $x = \pm 3$ m **(e)** $3\pi^2$ m/s^2

15 $4t^3 + 3t^2 - 7t + 15$ m; $24t + 6$ m/s^2

16 $-\frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t$; $-4t^2 + t + 3$

17 86.7 N; 107.8 N

18 76.3 N; 84.2 N

19 **(a)** 40 kg m/s **(b)** 0.25 m

20 **(a)** $v = -6\cos(1.2t - 4)$; $a = 7.2\sin(1.2t - 4)$

(b) $3\sqrt{3}$ m/s; 3.6 m/s^2

21 **(a)** $v(t) = \frac{-2}{1+t} + 6$; $x(t) = -2\log_e|1+t| + 6t$

(b) $v(t) = -\frac{3}{2}e^{-2t+1} + 2e$; $x(t) = \frac{3}{4}e^{-2t+1} + 2et - \frac{3}{4}e$

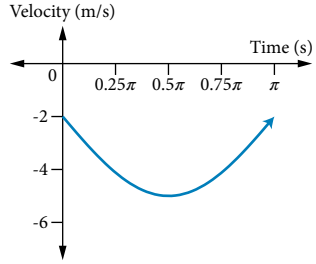
(c) $v(t) = \frac{1}{\pi}\tan(\pi t)$; $x(t) = -\frac{1}{\pi^2}\log_e|\cos(\pi t)|$

22 **(a)** 3; 1.58 **(b)** Between 0.99 s and 1.99 s

(c) 5 m/s^2

23 From the fourth floor it takes twice as long to reach the ground.

24 **(a)**



(b) $-6 - 2\pi$ m

25 **(a)** 665.1 N; 384 N

(b) 12.8 m/s^2 at an angle of 30° with the horizontal

(c) 204 N

26 11.6 s on Mars, 4.5 s on Jupiter, 17.7 s on the Moon and 7.1 s on Earth

27 **(a)** 14700 N **(b)** 1050 N

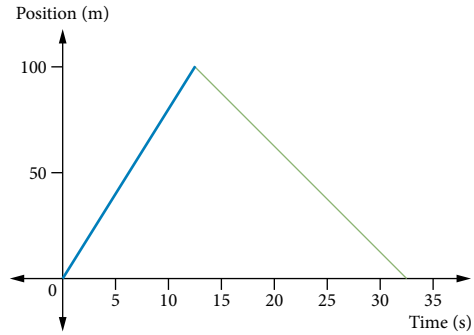
(c) $v = 19.6$ m/s **(d)** 1.4 m/s^2; 58.8 m/s

28 **(a)** $a = g\sin(\theta) - \mu g\cos(\theta)$ **(b)** $\tan(\theta) = \mu$

29 **(a)** 1.28 m/s^2 **(b)** $v = 16$ m/s

(c) -0.8 m/s^2 **(d)** 6.15 m/s

(e)



30 12 m/s; -6 m/s^2

31 **(a)** 24.50 N/m **(b)** 2.84 s; 0.35 cycles/s

(c) 0.3 m; 0.1 m; 0.1 m

(d) $x(t) = 0.1\sin\left(2.21t + \frac{\pi}{2}\right) + 0.2$; $\dot{x}(t) = 0.22\cos\left(2.21t + \frac{\pi}{2}\right)$

32 **(a)** $\dot{x}^2 = \frac{\pi^2}{9}\left(20 - x^2\right)$

$$\frac{1}{2}\dot{x}^2 = \frac{\pi^2}{18}\left(20 - x^2\right)$$

$$\ddot{x} = \frac{d\left(\frac{1}{2}\dot{x}^2\right)}{dx}$$

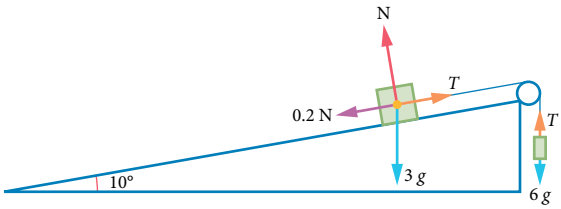
$$= \frac{\pi^2}{18}(-2x)$$

$$= -\frac{\pi^2}{9}x$$

This is of the form $\ddot{x} = -n^2x$, which defines simple harmonic motion.

(b) 6 s; $2\sqrt{5}$ m **(c)** $\frac{3}{2}$ s; $\frac{9}{2}$ s

33 **(a)**



(b) 26.9 N

34 $v^2 = -25\left((x-2)^2 - 1\right)$

$$2v\frac{dv}{dx} = -50(x-2)$$

$$a = v\frac{dv}{dx}$$

$$= -25(x-2)$$

Let $x - 2 = y$

$$\frac{dx}{dt} = \frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} = -25y$$

which represents simple harmonic motion.

35 278 m

36 **(a)** $v = -\frac{g}{0.2}e^{-0.1t} + \frac{g}{0.2}$ **(b)** 49

(c) $a = \frac{g}{2} - 0.1v$ **(d)** 0

Chapter 9: Statistical inference

RECALL

1 **(a)** discrete **(b)** continuous

(c) discrete **(d)** continuous

2 **(a)** The population is all Australian boys between the age of 10 and 12 years old.

