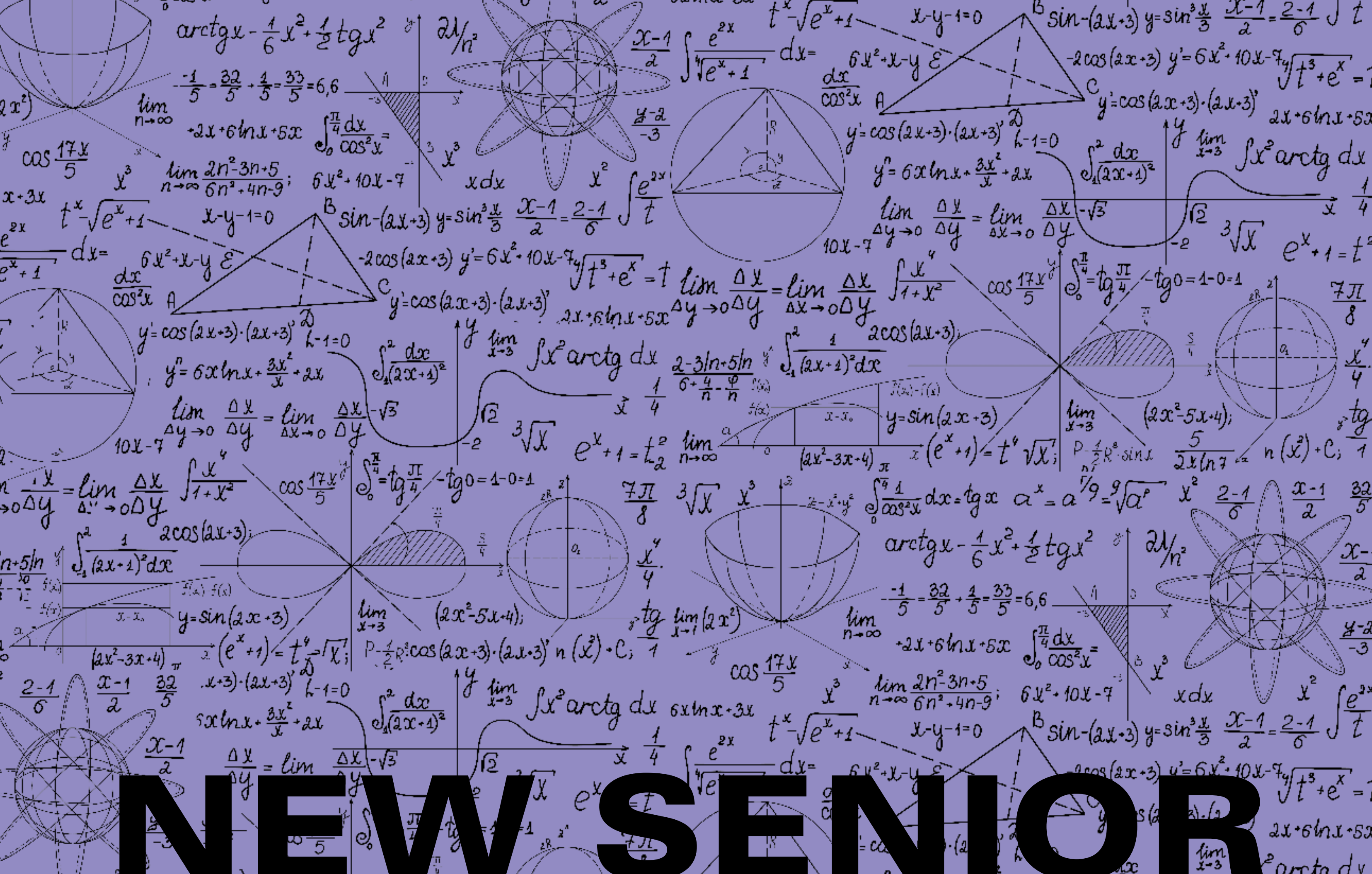

NEW SENIOR MATHEMATICS

EXTENSION 2

FOR YEAR 12

THIRD EDITION

J.B. FITZPATRICK
BOB AUS

NSW
STAGE 6

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INTRODUCTION AND DEDICATION

J.B. Fitzpatrick

It is interesting to wonder whether J. B. Fitzpatrick ('Bernie') realised in 1983 just how popular his book *New Senior Mathematics* would be. That first edition of *New Senior Mathematics* was to remain in print for almost 30 years. It has stood the test of time thanks to the quality, rigour and variety of its questions, its accuracy and its high mathematical standards.

As Fitzpatrick wrote in 1983: 'Mathematics, like many other things, is best learnt by doing. A student begins to appreciate the power of mathematics when he or she has achieved a mastery of basic techniques, not after reading lengthy explanations … The emphasis throughout the book is on the understanding of mathematical concepts' (Introduction, *New Senior Mathematics* 1984).

J. B. Fitzpatrick passed away in 2008. Fitzpatrick was a respected author, teacher and figurehead of mathematics education.

Bob Aus

Bob Aus taught in New South Wales high schools for 40 years, retiring in 2007. During that time Bob taught all courses from Years 7 to 12 up to Level 1 / 4-unit / Extension 2. He has marked HSC examination papers and has been involved in the standards setting process as judge and chief judge for the three Calculus-based courses over four years. He has also completed review work for the NSW Board of Studies and represented NSW at a week-long review and standards setting of the upper-level course from each state prior to the development of the Australian National Curriculum for senior students.

Bob spent time as Regional Vocational Education Consultant in the North Coast region and was a Mathematics consultant in the Hunter region. When he retired he was Head Teacher Mathematics at Merewether High School and enjoyed teaching an Extension 2 class with 24 students.

Bob's first publication was in 1983 and he has been involved with writing a range of textbooks and study guides since then, including revising and updating the *New Senior Mathematics* series 2nd edition in 2013.

Bob has presented talks on the three Calculus-based courses throughout the state. He has co-written the Years 6–9 Mathematics syllabus for the Abu Dhabi Education Authority, as well as managing the writing project for support material for this course. He also wrote the Years 10–12 syllabus for their Calculus-based course.

This updated third edition of *New Senior Mathematics* applies to the new Stage 6 HSC courses in NSW to be implemented in Year 11, 2019.

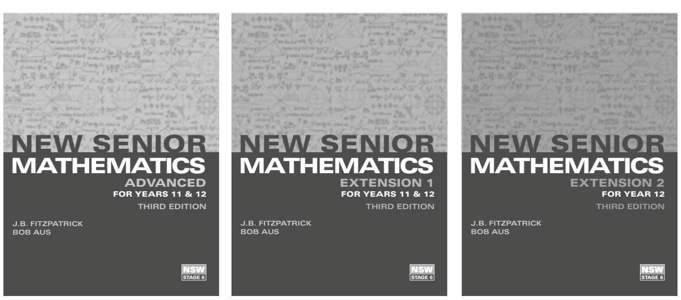
NEWSENIOR MATHEMATICS
THIRD EDITION

New Senior Mathematics Extension 2 for Year 12 is part of a new edition of the well-known Mathematics series for New South Wales. The series has been updated to address all requirements of the new Stage 6 syllabus. We have maintained our focus on mathematical rigour and challenging student questions, while providing new opportunities for students to consolidate their understanding of concepts and ideas with the aid of digital resources and activities.

Student Book

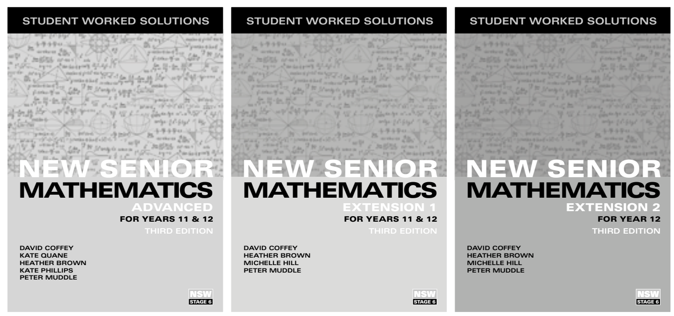
The first three chapters of the first student book contain revision material that provides the necessary foundation for the development of senior mathematics concepts. In the new edition you'll also find:

- content built on a rigorous, academic approach that promotes excellence and prepares students for higher education
- a simple, convenient approach with Year 11 and 12 content in one book for Advanced and Extension 1, and Year 12 content for Extension 2, with colour coding to distinguish year levels
- digital technology activities that promote a deeper understanding, allowing students to make connections, and visualise and manipulate data in real time.



Student Worked Solutions

The *New Senior Mathematics Extension 2 for Year 12 Student Worked Solutions* contains the full worked solutions for every second question in *New Senior Mathematics Extension 2 for Year 12*.



Reader+

Reader+, our next generation eBook, features content and digital activities, with technology such as graphing software and spreadsheets, to help students engage on their devices.

There are also teacher support materials, such as practice exams, question banks, investigation assignments, and fully worked solutions to cover all internal and external assessment items and save you time.

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FEATURES OF THE 3RD EDITION STUDENT BOOK/READER+

YEAR LEVELS

Year levels are indicated on each page for easy identification of Year 11 and 12 content.

YEAR 11

YEAR 12

MAKING CONNECTIONS

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their conceptual understanding.

MAKING CONNECTIONS

EXPLORING FURTHER

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skill base. These activities should take approximately 5–15 minutes to complete.

EXPLORING FURTHER

CHAPTER REVIEW

Each chapter contains a comprehensive review of chapter content.

CHAPTER REVIEW

SUMMARY PAGES

A comprehensive course summary is provided at the end of the book.

SUMMARY

CONTENTS

CHAPTER 1
Complex numbers

1.1 ARITHMETIC OF COMPLEX NUMBERS AND THE SOLUTION OF QUADRATIC EQUATIONS

The need for complex numbers

As society has developed over time, so has our need for a more comprehensive number system. Initially, the only numbers needed were the counting numbers (1, 2, 3, …). Later, people found a need for zero and for negative numbers, giving us the set of integers. Fractions and decimals gave us the set of rational numbers. With numbers such as $\sqrt{2}$ and π, the set of irrational numbers was developed. The rationals and the irrationals together form the set of real numbers.

These number systems have been developed by mathematicians to address new and different problems that have emerged.

For example, to solve different kinds of equations requires different kinds of numbers. Using only integers, you can solve equations such as $x + 5 = 2$ but you can't solve $5x = 2$. You need rational numbers for that. To solve $x^2 = 5$ you need irrational numbers.

There are other equations which can't be solved using any real numbers. The simplest example is $x^2 = -1$ or $x^2 + 1 = 0$. However, this equation can be solved by defining a number i such that $i^2 = -1$:

$$x^2 + 1 = 0$$

i.e. $x^2 - i^2 = 0$ where $i^2 = -1$

$(x - i)(x + i) = 0$ (difference of two squares)

$\therefore$ $x = i$ or $x = -i$

Example 1

Solve the quadratic equation $x^2 - 4x + 13 = 0$.

Solution

Note that the discriminant $\Delta = b^2 - 4ac = 16 - 52 = -36$. Hence the quadratic equation has no real roots and the parabola $y = x^2 - 4x + 13$ is entirely above the x-axis.

However, you can find solutions using **complex numbers**, which are of the form $a + bi$ where a and b are real numbers.

Method 1 Using the quadratic formula

$$x^2 - 4x + 13 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{4 \pm \sqrt{-36}}{2}$$
$$x = 2 \pm 3\sqrt{-1}$$
$$x = 2 \pm 3i$$

Method 2 Completing the square

$$x^2 - 4x + 13 = 0$$
$$x^2 - 4x + 4 + 9 = 0$$
$$(x - 2)^2 = -9$$
$$x - 2 = \pm 3i$$
$$x = 2 \pm 3i$$

The complex number system

Any number z of the form $x + iy$, where x and y are real numbers, is called a complex number.

x is the **real part** of z, denoted by $\text{Re}(z) = x$, and y is called the **imaginary part** of z (although it is not literally 'imaginary' in the usual sense of that word), denoted by $\text{Im}(z) = y$.

(Note that you use a single letter z to denote the complex number $x + iy$, to emphasise that $x + iy$ is a single number despite being written as a sum of two parts.)

If the imaginary part of z is zero, i.e. $y = 0$, then z is purely real. This means that the set of real numbers is a subset of the set of complex numbers.

If the real part of z is zero, i.e. $x = 0$, then z is purely imaginary, e.g. $3i$ or $-i$.

The following definitions apply to complex numbers.

Equality

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal:

$a + bi = c + di \quad$ if and only if $\quad a = c \quad$ and $\quad b = d$

The 'if and only if' in the above statement means that the statement applies forwards and backwards, in other words it is two statements in one.

If $a + bi = c + di$, then $a = c$ and $b = d$.

If $a = c$ and $b = d$, then $a + bi = c + di$.

Addition and subtraction

If $z = z_1 \pm z_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z = (x_1 + x_2) \pm i(y_1 + y_2)$.

Multiplication

If $z = z_1 \times z_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then:

$$\begin{aligned} z &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i^2y_1y_2 + ix_1y_2 + ix_2y_1 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

The conjugate of a complex number

If $z = x + iy$, then the **conjugate** of z is $\overline{z} = x - iy$. (This is similar to the conjugate of a surd.)

Note that the product of a complex number and its conjugate is a real number:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - i^2y^2 \\ &= x^2 + y^2 \end{aligned}$$

Division

To calculate the division $z = \dfrac{z_1}{z_2}$, multiply the numerator and denominator by the conjugate of z_2. This **realises** the denominator, i.e. makes the denominator real. This is similar to how you rationalise a denominator when dividing surds.

Example 2

If $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, find:

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $z_1 \times z_2$ (d) $z_2\overline{z}_2$ (e) ${z_1}^2$ (f) $z_1 \div z_2$

Solution

(a) $z_1 + z_2 = 2 + 3i + (-1 + 4i) = 1 + 7i$

(b) $z_1 - z_2 = 2 + 3i - (-1 + 4i) = 3 - i$

(c) $z_1 \times z_2 = (2 + 3i)(-1 + 4i) = -2 + 8i - 3i + 12i^2 = -2 + 5i - 12 = -14 + 5i$

(d) $z_2\overline{z}_2 = (-1+4i)(-1-4i) = (-1)^2 - 16i^2 = 1 + 16 = 17$

(e) $z_1^{\ 2} = (2+3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$

(f) $\dfrac{z_1}{z_2} = \dfrac{2+3i}{-1+4i} = \dfrac{(2+3i)}{(-1+4i)} \times \dfrac{(-1-4i)}{(-1-4i)} = \dfrac{-2-8i-3i-12i^2}{1-16i^2} = \dfrac{-2-11i+12}{1+16} = \dfrac{10-11i}{17} = \dfrac{10}{17} - \dfrac{11}{17}i$

Example 3

If $z_1 = 2 - 3i$ and $z_2 = -4 - 5i$, find:

(a) $z_1 + z_2$ (b) $z_2 - z_1$ (c) $z_1 + \overline{z}_1$ (d) $z_2 - \overline{z}_2$ (e) $\overline{z}_1 \times \overline{z}_2$ (f) $\dfrac{\overline{z}_2}{\overline{z}_1}$

Solution

(a) $z_1 + z_2 = 2 - 3i + (-4 - 5i) = -2 - 8i$

(b) $z_2 - z_1 = (-4 - 5i) - (2 - 3i) = -6 - 2i$

(c) $\overline{z}_1 = 2 + 3i$. $z_1 + \overline{z}_1 = 2 - 3i + (2 + 3i) = 4$

(d) $\overline{z}_2 = -4 + 5i$. $z_2 - \overline{z}_2 = -4 - 5i - (-4 + 5i) = -10i$

(e) $\overline{z}_1 \times \overline{z}_2 = (2+3i)(-4+5i) = -8 + 10i - 12i + 15i^2 = -8 - 15 - 2i = -23 - 2i$

(f) $\dfrac{\overline{z}_2}{\overline{z}_1} = \dfrac{-4+5i}{2+3i} = \dfrac{-4+5i}{2+3i} \times \dfrac{2-3i}{2-3i} = \dfrac{-8+12i+10i-15i^2}{4+9} = \dfrac{7+22i}{13} = \dfrac{7}{13} + \dfrac{22}{13}i$

Example 4

Express $z^3 + 64$ as the product of three linear factors. Hence find the three cube roots of -64.

Solution

$$\begin{aligned} z^3 + 64 &= (z+4)(z^2 - 4z + 16) && \text{(sum of two cubes)} \\ &= (z+4)(z^2 - 4z + 4 + 12) && \text{(complete the square)} \\ &= (z+4)((z-2)^2 - 12i^2) && \text{(construct the difference of two squares)} \\ &= (z+4)((z-2)^2 - (2\sqrt{3}\,i)^2) \\ &= (z+4)(z - 2 - 2\sqrt{3}\,i)(z - 2 + 2\sqrt{3}\,i) \end{aligned}$$

The cube roots of -64 are obtained from $z + 4 = 0$, $z - 2 - 2\sqrt{3}\,i = 0$, $z - 2 + 2\sqrt{3}\,i = 0$.

$\therefore$ The cube roots are -4, $2 - 2\sqrt{3}\,i$ and $2 + 2\sqrt{3}\,i$.

Square roots of a complex number

The general method for finding the square roots of a complex number is illustrated in the following example.

Example 5

Find the square roots of $3 + 4i$.

Solution

Let $z = x + iy$, where x, y are real, such that $z^2 = 3 + 4i$:

$$(x + iy)^2 = 3 + 4i$$

$$(x^2 - y^2) + 2xyi = 3 + 4i$$

Equating the real and imaginary parts of LHS and RHS:

$x^2 - y^2 = 3$ [1] $2xy = 4$ [2]

From [2], $y = \frac{2}{x}$, then substitute into [1]:

$$x^2 - \frac{4}{x^2} = 3$$
$$x^4 - 3x^2 - 4 = 0$$
$$(x^2 - 4)(x^2 + 1) = 0$$
$$x^2 = 4 \quad \text{or} \quad x^2 = -1$$

But x is real $\therefore\ x = \pm 2$ are the only solutions.

Substituting this into [2]: $y = \pm 1$

So the square roots of $3 + 4i$ are $2 + i$ and $-2 - i$, which can be written as $\pm(2 + i)$.

EXPLORING FURTHER

Arithmetic of complex numbers

Use technology to explore the arithmetic of complex numbers.

EXERCISE 1.1 ARITHMETIC OF COMPLEX NUMBERS AND THE SOLUTION OF QUADRATIC EQUATIONS

1 $i^5 = \ldots$

A 1 **B** -1 **C** i **D** $-i$

2 Solve the following equations.

(a) $z^2 + 9 = 0$ **(b)** $z^2 + 25 = 0$ **(c)** $z^2 + 2z + 17 = 0$
(d) $-z^2 + 2z - 5 = 0$ **(e)** $z^2 = 4z - 20$ **(f)** $-2z^2 + 2z - 13 = 0$
(g) $z^2 - z + 8 = 0$ **(h)** $z - 4 - z^2 = 0$

3 Simplify:

(a) i^3 **(b)** i^4 **(c)** i^6 **(d)** i^7 **(e)** i^8

4 If $z = 5 - 2i$, find:

(a) z^{-1} **(b)** $\bar{z}$ **(c)** $z\bar{z}$ **(d)** z^2 **(e)** $(z - \bar{z})^2$ **(f)** $\frac{z-1}{z-i}$

5 Simplify:

(a) $(3 + 5i) + (7 - 2i)$ **(b)** $(4 + 7i) - (-2 + 9i)$ **(c)** $(5 + 2i)(3 - 4i)$ **(d)** $(7 + 3i)(7 - 3i)$
(e) $(2 - 5i)^2$ **(f)** i^{17} **(g)** $(\sqrt{3} + 2i)(\sqrt{3} - 2i)$ **(h)** $\frac{1}{2+3i}$
(i) $\frac{8+5i}{4-3i}$ **(j)** $\frac{3i}{2+5i} + \frac{2}{2-5i}$ **(k)** $\frac{-8+3i}{-2-4i} - \frac{2+3i}{1+2i}$ **(l)** $\left(\frac{5+9i}{2-4i}\right)^2$

6 Find real numbers x and y such that:

(a) $(x + iy)(2 - 3i) = -13i$ **(b)** $(1 + i)x + (2 - 3i)y = 10$

7 If $z_1 = 3 + i$ and $z_2 = 2 - 3i$, find:

(a) $(z_1 - z_2)^2$ **(b)** $\overline{z_1} \times \overline{z_2}$ **(c)** $\overline{z_1 z_2}$ **(d)** $\frac{z_1 - z_2}{z_1 + z_2}$

8 Find the linear factors of the following expressions.

(a) $z^2 + 9$ **(b)** $z^2 + 36$ **(c)** $(z - 3)^2 + 16$ **(d)** $(2z + 3)^2 + 8$
(e) $z^2 + 2z + 26$ **(f)** $z^2 - 6z + 20$ **(g)** $2z^2 + 2z + 4$ **(h)** $z^3 + 1000$

9 Solve the equation: **(a)** $2z - 1 = (4 - i)^2$ **(b)** $\frac{z-2}{z} = 1 + i$

10 **(a)** Find the square roots of $-8+6i$.
(b) Hence solve $2z^2+(1-i)z+(1-i)=0$.
(c) Use your answer to part **(b)** to verify that the results for the sum of roots and for the product of roots of a quadratic equation are true when the coefficients and roots are complex numbers.

11 Find the square roots of the following: **(a)** $-8-6i$ **(b)** $-16i$ **(c)** $12+5i$

12 **(a)** Show that $\sqrt{3}-i$ is a root of the equation $z^3-(\sqrt{3}-i)z^2+9z-9\sqrt{3}+9i=0$.
(b) Find the other two solutions of the equation.
(c) Use your answer to part **(b)** to verify that the results for the sum of roots, for the sum of products of pairs of roots and for the product of roots of a cubic equation are true when the coefficients and roots are complex numbers.

13 Solve the following quadratic equations.
(a) $z^2-(3-2i)z+(1-3i)=0$ **(b)** $z^2-z+(4+2i)=0$
(c) $z^2-(2+2i)z+(-1+2i)=0$ **(d)** $z^2-(3+i)z+(2-3i)=0$

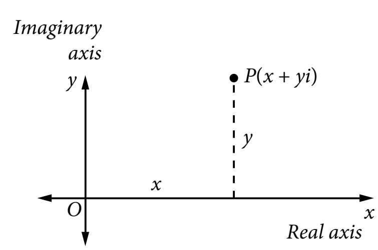
14 Let $z=a+ib$ where a, b are real. Prove that there are always two square roots of z except when $a=b=0$.

15 If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, show that the following equations are true.
(a) $z_1+\overline{z_1}=2\times\text{Re}(z_1)$ **(b)** $z_1-\overline{z_1}=2\times\text{Im}(z_1)\times i$ **(c)** $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$
(d) $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$ **(e)** $\overline{z_1\times z_2}=\overline{z_1}\times\overline{z_2}$

16 **(a)** Express z^3+125 as the product of three linear factors.
Let w be one of the non-real complex roots of the equation $z^3+125=0$.
(b) Show that $w^2=5w-25$. **(c)** Hence simplify $(5w-25)^3$.

1.2 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A POINT

As the complex number $z=x+iy$ is composed of two parts, it can be considered as an ordered pair (x,y), and so complex numbers can be represented as points in a plane. Any complex number $z=x+iy$ can be represented by the point P with coordinates (x,y) in a number plane in which the x-axis is the 'real' axis and the y-axis is the 'imaginary' axis. This Cartesian representation of complex numbers is called the **Argand diagram**, after the French mathematician Jean-Robert Argand (1768–1822). The number plane on which Argand diagrams are mapped is called the **complex number plane**.



Geometrical addition and subtraction of complex numbers

Example 6

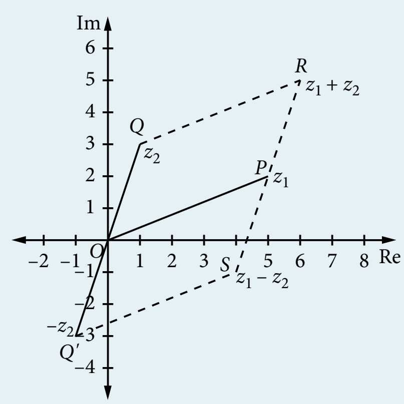
If $z_1=5+2i$ and $z_2=1+3i$, show z_1+z_2 and z_1-z_2 on an Argand diagram.

Solution

Algebraically, $z_1+z_2=6+5i$ and $z_1-z_2=4-i$.

On an Argand diagram, points P, Q and R represent z_1, z_2 and z_1+z_2 respectively. Note the location of R to complete the parallelogram $OPRQ$.

The diagram also shows points Q' and S, representing $-z_2$ and z_1-z_2 respectively. Note that z_1-z_2 has been calculated as $z_1+(-z_2)$. S completes the parallelogram $OPSQ'$.



Geometrical representation of multiplication by *i*

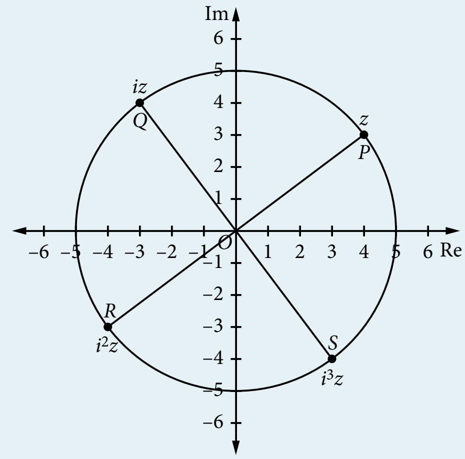
Example 7

If $z = 4 + 3i$, show iz, i^2z and i^3z on an Argand diagram.

Solution

Algebraically: $iz = -3 + 4i$, $i^2z = -4 - 3i$ and $i^3z = 3 - 4i$.

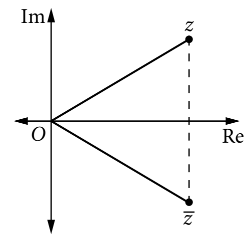
The Argand diagram shows that each multiplication by i causes the point z to be rotated anticlockwise about the origin by $\frac{\pi}{2}$ (90°).



Geometrical representation of conjugates

The points that represent a pair of complex conjugates are reflections in the real axis. (This is because a number and its complex conjugate are the same except that the imaginary part has changed from negative to positive, or vice versa.)

If $z = x + iy$, then $\overline{z} = x - iy$. The sign of the imaginary part has changed, while the sign of the real part has remained the same. On the Argand diagram this means a reflection in the x-axis (real axis).



MAKING CONNECTIONS

Geometric representation of conjugates

Use technology to explore the geometric representation of complex numbers and conjugates.

Modulus–argument form or polar form of a complex number

A point P on an Argand diagram may be located by Cartesian coordinates (i.e. an x-coordinate and y-coordinate, indicating horizontal displacement and vertical displacement respectively from the origin O), or alternatively by its modulus (plural 'moduli') and its argument:

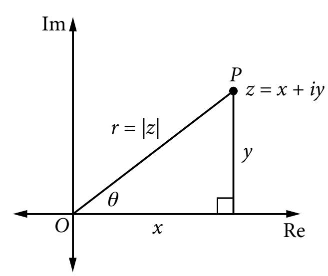
- The **modulus** is the distance from the origin O to P.
- The **argument** is the angle at which the ray OP is inclined to the positive direction of the real axis.

Specifically, you define the modulus of z as $\operatorname{mod} z = |z| = |x + iy| = \sqrt{x^2 + y^2} = r$

From this definition and the diagram, note that $x = r\cos\theta$ and $y = r\sin\theta$.

Therefore, for any complex number: $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

When a complex number is expressed in the form $r(\cos\theta + i\sin\theta)$, it is said to be in **mod–arg form** or **polar form**. The abbreviation $r\operatorname{cis}\theta$ may be used, where $r\operatorname{cis}\theta = r(\cos\theta + i\sin\theta)$.



The argument of $z = x + iy$ is then defined as $\arg z = \theta$ such that $x = r\cos\theta$ and $y = r\sin\theta$

Clearly, an infinite number of values of θ are possible for any complex number z, obtained by adding or subtracting multiples of 2π (or 360°). Therefore:

Define the **principal argument** to be θ such that $-\pi < \theta \le \pi$.

Results should always be given using the principal argument.

Note: $\arg 0$ is undefined.

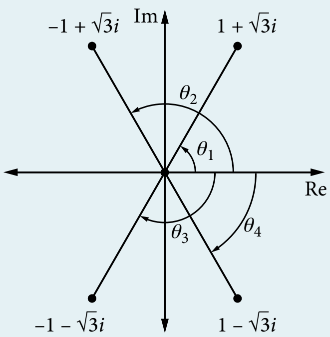
Example 8

Write each of the following in mod–arg form.

(a) $1+\sqrt{3}i$ (b) $-1+\sqrt{3}i$ (c) $-1-\sqrt{3}i$ (d) $1-\sqrt{3}i$

Solution

It is always helpful to show the complex numbers on an Argand diagram.



(a) $|z|=\sqrt{1^2+(\sqrt{3})^2}=2$

$2\cos\theta=1$ and $2\sin\theta=\sqrt{3}$, so θ is a first-quadrant angle.

$\therefore\ \arg z=\theta=\dfrac{\pi}{3}$

$\therefore\ 1+\sqrt{3}i=2\operatorname{cis}\dfrac{\pi}{3}$

(b) $|z|=\sqrt{(-1)^2+(\sqrt{3})^2}=2$

$2\cos\theta=-1$ and $2\sin\theta=\sqrt{3}$, so θ is a second-quadrant angle.

$\therefore\ \arg z=\theta=\dfrac{2\pi}{3}$

$\therefore\ -1+\sqrt{3}i=2\operatorname{cis}\dfrac{2\pi}{3}$

(c) $|z|=\sqrt{(-1)^2+(-\sqrt{3})^2}=2$

$2\cos\theta=-1$ and $2\sin\theta=-\sqrt{3}$, so θ is a third-quadrant angle.

$\therefore\ \arg z=\theta=-\dfrac{2\pi}{3}$ (Note the use of the principal argument.)

$\therefore\ -1-\sqrt{3}i=2\operatorname{cis}\left(-\dfrac{2\pi}{3}\right)$

(d) $|z|=\sqrt{1^2+(-\sqrt{3})^2}=2$

$2\cos\theta=1$ and $2\sin\theta=-\sqrt{3}$, so θ is a fourth-quadrant angle.

$\therefore\ \arg z=\theta=-\dfrac{\pi}{3}$ (Again, note the use of the principal argument.)

$\therefore\ 1-\sqrt{3}i=2\operatorname{cis}\left(-\dfrac{\pi}{3}\right)$

The result $z\times\bar{z}=|z|^2$

This useful result can be proved as follows. Let $z=x+iy$ so that $\bar{z}=x-iy$.

$\therefore$ LHS $=(x+iy)(x-iy)=x^2+y^2=$ RHS

Products in mod–arg form

Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$.

$$
\begin{aligned}
\text{Then } z_1\times z_2&=r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\\
&=r_1r_2(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2+i\cos\theta_1\sin\theta_2)\\
&=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))\\
&=r_1r_2\operatorname{cis}(\theta_1+\theta_2)
\end{aligned}
$$

This is a complex number in mod–arg form with modulus r_1r_2 and argument $(\theta_1 + \theta_2)$.

$\therefore |z_1z_2| = |z_1| \times |z_2|$

Also, note that $\arg z_1 + \arg z_2$ is one value of $\arg(z_1z_2)$, but not necessarily the principal value. (You may have to add or subtract a multiple of 2π to obtain the principal argument.)

$\arg(z_1z_2) = \arg z_1 + \arg z_2$ expressed in terms of the principal values

The modulus of a product is the product of the moduli.
The argument of a product is the sum of the arguments.

Example 9

If $z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $z_2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, find $z_1 \times z_2$ in mod–arg form and in Cartesian form.

Hence find the exact value of $\cos\frac{5\pi}{12}$.

Solution

$$z_1 \times z_2 = 2\sqrt{2}\left(\cos\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right) + i\sin\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)\right)$$

$$= 2\sqrt{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$ (which is in mod–arg form, but not using the principal argument)

$$= 2\sqrt{2}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right)$$ (subtracting 2π to find the principal arg)

To find $z_1 \times z_2$ in Cartesian form:

$$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}i$$

$$z_2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$$

$$\therefore z_1 \times z_2 = (-1 + \sqrt{3}i)(-1 + i) = (1 - \sqrt{3}) + (-1 - \sqrt{3})i$$

So $2\sqrt{2}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right) = (1 - \sqrt{3}) + (-1 - \sqrt{3})i$ in Cartesian form.

Equating the real parts (because you are trying to prove a result involving $\cos\frac{5\pi}{12}$):

$$2\sqrt{2}\cos\frac{-7\pi}{12} = 1 - \sqrt{3} \quad \therefore \cos\frac{-7\pi}{12} = \frac{1 - \sqrt{3}}{2\sqrt{2}}$$

But $\cos\frac{-7\pi}{12} = \cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12}$ (as $\cos x$ is an even function and $\cos(\pi - \theta) = -\cos\theta$)

$$\therefore -\cos\frac{5\pi}{12} = \frac{1 - \sqrt{3}}{2\sqrt{2}} \quad \text{and so} \quad \cos\frac{5\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

EXPLORING FURTHER

Polar form of a complex number

Use technology to explore the polar form of a complex number.

Quotients in mod–arg form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\begin{aligned}
\text{Then } \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \\
&= \frac{r_1\left((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right)}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\
&= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right) \\
&= \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)
\end{aligned}$$

This is a complex number in mod–arg form with modulus $\frac{r_1}{r_2}$ and argument $(\theta_1 - \theta_2)$.

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Also, note that $\arg z_1 - \arg z_2$ is one value of $\arg\frac{z_1}{z_2}$, but not necessarily the principal value. (You may have to add or subtract a multiple of 2π to find the principal argument.)

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2, z_2 \neq 0,$ expressed in terms of the principal values.

It follows from these results that $\left|\frac{1}{z}\right| = \frac{|1|}{|z|} = \frac{1}{|z|}$ and $\arg\left(\frac{1}{z}\right) = \arg 1 - \arg z = -\arg z$.

The modulus of a quotient is the quotient of the moduli.
The argument of a quotient is the difference of the arguments.

Example 10

If $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, find $\frac{z_1}{z_2}$ in mod–arg form.

Solution

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = 2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)$$

$$\therefore \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{4} - \frac{-\pi}{6}\right) = \frac{\sqrt{2}}{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

Two special results

1. If $z = r(\cos\theta + i\sin\theta)$ then the conjugate $\bar{z} = r\,\text{cis}(-\theta)$
2. If $z = r(\cos\theta + i\sin\theta)$ then $\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$

This second result can be proved as follows:

Method 1

$$\frac{1}{z} = \frac{1\,\text{cis}\,0}{r\,\text{cis}\,\theta} = \frac{1}{r}\text{cis}(0 - \theta) = \frac{1}{r}\text{cis}(-\theta)$$

Method 2

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{r\,\text{cis}(-\theta)}{r^2} = \frac{1}{r}\text{cis}(-\theta)$$

Powers using mod–arg form

De Moivre's theorem (named for the French mathematician Abraham de Moivre (1667–1754)) states:

> If $z = r(\cos\theta + i\sin\theta)$ and n is an integer,
> then $z^n = r^n(\cos n\theta + i\sin n\theta)$
> $\left|z^n\right| = |z|^n$ and $\arg z^n = n\arg z$

It follows from these results that $\left|\dfrac{1}{z^n}\right| = \dfrac{|1|}{\left|z^n\right|} = \dfrac{1}{|z|^n}$ and $\arg\left(\dfrac{1}{z^n}\right) = \arg 1 - \arg z^n = -n\arg z$.

You will prove this theorem in question **13** of Exercise 1.2 below.

Example 11

If $z = 1 + i$, express z^{-10} in Cartesian form $(x + iy)$.

Solution

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\begin{aligned}\therefore \quad z^{-10} &= \left(\sqrt{2}\right)^{-10}\left(\cos\left(\frac{-10\pi}{4}\right) + i\sin\left(\frac{-10\pi}{4}\right)\right)\\ &= \frac{1}{32}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)\\ &= -\frac{i}{32}\end{aligned}$$

Example 12

Let $z_1 = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$, $z_2 = \sqrt{2}\left(\cos\dfrac{-\pi}{4} + i\sin\dfrac{-\pi}{4}\right)$.

(a) Find the smallest positive integer n for which $\left(\dfrac{z_1}{z_2}\right)^n$ is a real number.

(b) If $z = \dfrac{z_1^3}{z_2^5}$, find z in Cartesian form.

Solution

(a) $\arg z_1 = \dfrac{\pi}{3}$ and $\arg z_2 = -\dfrac{\pi}{4}$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{3} - \frac{-\pi}{4} = \frac{7\pi}{12}$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right)^n = \frac{7n\pi}{12}$$

Now $\left(\dfrac{z_1}{z_2}\right)^n$ is a real number when $\arg\left(\dfrac{z_1}{z_2}\right)^n$ is an integer multiple of π, because that makes the argument zero (so the imaginary part is zero). $n = 12$ is the smallest positive value of n that makes this happen.

(b) $z = \dfrac{z_1^3}{z_2^5}$

$$\begin{aligned} &= \frac{2^3(\cos\pi + i\sin\pi)}{\left(\sqrt{2}\right)^5\left(\cos\frac{-5\pi}{4} + i\sin\frac{-5\pi}{4}\right)}\\ &= \sqrt{2}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)\\ &= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\\ &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\\ &= 1 + i\end{aligned}$$

Basic identities

- $|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2,\ z_2 \neq 0$
- $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$
- $\left|\frac{1}{z^n}\right| = \frac{1}{|z^n|}$ and $\arg\left(\frac{1}{z^n}\right) = -n\arg z,\ z \neq 0$
- $\overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}$
- $\overline{z}_1\overline{z}_2 = \overline{z_1z_2}$
- $z\overline{z} = |z|^2$
- $z + \overline{z} = 2\operatorname{Re}(z)$
- $z - \overline{z} = 2i\operatorname{Im}(z)$

EXERCISE 1.2 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A POINT

1 If $z = 2 + i$ and $w = -3 - 4i$, represent each of the following on the complex plane.

(a) z (b) $\overline{z}$ (c) $z\overline{z}$ (d) $3z$ (e) $-2z$ (f) $\frac{1}{z}$ (g) $z + w$
(h) $-w$ (i) $z - w$ (j) z^2 (k) $\operatorname{Re}(z)$ (l) $\operatorname{Im}(z)$

2 If $z = 2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$, then $z^4 = \ldots$

A $16\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$ B $16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

C $16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ D $16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

3 If $z = \overline{z}$, then $\arg z = \ldots$

A π B $\frac{\pi}{2}$ C 0 D 0 or π

4 Express each of the following in mod–arg form. (Give the argument in radians and in exact form.)

(a) $2 - 2i$ (b) $-\sqrt{3} + i$ (c) $-6 - 6i$ (d) $4i$ (e) -4
(f) $-3 - \sqrt{3}i$ (g) $2\sqrt{3} - 2i$ (h) $\sqrt{2} + \sqrt{2}i$

5 Convert each of the following into Cartesian form.

(a) $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (b) $8\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

(c) $6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (d) $2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$

6 For each of the following, find both zw and $\frac{z}{w}$ in mod–arg form.

(a) $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $w = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ (b) $z = 5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, $w = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(c) $z = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$, $w = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

7 If $z = x + iy$, prove the following.

(a) $|z| = |\overline{z}|$ (b) $z\overline{z} = |z|^2$ (c) $z + \frac{|z|^2}{z} = 2\operatorname{Re}(z)$

8 On an Argand diagram, mark points A, B and C to represent complex numbers z, w and $z + w$. Give a geometrical explanation to show that $|z + w| \leq |z| + |w|$.

9 Find the following in Cartesian form.

(a) $\left[2\left(\cos\frac{3\pi}{10}+i\sin\frac{3\pi}{10}\right)\right]^5$ **(b)** $\left[\sqrt{2}\left(\cos\frac{-3\pi}{4}+i\sin\frac{-3\pi}{4}\right)\right]^8$ **(c)** $(\sqrt{3}+i)^6$

(d) $(1-i)^5$ **(e)** $(\sqrt{3}-i)^4$ **(f)** $\frac{1}{(2\sqrt{3}+2i)^5}$ **(g)** $(-4-4\sqrt{3}i)^{-3}$

(h) $(1-i)^3(2+2i)^4$ **(i)** $\frac{(1+i)^3}{(1-i)^4}$ **(j)** $\frac{(\sqrt{3}+i)^6}{(1-i)^8}$

10 If $z=3-4i=5(\cos\theta+i\sin\theta)$, find the following in $x+iy$ form.

(a) $25(\cos 2\theta+i\sin 2\theta)$ **(b)** $5(\sin\theta-i\cos\theta)$ **(c)** $\frac{1}{5}(\cos\theta-i\sin\theta)$

11 If $z=r(\cos\theta+i\sin\theta)$, show that $\frac{z}{z^2+r^2}$ is real.

12 Let $z=\sqrt{3}+i$ and $w=z\times(\cos\theta+i\sin\theta)$ where $-\pi<\theta\le\pi$.

(a) Find the value of θ if w is purely imaginary and $\text{Im}(w)>0$.

(b) Find the value of $\arg(z+w)$.

13 **(a)** If $z=\cos\theta+i\sin\theta$, prove by induction that $z^n=\cos n\theta+i\sin n\theta$ for all positive integers n. (This is the proof of de Moivre's theorem for positive integers.)

(b) By writing $z^{-n}=\frac{1}{z^n}$, complete the proof of de Moivre's theorem for negative integers.

14 Use de Moivre's theorem to prove that the conjugate of a power is equal to the power of the conjugate, i.e. let $z=r(\cos\theta+i\sin\theta)$ and prove that $\overline{z^n}=(\overline{z})^n$.

15 We have already proved (earlier and in question **14**) that:

- $z+\overline{z}=2\text{Re}(z)$ and $z-\overline{z}=2\text{Im}(z)\times i$
- the conjugate of a sum is equal to the sum of the conjugates
- the conjugate of a difference is equal to the difference of the conjugates
- the conjugate of a product is equal to the product of the conjugates
- the conjugate of a quotient is equal to the quotient of the conjugates
- the conjugate of a power is equal to the power of the conjugate.
- It is also obvious that the conjugate of a real number is itself, i.e. if $z=x+0i$ then $\overline{z}=x-0i=z$.

Use these properties of conjugates to answer the following.

(a) Show that $z^n+(\overline{z})^n=2\text{Re}(z^n)$.

(b) Simplify $(1+\sqrt{3}i)^{10}+(1-\sqrt{3}i)^{10}$.

16 Consider the cubic polynomial $P(x)=ax^3+bx^2+cx+d$ for which all the coefficients a, b, c and d are real. Let the complex number z be a root of the equation $P(x)=0$. Show that $\overline{z}$ is also a root of $P(x)=0$.

1.3 OTHER REPRESENTATIONS OF COMPLEX NUMBERS

Euler's formula

A very useful result is Euler's formula, which states that $e^{ix}=\cos x+i\sin x$, for real x.

It may seem strange to have an exponential function as the sum of two trigonometric functions, so the result needs to be proved.

Proof

Let $f(x)=\cos x+i\sin x$ [1]

Differentiate with respect to x:

$$\begin{aligned}f'(x)&=-\sin x+i\cos x\\&=i(\cos x+i\sin x)\\&=i\,f(x)\end{aligned}$$

Hence $\frac{f'(x)}{f(x)} = i$.

Integrate both sides with respect to x:

$$\int \frac{f'(x)}{f(x)} dx = i \int dx$$

Hence $\log_e |f(x)| = ix + C$ [2]

Using [1] you have: $f(0) = \cos 0 + i\sin 0 = 1$

Substitute into [2]: $\log_e 1 = C$ so $C = 0$

[2] becomes: $\log_e |f(x)| = ix$

So $f(x) = e^{ix}$

Hence $e^{ix} = \cos x + i\sin x$

Thus $e^{-ix} = \cos(-x) + i\sin(-x)$

$= \cos x - i\sin x$

In general, when $|z| \neq 1, z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Example 13

Write each complex number in both polar and Cartesian form.

(a) $e^{\frac{i\pi}{6}}$ (b) $e^{\frac{-i\pi}{3}}$ (c) $e^{\frac{3\pi i}{4}}$ (d) $e^{\frac{-5\pi i}{6}}$ (e) $e^{-i\pi}$ (f) $e^{1+\frac{i\pi}{6}}$

Solution

(a) $e^{\frac{i\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

(b) $e^{\frac{-i\pi}{3}} = \cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right) = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

(c) $e^{\frac{3\pi i}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}}(-1+i)$

(d) $e^{\frac{-5\pi i}{6}} = \cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right) = -\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

(e) $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -\cos 0 + i\sin 0 = -1$

(f) $e^{1+\frac{i\pi}{6}} = e \times e^{\frac{i\pi}{6}} = e\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = e\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{e\sqrt{3}}{2} + \frac{e}{2}i$

Example 14

Write each complex number in the form $re^{i\theta}$, giving any decimal answers correct to two decimal places.

(a) $3(\cos 2 + i\sin 2)$ (b) $-1 + i\sqrt{3}$ (c) $2 + 3i$ (d) $2(\cos 1.5 - i\sin 1.5)$ (e) $-3 - 3i$

Solution

(a) $3(\cos 2 + i\sin 2) = 3e^{2i}$

(b) $-1 + i\sqrt{3} = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2e^{\frac{2\pi i}{3}}$

(c) $2+3i=\sqrt{13}\left(\frac{2}{\sqrt{13}}+\frac{3}{\sqrt{13}}i\right)=\sqrt{13}(\cos\theta+i\sin\theta)$ where $\theta=\tan^{-1}\left(\frac{3}{2}\right)\approx 0.98$

$=\sqrt{13}e^{0.98i}$

(d) $2(\cos 1.5-i\sin 1.5)=2(\cos(-1.5)+i\sin(-1.5))=2e^{-1.5i}$

(e) $-3-3i=3(-1-i)=3\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=3\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right)=3\sqrt{2}e^{\frac{-3\pi i}{4}}$

Powers of complex numbers

Since any complex number can be written in the exponential form, $e^{i\theta}$, it is easy to find powers of this number as $(e^{i\theta})^n=e^{in\theta}$.

Example 15

(a) Write $z=1+i$ in the form $re^{i\theta}$.

(b) Hence find the following in both polar form and Cartesian form.

(i) z^2 (ii) z^3 (iii) z^4 (iv) $\sqrt{z}$ (v) z^{-1}

Solution

(a) $z=1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}e^{\frac{i\pi}{4}}$

(b) (i) $z^2=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^2=2e^{\frac{i\pi}{2}}=2(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})=2i$

(ii) $z^3=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^3=2\sqrt{2}e^{\frac{3\pi i}{4}}=2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=-2+2i$

This answer could also have been obtained using $z^3=z^2\times z=2i(1+i)=-2+2i$.

(iii) $z^4=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^4=4e^{i\pi}=4(\cos\pi+i\sin\pi)=-4$

(iv) $\sqrt{z}=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{\frac{1}{2}}=\sqrt[4]{2}e^{\frac{i\pi}{8}}=\sqrt[4]{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)=\sqrt[4]{2}(0.9239+0.3827i)=1.099+0.4204i$

(v) $z^{-1}=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{-1}=\frac{1}{\sqrt{2}}e^{\frac{-i\pi}{4}}=\frac{1}{\sqrt{2}}\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\frac{1}{2}-\frac{1}{2}i$

Example 16

Given $z_1=2e^{\frac{i\pi}{6}}$, $z_2=3e^{\frac{-i\pi}{3}}$ and $z_3=e^{\frac{3i\pi}{4}}$, find the polar form for each of the following.

(a) $z_1\times z_2$ (b) $z_2\times z_3$ (c) $z_1^2\times z_2$ (d) $\frac{z_1}{z_2}$

(e) $\frac{z_2}{z_3}$ (f) $\frac{z_1^2\times z_2}{z_3}$ (g) On the Argand diagram, plot z_1, z_2 and $z_1\times z_2$.

(h) On the Argand diagram, plot z_2, z_3 and $\frac{z_2}{z_3}$.

Solution

(a) $z_1 \times z_2 = 2e^{\frac{i\pi}{6}} \times 3e^{\frac{-i\pi}{3}} = 6e^{\frac{i\pi}{6}+\frac{-i\pi}{3}} = 6e^{\frac{-i\pi}{6}}$

(b) $z_2 \times z_3 = 3e^{\frac{-i\pi}{3}} \times e^{\frac{3i\pi}{4}} = 3e^{\frac{-i\pi}{3}+\frac{3i\pi}{4}} = 3e^{\frac{5i\pi}{12}}$

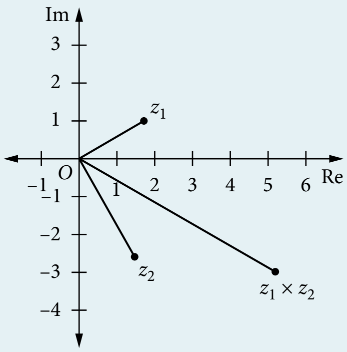
(c) $z_1{}^2 \times z_2 = 2^2 e^{\frac{2i\pi}{6}} \times 3e^{\frac{-i\pi}{3}} = 12e^{\frac{i\pi}{3}+\frac{-i\pi}{3}} = 12e^0 (= 12)$

(d) $\dfrac{z_1}{z_2} = \dfrac{2e^{\frac{i\pi}{6}}}{3e^{\frac{-i\pi}{3}}} = \dfrac{2}{3}e^{\frac{i\pi}{6}-\frac{-i\pi}{3}} = \dfrac{2}{3}e^{\frac{i\pi}{2}}\left(= \dfrac{2}{3}i\right)$

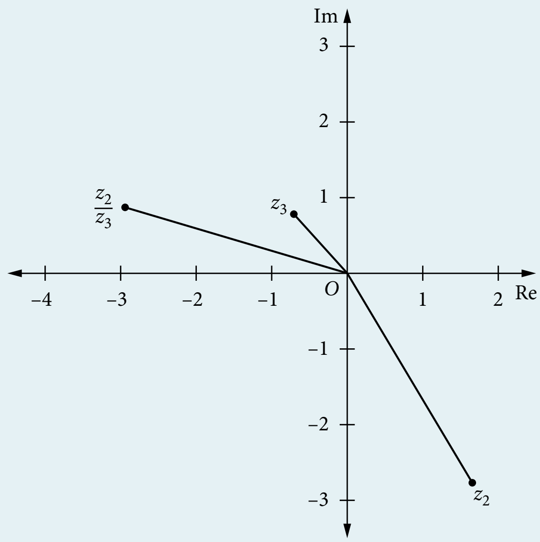
(e) $\dfrac{z_2}{z_3} = \dfrac{3e^{\frac{-i\pi}{3}}}{e^{\frac{3i\pi}{4}}} = 3e^{\frac{-i\pi}{3}-\frac{3i\pi}{4}} = 3e^{\frac{-13i\pi}{12}} = 3e^{\frac{11i\pi}{12}}$

(f) $\dfrac{z_1{}^2 \times z_2}{z_3} = \dfrac{2^2 e^{\frac{2i\pi}{6}} \times 3e^{\frac{-i\pi}{3}}}{e^{\frac{3i\pi}{4}}} = \dfrac{12e^0}{e^{\frac{3i\pi}{4}}} = 12e^{\frac{-3i\pi}{4}}$

(g)



(h)



Geometrical representation of products involving complex numbers—consolidation and summary

Multiplication of a complex number z by a real number k:

- $\arg(kz) = \arg k + \arg z$
  If k is a positive real number, then $\arg k = 0$, so $\arg(kz) = \arg z$.
  If k is a negative real number, then $\arg k = \pi$, so $\arg(kz) = \pi + \arg z = \pi + \arg z - 2\pi = -(\pi - \arg z)$.
  (Note that 2π is subtracted to find the principal argument.)
- $|kz| = |k| \times |z|$, i.e. there is a scaling by a factor of $|k|$
  If k is a negative real number, then the direction from the origin O to the point representing kz is opposite to the direction from O to the point representing z.

Multiplication of a complex number z by i:

- $\arg(iz) = \arg i + \arg z = \frac{\pi}{2} + \arg z$
- $|iz| = |i| \times |z| = |z|$ as $|i| = 1$
- Hence multiplication by i causes an anticlockwise rotation by $\frac{\pi}{2}$ about the origin O, with no change to the modulus.

Multiplication of a complex number z by ki, where k is a real number:

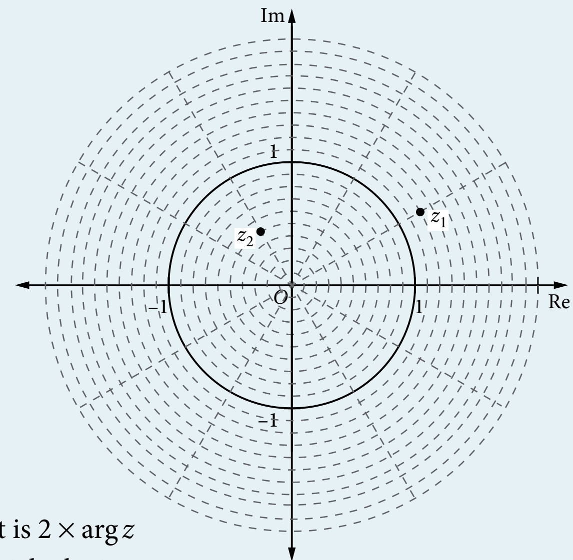
- This combines the two cases above.
- Rotate by $\frac{\pi}{2}$ anticlockwise about O and then scale by a factor of $|k|$, remembering also to reverse the direction if k is negative.

Multiplication of a complex number z by another complex number $r(\cos\theta + i\sin\theta)$:

- $\arg(z \times r\operatorname{cis}\theta) = \arg z + \arg(r\operatorname{cis}\theta) = \arg z + \theta$
- $|z \times r\operatorname{cis}\theta| = |z| \times |r\operatorname{cis}\theta| = |z| \times r$
- To multiply by $r\operatorname{cis}\theta$, rotate by θ anticlockwise about O and then scale by a factor of r.

Example 17

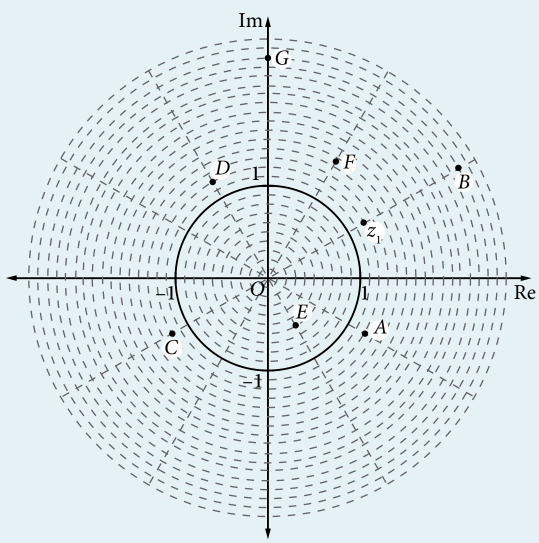
The Argand diagram at right shows the unit circle as well as points representing the complex numbers z_1 and z_2.
For **(a)** $z = z_1$ and **(b)** $z = z_2$, mark points A, B, C, D, E, F, G to represent $\bar{z}$, $2z$, $-z$, iz, $-\frac{1}{2}iz$, z^2 and $(1+\sqrt{3}i)z$.



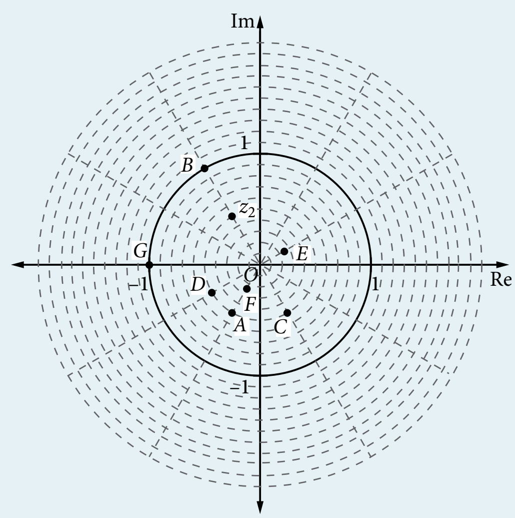
Solution

A: $\bar{z}$ is the reflection of z in the real axis
B: $2z$ is z scaled by a factor of 2
C: $-z$ is z scaled by a factor of -1 (i.e. reflected back through O)
D: iz is z rotated by $\frac{\pi}{2}$ anticlockwise about O
E: $-\frac{1}{2}iz$ is iz scaled by a factor of $-\frac{1}{2}$
F: z^2 has a modulus that is $(\operatorname{mod} z)^2$ and an argument that is $2 \times \arg z$
G: $1+\sqrt{3}i = 2\operatorname{cis}\frac{\pi}{3}$, so $(1+\sqrt{3}i)z$ is found by rotating anticlockwise by $\frac{\pi}{3}$ and then doubling the modulus.

(a)



(b)



Example 18

Let $OABC$ be a square on an Argand diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively.

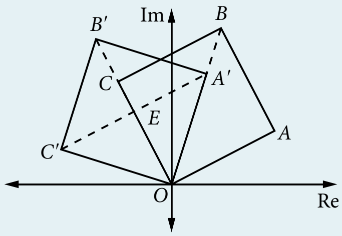
(a) Find the complex number represented by B.

(b) The square is now rotated anticlockwise 45° about O to form $OA'B'C'$. Find the complex numbers represented by A', B' and C'.

(c) E is the point of intersection of the diagonals of the square $OA'B'C'$. What complex number does E represent?

Solution

(a) B represents $z + iz$ (completion of the parallelogram represents the sum)



(b) **Method 1**

A' is formed by rotating A anticlockwise by $45°$ about O.

Hence A' represents $z \times 1(\cos 45° + i\sin 45°)$
$= \frac{z}{\sqrt{2}}(1+i)$

B' represents $(z+iz) \times 1(\cos 45° + i\sin 45°)$
$= \frac{z}{\sqrt{2}}(1+i)^2 = \frac{z}{\sqrt{2}} \times 2i = \sqrt{2}\,iz$

Method 2

In a square, the length of the diagonal is $\sqrt{2}$ times the length of a side. Also, the diagonals are inclined at $45°$ to the sides. Hence, when A is rotated by $45°$ to A', A' is the point on the diagonal OB which is $\frac{1}{\sqrt{2}}$ from O.

Thus A' represents the number $\frac{1}{\sqrt{2}} \times (z+iz)$
$= \frac{z}{\sqrt{2}}(1+i)$

B' is the point along the extension of OC such that OB' $= \sqrt{2} \times OC$.

Hence B' represents $\sqrt{2} \times iz = \sqrt{2}\,iz$

By either method, similarly C' is:
$iz \times 1(\cos 45° + i\sin 45°) = \frac{z}{\sqrt{2}}(-1+i)$

(c) The diagonals of a square bisect each other, so E is the midpoint of OB'.

Hence E represents: $\frac{1}{2} \times \sqrt{2}\,iz$

EXERCISE 1.3 OTHER REPRESENTATIONS OF COMPLEX NUMBERS

1 Write each complex number in both polar and Cartesian form.

(a) $e^{\frac{i\pi}{3}}$ (b) $e^{\frac{i\pi}{2}}$ (c) $e^{\frac{5\pi i}{6}}$ (d) $e^{\frac{i\pi}{4}}$ (e) $e^{\frac{-i\pi}{2}}$ (f) $e^{\frac{-2\pi i}{3}}$ (g) $e^{1-\frac{i\pi}{2}}$ (h) $e^{2+\frac{i\pi}{3}}$

2 Write each complex number in the form $re^{i\theta}$, giving any decimal answers correct to two decimal places, where necessary.

(a) $3(\cos 1.5 + i\sin 1.5)$ (b) $-\sqrt{3}+i$ (c) $3+2i$ (d) $4(\cos 2 - i\sin 2)$

(e) $2-2i$ (f) $4\left(-\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$ (g) $-2-2\sqrt{3}i$ (h) $\left(1+\sqrt{2}\right)+\left(1-\sqrt{2}\right)i$

3 If $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$, then the complex number $\frac{1}{2}\left(\sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}\right)$ is equal to:

A $e^{\frac{5\pi i}{8}}$ B $e^{\frac{-5\pi i}{8}}$ C $e^{\frac{i\pi}{8}}$ D $e^{\frac{-i\pi}{8}}$

4 (a) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, write an expression for $e^{-i\theta}$.

(b) Using part (a), obtain expressions for $\sin\theta$ and $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

5 (a) Write $z = \sqrt{3}+i$ in the form $re^{i\theta}$.

(b) Hence find the following in both polar form and Cartesian form.

(i) z^2 (ii) z^3 (iii) z^4 (iv) $\sqrt{z}$ (v) z^{-1}

6 (a) Write $z = 1 - \sqrt{3}i$ in the form $re^{i\theta}$.

(b) Hence find the following in both polar form and Cartesian form.

(i) z^2 (ii) z^3 (iii) z^5 (iv) $\sqrt{z}$ (v) $\frac{1}{z}$ (vi) $\frac{1}{z^3}$

7 (a) Write $z = -1 - i$ in the form $re^{i\theta}$.

(b) Hence find the following in both polar form and Cartesian form.

(i) z^2 (ii) z^3 (iii) z^4 (iv) $\sqrt{z}$ (v) $\frac{1}{z}$ (vi) $\frac{1}{z^2}$

8 (a) Given $e^{i\theta} = \cos\theta + i\sin\theta$, write expressions for $e^{2i\theta}$ and $\left(e^{i\theta}\right)^2$.

(b) Hence write expressions for $\cos 2\theta$ and $\sin 2\theta$.

9 (a) Given $e^{i\theta} = \cos\theta + i\sin\theta$, write expressions for $e^{3i\theta}$ and $\left(e^{i\theta}\right)^3$.

(b) Hence write expressions for $\cos 3\theta$ in terms of $\cos\theta$ and $\cos^3\theta$.

(c) Hence write expressions for $\sin 3\theta$ in terms $\sin\theta$ and $\sin^3\theta$.

10 Given that $\frac{1-3i}{1+2i} = re^{i\theta}$, $r > 0$ and for $-\pi < \theta \le \pi$, find the values of r and θ.

11 Given that $\left(\sqrt{3} - i\right)\left(-\sqrt{3} + i\right) = re^{i\theta}$, $r > 0$ and for $-\pi < \theta \le \pi$, find the values of r and θ.

12 Given $z_1 = 2e^{\frac{-i\pi}{6}}$, $z_2 = 3e^{\frac{2i\pi}{3}}$ and $z_3 = e^{\frac{i\pi}{4}}$, find the polar form for each of the following.

(a) $z_1 \times z_2$ (b) $z_2 \times z_3$ (c) $z_1{}^2 \times z_2$ (d) $\frac{z_1}{z_2}$ (e) $\frac{z_2}{z_3}$

(f) $\frac{z_1{}^2 \times z_2}{z_3}$ (g) On the Argand diagram, plot z_1, z_2 and $z_1 \times z_2$.

(h) On the Argand diagram, plot z_2, z_3 and $\frac{z_2}{z_3}$.

13 Given $z_1 = 2e^{\frac{i\pi}{4}}$, $z_2 = 3e^{\frac{5i\pi}{6}}$, $z_3 = e^{\frac{-i\pi}{3}}$ and $z_4 = \frac{1}{2}e^{\frac{-i\pi}{2}}$, find the polar form for each of the following, plotting each one on the Argand plane:

(a) $z_1 \times z_4$ (b) $z_2 \times z_3$ (c) $z_2{}^2 \times z_4$ (d) $\frac{z_2}{z_3}$

(e) $\frac{z_1}{z_4}$ (f) $\frac{z_1{}^2 \times z_2}{z_3}$ (g) $\sqrt{z_4} \times z_1$ (h) $\frac{z_1 \times z_2}{z_3 \times z_4}$

14 Given $z_1 = 2e^{\frac{i\pi}{8}}$, $z_2 = 3e^{\frac{5i\pi}{12}}$, $z_3 = \frac{1}{3}e^{\frac{-5i\pi}{6}}$ and $z_4 = \frac{1}{2}e^{\frac{-3i\pi}{4}}$, find the polar form for each of the following, plotting each one on the Argand plane.

(a) $z_1{}^2 \times z_4$ (b) $z_2 \times z_3$ (c) $z_1 \times z_2 \times z_3 \times z_4$ (d) $\frac{z_1{}^2}{z_4}$ (e) $\frac{\sqrt{z_3}}{z_2}$

15 Given $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, use this result to prove de Moivre's theorem, i.e. that $z^n = r^n(\cos n\theta + i\sin n\theta)$.

16 $OABC$ is a square on an Argand diagram. O represents 0, A represents $-4 + 2i$, B represents z, C represents w and D is the point where the diagonals of the square meet. Note that there are two squares that satisfy these requirements. For each square, find:

(a) the complex numbers represented by C and D in Cartesian form

(b) the value of $\arg\left(\frac{w}{z}\right)$.

17 On an Argand diagram, $OABC$ is a rectangle. The length of OC is twice the length of OA. The vertex A corresponds to the complex number z. Find the complex number represented by D, the point of intersection of the diagonals OB and AC.

1.4 DE MOIVRE'S THEOREM AND ITS APPLICATIONS

De Moivre's theorem

De Moivre's theorem has been stated previously for positive integers, n.

If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$

This theorem is proved using induction.

Proof

If $n = 1$, LHS $= z$

RHS $= r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta) = z =$ LHS

Hence the result is true when $n = 1$.

Assume the result is true for $n = k$, i.e. assume that $z^k = r^k(\cos k\theta + i\sin k\theta)$.

Prove the result is true for $n = k + 1$, i.e. prove that $z^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$.

$$\begin{aligned}
\text{LHS} &= r^{k+1}(\cos\theta + i\sin\theta)^{k+1} \\
&= r^{k+1}(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\
&= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\
&= r^{k+1}(\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta) \\
&= r^{k+1}(\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta) \\
&= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta) \\
&= \text{RHS}
\end{aligned}$$

Hence the result is true for $n = k + 1$ if it is true for $n = k$. Since the result is true for $n = 1$, by the principle of mathematical induction it is true for all integers $n \geq 1$.

If n is a negative integer, de Moivre's theorem states that $z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta))$.

Proof

If $z = r(\cos\theta + i\sin\theta)$, then $z^{-n} = \dfrac{1}{z^n} = \dfrac{1}{r^n(\cos n\theta + i\sin n\theta)}$ where n is a positive integer.

$$\begin{aligned}
z^{-n} &= \frac{1}{r^n(\cos n\theta + i\sin n\theta)} \\
&= \frac{r^{-n}(\cos n\theta - i\sin n\theta)}{(\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)} \\
&= \frac{r^{-n}(\cos n\theta - i\sin n\theta)}{\cos^2 n\theta + i\sin n\theta\cos n\theta - i\sin n\theta\cos n\theta - i^2\sin^2 n\theta} \\
&= \frac{r^{-n}(\cos n\theta - i\sin n\theta)}{\cos^2 n\theta - i^2\sin^2 n\theta} \\
&= \frac{r^{-n}(\cos n\theta - i\sin n\theta)}{\cos^2 n\theta + \sin^2 n\theta} \\
&= \frac{r^{-n}\left(\cos(-n\theta) + i\sin(-n\theta)\right)}{1} \\
&= r^{-n}\left(\cos(-n\theta) + i\sin(-n\theta)\right)
\end{aligned}$$

Hence the result is true when n is a negative integer.

In the Mathematics Extension 1 course, you learnt how to expand $(a + b)^n$ using the binomial theorem and Pascal's triangle (see Chapter 6 of *New Senior Mathematics Extension 1 for Years 11 & 12*). You will now use Pascal's triangle to expand expressions such as $(\cos\theta + i\sin\theta)^4$.

In the following table, you should note how the coefficients in the binomial expansions are equivalent to the terms of Pascal's triangle.

Expansion	Pascal's triangle
$(a+b)^0 = 1$	1
$(a+b)^1 = 1a + 1b$	1 1
$(a+b)^2 = 1a^2 + 2ab + 1b^2$	1 2 1
$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1

To write any binomial expansion, the coefficients can be found by continuing the pattern of Pascal's triangle. Each number in the triangle is the sum of the two numbers immediately above left and above right, except for the first term and last term of each row (which are always 1).
For example: $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Example 19

(a) Expand $(\cos\theta + i\sin\theta)^3$ using:
(i) de Moivre's theorem **(ii)** the binomial theorem (Pascal's triangle).

(b) Hence show that:

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **(ii)** $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **(iii)** $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(c) Solve $x^3 + 3x^2 - 3x - 1 = 0$ using the substitution $x = \tan\theta$.

(d) Show that $\tan\dfrac{\pi}{12}\tan\dfrac{5\pi}{12} = 1$.

Solution

(a) **(i)** $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

(ii)
$$
\begin{aligned}
(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta \times i\sin\theta + 3\cos\theta \times i^2\sin^2\theta + i^3\sin^3\theta \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) \\
&= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) + i(3(1-\sin^2\theta)\sin\theta - \sin^3\theta) \\
&= 4\cos^3\theta - 3\cos\theta + i(3\sin\theta - 4\sin^3\theta)
\end{aligned}
$$

(b) **(i)** Equating the real parts of **(a)**: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(ii) Equating the imaginary parts of **(a)**: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(iii)
$$
\begin{aligned}
\tan 3\theta &= \frac{\sin 3\theta}{\cos 3\theta} \\
&= \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} \\
&= \frac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta} \quad \text{(dividing numerator and denominator by } \cos^3\theta\text{)} \\
&= \frac{3\tan\theta\left(1+\tan^2\theta\right) - 4\tan^3\theta}{4 - 3\left(1+\tan^2\theta\right)} \\
&= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}
\end{aligned}
$$

(c) Let $x = \tan\theta$. The equation $x^3 + 3x^2 - 3x - 1 = 0$ becomes:

$$
\begin{aligned}
\tan^3\theta + 3\tan^2\theta - 3\tan\theta - 1 &= 0 \\
\tan^3\theta - 3\tan\theta &= 1 - 3\tan^2\theta \\
\frac{\tan^3\theta - 3\tan\theta}{1 - 3\tan^2\theta} &= 1 \quad \text{(dividing by } 1 - 3\tan^2\theta\text{, noting } \tan\theta \neq \pm\tfrac{1}{\sqrt{3}}\text{)} \\
\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} &= -1
\end{aligned}
$$

which from part **(b)(iii)** above is: $\tan 3\theta = -1$

Now solve $\tan 3\theta = -1$ as follows:

$\tan 3\theta = \tan\left(-\frac{\pi}{4}\right)$

$\therefore \quad 3\theta = -\frac{\pi}{4} + n\pi \qquad$ for any integer n

$3\theta = \ldots -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$

$\theta = \ldots -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{3\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots$

So the three solutions to the cubic equation $x^3 + 3x^2 - 3x - 1 = 0$ are given by:

$x = \tan\left(-\frac{\pi}{12}\right), \tan\frac{\pi}{4}, \tan\frac{7\pi}{12}$

$= -\tan\frac{\pi}{12}, 1, -\tan\frac{5\pi}{12}$

The other possible values of θ repeat these three values, e.g. $\theta = -\frac{5\pi}{12}$ produces the solution $x = \tan\left(-\frac{5\pi}{12}\right) = -\tan\frac{5\pi}{12}$.

(d) Product of the roots of a cubic is equal to $-\frac{d}{a}$, so: $\left(-\tan\frac{\pi}{12}\right) \times 1 \times \left(-\tan\frac{5\pi}{12}\right) = -\frac{-1}{1}$

$\tan\frac{\pi}{12}\tan\frac{5\pi}{12} = 1$

EXERCISE 1.4 DE MOIVRE'S THEOREM AND ITS APPLICATIONS

1 **(a)** Expand $(\cos\theta + i\sin\theta)^4$ by de Moivre's theorem and by the binomial theorem (Pascal's triangle) to show:

(i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ **(ii)** $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\sin^3\theta\cos\theta$

(b) Obtain an expression for $\tan 4\theta$ in terms of $\tan\theta$.

(c) By making suitable substitutions, solve the following. **(i)** $8x^4 - 8x^2 + 1 = 0$

(ii) $16x^4 - 16x^2 + 1 = 0$ **(iii)** $16x^4 - 16x^2 + 3 = 0$ **(iv)** $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

2 **(a)** Given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (see Example **19**, page 20), solve $8x^3 - 6x - 1 = 0$.

(b) Show that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$.

3 Let $z = \cos\theta + i\sin\theta$.

(a) Show that: **(i)** $z^n + z^{-n} = 2\cos n\theta$ **(ii)** $z^n - z^{-n} = 2i\sin n\theta$

(b) Show that $(z - z^{-1})^3 = (z^3 - z^{-3}) - 3(z - z^{-1})$ **(c)** Hence show that $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

4 **(a)** Let $z = \cos\theta + i\sin\theta$ and let $w = z + \frac{1}{z}$. Given $z^n + z^{-n} = 2\cos n\theta$, prove that

$w^3 - 2w^2 - w + 2 = \left(z^3 + \frac{1}{z^3}\right) - 2\left(z^2 + \frac{1}{z^2}\right) + 2\left(z + \frac{1}{z}\right) - 2.$

(b) Hence solve $\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1 = 0$ for $-\pi \le \theta \le \pi$.

5 Express $\cos 3\theta$ and $\cos 2\theta$ in terms of $\cos\theta$. Show that the equation $\cos 3\theta = \cos 2\theta$ can be expressed as $4x^3 - 2x^2 - 3x + 1 = 0$, where $x = \cos\theta$. By solving this equation for x, find the exact value of $\cos\frac{2\pi}{5}$.

6 **(a)** Use de Moivre's theorem to express $\cos 4\theta$ in terms of $\cos\theta$.

(b) Use your result from part **(a)** to solve the equation $8x^4 - 8x^2 + 1 = 0$.

(c) Show that: $\cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{7\pi}{8} = 0$.

(d) Show that: $\cos\frac{\pi}{8}\cos\frac{3\pi}{8}\cos\frac{5\pi}{8}\cos\frac{7\pi}{8} = \frac{1}{8}$.

7 **(a)** Use de Moivre's theorem to express $\cos 5\theta$ and $\sin 5\theta$ as powers of $\cos\theta$ and $\sin\theta$.

(b) Hence express $\tan 5\theta$ as a rational function of t where $t = \tan\theta$.

(c) By considering the roots of $\tan 5\theta = 0$, deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$.

1.5 COMPLEX NUMBERS AND POLYNOMIAL EQUATIONS

Quadratic equations

It is time to revisit the roots of quadratic equations and then consider polynomials.

The quadratic equation $x^2 + x + 1 = 0$ has real coefficients but no real roots. The roots are complex numbers and occur as conjugate pairs.

Example 20

(a) Solve $x^2 + x + 1 = 0$.

(b) Discuss the nature of the roots.

Solution

(a) Using the quadratic formula: $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2}$.

Write the answer in the form $a + bi$: $x = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$.

(b) The roots are $x = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ and $x = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$.

These complex numbers are of the form $a + bi$ and $a - bi$, that is they are conjugates.

The complex roots of a quadratic equation with real coefficients occur as conjugate pairs.

Polynomial equations

The introduction of the 'imaginary' number i, with the special property that $i^2 = -1$, allows the solution of all quadratic equations $ax^2 + bx + c = 0$ over the field of complex numbers. The two solutions to this equation can be real or complex numbers, depending on whether the discriminant $\Delta = b^2 - 4ac$ is non-negative or negative. If the discriminant is zero, then these two solutions are the same number.

In general, all polynomial equations $P(z) = 0$ of degree n have n solutions over the field of complex numbers. The proof of this statement is beyond the scope of this course.

A polynomial $P(z)$ of degree n in one variable is an expression of the form $a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$.

If any of the coefficients $a_n, a_{n-1}, \ldots$ are complex numbers then the polynomial is a polynomial over the set of complex numbers. If all of the coefficients are real numbers, then the polynomial is also a polynomial over the set of real numbers.

- $z^2 - 5z + 1$ is a 2nd-degree polynomial over the set of real numbers, the set of complex numbers and the set of integers.
- $z^3 + iz^2 + (2 - 3i)z + 2$ is a 3rd-degree polynomial over the set of complex numbers, but not over the set of real numbers.

Remember that the set of complex numbers contains the sets of real numbers, rational numbers and integers, so that any polynomial over the reals, rationals or integers is also a polynomial over the complex numbers.

Note: Even if the coefficients are all real numbers, the solutions of the polynomial equation may involve complex numbers. For example, $z^2 - 4z + 5 = 0$ is a quadratic equation with integer coefficients; but the solutions for z are complex numbers (as $\Delta < 0$).

Fundamental theorem of algebra

Every polynomial equation with complex coefficients, $P(z) = 0$, of degree n (where n is a positive integer), has a root that is a complex number.

This important theorem was first proved convincingly by the German scientist Carl Friedrich Gauss (1777–1855), and then more completely and rigorously a few years later by the French mathematician Jean-Robert Argand. You can use it to show, with the aid of the factor theorem, that a polynomial of degree n is reducible to n linear factors and that a polynomial equation has no more than n roots:

- Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$.
- By the fundamental theorem of algebra, the equation $P(z) = 0$ has a root z_1 such that $P(z_1) = 0$.
- Hence, by the factor theorem, $(z - z_1)$ is a factor of $P(z)$.

Thus $P(z) = (z - z_1)Q_{n-1}(z) = 0$, where $Q_{n-1}(z)$ is a polynomial of degree $n - 1$.

- Applying the fundamental theorem of algebra again, the equation $Q_{n-1}(z) = 0$ similarly has a root z_2.

Thus $P(z) = (z - z_1)(z - z_2)Q_{n-2}(z) = 0$ where $Q_{n-2}(z)$ is a polynomial of degree $n - 2$.

By continuing this application of the fundamental theorem of algebra, after n applications you have:

$P(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_n)$ where $a_n \neq 0$, as each factor is monic and the leading term of $P(z)$ is $a_n z^n$.

If z is any number different from $z_1, z_2, \ldots z_n$, then $P(z) \neq 0$. Thus $P(z)$ does not have any more than n zeros. (Note also that the complex numbers $z_1, z_2, \ldots z_n$ may not all be different from each other.)

Example 21

Reduce $z^4 + z^2 - 12$ to its linear factors over the complex numbers. Hence find the values of z for which $z^4 + z^2 - 12 = 0$.

Solution

$$\begin{aligned} z^4 + z^2 - 12 &= (z^2 - 3)(z^2 + 4) \\ &= (z - \sqrt{3})(z + \sqrt{3})(z - 2i)(z + 2i) \end{aligned}$$

Hence the roots of $z^4 + z^2 - 12 = 0$ are $z = \sqrt{3}, -\sqrt{3}, 2i, -2i$.

Notice that $z^4 + z^2 - 12$ is reduced to $(z^2 - 3)(z^2 + 4)$ over the rational numbers, and to $(z - \sqrt{3})(z + \sqrt{3})(z^2 + 4)$ over the real numbers.

Example 22

Reduce $2z^3 - 3z^2 + 8z + 5$ to its linear factors over the set of complex numbers. Hence find the values of z for which $2z^3 - 3z^2 + 8z + 5 = 0$.

Solution

$P(z) = 2z^3 - 3z^2 + 8z + 5$

If $(az - b)$ is a factor of $P(z)$, then you know from the factor theorem that $P\left(\frac{b}{a}\right) = 0$.

The coefficients of $P(z)$ are integers, so if there is to be a rational zero then b must be a factor of 5 and a must be a factor of 2. Thus the only possible values for $\frac{b}{a}$ are $\pm 1, \pm 5, \pm\frac{1}{2}, \pm\frac{5}{2}$.

Substitution of these values shows that the integer values do not work. However:

$$P\left(-\frac{1}{2}\right) = -\frac{1}{4} - \frac{3}{4} - 4 + 5 = 0$$

Hence $(2z+1)$ is a factor of $P(z)$.

$$\begin{array}{r}
z^2 - 2z + 5 \\
2z+1 \overline{) 2z^3 - 3z^2 + 8z + 5} \\
\underline{2z^3 + z^2} \\
-4z^2 + 8z \\
\underline{-4z^2 - 2z} \\
10z + 5 \\
\underline{10z + 5} \\
0
\end{array}$$

Thus $P(z) = (2z+1)(z^2 - 2z + 5)$.

Completing the square:

$$\begin{aligned}
P(z) &= (2z+1)\left[(z^2 - 2z + 1) + 4\right] \\
&= (2z+1)\left[(z-1)^2 + 4\right] \\
&= (2z+1)\left[(z-1)^2 - 4i^2\right] \\
&= (2z+1)\left[(z-1-2i)(z-1+2i)\right]
\end{aligned}$$

Hence $2z^3 - 3z^2 + 8z + 5 = 0$ when $z = -\frac{1}{2}, 1+2i, 1-2i$.

Notice that $z = -\frac{1}{2}$ is a real root and that $z = 1 + 2i$, $z = 1 - 2i$ are a conjugate pair (i.e. their sum and product are real).

You will only consider simple cases involving the following factorisation techniques that you have used before:

- quadratic trinomials
- sum and difference of two squares, $z^2 \pm a^2$
- sum and difference of two cubes, $z^3 \pm a^3$
- factor theorem (with at least one rational zero that can be found by trial and error)
- grouping.

EXERCISE 1.5 COMPLEX NUMBERS AND POLYNOMIAL EQUATIONS

1 Use the factor theorem to show that:

(a) $z - i$ is a factor of $z^3 + 2iz^2 + 3i$ **(b)** $z - 3$ is a factor of $z^2 - (5 - i)z + 6 - 3i$

(c) $z + 2 - i$ is a factor of $2z^3 + 3z^2 - (5 + 2i)z - 17 - 9i$

(d) $z - 3 + \sqrt{2}i$ is a factor of $2z^4 - 12z^3 + 23z^2 - 6z + 11$.

2 Given $P(z) = z^3 - z^2 - z + a$, what is the value of a if $P\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0$?

A 2 **B** 1 **C** −2 **D** −1

3 Given $P(z) = z^4 - 2z^3 + az - 9$, find the value of a if $P\left(1 + \sqrt{2}i\right) = 0$.

4 Given $P(z) = z^4 + az^3 + bz - 1$, find the value of a and b if i and $1 + \sqrt{2}$ are zeros of $P(z)$.

5 Given $P(z) = z^3 - az^2 + 6z + b$, find the value of a and b if $P(2) = 0$ and $P(1 - i) = 0$.

6 Factorise each polynomial over the set of complex numbers:

(a) $z^2 + 2z + 3$ **(b)** $2z^2 - 2z + 1$ **(c)** $2z^3 - 3z^2 + 2z - 3$ **(d)** $z^3 + 3z^2 + z + 3$

7 Factorise $2z^3 - 4z^2 + 3z - 1$ over:

(a) the set of integers **(b)** the set of complex numbers.

8 Factorise $z^4 - 16$ over:

(a) the set of integers **(b)** the set of complex numbers.

9 Factorise $z^4 + z^2 - 6$ over:

(a) the set of rational numbers **(b)** the set of real numbers **(c)** the set of complex numbers.

10 Factorise $z^3 - 4z^2 + 9z - 10$ over:

(a) the set of real numbers (b) the set of complex numbers.

11 Factorise $z^3 + 8$ over:

(a) the set of real numbers (b) the set of complex numbers.

12 When factorised over the set of complex numbers, $z^4 + 2z^2 + 1$ becomes:

A $(z^2 - 1)^2$ B $(z - i)^2(z + 1)^2$ C $(z^2 + 1)^2$ D $(z - i)^2(z + i)^2$

13 Factorise $z^6 - 1$ over:

(a) the set of real numbers (b) the set of complex numbers.

14 Factorise $z^5 + 3z^4 - z - 3$ over:

(a) the set of real numbers (b) the set of complex numbers.

15 Factorise $4z^3 - iz^2 - 4z + i$ over the set of complex numbers.

16 Factorise $z^3 + 2z^2 - 2z + 3$ over:

(a) the set of real numbers (b) the set of complex numbers.

17 Factorise $2z^3 - 7z^2 + 10z - 8$ over the set of complex numbers.

1.6 ZEROS OF A POLYNOMIAL

Conjugate root theorem

If a polynomial $P(x)$ has real coefficients and $P(a + bi) = 0$, then $P(a - bi) = 0$, i.e. complex zeros of real polynomials occur in conjugate pairs.

This theorem has been well illustrated in Examples **21** and **22**. You will now consider it in more general terms.

Consider a typical polynomial equation, e.g. the cubic equation (of which z is a root):

$$2z^3 + 3z^2 - 6z + 2 = 0 \qquad [1]$$

The coefficients of the terms in this equation are real numbers. Taking the complex conjugate of both sides, noting that the conjugate of a sum is the sum of the conjugates:

$$\overline{2z^3} + \overline{3z^2} - \overline{6z} + \overline{2} = \overline{0} \qquad [2]$$

Now $2 = 2 + 0i$ so that $\overline{2} = 2 - 0i = 2$. The other coefficients will similarly be real numbers, as $\overline{a} = a$ when a is real. It has also been established (see Exercise 1.2 question **14**, on page 12) that $\overline{z^3} = (\overline{z})^3$ and $\overline{z^2} = (\overline{z})^2$.

Thus equation [2] becomes $2(\overline{z})^3 + 3(\overline{z})^2 - 6(\overline{z}) + 2 = 0$.

This is the same equation as [1], because it has the same coefficients. Thus you see that z and $\overline{z}$ both satisfy the equation, as required by the theorem.

This result can be extended to the general polynomial equation $a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$ where the coefficients $a_0, a_1, \ldots a_n$ are all real numbers.

The expansion of $(z + 1)(z + i) = z^2 + (1 + i)z + i$. The coefficients on the right-hand side are not all real.

The roots of $z^2 + (1 + i)z + i$ are $z = -1$ and $z = -i$. They are not a conjugate pair as the coefficients of the polynomial are not all real.

If any of a polynomial's coefficients are not real, then the roots may not all occur in conjugate pairs.

Example 23

The polynomial $z^3 - 7z^2 + 25z - 39$ has one zero equal to $2 + 3i$. Write its three linear factors.

Solution

The coefficients are real numbers and $2 + 3i$ is a zero, so from the conjugate root theorem $2 - 3i$ must be another zero.

$$z^3 - 7z^2 + 25z - 39 = (z - (2 + 3i))(z - (2 - 3i))Q(z)$$
$$= (z^2 - 4z + 13)Q(z) \quad \text{where } Q(z) \text{ is a 1st-degree polynomial.}$$

Therefore you have $z^3 - 7z^2 + 25z - 39 = (z^2 - 4z + 13)(z - k)$.

Equating the constant terms: $-39 = 13 \times (-k)$

But $-39 = 13 \times (-3)$, so $Q(z) = (z - 3)$.

Hence $z^3 - 7z^2 + 25z - 39 = (z - (2 + 3i))(z - (2 - 3i))(z - 3)$.

Multiple zeros of a polynomial

A polynomial of degree n has n zeros, but they are not necessarily all different. You say that c is a zero of multiplicity r when the factor $(z - c)$ occurs r times.

For example, if $P(z) = (z - 1)^3(z - 5)^2(z - 6)$ then 1 is a zero of multiplicity three, 5 is a zero of multiplicity two and 6 is a zero of multiplicity one.

Furthermore, if $x = c$ is a zero of multiplicity r of the real polynomial $P(x)$, then $x = c$ is also a zero of multiplicity $(r - 1)$ of the derived polynomial $P'(x)$, a zero of multiplicity $(r - 2)$ of the second derived polynomial $P''(x)$, and so on:

If $\quad P(x) = (x - c)^r S(x) \quad$ where $r > 0$, $S(c) \neq 0$

then $\quad P'(x) = r(x - c)^{r-1} S(x) + (x - c)^r S'(x)$

$$= (x - c)^{r-1}\left[rS(x) + (x - c)S'(x)\right]$$
$$= (x - c)^{r-1} Q(x) \quad [1]$$

i.e. the polynomial $P'(x)$ has a zero $x = c$ of multiplicity $(r - 1)$.

Applying the product rule to $P'(x)$ in [1] produces the polynomial $P''(x)$ with a zero $x = c$ of multiplicity $(r - 2)$.

If $P(x)$ is a polynomial of degree n, then $P'(x)$ must be a polynomial of degree $(n - 1)$, $P''(x)$ a polynomial of degree $(n - 2)$ and so on. This property allows you to use calculus techniques to solve equations that are known to have multiple roots.

Example 24

Solve $z^4 - 6z^3 + 14z^2 - 30z + 45 = 0$ given that it has a real root of multiplicity 2.

Solution

Consider the polynomial $P(z) = z^4 - 6z^3 + 14z^2 - 30z + 45$

Differentiate: $P'(z) = 4z^3 - 18z^2 + 28z - 30$

The real root must be a factor of both 45 and 30, consider ± 1, ± 3, ± 5 and substitute first into $P'(z)$.

$P'(1) = -16$, $P'(-1) = -80$, $P'(3) = 0$, $P'(-3) = -60$, $P'(5) = 160$, $P'(-5) = -1120$

Now find P(3): P(3) = 0

Hence $z = 3$ is the double real root.

$$P(z) = (z - 3)^2(z^2 + bz + c)$$

Hence $9c = 45$ so $c = 5$ and $P(z) = (z - 3)^2(z^2 + bz + 5)$

$(z^2 - 6z + 9)(z^2 + bz + 5) = z^4 - 6z^3 + 14z^2 - 30z + 45$

Equate coefficients of z: $-30 + 9b = -30$ so $b = 0$

Hence $P(z) = (z - 3)^2(z^2 + 5)$

$$P(z) = 0\text{: } z = 3 \text{ or } z = \pm i\sqrt{5}$$

EXERCISE 1.6 ZEROS OF A POLYNOMIAL

1 Given that $(z + 2 - i)$ is a factor, factorise $z^4 + 4z^3 + 3z^2 - 8z - 10$ over:
(a) the set of real numbers **(b)** the set of complex numbers.

2 Solve the following for z as a complex number.
(a) $z^2 - 4z + 8 = 0$ **(b)** $z^3 + 2z^2 - 2z + 3 = 0$ **(c)** $z^6 + 7z^3 - 8 = 0$ **(d)** $z^4 + 2z^2 + 1 = 0$

3 Solve $z^3 + z^2 + 3z - 5 = 0$ for z as **(a)** a real number **(b)** a complex number.

4 Solve $z^5 + 3z^4 - z - 3 = 0$ for z as a real number.

5 What are the roots of $z^4 - 2z^3 - z + 2 = 0$ for z as a complex number?

A $1, 2$ **B** $1, 2, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ **C** $-1, -2$ **D** $-1, -2, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

6 Find the values of the real numbers a and b such that $1 + i$ is a root of the equation $z^3 + az + b = 0$.

7 Solve $z^4 + 2z^3 + z^2 - 4 = 0$ for z if z is a complex number.

8 Solve $3z^3 - 4z^2 - 13z - 6 = 0$ for z if z is a real number.

9 Solve $z^4 - z^3 + 6z^2 - z + 15 = 0$ for z given that $z = 1 - 2i$ is a root of the equation.

10 Solve $z^3 - 3z^2 + 7z - 5 = 0$ for z if z is a real number.

11 If z is a complex number, solve the following.
(a) $z^4 + 4z^3 - z - 4 = 0$ **(b)** $z^6 - 16z^3 + 64 = 0$

12 Write an equation of the lowest possible degree with **(i)** complex coefficients **(ii)** rational coefficients that includes the following among its roots.
(a) $2, 1 + i$ **(b)** $\sqrt{3} + 1, 2 - i$ **(c)** $3 + \sqrt{2}i$ **(d)** $2 - i$ (of multiplicity 2)
(e) $3 + 2i$ **(f)** $1 + \sqrt{3}i$ (of multiplicity 2)

13 Find the real values of a for which ai is a root of the polynomial equation $z^4 - 2z^3 + 7z^2 - 4z + 10 = 0$.

14 Find the values that the real numbers a and b must take for $z = 1$ to be a root of the complex equation $iz^2 + (ia - 1)z + (i - b) = 0$.

15 **(a)** Find the three roots of the equation $z^3 - z^2 + 2 = 0$.
(b) Find the remainder when $z^3 - z^2 + 2$ is divided by $(z - i)$.

16 Find the real numbers k such that $z = ki$ is a root of the equation $z^3 + (2 + i)z^2 + (2 + 2i)z + 4 = 0$. Hence, or otherwise, find the three roots of the equation.

17 Solve the following equations using a calculus method.
(a) $z^4 + 4z^3 + 5z^2 + 4z + 4 = 0$, given that it has a root of multiplicity 2.
(b) $z^4 + 2z^3 - 2z^2 - 6z + 5 = 0$, given that it has a root of multiplicity 2.

18 If z is a complex number, solve $z^4 - 2z^2 + 9 = 0$, given that $1 + 2\sqrt{2}i = (\sqrt{2} + i)^2$.

1.7 SOLVING QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS

The roots of the equation $ax^2 + bx + c = 0$, where a, b and c are real, may be found using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. When a, b and c are all real, the roots will be either real or complex. If they are complex, they will occur as a conjugate pair, z and $\overline{z}$.

If any of a, b or c are complex, this formula does not generate a simple solution.

Solve the equation by completing the square.

$ax^2 + bx + c = 0$ [1]

$a\left(x+\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a} - c$

$a\left(x+\dfrac{b}{2a}\right)^2 = \dfrac{b^2-4ac}{4a}$

$\left[2a\left(x+\dfrac{b}{2a}\right)\right]^2 = b^2 - 4ac$ [2]

If the original equation, [1], has complex roots, then equation [2] says that you have a complex number whose square is equal to $b^2 - 4ac$.