The sample is the 344 Australian boys.

The parameter is the average (mean) weight of all Australian boys between 10 and 12 years old.

The statistic is 37.6 kg.

(b) The population is all teachers in the Queensland public system.

The sample is the 1378 selected teachers from the public system.

The parameter is the unknown proportion of teachers that would not leave the public system.

The statistic is the proportion $\frac{923}{1378}$ or 0.67.

(c) The population is all 15-year-olds.

The sample is the 1783 Australian 15-year-olds.

The parameter is the true average number of hours of sleep that is required by a 15-year-old.

The statistic is the amount of sleep obtained, 7.8 hours.

(d) The population is all rolls with this die.

The sample is the 50 rolls.

The parameter is $\frac{1}{6}$ or 16.6%.

The statistic is $\frac{9}{50}$ or 18%.

3 **(a)** $\mu = 10, \sigma = 2$ **(b)** $\mu = 3955, \sigma = 282$

(c) $\mu = 148, \sigma = 10$

4 **(a)** independent **(b)** dependent **(c)** independent

5 **(a)** $E(X) = 3.05$, $\text{Var}(X) = 1.79$, $\sigma(X) = 1.34$

(b) $E(X) = 8.9$, $\text{Var}(X) = 80.19$, $\sigma(X) = 8.95$

(c) $E(X) = 8.73$, $\text{Var}(X) = 6.68$, $\sigma(X) = 2.58$

6 68% is captured by ±1 standard deviation from the mean.

95% is captured by ±2 standard deviations from the mean.

99.7% is captured by ±3 standard deviations from the mean.

7 One standard deviation from the mean would include scores ranging from 85 to 115.

Two standard deviations from the mean would include scores ranging from 70 to 130.

Three standard deviations from the mean would include scores ranging from 55 to 145.

EXERCISE 9.1

1 **(a)** $\bar{x} = 26$

sample var = 301

sd = 17.35

(b) $\bar{x} = 0.91$

sample var = 2.23

sd = 1.49

2 (a) $\text{sd}(\overline{X}) = 0.43$ (b) $\text{sd}(\overline{X}) = 1.69$ (c) $\text{sd}(\overline{X}) = 0.02$

3 (a) The population is all girls between the age of 10 and 15 years.

The sample is the 247 girls.

The parameter is the average height of all girls between 10 and 15 years.
The statistic is sample mean 136.9 cm.

(b) The population is all male teenagers in Australia.

The sample is the 1342 selected male teenagers.

The parameter is the population average of calcium consumed by male teenagers.

The statistic is 1121 mg.

(c) The population is all Year 9 students whose first language is not English.

The sample is the 423 Year 9 students who first language is not English.

The parameter is the unknown mean of all Year 9 students whose first language is not English.

The statistic is the mean score of the 423 Year 9 students whose first language is not English.

4 D

5 B

6 $E(\overline{X}) = 40.1$ mm, $\text{sd}(\overline{X}) = 0.006$ mm

7 (a) Mean weekly income for apprentices is \$701.20.

(b) $\text{sd}(\overline{X}) = \$38.26$

8 $E(\overline{X}) = -3.5\%$, $\text{sd}(\overline{X}) = 11.63\%$

9 The mean of the sample means in the study would be unchanged. That is, $E(\overline{X}) = \mu = 100$.

The standard deviation of the sample means would decrease from 4.74 (2 d.p.) to 3.35 (2 d.p.).

EXERCISE 9.2

1 A

2 $\overline{x} = 34$ years old, $S(\overline{X}) = 4.44$ years

3 $S(\overline{X}) = 0.89$

4 (a) $E(\overline{X}) = 75$ (b) $S(\overline{X}) = 3$

(c) The sampling distribution of $\overline{X}$ will be normal and centred about the population mean of 75.

5 B

6 B

7 A sample size of $n \geq 100$ is required.

8 (a) $\overline{x} = 1.2$ (b) $S(\overline{X}) = 0.252$

(c) From the table given in part (a) it is clear that the population distribution is unlikely to be normal, as the sample distribution has a distinct positive skew. Since $n = 25$ the general requirement of $n \geq 30$ has not been met. Therefore, the central limit theorem would not apply.

9 (a) $\mu = 71.6$ kg (b) $\sigma = 2.01$

(c) $E(\overline{X}) = 71.6$ kg (d) $S(\overline{X}) = 1.42$ kg

10 (a) No, the point estimate will vary from sample to sample, sometimes more, sometimes less.

(b) It is highly unlikely to be lower than \$45 000. A large difference in sample mean values could indicate a problem with the sampling process or bias.

(c) Statistical inference tells you that, under the right circumstances, the sample proportion is the best approximation for the population proportion. The level to which it occurs is unknown, but you can estimate it with a level of confidence. Therefore, you can expect that a point estimate of \$79 767 will estimate the true population mean.

(d) If a simple random sample is not used, if the sample is too small or there is a bias skewing the results.

EXERCISE 9.3

1 (a) The 68% confidence interval is (\$138, \$152).

(b) The 95% confidence interval is (\$131, \$159).

(c) The 99.7% confidence interval is (\$124, \$166).

2 (a) The margin of error for a 90% confidence interval is 0.198 min.

(b) The margin of error for a 95% confidence interval is 0.236 min.

(c) The margin of error for a 98% confidence interval is 0.280 min.

(d) The margin of error for a 99% confidence interval is 0.309 min.

3 B

4 (a) $E = 0.047$, $\overline{x} = 0.60$ (b) $E = 0.045$, $\overline{x} = 0.45$

(c) $E = 0.022$, $\overline{x} = 0.77$

5 C

6 Sample size of 1444 is needed.

7 (a) The 90% confidence interval is (244.32 h, 249.68 h).

(b) The 95% confidence interval is (243.81 h, 250.19 h).

(c) The 98% confidence interval is (243.21 h, 250.79 h).

(d) The 99% confidence interval is (242.81 h, 251.19 h).

8 (a) The 90% confidence interval is (105.17 km/h, 109.43 km/h).

(b) The 95% confidence interval is (104.77 km/h, 109.83 km/h).

(c) The 98% confidence interval is (104.29 km/h, 110.31 km/h).

(d) The 99% confidence interval is (103.97 km/h, 110.63 km/h).

9 (a) The margin of error for a 90% confidence interval is 1.993.

(b) The margin of error for a 95% confidence interval is 2.374.

(c) The margin of error for a 98% confidence interval is 2.818.

(d) The margin of error for a 99% confidence interval is 3.120.

10 (a) $\text{sd}\left(\overline{X}\right) = 0.36\%$

(b) Assuming a random sample, since $n \geq 30$ normality conditions are met. You can apply a normal approximation and construct a confidence interval.

(c) Margin of error: $E = 0.83\%$

(d) Confidence interval: (57.17%, 58.83%)

11 The required confidence interval is 80.22%.

12 37 or 38 more dives needed.

13 No, the machine does not need to be adjusted. A 99% confidence interval based on a mean of 944.58 mL and a standard deviation of 4.4 mL is 944.58 ± 2.27 or (942.31, 946.85). Both of these limits fall within the required limits of the machine.

14 (a) To cut the margin of error in half would require a new margin of error of 0.415%. Doubling 500 to 1000 gives 0.59%, not the 0.415% required. Therefore, doubling the sample size will not cut the interval width in half.

(b) A sample size of approximately 2001 would be needed to cut the width of the confidence interval in half.

CHAPTER REVIEW 9

1 (a) Sample means, sample proportions and other statistics calculated from single samples are random variables belonging to normal distributions under certain circumstances.

Hence, a statistic obtained from a simple random sample is a random variable and the statement is true.

(b) $\overline{x}$ is the point estimate (mean of a single sample), so the statement is true.

(c) The point estimate (a statistic) will vary from sample to sample. Each point estimate gives an approximation of the population mean μ (the parameter), which is fixed.

Hence, a parameter will not vary from sample to sample, so the statement is false.

(d) s is the sample standard deviation. $\frac{s}{\sqrt{n}}$ is the standard error (standard deviation of the sample means when σ is unknown).

Hence, the standard deviation of a sample means is not the same calculation as the standard deviation for a sample, so the statement is false.

2 (a) $S\left(\overline{X}\right) = 0.442$ (b) $S\left(\overline{X}\right) = 7.482$ (c) $S\left(\overline{X}\right) = 0.001$

3 (a) $\overline{x} = 384.50$, $s = 125.581$ (b) $\overline{x} = 2.16$, $s = 2.972$

4 The sample distribution is normal as the sample size is ≥ 30.

5 $S\left(\overline{X}\right) = 0.038$

6 98% confidence interval: ($130.95, $153.05)

7 $E \approx 0.014\%$

8 B

9 (71%, 79%)

10 $\overline{x} = 7.9$ years, hence mean age is 8.4 years; $S\left(\overline{X}\right) = 1.11$ years

11 (a) 90% confidence interval: (2.206 h, 2.794 h)

(b) 95% confidence interval: (2.150 h, 2.850 h)

(c) 98% confidence interval: (2.085 h, 2.915 h)

(d) 99% confidence interval: (2.040 h, 2.960 h)

12 $n = 9$

13 (a) 99% confidence interval: (86.34, 100.14), $E = 6.90$

(b) 99% confidence interval: (92.15, 94.33), $E = 1.09$

(c) 99% confidence interval: (92.89, 93.59), $E = 0.35$

14 (a) 95% confidence interval: (6.41 h, 7.79 h)

(b) You can be 95% confident that the average number of hours worked by all students is between 6.41 h and 7.79 h.

(c) A sample is used to calculate the mean and this value is used to estimate the true population mean. By calculating a confidence interval of 95%, you can also calculate a margin of error that will show how accurate you believe the figures are. A 100% confidence interval would require a census rather than a sample, which is not very practical.