Let this complex number be $\alpha + i\beta$.

Thus equation [2] becomes $(\alpha + i\beta)^2 = A + iB$ [3] and you have to obtain α and β as real numbers.

$(\alpha + i\beta)^2 = A + iB$

$\alpha^2 + 2i\alpha\beta - \beta^2 = A + iB$

$(\alpha^2 - \beta^2) + 2\alpha\beta i = A + Bi$

If two complex numbers are equal, then their real parts and their imaginary parts are equal.

$\therefore\ \alpha^2 - \beta^2 = A$ [i]

$2\alpha\beta = B$ [ii]

Solve [i] and [ii] for α and β.

Now $\left(\alpha^2+\beta^2\right)^2 = \alpha^4 + 2\alpha^2\beta^2 + \beta^4$

$= \alpha^4 - 2\alpha^2\beta^2 + \beta^4 + 4\alpha^2\beta^2$

$= \left(\alpha^2-\beta^2\right)^2 + (2\alpha\beta)^2$

$= A^2 + B^2$

Since $\alpha^2 + \beta^2 > 0$ then $\alpha^2 + \beta^2 = \sqrt{A^2+B^2}$ [iii]

[i] + [iii] $2\alpha^2 = A + \sqrt{A^2+B^2}$

$\alpha^2 = \dfrac{A+\sqrt{A^2+B^2}}{2}$

$\therefore\ \alpha = \pm\sqrt{\dfrac{A+\sqrt{A^2+B^2}}{2}}$

[ii] gives: $\beta = \dfrac{B}{2\alpha}$

If $\alpha = \sqrt{\dfrac{A+\sqrt{A^2+B^2}}{2}}$, then $\beta = \dfrac{B}{2}\sqrt{\dfrac{2}{A+\sqrt{A^2+B^2}}}$

and if: $\alpha = -\sqrt{\dfrac{A+\sqrt{A^2+B^2}}{2}}$, then $\beta = -\dfrac{B}{2}\sqrt{\dfrac{2}{A+\sqrt{A^2+B^2}}}$

Thus a quadratic equation with complex coefficients may be solved.

Example 25

Solve these equations.

(a) $x^2 + 2x + i = 0$ (b) $x^2 + 2(2-i)x + 6 = 0$ (c) $ix^2 + 3x - 4 = 0$

Solution

(a) $x^2 + 2x + i = 0$

Complete the square: $x^2 + 2x + 1 = 1 - i$

$(x+1)^2 = 1 - i$

Let $x + 1 = \alpha + \beta i$

$\therefore (\alpha + \beta i)^2 = 1 - i$

$\alpha^2 - \beta^2 + 2\alpha\beta i = 1 - i$

$\therefore \alpha^2 - \beta^2 = 1$

$2\alpha\beta = -1$

$$\text{Now } \left(\alpha^2 + \beta^2\right)^2 = \left(\alpha^2 - \beta^2\right)^2 + (2\alpha\beta)^2$$
$$= 1^2 + (-1)^2$$
$$= 2$$

$\therefore \alpha^2 + \beta^2 = \sqrt{2}$

and $\alpha^2 - \beta^2 = 1$

$$\therefore \quad 2\alpha^2 = \sqrt{2} + 1$$
$$\alpha^2 = \frac{\sqrt{2}+1}{2}$$
$$\alpha = \pm\sqrt{\frac{\sqrt{2}+1}{2}}$$

$$\beta = \frac{-1}{2\alpha}$$

When $\alpha = \sqrt{\frac{\sqrt{2}+1}{2}}$ then $\beta = \frac{-1}{2} \times \frac{\sqrt{2}}{\sqrt{\sqrt{2}+1}} = \frac{-1}{2} \times \sqrt{\frac{2}{\sqrt{2}+1}} = \frac{-1}{2} \times \sqrt{\frac{2\left(\sqrt{2}-1\right)}{2-1}} = -\sqrt{\frac{2\left(\sqrt{2}-1\right)}{4}} = -\sqrt{\frac{\sqrt{2}-1}{2}}$

and when $\alpha = -\sqrt{\frac{\sqrt{2}+1}{2}}$ then $\beta = \sqrt{\frac{\sqrt{2}-1}{2}}$

Hence $x + 1 = \sqrt{\frac{\sqrt{2}+1}{2}} - \sqrt{\frac{\sqrt{2}-1}{2}}i$ and $x + 1 = -\sqrt{\frac{\sqrt{2}+1}{2}} + \sqrt{\frac{\sqrt{2}-1}{2}}i$

$$x = \left(\sqrt{\frac{\sqrt{2}+1}{2}} - 1\right) - \sqrt{\frac{\sqrt{2}-1}{2}}i \qquad x = -\left(\sqrt{\frac{\sqrt{2}+1}{2}} + 1\right) + \sqrt{\frac{\sqrt{2}-1}{2}}i$$

(b) $x^2 + 2(2-i)x + 6 = 0$

$$\text{Complete the square: } x^2 + 2(2-i)x + (2-i)^2 = (2-i)^2 - 6$$
$$(x + 2 - i)^2 = 4 - 4i - 1 - 6$$
$$(x + 2 - i)^2 = -3 - 4i$$

Let $x + 2 - i = \alpha + \beta i$

$\therefore (\alpha + \beta i)^2 = -3 - 4i$

$\alpha^2 - \beta^2 + 2\alpha\beta i = -3 - 4i$

$\therefore \alpha^2 - \beta^2 = -3$

$2\alpha\beta = -4$

$$\text{Now } \left(\alpha^2 + \beta^2\right)^2 = \left(\alpha^2 - \beta^2\right)^2 + (2\alpha\beta)^2$$
$$= (-3)^2 + (-4)^2$$
$$= 25$$

$\therefore \alpha^2 + \beta^2 = 5$

and $\alpha^2 - \beta^2 = -3$

$2\alpha^2 = 2$

$\alpha^2 = 1$

$\alpha = 1, \beta = -2$

$\alpha = -1, \beta = 2$

The roots of the equation are $x + 2 - i = 1 - 2i$ and $x + 2 - i = -1 + 2i$

i.e. $x = -1 - i$ and $x = -3 + 3i$

(c) $ix^2 + 3x - 4 = 0$

Multiply by $-i$: $x^2 - 3ix + 4i = 0$

Complete the square: $x^2 - 3ix + \left(\frac{3i}{2}\right)^2 = \left(\frac{3i}{2}\right)^2 - 4i$

$\left(x - \frac{3i}{2}\right)^2 = -\frac{9}{4} - 4i$

Let $x - \frac{3i}{2} = \alpha + \beta i$

$\therefore \quad (\alpha + \beta i)^2 = -\frac{9}{4} - 4i$

$\alpha^2 - \beta^2 + 2\alpha\beta i = -\frac{9}{4} - 4i$

$\therefore \quad \alpha^2 - \beta^2 = -\frac{9}{4}$

$2\alpha\beta = -4$

$$\left(\alpha^2 + \beta^2\right)^2 = \left(\alpha^2 - \beta^2\right)^2 + (2\alpha\beta)^2$$
$$= \left(-\frac{9}{4}\right)^2 + (-4)^2$$
$$= \frac{337}{16}$$

Now $\quad \therefore \quad \alpha^2 + \beta^2 = \frac{\sqrt{337}}{4}$

$\alpha^2 - \beta^2 = -\frac{9}{4}$

$2\alpha^2 = \frac{\sqrt{337} - 9}{4}$

$\alpha^2 = \frac{\sqrt{337} - 9}{8}$

$$\alpha = \frac{\sqrt{\sqrt{337} - 9}}{2\sqrt{2}} \text{ and } \beta = \frac{-2}{\alpha} = \frac{-4\sqrt{2}}{\sqrt{\sqrt{337} - 9}} = \frac{-4\sqrt{2} \times \sqrt{\sqrt{337} + 9}}{\sqrt{337 - 81}} = \frac{-4\sqrt{2} \times \sqrt{\sqrt{337} + 9}}{\sqrt{256}} = \frac{-\sqrt{2} \times \sqrt{\sqrt{337} + 9}}{4}$$

Rationalise so that when $\alpha = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$, $\beta = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$

and when $\alpha = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$, $\beta = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$

Thus $x - \frac{3i}{2} = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4}i$ and $x - \frac{3i}{2} = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4}i$

$x = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{6 - \sqrt{2\left(\sqrt{337} - 9\right)}}{4}i \qquad x = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{6 + \sqrt{2\left(\sqrt{337} - 9\right)}}{4}i$

Square root of a complex number

The example below demonstrates the method for finding the square root of a complex number.

Example 26

Find the square root of $4 + 3i$

Solution

Let $z = x + iy$, where x, y are real, such that $z^2 = 4 + 3i$

Hence $(x + iy)^2 = 4 + 3i$

Expand: $x^2 - y^2 + 2xyi = 4 + 3i$

Equating the real and imaginary parts of the LHS and RHS gives:

$x^2 - y^2 = 4$ [1] and

$2xy = 3$ [2]

Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$: $(x^2 + y^2)^2 = 4^2 + 3^2 = 25$

$x^2 + y^2 = 5$ [3]

$x^2 - y^2 = 4$ [1]

[3] + [1]: $2x^2 = 9$

$$x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$$

From [2]: $y = \frac{3}{2x}$

$$y = \frac{3}{2} \times \left(\pm\frac{\sqrt{2}}{3}\right) = \pm\frac{\sqrt{2}}{2}$$

Hence the square roots of $4 + 3i$ are $\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ or $-\frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

EXERCISE 1.7 SOLVING QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS

1 Find the square root of the following complex numbers:

(a) $2i$ (b) $3 + 4i$ (c) $5 - 12i$ (d) $-8 + 15i$ (e) $-3 - 4i$ (f) $1 + i$

2 Solve the following statements.

(a) $x^2 + 2x + 2i = 0$ (b) $x^2 - 4x + 2 - i = 0$ (c) $x^2 + 2(2 + i)x + 3 = 0$

(d) $x^2 + (3 + 4i)x - 4 = 0$ (e) $x^2 + 2(1 - 3i)x + 2 + i = 0$ (f) $ix^2 - 4x + 3 = 0$

(g) $ix^2 + 2ix + 3 = 0$ (h) $(2 - i)x^2 + 2x + 1 = 0$

3 (a) Expand and simplify the expression $(x - 3)(x - 1 - i)(x - 1 + i)$.

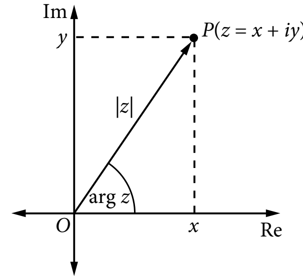
(b) Hence, or otherwise, solve the equation $x^3 - 5x^2 + 8x = 6$.

1.8 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A VECTOR

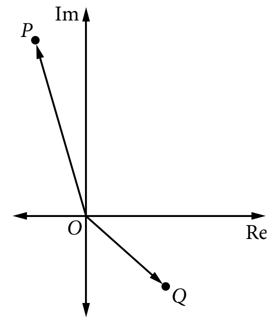
A **vector** is a mathematical object that has both magnitude and direction. For example, a 5° increase in temperature is a vector quantity, because it has both magnitude (5°) and direction ('increase').

Complex numbers can be represented as vectors for which the modulus gives the magnitude and the argument gives the direction.

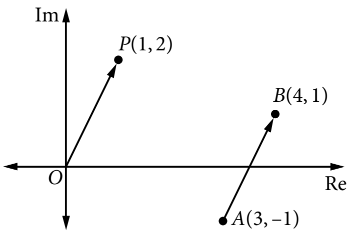
On an Argand diagram, let point P represent the complex number $z = x + iy$ (see diagram at right). This z is also represented by the vector $\overrightarrow{OP}$, where $\overrightarrow{OP}$ has length $|z|$ and direction given by a rotation from the positive direction of the real axis by $\arg z$.



In the diagram below right, point P represents the complex number $-1 + 3i$ and point Q represents the complex number $2\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$. Here the vector $\overrightarrow{OP}$ represents $-1 + 3i$ and the vector $\overrightarrow{OQ}$ represents $2\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$.



The vector to represent a complex number does not have to 'start' at the origin O. For example, the number $1 + 2i$ could be represented by any vector with length $\sqrt{5}$ that is inclined at an angle $\tan^{-1} 2$.



Both $\overrightarrow{OP}$ and $\overrightarrow{AB}$ represent $1 + 2i$.

Point P represents $1 + 2i$, but point B does not.

$\overrightarrow{OP}$ is called the **position vector**.

$\overrightarrow{AB}$ is called a **free vector**.

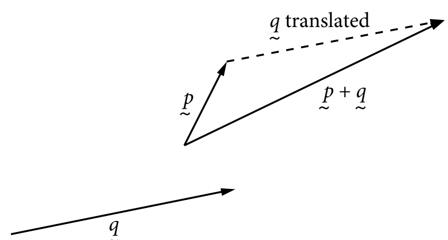
Note: $\overrightarrow{BA}$ does not represent $1 + 2i$. In fact, $\overrightarrow{BA}$ represents $-1 - 2i$.

Operations on complex numbers represented as vectors

Addition (to form the vector sum $\underset{\sim}{p} + \underset{\sim}{q}$):

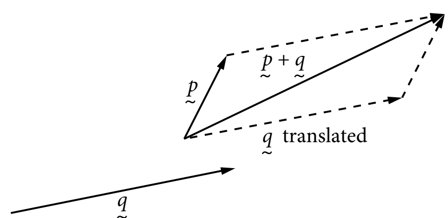
Method 1 (complete the triangle)

1 Translate $\underset{\sim}{q}$ so that its tail is located at the head of $\underset{\sim}{p}$.
2 The vector for the sum $\underset{\sim}{p} + \underset{\sim}{q}$ now goes from the tail of $\underset{\sim}{p}$ to the head of the translated $\underset{\sim}{q}$.



Method 2 (construct the diagonal of the parallelogram)

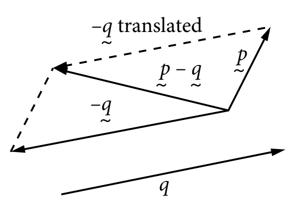
1 Translate $\underset{\sim}{q}$ so that its tail is located at the tail of $\underset{\sim}{p}$.
2 Locate the fourth vertex of the parallelogram of vectors.
3 The vector for the sum $\underset{\sim}{p} + \underset{\sim}{q}$ now goes from the common tail of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ translated to the fourth vertex of the parallelogram.



Subtraction (to form the vector difference $\underset{\sim}{p} - \underset{\sim}{q}$):

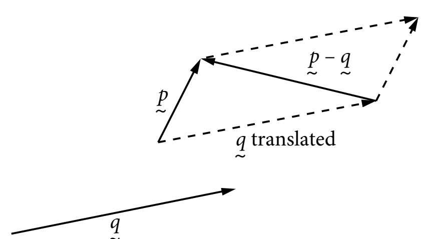
Method 1 (add the opposite vector)

1 Form the vector $-\underset{\sim}{q}$.
2 Form the vector sum $\underset{\sim}{p} + (-\underset{\sim}{q})$.



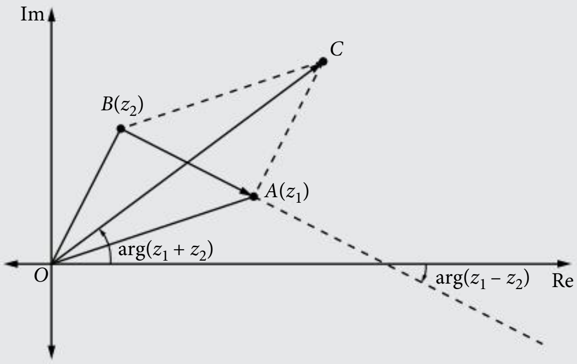
Method 2 (construct the other diagonal of the parallelogram)

1 Translate $\underset{\sim}{q}$ so that its tail is located at the tail of $\underset{\sim}{p}$.
2 Locate the fourth vertex of the parallelogram of vectors.
3 The vector for the sum $\underset{\sim}{p} - \underset{\sim}{q}$ is the diagonal of the parallelogram that goes *from* the head of $\underset{\sim}{q}$ *to* the head of $\underset{\sim}{p}$.



Let A represent z_1 and B represent z_2. When the parallelogram $OACB$ is formed:

- vector $\overrightarrow{OC}$ represents $z_1 + z_2$
- vector $\overrightarrow{AB}$ represents $z_2 - z_1$ and vector $\overrightarrow{BA}$ represents $z_1 - z_2$
- $|z_1 + z_2|$ and $|z_1 - z_2|$ are the lengths of the diagonals of the parallelogram
- $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$ are the angles at which the diagonals are inclined to the positive direction of the real axis.



Example 27

On an Argand diagram, let A represent $z_1 = 3 + 2i$ and B represent $z_2 = -2 + 4i$. Show the following:

(a) a point C that represents $z_1 + z_2$

(b) a vector that represents $z_1 + z_2$

(c) a vector that represents $z_1 - z_2$

(d) a point D that represents $z_1 - z_2$

(e) a point E that represents $\frac{1}{2}(z_1 + z_2)$.

(f) What complex number is represented by the vector $\overrightarrow{EB}$?

(g) What geometrical relationships are there between the intervals EB and DO?

Solution

(a) Locate C by completing the parallelogram.

(b) $\overrightarrow{OC}$

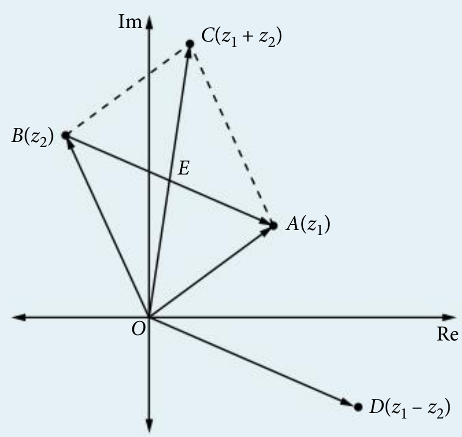
(c) $\overrightarrow{BA}$

(d) D is at the head of $\overrightarrow{OD}$, which is equal to $\overrightarrow{BA}$.

(e) E is the midpoint of the diagonal OC.

(f) $z_2 - \frac{1}{2}(z_1 + z_2) = \frac{1}{2}(z_2 - z_1)$

(g) $EB = \frac{1}{2}DO$, $EB \parallel DO$



MAKING CONNECTIONS

Operations on complex numbers represented as vectors

Use technology to explore operations on complex numbers represented as vectors.

Multiplication by i:

When working with vector representations of complex numbers, multiplication by i rotates the vector anticlockwise by $\frac{\pi}{2}$ (90°).

Multiplication by ki (where k is real):

To multiply by ki, rotate the vector anticlockwise by $\frac{\pi}{2}$ (90°) and scale by a factor (dilation) of k.

Multiplication of a complex number z by another complex number $r(\cos\theta + i\sin\theta)$:
To multiply z by $r\,\text{cis}\,\theta$, rotate the vector that represents z through an angle θ and scale by a factor of r.

Example 28

If k is real and $\frac{z_1 - z_2}{z_1 + z_2} = ki$, show that $|z_1| = |z_2|$.

Solution

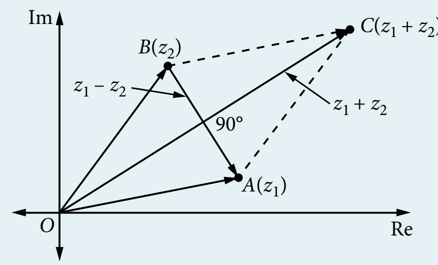
Show the problem on an Argand diagram. Let A represent z_1 and B represent z_2. Complete the parallelogram $OACB$ such that C represents $z_1 + z_2$.

As a result of this, the vectors along the diagonals of the parallelogram represent $z_1 + z_2$ (vector $\overrightarrow{OC}$) and $z_1 - z_2$ (vector $\overrightarrow{BA}$).

As $\frac{z_1 - z_2}{z_1 + z_2} = ki$, thus $\arg\left(\frac{z_1 - z_2}{z_1 + z_2}\right) = \arg(ki)$

$$\arg(z_1 - z_2) - \arg(z_1 + z_2) = \pm\frac{\pi}{2}$$

i.e. the diagonals meet at right angles.



Hence $OACB$ is a rhombus (parallelogram with perpendicular diagonals).

$\therefore |z_1| = |z_2|$ (sides of a rhombus are equal)

Hint: Whenever you see $z_1 + z_2$ and $z_1 - z_2$ in a problem, the problem can most likely be solved using the geometrical properties of parallelograms.

The triangle inequalities

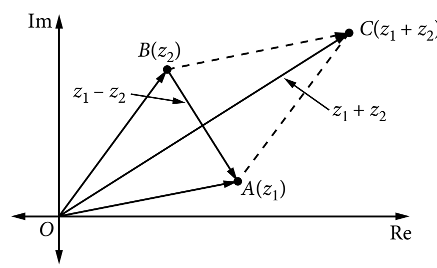
In a triangle OAC, $OC \le OA + AC$, with $OC = OA + AC$ when O, A and C are collinear.

$\therefore |z_1 + z_2| \le |z_1| + |z_2|$, with $|z_1 + z_2| = |z_1| + |z_2|$ when $z_2 = kz_1$ (where k is real).

In ΔOAB: $OA \le OB + BA$, with equality when O, B and A are collinear.

$\therefore |z_1| \le |z_2| + |z_1 - z_2|$

$\therefore |z_1 - z_2| \ge |z_1| - |z_2|$, with equality when $z_2 = kz_1$ (where k is real).



EXERCISE 1.8 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A VECTOR

1 On an Argand diagram, point A represents the complex number α. Point B is located so that the vector $\overrightarrow{OB}$ is the result of rotating $\overrightarrow{OA}$ anticlockwise by $\frac{2\pi}{3}$ and then halving its length. Which complex number represents point B?

A $\frac{\pi}{3}\alpha$ **B** $\alpha\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ **C** $\alpha\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ **D** $\frac{\alpha}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

2 On a complex plane, P represents $z = -3 + 4i$ and Q represents the complex number w. Find w so that triangle OPQ is:

(a) an isosceles right-angled triangle with the right angle at O
(b) an isosceles right-angled triangle with the right angle at P
(c) right-angled at O, with OQ twice the length of OP.

3 Point E is the centre of a square $ABCD$ (labelled anticlockwise) on an Argand diagram. E and A are the points corresponding to $-2+i$ and $1+5i$ respectively. Find the complex numbers represented by the points B, C and D.

4 **(a)** If $z_1 = 6+8i$ and $|z_2| = 15$, show that the greatest possible value of $|z_1 + z_2|$ is 25.

(b) If $|z_1 + z_2|$ takes this greatest value, find z_2 in Cartesian form.

5 On an Argand diagram, P represents $z = 1+i$ and Q represents q. Find the two possible values of q (in mod–arg form) such that ΔOPQ is equilateral.

6 On an Argand diagram, let A represent $z_1 = \sqrt{3}+i$, let B represent $z_2 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$, let C represent $z_3 = z_1 + z_2$ and let $\angle OAC = \theta$.

(a) Prove that $OACB$ is a rhombus. **(b)** Find z_3. **(c)** Find the value of θ.

7 On an Argand diagram, P represents the complex number z while w is the complex number $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. Points A and B represent wz and $\overline{w}z$ respectively. C is the midpoint of AB and represents the complex number α.

(a) Find α in terms of z.

(b) A point D is located so that $OADB$ is a parallelogram. Find, in terms of z, the complex number that D represents.

8 z_1 and z_2 are two complex numbers of equal moduli, with $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$. Use an Argand diagram to find the values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$ in terms of θ_1 and θ_2.

9 The points P and Q in the complex plane correspond to the complex numbers z and w respectively. Triangle OPQ is right-angled and isosceles with $OP = OQ$.

(a) Show that $w^2 + z^2 = 0$.

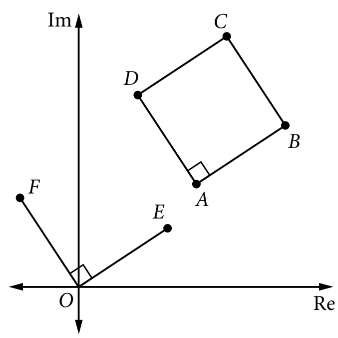
(b) If $OPRQ$ is a square, find (in terms of z) the complex number represented by E, the point of intersection of the diagonals of the square.

10 On an Argand diagram, $ABCD$ is a square. OE and OF are parallel to and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively.

(a) Explain why the point E corresponds to $w_2 - w_1$.

(b) What complex number corresponds to the point F?

(c) What complex number corresponds to the vertex D?



11 z_1 and z_2 are two complex numbers such that $\frac{z_1+z_2}{z_1-z_2} = 2i$.

(a) Show that $|z_1| = |z_2|$.

(b) If α is the angle between the vectors representing z_1 and z_2, show that $\tan\frac{\alpha}{2} = \frac{1}{2}$.

(c) Show that $z_2 = \frac{1}{5}(3+4i)z_1$.

12 The points P and Q in the complex plane correspond to the complex numbers z and w respectively, where $z = \frac{1+i}{1-i}$ and $w = \frac{\sqrt{2}}{1-i}$. The point R represents the complex number $z+w$.

(a) Find the modulus and argument of z and w.

(b) What type of quadrilateral is $OPRQ$?

(c) Deduce that $\tan\frac{3\pi}{8} = \sqrt{2}+1$.

13 z and w are two complex numbers.

(a) On an Argand diagram, show z, w, $z+w$ and $z-w$.

(b) **(i)** If $|z| = |w|$, show that $\frac{z+w}{z-w}$ is purely imaginary.

(ii) If $|z+w| = |z-w|$, show that $\frac{z}{w}$ is purely imaginary.

14 On an Argand diagram, the points P, Q and R represent the complex numbers w_1, w_2 and w_3 respectively such that $w_2 - w_1 = i(w_3 - w_1)$. Let S be the point that completes the parallelogram $PQSR$.

(a) What type of triangle is ΔPQR? Justify your answer.

(b) What complex numbers are represented by the vectors $\overrightarrow{PR}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$?

(c) Find, in terms of w_1, w_2 and w_3, the complex number that corresponds to the point S.

15 Points P and Q correspond to the complex numbers z_1 and z_2 respectively, where $z_1 = -\sqrt{2} + \sqrt{2}i$ and $z_2 = \sqrt{3} + i$. Point R corresponds to $z_1 + z_2$.

(a) Express z_1 and z_2 in mod–arg form.

(b) What type of quadrilateral is $OPRQ$? Justify your answer.

(c) Show that $\arg(z_1 + z_2) = \dfrac{11\pi}{24}$.

(d) Hence show that $\tan\dfrac{11\pi}{24} = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$.

16 (a) Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$ for any complex numbers z_1 and z_2.

(b) Complete the following sentence to give the geometrical interpretation of this result.
'In a parallelogram, the .... is equal to ....'

17 On an Argand diagram, the points A, B, C and D represent the complex numbers α, β, γ and δ respectively.

(a) Describe geometrically the point representing $\frac{1}{2}(\alpha + \gamma)$.

(b) If $\alpha + \gamma = \beta + \delta$ then what type of quadrilateral is $ABCD$? Justify your answer.

18 The four complex numbers z_1, z_2, z_3 and z_4 are represented on an Argand diagram by the points A, B, C and D respectively. If $z_1 - z_2 + z_3 - z_4 = 0$ and $z_1 - iz_2 - z_3 + iz_4 = 0$, determine the nature of quadrilateral $ABCD$.

19 z_1, z_2 and z_3 are three complex numbers represented on an Argand diagram by the points Z_1, Z_2 and Z_3 respectively. If $z_1 z_2 = (z_3)^2$, show that the moduli r_1, r_3 and r_2 are successive terms of a geometric series (in that order) and that OZ_3 bisects the angle Z_1OZ_2.

1.9 ROOTS OF COMPLEX NUMBERS

De Moivre's theorem provides an easy means of finding z^n given z:

If $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

You are now faced with the reverse problem: given z^n, find z (i.e. the n-th root of z^n). The examples below show that there are n such roots.

Example 29

Find the cube roots of 1 (i.e. solve the equation $z^3 = 1$).

Solution

Let $z = r(\cos\theta + i\sin\theta)$ be a root, where r is a positive real number.

Note that $1 = 1(\cos 0 + i\sin 0)$.

Then: $r^3(\cos 3\theta + i\sin 3\theta) = 1(\cos 0 + i\sin 0)$

$\therefore$ $r^3 = 1$ (i.e. $r = 1$)

and $\cos 3\theta + i\sin 3\theta = \cos 0 + i\sin 0$

$\therefore$ $\cos 3\theta = \cos 0$ and $\sin 3\theta = \sin 0$

If you considered only $\sin 3\theta = \sin 0$, you would conclude that $3\theta = 0 + k\pi$ (where $k = 0, \pm1, \pm2, \ldots$).
But $\cos 3\theta = \cos 0$ must also be true:

$$\therefore \quad 3\theta = 0 + 2k\pi \qquad \text{(where } k = 0, \pm1, \pm2, \ldots)$$

$$\therefore \quad \theta = \frac{2k\pi}{3}$$

Hence: $z = 1\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$ (where $k = 0, \pm1, \pm2, \ldots$)

For $k = 0$: $z_1 = 1$

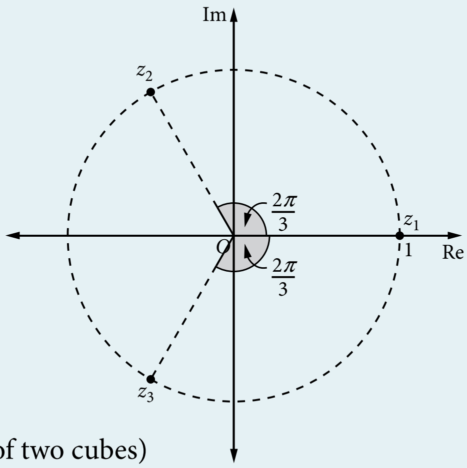
For $k = 1$: $z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

For $k = -1$: $z_3 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

It might appear that there are many more values of z obtained from $k = \pm2, \pm3, \ldots$ However, if you check them you will find that they simply repeat the values of z that have already been found (because their cosines are equivalent): e.g. for $k = 2$, $z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = z_3$.

There are therefore only **three** cube roots of unity, and their representation on the complex plane reveals an interesting pattern. Each has a modulus of 1, so they are all on the circumference of a unit circle.

Furthermore, the non-real roots are a pair of conjugates (as they are the roots of a polynomial equation with real coefficients), $z_2 = \overline{z_3}$. The three roots are equally spaced around the circle, each separated by an angle of $\frac{2\pi}{3}$ at the centre. They form the vertices of an equilateral triangle.



Alternatively:

The equation can be solved algebraically. $z^3 - 1 = 0$

$$(z - 1)(z^2 + z + 1) = 0 \qquad \text{(difference of two cubes)}$$

Hence $z = 1$ or $z = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ as above.

Example 30

Find the six 6th roots of -64 (i.e. solve $z^6 = -64$).

Solution

Let $z = r(\cos\theta + i\sin\theta)$ be a root.

Then: $r^6(\cos 6\theta + i\sin 6\theta) = 64(\cos\pi + i\sin\pi)$

$$\therefore \quad r^6 = 64$$

$$r = 2 \qquad (r > 0 \text{ because it is the modulus of } z)$$

and $6\theta = \pi + 2k\pi$ (where $k = 0, \pm1, \pm2, \ldots$)

$$\theta = \frac{\pi}{6} + \frac{2k\pi}{6}$$

Hence: $z = 2\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)\right)$ (where $k = 0, \pm 1, \pm 2, \ldots$)

For $k = 0$: $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$

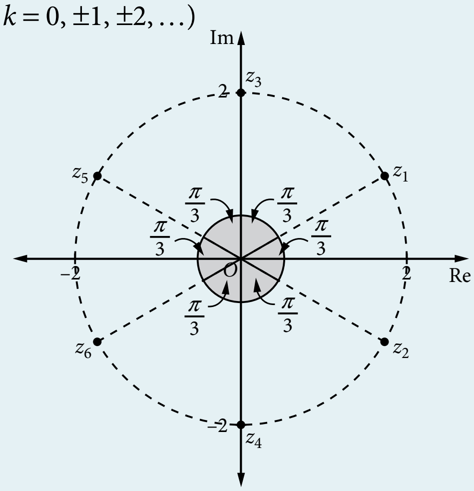
For $k = -1$: $z_2 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \sqrt{3} - i$

For $k = 1$: $z_3 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$

For $k = -2$: $z_4 = 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = -2i$

For $k = 2$: $z_5 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$

For $k = -3$: $z_6 = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) = -\sqrt{3} - i$



As each root has a modulus of 2, they are all on the circumference of a circle of radius 2. They are also equally spaced around the circumference, each separated by an angle of $\frac{2\pi}{6}$ (i.e. $\frac{\pi}{3}$) at the centre. The roots occur in conjugate pairs (as they are the roots of a polynomial equation with real coefficients). They form the vertices of a regular hexagon.

Example 31

Find the four 4th roots of $1 + \sqrt{3}i$ (i.e. solve $z^4 = 1 + \sqrt{3}i$). Answer in mod–arg form.

Solution

Let $z = r(\cos\theta + i\sin\theta)$ be a root.

Then: $r^4(\cos 4\theta + i\sin 4\theta) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore$ $r^4 = 2$

$r = \sqrt[4]{2}$ ($r > 0$ because it is the modulus of z)

and $4\theta = \frac{\pi}{3} + 2k\pi$ (where $k = 0, \pm 1, \pm 2, \ldots$)

$\theta = \frac{\pi}{12} + \frac{k\pi}{2}$

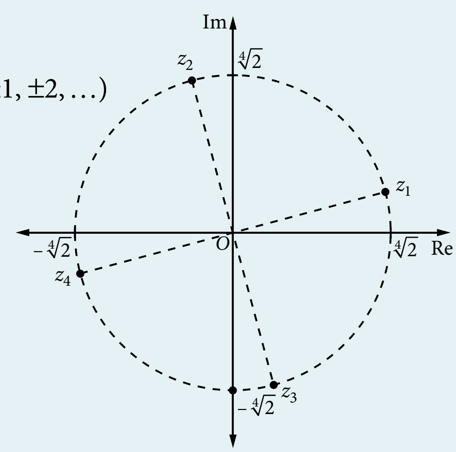
Hence: $z = \sqrt[4]{2}\left(\cos\left(\frac{\pi}{12} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$ (where $k = 0, \pm 1, \pm 2, \ldots$)

For $k = 0$: $z_1 = \sqrt[4]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

For $k = -1$: $z_2 = \sqrt[4]{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

For $k = 1$: $z_3 = \sqrt[4]{2}\left(\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right)$

For $k = -2$: $z_4 = \sqrt[4]{2}\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$



The roots are equally spaced around the circumference of a circle of radius $\sqrt[4]{2}$. Because the polynomial equation $z^4 = 1 + \sqrt{3}i$ has non-real coefficients, the roots do not occur as conjugate pairs.

To summarise:

- The n-th roots of **1** (i.e. the n roots of $z^n = 1$) are equally spaced around the circumference of the circle with centre O and radius 1, separated by an angle of $\frac{2\pi}{n}$ at the centre.
- One root is 1. If n is even, another root is -1. The other roots occur as non-real conjugate pairs.
- The roots are $1\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$ where $k = 0, \pm 1, \pm 2, \ldots$ until the n unique roots are identified.

- The n-th roots of **any complex number** $R\,\text{cis}\,\alpha$ are equally spaced around the circumference of the circle with centre O and radius $\sqrt[n]{R}$, separated by an angle of $\frac{2\pi}{n}$ at the centre.
- The roots are $\sqrt[n]{R}\left(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}\right)$ where $k = 0, \pm 1, \pm 2, \ldots$ until the n unique roots are identified.

Complex roots of unity

To investigate some properties of the n-th roots of 1, you can work with symbolic representations rather than the actual values of the roots.

An important result that is frequently used is this factorisation:

$$w^n - 1 = (w-1)(w^{n-1} + w^{n-2} + \ldots + w^2 + w + 1)$$

For example, $w^3 - 1 = (w-1)(w^2 + w + 1)$, which you already know, and $w^5 - 1 = (w-1)(w^4 + w^3 + w^2 + w + 1)$.

You can prove this general factorisation by expanding the RHS or by regarding $1 + w + w^2 + \ldots + w^{n-1}$ as the sum of a geometric series.

Example 32

If w is a non-real cube root of unity, show that:

(a) $\overline{w}$ is also a root **(b)** $1 + w + \overline{w} = 0$ **(c)** $1 + w + w^2 = 0$ **(d)** $(1 + w^2)^3(2 + 3w + 3w^2) = 1$

Solution

(a) w is a root of $z^3 = 1$, so $w^3 = 1$.

Take conjugates of both sides (if two complex numbers are equal then their conjugates are equal):

$\overline{w^3} = \overline{1 + 0i}$

$(\overline{w})^3 = 1$ (the conjugate of a power equals the power of the conjugate)

i.e. $\overline{w}$ also satisfies $z^3 = 1$, so $\overline{w}$ is also a root.

(b) From part **(a)**, you know the three roots of $z^3 - 1 = 0$ are 1, w and $\overline{w}$.

Sum of roots $= -\frac{b}{a}$ $\quad\therefore 1 + w + \overline{w} = 0$

(c) $w^3 = 1 \quad \therefore w^3 - 1 = 0$

Factorise: $(w-1)(w^2 + w + 1) = 0$

$\therefore w - 1 = 0$ or $w^2 + w + 1 = 0$

But w is non-real $\quad\therefore w^2 + w + 1 = 0$

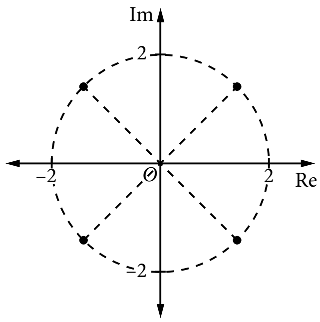
(d) Use the results $w^3 = 1$ and $w^2 + w + 1 = 0$ to obtain $1 + w^2 = -w$ and $w + w^2 = -1$:

$$\begin{aligned}(1 + w^2)^3(2 + 3w + 3w^2) &= (-w)^3\left(2(1 + w + w^2) + w + w^2\right) \\ &= -w^3(2 \times 0 + (-1)) \\ &= -1 \times -1 \\ &= 1\end{aligned}$$

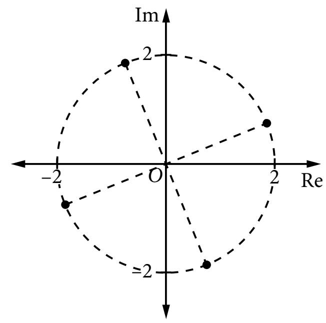
EXERCISE 1.9 ROOTS OF COMPLEX NUMBERS

1 Which Argand diagram best shows the fourth roots of $16i$?

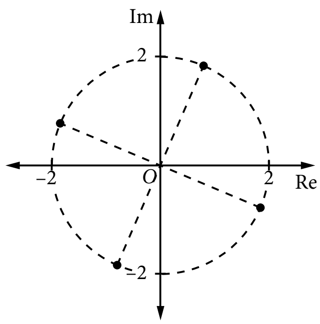
A



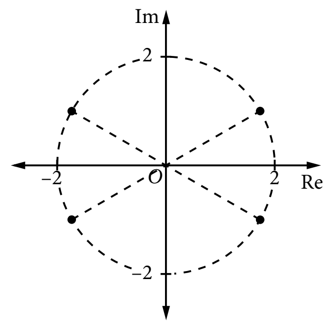
B



C



D



2 For each of the following, find the values of z (in mod–arg form) and plot them on the complex plane.

(a) $z^5 = 1$ **(b)** $z^4 + 1 = 0$ **(c)** $z^2 = i$ **(d)** $z^3 + 8i = 0$ **(e)** $z^4 = 8(\sqrt{3} + i)$ **(f)** $z^6 = i$

3 Find (in mod–arg form):

(a) the 5th roots of 32 **(b)** the 4th roots of -16

(c) the 5th roots of -1 **(d)** the 6th roots of $27\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

4 The point $1 + \sqrt{3}i$ and two other points are on the circumference of a circle with centre O and radius 2. The three points are the vertices of an equilateral triangle.

(a) Find the complex numbers represented by the two other points.
(b) Find the cubic equation that has these three complex numbers as its roots.

5 If 1, w_1 and w_2 are the cube roots of unity, prove the following:

(a) $w_1 = \overline{w_2} = w_2^2$ **(b)** $w_1 + w_2 = -1$ **(c)** $w_1 w_2 = 1$

6 If w is a non-real cube root of unity (i.e. w is a non-real root of $z^3 = 1$), show the following:

(a) $1 + w + w^2 = 0$ **(b)** $(1 - w)(1 - w^2) = 3$

Now evaluate the following:

(c) $(1 + w)^3$ **(d)** $(1 + 2w + 3w^2)(1 + 2w^2 + 3w)$ **(e)** $(w^2 + 2w + w^3)(2w^2 + w + w^3)$
(f) $(1 - w)(1 - w^2)(1 - w^4)(1 - w^5)(1 - w^7)(1 - w^8)$

7 **(a)** If w is a non-real root of $z^5 = 1$, show that the other non-real roots can be expressed as w^2, w^{-1} and w^{-2}.
(b) Hence show that $w^2 + w + 1 + w^{-1} + w^{-2} = 0$.
(c) Show that $\cos\frac{2\pi}{5} - \cos\frac{\pi}{5} = -\frac{1}{2}$.
(d) Deduce that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$.

8 **(a)** Find the roots of $z^7 = 1$ in mod–arg form and show them on an Argand diagram.
(b) If w is a non-real root, show that $w + w^2 + w^3 + w^4 + w^5 + w^6 = -1$.
(c) Show that the quadratic equation $z^2 + z + 2 = 0$ has roots $w + w^2 + w^4$ and $w^3 + w^5 + w^6$.
(d) Show that $\cos\frac{\pi}{7} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \frac{1}{2}$.

9 w is a non-real root of $z^6 = 1$.

(a) Show that $1 + w^2 + w^4 = 0$.

(b) Show that $w^{-2} = 1 + 2w^4 + w^8$.

10 (a) Show that the roots of $z^6 + z^3 + 1 = 0$ are also roots of $z^9 - 1 = 0$.

(b) Show the nine 9th roots of unity on an Argand diagram. Identify which of them are the roots of $z^6 + z^3 + 1 = 0$.

(c) Show that the sum of the six roots of $z^6 + z^3 + 1 = 0$ is $2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}\right)$.

(d) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$.

(e) Show that $z^6 + z^3 + 1 = \left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$.

(f) Hence evaluate $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} - \cos\frac{4\pi}{9}\cos\frac{\pi}{9} - \cos\frac{\pi}{9}\cos\frac{2\pi}{9}$.

11 (a) Show that $z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a root of $z^4 + z^3 + z^2 + z + 1 = 0$.

(b) Find all four roots of $z^4 + z^3 + z^2 + z + 1 = 0$.

(c) Show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

(d) Deduce that $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$.

12 $w_1, w_2, \ldots w_{10}$ are the ten 10th roots of unity and are represented by points $W_1, W_2, \ldots W_{10}$ on an Argand diagram. Point P represents the variable complex number z such that $|z| = 1$.

(a) Show this information on an Argand diagram.

(b) Show that $\sum_{i=1}^{10} w_i = \sum_{i=1}^{10} \overline{w_i} = 0$.

(c) Show that the distance from P to any of the points W_i is such that $[PW_i]^2 = 2 - z\overline{w_i} - \overline{z}w_i$.

(d) Show that $\sum_{i=1}^{10} [PW_i]^2 = 20$.

1.10 CURVES AND REGIONS ON THE ARGAND DIAGRAM

You have seen that the modulus and argument, while defined algebraically, can also be thought of geometrically. If $z = x + iy$, then:

- $|z| = \sqrt{x^2 + y^2}$ is the algebraic definition, while geometrically $|z|$ is the distance from the origin O to the point (x, y)
- algebraically $|z - (a + ib)| = \sqrt{(x-a)^2 + (y-b)^2}$, while geometrically $|z - (a + ib)|$ is the magnitude of the vector from (a, b) to the point representing z
- geometrically $\arg z$ is the angle made with the positive direction of the real axis by the vector from the origin O to the point representing z
- geometrically $\arg(z - z_1)$ is the angle made with the positive direction of the real axis by the vector from the point representing z_1 to the point representing z
- $\arg 0$ is undefined.

When you need to sketch a subset on an Argand diagram, you should try to interpret the information geometrically rather than algebraically, unless the question contains an obvious algebraic substitution. For example, the presence of $z + \overline{z}$ (which is $2x$) or $z - \overline{z}$ (which is $2yi$) would indicate an algebraic approach.

Example 33

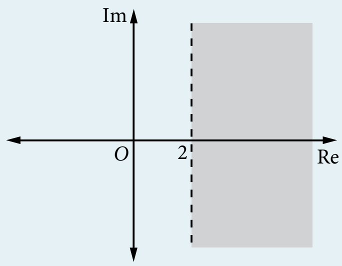
On an Argand diagram, sketch the subsets of z for each of the following.

(a) $z + \overline{z} > 4$ **(b)** $\mathrm{Im}(z) = 3$ **(c)** $2|z| = z + \overline{z} + 4$

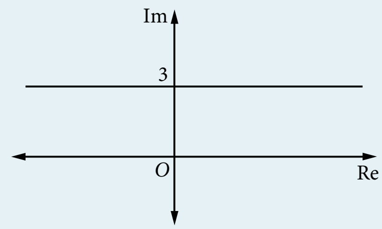
Solution

These need to be done algebraically. Let $z = x + iy$.

(a)
$$z + \overline{z} > 4$$
$$2x > 4$$
$$x > 2$$

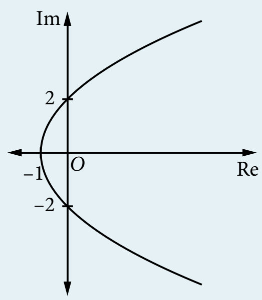


(b)
$$\mathrm{Im}(z) = 3$$
$$y = 3$$



(c)
$$2|z| = z + \overline{z} + 4$$
$$2\sqrt{x^2 + y^2} = 2x + 4$$
$$4x^2 + 4y^2 = 4x^2 + 16x + 16$$
$$y^2 = 4x + 4$$

The region is the set of points on the parabola.



Example 34

On an Argand diagram, sketch the subsets of z for each of the following.

(a) $|z| = 2$ **(b)** $|z + 2 - 2i| = 2$ **(c)** $|z - 2i| = |z + 1 - i|$

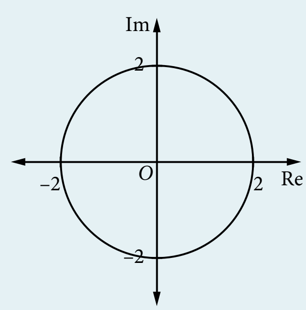
(d) For part **(b)**, find the possible values of $\arg z$ and the maximum and minimum values of $|z|$ on the locus.

Solution

These are best done geometrically, regarding each modulus as a distance. Parts **(a)** and **(c)** are also solved algebraically below (with $z = x + iy$).

(a) Geometrically:

The distance from the origin to the point representing z is equal to 2. (You can just say 'the distance from O to z is 2'.) The subset is the circle with centre O and radius 2.



Algebraically:

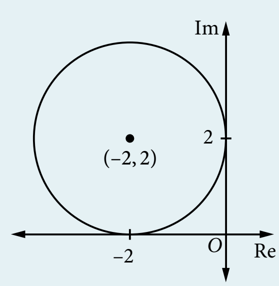
$$\sqrt{x^2 + y^2} = 2$$
$$x^2 + y^2 = 4$$

(b) $|z + 2 - 2i| = 2$

$|z - (-2 + 2i)| = 2$ (written in the form $|z - z_1|$)

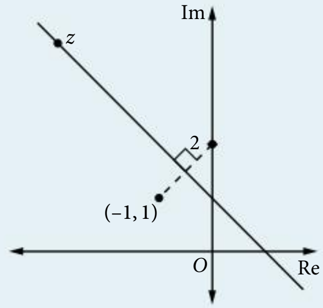
The distance from $-2 + 2i$ to z is 2.

The subset is the circle with centre $(-2, 2)$ and radius 2.



(c) Geometrically:

The distance from $2i$ to z is equal to the distance from $-1+i$ to z,
i.e. z is equidistant from $2i$ and $-1+i$.
This is the perpendicular bisector of the interval joining $(0,2)$ and $(-1,1)$.



Algebraically:

$$|x+(y-2)i| = |(x+1)+(y-1)i|$$
$$\sqrt{x^2+(y-2)^2} = \sqrt{(x+1)^2+(y-1)^2}$$
$$x^2+y^2-4y+4 = x^2+2x+1+y^2-2y+1$$
$$x+y-1=0$$

(d) As z moves around the circle, the smallest value of $\arg z$ occurs when z is at point B (where $\arg z = \frac{\pi}{2}$). The largest value of $\arg z$ occurs when z is at C (where $\arg z = \pi$).

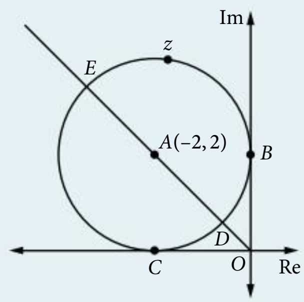
$\therefore \frac{\pi}{2} \le \arg z \le \pi$

The smallest value of $|z|$ occurs when z is at D and the largest value when z is at E.

Now $OA = 2\sqrt{2}$ (Pythagoras), $AD = 2$ (radius)

$\therefore$ minimum $|z| = OD = OA - AD = 2\sqrt{2} - 2$

maximum $|z| = OE = OA + AE = 2\sqrt{2} + 2$



Example 35

On an Argand diagram, sketch the region of the complex plane for each of the following.

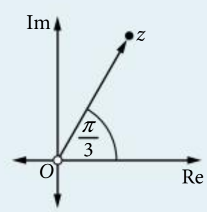
(a) $\arg z = \frac{\pi}{3}$ **(b)** $-\frac{\pi}{4} < \arg z \le \frac{\pi}{3}$ **(c)** $\arg(z-2+2i) = \frac{3\pi}{4}$

Solution

These can be done geometrically.

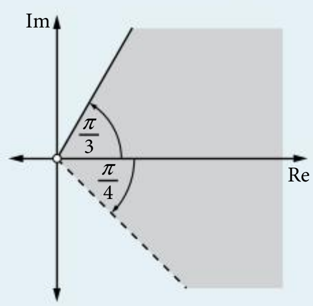
(a) The vector from O to the point representing z is inclined to the positive direction of the real axis at an angle of $\frac{\pi}{3}$.

Note the open circle at O: $z \ne 0$ as $\arg 0$ is undefined.



(b) This is the region between the vectors from O that have inclinations of $-\frac{\pi}{4}$ and $\frac{\pi}{3}$. z can be anywhere in the shaded region.

Note the dashed line on the vector inclined at $-\frac{\pi}{4}$ (due to the $<$ symbol) and the open circle at O (for the undefined $\arg 0$).

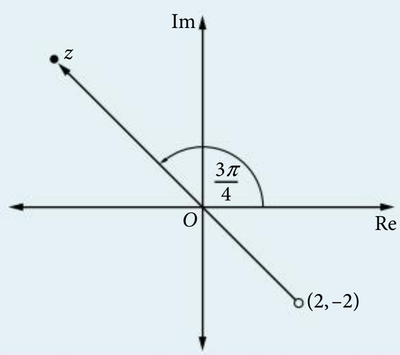


(c) $\arg(z-2+2i) = \frac{3\pi}{4}$

$\therefore \arg(z-(2-2i)) = \frac{3\pi}{4}$

The vector from $(2,-2)$ to the point representing z is inclined at $\frac{3\pi}{4}$ to the positive direction of the real axis.

Note that the vector passes through O. (You should try to notice details like this without being prompted.) Also note the open circle at $(2,-2)$, again for the undefined $\arg 0$.



Example 36

On an Argand diagram, sketch the subsets of z for each of the following.

(a) $\arg\left(\dfrac{z-2i}{z+3}\right)=0$ **(b)** $\arg\left(\dfrac{z-2i}{z+3}\right)=\pi$

Solution

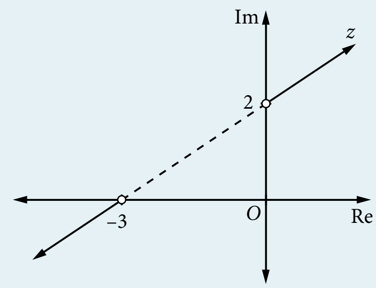
Use the result $\arg\left(\dfrac{z_1}{z_2}\right)=\arg z_1-\arg z_2$.

(a) $\arg(z-2i)-\arg(z+3)=0$

$\therefore\ \arg(z-2i)=\arg(z+3)$

The angle of inclination of the vector from $2i$ to z is equal to the angle of inclination of the vector from -3 to z. This means that z is along the straight line that joins $2i$ and -3, but z must be on the 'outer' parts of the line ('going away'). The direction from $2i$ to z is exactly the same direction as from -3 to z.

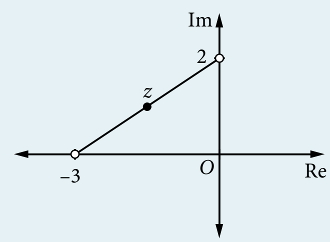
Note the open circles at $2i$ and -3.



(b) $\arg(z-2i)-\arg(z+3)=\pi$

The direction from $2i$ to z is exactly opposite to the direction from -3 to z. This means that z is anywhere along the 'inner' part of the straight line that joins $2i$ and -3.

Note the open circles at $2i$ and -3.



EXPLORING FURTHER

Regions on the complex plane

Use technology to explore regions on the complex plane.

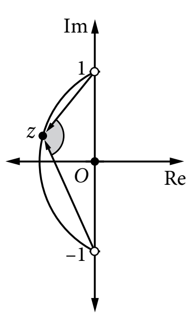
EXERCISE 1.10 CURVES AND REGIONS ON THE ARGAND DIAGRAM

1 If $z=x+iy$, the Cartesian equation $x-y=0$ represents:

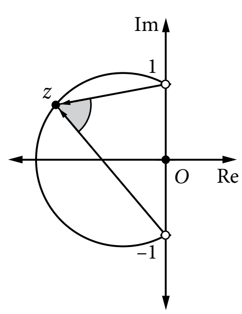
A $\arg z=\dfrac{\pi}{4}$ **B** $|z+2i|=|z+2|$ **C** $\arg z=-\dfrac{3\pi}{4}$ **D** $|z+2i|=|z-2|$

2 Which diagram represents z such that $\arg(z+i)-\arg(z-i)=\dfrac{2\pi}{3}$?

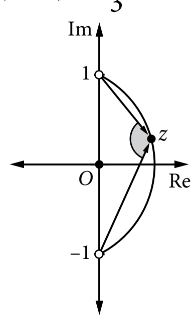
A



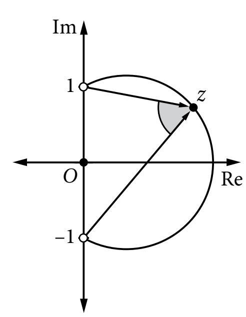
B



C



D



3 On Argand diagrams, show the curves or regions described by the following.

(a) $|z|=4$ **(b)** $|z|\le 2$ **(c)** $1\le|z|\le 3$ **(d)** $\left|z-(1+\sqrt{3}i)\right|=2$ **(e)** $|z-2+2i|=3$

(f) $|z+2|=|z|$ **(g)** $|z+2-i|=|z-2+i|$ **(h)** $|z+2|\ge|z-4|$ **(i)** $\dfrac{|z-3+3i|}{|z+1-i|}=1$

4 On Argand diagrams, show the curves or regions described by the following.

(a) $\arg z = \frac{\pi}{3}$ (b) $\arg z = \frac{2\pi}{3}$ (c) $-\frac{\pi}{3} \le \arg z \le \frac{2\pi}{3}$

(d) $\arg(z+2) = \frac{3\pi}{4}$ (e) $\arg(z-i) = 0$ (f) $\arg(z+1+i) = -\frac{2\pi}{3}$

5 Show the following on the complex plane.

(a) $\text{Re}(z) = 2$ (b) $\text{Im}(z) = -1$ (c) $\text{Re}(z) + \text{Im}(z) = 1$ (d) $\text{Re}(z) < \text{Im}(z)$ (e) $z + \overline{z} = 6$

(f) $z - \overline{z} = 4i$ (g) $2|z| = z + \overline{z} + 4$ (h) $|z^2 - (\overline{z})^2| \ge 16$ (i) $|z + 2 - 4i| = 2|z - 4 - i|$

(j) $|z - z_1|^2 - |z - z_2|^2 = 16$, where $z_1 = 2 + 3i$ and $z_2 = -2 - i$

6 On Argand diagrams, show:

(a) the region where $|z-1| \le 1$ and $\text{Im}(z) \ge 0$ are both true

(b) the intersection of $2 \le |z| \le 3$ and $-\frac{\pi}{2} \le \arg z \le \frac{\pi}{4}$

(c) the intersection of $-\frac{\pi}{3} \le \arg z \le \frac{\pi}{3}$ and $\text{Re}(z) < 2$

(d) the intersection of $|z| \le 3$ and $\text{Re}(z) + \text{Im}(z) \le 3$

(e) the region common to $z\overline{z} \le 4$ and $z + \overline{z} \le 2$.

CHAPTER REVIEW 1

1 If $z = 1 + 2i$ and $w = -3 - 4i$, find the following in $x + iy$ form:

(a) $3z + w$ (b) z^2 (c) $w\overline{w}$ (d) $\frac{z}{w}$ (e) the square roots of w.

2 Solve the following equations.

(a) $z^2 + 25 = 0$ (b) $z^2 + 2z + 2 = 0$ (c) $8z^2 - 20z + 17 = 0$ (d) $\frac{iz}{z-1} = 3 + 2i$

3 Find the real numbers p and q for each of the following.

(a) $\frac{p + 3qi}{2 + i} = \overline{1 - i}$ (b) $(2 - ip)(3 + iq) = 4 + 7i$

4 Convert the following complex numbers into mod–arg form.

(a) $-3 + 3i$ (b) $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

5 Express the following in Cartesian form:

(a) $4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ (b) $6\sqrt{2}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

6 (a) Evaluate the following, giving answers in both mod–arg form and $x + iy$ form.

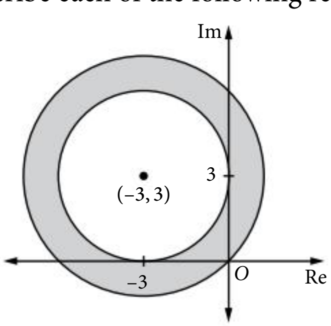
(i) $(\sqrt{3} - i)^3$ (ii) $\frac{(1 - \sqrt{3}i)^2}{(1+i)^3}$

(b) Use your answer to part **(a)(ii)** to show that $\cos\frac{7\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}$.

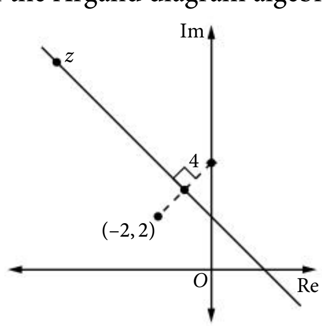
7 If $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ and $w = \cos\left(-\frac{3\pi}{10}\right) + i\sin\left(-\frac{3\pi}{10}\right)$, find $\frac{z^2}{w^5}$ in mod–arg form.

8 Describe each of the following regions of the Argand diagram algebraically.

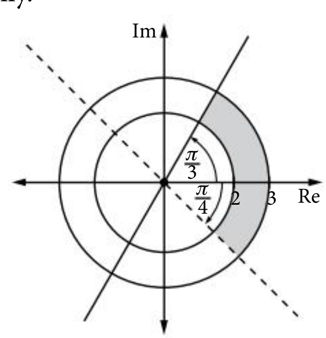
(a)



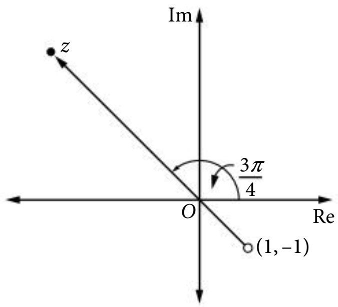
(b)



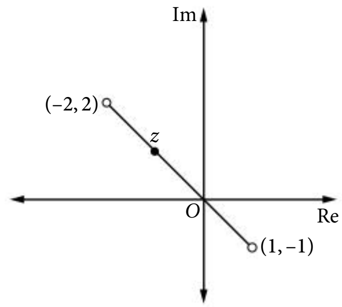
(c)



(d)



(e)



9 Solve the following, giving answers in mod–arg form.

(a) $z^4 = -1$ (b) $z^6 = 1$ (c) $1 + z + z^2 + z^3 + z^4 = -z^5$

10 Find the product of the five 5th roots of $\left(1+\sqrt{2}i\right)^3$.

11 If $z = \cos\theta + i\sin\theta$:

(a) Show that $\arg(z^2 + z^4) = 3\theta$. (b) Show that $z^2 + z^4 = 2\cos\theta(\cos 3\theta + i\sin 3\theta)$.

(c) Find the value(s) of θ for which $z^2 + z^4$ is purely imaginary, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

12 Find the range of values of $\arg(z)$ if $|z - 2i| = 1$.

13 (a) If w is a root of $z^{12} = i$, show that $-w$ is also a root.

(b) Let z_1 and z_2 be two distinct roots of $z^{12} = i$. Show that $|z_1 + z_2| < 2$.

14 (a) If $\arg z = \alpha$, show that $\arg iz = \frac{\pi}{2} + \alpha$.

(b) On an Argand diagram, show a non-zero complex number z and the corresponding position of iz.

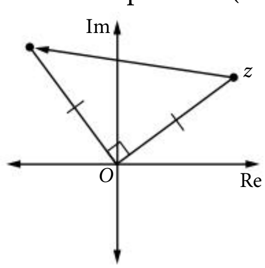
(c) Use a geometrical argument to show that $|z|^2 + |iz|^2 = |z - iz|^2$.

(d) Show that the area of the triangle formed on an Argand diagram by z, iz and the origin O is $\frac{1}{2}z\overline{z}$.

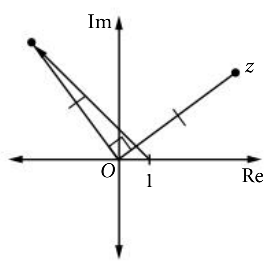
(e) If $z = 1 + 2i$, find the area between the triangle of part **(d)** (formed by z, iz and O) and the circle in which that triangle is inscribed.

15 On an Argand diagram, point Z is shown to represent the complex number z. Which diagram below shows the vector that represents $(1 - i)z$?

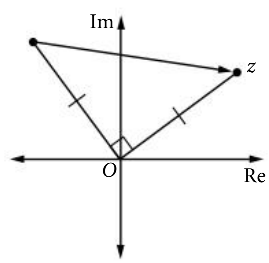
A



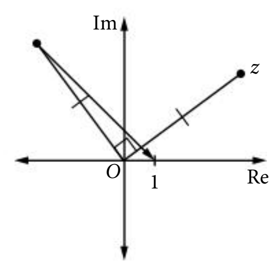
B



C



D



16 On an Argand diagram, the points A, B, C and D represent the complex numbers α, β, γ and δ respectively.

(a) Describe the point that represents $\frac{1}{2}(\alpha + \gamma)$.

(b) If $\alpha + \gamma = \beta + \delta$, deduce that $ABCD$ is a parallelogram.

17 Let $z = x + iy$ be any non-zero complex number such that $z + \frac{1}{z} = k$, where k is a real number.

(a) Prove that either $y = 0$ or $x^2 + y^2 = 1$. (b) If $y = 0$, show that $|k| \geq 2$.

(c) If $x^2 + y^2 = 1$, show that $|k| \leq 2$.

18 On an Argand diagram, the points A and C represent the complex numbers $3i$ and $4 - 5i$ respectively. $ABCD$ is a rhombus.

(a) Find the Cartesian equation of the diagonal BD.

(b) Show that the diagonal BD is also represented by the equation $(1 + 2i)z + (1 - 2i)\overline{z} - 8 = 0$.

19 If w is a non-real root of the equation $z^5 = 1$, show that:

(a) $1 + w + w^2 + w^3 + w^4 = 0$ **(b)** $(1-w)(1-w^2)(1-w^3)(1-w^4) = 5$

(c) $z_1 = w + w^4$ and $z_2 = w^2 + w^3$ are the roots of the quadratic equation $z^2 + z - 1 = 0$.

20 **(a)** Find the cube roots of -8 in mod–arg form.

(b) If w_1 and w_2 are the non-real roots of -8, show that ${w_1}^{6n} + {w_2}^{6n} = 2^{6n+1}$ for all integers n.

21 On an Argand diagram, A represents the complex number $z = \cos\theta + i\sin\theta$. B represents wz, where $w = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. M is the midpoint of OB.

(a) Show that $\overrightarrow{AM} = \frac{1}{2}wz - z$. **(b)** Show that $\left|\frac{1}{2}wz - z\right| = \sqrt{2-\sqrt{2}}$.

(c) Show that $\arg\left(\frac{1}{2}wz - z\right) = \frac{5\pi}{8} + \theta$.

22 **(a)** Show that $(1 + \cos\theta + i\sin\theta)(1 - (\cos\theta + i\sin\theta)) = 1 - (\cos 2\theta + i\sin 2\theta)$.

(b) Prove by induction that $1 + \text{cis}\,\theta + \text{cis}\,2\theta + \ldots + \text{cis}\,n\theta = \dfrac{1 - \text{cis}(n+1)\theta}{1 - \text{cis}\,\theta}$ provided $\cos\theta + i\sin\theta \neq 1$.

(c) Hence evaluate $1 + \text{cis}\frac{\pi}{36} + \text{cis}\frac{2\pi}{36} + \ldots + \text{cis}\frac{71\pi}{36}$.

(d) Interpret part **(c)** as a statement about the roots of $z^{72} = 1$.

23 **(a)** Show that $\cos\frac{\pi}{12} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$. (Recall that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.)

(b) Show that $\sin\frac{\pi}{12} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$. **(c)** Evaluate $\left(\dfrac{\sqrt{2+\sqrt{3}}}{2} + \dfrac{\sqrt{2-\sqrt{3}}}{2}i\right)^4$.

24 **(a)** Given that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ (see Example **17**, page 16), solve $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$.

(b) Show that $\tan\frac{\pi}{9} - \tan\frac{2\pi}{9} + \tan\frac{4\pi}{9} = 3\sqrt{3}$.

25 Solve $\left(\dfrac{z}{z-i}\right)^4 = -4$. (Hint: Let $w = \dfrac{z}{z-i}$.)

26 One root of $8x^3 + 8x^2 - 1 = 0$ is $-\frac{1}{2}$. Find the other roots.

27 **(a)** Find the square root of $5 + 12i$.

(b) Hence, or otherwise, solve the equation $z^2 + 3z + 1 - 3i = 0$.

28 Let $P(x) = x^3 + ax^2 + bx + 2$ where a and b are real numbers. Find the values of a and b given that $(x-1)^2$ is a factor of $P(x)$.

29 Two of the zeros of $P(x) = x^4 - 4x^3 + 26x^2 - 44x + 85$ are $a + ib$ and $a + 2ib$, where a and b are real and $b > 0$.

(a) Find the values of a and b.

(b) Hence, or otherwise, express $P(x)$ as the product of quadratic factors with real coefficients.

30 The polynomial $P(x) = ax^3 + bx + c$ has a multiple zero at $x = 2$ and has a remainder of 20 when divided by $x + 2$. Find a, b and c.

CHAPTER 2
The nature of proof

2.1 THE LANGUAGE AND LOGIC OF PROOF

Mathematical knowledge is unlike the knowledge of other scientific disciplines. As opposed to consisting of theories based on observation and evidence that can possibly be falsified, it consists of *theorems*: significant mathematical statements that have been proved to be true: (think of Pythagoras' theorem, for instance). Scientific theories are analogous to mathematical *conjectures*. A conjecture is a statement that mathematicians have reason to believe may be true, but which has not been proved definitively. One of the most famous mathematical conjectures is the *Goldbach conjecture*, named after the eighteenth-century mathematician Christian Goldbach:

Every even integer greater than 2 can be expressed as the sum of two prime numbers.

The Goldbach conjecture seems likely to be true; in fact, it has been shown to be true for every integer up to 4×10^{18}. However, it is, at present, unproven—there may, in fact, be a very large even integer that cannot be expressed as the sum of two prime numbers.

The idea of mathematical proof is extremely powerful. It enables mathematics to be a rich and robust system of knowledge that will never be falsified; further, any result that has been proved can safely be used to help establish further useful results, adding to this system of knowledge.

To prove mathematical statements, it is important to use clear, unambiguous language and valid logic. The focus of this section is on the language and logic used to construct, combine, and evaluate the truth or falsehood of mathematical statements about numbers.

Note that the word 'statement' will be used to not only refer to an assertion that is true or false (such as 'the number 7 is prime' or 'all multiples of 10 are also multiples of 5'), but also for an assertion involving one or more variables that becomes true or false whenever values are substituted for the variable (such as 'n is a multiple of 5' or '$x^2 < 20$').

Negating statements

The negation of a mathematical statement is the statement that is true precisely when the original statement is false, and vice versa. As an example, the negation of the statement $x > 0$ is $x \leq 0$. Notice that for any number that is substituted in place of x, if the statement $x > 0$ is true, then the negation $x \leq 0$ will be false, and vice versa. As a general rule, the negation of a statement can be obtained by preceding the statement with the phrase 'it is not the case that …' For example, if n represents an integer, then the negation of the statement 'n is an even number' is 'it is not the case that n is an even number', which is equivalent to saying that n is an odd number. If P represents any statement, then the negation of P can be written as $\neg P$, $\sim P$, or simply '*not P*'.

The negation of statements involving the words 'and' or 'or' can sometimes cause confusion. Consider negating the statement 'either $x = 5$ or $x = 7$'. If it is not the case that x is equal to 5 or 7, then it must be the case that $x \neq 5$ *and* $x \neq 7$.

As another example, consider the negation of the statement: $x > 0$ and $x < 10$. If it is not the case that x is between 0 and 10, then either $x \leq 0$ *or* $x \geq 10$.

- The negation of 'P and Q' is '*not P* or *not Q*'.
- The negation of 'P or Q' is '*not P* and *not Q*'.

The preceding rules about negating AND and OR statements are known as de Morgan's laws.

Example 1

Negate the following statements.

(a) n is divisible by 2 or n is divisible by 3

(b) $x > 0$ and $x < 5$.

Solution

(a) The opposite (negative of) 'divisible' is 'not divisible'.
n is not divisible by 2 and n is not divisible by 3.
Or, n is divisible by neither 2 or 3.

(b) The opposite of 'greater than' is 'not greater than' or 'less than or equal to'.
The opposite of 'less than' is 'not less than' or 'greater than or equal to'.
Hence $x \le 0$ or $x \ge 5$.

Statements involving quantifiers

Often, mathematical statements involving variables are true for certain values of the variable, and false for others. For example, the statement $(x - 2)(x - 5) = 0$ is true if x is equal to either 2 or 5, but false otherwise.

But consider the following statement, where x represents a real number: $x^2 \ge 0$. This is true for every possible real number that can be substituted in place of x. Therefore, you could say:

'For all real numbers x, $x^2 \ge 0$'.

Statements that assert that some property is true for all possible values of a variable are so common that a special symbol is often used to enable such statements to be written in a more compact way. The symbol $\forall$ is known as the universal quantifier and is used as shorthand for the phrase 'for all'. Thus, the previous statement could be written as follows:

'$\forall$ real numbers x, $x^2 \ge 0$'.

This can be condensed further by making use of the notation $x \in R$ to signify that the variable x is an element of the set of real numbers. Specifically, you could write the statement as follows:

'$\forall\ x \in R, x^2 \ge 0$'.

In some situations, you are not interested in whether some property is true for all possible values of a variable, but rather, whether there is at least one possible value of the variable that makes it true. For example, consider the statement: $n^2 + 2n$ is a prime number. This is certainly not true for all integer values of n, but there is at least one value of n that makes it true, (for example, $n = 1$). Therefore, you could say:

'There exists an integer n, such that $n^2 + 2n$ is a prime number'.

Notice that this doesn't preclude the possibility that there is more than one value of n that satisfies the statement; it simply guarantees the existence of at least one value of n for which the statement is true.

Statements that assert that some property is true for at least one possible value of a variable are also common in mathematics. For such statements, the symbol $\exists$ (known as the existential quantifier), can be used as shorthand for the phrase 'there exists'. Thus, the previous statement could be written as follows:

'$\exists$ an integer, n such that $n^2 + 2n$ is a prime number'.

The symbol $\forall$ (known as the *universal quantifier*) is used to mean 'for all'.

The symbol $\exists$ (known as the *existential quantifier*) is used to mean 'there exists'.

Example 2

Translate the following statements into everyday language. Also determine whether the statement is true or false, justifying your answer.

(a) $\exists\, x \in R$ such that $x^2 = \sqrt{x}$.

(b) $\forall$ integers n, the number $5n$ is even.

Solution

(a) There is at least one real number whose square is equal to its square root.
This is true as the number 1 satisfies this property.

(b) Multiplying any integer by 5 results in an even number.
This is false as, for example, $5 \times 3 = 15$, which is not even.

Example 3

Rewrite the following statements using the symbols $\forall$ and $\exists$. Also, state whether the statement is true or false, justifying your answer.

(a) The square root of any positive integer is less than or equal to the integer.

(b) There is at least one real number which, when squared, results in a smaller number.

Solution

(a) $\forall$ positive integers n, $\sqrt{n} \leq n$.
This is a true statement as the square root of any number greater than or equal to 1 is less than or equal to the number itself.

(b) $\exists$ a real number x, such that $x^2 < x$.
This is true for $0 < x < 1$; for example, $0.5^2 = 0.25 < 0.5$.

Note that the symbols $\forall$ and $\exists$ may be used together in a single statement; however, the order in which they appear is important. As an example, consider the following two statements:

$\forall$ integers n, $\exists$ an integer m such that $n + m$ is a multiple of 5.

$\exists$ an integer n, such that $\forall$ integers m, $n + m$ is a multiple of 5.

The first statement is true as it says that 'for every integer, you can find another integer to add to it to give a sum that is a multiple of 5'.

The second statement is false as it says that 'there is a special integer that has the property that when you add any other integer to it, you always obtain a multiple of 5'.

When the symbols $\forall$ and $\exists$ appear together in the same statement, the order in which they appear is important.

Examples and counterexamples

In part **(a)** of Example **2**, it was claimed that the statement, '$\exists\, x \in R$ such that $x^2 = \sqrt{x}$' is true, and to justify this claim, a single example was provided of a real number whose square was equal to its square root (namely, $x = 1$). Providing a single example is always sufficient to prove that a 'there exists' statement is true.

In part **(b)** of Example **2**, it was claimed that the statement, '$\forall$ integers n, the number $5n$ is even' is false, and to justify this claim, a single example was provided of an integer which, when multiplied by 5, gives a number that is not even (namely, $n = 3$). An example, such as this, that demonstrates the falsehood of a statement is known as a counterexample. Providing a single counterexample is always sufficient to prove that a 'for all' statement is false.

- Providing a single example is always sufficient to prove that a 'there exists' statement is true.
- Providing a single counter example is always sufficient to prove that a 'for all' statement is false.

In part **(a)** of Example **3**, it was claimed that the statement '$\forall$ positive integers n, $\sqrt{n} \leq n$' is true. It is easy to check that this result holds when $n = 1, 2, 3, 4$ and 5, and you would expect it to hold for larger values of n as it appears as though the larger the value of n is, the smaller $\sqrt{n}$ is in comparison to n. However, simply providing several example values of n for which the inequality holds does not actually prove that it must hold for all positive integers. Proving a statement is true for infinitely many values of a variable requires a general proof. Later, several strategies for constructing general proofs of results such as this are discussed.

Negating statements involving quantifiers

Consider the negation of the statement '$\forall$ real numbers x, $x^2 \geq 0$'. If it is not the case that the square of every real number is greater than or equal to zero, then it must mean that there is at least one real number whose square is less than zero. Thus, the negation is '$\exists$ a real number x such that $x^2 < 0$'. Notice how the negation of a 'for all' statement is a 'there exists' statement. The reverse is also true. For example, the negation of the statement: '$\exists$ a real number x such that $3x = x$' is '$\forall$ real numbers x, $3x \neq x$'.

The negation of a 'for all' statement is a 'there exists' statement. Similarly, the negation of a 'there exists' statement is a 'for all' statement.

Example 4

Determine the negation of each of the following statements. Also state whether the original statement or the negation is true or false, justifying your answer.

(a) $\forall$ integers n, $2n$ is even.

(b) $\exists$ a real number x, such that $x^2 = -1$.

(c) $\exists$ an integer n, such that n is even and n is prime.

Solution

(a) $\exists$ an integer n such that $2n$ is odd.
Original statement is true as 2 multiplied by an integer is, by definition, even.

(b) $\forall$ real numbers x, $x^2 \neq -1$.
Negation is true, as the square of any real number is greater than or equal to zero.

(c) $\forall$ integers n, n is odd or n is not prime.
Original statement is true, as the number 2 is even and prime.

Conditional statements

Consider the following statement: if n is a multiple of 10, then n is an even number. This is an example of a conditional statement. A conditional statement (also known as an 'if-then' statement, or an 'implication') is one that asserts that *if* some condition holds, *then* it must be the case that some property is true. Conditional statements are so common in mathematics that there is a variety of ways to express them. The previous example, for instance, could be represented in any of the following ways:

If n is a multiple of 10, then n is an even number.

n is an even number if n is a multiple of 10.

n being a multiple of 10 is a *sufficient* condition to conclude that n is even.

n being even is *necessary* if n is a multiple of 10.

n is a multiple of 10 *implies that* n is an even number.

Finally, the implication symbol, $\Rightarrow$, is used to mean 'implies that'. Thus, the statement could be also written as follows:

n is a multiple of 10 $\Rightarrow$ n is an even number.

Each of the following means the same as $P \Rightarrow Q$:

If P, then Q

Q if P

P is a *sufficient* condition to conclude that Q

Q is *necessary* if P

P implies that Q

Example 5

Rewrite the following conditional statements using the implication symbol, $\Rightarrow$.

(a) If n ends in a zero, then n is even.

(b) $\forall$ integers n, $n > 3$ is a sufficient condition to conclude that n is positive.

(c) $n > 3$ is necessary if n is greater than 4.

Solution

(a) 'If p, then q' can be written as $p \Rightarrow q$.
If n ends in a zero $\Rightarrow$ n is even.

(b) 'p is a sufficient condition to conclude q' means the same as 'if p then q'.
$n > 3 \Rightarrow n$ positive.

(c) 'p is a necessary condition if q' means the same as 'if q, then p'.
$n > 4 \Rightarrow n > 3$

The *converse* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol. For example, consider the conditional statement previously introduced:

original: n is a multiple of 10 $\Rightarrow$ n is an even number

converse: n is an even number $\Rightarrow$ n is a multiple of 10

Notice that the converse is not saying the same thing as the original. The original statement is claiming that if a number is a multiple of 10, then it must be even (which is true). But the converse is claiming that if a number is even, then it must be a multiple of 10 (which is definitely not true).

The *contrapositive* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol, and also negating both statements. Again, consider the conditional statement previously introduced:

original: n is a multiple of 10 $\Rightarrow$ n is an even number

contrapositive: n is not an even number $\Rightarrow$ n is not multiple of 10

Notice that the contrapositive *is* actually true, just like the original. In a sense, it is saying the exact same thing as the original statement.

As a classic illustrative real-life example, the contrapositive of the statement 'if an animal is a poodle, then it is a dog' is 'if an animal is not a dog, then it is not a poodle'. Notice, again, how the original and the contrapositive statements are essentially saying the same thing.

Now consider the *negation* of the statement 'n is a multiple of 10 $\Rightarrow$ n is an even number'. Remember that that this statement is essentially saying that for every integer that is a multiple of 10, this integer must also be even.

If this were not the case, it would mean that there must exist some integer that is a multiple of 10 but is not even. In general, the negation of a conditional statement of the form $P \Rightarrow Q$ that involves some variable is 'there exists some value of the variable for which P is true, but Q is false'.

Using the real-life example from earlier, the negation of 'if an animal is a poodle, then it is a dog' would be 'there exists some animal that is a poodle, but not a dog'.

Notice that the negation of a conditional statement is different from both the converse, and the contrapositive.

For a statement of the form $P \Rightarrow Q$ that involves some variable:

The converse is the statement $Q \Rightarrow P$;

The contrapositive is the statement *not* $Q \Rightarrow$ *not* P;

The negation is the statement 'there exists some value of the variable for which P is true, but Q is false'.

The contrapositive of a conditional statement essentially says the same thing as the original statement, and thus, will be true whenever the original statement is true.

Example 6

Write the converse, the contrapositive, and the negation of the following conditional statement.

If n is a perfect square, then n is divisible by 3.

Determine whether each of the original, converse, contrapositive, and negation are true or false, justifying your answer.

Solution

Is the original statement true or false? The original statement is false as, for example, 16 is a perfect square but it is not divisible by 3.

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$: Converse is 'if n is divisible by 3 then n is a perfect square'. For example, 12 is divisible by 3 but it is not a perfect square.

The contrapositive of a statement of the form $P \Rightarrow Q$ is *not* $Q \Rightarrow$ *not* P: Contrapositive is 'if n is not divisible by 3 then n is not a perfect square'. Like the original statement, this statement is false.

The negation of a statement of the form $P \Rightarrow Q$ is 'there exists some value of the variable for which P is true, but Q is false': Negation is 'there exists an integer n with the property that n is a perfect square, but n is not divisible by 3'. This statement must be true as the original statement was false. $n = 16$ has this property.

Notice that in the previous worked example, a counterexample was provided to justify the claim that the statement 'If n is a perfect square, then n is divisible by 3' is false. This should make sense, as conditional statements such as these are making a claim about all possible values of a variable that satisfy some condition, and are therefore similar to 'for all' statements (which are proved false by providing a counterexample). Incidentally, this statement could be rephrased as 'For all perfect square integers n, n is divisible by 3'.

Logical equivalent statements

Recall that the conditional statement 'n is a multiple of 10 $\Rightarrow$ n is an even number' is true, however, its converse 'n is an even number $\Rightarrow$ n is a multiple of 10' is not. Sometimes, however, a conditional statement and its converse are both true. As an example, notice that if $x = 5$, then $2x = 10$, and conversely, if $2x = 10$, then $x = 5$. This means that for the two statements, $x = 5$ and $2x = 10$, whenever one is of these is true, the other must be. Such statements are said to be *logically equivalent*.

Two statements are logically equivalent if whenever one is true, the other must be true.

There are a variety of ways to represent the fact that $x = 5$ and $2x = 10$ are logically equivalent:

$x = 5$ is *necessary* and *sufficient* for $2x = 10$

$x = 5$ *if and only if* $2x = 10$

$x = 5 \Rightarrow 2x = 10$ and $2x = 10 \Rightarrow x = 5$

Finally, the symbol $\Leftrightarrow$ is often used to denote logical equivalence. Thus, we could write $x = 5 \Leftrightarrow 2x = 10$ (or equivalently, $2x = 10 \Leftrightarrow x = 5$).

Each of the following can be used to express the fact that P and Q are logically equivalent:

P is *necessary* and *sufficient* for Q

P *if and only if* Q

$P \Rightarrow Q$ and $Q \Rightarrow P$

$P \Leftrightarrow Q$

Example 7

Rewrite the following statement using the logical equivalence symbol, $\Leftrightarrow$:

For n to be divisible by 5, it is both necessary and sufficient that n end in either 0 or 5.

Solution

'P is a necessary and sufficient condition for Q' means that P and Q are logically equivalent: n is divisible by 5 $\Leftrightarrow$ n ends in 0 or 5.

EXERCISE 2.1 THE LANGUAGE AND LOGIC OF PROOF

1 Determine the negation of each of the following statements.

- **(a)** p and q are both even.
- **(b)** $x > 5$ or $x < -5$.
- **(c)** x is divisible by either 7 or 8.
- **(d)** $x = 0$ or $y = 0$.

2 Translate the following statements into everyday language. Also, determine whether the statement is true or false, justifying your answer where appropriate.

- **(a)** $\forall$ integers n, the number $2n + 3$ is odd.
- **(b)** $\exists$ a real number x such that $\frac{1}{x} = x$.
- **(c)** $\forall$ real numbers x, $x^2 > 0$.
- **(d)** $\exists\, x \in R$ such that $x^2 = -1$.
- **(e)** $\forall\, n \in$ integers, the number $n(n + 1)$ is divisible by 3.
- **(f)** $\forall$ real numbers x and y, $x - y > 0$.
- **(g)** $\forall$ real numbers x, $\exists$ a real number y such that $x + y = 0$.
- **(h)** $\exists$ a real number x such that $\forall$ real numbers y, $xy = y$.

3 Rewrite the following statements using the symbols $\forall$ and $\exists$. Also, determine whether the statement is true or false, justifying your answer where appropriate.

- **(a)** The square of any integer is greater than the integer.
- **(b)** There is a real number which, when multiplied by 5 gives an answer of 0.
- **(c)** The sum of any two consecutive integers is odd.
- **(d)** There is a real number equal to its square.
- **(e)** The sum of the squares of any two real numbers is less than the product of the numbers.
- **(f)** There is a special real number with the property that whenever another real number is divided by this special number, this other real number is obtained as a result.
- **(g)** Every integer is divisible by at least one integer.

4 Determine the negation of each of the following statements. Also state whether the original statement or the negation is true, justifying your answer where appropriate.

(a) $\forall$ real numbers x, $x^2 > 0$.
(b) $\exists$ a real number x such that $x^2 = x$.
(c) $\forall$ positive integers n, $10n > n$.
(d) $\forall$ real numbers x, x is either positive or negative.
(e) $\exists$ an integer n such that $n \neq 0$ and $n^2 < 1$.
(f) $\forall$ integers n, either $(-1)^n = 1$ or $(-1)^n = -1$.

5 Rewrite the following statements using the implication symbol $\Rightarrow$.

(a) If $x > 3$, then $x^2 > 9$.
(b) If n is divisible by 9, then n is divisible by 3.
(c) $n > 5$ implies that $n > 4$.
(d) $7p$ is positive if $p > 3$.
(e) q is even if $2q$ is a perfect square.
(f) m is a multiple of 6 is a sufficient condition to conclude that m is divisible by 3.
(g) It is necessary that $x^2 > 2$ if $x < -2$.
(h) n is even and greater than 2 is a sufficient condition to conclude that n is not prime.

6 Write the converse, the contrapositive, and the negation of each of the following conditional statements. Determine whether each of the original, converse, contrapositive, and negation are true or false, justifying your answer where appropriate.

(a) If n is divisible by 20, then n is divisible by 5.
(b) If n is divisible by 3, then n^2 is divisible by 3.
(c) If $x > 7$, then $10x > 70$.
(d) If $xy = 0$, then either $x = 0$ or $y = 0$.
(e) If n is divisible by 5, then the final digit of n is 5.
(f) If $x = 4$ and $y = 4$, then $xy = 16$.
(g) If n is divisible by 24, then n is even and n is divisible by 3.

7 Rewrite the following statements using the logical equivalence symbol, $\Leftrightarrow$.

(a) n is even if and only if n^2 is even.
(b) $x + y = 0$ if and only if $x = -y$.
(c) n being even and divisible by 3 is necessary and sufficient for n to be divisible by 6.

8 For each of the following statements, provide three examples that are consistent with the statement, and then a single counter example to prove that the statement is actually false.

(a) If a positive integer is divisible by 7, then it is not divisible by 3.
(b) $\forall\, x \in R$, $x^2 \geq x$.
(c) If p is a prime number, then $2p - 1$ is prime.
(d) If a positive integer is divisible by both 10 and 6, then it is also divisible by 60.
(e) Suppose that x and y are real numbers. If $x \geq 3$, then $x^2 - 2y > 5$.
(f) Suppose that x and y are positive real numbers. Then $xy > x + y$.

9 Which statement is true?

A $\forall\, x \in R, \exists\, y \in R$ such that $xy = 6$.
B $\exists\, x \in R$ such that $\forall\, y \in R$, $xy = 6$.
C $\exists\, x \in R$ such that $\forall\, y \in R$, $x + y = 6$.
D $\forall\, x \in R, \exists\, y \in R$ such that $x + y = 6$.

10 The negation of the statement $\forall\, x \in R, \exists\, y \in R$ such that $x + y = 6$, is:

A $\forall\, x \in R, \exists\, y \in R$ such that $x + y \neq 6$.
B $\exists\, x \in R$ such that $\forall\, y \in R$, $x + y \neq 6$.
C $\forall\, x \in R, \forall\, y \in R$, $x + y \neq 6$.
D $\exists\, x \in R, \exists\, y \in R$ such that $x + y \neq 6$.

11 Write the negation of the following statement, where x represents a real number: $x > 0$ and $x < 10 \Rightarrow x \geq 0$ and $x \leq 10$. Also, determine whether the original or the negation is true.

12 Consider the following conjecture:
Start with any positive integer. If the integer is even, halve it. If the integer is odd, triple it and add one. Repeat this process. Eventually, the integer 1 will be obtained.
This is known as the '$3x + 1$' conjecture. It is yet to be proved but has been shown to be true for all integers up to roughly 10^{14}. Verify this conjecture for the following positive integers:

(a) 6 (b) 13 (c) 7

2.2 METHODS OF PROOF

Aside from examples and counterexamples (which can be used to prove the truth of a 'there exists' statement or the falsehood of a 'for all' statement), a mathematical proof typically consists of a sequence of statements with each statement following directly from either definitions, previous steps, or other established results. In this section, several common strategies of constructing proofs are illustrated.

As many of the statements proved are concerned with even and odd numbers, and divisibility more generally, it is necessary to be familiar with the following definitions:

An integer n is said to be even if $n = 2k$ for some integer k. (For example: 10 is even since 10 can be written as 2 multiplied by some integer, namely 5.)

Similarly, an integer n is said to be odd if $n = 2k + 1$ for some integer k. (For example: 11 is odd since 11 can be written as one more than 2 multiplied by some integer, namely, 5.)

Finally, an integer n is said to be divisible by the integer m if $n = mk$ for some integer k. (For example: the number 15 is divisible by 5 since 15 can be expressed as 5 multiplied by some integer, namely 3.)

Direct proof

The most straightforward way to prove a statement is to use a direct proof. A direct proof typically starts by introducing any relevant variables, clearly states any assumptions, and then establishes the desired result via a logical sequence of valid statements. Note that if the statement to be proved has the form 'if P, then Q', then you assume that P is true, and then proceed to show that Q must be true.

Example 8

Use a direct proof to prove that if a number is odd, then its square is also odd.

Solution

Let p be an odd integer.

Hence $p = 2k + 1$ for some integer k.

Consider p^2, which is to be proved odd:

$$\begin{aligned} p^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

As $2k^2 + 2k$ is an integer then $2(2k^2 + 2k)$ is even and $2(2k^2 + 2k) + 1$ is odd.

Hence p^2 is odd.

Proof by contraposition

Recall that the contrapositive of the statement $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. As the contrapositive is logically equivalent to the original statement, the original statement, $P \Rightarrow Q$, can be proved indirectly by proving $\neg Q \Rightarrow \neg P$; that is, by assuming that Q is false, and then proceeding to show that P must be false.

Example 9

Use a contrapositive proof to prove that if $5n + 3$ is odd, then n is even.

Solution

The contrapositive statement is: if n is not even, then $5n + 3$ is not odd. In other words, if n is odd, then $5n + 3$ is even.

Let n be an odd integer.

Hence $n = 2k + 1$ for some integer k.

$$\begin{aligned}5n + 3 &= 5(2k + 1) + 3\\ &= 10k + 5 + 3\\ &= 10k + 8\\ &= 2(5k + 4)\end{aligned}$$

Since $5k + 4$ is an integer then $2(5k + 4)$ must be an even integer.

Hence $5n + 3$ is even.

Proof by contradiction

Another form of indirect proof, but one that is not restricted to proving conditional statements, is 'proof by contradiction'. The basic idea of such a proof is to assume that the statement needing to be proved is *false*, and then show that this assumption leads to an absurd and impossible result; this then must mean that the initial assumption that the result was false cannot be true, meaning that it must be true!

Two proofs by contradiction are illustrated in the following example. The first is a famous proof that $\sqrt{2}$ is irrational. The second is a proof of the same statement from the previous worked example, only using contradiction instead of contraposition.

Example 10

Use a proof by contradiction to prove each of the following statements.

(a) $\sqrt{2}$ is irrational.   **(b)** If $5n + 3$ is odd, then n is even.

Solution

(a) Assume, for a contradiction, that $\sqrt{2}$ is rational.

Let $\sqrt{2} = \frac{p}{q}$ for integers p and q, with p and q having no common factors other than 1.

(p and q are relatively prime.)

Square both sides: $2 = \frac{p^2}{q^2}$

Rearrange: $p^2 = 2q^2$ and hence p^2 is divisible by 2.

If p^2 is divisible by 2 then p is divisible by 2.

Therefore, you can write $p = 2m$ for some integer m.

Substitute in $p^2 = 2q^2$: $4m^2 = 2q^2$

$2m^2 = q^2$

Hence q^2 is divisible by 2, so q is divisible by 2.

Since p and q are divisible by 2, this gives a contradiction to the original assumption that p and q had no common factors other than 1.

Hence the original assumption that $\sqrt{2}$ is rational is false.