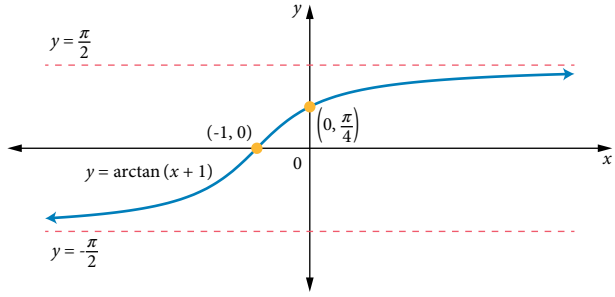
(d) Decrease the confidence level or increase the sample size.

15 The sample size must be increased to at least 68 people to reduce the margin of error to less than $E = 2\%$.

Exam review: Unit 4

PAPER 1: TECHNOLOGY-FREE

1 domain: $\mathbb{R}$, range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



2 (a) $\frac{5}{1 + 25x^2}$

(b) $\arccos\left(\frac{1}{x}\right) + \frac{-1}{\sqrt{x^2 - 1}}$

3 **(a)** $\arcsin\left(\frac{3x}{\sqrt{2}}\right)+c$ or $\arcsin\left(\frac{3\sqrt{2}x}{2}\right)+c$

(b) $\frac{1}{\sqrt{10}}\arctan\left(\frac{x}{\sqrt{10}}\right)+c$ or $\frac{\sqrt{10}}{10}\arctan\left(\frac{\sqrt{10}x}{10}\right)+c$

4 **(a)** $\frac{1}{3\sqrt{5}}\arctan\left(\frac{x^3+4}{\sqrt{5}}\right)+c$ **(b)** $\frac{1}{3}\left(8^{\frac{3}{2}}-1\right)$

5 Using Simpson's rule,

$$\int_1^2 f(x)\,dx \approx \frac{0.25}{3}\sum f(x)\times M$$
$$= 0.386 \text{ (3 d.p.)}$$

Using technology, $\int_1^2 \ln(x)\,dx = 0.38629$ (5 d.p.). Therefore, the approximation obtained is less than 0.001 from this value, and so it is reasonable.

6 **(a)**
$$y = \sin^{-1}(x)+7$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$
$$\sqrt{1-x^2}\,\frac{dy}{dx} = 1$$

$y = \sin^{-1}(x)+7$ is a solution

(b)
$$\frac{dy}{dx} = mbx^{m-1}$$
$$x\frac{dy}{dx} = xmbx^{m-1}$$
$$= mbx^m$$
$$= my$$

$y = bx^m$ is a solution

(c)
$$2y\frac{dy}{dx} - 8x = 0$$
$$\frac{dy}{dx} = \frac{8x}{2y}$$
$$= \frac{4x}{y}$$

$y^2 - 4x^2 = 3$ is a solution

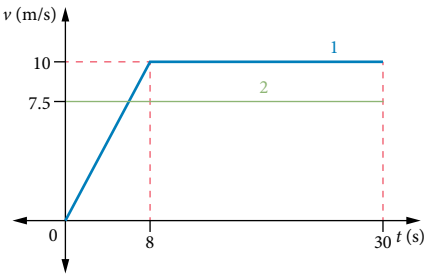
(d)
$$\frac{dy}{dx} = \frac{1}{\tan(x)}\times\sec^2(x)$$
$$= \frac{\cos(x)}{\sin(x)\cos^2(x)}$$
$$= \frac{2}{2\sin(x)\cos(x)}$$
$$= 2\operatorname{cosec}(2x)$$

$y = \ln\left|\tan(x)\right|$ is a solution.

7 **(a)** $y = Ae^{-t}$ **(b)** $y = \frac{x}{2}+\frac{1}{4}\sin(2x)+5$

(c) $y = \tan\left(\frac{1}{x}+c\right)$ **(d)** $y = \log_e\left(\frac{x^2}{2}+e\right)$

8 **(a)**



(b) The first cyclist is ahead.

9 **(a)** $\ddot{x} = -\frac{18}{\pi^2}\sin\left(\frac{3t}{\pi}\right)$; $\ddot{x} = -\frac{9}{\pi^2}x$

(b) $\frac{2}{3}\pi^2$

10 Let $u = \arctan\left(\frac{x}{a}\right)$ and $v' = 1$.

Then $u' = \frac{a}{a^2+x^2}$ and $v = x$.

$$\int \arctan\left(\frac{x}{a}\right)dx = \int uv'\,dx$$
$$= uv - \int u'v\,dx$$
$$= x\arctan\left(\frac{x}{a}\right) - \int \frac{ax}{a^2+x^2}\,dx$$
$$= x\arctan\left(\frac{x}{a}\right) - a\int \frac{x}{a^2+x^2}\,dx$$

For $\int \frac{x}{a^2+x^2}\,dx$, let $u = a^2+x^2$. Then $\frac{du}{dx} = 2x$.

$$\int \frac{x}{a^2-x^2}\,dx = \int \frac{1}{u}\times\frac{1}{2}\frac{du}{dx}\,dx$$
$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| + c$$
$$= \frac{1}{2}\ln\left|a^2+x^2\right| + c$$
$$= \frac{1}{2}\ln\left(a^2+x^2\right) + c$$

$$\int \arctan\left(\frac{x}{a}\right)dx = x\arctan\left(\frac{x}{a}\right) - a\int \frac{x}{a^2+x^2}\,dx$$
$$= x\arctan\left(\frac{x}{a}\right) - a\left(\frac{1}{2}\ln\left(a^2+x^2\right)+c\right)$$
$$= x\arctan\left(\frac{x}{a}\right) - \frac{a}{2}\ln\left(a^2+x^2\right) + c_1 \text{ where } c_1 = -ac$$

$$\int_0^a \arctan\left(\frac{x}{a}\right)dx = \left[x\arctan\left(\frac{x}{a}\right) - \frac{a}{2}\ln\left(a^2+x^2\right)\right]_0^a$$
$$= a\arctan(1) - \frac{a}{2}\ln\left(2a^2\right) - \left(0\arctan(0) - \frac{a}{2}\ln\left(a^2\right)\right)$$
$$= \frac{\pi}{4}a - \frac{a}{2}\left(\ln(2)+\ln\left(a^2\right)\right) + \frac{a}{2}\ln\left(a^2\right)$$
$$= \frac{a}{2}\left(\frac{\pi}{2} - \ln(2)\right)$$

11 $-\ln\left(\frac{\sqrt{2}}{2}\right)$ units^2

12 (a) $P(X > 3) = e^{-\frac{3}{2}}$

(b) The manufacturer can truthfully claim that 90% of batteries will last beyond $d = 76$ days.

13 $\frac{d^2y}{dx^2} = \frac{24xy^2 - 18x^4}{8y^3}$

14 (a) $-\frac{40}{27}$ m/s^2 (b) $-\frac{5}{8}$ N (c) $\frac{10}{3}$ m

15 $\frac{1}{2}\cos^8(x) - \frac{2}{3}\cos^6(x) + c$

16 -7

17 $v = \pm 2\sqrt{x^2 + x}$

PAPER 2: TECHNOLOGY-ACTIVE

1 $\frac{127\pi}{7}$ units^3

2 (a) $\frac{dy}{dx} = -\frac{2\cos(2x) + y^3}{3xy^2}$ (b) $\frac{dy}{dx} = 1$

(c) $\frac{dy}{dx} = -\frac{3y}{2}$

3 11.3 km/h; 8.7 km/h at angle 63.4° east of north

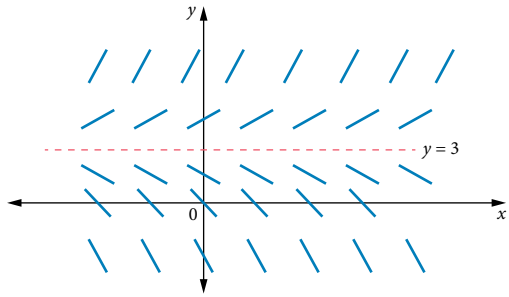
4 $S(\overline{X}) = 2.83$

5 (a) $E = 0.32$, $\overline{x} = 2.10$ (b) $E = 1.77$, $\overline{x} = 12.35$

6 1.74 units^3

7 The volume is decreasing at 135.2 m^3/min.

8



9 (a) 0.405 (b) 1.47 m/s^2

10 $E(\overline{X}) = 250$ g, $\text{sd}(\overline{X}) = 1.03$ g

11 Sample size of 5201 is needed.

12 $\frac{dQ}{dt} = 24 - \frac{3Q}{40}$

Exam review: Units 3 & 4

PAPER 1: TECHNOLOGY-FREE

1 This statement can be rewritten as $n(n^2 + 5) = 6A$ where $A \in \mathbb{Z}$.

When $n = 5$:

LHS $= 5(5^2 + 5)$

$= 150$

$= 6 \times 25$

$= \text{RHS}$

So the statement is true for $n = 5$.

When $n = 2$:

LHS $= 2(2^2 + 5)$

$= 18$

$= 3 \times 6$

$= \text{RHS}$

So the statement is true for $n = 2$.

When $n = 1$:

LHS $= 1(1^2 + 5)$

$= 6$

$= 6 \times 1$

$= \text{RHS}$

So the statement is true for $n = 1$.

2 (a) $3\hat{i} + \hat{j} + 2\hat{k}$ (b) $17\hat{i} + 7\hat{j} - 10\hat{k}$

3 $|-3a + 4b| = \sqrt{806}$

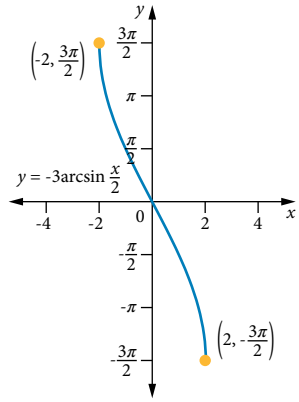
4 $\overrightarrow{BA} = -5\hat{i} - 5\hat{j} - 5\hat{k}$