Hence $\sqrt{2}$ is irrational.

(b) Assume, for a contradiction that $5n + 3$ is odd and n is odd.

Since n is odd: $n = 2k + 1$ for some integer k.

$$\begin{aligned}5n + 3 &= 5(2k + 1) + 3\\ &= 10k + 5 + 3\\ &= 10k + 8\\ &= 2(5k + 4)\end{aligned}$$

Since $5k + 4$ is an integer then $2(5k + 4)$ is even.

Hence $5n + 3$ is even and it can't be both odd and even.

Hence the assumption for n must be wrong, n cannot be odd so it must be even.

Note that the proof in part **(a)** of the previous example used the fact that if the square of an integer is divisible by 2, then the original integer must also be divisible by 2. This fact is true not just for the number 2 but for any integers with no perfect square factors other than 1. You may use this fact when modifying the previous proof to prove the irrationality of other surds in the subsequent exercise.

You may have noticed that the contradiction proof in part **(b)** looks very similar to the contraposition proof from Example **9**. The actual logic used is almost identical—the difference is that there is no actual contradiction obtained in the contrapositive proof since it was never assumed that $5n + 3$ was odd. Note also that a contradiction proof does not aim for a *particular* contradiction—any contradiction is sufficient. In a situation such as this, when a contradiction proof is very similar to a contrapositive proof, the contrapositive one is considered more efficient and elegant.

Proving logical equivalences

The simplest method to prove a statement of the form $P \Leftrightarrow Q$ is to separately prove both $P \Rightarrow Q$ and $Q \Rightarrow P$, as demonstrated in the following example.

Example 11

Let n be a positive integer. Prove that $n + 9$ is odd if and only if $n - 8$ is even.

Solution

(i) Assume that $n + 9$ is odd.
Thus $n + 9 = 2k + 1$ for some integer k.

$$\begin{aligned} n - 8 &= n + 9 - 17 \\ &= 2k + 1 - 17 \\ &= 2k - 16 \\ &= 2(k - 8) \end{aligned}$$

Hence $n - 8$ is even as $2(k - 8)$ is even.

(ii) Conversely, assume that $n - 8$ is even.
Thus $n - 8 = 2k$ for some integer k.

$$\begin{aligned} n + 9 &= n - 8 + 17 \\ &= 2k + 17 \\ &= 2(k + 8) + 1 \end{aligned}$$

Hence $n + 9$ is odd since as $(2(k + 8) + 1)$ is odd.

The final example presented is a more complex proof of a well-known divisibility result.

Example 12

Prove that a three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Solution

(i) Let a, b and c be the digits, in order, of a three-digit number, N.
The number is: $N = 100a + 10b + c$
If N is divisible by 3, then $100a + 10b + c = 3k$ for some integer k.
Rearrange to create factors of 3:

$$\begin{aligned} 99a + 9b + a + b + c &= 3k \\ a + b + c &= 3k - 99a - 9b \\ a + b + c &= 3(k - 33a - 3b) \end{aligned}$$

Hence the sum of the digits is divisible by 3.

(ii) Conversely, assume that the sum of the digits is divisible by 3:
$a + b + c = 3k$ for some integer k.

$$\begin{aligned} N &= 100a + 10b + c \\ &= 99a + 9b + a + b + c \\ &= 99a + 9b + 3k \\ &= 3(33a + 3b + k) \end{aligned}$$

Hence N is divisible by 3.

EXERCISE 2.2 METHODS OF PROOF

1 Use a direct proof to prove each of the following statements.

(a) The sum of any two odd integers is even.
(b) The sum of an odd integer and an even integer is always odd.
(c) The product of two odd integers is odd.
(d) The sum of two consecutive odd numbers is divisible by 4.
(e) The sum of the squares of five consecutive integers is divisible by 5.
(f) The product of two rational numbers is rational.
(g) The sum of two rational numbers is rational.
(h) If n is odd, then n^2 is odd.
(i) If n is divisible by 7, then n^2 is divisible by 7.
(j) If $m + n$ and $n + p$ are even, where m, n, p are integers, then $m + p$ is even.

2 Use a contrapositive proof to prove each of the following statements.

(a) Let n be an integer. If $3n + 2$ is even, then n is even.
(b) If a and b are integers and ab is even, then at least one of a and b is even.
(c) Let n be an integer. If $n^3 + 5$ is odd, then n is even.
(d) If x is irrational, then $\sqrt{x}$ is irrational.
(e) If x is irrational, then $\frac{1}{x}$ is irrational.

3 Use a proof by contradiction to prove each of the following statements.

(a) $\sqrt{3}$ is irrational.
(b) $\sqrt{5}$ is irrational.
(c) The sum of a rational and an irrational number is irrational.
(d) The product of a rational and an irrational number is irrational.
(e) There are no integers a and b such that $18a + 6b = 1$.

4 Prove each of the following logical equivalences.

(a) Let n be a positive integer. $n + 9$ is even if and only if $n + 6$ is odd.
(b) Let n be a positive integer. $n - 3$ is odd if and only if $n + 2$ is even.
(c) Let n be a positive integer. n is even if and only if $13n + 4$ is even.
(d) Let n be a positive integer. n is odd if and only if $7n + 6$ is odd.
(e) Let n be a positive integer. n is even if and only if n^2 is even.

5 Consider the following statement:

'If two integers have an even product, then at least one of the two integers must be even.'

To prove this statement by contraposition, it would be necessary to:

A suppose that at least one of the two integers is even, and then show that the product must be even.
B suppose that at least one of the integers is odd, and then show that the product must be odd.
C suppose that both integers are odd, and then show that the product must be odd.
D suppose that the two integers have an even product and that both integers are odd, and then show that a contradiction arises.

6 Prove that the number $1 - 5\sqrt{2}$ is irrational.

7 Let a, b, c be positive real numbers such that $ab = c$. Prove that $a \le \sqrt{c}$ or $b \le \sqrt{c}$.

8 Prove that if $a^2 - 2a + 7$ is even, then a is odd.

9 Prove that a four-digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

10 **(a)** Prove that for all real numbers a and b, $a^2 + b^2 \geq 2ab$.

(b) Using the result from part **(a)** or otherwise, prove that for all positive real numbers a and b, that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

(c) Using the result from part **(a)** or otherwise, prove that for all positive real numbers a and b, that $\frac{a+b}{2} \geq \sqrt{ab}$.

11 Prove that every odd integer can be expressed as the difference between two perfect squares.

12 Prove that if a, b are integers, then $a^2 - 4b - 3 \neq 0$.

13 Let k be a positive integer. Prove that if $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+3} + 3^{3k+3}$ is also divisible by 5.

14 It is known that $\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square. Use this result to help prove that $\sqrt{6} + \sqrt{10}$ is irrational.

15 Use a proof by contradiction to show that there is no rational solution to the equation $x^3 + x + 1 = 0$. As a hint, start by supposing, for a contradiction that $r = \frac{p}{q}$ is a rational solution to the equation, where p, q are integers with no common factor other than 1 and with $q \neq 0$. Then consider what would happen if both p and q were odd, or if one of them was even and the other odd.

2.3 INEQUALITIES

The relation $a > b$ is equivalent to the statement that $a - b$ is positive, i.e. $a - b > 0$. Similarly, you can interpret $a < b$ to mean either that $b > a$ or that $a - b$ is negative. The following properties are stated for inequalities with a '>' relation; similar properties exist for '<'.

Properties of inequalities

1. If $a > b$ then $a + x > b + x$ for all x.
   You can add or subtract the same amount to both sides of an inequality.
2. If $a > b$ then $ax > bx$ for $x > 0$.
   You can multiply (or divide) both sides of an inequality by a positive number.
3. If $a > b$ then $ax < bx$ for $x < 0$.
   The inequality relation changes when you multiply (or divide) both sides of an inequality by a negative number.
4. If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$.
   When you take reciprocals of both sides of an inequality and both sides are positive, the inequality is reversed.
5. If $a > b > 0$ then $a^2 > b^2$.
   Inequalities in which both sides are positive can be squared.
6. If $a > b$ and $b > c$ then $a > c$.
   Inequalities of the same type can be linked together.
7. If $a > b > 0$ and $c > d > 0$ then $ac > bd$.
   Inequalities of the same type involving positive numbers can be multiplied together.

   Proof: $a > b$ and $c > 0$ $\quad \therefore ac > bc$
   $c > d$ and $b > 0$ $\quad \therefore bc > bd$
   $\therefore ac > bc > bd$ $\quad \therefore ac > bd$
8. If $a > b$ and $c > d$ then $a + c > b + d$.
   Inequalities of the same type can be added together.

   Proof: $a - b > 0$ and $c - d > 0$ $\quad \therefore a - b + c - d > 0$
   $\therefore a + c - (b + d) > 0$
   $\therefore a + c > b + d$

9 As a general rule, inequalities **cannot** be subtracted from one another, nor divided into each other. For example:

Consider the pair of inequalities	$12 > 6$	[1]
and	$4 > 3$	[2]
[1] − [2] gives a true result:	$8 > 3$	
[1] ÷ [2] gives a true result:	$3 > 2$	
But now consider the inequalities	$12 > 10$	[3]
and	$100 > 3$	[4]
[3] − [4] gives a false result:	$-88 > 7$	
[3] ÷ [4] gives a false result:	$\frac{3}{25} > 3\frac{1}{3}$	

Techniques for proving inequalities

1 Use the properties of inequalities listed above.

2 Use proof by contradiction.

3 Use a known fact.
For example: $(a-b)^2 \geq 0$ for real a, b; or a variation of this, such as $(\sqrt{a}-\sqrt{b})^2 \geq 0$.
Another useful identity is $(a+b)^2 = (a-b)^2 + 4ab$, which enables the statement $(a+b)^2 \geq 4ab$ (as $(a-b)^2 \geq 0$ for real a, b).

4 Substitute different expressions into known inequalities.

5 If the inequality to be proved involves trigonometric, logarithmic or exponential terms, then a calculus-based approach is probably needed.

Example 13

If $a > -1$, show that $a^3 + 1 > a^2 + a$.

Solution

Technique: To prove $X > Y$, prove that $X - Y > 0$.

$$\begin{aligned} a^3 - a^2 - a + 1 &= a^2(a-1) - (a-1) \\ &= (a-1)(a^2-1) \\ &= (a-1)^2(a+1) \\ &> 0 \text{ as } a > -1, \text{ so } (a+1) \text{ is positive; also, } (a-1)^2 \text{ is positive} \end{aligned}$$

$\therefore\ a^3 + 1 > a^2 + a$.

Example 14

a, b, c and d are positive real numbers. Prove that:

(a) $a^2 + b^2 \geq 2ab$

(b) $a^2 + b^2 + c^2 \geq ab + bc + ca$

(c) $a^2 + b^2 + c^2 + d^2 \geq 2(ab + cd)$

(d) if $a + b + c = 1$, then $ab + bc + ca \leq \frac{1}{3}$, and state the condition for which equality is true

(e) $(a+b)^2 \geq 4ab$, and hence that $\left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^2 \geq 4$ for all real x except 0.

Solution

(a) **Method 1**

$$(a-b)^2 \geq 0 \text{ for real } a, b$$
$$\therefore\ a^2 - 2ab + b^2 \geq 0$$
$$a^2 + b^2 \geq 2ab$$

Method 2—*Proof by contradiction*

Assume that $a^2 + b^2 < 2ab$ for all real a, b.

$$\therefore\ a^2 - 2ab + b^2 < 0$$
$$\therefore\ (a-b)^2 < 0 \text{ for real } a, b$$

This contradicts the fact that the square of a real number is non-negative.

$\therefore$ Assumption is false, so: $a^2 + b^2 \geq 2ab$

(b)

$$a^2 + b^2 \geq 2ab \quad [1]$$

Similarly: $a^2 + c^2 \geq 2ac$ [2]

Similarly: $b^2 + c^2 \geq 2bc$ [3]

[1] + [2] + [3]: $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

This shows how to extend from a known result involving two variables to a similar result involving three variables: use the known result to generate each of the three paired results, then add.

(c)

$$a^2 + b^2 \geq 2ab \quad [1]$$

Similarly: $c^2 + d^2 \geq 2cd$ [4]

[1] + [4]: $a^2 + b^2 + c^2 + d^2 \geq 2(ab + cd)$

This shows a method to extend from a known result involving two variables to a similar result involving four variables: use the known result to generate two suitable paired results, then add.

(d) The question now involves $a + b + c$, which suggests you examine $(a + b + c)^2$.

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$
$$\geq 3(ab + bc + ca) \quad \text{using the result of part (b)}$$

But $a + b + c = 1$: $1^2 \geq 3(ab + bc + ca)$

$$\therefore\ ab + bc + ca \leq \frac{1}{3}$$

Note that in the inequality $a^2 + b^2 \geq 2ab$ [1], equality occurs when $a = b$. Similarly, equality occurs in [2] when $a = c$. Hence the equality of this result in part **(d)** occurs when $a = b = c$. As $a + b + c = 1$, equality occurs when $a = b = c = \frac{1}{3}$.

(e) Using $(a+b)^2 = (a-b)^2 + 4ab$: you have $(a-b)^2 \geq 0$, so $(a+b)^2 \geq 4ab$.

Letting $a = \sqrt[3]{x}$ and $b = \frac{1}{\sqrt[3]{x}}$, you obtain: $\left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^2 \geq 4 \times \sqrt[3]{x} \times \frac{1}{\sqrt[3]{x}}$

$$\therefore\ \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^2 \geq 4$$

Example 15

a, b, c are positive real numbers. Prove that:

(a) $a + \frac{1}{a} \geq 2$ **(b)** $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$ **(c)** $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$

(d) Extend this to the general case. If $x_1, x_2, x_3, \ldots x_n$ are positive real numbers, prove that:

$$(x_1 + x_2 + x_3 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}\right) \geq n^2$$

Solution

(a) $\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 \geq 0$

$\therefore\ a-2+\frac{1}{a} \geq 0$

$a+\frac{1}{a} \geq 2$ This could also have been done very well by contradiction.

(b) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=1+\frac{a}{b}+\frac{b}{a}+1$

$=2+\left(A+\frac{1}{A}\right)$ with $A=\frac{a}{b}$

≥ 4 from part **(a)**

(c) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+1$

$=3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)$

$\geq 3+2+2+2$ using part **(a)**

≥ 9

(d) $(x_1+x_2+x_3+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}\right)$

$=1+\frac{x_1}{x_2}+\frac{x_1}{x_3}+\ldots+\frac{x_1}{x_n}+\frac{x_2}{x_1}+1+\frac{x_2}{x_3}+\ldots+\frac{x_2}{x_n}+\ldots+\frac{x_n}{x_{n-1}}+1$

$=n\times 1+$ (every possible pairing of the form $\frac{x_i}{x_j}+\frac{x_j}{x_i}$ where $i\neq j$)

$\geq n+{}^nC_2\times 2$ (from the n different x terms, there are ${}^nC_2$ pairs)

$\geq n+\frac{1}{2}\times n(n-1)\times 2$

$\geq n^2$

$\therefore\ (x_1+x_2+x_3+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}\right)\geq n^2$

Arithmetic and geometric means

The arithmetic mean of a and b is $\frac{a+b}{2}$, the average of a and b.

The geometric mean of a and b is $\sqrt{ab}$. The numbers a, $\sqrt{ab}$, b form a geometric sequence.

Example 16

Prove that the arithmetic mean of two positive real numbers, a and b, is equal to or greater than the geometric mean, i.e. prove that $\frac{a+b}{2}\geq\sqrt{ab}$.

Solution

Consider the expression $\left(\frac{a+b}{2}\right)^2-ab$:

$$=\frac{a^2+2ab+b^2}{4}-ab$$

$$=\frac{a^2+2ab+b^2-4ab}{4}$$

$$=\frac{a^2-2ab+b^2}{4}$$

$$=\frac{(a-b)^2}{4}$$

$$\geq 0$$

Hence $\left(\frac{a+b}{2}\right)^2 \geq ab$

Taking the positive square root of both sides gives $\frac{a+b}{2} \geq \sqrt{ab}$.

The triangle inequality

The triangle inequality states that $|x|+|y| \geq |x+y|$.

This result was proved geometrically in Chapter 1, section 1.8 on page 31.

Proof

Remember that if $|x| \leq a$ then $-a \leq x \leq a$.

If x and y are either both positive or both negative, then $|x|+|y| = |x+y|$.

Now $-|x| \leq x \leq |x|$ since x equals either $|x|$ or $-|x|$.

Similarly, $-|y| \leq y \leq |y|$ since y equals either $|y|$ or $-|y|$.

Adding these two results gives: $-|x|-|y| \leq x+y \leq |x|+|y|$

$$-(|x|+|y|) \leq x+y \leq |x|+|y|$$

Let $a = x+y$ and $b = |x|+|y|$: $-b \leq a \leq b$

Hence $|a| \leq b$

So $|x+y| \leq |x|+|y|$

Or $|x|+|y| \geq |x+y|$ as required.

EXERCISE 2.3 INEQUALITIES

1 Show that if $a \geq 0$ and $b \geq 0$ then $ab(a^2+b^2) \geq 2a^2b^2$.

2 If $0 < x < y$, prove that $x^2 < xy < y^2$.

3 **(a)** For positive x and y, prove that $\frac{x}{y} + \frac{y}{x} \geq 2$.

(b) Hence prove that $x^2 - xy + y^2 \geq xy$.

(c) Factorise $x^3 + y^3$ and show that $x^3 + y^3 \geq xyz\left(\frac{x}{z} + \frac{y}{z}\right)$ for $x, y, z > 0$.

(d) Write similar expressions for $y^3 + z^3$ and $z^3 + x^3$.

(e) Using results from parts **(c)** and **(d)**, prove that $x^3 + y^3 + z^3 \geq 3xyz$.

(f) If a, b, c, d are positive, deduce that:

(i) $a+b+c \geq 3\sqrt[3]{abc}$

(ii) $(a+b+c)(a+b+d)(a+c+d)(b+c+d) \geq 81abcd$

4 **(a)** Show that $a^2+b^2+c^2 \geq ab+bc+ca$ for real a, b, c.

(b) Hence show that $(a+b+c)^2 \geq 3(ab+bc+ca)$.

5 Given that $a^2+b^2+c^2 \geq ab+bc+ca$ for real a, b, c, show that $x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x+y+z)$.

6 **(a)** Prove that $\frac{a+b}{2} \geq \sqrt{ab}$. Hence prove that $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ for positive a, b, c, d.

(b) Let $d = \frac{a+b+c}{3}$. Show that $abc \leq \left(\frac{a+b+c}{3}\right)^3$.

7 For each of the following, write an inequality that describes the statement and then identify the question above (from questions **1–6**) that involves the proof of the statement.

(a) The area of a rectangle cannot be greater than the square of the average length of its two sides.
(b) The volume of a rectangular prism cannot be greater than the cube of the average length of its three sides.

8 If $x > 0$ and $y > 0$, prove that: (a) $\frac{1}{x}+\frac{1}{y} \ge \frac{4}{x+y}$ (b) $\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{8}{(x+y)^2}$

9 (a) If a and b are real numbers, prove that $4a^2 - 6ab + 4b^2 \ge a^2 + b^2$.
(b) Write the binomial expansion of $(a-b)^4$ and prove that $a^4 + b^4 \ge a^3b + ab^3$ if $a > 0$ and $b > 0$.

10 Prove that the average of the squares of two unequal positive numbers is greater than the square of the average of those two numbers.

11 Use the results $\frac{a+b}{2} \ge \sqrt{ab}$ and $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$ to prove that:

(a) $(b+c)(c+a)(a+b) \ge 8abc$
(b) $bc(b+c) + ca(c+a) + ab(a+b) \ge 6abc$
(c) $b^2c^2 + c^2a^2 + a^2b^2 \ge abc(a+b+c)$
(d) $(ab+xy)(ax+by) \ge 4abxy$
(e) $ax + by \le 1$ if $a^2 + b^2 = 1$ and $x^2 + y^2 = 1$.

12 (a) Use $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$ to show that: (i) $x^2y + y^2z + z^2x \ge 3xyz$ (ii) $xy^2 + yz^2 + zx^2 \ge 3xyz$
(b) Write the expansion of $(x+y+z)^3$ and hence show that $(x+y+z)^3 \ge 27xyz$.

13 For positive a, b, c, d, prove that:

(a) $ab + cd \le \sqrt{(a^2+c^2)(b^2+d^2)}$
(b) $(a+5b)(a+2b) \ge 9b(a+b)$
(c) $\frac{a^2}{b^2} + \frac{b^2}{a^2} + 6 \ge \frac{4a}{b} + \frac{4b}{a}$
(d) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \ge 4$

14 The area of a triangle is given by Heron's formula as $A = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b and c are the lengths of the sides and $s = \frac{1}{2}(a+b+c)$. Given that $\sqrt{ab} \le \frac{a+b}{2}$ and $ab + bc + ca \le a^2 + b^2 + c^2$, show that: $A \le \frac{a^2+b^2+c^2}{6}$.

15 Let p be an integer greater than 1. Show that $x^p + (p-1) \ge px$ for all $x > 0$. For what values of x does the equality hold?

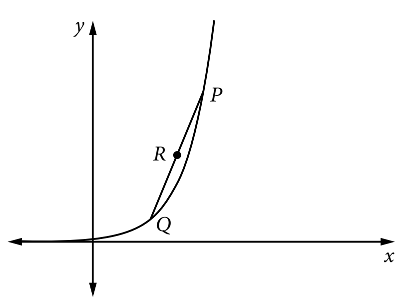
16 If a is any constant, prove that $e^x - e^a \ge e^a(x-a)$.

17 Let $g(x) = \sin x - x$.

(a) Show that $g(0) = 0$ and $g'(0) = 0$.
(b) Show that $-2 \le g'(x) \le 0$ for all x.
(c) Hence explain why $g(x) \le 0$ for $x \ge 0$.
(d) Explain why $\sin x < x$ for $x > 0$.

18 On the curve $y = e^x$, P and Q are two points with x-coordinates a and b respectively, $a > b$. Let R be the midpoint of chord PQ.

(a) Prove that $y = e^x$ is concave up for all x.
(b) Explain why $\frac{e^a + e^b}{2} > e^{\frac{a+b}{2}}$.
(c) Hence show that $e^a + e^b + e^c + e^d > 4e^{\frac{a+b+c+d}{4}}$ if $a > b > c > d$.



19 By letting $a = \frac{1}{x}$ and $b = \frac{1}{y}$ in $\frac{a+b}{2} \ge \sqrt{ab}$, prove that:

(a) $\frac{1}{x}+\frac{1}{y} \ge \frac{2}{\sqrt{xy}}$
(b) $\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{2}{xy}$

20 If $1 \le x \le 4$, show that: $\frac{1}{3} \le \frac{1}{1+\sqrt{x}} \le \frac{1}{2}$

21 In this exercise you have proved each of the following:

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad \frac{a+b+c}{3} \geq \sqrt[3]{abc} \qquad \frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

These are the first three instances of a general result, which says that the arithmetic mean of a set of n non-negative numbers is greater than or equal to the geometric mean of the numbers:

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$$ Note that equality occurs when all of the x_i are equal.

There are many ways of proving this general result. This question will develop two proofs.

(a) **(i)** Show that $f(x) = x - e^{x-1}$ has only one stationary point, which is a maximum turning point at $(1, 0)$.

(ii) Noting that $f(x)$ is continuous for all x, explain why $x \leq e^{x-1}$ for all real x.

(iii) Let $x_1, x_2, \ldots x_n$ be non-negative numbers with arithmetic mean X, i.e. $X = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$.

Using the fact $\dfrac{x_1}{X} \leq e^{\frac{x_1}{X} - 1}$ from **(ii)**, show that $\dfrac{x_1 x_2 \ldots x_n}{X^n} \leq 1$ and hence complete the proof of the general AM–GM ('arithmetic mean–geometric mean') inequality.

(b) For a second proof, you take Jensen's theorem as the starting point. (You will not prove Jensen's theorem; if you are interested, you can find its proof from an internet search.) Reworded for simplicity, Jensen's theorem states that in a function that is concave down for some domain, the average of n function values in that domain is less than or equal to the function value at the average of the n values:

$$\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

For a concave-up function, this inequality is reversed.

(i) Prove that $y = \log_e x$ is concave down for all $x > 0$.

(ii) Let $x_1, x_2, \ldots x_n$ be non-negative numbers with arithmetic mean X, i.e. $X = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$.
Write Jensen's theorem as it applies to the function $y = \log_e x$.

(iii) Hence complete this proof of the AM–GM inequality.

(c) By substitution into the result $\dfrac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$, prove that $n! \leq \left(\dfrac{n+1}{2}\right)^n$ for all positive integers n.

2.4 MATHEMATICAL INDUCTION, HARDER QUESTIONS

In this section, you will construct harder mathematical induction proofs in a variety of situations. You will also study a slightly different form of induction in which you need to assume $S(1)$, $S(2)$, …, $S(k)$ are true in order to prove that $S(k + 1)$ is true.

Induction questions for $n > 1$

Example 17

Use mathematical induction to prove that $n^3 - n$ is a multiple of 6 for $n \geq 2$.

Solution

Step 1 $n = 2$: Exp $= 2^3 - 2 = 8 - 2 = 6$, which is a multiple of 6.

Hence the result is true for $n = 2$.

Step 2 Assume the result is true for $n = k$, i.e. assume that $k^3 - k = 6M$, where M is a positive integer.

Prove the result is true for $n = k + 1$, i.e. prove that $(k + 1)^3 - (k + 1)$ is a multiple of 6.

$$\begin{aligned} \text{Exp} &= (k+1)^3 - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 6M + 3k(k+1) \end{aligned}$$

Now since k is a positive integer then $k(k+1)$ is even so that $k(k+1) = 2N$, where N is an integer.

Hence Exp $= 6M + 3 \times 2N$

$= 6(M + N)$ which is a multiple of 6.

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 2$, hence it is true for $n = 2 + 1$ and by the principle of mathematical induction it is true for all $n \geq 2$.

Sigma notation

Series questions to be proved by induction are often written using sigma notation, Σ, to save space.

For example: $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n \times (n+1) = \sum_{r=1}^{n} r(r+1)$.

Hence $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + k \times (k+1) = \sum_{r=1}^{k} r(r+1)$

and $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + k \times (k+1) + (k+1)(k+2) = \sum_{r=1}^{k} r(r+1) + (k+1)(k+2) = \sum_{r=1}^{k+1} r(r+1)$.

Example 18

Prove that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$.

Solution

Step 1 $n = 1$: LHS $= \sum_{r=1}^{1} r(r+1) = 1 \times 2 = 2$;

RHS $= \frac{1 \times 2 \times 3}{3} = 2 =$ LHS

Result is true when $n = 1$.

Step 2 Assume the result is true for $n = k$, i.e. assume that $\sum_{r=1}^{k} r(r+1) = \frac{k(k+1)(k+2)}{3}$.

Prove the result is true for $n = k + 1$, i.e. prove that $\sum_{r=1}^{k+1} r(r+1) = \frac{(k+1)(k+2)(k+3)}{3}$.

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k+1} r(r+1) \\
&= \sum_{r=1}^{k} r(r+1) + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\
&= (k+1)(k+2)\left(\frac{k+3}{3}\right) \\
&= \frac{(k+1)(k+2)(k+3)}{3} \\
&= \text{RHS}
\end{aligned}$$

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

Example 19

Prove by induction that $(2^2-1)(3^2-1)\ldots(n^2-1)=\dfrac{(n!)^2(n+1)}{2n}$ for all integers $n \geq 2$.

Solution

Note that this involves a product of terms rather than a sum of terms.

Let $S(n)$ be the statement that $(2^2-1)(3^2-1)\ldots(n^2-1)=\dfrac{(n!)^2(n+1)}{2n}$ for integers $n \geq 2$.

Step 1 Prove that $S(2)$ is true.

$$\text{LHS}=(2^2-1)=3 \qquad \text{RHS}=\frac{(2!)^2(2+1)}{2\times 2}=3$$

$$\text{LHS}=\text{RHS} \qquad \therefore\ S(2)\text{ is true}$$

Step 2 Assume $S(k)$ is true for a positive integer $k \geq 2$.

i.e. assume that $\boxed{(2^2-1)(3^2-1)\ldots(k^2-1)}=\dfrac{(k!)^2(k+1)}{2k}$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $(2^2-1)(3^2-1)\ldots([k+1]^2-1)=\dfrac{((k+1)!)^2((k+1)+1)}{2(k+1)}$

$$=\frac{((k+1)!)^2(k+2)}{2(k+1)}$$

$$\text{LHS}=\boxed{(2^2-1)(3^2-1)\ldots(k^2-1)}\,([k+1]^2-1)$$

using [a]:

$$=\frac{(k!)^2(k+1)}{2k}\times(k^2+2k)$$

$$=\frac{(k!)^2(k+1)}{2k}\times k(k+2)$$

$$=\frac{(k!)^2(k+1)(k+2)}{2}$$

$$=\frac{(k!)^2(k+1)(k+2)}{2}\times\frac{(k+1)}{(k+1)}$$

$$=\frac{((k+1)!)^2(k+2)}{2(k+1)}=\text{RHS}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

If the question in Example **19** had been to prove by induction that $(2^2-1)(3^2-1)\ldots(n^2-1)=\dfrac{(n!)^2(n+1)}{2n}$ for all integers $n \geq 1$, then the proof would not have worked.

Step 1: $n=1$, LHS $=(1^2-1)=0$

RHS $=\dfrac{(1!)^2\times 2}{2}=1$, which is not equal to the LHS.

Since the result cannot be proved for $n=1$ then the process of proof by mathematical induction will not work.

EXERCISE 2.4 MATHEMATICAL INDUCTION, HARDER QUESTIONS

Use mathematical induction to prove the following results.

1 Prove that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for n a positive integer.

2 Prove that $n^3 - n$ is a multiple of 6 for $n > 1$, n a positive integer.

3 Prove that $n^2 + 2n$ is divisible by 8 if n is an even integer.

4 Prove that $3^{4n} - 1$ is divisible by 80 for n a positive integer.

5 Prove that $\sum_{r=1}^{n} (2r-1) = n^2$.

6 Prove that $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(n+1)(n+2)}$.

7 Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ for $n \geq 2$.

8 Prove that $\sum_{k=1}^{n} (-1)^{k-1}k^2 = (-1)^{n-1}\left[\frac{n(n+1)}{2}\right]$.

9 Prove that $\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right) = \log(n+1)$.

10 Prove that $x^n - y^n$ is divisible by $(x - y)$ for n a positive integer.

11 Prove that $x^n - 1$ is divisible by $(x - 1)$ for n a positive integer. Use the result that $\frac{x^n - 1}{x-1} = x^{n-1} + \frac{x^{n-1} - 1}{x-1}$.

12 Prove that $\sum_{r=1}^{n} r\log\left(\frac{r+1}{r}\right) = \log\frac{(n+1)^n}{n!}$.

2.5 OTHER INDUCTION QUESTIONS

There is quite a range of situations in which you will use mathematical induction to prove results.

Example 20

Prove by induction that $3^n > 1 + 2n$ for all integers $n > 1$ (i.e. prove that $3^n - 1 - 2n > 0$).

Solution

Let $S(n)$ be the statement that $3^n - 1 - 2n > 0$ for integer n.

Step 1 Prove that $S(2)$ is true. (Note that $n = 2$ is the first case.)

$$\begin{aligned} \text{LHS} &= 3^2 - 1 - 4 \\ &= 4 \\ &> 0 \end{aligned} \qquad \therefore S(2) \text{ is true.}$$

Step 2 Assume $S(k)$ is true for an integer $k \geq 2$.

i.e. assume that $3^k - 1 - 2k > 0$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $3^{k+1} - 1 - 2(k+1) > 0$

$$\begin{aligned} \text{LHS} &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3 \times 3^k - 2k - 3 \end{aligned}$$

We need to link [a] to this, so we need to group $3^k - 1 - 2k$ together. However, the term in 3^k is multiplied by 3, so we need $3(3^k - 2k - 1)$. Form this group and 'pay back' the extra terms as required:

$$\begin{aligned} \text{LHS} &= 3 \times 3^k - 6k - 3 + 4k \\ &= 3(3^k - 2k - 1) + 4k \\ &> 0 \qquad \text{as } 3^k - 1 - 2k > 0 \quad \text{from [a] and } k > 0 \end{aligned}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

Example 21

Prove that $n^2 \geq 2n + 1$ for positive integers $n \geq 3$.

Solution

Let $S(n)$ be the statement that $n^2 - (2n+1) > 0$.

Step 1 Prove that $S(3)$ is true.

$$\begin{aligned} \text{LHS} &= 3^2 - 6 - 1 \\ &= 9 - 7 \\ &= 2 > 0 \end{aligned}$$

Hence $S(3)$ is true.

Step 2 Assume that $S(k)$ is true, i.e. assume that $k^2 - 2k - 1 > 0$.

Prove that $S(k+1)$ is true if $S(k)$ is true, i.e. $(k+1)^2 - 2(k+1) - 1 > 0$.

$$\begin{aligned} \text{LHS} &= (k+1)^2 - 2(k+1) - 1 \\ &= k^2 + 2k + 1 - 2k - 2 - 1 \\ &= k^2 - 2k - 1 + 2k - 1 \\ &> 0 + 2k - 1 \\ &> 0 \text{ as } 2k - 1 > 5 \text{ when } k \geq 3. \end{aligned}$$

Hence $S(k+1)$ is true if $S(k)$ is true.

Step 3 But $S(3)$ is true so by the principle of mathematical induction $S(n)$ is true for all $n \geq 3$.

Example 22

Prove that $3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n.

Solution

Step 1 $n = 1$: LHS $= 3^6 - 2^2 = 729 - 4 = 725$, which is divisible by 5.

Hence the result is true when $n = 1$.

Step 2 Assume the result is true for $n = k$, i.e. assume that $3^{2k+4} - 2^{2k} = 5M$, where M is a positive integer.

Prove the result is true for $n = k + 1$, i.e. prove that $3^{2k+6} - 2^{2k+2}$ is divisible by 5.

$$\begin{aligned}\text{Exp} &= 3^{2k+6} - 2^{2k+2}\\ &= 9 \times 3^{2k+4} - 4 \times 2^{2k}\\ &= 9 \times 3^{2k+4} - 9 \times 2^{2k} + 5 \times 2^{2k}\\ &= 9 \times 5M + 5 \times 2^{2k}\\ &= 5(9M + 2^{2k}),\text{ which is divisible by 5.}\end{aligned}$$

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

Example 23

Prove that every integer greater than 1 is either prime or a product of primes.

Solution

Let $S(n)$ be the proposition that n is either prime or a product of primes, $n \geq 2$.

Step 1 2 is a prime $\therefore$ $S(2)$ is true.

Step 2 Assume $S(2), S(3), S(4), \ldots S(k)$ are true. Thus prove that $S(k + 1)$ is true.

i.e. assume 3 is either prime or a product of primes
assume 4 is either prime or a product of primes
assume 5 is either prime or a product of primes
…
assume k is either prime or a product of primes.

Now $k + 1$ is either prime, in which case $S(k + 1)$ is true, or $k + 1$ is composite, in which case $k + 1 = p \times q$ where p and q are integers less than k (so that both p and q are either prime or a product of primes, because both p and q are in the set of assumed primes or products of primes.)

$\therefore$ $k + 1 = p \times q$ is a product of primes and $S(k + 1)$ is true.

Step 3 *Conclusion*

$S(k + 1)$ is true if $S(2), S(3), S(4), \ldots S(k)$ are true.
$S(2)$ is true.
$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

Example 24

Construct a proof by induction of the geometrical property that 'the angle sum of an n-sided polygon is $(n - 2) \times 180°$ for all integers $n \geq 3$'.

Solution

Let $S(n)$ be the statement that the angle sum of an n-sided polygon is $(n - 2) \times 180°$ for integer n.

Step 1 Prove that $S(3)$ is true.

When $n = 3$, angle sum $= (3 - 2) \times 180° = 180°$, which is the angle sum of a triangle.
$\therefore$ $S(3)$ is true.

Step 2 Assume $S(k)$ is true for an integer $k > 3$.

i.e. assume that the angle sum of a k-sided polygon is $(k - 2) \times 180°$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that the angle sum of a $(k+1)$-sided polygon is $([k+1]-2)\times 180°$

$= (k-1)\times 180°$

In the $(k+1)$-sided polygon, construct a diagonal that divides the polygon into a k-sided polygon and a triangle. (This can always be done.)

Angle sum of $(k+1)$-sided polygon = (angle sum of k-sided polygon) + (angle sum of triangle)

using [a]: $= (k-2)\times 180° + 180°$

$= (k-1)\times 180°$ (as required)

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(3)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \ge 3$.

EXERCISE 2.5 OTHER INDUCTION QUESTIONS

Use mathematical induction to prove the following results.

1 $n^2 - 11n + 30 \ge 0$ for all integers $n \ge 6$.

2 $n^2 > -5n + 14$ for all integers $n > 2$.

3 $12^n > 7^n + 5^n$ for all integers $n \ge 2$.

4 $10^n \ge 2^n + 3^n + 5^n$ for all positive integers n.

5 If $x > 0$, prove that $(1+x)^n \ge 1 + nx$ for all positive integers n.

6 Prove that $2^n \ge n^2$ for positive integers $n \ge 4$. (Hint: Use the result from Example **21** page 70.)

7 **(a)** If $F(n) = n^3 + 20n$, find $F(n+2) - F(n)$.

(b) Hence prove that $n^3 + 20n$ is divisible by 48 if n is even.

8 **(a)** Prove that $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ for any positive integer n by:

(i) first proving S(1) that $\frac{d}{dx}(x) = 1$

(ii) then writing $x^{n+1} = x \times x^n$ and using the product rule to prove that $S(k+1)$ is true.

(b) Summarise your results to give the proof of the result by induction.

9 Prove that the nth odd number is $2n - 1$.

10 If n and r are positive integers, prove that

$$1 + n + \frac{n(n+1)}{2!} + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!} = \frac{(n+1)(n+2)\ldots(n+r)}{r!}.$$

11 Prove that the sum of the exterior angles of a convex polygon with n sides is 360° for $n \ge 3$.

12 Prove that $(n+1)^2 + (n+2)^2 + (n+3)^2 + \ldots + (2n)^2 = \frac{n(2n+1)(7n+1)}{6}$ for positive integers n.

13 Prove that $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$ for positive integers n.

14 Prove that $\frac{d^n}{dx^n}\left(x^n\right) = n!$ for integral n, $n \ge 0$.

15 The binomial theorem states that if n is an integer, $n \ge 1$, then $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^r a^{n-r}$. Use mathematical induction to prove this result.

16 At a mathematics conference, each of the n mathematicians attending wishes to shake hands with all the others.

(a) Work out how many handshakes there are.

(b) Use mathematical induction to prove your hypothesis in part **(a)**.

17 Prove that the number of diagonals of a convex polygon with n vertices is $\frac{n(n-3)}{2}$ for $n \geq 4$.

18 It is given that $A > 0$, $B > 0$ and n is a positive integer.

(a) Factorise $A^{n+1} - A^nB + B^{n+1} - B^nA$.

(b) Hence show that $A^{n+1} + B^{n+1} \geq A^nB + B^nA$.

(c) Prove by induction that $\left(\frac{A+B}{2}\right)^n \leq \frac{A^n + B^n}{2}$ for all positive integers n.

19 Prove by induction that the greatest number of regions that a circle can be divided into by n straight lines is $\frac{1}{2}(n^2 + n + 2)$ for all positive integers n.

20 Prove by induction that: $\sin(x + n\pi) = (-1)^n \sin x$ for all integers $n \geq 1$.

21 Suppose there is a country in which the only currency is \$3 coins and \$5 coins.

(a) Construct a table to show how the amounts of money from \$1 to \$15 (in whole dollars) can be made using only \$3 and \$5 coins. Note that some amounts are impossible to achieve.

(b) Complete the following statement, then prove it by induction:
'Using only \$3 and \$5 coins, it is possible to make \$$n$ for all integers $n \geq \ldots$'

22 By induction, prove that for each positive integer n:

(a) there are unique positive integers p_n and q_n such that $(1+\sqrt{3})^n = p_n + q_n\sqrt{3}$

(b) there are unique positive integers p_n and q_n such that $p_n^{\ 2} - 3q_n^{\ 2} = (-2)^n$

23 (a) Write the binomial expansion of $(k+1)^p$ where p is a positive integer.

(b) If p is a prime number, identify which of the terms in the expansion do not have a factor of p.

(c) Prove by induction on n that if n is a positive integer and p is a prime number, then $n^p - n$ is a multiple of p.

24 (a) Show that: $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$.

(b) Prove by induction that $\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n$ for all integers $n \geq 1$.

25 (a) Show that $2k + 3 > 2\sqrt{(k+1)(k+2)}$ for $k > 0$.

(b) Hence prove by induction that: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$.

(c) Consider the statement that $\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10}$ for all positive integers N. Is this statement correct? Justify your answer.

2.6 USING INDUCTION TO PROVE FIRST-ORDER RECURSIVE FORMULAE

A recursive formula is when one term is defined in terms of one or more preceding terms.

A **recursive** definition of a function allows you to evaluate the function at a certain value of n using value(s) of the function at some other value(s) of n.

A **closed form** definition allows a function to be evaluated directly from the required value of n.

Example 25

A sequence $\{u_n\}$ is defined recursively as $u_1 = 2$, $u_2 = 6$, $u_n = 6u_{n-1} - 5u_{n-2}$ for $n \geq 3$.

Prove by induction the closed form definition $u_n = 5^{n-1} + 1$ for all integers $n \geq 1$.

Solution

Step 1 Prove true for the two initial cases.

$n = 1$: $u_1 = 5^{1-1} + 1 = 2$, which agrees with the recursive definition.

$n = 2$: $u_2 = 5^{2-1} + 1 = 6$, which agrees with the recursive definition.

The proposition is true for $n = 1$ and $n = 2$.

Step 2 Assume the proposition is true for all integers from 1 to k. Thus prove that it is true for $n = k + 1$.

i.e. assume: $u_1 = 5^{1-1} + 1$

$u_2 = 5^{2-1} + 1$

$u_3 = 5^{3-1} + 1$

$\dots$

$u_{k-1} = 5^{(k-1)-1} + 1 = 5^{k-2} + 1$

$u_k = 5^{k-1} + 1$

For $n = k + 1$: $u_{k+1} = 6u_{(k+1)-1} - 5u_{(k+1)-2}$ (using the recursive definition)

$= 6u_k - 5u_{k-1}$

$= 6(5^{k-1} + 1) - 5(5^{k-2} + 1)$

$= 6 \times 5^{k-1} + 6 - 1 \times 5^{k-1} - 5$

$= 5 \times 5^{k-1} + 1$

$= 5^k + 1$ which is the closed form definition.

Step 3 Conclusion

The proposition is true for $n = k + 1$ if it is true for $n = 1, 2, \dots k$.
It is true for $n = 1, 2$.
$\therefore$ by induction, it is true for all integers $n \geq 1$.

Example 26

(a) If $u_{r+1} = 2u_r + 1$ for all positive integer values of r, prove that $u_n + 1 = 2^{n-1}(u_1 + 1)$.

(b) Find the value of $\sum_{r=1}^{n} u_r$ if $u_1 = 1$.

Solution

(a) $r = 1$: $u_2 = 2u_1 + 1$

Consider $u_n + 1 = 2^{n-1}(u_1 + 1)$.

Step 1 When $n = 1$: LHS $= u_1 + 1$; RHS $= 2^0(u_1 + 1) = u_1 + 1 =$ LHS

Hence the result is true for $n = 1$.

When $n = 2$: LHS $= u_2 + 1 = 2u_1 + 2$; RHS $= 2^1(u_1 + 1) = 2u_1 + 2 =$ LHS

Hence the result is true for $n = 2$.

Step 2 Assume the result is true for $n = k$, given $u_{k+1} = 2u_k + 1$, i.e. assume that $u_k + 1 = 2^{k-1}(u_1 + 1)$.

Prove the result is true for $n = k + 1$, i.e. prove that $u_{k+1} + 1 = 2^k(u_1 + 1)$.

LHS $= u_{k+1} + 1$

$= 2u_k + 1 + 1$

$= 2(u_k + 1)$

$= 2\left(2^{k-1}(u_1 + 1)\right)$

$= 2^k(u_1 + 1)$

$=$ RHS

Step 3 The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

(b) $u_r = 2^{r-1}(1+1)-1$

$= 2^r - 1$

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left(2^r - 1\right) = \sum_{r=1}^{n} 2^r - n$$

$$= \frac{2\left(2^n - 1\right)}{2-1} - n$$

$$= 2^{n+1} - n - 2$$

EXERCISE 2.6 USING INDUCTION TO PROVE FIRST-ORDER RECURSIVE FORMULAE

1 If $u_{n+1} = 3u_n + 4$ and $u_1 = 1$, use mathematical induction to prove that $u_n = 3^n - 2$ for all positive integers n.

2 If $u_1 = 6$ and $u_{n+1} = u_n + 2^n + 4$, use mathematical induction to prove that $u_n = 2^n + 4n$ for all positive integers n.

3 A sequence is defined recursively as $u_1 = 1$, $u_2 = 3$, $u_{n+1} = 2u_n - u_{n-1}$ for $n \geq 2$. By induction, prove that $u_n = 2n - 1$ for all positive integers n.

4 A sequence is defined recursively as $u_1 = 5$, $u_2 = 13$, $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geq 3$. By induction, prove that $u_n = 2^n + 3^n$ for all positive integers n.

5 A sequence is defined recursively as $u_1 = 3$, $u_2 = 33$, $u_n = 11u_{n-1} - 28u_{n-2}$ for $n \geq 3$. By induction, prove that $u_n = 7^n - 4^n$ for all positive integers n.

6 A sequence is defined recursively as $u_1 = 0$, $u_2 = 25$, $u_n = 10u_{n-1} - 25u_{n-2}$ for $n \geq 3$. By induction, prove that $u_n = (n-1)5^n$ for all positive integers n.

7 The Fibonacci sequence is defined as $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ for all positive integers $n \geq 1$.

Prove by induction that $u_n < \left(\frac{5}{3}\right)^n$ for all positive integers n.

8 If $a_0 = 1$, $a_1 = 6$ and $a_n = 6a_{n-1} - 9a_{n-2}$, use mathematical induction to prove that $a_n = 3^n + n3^n$.

9 (a) Prove by contradiction that $(4k+3)\sqrt{k} \leq (4k+1)\sqrt{k+1}$ for all $k \geq 0$.

(b) Prove by induction that $\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \leq \frac{(4n+3)\sqrt{n}}{6}$ for all integers $n \geq 1$.

10 If $4 = \frac{3}{u_1} = u_1 + \frac{3}{u_2} = u_2 + \frac{3}{u_3} = u_3 + \frac{3}{u_4} = \ldots$, prove by induction that $u_n = \frac{3^{n+1} - 3}{3^{n+1} - 1}$ for all positive integers n.

11 If $u_1 = 0$ and if $u_{n+1} = (1-x)u_n + nx$ for all positive integers n, prove that $u_n = \frac{1}{x}\left(nx - 1 + (1-x)^n\right)$.

CHAPTER REVIEW 2

1 Use mathematical induction to prove the following results.

(a) $\displaystyle\sum_{r=1}^{n}(2r-1)^2 = \frac{n}{3}\left(4n^2 - 1\right)$

(b) $\displaystyle\sum_{r=1}^{n}\frac{1}{(3r-2)(3r+1)} = \frac{n}{3n+1}$

(c) $\displaystyle\sum_{r=1}^{n}\left(r^2 + 1\right)r! = n \times (n+1)!$

(d) $\displaystyle\sum_{r=1}^{n}\left(r^3 + 3r^5\right) = \frac{n^3(n+1)^3}{2}$

2 **(a)** Simplify $\dfrac{1}{r(r+1)}-\dfrac{1}{(r+1)(r+2)}$. **(b)** Hence evaluate $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$.

(c) Use mathematical induction to prove that $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$ equals the result that you obtained in part **(b)**.

3 Consider the sequence of numbers defined by $T_1 = 3$, $T_n = 2 \times T_{n-1} + 3$ for all $n \geq 2$.

(a) List the first five terms of this sequence.

(b) Prove by induction that $T_n = 3(2^n - 1)$ for all integers $n \geq 1$.

4 **(a)** If $u_{n+1} = 2u_n + 1$ for all positive integral values of n, use mathematical induction to prove that $u_n + 1 = 2^{n-1}(u_1 + 1)$.

(b) If $u_1 = 1$, find the value of $\displaystyle\sum_{r=1}^{n} u_r$.

5 If $u_1 = 0$ and if $u_{r+1} = (1 + x)u_r - rx$ for all positive integral values of r, prove by induction that $u_n = \dfrac{1}{x}\left[1 + nx - (1 + x)^n\right]$.

6 If $x > 0$ and $y > 0$, prove by induction that $(x + y)^n > x^n + y^n$ for all integers $n \geq 2$.

7 **(a)** By writing $\cos([2k + 1]x)$ as $\cos(2kx + x)$, and remembering that $\cos 2x = 1 - 2\sin^2 x$, show that:

$$\frac{\sin 2kx}{2\sin x} + \cos(2k+1)x = \frac{\sin(2(k+1)x)}{2\sin x}$$

(b) Use the result of part **(a)** to prove by induction that $\cos x + \cos 3x + \ldots + \cos([2n - 1]x) = \dfrac{\sin(2nx)}{2\sin x}$ for all positive integers n.

8 If $x_1, x_2, x_3, \ldots x_n$ are positive real numbers, prove by induction that:

$(x_1 + x_2 + x_3 + \ldots + x_n)\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3} + \ldots + \dfrac{1}{x_n}\right) \geq n^2$ for all integers $n \geq 1$.

9 **(a)** Prove that $x + \sqrt{x} \geq \sqrt{x(x+1)}$ for all real $x \geq 0$.

(b) A sequence is defined as $u_1 = 1$, $u_2 = 2$, $u_n = u_{n-1} + (n - 1)u_{n-2}$ for $n \geq 3$. Prove by induction that $u_n \geq \sqrt{n!}$.

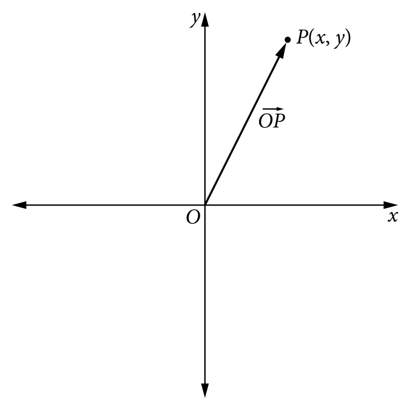
CHAPTER 3
Further work with vectors

Vectors in two dimensions were introduced in *New Senior Mathematics Extension 1*, Chapter 10. In this chapter, you will extend the use of vector notation to three dimensions.

OVERVIEW OF VECTORS IN TWO-DIMENSIONAL SPACE

Position vector

A **position vector** can be represented by a coordinate pair (a, b). This represents the vector that is a units from O in the positive x direction and b units from O in the positive y direction. This can be represented as the column vector $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$ or by the coordinates (a, b). Any vector that is equivalent to a translation of a units in the positive x direction and b units in the positive y direction can be represented in this way. When the reference point is not the origin, the term **relative position vector** is used.



The magnitude of a position vector

The magnitude of a position vector $\overrightarrow{OA} = \begin{pmatrix} a \\ b \end{pmatrix}$ can be calculated using Pythagoras' theorem, $\left|\overrightarrow{OA}\right| = \sqrt{a^2 + b^2}$.

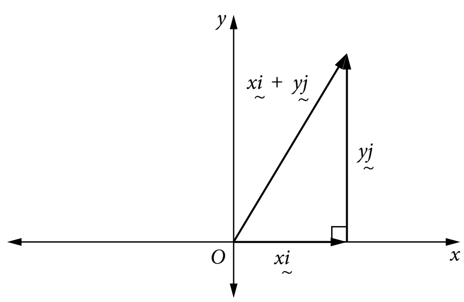
Unit vector

The unit vector in the direction of $\underset{\sim}{a}$ is denoted $\hat{\underset{\sim}{a}}$, where $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ and $|\hat{\underset{\sim}{a}}| = 1$.

Vectors in component form

The form $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is called **component form** or $\underset{\sim}{i}, \underset{\sim}{j}$ **form** of a vector.

The vector $\underset{\sim}{a}$ may also be represented in column vector form as $\begin{pmatrix} x \\ y \end{pmatrix}$.

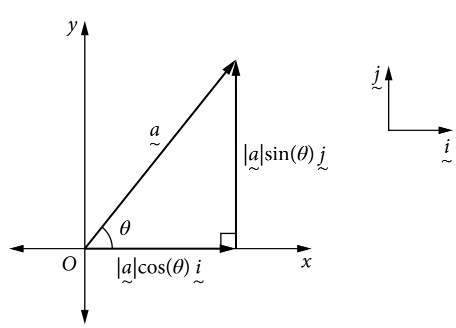


Resolving vectors into component form

If a vector $\underset{\sim}{a}$ of magnitude $|\underset{\sim}{a}|$ makes an angle θ with the positive x-axis, then $\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta\underset{\sim}{i} + |\underset{\sim}{a}|\sin\theta\underset{\sim}{j}$.

The horizontal component is of the vector $\underset{\sim}{a}$ is $|\underset{\sim}{a}|\cos\theta\underset{\sim}{i}$ and the vertical component is $|\underset{\sim}{a}|\sin\theta\underset{\sim}{j}$.

The process of specifying a vector of known magnitude and direction in component form is called **resolving the vector**.



Addition and subtraction of vectors in component form

For $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$,

$$\begin{aligned}\underset{\sim}{a} + \underset{\sim}{b} &= \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\right) + \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j}\right)\\ &= x_1\underset{\sim}{i} + x_2\underset{\sim}{i} + y_1\underset{\sim}{j} + y_2\underset{\sim}{j}\\ &= \left(x_1 + x_2\right)\underset{\sim}{i} + \left(y_1 + y_2\right)\underset{\sim}{j}\end{aligned}$$

In column vector notation, this can be written as $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$ respectively.

Scalar multiplication of vectors in component form

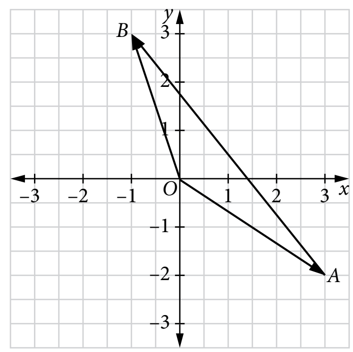
If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $k\underset{\sim}{a} = kx\underset{\sim}{i} + ky\underset{\sim}{j}$, where k is a real number.

Equality of vectors in component form

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$, then $\underset{\sim}{a} = \underset{\sim}{b}$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Relative position vectors

You have already looked at position vectors that represent the position of one point in relation to the origin. A **relative position vector** represents a point's position in relation to another point.



The position vector of B relative to (as seen from) A is given by $\overrightarrow{AB}$.

In the diagram shown, $\overrightarrow{OA} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\overrightarrow{OB} = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

The position vector of B relative to A is $\overrightarrow{AB}$, where $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.

Now, $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$$\begin{aligned}\therefore\quad \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= \left(-\underset{\sim}{i} + 3\underset{\sim}{j}\right) - \left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)\\ &= -4\underset{\sim}{i} + 5\underset{\sim}{j}\end{aligned}$$

Parallel vectors

If $\underset{\sim}{b} = k\underset{\sim}{a}$, where k is a real number, then $\underset{\sim}{b}$ is parallel to $\underset{\sim}{a}$.

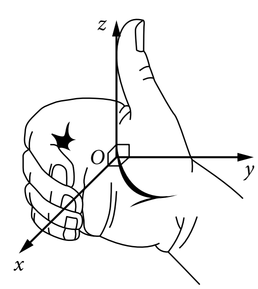
Unit vectors in component form

If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $\hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{x^2 + y^2}}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$ or $\hat{\underset{\sim}{a}} = \dfrac{\sqrt{x^2 + y^2}}{x^2 + y^2}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

3.1 VECTORS IN THREE DIMENSIONS

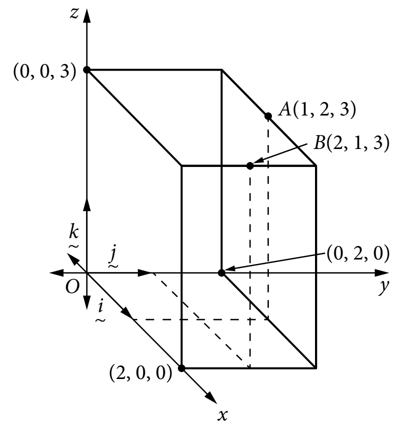
The work that follows requires you to be able to visualise objects in three-dimensional space. You need to be able to draw cubes and rectangular prisms with some of the edges forming the axes of the Cartesian system.

To describe the position of a point in three-dimensional space, you need three coordinates. The Cartesian system of two coordinates, X and Y, is extended by means of a third axis, OZ, which is perpendicular to the plane OXY. The positive direction of OZ is towards the top of the page, the positive direction of OX is out of the page and the positive direction of OY is horizontally to the right.



This is shown in the diagram on the right. It is called a right-hand system of axes. Here, the fingers of the right hand point in the direction from the positive x-axis to the positive y-axis, so that the thumb points upwards in the direction of the positive z-axis.

In the following diagram, the x–y plane is horizontal, the x–z and y–z planes are vertical. In this $OXYZ$ system, the points $A(1, 2, 3)$ and $B(2, 1, 3)$ are shown.



The vector $\overrightarrow{OA}$ may be written as $\overrightarrow{OA} = (1,2,3)$, $\overrightarrow{OA} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ or $\overrightarrow{OA} = \underset{\sim}{a}$, as well as in component form.

The components of a vector in three dimensions use $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ as the unit vectors parallel to the x, y and z axes respectively.

Thus $\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

Using your knowledge of two-dimensional vectors, it should seem reasonable to find $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= \underset{\sim}{b} - \underset{\sim}{a} = \left(2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\right) - \left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)\\ &= (2-1)\underset{\sim}{i} + (1-2)\underset{\sim}{j} + (3-3)\underset{\sim}{k}\\ &= \underset{\sim}{i} - \underset{\sim}{j}\end{aligned}$$

$\overrightarrow{AB}$ has no $\underset{\sim}{k}$ component, so it lies in a plane that is parallel to the x–y plane. It is the plane given by the equation $z = 3$.

In particular, points in the x–y plane are of the form $(x, y, 0)$, points in the y–z plane are of the form $(0, y, z)$ and points in the x–z plane are of the form $(x, 0, z)$.

In two-dimensional space, the coordinate axes divide the plane into four regions or quadrants. In three-dimensional space, the coordinate axes divide the space into eight regions or octants. The sign of the coordinates indicates in which octant the point is located.

Example 1

Given $\underset{\sim}{a} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$, $\underset{\sim}{b} = (-1,3,2)$ and $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, find each of the following vectors, expressing your answer in component form.

(a) $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ **(b)** $\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c}$ **(c)** $\underset{\sim}{a} - \underset{\sim}{b} - \underset{\sim}{c}$ **(d)** $4\underset{\sim}{a}$ **(e)** $\underset{\sim}{a} - 2\underset{\sim}{b} + 3\underset{\sim}{c}$

Solution

The answers are to be given in component form, so rewrite each vector in component form.

$\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$.

(a)
$$\begin{aligned}\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} &= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + \left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\\ &= 2\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}\end{aligned}$$

(b)
$$\begin{aligned}\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c} &= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} - \left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\\ &= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + \underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k} + \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\\ &= 4\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}\end{aligned}$$

(c)
$$\begin{aligned}\underset{\sim}{a} - \underset{\sim}{b} - \underset{\sim}{c} &= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} - \left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) - \left(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)\\ &= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + \underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k} - \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\\ &= 2\underset{\sim}{i} - 6\underset{\sim}{j} + 5\underset{\sim}{k}\end{aligned}$$

(d)
$$\begin{aligned}4\underset{\sim}{a} &= 4\left(2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}\right)\\ &= 8\underset{\sim}{i} - 4\underset{\sim}{j} + 16\underset{\sim}{k}\end{aligned}$$

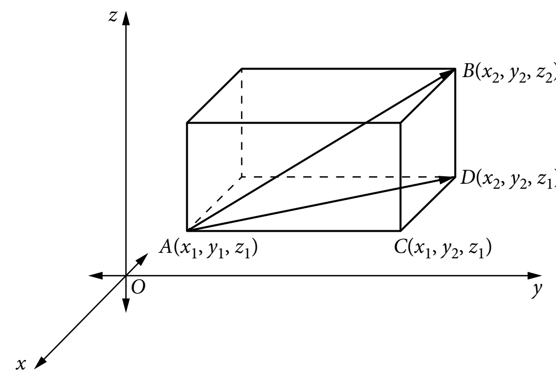
(e) $\underset{\sim}{a} - 2\underset{\sim}{b} + 3\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} - 2\left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) + 3\left(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

$= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k} + 2\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k} + 3\underset{\sim}{i} + 6\underset{\sim}{j} - 9\underset{\sim}{k}$

$= 7\underset{\sim}{i} - \underset{\sim}{j} - 9\underset{\sim}{k}$

Magnitude of vectors in three-dimensional space

To obtain the magnitude of vectors in three-dimensional space you use an extension of Pythagoras' theorem. This is similar to the method used to find the length of a diagonal of a cuboid in solid geometry.



Suppose you wish to find the length of the line segment joining $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$, as shown in the diagram.

Construct the planes ACD and CDB that are parallel to the x–y and x–z planes respectively. Then C, which has the same x and z coordinates as A, will be $C(x_1, y_2, z_1)$, while D, which has the same x and y coordinates as B, will be $D(x_2, y_2, z_1)$.

Now $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{DB}\right|^2$

$= \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{CD}\right|^2 + \left|\overrightarrow{DB}\right|^2$

And $\left|\overrightarrow{AC}\right| = |y_2 - y_1|$

$\left|\overrightarrow{CD}\right| = |x_2 - x_1|$

$\left|\overrightarrow{DB}\right| = |z_2 - z_1|$

Hence, $\left|\overrightarrow{AB}\right|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ and thus $\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Referring to diagram on page 79:

$\left|\overrightarrow{OA}\right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$\left|\overrightarrow{OB}\right| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$

$\left|\overrightarrow{AB}\right| = \sqrt{(2-1)^2 + (1-2)^2 + (3-3)^2} = \sqrt{2}$

Unit vectors in three-dimensional space

Given $\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, then since $\left|\overrightarrow{OA}\right| = \sqrt{14}$ and $\left|\overrightarrow{OB}\right| = \sqrt{14}$, the corresponding unit vectors can be found using the fact that $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

$\hat{\underset{\sim}{a}} = \dfrac{1}{\sqrt{14}}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right) = \dfrac{\sqrt{14}}{14}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ and $\hat{\underset{\sim}{b}} = \dfrac{1}{\sqrt{14}}\left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right) = \dfrac{\sqrt{14}}{14}\left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$.

Thus $\hat{\underset{\sim}{a}} = \dfrac{\sqrt{14}}{14}\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ is the unit vector parallel to $\overrightarrow{OA}$ and $\hat{\underset{\sim}{b}} = \dfrac{\sqrt{14}}{14}\left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$ is the unit vector parallel to $\overrightarrow{OB}$.

From the given vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ you found that $\overrightarrow{AB} = \underset{\sim}{i} - \underset{\sim}{j}$ and that $\left|\overrightarrow{AB}\right| = \sqrt{2}$ so the unit vector parallel to $\overrightarrow{AB}$ is given by $\dfrac{\sqrt{2}}{2}\left(\underset{\sim}{i} - \underset{\sim}{j}\right)$. Since this vector does not contain a $\underset{\sim}{k}$ component, it lies in a plane parallel to the x–y plane.

Example 2

(a) On a set of Cartesian axes, mark the terminal points $A(1, 3, 4)$, $B(-1, 3, 2)$ of vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$.

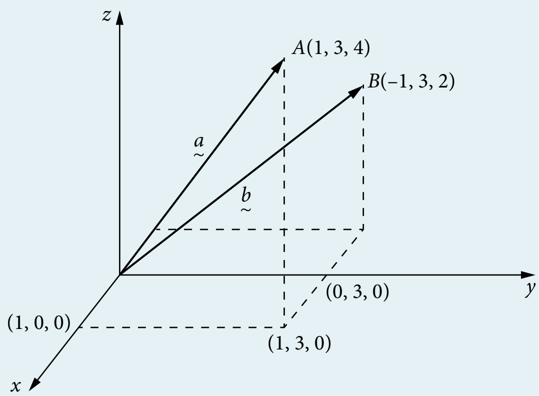
(b) Write the vectors $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, in terms of $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$. Similarly, write an expression for $\underset{\sim}{e} = \underset{\sim}{b} - \underset{\sim}{a}$.

(c) Find the magnitudes of vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{e}$.

(d) Find unit vectors in the direction of the vectors given in part **(c)**.

Solution

(a)



Note: **(i)** $\underset{\sim}{a}$ points out from the page upwards towards the reader.

(ii) $\underset{\sim}{b}$ points into the page.

(b) $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{e} = \overrightarrow{AB} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} - (\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}) = -2\underset{\sim}{i} - 2\underset{\sim}{k}$.

(c) $|\underset{\sim}{a}| = \sqrt{1+9+16} = \sqrt{26}$, $|\underset{\sim}{b}| = \sqrt{1+9+4} = \sqrt{14}$, $|\underset{\sim}{e}| = \sqrt{4+4} = 2\sqrt{2}$.

(d) $\hat{\underset{\sim}{a}} = \dfrac{\sqrt{26}}{26}\left(\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}\right)$, $\hat{\underset{\sim}{b}} = \dfrac{\sqrt{14}}{14}\left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)$, $\hat{\underset{\sim}{e}} = \dfrac{\sqrt{2}}{4}\left(-\underset{\sim}{i} - \underset{\sim}{k}\right)$.

Position vectors in three dimensions

A **vector** defines the position of one point relative to another point. When the reference point is the origin, the vector is simply called a ‘position vector’. When the reference point is not the origin, the term **relative position vector** is used.

Given the position vector of A is $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and the position vector of B is $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, the position vector of B *relative to* A (i.e. B as seen from A) is

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{b} - \underset{\sim}{a}\\ &= -2\underset{\sim}{i} - 2\underset{\sim}{k}.\end{aligned}$$

The position vector of A relative to B (i.e. A as seen from B) is $\overrightarrow{BA} = 2\underset{\sim}{i} + 2\underset{\sim}{k}$.

Note that the position vector of B relative to A = position vector of B − position vector of A.

Algebra of vectors expressed in component form

(i) Equality: Two vectors are equal if and only if the corresponding components are equal;

i.e. $a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k} = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

(This is true because the $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ representation of a vector is unique.)

(ii) The components of the sum (difference) of two vectors are equal to the sum (difference) of the corresponding components of the vectors;

e.g. If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, then $\underset{\sim}{a} \pm \underset{\sim}{b} = (x_1 \pm x_2)\underset{\sim}{i} + (y_1 \pm y_2)\underset{\sim}{j} + (z_1 \pm z_2)\underset{\sim}{k}$.

(iii) The components of a scalar multiple of a vector are equal to the scalar multiple of the corresponding components of the vector;

e.g. If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ and $\lambda \in R$, then $\lambda\underset{\sim}{a} = (\lambda x)\underset{\sim}{i} + (\lambda y)\underset{\sim}{j} + (\lambda z)\underset{\sim}{k}$.

Example 3

Given $A(6, 4, 8)$ and $B(10, 6, 8)$ show that $\overrightarrow{AB}$ is parallel to the x–y plane.

Solution

$$\overrightarrow{AB} = (10-6)\underset{\sim}{i} + (6-4)\underset{\sim}{j} + (8-8)\underset{\sim}{k}$$
$$= 4\underset{\sim}{i} + 2\underset{\sim}{j}$$

Since $\overrightarrow{AB}$ has no $\underset{\sim}{k}$ component, it is parallel to the x–y plane.

Example 4

Determine whether the points $A(0, 2, 2)$, $B(4, 10, 18)$, $C(6, 14, 26)$ are collinear.

Solution

$$\overrightarrow{AB} = (4-0)\underset{\sim}{i} + (10-2)\underset{\sim}{j} + (18-2)\underset{\sim}{k}$$
$$= 4\underset{\sim}{i} + 8\underset{\sim}{j} + 16\underset{\sim}{k}$$
$$= 4\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}\right)$$

$$\overrightarrow{BC} = (6-4)\underset{\sim}{i} + (14-10)\underset{\sim}{j} + (26-18)\underset{\sim}{k}$$
$$= 2\underset{\sim}{i} + 4\underset{\sim}{j} + 8\underset{\sim}{k}$$
$$= 2\left(\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}\right)$$

Hence $\overrightarrow{AB} = 2\overrightarrow{BC}$.

Thus $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel and have point B in common.

Hence ABC is a straight line and the points A, B and C are collinear.

Example 5

Find the distance of the point $P(2, 3, 5)$:

(a) from the y–z plane

(b) from the x-axis.

Solution

(a) The line segment from P normal to the y–z plane intersects the y–z plane at $Q(0, 3, 5)$.

Hence $\left|\overrightarrow{PQ}\right| = |0-2| = 2$.

(b) The line segment from P perpendicular to the x-axis meets the x-axis at $R(2, 0, 0)$.

Thus $\left|\overrightarrow{PR}\right| = \sqrt{0^2 + 3^2 + 5^2} = \sqrt{34}$.

MAKING CONNECTIONS

Vectors in three dimensions

Use technology to plot vectors in three dimensions.

EXERCISE 3.1 VECTORS IN THREE DIMENSIONS

1 For each of the following vectors, find:

(i) the magnitude of the vector (ii) a unit vector parallel to the given vector.

(a) $3\underset{\sim}{i}+4\underset{\sim}{j}$ (b) $-5\underset{\sim}{i}+12\underset{\sim}{j}$ (c) $-7\underset{\sim}{i}-24\underset{\sim}{j}$ (d) $9\underset{\sim}{i}-12\underset{\sim}{j}$

2 For each of the points P whose coordinates are given, find:

(i) an $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ representation for the position vector $\overrightarrow{OP}$ (ii) the magnitude of $\overrightarrow{OP}$

(iii) a unit vector in the direction of $\overrightarrow{OP}$.

(a) $P(-1, 4)$ (b) $P(6, 8)$ (c) $P(2, 2, 1)$ (d) $P(-3, 4, 5)$ (e) $P(4, 0, 0)$ (f) $P(1, 1, 1)$

3 Given vectors $\underset{\sim}{a} = 6\underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k}$, $\underset{\sim}{b} = -4\underset{\sim}{i}+3\underset{\sim}{j}+\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i}+3\underset{\sim}{k}$, write:

(a) $\underset{\sim}{a}+\underset{\sim}{b}$ (b) $\underset{\sim}{a}+\underset{\sim}{b}-\underset{\sim}{c}$ (c) $2\underset{\sim}{c}+3\underset{\sim}{a}-5\underset{\sim}{b}$ (d) $3\left(\underset{\sim}{c}-\underset{\sim}{a}\right)$ (e) $5\left(\underset{\sim}{a}-\underset{\sim}{b}\right)+6\underset{\sim}{c}$ (f) $5\underset{\sim}{a}+4\left(\underset{\sim}{b}-3\underset{\sim}{c}\right)$

4 The position vectors of points P and Q are $2\underset{\sim}{i}-3\underset{\sim}{j}+4\underset{\sim}{k}$ and $3\underset{\sim}{i}-7\underset{\sim}{j}+12\underset{\sim}{k}$ respectively. Find the length of $\overrightarrow{PQ}$ and a unit vector parallel to $\overrightarrow{PQ}$.

5 Given $\underset{\sim}{a} = 12\underset{\sim}{i}+3\underset{\sim}{j}-4\underset{\sim}{k}$, $\underset{\sim}{b} = 4\underset{\sim}{i}-3\underset{\sim}{j}-5\underset{\sim}{k}$, $\underset{\sim}{c} = -6\underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k}$, find unit vectors $\hat{\underset{\sim}{a}}$, $\hat{\underset{\sim}{b}}$, $\hat{\underset{\sim}{c}}$.

6 The position vectors of the points P, Q, R and S are respectively $4\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k}$, $5\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$, $2\underset{\sim}{i}-2\underset{\sim}{j}-3\underset{\sim}{k}$, $4\underset{\sim}{i}-4\underset{\sim}{j}+3\underset{\sim}{k}$. Show that PQ is parallel to RS.

7 Given that $\underset{\sim}{u} = 3\underset{\sim}{i}+4\underset{\sim}{j}+5\underset{\sim}{k}$, $\underset{\sim}{v} = 2\underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k}$, $\underset{\sim}{w} = 6\underset{\sim}{i}-7\underset{\sim}{j}-8\underset{\sim}{k}$, find the vectors:

(a) $\underset{\sim}{u}+\underset{\sim}{v}$ (b) $3\underset{\sim}{u}+\underset{\sim}{v}-3\underset{\sim}{w}$ (c) $-\underset{\sim}{u}+\underset{\sim}{v}-\underset{\sim}{w}$ (d) $\underset{\sim}{v}+\underset{\sim}{w}$.

(e) If $\underset{\sim}{d} = 7\underset{\sim}{i}+23\underset{\sim}{j}+29\underset{\sim}{k}$, find r, s and $t \in R$ such that $\underset{\sim}{d} = r\underset{\sim}{u}+s\underset{\sim}{v}+t\underset{\sim}{w}$.

8 Given that $\underset{\sim}{a} = 4\underset{\sim}{i}-5\underset{\sim}{j}+8\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i}-3\underset{\sim}{j}-4\underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$, $\underset{\sim}{u} = \underset{\sim}{a}+m\underset{\sim}{b}+n\underset{\sim}{c}$, $\underset{\sim}{v} = -2\underset{\sim}{i}-\underset{\sim}{j}-3\underset{\sim}{k}$. Find $m, n \in R$ such that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are opposite in direction.

9 $P(6, 3, -4)$, $Q(3, 1, 1)$ and $R(2, -1, 3)$ are the vertices of a triangle. Show that $\left|\overrightarrow{RP}\right| = 3\left|\overrightarrow{RQ}\right|$.

10 If $\underset{\sim}{a} = 2\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i}-4\underset{\sim}{j}+5\underset{\sim}{k}$, $\underset{\sim}{c} = -\underset{\sim}{i}-4\underset{\sim}{j}+2\underset{\sim}{k}$, find the values of p and q such that $\underset{\sim}{a}+p\underset{\sim}{b}+q\underset{\sim}{c}$ is parallel to the x-axis.

11 Find the distance of the point $P(1, 4, 3)$ from:

(a) the y–z plane (b) the x–z plane (c) the x–y plane

(d) the x-axis (e) the y-axis (f) the z-axis.

3.2 SCALAR PRODUCT OF VECTORS IN THREE DIMENSIONS

The scalar product (dot product) has been defined and used for vectors in two dimensions. (Refer to *New Senior Mathematics Extension 1*, Section 10.4.)

In component form, it is written $\underset{\sim}{a}\bullet\underset{\sim}{b} = \left(x_1\underset{\sim}{i}+y_1\underset{\sim}{j}\right)\bullet\left(x_2\underset{\sim}{i}+y_2\underset{\sim}{j}\right) = x_1x_2+y_1y_2$.

The proof of this result is similar to the proof in two dimensions.

$$\underset{\sim}{a}\bullet\underset{\sim}{b} = \left(x_1\underset{\sim}{i}+y_1\underset{\sim}{j}+z_1\underset{\sim}{k}\right)\bullet\left(x_2\underset{\sim}{i}+y_2\underset{\sim}{j}+z_2\underset{\sim}{k}\right)$$

$$= (x_1x_2)\left(\underset{\sim}{i}\bullet\underset{\sim}{i}\right)+(x_1y_2)\left(\underset{\sim}{i}\bullet\underset{\sim}{j}\right)+\left(x_1z_2\left(\underset{\sim}{i}\bullet\underset{\sim}{k}\right)\right)+(y_1x_2)\left(\underset{\sim}{j}\bullet\underset{\sim}{i}\right)+(y_1y_2)\left(\underset{\sim}{j}\bullet\underset{\sim}{j}\right)+(y_1z_2)\left(\underset{\sim}{j}\bullet\underset{\sim}{k}\right)$$

$$+(z_1x_2)\left(\underset{\sim}{k}\bullet\underset{\sim}{i}\right)+(z_1y_2)\left(\underset{\sim}{k}\bullet\underset{\sim}{j}\right)+(z_1z_2)\left(\underset{\sim}{k}\bullet\underset{\sim}{k}\right)$$

$$= x_1x_2+y_1y_2+z_1z_2 \text{ as } \underset{\sim}{i}\bullet\underset{\sim}{i} = \underset{\sim}{j}\bullet\underset{\sim}{j} = \underset{\sim}{k}\bullet\underset{\sim}{k} = 1 \text{ and } \underset{\sim}{i}\bullet\underset{\sim}{j} = \underset{\sim}{i}\bullet\underset{\sim}{k} = \underset{\sim}{j}\bullet\underset{\sim}{k} = \underset{\sim}{j}\bullet\underset{\sim}{i} = \underset{\sim}{k}\bullet\underset{\sim}{i} = \underset{\sim}{k}\bullet\underset{\sim}{j} = 0.$$

It is sometimes useful to be able to use sigma notation for the scalar product.

If $\underset{\sim}{u} = x_1\underset{\sim}{i} + x_2\underset{\sim}{j} + x_3\underset{\sim}{k}$ and $\underset{\sim}{v} = y_1\underset{\sim}{i} + y_2\underset{\sim}{j} + y_3\underset{\sim}{k}$, then $\underset{\sim}{u} \bullet \underset{\sim}{v} = x_1y_1 + x_2y_2 + x_3y_3 = \sum_{i=1}^{3} x_i y_i$.

Example 6

Given $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, find:

(a) $\underset{\sim}{a} \infty \underset{\sim}{b}$ **(b)** $\underset{\sim}{b} \infty \underset{\sim}{a}$ **(c)** $|\underset{\sim}{a}|$ **(d)** $\hat{\underset{\sim}{a}}$

Solution

(a)
$$\begin{aligned}\underset{\sim}{a} \bullet \underset{\sim}{b} &= \left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) \bullet \left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)\\ &= 2 \times 1 + (-1) \times 2 + 3 \times (-1)\\ &= 2 - 2 - 3\\ &= -3\end{aligned}$$

(b)
$$\begin{aligned}\underset{\sim}{b} \bullet \underset{\sim}{a} &= \left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) \bullet \left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)\\ &= 1 \times 2 + 2 \times (-1) + (-1) \times 3\\ &= 2 - 2 - 3\\ &= -3\end{aligned}$$

(c) $|\underset{\sim}{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$

(d) $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \dfrac{1}{\sqrt{14}}\left(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)$

Angle between two vectors

Just as the scalar product can be used to find the angle between two-dimensional vectors, it can also be used to find the angle between two vectors in three dimensions.

The scalar product says $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, where θ is the angle between the two vectors.

Hence

$$\cos\theta = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^{\,2} + y_1^{\,2} + z_1^{\,2}} \times \sqrt{x_2^{\,2} + y_2^{\,2} + z_2^{\,2}}}$$

Example 7

Find the angle, in degrees, between the vectors $\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$.

Solution

$\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$: $|\underset{\sim}{a}| = \sqrt{4 + 1 + 1} = \sqrt{6}$

$\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$: $|\underset{\sim}{b}| = \sqrt{1 + 4 + 1} = \sqrt{6}$

$$\begin{aligned}\underset{\sim}{a} \bullet \underset{\sim}{b} &= \left(-2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right) \bullet \left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)\\ &= (-2) \times 1 + (-1) \times 2 + (-1) \times (-1)\\ &= -3\end{aligned}$$

$$\cos\theta = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{-3}{6} = -\frac{1}{2}$$

As $\cos\theta$ is negative, the angle must be obtuse, and hence in the second quadrant so $\theta = 120°$.

Example 8

Show that the vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{v} = 5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ are perpendicular to each other.

Solution

Consider $\underset{\sim}{u} \bullet \underset{\sim}{v}$:

$$\begin{aligned}\underset{\sim}{u} \bullet \underset{\sim}{v} &= \left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}\right) \bullet \left(5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right) \\ &= 10 - 6 - 4 \\ &= 0\end{aligned}$$

Since the scalar product is zero, the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular. (Remember, $\cos 90° = 0$.)

Example 9

Find a vector perpendicular to the vector $3\underset{\sim}{i} - 4\underset{\sim}{j}$.

Solution

Let a unit vector perpendicular to $\underset{\sim}{v} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ be $\hat{\underset{\sim}{u}} = x\underset{\sim}{i} + y\underset{\sim}{j}$.

Since $\hat{\underset{\sim}{u}}$ is a unit vector, then $\quad x^2 + y^2 = 1 \quad [1]$

Since $\underset{\sim}{v} \bullet \hat{\underset{\sim}{u}} = 0$: $\quad 3x - 4y = 0 \quad [2]$

[2] becomes $\quad y = \frac{3x}{4}$

Substitute in [1]:

$$\begin{aligned}x^2 + \frac{9x^2}{16} &= 1 \\ 25x^2 &= 16 \\ x = \pm\frac{4}{5}, \quad y &= \pm\frac{3}{5}\end{aligned}$$

Hence $\frac{1}{5}\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ and $-\frac{1}{5}\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ are unit vectors perpendicular to $3\underset{\sim}{i} - 4\underset{\sim}{j}$.

Since any scalar multiples of these unit vectors are also perpendicular to $\underset{\sim}{v}$, two possible answers are $4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $-\left(4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$.

Algebraic properties of the dot product

These results have been used before in the Mathematics Extension 1 course.

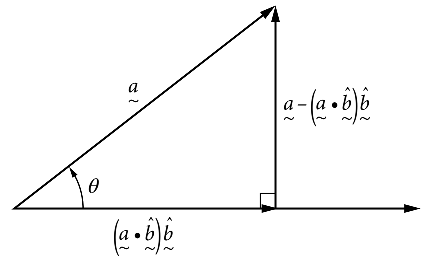
- $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a}$
- $\underset{\sim}{a} \bullet \underset{\sim}{a} = \left|\underset{\sim}{a}\right|^2$
- $\underset{\sim}{a} \bullet \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c}$
- $\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \bullet \left(\underset{\sim}{c} + \underset{\sim}{d}\right) = \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{d} + \underset{\sim}{b} \bullet \underset{\sim}{c} + \underset{\sim}{b} \bullet \underset{\sim}{d}$
- $\left(m\underset{\sim}{a}\right) \cdot \underset{\sim}{b} = m\left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)$ where m is a real number.

Geometric properties of the dot product

- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, $\underset{\sim}{a} \perp \underset{\sim}{b}$
- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors
- If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$
- The angle between two vectors, θ, is given by $\cos\theta = \frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|}$, $\underset{\sim}{a} \neq 0$, $\underset{\sim}{b} \neq 0$
- By convention, $0° \leq \theta \leq 180°$.

Scalar and vector projections of vectors

In *New Senior Mathematics Extension 1*, expressions were given for the scalar and vector projection of the two-dimensional vector, $\underset{\sim}{a}$, onto the two-dimensional vector, $\underset{\sim}{b}$.



- The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\underset{\sim}{a} \infty \hat{\underset{\sim}{b}}$, where $\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}$.
- The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\left(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$ or $\dfrac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\underset{\sim}{b}\bullet\underset{\sim}{b}}\underset{\sim}{b}$.
- The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a}-\dfrac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\underset{\sim}{b}\bullet\underset{\sim}{b}}\underset{\sim}{b}$, or $\underset{\sim}{a}-\left(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

These results also apply to three-dimensional vectors.

Example 10

Given the vectors $\underset{\sim}{a} = 4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, find:

(a) the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

(b) the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

(c) the vector projection of $\underset{\sim}{a}$ onto the x-axis. Hence write the vector projection on the y- and z-axes

(d) the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.

Solution

$\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}: \left|\underset{\sim}{b}\right| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3,\ \hat{\underset{\sim}{b}} = \frac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

(a) Scalar projection

$$\text{onto } \underset{\sim}{b} = \frac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|} = \frac{\left(4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}\right)\bullet\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)}{3} = \frac{8-10-3}{3} = -\frac{5}{3}$$

(b) Vector projection

$$\text{onto } \underset{\sim}{b} = \left(\underset{\sim}{a}\bullet\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\left|\underset{\sim}{b}\right|}\hat{\underset{\sim}{b}} = -\frac{5}{3}\times\frac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) = \frac{5}{9}\left(-2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

(c) $\underset{\sim}{i}$ is the unit vector in the direction of the x-axis: vector projection $= \left(\underset{\sim}{a}\bullet\underset{\sim}{i}\right)\underset{\sim}{i}$

$$= \left(\left(4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}\right)\bullet\underset{\sim}{i}\right)\underset{\sim}{i} = 4\underset{\sim}{i}$$

The vector projection of $\underset{\sim}{a}$ on the y- and z-axes is $5\underset{\sim}{j}$ and $-3\underset{\sim}{k}$.

(d) Vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b} = \underset{\sim}{a} - \dfrac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\underset{\sim}{b}\bullet\underset{\sim}{b}}\underset{\sim}{b}$

$$= \left(4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}\right) - \frac{\left(4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}\right)\bullet\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)}{9}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$$

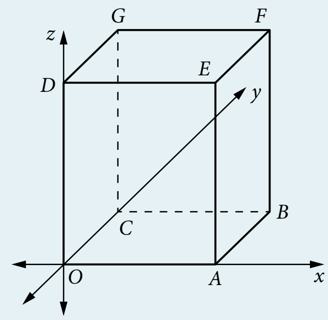
$$= \left(4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}\right) - \frac{-5}{9}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$$

$$= 4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k} + \frac{10}{9}\underset{\sim}{i} - \frac{10}{9}\underset{\sim}{j} + \frac{5}{9}\underset{\sim}{k}$$

$$= \frac{1}{9}\left(46\underset{\sim}{i} + 35\underset{\sim}{j} - 22\underset{\sim}{k}\right)$$

Example 11

$OABCDEFG$ is a cube of side length 1 unit. With reference to O as the origin:



(a) write the position vectors for $\overrightarrow{CB}$, $\overrightarrow{AB}$ and $\overrightarrow{CG}$

(b) find the position vectors of F and G

(c) find vectors $\overrightarrow{OF}$ and $\overrightarrow{AG}$

(d) calculate the acute angle at which the diagonals OF and AG intersect.

Solution

$\overrightarrow{OA} = \underset{\sim}{i}$, $\overrightarrow{OC} = \underset{\sim}{j}$ and $\overrightarrow{OD} = \underset{\sim}{k}$.

(a) $\overrightarrow{CB} = \overrightarrow{OA} = \underset{\sim}{i}$, $\overrightarrow{AB} = \overrightarrow{OC} = \underset{\sim}{j}$, $\overrightarrow{CG} = \overrightarrow{OD} = \underset{\sim}{k}$

(b) $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF}$

$= \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{OG} = \overrightarrow{OC} + \overrightarrow{CG}$

$= \underset{\sim}{j} + \underset{\sim}{k}$

The position vector of F is $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$;

the position vector of G is $\underset{\sim}{j} + \underset{\sim}{k}$.

(c) $\overrightarrow{OF} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{AG} = \overrightarrow{AO} + \overrightarrow{OC} + \overrightarrow{CG}$

$= -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

(d) $\overrightarrow{OF} \bullet \overrightarrow{AG} = \left|\overrightarrow{OF}\right|\left|\overrightarrow{AG}\right|\cos\theta$

$\left|\overrightarrow{OF}\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$,

$\left|\overrightarrow{AG}\right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) \bullet \left(-\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) = \sqrt{3}\sqrt{3}\cos\theta$

$-1 + 1 + 1 = 3\cos\theta$

$\cos\theta = \frac{1}{3}$

$\theta = 70°32$

The angle between the two vectors is 70°32′.

Example 12

Find unit vectors perpendicular to both $\underset{\sim}{u} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$ and $\underset{\sim}{v} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

Solution

Let $\underset{\sim}{w} = p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$ be a unit vector that is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Use the fact that $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$ when $\underset{\sim}{a} \perp \underset{\sim}{b}$.

$\underset{\sim}{u} \bullet \underset{\sim}{w} = 0$: $\quad 2p - 3q + 6r = 0 \quad$ [1]

$\underset{\sim}{v} \bullet \underset{\sim}{w} = 0$: $\quad -6p + 2q + 3r = 0 \quad$ [2]

$\underset{\sim}{w}$ is a unit vector: $\quad p^2 + q^2 + r^2 = 1 \quad$ [3]

$[1] - 2 \times [2]$: $\quad 14p - 7q = 0$

$q = 2p$

$2 \times [1] + 3 \times [2]$: $\quad -14p + 21r = 0$

$r = \frac{2p}{3}$

Substitute in [3]: $\quad p^2 + 4p^2 + \frac{4p^2}{9} = 1$

$49p^2 = 9$

$p = \pm\frac{3}{7}, \quad q = \pm\frac{6}{7}, \quad r = \pm\frac{2}{7}$

Hence $\pm\frac{1}{7}\left(3\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ are unit vectors perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Since any scalar multiple of these vectors is also perpendicular, then $\pm\left(3\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ are vectors perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

Since $\underset{\sim}{w}$ is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$, then it is perpendicular to the plane containing $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

EXERCISE 3.2 SCALAR PRODUCT OF VECTORS IN THREE DIMENSIONS

1 For each of the following pairs of vectors, find **(i)** $\underset{\sim}{a} \infty \underset{\sim}{b}$ **(ii)** the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}, \underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j}$

(b) $\underset{\sim}{a} = -2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}, \underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{k}$

(c) $\underset{\sim}{a} = 4\underset{\sim}{i} - 5\underset{\sim}{j} + 7\underset{\sim}{k}, \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

(d) $\underset{\sim}{a} = 6\underset{\sim}{i} - \underset{\sim}{j}, \underset{\sim}{b} = 2\underset{\sim}{j} - \underset{\sim}{k}$

2 A, B, C, D are four points in space with respective coordinates (0, 0, 0), (1, 2, 3), (–3, 4, 6), (2, –6, –4).

Find:

(a) the position vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$

(b) the magnitude of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$

(c) the position vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$

(d) the magnitude of the angle between $\overrightarrow{BC}$ and $\overrightarrow{AD}$

(e) the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$

(f) the scalar projection of $\overrightarrow{CD}$ on $\overrightarrow{AB}$.

3 **(a)** If $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}, \underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = -6\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$, verify that $\underset{\sim}{a} \bullet \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c}$.

(b) Simplify $\left(\underset{\sim}{a} \bullet \underset{\sim}{b}\right)\underset{\sim}{c} + \left(\underset{\sim}{a} \bullet \underset{\sim}{c}\right)\underset{\sim}{b}$.

4 If $\underset{\sim}{u} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$, find:

(a) $\underset{\sim}{u} \infty \underset{\sim}{v}$ **(b)** $\underset{\sim}{u} \infty \underset{\sim}{u}$ **(c)** $\underset{\sim}{v} \infty \underset{\sim}{v}$

(d) a unit vector in the direction of $\underset{\sim}{v}$

(e) the components of $\underset{\sim}{u}$ **(i)** parallel to $\underset{\sim}{v}$, **(ii)** perpendicular to $\underset{\sim}{v}$.

5 If $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}, \underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{c} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, find:

(a) $\left(\underset{\sim}{c} - \underset{\sim}{a}\right) \bullet \underset{\sim}{b}$ **(b)** $\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \bullet \underset{\sim}{c}$ **(c)** the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

6 If $\underset{\sim}{a} = 3\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, find:

(a) a unit vector parallel to $2\underset{\sim}{a} + \underset{\sim}{b}$

(b) a unit vector perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

7 If $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$, simplify the following expressions.

(a) $\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \bullet \left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

(b) $\left(\underset{\sim}{a} + 2\underset{\sim}{b}\right) \bullet \left(2\underset{\sim}{a} - \underset{\sim}{b}\right)$

8 For the following vectors, find **(i)** the scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ and **(ii)** the vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

(a) $\underset{\sim}{a} = 6\underset{\sim}{i} + 12\underset{\sim}{j}, \underset{\sim}{b} = -3\underset{\sim}{i} - 2\underset{\sim}{j}$

(b) $\underset{\sim}{a} = 4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}, \underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

(c) $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}, \underset{\sim}{b} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

(d) $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j}, \underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{k}$

9 If $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} - \underset{\sim}{j} - 5\underset{\sim}{k}$, find:

(a) $\underset{\sim}{a} \infty \underset{\sim}{b}, \underset{\sim}{a} \infty \underset{\sim}{a}$ and $\underset{\sim}{b} \infty \underset{\sim}{b}$

(b) the cosine of the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(c) the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ and $\underset{\sim}{b}$ on $\underset{\sim}{a}$

(d) a unit vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

10 If $\underset{\sim}{a} = 3\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, find:

(a) $\underset{\sim}{a} \infty \underset{\sim}{b}, \underset{\sim}{a} \infty \underset{\sim}{a}$ and $\underset{\sim}{b} \infty \underset{\sim}{b}$

(b) the cosine of the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(c) the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ and $\underset{\sim}{b}$ on $\underset{\sim}{a}$.

11 If $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$, find:

(a) $\hat{\underset{\sim}{u}}$ **(b)** $\hat{\underset{\sim}{v}}$

(c) a unit vector in the direction $2\underset{\sim}{u} - \underset{\sim}{v}$

(d) **(i)** the vector projection of $\underset{\sim}{u}$ parallel to $\underset{\sim}{v}$

(ii) the vector projections of $\underset{\sim}{u}$ perpendicular to $\underset{\sim}{v}$.

12 Show that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel if $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$.

13 Find the value of p for which $\underset{\sim}{i} - 2p\underset{\sim}{j} + 3\underset{\sim}{k}$ and $p\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$ are perpendicular.

14 If $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j}$, find:

(a) a unit vector parallel to $\underset{\sim}{b}$

(b) the components of $\underset{\sim}{a}$ parallel to and perpendicular to $\underset{\sim}{b}$. (This is another way of saying 'find the vector projection of $\underset{\sim}{a}$ parallel to and perpendicular to $\underset{\sim}{b}$'.)

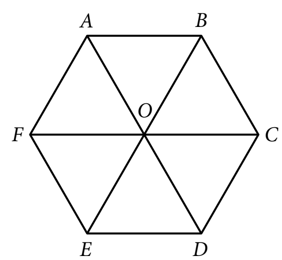
15 If $\underset{\sim}{a} = (6, -2, 6)$ and $\underset{\sim}{b} = (-6, -2, 1)$, find:

(a) $|\underset{\sim}{a}|$ **(b)** the scalar projection of $\underset{\sim}{b}$ on to $\underset{\sim}{a}$

(c) $|\underset{\sim}{a} - \underset{\sim}{b}|$ **(d)** the magnitude of the projection of $\underset{\sim}{b}$ on to $\underset{\sim}{a} - \underset{\sim}{b}$.

16 If $\underset{\sim}{a} = \begin{matrix} 1 \\ 2 \\ -4 \end{matrix}$, $\underset{\sim}{b} = \begin{matrix} 3 \\ -2 \\ 0 \end{matrix}$ and $\underset{\sim}{c} = \begin{matrix} 0 \\ 1 \\ -5 \end{matrix}$, find the projection of each of $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{b} + \underset{\sim}{c}$ on $\underset{\sim}{a}$. What do you observe?

17 *ABCDEF* is a regular hexagon with centre *O*. If $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$, express each of the following in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$:



(a) $\overrightarrow{AB}$ **(b)** $\overrightarrow{BC}$

(c) $\overrightarrow{CD}$ **(d)** $\overrightarrow{BD}$

(e) $\overrightarrow{FC}$ **(f)** Prove that $\overline{BD}$ and $\overline{FC}$ are perpendicular.

18 Given $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$, find two vectors $\underset{\sim}{c}$ and $\underset{\sim}{d}$ such that $\underset{\sim}{a} = \underset{\sim}{c} + \underset{\sim}{d}$, $\underset{\sim}{c}$ is parallel to $\underset{\sim}{b}$, $\underset{\sim}{d}$ is perpendicular to $\underset{\sim}{b}$.

19 The points *P*, *Q* and *R* have position vectors $2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $-\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ respectively. Find:

(a) the length of *QR*

(b) the vector projection of $\overline{PQ}$ in the direction of $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

20 If $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, find:

(a) the scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$

(b) a unit vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

21 Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are defined by $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$. Find:

(a) $2\underset{\sim}{a} - \underset{\sim}{b}$ **(b)** $\hat{\underset{\sim}{a}}$ **(c)** $\underset{\sim}{a} \infty \underset{\sim}{b}$

(d) the scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

22 *ABCD* is a rectangle with vector $\overrightarrow{AB} = 3\underset{\sim}{i}$ and vector $\overrightarrow{AD} = 2\underset{\sim}{j}$.

(a) Express the diagonal vectors $\overline{AC}$ and $\overline{DB}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

(b) Calculate, to the nearest degree, the angle between the diagonals.

23 If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are vectors defined by $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j} + \sqrt{2}\underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} + \sqrt{2}\underset{\sim}{k}$, find:

(a) a unit vector parallel to $\underset{\sim}{u}$

(b) the angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$

(c) the vector projection of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$.

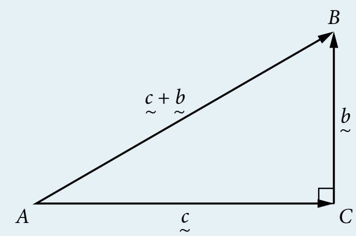
24 The position vectors of the points *P*, *Q* and *R* are $8\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$, $6\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$ and $7\underset{\sim}{i} + 5\underset{\sim}{j} - 5\underset{\sim}{k}$ respectively. Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

3.3 USING VECTORS IN GEOMETRIC PROOFS

The scalar product is used in geometrical proofs involving parallel and perpendicular lines, and squares on the sides of figures.

Example 13

Prove the theorem of Pythagoras in the right-angled triangle ABC, given in the diagram.



Solution

You have to prove that $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{CB}\right|^2$.

Let $\overrightarrow{AC} = \underset{\sim}{c}$ and $\overrightarrow{CB} = \underset{\sim}{b}$.

Hence $\overrightarrow{AB} = \underset{\sim}{c} + \underset{\sim}{b}$

Since $\overrightarrow{AC} \perp \overrightarrow{CB}$, then $\underset{\sim}{c} \bullet \underset{\sim}{b} = 0$

Now $\left(\underset{\sim}{c} + \underset{\sim}{b}\right) \bullet \left(\underset{\sim}{c} + \underset{\sim}{b}\right) = \underset{\sim}{c} \bullet \underset{\sim}{c} + 2\underset{\sim}{c} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{b}$

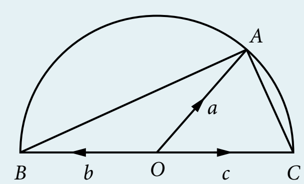
$$\left|\underset{\sim}{c} + \underset{\sim}{b}\right|^2 = \left|\underset{\sim}{c}\right|^2 + 0 + \left|\underset{\sim}{b}\right|^2$$

$$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{CB}\right|^2$$

Hence the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Example 14

Prove that an angle inscribed in a semicircle is a right angle.



Solution

The diagram shows the semicircle ACB with centre O and diameter BOC.
A is a point on the circumference.

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$, where $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right| = \left|\underset{\sim}{c}\right|$ as they are all radii.

Now $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\underset{\sim}{a} + \underset{\sim}{b}$

$= \underset{\sim}{b} - \underset{\sim}{a}$

And $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\underset{\sim}{a} + \underset{\sim}{c}$

$= \underset{\sim}{c} - \underset{\sim}{a}$

Thus $\overrightarrow{AB} \bullet \overrightarrow{AC} = \left(\underset{\sim}{b} - \underset{\sim}{a}\right) \bullet \left(\underset{\sim}{c} - \underset{\sim}{a}\right)$

$= \underset{\sim}{b} \bullet \underset{\sim}{c} - \underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{a}$

But $\underset{\sim}{c} = -\underset{\sim}{b}$

So $\overrightarrow{AB} \bullet \overrightarrow{AC} = \underset{\sim}{b} \bullet \left(-\underset{\sim}{b}\right) - \underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet \left(-\underset{\sim}{b}\right) + \underset{\sim}{a} \bullet \underset{\sim}{a}$

$= -\underset{\sim}{b} \bullet \underset{\sim}{b} - \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{a}$

$= \left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{b}\right|^2$

$= 0$ since $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$

Hence $\overrightarrow{AC} \perp \overrightarrow{AB}$ so $\angle BAC = 90°$

Example 15

The position vectors of the points P, Q, R and S are respectively $4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $5\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$, $2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$ and $4\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$.

(a) Show that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

(b) Is $PQRS$ a parallelogram?

Solution

(a)

$$\overrightarrow{PQ} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} - \left(4\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\right)$$
$$= \underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\overrightarrow{RS} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k} - \left(2\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}\right)$$
$$= 2\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$$
$$= 2\left(\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right)$$

Since $\overrightarrow{RS} = 2\overrightarrow{PQ}$, then $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

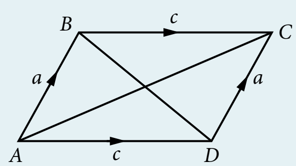
(b)

$$\left|\overrightarrow{PQ}\right| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$$
$$\left|\overrightarrow{RS}\right| = \sqrt{2^2 + 2^2 + 6^2} = 2\sqrt{11}$$

Since $\left|\overrightarrow{PQ}\right| \uparrow \left|\overrightarrow{RS}\right|$, then $PQRS$ is not a parallelogram, it is a trapezium.

Example 16

Prove that the sum of the squares of the lengths of the diagonals of any parallelogram is equal to the sum of the squares of the lengths of the sides.



Solution

It is required to show that: $\left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CD}\right|^2 + \left|\overrightarrow{DA}\right|^2$

$$= 2\left(\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2\right)$$

Denote $\overrightarrow{AB}$ and $\overrightarrow{DC}$ by $\underset{\sim}{a}$, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ by $\underset{\sim}{c}$, as shown in the diagram.

From BCD: $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \underset{\sim}{c} - \underset{\sim}{a}$

From ACD: $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \underset{\sim}{c} + \underset{\sim}{a}$

$$\left|\overrightarrow{BD}\right|^2 + \left|\overrightarrow{AC}\right|^2 = \overrightarrow{BD} \bullet \overrightarrow{BD} + \overrightarrow{AC} \bullet \overrightarrow{AC}$$
$$= \left(\underset{\sim}{c} - \underset{\sim}{a}\right) \bullet \left(\underset{\sim}{c} - \underset{\sim}{a}\right) + \left(\underset{\sim}{c} + \underset{\sim}{a}\right) \bullet \left(\underset{\sim}{c} + \underset{\sim}{a}\right)$$
$$= \underset{\sim}{c} \bullet \underset{\sim}{c} - 2\underset{\sim}{c} \bullet \underset{\sim}{a} + \underset{\sim}{a} \bullet \underset{\sim}{a} + \underset{\sim}{c} \bullet \underset{\sim}{c} + 2\underset{\sim}{c} \bullet \underset{\sim}{a} + \underset{\sim}{a} \bullet \underset{\sim}{a}$$
$$= 2\left(\underset{\sim}{c} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{a}\right)$$
$$= 2\left(c^2 + a^2\right)$$
$$= 2\left(\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2\right)$$

which is the required result.

EXERCISE 3.3 USING VECTORS IN GEOMETRIC PROOFS

1 Given the points $A(0, 2)$, $B(4, 10)$ and $C(6, 14)$:

(a) find the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$

(b) show that $\overrightarrow{AC} = 3\overrightarrow{BC}$

(c) show that A, B and C are collinear.

2 In OAB, $\overrightarrow{OA} = 5\underset{\sim}{i}$ and $\overrightarrow{OB} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$.

(a) Find $\overrightarrow{AB}$.

(b) Show that OAB is isosceles.

(c) Find $\overrightarrow{OC}$, where C is the midpoint of AB.

3 $OABC$ is a parallelogram in which $\overrightarrow{OA} = 6\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{i} + 3\underset{\sim}{j}$. Find:

(a) $\overrightarrow{AB}$ and $\overrightarrow{CB}$

(b) the diagonal vectors $\overrightarrow{OB}$ and $\overrightarrow{CA}$

(c) the vectors $\overrightarrow{ON}$ and $\overrightarrow{OM}$, where N is the midpoint of OB and N is the midpoint of CA. What conclusion can you make?

(d) the vectors $\overrightarrow{CP}$ and $\overrightarrow{BP}$, where P is the midpoint of OA.

4 In OAB, $\overrightarrow{OA} = 8\underset{\sim}{i}$ and $\overrightarrow{OB} = 6\underset{\sim}{i} + 4\underset{\sim}{j}$. If C is the midpoint of AB, and D is the midpoint of OB, find:

(a) $\overrightarrow{AB}$ and $\overrightarrow{OC}$

(b) $\overrightarrow{DC}$, and show that DC is parallel to OA and equal to half its length.

5 The points A, B and C have position vectors $\underset{\sim}{i}$, $2\underset{\sim}{i} + 2\underset{\sim}{j}$ and $4\underset{\sim}{i} + \underset{\sim}{j}$ respectively.

(a) Find $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$.

(b) By finding the magnitude of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$, prove that $\angle ABC$ is a right angle.

(c) Find the position vector of the point D such that $ABCD$ is a square.

6 $OABC$ is a quadrilateral, $\overrightarrow{OA} = 4\underset{\sim}{i}$, $\overrightarrow{OB} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\overrightarrow{OC} = 8\underset{\sim}{j}$.

(a) If P and Q are the midpoints of AB and BC respectively, find $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{PQ}$.

(b) Show that $\overrightarrow{PQ} = k\overrightarrow{AC}$. What geometrical conclusion can you now make?

7 The position vectors of the vertices A, B, C and D of a quadrilateral are $\underset{\sim}{i} + 2\underset{\sim}{j}$, $5\underset{\sim}{i} + 2\underset{\sim}{j}$, $4\underset{\sim}{i} - \underset{\sim}{j}$ and $2\underset{\sim}{i} - \underset{\sim}{j}$ respectively. Show that the diagonals intersect at right angles.

8 $OXAY$ is a square. M is the midpoint of XA, and N is the midpoint of YA.

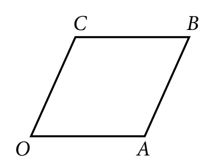
(a) If $\overrightarrow{OX} = \underset{\sim}{i}$ and $\overrightarrow{OY} = \underset{\sim}{j}$, find $\overrightarrow{OM}$ and $\overrightarrow{ON}$.

(b) Hence find the angle between OM and ON.

9 In the rhombus $OABC$, shown in the diagram, $\overrightarrow{OA} = m\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, where $m \in R$.

(a) Find the value of m.

(b) Show that the diagonals of $OABC$ are perpendicular.



10 OPQ is an equilateral triangle that has sides of length $2m$, and lies in the x–y plane so that O coincides with the origin, P is on the positive x-axis and Q is in the first quadrant.

(a) Express $\overrightarrow{PQ}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

(b) If M is the midpoint of PQ, express $\overrightarrow{OM}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

(c) Show that $\overrightarrow{OM} \perp \overrightarrow{PQ}$.

11 The respective position vectors of the points P, Q, R and S are $\underset{\sim}{i} + \underset{\sim}{j}$, $5\underset{\sim}{i} + 4\underset{\sim}{j}$, $2\underset{\sim}{i} + 8\underset{\sim}{j}$ and $-2\underset{\sim}{i} + 5\underset{\sim}{j}$. Show that $PQRS$ is a square.

12 The points A, B and C have position vectors relative to a fixed point, O, such that $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\overrightarrow{OB} = 3\underset{\sim}{i} - \underset{\sim}{j}$ and $\overrightarrow{OC} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$. Prove that AC bisects OB at right angles.

13 The position vectors of points P, Q and R are $4\underset{\sim}{i} + \underset{\sim}{j}$, $3\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ respectively.

(a) Show that PQR is right-angled.

(b) Find the area of the triangle.

14 The position vectors of points A, B and C are $2\underset{\sim}{i} - 3\underset{\sim}{j}$, $5\underset{\sim}{i} + 2\underset{\sim}{j}$ and $-3\underset{\sim}{i}$ respectively.

(a) Find $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$.

(b) Show that ABC is right-angled.

15 P and Q are the midpoints of the sides OA and OB of triangle OAB. Use a vector method to prove that PQ is parallel to AB and half its length.

16 If $\overrightarrow{AC} = 2\underset{\sim}{i} + 4\underset{\sim}{j}$ represents the hypotenuse of a right-angled triangle ABC, and $\overrightarrow{AB}$ is parallel to $\underset{\sim}{i} + \underset{\sim}{j}$, find $\overrightarrow{AB}$.

17 O is any point on the diagonal BD of the parallelogram $ABCD$. Prove that $\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD}$.

18 $ABCD$ is a trapezium in which the sides AB and DC are parallel. P and Q are the midpoints of the sides AD and BC respectively.

(a) Express $\overrightarrow{PQ}$ in two different ways as the sum of vectors along the sides of the trapezium.

(b) Use your answer to prove that PQ is parallel to two of the sides and that its length is equal to $\frac{1}{2}(AB + DC)$.

19 The sides of a parallelogram are represented by vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Show that, if the diagonals are perpendicular to one another, $\underset{\sim}{u} \bullet \underset{\sim}{u} = \underset{\sim}{v} \bullet \underset{\sim}{v}$.

20 ABC is a right-angled triangle with a right angle at B. If $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are the position vectors of A, B and C respectively, show that $\underset{\sim}{b}^2 = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{c} - \underset{\sim}{c} \bullet \underset{\sim}{a}$.

21 $ABCD$ is a trapezium in which $AB = x$, $DC = 2x$, $DA = y$. If E is a point in BC such that $BE = \frac{1}{3}BC$, prove that $\overrightarrow{AC} \bullet \overrightarrow{DE} = \frac{2}{3}\left(4x^2 - y^2\right)$.

22 The position vectors of the points A, B, C relative to a point O are $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. If $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{AC}$, show that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$.

23 The diagonals of a parallelogram are given by $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$.

(a) Show that the parallelogram is a rhombus.

(b) Find the length of each of its sides and the size of each of its angles.

24 Find the area of the parallelogram having diagonals $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$.

Prove each of the following results using vector methods.

25 The diagonals of a rhombus are perpendicular.

26 The midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices.

27 If the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle.

28 The line segments joining the midpoints of consecutive sides of a rhombus form a rectangle.

29 The line segments joining the midpoints of consecutive sides of a square form a square.

30 The line segments joining the midpoints of consecutive sides of a rectangle form a rhombus.

31 For any triangle PQR, $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{PR}\right| \cos P + \left|\overrightarrow{QR}\right| \cos Q$.

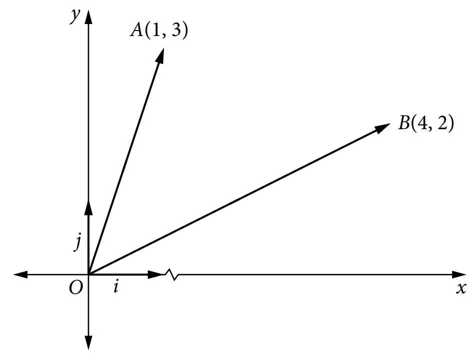
32 The sum of the squares of the distances from a point A to two opposite vertices of a rectangle is equal to the sum of the squares of the distances from A to the remaining two vertices.

3.4 CARTESIAN COORDINATES IN THREE-DIMENSIONAL SPACE

You have used the Cartesian system for vectors in two dimensions.

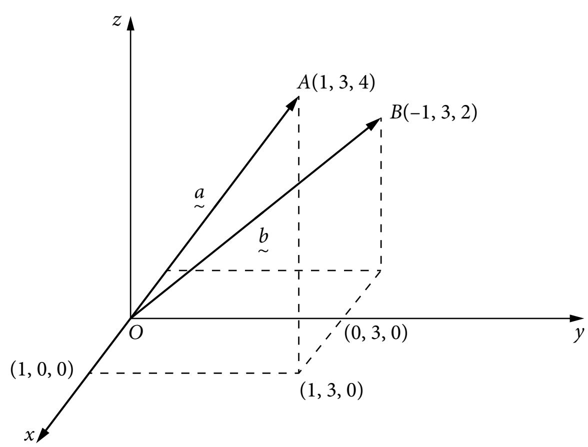
In the diagram, the vectors can be written in several forms,

such as $\overrightarrow{OA} = (1,3) = \underset{\sim}{i} + 3\underset{\sim}{j} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = (4,2) = 4\underset{\sim}{i} + 2\underset{\sim}{j} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.



A similar approach has been used in three-dimensional space. In this diagram, $\overrightarrow{OA} = \underset{\sim}{a} = (1, 3, 4) = \underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{j} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

and $\overrightarrow{OB} = \underset{\sim}{b} = (-1, 3, 2) = -\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.



Example 17

(a) Show by calculation that the points $A(1, -1, 3)$, $B(2, -4, 5)$ and $C(5, -13, 11)$ are collinear.

(b) Using vectors, show that ABC is a straight line.

Solution

(a) $AB = \sqrt{(2-1)^2 + (-4+1)^2 + (5-3)^2} = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$

$BC = \sqrt{3^2 + 9^2 + 6^2} = \sqrt{126} = 3\sqrt{14}$

$AC = \sqrt{4^2 + 12^2 + 8^2} = \sqrt{224} = 4\sqrt{14}$

$AB + BC = \sqrt{14} + 3\sqrt{14} = 4\sqrt{14} = AC$

Since the length of AC is the sum of the lengths on AB and BC, and the two intervals have the point B in common, then A, B and C are collinear.

(b) $\overrightarrow{AB} = (2-1, -4+1, 5-3) = (1, -3, 2)$

$\overrightarrow{BC} = (5-2, -13+4, 11-5) = (3, -9, 6) = 3(1, -3, 2)$

Hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ and they have a point, B, in common therefore ABC is a straight line.

Equation of a sphere

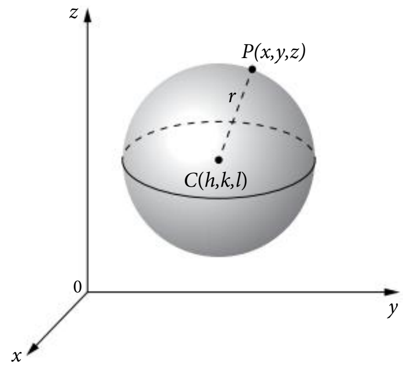
A circle is defined as the set of points in a plane equidistant from a fixed point in the plane. The equation of a circle centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

A sphere is defined as the set of points in three-dimensional space equidistant from a fixed point in space. If the point $P(x, y, z)$ is a point in space and $C(h, k, l)$ is the fixed point in space, then $PC = \sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2}$ using the distance formula.

Let $PC = r$ so that $\sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2} = r$

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

which is the equation of the sphere centre (h, k, l) with radius, r.



Any plane that intersects the sphere will do so in a circle. In particular, the plane $z = l$ intersects the sphere in the circle $(x - h)^2 + (y - k)^2 = r^2$.

Example 18

Show that $x^2 + y^2 + z^2 + 6x - 4y + 2z + 6 = 0$ is the equation of a sphere and find the coordinates of its centre and its radius. Hence sketch this sphere.

Solution

$x^2 + y^2 + z^2 + 6x - 4y + 2z + 6 = 0$

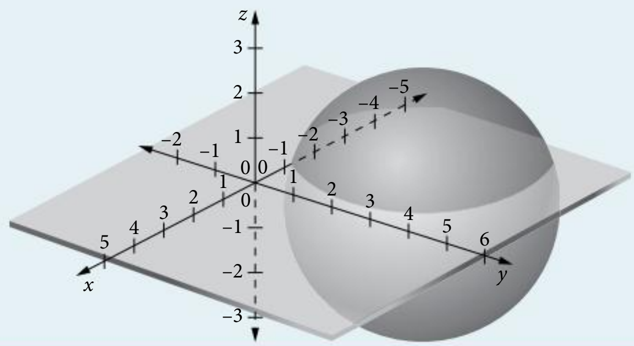
Rearrange the equation: $x^2 + 6x + y^2 - 4y + z^2 + 2z = -6$

Complete the square:

$x^2 + 6x + 9 + y^2 - 4y + 4 + z^2 + 2z + 1 = -6 + 9 + 4 + 1$

$(x + 3)^2 + (y - 2)^2 + (z + 1)^2 = 8$

This is a sphere with centre $(-3, 2, -1)$ and radius $2\sqrt{2}$.



Example 19

The spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + (z - 4)^2 = 16$ intersect. Find:

(a) the value of z when they intersect

(b) the equation of the circle in which they intersect, giving the coordinates of the centre and the radius.

Solution

(a) Solve the equations simultaneously by subtracting the first equation from the second equation:

$x^2 + y^2 + (z - 4)^2 - (x^2 + y^2 + z^2) = 16 - 9$

$z^2 - 8z + 16 - z^2 = 7$

$8z = 9$

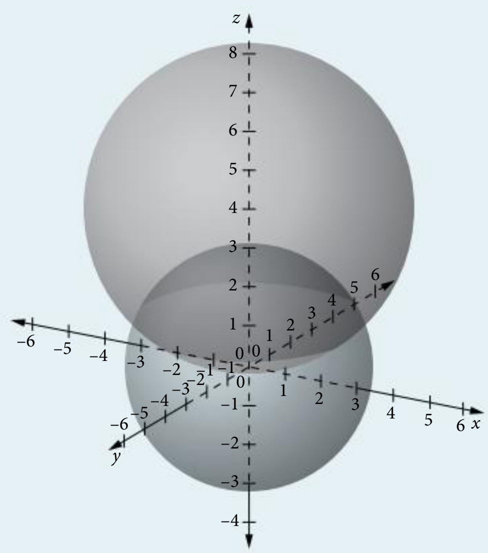
$z = \frac{9}{8}$, so they intersect on the horizontal plane $z = \frac{9}{8}$.

(b) Substitute into the first equation: $x^2 + y^2 + \frac{81}{64} = 9$

$$x^2 + y^2 = \frac{495}{64}$$

$$x^2 + y^2 = \frac{3\sqrt{55}}{8}^{\,2}$$

The circle has centre $\left(0, 0, \frac{9}{8}\right)$, and radius $\frac{3\sqrt{55}}{8}$.



MAKING CONNECTIONS

Equation of a sphere

Move the sliders to explore the effect of changing the values h, k and l on the sphere given by equation $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

EXERCISE 3.4 CARTESIAN COORDINATES IN THREE-DIMENSIONAL SPACE

1 Find the lengths of the sides of the triangle ABC and determine whether it is isosceles, right-angled, equilateral or none of these.

(a) $A(5, 5, 1), B(3, 3, 2), C(1, 4, 4)$
(b) $A(3, 4, -1), B(-5, 3, 0), C(6, -7, 4)$
(c) $A(4, -2, 3), B(6, 1, 4), C(5, 4, -3)$
(d) $A(1, 2, 3), B(2, 5, 7), C(-2, 6, 4)$

2 Determine whether the given sets of points are collinear.

(a) $A(1, 2, 3), B(3, 4, 1), C(5, 6, -2)$
(b) $D(3, 1, -4), E(-2, 3, 2), F(1, 3, 5)$
(c) $K(-4, 3, 5), L(2, -1, 3), M(5, -3, 2)$
(d) $P(0, 5, -1), Q(-3, -1, -10), R(-2, 1, -7)$

3 Find the equation of the sphere with centre C and radius r.

(a) $C(1, 2, 3), r = 4$
(b) $C(-1, 0, 5), r = \sqrt{3}$
(c) $C(3, -2, 4), r = 1.5$
(d) $C(0, -1, 2), r = 3\sqrt{2}$

4 Show that the given equation is the equation of a sphere and find the coordinates of its centre and the radius. Sketch the sphere in parts **(a)** and **(c)**.

(a) $x^2 + y^2 + z^2 + 4x - 2y + 6z + 4 = 0$
(b) $x^2 + y^2 + z^2 - 8x + 4y + 6 = 0$
(c) $x^2 + y^2 + z^2 + 2y - 2z = 0$
(d) $x^2 + y^2 + z^2 + x - 3y + 2z + 2 = 0$

5 Find the equation of the sphere with the end points of a diameter $(2, 3, 5)$ and $(4, -3, 9)$.

6 Find the equation of the sphere with centre $(4, 5, -2)$ that passes through the point $(1, 0, 0)$.

7 ABC has vertices $A(1, -1, 0)$, $B(2, 1, -1)$ and $C(-1, 1, 2)$.

(a) Use the scalar product to determine which of the angles of ABC is a right angle.
(b) Hence determine the area of ABC.

8 Given $D(4, -2, 3)$, $E(6, 1, 7)$ and $F(5, 4, -2)$, prove that DEF is right-angled in two different ways.

9 Find the equation of the sphere, centre $(2, 2, 3)$ that:

(a) touches both the x–z and y–z planes
(b) touches the x–y plane only.

10 The spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + (z - 4)^2 = 25$ intersect. Find:

(a) the value of z when they intersect
(b) the equation of the circle in which they intersect, giving the coordinates of the centre and the radius.

11 The spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + (y - 2)^2 + z^2 = 9$ intersect. Find:

(a) the value of y when they intersect
(b) the equation of the circle in which they intersect, giving the coordinates of the centre and the radius.

12 The spheres $(x - 1)^2 + y^2 + z^2 = 16$ and $(x - 5)^2 + y^2 + z^2 = 49$ intersect. Find:

(a) the value of x when they intersect
(b) the equation of the circle in which they intersect, giving the coordinates of the centre and the radius.

3.5 PARAMETRIC AND CARTESIAN EQUATIONS

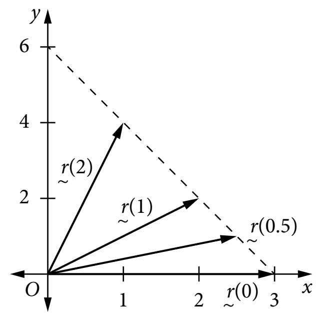
The position of any point P, relative to an origin O, is uniquely specified by the vector $\overrightarrow{OP}$, which is termed the *position vector* of P (relative to O).

Consider a body moving in the $\underset{\sim}{i}-\underset{\sim}{j}$ plane so that its position at time, t, is given by the position vector $\underset{\sim}{r}(t) = (3-t)\underset{\sim}{i} + 2t\underset{\sim}{j},\ t \geq 0$.

By substituting values of t into $\underset{\sim}{r}(t)$, you can determine where the body is at these times.

For example:

t	$\underset{\sim}{r}(t)$	position
0	$3\underset{\sim}{i}$	(3, 0)
0.5	$2.5\underset{\sim}{i} + \underset{\sim}{j}$	(2.5, 1)
1	$2\underset{\sim}{i} + 2\underset{\sim}{j}$	(2, 2)
2	$\underset{\sim}{i} + 4\underset{\sim}{j}$	(1, 4)



The figure on the right shows these four position vectors and the path that the body appears to be taking.

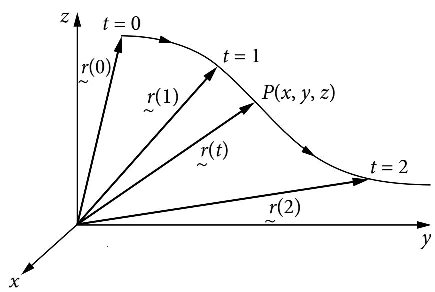
The graph of a vector equation is the set of points determined by $\underset{\sim}{r}(t)$ as t varies.

For $t \geq 0$, $\underset{\sim}{r}(0)$ gives us the initial position of the body.

Given that $\underset{\sim}{r}(0) = 3\underset{\sim}{i}$, you conclude that the body begins its motion at (3, 0).

Substituting increasing values of t into $\underset{\sim}{r}(t)$ can help you determine the body's direction of motion.

The body starts at (3, 0) and appears to be moving along the line with equation $y = 6 - 2x$.



In general, when a body moves in space, its position vector varies with time.

Having the vector equation of the paths means that you are only working with one variable, t, rather than three variables x, y and z.

This is illustrated in the diagram to the right where the position vector is denoted by $\underset{\sim}{r}(t)$ to indicate its dependence on time, t.

> If the coordinates of the variable point $P(x, y, z)$ are expressed parametrically as $x = f(t)$, $y = g(t)$ and $z = h(t)$, then the **vector equation** of the path is given by the position vector $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j} + h(t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the x-, y- and z-axes respectively.
>
> The equations $x = f(t)$, $y = g(t)$ and $z = h(t)$, are called **parametric equations** and t is called the **parameter**.

In the introductory example, the parametric equations are $x = 3 - t$ and $y = 2t$.

Rearranging the first equation to make the parameter t the subject gives $t = 3 - x$.

Substituting $t = 3 - x$ into $y = 2t$ gives $y = 2(3 - x) = 6 - 2x$.

So, confirming the earlier assertion, $y = 6 - 2x$ is the Cartesian equation that describes the path of the body.

Example 20

State the vector equation of a curve that has parametric equations given by $x = 2\cos t$, $y = 2\sin t$ and $z = 1$ for $t \geq 0$.

Solution

The general form is $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$.

The vector equation of the curve is $\underset{\sim}{r}(t) = (2\cos t)\underset{\sim}{i} + (2\sin t)\underset{\sim}{j} + \underset{\sim}{k},\ t \geq 0$.

Example 21

The position vector of a body at time t is given by $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$.

(a) Find the Cartesian equation of the path of the body and state the domain.

(b) Sketch the path of the body.

Solution

(a) State and number the parametric equations: $x = 1 - t$ [1]

$y = t^2$ [2]

Express t in terms of x, from [1]: $t = 1 - x$

State the allowed values of x: If $t \geq 0$, then $x \leq 1$.

Substitute into [2] to eliminate t: $y = (1-x)^2$

State the Cartesian equation with domain: $y = (1-x)^2$, $x \leq 1$.

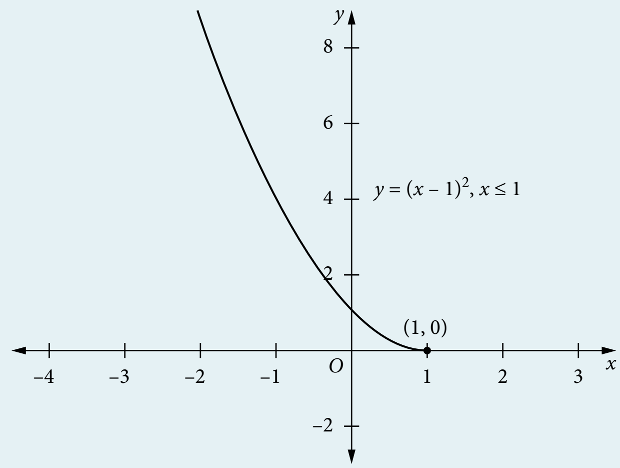
(b) Sketch the parabola with correct domain:
The path is that of the parabola $y = (1-x)^2$ for which $x \leq 1$.

To summarise from part **(a)**:

1 The parametric equations are $x = 1 - t$ and $y = t^2$, $t \geq 0$.

2 The Cartesian equation is $y = (1-x)^2$, $x \leq 1$.

3 The vector equation is $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$.

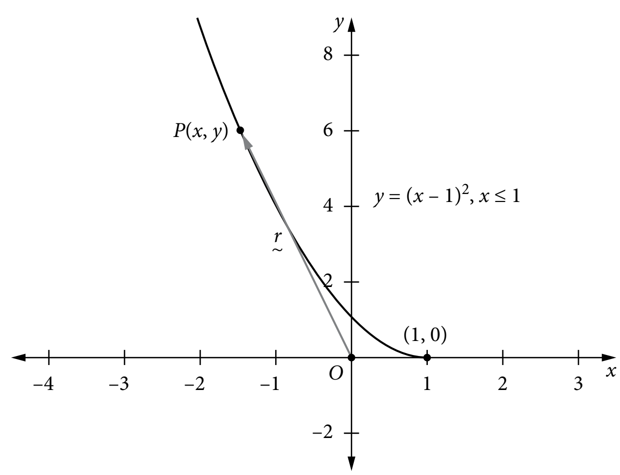


As shown in Example **21**, elimination of the parameter t gives the Cartesian equation of the body's path shown in the figure on the right.

The vector equation $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$ is an example of a **vector function of the scalar** (real) variable t because each element in the domain gives a unique value of $\underset{\sim}{r}$.

Consequently, the function defined by $\underset{\sim}{r}$ is a **one-to-one** function.

To specify a vector function, it is sufficient to state its rule and the domain, i.e. $\underset{\sim}{r}(t) = (1-t)\underset{\sim}{i} + t^2\underset{\sim}{j}$, $t \geq 0$.



Example 22

The parametric equations of a curve are $x = \cos 2t$, $y = \sin t$, $t \geq 0$.

(a) Find the Cartesian equation of the curve.

(b) Find the vector equation of the curve.

(c) Sketch the curve.

Solution

(a) $x = \cos 2t,\ y = \sin t,\ t \geq 0$

$x = \cos 2t$

$= 1 - 2\sin^2 t$

$= 1 - 2y^2$

$2y^2 = 1 - x$

$y^2 = \dfrac{1-x}{2},\ -1 \leq x \leq 1$ is the Cartesian equation of the curve.

(b) $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$\underset{\sim}{r} = (\cos 2t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j},\ t \geq 0$ is the vector equation of the curve.

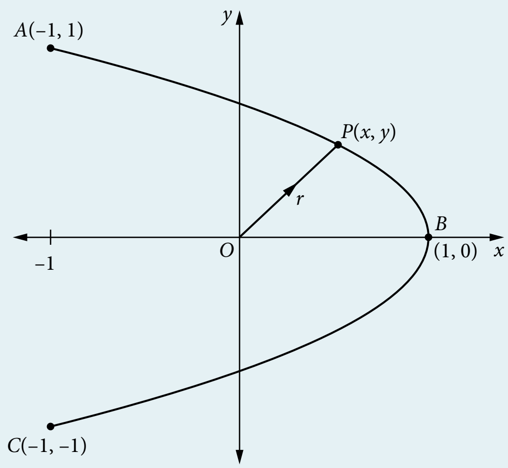
(c) Since $-1 \leq \cos 2t \leq 1$ for all t, it follows that $-1 \leq x \leq 1$ and so the equation represents only the arc ABC of the parabola as shown in the diagram.

When $t = 0$, $x = 1$, $y = 0$ and $\underset{\sim}{r} = \underset{\sim}{i}$.

When $t = \dfrac{\pi}{2}$, $x = -1$, $y = 1$ and $\underset{\sim}{r} = -\underset{\sim}{i} + \underset{\sim}{j}$.

When $t = \pi$, $x = 1$, $y = 0$ and $\underset{\sim}{r} = \underset{\sim}{i}$.

When $t = \dfrac{3\pi}{2}$, $x = -1$, $y = -1$ and $\underset{\sim}{r} = -\underset{\sim}{i} - \underset{\sim}{j}$.



From this you can see that if you consider the curve as a path traced out by a moving point, the point starts at B, moves to A, then back to B, then to C and back to B as t increases from 0 to 2π. Since x and y are periodic functions of t, the curve retraces itself as t increases beyond 2π.

EXPLORING FURTHER

Parametric and Cartesian equations

Use technology to explore parametric and Cartesian equations.

EXERCISE 3.5 PARAMETRIC AND CARTESIAN EQUATIONS

1 For the curves whose parametric equations are given, find:

(i) the Cartesian equation (ii) the vector equation.

(a) $x = 2t,\ y = t + 2,\ t \geq 0$

(b) $x = t,\ y = \dfrac{1}{t},\ t > 0$

(c) $x = 2\cos\theta,\ y = 2\sin\theta,\ 0 \leq \theta \leq 2\pi$

(d) $x = t + 3,\ y = t^2 - 5,\ t \geq 0$

(e) $x = u^3,\ y = 1 - u^2,\ -1 \leq u \leq 1$

(f) $x = s^2 + s,\ y = s^2 - s,\ s > 0$

(g) $x = \cos 2\theta,\ y = \cos\theta,\ 0 \leq \theta \leq 2\pi$

(h) $x = 2\cos t,\ y = \sqrt{3}\sin t,\ t \geq 0$

2 For the curves whose parametric equations are given, find:

(i) the Cartesian equation (ii) the vector equation.

(a) $x = \dfrac{2t}{1+t^2},\ y = \dfrac{1-t^2}{1+t^2},\ t \in R$

(b) $x = a\sin\phi,\ y = b\cos\phi,\ \phi \in R$

(c) $x = \sin t,\ y = \sin 2t,\ 0 \leq t \leq 2\pi$

(d) $x = 2 + \sin\theta,\ y = 1 - \cos\theta,\ \theta \geq 0$

(e) $x = u + \dfrac{1}{u},\ y = u - \dfrac{1}{u},\ u \neq 0$

(f) $x = 3\cos^2 t,\ y = 4\sin^2 t,\ t \geq 0$

(g) $x = \dfrac{4t}{1+t^2},\ y = \dfrac{3\left(1-t^2\right)}{1+t^2},\ t \in R$

3 For the curves whose vector equations are given, find:

(i) the parametric equation **(ii)** the Cartesian equation.

(a) $\underset{\sim}{r} = 2t\underset{\sim}{i} + t^2\underset{\sim}{j},\ t \geq 0$

(b) $\underset{\sim}{r} = \cos 2\theta\underset{\sim}{i} + \sin 2\theta\underset{\sim}{j},\ \theta \in R$

(c) $\underset{\sim}{r} = 4\cos^2\theta\underset{\sim}{i} + 3\sin^2\theta\underset{\sim}{j},\ \theta \in R$

(d) $\underset{\sim}{r} = t\underset{\sim}{i} + 4at^2\underset{\sim}{j},\ t \geq 0$

(e) $\underset{\sim}{r} = (2 - \sin\theta)\underset{\sim}{i} + (1 + \cos\theta)\underset{\sim}{j},\ 0 \leq \theta \leq 2\pi$

(f) $\underset{\sim}{r} = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j},\ t \neq 0$

(g) $\underset{\sim}{r} = (u + 3)\underset{\sim}{i} + \left(u^2 - 5\right)\underset{\sim}{j},\ u \in R$

(h) $\underset{\sim}{r} = (2\sin t)\underset{\sim}{i} + (4\cos t)\underset{\sim}{j},\ t \in R$

(i) $\underset{\sim}{r} = \left(t - \frac{1}{t}\right)\underset{\sim}{i} + \ t^2 + \frac{1}{t^2}\ \underset{\sim}{j},\ t \neq 0$

4 The vector equation is given by $\underset{\sim}{r} = (a\cos t)\underset{\sim}{i} + (a\sin t)\underset{\sim}{j},\ a > 0$. Find the Cartesian equation of the curve and show that it is a circle.

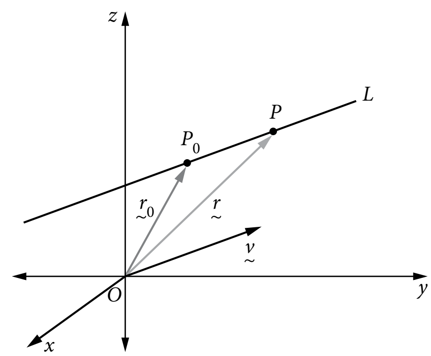
5 The position of a particle at any time, t, is $\underset{\sim}{r}(t) = (4\cos 3t)\underset{\sim}{i} + (4\sin 3t)\underset{\sim}{j}$.

(a) Show that the path is circular.

(b) Find the Cartesian equation of the path.

(c) Find the value of $|\underset{\sim}{r}|$.

3.6 VECTOR EQUATION OF A LINE



The position of a line in the x–y plane is located when a point on the line and its direction, given by either its gradient or the angle of inclination, are known, or by the location of two points through which it passes.

In three-dimensional space, the position of a line is located when you have a point on the line and its direction, this direction given by a vector, or by the location of two points through which it passes.

The diagram to the right shows the line, L, passing through the point $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$. L may also be defined by the point P_0 and the vector $\overrightarrow{P_0P}$, which determines the direction of the line.

Consider the position vector $\underset{\sim}{v} = l\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k}$ that is parallel to $\overrightarrow{P_0P}$.

Since $\underset{\sim}{v}$ and $\overrightarrow{P_0P}$ are parallel, then $\overrightarrow{P_0P} = \lambda\underset{\sim}{v}$, where λ is a real number.

Given that the location of P is specified by position vector $\underset{\sim}{r}$, then $\overrightarrow{OP} = \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$.

Hence $\underset{\sim}{r} = \underset{\sim}{r}_0 + \overrightarrow{P_0P}$

$= \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$

Thus, the vector equation of the line L is $\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$.

This equation consists of the position vector $\underset{\sim}{r}_0$, a fixed point P_0 on L, and a fixed vector $\underset{\sim}{v}$ that determines the direction of L.

From this equation, you can obtain the parametric equation of the line L, namely

$x = x_0 + a\lambda,\ y = y_0 + b\lambda,\ z = z_0 + c\lambda.$

They represent the parametric equations of the line L through the point (x_0, y_0, z_0), and parallel to the vector $\underset{\sim}{v} = (a, b, c)$.

Also, the values a, b, c are called the **direction numbers** of the line as they give its direction.

The vector equation of the line L gives the coordinates of a point for each value of λ.

For example, if $\lambda = 0$, $\underset{\sim}{r} = \underset{\sim}{r}_0$, then the point is P_0.

If $\lambda = 1$, $\underset{\sim}{r} = \underset{\sim}{r}_0 + \underset{\sim}{v}$, then the point is $|\underset{\sim}{v}|$ units to the right of P_0. ($\underset{\sim}{v}$ is positive to the right in the diagram.)

If $\lambda = -2$, $\underset{\sim}{r} = \underset{\sim}{r}_0 - 2\underset{\sim}{v}$, then the point is $2|\underset{\sim}{v}|$ units to the left of P_0.

Remember, a vector equation generates the infinite set of points that make up the line, each point corresponding to a real value of λ.

Example 23

For the straight line with equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$ where $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$, find the coordinates of the points on the line for which:

(a) $\lambda = 0$ **(b)** $\lambda = 3$ **(c)** $\lambda = 100$ **(d)** $\lambda = -5$.

Solution

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = \left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right) + \lambda\left(\underset{\sim}{i} - \underset{\sim}{j}\right)$

$x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = (1+\lambda)\underset{\sim}{i} + (1-\lambda)\underset{\sim}{j} + \underset{\sim}{k}$

Equating components: $x = 1 + \lambda$, $y = 1 - \lambda$, $z = 1$

(a) $\lambda = 0$: $x = 1$, $y = 1$, $z = 1$ so the point is $(1, 1, 1)$

(b) $\lambda = 3$: $x = 4$, $y = -2$, $z = 1$ so the point is $(3, -2, 1)$

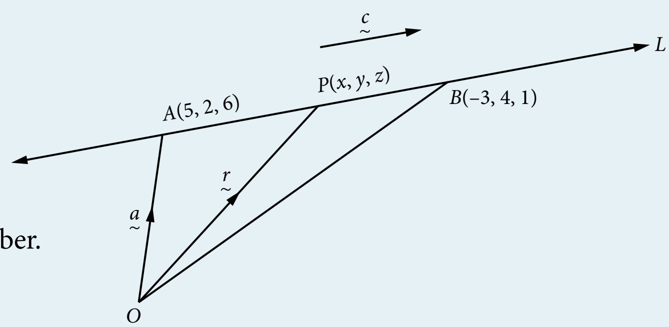
(c) $\lambda = 100$: $x = 101$, $y = -99$, $z = 1$ so the point is $(101, -99, 1)$

(d) $\lambda = -5$: $x = -4$, $y = 6$, $z = 1$ so the point is $(-4, 6, 1)$.

Example 24

Find the vector equation to the straight line joining the points $A(5, 2, 6)$ and $B(-3, 4, 1)$ and find the coordinates of the point on AB where $\lambda = 2$.

Solution



$\overrightarrow{OA} = \underset{\sim}{a} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$\overrightarrow{AB} = \underset{\sim}{c} = (-3-5)\underset{\sim}{i} + (4-2)\underset{\sim}{j} + (1-6)\underset{\sim}{k}$

$\underset{\sim}{c} = -8\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}$

The equation of L is given by $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{c}$, λ is a real number.

Thus $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = \left(5\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}\right) + \lambda\left(-8\underset{\sim}{i} + 2\underset{\sim}{j} - 5\underset{\sim}{k}\right)$

$= (5-8\lambda)\underset{\sim}{i} + (2+2\lambda)\underset{\sim}{j} + (6-5\lambda)\underset{\sim}{k}$

This gives $x = 5 - 8\lambda$, $y = 2 + 2\lambda$, $z = 6 - 5\lambda$

$\lambda = 2$: $x = -11$, $y = 6$, $z = -4$ so the coordinates of the point are $(-11, 6, -4)$.

Example 25

Find the vector and parametric equations of the line:

(a) through the point $A(-2, 1, 4)$ parallel to the vector $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

(b) through the point $A(1, 3, 2)$ parallel to the line through $B(1, 2, 3)$ and $C(3, 4, 1)$.

Solution

(a) $A(-2, 1, 4)$: $\overrightarrow{OA} = \underset{\sim}{a} = -2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = -2\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

Hence the vector equation of the line is $\underset{\sim}{r} = (-2+2\lambda)\underset{\sim}{i} + (1+\lambda)\underset{\sim}{j} + (4-2\lambda)\underset{\sim}{k}$.

The parametric equation is $x = -2 + 2\lambda$, $y = 1 + \lambda$, $z = 4 - 2\lambda$. The direction numbers are 2, 1, −2 (the coefficients of λ).

(b) $A(1, 3, 2)$: $\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{b} = \overrightarrow{BC} = (3-1)\underset{\sim}{i} + (4-2)\underset{\sim}{j} + (1-3)\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

Hence the vector equation of the line is $\underset{\sim}{r} = (1+2\lambda)\underset{\sim}{i} + (3+2\lambda)\underset{\sim}{j} + (2-2\lambda)\underset{\sim}{k}$.

The parametric equation is $x = 1 + 2\lambda$, $y = 3 + 2\lambda$, $z = 2 - 2\lambda$.

Example 26

Given $P_1(2, 1, 2)$ and $P_2(-1, 3, 3)$. With P_1 as the fixed point, find:

(a) the vector equation of the line L that passes through P_1 and P_2

(b) the parametric equation of the line L that passes through P_1 and P_2.

With P_2 as the fixed point, find:

(c) the vector equation of the line L that passes through P_1 and P_2

(d) the parametric equation of the line L that passes through P_1 and P_2.

(e) Discuss your answers to parts **(b)** and **(d)**.

Solution

(a) $\underset{\sim}{v} = \overrightarrow{P_1P_2} = (-1-2)\underset{\sim}{i} + (3-1)\underset{\sim}{j} + (3-2)\underset{\sim}{k}$

$= -3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$P_1(2, 1, 2)$: $\underset{\sim}{r}_0 = \overrightarrow{OP_1} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

L is parallel to $\overrightarrow{P_1P_2}$, so the vector equation of L is $\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$, where $\underset{\sim}{r} = (x, y, z)$.

Hence the equation of L is $\underset{\sim}{r} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} + \lambda\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

$\underset{\sim}{r} = \left(2-3\lambda\right)\underset{\sim}{i} + \left(1+2\lambda\right)\underset{\sim}{j} + \left(2+\lambda\right)\underset{\sim}{k}$

(b) The parametric equations of L are: $x = 2 - 3\lambda$, $y = 1 + 2\lambda$, $z = 2 + \lambda$.

(c) $P_2(-1, 3, 3)$: $\underset{\sim}{r}_0 = \overrightarrow{OP_2} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{v} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$: $\underset{\sim}{r} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda\left(-3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

$\underset{\sim}{r} = \left(-1-3\lambda\right)\underset{\sim}{i} + \left(3+2\lambda\right)\underset{\sim}{j} + \left(3+\lambda\right)\underset{\sim}{k}$

(d) The parametric equations of L are: $x = -1 - 3\lambda$, $y = 3 + 2\lambda$, $z = 3 + \lambda$.

(e) The equations of the line L are different when they have the different fixed points. To show that each equation represents the same line, find values of λ that generate the points P_1 and P_2 for each equation. This is most easily done from the parametric equation. Obviously, $\lambda = 0$ will generate the fixed point in each case.

For part **(b)**: $x = 2 - 3\lambda$, $y = 1 + 2\lambda$, $z = 2 + \lambda$, for P_2: $2 - 3\lambda = -1$, $\lambda = 1$.
Substitute for y and z: $y = 1 + 2 = 3$. $z = 2 + 1 = 3$. Hence the point is $(-1, 3, 3)$, which is P_2.

For part **(d)**: $x = -1 - 3\lambda$, $y = 3 + 2\lambda$, $z = 3 + \lambda$, for P_1: $-1 - 3\lambda = 2$, $\lambda = -1$.
Substitute for y and z: $y = 3 - 2 = 1$. $z = 3 - 1 = 2$. Hence the point is $(2, 1, 2)$, which is P_1.
Thus, both sets of equations represent the line L through the points P_1 and P_2.

If A, B and R are points on a straight line through A and B such that $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{AB}$ and $\underset{\sim}{r} = \overrightarrow{OR}$, then the vector equation of the line AB is given by $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$, where λ is a parameter.

When starting with the coordinates of two points to form the vector equation of a line, the final equation depends on which point you fix to obtain $\underset{\sim}{a}$. Both choices will lead to a correct vector equation defining the line.

Hence the vector (or parametric) equation of a line through two points is not unique.

Vector equations of a line in two dimensions

Gradient intercept form

Consider the equation of the line given by $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$, where $\underset{\sim}{r} = (x, y)$, $\underset{\sim}{a} = \left(x_0, y_0\right)$ and $\underset{\sim}{b} = \left(x_1, y_1\right)$.
The vector equation becomes $\underset{\sim}{r} = \left(x_0 + x_1\lambda\right)\underset{\sim}{i} + \left(y_0 + y_1\lambda\right)\underset{\sim}{j}$.
The parametric equations are $x = x_0 + x_1\lambda$, $y = y_0 + y_1\lambda$.

Rearranging the first equation gives $\lambda = \dfrac{x - x_0}{x_1}$

Substitute in the second equation for y: $y = y_0 + \dfrac{y_1(x - x_0)}{x_1}$

$$y = y_0 + \frac{y_1}{x_1}x - \frac{x_0 y_1}{x_1}$$

$$y = \frac{y_1}{x_1}x + \frac{x_1 y_0 - x_0 y_1}{x_1}$$

Comparing this answer with the gradient–intercept form of the straight line, $y = mx + c$, gives $m = \dfrac{y_1}{x_1}$ and $c = \dfrac{x_1 y_0 - x_0 y_1}{x_1}$.

Thus, the gradient of the line is $\dfrac{y_1}{x_1}$ and the y-intercept is $\dfrac{x_1 y_0 - x_0 y_1}{x_1}$.

The vector $\underset{\sim}{b} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ is a vector parallel to the line with gradient $\dfrac{y_1}{x_1}$. You would expect the gradients of the line and the vector to be the same.

Two-point form

In two-dimensional coordinate geometry, the equation of the line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$, provided that $x_2 \neq x_1$.

The vector equation of the line AB is $x\underset{\sim}{i} + y\underset{\sim}{j} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + \lambda\left((x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j}\right)$ as $(x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j}$ is a vector parallel to $\overrightarrow{AB}$.

Thus $x = x_1 + \lambda(x_2 - x_1)$

$y = y_1 + \lambda(y_2 - y_1)$

Solving both of these equations for λ gives $\dfrac{x - x_1}{x_2 - x_1} = \lambda = \dfrac{y - y_1}{y_2 - y_1}$ [1]

This means that $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$, which is the two-point form of the Cartesian equation of a line.

If $\lambda = 0$, then equation [1] gives $x = x_1$, $y = y_1$, or (x_1, y_1), the coordinates of the fixed point for the vector equation.

Example 27

Find the vector and Cartesian equations of the line through the point $A(2, -1)$ parallel to the vector $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j}$.

Solution

$\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j}$

Vector equation: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$: $\underset{\sim}{r} = 2\underset{\sim}{i} - \underset{\sim}{j} + \lambda\left(\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

$\underset{\sim}{r} = (2 + \lambda)\underset{\sim}{i} + (-1 + 2\lambda)\underset{\sim}{j}$

Parametric equations: $x = 2 + \lambda$, $y = -1 + 2\lambda$

$\lambda = x - 2$ so $y = -1 + 2(x - 2)$

$y = 2x - 5$ is the Cartesian equation of the line.

EXPLORING FURTHER

Vector equation of a line

Use technology to explore the vector equation of a line.

EXERCISE 3.6 VECTOR EQUATION OF A LINE

1 **(a)** Find the vector equation of the line through (1, 2) parallel to the vector $\underset{\sim}{i} + \underset{\sim}{j}$.
(b) Find the points corresponding to: **(i)** $\lambda = 0$ **(ii)** $\lambda = -1$ **(iii)** $\lambda = 2$.

2 **(a)** Find the vector equation of the line through (−1, 4, 6) parallel to the vector $\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$.
(b) Find the points corresponding to: **(i)** $\lambda = \frac{1}{2}$ **(ii)** $\lambda = 6$ **(iii)** $\lambda = -3$.

3 **(a)** Find the vector equation of the line through (4, 2) parallel to the line joining the points (−1, 3) and (3, 7).
(b) Find the points corresponding to: **(i)** $\lambda = 2$ **(ii)** $\lambda = 4$ **(iii)** $\lambda = 8$.

4 **(a)** Find the vector equation of the line through (2, 3, 4) parallel to the line joining the points (0, 2, 4) and (−5, −3, 6).
(b) Find the points corresponding to: **(i)** $\lambda = -1$ **(ii)** $\lambda = 0$ **(iii)** $\lambda = 1$.

5 Find the vector equation of the line through (4, 12, −3) parallel to the vector $2\underset{\sim}{i} - 3\underset{\sim}{k}$.

6 Find the vector equation of the line through $A(3, 5, 7)$ and $B(6, 4, 5)$.

3.7 PARALLEL AND PERPENDICULAR LINES IN THREE DIMENSIONS

In two dimensions, two lines can either intersect, be parallel to each other, or be coincident (the same line).
A special case of intersecting is when they intersect at an angle of 90° when they are said to be *perpendicular* to each other. If two or more lines lie on the same plane, they called coplanar.
In two dimensions, all lines lie in the same plane, the x–y plane. All these lines are *coplanar*.

Given line L_1 with equation $y = m_1x + c_1$ and line L_2 with equation $y = m_2x + c_2$, then:

$L_1 \parallel L_2$ if $m_1 = m_2$, $L_1 \perp L_2$ if $m_1 \times m_2 = -1$ and the lines are coincident if $m_1 = m_2$ and $c_1 = c_2$.

Given line L_1 with equation $a_1x + b_1y + c_1 = 0$ and line L_2 with equation $a_2x + b_2y + c_2 = 0$, then:

$L_1 \parallel L_2$ if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\frac{c_1}{c_2} \neq \frac{a_1}{a_2}$

$L_1 \perp L_2$ if $\frac{a_1}{b_1} \times \frac{a_2}{b_2} = -1$

and the lines are coincident if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\frac{c_1}{c_2} = \frac{a_1}{a_2}$.

In three dimensions, coplanar lines may intersect, be parallel or be coincident. If they intersect, then they could be perpendicular to each other (orthogonal).
If two lines in three dimensions do not intersect, and are not parallel or coincident, they are called skew lines and are not coplanar. They may still be perpendicular as there may exist a pair of intersecting perpendicular lines, each of which is parallel to one of the given lines.

Given line L_1 with parametric equations $x = a_1 + b_1t$, $y = c_1 + d_1t$, $z = e_1 + f_1t$ and line L_2 with parametric equations $x = a_2 + b_2s$, $y = c_2 + d_2s$, $z = e_2 + f_2s$, then:

- $L_1 \parallel L_2$ if $\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$
- L_1 and L_2 intersect if a unique value of t and s satisfies the three equations $a_1 + bt = a_2 + b_2s$, $c_1 + d_1t = c_2 + d_2s$, $e_1 + f_1t = e_2 + f_2s$ simultaneously,
- $L_1 \perp L_2$ if $b_1b_2 + d_1d_2 + f_1f_2 = 0$ ($\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$)
- they are skew if they are not parallel, coincident or intersecting.

Note: In three dimensions, coincident lines will be parallel and have a point in common. This means that for L_1 and L_2 above to be coincident, $\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2} = \frac{a_1}{a_2} = \frac{c_1}{c_2} = \frac{e_1}{e_2}$.

Use the parametric form of the equation of a straight line to determine which of the above occur in each case.

Strategy:

1. Determine if the lines are parallel. If yes, determine if they are coincident.
2. Determine if the lines intersect.
3. If they are not parallel or coincident and do not intersect, then they are skew.
4. Determine if they are perpendicular.

MAKING CONNECTIONS

Parallel and perpendicular lines

Use technology to explore parallel, perpendicular and skew lines in three dimensions.

Example 28

For the lines $x = -2 + 2t$, $y = 1 + t$, $z = 4 - 2t$ and $x = 1 + 2s$, $y = 3 + 2s$, $z = 2 - 2s$, determine whether they: **(a)** are parallel, **(b)** intersect, **(c)** are skew, **(d)** are perpendicular.

Solution

(a) Write the ratio of the direction numbers $\frac{2}{2}, \frac{1}{2}, \frac{-2}{-2}$

Since $\frac{2}{2} \neq \frac{1}{2} \neq \frac{-2}{-2}$, the lines are not parallel.

(b) Equate the x, y and z values for each line to obtain 3 equations:

$-2 + 2t = 1 + 2s \quad [1]$

$1 + t = 3 + 2s \quad [2]$

$4 - 2t = 2 - 2s \quad [3]$

Rewrite [1]: $2s = 2t - 3$

Substitute in [2]: $1 + t = 3 + 2t - 3$

$t = 1$

$s = -\frac{1}{2}$

Substitute in [3]: LHS $= 2 - 2 = 0$, RHS $= 2 - 1 = 1$

LHS $\neq$ RHS, so the two lines do not intersect as there is not a unique pair of values for t and s that satisfies all three equations simultaneously.

(c) Hence the lines are skew.

(d) $\underset{\sim}{a} = (2, 1, -2)$, $\underset{\sim}{b} = (2, 2, -2)$ represent the direction vectors of the two lines.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2, 1, -2) \cdot (2, 2, -2)$$
$$= 4 + 2 + 4 = 10 \neq 0$$

$\underset{\sim}{a} \cdot \underset{\sim}{b} \neq 0$, hence the lines are not perpendicular.

Example 29

L_1 is $x = 1 + 3t$, $y = 2 + 6t$, $z = 1 - 3t$. L_2 is $x = 2 - 2s$, $y = 1 - 4s$, $z = 2 + 2s$. L_3 is $x = 3 - r$, $y = -1 + r$, $z = -1 + r$.

(a) Show that $L_1 \parallel L_2$.

(b) Show that L_1 and L_3 intersect.

(c) Is L_1 perpendicular to L_3?

(d) What can you say about L_2 and L_3?

Solution

(a) Write the ratio of the direction numbers:

$\frac{3}{-2}, \frac{6}{-4}, \frac{-3}{2}$

In each case, $\frac{t}{s} = -\frac{3}{2}$ so the lines are parallel.

(b) L_1: $x = 1 + 3t, y = 2 + 6t, z = 1 - 3t$

L_3: $x = 3 - r, y = -1 + r, z = -1 + r$

$1 + 3t = 3 - r$ [1]

$2 + 6t = -1 + r$ [2]

$1 - 3t = -1 + r$ [3]

[1] $\Rightarrow$: $r = 2 - 3t$

Substitute in [2]: $2 + 6t = -1 + 2 - 3t$

$9t = -1$

$t = -\frac{1}{9}$ $r = 2\frac{1}{3}$

Substitute in [3]: LHS $= 1 + \frac{3}{9} = 1\frac{1}{3}$,

RHS $= -1 + 2\frac{1}{3} = 1\frac{1}{3}$

LHS = RHS so the two lines intersect.

(c) L_1: $\underset{\sim}{a} = (3, 6, -3)$, L_3: $\underset{\sim}{b} = (-1, 1, 1)$. These are the direction vectors for each line.

$\underset{\sim}{a} \bullet \underset{\sim}{b} = (3, 6, -3) \bullet (-1, 1, 1) = -3 + 6 - 3 = 0$

Hence $L_1 \perp L_3$

(d) Since $L_1 \parallel L_2$ and $L_1 \perp L_3$, then $L_2 \perp L_3$.

Vector equations of a line

Two straight lines will be parallel if their direction vectors are parallel. Similarly, two straight lines will be perpendicular if their direction vectors are perpendicular.

Consider two straight lines, L_1 and L_2, with vector equations $\underset{\sim}{r}_1 = \underset{\sim}{a}_1 + \lambda\underset{\sim}{b}_1$ and $\underset{\sim}{r}_2 = \underset{\sim}{a}_2 + \lambda\underset{\sim}{b}_2$ respectively and with direction vectors $\underset{\sim}{b}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$ respectively.

- If the two lines are parallel, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$

Then, eliminating the constant k, if L_1 and L_2 are parallel, then $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

- If the two lines are perpendicular, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$ and so $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

Example 30

Line L_1 passes through the points (1, –2, 4) and (5, 3, –2), while line L_2 passes through the points (3, 8, –2) and (a, –2, 10), where $a \in R$.

Find the value(s) of a if: **(a)** L_1 is parallel to L_2. **(b)** L_1 is perpendicular to L_2.

Solution

(a) Find the direction vectors for each of the lines.

Line L_1 passes through the points (1, –2, 4) and (5, 3, –2) and so the direction vector is:

$\underset{\sim}{b}_1 = (5-1)\underset{\sim}{i} + (3--2)\underset{\sim}{j} + (-2-4)\underset{\sim}{k}$

$= 4\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k}$

Line L_2 passes through the points (3, 8, –2) and (a, –2, 10) and so the direction vector is:

$\underset{\sim}{b}_2 = (a-3)\underset{\sim}{i} + (-2-8)\underset{\sim}{j} + (10--2)\underset{\sim}{k}$

$= (a-3)\underset{\sim}{i} - 10\underset{\sim}{j} + 12\underset{\sim}{k}$

If the two lines are parallel, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$.

If $L_1 \parallel L_2$, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$.

$4\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k} = k\left((a-3)\underset{\sim}{i} - 10\underset{\sim}{j} + 12\underset{\sim}{k}\right)$ and $\frac{4}{a-3} = \frac{5}{-10} = \frac{-6}{12}$

$\frac{4}{a-3} = -\frac{1}{2}$

$a - 3 = -8$

$a = -5$

(b) If the two lines are perpendicular, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$ and so $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

If $L_1 \perp L_2$, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$: $\left(4\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k}\right) \bullet \left((a-3)\underset{\sim}{i} - 10\underset{\sim}{j} + 12\underset{\sim}{k}\right) = 0$

$$4 \times (a-3) + 5 \times (-10) + (-6) \times 12 = 0$$
$$4a - 134 = 0$$
$$4a = 134$$
$$a = \frac{67}{2}$$

Example 31

(a) Find the parametric equations of the line through (–2, 1, 2) that is parallel to $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$.

(b) Do either of the the points (1, 2, 3) or (–6, 3, –4) lie on this line?

(c) Find the coordinates of two points that lie on the line in part **(a)**.

Solution

(a) Equation of the line is: $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$$\underset{\sim}{r} = (-2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$$
$$= (-2 + 2\lambda)\underset{\sim}{i} + (1 - \lambda)\underset{\sim}{j} + (2 + 3\lambda)\underset{\sim}{k}$$

The parametric equations of the line are $x = -2 + 2\lambda$, $y = 1 - \lambda$, $z = 2 + 3\lambda$.

(b) Substitute coordinates (1, 2, 3) into one part of the parametric equations to find λ, then see if that value works in the other two parts: $1 = -2 + 2\lambda$ so $\lambda = \frac{3}{2}$.

This gives $y = 1 - \frac{3}{2} = -\frac{1}{2}$ and $z = 2 + \frac{9}{2} = 6\frac{1}{2}$. In the given point, $y = 2$ and $z = 3$ so the point (1, 2, 3) does not lie on the line.

For (–6, 3, –4): $-6 = -2 + 2\lambda$ so $\lambda = -2$.

This gives $y = 1 + 2 = 3$ and $z = 2 - 6 = -4$, which are the corresponding coordinates of the given point. Hence the point (–6, 3, –4) lies on the line.

(c) Give λ two different values: $\lambda = 1$, $x = -2 + 2 = 0$, $y = 1 - 1 = 0$, $z = 2 + 3 = 5$. Hence the point (0, 0, 5) lies on the line.

$\lambda = 2$: $x = 2$, $y = -1$, $z = 8$. Hence the point (2, –1, 8) lies on the line.

Example 32

Find the coordinates of the points where the line $\underset{\sim}{r} = (1 + \lambda)\underset{\sim}{i} + (4 - 2\lambda)\underset{\sim}{j} + (3 + \lambda)\underset{\sim}{k}$ cuts the coordinate planes.

Solution

In the x–y plane, $z = 0$: $3 + \lambda = 0$ so $\lambda = -3$. The point is (–2, 10, 0).

In the x–z plane, $y = 0$: $4 - 2\lambda = 0$ so $\lambda = 2$. The point is (3, 0, 5).

In the y–z plane, $x = 0$: $1 + \lambda = 0$ so $\lambda = -1$. The point is (0, 6, 2).

Summary of formulae

Distance formula

Given $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then $\overrightarrow{AB} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j} + (z_1 - z_2)\underset{\sim}{k}$ and

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Unit vectors

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$$

Scalar product

$$\underset{\sim}{a} \bullet \underset{\sim}{b} = \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}\right) \bullet \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}\right) = x_1x_2 + y_1y_2 + z_1z_2$$

Algebraic properties of the dot product

- $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a}$
- $\underset{\sim}{a} \bullet \underset{\sim}{a} = |\underset{\sim}{a}|^2$
- $\underset{\sim}{a} \bullet (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c}$
- $(\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{c} + \underset{\sim}{d}) = \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{d} + \underset{\sim}{b} \bullet \underset{\sim}{c} + \underset{\sim}{b} \bullet \underset{\sim}{d}$
- $(m\underset{\sim}{a}) \bullet \underset{\sim}{b} = m(\underset{\sim}{a} \bullet \underset{\sim}{b})$ where m is a real number

Geometric properties of the dot product

- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, $\underset{\sim}{a} \perp \underset{\sim}{b}$.
- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors.
- If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$.
- The angle between two vectors, θ, is given by $\cos\theta = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$, $\underset{\sim}{a} \uparrow 0$, $\underset{\sim}{b} \uparrow 0$.
- By convention, $0° \le \theta \le 180°$.

Projections

The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\underset{\sim}{a} \infty \hat{\underset{\sim}{b}}$, where $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{b}|}$.

The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$ or $\dfrac{\underset{\sim}{a} \infty \underset{\sim}{b}}{\underset{\sim}{b} \infty \underset{\sim}{b}}\underset{\sim}{b}$.

The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a} - \left(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$ or $\underset{\sim}{a} - \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{\underset{\sim}{b} \bullet \underset{\sim}{b}}\underset{\sim}{b}$.

Equation of a sphere

$$(x - h)^2 + (y - j)^2 + (z - k)^2 = r^2$$

Vector equation of a line

$\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$, this vector equation consists of the position vector, $\underset{\sim}{r}_0$, of a fixed point P_0 on L and a fixed vector, $\underset{\sim}{r}$, which determines the direction of L.

Parametric equations of a line: $x = x_0 + a\lambda$, $y = y_0 + b\lambda$, $z = z_0 + c\lambda$, represent the parametric equations of the line L through the point (x_0, y_0, z_0) parallel to the vector $\underset{\sim}{v} = (a, b, c)$.

Properties of lines in three dimensions

Given line L_1 with parametric equation $x = a_1 + b_1t$, $y = c_1 + d_1t$, $z = e_1 + f_1t$ and line L_2 with parametric equation $x = a_2 + b_2s$, $y = c_2 + d_2s$, $z = e_2 + f_2s$, then:

- $L_1 \parallel L_2$ if $\dfrac{b_1}{b_2} = \dfrac{d_1}{d_2} = \dfrac{f_1}{f_2}$
- L_1 and L_2 intersect if unique values of t and s satisfy the three equations $a_1 + b_1t = a_2 + b_2s$, $c_1 + d_1t = c_2 + d_2s$, $e_1 + f_1t = e_2 + f_2s$ simultaneously
- $L_1 \perp L_2$ if $b_1b_2 + d_1d_2 + f_1f_2 = 0$ (direction vector dot product, $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$)
- They are skew if none of the preceding events occur.

Note: In three dimensions, coincident lines will be parallel and have a point in common. This means that for L_1 and L_2 above to be coincident, $\dfrac{b_1}{b_2} = \dfrac{d_1}{d_2} = \dfrac{f_1}{f_2} = \dfrac{a_1}{a_2} = \dfrac{c_1}{c_2} = \dfrac{e_1}{e_2}$.

Parallel and perpendicular lines

Consider two straight lines, L_1 and L_2, with vector equations $\underset{\sim}{r}_1 = \underset{\sim}{a}_1 + \lambda \underset{\sim}{b}_1$ and $\underset{\sim}{r}_2 = \underset{\sim}{a}_2 + \lambda \underset{\sim}{b}_2$ respectively and with direction vectors $\underset{\sim}{b}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$ respectively.

- If the two lines are parallel, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$.

Then, eliminating the constant k, and if L_1 and L_2 are parallel, then $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

- If the two lines are perpendicular, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$ and so $x_1x_2 + y_1y_2 + z_1z_2 = 0$.

EXERCISE 3.7 PARALLEL AND PERPENDICULAR LINES IN THREE DIMENSIONS

1 Given L_1 has equation $x = 1 + 2t$, $y = 2 - t$, $z = 3 + t$ and L_2 has equation $x = 2 + s$, $y = -1 + 2s$, $z = 1 - 3s$, then L_1 and L_2:

A are parallel **B** intersect **C** are perpendicular **D** are skew

2 Given L_1 has equation $x = 1 + 2t$, $y = 2 - t$, $z = 3 + t$ and L_2 has equation $x = 2 + s$, $y = -1 + 2s$, $z = 1 + 3s$, then L_1 and L_2:

A are parallel **B** intersect **C** are perpendicular **D** are skew

3 Given L_1 has equation $x = 1 + 2t$, $y = 2 - 4t$, $z = 3 - 2t$ and L_2 has equation $x = 2 + s$, $y = 1 - 2s$, $z = 1 - s$, then L_1 and L_2:

A are parallel **B** intersect **C** are perpendicular **D** are skew

4 Given L_1 has equation $x = 2 + 2t$, $y = 2 + t$, $z = 3 - t$ and L_2 has equation $x = 2 + s$, $y = -1 + 2s$, $z = -6 + 4s$, then L_1 and L_2:

A are parallel **B** intersect **C** are perpendicular **D** are skew

5 Line L_1 passes through the points $(1, 2, -1)$ and $(4, -1, 2)$ while line L_2 passes through the points $(2, 6, -2)$ and $(a, -1, 5)$, where $a \in R$.

Find the value(s) of a, if:

(a) L_1 is parallel to L_2 **(b)** L_1 is perpendicular to L_2.

6 **(a)** Find the parametric equations of the line through $(-1, 2, 1)$ that is parallel to $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$.
(b) Do either of the the points $(1, 2, 3)$ or $(3, 0, 7)$ lie on this line?
(c) Find the coordinates of two points that lie on the line in part **(a)**.

7 Find the coordinates of the points where the line $\underset{\sim}{r} = (1 - \lambda)\underset{\sim}{i} + (4 + 2\lambda)\underset{\sim}{j} + (3 - \lambda)\underset{\sim}{k}$ cuts the coordinate planes.

8 Given points A, B and C, determine whether point C lies on the line AB.
(a) $A(1, 0, 1)$, $B(1, 1, 1)$, $C(2, 1, 2)$
(b) $A(3, 3, -2)$, $B(-2, 3, -2)$, $C(2, 3, -2)$
(c) $A(2, 1, -1)$, $B(3, -2, 0)$, $C(1, 4, -2)$

9 **(a)** Show that the line through the points $(2, 1, -1)$ and $(3, -2, 0)$ is parallel to the line through the points $(-2, 3, 2)$ and $(2, 15, 6)$.
(b) Show that the point $(1, 4, -2)$ lies on the first line and the point $(6, -21, 10)$ lies on the second line.

10 Show that the line through the points $(2, 1, -1)$ and $(3, -2, 3)$ is perpendicular to the line through the points $(1, 3, 2)$ and $(-1, 5, 4)$.

11 **(a)** Find the equation of the line L_1 through the point $(2, 1, -2)$ parallel to the vector $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$.
(b) Find the equation of the line L_2 through the points $(1, -2, 1)$ and $(0, 2, -2)$.
(c) Determine whether **(i)** $L_1 \parallel L_2$, **(ii)** $L_1 \perp L_2$, **(iii)** L_1 and L_2 intersect.

12 If $\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -m\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, where $m \in R$, then the vector $\underset{\sim}{a} - \underset{\sim}{b}$ will be perpendicular to vector $\underset{\sim}{b}$ when m equals:

A 0 only **B** 2 only **C** 0 or 2 **D** 0 or −2

CHAPTER REVIEW 3

1 For each of the points, P, whose coordinates are given, find:

(i) an $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ representation for the position vector $\overrightarrow{OP}$ **(ii)** the magnitude of $\overrightarrow{OP}$

(iii) a unit vector in the direction of $\overrightarrow{OP}$.

(a) $P(-1, 4, 2)$ **(b)** $P(3, 6, 8)$ **(c)** $P(-2, 2, -1)$

2 Given $A(3, 3, 1)$, $B(-2, 1, -1)$, $C(1, 1, 1)$ and $D(2, 1, -2)$, find:

(a) the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ **(b)** the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$

(c) the angle between $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

3 Determine whether the given sets of points are collinear.

(a) $A(1, 3, 2)$, $B(3, 1, 4)$, $C(5, -2, -6)$ **(b)** $D(1, 3, -4)$, $E(3, -2, 2)$, $F(3, 1, 5)$

4 Given $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k}$, find unit vectors $\hat{\underset{\sim}{a}}$, $\hat{\underset{\sim}{b}}$, $\hat{\underset{\sim}{c}}$.

5 If $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = 4\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -5\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, simplify:

(a) $(\underset{\sim}{a} \bullet \underset{\sim}{b})\underset{\sim}{c} + (\underset{\sim}{a} \bullet \underset{\sim}{c})\underset{\sim}{b}$ **(b)** $(\underset{\sim}{c} - \underset{\sim}{a}) \bullet \underset{\sim}{b}$ **(c)** $(\underset{\sim}{a} - \underset{\sim}{b}) \bullet (\underset{\sim}{b} - \underset{\sim}{c})$

6 The position vectors of the points P, Q and R are $8\underset{\sim}{i} - 4\underset{\sim}{j} - 3\underset{\sim}{k}$, $6\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$ and $7\underset{\sim}{i} + 5\underset{\sim}{j} - 5\underset{\sim}{k}$ respectively. Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

7 Find a vector perpendicular to both $\underset{\sim}{u} = 4\underset{\sim}{i} - 7\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{v} = -7\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k}$.

8 Show that each given equation is the equation of a sphere and find the coordinates of its centre and the radius.

(a) $x^2 + y^2 + z^2 + 14x - 12y + 2z + 5 = 0$ **(b)** $x^2 + y^2 + z^2 - 6x + 2z + 6 = 0$

9 For the curves whose parametric equations are given, find:

(i) the Cartesian equation **(ii)** the vector equation.

(a) $x = 2t, y = t^2, t \in R$ **(b)** $x = \sec\theta, y = \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

10 For the curves whose vector equations are given, find:

(i) the parametric equation **(ii)** the Cartesian equation.

(a) $\underset{\sim}{r} = 4t^2\underset{\sim}{i} + 2t\underset{\sim}{j}$ **(b)** $\underset{\sim}{r} = \operatorname{cosec} 2\theta\underset{\sim}{i} + \sec 2\theta\underset{\sim}{j}, \theta \in R$

11 Find the vector equation of the line through $(2, 19, -31)$ parallel to the vector $2\underset{\sim}{i} - 3\underset{\sim}{j}$.

12 Find the vector equation of the line through $A(4, 3, 6)$ and $B(2, 5, 3)$.

13 Show that the line through the points $(1, -1, 1)$ and $(5, 3, 3)$ is perpendicular to the line through the points $(1, 1, 2)$ and $(4, -4, 6)$.

14 Given $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, find two vectors $\underset{\sim}{c}$ and $\underset{\sim}{d}$ such that $\underset{\sim}{a} = \underset{\sim}{c} + \underset{\sim}{d}$, $\underset{\sim}{c}$ is parallel to $\underset{\sim}{b}$ and $\underset{\sim}{d}$ is perpendicular to $\underset{\sim}{b}$.

15 If $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$, $\underset{\sim}{b} = 5\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$, find the values of p and q such that $\underset{\sim}{a} + p\underset{\sim}{b} + q\underset{\sim}{c}$ is parallel to the y-axis.

16 **(a)** Show that the points $O(0, 0, 0)$, $A(1, 1, 0)$, $B(1, 0, 1)$ and $C(0, 1, 1)$ are the vertices of a regular tetrahedron by finding the lengths of each of the six edges.

(b) Use the dot product to find the angle between any two edges.

(c) If M is the midpoint of BC, find the size of $\angle AMB$.

17 Relative to a fixed origin, the points A, B and C are defined respectively by the position vectors $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = m\underset{\sim}{i}$, where m is a real constant.

(a) If $\angle ABC = \frac{\pi}{3}$, find m. **(b)** If $\angle ABC = \frac{\pi}{2}$, find m.

CHAPTER 4
Integration by substitution

This chapter is an extension of your study of integration techniques from the Mathematics Advanced course (see *New Senior Mathematics Advanced for Years 11 & 12*) and the Mathematics Extension 1 course (see *New Senior Mathematics Extension 1 for Years 11 & 12*).

Integration by substitution

In the Mathematics Extension 1 course, you were introduced to integration using a substitution where the substitution was given.

In the Mathematics Extension 2 course, you are expected to be able to use integration by substitution where you have to decide on the appropriate substitution. In the following sections, various possible substitutions will be introduced and their use developed. In some cases, you may find that there is more than one substitution that can be used. You will find that some substitutions are more efficient than others, and with practice you will be able to choose the best one.

4.1 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

The simplest integrals involving the trigonometric functions are:

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C \qquad \int \sin ax\,dx = -\frac{1}{a}\cos ax + C \qquad \int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

The identity $\tan^2 x = \sec^2 x - 1$ allows $\int \tan^2 x\,dx$ to be found.

The double-angle results are:

- $\sin 2x = 2\sin x\cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x,$

which can be rewritten as $\cos^2 x = \dfrac{1+\cos 2x}{2}$ and $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

These results can be used to reduce trigonometric expressions to the simpler forms given as the standard integrals, which allows you to find integrals such as:

$$\int \sin x\cos x\,dx = \frac{1}{2}\int \sin 2x\,dx \qquad \int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx \qquad \int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$

Integrals $\int \sin^m x\,dx$ and $\int \cos^m x\,dx$, *m* an even positive integer

The $\cos 2x$ results can be generalised as $\cos 2ax = 2\cos^2 ax - 1 = 1 - 2\sin^2 ax$.

Consider this for the following values of a:

$$a = 1: \begin{cases} \cos^2 x = \frac{1}{2}(1+\cos 2x) \\ \sin^2 x = \frac{1}{2}(1-\cos 2x) \end{cases} \qquad a = 2: \begin{cases} \cos^2 2x = \frac{1}{2}(1+\cos 4x) \\ \sin^2 2x = \frac{1}{2}(1-\cos 4x) \end{cases} \qquad a = \frac{1}{2}: \begin{cases} \cos^2 \frac{x}{2} = \frac{1}{2}(1+\cos x) \\ \sin^2 \frac{x}{2} = \frac{1}{2}(1-\cos x) \end{cases}$$

Thus you can write:

$$\begin{aligned} \cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 &= \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right) \\ &= \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right) \\ &= \frac{1}{8}(3+4\cos 2x+\cos 4x) \end{aligned}$$

Example 1

Find:

(a) $\int \cos^2 x\,dx$ (b) $\int \sin^4 x\,dx$ (c) $\int \sin^2 2x\,dx$ (d) $\int \sin^2 x\cos^2 x\,dx$

Solution

(a) $\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

(b) $\sin^4 x = \frac{1}{4}(1-\cos 2x)^2$ as $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right)$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)$$

$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

Hence: $\int \sin^4 x\,dx = \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\,dx$

$$= \frac{1}{8}\left(3x - 2\sin 2x + \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

(c) $\int \sin^2 2x\,dx = \frac{1}{2}\int (1-\cos 4x)\,dx = \frac{x}{2} - \frac{1}{8}\sin 4x + C$

(d) As $\sin x\cos x = \frac{1}{2}\sin 2x$:

$\sin^2 x\cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{8}(1-\cos 4x)$

Hence: $\int \sin^2 x\cos^2 x\,dx = \frac{1}{8}\int (1-\cos 4x)\,dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

Alternatively: $\sin^2 x\cos^2 x = \sin^2 x(1-\sin^2 x) = \sin^2 x - \sin^4 x$

Now $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

and $\int \sin^4 x\,dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ from part **(b)**

Hence: $\int \sin^2 x\cos^2 x\,dx = \int \sin^2 x\,dx - \int \sin^4 x\,dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

Use of $\int f(u)\,du,\ u = g(x)$ for trigonometric integrals

Example 2

Find: (a) $\int \cos x\sin x\,dx$ (b) $\int \cos x\sin^3 x\,dx$ (c) $\int x\cos x^2\,dx$

Solution

(a) Let $u = \sin x$ so that $du = \cos x\,dx$

$$\int \cos x\sin x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}\sin^2 x + C$$

This method may be used instead of writing the integrand in terms of $\sin 2x$ as mentioned earlier.

(b) Let $u = \sin x$ so that $du = \cos x\, dx$:

$$\int \cos x \sin^3 x\, dx = \int u^3\, du = \frac{1}{4}u^4 + C$$
$$= \frac{1}{4}\sin^4 x + C$$

Alternatively:

$\int \cos x \sin^3 x\, dx = \int \sin^3 x \cos x\, dx$ is of the form $\int [f(x)]^n f'(x)\, dx = \frac{1}{n+1}[f(x)]^{n+1} + C$

∴ This integral is the reverse of the function-of-a-function rule, so:

$$\int \cos x \sin^3 x\, dx = \frac{1}{4}\sin^4 x + C$$

(c) Let $u = x^2$ so that $du = 2x\, dx$:

$$\int x \cos x^2\, dx = \frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + C$$
$$= \frac{1}{2}\sin x^2 + C$$

Integrals $\int \sin^m x\, dx$ and $\int \cos^m x\, dx$, m an odd positive integer

This method requires the use of a substitution.

Example 3

Find: (a) $\int \cos^3 x\, dx$ (b) $\int \sin^3 x\, dx$ (c) $\int \sin^2 x \cos^3 x\, dx$

Solution

(a)
$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$$
$$= \int (1 - \sin^2 x)\cos x\, dx$$
$$= \int (\cos x - \sin^2 x \cos x)\, dx$$

Now use $\int [f(x)]^2 f'(x)\, dx = \frac{1}{3}[f(x)]^3 + C$ where $f(x) = \sin x$ so that $f'(x) = \cos x$

$$\therefore \int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$$

(b)
$$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$$
$$= \int (1 - \cos^2 x)\sin x\, dx$$
$$= \int (\cos^2 x - 1)(-\sin x)\, dx$$
$$= \int (-\sin x \cos^2 x + \sin x)\, dx$$
$$= \frac{1}{3}\cos^3 x - \cos x + C$$

Alternatively: Let $u = \cos x$ so that $du = -\sin x\, dx$:

$$\int \sin^3 x\, dx = \int (u^2 - 1)\, du$$
$$= \frac{1}{3}u^3 - u + C$$
$$= \frac{1}{3}\cos^3 x - \cos x + C$$

(c)
$$\int \sin^2 x \cos^3 x\, dx = \int (1 - \cos^2 x)\cos^3 x\, dx$$
$$= \int (\cos^3 x - \cos^5 x)\, dx$$

Now $\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$ from **(a)**

and $\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx$

$$= \int (1-\sin^2 x)^2 \cos x\,dx$$

$$= \int (1-2\sin^2 x + \sin^4 x)\cos x\,dx$$

$$= \int (\cos x - 2\sin^2 x\cos x + \sin^4 x\cos x)\,dx$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Thus: $\int (\cos^3 x - \cos^5 x)\,dx = \sin x - \frac{1}{3}\sin^3 x - \left(\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right) + C$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

Alternatively: $\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cos x\,dx$

Let $u = \sin x$ so that $du = \cos x\,dx$ and $\cos^2 x = 1 - u^2$:

$$\int \sin^2 x \cos^3 x\,dx = \int u^2(1-u^2)\,du$$

$$= \int (u^2 - u^4)\,du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

Example 4

Evaluate: **(a)** $\int_0^{\frac{\pi}{4}} \cos^2 2x\,dx$ **(b)** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x\,dx$ **(c)** $\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx$

Solution

(a) $\int_0^{\frac{\pi}{4}} \cos^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 4x)\,dx$

$$= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + 0 - 0\right)$$

$$= \frac{\pi}{8}$$

(b) Let $u = \cos x$, $du = -\sin x\,dx$.

Calculate the new limits of the definite integral, $x = \frac{\pi}{3}$: $u = \cos\frac{\pi}{3} = \frac{1}{2}$ $x = \frac{\pi}{2}$: $u = \cos\frac{\pi}{2} = 0$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x\,dx = -\int_{\frac{1}{2}}^{0} u^3\,du$$

$$= -\left[\frac{u^4}{4}\right]_{\frac{1}{2}}^{0}$$

$$= \frac{1}{64}$$

or

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x\,dx = -\frac{1}{4}\left[\cos^4 x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= -\frac{1}{4}\left(0 - \frac{1}{16}\right)$$

$$= \frac{1}{64}$$

(c) $\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx = \int_{\frac{3\pi}{4}}^{\pi} \sin^2 x \sin x\,dx = \int_{\frac{3\pi}{4}}^{\pi} (1-\cos^2 x)\sin x\,dx$

Let $u = \cos x$, $du = -\sin x\,dx$.

$x = \frac{3\pi}{4}$: $u = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ $\quad x = \pi$: $u = \cos\pi = -1$

$$\begin{aligned}\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx &= -\int_{-\frac{1}{\sqrt{2}}}^{-1} (1-u^2)\,du \\ &= \int_{-\frac{1}{\sqrt{2}}}^{-1} (u^2-1)\,du \\ &= \left[\frac{u^3}{3} - u\right]_{-\frac{1}{\sqrt{2}}}^{-1} \\ &= \left(-\frac{1}{3} + 1 - \left(-\frac{1}{6\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right) \\ &= \frac{2}{3} - \frac{5\sqrt{2}}{12}\end{aligned}$$

or

$$\begin{aligned}\int_{\frac{3\pi}{4}}^{\pi} \sin^3 x\,dx &= \int_{\frac{3\pi}{4}}^{\pi} (\sin x - \sin x\cos^2 x)\,dx \\ &= \left[-\cos x + \frac{1}{3}\cos^3 x\right]_{\frac{3\pi}{4}}^{\pi} \\ &= 1 - \frac{1}{3} - \left(\frac{1}{\sqrt{2}} + \frac{1}{3}\times\left(-\frac{1}{\sqrt{2}}\right)^3\right) \\ &= \frac{2}{3} - \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) \\ &= \frac{2}{3} - \frac{5}{6\sqrt{2}} = \frac{2}{3} - \frac{5\sqrt{2}}{12}\end{aligned}$$

MAKING CONNECTIONS

Integration of trigonometric functions

Use technology to explore integration by substitution of trigonometric functions.

EXERCISE 4.1 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

1 Find: **(a)** $\int \sin^2 x\,dx$ **(b)** $\int \cos^2 2x\,dx$ **(c)** $\int \sin^2 \frac{x}{2}\,dx$ **(d)** $\int \cos^2 3x\,dx$

(e) $\int \sin^2 3x\,dx$ **(f)** $\int \sin^2 4x\,dx$ **(g)** $\int \cos^2 \frac{3x}{2}\,dx$ **(h)** $\int \cos^2 \frac{5x}{2}\,dx$

2 $\int \sin x \cos^2 x\,dx = \ldots$

A $\frac{\sin^3 x}{3} + C$ **B** $\frac{\cos^3 x}{3} + C$ **C** $-\frac{\sin^3 x}{3} + C$ **D** $-\frac{\cos^3 x}{3} + C$

3 Find: **(a)** $\int \sin^2 x \cos x\,dx$ **(b)** $\int \tan x \sec^2 x\,dx$ **(c)** $\int \cos^3 x \sin x\,dx$

(d) $\int \cos x \sin^4 x\,dx$ **(e)** $\int (1+\cos 2x)\sin x\,dx$ **(f)** $\int \sin^2\left(x - \frac{\pi}{4}\right)dx$

(g) $\int \sin x \cos^4 x\,dx$ **(h)** $\int \sec^2 x \sin x\,dx$ **(i)** $\int \operatorname{cosec}^2 x \cos x\,dx$

4 Find: **(a)** $\int (1+\tan^2 x)\,dx$ **(b)** $\int \left(1+\tan^2 \frac{x}{2}\right)dx$ **(c)** $\int (1+\tan^2 3x)\,dx$

(d) $\int \frac{1}{1+\cos 2x}\,dx$ **(e)** $\int (\sin^2 2x - 1)\,dx$ **(f)** $\int \sin x \cos x\,dx$

(g) $\int 2\cos^2 \frac{x}{2}\,dx$ **(h)** $\int \sin^2\left(\frac{\pi}{2} - x\right)dx$ **(i)** $\int \sin x \cos 2x\,dx$

(j) $\int \frac{1}{1+\cos x}\,dx$ (Hint: Use double-angle formulae.)

5 Find: (a) $\int \sin^2 2x\,dx$ (b) $\int \cos^3 2x\,dx$ (c) $\int \sin^2 2x\cos^2 2x\,dx$

(d) $\int \cos^4 x\,dx$ (e) $\int \sin^5 x\,dx$ (f) $\int \cos 2x\cos 4x\,dx$

(g) $\int \cos^5 x\,dx$ (h) $\int \sin^3 x\cos^2 x\,dx$ (i) $\int \tan^4 x\,dx$

6 Find: $\int \cos^4 x\sin^3 x\,dx$

7 Evaluate: (a) $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$ (c) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\cos^3 x\,dx$

(d) $\int_{-\pi}^{\pi} \sin^3 x\cos x\,dx$ (e) $\int_0^{\frac{\pi}{4}} \tan x\sec^2 x\,dx$ (f) $\int_{\pi}^{\frac{3\pi}{2}} \sin x\cos x\,dx$

(g) $\int_0^{\pi} 2\sin\theta\cos^2\theta\,d\theta$ (h) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\left(x-\frac{\pi}{4}\right)dx$ (i) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos\theta\sin^4\theta\,d\theta$ (j) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2\theta\sin\theta\,d\theta$

4.2 INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

In the Mathematics Extension 1 course, you established the derivatives of the inverse trigonometric functions (see *New Senior Mathematics Extension 1 for Years 11 & 12*, Chapter 11). These lead to the following integrals.

- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+C$ for $-a<x<a$
- $\int \frac{a}{a^2+x^2}dx = \tan^{-1}\frac{x}{a}+C$ for all x
- $\int \frac{-1}{\sqrt{a^2-x^2}}dx = \cos^{-1}\frac{x}{a}+C$ for $-a<x<a$

The integrals in the following example were first considered in the Mathematics Extension 1 course.

Example 5

Find: (a) $\int \frac{dx}{\sqrt{4-x^2}}$ (b) $\int \frac{-1}{\sqrt{9-x^2}}dx$ (c) $\int \frac{2}{4+x^2}dx$ (d) $\int \frac{dx}{\sqrt{3-x^2}}$

(e) $\int \frac{dx}{\sqrt{1-9x^2}}$ (f) $\int \frac{dx}{1+4x^2}$ (g) $\int \frac{2}{\sqrt{4-25x^2}}dx$

Solution

(a) $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}+C$

(b) $\int \frac{-1}{\sqrt{9-x^2}}dx = \cos^{-1}\frac{x}{3}+C$

(c) $\int \frac{2}{4+x^2}dx = \tan^{-1}\frac{x}{2}+C$

(d) $\int \frac{dx}{\sqrt{3-x^2}} = \sin^{-1}\frac{x}{\sqrt{3}}+C$

(e) Write: $\sqrt{1-9x^2} = \sqrt{9\left(\frac{1}{9}-x^2\right)} = 3\sqrt{\left(\frac{1}{3}\right)^2-x^2}$

Thus: $\int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2-x^2}}dx = \frac{1}{3}\sin^{-1}\frac{x}{\left(\frac{1}{3}\right)}+C = \frac{1}{3}\sin^{-1}3x+C$

(f) Write: $1+4x^2 = 4\left(\frac{1}{4}+x^2\right) = 4\left(\left(\frac{1}{2}\right)^2+x^2\right)$

Thus: $\int \frac{dx}{1+4x^2} = \frac{1}{4}\int \frac{1}{\left(\frac{1}{2}\right)^2+x^2}dx = \frac{1}{2}\int \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2+x^2}dx = \frac{1}{2}\tan^{-1}\frac{x}{\left(\frac{1}{2}\right)}+C = \frac{1}{2}\tan^{-1}2x+C$

(g) Write: $\sqrt{4-25x^2} = \sqrt{25\left(\frac{4}{25}-x^2\right)} = 5\sqrt{\left(\frac{2}{5}\right)^2 - x^2}$

Thus: $\int \frac{2}{\sqrt{4-25x^2}}\,dx = \frac{2}{5}\int \frac{1}{\sqrt{\left(\frac{2}{5}\right)^2 - x^2}}\,dx = \frac{2}{5}\sin^{-1}\frac{x}{\left(\frac{2}{5}\right)} + C = \frac{2}{5}\sin^{-1}\frac{5x}{2} + C$

Integration by substitution

The integrals in the previous example were found by the standard trigonometric integrals. They can be found explicitly by the method of change of variable, also called 'integration by substitution', i.e. the process:

$$\int f(x)\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

An expression like $\sqrt{a^2 - x^2}$ can be integrated with a substitution of $x = a\sin\theta$ or $x = a\cos\theta$, while an expression like $a^2 + x^2$ can be integrated with a substitution of $x = a\tan\theta$, as shown in the following examples.

Example 6

Find: (a) $\int \frac{dx}{\sqrt{4-x^2}}$ (b) $\int \frac{-dx}{\sqrt{1-9x^2}}$ (c) $\int \frac{dx}{4+x^2}$

Solution

(a) $\sqrt{4-x^2} = \sqrt{a^2-x^2}$ with $a = 2$, so use the substitution $x = 2\sin\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$(x \neq \pm\frac{\pi}{2}$ as these values make the denominator of the integrand zero)

Thus: $\frac{dx}{d\theta} = 2\cos\theta,\ dx = 2\cos\theta\,d\theta$

and $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$ as $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (i.e. $\cos\theta > 0$)

Hence: $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta\,d\theta}{2\cos\theta} = \int d\theta$

$= \theta + C$

$= \sin^{-1}\frac{x}{2} + C$

(b) $\sqrt{1-9x^2} = 3\sqrt{\frac{1}{9}-x^2} = 3\sqrt{\left(\frac{1}{3}\right)^2 - x^2} = 3\sqrt{a^2-x^2}$, so put $a = \frac{1}{3}$ and use the substitution

$x = \frac{1}{3}\cos\theta$ for $0 < \theta < \pi$ $(x \neq 0, \pi$ as these values make the denominator of the integrand zero).

Thus: $\frac{dx}{d\theta} = -\frac{1}{3}\sin\theta,\ dx = -\frac{1}{3}\sin\theta\,d\theta$

and $\sqrt{1-9x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$

Hence: $\int \frac{-dx}{\sqrt{1-9x^2}} = \int \frac{-1}{\sin\theta}\times\left(-\frac{1}{3}\sin\theta\right)d\theta = \frac{1}{3}\int d\theta$

$= \frac{\theta}{3} + C$

$= \frac{1}{3}\cos^{-1}3x + C$

(c) $4 + x^2 = a^2 + x^2$ with $a = 2$, so use the substitution $x = 2\tan\theta$

Thus: $\frac{dx}{d\theta} = 2\sec^2\theta,\ dx = 2\sec^2\theta\, d\theta$

and $4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta$

Hence:
$$\int \frac{dx}{4 + x^2} = \int \frac{1}{4\sec^2\theta} \times 2\sec^2\theta\, d\theta = \frac{1}{2}\int d\theta$$
$$= \frac{\theta}{2} + C$$
$$= \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

In Example **6(a)**, the restriction that θ is in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ has several implications.
It means that $\sqrt{4 - x^2} = \sqrt{4\cos^2\theta}$ requires you to find the positive square root of the expression, because $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ implies $\cos\theta > 0$ and hence $\sqrt{4 - x^2} = 2\cos\theta$.
If the restriction had been $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ then $\cos\theta < 0$ and so $\sqrt{4 - x^2} = -2\cos\theta$.
The domain of the substitution should be chosen to coincide with the corresponding inverse function:

- for the substitution $x = a\sin\theta$, set $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- for the substitution $x = a\cos\theta$, set $0 < \theta < \pi$.

The endpoints of the domain are not included as $\frac{1}{\sqrt{a^2 - x^2}}$ is undefined there.

For the substitution $x = \tan\theta$, no restriction on θ is required: $1 + x^2 > 0$ for all values of x, so the denominator of the integrand will never be zero for real values of x.

Example 7

Find: **(a)** $\int \frac{dx}{(x+2)^2 + 9}$ **(b)** $\int \sqrt{4 - x^2}\, dx$ for $|x| \le 2$ **(c)** $\int x\sqrt{4 - x^2}\, dx$ for $|x| \le 2$

Solution

(a) Let $(x+2)^2 + 9 = 9 + u^2$ where $u = x + 2$. Thus $du = dx$.

Hence:
$$\int \frac{dx}{(x+2)^2 + 9} = \int \frac{du}{9 + u^2} = \frac{1}{3}\int \frac{3}{9 + u^2}\, du$$
$$= \frac{1}{3}\tan^{-1}\frac{u}{3} + C$$
$$= \frac{1}{3}\tan^{-1}\frac{x+2}{3} + C$$

(b) $\sqrt{4 - x^2}$ suggests the substitution $x = 2\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

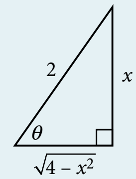
Note that $\theta = \pm\frac{\pi}{2}$ is allowed in this case as $\sqrt{4 - x^2}$ does not occur in the denominator.

Let $x = 2\sin\theta$ so that $dx = 2\cos\theta\, d\theta$ and: $\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\theta} = 2\cos\theta$

Hence:
$$\int \sqrt{4 - x^2}\, dx = \int 2\cos\theta \times 2\cos\theta\, d\theta = 2\int 2\cos^2\theta\, d\theta$$
$$= 2\int (1 + \cos 2\theta)\, d\theta$$
$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= 2\theta + \sin 2\theta + C$$
$$= 2\theta + 2\sin\theta\cos\theta + C$$

Now $x = 2\sin\theta$, so $\theta = \sin^{-1}\dfrac{x}{2}$ and $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$

$$\therefore \int \sqrt{4-x^2}\,dx = 2\sin^{-1}\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C \quad \text{for } |x| \le 2$$



(c) $x\sqrt{4-x^2}$ is of the form $f'(x)[f(x)]^n$, which suggests a substitution of the form $u = 4 - x^2$ rather than a trigonometric substitution.

Let $u = 4 - x^2$, so $du = -2x\,dx$, $x\,dx = -\dfrac{du}{2}$

Hence:
$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\,du$$
$$= -\frac{1}{3}u^{\frac{3}{2}} + C$$
$$= -\frac{1}{3}(4-x^2)^{\frac{3}{2}} + C \quad \text{for } |x| \le 2$$

The substitution $x = 2\sin\theta$ would have found the same answer, but it would have taken more steps.

Example 8

Evaluate: **(a)** $\displaystyle\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}}$ **(b)** $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}\,dx$ **(c)** $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ **(d)** $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

Solution

(a) $$\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}} = \left[\sin^{-1}\frac{x}{5}\right]_0^{2.5} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6}$$

Alternatively: Use the substitution $x = 5\sin\theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$:

$x = 5\sin\theta$, $dx = 5\cos\theta\,d\theta$: $\sqrt{25-x^2} = \sqrt{25 - 25\sin^2\theta} = 5\cos\theta$

Limits of integration are $x = 0$: $\theta = 0$ $x = 2.5$: $\sin\theta = \dfrac{1}{2}$, $\theta = \dfrac{\pi}{6}$

Hence:
$$\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}} = \int_0^{\frac{\pi}{6}} \frac{1}{5\cos\theta} \times 5\cos\theta\,d\theta = \int_0^{\frac{\pi}{6}} d\theta = [\theta]_0^{\frac{\pi}{6}} = \frac{\pi}{6}$$

(b) $$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}\,dx = \left[\cos^{-1}\frac{x}{2}\right]_{-\sqrt{3}}^{\sqrt{3}} = \cos^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{\pi}{6} - \frac{5\pi}{6} = -\frac{2\pi}{3}$$

Alternatively: The substitution $x = 2\cos\theta$ could have been used.

(c) $$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

Alternatively: The substitution $x = \tan\theta$ could have been used.

(d) Let $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$:

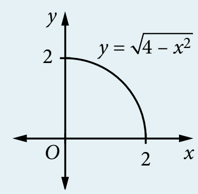
$x = 2\sin\theta,\ dx = 2\cos\theta\, d\theta$: $\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\theta} = 2\cos\theta$

Limits of integration are $x = 0$: $\theta = 0$ $\quad x = 2$: $\sin\theta = 1,\ \theta = \frac{\pi}{2}$

Hence:
$$\int_0^2 \sqrt{4 - x^2}\, dx = \int_0^{\frac{\pi}{2}} 2\cos\theta \times 2\cos\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$
$$= [2\theta + \sin 2\theta]_0^{\frac{\pi}{2}} = (\pi + \sin\pi) - (0 + \sin 0) = \pi$$

The graph of $y = \sqrt{4 - x^2}$ for the domain $0 \le x \le 2$ is the quadrant of the circle $x^2 + y^2 = 4$ that is in the first quadrant. The formula for the area of a circle could have been used to evaluate the integral:

Area $= \frac{1}{4} \times \pi \times 2^2 = \pi$ $\quad$ (as $r = 2$)



Sum or difference of two squares

All quadratic expressions of the form $ax^2 + bx + c$ can be expressed as the sum or the difference of two squares, depending on whether their discriminant $\Delta = b^2 - 4ac$ is positive or negative. This can be achieved by completing the square of the quadratic.

After the quadratic expression is written as the sum or difference of squares, its associated integral can usually be found by using an appropriate substitution.

Example 9

Find: **(a)** $\int \frac{dx}{x^2 + 4x + 13}$ **(b)** $\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$ **(c** $\int \frac{dx}{\sqrt{2x - x^2}}$

Solution

(a) Completing the square:

$$x^2 + 4x + 13 = x^2 + 4x + 4 + 9$$
$$= (x + 2)^2 + 9$$

Let $u = x + 2$ so $du = dx$

Hence $(x + 2)^2 + 9 = 9 + u^2$:

$$\int \frac{dx}{x^2 + 4x + 13} = \int \frac{1}{9 + u^2}\, du$$
$$= \frac{1}{3}\tan^{-1}\frac{u}{3} + C$$
$$= \frac{1}{3}\tan^{-1}\left(\frac{x + 2}{3}\right) + C$$

(b) Completing the square:

$$9 + 16x - 4x^2 = -4\left[x^2 - 4x - \frac{9}{4}\right]$$
$$= -4\left[(x^2 - 4x + 4) - 4 - \frac{9}{4}\right]$$
$$= 25 - 4(x - 2)^2$$

Let $u = x - 2$ so $du = dx$

Hence $25 - 4(x - 2)^2 = 25 - 4u^2$:

$$\int \frac{dx}{\sqrt{9 + 16x - 4x^2}} = \int \frac{du}{\sqrt{25 - 4u^2}}$$
$$= \frac{1}{2}\int \frac{du}{\sqrt{\frac{25}{4} - u^2}}$$
$$= \frac{1}{2}\sin^{-1}\frac{2u}{5} + C$$
$$= \frac{1}{2}\sin^{-1}\frac{2(x - 2)}{5} + C$$

(c) Completing the square: $2x - x^2 = -[(x^2 - 2x + 1) - 1]$
$= 1 - (x-1)^2$

Let $u = x - 1$ so $du = dx$
Hence $1 - (x-1)^2 = 1 - u^2$:

$$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{du}{\sqrt{1-u^2}}$$
$$= \sin^{-1} u + C$$
$$= \sin^{-1}(x-1) + C$$

EXERCISE 4.2 INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

1 Find: (a) $\int \frac{dx}{\sqrt{9-x^2}}$ (b) $\int \frac{dx}{9+x^2}$ (c) $\int \frac{-1}{\sqrt{16-x^2}}\,dx$

2 Find: (a) $\int \frac{dx}{\sqrt{3-x^2}}$ (b) $\int \frac{-1}{\sqrt{16-9x^2}}\,dx$ (c) $\int \frac{-1}{\sqrt{25-4x^2}}\,dx$

3 $\int \frac{x^2+2}{x^2+1}\,dx = \ldots$

A $x + \ln(x^2+1) + C$ B $x + \tan^{-1} x + C$ C $x - \ln(x^2+1) + C$ D $x - \tan^{-1} x + C$

4 Find: (a) $\int \frac{dx}{x^2+4x+5}$ (b) $\int \frac{dx}{x^2-6x+10}$ (c) $\int \frac{dx}{\sqrt{3+2x-x^2}}$

(d) $\int \frac{dx}{\sqrt{15+2x-x^2}}$ (e) $\int x\sqrt{16-x^2}\,dx$ (f) $\int \sqrt{9-x^2}\,dx$

(g) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ (h) $\int \frac{x+1}{\sqrt{4-x^2}}\,dx$ (i) $\int \frac{dx}{x^2+4x+6}$

5 Evaluate: (a) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2}$ (b) $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$ (c) $\int_0^1 x\sqrt{1-x^2}\,dx$

(d) $\int_0^{\sqrt{3}} \frac{dx}{(1+x^2)^{\frac{5}{2}}}$ (e) $\int_{-2}^{-1} \frac{dx}{x^2+4x+5}$ (f) $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dx}{\sqrt{2x-x^2}}$ (g) $\int_0^{\frac{5}{2}} \sqrt{25-4x^2}\,dx$

(h) $\int_0^{\frac{1}{2}} \frac{dx}{(1-x^2)^{\frac{3}{2}}}$ (i) $\int_0^4 \frac{dx}{x^2-2x+4}$ (j) $\int_{-1}^1 \frac{dx}{x^2-2x+5}$ (k) $\int_0^1 \frac{dt}{2-2t+t^2}$

(l) $\int_{-1}^0 \frac{dx}{x^2+2x+4}$ (m) $\int_{-\frac{7}{4}}^{\frac{3}{4}} \frac{dx}{\sqrt{6-x-x^2}}$ (n) $\int_0^a \sqrt{a^2-x^2}\,dx$ (o) $\int_{\tan x}^{\cot x} \frac{dt}{1+t^2},\ 0 < x < \frac{\pi}{2}$

6 $\int_1^3 \frac{dx}{\sqrt{4x-x^2}} = \ldots$

A 0 B $\frac{\pi}{6}$ C $\frac{\pi}{3}$ D $\frac{\pi}{2}$

4.3 INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

You have studied the derivative of the logarithmic function in the Mathematics Advanced course (see *New Senior Mathematics Advanced for Years 11 & 12*, Chapter 13). For example:

$$\frac{d}{dx}\log_e(x^3+1) = \frac{3x^2}{x^3+1} \qquad \frac{d}{dx}\log_e(x^2+2x+1) = \frac{2x+2}{x^2+2x+1} \qquad \frac{d}{dx}\log_e(\sin x) = \frac{\cos x}{\sin x},\ \sin x > 0$$

These results are all of the form $\frac{d}{dx}\log_e\left(f(x)\right) = \frac{f'(x)}{f(x)}$.

Consider $f(x) = \log_e[h(x)]$. Let $u = h(x)$ so $\frac{du}{dx} = h'(x)$:

$$f'(x) = \frac{1}{h(x)} \times \frac{du}{dx}$$
$$= \frac{h'(x)}{h(x)}$$

Hence it follows that: $\int \frac{dx}{x} = \log_e x + C, x > 0$

$$\int \frac{h'(x)}{h(x)}\,dx = \log_e[h(x)] + C, h(x) > 0$$

Important consideration:

The function where $f(x) = \log_e(-x)$ is a logarithmic function defined for $x < 0$.

Let $u = -x$ so that $y = \log_e u$: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ now gives $\frac{dy}{dx} = \frac{1}{u} \times (-1) = \frac{-1}{u} = \frac{-1}{-x} = \frac{1}{x}$

This is the same result as previously obtained for the function defined for $x > 0$.

This means that: $\int \frac{dx}{x} = \begin{cases} \log_e x + C, & x > 0 \\ \log_e(-x) + C, & x < 0 \end{cases}$

i.e. $\int \frac{dx}{x} = \log_e|x| + C$

Example 10

Write a primitive (antiderivative) of each function, stating any restrictions that must be placed on x.

(a) $\frac{4}{2x-3}$ **(b)** $\frac{6x^2+3}{2x^3+3x-5}$ **(c)** $\frac{\sin x}{1+\cos x}$ **(d)** $\frac{x}{x-1}$ **(e)** $\frac{x^2}{x-1}$

Solution

(a) $\int \frac{4}{2x-3}\,dx = 2\int \frac{2}{2x-3}\,dx = 2\int \frac{h'(x)}{h(x)}\,dx$ where $h(x) = 2x-3$

$= 2\log_e(2x-3) + C, x > 1.5$

or $2\log_e|2x-3| + C$ without the restriction on x.

(b) $\int \frac{6x^2+3}{2x^3+3x-5}\,dx = \int \frac{h'(x)}{h(x)}\,dx$ where $h(x) = 2x^3+3x-5$

$= 2\log_e(2x^3+3x-5) + C, x > 1$

It is not obvious that $2x^3+3x-5 > 0$ when $x > 1$, so it is better to write this answer as:

$= 2\log_e|2x^3+3x-5| + C$

(c) $\int \frac{\sin x}{1+\cos x}\,dx = -\int \frac{-\sin x}{1+\cos x}\,dx = -\int \frac{h'(x)}{h(x)}\,dx$ where $h(x) = 1+\cos x$

$= -\log_e(1+\cos x) + C, \cos x \neq -1$

or $= -\log_e|1+\cos x| + C$

(d) The power of the numerator equals the power of the denominator, so division gives:

$$\frac{x}{x-1} = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}$$

$$\therefore \int \frac{x}{x-1}\,dx = \int\left(1+\frac{1}{x-1}\right)dx = x + \ln|x-1| + C$$

(e) The power of the numerator is greater than the power of the denominator, so division gives:

$$\frac{x^2}{x-1}=\frac{(x^2-x)+(x-1)+1}{x-1}=x+1+\frac{1}{x-1}$$ Long division requires $x-1\overline{)x^2+0x+0}$.

$$\therefore \int\frac{x^2}{x-1}\,dx=\int\left(x+1+\frac{1}{x-1}\right)dx=\frac{x^2}{2}+x+\log_e(x-1)+C,\ x>1$$

$$\text{or } \frac{x^2}{2}+x+\log_e|x-1|+C$$

Example 11

Evaluate: (a) $\int_1^2\frac{x}{1+x^2}\,dx$ (b) $\int_{\frac{\pi}{2}}^{\pi}\frac{\sin x}{1-\cos x}\,dx$ (c) $\int_0^1\frac{e^x}{1+e^x}\,dx$

Solution

(a) $$\int_1^2\frac{x}{1+x^2}\,dx=\frac{1}{2}\int_1^2\frac{2x}{1+x^2}\,dx$$
$$=\frac{1}{2}\left[\log_e(1+x^2)\right]_1^2$$
$$=\frac{1}{2}(\log_e 5-\log_e 2)$$
$$=\frac{1}{2}\log_e 2.5$$

(b) $$\int_{\frac{\pi}{2}}^{\pi}\frac{\sin x}{1-\cos x}\,dx=\left[\log_e(1-\cos x)\right]_{\frac{\pi}{2}}^{\pi}$$
$$=\log_e 2-\log_e 1$$
$$=\log_e 2$$

(c) $$\int_0^1\frac{e^x}{1+e^x}\,dx=\left[\log_e(1+e^x)\right]_0^1$$
$$=\log_e(1+e)-\log_e 2$$
$$=\log_e\frac{1+e}{2}$$

EXERCISE 4.3 INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

1 Find $f(x)$ and the domain of f for the following.

(a) $f'(x)=\frac{1}{x+1}$ (b) $f'(x)=\frac{2}{2x+1}$ (c) $f'(x)=\frac{x^2}{8-x^3}$ (d) $f'(x)=\frac{1}{2-4x}$

(e) $f'(x)=\frac{x+1}{x^2+2x+3}$ (f) $f'(x)=\frac{2x+5}{x}$ (g) $f'(x)=\frac{\cos x-\sin x}{\sin x+\cos x}$ (h) $f'(x)=\frac{x}{1+x^2}$

(i) $f'(x)=\frac{x}{(1+x^2)^2}$ (j) $f'(x)=\frac{x}{\sqrt{1+x^2}}$ (k) $f'(x)=\frac{1}{2x+5}$ (l) $f'(x)=\frac{1}{(2x+5)^2}$

(m) $f'(x)=\frac{2x+1}{2x-1}$ (n) $f'(x)=\tan x$ (o) $f'(x)=\frac{2\cos x}{1+\sin x}$ (p) $f'(x)=\frac{x^2+x+1}{x^2+1}$

(q) $f'(x)=\frac{x^2-5x+1}{x-2}$ (r) $f'(x)=\frac{x^3}{x+1}$ (s) $f'(x)=\frac{x+3}{x^2+6x-7}$ (t) $f'(x)=\cot x$

2 $\int\frac{x^3+2x^2+3x+2}{x^2+1}\,dx=\ldots$

A $\frac{x^2}{2}+2x+\log_e(x^2+1)+C$ B $\frac{x^2}{2}+2x+\tan^{-1}x+C$

C $\frac{x^2}{2}+2x+\log_e\sqrt{x^2+1}+C$ D $\frac{x^2}{2}+2x+\tan^{-1}\frac{x}{2}+C$

3 Evaluate:

(a) $\int_0^2\frac{dx}{x+1}$ (b) $\int_2^4\frac{3}{4x-2}\,dx$ (c) $\int_0^2\frac{2x+1}{x^2+x+1}\,dx$ (d) $\int_1^2\frac{2x+1}{2x-1}\,dx$

(e) $\int_0^3\frac{x^2+x+1}{x^2+1}\,dx$ (f) $\int_0^2\frac{x^3}{2x+1}\,dx$ (g) $\int_3^4\frac{x^2+3x+1}{x-2}\,dx$ (h) $\int_1^3\frac{3x}{x^2+1}\,dx$

(i) $\int_2^3 \frac{x^2}{x-1}\,dx$ (j) $\int_0^2 \frac{x^2-3x}{2x+1}\,dx$ (k) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x\,dx$ (l) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin\theta}{1-\cos\theta}\,d\theta$

(m) $\int_2^3 \frac{x}{\sqrt{1+x^2}}\,dx$ (n) $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$ (o) $\int_2^4 \frac{x^2+1}{x^2}\,dx$ (p) $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin\theta}\,d\theta$

(q) $\int_1^2 \left(x-\frac{1}{x^2}\right)^2 dx$ (r) $\int_0^3 \frac{2x}{x^2+9}\,dx$ (s) $\int_0^1 \frac{e^x}{1+e^x}\,dx$ (t) $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{2+\cos x}\,dx$

4 (a) Differentiate $y=\log_e\left(x+\sqrt{x^2-a^2}\right)$, $x>|a|$ with respect to x.

(b) Hence find $\int \frac{dx}{\sqrt{x^2-a^2}}$, $x>|a|$.

5 (a) Differentiate $y=\log_e\left(x+\sqrt{x^2+a^2}\right)$ with respect to x. (b) Hence find $\int \frac{dx}{\sqrt{x^2+a^2}}$.

6 Use the integrals in questions **4** and **5** to find the following.

(a) $\int \frac{dx}{\sqrt{x^2-1}}$ (b) $\int \frac{dx}{\sqrt{x^2+1}}$ (c) $\int \frac{dx}{\sqrt{x^2-4x+3}}$

(d) $\int \frac{dx}{\sqrt{x^2+6x+13}}$ (e) $\int \frac{dx}{\sqrt{x^2-5x+7}}$ (f) $\int \frac{dx}{\sqrt{x^2+x+1}}$

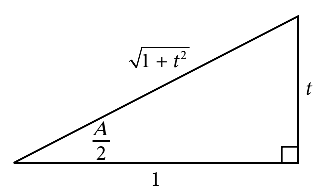
7 Evaluate:

(a) $\int_3^4 \frac{dx}{\sqrt{x^2-4}}$ (b) $\int_0^1 \frac{dx}{\sqrt{x^2+9}}$ (c) $\int_{-1}^1 \frac{dx}{\sqrt{x^2+8x+17}}$ (d) $\int_{2.5}^3 \frac{dx}{\sqrt{x^2-3x+2}}$

4.4 THE SUBSTITUTION $t=\tan\frac{A}{2}$

The substitution $t=\tan\frac{A}{2}$ enables you to express $\sin A$ and $\cos A$ in terms of t, which then enables you to express any rational function of $\sin A$ and $\cos A$ as a rational algebraic function of t. This then allows you to use standard techniques of integration such as partial fractions or integration by parts, covered in Chapter 5.

If $t=\tan\frac{A}{2}$, then it follows from the right-angled triangle that:



$$\sin\frac{A}{2}=\frac{t}{\sqrt{1+t^2}}$$

$$\cos\frac{A}{2}=\frac{1}{\sqrt{1+t^2}}$$

Now
$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

and
$$\cos A=\cos^2\frac{A}{2}-\sin^2\frac{A}{2} = \frac{1}{1+t^2}-\frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

As $t=\tan\frac{A}{2}$: $\frac{dt}{dA}=\frac{1}{2}\sec^2\frac{A}{2}=\frac{1+t^2}{2}$ $\quad\therefore \frac{dA}{dt}=\frac{2}{1+t^2}$

Summary—*t* formulae

- $t=\tan\frac{A}{2}$
- $\sin A=\frac{2t}{1+t^2}$
- $\cos A=\frac{1-t^2}{1+t^2}$
- $\frac{dA}{dt}=\frac{2}{1+t^2}$

Example 12

Find: $\displaystyle\int \frac{dx}{1+\cos x}$

Solution

Let $t = \tan\frac{x}{2}$ so that: $\quad 1+\cos x = 1+\frac{1-t^2}{1+t^2} = \frac{1+t^2+1-t^2}{1+t^2} = \frac{2}{1+t^2}$

Use $dx = \frac{2}{1+t^2}dt$: $\quad \displaystyle\int \frac{dx}{1+\cos x} = \int \frac{1+t^2}{2} \times \frac{2}{1+t^2}dt$

$$= \int dt$$

$$= t + C$$

$$= \tan\frac{x}{2} + C$$

Alternatively:

From the double-angle results: $\quad 2\cos^2\frac{x}{2} = 1+\cos x$

$$\therefore \int \frac{dx}{1+\cos x} = \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}} = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx = \tan\frac{x}{2} + C$$

Example 13

Evaluate: $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x}$

Solution

Let $t = \tan\frac{x}{2}$ so that: $\quad 2-\sin x = 2-\frac{2t}{1+t^2} = \frac{2(1-t+t^2)}{1+t^2}$

Use $dx = \frac{2}{1+t^2}dt$: $\quad \displaystyle\int \frac{dx}{2-\sin x} = \int \frac{1+t^2}{2(1-t+t^2)} \times \frac{2}{1+t^2}dt$

$$= \int \frac{dt}{t^2-t+1}$$

Complete the square: $\quad \displaystyle= \int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}$

$$= \int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

But $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$: $\quad \displaystyle= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

As $t = \tan\frac{x}{2}$: $\quad \displaystyle\int \frac{dx}{2-\sin x} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}\right) + C$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{dx}{2-\sin x} = \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2-1}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right)$$

$$= \frac{2\pi}{3\sqrt{3}}$$

EXERCISE 4.4 THE SUBSTITUTION $t = \tan\frac{A}{2}$

1 $\displaystyle\int \frac{dx}{1-\sin x} = \ldots$

A $\dfrac{2}{1-\tan\frac{x}{2}} + C$ **B** $\dfrac{2}{\tan\frac{x}{2}-1} + C$ **C** $\dfrac{1}{1-t} + C$ **D** $2\log_e\left(\tan\frac{x}{2}-1\right) + C$

2 Find: **(a)** $\displaystyle\int \frac{\sin x}{2-\cos x}\,dx$ **(b)** $\displaystyle\int \frac{dx}{3+2\cos x}$ **(c)** $\displaystyle\int \frac{dx}{1+\sin x}$

3 Evaluate: **(a)** $\displaystyle\int_0^{\frac{\pi}{3}} \sec x\,dx$ **(b)** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{2+\cos\theta}\,d\theta$ **(c)** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$

4 Find: **(a)** $\displaystyle\int \frac{dx}{5+4\cos x}$ **(b)** $\displaystyle\int \frac{\cos\theta}{2-\cos\theta}\,d\theta$ **(c)** $\displaystyle\int \frac{\sin\theta}{2+\sin\theta}\,d\theta$ **(d)** $\displaystyle\int \frac{dx}{3-\cos x}$

(e) $\displaystyle\int \frac{\cos x}{\sin x+1}\,dx$ **(f)** $\displaystyle\int \frac{dx}{1+\cos x}$ **(g)** $\displaystyle\int \frac{\tan x}{1+\cos x}\,dx$

5 Using an appropriate substitution of the type $t = \tan x$, find:

(a) $\displaystyle\int \frac{dx}{1+\sin 2x}$ **(b)** $\displaystyle\int \frac{\tan 2x}{1+\cos 2x}\,dx$

6 Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of:

(a) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{dx}{\sin x}$ **(b)** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$ **(c)** $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}\,dx$ **(d)** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3-\cos x-2\sin x}$

7 Use the substitution $t = \tan\frac{x}{2}$ to find $\displaystyle\int \frac{d\theta}{1+\cos\theta+\sin\theta}$.

8 Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{1+\cos x-\sin x}$.

9 Use the substitution $t = \tan\theta$ to find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \frac{d\theta}{2+\sin 2\theta}$.

10 Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{12\sin x-5\cos x+13}$.

CHAPTER REVIEW 4

1 Evaluate: (a) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$ (c) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$ (d) $\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx$

2 Find: (a) $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$ (b) $\int \frac{x^3+1}{x^2+1}\,dx$ (c) $\int \frac{x^3}{x^2+2x+1}\,dx$

3 Evaluate: (a) $\int_0^2 \frac{x}{(x^2+2)^2}\,dx$ (b) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,dx$ (c) $\int_0^1 \frac{e^x}{1+2e^x}\,dx$

(d) $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sin^2 x \cos^2 x\,dx$ (e) $\int_{\frac{3}{2}}^{4} \sqrt{2x+1}\,dx$

4 Evaluate: (a) $\int_0^4 \frac{dx}{x^2-2x+4}$ (b) $\int_0^{\frac{\pi}{4}} \cos^4\theta\,d\theta$ (c) $\int_{\frac{a}{2}}^{2} \frac{\sqrt{a^2-x^2}}{x^2}\,dx$

5 Evaluate: $\int_0^3 x^2\sqrt{9-x^2}\,dx$

6 Find:

(a) $\int \frac{1-4x^2}{x}\,dx$ (b) $\int (\sin x+\cos x)^2\,dx$ (c) $\int \sin^2 x\cos x\,dx$ (d) $\int \sin x\sec^2 x\,dx$

(e) $\int \frac{\sin^2 x}{\cos^2 x}\,dx$ (f) $\int \sin^2 x\cos^2 x\,dx$ (g) $\int \cos^2 x\,dx$ (h) $\int \cos^4 x\,dx$

7 Evaluate: (a) $\int_0^{\frac{1}{2}} x\sqrt{1-4x^2}\,dx$ (b) $\int_1^2 \frac{x}{1+4x^2}\,dx$ (c) $\int_{\frac{3}{5}}^{4} \frac{dx}{1+x^2}$

(d) $\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-\sin^2 x}\,dx$ (e) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ (f) $\int_0^{\frac{\pi}{2}} \sin^2 2x\,dx$

8 Find: (a) $\int \sin x\, e^{\cos x}\,dx$ (b) $\int x\sqrt{x-3}\,dx$ (c) $\int (x-3)\sqrt{x}\,dx$

(d) $\int \frac{x}{(2x^2-4)^3}\,dx$ (e) $\int \cos^2 x\sin x\,dx$ (f) $\int \cot x\,dx$ (g) $\int \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$

9 Find: (a) $\int \frac{\cos 2\theta}{\sin^2 2\theta}\,d\theta$ (b) $\int xe^{-x^2}\,dx$ (c) $\int \frac{2x}{x^2+1}\,dx$

(d) $\int x\cos(x^2)\,dx$ (e) $\int \sec^2 x\tan^2 x\,dx$

10 Evaluate: (a) $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ (b) $\int_0^{\frac{\pi}{2}} \sec^2\frac{\theta}{2}\,dx$ (c) $\int_0^1 \frac{x^2}{x^2+1}\,dx$

(d) $\int_1^2 \frac{3}{x^2+5x+4}\,dx$ (e) $\int_1^2 \frac{2x+5}{x^2+5x+4}\,dx$

11 Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

12 (a) Sketch the graph of the curve with equation: $y=\frac{x(3-x)}{x-1}$.

(b) Calculate the area of the region enclosed by the curve and the straight lines $x+3=0$, $x=0$ and $y=2-x$.

13 Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of $\int_0^{\frac{\pi}{2}} \frac{1}{3\sin x + 4\cos x + 5}\,dx$.

14 Sketch the graph of $y = \frac{\cos x}{\sin^2 x}$ for $0 < x < \pi$. The graph meets the x-axis at A and meets the line $y = \frac{2}{3}$ at B. Find the area enclosed between the arc AB and its chord.

15 Find the area of the region enclosed by the x-axis and the curve with equation $y = \frac{1-x^2}{1+x^2}$.

16 By integration, find the area of the ellipse $x = 2\cos\theta$, $y = 3\sin\theta$.

17 (a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$.

(b) Hence find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx$.

CHAPTER 5
Further integration

5.1 PARTIAL FRACTIONS, LINEAR FACTORS

You will now develop the technique of partial fractions. This will be applied for use in integrating certain rational functions. In this section you may benefit from working through the numerical examples before looking at the theory.

Identity of polynomial expressions

If two polynomials of the n-th degree are equal for more than n values of the variable then they are **identically equal**, i.e. equal for all values of the variable.

You use this result to find the numerator of partial fractions.

Example 1

Express $\frac{5x+1}{(x-1)(x+2)}$ in partial fractions.

Solution

Method 1

Let $\frac{5x+1}{(x-1)(x+2)} \equiv \frac{a}{x-1} + \frac{b}{x+2}$, $x \neq 1, -2$.

Write with common denominator: $\frac{5x+1}{(x-1)(x+2)} \equiv \frac{a(x+2)+b(x-1)}{(x-1)(x+2)}$, $x \neq 1, -2$

Write the numerators: $5x + 1 \equiv a(x + 2) + b(x - 1)$

Now use the identity property of polynomials and substitute two values for x. As we are now dealing only with the numerators, we can use $x = 1$ and $x = -2$ as the two values for x.

Let $x = 1$: $6 = 3a + 0$

$a = 2$

Let $x = -2$: $-9 = -3b$

$b = 3$

Hence: $\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$

Method 2

Write the numerators: $5x + 1 \equiv a(x + 2) + b(x - 1)$

Expand and simplify RHS: $5x + 1 \equiv (a + b)x + 2a - b$

Equate coefficients: $5 = a + b$ [1]

$1 = 2a - b$ [2]

[1] + [2]: $6 = 3a \quad \therefore a = 2$

Substitute into [1]: $5 = 2 + b \quad \therefore b = 3$

Hence: $\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$

Rational functions

A rational function $f(x)$ is the ratio of two polynomials, $f(x)=\frac{A(x)}{B(x)}$, defined for all values of x except those for which $B(x)=0$.

If the polynomial degree of $A(x) \geq$ degree of $B(x)$, then you can divide $B(x)$ into $A(x)$:

$A(x)=B(x)\times Q(x)+R(x)$ where degree of $R(x) <$ degree of $B(x)$

which leads to:

$$f(x)=Q(x)+\frac{R(x)}{B(x)}$$

Consider $f(x)=\frac{x^2+x-1}{x^2-1}$. By division you find $f(x)=1+\frac{x}{x^2-1}$, so that $Q(x)=1$, $R(x)=x$ and $B(x)=x^2-1$.

This means that $(x^2+x-1)=(x^2-1)\times 1+x$.

As $x^2-1=(x-1)(x+1)$, you have $B(x)=(x-1)(x+1)$.

The problem of partial fraction decomposition arises when $B(x)$ is a product of polynomials of lower degree, i.e. $B(x)=B_1(x)\times B_2(x)$ with degree $B_1(x)>0$, degree $B_2(x)>0$.

You wish to find polynomials $m_1(x)$, $m_2(x)$ such that: $\frac{R(x)}{B(x)}=\frac{m_1(x)}{B_1(x)}+\frac{m_2(x)}{B_2(x)}$

Now $m_1(x)$ and $m_2(x)$ can be found if: $R(x)\equiv m_1(x)\times B_2(x)+m_2(x)\times B_1(x)$

Comparison of degrees shows that you can suppose degree $m_1(x) <$ degree $B_1(x)$, degree $m_2(x) <$ degree $B_2(x)$.

Considering $\frac{R(x)}{B(x)}=\frac{x}{(x-1)(x+1)}=\frac{m_1(x)}{(x-1)}+\frac{m_2(x)}{(x+1)}$ you can write $m_1(x)=a$ and $m_2(x)=b$. These must be constants: the degree of the denominator is one, so the degree of the numerator must be zero.

Hence $\frac{R(x)}{B(x)}=\frac{x}{(x-1)(x+1)}=\frac{a}{(x-1)}+\frac{b}{(x+1)}$ and you now need to find the values of a and b.

You have: $x\equiv a(x+1)+b(x-1)$

i.e. $x\equiv(a+b)x+a-b$

Equate coefficients: $(a+b)=1$ and $a-b=0$, $\therefore\ a=b$

$\therefore$ $2a=1$, $a=0.5$ and $b=0.5$

Hence: $\frac{R(x)}{B(x)}=\frac{x}{(x-1)(x+1)}=\frac{0.5}{(x-1)}+\frac{0.5}{(x+1)}$

Thus you can write $f(x)=1+\frac{x}{x^2-1}$ as: $f(x)=1+\frac{0.5}{x-1}+\frac{0.5}{x+1}$

Linear factors

Consider the general case, where $B(x)$ is a product of distinct linear factors:

$B(x)=k(x-a_1)(x-a_2)\ldots(x-a_n)$

You want to discover if constants $c_1, c_2, \ldots c_n$ exist so that: $\frac{R(x)}{B(x)}=\frac{c_1}{x-a_1}+\frac{c_2}{x-a_2}+\ldots+\frac{c_n}{x-a_n}$

Linear factors—Method 1 (equating coefficients)

Consider the monic case when $n=2$, so that $k=1$ and $B(x)$ is a quadratic.

$R(x)=dx+e$ and $B(x)=(x-a_1)(x-a_2)$: $\frac{R(x)}{B(x)}=\frac{dx+e}{(x-a_1)(x-a_2)}=\frac{c_1}{x-a_1}+\frac{c_2}{x-a_2}$

Write with common denominator: $\frac{dx+e}{(x-a_1)(x-a_2)}=\frac{c_1(x-a_2)+c_2(x-a_1)}{(x-a_1)(x-a_2)}$

Write the numerators: $dx+e\equiv c_1(x-a_2)+c_2(x-a_1)$

Using identity property of polynomials, equate coefficients: $d=c_1+c_2$

$e=-(a_2c_1+a_1c_2)$

As $a_1 \neq a_2$, these two equations can be solved for c_1 and c_2.
In the general case, $a_1, a_2, \ldots a_n$ are distinct, so the coefficient equations can be solved for $c_1, c_2, \ldots c_n$.
This is best demonstrated with a numerical example.

Example 2

Reduce $\frac{x+1}{(x-2)(x-3)}$ to its partial fractions using linear factors method 1 (equating coefficients).

Solution

Let $\frac{x+1}{(x-2)(x-3)} = \frac{c_1}{x-2} + \frac{c_2}{x-3}$

Write with common denominator: $\frac{x+1}{(x-2)(x-3)} = \frac{c_1(x-3)+c_2(x-2)}{(x-2)(x-3)}$

Write the numerators: $x+1 \equiv (c_1+c_2)x - (3c_1+2c_2)$

Using identity property, equate coefficients: $c_1 + c_2 = 1$ [1]

$3c_1 + 2c_2 = -1$ [2]

$2 \times [1]$: $2c_1 + 2c_2 = 2$ [3]

$[2] - [3]$: $c_1 = -3$

Substitute into [1]: $c_2 = 4$

Hence: $\frac{x+1}{(x-2)(x-3)} = \frac{4}{x-3} - \frac{3}{x-2}$

Linear factors—Method 2 (substitution)

Again consider the monic case when $n = 2$, so that $k = 1$ and $B(x)$ is a quadratic.

$R(x) = dx + e$ and $B(x) = (x-a_1)(x-a_2)$: $\frac{R(x)}{B(x)} = \frac{dx+e}{(x-a_1)(x-a_2)} = \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2}$

Write with common denominator: $\frac{dx+e}{(x-a_1)(x-a_2)} = \frac{c_1(x-a_2)+c_2(x-a_1)}{(x-a_1)(x-a_2)}$

Write the numerators: $dx + e \equiv c_1(x-a_2) + c_2(x-a_1)$

Let $x = a_1$: $da_1 + e = c_1(a_1 - a_2)$

Solve for c_1: $c_1 = \frac{da_1+e}{a_1-a_2}$

Let $x = a_2$: $da_2 + e = c_2(a_2 - a_1)$

Solve for c_2: $c_2 = \frac{da_2+e}{a_2-a_1}$

You should always check your answer by writing the partial fractions over a common denominator again.

Example 3

Reduce $\frac{2x+1}{(x-2)(x+3)}$ to partial fractions using linear factors method 2 (substitution).

Solution

Let $\frac{2x+1}{(x-2)(x+3)} = \frac{c_1}{x-2} + \frac{c_2}{x+3}$

Write with common denominator: $\frac{2x+1}{(x-2)(x+3)} = \frac{c_1(x+3)+c_2(x-2)}{(x-2)(x+3)}$

Write the numerators: $2x+1 \equiv c_1(x+3) + c_2(x-2)$

Let $x = 2$: $5 = 5c_1 + 0$

$c_1 = 1$

Let $x = -3$: $\quad -5 = 0 - 5c_2$

$c_2 = 1$

Hence: $\dfrac{2x+1}{(x-2)(x+3)} = \dfrac{1}{x-2} + \dfrac{1}{x+3}$

Linear factors—Method 3 (limits)

Multiply through the expression $\dfrac{R(x)}{B(x)}$ and note that $B(a_1) = 0$. Hence $B(x) = B(x) - B(a_1)$.

Thus $\dfrac{R(x)}{B(x)} = \dfrac{c_1}{x-a_1} + \dfrac{c_2}{x-a_2} + \ldots + \dfrac{c_n}{x-a_n}$ becomes: $\dfrac{R(x)(x-a_1)}{B(x)-B(a_1)} = c_1 + \dfrac{c_2(x-a_1)}{x-a_2} + \ldots + \dfrac{c_n(x-a_1)}{x-a_n}$

Let $x \to a_1$ and recall that $\lim\limits_{x\to a_1} \dfrac{x-a_1}{B(x)-B(a_1)} = \dfrac{1}{B'(a_1)}$ (from the definition of differentiation from first principles).

You have: $\quad$ RHS $\to c_1$

LHS $\to \dfrac{R(a_1)}{B'(a_1)}$

Repeating this process for each of the other linear factors allows the other c_i to be found.
Remember that $B(x) = k(x-a_1)(x-a_2)\ldots(x-a_n)$, i.e. each factor is monic.

Example 4

Reduce $\dfrac{3x-4}{(x+2)(x-3)}$ to its partial fractions using linear factors method 3 (limits).

Solution

Let $\dfrac{3x-4}{(x+2)(x-3)} = \dfrac{c_1}{x+2} + \dfrac{c_2}{x-3}$

$R(x) = 3x - 4 \qquad B(x) = x^2 - x - 6 \qquad B'(x) = 2x - 1$

$c_1 = \dfrac{R(-2)}{B'(-2)} = \dfrac{-10}{-4-1} = 2 \qquad c_2 = \dfrac{R(3)}{B'(3)} = \dfrac{5}{6-1} = 1$

Hence: $\dfrac{3x-4}{(x+2)(x-3)} = \dfrac{2}{x+2} + \dfrac{1}{x-3}$

You should try using each of these methods to find partial fractions. You may find that you end up preferring one of the methods, but it is best to be able to use whichever is the most efficient method for a question.

Example 5

Reduce $\dfrac{3x}{(2x-1)(x+1)}$ to its partial fractions.

Solution

Let $\dfrac{3x}{(2x-1)(x+1)} = \dfrac{a}{2x-1} + \dfrac{b}{x+1}$

Write with common denominator: $\quad \dfrac{3x}{(2x-1)(x+1)} = \dfrac{a(x+1)+b(2x-1)}{(2x-1)(x+1)}$

Write the numerators: $\quad 3x \equiv a(x+1) + b(2x-1)$

To find a, let $x = \dfrac{1}{2}$: $\quad \dfrac{3}{2} = a \times \dfrac{3}{2} + 0$

$a = 1$

To find b, let $x = -1$: $\quad -3 = 0 + b \times (-3)$

$b = 1$

Hence: $\dfrac{3x}{(2x-1)(x+1)} = \dfrac{1}{2x-1} + \dfrac{1}{x+1}$

Example 6

Express $\dfrac{54}{(x^2+x-20)(x-1)}$ using partial fractions.

Solution

Factorise the quadratic expression: $x^2+x-20=(x+5)(x-4)$

$$\therefore \quad \frac{54}{(x+5)(x-4)(x-1)}=\frac{c_1}{x+5}+\frac{c_2}{x-4}+\frac{c_3}{x-1}$$

Write with common denominator: $\dfrac{54}{(x+5)(x-4)(x-1)}=\dfrac{c_1(x-4)(x-1)+c_2(x+5)(x-1)+c_3(x+5)(x-4)}{(x+5)(x-4)(x-1)}$

Write the numerators: $54 \equiv c_1(x-4)(x-1)+c_2(x+5)(x-1)+c_3(x+5)(x-4)$

Let $x=-5$: $54=54c_1 \quad \therefore c_1=1$

Let $x=4$: $54=27c_2 \quad \therefore c_2=2$

Let $x=1$: $54=-18c_3 \quad \therefore c_3=-3$

Hence: $\dfrac{54}{(x^2+x-20)(x-1)}=\dfrac{1}{x+5}+\dfrac{2}{x-4}-\dfrac{3}{x-1}$

Example 7

Reduce $\dfrac{x^2}{x^2+3x+2}$ to its partial fractions.

Solution

As degree of numerator = degree of denominator, first divide by the denominator.

As a quicker way of doing the division, consider rewriting:

$$\frac{x^2}{x^2+3x+2}=\frac{(x^2+3x+2)-(3x+2)}{x^2+3x+2}=1-\frac{3x+2}{x^2+3x+2}$$

We now need to find c_1 and c_2 for $\dfrac{3x+2}{x^2+3x+2}=\dfrac{c_1}{x+1}+\dfrac{c_2}{x+2}$.

$R(x)=3x+2 \qquad B(x)=x^2+3x+2 \qquad B'(x)=2x+3$

$x=-1$: $c_1=\dfrac{R(-1)}{B'(-1)}=\dfrac{-3+2}{-2+3}=-1 \qquad x=-2$: $c_2=\dfrac{R(-2)}{B'(-2)}=\dfrac{-6+2}{-4+3}=4$

Hence: $\dfrac{3x+2}{x^2+3x+2}=\dfrac{-1}{x+1}+\dfrac{4}{x+2} \quad \therefore \dfrac{x^2}{x^2+3x+2}=1+\dfrac{1}{x+1}-\dfrac{4}{x+2}$

EXERCISE 5.1 PARTIAL FRACTIONS, LINEAR FACTORS

1 If $\dfrac{2x-11}{(x+2)(x-1)}=\dfrac{a}{x+2}+\dfrac{b}{x-1}$ then the values of a and b are:

A $a=-5, b=3$ **B** $a=-5, b=-3$ **C** $a=5, b=3$ **D** $a=5, b=-3$

2 Reduce each rational function to its partial fractions.

(a) $\dfrac{4}{(x-1)(x+3)}$ **(b)** $\dfrac{2x-1}{(x+2)(x-3)}$ **(c)** $\dfrac{3x+1}{x(x+1)}$ **(d)** $\dfrac{2x^2-6x-2}{x(x-1)(x+2)}$

3 Reduce each rational function to its partial fractions.

(a) $\dfrac{3}{x^2+x-2}$ (b) $\dfrac{3x-1}{x^2+x-6}$ (c) $\dfrac{x+10}{x^2-4}$ (d) $\dfrac{18x-12}{x(x^2+x-12)}$

4 $\dfrac{x^2}{x^2+3x+2}$ is equal to:

A $1+\dfrac{4}{x+2}+\dfrac{1}{x+1}$ B $1-\dfrac{4}{x+2}-\dfrac{1}{x+1}$ C $1-\dfrac{4}{x+2}+\dfrac{1}{x+1}$ D $1+\dfrac{4}{x+2}-\dfrac{1}{x+1}$

5 Reduce each rational function to its partial fractions.

(a) $\dfrac{x^2}{(x+1)(x-2)}$ (b) $\dfrac{x^2+2}{x^2-6x+8}$ (c) $\dfrac{x^2}{x^2-9}$ (d) $\dfrac{x^2}{x(x^2-x-12)}$

6 Reduce each rational function to its partial fractions.

(a) $\dfrac{3x-19}{(x+3)(2x-1)}$ (b) $\dfrac{5x}{x^2+x-6}$ (c) $\dfrac{15x+28}{3x^2+25x+8}$ (d) $\dfrac{x^2-19x-32}{(x+1)(x+2)(x-3)}$

(e) $\dfrac{3(3x+1)}{x^2-9}$ (f) $\dfrac{1-2x}{2x^2+7x+6}$ (g) $\dfrac{2x^2+x+6}{x^2-4}$ (h) $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

7 Reduce $\dfrac{5x^2+26x+29}{x^3+6x^2+11x+6}$ to its partial fractions.

5.2 PARTIAL FRACTIONS, QUADRATIC FACTORS

Consider the case when $B(x)$ is a product of distinct linear factors and a simple quadratic factor, so that the decomposition is of the form: $\dfrac{R(x)}{B(x)}=\dfrac{c_1}{x-a_1}+\dfrac{c_2}{x-a_2}+\ldots+\dfrac{c_n}{x-a_n}+\dfrac{dx+e}{x^2+bx+c}$, with degree of $R(x)<n+2$ (as the numerator has a degree at least one less than the denominator).

As for partial fractions with linear factors, there are many different methods that may be used.

Example 8

Reduce $\dfrac{13}{(x-3)(x^2+4)}$ to its partial fractions.

Solution

Let $\dfrac{13}{(x-3)(x^2+4)}=\dfrac{c}{x-3}+\dfrac{dx+e}{x^2+4}$

$R(x)=13$ $B(x)=x^3-3x^2+4x-12$ $B'(x)=3x^2-6x+4$

For c, let $x=3$: $c=\dfrac{R(3)}{B'(3)}=\dfrac{13}{27-18+4}=1$

For e, let $x=0$: $\dfrac{13}{(-3)(4)}=\dfrac{1}{(-3)}+\dfrac{e}{4}$ $\therefore\ e=-3$

Hence: $\dfrac{13}{(x-3)(x^2+4)}=\dfrac{1}{x-3}+\dfrac{dx-3}{x^2+4}$

For d, multiply through by x: $\dfrac{13x}{(x-3)(x^2+4)}=\dfrac{x}{x-3}+\dfrac{dx^2-3x}{x^2+4}$

Divide by largest power of x in each numerator: $\dfrac{13}{\left(1-\frac{3}{x}\right)\left(x^2+4\right)}=\dfrac{1}{1-\frac{3}{x}}+\dfrac{d-\frac{3}{x}}{1+\frac{4}{x^2}}$

Find $\lim_{x\to\infty}$ of both sides: $\frac{13}{\infty} = \frac{1}{1} + \frac{d}{1}$

$$0 = 1 + d$$
$$d = -1$$

Thus: $\frac{13}{(x-3)(x^2+4)} = \frac{1}{x-3} - \frac{x+3}{x^2+4}$

The process behind the method of Example 8 is as follows:

$B(x)$ is a product of distinct linear factors and a simple quadratic factor, such that:
$\frac{R(x)}{B(x)} = \frac{c_1}{x-a_1} + \frac{c_2}{x-a_2} + \ldots + \frac{c_n}{x-a_n} + \frac{dx+e}{x^2+bx+c}$, with degree $R(x) < n+2$. As for partial fractions with linear factors, here the numbers $c_1, c_2, \ldots c_n, d, e$ can be found by comparing coefficients or by combining that method with others.

In other words, $c_1, c_2, \ldots c_n$ can be found using any of the previous methods for partial fractions with linear factors. You then only need to find d and e as follows.

- If none of $a_1, a_2, \ldots a_n$ is zero, then let $x = 0$ to find the value of e.
- Multiply through by x and let $x \to \infty$ (which will require dividing each fraction by the largest power of x in the numerators) to find the value of d.
- If e.g. $a_1 = 0$, first find d and then select a small integer value for x distinct from $a_1, a_2, \ldots a_n$ to give a simple equation that can be solved for e.

Example 9

Express $\frac{x^2+6x+5}{(x-2)(x^2+x+1)}$ using partial fractions.

Solution

x^2+x+1 has no real linear factors; degree of numerator < degree of denominator.

Let $\frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{c}{x-2} + \frac{dx+e}{x^2+x+1}$

Write with common denominator: $\frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{c(x^2+x+1)+(dx+e)(x-2)}{(x-2)(x^2+x+1)}$

Write the numerators: $x^2+6x+5 \equiv c(x^2+x+1) + (dx+e)(x-2)$ [1]

Let $x = 2$ in [1]: $21 = 7c$ $\therefore c = 3$

Let $x = 0$ in [1]: $5 = 3 - 2e$ $\therefore e = -1$

Equate coefficients of x^2 in [1]: $1 = c + d$

$\therefore$ $1 = 3 + d$ $\therefore d = -2$

Hence: $\frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{3}{x-2} - \frac{2x+1}{x^2+x+1}$

Example 10

Reduce $\frac{2x+7}{x(x^2+3x+1)}$ to its partial fractions.

Solution

x^2+3x+1 has no real linear factors.

Let $\frac{2x+7}{x(x^2+3x+1)} = \frac{c}{x} + \frac{dx+e}{x^2+3x+1}$ [1]

$R(x) = 2x + 7 \qquad B(x) = x^3 + 3x^2 + x \qquad B'(x) = 3x^2 + 6x + 1$

For c, let $x = 0$: $\quad c = \dfrac{R(0)}{B'(0)} = \dfrac{7}{1} = 7$

Write [1] with common denominator: $\quad \dfrac{2x+7}{x(x^2+3x+1)} = \dfrac{7(x^2+3x+1)+x(dx+e)}{x(x^2+3x+1)}$

Write the numerators: $\quad 2x + 7 \equiv 7(x^2 + 3x + 1) + x(dx + e)$

Let $x = 1$: $\quad 9 = 35 + d + e$ [2]

Let $x = -1$: $\quad 5 = -7 + d - e$ [3]

[2] + [3]: $\quad 14 = 28 + 2d \quad \therefore d = -7$

Substitute into [2]: $\quad -26 = -7 + e \quad \therefore e = -19$

Hence: $\dfrac{2x+7}{x(x^2+3x+1)} = \dfrac{7}{x} - \dfrac{7x+19}{x^2+3x+1}$

Example 11

Reduce $\dfrac{7}{(x^2+9)(x^2+16)}$ to its real partial fractions.

Solution

Neither $x^2 + 9$ nor $x^2 + 16$ have real linear factors.

Let $\dfrac{7}{(x^2+9)(x^2+16)} = \dfrac{cx+d}{x^2+9} + \dfrac{ex+f}{x^2+16}$

Equate numerators: $\quad 7 \equiv (cx+d)(x^2+16) + (ex+f)(x^2+9)$ [1]

Let $x = 0$ in [1]: $\quad 7 = 16d + 9f$ [2]

Let $x = 1$ in [1]: $\quad 7 = 17c + 17d + 10e + 10f$ [3]

Let $x = -1$ in [1]: $\quad 7 = -17c + 17d - 10e + 10f$ [4]

Let $x = 2$ in [1]: $\quad 7 = 40c + 20d + 26e + 13f$ [5]

[3] + [4]: $\quad 14 = 34d + 20f$

$\quad 7 = 17d + 10f$ [6]

[6] − [2]: $\quad 0 = d + f \quad \therefore f = -d$ [7]

Substitute [7] into [2]: $\quad 7 = 16d - 9d \quad \therefore d = 1 \quad \therefore f = -1$

[3] − [4]: $\quad 0 = 34c + 20e \quad \therefore e = -1.7c$ [8]

Substitute [8] into [5]: $\quad 7 = 40c + 20 - 44.2c - 13 \quad \therefore c = 0 \quad \therefore e = 0$

Hence: $\dfrac{7}{(x^2+9)(x^2+16)} = \dfrac{1}{x^2+9} - \dfrac{1}{x^2+16}$

With experience you will notice that $(x^2 + 16) - (x^2 + 9) = 7$, which is the numerator of the given fraction. This can make finding the partial fraction numerators easier with quadratic denominators of this type and a numerical numerator.

For fractions with quadratic denominators as in the example above, you can write:

$\dfrac{k}{(x^2+l)(x^2+m)} = \dfrac{a}{x^2+l} + \dfrac{b}{x^2+m}$ where a, b, k, l and m are constants.

This can be done because the original numerator (k) has no terms in x.

Equating numerators: $\quad k = (a + b)x^2 + (am + bl)$

Hence $a = -b$ and $am + bl = k$. This means that a and b must have the same magnitude but opposite sign, so that:

$\dfrac{k}{(x^2+l)(x^2+m)} = \dfrac{a}{x^2+l} - \dfrac{a}{x^2+m}$ where $a = \dfrac{k}{m-l}$

Example 12

Reduce $\frac{x^2-6}{(x^2+9)(x^2+4)}$ to its real partial fractions.

Solution

Neither x^2+9 nor x^2+4 have real linear factors.

Because the numerator does not contain an odd power of x, it seems reasonable to assume that the numerator of each partial fraction will not contain a term with an odd power of x (as was seen in the previous example). Try to use only c and d as the numerators of the partial fractions.

Let $\frac{x^2-6}{(x^2+9)(x^2+4)} = \frac{c}{x^2+9} + \frac{d}{x^2+4}$

$\therefore \quad \frac{x^2-6}{(x^2+9)(x^2+4)} = \frac{c(x^2+4)+d(x^2+9)}{(x^2+9)(x^2+4)}$

Equate numerators: $x^2-6 \equiv (c+d)x^2+4c+9d$

$\therefore \quad 1 = c+d$ [1]

$-6 = 4c+9d$ [2]

Rewrite [1]: $c = 1-d$ [3]

Substitute into [2]: $-6 = 4-4d+9d$

$5d = -10$

$d = -2$

Substitute into [3]: $c = 3$

Hence: $\frac{x^2-6}{(x^2+9)(x^2+4)} = \frac{3}{x^2+9} - \frac{2}{x^2+4}$

When finding partial fractions for pairs of quadratic denominators like (x^2+a^2) and (x^2+b^2) and the numerator does not have odd powers of x, Examples **11** and **12** show that you can write the first step as:

$\frac{cx^2+d}{(x^2+a^2)(x^2+b^2)} = \frac{e}{x^2+a^2} + \frac{f}{x^2+b^2}$ where either c or d could be zero.

It is an interesting task to show this algebraically.

Repeated linear factors

This Mathematics Extension 2 course does not require you to find partial fractions with repeated linear factors, as in a question such as 'Reduce $\frac{1}{(x-1)(x-2)^2}$ to its linear factors'. However, you may be asked to show a repeated factors result such as in the next example.

Example 13

Show that: $\frac{1}{(x-1)(x-2)^2} = \frac{1}{x-1} - \frac{1}{x-2} + \frac{1}{(x-2)^2}$

Solution

$$\text{RHS} = \frac{1}{x-1} - \frac{1}{x-2} + \frac{1}{(x-2)^2}$$

$$= \frac{(x-2)^2-(x-1)(x-2)+(x-1)}{(x-1)(x-2)^2}$$

$$= \frac{x^2 - 4x + 4 - (x^2 - 3x + 2) + x - 1}{(x-1)(x-2)^2}$$

$$= \frac{1}{(x-1)(x-2)^2} = \text{LHS}$$

EXERCISE 5.2 PARTIAL FRACTIONS, QUADRATIC FACTORS

1 If $\dfrac{x^2+2x+2}{(x-1)(x^2+4)} = \dfrac{a}{x-1} + \dfrac{b}{x^2+4}$, then the values of a and b are:

A $a = 1, b = -2$ **B** $a = -1, b = -2$ **C** $a = 1, b = 2$ **D** $a = -1, b = 2$

2 Reduce each rational function to its partial fractions.

(a) $\dfrac{3}{(x^2+1)(x^2+4)}$ **(b)** $\dfrac{8}{(x^2+1)(x^2+9)}$ **(c)** $\dfrac{3x^2}{(x^2+1)(x^2+4)}$ **(d)** $\dfrac{2x^2+5}{(x^2+1)(x^2+2)}$

3 Reduce each rational function to its partial fractions.

(a) $\dfrac{x^2-8x+2}{(x-2)(x^2+1)}$ **(b)** $\dfrac{x^2+4x+1}{(x+2)(x^2+x+1)}$ **(c)** $\dfrac{2x^3+7x^2-4x+3}{x^4-1}$ **(d)** $\dfrac{3x^2+5x+4}{x^3+x^2+x}$

4 $\dfrac{4x^2-2x}{(x+1)(x^2+1)}$ is equal to:

A $\dfrac{x-3}{x^2+1} - \dfrac{3}{x+1}$ **B** $\dfrac{3}{x+1} + \dfrac{x-3}{x^2+1}$ **C** $\dfrac{3}{x+1} + \dfrac{3-x}{x^2+1}$ **D** $\dfrac{3-x}{x^2+1} - \dfrac{3}{x+1}$

5 Reduce each rational function to its partial fractions.

(a) $\dfrac{x^2+9}{(x+2)(x^2-2x+5)}$ **(b)** $\dfrac{1+2x+3x^2-4x^3}{1-x^4}$ **(c)** $\dfrac{x^4-x^2-2x+2}{(x+1)(x^2+1)}$

(d) $\dfrac{x^4+3x^2-8x}{x^3-8}$ **(e)** $\dfrac{x^2+1}{x^3-1}$ **(f)** $\dfrac{x+3}{(2x+1)(x^2+1)}$

6 Find the values of a and b if:

(a) $\dfrac{4x-7}{(x-2)^2} = \dfrac{a}{x-2} + \dfrac{b}{(x-2)^2}$ **(b)** $\dfrac{3x+5}{x^2+6x+9} = \dfrac{a}{x+3} + \dfrac{b}{(x+3)^2}$

(c) $\dfrac{3x^2+10x-5}{(x-2)(x+1)^2} = \dfrac{a}{x-2} + \dfrac{b}{(x+1)^2}$ **(d)** $\dfrac{18x^2-7x+12}{(x+2)(3x-1)^2} = \dfrac{a}{x+2} + \dfrac{b}{(3x-1)^2}$

7 Show that each result is true.

(a) $\dfrac{42-4x-5x^2}{(2x+3)(x-2)^2} = \dfrac{3}{2x+3} - \dfrac{4}{x-2} + \dfrac{2}{(x-2)^2}$ **(b)** $\dfrac{2x^2+13x+16}{(x-2)(2x+1)^2} = \dfrac{2}{x-2} - \dfrac{3}{2x+1} - \dfrac{4}{(2x+1)^2}$

8 Find the values of a, b and c if:

(a) $\dfrac{2x^2+19x-36}{(x+3)(x^2-4x+4)} = \dfrac{a}{x+3} + \dfrac{b}{x-2} + \dfrac{c}{(x-2)^2}$

(b) $\dfrac{4x}{(1+x)(1-x)^2} = \dfrac{a}{1+x} + \dfrac{b}{1-x} + \dfrac{c}{(1-x)^2}$

5.3 USING PARTIAL FRACTIONS TO FIND INTEGRALS

You are now going to use this technique to find and evaluate integrals.

Example 14

Find: (a) $\displaystyle\int \frac{dx}{(x-2)(x-1)}$ (b) $\displaystyle\int \frac{2x-1}{(x+2)(x-3)}dx$

Solution

(a) Partial fractions: $\dfrac{1}{(x-2)(x-1)} = \dfrac{a}{x-2} + \dfrac{b}{x-1} = \dfrac{a(x-1)+b(x-2)}{(x-2)(x-1)}$

Equate numerators: $1 \equiv a(x-1) + b(x-2)$

Let $x = 1$: $1 = -b \quad \therefore b = -1$

Let $x = 2$: $a = 1$

Hence:
$$\int \frac{dx}{(x-2)(x-1)} = \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx$$
$$= \log_e(x-2) - \log_e(x-1) + C$$
$$= \log_e\left|\frac{x-2}{x-1}\right| + C$$

(b) Partial fractions: $\dfrac{2x-1}{(x+2)(x-3)} = \dfrac{a}{x+2} + \dfrac{b}{x-3} = \dfrac{a(x-3)+b(x+2)}{(x+2)(x-3)}$

Equate coefficients: $2 = a + b$ [1]

$-1 = -3a + 2b$ [2]

Rewrite [1]: $b = 2 - a$

Substitute [1] into [2]: $-1 = -3a + 4 - 2a$

$5a = 5 \quad \therefore a = 1$

Substitute into [1]: $b = 1$

Hence:
$$\int \frac{2x-1}{(x+2)(x-3)}dx = \int \left(\frac{1}{x+2} + \frac{1}{x-3}\right)dx$$
$$= \log_e(x+2) + \log_e(x-3) + C$$
$$= \log_e(x+2)(x-3) + C \qquad (x > 3,\ x < -1)$$
$$= \log_e|(x+2)(x-3)| + C$$

Example 15

Find: (a) $\displaystyle\int \frac{1-2x}{(x+2)(x^2+1)}dx$ (b) $\displaystyle\int \frac{x^2+4x}{(x-1)(x^2+4)}dx$

Solution

(a) Partial fractions: $\dfrac{1-2x}{(x+2)(x^2+1)} = \dfrac{a}{x+2} + \dfrac{bx+c}{x^2+1} = \dfrac{a(x^2+1)+(bx+c)(x+2)}{(x+2)(x^2+1)}$

$$= \frac{ax^2 + a + bx^2 + 2bx + cx + 2c}{(x+2)(x^2+1)}$$
$$= \frac{(a+b)x^2 + (2b+c)x + a + 2c}{(x+2)(x^2+1)}$$

Equate coefficients:	$0 = a + b$	[1]
	$-2 = 2b + c$	[2]
	$1 = a + 2c$	[3]
Rewrite [1] and [2]:	$b = -a, \quad c = -2 - 2b$	
$\therefore$	$c = -2 + 2a$	
Substitute into [3]:	$1 = a - 4 + 4a$	
	$a = 1 \quad \therefore b = -1, \quad c = 0$	

Hence:
$$\int \frac{1-2x}{(x+2)(x^2+1)}\,dx = \int \left(\frac{1}{x+2} - \frac{x}{x^2+1}\right) dx$$
$$= \log_e(x+2) - \frac{1}{2}\log_e(x^2+1) + C$$

(b) Partial fractions:
$$\frac{x^2+4x}{(x-1)(x^2+4)} = \frac{a}{x-1} + \frac{bx+c}{x^2+4} = \frac{a(x^2+4)+(bx+c)(x-1)}{(x-1)(x^2+4)}$$
$$= \frac{(a+b)x^2+(c-b)x+4a-c}{(x-1)(x^2+4)}$$

Equate coefficients:	$1 = a + b$	[1]
	$4 = c - b$	[2]
	$0 = 4a - c$	[3]
Rewrite [2] and [3]:	$b = c - 4, \; c = 4a$	
$\therefore$	$b = 4a - 4$	
Substitute into [1]:	$1 = a + 4a - 4$	
	$a = 1 \quad \therefore b = 0, \quad c = 4$	

Hence:
$$\int \frac{x^2+4x}{(x-1)(x^2+4)}\,dx = \int \left(\frac{1}{x-1} + \frac{4}{x^2+4}\right) dx$$
$$= \log_e(x-1) + 2\tan^{-1}\frac{x}{2} + C$$

Recall that in Examples **11** and **12** (see pages 136–137), you found that an expression $\dfrac{cx^2+d}{(x^2+a^2)(x^2+b^2)}$ (where either of c or d can be zero, but the numerator does not contain odd powers of x) can be written in partial fractions as: $\dfrac{cx^2+d}{(x^2+a^2)(x^2+b^2)} = \dfrac{e}{x^2+a^2} + \dfrac{f}{x^2+b^2}$.

Example 16

Find: **(a)** $\displaystyle\int \frac{5}{(x^2+4)(x^2+9)}\,dx$ **(b)** $\displaystyle\int \frac{3x^2}{(x^2+1)(x^2+4)}\,dx$

Solution

(a) Partial fractions:
$$\frac{5}{(x^2+4)(x^2+9)} = \frac{a}{x^2+4} + \frac{b}{x^2+9} = \frac{a(x^2+9)+b(x^2+4)}{(x^2+4)(x^2+9)}$$

Equate coefficients:	$0 = a + b$	[1]
	$5 = 9a + 4b$	[2]
Rewrite [1]:	$b = -a$	
Substitute into [2]:	$5 = 9a - 4a$	
	$a = 1 \quad \therefore b = -1$	

Hence:
$$\int \frac{5}{(x^2+4)(x^2+9)}\,dx = \int \left(\frac{1}{x^2+4} - \frac{1}{x^2+9}\right) dx$$
$$= \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

(b) Partial fractions: $\dfrac{3x^2}{(x^2+1)(x^2+4)} = \dfrac{a}{x^2+1} + \dfrac{b}{x^2+4} = \dfrac{a(x^2+4)+b(x^2+1)}{(x^2+1)(x^2+4)}$

Equate coefficients: $3 = a + b$ [1]

$0 = 4a + b$ [2]

Rewrite [2]: $b = -4a$

Substitute into [1]: $3 = a - 4a$

$a = -1 \quad \therefore b = 4$

Hence: $\displaystyle\int \frac{3x^2}{(x^2+1)(x^2+4)}\,dx = \int\left(\frac{4}{x^2+4} - \frac{1}{x^2+1}\right)dx$

$= 2\tan^{-1}\dfrac{x}{2} - \tan^{-1}x + C$

Example 17

(a) If $\dfrac{54}{(x^2+x-20)(x-1)} = \dfrac{a}{x+5} + \dfrac{b}{x-4} + \dfrac{c}{x-1}$ then find the values of a, b and c and write the expression in terms of its partial fractions.

(b) Hence find: $\displaystyle\int \frac{54}{(x^2+x-20)(x-1)}\,dx$

Solution

(a) $54 \equiv a(x-4)(x-1) + b(x+5)(x-1) + c(x+5)(x-4)$

Let $x = 4$: $54 = 0 + b \times 9 \times 3 + 0$

$b = 2$

Let $x = 1$: $54 = 0 + 0 + c \times 6 \times (-3)$

$c = -3$

Let $x = -5$: $54 = a \times (-9) \times (-6)$

$a = 1$

Hence: $\dfrac{54}{(x^2+x-20)(x-1)} = \dfrac{1}{x+5} + \dfrac{2}{x-4} - \dfrac{3}{x-1}$

(b) $\displaystyle\int \frac{54}{(x^2+x-20)(x-1)}\,dx = \int\left(\frac{1}{x+5} + \frac{2}{x-4} - \frac{3}{x-1}\right)dx$

$= \log_e(x+5) + 2\log_e(x-4) - 3\log_e(x-1) + C$

$= \log_e\left|\dfrac{(x+5)(x-4)^2}{(x-1)^3}\right| + C$

Example 18

(a) If $\dfrac{3x^2-6x+2}{(x-1)^2(x-2)} = \dfrac{a}{x-1} + \dfrac{b}{(x-1)^2} + \dfrac{c}{x-2}$, then find the values of a, b and c.

(b) Hence evaluate: $\displaystyle\int_3^4 \frac{3x^2-6x+2}{(x-1)^2(x-2)}\,dx$

Solution

(a) $3x^2 - 6x + 2 \equiv a(x-1)(x-2) + b(x-2) + c(x-1)^2$

Let $x = 1$: $3 - 6 + 2 = b \times (-1)$

$b = 1$

Let $x = 2$: $\quad 12 - 12 + 2 = c$

$c = 2$

Let $x = 0$: $\quad 2 = 2a - 2b + c$

$2a = 2 + 2 - 2$

$a = 1$

(b) Hence:
$$\int_3^4 \frac{3x^2 - 6x + 2}{(x-1)^2(x-2)}\,dx = \int_3^4 \left(\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{x-2}\right)dx$$
$$= \left[\log_e(x-1) - \frac{1}{x-1} + 2\log_e(x-2)\right]_3^4$$
$$= \left(\log_e 3 - \frac{1}{3} + 2\log_e 2\right) - \left(\log_e 2 - \frac{1}{2} + 0\right)$$
$$= \frac{1}{6} + \log_e 6$$

Example 19

Find: $\displaystyle\int \frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4}\,dx$

Solution

Instead of using long division, it can be faster to regroup the terms in the numerator to produce factors of $(x^2 - 4)$:

$$\frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4} = \frac{x^3 - 4x - 2x^2 + 8 + x + 1}{x^2 - 4}$$
$$= \frac{x(x^2 - 4) - 2(x^2 - 4) + x + 1}{x^2 - 4}$$
$$= x - 2 + \frac{x+1}{(x-2)(x+2)}$$

$$\begin{array}{r|l} & x \quad -2 \\ \hline x^2 - 4 & x^3 - 2x^2 - 3x + 9 \\ & x^3 \qquad\quad -4x \\ \hline & -2x^2 + x + 9 \\ & -2x^2 \qquad + 8 \\ \hline & x + 1 \end{array}$$

Now let $\dfrac{x+1}{(x-2)(x+2)} = \dfrac{a}{x-2} + \dfrac{b}{x+2}$: $\qquad x + 1 \equiv a(x+2) + b(x-2)$

Let $x = -2$: $\quad -1 = -4b \quad$ so $b = \dfrac{1}{4}$

Let $x = 2$: $\quad 3 = 4a \quad$ so $a = \dfrac{3}{4}$

$$\therefore \frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4} = x - 2 + \frac{3}{4(x-2)} + \frac{1}{4(x+2)}$$

Hence:
$$\int \frac{x^3 - 2x^2 - 3x + 9}{x^2 - 4}\,dx = \int \left(x - 2 + \frac{3}{4(x-2)} + \frac{1}{4(x+2)}\right)dx$$
$$= \frac{x^2}{2} - 2x + \frac{3}{4}\log_e|x-2| + \frac{1}{4}\log_e|x+2| + C$$

Example 20

(a) If $\dfrac{3x^2 - 2x + 1}{(x^2+1)(x^2+2)} = \dfrac{ax+b}{x^2+1} + \dfrac{cx+d}{x^2+2}$ then find the values of a, b, c and d.

(b) Hence evaluate: $\displaystyle\int_0^1 \frac{3x^2 - 2x + 1}{(x^2+1)(x^2+2)}\,dx$

Solution

(a) $3x^2 - 2x + 1 \equiv (ax+b)(x^2+2) + (cx+d)(x^2+1)$

As there are four variables, select four values of x and substitute them into the identity.

Let $x = 0$:	$1 = 2b + d$	
	$d = 1 - 2b$	[1]
Let $x = 1$:	$2 = 3(a+b) + 2(c+d)$	[2]
Let $x = 2$:	$9 = 6(2a+b) + 5(2c+d)$	[3]
Let $x = -1$:	$6 = 3(b-a) + 2(d-c)$	[4]
[2] + [4]:	$8 = 6b + 4d$	
Substitute for d from [1]:	$8 = 6b + 4(1-2b)$	
	$8 = 4 - 2b$	
	$b = -2$	
$\therefore$	$d = 5$	
[3] becomes:	$9 = 12a - 12 + 10c + 25$	
	$6a + 5c = -2$	[5]
[2] − [4]:	$-4 = 6a + 4c$	
	$3a = -2 - 2c$	[6]
Substitute [6] into [5]:	$2(-2c-2) + 5c = -2$	
	$-4c - 4 + 5c = -2$	
	$c = 2$	
[6] becomes:	$3a = -4 - 2$	
	$a = -2$	

$$\therefore \frac{3x^2-2x+1}{(x^2+1)(x^2+2)} = \frac{-2x-2}{x^2+1} + \frac{2x+5}{x^2+2}$$

(b) Hence:

$$\int_0^1 \frac{3x^2-2x+1}{(x^2+1)(x^2+2)}\,dx = \int_0^1 \left(\frac{2x+5}{x^2+2} - \frac{2x+2}{x^2+1}\right)dx$$

$$= \int_0^1 \left(\frac{2x}{x^2+2} + \frac{5}{x^2+2} - \frac{2x}{x^2+1} - \frac{2}{x^2+1}\right)dx$$

$$= \left[\log_e(x^2+2) + \frac{5}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \log_e(x^2+1) - 2\tan^{-1}x\right]_0^1$$

$$= \left(\log_e 3 + \frac{5}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} - \log_e 2 - 2\tan^{-1}1\right) - \left(\log_e 2 + 0 - \log_e 1 - 0\right)$$

$$= \log_e \frac{3}{4} + \frac{5}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} - \frac{\pi}{2}$$

Example 21

Find: $\displaystyle\int \frac{d\theta}{\cos\theta}$

Solution

Let $t = \tan\frac{\theta}{2}$ so that $\cos\theta = \dfrac{1-t^2}{1+t^2}$ and $d\theta = \dfrac{2}{1+t^2}dt$.

Hence:

$$\int \frac{d\theta}{\cos\theta} = \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1-t^2}\,dt$$

Partial fractions:
$$= \int \left(\frac{1}{1+t} + \frac{1}{1-t} \right) dt$$
$$= \log_e |1+t| - \log_e |1-t| + C$$
$$= \log_e \left| \frac{1+t}{1-t} \right| + C$$
$$= \log_e \left| \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} \right| + C$$

Example 22

Find $\int \frac{dx}{\sin 2x + \cos 2x}$ using the substitution $t = \tan x$.

Solution

$\sin 2x = \frac{2t}{1+t^2}$, $\cos 2x = \frac{1-t^2}{1+t^2}$

You must recalculate the expression for dx, as $t = \tan x$ not $\tan \frac{x}{2}$:

$$\frac{dt}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + t^2$$

Hence $dx = \frac{dt}{1+t^2}$:
$$\int \frac{dx}{\sin 2x + \cos 2x} = \int \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2}$$
$$= \int \frac{dt}{1+2t-t^2}$$
$$= \int \frac{dt}{2-(t^2-2t+1)}$$
$$= \int \frac{dt}{2-(t-1)^2}$$
$$= \int \frac{dt}{\left(\sqrt{2}-(t-1)\right)\left(\sqrt{2}+(t-1)\right)}$$
$$= \frac{1}{2\sqrt{2}} \int \left(\frac{1}{\left(\sqrt{2}+1-t\right)} + \frac{1}{\left(\sqrt{2}-1+t\right)} \right) dt$$
$$= \frac{1}{2\sqrt{2}} \left(-\log_e \left|\sqrt{2}+1-t\right| + \log_e \left|\sqrt{2}-1+t\right| \right) + C$$
$$= \frac{1}{2\sqrt{2}} \log_e \left| \frac{\sqrt{2}-1+t}{\sqrt{2}+1-t} \right| + C$$
$$= \frac{1}{2\sqrt{2}} \log_e \left| \frac{\sqrt{2}-1+\tan x}{\sqrt{2}+1-\tan x} \right| + C$$

Note: When using a substitution of $t = \tan f(x)$, $f(x) \neq \frac{x}{2}$, differentiate the substitution to find the link between dx and dt.

EXERCISE 5.3 USING PARTIAL FRACTIONS TO FIND INTEGRALS

1 Find: (a) $\int \frac{2\,dx}{(x-3)(x-1)}$ (b) $\int \frac{3\,dx}{(x-2)(x+1)}$ (c) $\int \frac{dx}{(x-4)(x+2)}$

(d) $\int \frac{2x\,dx}{(x-1)(x+1)}$ (e) $\int \frac{3x\,dx}{(x+2)(x-1)}$ (f) $\int \frac{(2x-5)\,dx}{(x-3)(x-2)}$

2 $\int \frac{5x}{x^2+x-6}dx = \ldots$

A $\log_e\left|\frac{(x-2)^2}{(x+3)^3}\right| + C$

B $\log_e\left|(x-2)^3(x+3)^2\right| + C$

C $6\log_e|(x-2)(x+3)| + C$

D $\log_e\left|(x-2)^2(x+3)^3\right| + C$

3 Find: (a) $\int \frac{1-x}{(x+1)(x^2+1)}dx$ (b) $\int \frac{x^2-2x}{(x+2)(x^2+4)}dx$ (c) $\int \frac{x^2+x}{(x-1)(x^2+1)}dx$

(d) $\int \frac{x^2+x+12}{(x+3)(x^2+9)}dx$ (e) $\int \frac{x^2+x}{(x-2)(x^2+2)}dx$ (f) $\int \frac{5x+3}{(x-5)(x^2+3)}dx$

4 Find: (a) $\int \frac{8}{(x^2+1)(x^2+9)}dx$ (b) $\int \frac{15x^2}{(x^2+16)(x^2+1)}dx$ (c) $\int \frac{x^2-2}{(x^2+1)(x^2+4)}dx$

(d) $\int \frac{dx}{(x^2+2)(x^2+3)}$ (e) $\int \frac{x}{(x^2+4)(x^2+5)}dx$ (f) $\int \frac{x^3-x^2+3x-9}{(x^2+9)(x^2+3)}dx$

5 Find: (a) $\int \frac{2x-11}{(x+2)(x-1)}dx$ (b) $\int \frac{3(3x+1)}{x^2-9}dx$ (c) $\int \frac{4x^2-2x}{(x+1)(x^2+1)}dx$

(d) $\int \frac{2x^2+x+6}{x^2-4}dx$ (e) $\int \frac{2x^2+3x-1}{x^3-x^2+x-1}dx$ (f) $\int \frac{4x^2+11x+15}{x^3+3x^2-x-3}dx$

(g) $\int \frac{3x-19}{(x+3)(2x-1)}dx$ (h) $\int \frac{9x-2}{2x^2-7x+3}dx$ (i) $\int \frac{e^x}{e^{2x}+3e^x+2}dx$

6 Evaluate: (a) $\int_2^3 \frac{x}{(x+5)(x-1)}dx$ (b) $\int_3^4 \frac{dx}{x^2+x}$ (c) $\int_0^2 \frac{1-x}{(x+1)(x^2+1)}dx$

(d) $\int_0^{\frac{1}{2}} \frac{dx}{1-x^2}$ (e) $\int_0^1 \frac{dx}{4-x^2}$ (f) $\int_3^4 \frac{3x-2}{x^2-4}dx$

7 Evaluate: (a) $\int_4^5 \frac{x^2-5}{x^2-2x-3}dx$ (b) $\int_1^3 \frac{dx}{(x^2+1)(x^2+3)}$ (c) $\int_2^3 \frac{5x}{x^2+3x-4}dx$

8 Given that $\frac{1}{(x+2)(x+1)^2} = \frac{1}{x+2} - \frac{1}{x+1} + \frac{1}{(x+1)^2}$, evaluate: $\int_0^1 \frac{1}{(x+2)(x+1)^2}dx$

9 (a) Show that: $\frac{2}{x^3+x^2+x+1} = \frac{1}{x+1} + \frac{1-x}{x^2+1}$

(b) Hence evaluate: $\int_1^2 \frac{2}{x^3+x^2+x+1}dx$

10 (a) Find real numbers a, b and c such that: $\frac{1}{(x+1)x^2} = \frac{a}{x+1} + \frac{b}{x} + \frac{c}{x^2}$

(b) Hence find: $\int \frac{dx}{(x+1)x^2}$

11 (a) Given that $\frac{4x-7}{(x-2)^2}$ can be written as $\frac{a}{x-2} + \frac{b}{(x-2)^2}$ where a and b are real numbers, find a and b.

(b) Hence find: $\int \frac{4x-7}{(x-2)^2}dx$

12 (a) Find real numbers a, b and c such that: $\dfrac{2x^2+19x-36}{(x+3)(x-2)^2}=\dfrac{a}{x+3}+\dfrac{b}{x-2}+\dfrac{c}{(x-2)^2}$

(b) Hence find: $\displaystyle\int\frac{2x^2+19x-36}{(x+3)(x-2)^2}\,dx$

13 (a) Find real numbers a, b and c such that: $\dfrac{3x^2+10x-5}{(x+1)^2(x-2)}=\dfrac{a}{x+1}+\dfrac{b}{(x+1)^2}+\dfrac{c}{x-2}$

(b) Hence find: $\displaystyle\int\frac{3x^2+10x-5}{(x+1)^2(x-2)}\,dx$

14 (a) Find real numbers a, b and c such that: $\dfrac{1}{x(x+1)^2}=\dfrac{a}{x}+\dfrac{b}{x+1}+\dfrac{c}{(x+1)^2}$ (b) Hence evaluate: $\displaystyle\int_1^2\frac{dx}{x(x+1)^2}$

15 It can be shown that $\dfrac{-5x^2+18x-12}{(4-x^2)(5-4x+x^2)}=\dfrac{2x}{4-x^2}-\dfrac{3-2x}{5-4x+x^2}$. Use this result to evaluate:

$\displaystyle\int_0^1\frac{-5x^2+18x-12}{(4-x^2)(5-4x+x^2)}\,dx$

16 (a) Show that: $\dfrac{1}{(x-3)x^2}=\dfrac{1}{9(x-3)}-\dfrac{1}{9x}-\dfrac{1}{3x^2}$

(b) Calculate the area bounded by the curve $y=\dfrac{1}{x^2(x-3)}$, the x-axis and the ordinates $x=4$ and $x=6$.

17 Calculate the area bounded by the curve $y=\dfrac{1-2x}{2x^2+7x+6}$, the x-axis and the ordinates $x=-\frac{1}{2}$ and $x=\frac{1}{2}$.

18 (a) Given that $\dfrac{7x-4}{(x-2)^2(x+3)}$ can be written as $\dfrac{a}{(x-2)^2}+\dfrac{b}{x-2}+\dfrac{c}{x+3}$ where a, b and c are real numbers, find a, b and c.

(b) Hence find: $\displaystyle\int\frac{7x-4}{(x-2)^2(x+3)}\,dx$

19 It can be shown that $\dfrac{2(4-7x)}{(5-2x+x^2)(2-x^2)}=\dfrac{4-2x}{5-2x+x^2}-\dfrac{2x}{2-x^2}$. (Do not prove this.)

Use this result to evaluate: $\displaystyle\int_{-1}^1\frac{2(4-7x)}{(5-2x+x^2)(2-x^2)}\,dx$

20 (a) Find real numbers a, b and c such that: $\dfrac{3x+2}{x^2(x+1)}=\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x+1}$ (b) Hence find: $\displaystyle\int\frac{3x+2}{x^2(x+1)}\,dx$

21 Using an appropriate substitution of the type $t=\tan x$, find: $\displaystyle\int\frac{dx}{\sin 2x-\cos 2x}$

22 Use the substitution $t=\tan\frac{x}{2}$ to find $\displaystyle\int\frac{dx}{1+3\sin x}$.

23 Use the substitution $t=\tan x$ to find $\displaystyle\int\frac{1+\sin^2 x}{1+\cos^2 x}\,dx$.

5.4 INTEGRATION BY PARTS

The product rule for differentiation is associated with the technique known as **integration by parts**, which comes from rearranging the product rule. This rule is useful for solving integrals that cannot be found in easier ways.

If $u(x)$ and $v(x)$ are differentiable functions of x, then the product rule tells you that:

$$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}\quad\text{or}\quad\frac{d}{dx}(uv)=vu'+uv'$$

Rewriting this:

$$u\frac{dv}{dx}=\frac{d}{dx}(uv)-v\frac{du}{dx}\qquad uv'=\frac{d}{dx}(uv)-vu'$$

Hence: $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$ $\qquad \displaystyle\int uv'\,dx = uv - \int vu'\,dx$

$\therefore$ $\displaystyle\int u\,dv = uv - \int v\,du$

The integrand on the left-hand side is seen to be a product of two expressions involving x: one of these is denoted by u and the other by $\frac{dv}{dx}$ (or just dv). The choice of which expressions to label as u and dv is made so that the integral on the right-hand side can be easily found by normal techniques, such as change of variable (substitution). The arbitrary constant C is inserted into the solution at the appropriate point.

As a general rule, dv should be a function that is easy to integrate and u should be the other function. If they are both easy to integrate, then you should make u the function that will be of a lesser degree (i.e. simpler) after differentiation. This is illustrated in the following examples.

Note that for complex integrals the rule sometimes needs to be applied more than once. You may also need to rearrange terms to solve the desired integral.

Example 23

Find: $\displaystyle\int x\cos x\,dx$

Solution

Let $u = x$, $\frac{dv}{dx} = \cos x$ as this gives $\frac{du}{dx} = 1$, which is easy to work with in the resulting integral.

Thus: $\displaystyle\frac{du}{dx} = 1, \quad v = \sin x$

Integration by parts: $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Hence:
$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + C$$

The constant of integration is added after the last integration is performed.

Alternatively: Another choice of variable could have been $u = \cos x$, $\frac{dv}{dx} = x$.

Thus: $\displaystyle\frac{du}{dx} = -\sin x, \quad v = \frac{x^2}{2}$

Hence: $\displaystyle\int x\cos x\,dx = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)\,dx$

This integral is now more complicated than the original integral.

Example 24

Find: $\displaystyle\int \cos^{-1} x\,dx$

Solution

Rewrite as: $\displaystyle\int 1\times\cos^{-1} x\,dx$

Let $u = \cos^{-1} x$, $\frac{dv}{dx} = 1$. This gives $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$, $|x| < 1$, and $v = x$.

Integration by parts: $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Hence:
$$\int \cos^{-1} x\,dx = x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= x\cos^{-1} x - (1-x^2)^{\frac{1}{2}} + C \text{ for } |x| < 1$$

The method of integration by parts should be used mostly as a last resort, when other known techniques fail. It can be used very effectively to integrate products of different kinds of expressions, for example:

- algebraic and trigonometric functions such as $x\cos x$
- algebraic and logarithmic functions such as $x\log_e x$
- inverse trigonometric functions and logarithmic functions.

Example 25

Find **(a)** $\int \log_e x\,dx$ **(b)** $\int xe^x\,dx$ **(c)** $\int x^2e^x\,dx$

Solution

(a) Let $u = \log_e x$, $\frac{dv}{dx} = 1$. This gives $\frac{du}{dx} = \frac{1}{x}$, $v = x$.

Integration by parts: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Hence:
$$\begin{aligned}\int \log_e x\,dx &= x\log_e x - \int x \times \frac{1}{x}\,dx\\ &= x\log_e x - x + C, \qquad x > 0\\ &= x\log_e |x| - x + C\end{aligned}$$

(b) Let $u = x$, $\frac{dv}{dx} = e^x$. This gives $\frac{du}{dx} = 1$, $v = e^x$.