5 (a) $11\hat{i} - 9\hat{j} - 5\hat{k}$ (b) $\hat{i} + 2\hat{k}$

6 $x = (y + 4)^2$, $y \geq -3$

7 $\frac{2\sqrt{2}}{3}$

8 domain: $[-2, 2]$, range: $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$



9 (a) $\arccos\left(\frac{x}{\sqrt{5}}\right) + c$ or $-\arcsin\left(\frac{x}{\sqrt{5}}\right) + c$

(b) $\frac{\sqrt{5}}{5\sqrt{2}}\arctan\left(\frac{\sqrt{5}x}{\sqrt{2}}\right) + c$

10 (a) $\ln|\sin(x)| + c$ (b) $2e^{\sqrt{x}} + c$

11 (a) $2\arcsin\left(\frac{\sqrt{x} + 1}{2}\right) + c$ (b) $\frac{e^9 - 1}{2}$

(c) $\sqrt{3} - 2$

12 (a) $\sin(x) + \frac{1}{2}x - \frac{1}{8}\sin(4x) + c$

(b) $3x - \frac{1}{3}\tan(3x) + c$

13 $\dfrac{3x-1}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3}$

$= \dfrac{A(x+3)+B(x-2)}{(x-2)(x+3)}$

$\therefore 3x-1 = A(x+3)+B(x-2)$

Setting $x = 2$:

$5 = 5A$

$A = 1$

Setting $x = -3$:

$-10 = -5B$

$B = 2$

$\therefore \dfrac{3x-1}{(x-2)(x+3)} = \dfrac{1}{x-2} + \dfrac{2}{x+3}$

$\displaystyle\int_3^4 \frac{3x-1}{(x-2)(x+3)}\,dx = \int_3^4 \frac{1}{x-2} + \frac{2}{x+3}\,dx$

$= \left[\ln|x-2| + 2\ln|x+3|\right]_3^4$

$= \left[\ln\left|(x-2)(x+3)^2\right|\right]_3^4$

$= \ln(98) - \ln(36)$

$= \ln\left(\dfrac{49}{18}\right)$

14 $\dfrac{x+5}{(x+4)^2} = \dfrac{A}{x+4} + \dfrac{B}{(x+4)^2}$

$= \dfrac{A(x+4)+B}{(x+4)^2}$

$\therefore x+5 = A(x+4)+B$

Setting $x = -4$:

$1 = B$

Setting $x = -3$:

$2 = A + B$

$2 = 1 + B$

$B = 1$

$\therefore \dfrac{x+5}{(x+4)^2} = \dfrac{1}{x+4} + \dfrac{1}{(x+4)^2}$

$\displaystyle\int_0^1 \frac{x+5}{(x+4)^2}\,dx = \int_0^1 \frac{1}{x+4} + \frac{1}{(x+4)^2}\,dx$

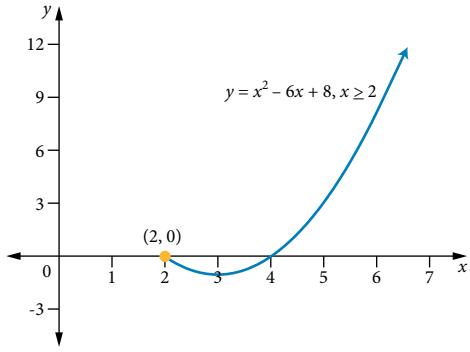
$= \left[\ln|x+4| - \dfrac{1}{x+4}\right]_0^1$

$= \ln(5) - \dfrac{1}{5} - \left(\ln(4) - \dfrac{1}{4}\right)$

$= \ln(5) - \ln(4) + \dfrac{1}{20}$

$= \ln\left(\dfrac{5}{4}\right) + \dfrac{1}{20}$

15 $y = x^2 - 6x + 8, x \geq 2$



16 $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = m\cos(t)\hat{\boldsymbol{i}} - m\sin(t)\hat{\boldsymbol{j}} + 2nt\hat{\boldsymbol{k}}$

17 (a) $\text{cis}(7\theta)$ (b) $-22 + 19i$

18 $4\text{cis}(0), 4\text{cis}\left(\dfrac{2\pi}{3}\right), 4\text{cis}\left(-\dfrac{2\pi}{3}\right)$

19 (a) $\left(z - 3\sqrt{3}\,i\right)\left(z + 3\sqrt{3}\,i\right)$ (b) $(z+8-6i)(z+8+6i)$

(c) $(x+1)(x+4)(x-5)$ (d) $(z-4i)(z+4i)(z-2)$

20 $1-2i,\ \dfrac{1}{2} + \left(\dfrac{\sqrt{3}}{2} - 2\right)i,\ \dfrac{1}{2} - \left(\dfrac{\sqrt{3}}{2} + 2\right)i$

21 $\left\{z : \text{Arg}(z) \leq \dfrac{\pi}{3}\right\} \cap \{z : \text{Re}(z) + \text{Im}(z) = 1\}$

22 $x^2 + y^2 = 2^n$

23 $\dfrac{7}{2}$

24 (a) $y = 2e^{-x}$ is a solution. (b) $-\dfrac{y^2 + x^2}{y^3}$

25 (a) $-2e^{-2t}; 4e^{-2t}$ (b) $-\dfrac{1}{2\sqrt{3-t}}; -\dfrac{1}{4(3-t)^{\frac{3}{2}}}$

(c) $4\pi\cos(\pi t) - 3\pi\sin(\pi t); 4\pi^2\sin(\pi t) - 3\pi^2\cos(\pi t)$

26 (a) -33 (b) $\dfrac{1}{5}$

27 The statement given can be rewritten as: $3^{2n-1} + 2^{n+1} = 7A$ where $A \in \mathbb{Z}$.

When $n = 1$:

LHS $= 3^{2\times1-1} + 2^{1+1}$

$= 7$

$= 7 \times 1$

$=$ RHS

So the statement is true for $n = 1$.