Hence:
$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + C\\ &= (x-1)e^x + C\end{aligned}$$

(c) Let $u = x^2$, $\frac{dv}{dx} = e^x$. This gives $\frac{du}{dx} = 2x$, $v = e^x$.

Hence: $\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$

You now need to find $\int xe^x\,dx$ by applying integration by parts again, as in part **(b)**:

Let $u = x$, $\frac{dv}{dx} = e^x$. This gives $\frac{du}{dx} = 1$, $v = e^x$.

Hence:
$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + C\\ &= (x-1)e^x + C\end{aligned}$$

Thus:
$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2(x-1)e^x + C\\ &= (x^2 - 2x + 2)e^x + C\end{aligned}$$

Sometimes integration by parts may need to be applied more than once. When there is an integer power of x in the integrand, as in part **(c)** of the example above, then each time it is differentiated the power will be reduced by one until eventually the function becomes a constant. You let u equal this power of x and apply integration by parts successively.

Useful results for integration by parts: $\int \cos x\,dx = \sin x$ $\quad \int \sin x\,dx = -\cos x$ $\quad \int e^x\,dx = e^x$

Example 26

Find: $I = \int e^x \sin x\, dx$

Solution

Let $u = e^x$, $\dfrac{dv}{dx} = \sin x$. This gives $\dfrac{du}{dx} = e^x$, $v = -\cos x$.

$$\therefore\ I = \int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

Now you need to find $\int e^x \cos x\, dx$.

Let $u = e^x$, $\dfrac{dv}{dx} = \cos x$. This gives $\dfrac{du}{dx} = e^x$, $v = \sin x$.

$$\therefore \quad \int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

Thus:
$$I = -e^x \cos x + e^x \sin x - I$$
$$2I = e^x \sin x - e^x \cos x$$
$$\therefore \quad I = \int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

The arbitrary constant C did not need to be included until the last line.

Example 27

Evaluate: **(a)** $I = \int_0^{\frac{1}{2}} \sin^{-1} x\, dx$ **(b)** $I = \int_0^{\pi} x^2 \sin x\, dx$

Solution

(a) Let $u = \sin^{-1} x$, $\dfrac{dv}{dx} = 1$. This gives $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}}$, $v = x$.

Hence:
$$I = \int_0^{\frac{1}{2}} \sin^{-1} x\, dx = \left[x \sin^{-1} x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - 0 - \left[-(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

(b) Let $u = x^2$, $\dfrac{dv}{dx} = \sin x$. This gives $\dfrac{du}{dx} = 2x$, $v = -\cos x$.

Hence:
$$I = \int_0^{\pi} x^2 \sin x\, dx = \left[-x^2 \cos x\right]_0^{\pi} + 2\int_0^{\pi} x \cos x\, dx$$
$$= \pi^2 + 2\int_0^{\pi} x \cos x\, dx$$

Now let $u = x$, $\dfrac{dv}{dx} = \cos x$. This gives $\dfrac{du}{dx} = 1$, $v = \sin x$.

Hence:
$$\int_0^{\pi} x \cos x\, dx = [x \sin x]_0^{\pi} - \int_0^{\pi} \sin x\, dx$$
$$= 0 + [\cos x]_0^{\pi}$$
$$= -2$$
$$\therefore \quad I = \int_0^{\pi} x^2 \sin x\, dx = \pi^2 - 4$$

Example 28

Evaluate: (a) $I=\int_1^2 x\log_e x\,dx$ (b) $I=\int_0^1 \tan^{-1}x\,dx$

Solution

(a) Let $u=\log_e x$, $\frac{dv}{dx}=x$. This gives $\frac{du}{dx}=\frac{1}{x}$, $v=\frac{x^2}{2}$.

$$\therefore I=\int_1^2 x\log_e x\,dx=\left[\frac{x^2}{2}\log_e x\right]_1^2-\frac{1}{2}\int_1^2 x^2\times\frac{1}{x}\,dx$$
$$=2\log_e 2-0-\frac{1}{2}\int_1^2 x\,dx$$
$$=2\log_e 2-\frac{1}{2}\left[\frac{x^2}{2}\right]_1^2$$
$$=2\log_e 2-1+\frac{1}{4}$$
$$=2\log_e 2-\frac{3}{4}$$

(b) Let $u=\tan^{-1}x$, $\frac{dv}{dx}=1$. This gives $\frac{du}{dx}=\frac{1}{1+x^2}$, $v=x$.

$$\therefore I=\int_0^1 \tan^{-1}x\,dx=\left[x\tan^{-1}x\right]_0^1-\int_0^1\frac{x}{1+x^2}\,dx$$
$$=\frac{\pi}{4}-0-\frac{1}{2}\left[\log_e(1+x^2)\right]_0^1$$
$$=\frac{\pi}{4}-\frac{1}{2}\log_e 2$$

EXERCISE 5.4 INTEGRATION BY PARTS

1 $\int \sin^{-1}x\,dx=\ldots$

A $\sin^{-1}x+x\sqrt{1-x^2}+C,\ |x|<1$

B $\sin^{-1}x-x\sqrt{1-x^2}+C,\ |x|<1$

C $x\sin^{-1}x+\sqrt{1-x^2}+C,\ |x|<1$

D $x\sin^{-1}x-\sqrt{1-x^2}+C,\ |x|<1$

2 Find: (a) $\int x\sqrt{4-x}\,dx$ (b) $\int x\tan^{-1}x\,dx$ (c) $\int x\sin x\,dx$

(d) $\int x\sin 2x\,dx$ (e) $\int x\cos 2x\,dx$ (f) $\int \sin^{-1}2x\,dx$

(g) $\int x\cos(2x+1)\,dx$ (h) $\int \cos^{-1}2x\,dx$ (i) $\int x\sin^{-1}x\,dx$

3 Find: (a) $\int x\sqrt{x+1}\,dx$ (b) $\int x^2\sin x\,dx$ (c) $\int x\cos^{-1}x\,dx$

(d) $\int x^2\cos x\,dx$ (e) $\int xe^{-x}\,dx$ (f) $\int x\log_e x\,dx$

4 Find: (a) $\int e^{-x}\sin x\,dx$ (b) $\int e^{x}\cos x\,dx$ (c) $\int e^{-x}\cos x\,dx$

(d) $\int \sin(\log_e x)\,dx$ (e) $\int \cos(\log_e x)\,dx$ (f) $\int \tan^{-1}x\,dx$

5 Evaluate: (a) $\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin x\,dx$ (c) $\int_0^1 x\sin^{-1} x\,dx$

(d) $\int_1^{\sqrt{3}} x\tan^{-1} x\,dx$ (e) $\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx$ (f) $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} x\sin 2x\,dx$

6 Evaluate: (a) $\int_{\frac{1}{4}}^{\frac{1}{2}} \cos^{-1} 2x\,dx$ (b) $\int_0^{\frac{\pi}{2}} e^x \sin x\,dx$ (c) $\int_1^2 te^{-t}\,dt$

(d) $\int_1^e \log_e x\,dx$ (e) $\int_1^2 x\log_e x\,dx$ (f) $\int_0^{\frac{\pi}{2}} e^x \cos x\,dx$

(g) $\int_1^2 x^2 \log_e 2x\,dx$ (h) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x\sec^2 x\,dx$ (i) $\int_{\frac{\pi}{2}}^{\pi} e^{-x}\sin 2x\,dx$

7 $\int_{-\pi}^{\pi} x^2 \sin x\,dx = \ldots$

A $2\pi^2 - 2$ B 0 C 2 D $2\pi^2$

8 Find the area of the region bounded by the curve $y = \log_e x$ $(x > 0)$, the x-axis and the line $x = a$ $(a > 1)$.

5.5 RECURRENCE RELATIONS

Integration by parts can also be used to find a **recurrence relation** that will allow the evaluation of an integral. Given $I_n = \int_a^b [f(x)]^n\,dx$, you find a relationship of the form $I_n = kI_{n-1}$, where $I_{n-1} = \int_a^b [f(x)]^{n-1}\,dx$. After I_1 or another of the other easy I integrals can be found, the recurrence relation is then used to evaluate the original integral.

Example 29

Find the recurrence relation for $\int \cos^n x\,dx$ and use it to evaluate: $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$

Solution

Let $I_n = \int \cos^n x\,dx$ and write $\cos^n x = \cos^{n-1} x \times \cos x$:

$$I_n = \int \cos^{n-1} x\cos x\,dx$$

Let $u = \cos^{n-1} x$, $\frac{dv}{dx} = \cos x$. This gives $\frac{du}{dx} = (n-1)\cos^{n-2} x \times (-\sin x)$, $v = \sin x$.

Hence: $$I_n = \int \cos^n x\,dx = \cos^{n-1} x \times \sin x + (n-1)\int \sin^2 x\cos^{n-2} x\,dx$$

$$I_n = \cos^{n-1} x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx$$

$$I_n = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$I_n = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)I_n$$

$$(1+n-1)I_n = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$

$$nI_n = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}I_{n-2}$$

This relation between I_n and I_{n-2} is a recurrence relation.

To find $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$, write: $I_4 = \int_0^{\frac{\pi}{2}} \cos^4 x\,dx$

From the recurrence relation for $n = 4$, we have: $I_4 = \left[\frac{\cos^3 x \sin x}{4}\right]_0^{\frac{\pi}{2}} + \frac{3}{4}I_2 = \frac{3}{4}I_2$

and $I_2 = \left[\frac{\cos x \sin x}{2}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}I_0 = 0 + \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$

Thus: $I_2 = \frac{1}{2}[x]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ and so $I_4 = \frac{3}{4}I_2 = \frac{3}{4} \times \frac{\pi}{4} = \frac{3\pi}{16}$

Hence: $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \frac{3\pi}{16}$

Example 30

(a) Let $I_n = \int_0^x \tan^n t\,dt$ where $0 \le x < \frac{\pi}{2}$. Show that: $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$

(b) Hence find the exact value of: $\int_0^{\frac{\pi}{4}} \tan^5 t\,dt$

Solution

(a) Write $\tan^n t = \tan^2 t \times \tan^{n-2} t = (\sec^2 t - 1)\tan^{n-2} t$. Thus:

$$I_n = \int_0^x \tan^n t\,dt = \int_0^x (\sec^2 t - 1)\tan^{n-2} t\,dt$$

$$= \int_0^x \sec^2 t\,\tan^{n-2} t\,dt - \int_0^x \tan^{n-2} t\,dt$$

$$= \int_0^x \sec^2 t\,\tan^{n-2} t\,dt - I_{n-2}$$

Now consider: $\int_0^x \sec^2 t\,\tan^{n-2} t\,dt$

It is tempting to try integration by parts here, but using the substitution $u = \tan t$ is much easier. Let $u = \tan t$ so that $du = \sec^2 t\,dt$. Limits are $t = 0$: $u = 0$ $\quad t = x$: $u = \tan x$

$$\therefore \int_0^x \sec^2 t\,\tan^{n-2} t\,dt = \int_0^{\tan x} u^{n-2}\,du$$

$$= \left[\frac{u^{n-1}}{n-1}\right]_0^{\tan x}$$

$$= \frac{1}{n-1}\tan^{n-1} x$$

Thus: $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$

(b) $I_5 = \int_0^{\frac{\pi}{4}} \tan^5 t\,dt \quad \therefore I_5 = \frac{1}{4}\left[\tan^4 x\right]_0^{\frac{\pi}{4}} - I_3 = \frac{1}{4} - I_3$

and $I_3 = \frac{1}{2}\left[\tan^2 x\right]_0^{\frac{\pi}{4}} - I_1 = \frac{1}{2} - I_1$

Now: $I_1 = \int_0^{\frac{\pi}{4}} \tan t\,dt = -\left[\log_e(\cos t)\right]_0^{\frac{\pi}{4}} = -\left(\log_e\left(\frac{1}{\sqrt{2}}\right) - \log_e 1\right) = \frac{1}{2}\log_e 2$

Hence: $I_3 = \frac{1}{2} - \frac{1}{2}\log_e 2$ and $I_5 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\log_e 2\right) = \frac{1}{2}\log_e 2 - \frac{1}{4}$

This gives: $\int_0^{\frac{\pi}{4}} \tan^5 t\,dt = \frac{1}{2}\log_e 2 - \frac{1}{4}$

Example 31

Obtain a recurrence relation for $\int e^{ax}\cos bx\,dx$ and use it to find $\int e^{3x}\cos 4x\,dx$.
(This example is just outside the scope of this course, but it is worth looking at as a demonstration of just how useful this process can be.)

Solution

Write: $I = \int e^{ax}\cos bx\,dx$

Let $u = e^{ax}$, $\dfrac{dv}{dx} = \cos bx$. This gives $\dfrac{du}{dx} = ae^{ax}$, $v = \dfrac{1}{b}\sin bx$.

$$\therefore\ I = \int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

Now write: $H = \int e^{ax}\sin bx\,dx$

Let $u = e^{ax}$, $\dfrac{dv}{dx} = \sin bx$. This gives $\dfrac{du}{dx} = ae^{ax}$, $v = -\dfrac{1}{b}\cos bx$.

$$\therefore\ H = \int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

Substitute this expression for H into the expression for I:

$$I = \int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left(-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right)$$

$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$$

$$\left(1+\frac{a^2}{b^2}\right)I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

$$\therefore\ I = \int e^{ax}\cos bx\,dx = \frac{b^2}{b^2+a^2}\left(\frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx\right) + C$$

$$= \frac{e^{ax}}{b^2+a^2}(b\sin bx + a\cos bx) + C$$

In $\int e^{3x}\cos 4x\,dx$ you have $a = 3$ and $b = 4$, thus: $\int e^{3x}\cos 4x\,dx = \dfrac{e^{3x}}{25}(4\sin 4x + 3\cos 4x) + C$

EXERCISE 5.5 RECURRENCE RELATIONS

1 **(a)** Differentiate $\sin x\cos^{n-1}x$ with respect to x to show that if $I_n = \int \cos^n x\,dx$, then:
$I_n = \dfrac{1}{n}\sin x\cos^{n-1}x + \dfrac{n-1}{n}I_{n-2}$

(b) Hence evaluate:

(i) $\int_0^{\frac{\pi}{2}}\cos^6 x\,dx$ **(ii)** $\int_0^{\frac{\pi}{2}}\sin^6 x\,dx$ **(iii)** $\int_0^{\frac{\pi}{2}}\cos^4 x\sin^2 x\,dx$ **(iv)** $\int_0^{\frac{\pi}{4}}\sec^4 x\,dx$

2 Show that $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$ and hence find: $\int x^3 e^x\,dx$

3 **(a)** Find the derivative of $x^n\log_e x$.
(b) Hence find (correct to three decimal places) the value of:

(i) $\int_1^2 x^2\log_e x\,dx$ **(ii)** $\int_1^2 x^3\log_e x\,dx$

4 **(a)** Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$, prove that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ where $n \geq 2$ is an integer.

(b) Hence evaluate: $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$

5 **(a)** Show that: $\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

(b) Hence evaluate: $\int_0^{\frac{\pi}{6}} \sin^4 x\, dx$

6 **(a)** If $I_n = \int \sec^n x\, dx$ show that: $I_n = \frac{1}{n-1}(\sec^{n-2} x \tan x) + \frac{n-2}{n-1} I_{n-2}$

(b) Hence evaluate: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^4 x\, dx$

7 **(a)** If $I_n = \int_0^{\frac{\pi}{4}} \tan^{2n} x\, dx$ show that $I_n = \frac{1}{2n-1} - I_{n-1}$ for $n \geq 1$.

(b) Hence evaluate I_2.

8 **(a)** Given that $I_n = \int_0^1 x^{2n-1} e^{x^2}\, dx$ for each integer $n \geq 1$, show that: $I_n = \frac{e}{2} - (n-1)I_{n-1}$

(b) Hence, or otherwise, calculate I_2.

9 **(a)** If $I_n = \int x^5 (\log_e x)^n\, dx$ show that: $I_n = \frac{x^6}{6}(\log_e x)^n - \frac{n}{6} I_{n-1}$

(b) Hence find: $\int x^5 (\log_e x)^2\, dx$

10 **(a)** Show that: $\int e^{ax} \sin x\, dx = \frac{1}{a^2+1} e^{ax}(a \sin x - \cos x)$

(b) Hence find: **(i)** $\int e^x \sin x\, dx$ **(ii)** $\int e^{3x} \sin x\, dx$

11 If $I_n = \int_0^x \frac{t^n}{1+t}\, dt$ show that: $I_n + I_{n-1} = \frac{x^n}{n}$

5.6 OTHER USEFUL TECHNIQUES

Properties of the definite integral

1. $\int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx = \int_a^b \big(f(x) \pm g(x)\big)\, dx$
2. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
3. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
4. $\int_a^b f(x)\, dx = \int_a^b f(u)\, du$ (i.e. the value of the integral is independent of the variable of integration)
5. If $f(-x) = f(x)$ (i.e. if $f(x)$ is an **even** function), then: $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$.
6. If $f(-x) = -f(x)$ (i.e. if $f(x)$ is an **odd** function), then: $\int_{-a}^a f(x)\, dx = 0$.
7. $\int_{-a}^a \sqrt{a^2 - x^2}\, dx$ represents the area of a semicircle of radius a, so it can be quickly evaluated using $A = \pi r^2$.

Combinations of even and odd functions

- The product of two **odd** functions or two **even** functions is an **even** function.
- The product of an **odd** function and an **even** function is an **odd** function.

Example 32

Without evaluating the definite integral, find a simpler expression for each definite integral.

(a) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx$ (b) $\int_{-1}^{1} x \sin^{-1} x \, dx$

Solution

(a) $f(x) = x^3$ is an odd function.
$g(x) = \cos x$ is an even function.
Hence $f(x) \times g(x) = [\text{odd}] \times [\text{even}] = [\text{odd}]$

$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \cos x \, dx = 0$

(b) $f(x) = x$ is an odd function.
$g(x) = \sin^{-1} x$ is an odd function.
Hence $f(x) \times g(x) = [\text{odd}] \times [\text{odd}] = [\text{even}]$

$\therefore \int_{-1}^{1} x \sin^{-1} x \, dx = 2\int_{0}^{1} x \sin^{-1} x \, dx$

Example 33

Use the properties of odd and even functions to evaluate: $\int_{-1}^{1} (1 + x^3)^3 \, dx$

Solution

$f(x) = (1 + x^3)^3$ $f(-x) = (1 - x^3)^3$ $\therefore f$ is neither odd nor even.

Expand: $(1 + x^3)^3 = 1 + 3x^3 + 3x^6 + x^9$

$$\begin{aligned}
\therefore \int_{-1}^{1} (1 + x^3)^3 \, dx &= \int_{-1}^{1} (1 + 3x^3 + 3x^6 + x^9) \, dx \\
&= \int_{-1}^{1} dx + 3\int_{-1}^{1} x^3 \, dx + 3\int_{-1}^{1} x^6 \, dx + \int_{-1}^{1} x^9 \, dx \\
&= [\text{even}] + [\text{odd}] + [\text{even}] + [\text{odd}] \\
&= 2\int_{0}^{1} dx + 0 + 2 \times 3\int_{0}^{1} x^6 \, dx + 0 \\
&= 2\int_{0}^{1} (1 + 3x^6) \, dx \\
&= 2\left[x + \frac{3x^7}{7}\right]_0^1 = 2\left(1 + \frac{3}{7}\right) = \frac{20}{7}
\end{aligned}$$

Example 34

Evaluate: $\int_{-\pi}^{\pi} \sin^5 x \cos^8 x \, dx$

Solution

This looks formidable, but the properties of odd and even functions make it quite simple to evaluate.

$f(x) = \sin^5 x$ is an odd function, $g(x) = \cos^8 x$ is an even function:
$f(x) \times g(x) = [\text{odd}] \times [\text{even}] = [\text{odd}]$

Hence: $\int_{-\pi}^{\pi} \sin^5 x \cos^8 x \, dx = 0$

Example 35

(a) Show that: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(b) Hence evaluate: $\int_0^\pi x\sin^2 x\,dx$

Solution

(a) **Method 1**

If $\int f(x)\,dx = F(x) + C$ then $\int_0^a f(x)\,dx = [F(x)]_0^a = F(a) - F(0)$

and $\int_0^a f(a-x)\,dx = [-F(a-x)]_0^a = -F(0) + F(a) = F(a) - F(0) = \int_0^a f(x)\,dx$

Method 2

Let $y = a - x$ so that $dy = -dx$. Limits are $x = 0$: $y = a$ $\quad x = a$: $y = 0$

$$\int_0^a f(a-x)\,dx = \int_a^0 f(y)(-dy) = \int_0^a f(y)\,dy = \int_0^a f(x)\,dx$$

(b) You can write: $\int_0^\pi x\sin^2 x\,dx = \int_0^\pi (\pi - x)\sin^2(\pi - x)\,dx$

Now $\int_0^\pi (\pi - x)\sin^2(\pi - x)\,dx = \int_0^\pi (\pi - x)\sin^2 x\,dx$

$$= \int_0^\pi \pi\sin^2 x\,dx - \int_0^\pi x\sin^2 x\,dx$$

Hence: $\int_0^\pi x\sin^2 x\,dx = \pi\int_0^\pi \sin^2 x\,dx - \int_0^\pi x\sin^2 x\,dx$

$$2\int_0^\pi x\sin^2 x\,dx = \frac{\pi}{2}\int_0^\pi (1 - \cos 2x)\,dx$$

$$\int_0^\pi x\sin^2 x\,dx = \frac{\pi}{4}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$

$$= \frac{\pi}{4}(\pi - 0) = \frac{\pi^2}{4}$$

Warning

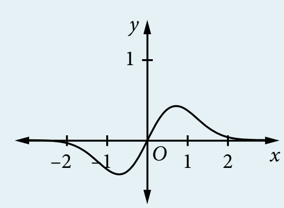
Be aware of the difference between *evaluating the integral* $\int_{-a}^a f(x)\,dx$ and *finding the area* under the curve $y = f(x)$ between the ordinates $x = -a$ and $x = a$.

Example 36

(a) Evaluate: $\int_{-2}^2 xe^{-x^2}\,dx$

(b) Find the area bounded by the curve $y = xe^{-x^2}$, the x-axis and the ordinates $x = -2$ and $x = 2$.

Solution



(a) xe^{-x^2} is an odd function, so: $\int_{-2}^2 xe^{-x^2}\,dx = 0$

(b) Area $= \int_{-2}^2 \left|xe^{-x^2}\right| dx = 2\int_0^2 xe^{-x^2}\,dx = \left[-e^{-x^2}\right]_0^2 = 1 - e^{-4}$ units^2

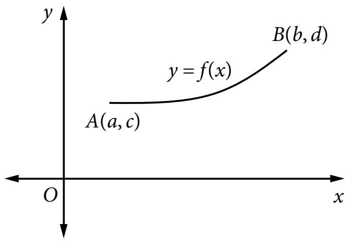
Formulae involving integration

The length of the arc of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by:

$$\text{Arc length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_{u_1}^{u_2} \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}\,du$$



Here $u = u_1$ and $u = u_2$ are the parameters of A and B respectively.

Example 37

Calculate the arc length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$ using the formula:

$$\text{Arc length} = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Solution

$$\frac{dy}{dx} = \frac{x^3}{2} - \frac{1}{2x^3} \qquad \therefore \text{ Arc length} = \int_1^2 \sqrt{1+\left(\frac{x^3}{2}-\frac{1}{2x^3}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1+\frac{x^6}{4}-\frac{1}{2}+\frac{1}{4x^6}}\,dx$$

$$= \int_1^2 \sqrt{\frac{x^6}{4}+\frac{1}{2}+\frac{1}{4x^6}}\,dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^3}{2}+\frac{1}{2x^3}\right)^2}\,dx$$

$$= \int_1^2 \left(\frac{x^3}{2}+\frac{1}{2x^3}\right)dx = \left[\frac{x^4}{8}-\frac{1}{4x^2}\right]_1^2 = \frac{33}{16}$$

EXERCISE 5.6 OTHER USEFUL TECHNIQUES

1 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx = \ldots$

A $2\int_0^{\frac{\pi}{2}} x\cos x\,dx$ **B** 0 **C** $\pi - 2$ **D** $\frac{\pi}{2} - 1$

2 Evaluate $\int_0^{\frac{1}{2}} \sqrt{\frac{1+x}{1-x}}\,dx$ using: $\frac{1+x}{1-x} = \frac{(1+x)^2}{1-x^2}$

3 **(a)** Write $(x-1)(3-x)$ in the form $b^2 - (x-a)^2$ where a and b are real numbers.

(b) Using the values of a and b from part **(a)** and making the substitution $x - a = b\sin\theta$, or otherwise, evaluate: $\int_1^3 \sqrt{(x-1)(3-x)}\,dx$

4 **(a)** Use the substitution $t = \tan\frac{x}{2}$ to find: $\int \sec x\, dx$

(b) By rewriting $\sec x$ as $\dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$, find: $\int \sec x\, dx$

5 **(a)** Find: $\int \sec x \tan x\, dx$

(b) Using the substitution $u = \frac{\pi}{2} - x$, find: $\int \operatorname{cosec} x \cot x\, dx$

6 Find the length of the circumference of the circle $x^2 + y^2 = r^2$ using the arc length formula on the circle's first quadrant, i.e. arc length $= \int_0^r \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$.

7 Let $I = \int_1^2 \dfrac{\cos^2\left(\frac{\pi}{6}x\right)}{x(3-x)}\, dx$.

(a) Use the substitution $u = 3 - x$ to show that: $I = \int_1^2 \dfrac{\sin^2\left(\frac{\pi}{6}u\right)}{u(3-u)}\, du$

(b) Hence find the value of I.

8 **(a)** Differentiate $x^2 \tan^{-1} x$ with respect to x. **(b)** Hence find: $\int 2x \tan^{-1} x\, dx$

9 Show that: **(a)** $\int_{-1}^{1} x^3(1-x^2)^2\, dx = 0$ **(b)** $\int_{-1}^{1} x^2(1-x^2)^3\, dx = 2\int_0^1 x^2(1-x^2)^3\, dx$

10 **(a)** By using an appropriate substitution, show that: $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

(b) Hence show that: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x\, dx$

11 **(a)** By using an appropriate substitution, show that $\int_0^{\pi} x \sin x\, dx = \int_0^{\pi} (\pi - x)\sin x\, dx$ and hence show that: $\int_0^{\pi} x \sin x\, dx = \frac{\pi}{2}\int_0^{\pi} \sin x\, dx$

(b) Hence evaluate: $\int_0^{\pi} x \sin x\, dx$

12 **(a)** If $x > 0$ and $1 < u < 1 + x$, show that: $\dfrac{1}{1+x} < \dfrac{1}{u} < 1$

(b) By integrating each term of $\dfrac{1}{1+x} < \dfrac{1}{u} < 1$ with respect to u between 1 and $(1 + x)$, show that: $\dfrac{x}{1+x} < \log_e(1+x) < x$

5.7 USES OF INTEGRATION

Example 38

Calculate the area of the region bounded by the curve $y = \dfrac{1}{x^2 + x}$, the x-axis and the ordinates $x = 2$ and $x = 3$.

Solution

Area $= \int_2^3 \dfrac{1}{x^2 + x}\, dx$

Use partial fractions: $\dfrac{1}{x^2 + x} = \dfrac{1}{x(x+1)} = \dfrac{a}{x} + \dfrac{b}{x+1}$

By inspection you can see that $a = 1$ and $b = -1$.

Or: $1 = a(x + 1) + bx$

$x = -1$: $1 = -b \Rightarrow b = -1$

$x = 0$: $1 = a$

Hence

$$\text{Area} = \int_2^3 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \left[\log_e x - \log_e (x+1)\right]_2^3 \text{ since } x > 0$$

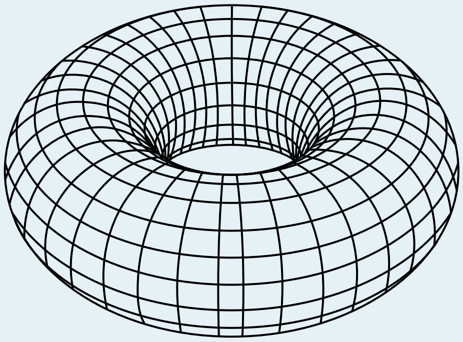
$$= \left[\log_e \frac{x}{x+1}\right]_2^3$$

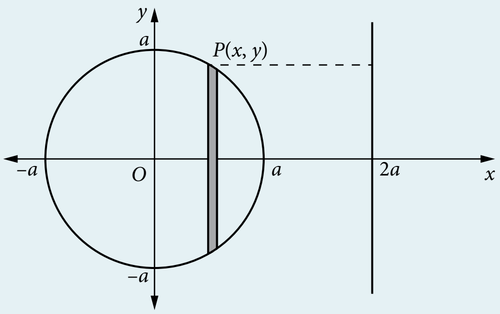
$$= \log_e \frac{3}{4} - \log_e \frac{2}{3}$$

$$= \log_e \frac{9}{8} \text{ units}^2$$

Example 39

The region enclosed within the circle $x^2 + y^2 = a^2$ is rotated about the line $x = 2a$ to form a solid of revolution called a torus. (A torus is a ring-shaped object like a doughnut.)





The volume of this torus is given by $V = 8\pi a\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx - 4\pi\int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx$.

(a) Evaluate $8\pi a\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$.

(b) Evaluate $\int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx$.

(c) Hence find the volume of the torus.

Solution

(a) The integral $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$ is the area of a semicircle of radius a.

Hence $8\pi a\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 8\pi a \times \frac{\pi a^2}{2} = 4\pi^2 a^3$.

(b) $$\int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx = -\frac{1}{2}\int_{-a}^{a} (-2x)\sqrt{a^2 - x^2}\, dx$$

$$= -\frac{1}{3}\left[\left(a^2 - x^2\right)^{\frac{3}{2}}\right]_{-a}^{a}$$

$$= 0$$

(c) $V = 4\pi^2 a^3 - 4\pi \times 0 = 4\pi^2 a^3$ units^3

In part **(a)**, if you did not recognise that the integral represented the area of a semicircle you would use the substitution $x = a\sin\theta$ to evaluate the integral. This would involve much more work, as shown below.

$x = a\sin\theta,\ dx = a\cos\theta.\quad x = -a, \theta = -\frac{\pi}{2}.\quad x = a, \theta = \frac{\pi}{2}.$

$$\begin{aligned}
\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos\theta \times a\cos\theta\, d\theta \\
&= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta \\
&= \frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta \\
&= \frac{a^2}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{a^2}{2} \left(\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2}\right) + 0\right) \\
&= \frac{\pi a^2}{2}
\end{aligned}$$

Hence $8\pi a \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = 8\pi a \times \frac{\pi a^2}{2} = 4\pi^2 a^3.$

The solution of differential equations is all about integrating the given function and substituting the initial conditions to find a particular solution. You now have new integration skills, so this extends the types of differential equations that you can solve.

Example 40

Find the particular solution of $\frac{dy}{dx} = x^2 \cos x$, given that $y = 0$ when $x = 0$.

Solution

$\frac{dy}{dx} = x^2 \cos x: \quad y = \int x^2 \cos x\, dx$

Use integration by parts: $y = x^2 \sin x - \int 2x \sin x\, dx$

Use integration by parts: $y = x^2 \sin x + 2x\cos x - \int 2\cos x\, dx$

$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

$x = 0$: $0 = C$

Hence the particular solution is $y = x^2 \sin x + 2x\cos x - 2\sin x$.

EXERCISE 5.7 USES OF INTEGRATION

1 Without evaluating the integral, explain why $\int_{-1}^{1} \tan^{-1} x\, dx$ is equal to zero.

2 Calculate the area of the region bounded by the curve $y = \dfrac{1}{x(3-x)}$, the x-axis and the lines $x = 1$ and $x = 2$.

3 Find the area of the region bounded by the curve $y = \dfrac{x^2 - 6x + 8}{x^2 - 3x}$, the x-axis and the ordinates $x = 4$ and $x = 5$.

4 Calculate the area of the region enclosed by the curve $y = \dfrac{1-x^2}{1+x^2}$ and the x-axis.

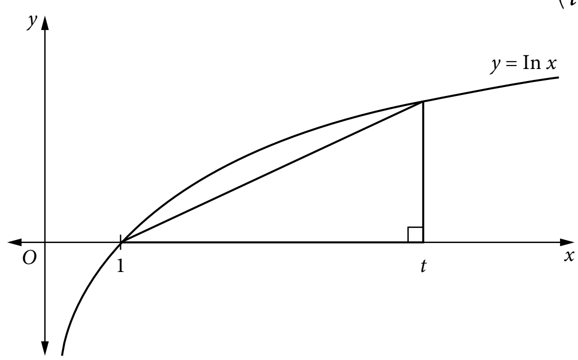
5 Find the volume of the solid generated when the part of the curve $y = \dfrac{1}{\sqrt{4-x^2}}$ between $x = -1$ and $x = 1$ is rotated about the x-axis.

6 Find the volume of the solid generated when the curve with equation $y = \dfrac{x}{\sqrt{x^2-1}}$ between $x = 2$ and $x = 3$ is rotated about the x-axis.

7 The diagram shows the graph of $y = \ln x$.

(a) Find $\int_{1}^{t} \ln x\, dx$.

(b) By comparing relevant areas in the diagram, or otherwise, show that $\ln t > 2\left(\dfrac{t-1}{t+1}\right)$, for $t > 1$.

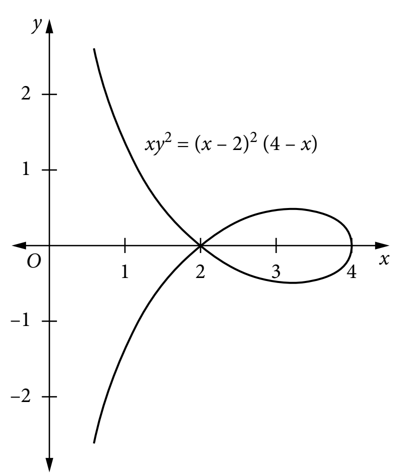


8 (a) Differentiate $\sin^{-1} x - \sqrt{1-x^2}$.

(b) Hence find $\int_{0}^{\theta} \left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}} dx = \sin^{-1}\theta + 1 - \sqrt{1-\theta^2}$ for $0 < \theta < 1$.

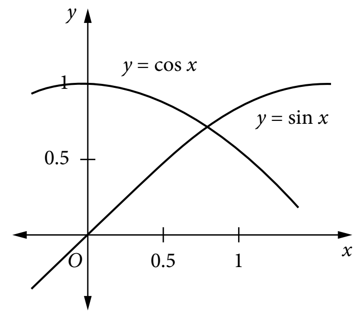
9 Find the volume of the solid of revolution formed by rotating the region between the curve $y = \sin x$ and the x-axis for $0 \le x \le \pi$ about the x-axis.

10 The diagram shows the graph of $xy^2 = (x-2)^2(4-x)$. Find the volume of the solid formed by rotating the loop in the graph of $xy^2 = (x-2)^2(4-x)$ about the x-axis.



11 **(a)** The region bounded by the curves $y = \sin x$, $y = \cos x$ and the y-axis is rotated about the x-axis. Calculate the volume of the solid of revolution formed.

(b) The region in part **(a)** is rotated about the y-axis. Write the integral that would give the volume of the solid of revolution formed. Use technology to find the value of this integral.



12 When the circle $x^2 + y^2 = a^2$ is rotated about the line $x = b$ $(b > a)$ to generate a torus, the volume of the solid of revolution formed is given by $V = 4\pi\int_{-a}^{a}(b-x)\sqrt{a^2 - x^2}\,dx$. Calculate this volume.

13 Find the volume of the solid of revolution generated when the region bounded $y = \sin x$, $0 \le x \le \frac{\pi}{2}$, the x-axis and the line $x = \frac{\pi}{2}$ is rotated about:

(a) the x-axis

(b) the y-axis, using technology to evaluate your integral.

(c) The volume of the solid in part **(b)** is also given by $V = 2\pi\int_{0}^{\frac{\pi}{2}} x\sin x\,dx$. Find the exact value of this volume and compare it to your answer in part **(b)**.

14 Find the particular solution of $\frac{dy}{dx} = x^2\sin x + 2x\cos x - 2\sin x$, given that $y = 6$ when $x = 0$.

15 Find the particular solution of $\frac{dy}{dx} = e^{-x}\sin x$, given that $y = -\frac{1}{2}$ when $x = 0$.

16 Find the particular solution of $\frac{dy}{dx} = -\frac{2x}{x^2+1}$, given that $y = 2$ when $x = 0$.

17 Find the particular solution of $\frac{dy}{dx} = -\frac{e^{-x}}{2}(\cos x + \sin x)$, given that $y(0) = \frac{1}{2}$.

18 When the region bounded by $y = e^{-x}$ and the lines $x = 1$, $y = 1$ is rotated about the line $x = 1$, the volume of the solid generated is given by $V = 2\pi\int_{0}^{1}\left(1-e^{-x}\right)(1-x)\,dx$. Calculate the exact value of this integral.

19 **(a)** The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the x-axis. Show that the volume of the solid generated is $\frac{1}{3}4\pi ab^3$ cubic units.

(b) When the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the line $x = a$, the volume of the solid of revolution is given by $V = 2\pi\int_{-a}^{a} 2y(a-x)\,dx$. Calculate the volume of this solid of revolution.

20 Find the particular solution of $\frac{dy}{dx} = 2 - \log_e\left(x^2+1\right)$, given that $y = 1$ when $x = 0$.

21 Find the particular solution of $\frac{dy}{dx} = -\frac{2x}{\left(x^2+1\right)^2}$, given that $y(0) = 2$.

22 **(a)** Let $I_n = \int_{1}^{2}\left(\log_e x\right)^n dx$ where n is a positive integer. Show that $I_n = 2\left(\log_e 2\right)^n - nI_{n-1}$.

(b) Let $f(x) = \left(\log_e x\right)^n$ where integer $n \ge 2$. By considering the first and second derivatives, show that $f(x)$ is increasing and concave up over the domain $1 < x \le 2$. Sketch the curve $y = \left(\log_e x\right)^n$ (where $n \ge 2$) over the domain $1 \le x \le 2$.

(c) By considering the area of a triangle that acts as an upper bound, show that: $nI_{n-1} > \frac{3}{2}\left(\log_e 2\right)^n$.

(d) Hence show that $\frac{2}{3} < \log_e 2 < 2$.

23 It is given that $x^4 - 8x^2 + 14 = (x^2 + 4x + 43)(x^2 - 4x + 4)$.

(a) Find a and b so that $\dfrac{64}{x^4 - 8x^2 + 16} = \dfrac{2x + a}{x^2 + 4x + 4} - \dfrac{2x - b}{x^2 - 4x + 4}$.

(b) Hence, or otherwise, show that for any real number $0 \le m < 2$,

$$\int_0^m \frac{64}{x^4 - 8x^2 + 16}\,dx = 2\log_e\left(\frac{m+2}{m-2}\right) - \frac{8m}{m^2 - 4}.$$

24 It is given that $x^4 - 3x^2 + 9 = (x^2 + 3x + 3)(x^2 - 3x + 3)$.

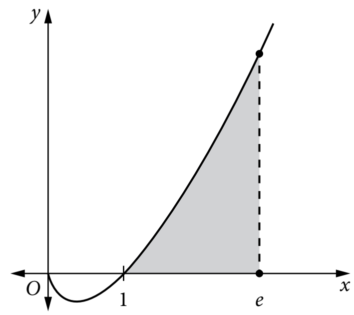
(a) Find a and b so that $\dfrac{36}{x^4 - 3x^2 + 9} = \dfrac{2x + a}{x^2 + 3x + 3} - \dfrac{2x - b}{x^2 - 3x + 3}$.

(b) Hence, or otherwise, show that for any real number p,

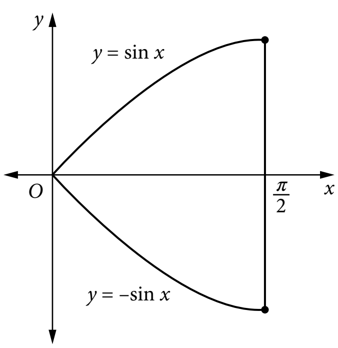
$$\int_0^p \frac{36}{x^4 - 3x^2 + 9}\,dx = \log_e\left(\frac{p^2 + 3p + 3}{p^2 - 3p + 3}\right) + 2\sqrt{3}\left(\tan^{-1}\left(\frac{2p+3}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2p-3}{\sqrt{3}}\right)\right).$$

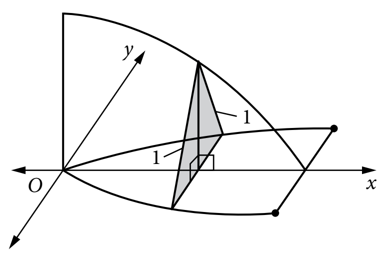
(c) Find the limiting value as $p \to \infty$ as $\displaystyle\int_0^p \frac{36}{x^4 - 3x^2 + 9}\,dx$.

25 The shaded region in the diagram is bounded by $y = x\log_e x$, $y = 0$ and $x = e$. When this region is rotated about the y-axis, the volume of the solid formed is given by $V = 2\pi\displaystyle\int_1^e x\,y\,dx$ where $y = x\log_e x$. Calculate the exact volume of this solid.



26 The base of a solid is formed by the area bounded by $y = \sin x$ and $y = -\sin x$ for $0 < x < \dfrac{\pi}{2}$ and the line $x = \dfrac{\pi}{2}$. Vertical cross-sections of the solid taken parallel to the y-axis are isosceles triangles with equal sides of length 1 unit, as shown. The volume of this solid is given by $V = \displaystyle\int_0^{\frac{\pi}{2}} y\sqrt{1 - y^2}\,dx$ where $y = \sin x$. Calculate the exact volume of this solid.





CHAPTER REVIEW 5

1 Evaluate: **(a)** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x\cos x\,dx$ **(b)** $\displaystyle\int_3^4 \frac{5x - 7}{x^2 - 3x + 2}\,dx$

2 Find: **(a)** $\displaystyle\int \log_e 2x\,dx$ **(b)** $\displaystyle\int \frac{x + 2}{x^2 - 1}\,dx$

3 Evaluate: $\displaystyle\int_2^3 \frac{dx}{x^2 + x}$

4 Evaluate: **(a)** $\int_1^{\sqrt{3}} \tan^{-1} x\, dx$ **(b)** $\int_{-2}^{2} \frac{6}{9-x^2}\, dx$

5 Find the derivative of $\log_e(\operatorname{cosec} x + \cot x)$ and deduce the value of:

(a) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}\frac{\theta}{2}\, d\theta$ **(b)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec u\, du$

6 **(a)** Find the derivative of: $\frac{\sin x}{1-\sin^2 x} + \log_e\sqrt{\frac{1+\sin x}{1-\sin x}}$

(b) Hence evaluate: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^3\theta\, d\theta$

7 **(a)** Write $(x-1)(7-x)$ in the form $b^2-(x-a)^2$, where a and b are real numbers.
(b) Using the values of a and b from part **(a)** and making the substitution $x-a=b\sin\theta$, or otherwise, evaluate: $\int_1^7 \sqrt{(x-1)(7-x)}\, dx$

8 Find real numbers a, b and c such that: $\frac{1}{x^2(x+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1}$

9 Reduce each rational function to its partial fractions.

(a) $\frac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ **(b)** $\frac{x^2+10x+16}{(x-1)(x^2-4)}$ **(c)** $\frac{x^3-4x^2+2x+6}{(x-3)(x-2)}$ **(d)** $\frac{6x^2-53x+42}{(2x-3)(2x^2-5x-3)}$

10 Express each rational function as partial fractions.

(a) $\frac{x^2-6x+14}{(x-3)(x-2)(2x-1)}$ **(b)** $\frac{2x^2+x+6}{x^2-4}$

11 **(a)** Show that: $\frac{2(5x+6)}{(x^2+2x+2)(3-x^2)} = \frac{2x+4}{x^2+2x+2} + \frac{2x}{3-x^2}$

(b) Show that: $\frac{2x+4}{x^2+2x+2} = \frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1}$

12 Given that $\frac{11x-28}{(x+3)^2(x-2)} = \frac{a}{(x+3)^2} + \frac{b}{x+3} + \frac{c}{x-2}$ where a, b and c are real numbers, find a, b and c.

13 Evaluate:

(a) $\int_1^3 \frac{2x^2+2x+5}{(x^2+3)(2x-1)}\, dx$ **(b)** $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\cos x\, dx$

14 Differentiate $\log_e x - \log_e\left(a+\sqrt{a^2-x^2}\right)$ where $a>0$ and deduce the value of: $\int_3^4 \frac{dx}{x\sqrt{25-x^2}}$

15 Evaluate: **(a)** $\int_1^4 (x+1)\sqrt{x}\, dx$ **(b)** $\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x}\, dx$ **(c)** $\int_5^6 \frac{dx}{x^2-16}$

(d) $\int_{-1}^{1} (2x-1)\sin x\, dx$ **(e)** $\int_0^3 x^2\sqrt{9-x^2}\, dx$

16 Find: **(a)** $\int \frac{dx}{1-4x^2}$ **(b)** $\int \frac{x}{1-4x^2}\, dx$ **(c)** $\int \frac{x^2}{1-4x^2}\, dx$

(d) $\int \frac{x}{\sqrt{1-4x^2}}\, dx$ **(e)** $\int \frac{dx}{\sqrt{1-4x^2}}$ **(f)** $\int \frac{dx}{1+4x^2}$

17 Find:

(a) $\int \frac{dx}{\sin x + \tan x}$ **(b)** $\int \frac{dx}{5+4\cos 2x}$ **(c)** $\int \frac{d\theta}{4\cos\theta - 3\sin\theta}$ **(d)** $\int \frac{\cos\theta}{5-3\cos\theta}\, d\theta$

18 Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of $\int_0^{\frac{\pi}{3}} \frac{1}{4+5\cos x}dx$.

19 Find: $\int x\log_e 2x\,dx$

20 Find:

(a) $\int \frac{x^2+1}{x^3+3x}dx$ (b) $\int \frac{dx}{x^2+2x+1}$ (c) $\int \frac{x^3+1}{x}dx$ (d) $\int \frac{x+1}{\sqrt{x^2+2x-3}}dx$

(e) $\int \frac{x+4}{x^3+4x}dx$ (f) $\int \frac{dx}{x^3-1}$ (g) $\int x\sin^{-1}x\,dx$ (h) $\int x^2\tan^{-1}x\,dx$

21 Find:

(a) $\int \frac{dx}{x^2-4x-1}$ (b) $\int \frac{dx}{3x^2+6x+10}$ (c) $\int \frac{dx}{\sqrt{x^2-4x+1}}$ (d) $\int \frac{dx}{\sqrt{x^2+16}}$

22 (a) Sketch the graph of the curve with equation $y = \frac{1}{x(3-x)}$ for $0 < x < 3$.

(b) Find the area bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$.

23 Calculate the area of the region bounded by the curve $y = xe^{-x}$, the x-axis and the line $x = 1$.

24 Sketch the graph of $y = \frac{\cos x}{1+\cos x}$ for $-\pi < x < \pi$, stating the coordinates of its intersection with the x-axis and of the turning point. Find the area of the region bounded by the curve and the x-axis.

25 Find the particular solution of $\frac{dy}{dx} = \frac{1}{x^2+1} + 1$ given that $y(0) = 1$.

26 Find the general solution of $\frac{dy}{dx} = (2-y)(2+y)$ given that $y(0) = 1$.

27 (a) Using the substitution $u = a - x$, or otherwise, prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

(b) Hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

CHAPTER 6
Mechanics

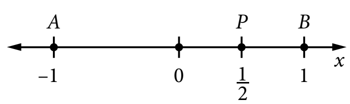
6.1 VELOCITY AND ACCELERATION AS FUNCTIONS OF *x*

The expressions for displacement $x(t)$, velocity $v(t) = \dot{x}(t)$ and acceleration $a(t) = \ddot{x}(t) = \dot{v}(t)$ all clearly define functions of time t. This makes sense because an ordinary particle cannot be in two different places at the same time: for each value of t there corresponds only one value of $x(t)$ that satisfies the function. Similarly, $v(t)$ and $a(t)$ define functions because a particle cannot be moving in two different ways (that is, cannot have more than one velocity and one acceleration) at any given time.

However, sometimes it is still useful to express acceleration in terms of the displacement x or in terms of the velocity v, rather than in terms of time t. In this case you must be careful, because equations relating a, v and x will not necessarily define functions.

Consider a particle that moves so that its displacement at time $t \geq 0$ is given by $x(t) = \sin t$. This means that the particle is moving back and forth between the limits $x = \pm 1$. (This is an example of simple harmonic motion, an important kind of motion that is explored in more detail later in this chapter.)

The particle is at $x = \frac{1}{2}$ whenever $\sin t = \frac{1}{2}$, i.e. when $t = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, \ldots$ so that $x(t)$ defines a many-to-one function of time. Similarly, $v(t) = \dot{x}(t) = \cos t$ defines a many-to-one function of time.



Note that $v^2 = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. Hence the equation $v^2 = 1 - x^2$ expresses v in terms of x. However, this equation is not a function of x, because (for example) at $x = \frac{1}{2}$ you may have $v = \pm\frac{\sqrt{3}}{2}$. The point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ corresponds to $t = \frac{\pi}{6}$, while $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ corresponds to $t = \frac{5\pi}{6}$. There is an infinite number of values of t for each value of x. As the particle moves to and fro between A and B, its velocity has two possible values at any point P in between, depending on whether the particle is moving towards A or towards B at the time. At the extremes A and B, the velocity is zero, so at these points you can say the particle is instantaneously at rest.

Remember that differential calculus needs to be applied to functions. In this context, $\frac{dv}{dx}$ can only have a useful meaning if $v = v(x)$ defines a function. Hence if you want to differentiate the equation $v^2 = 1 - x^2$ you must restrict v to be either positive or negative. This means that in the equation $v^2 = 1 - x^2$ you consider v to apply only to velocities as the particle moves from A to B, i.e. $v > 0$, or only from B to A, i.e. $v < 0$.

If $v(x)$ specifies a velocity function of x according to one of these restrictions, then you can use the chain rule of differentiation to calculate acceleration:

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{dx} \times \frac{dx}{dt} \\ &= v\frac{dv}{dx} \quad \text{as } v = \frac{dx}{dt} \\ &= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx} \\ &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\end{aligned}$$

Hence acceleration may be expressed in any of the forms $\frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

The form to use in a particular problem will depend on the form of the equation that defines acceleration:

- Given $a = f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a = g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

- Given $a = h(v)$, use $\begin{cases} \dfrac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\dfrac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v \end{cases}$

It is customary to write derivatives with respect to time using dots above the dependent variable,
e.g. $\dot{x} = \dfrac{dx}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, $\dot{v} = \dfrac{dv}{dt}$.

Example 1

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If the acceleration is $3 - 2x$, find v in terms of x given that $v = 2$ when $x = 1$.

Solution

$\ddot{x} = 3 - 2x$ gives the acceleration as a function of x and the initial conditions are v and x, so to find v as a function of x:

Use $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$: $\quad \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 3 - 2x$

Integrate with respect to x: $\quad \dfrac{1}{2}v^2 = \int (3 - 2x)\,dx$

$$\frac{1}{2}v^2 = 3x - x^2 + C$$

At $x = 1$, $v = 2$: $\quad 2 = 3 - 1 + C$

$$C = 0$$

$$\therefore \quad \frac{1}{2}v^2 = 3x - x^2$$

$$v^2 = 6x - 2x^2$$

$$v = \pm\sqrt{6x - 2x^2}$$

The conditions include $x = 1$, $v = 2$, which means that v is positive when $x = 1$.
To satisfy the initial conditions you must take the positive square root.

$$\therefore \quad v = \sqrt{6x - 2x^2}$$

It is worth noting the domain of the velocity function. For v to exist, $6x - 2x^2 \geq 0$, so $0 \leq x \leq 3$: this means that the motion exists only for $0 \leq x \leq 3$.

Example 2

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = 3x^2$ and $v = -\sqrt{2}$, $x = 1$ when $t = 0$, find x as a function of t.

Solution

$\ddot{x} = 3x^2$ gives acceleration as a function of x, so to find v as a function of x:

Use $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$: $\quad \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 3x^2$

Integrate with respect to x: $\frac{1}{2}v^2 = \int 3x^2\,dx$

$$\frac{1}{2}v^2 = x^3 + C_1$$

At $x = 1$, $v = -\sqrt{2}$: $1 = 1 + C_1$

$$C_1 = 0$$

$\therefore$ $\frac{1}{2}v^2 = x^3$

$$v^2 = 2x^3$$

$$v = \pm\sqrt{2x^3}$$

The conditions include $x = 1$, $v = -\sqrt{2}$, which means that v is negative when $x = 1$.
To satisfy the initial conditions you must take the negative square root.

$\therefore$ $v = -\sqrt{2}x^{\frac{3}{2}}$

Now $v = \frac{dx}{dt}$: $\frac{dx}{dt} = -\sqrt{2}x^{\frac{3}{2}}$

Reciprocal of both sides: $\frac{dt}{dx} = -\frac{1}{\sqrt{2}}x^{-\frac{3}{2}}$

Integrate with respect to t: $t = -\frac{1}{\sqrt{2}}\int x^{-\frac{3}{2}}\,dx$

$$t = -\frac{1}{\sqrt{2}} \times \left(-\frac{2}{1}\right)x^{-\frac{1}{2}} + C_2$$

$$t = \sqrt{2}x^{-\frac{1}{2}} + C_2$$

At $x = 1$, $t = 0$: $0 = \sqrt{2} + C_2$

$$C_2 = -\sqrt{2}$$

$\therefore$ $t = \sqrt{2}x^{-\frac{1}{2}} - \sqrt{2}$

Find x in terms of t: $\sqrt{2}x^{-\frac{1}{2}} = t + \sqrt{2}$

$$x^{-\frac{1}{2}} = \frac{t+\sqrt{2}}{\sqrt{2}}$$

$$x^{\frac{1}{2}} = \frac{\sqrt{2}}{t+\sqrt{2}}$$

$$x = \frac{2}{\left(t+\sqrt{2}\right)^2}$$

Example 3

The velocity of a particle is $v = 3x + 7\text{ m s}^{-1}$.

(a) Find an expression for the acceleration.

(b) If the initial displacement is 1 m to the right of the origin, find the displacement as a function of time.

Solution

(a)

$$v = 3x + 7$$

$$\frac{dv}{dx} = 3$$

Use $\ddot{x} = v\frac{dv}{dx}$: $\quad \ddot{x} = (3x+7)\times 3$

$$= 3(3x+7)\ \mathrm{m\,s^{-2}}$$

(b)

$$\frac{dx}{dt} = 3x+7$$

Reciprocal of both sides: $\quad \frac{dt}{dx} = \frac{1}{3x+7}$

Integrate with respect to x: $\quad t = \int \frac{dx}{3x+7}$

$$t = \frac{1}{3}\log_e(3x+7) + C$$

At $t = 0$, $x = 1$: $\quad C = -\frac{1}{3}\log_e 10$

$\therefore \quad t = \frac{1}{3}\log_e(3x+7) - \frac{1}{3}\log_e 10$

$$t = \frac{1}{3}\log_e\left(\frac{3x+7}{10}\right)$$

$$3t = \log_e\left(\frac{3x+7}{10}\right)$$

$$\left(\frac{3x+7}{10}\right) = e^{3t}$$

$$3x + 7 = 10e^{3t}$$

$$x = \frac{1}{3}\left(10e^{3t} - 7\right)$$

Example 4

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = -e^{-\frac{x}{2}}$ and $v = 2$, $x = 0$ when $t = 0$, find x as a function of t.

Solution

Use $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\quad \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-\frac{x}{2}}$

Integrate with respect to x: $\quad \frac{1}{2}v^2 = -\int e^{-\frac{x}{2}}\,dx$

$$\frac{1}{2}v^2 = 2e^{-\frac{x}{2}} + C_1$$

At $x = 0$, $v = 2$: $\quad 2 = 2 + C_1$

$$C_1 = 0$$

$\therefore \quad \frac{1}{2}v^2 = 2e^{-\frac{x}{2}}$

$$v^2 = 4e^{-\frac{x}{2}}$$

$$v = \pm 2e^{-\frac{x}{4}}$$

The conditions include $x = 0$, $v = 2$, which means that v is positive when $x = 0$; also $e^{-\frac{x}{4}} > 0$ for all x. To satisfy the initial conditions you must take the positive square root.

$\therefore \quad v = 2e^{-\frac{x}{4}}$

Now $v = \frac{dx}{dt}$: $\quad \frac{dx}{dt} = 2e^{-\frac{x}{4}}$

Reciprocal of both sides: $\quad \frac{dt}{dx} = \frac{1}{2}e^{\frac{x}{4}}$

Integrate with respect to t:	$t=\frac{1}{2}\int e^{\frac{x}{4}}\,dx$
	$t=\frac{1}{2}\times 4e^{\frac{x}{4}}+C_2$
	$t=2e^{\frac{x}{4}}+C_2$
At $t=0$, $x=0$:	$0=2+C_2$
	$C_2=-2$
$\therefore$	$t=2e^{\frac{x}{4}}-2$
Find x in terms of t:	$e^{\frac{x}{4}}=\frac{t+2}{2}$
	$\frac{x}{4}=\log_e\left(\frac{t+2}{2}\right)$
	$x=4\log_e\left(\frac{t+2}{2}\right)$

EXERCISE 6.1 VELOCITY AND ACCELERATION AS FUNCTIONS OF *x*

1 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If its acceleration is given by $\ddot{x}=4+x$ and $v=1$ when $x=0$, find v when $x=1$.

2 The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2}=7-2x$ and the particle starts from rest at the point where $x=0$. Find the speed in terms of x. What values can x take?

3 At time t, the displacement of a particle moving in a straight line is x. If the acceleration is given by $\frac{d^2x}{dt^2}=3-4x$ and the particle starts from rest at $x=1$, find its velocity at any position. At what other point, if any, does the particle come to rest?

4 The acceleration of a particle moving in a straight line is given by $\ddot{x}=kx$ where $k=1$. If $\dot{x}=10\text{ m s}^{-1}$ when $x=6\text{ m}$, the velocity of the particle when it is 15 m from the origin is:

A -17 m s^{-1} **B** $\sqrt{79}\text{ m s}^{-1}$ **C** 17 m s^{-1} **D** 23 m s^{-1}

5 A particle leaves O with velocity 2 m s^{-1}. Its acceleration is $-\frac{1}{6}\sqrt[3]{x}\text{ m s}^{-2}$ when its displacement from O is x metres. Find its displacement when it first comes to rest.

6 A particle moves in a straight line and its acceleration at any time t is $\cos x$. If $v=0$ and $x=0$ when $t=0$, express v in terms of x.

7 A particle moves in a straight line and its acceleration at any time is given by $\frac{d^2x}{dt^2}=\sin^2 x$. Find $\frac{dx}{dt}$ given that $\frac{dx}{dt}=1$ when $x=0$.

8 If $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=2x-3x^2$ and $v=2$ when $x=0$, find v in terms of x.

9 A particle moves in a straight line. At time t its displacement from a fixed origin is x. If $\ddot{x}=x-3$ and $\dot{x}=-2$ when $x=1$, find $\dot{x}$ when $x=0$.

10 The velocity of a particle is given by $v=4+x^2\text{ m s}^{-1}$.

(a) Find the acceleration as a function of x.

(b) If initially $x=-2\text{ m}$, what is the displacement after $\frac{\pi}{4}$ seconds?

11 If $\frac{dx}{dt}=\frac{1}{x+4}$ and $x=0$ when $t=0$, find t when $x=2$.

12 If $\frac{dx}{dt}=x+4$ and $x=-3$ when $t=0$, express x in terms of t.

13 The velocity of a particle at any time t is $2(4-x)^2$. Find its position $x(t)$ at any time t given $x(0)=0$.

14 If $\frac{dx}{dt}=(3-x)^2$ and $x=2$ when $t=0$, find: **(a)** x as a function of t **(b)** $\frac{d^2x}{dt^2}$ as a function of x.

15 A particle moves in a straight line. At time t its displacement from a fixed origin is x. If $\dot{x}=x+3$:

(a) express $\ddot{x}$ in terms of x **(b)** find x when $t=1$, given that $x=-2$ when $t=0$.

16 The acceleration of a body moving under gravitational attraction towards a planet varies inversely as the square of its distance from the centre of the planet. This can be written as $\frac{d^2x}{dt^2}=-\frac{k}{x^2}$ where x is the distance from the centre of the planet and k is a constant. If the body starts from rest at a distance a from the centre of the planet, show that its speed at x (before it hits the planet) is given by $\frac{dx}{dt}=\sqrt{\frac{2k(a-x)}{ax}}$.

17 A particle is moving in a straight line with its acceleration as a function of x given by $\ddot{x}=-e^{-2x}$. It is initially at the origin and travelling with a velocity of 1 metre per second.

(a) Show that $\dot{x}=e^{-x}$. **(b)** Hence show that $x=\log_e(t+1)$.

18 A particle is moving so that $\ddot{x}=32x^3+48x^2+16x$. Initially $x=-2$ and the velocity v is -8.

(a) Show that $v^2=16x^2(1+x)^2$. **(b)** Hence, or otherwise, show that $-4t=\int\frac{1}{x(1+x)}dx$.

(c) It can be shown that for some constant C, $\log_e\left(1+\frac{1}{x}\right)=4t+C$. Using this equation and the initial conditions, find x as a function of t.

19 The acceleration of a particle P is given by the equation $\frac{d^2x}{dt^2}=32x(x^2+9)$, where x metres is the displacement of P from a fixed point O after t seconds. Initially the particle is at O and has a velocity 36 m s^{-1} in the positive direction.

(a) Show that the speed at any position x is given by $4(x^2+9)\text{ m s}^{-1}$.
(b) Hence find the time taken for the particle to travel 3 metres from O.

20 A body falls from rest so that its velocity v metres per second after t seconds is $v=80(1-e^{-0.4t})$.

(a) Show that the acceleration is proportional to $(80-v)$.
(b) Calculate the distance fallen in the first five seconds.
(c) Calculate the distance fallen when $v=60$.

21 A particle is brought to top speed by an acceleration that varies linearly with the distance travelled, i.e. $\ddot{x}=kx+C$ where k and C are constants. It starts from rest with an acceleration of 3 m s^{-2} and reaches top speed in a distance of 160 metres. Find:

(a) the top speed **(b)** the speed when the particle has moved 80 metres.

6.2 SIMPLE HARMONIC MOTION (SHM)

Consider the motion of a particle moving along a straight line back and forth, so that its displacement from a fixed point at time t is given by a sine or cosine function.

Example 5

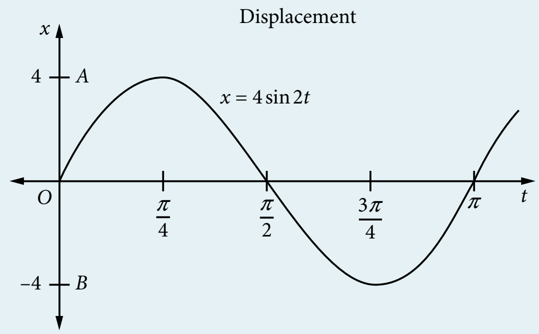
A particle moves in a straight line so that its displacement x m from a fixed point O at time t seconds is defined by $x = 4\sin 2t$. After considering properties of the graph of $x = 4\sin 2t$ to analyse the motion of the particle, find expressions for:

(a) the velocity **(b)** the acceleration. **(c)** Discuss the motion of the particle.

Solution

Look at the properties of the sine curve to help determine the nature of the movement of the particle.

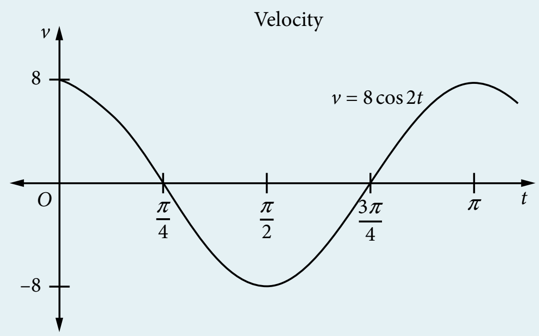
- When $t = 0$, $x = 0$
- When $t = \frac{\pi}{4}$, $x = 4$
- When $t = \frac{\pi}{2}$, $x = 0$
- When $t = \frac{3\pi}{4}$, $x = -4$
- When $t = \pi$, $x = 0$



Remember that the particle is moving in a straight line along the x-axis. It starts at $x = 0$ and takes $\frac{\pi}{4}$ seconds (approximately 0.8 s) to move to A, another $\frac{\pi}{4}$ seconds to return to O, another $\frac{\pi}{4}$ seconds to move to B and then another $\frac{\pi}{4}$ seconds to return to O again. This pattern of movement repeats every π seconds. You say that the particle **oscillates** from A to B about the point O, the centre of the motion, with a **period** of π. The distance between O and the extreme positions A and B (in this case 4 m) is called the **amplitude**.

(a) $v = \frac{dx}{dt} = 8\cos 2t$ [1]

- When $t = 0$, $v = 8$ at $x = 0$
- When $t = \frac{\pi}{4}$, $v = 0$ at $x = 4$
- When $t = \frac{\pi}{2}$, $v = -8$ at $x = 0$
- When $t = \frac{3\pi}{4}$, $v = 0$ at $x = -4$
- When $t = \pi$, $v = 8$ at $x = 0$



In the original displacement diagram above, the particle is at rest at A and B because $v = 0$ when $t = \frac{\pi}{4}$ at $x = 4$ and when $t = \frac{3\pi}{4}$ at $x = -4$.

The velocity diagram above right shows that $v = 0$ when $t = \frac{\pi}{4}$ and when $t = \frac{3\pi}{4}$.

When $t = 0$, $x = 0$ and $v = 8$. When $t = \frac{\pi}{2}$, $x = 0$ and $v = -8$. This means that at O, $v = 8$ when the particle is travelling in the positive direction (towards A) but $v = -8$ when the particle is travelling in the negative direction (towards B).

Squaring [1]: $v^2 = 64\cos^2 2t$

$$v^2 = 64(1 - \sin^2 2t)$$

$$v^2 = 64\left(1 - \frac{x^2}{16}\right)$$

$$v^2 = 4(16 - x^2) \qquad [2]$$

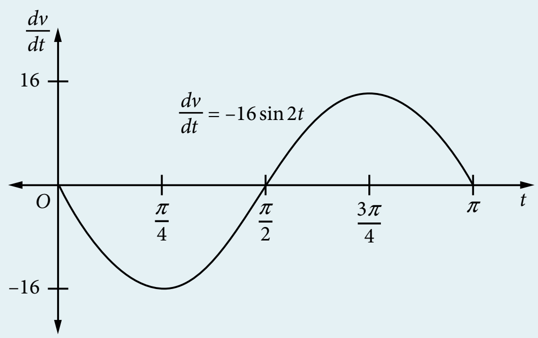
Equation [2] gives the velocity in terms of x.

(b) $\ddot{x} = \dfrac{dv}{dt} = -16\sin 2t \qquad [3]$

$\ddot{x} = -4x \qquad [4]$

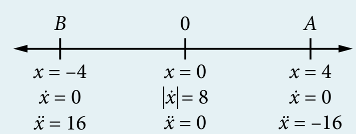
Equation [3] gives the acceleration in terms of t. Equation [4] gives the acceleration in terms of x. $\ddot{x} = -4x$ tells us that when $x = 0$, $\ddot{x} = 0$; when $x = 4$ (at A), $\ddot{x} = -16$; and when $x = -4$ (at B), $\ddot{x} = 16$. $\ddot{x}$ and x always have the opposite sign.

- When $t = 0$, $\ddot{x} = 0$ at $x = 0$
- When $t = \frac{\pi}{4}$, $\ddot{x} = -16$ at $x = 4$
- When $t = \frac{\pi}{2}$, $\ddot{x} = 0$ at $x = 0$
- When $t = \frac{3\pi}{4}$, $\ddot{x} = 16$ at $x = -4$
- When $t = \pi$, $\ddot{x} = 0$ at $x = 0$



(c) In this type of motion, the particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and the magnitude of this acceleration is proportional to its distance from the fixed point.

Taking O as the fixed point, the above description tells you that when the particle is at A ($x > 0$) its acceleration is towards O ($\ddot{x} < 0$), while when the particle is at B ($x < 0$) its acceleration is again towards O ($\ddot{x} > 0$).



Note that the equation of motion $x = 4\sin 2t$ in this example could also be written as $x = 4\cos\left(2t - \frac{\pi}{2}\right)$.

In a situation such as Example **5**, $\ddot{x}$ and x are always opposite in sign and the magnitude of the acceleration is always proportional to the distance from O. This means you can take the differential equation $\ddot{x} = -4x$ from this example and write it as a general equation $\ddot{x} = -kx$, where k can be any positive constant, to define all motion of this type.

This type of motion is called **simple harmonic motion** (SHM) and it can be applied to many real-life physical situations. Because k is a positive constant, it is usually replaced by n^2 so that $\ddot{x} = -n^2 x$ is the basic equation of SHM.

In general, for simple harmonic motion you have:

- displacement x: $x = a\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$ OR $x = a\sin(nt + \alpha)$, $\alpha > 0$, $n > 0$
- velocity, $\dot{x}$: $\dot{x} = -an\sin(nt + \alpha)$; $\dot{x} = an\cos(nt + \alpha)$
- acceleration $\ddot{x}$: $\ddot{x} = -an^2\cos(nt + \alpha)$; $\ddot{x} = -an^2\sin(nt + \alpha)$
  or $\ddot{x} = -n^2 x$; $\ddot{x} = -n^2 x$
- Squaring the velocity:

$$v = -an\sin(nt + \alpha) \qquad v = -an\cos(nt + \alpha)$$

$$v^2 = a^2n^2\sin^2(nt + \alpha) \qquad v^2 = a^2n^2\cos^2(nt + \alpha)$$

$$= n^2\left(a^2 - a^2\cos^2(nt + \alpha)\right) \qquad = n^2\left(a^2 - a^2\sin^2(nt + \alpha)\right)$$

$$= n^2\left(a^2 - x^2\right) \qquad = n^2\left(a^2 - x^2\right)$$

So, regardless of your starting point you have the very useful result: $v^2 = n^2(a^2 - x^2)$.

As you work through this chapter, you will discover which form (sine or cosine) is best to use in a particular situation.

Of course, if you start with the differential equation $\ddot{x} = -n^2x$ and use either form, the forms will only differ by a constant, which will be given by the initial conditions.

Important results

1 When $\dot{x} = 0$ (e.g. at A and B in Example **5**) the magnitude of the acceleration is greatest.

2 When $\ddot{x} = 0$ (i.e. at O, the centre of the motion), the speed is greatest (i.e. the velocity has its greatest or least value).

The general equation $x = a\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$

If when $t = 0$, $x = a$, the particle is initially at A (the extreme point) and $a = a\cos\alpha$, then $\cos\alpha = 1$ and $\alpha = 0$. The equation of motion can then be written $x = a\cos nt$.

$$x = a\cos nt \qquad [1]$$

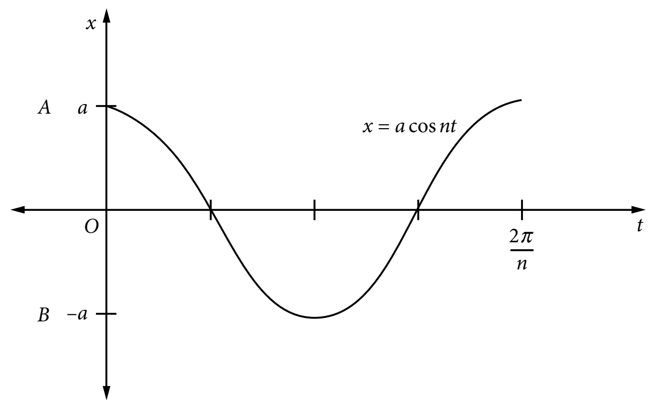
$$v = \dot{x} = -na\sin nt \qquad [2]$$

$$v^2 = n^2a^2\sin^2 nt$$
$$= n^2a^2(1 - \cos^2 nt)$$
$$= n^2a^2\left(1 - \frac{x^2}{a^2}\right)$$

$$v^2 = n^2(a^2 - x^2) \qquad [3]$$

$$\frac{dv}{dt} = \ddot{x} = -n^2a\cos nt$$

$$\ddot{x} = -n^2x \qquad [4]$$



- The period T is the time for one complete oscillation: $T = \dfrac{2\pi}{n}$
- The frequency f is the number of oscillations per unit time: $f = \dfrac{1}{T} = \dfrac{n}{2\pi}$
- The amplitude a is the distance from the centre of motion O to either of the extreme points A or B.

Simple harmonic motion problems are usually solved using either $x = a\cos(nt + \alpha)$ or $\ddot{x} = -n^2x$, or occasionally $v^2 = n^2(a^2 - x^2)$. The starting point will depend on the information given.

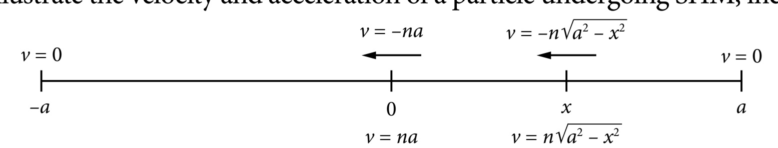
Simple harmonic motion (SHM)—summary

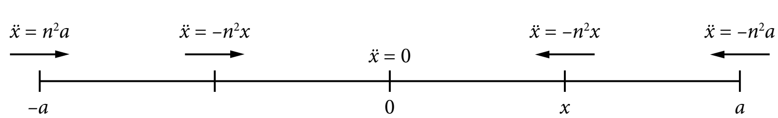
$x = a\cos(nt + \alpha)$	$\dfrac{dv}{dt} = \ddot{x} = -n^2a\cos nt$	$v^2 = n^2(a^2 - x^2) \quad -a \le x \le a$
$= a\cos nt \quad \text{if } x(0) = a$		$T = \dfrac{2\pi}{n} = \dfrac{1}{f}$
$v = \dot{x} = -na\sin nt$	$\ddot{x} = -n^2x$	

Note:

- $x = a\sin(nt + \alpha)$ can also describe the displacement function, but it is more conventional to use $x = a\cos(nt + \alpha)$.
- In problems involving a pendulum, the motion usually starts at the maximum displacement, so $\alpha = 0$ and the equation of motion becomes $x = a\cos nt$.

The following diagrams illustrate the velocity and acceleration of a particle undergoing SHM, including the extreme values.





Note that to the left of O, x is negative, so $-n^2x$ (the acceleration) is positive.

The equation $\ddot{x} = -n^2x$ describes the motion of a particle under the influence of a force that is directed towards the origin O and is proportional to the distance of the particle from O. The force (and hence the acceleration) is zero at O, where the speed is greatest. The magnitude of the force (and hence the acceleration) is greatest at the extreme points, where the speed is zero.

This type of motion occurs in real physical situations (either approximately or exactly) where a particle oscillates about an equilibrium position. For example:

- the to-and-fro motion of a pendulum bob
- the up-and-down motion of a mass attached to a spring
- the bobbing motion of a buoy floating on water.

MAKING CONNECTIONS

Simple harmonic motion

Use technology to explore the representation of a point moving along a straight line as a trigonometric function.

Example 6

The displacement x m of a particle moving in a straight line is given by $x = 4\cos 6t$. Discuss the motion of the particle.

Solution

$x = 4\cos 6t$ is of the form $x = a\cos(nt + \alpha)$, so the motion is simple harmonic about the origin.

Amplitude $a = 4$ $\quad$ $n = 6$ $\quad$ Period $= \frac{2\pi}{n} = \frac{2\pi}{6} = \frac{\pi}{3}$

When $t = 0$: $x = 4\cos 0 = 4$

$\therefore$ The particle starts 4 m to the right of O.

Velocity: $\dot{x} = -24\sin 6t$

When $t = 0$: $\dot{x} = -24\sin 0 = 0$

$\therefore$ The particle is initially at rest.

Acceleration: $\ddot{x} = -144\cos 6t$

$\ddot{x} = -36x$

When $t = 0$: $\ddot{x} = -144$

$\therefore$ The initial acceleration is 144 m s^{-2} towards O.

The motion is simple harmonic with amplitude 4 m, period $\frac{\pi}{3}$ seconds, initially at rest 4 m to the right of O with an acceleration of 144 m s^{-2} towards O.

Example 7

The motion of a particle moving along a straight line is given by the equation $\frac{d^2x}{dt^2} = -16x$.

If $x = 0$ and $v = 4$ when $t = 0$, find its displacement at any time t and state the period and amplitude.

Solution

Because $\frac{d^2x}{dt^2} = -16x = -n^2x$ where $n = 4$, the motion is simple harmonic.

$\therefore$ $x = a\cos(nt + \alpha)$

For $n = 4$: $x = a\cos(4t + \alpha)$ $\quad$ [1]

$v = \dot{x} = -4a\sin(4t + \alpha)$ $\quad$ [2]

When $t = 0$, $x = 0$ in [1]: $0 = a\cos\alpha$

$\alpha = \frac{\pi}{2}$

When $t=0$, $v=4$ in [2]: $4=-4a\sin\frac{\pi}{2}$

$$a=-1$$

$$\therefore\quad x=-\cos\left(4t+\frac{\pi}{2}\right)$$

You can use the identity $-\cos\theta=\cos(\pi-\theta)$ to remove the negative sign and write the answer in a more familiar form:

$$-\cos\left(4t+\frac{\pi}{2}\right)=\cos\left(\pi-4t-\frac{\pi}{2}\right)$$

$$=\cos\left(\frac{\pi}{2}-4t\right)$$

$$=\cos\left(4t-\frac{\pi}{2}\right)$$

$$\therefore\quad x=\cos\left(4t-\frac{\pi}{2}\right)\qquad \text{Period}=\frac{2\pi}{n}=\frac{2\pi}{4}=\frac{\pi}{2}\qquad \text{Amplitude}=a=1$$

It is worth noticing that $\cos\left(\frac{\pi}{2}-4t\right)=\sin 4t$, so the equation of motion could be written as $x=\sin 4t$.

This would make it easier to sketch the function.

Example 8

A particle moves in a straight line so that its acceleration at any time is given by $\ddot{x}=-4x$. Find its period, amplitude and displacement at time t given that at $t=0$, $x=3$ and $v=-6\sqrt{3}$.

Solution

$\ddot{x}=-4x=-n^2x$ where $n=2$, so the motion is simple harmonic.

$$x=a\cos(nt+\alpha)$$

For $n=2$: $x=a\cos(2t+\alpha)$

When $t=0$, $x=3$: $3=a\cos\alpha$ [1]

Velocity: $\dot{x}=-2a\sin(2t+\alpha)$

When $t=0$, $v=-6\sqrt{3}$: $-6\sqrt{3}=-2a\sin\alpha$

$3\sqrt{3}=a\sin\alpha$ [2]

[2] ÷ [1]: $\frac{a\sin\alpha}{a\cos\alpha}=\frac{3\sqrt{3}}{3}$

$$\tan\alpha=\sqrt{3}$$

As $a>0$: $\sin\alpha>0$ (from [2]) and $\cos\alpha>0$ (from [1]), so α is in the first quadrant and is an acute angle.

Hence: $\alpha=\frac{\pi}{3}$

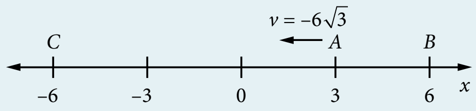
From [1]: $3=a\cos\frac{\pi}{3}$

$$a=6$$

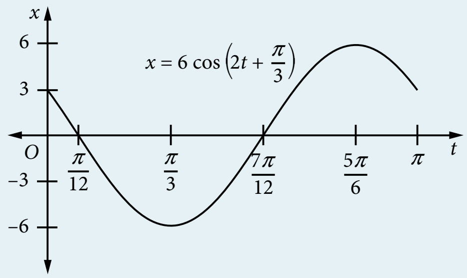
$$\therefore\quad x=6\cos\left(2t+\frac{\pi}{3}\right)$$

Hence the period $=\frac{2\pi}{2}=\pi$, amplitude $=6$ and displacement is given by $x=6\cos\left(2t+\frac{\pi}{3}\right)$.

This diagram shows the motion of the particle at $t = 0$.
It starts at A and moves towards the centre of the motion.
C and B are the extreme points of the motion.



This diagram shows the displacement x for any time t in the domain $0 \le t \le \pi$.



Example 9

The speed v m s^{-1} of a particle moving in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where the magnitude of its displacement from a fixed point O is x m. Show that the motion is simple harmonic and find:

(a) the centre of the motion **(b)** the period **(c)** the amplitude.

Solution

$v^2 = f(x)$, so it seems likely that using $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ might help.

$$v^2 = 6 + 4x - 2x^2$$

$$\frac{1}{2}v^2 = 3 + 2x - x^2$$

Differentiate with respect to x: $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 2 - 2x$

Hence: $\ddot{x} = -2(x - 1)$

With $y = x - 1$: $\ddot{y} = -2y$ (as $\ddot{y} = \ddot{x}$)

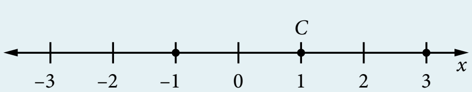
This is simple harmonic motion about $y = 0$.

(a) With $y = x - 1$, SHM about $y = 0$ is the same as SHM about $x = 1$.

(b) $n^2 = 2$, so $n = \sqrt{2}$ and the period $= \dfrac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$.

(c) The extreme positions of the particle are found where $v = 0$.

$$6 + 4x - 2x^2 = 0$$
$$-2(x^2 - 2x - 3) = 0$$
$$(x + 1)(x - 3) = 0$$
$$x = -1 \text{ or } 3$$



The particle oscillates between $x = -1$ and $x = 3$ about the centre $x = 1$, so the amplitude is 2 m.

Example 10

A particle moves in a straight line so that its position at any time t is given by $x = 3\cos 2t + 4\sin 2t$.

(a) Show that $3\cos 2t + 4\sin 2t = 5\cos(2t - \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $\tan\alpha = \frac{4}{3}$.

(b) Show that the motion is simple harmonic and find its greatest speed in metres per second.

Solution

(a)

$$\text{Expression} = 3\cos 2t + 4\sin 2t$$

$$\sqrt{3^2+4^2} = 5: \quad = 5\left(\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t\right)$$

$$\cos\alpha = \frac{3}{5},\ \sin\alpha = \frac{4}{5}: \quad = 5(\cos 2t\cos\alpha + \sin 2t\sin\alpha)$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{4}{3}: \quad = 5\cos(2t - \alpha) \text{ where } 0 < \alpha < \frac{\pi}{2} \text{ and } \tan\alpha = \frac{4}{3}$$

(b)

$$x = 5\cos(2t - \alpha)$$

$$\text{Velocity:}\quad \dot{x} = -10\sin(2t - \alpha)$$

$$\text{Acceleration:}\quad \ddot{x} = -20\cos(2t - \alpha)$$

$$= -4x$$

$\ddot{x} = -n^2x$ with $n = 2$, so the motion is simple harmonic.

The greatest speed occurs when $\ddot{x} = 0$, i.e. when $\cos(2t - \alpha) = 0$.

$$\text{Hence:}\quad 2t - \alpha = \frac{\pi}{2}$$

$$\text{Thus:}\quad \dot{x} = -10\sin\frac{\pi}{2}$$

$$= -10$$

The greatest speed is 10 m s^{-1}.

Example 11

A particle moving in a straight line with SHM has a speed of 15 m s^{-1} when passing through its mean position (the centre of the motion). Find the amplitude of the motion and the acceleration in the extreme positions, given that the period of the motion is 2 seconds.

Solution

$x = 0 \qquad |v| = 15 \qquad T = 2 = \frac{2\pi}{n}$ hence $n = \pi$.

The values of x, v and n are known. Use the result $v^2 = n^2(a^2 - x^2)$ to find a.

$$225 = \pi^2(a^2 - 0)$$

$$a^2 = \frac{225}{\pi^2}$$

$$a = \frac{15}{\pi} \quad \text{as } a > 0$$

Motion is SHM, so: $\ddot{x} = -n^2 x$

Hence: $\ddot{x} = -\pi^2 x$

At the extreme position, $x = a = \frac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \frac{15}{\pi} = -15\pi$

At the other extreme position, $x = -\frac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \left(-\frac{15}{\pi}\right) = 15\pi$

Thus the amplitude of the motion is $\frac{15}{\pi}$ metres and the magnitude of the acceleration is 15π m s^{-2}.

Example 12

A particle is moving in a straight line with SHM. The velocity of the particle is respectively $\sqrt{5}$ m s^{-1} and 2 m s^{-1} at distances of 1 m and 2 m from the centre of motion. Find:

(a) the length of the path **(b)** the period of the motion.

Solution

(a) Given $x = 1, v = \sqrt{5}$; $x = 2, v = 2$.

Hence use: $v^2 = n^2(a^2 - x^2)$

At $x = 1, v = \sqrt{5}$: $5 = n^2(a^2 - 1)$ [1]

At $x = 2, v = 2$: $4 = n^2(a^2 - 4)$ [2]

[1] ÷ [2]: $\frac{5}{4} = \frac{n^2(a^2 - 1)}{n^2(a^2 - 4)}$

$5a^2 - 20 = 4a^2 - 4$

$a^2 = 16$

$a = 4$ as $a > 0$

The amplitude of the motion is 4 m, so the length of the path ($2a$) is 8 m.

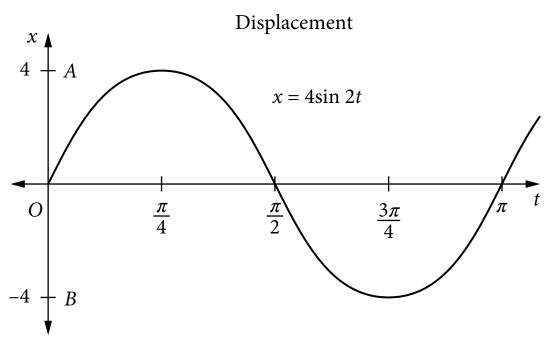
(b) Substitute $a = 4$ into [1]: $5 = 15n^2$

$\therefore\ n = \frac{1}{\sqrt{3}}$ as $n > 0$

Thus the period of the motion is $T = \frac{2\pi}{n} = 2\pi\sqrt{3}$ seconds.

Simple harmonic motion about the point $x = c$

In Example 5, you considered a particle moving in a straight line with displacement x m from a fixed point O at time t seconds defined by the equation $x = 4\sin 2t$. The graph of the displacement is given in the following diagram.



Simple harmonic motion about the point $x = c$ will involve a vertical translation of the trigonometric graph. This will be considered in the following example.

Example 13

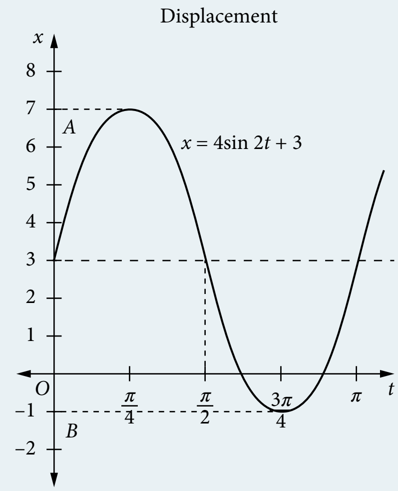
The equation of motion changes to $x = 4\sin 2t + 3$. By considering the graph of this function, analyse the motion of the particle, and find expressions for **(a)** velocity, **(b)** the acceleration. **(c)** Discuss the motion of the particle.

Solution

The effect of the +3 on the equation $x = 4\sin 2t$ is to translate the graph 3 units upwards with no horizontal translation.

The graph shows the following points.

- When $t = 0$, $x = 3$.
- When $t = \frac{\pi}{4}$, $x = 7$.
- When $t = \frac{\pi}{2}$, $x = 3$.
- When $t = \frac{3\pi}{4}$, $x = -1$.
- When $t = \pi$, $x = 3$.



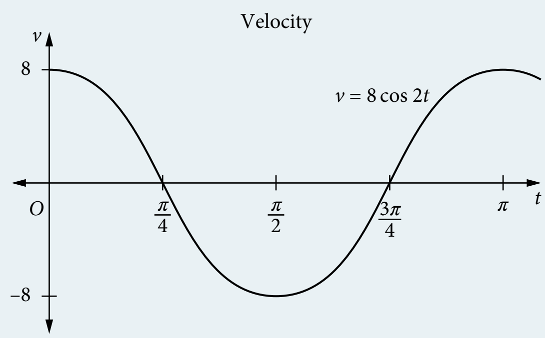
The particle is moving in a straight line along the x-axis. It starts at $x = 3$ and takes $\frac{\pi}{4}$ seconds to move to A, a distance of 4 m. It takes another $\frac{\pi}{4}$ seconds to return to $x = 3$ and then another $\frac{\pi}{4}$ seconds to move to B, a point 4 m below $x = 3$. It then takes $\frac{\pi}{4}$ seconds to return to $x = 3$.

The pattern repeats every π seconds and the particle oscillates about $x = 3$, the centre of the motion, with a period of π. The amplitude of the motion is 4.

(a) $x = 4\sin 2t + 3$

$$\frac{dx}{dt} = 8\cos 2t$$

This is the same as the velocity function in Example **5** (page 172), so shifting the centre of the motion makes no change to the velocity of the motion as a function of time.



The particle is instantaneously at rest at A and B because $v = 0$ when $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$.

When $t = 0$, $x = 3$, $v = 8$ so the particle is travelling towards A with its greatest velocity.

When $t = \frac{\pi}{2}$, $x = 3$, $v = -8$ so the particle is travelling towards B with its least velocity.

Squaring the velocity function gives $v^2 = 64\cos^2 2t$

$$v^2 = 64\left(1 - \sin^2 2t\right)$$

But $\sin 2t = \frac{x-3}{4}$: $v^2 = 64\left(1 - \left(\frac{x-3}{4}\right)^2\right)$

$$v^2 = 4\left(16 - (x-3)^2\right)$$

This gives the velocity in terms of x and reflects the shift of 3 units in the centre of the motion.

(b) $\dot{x} = v = 8\cos 2t$

$$\ddot{x} = \frac{dv}{dt} = -16\sin 2t$$

This is the same as the acceleration function in Example **5** (page 172), so shifting the centre of the motion makes no change to the acceleration of the motion as a function of time.

But $\sin 2t = \frac{x-3}{4}$: $\ddot{x} = -16 \times \frac{x-3}{4}$

$$\ddot{x} = -4(x-3)$$

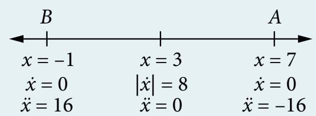
This gives the acceleration in terms of x and reflects the shift of 3 units in the centre of the motion. It shows that the acceleration is always directed towards the centre of motion.

Thus when $x = 3$, $\ddot{x} = 0$: the acceleration is zero when the particle passes through the centre of the motion.

When $x = 7$, $\ddot{x} = -16$: the acceleration is least when the displacement is greatest.

When $x = -1$, $\ddot{x} = 16$: the acceleration is greatest when the displacement is least.

(c) In this motion, the particle moves in a straight line so that its acceleration is always directed towards the fixed point in the line that is the centre of the motion. The magnitude of this acceleration is proportional to its distance from the centre of the motion.



$x = 3$ is the centre of the motion, when the particle is at A $(x > 3)$ its acceleration is towards $x = 3$ $(\ddot{x} < 0)$, while when the particle is at B $(x < 3)$ its acceleration is again towards $x = 3$ $(\ddot{x} > 0)$.

Thus the equations of motion for simple harmonic motion about the point $x = c$ may be written as:

Displacement:	$x = a\sin(nt+\alpha)+c$	$x = a\cos(nt+\alpha)+c$
Velocity:	$\dot{x} = an\cos(nt+\alpha)$	$\dot{x} = -an\sin(nt+\alpha)$
Acceleration:	$\ddot{x} = -an^2\sin(nt+\alpha)$	$\ddot{x} = -an^2\cos(nt+\alpha)$
or:	$\ddot{x} = -n^2(x-c)$	

where $x = c$ is the centre of the motion, a is the amplitude, $\frac{2\pi}{n}$ is the period and $\frac{\alpha}{n}$ the phase shift.

Example 14

A particle is moving with SHM about the point $x = 5$ cm. The particle starts from rest at the point $x = 14$ cm with a period of 4π seconds. Calculate:

(a) the amplitude **(b)** the acceleration when $t = 4$.

Solution

(a) The centre of motion is $x = 5$ so the displacement is given by: $x = a\cos(nt+\alpha)+5$

$t = 0$, $x = 14$: $14 = a\cos\alpha + 5$

$$a\cos\alpha = 9$$

Velocity is given by: $\dot{x} = -an\sin(nt+\alpha)$

$t = 0$, $\dot{x} = 0$: $0 = -an\sin\alpha$

$$\alpha = 0$$

$$a\cos 0 = 9$$

$$a = 9$$

The amplitude is 9.

(b) Period: $4\pi = \frac{2\pi}{n}$

$$n = \frac{1}{2}$$

$$\dot{x} = -9n\sin nt$$

$n = \frac{1}{2}$: $\dot{x} = -\frac{9}{2}\sin\frac{t}{2}$

$$\ddot{x} = -\frac{9}{4}\cos\frac{t}{2}$$

$t = 4$: $\ddot{x} = -\frac{9}{4}\cos 2$

$$\approx 0.936 \text{ cm s}^{-2}$$

Example 15

Assume that the tides of the ocean rise and fall in SHM. The depth of water y metres in a harbour channel is given by $y = 10 - 2\cos\frac{\pi t}{6}$ where t is the number of hours after low tide. On a particular day, low tide occurs at 7 a.m. when the channel is 8 metres deep. On the same day, high tide occurs at 1 p.m. when the channel is 12 metres deep.

(a) Show that these values satisfy the equation $y = 10 - 2\cos\frac{\pi t}{6}$.

(b) What is the depth of the channel at 10 a.m.?

(c) State the amplitude and period of the motion.

(d) Draw the graph of $y = 10 - 2\cos\frac{\pi t}{6}$ for $0 \le t \le 12$.

(e) A ship needs at least 9 metres of water in the channel to be able to pass through safely. Use your graph to find when the ship can safely pass through the channel.

(f) Find the answer to part **(e)** algebraically.

Solution

(a) At 7 a.m., $t = 0$:

$$y = 10 - 2\cos(0) = 10 - 2 = 8\text{ m}$$

The depth at 7 a.m. is 8 metres.

At 1 p.m., $t = 6$:

$$y = 10 - 2\cos(\pi) = 10 - 2 \times (-1) = 10 + 2 = 12\text{ m}$$

The depth at 1 p.m. is 12 metres.

(b) 10 a.m., $t = 3$:

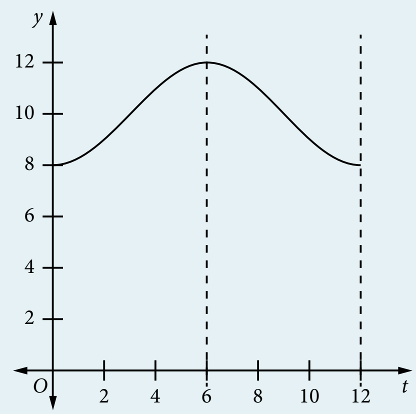
$$y = 10 - 2\cos\left(\frac{\pi}{2}\right) = 10 - 0 = 10\text{ m}$$

The depth at 10 a.m. is 10 metres.

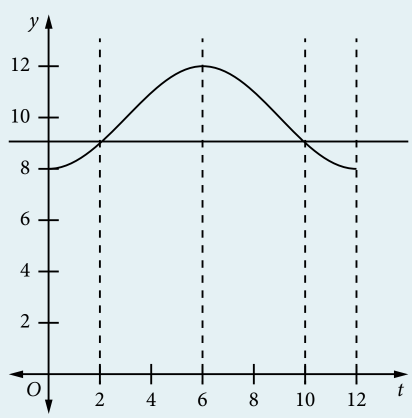
(This is halfway between low tide and high tide.)

(c) Amplitude = 2 m; period = 12 hours

(d)



(e) Draw the line $y = 9$ on the graph. When the curve is above the line it is safe to pass through the channel.



It is safe to pass through the channel from 2 hours after low tide (i.e. after 9 a.m.) until 10 hours after low tide (i.e. before 5 p.m.).

(f) To find the answer algebraically, solve $10 - 2\cos\left(\frac{\pi t}{6}\right) \ge 9$ as an equation and interpret the result:

$$2\cos\left(\frac{\pi t}{6}\right) = 1$$

$$\cos\left(\frac{\pi t}{6}\right) = \frac{1}{2}$$

$$\frac{\pi t}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\therefore \quad t = 2, 10$$

EXERCISE 6.2 SIMPLE HARMONIC MOTION (SHM)

1 The displacement x m of a particle moving in a straight line is given by $x = 6\cos 4t$. Describe the motion of the particle.

2 The equation of motion of a particle moving with simple harmonic motion is $\ddot{x} = -9x$. Find its period, amplitude and greatest speed if: **(a)** $x = 0$, $\dot{x} = 2$ when $t = 0$ **(b)** $x = 2$, $\dot{x} = 2$ when $t = 0$.

3 A particle is moving in a straight line. If x metres is its displacement at time t seconds and $\left(\frac{dx}{dt}\right)^2 = 5\left(4 - x^2\right)$, find the acceleration in terms of x only. Show that the motion is simple harmonic and find its period and amplitude.

4 The velocity v metres per second of a particle moving in simple harmonic motion along the x-axis is given by the equation $v^2 = 100 - 4x^2$. The amplitude in metres of the motion of the particle is:

A 2 **B** 5 **C** 10 **D** 50

5 The displacement of a particle at time t is given by $x = 8\sin 4t + 15\cos 4t$. What is the maximum velocity of the particle?

A $2\sqrt{17}$ **B** 17 **C** 34 **D** 68

6 A particle undergoing simple harmonic motion in a straight line has an acceleration given by $\ddot{x} = 36 - 6x$, where x is the displacement after t seconds. Where is the centre of the motion?

A $x = 0$ **B** $x = 6$ **C** $x = 12$ **D** $x = 18$

7 The displacement of a particle at time t is given by $x = 5\sin 2t + 12\cos 2t$. What is the maximum acceleration of the particle?

A 52 **B** 26 **C** 13 **D** 4

8 A particle is moving along the x-axis in simple harmonic motion centred at the origin.
When $x = 2$, the velocity of the particle is 5.
When $x = 5$, the velocity of the particle is 4. Find:

(a) the amplitude of the motion **(b)** the period of the motion.

9 A particle moves in a straight line. At time t seconds, its displacement x cm from a fixed point O in the line is given by $x = 5\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$. Express the acceleration in terms of x only and hence show that the motion is simple harmonic. Find:

(a) the period **(b)** the amplitude **(c)** the speed when $x = -2.5$ **(d)** the acceleration when $x = -2.5$.

10 The speed of a point moving along the x-axis is v m s^{-1} and the displacement of the point from the origin is x m. Prove in each case below that the motion is simple harmonic, finding **(i)** the centre of motion and **(ii)** the period and amplitude, when v is given by:

(a) $v^2 = 300 + 100x - 25x^2$ **(b)** $v^2 = 128 - 32x - 16x^2$ **(c)** $v^2 = 6 + 4x - 2x^2$

11 A particle is moving in SHM along the x-axis. Its velocity v at position x is given by $v^2 = 30 - 4x - 2x^2$. Find:

(a) all values of x for which the particle is at rest
(b) an expression for the acceleration of the particle in terms of x
(c) the maximum speed of the particle.

12 The displacement x metres at time t seconds of a point moving in a straight line is given by $x = a\cos(nt + \varepsilon)$. Find the form that this expression takes if initially:

(a) $\dot{x} = 0$ and $x = -5$ **(b)** $x = 0$ and the velocity is negative.

13 The velocity of a particle moving in a straight line is $3\sqrt{16 - x^2}$ m s^{-1} where x m is the displacement of the particle from a fixed point in the line. Is the motion simple harmonic? Find the acceleration when $x = 4$. What is the maximum speed?

14 Solve the differential equation $\frac{d^2x}{dt^2} + 16x = 0$ subject to the conditions $x = 3$ and $\frac{dx}{dt} = 16$ when $t = 0$. Find the maximum displacement and the maximum speed if x metres is the displacement of the particle moving in a straight line at time t seconds.

15 A particle moves with simple harmonic motion. If the particle starts from the equilibrium position (centre of motion) with velocity 4 m s^{-1} towards the origin and the period is $\frac{\pi}{2}$ seconds, find:

(a) the displacement at time t **(b)** the amplitude.

16 A particle moves with simple harmonic motion. When it is 2 m from its equilibrium position, its velocity is 6 m s^{-1}; when it is 3 m from equilibrium, its velocity is 4 m s^{-1}. Find the period of this motion and the amplitude.

17 If $x = a \sin nt + b \cos nt$, find the velocity and acceleration of a particle whose displacement from a fixed point O is x at time t. Show that $\ddot{x} = -n^2x$ and find the amplitude and maximum speed.

18 The position x m of a particle relative to a fixed point O at any time t seconds is $x = 5 - 2\cos^2 t$. By finding the acceleration in terms of x, show that the motion is simple harmonic. Find:

(a) the centre of motion **(b)** the period **(c)** the amplitude.

19 A particle executes simple harmonic motion of period 8 seconds and amplitude 10 m. Calculate the velocity and acceleration when its displacement is 6 m from the centre of motion. Find also its maximum acceleration.

20 A particle moves in a straight line so that its position x metres from a fixed point O at time t seconds is given by $x = 10 + 8\sin 2t + 6\cos 2t$. Prove that the motion is simple harmonic. Find the period and amplitude.

21 A particle moves in a straight line. At time t seconds its distance from a point O on the line is x metres. The following is an incomplete table of observations:

t	0	7	9	11	18
x	0			0.5	0

Complete the table using two different assumptions, namely:

(a) that the particle moves with uniform acceleration

(b) that the particle performs simple harmonic motion with a period of 12 seconds.

22 A floating buoy oscillates up and down with the waves, rising and falling 2 metres about its mean position. Find its greatest velocity and acceleration if the period of the motion is 3 seconds.

23 A particle moving with SHM starts from rest at $x = 5$ and after two seconds reaches $x = 2.5$. Find:

(a) an expression for the displacement at time $t \geq 0$ **(b)** the speed at $x = 0$

(c) the amplitude, frequency and period **(d)** the maximum speed **(e)** the maximum acceleration.

24 The amplitude of a particle moving with SHM is 5 m and the acceleration when 2 m from the mean position is 4 m s^{-2}. Find the speed of the particle when at the mean position and when 4 m from the mean position.

25 A particle is moving with SHM of period π seconds and a maximum velocity of 8 metres per second. Find the amplitude and the velocity at a distance of 3 metres from the central position.

26 A particle executes SHM of period $\frac{2\pi}{\sqrt{5}}$ seconds. If it starts from rest with displacement 10 metres, find:

(a) the frequency **(b)** the amplitude.

27 A point moving with SHM has a speed of 5 m s^{-1} when passing through its mean position O. Find the speed and acceleration when it is 1.5 m from O, given that the period is π seconds.

28 A particle is moving with SHM of amplitude 10 metres. Find how much time it takes to travel 6 metres from its mean position if the period is 10 seconds.

29 A point moves with SHM in such a way that its speed is 8 and 6 m s^{-1} respectively at distances 3 and 4 m from the mean position. Calculate the period of the motion and the magnitude of the greatest acceleration.

30 Assume that the tides rise and fall in SHM. At low tide the channel in a harbour is 9 m deep and at high tide it is 12 m deep. Low tide is at 9 a.m. and high tide is at 4 p.m.

(a) What is the depth of the channel at 12:30 p.m.? **(b)** What is the amplitude of the motion?

(c) What is the period of the motion?

(d) Show that the depth y m of the water in the channel is given by $y = 10.5 - 1.5\cos\frac{\pi t}{7}$ where t is the number of hours after low tide.

(e) A ship needs at least 10 m of water to pass through the channel safely. Between what times can the ship safely navigate the channel?

31 Assume that over several days of constant weather, the cycle of changing temperature each day is simple harmonic between 13°C at 4 a.m. and 23°C at 4 p.m.

(a) What is the temperature at 10 a.m.?

(b) Show that the pattern of temperature can be represented by $T = 18 - 5\cos\frac{\pi t}{12}$ where T is the temperature in °C and t is the time in hours after 4 a.m.

(c) At what time of day would the temperature be: (i) 18°C (ii) 15°C (iii) 21°C?

32 A particle is moving in simple harmonic motion in a straight line. Its maximum speed is 3 m s^{-1} and its maximum acceleration is 8 m s^{-2}. Find the amplitude and period of the motion.

33 (a) A particle is travelling in a straight line. Its displacement from the origin is x metres at time t seconds. If $x = \cos 2t - \sqrt{3}\sin 2t$, express x in the form $R\cos(2t + \alpha)$, where $R > 0$ and $0 \le \alpha < 2\pi$.

(b) Find the maximum speed of the particle and the time at which it first occurs.

34 A particle moves in a straight line and its position at time t is given by $x = 4 + \sqrt{3}\sin 3t - \cos 3t$.

(a) If $\sqrt{3}\sin 3t - \cos 3t = R\sin(3t - \alpha)$, where α is in radians, find the value of R and α.

(b) Prove that the particle is undergoing simple harmonic motion.

(c) Find the amplitude and the centre of the motion.

(d) Find when the particle is first at its minimum displacement.

35 A particle with displacement x and velocity v is moving in simple harmonic motion with acceleration, $\ddot{x}$, given by $\ddot{x} = -12x$. The particle is initially at rest at $x = -4$.

(a) What is the period of the motion? (b) Show that $v^2 = 12(16 - x^2)$. (c) Find x as a function of time.

36 A particle moves along a straight line and its displacement, x centimetres, from a fixed point O at a given time t seconds is given by $x = 4 + \cos^2 t$.

(a) Show that its acceleration is given by $\ddot{x} = 18 - 4x$.

(b) Explain why the motion is simple harmonic.

(c) Find the centre, amplitude and period of the motion.

37 A particle is moving in a straight line under simple harmonic motion. It has a displacement of x metres from a point O, on the line, at time t seconds given by $x = 1 + 2\cos\left(2t - \frac{\pi}{4}\right)$.

(a) Show that $\ddot{x} = -4(x - 1)$.

(b) Find the centre of the motion and the time taken for the particle to first reach maximum speed.

(c) Find the amplitude of the motion and when the particle is first at rest.

38 The tide can be modelled using simple harmonic motion. At a particular location, the depth at high tide is 5 metres and the depth at low tide is 1 metre. At this location, the tide completes two full periods every 25 hours. Let x represent the depth in metres and t be the time in hours after the first low tide of the day.

(a) If this depth of this tide can be modelled by the function $x = a\cos nt + c$, find the values of a, n and c. The first low tide today is at 2 a.m.

(b) At what time is the first high tide today?

(c) At what time this evening is the depth of water increasing at the fastest rate?

6.3 OTHER EXAMPLES OF MOTION

Displacement, velocity, acceleration—important connections

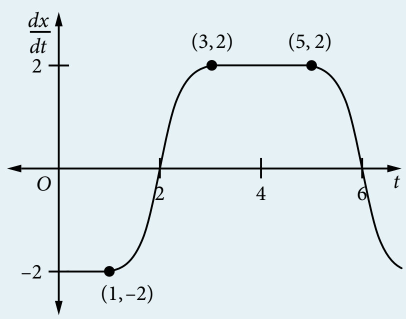
Given the displacement function, you can find the velocity and acceleration functions by differentiating with respect to time.

Given the velocity function, you can find the displacement function by integrating with respect to time, and you can find the acceleration function by differentiating with respect to time.

If the information is given as a graph, then you must remember that the definite integral can be seen as the area under the graph, while the derivative can be seen as the gradient of the curve. If you have the graph of the velocity $\frac{dx}{dt}$ against time, then the value of the definite integral gives the displacement and the slope of the curve gives the acceleration.

Example 16

The graph shows the velocity $\frac{dx}{dt}$ of a particle as a function of time. Initially the particle is at the origin.



(a) At what times is the velocity zero?

(b) At what time is the displacement x from the origin a maximum?

(c) What is the displacement when $t = 4$? What does this tell you?

(d) When does the particle have zero acceleration?

(e) At what time is the acceleration the greatest?

(f) Use the trapezoidal rule to estimate the displacement when $t = 6$.

Solution

(a) $\frac{dx}{dt} = 0$: graph shows this at $t = 2, 6$.

Hence the velocity is zero (particle at rest) at 2 seconds and again at 6 seconds.

(b) Maximum and minimum displacement occur when $\frac{dx}{dt} = 0$.

From the graph, the value of $x = \int_0^2 \frac{dx}{dt}\,dt < 0$ so the displacement at $t = 2$ is a minimum.

The value of $x = \int_0^6 \frac{dx}{dt}\,dt > 0$ so the displacement at $t = 6$ is a maximum.

(c) From the symmetry of the graph, $x = \int_0^4 \frac{dx}{dt}\,dt = 0$, so the displacement is zero and the particle is again at the origin at $t = 4$.

(d) The particle has zero acceleration when the velocity is constant, i.e. when the graph is horizontal. Acceleration is zero for $0 \le t \le 1$, $3 \le t \le 5$.

(e) The velocity graph is steepest at $t = 2$ and $t = 6$. At $t = 2$, the slope > 0; at $t = 6$, the slope < 0. Greatest acceleration is at $t = 2$.

(f) As the particle is back at the origin when $t = 4$, the displacement when $t = 6$ is given by $x = \int_4^6 \frac{dx}{dt}\,dt$.

From the graph, you have the points $(4, 2)$, $(5, 2)$, $(6, 0)$: $x = \int_4^6 \frac{dx}{dt}\,dt \approx \frac{1}{2}\left(2 + 2 \times 2 + 0\right) = 3$.

The particle is about 3 units on the positive side of the origin.

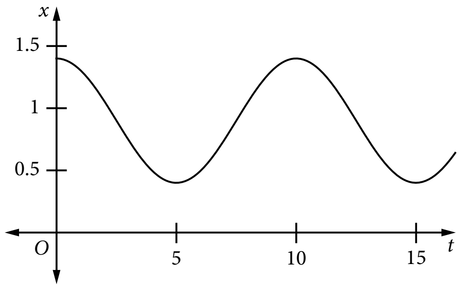
EXERCISE 6.3 OTHER EXAMPLES OF MOTION

1 The displacement x of a particle at time t is given by $x = 6\cos 4t + 3$. Find:

(a) the velocity and acceleration at any time t **(b)** the position of the particle when $t = 0$

(c) the values that x can take **(d)** the time when the particle first reaches the position $x = 0$.

2 Consider the graph.



Which of the following functions does this graph represent?

A $x = 0.9 + \cos\dfrac{\pi t}{10}$ **B** $x = 0.9 + \cos\dfrac{\pi t}{5}$

C $x = 0.9 + 0.5\cos\dfrac{\pi t}{10}$ **D** $x = 0.9 + 0.5\cos\dfrac{\pi t}{5}$

3 The position x metres of a particle relative to a fixed point O at any time t is $x = 4 - \cos 2t$.

(a) Sketch the graph of x as a function of t in the domain $0 \le t \le 2\pi$.

(b) Find the times when the particle is at rest.

(c) Express the acceleration in terms of: **(i)** t **(ii)** x

4 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If its acceleration is $2\sin t$, and $v = 1$ and $x = 1$ when $t = 0$, find x as a function of t.

5 A particle moves in a straight line. At time t seconds, its displacement x cm from a fixed point O in the line is given by $x = 5\sin\dfrac{\pi t}{2}$, $0 \le t \le 4$.

(a) Sketch the graph of its displacement at any time t. **(b)** At what times is the particle at rest?

(c) Find the speed when $t = 2.5$. **(d)** Express the acceleration in terms of: **(i)** t **(ii)** x

6 A particle moves in a straight line so that at time t its displacement from a fixed origin O is x, where $x = 2 + t - 2\cos t$.

(a) Write the velocity and acceleration at any time t.

(b) Find its initial displacement, velocity and acceleration.

7 A particle moves in a straight line so that its displacement x from a fixed origin at any time t is given by $x(t) = 2(1 - e^{-t})$.

(a) Find $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$. **(b)** Sketch the graph of $x(t)$. **(c)** Find t when $x(t) = 1$.

8 A particle moves in a straight line. At time t its displacement from a fixed origin on the line is x, where $x = 2 - 2\sin 2t$, $0 \le t \le 2\pi$.

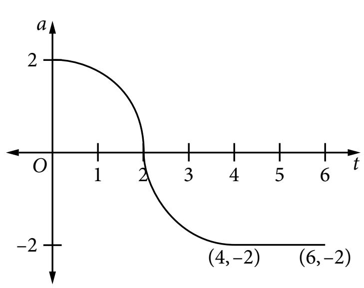
(a) Draw the graph of x as a function of t.

(b) Show that the particle oscillates between $x = 0$ and $x = 4$.

(c) For what values of t is the velocity zero?

(d) Express the acceleration in terms of: **(i)** t **(ii)** x

9 A particle moves along the x-axis. Initially it is at rest at the origin. The graph shows the acceleration a of the particle as a function of time t for $0 \le t \le 6$.

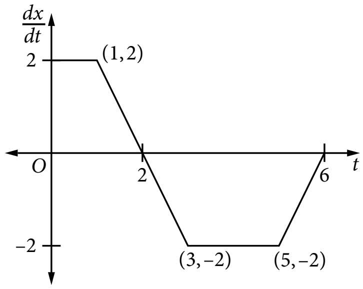


(a) Write the time at which the velocity of the particle is a maximum.

(b) At what time during the interval $0 < t \le 6$ is the particle at rest?

(c) At what time during the interval $0 \le t \le 6$ is the particle farthest from the origin? Give brief reasons for your answer.

10 The graph shows the velocity $\frac{dx}{dt}$ of a particle as a function of time. Initially the particle is at the origin.



(a) At what time is the displacement x from the origin a maximum?

(b) At what time does the particle return to the origin? Justify your answer.

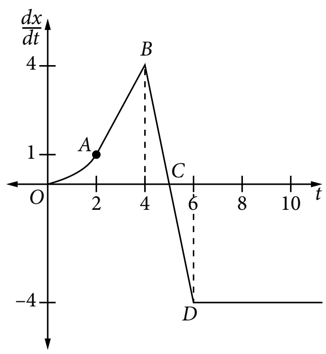
(c) Draw a sketch of the acceleration $\frac{d^2x}{dt^2}$ as a function of time for $0 \le t \le 6$.

11 The table shows the velocity (in metres per second) of a moving object, evaluated at one-second intervals.

t	0	1	2	3	4	5	6
v	0	4.6	5.7	8.0	9.9	12.7	18.2

Use the trapezoidal rule to estimate the distance travelled over the time interval $0 \le t \le 6$.

12 An object is moving on the x-axis. The graph shows the velocity $\frac{dx}{dt}$ of the object as a function of time t. The coordinates of the points shown on the graph are $A(2, 1)$, $B(4, 4)$, $C(5, 0)$ and $D(6, -4)$. The velocity is constant for $t \ge 6$.



(a) Using the trapezoidal rule, estimate the distance travelled between $t = 0$ and $t = 4$.

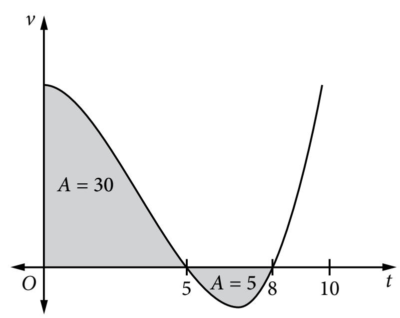
(b) How far is it from B to C?

(c) The object is initially at the origin. During which time(s) is the displacement of the object decreasing?

(d) Estimate the time at which the object returns to the origin. Justify your answer.

(e) Sketch the displacement x as a function of time.

13 The velocity–time graph at right shows the first 10 seconds of motion of an object moving in a straight line. The graph also shows the areas between the curve and the t-axis from $t = 0$ to $t = 5$ and from $t = 5$ to $t = 8$.

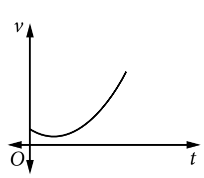


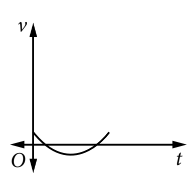
(a) Find the distance travelled by the object before it first comes to rest.

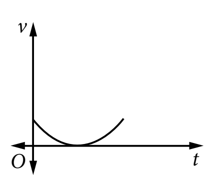
(b) Find the total distance travelled by the object before it comes to rest for the second time.

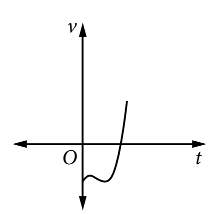
(c) If the initial displacement is 20 metres, find the displacement when the object comes to rest for the second time.

14 Each graph below shows the velocity–time relationship for an object moving in a straight line. In which case does the object change its direction only once?

A


B


C


D


6.4 MATHEMATICAL REPRESENTATION OF MOTION IN PHYSICAL TERMS

Mechanics is a topic that includes both kinematics and dynamics.

- **Kinematics** is the study of the motion of bodies without reference to the causes of their motion.
- **Dynamics** is the study of the effects of forces that cause bodies at rest to move, or that cause bodies in motion to have their state of motion altered.

Forces produce accelerations, so you will be using the principles of kinematics in much of the dynamics material of this chapter.

In this topic, all bodies are treated as particles, which means that all forces are regarded as acting through a single point in the body (as if the body were just a single point). Hence the terms 'particle', 'object' and 'body' will be interchangeable.

Newton's first law of motion

Consider a book resting on the top of your desk. It will remain there in that state of rest unless some external force or forces are applied to change that state. The book is unable to alter its state of rest by itself.

Similarly, consider a marble rolling along a smooth horizontal floor at a constant speed. By itself, this marble is unable to increase or decrease its speed or to change its direction.

This property of bodies is summed up in Newton's first law of motion:

> A body remains at rest or in uniform motion in a straight line unless it is acted on by a non-zero resultant force.

A body can be acted on by several forces that balance each other, so that the resultant force is zero. (For example, a book resting on a horizontal desk is acted on by two forces: the weight force of gravity that acts vertically downwards and the reaction force of the desk that acts vertically upwards.) A zero resultant force is the equivalent of no force acting, so the body remains stationary or in its original state of motion.

Newton's second law of motion

Experience suggests that a given force will produce different accelerations in different bodies. For example, on a smooth horizontal floor the same amount of rolling force applied to a marble and to a heavy steel ball would produce a larger acceleration in the marble. The steel ball is more massive than the marble. Similarly, a piano is more massive than a school desk, as is shown by the fact that it is easier to make the desk move than it is to get the piano to move.

This property of bodies that determines their response to an applied force is called their **inertial mass**. Inertial mass is a measure of a body's resistance to acceleration. This mass is essentially related to the amount of matter that makes up the body. It can be shown experimentally that the ratio of the accelerations produced in two bodies by the same force is the inverse ratio of their masses,

i.e. $\frac{a_1}{a_2} = \frac{m_2}{m_1}$ so that $m_1 a_1 = m_2 a_2 =$ a constant, proportional to the same force.

The standard unit of mass is the kilogram (kg). Mass is a scalar quantity.

The **momentum** p of a body is the product of its mass and velocity: $p = mv$

Because the standard unit of mass is the kilogram (kg) and the standard unit of velocity is the metre per second (m s^{-1}), the standard unit of momentum is the kilogram metre per second (kg m s^{-1}). This is a vector quantity that has the same direction as the velocity.

For example, if a body of mass 5 kg is moving with a velocity of $10\ \text{m s}^{-1}$, its momentum is $50\ \text{kg m s}^{-1}$. According to Newton's first law of motion, this body is unable by itself to change its velocity, and hence is also unable to change its momentum, unless it is acted on by a non-zero resultant force.

This leads to a statement of Newton's second law of motion:

> The rate of change of momentum is proportional to the applied force and occurs in the direction of the force.

If a body of mass m is acted on by a non-zero resultant force F, then:

$$\begin{aligned} F &\propto \text{rate of change of momentum} \\ &\propto \frac{d}{dt}(mv) \\ &\propto m\frac{dv}{dt} \quad \text{if } m \text{ is constant} \\ \therefore \quad F &\propto ma \quad \text{where } a = \frac{dv}{dt} \end{aligned}$$

That is, $F = kma$ where k is a constant.

By a suitable selection of units, you can make $k = 1$. If 1 unit of force is defined as the amount of force required to produce an acceleration of 1 m s^{-1} in a body of mass 1 kg, then: $1 = k \times 1 \times 1 \quad \therefore k = 1$

Hence $F = ma$.

This standard unit of force is called a newton (N) where $1\text{ N} = 1\text{ kg m s}^{-2}$.

Alternative definition for Newton's second law of motion

> The acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body.

Look at the formula $F = kma$ from the previous definition.

If k and m are constant, then $F \propto a$ and $a \propto F$, which satisfies the alternative definition.

If F and k are constant, then since ma = constant then $a \propto \frac{1}{m}$, which satisfies the alternative definition.

This definition says that 'The acceleration of a body is proportional to the resultant force that acts on the body', which can be written $a \propto F$, or removing the proportionality that $F = Ka$, where K is a constant.

The m, which is a constant in a given situation, is introduced by taking $K = km$, as this makes later calculations easier.

Newton's third law of motion

> When two objects exert force on each other, the forces are equal in magnitude but opposite in direction. In other words: For every action there is an equal but opposite reaction.

Dynamics of a particle

You will consider all bodies as particles, so that all external forces acting on a body are regarded as acting through a single point in the body and producing only a translational effect (i.e. no rotation).

You have seen in section 6.1 that acceleration can take different forms apart from $\frac{dv}{dt}$. For example, using the chain rule:

$$\begin{aligned} \frac{dv}{dt} &= \frac{dv}{dx} \times \frac{dx}{dt} \\ &= v\frac{dv}{dx} \qquad \text{as } v = \frac{dx}{dt} \\ &= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx} \\ &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \end{aligned}$$

Hence acceleration may be expressed in any of the forms: $\frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}, \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

The form to use in a particular problem will depend on the form of the equation that defines acceleration or force:

- Given $a = f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a = g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- Given $a = h(v)$, use $\begin{cases} \frac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\frac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v \end{cases}$

Derivatives with respect to time are often written using dots above the dependent variable,
e.g. $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{v} = \frac{dv}{dt}$

Note also that for constant m you have $F = m\frac{dv}{dt}$ and so you obtain: $\frac{dv}{dt} = \frac{F}{m}$

The derivative on the left-hand side is the acceleration, which forms the basis for the solution of a differential equation. As m is constant, the problem is converted to a kinematics problem, for which previous methods can be applied.

Some problems may be set in terms of acceleration and some in terms of force.

Example 17

A particle of mass 4 kg is acted on by a force whose direction is constant and whose magnitude at time t seconds is $(12t - 3t^2)$ newtons. If the particle has an initial velocity of 2 m s^{-1} in the direction of the force, find the velocity after 4 seconds.

Solution

There is only one external force acting on the particle, so $F = ma$ becomes:

$$12t - 3t^2 = 4\frac{dv}{dt}$$

where the particle moves with a variable velocity of magnitude $v\text{ m s}^{-1}$ in the direction of the force. Hence:

$$\frac{dv}{dt} = 3t - \frac{3}{4}t^2$$

There are two different methods to complete the solution for the given initial conditions:

Method 1

$$v = \int\left(3t - \frac{3}{4}t^2\right)dt$$

$$= \frac{3}{2}t^2 - \frac{1}{4}t^3 + C$$

When $t = 0$: $v = 2$ and so $C = 2$.

$$\therefore \quad v = \frac{3}{2}t^2 - \frac{1}{4}t^3 + 2$$

When $t = 4$: $v = 10$

Method 2

$$dv = \left(3t - \frac{3}{4}t^2\right)dt$$

$$\int_{\text{initial } v}^{\text{final } v} dv = \int_{\text{initial } t}^{\text{final } t}\left(3t - \frac{3}{4}t^2\right)dt$$

$$\int_2^v dv = \int_0^4\left(3t - \frac{3}{4}t^2\right)dt$$

$$[v]_2^v = \left[\frac{3}{2}t^2 - \frac{1}{4}t^3\right]_0^4$$

$$v - 2 = 24 - 16 - (0 - 0)$$

$$v = 10$$

Hence the velocity after 4 seconds has a magnitude of 10 m s^{-1}.

Example 18

A particle of mass 2 kg moves in a straight line so that at time t seconds its displacement from a fixed origin is x metres and its velocity is $v\text{ m s}^{-1}$. If the resultant force (in newtons) that acts on the particle is:

(a) $6 - 4x$, find v in terms of x given that $v = 2$ when $x = 1$

(b) $8 - 2v^2$, find t in terms of v given that the particle is initially at rest

(c) $8 - 2v^2$, find x in terms of v given that the particle is initially at the origin.

Solution

(a) $F = m\ddot{x}$: $\quad 2\ddot{x} = 6 - 4x$

Hence the equation of motion is $\ddot{x} = 3 - 2x$. As the question requires v in terms of x and the initial conditions are in v and x, you can use either $\ddot{x} = v\dfrac{dv}{dx}$ or $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$:

Method 1

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 2x$$

$$\frac{1}{2}v^2 = \int (3 - 2x)\,dx$$

$$\frac{1}{2}v^2 = 3x - x^2 + C$$

When $v = 2$, $x = 1$

$\therefore\ 2 = 3 - 1 + C$, so $C = 0$:

$$\frac{1}{2}v^2 = 3x - x^2$$

$$v^2 = 6x - 2x^2$$

$$v = \pm\sqrt{6x - 2x^2}$$

Method 2

$$v\frac{dv}{dx} = 3 - 2x$$

$$v\,dv = (3 - 2x)\,dx$$

From start $v = 2$, $x = 1$ to end $v = v$, $x = x$:

$$\int_2^v v\,dv = \int_1^x (3 - 2x)\,dx$$

$$\left[\frac{v^2}{2}\right]_2^v = \left[3x - x^2\right]_1^x$$

$$\frac{v^2}{2} - 2 = 3x - x^2 - (3 - 1)$$

$$\frac{v^2}{2} = 3x - x^2$$

$$v^2 = 6x - 2x^2$$

$$v = \pm\sqrt{6x - 2x^2}$$

Which solution for the velocity is valid—positive or negative?

On a quick inspection, you might say that because the initial condition is $v = 2$ (i.e. positive), then you should take the positive square root. However, you should recognise from the start of the question that the motion is simple harmonic, as: $\ddot{x} = -2\left(x - \dfrac{3}{2}\right)$

Hence the particle moves both left *and* right, and so both solutions are valid: $v = \pm\sqrt{6x - 2x^2}$

In future examples, Method 2 will be used.

(b) $m\ddot{x} = 8 - 2v^2$ and $m = 2$, so the equation of motion is: $\ddot{x} = 4 - v^2$

You require t in terms of v, hence: $\dfrac{dv}{dt} = 4 - v^2$

$$\frac{dt}{dv} = \frac{1}{4 - v^2}, \qquad v \neq \pm 2$$

$$\int_0^t dt = \int_0^v \frac{dv}{4 - v^2}$$

Use partial fractions: $\dfrac{1}{4 - v^2} = \dfrac{1}{(2 - v)(2 + v)} = \dfrac{1}{4}\left(\dfrac{1}{2 - v} + \dfrac{1}{2 + v}\right)$

$$[t]_0^t = \frac{1}{4}\left[\log_e\left|\frac{2 + v}{2 - v}\right|\right]_0^v$$

$$t = \frac{1}{4}\log_e\frac{2 + v}{2 - v}, \qquad -2 < v < 2$$

(c) $m\ddot{x} = 8 - 2v^2$ and $m = 2$, so the equation of motion is: $\ddot{x} = 4 - v^2$

You require x in terms of v, hence:

$$v\frac{dv}{dx} = 4 - v^2$$

$$\frac{dv}{dx} = \frac{4 - v^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{4 - v^2}$$

$$\int_0^x dx = \int_0^v \frac{v}{4 - v^2}\,dv \quad \text{(from start } x = 0,\ v = 0 \text{ to end } x = x,\ v = v\text{)}$$

$$[x]_0^x = -\frac{1}{2}\left[\log_e\left|4 - v^2\right|\right]_0^v$$

$$x = -\frac{1}{2}\left(\log_e\left|4 - v^2\right| - \log_e 4\right)$$

$$x = \frac{1}{2}\log_e\left|\frac{4}{4 - v^2}\right|$$

Example 19

Assume that Earth is a sphere of radius R and that at any point $x \geq R$ distant from the centre of Earth, the acceleration due to gravity is proportional to x^{-2} and is directed towards Earth's centre. Ignore all forces other than Earth's gravity.

A body is projected vertically upwards from the surface of Earth with initial speed V.

(a) Show that the equation of motion of the particle is $\ddot{x} = -\frac{gR^2}{x^2}$, where g is the acceleration due to gravity at Earth's surface.

(b) Show that the velocity v of the particle during its flight is given by: $v^2 = V^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

(c) Prove that the body's 'escape velocity' is $\sqrt{2gR}$ (i.e. prove that if the particle's initial speed is $V \geq \sqrt{2gR}$, then the particle will escape from Earth and never return).

(d) If $V = \sqrt{2gR}$, prove that the time taken to rise to a height R above Earth's surface is $\frac{1}{3}(4 - \sqrt{2})\sqrt{\frac{R}{g}}$.

Solution

(a) Take O as the centre of Earth and define motion away from O as being in the positive direction.

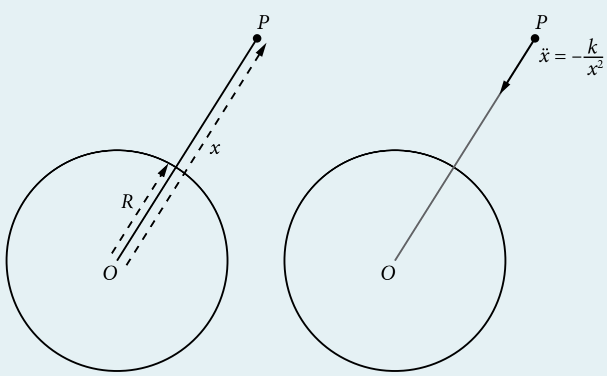
Then: $\ddot{x} = -\frac{k}{x^2}$ [1]

But at $x = R$, $\ddot{x} = -g$

(i.e. at the surface, acceleration due to gravity is g):

$$-g = -\frac{k}{R^2} \quad \therefore k = gR^2$$

Substitute into [1]: $\ddot{x} = -\frac{gR^2}{x^2}$



(b) $$v\frac{dv}{dx} = -gR^2x^{-2}$$

$$\int_V^v v\,dv = -gR^2\int_R^x x^{-2}\,dx \qquad \text{(from start } v = V, x = R \text{ to end } v = v, x = x\text{)}$$

$$\left[\frac{1}{2}v^2\right]_V^v = gR^2\left[x^{-1}\right]_R^x$$

$$\frac{1}{2}v^2 - \frac{1}{2}V^2 = gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$v^2 = V^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

(c) If the particle escapes, then $x \to \infty$ and so $\frac{1}{x} \to 0$

From the solution of part **(b)**: $v^2 \to V^2 + 2gR^2\left(0 - \frac{1}{R}\right)$ or $v^2 \to V^2 - 2gR$

But $v^2 \geq 0$ and hence $V^2 - 2gR \geq 0$

i.e. $V^2 \geq 2gR$

As the particle is escaping (i.e. always only moving away from the centre of Earth):

$$V \geq \sqrt{2gR}$$

(d) If $V = \sqrt{2gR}$, the solution of part **(b)** becomes: $v^2 = 2gR + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

i.e. $v^2 = \frac{2gR^2}{x}$

As you are only concerned with motion away from Earth:

$$v = \sqrt{2g}\,Rx^{-\frac{1}{2}}$$

$$\frac{dx}{dt} = \sqrt{2g}\,Rx^{-\frac{1}{2}}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{2g}\,R}x^{\frac{1}{2}}$$

$$\int_0^t dt = \frac{1}{\sqrt{2g}\,R}\int_R^{2R} x^{\frac{1}{2}}\,dx$$

(Note the limits of the integral on the RHS: from Earth's surface to distance R above Earth's surface.)

$$t = \frac{1}{\sqrt{2g}\,R} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_R^{2R}$$

$$t = \frac{\sqrt{2}}{3R\sqrt{g}}(2R\sqrt{2R} - R\sqrt{R})$$

$$t = \frac{\sqrt{2}}{3R\sqrt{g}}(2\sqrt{2} - 1)R\sqrt{R}$$

$$t = \frac{(4 - \sqrt{2})R\sqrt{R}}{3R\sqrt{g}}$$

$$t = \frac{1}{3}(4 - \sqrt{2})\sqrt{\frac{R}{g}}$$

Example 20

A particle of mass 1 kg is fixed in position, being suspended from the ceiling by two light rods AP and BP. The lengths of AP and BP are in the ratio 3 : 2, and AP is inclined at an angle of 30° below the ceiling. The tension forces in the rods AP and BP are T_1 and T_2 respectively.

(a) Show that the rod BP is inclined at an angle α below the ceiling, where $\sin\alpha = \frac{3}{4}$.

(b) Draw a diagram to show the three forces (T_1, T_2 and the weight force) that act on the particle.

(c) Resolve the forces into horizontal and vertical components.

(d) Use Newton's first law of motion to calculate the values of T_1 and T_2 correct to 1 decimal place. (Use $g = 9.8\text{ m s}^{-2}$.)

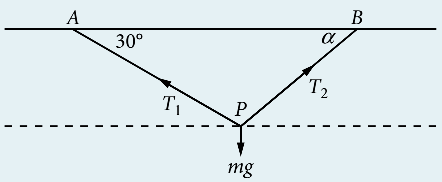
Solution

(a) Using the sine rule in triangle ABP:

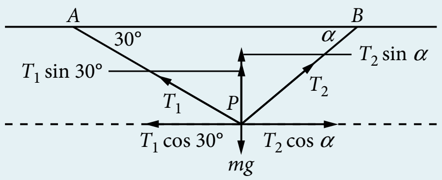
$$\frac{\sin\alpha}{3} = \frac{\sin 30°}{2}$$

$$\sin\alpha = \frac{3}{4}$$

(b)



(c)



(d) The particle is fixed in position, so according to Newton's first law of motion there is a zero resultant force acting horizontally and a zero resultant force acting vertically.

Horizontally: $T_1\cos 30° = T_2\cos\alpha$

$\sin\alpha = \frac{3}{4}$ $\quad\therefore\ \cos\alpha = \frac{\sqrt{7}}{4}$: $\quad \frac{\sqrt{3}}{2}T_1 = \frac{\sqrt{7}}{4}T_2 \qquad [1]$

Vertically (mass $m = 1$): $T_1\sin 30° + T_2\sin\alpha = 9.8$

$$\frac{1}{2}T_1 + \frac{3}{4}T_2 = 9.8 \qquad [2]$$

From [1]: $T_1 = \frac{\sqrt{21}}{6}T_2$

Substitute into [2]: $\frac{\sqrt{21}}{12}T_2 + \frac{3}{4}T_2 = 9.8$

$$T_2(\sqrt{21} + 9) = 117.6$$

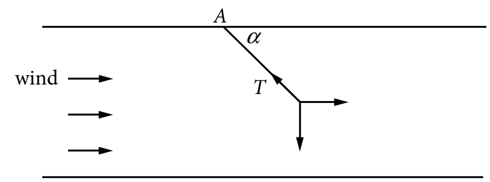
$$T_2 = 8.65815\ldots$$

Substitute into [1]: $T_1 = 6.61277\ldots$

$\therefore$ Correct to 1 decimal place: $T_1 = 6.6$ and $T_2 = 8.7$

EXERCISE 6.4 MATHEMATICAL REPRESENTATION OF MOTION IN PHYSICAL TERMS

1 A 1 kg object in a wind tunnel is suspended from a point A on the ceiling by a light rope. A wind equivalent to a horizontal force of 4.9 N is directed down the tunnel. This causes the rope and the object to move from their position vertically below A to be inclined at an angle α to the ceiling. The object remains in this position while the wind is blowing.



The diagram shows the three forces (tension T newtons in the rope, the wind force and the weight force) acting on the object. Resolve the forces into horizontal and vertical components. Noting that the object is stationary, use Newton's first law of motion to find the values of α and T (Use $g = 9.8\ \text{m s}^{-2}$). The correct equations are:

A $\alpha = \tan^{-1} 2$ and $T = \dfrac{2g}{\sqrt{5}}$ **B** $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$ and $T = \dfrac{2g}{\sqrt{5}}$

C $\alpha = \tan^{-1} 2$ and $T = \dfrac{\sqrt{5}\,g}{2}$ **D** $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$ and $T = \dfrac{\sqrt{5}\,g}{2}$

2 A particle of mass 1 kg moves in a straight line such that at time t seconds its displacement from a fixed origin is x metres and its velocity is $v\ \text{m s}^{-1}$. If the resultant force is $3 + 2t$ (in newtons), find its displacement in terms of t given that $v = 1$ and $x = 2$ when $t = 0$.

3 The velocity $v\ \text{m s}^{-1}$ of a particle of mass 2 kg that is moving in a straight line is $v = t^2 - 6t + 8$ at any time t s.

(a) Find the resultant force in terms of t. **(b)** Find the force when the particle is stationary.

4 A particle of mass 10 kg starts from rest at a point A and moves in a straight line under the action of a force that decreases uniformly from 20 N to zero in 20 seconds. The particle then travels with constant velocity for a further 20 seconds. After this, the particle moves under the action of a retarding force of 40 N until it comes to rest at point B.

(a) Express the force that applies for the first 20 seconds as a function of t.
(b) Find the velocity of the particle at $t = 20$ seconds.
(c) Find the time taken for the retarding force to stop the motion of the particle.
(d) Find the maximum speed attained during the motion.
(e) Find the total distance travelled during the motion.
(f) Find the average speed during the motion.

5 A particle of mass 10 kg moves under the action of a force so that its velocity $v\ \text{m s}^{-1}$ is given by $v = \sqrt{x^2 - 6x + 5}$. Find the force F in terms of the displacement x.

6 A particle of mass m moves so that its velocity v is given by $v = f(x)$. The resultant force that causes this motion is given by:

A $mf'(x)$ **B** $mxf'(x)$ **C** $mf'\left(\dfrac{1}{2}x^2\right)$ **D** $mf(x)f'(x)$

7 A particle of mass 8 kg is acted on by a force whose magnitude is $2(25 + 60x - 6x^2)$ newtons, where x is the displacement from a fixed point O. If the particle is initially at rest at O, find its speed when it is 10 metres from O.

8 A particle of mass 2 kg moves in a straight line. At time t, its displacement from a fixed origin is x metres and its velocity is $v\ \text{m s}^{-1}$. If the resultant force (in newtons) acting on the particle is:

(a) $6\cos t$, and $v = 2$ and $x = 0$ when $t = 0$, then find x in terms of t
(b) $2 + 4x$, and $v = 2$ when $x = 0$, then find v when $x = 2$
(c) $4 - 2v$, and $v = 0$ when $t = 0$, then find the time when $v = 1$.

9 At time t a particle of mass m is moving in a straight line under the action of a force given by $F = \dfrac{m(3-5x)}{x^3}$. The particle starts from rest at $x = \dfrac{1}{3}$.

(a) Find its velocity in terms of x.
(b) At what other point, if any, does the particle come to rest?

10 A particle of unit mass is acted on by a force $F = v^2 \log_e v$, where $v\ \text{m s}^{-1}$ is the velocity of the particle. The motion starts from O with a velocity of $e\ \text{m s}^{-1}$. Find the displacement when the velocity is $e^2\ \text{m s}^{-1}$.

11 The acceleration of an object moving towards a planet under gravitational attraction varies inversely as the square of the distance from the centre of the planet (i.e. $\ddot{x} = -\dfrac{k}{x^2}$ where x is the displacement from the centre of the planet and k is a constant). Show that if the object starts from rest at a distance a from the centre of the planet, its speed at distance x from the centre of the planet is: $\sqrt{\dfrac{2k(a-x)}{ax}}$

12 A particle of unit mass starts from rest with displacement b (where $b > 0$) and is attracted towards the origin O with an acceleration of magnitude $\dfrac{k}{x^2}$ where x is the displacement from the origin and k is a positive constant.

(a) Explain why $\ddot{x} = -\dfrac{k}{x^2}$.

(b) Show that the velocity v is given by: $v^2 = 2k\left(\dfrac{1}{x} - \dfrac{1}{b}\right)$

(c) Use the substitution $x = b\cos^2\theta$ to show that: $\displaystyle\int \sqrt{\frac{x}{b-x}}\,dx = -\sqrt{bx - x^2} - \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right) + C$

(d) Hence show that the time required for the particle to reach the origin is: $\pi\left(\dfrac{b^3}{8k}\right)^{\frac{1}{2}}$

13 An object of mass m moves in a straight line under a force of magnitude $mk^2\left(x + \dfrac{a^4}{x^3}\right)$, $k > 0$ towards the origin O. If the particle started from rest at a distance a units from O, show that its speed when $x = \dfrac{a}{2}$ is $\dfrac{\sqrt{15}\,ka}{2}$.

14 The deck of a ship is 2.4 m below the level of a wharf at low tide and 0.6 m above wharf level at high tide. Low tide is at 8:30 a.m. and high tide is at 2:35 p.m. Find the time when the deck is level with the wharf, assuming the motion of the tides is simple harmonic.

15 A particle moves in a straight line with simple harmonic motion. Its speed at distances x_1, x_2 from the centre of its motion are v_1, v_2 respectively. Show that:

(a) the period of the motion is $2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

(b) the amplitude is $\sqrt{\dfrac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$

16 A particle moves in a straight line under a force such that it describes simple harmonic motion of amplitude a about a point O. The time of a complete oscillation is $5T$. The particle is released from rest at a point A, where $OA = a$. At time T, another particle describing an exactly similar motion is released from rest at A.

(a) Show that the two particles first meet at time $2T$ after the release of the second particle.
(b) Find the distance from O to the point where the particles first meet.

17 A vertical pole subtends an angle α at a point P in the same horizontal plane as the foot of the pole. Two particles are projected at the same instant from P in directions that make angles α_1 and α_2 with the horizontal, with initial speeds v_1 and v_2, so that the first particle hits the top of the pole at the same instant that the second particle hits the bottom.

(a) Show that: $v_1\cos\alpha_1 = v_2\cos\alpha_2$

(b) Show that the time of flight is: $\dfrac{2v_2\sin\alpha_2}{g}$

(c) Hence prove that: $\tan\alpha = \tan\alpha_1 - \tan\alpha_2$

18 A projectile A is projected from O with speed u at an angle α above the horizontal. A package B is parachuting vertically downwards at a constant speed equal to $-u\sin\alpha$. At the moment when A is projected, the package B is at the point $Q\left(\frac{u^2}{g}\sin 2\alpha, \frac{2u^2}{g}\sin^2\alpha\right)$.

(a) Find the coordinates of A and B at time t after A is projected.

(b) Show that a searchlight, which is located at O and which is moved so that its beam is continually directed at A, will always have B in its beam.

19 Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection α (upward inclination to the horizontal), but with different velocities u, v metres per second ($u < v$).

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant, the faster stone just clears a wall ZQ of height h metres above the level of projection as its (downward) path makes an angle β with the horizontal.

(a) Show that while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time.

(b) Hence express the horizontal distance from P to Z (the foot of the wall) in terms of h and α.

(c) Show that $v(\tan\alpha + \tan\beta) = 2u\tan\alpha$.

(d) Deduce that if $\beta = \frac{1}{2}\alpha$ then $u < \frac{3}{4}v$.

20 A ball thrown from a point A with speed V at an inclination α to the horizontal reaches a point B after t seconds.

(a) Find the position of B relative to A.

(b) Show that if AB is inclined at θ to the horizontal then the direction of motion of the ball when at B is inclined to the horizontal at an angle β, given by $\tan\beta = 2\tan\theta - \tan\alpha$.

21 A particle is projected, with speed V and inclination α above the horizontal, from a point O on a plane inclined at 30° to the horizontal (so that $\alpha > 30°$). It lands at a point Q on the inclined plane, at right angles to the plane.

(a) Find the coordinates of Q.

(b) Find the horizontal and vertical components of the velocity when the particle is at Q.

(c) Show that the time of flight is: $t = \frac{V(\sin\alpha + \sqrt{3}\cos\alpha)}{g}$

(d) Show that: $\tan\alpha = \frac{5\sqrt{3}}{3}$

(e) Hence show that the range on the inclined plane (i.e. OQ) is: $\frac{4V^2}{7g}$

6.5 RESISTED MOTION

Whenever a body moves through a medium (such as air, water, oil etc.), it is subjected to a resistance that acts in the opposite direction to the motion. If the body falls vertically downwards, the resistance acts upwards; if the body is projected vertically upwards, the resistance acts downwards. (Of course the gravitational weight force mg always acts vertically downwards.)

In general, the faster the body moves, the greater the resistance. Air resistance is typically proportional to some power of the speed, so that air resistance $= kv^n$. This topic will consider this for $n = 1$ and $n = 2$, i.e. air resistance proportional to the speed of the body ($n = 1$) and air resistance proportional to the square of the speed of the body ($n = 2$). These are the two most useful models for air resistance.

It is important to realise that the effect of air resistance (or other resistance) is always one of retardation:

In problems involving resisted motion, you should always define the positive direction to be the direction in which motion is actually occurring and take the origin O as the point from which the motion begins.

Resisted motion in a horizontal line

Example 21

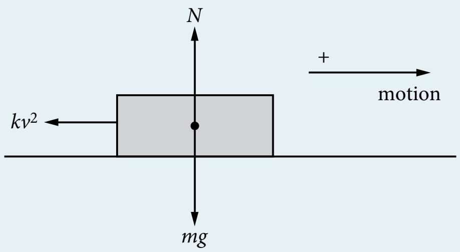
A particle of mass m and initial speed v_0 moves on a horizontal surface against a resistance proportional to the square of the speed. Express the velocity in terms of the distance travelled.

Solution

Take the origin O as the particle's initial position. Take the particle's direction of motion as the positive direction.

Vertically, there is no motion (i.e. the particle moves always at the same horizontal level), so there is a zero resultant force acting in the vertical direction: $\therefore N - mg = 0$

Horizontally, there is motion. The only force acting horizontally is the resistance:



$$m\ddot{x} = -kv^2$$

$$\ddot{x} = -\frac{k}{m}v^2$$

$$v\frac{dv}{dx} = -\frac{k}{m}v^2$$

$$\frac{dv}{dx} = -\frac{k}{m}v$$

$$\therefore \frac{dx}{dv} = -\frac{m}{kv}, \quad v \neq 0$$

$$\int_0^x dx = -\frac{m}{k}\int_{v_0}^{v}\frac{dv}{v} \qquad \text{(from start } x = 0,\ v = v_0 \text{ to end } x = x,\ v = v\text{)}$$

$$x = -\frac{m}{k}\left[\log_e v\right]_{v_0}^{v}$$

$$x = -\frac{m}{k}\log_e \frac{v}{v_0}$$

$$v = v_0 e^{-\frac{kx}{m}} \qquad \text{(making } v \text{ the subject)}$$

Motion of a particle falling downwards in a resisting medium with gravity

Example 22

A body of mass 5 kg is dropped from a great height under a constant gravitational acceleration $g\,\text{m s}^{-2}$ and air resistance proportional to the speed $v\,\text{m s}^{-1}$. If the constant of proportionality is $\frac{1}{8}$:

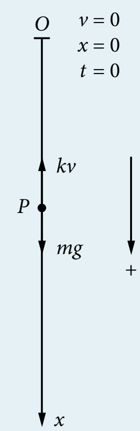
(a) find the velocity at time t

(b) sketch the velocity–time graph

(c) find the terminal (i.e. maximum) velocity

(d) find the distance the particle has fallen at time t.

Solution



(a) Take O as the point of release of the particle P. Take motion downwards as positive. There are two forces acting on P: its weight force of $5g$ acting downwards and its air resistance of $\frac{1}{8}v$ acting upwards (i.e. opposing the motion).

The resultant force on P is $5\ddot{x} = 5g - \frac{1}{8}v$ and this is the equation of motion:

$$\ddot{x} = g - \frac{v}{40}$$

$$\frac{dv}{dt} = \frac{40g - v}{40}$$

$$\frac{dt}{dv} = \frac{40}{40g - v}, \qquad v \neq 40g$$

$$\int_0^t dt = \int_0^v \frac{40}{40g - v}\,dv \qquad \text{(from } t = 0, v = 0 \text{ to } t = t, v = v\text{)}$$

$$t = -40\left[\log_e(40g - v)\right]_0^v, \quad 0 \le v < 40g$$

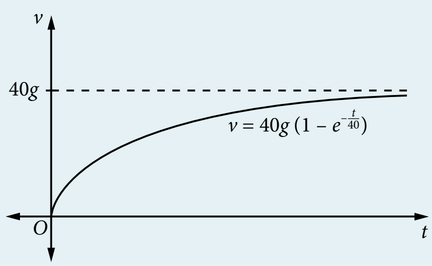
$$t = 40\log_e \frac{40g}{40g - v}$$

$$e^{\frac{t}{40}} = \frac{40g}{40g - v}$$

$$40g - v = 40ge^{-\frac{t}{40}}$$

$$v = 40g\left(1 - e^{-\frac{t}{40}}\right)$$

(b)



(c) As t increases, the gravitational force acting on P causes it to accelerate downwards. However, as the speed increases, the resistance also increases until eventually the downwards weight force and the resistance force are equal in magnitude but opposite in direction. At this stage the resultant force is zero: according to Newton's first law of motion, P will now move in a straight line at constant speed, i.e. the terminal velocity.

The terminal velocity occurs when the acceleration is equal to zero.

Method 1 (graphically):
In the velocity–time graph in part **(b)**, the horizontal asymptote at $v = 40g$ indicates the terminal velocity.

Method 2 (algebraically):
As $\ddot{x} = g - \frac{v}{40}$ the terminal velocity occurs when $g - \frac{v}{40} = 0$, i.e. when $v = 40g$.

(d)
$$v = 40g\left(1 - e^{-\frac{t}{40}}\right)$$

$$\frac{dx}{dt} = 40g\left(1 - e^{-\frac{t}{40}}\right)$$

$$\int_0^x dx = 40g\int_0^t \left(1 - e^{-\frac{t}{40}}\right)dt \qquad \text{(from } x = 0, t = 0 \text{ to } x = x, t = t\text{)}$$

$$x = 40g\left[t + 40e^{-\frac{t}{40}}\right]_0^t$$

$$x = 40g\left(t + 40e^{-\frac{t}{40}}\right) - 40g \times 40$$

$$x = 40g\left(t + 40e^{-\frac{t}{40}} - 40\right)$$

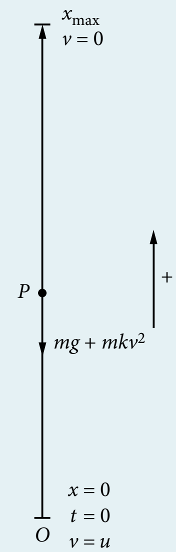
Motion of a particle moving upwards in a resisting medium with gravity

Example 23

A particle is projected vertically upwards with a velocity of u m s^{-1} under the influence of gravity and a resistance force proportional to the square of the speed. Find:

(a) the greatest height reached

(b) the time taken to reach the maximum height.



Solution

(a) Take O as the point of release of the particle P. Take motion upward as positive. There are two forces acting on P: its weight force of mg acting downwards and its resistance of mkv^2 acting downwards (i.e. opposing the motion).

(Note that although you do not have a numerical value for m, you write a factor of m into the constant of proportionality for the resistance. This will simplify later calculations where factors of m can be cancelled in the equation of motion.)

The resultant force on P is $m\ddot{x} = -mg - mkv^2$ and this is the equation of motion. You need to find x when $v = 0$:

$$\ddot{x} = -(g + kv^2)$$

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$\int_0^x dx = -\int_u^0 \frac{v}{g + kv^2}\,dv \text{ (from } x = 0,\ v = u \text{ to } x = x,\ v = 0)$$

$$x = -\frac{1}{2k}\left[\log_e(g + kv^2)\right]_u^0$$

$$x = -\frac{1}{2k}\log_e g + \frac{1}{2k}\log_e(g + ku^2)$$

$$x = \frac{1}{2k}\log_e\left(\frac{g + ku^2}{g}\right)$$

The maximum height attained has a value of: $\frac{1}{2k}\log_e\left(\frac{g + ku^2}{g}\right)$ m

(b) You now need to find t when $v = 0$:

$$\ddot{x} = -(g + kv^2)$$

$$\frac{dv}{dt} = -(g + kv^2)$$

$$\int_0^t dt = -\int_u^0 \frac{1}{g + kv^2}\,dv \text{ (from } t = 0,\ v = u \text{ to } t = t,\ v = 0)$$

$$t = -\frac{1}{k}\int_u^0 \frac{1}{\left(\frac{g}{k} + v^2\right)}\,dv$$

$$t = -\frac{1}{k} \times \sqrt{\frac{k}{g}}\left[\tan^{-1}\left(\sqrt{\frac{k}{g}}\,v\right)\right]_u^0$$

$$t = -\frac{1}{\sqrt{kg}}\left(\tan^{-1} 0 - \tan^{-1}\sqrt{\frac{k}{g}}\,u\right)$$

$$t = \frac{1}{\sqrt{kg}}\tan^{-1}\sqrt{\frac{k}{g}}\,u$$ seconds, this is the time taken to reach the maximum height.

The next example looks at the motion of a particle that goes upwards and then downwards. This requires the solution to take a new O and a new positive direction when you analyse the downwards motion.

Example 24

A particle of mass 1 kg is projected vertically upwards from the ground at a speed of V m s^{-1}. The particle is acted on by both gravity and a resistance of magnitude $0.02v^2$, where v is the velocity of the particle at time t.

- **(a)** Explain why the equation of motion while the particle is moving upwards is: $\ddot{x} = -(g + 0.02v^2)$
- **(b)** Find the greatest height h reached by the particle.
- **(c)** Find the time taken to reach this greatest height.

Having reached its maximum height, the particle falls back down towards its initial point of projection.

- **(d)** Write the equation of motion for the downwards journey.
- **(e)** Find the speed of the particle when it hits the ground.
- **(f)** Determine whether the particle's speed on return is less than, equal to, or greater than its initial speed on projection.

Solution

(a) Take O as the point of projection and take upwards motion as positive.

The particle is acted on by gravitational force mg downwards and by resistance force $0.02v^2$ downwards (i.e. opposite to the motion).

Resultant force on the particle: $m\ddot{x} = -mg - 0.02v^2$

But $m = 1$, so: $\ddot{x} = -(g + 0.02v^2)$

(b)

$$\ddot{x} = -(g + 0.02v^2)$$

$$v\frac{dv}{dx} = -\left(\frac{50g + v^2}{50}\right)$$

$$\frac{dv}{dx} = -\left(\frac{50g + v^2}{50v}\right)$$

$$\frac{dx}{dv} = -\frac{50v}{50g + v^2}$$

$$\int_0^h dx = -\int_V^0 \frac{50v}{50g + v^2}\,dv$$ (from $x = 0$, $v = V$ to $x = h$, $v = 0$)

$$h = -25\left[\log_e(50g + v^2)\right]_V^0$$

$$h = -25\left(\log_e 50g - \log_e(50g + V^2)\right)$$

$$h = 25\log_e\left(\frac{50g + V^2}{50g}\right)$$

(c) $\frac{dv}{dt} = -\left(\frac{50g + v^2}{50}\right)$

$\frac{dt}{dv} = -\frac{50}{50g + v^2}$

$\int_0^t dt = -\int_V^0 \frac{50}{50g + v^2}\,dv$ (from $t = 0$, $v = V$ to $t = t$, $v = 0$)

$t = -\frac{50}{\sqrt{50g}}\left[\tan^{-1}\left(\frac{v}{\sqrt{50g}}\right)\right]_V^0$

$t = \sqrt{\frac{50}{g}}\tan^{-1}\left(\frac{V}{\sqrt{50g}}\right)$

(d) For the downwards journey it is necessary to take a new O and a new positive direction. Remember:

> In problems involving resisted motion, you should always define the positive direction to be the direction in which motion is actually occurring and take the origin O as the point from which the motion begins.
>
> This simplifies the calculations required to solve the problem.

Take O as the maximum height and take downwards motion as positive.

The particle is acted on by gravitational force mg downwards and by resistance force $0.02v^2$ upwards (i.e. opposite to the motion).

Resultant force on the particle: $m\ddot{x} = mg - 0.02v^2$

But $m = 1$, so: $\ddot{x} = g - 0.02v^2$

(e) $v\frac{dv}{dx} = \frac{50g - v^2}{50}$

$\frac{dv}{dx} = \frac{50g - v^2}{50v}$

$\frac{dx}{dv} = \frac{50v}{50g - v^2}$

You need to find v when $x = h$:

$\int_0^h dx = \int_0^v \frac{50v}{50g - v^2}\,dv$

$h = -25\left[\log_e\left|50g - v^2\right|\right]_0^v$

$h = 25\log_e\left|\frac{50g}{50g - v^2}\right|$ But from **(b)**: $h = 25\log_e\left|\frac{50g + V^2}{50g}\right|$

$\therefore 25\log_e\left|\frac{50g + V^2}{50g}\right| = 25\log_e\left|\frac{50g}{50g - v^2}\right|$

$\frac{50g + V^2}{50g} = \frac{50g}{50g - v^2}$

$2500g^2 + 50gV^2 - 50gv^2 - v^2V^2 = 2500g^2$

$50gV^2 = v^2(50g + V^2)$

$v^2 = \frac{50gV^2}{50g + V^2}$ and so the speed on return is: $V\sqrt{\frac{50g}{50g + V^2}}$

(f) $V^2 > 0$, so $50g + V^2 > 50g$ and $\sqrt{\frac{50g}{50g + V^2}} < 1$

$\therefore$ Speed on return is less than V, i.e. the speed on return is less than the speed of projection.

EXERCISE 6.5 RESISTED MOTION

1 A particle of unit mass moves horizontally under a retarding force that is proportional to the cube of the speed. Find:

(a) the speed of the particle when it has travelled a distance d after the instant when its speed is u

(b) the time taken to travel a distance d after the instant when its speed is u.

2 A particle has an initial velocity U. After travelling a distance d in time T along a straight horizontal path, its velocity is V. The retardation of the particle at any time is proportional to its velocity at that time.

Show that: (a) $V = U - kd$ (b) $U = Ve^{kT}$ (c) $U = Ve^{\frac{T(U-V)}{d}}$

3 As a truck moves along a horizontal straight road it experiences a horizontal driving force of T newtons (from the engine) and a resistance force. The resistance force is mkv^2 newtons, where m kg is the mass of the truck, v m s^{-1} is its speed at time t seconds and k is a positive constant. The equation of motion of the truck is:

A $m\dfrac{dv}{dt} = T - kv^2$ B $m\dfrac{dv}{dt} = T - mkv^2$ C $\dfrac{dv}{dx} = T - kv$ D $m\dfrac{dv}{dx} = T - mkv^2$

4 An object of mass m falls from rest under constant gravitational force and against air resistance equal to kv, where v is the speed and k is a positive constant.

(a) Find its velocity at any time t.

(b) Sketch the velocity–time graph.

(c) Find the terminal velocity. Find the time taken to reach a speed v_1 where v_1 is one-quarter of the terminal velocity.

(d) Find the distance travelled when the speed v_1 is reached.

5 A particle of mass 10 kg falls from rest and is subject to a force of $(98 - 2v)$ newtons, where v is the speed of the particle at time t seconds.

(a) In writing the force as '$(98 - 2v)$' newtons, the '98' represents the weight force of the particle. What is the physical meaning of the '$2v$'?

(b) Find the terminal velocity.

(c) Find the distance fallen in the first 5 seconds.

6 A particle falls from rest under constant gravity and a resistance force. If the retardation due to the resistance force varies as the square of the velocity, find:

(a) the equation of motion

(b) the terminal velocity

(c) the distance fallen as a function of the velocity

(d) the distance fallen when half the terminal velocity is reached

(e) the time taken to reach half the terminal velocity.

7 An object falls towards the Earth with constant gravitational acceleration g and against a resistance that produces retardation proportional to the velocity.

(a) State the equation of motion.

(b) Express the velocity v as a function of time t.

(c) State the terminal velocity.

If the constant of proportionality $k = 0.2$ and $g = 9.8$, find:

(d) the velocity after 5 seconds

(e) the time required to reach half the terminal velocity.

8 A particle is dropped from a height of 1000 metres in a medium whose resistance provides a retardation of $0.004v^2$, where v is the velocity.

(a) Find v^2 in terms of x, the distance fallen.

(b) Find the speed at which the particle reaches the lowest point.

9 A parachutist jumps from a stationary balloon at a great height. The parachute opens after 10 seconds. Assume that air resistance produces a retardation proportional to the velocity, with a constant of proportionality $k = 0.1$ for the first 10 seconds (i.e. during freefall) and $k = 2$ after the parachute opens. Find:

(a) the parachutist's velocity after 10 seconds

(b) the parachutist's velocity after 15 seconds

(c) the parachutist's terminal velocity, i.e. the approximate velocity while floating to the ground.

10 A particle is projected vertically upwards against air resistance. Its acceleration at any time t seconds after projection is given by $\ddot{x} = -\left(g + \frac{1}{10}v^2\right)$, where v m s^{-1} is the velocity. If the initial velocity is 20 m s^{-1}, find:

(a) the greatest height reached

(b) the time taken to reach the greatest height.

11 A particle is projected vertically upwards with initial speed u. Its acceleration is given by the differential equation $\ddot{x} = -(g + kv)$ where v is the speed at any time t, k is a positive constant and kv is the retardation due to air resistance.

(a) Find the maximum height reached by the particle.

(b) Find the time taken to reach the maximum height.

(c) Write the differential equation for the downward motion.

(d) Show that the particle returns to its point of projection with a speed V given by:

$$k(u + V) = g\log_e\left[\frac{g + ku}{g - kV}\right]$$

12 An object of mass m is projected vertically upwards with speed u. Air resistance is equal to k times the square of the speed, where k is a positive constant.

(a) Find the maximum height reached by the object.

(b) Find the speed V of the object when it returns to the point of projection.

(c) Show that $V < u$.

13 An object, projected vertically upwards with speed U, returns to the point of projection with speed V. Assuming constant gravity and air resistance proportional to the square of the speed, find the total time taken in terms of U and V.

14 A particle of unit mass moves in a horizontal straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(a) Show that: $\tan^{-1}Q - \tan^{-1}v = \tan^{-1}\left[\frac{Q - v}{1 + Qv}\right]$

(b) Show that $x = \tan^{-1}\left[\frac{Q - v}{1 + Qv}\right]$, where x is the displacement.

(c) Show that $t = \frac{1}{2}\log_e\left[\frac{Q^2(1 + v^2)}{v^2(1 + Q^2)}\right]$, where t is the elapsed time when the particle is travelling with velocity v.

(d) Find v^2 as a function of t.

(e) Find the limiting values of v and x as $t \to \infty$.

15 A particle of unit mass is projected vertically upwards in a medium in which the retardation due to resistance is $0.1v$. It is allowed to fall back to its point of projection. The initial speed of projection is V_0 and the final speed on return is V_F. Show that:

(a) the equation of motion on the upwards journey is $\ddot{x} = -(g + 0.1v)$

(b) the maximum height reached is $h = 10V_0 + 100g\log_e\left(\frac{10g}{10g + V_0}\right)$

(c) the time taken to reach the highest point is $T_1 = 10\log_e\left(\frac{10g + V_0}{10g}\right)$

(d) the equation of motion on the downwards journey is $\ddot{x} = g - 0.1v$

(e) the time taken on the downwards journey is $T_2 = 10\log_e\left(\frac{10g}{10g - V_F}\right)$

(f) by analysis of the downwards journey, $h = -10V_F + 100g\log_e\left(\frac{10g}{10g - V_F}\right)$

(g) the total time of the motion is $T = \frac{V_0 + V_F}{g}$.

16 A plane of mass 20 tonnes touches down on a runway while moving at a horizontal speed of $60\,\text{m s}^{-1}$. As the plane moves along the runway, its speed at time t seconds after touchdown is $v\,\text{m s}^{-1}$. From the instant that it touches down, it is being slowed by a reverse thrust of 40 000 newtons supplied by the engines and a force of $4v^2$ newtons supplied by the braking effect of wing flaps and other frictional forces. After 10 seconds of motion along the runway, wheel brakes are applied which supply an additional $300(60 - v)$ newtons of braking force.

(a) Show that: $\ddot{x} = \begin{cases} -\left(2 + \dfrac{v^2}{5000}\right) & 0 \le t \le 10 \\ -\left(2.9 + \dfrac{v^2}{5000} - \dfrac{3v}{200}\right) & t > 10 \end{cases}$

(b) Find the speed of the plane 10 seconds after touchdown. Answer in m s^{-1} correct to one decimal place.

(c) Find the distance in metres (correct to one decimal place) travelled in the first 10 seconds after touchdown.

(d) Find the total time in seconds (correct to one decimal place) from touchdown until the plane stops. Without proof, you may assume $\displaystyle\int_0^{35.4} \frac{dv}{v^2 - 75v + 14500} = 0.00260721$ (correct to six significant figures).

(e) Find the total distance that the plane travels along the runway before it stops. Without proof, you may assume $\displaystyle\int_0^{35.4} \frac{v\,dv}{v^2 - 75v + 14500} = 0.0469319$ (correct to six significant figures).

(f) By integrating, verify the integral values given in parts **(d)** and **(e)**.

6.6 PROJECTILES AND RESISTED MOTION

Review of projectile motion with no resistance

In *New Senior Mathematics Extension 1 for Years 11 & 12*, Chapter 13, you looked at projectile motion. Unless otherwise stated, any air resistance is ignored when investigating projectile motion and it is assumed that the only force acting on the projectile is the force of gravity. Furthermore, it is assumed that the gravitational force is constant throughout the motion. The following examples **25–27** review projectile motion with no resistance.

Use $\ddot{\underset{\sim}{r}}$ for the acceleration vector, $\dot{\underset{\sim}{r}}$ for the velocity vector and $\underset{\sim}{r}$ for the displacement vector.

Example 25

A golf ball of mass m kg is hit down the middle of a fairway with an initial speed of $25\,\text{m s}^{-1}$ at an angle of projection to the horizontal α, where $\tan\alpha = \frac{4}{3}$.

(a) Taking unit vectors $\underset{\sim}{i}$ horizontally in the direction of motion and $\underset{\sim}{j}$ vertically upward, find an expression for the initial velocity $\underset{\sim}{u}\ \text{m s}^{-1}$ of the ball.

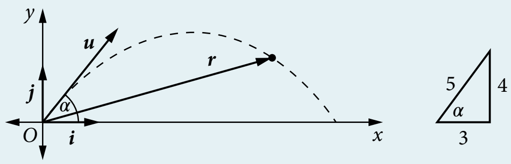
(b) Taking the origin at the point of projection, find an expression for the position vector $\underset{\sim}{r}$ of the ball after t seconds.

(c) Assuming the fairway to be horizontal, find the horizontal distance that the ball travels before hitting the ground. Give your answer correct to one decimal place.

(d) Find the maximum height reached by the ball during its flight. Give your answer correct to one decimal place.

Solution

(a) Draw a diagram.



Express $\underset{\sim}{u}$ in component form: $\underset{\sim}{u} = u\cos\alpha\,\underset{\sim}{i} + u\sin\alpha\,\underset{\sim}{j}$

$$= 25\left(\frac{3}{5}\right)\underset{\sim}{i} + 25\left(\frac{4}{5}\right)\underset{\sim}{j}$$

$$= 15\underset{\sim}{i} + 20\underset{\sim}{j}$$

(b) Determine the equation of motion: If the ball has a mass of m kg, its equation of motion is $-mg\,\underset{\sim}{j} = m\ddot{\underset{\sim}{r}} \quad \therefore\ \ddot{\underset{\sim}{r}} = -g\,\underset{\sim}{j}$

Integrate each component with respect to t to find $\dot{\underset{\sim}{r}}$: $\quad \dot{\underset{\sim}{r}} = -gt\,\underset{\sim}{j} + \underset{\sim}{c}$

Apply the initial condition to find $\underset{\sim}{c}$: $\quad$ At $t = 0$, $\dot{\underset{\sim}{r}} = \underset{\sim}{u}$ so $\underset{\sim}{c} = 15\underset{\sim}{i} + 20\underset{\sim}{j}$

Hence $\dot{\underset{\sim}{r}} = 15\underset{\sim}{i} + (20 - gt)\underset{\sim}{j}$

Integrate each component with respect to t to find $\underset{\sim}{r}$: $\quad \underset{\sim}{r} = 15t\underset{\sim}{i} + \left(20t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

Apply the initial condition to find $\underset{\sim}{d}$: $\quad$ At $t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$ so $\underset{\sim}{d} = \underset{\sim}{0}$

Hence $\underset{\sim}{r} = 15t\underset{\sim}{i} + \left(20t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$

(c) The ball hits the ground when the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ is zero: $20t - 4.9t^2 = 0$

$$t(20 - 4.9t) = 0$$

$$t = 0 \text{ or } t = \frac{20}{4.9} \approx 4.08\text{ s}$$

The horizontal distance that the ball has covered is the $\underset{\sim}{i}$ component of $\underset{\sim}{r}$:

$$\begin{aligned}\text{Distance} &= 15t \\ &= 15 \times \frac{20}{4.9} \\ &= 61.2\text{ m}\end{aligned}$$

The ball travels 61.2 m before hitting the ground.

(d) The ball reaches its maximum height when the $\underset{\sim}{j}$ component of $\dot{\underset{\sim}{r}}$ is zero: $20 - 9.8t = 0$

$$t = \frac{20}{9.8}$$

$$t = 2.04\text{s}$$

Its height is given by the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$: $h_{max} = h(2.04)$

$$= 2.04\left(20 - 4.9 \times \frac{20}{9.8}\right)$$

$$= 2.04 \times 10$$

$$= 20.4\text{ m}$$

The ball reaches a maximum height of 20.4 m.

Example 26

A football is kicked towards goal with an initial speed of $26\,\text{m s}^{-1}$ at an angle of projection to the horizontal α, where $\tan\alpha = \dfrac{5}{12}$.

(a) Find an expression for the initial velocity $\underset{\sim}{u}$ m s^{-1} of the football.

(b) Taking the origin at the point of projection, find an expression for the position vector $\underset{\sim}{r}$ of the football after t seconds.

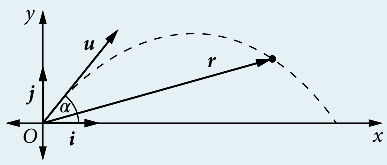
(c) Write the parametric equations of the path of the football and use them to find the Cartesian equation of the path.

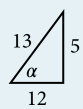
(d) Assuming the ground to be horizontal, find the horizontal distance that the football travels before hitting the ground. Give your answer correct to the nearest metre.

(e) The horizontal crossbar of the goal is 2.44 m above the ground and is 0.1 m wide. If the football is heading towards the goal, which is 40 m from where the football is kicked, will the football pass below the crossbar?

Solution

(a)





$u = 26\,\text{m s}^{-1}$, $\tan\alpha = \dfrac{5}{12}$ so $\sin\alpha = \dfrac{5}{\sqrt{5^2+12^2}} = \dfrac{5}{13}$, $\cos\alpha = \dfrac{12}{13}$

$$\begin{aligned}\underset{\sim}{u} &= u\cos\alpha\,\underset{\sim}{i} + u\sin\alpha\,\underset{\sim}{j}\\ &= 26\times\frac{12}{13}\underset{\sim}{i} + 26\times\frac{5}{13}\underset{\sim}{j}\\ &= 24\underset{\sim}{i} + 10\underset{\sim}{j}\end{aligned}$$

(b) $\ddot{\underset{\sim}{r}} = -g\,\underset{\sim}{j}$ as gravity is acting against the motion in the vertical direction.

Integrate with respect to t: $\quad \dot{\underset{\sim}{r}} = -gt\,\underset{\sim}{j} + \underset{\sim}{c}$

$t = 0$, $\underset{\sim}{u} = 24\underset{\sim}{i} + 10\underset{\sim}{j}$: $\quad 24\underset{\sim}{i} + 10\underset{\sim}{j} = \underset{\sim}{c}$

$$\begin{aligned}\dot{\underset{\sim}{r}} &= -gt\,\underset{\sim}{j} + 24\underset{\sim}{i} + 10\underset{\sim}{j}\\ &= 24\underset{\sim}{i} + (10 - gt)\underset{\sim}{j}\end{aligned}$$

Integrate with respect to t: $\quad \underset{\sim}{r} = 24t\underset{\sim}{i} + \left(10t - \dfrac{gt^2}{2}\right)\underset{\sim}{j} + \underset{\sim}{d}$

$t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$: $\quad \underset{\sim}{d} = \underset{\sim}{0}$

$$\underset{\sim}{r} = 24t\underset{\sim}{i} + \left(10t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$$

(c) $x = 24t$, $y = 10t - \dfrac{1}{2}gt^2$ are the parametric equations of the path.

$t = \dfrac{x}{24}$, $y = \dfrac{10x}{24} - \dfrac{g}{2}\times\left(\dfrac{x}{24}\right)^2$

If $g = 9.8\,\text{m s}^{-2}$: $\quad y = \dfrac{10x}{24} - \dfrac{4.9x^2}{576}$ is the Cartesian equation of the path.

(d) Hits the ground when $y = 0$: $\dfrac{10x}{24} - \dfrac{4.9x^2}{576} = 0$

$$x(240 - 4.9x) = 0$$

$$x = 0 \text{ or } x = \frac{240}{4.9} \approx 49\,\text{m}$$

The ball hits the ground 49 metres from where it was kicked.

(e) $x = 40,\ y = \dfrac{10}{24} \times 40 - \dfrac{4.9}{576} \times 40^2$

$$\approx 3.06\,\text{m}$$

The ball will pass over the crossbar.

Example 27

A particle is projected from level ground with a velocity of $7\underset{\sim}{i} + 24\underset{\sim}{j}$ m s^{-1}, where $\underset{\sim}{i}$ is horizontal and $\underset{\sim}{j}$ is vertically up. Use $g = 9.8$ m s^{-2}.

(a) Find the initial speed and angle of projection of the particle. Give the angle of projection correct to the nearest tenth of a degree.

(b) Find the time of flight of the particle. Give your answer correct to one decimal place.

(c) Find the horizontal distance travelled by the particle, correct to one decimal place.

(d) Find the maximum height reached by the particle, correct to one decimal place.

(e) Determine whether the particle is ever travelling in a direction perpendicular to its initial velocity.

Solution

(a) $\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$: $\quad |\underset{\sim}{u}| = \sqrt{7^2 + 24^2} = 25$ ms^{-1}

Angle of projection θ is $\theta = \tan^{-1}\left(\dfrac{24}{7}\right) = 73.7°$ to the horizontal

(b) $\ddot{\underset{\sim}{r}} = -g\underset{\sim}{j}$

Integrate with respect to t: $\quad \dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + \underset{\sim}{c}$

$t = 0,\ \underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$: $\quad 7\underset{\sim}{i} + 24\underset{\sim}{j} = \underset{\sim}{c}$

$$\dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + 7\underset{\sim}{i} + 24\underset{\sim}{j}$$
$$= 7\underset{\sim}{i} + (24 - gt)\underset{\sim}{j}$$

Integrate with respect to t: $\quad \underset{\sim}{r} = 7t\underset{\sim}{i} + \left(24t - \dfrac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\quad \underset{\sim}{d} = \underset{\sim}{0}$

$$\underset{\sim}{r} = 7t\underset{\sim}{i} + \left(24t - \frac{1}{2}gt^2\right)\underset{\sim}{j} = 7t\underset{\sim}{i} + \left(24t - 4.9t^2\right)\underset{\sim}{j}$$

For the time of flight, find when the vertical component of $\underset{\sim}{r}$ is zero: $24t - 4.9t^2 = 0$

$$t(24 - 4.9t) = 0$$

$$t = 0,\ t = \frac{24}{4.9} = 4.9\,\text{s}$$

(c) Substitute $t = 4.9$ into the horizontal component of $\underset{\sim}{r}$: Distance $= 7 \times 4.9 = 34.3$ m

(d) The time to the greatest height is half the time of flight: $t = 2.45$

Substitute this value of t into the vertical component of $\underset{\sim}{r}$:
Greatest height $= 24 \times 2.45 - 4.9 \times 2.45^2 \approx 29.4$ m

(e) The direction at any instant is given by the velocity function as it is tangential to the position function at any point in the path.

$\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$, $\dot{\underset{\sim}{r}} = 7\underset{\sim}{i} + (24 - 4.9t)\underset{\sim}{j}$. Find when $\underset{\sim}{u} \bullet \dot{\underset{\sim}{r}} = 0$: $\left(7\underset{\sim}{i} + 24\underset{\sim}{j}\right) \bullet \left(7\underset{\sim}{i} + (24 - 9.8t)\underset{\sim}{j}\right) = 0$

$$49 + 24(24 - 9.8t) = 0$$

$$49 + 576 - 235.2t = 0$$

$$t = \frac{625}{235.2} = 2.66\text{ s}$$

The particle is travelling in a direction perpendicular to the original direction at 2.66 seconds.

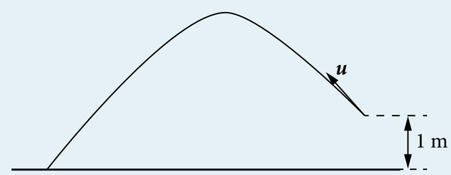
As considered in section 6.5, projectile motion with no air resistance is generally not a realistic model because real projectiles experience resistance due to the medium that they move through. This resistance is a force that can often be approximated as being proportional to the velocity, or proportional to the square of the velocity of the projectile, depending on the conditions of the medium. In this section you will consider resistance proportional to the velocity of the projectile.

Resistance as well as gravity—a vector approach

If there is any resistance other than gravity, it needs to be incorporated into the acceleration vector. The new acceleration vector can then be used to form the other equations of motion.

Example 28

During a game of badminton at the beach, a shuttlecock is hit at a height of 1 m with a velocity of $2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$ m s^{-1}, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively. The acceleration of the shuttlecock due to the combined effect of gravity, air resistance and wind is $2\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k}$ m s^{-2}. Assume the sand to be horizontally flat and take the origin to be at sand level, directly below the point of projection.



(a) Find the time of flight of the shuttlecock. Give your answer correct to two decimal places.

(b) Find where the shuttlecock will land.

(c) Find the maximum height reached by the shuttlecock.

Solution

(a) Define $\ddot{\underset{\sim}{r}}$: $\ddot{\underset{\sim}{r}} = 2\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k}$.

Integrate with respect to t: $\dot{\underset{\sim}{r}} = \int \ddot{\underset{\sim}{r}}\, dt$

$$= \int \left(2\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k}\right) dt$$

$$= 2t\underset{\sim}{i} - t\underset{\sim}{j} - 8t\underset{\sim}{k} + \underset{\sim}{c}$$

$t = 0$, $\dot{\underset{\sim}{r}} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$: $2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k} = \underset{\sim}{c}$

Hence $\dot{\underset{\sim}{r}} = 2t\underset{\sim}{i} - t\underset{\sim}{j} - 8t\underset{\sim}{k} + 2\underset{\sim}{i} + 2\underset{\sim}{j} + 8\underset{\sim}{k}$

$$= (2 + 2t)\underset{\sim}{i} + (2 - t)\underset{\sim}{j} + (8 - 8t)\underset{\sim}{k}$$

Integrate with respect to t: $\underset{\sim}{r} = \int \left((2+2t)\underset{\sim}{i} + (2-t)\underset{\sim}{j} + (8-8t)\underset{\sim}{k} \right) dt$

$$= \left(2t + t^2\right)\underset{\sim}{i} + \left(2t - \frac{t^2}{2}\right)\underset{\sim}{j} + \left(8t - 4t^2\right)\underset{\sim}{k} + \underset{\sim}{d}$$

$t = 0$, $\underset{\sim}{r} = \underset{\sim}{k}$: $\underset{\sim}{k} = \underset{\sim}{d}$

Hence $\underset{\sim}{r} = \left(2t + t^2\right)\underset{\sim}{i} + \left(2t - \frac{t^2}{2}\right)\underset{\sim}{j} + \left(1 + 8t - 4t^2\right)\underset{\sim}{k}$

The shuttlecock hits the sand when the vertical component ($\underset{\sim}{k}$) of the motion is zero: $1 + 8t - 4t^2 = 0$

Rewrite as $4t^2 - 8t - 1 = 0$ and solve: $t = \frac{8 \pm \sqrt{80}}{8} = \frac{2 \pm \sqrt{5}}{2}$

As $t > 0$, the only solution is $t = \frac{2+\sqrt{5}}{2} \approx 2.12$ s

The time of flight of the shuttlecock is 2.12 seconds.

(b) Find $\underset{\sim}{r}$ when $t = 2.12$ s: $\underset{\sim}{r} = \left(4.24 + 2.12^2\right)\underset{\sim}{i} + \left(4.24 - \frac{2.12^2}{2}\right)\underset{\sim}{j} + 0\underset{\sim}{k}$

$$= 8.73\underset{\sim}{i} + 1.99\underset{\sim}{j}$$

The shuttlecock lands approximately 8.7 m east and 2 m north of its point of projection.

(c) The maximum height is achieved when the $\underset{\sim}{k}$ component of the velocity vector is zero: $8 - 8t = 0$

$t = 1$ s

Hence the maximum height can be found by substituting $t = 1$ into the $\underset{\sim}{k}$ component of the displacement: $h = 1 + 8(1) - 4(1)^2 = 5$ m.

The shuttlecock reaches a maximum height of 5 metres.

When the resistance to the motion is included in the acceleration vector, the solution to the problem follows the same approach as used earlier. Note that if you are working in three dimensions, coordinates will be needed to locate points in a plane.

Resistance as well as gravity—a Cartesian approach

A particle of mass m is launched at time $t = 0$, from ground level on a flat plane, with an initial velocity of u m s^{-1} at an angle of θ to the horizontal. In addition to gravity, there is an air resistance force, which acts in the opposite direction to the instantaneous direction of motion. The magnitude of this resistance force is directly proportional to the particle's instantaneous speed.

Use standard Cartesian coordinates with the x-axis horizontal and the y-axis vertical. Let the components of the acceleration be $\ddot{x}$ and $\ddot{y}$, so that the components of the velocity are $\dot{x}$ and $\dot{y}$ and so the components of the displacement are x and y.

Initially: $x = 0$, $\dot{x} = u_x = u\cos\theta$, $y = 0$, $\dot{y} = u_y = u\sin\theta$.

Now $m\ddot{x} = -mk\dot{x}$ and $m\ddot{y} = -mg - mk\dot{y}$, where k is a positive constant.

Dividing by m reduces these equations to: $\ddot{x} = -k\dot{x}$, $\ddot{y} = -g - k\dot{y}$

Consider the horizontal motion and let $v_x = \dot{x}$ so that $\ddot{x} = \frac{dv_x}{dt}$: $\frac{dv_x}{dt} = -kv_x$

$$\frac{dv_x}{v_x} = -kdt$$

Integrate with respect to t: $\int_{u_x}^{v_x} \frac{dv_x}{v_x} = -k\int_0^t dt$

$$\left[\log_e(v_x)\right]_{u_x}^{v_x} = -kt$$

$$\log_e\left(\frac{v_x}{u_x}\right) = -kt$$

$$v_x = u_x e^{-kt}$$

Now $v_x = \dot{x}$: $\dfrac{dx}{dt} = u_x e^{-kt}$

Integrate with respect to t: $x = u_x \displaystyle\int_0^t e^{-kt}\,dt$

$$x = -\frac{u_x}{k}\left[e^{-kt}\right]_0^t$$

$$x = -\frac{u_x}{k}\left(e^{-kt} - 1\right)$$

$$x = \frac{u_x}{k}\left(1 - e^{-kt}\right)$$

Consider the vertical motion and let $v_y = \dot{y}$ so that $\ddot{y} = \dfrac{dv_y}{dt}$: $\quad \dfrac{dv_y}{dt} = -g - kv_y$

$$\frac{dv_y}{g + kv_y} = -dt$$

Integrate with respect to t: $\quad \displaystyle\int_{u_y}^{v_y} \frac{dv_y}{g + kv_y} = -\int_0^t dt$

$$\left[\frac{1}{k}\log_e\left(g + kv_y\right)\right]_{u_y}^{v_y} = -t$$

$$\log_e\left(g + kv_y\right) - \log_e\left(g + ku_y\right) = -kt$$

$$\log_e\left(\frac{g + kv_y}{g + ku_y}\right) = -kt$$

$$\frac{g + kv_y}{g + ku_y} = e^{-kt}$$

$$kv_y = (g + ku_y)e^{-kt} - g$$

$$v_y = \frac{1}{k}\left((g + ku_y)e^{-kt} - g\right)$$

Now $v_y = \dot{y}$ and $\dfrac{dy}{dt} = \dfrac{1}{k}\left((g + ku_y)e^{-kt} - g\right)$.

Integrate with respect to t: $y = \dfrac{1}{k}\displaystyle\int_0^t \left((g + ku_y)e^{-kt} - g\right)dt$

$$= \frac{1}{k}\left[\frac{g + ku_y}{-k}e^{-kt} - gt\right]_0^t$$

$$= -\frac{g + ku_y}{k^2}e^{-kt} - \frac{gt}{k} + \frac{g + ku_y}{k^2}$$

$$= \frac{g + ku_y}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$$

As a result of all of this, you have obtained the parametric equations of the velocity: $\dot{x} = u_x e^{-kt}$, $\dot{y} = \dfrac{1}{k}\left((g + ku_y)e^{-kt} - g\right)$.

The parametric equations of the path are $x = \dfrac{u_x}{k}\left(1 - e^{-kt}\right)$, $y = \dfrac{\left(g + ku_y\right)}{k^2}\left(1 - e^{-kt}\right) - \dfrac{gt}{k}$.

Summary of important results

Projectile motion in a medium whose resistance is proportional to the velocity of the particle:

$t = 0$, $x = 0$, $y = 0$, $\dot{x} = u_x$, $\dot{y} = u_y$ where $u_x = u\cos\theta$, $u_y = u\sin\theta$, k is the constant of proportionality in the resistance, θ is the angle of projection relative to the horizontal axis, and u is the initial velocity.