Let $k \geq 1$ and assume statement is true for $n = k$. That is, assume

$3^{2k-1} + 2^{k+1} = 7A$ where $A \in \mathbb{Z}$ [1]

Must demonstrate statement is true when $n = k+1$. That is, must demonstrate

$3^{2(k+1)-1} + 2^{k+1+1} = 7B$ where $B \in \mathbb{Z}$:

$$\begin{aligned}\text{LHS} &= 3^{2(k+1)-1} + 2^{k+1+1}\\ &= 3^{2k+1} + 2^{k+2}\\ &= 3^2 \times 3^{2k-1} + 2^{k+2}\\ &= 9\left(7A - 2^{k+1}\right) + 2^{k+2} \text{ using [1]}\\ &= 63A - 9 \times 2^{k+1} + 2 \times 2^{k+1}\\ &= 63A - 7 \times 2^{k+1}\\ &= 7\left(9A - 2^{k+1}\right)\\ &= \text{RHS (since } A, k \in \mathbb{Z} \Rightarrow 9A - 2^{k+1} \in \mathbb{Z})\end{aligned}$$

Hence, by induction, the original statement must be true for all integers $n \geq 1$.

28 $x = 3$ and $y = -1$

29 $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

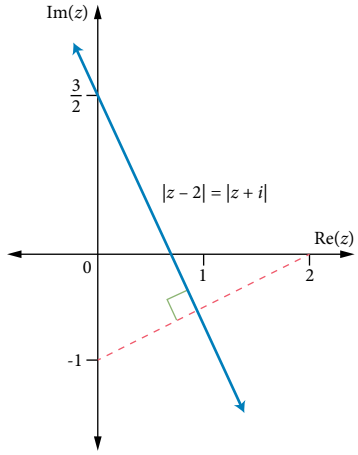
30 Let X be exponentially distributed with rate λ. The probability density function is therefore $f(t) = \lambda e^{-\lambda t}$.

Let $t_{0.25} = x$. Then

$$\begin{aligned}P(X > x) &= 0.25\\ P(X < x) &= 0.75\\ \int_0^x \lambda e^{-\lambda t}\, dt &= 0.75\\ \left[-e^{-\lambda t}\right]_0^x &= 0.75\\ -e^{-\lambda x} - -e^0 &= 0.75\\ -e^{-\lambda x} + 1 &= 0.75\\ e^{-\lambda x} &= 0.25\\ -\lambda x &= \ln\left(\frac{1}{4}\right)\\ -\lambda x &= \ln\left(2^{-2}\right)\\ -\lambda x &= -2\ln(2)\\ x &= \frac{2\ln(2)}{\lambda}\end{aligned}$$

$\therefore\ t_{0.25} = \frac{2\ln(2)}{\lambda}$

31



A straight line with gradient -2 and imaginary axis intercept $\frac{3}{2}i$, which is the perpendicular bisector or the line segment joining the points $2 + 0i$ and $0 - i$.

32 $y + \frac{\pi}{3} = 2\sqrt{3}(x - 3)$

33 (a) $y = Ae^{x^2} + 11$ (b) $y = \pm 3e^{\cos(x)}$

(c) $y = \frac{3\left(Ae^{6x} - 1\right)}{Ae^{6x} + 1}$

34 Let $u = \sin^{n-1}(x)$ and $v' = \sin(x)$.

Then $u' = (n-1)\sin^{n-2}(x)\cos(x)$ and $v = -\cos(x)$.

$$\begin{aligned}\int \sin^n(x)\, dx &= \int uv'\, dx\\ &= -\cos(x)\sin^{n-1}(x) - \int -(n-1)\cos^2(x)\sin^{n-2}(x)\, dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \left(1 - \sin^2(x)\right)\sin^{n-2}(x)\, dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x) - \sin^n(x)\, dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\, dx\\ &\quad -(n-1)\int \sin^n(x)\, dx\end{aligned}$$

$$\begin{aligned}\int \sin^n(x)\, dx + (n-1)\int \sin^n(x)\, dx &= -\cos(x)\sin^{n-1}(x)\\ &\quad + (n-1)\int \sin^{n-2}(x)\, dx\\ n\int \sin^n(x)\, dx &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\, dx\\ \int \sin^n(x)\, dx &= -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{(n-1)}{n}\int \sin^{n-2}(x)\, dx\end{aligned}$$

35 $x^2\ln(x+1) - \frac{1}{2}(x+1)^2 + 2(x+1) - \ln|x+1| + c$

36 $a = 3$

PAPER 2: TECHNOLOGY-ACTIVE

1 Using Simpson's rule,

$$\begin{aligned}\int_0^1 f(x)\, dx &\approx \frac{0.25}{3}\sum f(x) \times M\\ &= 0.585 \text{ (3 d.p.)}\end{aligned}$$

The actual value of this definite integral, using technology, is equal to 0.57080. Thus, the result of 0.585 is within 0.02 of this and is therefore reasonable.

2 23.678

3 (a) $\mu = 20$ seconds, $\lambda = 0.05$ mistakes per second, $f(t) = 0.05e^{-0.05t}$

(b) 0.050

4 $\frac{dy}{dt} = 0.31$ m/s

5 117.2

6 98% confidence interval: ($60.64, $69.36)

7 (a) $-\frac{4}{11}\left(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}\right)$ (b) $-\frac{4}{30}\left(-5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)$

8 $\frac{d\boldsymbol{r}}{dt} = -10t\hat{\boldsymbol{i}} + 50\hat{\boldsymbol{j}} - \frac{1}{2\sqrt{t}}\hat{\boldsymbol{k}}$ and $\frac{d^2\boldsymbol{r}}{dt^2} = -10\hat{\boldsymbol{i}} - \frac{1}{4t^{\frac{3}{2}}}\hat{\boldsymbol{k}}$

9 $x + 5y + z = 4$

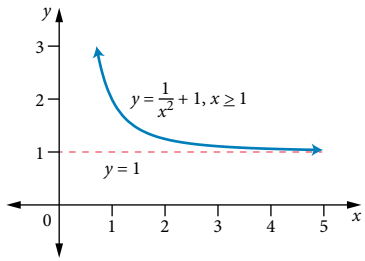
10 $\left(\frac{21}{4}, \frac{5}{8}\right)$

11 $x = -2$, $y = 1$ and $z = 4$.

12 $\theta = 70°$

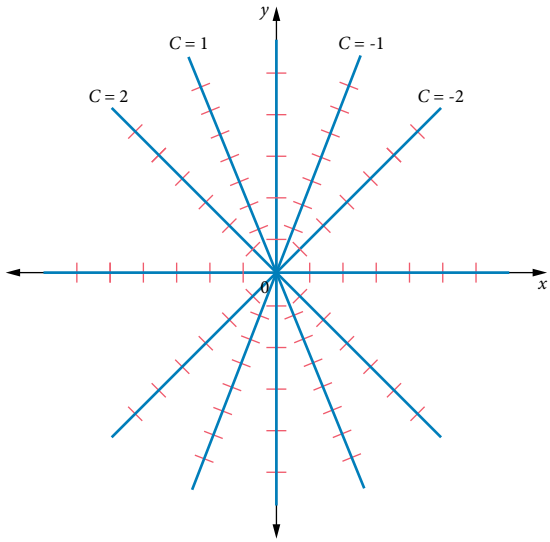
13 $\boldsymbol{r}(t) = \left(-2\sin(2t) + 8t\right)\hat{\boldsymbol{i}} + -4t\,\hat{\boldsymbol{j}} + -2\cos(2t)\hat{\boldsymbol{k}}$

14 The Cartesian equation is $y = \frac{1}{x^2} + 1,\ x \geq 1$.



15 **(a)** $\frac{dy}{dx} = -2$ **(b)** $y = -2x + c$

(c)



(d) An ellipse, centre $(0, 0)$

16 **(a)** 10.5 kg m/s **(b)** 2.5 m

17 $a = \frac{g}{3};\ T = \frac{4mg}{3}$

18 $n = 9$

19 A vector perpendicular to the plane $4x + y - 2z = 4$ will be $n\left(4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right)$, where $n \in \mathbb{R}$.

20 A vector perpendicular to the plane $2x + y + 3z = 18$ will be $n\left(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\right)$, where $n \in \mathbb{R}$.

21 **(a)** $\dot{\boldsymbol{r}}(t) = 4\hat{\boldsymbol{i}} + 6t^2\hat{\boldsymbol{j}} + e^t\hat{\boldsymbol{k}}$ **(b)** Initial speed is $\sqrt{17}$

(c) $\boldsymbol{a}_t(1) = \frac{72 + e^2}{52 + e^2}\left(4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} + e\hat{\boldsymbol{k}}\right)$

22 $6283\,\text{cm}^3$

23 **(a)** $\frac{10}{3}\hat{\boldsymbol{i}} + \frac{16}{3}\hat{\boldsymbol{j}} + \frac{11}{3}\hat{\boldsymbol{k}}$ **(b)** 7.28 km

24 **(a)** $\frac{dm}{dt} = \frac{16000 - m}{40}$ **(b)** $m = 16000 + 14000\,e^{-\frac{1}{40}t}$

(c) 16 tonnes

25 **(a)** $v(x) = -10x + 2$

(b) $v = 2e^{-10t};\ x = 0.2\left(1 - e^{-10t}\right)$