$\ddot{x} = -k\dot{x}$, $\ddot{y} = -g - k\dot{y}$

$$\dot{x} = u_x e^{-kt} = u\cos\theta e^{-kt},\ \dot{y} = \frac{1}{k}\left((g + ku_y)e^{-kt} - g\right) = \frac{1}{k}\left((g + ku\sin\theta)e^{-kt} - g\right) = u\sin\theta e^{-kt} - \frac{g}{k}\left(1 - e^{-kt}\right)$$

$$x = \frac{u_x}{k}\left(1 - e^{-kt}\right) = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right),\ y = \frac{\left(g + ku_y\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k} = \frac{(g + ku\sin\theta)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$$

With no air resistance, the equations are instead:

Acceleration: $\ddot{x} = 0$, $\ddot{y} = -g$

Velocity: $\dot{x} = u\cos\theta$, $\dot{y} = u\sin\theta - gt$

Displacement: $x = u\cos\theta t$, $y = u\sin\theta t - \frac{1}{2}gt^2$

When there is no air resistance, the horizontal velocity is $\dot{x} = u\cos\theta$, whereas with air resistance the horizontal velocity becomes $\dot{x} = u\cos\theta e^{-kt}$, which means the horizontal velocity is reducing over time, decaying exponentially. This means that with air resistance, the particle slows down and will not travel as far.

When there is no air resistance, the vertical velocity is $\dot{y} = u\sin\theta - gt$, whereas with air resistance the vertical velocity becomes $\dot{y} = u\sin\theta e^{-kt} - \frac{g}{k}\left(1 - e^{-kt}\right)$. With no air resistance the greatest height is achieved after $\frac{u\sin\theta}{g}$ seconds, but with air resistance it is after $\frac{1}{k}\log_e\left|\frac{ku\sin\theta}{g} + 1\right|$ seconds.

With no air resistance, the greatest height attained is $\frac{u^2\sin^2\theta}{2g}$ m; with air resistance it is $\frac{u\sin\theta}{k} - \frac{g}{k^2}\log_e\left|1 + \frac{ku\sin\theta}{g}\right|$, which is always less (for sensible positive values of the constants).

This air resistance behaviour is best considered using a numerical example.

Example 29

A particle is projected from a point on the horizontal plane with an initial velocity given by the components $u_x = 3\text{ m s}^{-1}$, $u_y = 8\text{ m s}^{-1}$. Use $g = 9.8\text{ m s}^{-2}$.

(a) If the only resistance to the motion is gravity, find the parametric equations of the trajectory of the particle. When does the particle hit the ground?

The particle is projected again into a medium which resists the motion, where the resistance to the motion is directly proportional to the velocity of the particle. Let the constant of proportionality be k.

The parametric equations of the trajectory are given as $x = \frac{u_x}{k}\left(1 - e^{-kt}\right)$, $y = \frac{\left(g + ku_y\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$.

(b) Find the parametric equations of the trajectory when **(i)** $k = 0.5$ **(ii)** $k = 0.1$.

(c) Using technology, draw on the same set of axes the graphs of the three trajectories in parts **(a)** and **(b)**.

(d) Discuss what your graphs in part **(c)** tell you about the motion in each case.

Solution

(a) No air resistance: $u_x = 3$, $u_y = 8$, $g = 9.8$.

Initial equations of motion:	$\ddot{x} = 0$	$\ddot{y} = -g = -9.8$
Integrate with respect to t:	$\dot{x} = C_1$	$\dot{y} = -9.8t + C_2$
$t = 0, u_x = 3, u_y = 8$:	$\dot{x} = 3$	$\dot{y} = 8 - 9.8t$
Integrate with respect to t:	$x = 3t + C_3$	$y = 8t - 4.9t^2 + C_4$
$t = 0, x = 0, y = 0$:	$x = 3t$	$y = 8t - 4.9t^2$
Hits the ground when $y = 0$:	$8t - 4.9t^2 = 0$	

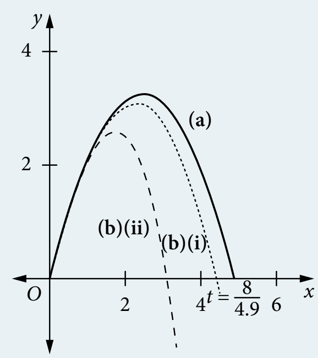
$$t = \frac{8}{4.9} \text{ s}$$

(b) **(i)** $k = \frac{1}{2}$: $x = \frac{3}{0.5}\left(1 - e^{-\frac{t}{2}}\right)$ $\quad y = \frac{9.8 + 4}{0.5^2}\left(1 - e^{-\frac{t}{2}}\right) - \frac{9.8t}{0.5}$

$x = 6\left(1 - e^{-\frac{t}{2}}\right)$ $\quad y = 55.2\left(1 - e^{-\frac{t}{2}}\right) - 19.6t$

(ii) $k = 0.1$: $x = \frac{3}{0.1}\left(1 - e^{-\frac{t}{10}}\right)$ $\quad y = \frac{9.8 + 4}{0.1^2}\left(1 - e^{-\frac{t}{10}}\right) - \frac{9.8t}{0.1}$

$x = 30\left(1 - e^{-\frac{t}{10}}\right)$ $\quad y = 1380\left(1 - e^{-\frac{t}{10}}\right) - 98t$



(c) In the diagram, the solid line represents part **(a)**, the dotted line represents part **(b)(i)** and the dashed line represents part **(b)(ii)**. Each line has been completed for $t = \frac{8}{4.9}$ seconds, the time taken for the particle without air resistance to return to the ground.

Two of the lines show $y < 0$, but this is just so the line lengths can be compared for the largest value of t in part **(a)**. After each particle hits the ground, it stops, so the parts of the paths below zero do not actually exist.

(d) At the beginning of the motion the paths are similar. The trajectory in part **(a)** is symmetrical about its greatest height.

The larger the air resistance, the sooner the trajectory falls below the path in part **(a)**. The particles in part **(b)** hit the ground before the particle in part **(a)**.

The larger the air resistance, the shorter the range.

The larger the air resistance, the lower the greatest height and the steeper the fall after the particle reaches its greatest height.

EXPLORING FURTHER

Projectile motion and initial velocity

Use technology to observe the effect of changing the initial velocity on a model of projectile motion.

Example 30

A projectile is fired from the origin O with an initial velocity u m s^{-1} at an angle θ to the horizontal in a medium whose resistance is proportional to the velocity.

The parametric equations of the trajectory are $x = \dfrac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$ and $y = \dfrac{(10 + ku\sin\theta)}{k^2}\left(1 - e^{-kt}\right) - \dfrac{10t}{k}$,

where k is the constant of proportionality of the resistance.

The projectile is fired at an angle of 60° to the horizontal with an initial velocity of $10\sqrt{3}$ m s^{-1}, $k = 0.4$.

(a) Find when the projectile reaches its greatest height, correct to two decimal places.

(b) Find the greatest height that is reached, correct to two decimal places.

(c) Show that the projectile hits the ground when $t \approx 2.6$ s **(i)** graphically **(ii)** by substitution.

(d) Find the horizontal range of the projectile.

(e) Graph the path of the projectile.

Solution

Write the parametric equations of the trajectory using the information provided.

$$x = \frac{10\sqrt{3} \times \frac{1}{2}}{0.4}\left(1 - e^{-0.4t}\right) = 12.5\sqrt{3}\left(1 - e^{-0.4t}\right)$$

$$y = \frac{10 + 0.4 \times 10\sqrt{3} \times \frac{\sqrt{3}}{2}}{0.4^2}\left(1 - e^{-0.4t}\right) - \frac{10t}{0.4} = 100\left(1 - e^{-0.4t}\right) - 25t$$

(a) Greatest height when $\dot{y} = 0$: $y = 100\left(1 - e^{-0.4t}\right) - 25t$

$$\begin{aligned}
\dot{y} &= 40e^{-0.4t} - 25 \\
40e^{-0.4t} - 25 &= 0 \\
e^{-0.4t} &= \frac{5}{8} \\
-0.4t &= \log_e\left(\frac{5}{8}\right) \\
t &= 2.5\log_e 1.6 \\
t &= 1.18\text{ s}
\end{aligned}$$

(b) $t = 1.18$: $y = 100\left(1 - e^{-0.4\times1.18}\right) - 25 \times 1.18$

$= 8.12$ m

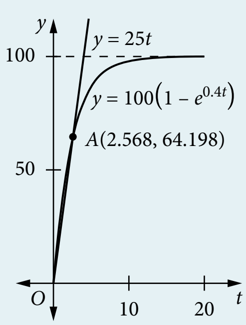
The greatest height is 8.12 metres.

(c) **(i)** Hits the ground when $y = 0$: $y = 100\left(1 - e^{-0.4t}\right) - 25t$

$$\begin{aligned}
100\left(1 - e^{-0.4t}\right) - 25t &= 0 \\
100\left(1 - e^{-0.4t}\right) &= 25t
\end{aligned}$$

Hence graph $y = 100\left(1 - e^{-0.4t}\right)$ and $y = 25t$.

It hits the ground at $t = 2.6$ seconds.



(ii) $t = 2.6$ s, $y = 100\left(1 - e^{-0.4t}\right) - 25t$

$$\begin{aligned}
&= 100\left(1 - e^{-1.04}\right) - 25 \times 2.6 \\
&= -0.345 \approx 0
\end{aligned}$$

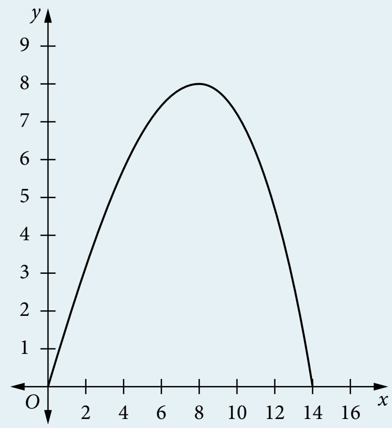
It hits the ground at approximately $t = 2.6$ seconds.

(d) $t = 2.6,\ x = 12.5\sqrt{3}\left(1 - e^{-0.4\times 2.6}\right)$

$= 12.5\sqrt{3}\left(1 - e^{-1.04}\right)$

$= 13.998 \approx 14.0\,\text{m}$

(e)



Using terminal velocity

Given the terminal velocity $v_T = \frac{g}{k}$, consider what this means in the equations involving air resistance. Remember, the terminal velocity for a projectile falling vertically downwards is the velocity at which the drag force (air resistance) balances the gravitational force.

Now $v_T = \frac{g}{k}$, $k = \frac{g}{v_T}$ or $\frac{1}{k} = \frac{v_T}{g}$ can be substituted in each of the previous equations for air resistance motion.

Horizontally: $\ddot{x} = -kx = -\frac{g\dot{x}}{v_T}$, $\dot{x} = u_x e^{-kt} = u_x e^{\frac{-gt}{v_T}}$, $x = \frac{u_x}{k}\left(1 - e^{-kt}\right) = \frac{u_x v_T}{g}\left(1 - e^{\frac{-gt}{v_T}}\right)$.

When there is no air resistance, the horizontal velocity is $\dot{x} = u_x$, whereas with air resistance the horizontal velocity has become $\dot{x} = u_x e^{\frac{-gt}{v_T}}$, which is reducing over time, decaying exponentially.

Vertically: $\ddot{y} = -g - k\dot{y} = -g\left(1 + \frac{\dot{y}}{v_T}\right)$,

$$\dot{y} = \frac{1}{k}\left((g + ku_y)e^{-kt} - g\right) = \left(v_T + u_y\right)e^{\frac{-gt}{v_T}} - v_T = u_y e^{\frac{-gt}{v_T}} - v_T\left(1 - e^{\frac{-gt}{v_T}}\right),$$

$$y = \frac{g + ku_y}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k} = \frac{v_T}{g}\left(v_T + u_y\right)\left(1 - e^{\frac{-gt}{v_T}}\right) - v_T t.$$

When there is no air resistance, the vertical velocity is $\dot{y} = u_y - gt$, whereas with air resistance the vertical velocity has become $\dot{y} = u_y e^{\frac{-gt}{v_T}} - v_T\left(1 - e^{\frac{-gt}{v_T}}\right)$.

Consider some values for t: $t = \frac{v_T}{g}$, $\dot{x} = u_x e^{-1} \approx 0.37u_x$, $\dot{y} = u_y e^{-1} - v_T\left(1 - e^{-1}\right) \approx 0.37u_y - 0.63v_T$

$t = \frac{2v_T}{g}$, $\dot{x} = u_x e^{-2} \approx 0.14u_x$, $\dot{y} = u_y e^{-2} - v_T\left(1 - e^{-2}\right) \approx 0.14u_y - 0.86v_T$

$t = \frac{3v_T}{g}$, $\dot{x} = u_x e^{-3} \approx 0.05u_x$, $\dot{y} = u_y e^{-3} - v_T\left(1 - e^{-3}\right) \approx 0.05u_y - 0.95v_T$

Because $v_T < 0$, as it is downwards, therefore as t increases above $t = \frac{v_T}{g}$, the horizontal velocity decreases exponentially and becomes negligible, so that the particle appears to be falling vertically.

When $t < \frac{v_T}{g}$, the equations of motion with air resistance approximate the equations of motion where air resistance is ignored.

EXERCISE 6.6 PROJECTILES AND RESISTED MOTION

1 An aircraft drops a package of emergency rations to a family stranded in the floods. The aircraft is travelling horizontally at $45.0\,\text{m s}^{-1}$ and is 100 m above the ground. A parachute allows the package to fall with constant speed and hit the ground 10 s after release. Air resistance can be ignored.

(a) Find where the package hits the ground relative to the point from where it was dropped, to the nearest metre.
(b) Find the velocity of the package just before it hits the ground, correct to one decimal place.

2 A particle is projected from level ground with a velocity of $7\underset{\sim}{i} + 24\underset{\sim}{j}\,\text{m s}^{-1}$, where $\underset{\sim}{i}$ is horizontal and $\underset{\sim}{j}$ is vertically up.

(a) Find the initial speed and angle of projection of the particle. Give the angle of projection, correct to the nearest tenth of a degree. Air resistance can be ignored.
(b) Find the time of flight of the particle. Give your answer correct to two decimal places.
(c) Find the horizontal distance travelled by the particle. Give your answer correct to one decimal place.
(d) Find the maximum height reached by the particle. Give your answer correct to one decimal place.
(e) Determine whether the particle is ever travelling in a direction perpendicular to its initial velocity.

3 A cricket ball is thrown from a height of 1 metre with a speed of $30\,\text{m s}^{-1}$ and at an angle of 60° to the horizontal.

(a) Taking the origin at the point of projection and assuming the ground to be level, find the horizontal distance travelled by the ball before it lands. Give your answer correct to one decimal place.
(b) Find the maximum height of the ball above the ground. Give your answer correct to one decimal place.
(c) Find the Cartesian equation of the path of the ball and sketch its path.

4 A baseball is hit horizontally with a speed of $200\,\text{km h}^{-1}$ from a height of 1.225 m. Find how far it travels horizontally before landing, assuming that the baseball field is level. Give your answer correct to one decimal place.

5 A projectile is launched at an angle of 45° to the horizontal. Determine the launching speed, correct to one decimal place, needed to achieve the following (relative to a horizontal plane through the point of projection):

(a) a range of 100 m (b) a height of 25 m.

6 A particle is projected from level ground with a speed of $19.6\,\text{m s}^{-1}$ at an angle of projection of α.

(a) Show that the particle reaches its maximum height halfway through its flight.
(b) Find the angle α for which the range of the particle is a maximum and determine this maximum range. Express the maximum range correct to one decimal place.

7 A projectile is projected from the origin with initial speed u and angle of projection α.

(a) Show that the Cartesian equation of its path is $y = x\tan\alpha - \dfrac{gx^2}{2u^2}\sec^2\alpha$.
(b) Hence find an expression for the projectile's range R.
(c) Hence find an expression for the projectile's maximum height y_{max}.
(d) Show that for a given initial speed, the maximum range of the projectile occurs when $\alpha = 45°$ and is given by $R = \dfrac{u^2}{g}$.

8 A golf ball is hit from a point O with initial velocity V at an angle θ to the horizontal ground. The ball first hits the ground at a point P, which is at the same horizontal level as O.

(a) Given that $V\cos\theta = 6u$ and $V\sin\theta = \dfrac{5u}{2}$, find the time taken to travel from O to P, in terms of u and g.
(b) Find the range R of the golf ball in terms of u and g.
(c) Express V in terms of u.
(d) State the golf ball's minimum speed during its flight.

9 The velocity (in m s^{-1}) at time t seconds of a ball hit from a height of 2 metres above ground level is given by $\dot{\underset{\sim}{r}} = 12\underset{\sim}{i} + 9\underset{\sim}{j} + (30 - 9.8t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively and the origin is at ground level. Find the ball's height above ground level, in metres, after 2 seconds.

10 A golf ball is hit from the ground with a velocity $20\underset{\sim}{i}+0\underset{\sim}{j}+15\underset{\sim}{k}\ \text{m s}^{-1}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors horizontally forward, horizontally to the left, and vertically upwards, respectively. After being hit, the ball has a gravitational acceleration of $-10\underset{\sim}{k}\ \text{m s}^{-2}$ and also has a 'hook' (i.e. a horizontal acceleration to the left) of $4\underset{\sim}{j}\ \text{m s}^{-2}$. Air resistance can be ignored.

(a) Find the expression for $\underset{\sim}{r}$, the position vector of the ball at time t.

(b) When the ball hits the ground, how far will it be to the left of the line along the horizontally forward direction?

(c) Find when the speed of the ball is a minimum. Give your answer correct to two decimal places.

(d) Calculate the minimum speed. Give your answer correct to one decimal place.

11 A particle is projected from a point on the horizontal plane with an initial velocity given by the components $u_x = 4\ \text{m s}^{-1}$, $u_y = 6\ \text{m s}^{-1}$. Use $g = 9.8\ \text{m s}^{-2}$.

(a) If the only resistance to the motion is gravity, find the parametric equations of the trajectory of the particle. When does the particle hit the ground?

The particle is projected again into a medium which resists the motion, where the resistance to the motion is directly proportional to the velocity of the particle. Let the constant of proportionality be k.

The parametric equations of the trajectory are given as $x = \dfrac{u_x}{k}\left(1-e^{-kt}\right)$, $y = \dfrac{\left(g+ku_y\right)}{k^2}\left(1-e^{-kt}\right)-\dfrac{gt}{k}$.

(b) Find the parametric equations of the trajectory when **(i)** $k = 0.4$ **(ii)** $k = 0.2$.

(c) Using technology, draw on the same set of axes the graphs of the three trajectories in parts **(a)** and **(b)**.

(d) Discuss what your graphs in part **(c)** tell you about the motion in each case.

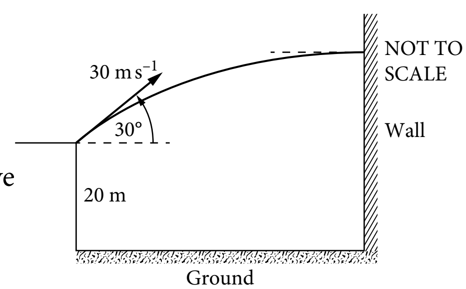
12 The velocity (in m s^{-1}) at time t seconds of a ball thrown at a height of 12 metres above ground level is given by $\dot{\underset{\sim}{r}} = 8\underset{\sim}{i}+3\underset{\sim}{j}+(20-9.8t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively and the origin is at ground level. Find when the ball hits the ground.

13 The trajectory of a projectile fired with speed $u\ \text{m s}^{-1}$ at an angle θ to the horizontal, in a medium whose resistance to the projectile's motion is proportional to the projectile's velocity, is represented by the parametric equations $x = \dfrac{u\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \dfrac{\left(10+ku\sin\theta\right)}{k^2}\left(1-e^{-kt}\right)-\dfrac{10t}{k}$, where k is the constant of proportionality of the resistance.

(a) Show that the greatest height is reached when $t = \dfrac{1}{k}\log_e\left(\dfrac{10+ku\sin\theta}{10}\right)$.

(b) Find the greatest height.

A ball is thrown from a point 20 m above the horizontal ground in the same medium as mentioned above. It is thrown with speed $30\ \text{m s}^{-1}$ at an angle of 30° to the horizontal. At its highest point it hits a wall as shown in the diagram.



(c) If $k = 0.4$, show that the ball hits the wall at a height of 25.8 m above the ground.

(d) What is the horizontal distance of the wall from the point of projection?

14 A projectile is fired from the origin O with an initial velocity $V\ \text{m s}^{-1}$ at an angle θ to the horizontal in a medium whose resistance is proportional to the velocity.

The parametric equations of the trajectory are $x = \dfrac{V\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \dfrac{(10+kV\sin\theta)}{k^2}\left(1-e^{-kt}\right)-\dfrac{10t}{k}$, where $k = 0.2$ is the constant of proportionality of the resistance.

The projectile is fired at an angle of 45° to the horizontal with an initial velocity of $20\sqrt{2}\ \text{m s}^{-1}$.

(a) Find when the projectile reaches its greatest height, correct to one decimal place.

(b) Find the greatest height attained, correct to one decimal place.

(c) Show that the projectile hits the ground when $t \approx 3.6$ s **(i)** graphically **(ii)** by substitution.

(d) Find the horizontal range of the projectile.

6.7 RESISTANCE PROPORTIONAL TO THE SQUARE OF THE VELOCITY

In section 6.6 you considered projectile motion with resistance proportional to the velocity. In this section you will extend this to consider projectile motion with resistance proportional to the square of the velocity.

For air resistance that is proportional to the square of the velocity, the resistance can be represented as a vector that is opposite in direction to the velocity vector, with a magnitude equal to the magnitude of the velocity vector squared, multiplied by a constant k.

Using this definition to derive the parametric equations of the resistance in the two-dimensional case:

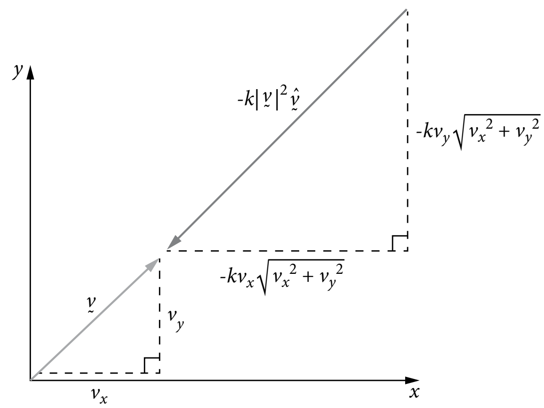
$$\underset{\sim}{R} = -k|\underset{\sim}{v}|^2\,\hat{\underset{\sim}{v}} \quad \text{where } \hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} \text{ and so}$$

$$\underset{\sim}{R} = -k\left(\sqrt{v_x^{\,2}+v_y^{\,2}}\right)^2 \frac{\underset{\sim}{v}}{\sqrt{v_x^{\,2}+v_y^{\,2}}}$$

$$\underset{\sim}{R} = -k\sqrt{v_x^{\,2}+v_y^{\,2}}\;\underset{\sim}{v}$$

$$\therefore\; R_x = -kv_x\sqrt{v_x^{\,2}+v_y^{\,2}}$$

$$R_y = -kv_y\sqrt{v_x^{\,2}+v_y^{\,2}}$$



The horizontal and vertical components of the air resistance are given by $-kv_x\sqrt{v_x^{\,2}+v_y^{\,2}}$ and $-kv_y\sqrt{v_x^{\,2}+v_y^{\,2}}$, respectively. These components are not independent of each other, which means there is no straightforward way to find the position vectors by integration. Instead, projectile motion can be examined by using a mathematical model where the components of the resistance are simplified and approximated to be simply proportional to the velocity vector components squared. That is, $-kv_x\sqrt{v_x^{\,2}+v_y^{\,2}}$ is approximated as $-k(v_x)^2$ and $-kv_y\sqrt{v_x^{\,2}+v_y^{\,2}}$ is approximated as $-k(v_y)^2$.

A mathematical model for air resistance with the square of the velocity

A particle of mass m is launched at time $t = 0$, from ground level on a flat plane, at an angle of θ to the horizontal with an initial velocity of u m s^{-1}. In addition to gravity, there is an air resistance force that acts in the opposite direction to the instantaneous direction of motion. The magnitude of this resistance force is directly proportional to the square of its instantaneous speed.

Use the standard Cartesian coordinates with the x-axis horizontal and the y-axis vertical.

Let the components of the acceleration be $\ddot{x}$ and $\ddot{y}$, the components of the velocity be $\dot{x}$ and $\dot{y}$, and the components of the displacement be x and y.

Initially, $x = 0$, $\dot{x} = u_x = u\cos\theta$, $y = 0$, $\dot{y} = u_y = u\sin\theta$.

Now the model is $m\ddot{x} = -mk(\dot{x})^2$ and $m\ddot{y} = -mg - mk(\dot{y})^2$, where k is the constant of proportionality of the resistance.

Dividing by m reduces these equations to: $\ddot{x} = -k(\dot{x})^2$, $\ddot{y} = -g - k(\dot{y})^2$.

Consider the horizontal motion and let $v_x = \dot{x}$ so that $\ddot{x} = \frac{dv_x}{dt}$: $\frac{dv_x}{dt} = -kv_x{}^2$

$$\frac{dv_x}{v_x{}^2} = -kdt$$

Integrate both sides: $\int_{u_x}^{v_x} \frac{dv_x}{v_x{}^2} = -k\int_0^t dt$

$$\left[\frac{-1}{v_x}\right]_{u_x}^{v_x} = -kt$$

$$\frac{-1}{v_x} + \frac{1}{u_x} = -kt$$

$$\frac{1}{v_x} = \frac{1+ku_x t}{u_x}$$

$$v_x = \frac{u_x}{1+ku_x t}$$

Now $v_x = \dot{x}$: $\frac{dx}{dt} = \frac{u_x}{1+ku_x t}$

Integrate with respect to t: $x = u_x \int_0^t \frac{dt}{1+ku_x t}$

$$x = \frac{1}{k}\Big[\log_e(1+ku_x t)\Big]_0^t$$

$$x = \frac{1}{k}\log_e(1+ku_x t)$$

Consider the vertical motion and let $v_y = \dot{y}$ so that $\ddot{y} = \frac{dv_y}{dt}$:

$$\frac{dv_y}{dt} = -g - k(v_y)^2$$

$$\frac{dv_y}{g + k(v_y)^2} = -dt$$

Integrate both sides: $\int_{u_y}^{v_y} \frac{dv_y}{g + k(v_y)^2} = -\int_0^t dt$

$$\frac{1}{k}\int_{u_y}^{v_y} \frac{dv_y}{\frac{g}{k} + (v_y)^2} = -\int_0^t dt$$

$$\frac{1}{k}\left[\sqrt{\frac{k}{g}}\tan^{-1}\left(\frac{v_y}{\sqrt{\frac{g}{k}}}\right)\right]_{u_y}^{v_y} = -t$$

$$\frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(\frac{\sqrt{k}v_y}{\sqrt{g}}\right) - \tan^{-1}\left(\frac{\sqrt{k}u_y}{\sqrt{g}}\right)\right) = -t$$

$$\sqrt{kg}\,t = \tan^{-1}\left(\frac{\sqrt{k}u_y}{\sqrt{g}}\right) - \tan^{-1}\left(\frac{\sqrt{k}v_y}{\sqrt{g}}\right)$$

Now $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$ and $A=\tan^{-1}\left(\dfrac{\sqrt{k}u_y}{\sqrt{g}}\right)$, $B=\tan^{-1}\left(\dfrac{\sqrt{k}v_y}{\sqrt{g}}\right)$

$$\text{So: }\tan\left(\sqrt{kg}\,t\right)=\frac{\left(\dfrac{\sqrt{k}u_y}{\sqrt{g}}\right)-\left(\dfrac{\sqrt{k}v_y}{\sqrt{g}}\right)}{1+\left(\dfrac{\sqrt{k}u_y}{\sqrt{g}}\right)\left(\dfrac{\sqrt{k}v_y}{\sqrt{g}}\right)}$$

$$\tan\left(\sqrt{kg}\,t\right)=\frac{g}{\sqrt{g}}\times\frac{\sqrt{k}u_y-\sqrt{k}v_y}{g+\sqrt{k}u_y\sqrt{k}v_y}$$

$$\tan\left(\sqrt{kg}\,t\right)\left(g+\sqrt{k}u_y\sqrt{k}v_y\right)=\sqrt{g}\left(\sqrt{k}u_y-\sqrt{k}v_y\right)$$

$$g\tan\left(\sqrt{kg}\,t\right)+kv_yu_y\tan\left(\sqrt{kg}\,t\right)=\sqrt{gk}u_y-\sqrt{gk}v_y$$

$$\sqrt{gk}v_y+kv_yu_y\tan\left(\sqrt{kg}\,t\right)=\sqrt{gk}u_y-g\tan\left(\sqrt{kg}\,t\right)$$

$$v_y=\frac{\sqrt{gk}u_y-g\tan\left(\sqrt{kg}\,t\right)}{\sqrt{gk}+ku_y\tan\left(\sqrt{kg}\,t\right)}$$

Now $v_y=\dot{y}$ and $\dfrac{dy}{dt}=\dfrac{\sqrt{gk}u\sin\theta-g\tan\left(\sqrt{gk}\,t\right)}{\sqrt{gk}+ku\sin\theta\tan\left(\sqrt{gk}\,t\right)}$, so that $y=\displaystyle\int\frac{\sqrt{gk}u\sin\theta-g\tan\left(\sqrt{gk}t\right)}{\sqrt{gk}+ku\sin\theta\tan\left(\sqrt{gk}t\right)}dt$.

Let $\alpha=\sqrt{gk}$ so $\alpha^2=gk$ or $g=\dfrac{\alpha^2}{k}$. Thus: $y=\displaystyle\int\frac{\alpha u\sin\theta-g\tan(\alpha t)}{\alpha+ku\sin\theta\tan(\alpha t)}dt$

Rewriting this integrand in terms of $\sin(\alpha t)$ and $\cos(\alpha t)$:

$$y=\int\frac{\alpha u\sin\theta-\dfrac{g\sin(\alpha t)}{\cos(\alpha t)}}{\alpha+\dfrac{ku\sin\theta\sin(\alpha t)}{\cos(\alpha t)}}dt$$

$$=\int\frac{\alpha u\sin\theta\cos(\alpha t)-g\sin(\alpha t)}{\alpha\cos(\alpha t)+ku\sin\theta\sin(\alpha t)}dt$$

Now differentiate the denominator:

$$\frac{d}{dt}(\alpha\cos(\alpha t)+ku\sin\theta\sin(\alpha t))=-\alpha^2\sin(\alpha t)+\alpha ku\sin\theta\cos(\alpha t)$$

$$=k\left(\alpha u\sin\theta\cos(\alpha t)-\frac{\alpha^2}{k}\sin(\alpha t)\right)$$

$$=k\left(\alpha u\sin\theta\cos(\alpha t)-g\sin(\alpha t)\right)$$

Thus you can write $y=\dfrac{1}{k}\displaystyle\int\frac{f'(t)}{f(t)}dt$ where $f(t)=\alpha\cos(\alpha t)+ku\sin\theta\sin(\alpha t)$, so that $y=\log_e\left|f(t)\right|+C$.

Hence: $y=\displaystyle\int\frac{\alpha u\sin\theta\cos(\alpha t)-g\sin(\alpha t)}{\alpha\cos(\alpha t)+ku\sin\theta\sin(\alpha t)}dt$

$$=\frac{1}{k}\int\frac{k\left(\alpha u\sin\theta\cos(\alpha t)-g\sin(\alpha t)\right)}{\alpha\cos(\alpha t)+ku\sin\theta\sin(\alpha t)}dt$$

$$=\frac{1}{k}\log_e\left|\alpha\cos(\alpha t)+ku\sin\theta\sin(\alpha t)\right|+C$$

Rewriting this with $\alpha = \sqrt{gk}$ now gives $y = \frac{1}{k}\log_e\left|\sqrt{gk}\cos\left(\sqrt{gk}t\right) + ku\sin\theta\sin\left(\sqrt{gk}t\right)\right| + C$.

When $t = 0$, $y = 0$, so $0 = \frac{1}{k}\log_e\left|\sqrt{gk}\cos 0 + ku\sin\theta\sin 0\right| + C$ and so $C = -\frac{1}{k}\log_e\left|\sqrt{gk}\right|$.

Hence:
$$\begin{aligned} y &= \frac{1}{k}\log_e\left|\sqrt{gk}\cos\left(\sqrt{gk}t\right) + ku\sin\theta\sin\left(\sqrt{gk}t\right)\right| - \frac{1}{k}\log_e\left|\sqrt{gk}\right| \\ &= \frac{1}{k}\log_e\left|\cos\left(\sqrt{gk}t\right) + \frac{ku\sin\theta\sin\left(\sqrt{gk}t\right)}{\sqrt{gk}}\right| \\ &= \frac{1}{k}\log_e\left|\cos\left(\sqrt{gk}t\right) + \sqrt{\frac{k}{g}}u\sin\theta\sin\left(\sqrt{gk}t\right)\right| \end{aligned}$$

Thus the parametric equations of the path are:

$x = \frac{1}{k}\log_e\left(1 + (ku\cos\theta)t\right)$ and $y = \frac{1}{k}\log_e\left|\cos\left(\sqrt{gk}t\right) + \sqrt{\frac{k}{g}}u\sin\theta\sin\left(\sqrt{gk}t\right)\right|$

To obtain y as a function of x, first solve x for t:

$$\begin{aligned} kx &= \log_e\left(1 + (ku\cos\theta)t\right) \\ e^{kx} &= 1 + (ku\cos\theta)t \\ (ku\cos\theta)t &= e^{kx} - 1 \\ t &= \frac{e^{kx} - 1}{ku\cos\theta} \end{aligned}$$

So $y = \frac{1}{k}\log_e\left|\cos\left(\frac{\sqrt{gk}\left(e^{kx} - 1\right)}{ku\cos\theta}\right) + \sqrt{\frac{k}{g}}u\sin\theta\sin\left(\frac{\sqrt{gk}\left(e^{kx} - 1\right)}{ku\cos\theta}\right)\right|$ is the equation of the path.

Example 31

A projectile is fired from the origin O with an initial velocity u m s^{-1} at an angle θ to the horizontal in a medium whose resistance is proportional to the square of the velocity. Use $g = 10$ m s^{-2}.

The parametric equations of the motion are: $\ddot{x} = -k(\dot{x})^2$, $\ddot{y} = -10 - k(\dot{y})^2$

$$\dot{x} = \frac{u\cos\theta}{1 + ku\cos\theta t}, \quad \dot{y} = \frac{\sqrt{10k}u\sin\theta - 10\tan\left(\sqrt{10k}t\right)}{\sqrt{10k} + ku\sin\theta\tan\left(\sqrt{10k}t\right)}$$

$$x = \frac{1}{k}\log_e\left(1 + ku\cos\theta t\right), \quad y = \frac{1}{k}\log_e\left|\cos\left(\sqrt{10k}t\right) + \sqrt{\frac{k}{10}}u\sin\theta\sin\left(\sqrt{10k}t\right)\right|$$

The projectile is fired at an angle of 60° to the horizontal with an initial velocity of $10\sqrt{3}$ m s^{-1}. $k = 0.4$.

- **(a)** Find when the projectile reaches its greatest height (correct to two decimal places).
- **(b)** Find the greatest height (correct to two decimal places).
- **(c)** Find when the projectile hits the ground (correct to two decimal places).
- **(d)** Graph the path of the projectile.

Solution

(a) Greatest height when $\dot{y}=0$: $\dot{y}=\dfrac{\sqrt{10k}u\sin\theta-10\tan\left(\sqrt{10k}t\right)}{\sqrt{10k}+ku\sin\theta\tan\left(\sqrt{10k}t\right)}$

$$=\frac{\sqrt{4}\times10\sqrt{3}\times\frac{\sqrt{3}}{2}-10\tan2t}{\sqrt{4}+0.4\times10\sqrt{3}\times\frac{\sqrt{3}}{2}\tan2t}$$

$$=\frac{30-10\tan2t}{2+6\tan2t}$$

$$=\frac{5(3-\tan2t)}{1+3\tan2t}$$

Solve: $5(3-\tan2t)=0$

$\tan2t=3$

$2t=1.249$

$t=0.62\text{s}$

Greatest height is at 0.62 seconds.

(b) $t=0.62$:

$$y=\frac{10}{4}\log_e\left|\cos\left(\sqrt{4}\times0.62\right)+\sqrt{\frac{0.4}{10}}\times10\sqrt{3}\times\frac{\sqrt{3}}{2}\sin\left(\sqrt{4}\times0.62\right)\right|=2.5\log_e\left|\cos1.24+3\sin1.24\right|=2.878\approx2.88\text{ m}$$

(c) $y=0$: $2.5\log_e\left|\cos2t+3\sin2t\right|=0$

$\cos2t+3\sin2t=-1$

$\sqrt{10}\left(\dfrac{3}{\sqrt{10}}\sin2t+\dfrac{1}{\sqrt{10}}\cos2t\right)=1$

$\sin(2t+\alpha)=\dfrac{1}{\sqrt{10}}$ where $\tan\alpha=\dfrac{1}{3}$

$2t+0.3218=0.3218,\quad 2.82$

$2t=2.4982$

$t=1.25\text{ s}$

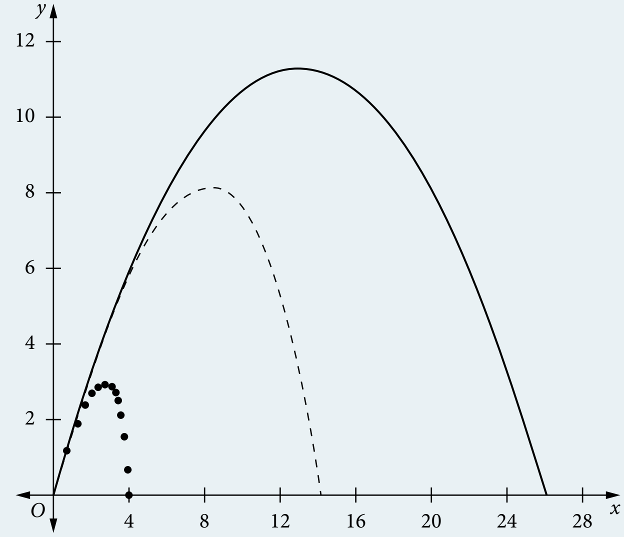
(d) The dotted line uses the parametric equations $x=2.5\log_e\left(1+2\sqrt{3}t\right)$, $y=2.5\log_e\left|\cos2t+3\sin2t\right|$

or the Cartesian equation of the path $y=2.5\log_e\left|\cos\left(\dfrac{2\left(e^{0.4x}-1\right)}{2\sqrt{3}}\right)+3\sin\left(\dfrac{2\left(e^{0.4x}-1\right)}{2\sqrt{3}}\right)\right|$.

The solid line represents no resistance, the dashed line represents resistance proportional to the velocity and the dotted line represents resistance proportional to the square of the velocity.

The solid graph uses the parametric equations $x=5\sqrt{3}t$, $y=15t-5t^2$ or $y=\sqrt{3}x-\dfrac{x^2}{15}$.

The dashed graph uses the parametric equations $x=12.5\sqrt{3}\left(1-e^{-0.4t}\right)$ or $y=100\left(1-e^{-0.4t}\right)-25t$.



EXPLORING FURTHER

Resistance proportional to the square of the velocity

Use technology to explore a model for projectile motion with resistance.

Summary of equations for projectile motion at an angle θ to the horizontal

A particle is projected from the ground with an initial velocity u at an angle θ to the horizontal where $u_x = u\cos\theta$, $u_y = u\sin\theta$. If projected from above the ground, then this initial height of projection needs to be added to the equation for y. k is the constant of proportionality for any resistance.

No resistance

$\ddot{x} = 0$ $\qquad\qquad$ $\ddot{y} = -g$

$\dot{x} = u\cos\theta$ $\qquad\qquad$ $\dot{y} = u\sin\theta - gt$

$x = u\cos\theta t$ $\qquad\qquad$ $y = u\sin\theta t - \frac{1}{2}gt^2$

Resistance proportional to the velocity

$\ddot{x} = -k\dot{x}$ $\qquad\qquad$ $\ddot{y} = -g - k\dot{y}$

$\dot{x} = u\cos\theta e^{-kt}$ $\qquad\qquad$ $\dot{y} = \frac{1}{k}\left((g + ku\sin\theta)e^{-kt} - g\right)$

$x = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$ $\qquad\qquad$ $y = \frac{\left(g + ku\sin\theta\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$

Resistance proportional to the square of the velocity—mathematical model

$\ddot{x} = -k(\dot{x})^2$ $\qquad\qquad$ $\ddot{y} = -g - k(\dot{y})^2$

$\dot{x} = \frac{u\cos\theta}{1 + k\cos\theta t}$ $\qquad\qquad$ $\dot{y} = \frac{\sqrt{gk}u\sin\theta - 10\tan\left(\sqrt{gk}t\right)}{\sqrt{gk} + ku\sin\theta\tan\left(\sqrt{gk}t\right)}$

$x = \frac{1}{k}\log_e\left(1 + ku\cos\theta t\right)$ $\qquad\qquad$ $y = \frac{1}{k}\log_e\left|\cos\left(\sqrt{gkt}\right) + \sqrt{\frac{k}{g}}u\sin\theta\sin\left(\sqrt{gkt}\right)\right|$

EXERCISE 6.7 RESISTANCE PROPORTIONAL TO THE SQUARE OF THE VELOCITY

Use the summary of equations given above and appropriate graphing technology, where necessary, to answer the following questions. Use $g = 10\text{ m s}^{-2}$.

1 A projectile is fired at an angle of 45° to the horizontal with an initial velocity of $10\sqrt{2}\text{ m s}^{-1}$.

(a) Write the equation of the trajectory if there is no air resistance.
(b) If air resistance is proportional to the velocity of the projectile, with $k = 0.01$, write the equation of the trajectory in parametric and Cartesian form.
(c) If air resistance is proportional to the square of the velocity of the projectile, with $k = 0.01$, write the equation of the trajectory in parametric form.
(d) Determine, by calculation, the greatest height of attained.
(e) Determine, by calculation, the range of the projectile.
(f) Graph the path of the projectile in each case.
(g) From your graph, determine the greatest height attained in each case.
(h) From your graph, determine the range of the projectile in each case.
(i) Discuss the significance of your answers.

2 Repeat question **1** with $k = 0.05$.

3 Repeat question **1** with $k = 0.1$. Compare your answers to questions **1**, **2** and **3**.

4 A projectile is fired at an angle of 30° to the horizontal with an initial velocity of $6\,\text{m s}^{-1}$.

(a) Write the equation of the trajectory if there is no air resistance.
(b) If air resistance is proportional to the velocity of the projectile, with $k = 0.02$, write the equation of the trajectory in both parametric form and Cartesian form.
(c) If air resistance is proportional to the square of the velocity of the projectile, with $k = 0.02$, write the equation of the trajectory in parametric form and Cartesian form.
(d) Graph the path of the projectile in each case.
(e) From your graph, determine the greatest height attained in each case.
(f) From your graph, determine the range of the projectile in each case.
(g) Discuss the significance of your answers.

5 A projectile is fired at an angle of 60° to the horizontal, from a point 2 metres above ground level, with an initial velocity of $10\,\text{m s}^{-1}$.

(a) Write the equation of the trajectory if there is no air resistance.
(b) If air resistance is proportional to the velocity of the projectile, with $k = 0.005$, write the equation of the trajectory in parametric form.
(c) If air resistance is proportional to the square of the velocity of the projectile, with $k = 0.005$, write the equation of the trajectory in parametric form and Cartesian form.
(d) Graph the path of the projectile in each case.
(e) From your graph, determine the greatest height attained in each case.
(f) From your graph, determine the range of the projectile in each case.
(g) Discuss the significance of your answers.

CHAPTER REVIEW 6

1 A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 2$. Its acceleration is given by $\ddot{x} = x + 6$. Using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, find the speed of the particle at $x = 3$.

2 A particle moves along the x-axis, starting from a position 2 metres to the right of the origin (i.e. $x = 2$ when $t = 0$), with an initial velocity of $\frac{5\sqrt{2}}{2}\,\text{m s}^{-1}$ and an acceleration $\ddot{x} = x^3 + x$.

(a) Show that $\dot{x} = \frac{x^2 + 1}{\sqrt{2}}$.
(b) Hence find an expression for x in terms of t.

3 The equation of motion for a particle moving in simple harmonic motion is given by $\frac{d^2x}{dt^2} = -n^2x$ where n is a positive constant and x is the displacement of the particle at time t.

(a) Show that the square of the velocity of the particle is $v^2 = n^2(a^2 - x^2)$, where $v = \frac{dx}{dt}$ and a is the amplitude of the motion.
(b) Find the maximum speed of the particle.
(c) Find the maximum acceleration of the particle.
(d) The particle is initially at the origin. Write a formula for x as a function of t. Hence find the first time that the particle's speed is a quarter of its maximum speed.

4 A particle moves with simple harmonic motion on the x-axis about the origin. It is initially at its extreme negative position. The amplitude of the motion is 16 and the particle returns to its initial position every 5 seconds.

(a) Write an equation for the position of the particle at time t seconds.
(b) How much time does the particle take to move from a rest position to the point halfway between the rest position and the equilibrium position?

5 A particle moves in a straight line. Its displacement x metres after t seconds is $x=\sin 2t-\sqrt{3}\cos 2t+3$.

(a) Prove that the particle is moving in simple harmonic motion about $x=3$ by showing that $\ddot{x}=-4(x-3)$.
(b) What is the period of the motion?
(c) Express the velocity of the particle in the form $\dot{x}=A\cos(2t-\alpha)$, where α is in radians.
(d) Hence, or otherwise, find all times within the first π seconds when the particle is moving at 2 metres per second in either direction.

6 A particle is moving along the x-axis and is initially at the origin. Its velocity v metres per second at time t seconds is given by $v=\dfrac{2t}{9+t^2}$.

(a) What is the initial velocity of the particle?
(b) Find an expression for the acceleration of the particle.
(c) When is the acceleration zero?
(d) What is the maximum velocity attained by the particle and when does it occur?
(e) Find the position of the particle when $t=3$.

7 A particle of mass 5 kg moves in a straight line under the action of a force whose magnitude after t seconds is $50-10t$ N. Initially the particle is at the origin O with velocity $24\,\text{m s}^{-1}$.

(a) At what time is the particle momentarily at rest?
(b) What is its position at that time?
(c) Describe the motion.

8 An object of mass 10 kg is at rest at the origin. It is acted on by a force that decreases uniformly with the distance travelled by the object, from 50 N at the start to 10 N when the distance travelled is 25 m.

(a) Write the function for this force F in terms of displacement x.
(b) Find the velocity of the object when its displacement is 25 m.

9 A particle of mass 5 kg moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v.

(a) If the resultant force (in newtons) on the particle is $10\sin t$, and $v=1$ and $x=1$ when $t=0$, then find x as a function of t.
(b) If the resultant force (in newtons) on the particle is $15+5v$, and $v=0$ when $t=0$, then find v as a function of t.

10 A parasailing waterskier is being towed horizontally at a constant speed. The tow rope from the boat makes an angle of 20° above the horizontal and there is tension of 300 N in the tow rope. The waterskier has a mass of 100 kg. A resistance force of 120 N acts against the waterskier in a horizontal direction. A parachute is attached to the skier by a cord that is inclined at an angle α above the horizontal. There is tension of T newtons in the parachute cord. (Use $g=9.8\,\text{m s}^{-2}$.)

(a) Draw a diagram to show the four forces acting on the waterskier, W.
(b) Explain why the resultant force on the waterskier is zero.
(c) Find T correct to one decimal place and find α correct to the nearest degree.

11 An object is fired vertically from the surface of the Moon with initial velocity v_0 under a gravitational acceleration such that $\ddot{x}=-\dfrac{k}{x^2}$, where x is the displacement from the centre of the Moon and k is a constant. Let the radius of the Moon be R. The gravitational acceleration at the surface of the Moon is $\dfrac{g}{6}$.

(a) Find the velocity of the object in terms of its distance x from the centre of the Moon.
(b) Find the value of v_0 for which the object travels a distance of $2R$ from launch before it starts to fall back.
(c) Find the escape velocity.

12 A particle moves from the initial conditions $t=t_0$, $x=x_0$, $v=v_0$ to the final conditions $t=t_F$, $x=x_F$, $v=v_F$ under a force that produces acceleration $\ddot{x}$. Which of the following is incorrect?

A If $\ddot{x}=f(v)$, then $x_F=\displaystyle\int_{v_0}^{v_F}\frac{v}{f(v)}dv+x_0$

B If $\ddot{x}=f(v)$, then $x_F=\displaystyle\int_{v_0}^{v_F}\frac{f(v)}{v}dv+x_0$

C If $\ddot{x}=f(v)$, then $t_F=\displaystyle\int_{v_0}^{v_F}\frac{dv}{f(v)}+t_0$

D If $\ddot{x}=f(x)$, then $v_F^2=2\displaystyle\int_{x_0}^{x_F}f(x)\,dx+v_0^2$

13 **(a)** Show that the range on a horizontal plane of a particle projected upwards at an angle α to the plane and with velocity V metres per second is $\dfrac{V^2 \sin 2\alpha}{g}$ metres, and that the maximum range is $\dfrac{V^2}{g}$.

A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second in a circular pattern. The direction of the spray varies continuously between angles of 15° and 60° to the horizontal.

(b) Prove that the sprinkler, from a fixed position on level ground, will wet the surface of an annular region with centre O and with internal and external radii $\dfrac{V^2}{2g}$ metres and $\dfrac{V^2}{g}$ metres respectively.

(c) Deduce that if the sprinkler is placed appropriately relative to a rectangular garden bed of size 6 m by 3 m, then the entire garden bed may be watered, provided that $\dfrac{V^2}{2g} \geq 1 + \sqrt{7}$.

14 An underwater camera of mass 0.5 kg is allowed to fall vertically from the ocean surface into a deep ocean trench. As it falls to the ocean floor, it is acted upon by gravity and by a resistance of $2v$ newtons, where v m s^{-1} is the velocity of the camera t seconds after beginning its descent.

(a) Show that the equation of motion of the camera is $\ddot{x} = g - 4v$.

(b) Find v as a function of t.

(c) Find the terminal velocity of the camera.

(d) Find the time taken for the camera to reach half of its terminal velocity.

(e) It takes 50 seconds for the camera to reach the ocean floor. Find the depth of the ocean at that point.

15 A particle moves so that its position vector $\underset{\sim}{r}$ at time t is given by $\underset{\sim}{r} = 3\cos 2t\,\underset{\sim}{i} + 3\sin 2t\,\underset{\sim}{j}$, $t \geq 0$.

(a) Show that the particle moves in a circle and find the Cartesian equation of its path.

(b) Show that the particle moves with constant speed.

(c) Show that the particle's acceleration has constant magnitude and is perpendicular to the direction of motion of the particle.

16 The position vector of a particle at time t seconds, $t \geq 0$, is $\underset{\sim}{r} = (1 + \sin 4t)\underset{\sim}{i} + (2 - \cos 4t)\underset{\sim}{j}$ metres.

(a) Show that the particle moves in a circle and sketch its path.

(b) Show that the particle's acceleration is always perpendicular to its velocity.

17 The position vector of a particle at time t, $t \geq 0$, is $\underset{\sim}{r} = 2\cos 3t\,\underset{\sim}{i} + 2\sin 3t\,\underset{\sim}{j} + 3t\,\underset{\sim}{k}$.

Show that the magnitudes of the particle's velocity and its acceleration are constant.

18 A particle moves so that its position vector at time t is given by $\underset{\sim}{r} = 3\cos t\,\underset{\sim}{i} + 2\sin t\,\underset{\sim}{j}$, $0 \leq t \leq 2\pi$.

(a) Find the Cartesian equation of the path of the particle and sketch the path.

(b) Find when the velocity of the particle is perpendicular to its position vector and hence find the position vectors at these times.

(c) Sketch the graph of the speed function and find the maximum and minimum speeds of the particle.

(d) Show that the particle's acceleration is directed towards the origin and is equal in magnitude to the particle's distance from the origin.

(e) Find when the acceleration is perpendicular to the velocity.

SUMMARY

1 COMPLEX NUMBERS

The complex number system

Any number z of the form $x + iy$, where x, y are real, is called a complex number.

x is the **real part** of z, $\text{Re}(z) = x$, and
y is the **imaginary part** of z, $\text{Im}(z) = y$.

If $y = 0$, then z is purely real. The set of real numbers is a subset of the set of complex numbers.
If $x = 0$, then z is purely imaginary.

Equality
Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal:

$a + bi = c + di$ if and only if $a = c$ and $b = d$

Addition and subtraction
If $z = z_1 \pm z_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z = (x_1 + x_2) \pm i(y_1 + y_2)$.

Multiplication
If $z = z_1 \times z_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then:

$$\begin{aligned} z &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i^2y_1y_2 + ix_1y_2 + ix_2y_1 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

The conjugate of a complex number
If $z = x + iy$, then the **conjugate** of z is $\bar{z} = x - iy$.

The product of a complex number and its conjugate is a real number:

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) \\ &= x^2 - i^2y^2 \\ &= x^2 + y^2 \end{aligned}$$

Division
To calculate the division $z = \dfrac{z_1}{z_2}$, multiply the numerator and denominator by the conjugate of z_2.
This **realises** the denominator, i.e. makes it real.

$$z = \frac{z_1\bar{z}_2}{z_2\bar{z}_2}$$

Square roots of a complex number

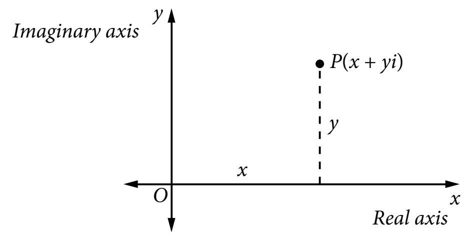
To find the square roots of $a + ib$, let $z = x + iy$ such that $z^2 = a + ib$, with a, b real. Equating the real parts and the imaginary parts:

$x^2 - y^2 = a \qquad 2xy = b$

Substitute to solve for x and y.

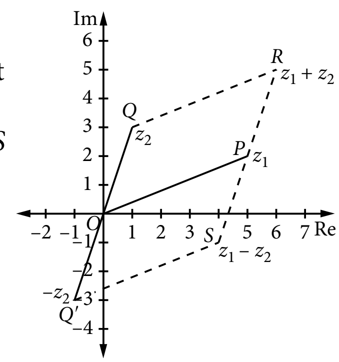
Geometrical representation of a complex number as a point

Any complex number $z = x + iy$ can be represented by the point P with coordinates (x, y) in a number plane in which the x-axis is the 'real' axis and the y-axis is the 'imaginary' axis. This Cartesian representation of complex numbers is called the **Argand diagram**, mapped on the **complex number plane.**



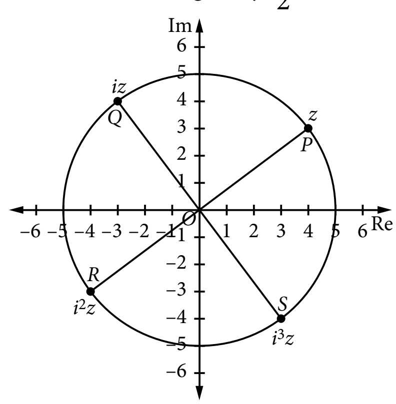
Geometrical addition and subtraction of complex numbers

On the Argand diagram, points P, Q and R represent z_1, z_2 and $z_1 + z_2$ respectively. Points Q' and S represent $-z_2$ and $z_1 - z_2$ respectively. Note that $z_1 - z_2$ is calculated as $z_1 + (-z_2)$. $OPRQ$ and $OPSQ'$ are parallelograms.



Geometrical representation of multiplication by i

The Argand diagram shows that each multiplication by i causes a complex number z to be rotated anticlockwise about the origin by $\frac{\pi}{2}$ (90°).



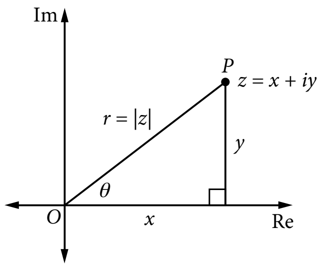
Geometrical representation of conjugates

The points that represent a pair of complex conjugates are reflections in the real axis.

Modulus–argument form of a complex number

A point P on an Argand diagram may be located by Cartesian coordinates (x, y), or by its modulus and argument:

- The **modulus** is the distance from the origin O to P: $\text{mod}\, z = |z| = |x + iy| = \sqrt{x^2 + y^2} = r$
- The **argument** is the angle at which the ray OP is inclined to the positive direction of the real axis: $\arg z = \theta$ such that $x = r\cos\theta$ and $y = r\sin\theta$
- The **principal argument** is θ such that $-\pi < \theta \le \pi$.



For any complex number:
$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

A complex number expressed as $r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$ is in **mod–arg form**.

The result $z \times \bar{z} = |z|^2$

The product of a complex number and its conjugate is the modulus squared.

Products in mod–arg form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then $z_1 \times z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$= r_1 r_2\,\text{cis}\,(\theta_1 + \theta_2)$

$\therefore |z_1 z_2| = |z_1| \times |z_2|$

Also: $\arg z_1 + \arg z_2$ is one value of $\arg(z_1 z_2)$, not necessarily the principal value.

- The modulus of a product is the product of the moduli.
- The argument of a product is the sum of the arguments.

Quotients in mod–arg form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \dfrac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

$\therefore \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

Also: $\arg z_1 - \arg z_2$ is one value of $\arg\dfrac{z_1}{z_2}$, not necessarily the principal value.

- The modulus of a quotient is the quotient of the moduli.
- The argument of a quotient is the difference of the arguments.

Two special results

1 If $z = r(\cos\theta + i\sin\theta)$, then the conjugate $\bar{z} = r\,\text{cis}\,(-\theta)$

2 If $z = r(\cos\theta + i\sin\theta)$, then $\dfrac{1}{z} = \dfrac{1}{r}\text{cis}(-\theta)$

Basic identities

- $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and $\arg\dfrac{z_1}{z_2} = \arg z_1 - \arg z_2,\ z_2 \ne 0$
- $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$
- $\left|\dfrac{1}{z^n}\right| = \dfrac{1}{|z^n|}$ and $\arg\dfrac{1}{z^n} = -n\arg z,\ z \ne 0$
- $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$
- $\bar{z}_1\bar{z}_2 = \overline{z_1 z_2}$
- $z\bar{z} = |z|^2$
- $z + \bar{z} = 2\,\text{Re}(z)$
- $z - \bar{z} = 2i\,\text{Im}(z)$

Powers using mod–arg form

De Moivre's theorem:

If $z = r(\cos\theta + i\sin\theta)$ and n is an integer,
then $z^n = r^n(\cos n\theta + i\sin n\theta)$

De Moivre's theorem for negative integers

If n is a negative integer, de Moivre's theorem states that $z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta))$.

Euler's formula

In general, when $|z| \uparrow 1$, $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Powers of complex numbers

Since any complex number can be written in the exponential form, $e^{i\theta}$, it is easy to find powers of this number as $\left(e^{i\theta}\right)^n = e^{in\theta}$.

Geometrical representation of products involving complex numbers—consolidation and summary

Multiplication of a complex number z by a real number k:

- $\arg(kz) = \arg k + \arg z$
  If k is a positive real number, then $\arg k = 0$, so $\arg(kz) = \arg z$.
  If k is a negative real number, then $\arg k = \pi$, so $\arg(kz) = \pi + \arg z = \pi + \arg z - 2\pi = -(\pi - \arg z)$.
  (Note that 2π is subtracted to find the principal argument.)
- $|kz| = |k| \times |z|$, i.e. there is a scaling by a factor of $|k|$.
  If k is a negative real number, then the direction from the origin O to the point representing kz is opposite to the direction from O to the point representing z.

Multiplication of a complex number z by i:

- $\arg(iz) = \arg i + \arg z = \frac{\pi}{2} + \arg z$
- $|iz| = |i| \times |z| = |z|$ as $|i| = 1$
- Hence multiplication by i causes an anticlockwise rotation by $\frac{\pi}{2}$ about the origin O, with no change to the modulus.

Multiplication of a complex number z by ki, where k is a real number:

- This combines the two cases above.
- Rotate by $\frac{\pi}{2}$ anticlockwise about O and then scale by a factor of $|k|$, remembering also to reverse the direction if k is negative.

Multiplication of a complex number z by another complex number $r(\cos\theta + i\sin\theta)$:

- $\arg(z \times r\operatorname{cis}\theta) = \arg z + \arg(r\operatorname{cis}\theta) = \arg z + \theta$
- $|z \times r\operatorname{cis}\theta| = |z| \times |r\operatorname{cis}\theta| = |z| \times r$
- To multiply by $r\operatorname{cis}\theta$, rotate by θ anticlockwise about O and then scale by a factor of r.

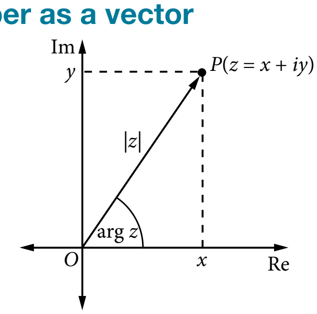
Properties of complex conjugates

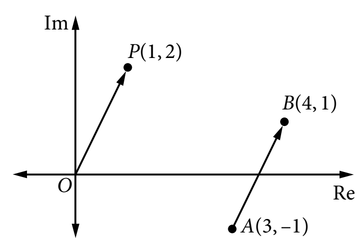
- $z + \overline{z} = 2\operatorname{Re}(z)$ and $z - \overline{z} = 2\operatorname{Im}(z) \times i$
- the conjugate of a sum is equal to the sum of the conjugates
- the conjugate of a difference is equal to the difference of the conjugates
- the conjugate of a product is equal to the product of the conjugates
- the conjugate of a quotient is equal to the quotient of the conjugates
- the conjugate of a power is equal to the power of the conjugate
- the conjugate of a real number is itself, i.e. if $z = x + 0i$, then $\overline{z} = x - 0i = z$.

Geometrical representation of a complex number as a vector

Complex numbers can be represented as vectors for which the modulus gives the magnitude and the argument gives the direction.

On an Argand diagram, let point P represent the complex number $z = x + iy$. This z is also represented by the **position vector** $\overrightarrow{OP}$, where $\overrightarrow{OP}$ has length $|z|$ and direction given by a rotation from the positive direction of the real axis by $\arg z$.





The vector to represent a complex number does not have to 'start' at the origin O; a complex number can be represented by any vector with the same modulus (length) and argument (angle), called a **free vector** (if it is not a position vector from the origin).

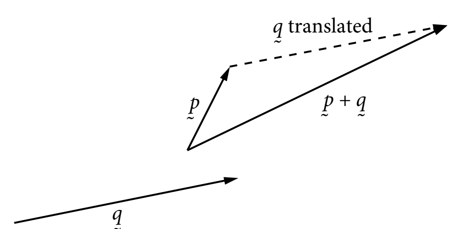
Operations on complex numbers represented as vectors

Addition (to form the vector sum $\underset{\sim}{p} + \underset{\sim}{q}$):

Method 1 (complete the triangle)

1 Translate $\underset{\sim}{q}$ so that its tail is located at the head of $\underset{\sim}{p}$.

2 The vector for the sum $\underset{\sim}{p} + \underset{\sim}{q}$ now goes from the tail of $\underset{\sim}{p}$ to the head of the translated $\underset{\sim}{q}$.

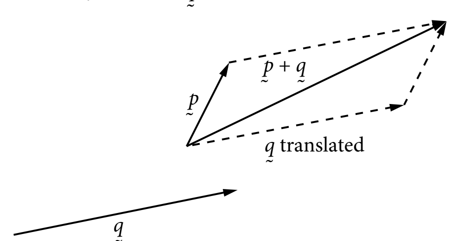


Method 2 (construct the diagonal of the parallelogram)

1 Translate $\underset{\sim}{q}$ so that its tail is located at the tail of $\underset{\sim}{p}$.

2 Locate the fourth vertex of the parallelogram of vectors.

3 The vector for the sum $\underset{\sim}{p} + \underset{\sim}{q}$ now goes from the common tail of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ to the fourth vertex of the parallelogram.

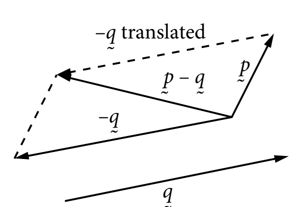


Subtraction (to form the vector difference $\underset{\sim}{p} - \underset{\sim}{q}$):

Method 1 (add the opposite vector)

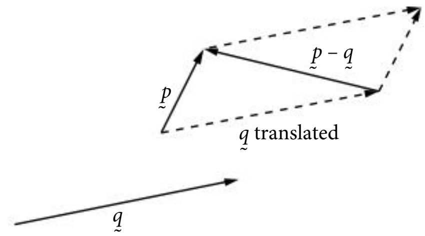
1 Form the vector $-\underset{\sim}{q}$.

2 Form the vector sum $\underset{\sim}{p} + (-\underset{\sim}{q})$.



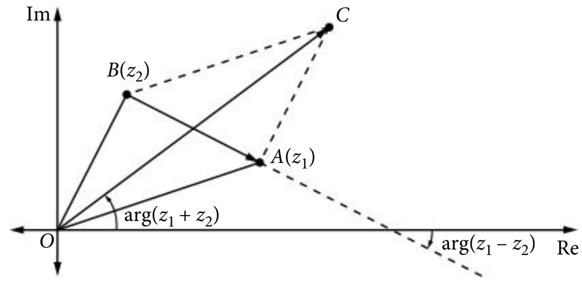
Method 2 (construct the other diagonal of the parallelogram)

1 Translate $\underset{\sim}{q}$ so that its tail is located at the tail of $\underset{\sim}{p}$.
2 Locate the fourth vertex of the parallelogram of vectors.
3 The vector for the sum $\underset{\sim}{p} - \underset{\sim}{q}$ is the diagonal of the parallelogram that goes from the tip of $\underset{\sim}{q}$ to the tip of $\underset{\sim}{p}$.



Let A represent z_1 and B represent z_2.
When the parallelogram $OACB$ is formed:

- vector $\overrightarrow{OC}$ represents $z_1 + z_2$
- vector $\overrightarrow{AB}$ represents $z_2 - z_1$ and vector $\overrightarrow{BA}$ represents $z_1 - z_2$
- $|z_1 + z_2|$ and $|z_1 - z_2|$ are the lengths of the diagonals of the parallelogram
- $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$ are the angles at which the diagonals are inclined to the positive direction of the real axis.



Multiplication by i:

When working with vector representations of complex numbers, multiplication by i rotates the vector anticlockwise by $\frac{\pi}{2}$ (90°).

Multiplication by ki (where k is real):

To multiply by ki, rotate the vector anticlockwise by $\frac{\pi}{2}$ (90°) and scale by a factor of k.

Multiplication of a complex number z by another complex number $r(\cos\theta + i\sin\theta)$:

To multiply z by $r\operatorname{cis}\theta$, rotate the vector that represents z through an angle θ and scale by a factor of r.

Hint: Whenever you see $z_1 + z_2$ and $z_1 - z_2$ in a problem, the problem can most likely be solved using the geometrical properties of parallelograms.

The triangle inequalities

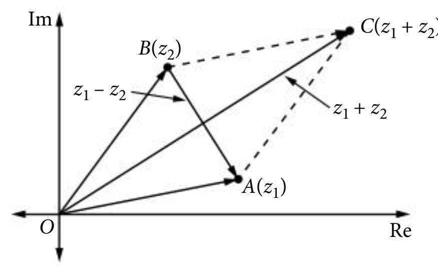
In a triangle OAC: $OC \le OA + AC$, with $OC = OA + AC$ when O, A and C are collinear.

$\therefore |z_1 + z_2| \le |z_1| + |z_2|$, with $|z_1 + z_2| = |z_1| + |z_2|$ when $z_2 = kz_1$ (where k is real).

In ΔOAB: $OA \le OB + BA$, with equality when O, B and A are collinear.

$\therefore |z_1| \le |z_2| + |z_1 - z_2|$

$\therefore |z_1 - z_2| \ge |z_1| - |z_2|$, with equality when $z_2 = kz_1$ (where k is real).



Roots of complex numbers

De Moivre's theorem can be used in reverse to find roots of complex numbers.

- The n-th roots of 1 (i.e. the n roots of $z^n = 1$) are equally spaced around the circumference of the circle with centre O and radius 1, separated by an angle of $\frac{2\pi}{n}$ at the centre.
- One root is 1. If n is even, another root is -1. The other roots occur as non-real conjugate pairs.
- The roots are $1\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$ where $k = 0, \pm1, \pm2, \ldots$ until the n unique roots are identified.
- The n-th roots of **any complex number** $R\operatorname{cis}\alpha$ are equally spaced around the circumference of the circle with centre O and radius $\sqrt[n]{R}$, separated by an angle of $\frac{2\pi}{n}$ at the centre.
- The roots are $\sqrt[n]{R}\left(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}\right)$ where $k = 0, \pm1, \pm2, \ldots$ until the n unique roots are identified.

Complex roots of unity

To investigate properties of the n-th roots of 1, you can work with symbolic representations rather than the actual values of the roots.

An important result is the factorisation (where w is a root):

$w^n - 1 = (w-1)(w^{n-1} + w^{n-2} + \ldots + w^2 + w + 1)$

Curves and regions on the Argand diagram

If $z = x + iy$, then:

- $|z| = \sqrt{x^2 + y^2}$ is the algebraic definition; geometrically $|z|$ is the distance from the origin O to the point (x, y)
- algebraically $|z - (a+ib)| = \sqrt{(x-a)^2 + (y-b)^2}$; geometrically $|z - (a+ib)|$ is the magnitude of the vector from (a, b) to the point representing z
- geometrically $\arg z$ is the angle made with the positive direction of the real axis by the vector from the origin O to the point representing z
- geometrically $\arg(z - z_1)$ is the angle made with the positive direction of the real axis by the vector from the point representing z_1 to the point representing z
- $\arg 0$ is undefined.

2 THE NATURE OF PROOF

- The negation of '***P*** and ***Q***' is '***not P*** or ***not Q***'.
- The negation of '***P*** or ***Q***' is '***not P*** and ***not Q***'.

The symbol ∀ (known as the *universal quantifier*) is used to mean 'for all'.

The symbol ∃ (known as the *existential quantifier*) is used to mean 'there exists'.

When the symbols ∀ and ∃ appear together in the same statement, the order in which they appear is important.

- Providing a single example is always sufficient to prove that a 'there exists' statement is true.
- Providing a single counter example is always sufficient to prove that a 'for all' statement is false.

The negation of a 'for all' statement is a 'there exists' statement. Similarly, the negation of a 'there exists' statement is a 'for all' statement.

Two statements are logically equivalent if whenever one is true, the other must be.

Each of the following means the same as $P \Rightarrow Q$:

- If P, then Q
- Q if P
- P is a *sufficient* condition to conclude that Q
- Q is *necessary* if P
- P *implies that* Q.

For a statement of the form $P \Rightarrow Q$ that involves some variable:

- The converse is the statement $Q \Rightarrow P$
- The contrapositive is the statement *not* $Q \Rightarrow$ *not* P
- The negation is the statement 'there exists some value of the variable for which P is true, but Q is false'.

The contrapositive of a conditional statement essentially says the same thing as the original statement, and thus, will be true whenever the original statement is true.

Two statements are logically equivalent if whenever one is true, the other must be.

Each of the following can be used to express the fact that P and Q are logically equivalent:

- P is *necessary* and *sufficient* for Q
- P *if and only if* Q
- $P \Rightarrow Q$ and $Q \Rightarrow P$
- $P \Leftrightarrow Q$.

3 FURTHER WORK WITH VECTORS

Algebra of vectors expressed in component form

(i) Equality: Two position vectors are equal if and only if the corresponding components are equal.

i.e. $a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k} = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}$ if and only if

$a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

(This is true because the $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ representation of a vector is unique.)

(ii) The components of the sum (difference) of two position vectors are equal to the sum (difference) of the corresponding components of the vectors; e.g. If

$\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, then

$\underset{\sim}{a} \pm \underset{\sim}{b} = (x_1 \pm x_2)\underset{\sim}{i} + (y_1 \pm y_2)\underset{\sim}{j} + (z_1 \pm z_2)\underset{\sim}{k}$.

(iii) The components of a scalar multiple of a position vector are equal to the scalar multiple of the corresponding components of the vector.

e.g. If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ and $\lambda \in R$, then

$\lambda\underset{\sim}{a} = (\lambda x)\underset{\sim}{i} + (\lambda y)\underset{\sim}{j} + (\lambda z)\underset{\sim}{k}$.

Summary of formulae

Distance formula

Given $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then

$\overrightarrow{AB} = (x_2 - x_1)\underset{\sim}{i} + (y_2 - y_1)\underset{\sim}{j} + (z_1 - z_2)\underset{\sim}{k}$ and

$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Unit vectors

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$$

Scalar product

$$\underset{\sim}{a} \bullet \underset{\sim}{b} = (x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}) \bullet (x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k})$$
$$= x_1x_2 + y_1y_2 + z_1z_2$$

Algebraic properties of the dot product

- $\underset{\sim}{a} \bullet \underset{\sim}{b} = \underset{\sim}{b} \bullet \underset{\sim}{a}$
- $\underset{\sim}{a} \bullet \underset{\sim}{a} = |\underset{\sim}{a}|^2$
- $\underset{\sim}{a} \bullet (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c}$
- $(\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{c} + \underset{\sim}{d}) = \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{d} + \underset{\sim}{b} \bullet \underset{\sim}{c} + \underset{\sim}{b} \bullet \underset{\sim}{d}$
- The angle between two vectors, θ, is given by $\cos\theta = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$, $\underset{\sim}{a} \uparrow 0$, $\underset{\sim}{b} \uparrow 0$.

Geometric properties of the dot product

- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, $\underset{\sim}{a} \perp \underset{\sim}{b}$
- If $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors
- If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$
- By convention, $0° \leq \theta \leq 180°$.

Projections

The scalar projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $\underset{\sim}{a} \infty \hat{\underset{\sim}{b}}$, where $\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{b}|}$.

The vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ is $(\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.

The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$ or $\underset{\sim}{a} - \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{\underset{\sim}{b} \bullet \underset{\sim}{b}}\underset{\sim}{b}$.

Equation of a sphere

$$(x - h)^2 + (y - k)^2 + (z - k)^2 = r^2$$

Equation of a line

Vector equation of a line: $\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda\underset{\sim}{v}$, this equation consists of the position vector, $\underset{\sim}{r}_0$, of a fixed point P_0 on L and a fixed vector, $\underset{\sim}{r}$, that determines the direction of L.

Parametric equations of a line: $x = x_0 + a\lambda$, $y = y_0 + b\lambda$, $z = z_0 + c\lambda$, represent the parametric equations of the line L through the point (x_0, y_0, z_0) parallel to the vector $\underset{\sim}{v} = (a, b, c)$.

Properties of lines in three dimensions

Given line L_1 with equation $x = a + bt$, $y = c + dt$, $z = e + ft$ and line L_2 with equation $x = g + hs$, $y = j + ks$, $z = l + ns$, then:

- $L_1 \parallel L_2$ if $\dfrac{b}{h} = \dfrac{d}{k} = \dfrac{f}{n}$
- L_1 and L_2 intersect if a unique value of t and s satisfies the three equations $a + bt = g + hs$, $c + dt = j + ks$, $e + ft = l + ns$ simultaneously
- $L_1 \perp L_2$ if $bh + dk + fn = 0$ ($\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$)
- If they are not parallel or coincident and do not intersect, then they are skew.

Note: In three dimensions coincident lines will be parallel and have a point in common. This means that for L_1 and L_2 above to be coincident, $\dfrac{b}{h} = \dfrac{d}{k} = \dfrac{f}{n} = \dfrac{a}{g} = \dfrac{c}{j} = \dfrac{e}{l}$.

Vector equation of a line

Consider two straight lines, L_1 and L_2, with vector equations $\underset{\sim}{r}_1 = \underset{\sim}{a}_1 + \lambda\underset{\sim}{b}_1$ and $\underset{\sim}{r}_2 = \underset{\sim}{a}_2 + \lambda\underset{\sim}{b}_2$ respectively and with direction vectors $\underset{\sim}{b}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{b}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$ respectively.

- If the two lines are parallel, then $\underset{\sim}{b}_1 = k\underset{\sim}{b}_2$, $k \neq 0$, and so $x_1 = kx_2$, $y_1 = ky_2$ and $z_1 = kz_2$.

Then, eliminating the constant k, you have that if L_1 and L_2 are parallel then $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2} = \dfrac{z_1}{z_2}$.

- If the two lines are perpendicular, then $\underset{\sim}{b}_1 \bullet \underset{\sim}{b}_2 = 0$ and so $x_1x_2 + y_1y_2 + z_1z_2 = 0$.
- Providing a single example is always sufficient to prove that a 'there exists' statement is true.

4 INTEGRATION BY SUBSTITUTION

Trigonometric integrals

Using double-angle formulae:

$$\int \sin x \cos x\, dx = \frac{1}{2}\int \sin 2x\, dx$$
$$\int \cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx$$
$$\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$$

- For $\int \sin^m x\, dx$ and $\int \cos^m x\, dx$, m an even positive integer:

  Use $\cos^{2n} x = \left(\dfrac{1 + \cos 2x}{2}\right)^n$ and $\sin^{2n} x = \left(\dfrac{1 - \cos 2x}{2}\right)^n$.
- For $\int \sin^m x\, dx$ and $\int \cos^m x\, dx$, m an odd positive integer:

  Rewrite as $\int \sin^{2n} x \sin x\, dx$ or $\int \cos^{2n} x \cos x\, dx$ where $m = 2n + 1$, then substitute $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$ and work towards creating an integral of the type $\int f(u)\, du$, $u = g(x)$ that can be found.

Inverse trigonometric functions

- $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$ for $-a < x < a$
- $\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\frac{x}{a} + C$ for $-a < x < a$
- $\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\frac{x}{a} + C$ for all x

You may need to complete the square to put the integrand into the required form.

Integration by substitution

This is also called 'integration by change of variable':

$$\int f(x)\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

An expression like $\sqrt{a^2 - x^2}$ can be integrated with a substitution of $x = a\sin\theta$ or $x = a\cos\theta$, while an expression like $a^2 + x^2$ can be integrated with a substitution of $x = a\tan\theta$.

The domain of the substitution should be chosen to coincide with the corresponding inverse function:

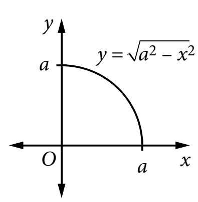
- for the substitution $x = a\sin\theta$, set $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- for the substitution $x = a\cos\theta$, set $0 < \theta < \pi$.

The endpoints of the domain are not included as $\frac{1}{\sqrt{a^2 - x^2}}$ is undefined there.

For the substitution $x = \tan\theta$, no restriction on θ is required: $1 + x^2 > 0$ for all values of x, so the integrand's denominator will never be zero for real x.

Special case: evaluating $\int_0^a \sqrt{a^2 - x^2}\,dx,\ a > 0$

The graph of $y = \sqrt{a^2 - x^2}$, $a > 0$ for the domain $0 \le x \le a^2$ is the first quadrant of the circle $x^2 + y^2 = a^2$.



The formula for the area of a circle can be used to evaluate the integral:

$$\text{Area} = \frac{1}{4} \times \pi \times a^2 = \frac{\pi a^2}{4} \quad (\text{as } r = a)$$

Sum or difference of two squares

All quadratic expressions of the form $ax^2 + bx + c$ can be expressed as the sum or the difference of two squares, depending on whether their discriminant $\Delta = b^2 - 4ac$ is positive or negative. This can be achieved by completing the square of the quadratic.

After this is done, the associated integral can usually be found by using an appropriate substitution.

Logarithmic functions

- $\frac{d}{dx}\log_e\left(f(x)\right) = \frac{f'(x)}{f(x)}$
- $\int \frac{dx}{x} = \log_e|x| + C$
- $\int \frac{h'(x)}{h(x)}\,dx = \log_e|h(x)| + C$

The substitution $t = \tan\frac{\theta}{2}$

The substitution $t = \tan\frac{\theta}{2}$ enables us to express $\sin\theta$ and $\cos\theta$ in terms of t, which then enables us to express any rational function of $\sin\theta$ and $\cos\theta$ as a rational algebraic function of t. This then allows standard techniques of integration such as partial fractions or integration by parts.

$$t = \tan\frac{\theta}{2} \qquad \sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2} \qquad \frac{d\theta}{dt} = \frac{2}{1+t^2}$$

If using a substitution of $t = \tan f(\theta)$, $f(\theta) \ne \frac{\theta}{2}$, differentiate the substitution to find the link between $d\theta$ and dt.

5 FURTHER INTEGRATION

Rational functions

A rational function $f(x)$ is the ratio of two polynomials, $f(x) = \frac{A(x)}{B(x)}$, defined for all values of x except those for which $B(x) = 0$.

If the polynomial degree of $A(x) \ge$ degree of $B(x)$, then: $\quad A(x) = B(x) \times Q(x) + R(x)$

where degree of $R(x) <$ degree of $B(x)$

which leads to: $\quad f(x) = Q(x) + \frac{R(x)}{B(x)}$

The problem of partial fraction decomposition arises when $B(x)$ is a product of polynomials of lower degree, i.e. $B(x) = B_1(x) \times B_2(x)$ with degree $B_1(x) > 0$, degree $B_2(x) > 0$.

You wish to find polynomials $m_1(x)$, $m_2(x)$ such that:

$$\frac{R(x)}{B(x)} = \frac{m_1(x)}{B_1(x)} + \frac{m_2(x)}{B_2(x)}$$

Now $m_1(x)$ and $m_2(x)$ can be found if: $R(x) \equiv m_1(x) \times B_2(x) + m_2(x) \times B_1(x)$.

Comparison of degrees shows that you can suppose degree $m_1(x) <$ degree $B_1(x)$, degree $m_2(x) <$ degree $B_2(x)$.

Linear factors

Consider the general case, where $B(x)$ is a product of distinct linear factors: $B(x) = k(x - a_1)(x - a_2)\ldots(x - a_n)$

You want to discover if constants $c_1, c_2, \ldots c_n$ exist so that: $\dfrac{R(x)}{B(x)} = \dfrac{c_1}{x - a_1} + \dfrac{c_2}{x - a_2} + \ldots + \dfrac{c_n}{x - a_n}$

Linear factors—Method 1 (equating coefficients)

Consider the monic case when $n = 2$, so that $k = 1$ and $B(x)$ is a quadratic.

$R(x) = dx + e$ and $B(x) = (x - a_1)(x - a_2)$: $\quad \dfrac{R(x)}{B(x)} = \dfrac{dx + e}{(x - a_1)(x - a_2)} = \dfrac{c_1}{x - a_1} + \dfrac{c_2}{x - a_2}$

Write with common denominator: $\quad \dfrac{dx + e}{(x - a_1)(x - a_2)} = \dfrac{c_1(x - a_2) + c_2(x - a_1)}{(x - a_1)(x - a_2)}$

Write the numerators: $\quad dx + e \equiv c_1(x - a_2) + c_2(x - a_1)$

Using identity property of polynomials, equate coefficients: $\quad d = c_1 + c_2$

$e = -(a_2c_1 + a_1c_2)$

As $a_1 \neq a_2$, these two equations can be solved for c_1 and c_2.

In the general case, $a_1, a_2, \ldots a_n$ are distinct, so the coefficient equations can be solved for $c_1, c_2, \ldots c_n$.

Linear factors—Method 2 (substitution)

Again consider the monic case when $n = 2$, so that $k = 1$ and $B(x)$ is a quadratic.

$R(x) = dx + e$ and $B(x) = (x - a_1)(x - a_2)$: $\quad \dfrac{R(x)}{B(x)} = \dfrac{dx + e}{(x - a_1)(x - a_2)} = \dfrac{c_1}{x - a_1} + \dfrac{c_2}{x - a_2}$

Write with common denominator: $\quad \dfrac{dx + e}{(x - a_1)(x - a_2)} = \dfrac{c_1(x - a_2) + c_2(x - a_1)}{(x - a_1)(x - a_2)}$

Write the numerators: $\quad dx + e \equiv c_1(x - a_2) + c_2(x - a_1)$

Let $x = a_1$: $\quad da_1 + e = c_1(a_1 - a_2)$

Solve for c_1: $\quad c_1 = \dfrac{da_1 + e}{a_1 - a_2}$

Let $x = a_2$: $\quad da_2 + e = c_2(a_2 - a_1)$

Solve for c_2: $\quad c_2 = \dfrac{da_2 + e}{a_2 - a_1}$

You should always check your answer by writing the partial fractions over a common denominator again.

Partial fractions, quadratic factors

$B(x)$ is a product of distinct linear factors and a simple quadratic factor such that

$\dfrac{R(x)}{B(x)} = \dfrac{c_1}{x - a_1} + \dfrac{c_2}{x - a_2} + \ldots + \dfrac{c_n}{x - a_n} + \dfrac{dx + e}{x^2 + bx + c}$, with degree $R(x) < n + 2$.

As for partial fractions with linear factors, here $c_1, c_2, \ldots c_n$ can be found using any of the previous methods for partial fractions with linear factors. You then only need to find d and e as follows.

- If none of $a_1, a_2, \ldots a_n$ is zero, then let $x = 0$ to find the value of e.
- Multiply through by x and let $x \to \infty$ (which will require dividing each fraction by the largest power of x in the numerators) to find the value of d.
- If e.g. $a_1 = 0$, first find d and then select a small integer value for x distinct from $a_1, a_2, \ldots a_n$ to give a simple equation that can be solved for e.

Two quadratic denominators

When finding partial fractions for pairs of quadratic denominators like $(x^2 + a^2)$ and $(x^2 + b^2)$ and the numerator does not have odd powers of x, you can write the first step as:

$\dfrac{cx^2 + d}{(x^2 + a^2)(x^2 + b^2)} = \dfrac{e}{x^2 + a^2} + \dfrac{f}{x^2 + b^2}$ where either c or d could be zero.

Repeated linear factors

This course does not require you to find partial fractions with repeated linear factors. However, you may be asked to show a repeated factors result.

Partial fractions

- $\int \frac{dx}{(x-a)(x-b)} = \int\left(\frac{c}{x-a}+\frac{d}{x-b}\right)dx$
- $\int \frac{cx+d}{(x+a)(x^2+b^2)}\,dx = \int\left(\frac{e}{x+a}+\frac{fx+g}{x^2+b^2}\right)dx$
- $\int \frac{c}{(x^2+a^2)(x^2+b^2)}\,dx = \int\left(\frac{d}{x^2+a^2}+\frac{e}{x^2+b^2}\right)dx$
- $\int \frac{cx^2+d}{(x^2+a^2)(x^2+b^2)}\,dx = \int\left(\frac{e}{x^2+a^2}+\frac{f}{x^2+b^2}\right)dx$

Integration by parts

This comes from rearranging the product rule.

If $u(x)$ and $v(x)$ are differentiable functions of x, then, by the product rule:

$$\frac{d}{dx}(uv) = v\frac{du}{dx}+u\frac{dv}{dx} \qquad \text{or} \qquad \frac{d}{dx}(uv) = vu'+uv'$$

Hence: $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \qquad \int uv'\,dx = uv - \int vu'\,dx$

$\therefore \quad \int u\,dv = uv - \int v\,du$

The LHS integrand is a product of two expressions involving x: one denoted by u and the other by $\frac{dv}{dx}$ (or just dv). The choice of which parts to label as u and dv is made so that the RHS integral can be easily found. The arbitrary constant C is inserted into the solution at the appropriate point.

As a general rule, dv should be a function that is easy to integrate and u should be the other function. If they are both easy to integrate, then u should be the function that will be simpler after differentiation.

Note that for complex integrals, the rule sometimes has to be applied more than once. You may also need to rearrange terms to solve for the desired integral.

Recurrence relations

These are found by integration by parts. Given $I_n = \int_a^b [f(x)]^n\,dx$ a relationship of the form $I_n = kI_{n-1}$ is found, where $I_{n-1} = \int_a^b [f(x)]^{n-1}\,dx$. After I_1 or another of the other easy I integrals can be found, the recurrence relation is then used to evaluate the original integral.

Properties of the definite integral

1. $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b \big(f(x)\pm g(x)\big)\,dx$
2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$
   (i.e. the value of the integral is independent of the variable of integration)
5. If $f(-x) = f(x)$ (i.e. $f(x)$ is an **even** function), then:
   $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
6. If $f(-x) = -f(x)$ (i.e. $f(x)$ is an **odd** function), then:
   $\int_{-a}^a f(x)\,dx = 0$
7. $\int_{-a}^a \sqrt{a^2-x^2}\,dx$ represents the area of a semicircle of radius a, so it can be quickly evaluated using $A = \pi r^2$.

Combinations of even and odd functions

- The product of two **odd** functions or two **even** functions is an **even** function.
- The product of an **odd** function and an **even** function is an **odd** function.

Integrals and areas

Be aware of the difference between *evaluating the integral* $\int_{-a}^a f(x)\,dx$ and *finding the area* under the curve $y = f(x)$ between the ordinates $x = -a$ and $x = a$.

Velocity and acceleration as functions of x

The expressions $x(t)$, $v(t)=\dot{x}(t)$ and $a(t)=\ddot{x}(t)=\dot{v}(t)$, which denote displacement, velocity and acceleration respectively, define functions of time t. For each t there corresponds only one value of $x(t)$ that satisfies the function, because the particle can only be in one place at one time. Similarly $(t, v(t))$ and $(t, a(t))$ define functions because a particle cannot have more than one velocity and one acceleration at any given time.

However, sometimes it is useful to express acceleration in terms of the displacement x or the velocity v. In this case you must be careful, because equations relating a, v and x will not necessarily define functions.

Remember that calculus only makes sense when applied to functions, so $\frac{dv}{dx}$ can only have meaning if $v=v(x)$ defines a function. Hence if you want to differentiate the equation $v^2=1-x^2$ you must restrict v to be either positive or negative, so you consider v to apply only to velocities as the particle moves from A to B, i.e. $v>0$, or only from B to A, i.e. $v<0$. If $v(x)$ specifies a function according to one of these restrictions, then the chain rule of differentiation can be used:

$$\begin{aligned}\frac{dv}{dt}&=\frac{dv}{dx}\times\frac{dx}{dt}\\&=v\frac{dv}{dx}\quad\text{as } v=\frac{dx}{dt}\\&=\frac{d}{dv}\left(\frac{1}{2}v^2\right)\times\frac{dv}{dx}\\&=\frac{d}{dx}\left(\frac{1}{2}v^2\right)\end{aligned}$$

Hence acceleration may be expressed in any of the forms $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

The form to use in a particular problem will depend on the form of the equation that defines acceleration:

- Given $a=f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a=g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- Given $a=h(v)$, use $\frac{dv}{dt}$ if initial conditions are values for t and v; $v\frac{dv}{dx}$ if initial conditions are values for x and v

It is customary to write derivatives with respect to time using dots above the dependent variable, e.g. $\dot{x}=\frac{dx}{dt}$, $\ddot{x}=\frac{d^2x}{dt^2}$, $\dot{v}=\frac{dv}{dt}$.

Simple harmonic motion (SHM)

SHM is the motion of a point particle moving along a straight line so that its displacement x from a fixed point at time t is given by a sine or cosine function:

- displacement x: $x=a\cos(nt+\alpha)$, $\alpha>0$, $n>0$
- velocity $\dot{x}$: $\dot{x}=-an\sin(nt+\alpha)$
- acceleration $\ddot{x}$: $\ddot{x}=-an^2\cos(nt+\alpha)$ or $\ddot{x}=-n^2x$

Important results:

1 When $\dot{x}=0$ (i.e. at the endpoints of the motion), the acceleration is greatest.

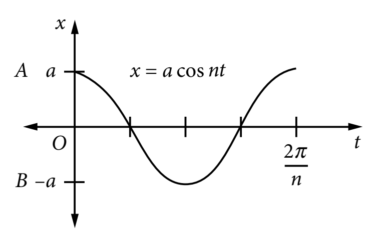
2 When $\ddot{x}=0$ (i.e. at O, the centre of the motion), the speed is greatest (i.e. velocity has its greatest or least value).

The general equation $x=a\cos(nt+\alpha)$, $\alpha>0$, $n>0$

If when $t=0$, $x=a$, the particle is initially at the extreme point A and $a=a\cos\alpha$, then $\cos\alpha=1$ and $\alpha=0$. The equation of motion can then be written $x=a\cos nt$.

$$\begin{aligned}x&=a\cos(nt+\alpha)\\&=a\cos nt\qquad\text{if } x(0)=a\\v&=\dot{x}=-na\sin nt\\\frac{dv}{dt}&=\ddot{x}=-n^2a\cos nt\\\ddot{x}&=-n^2x\end{aligned}$$

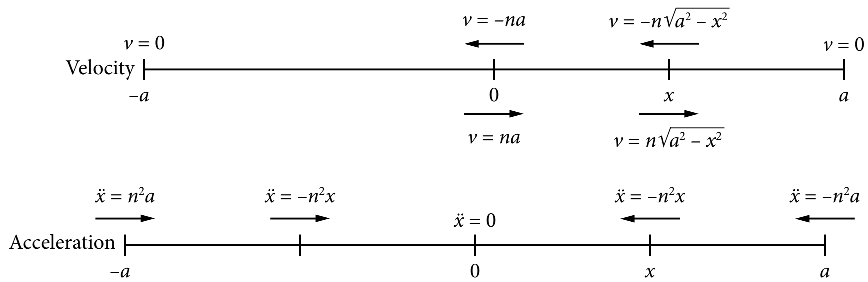
$v^2=n^2(a^2-x^2)$, $-a\le x\le a$ $\qquad T=\frac{2\pi}{n}=\frac{1}{f}$



The amplitude a is the distance from the centre of motion O to either of the extreme points A or B.

SHM problems are usually solved using either $x = a\cos(nt + \alpha)$ or $\ddot{x} = -n^2x$, or occasionally $v^2 = n^2(a^2 - x^2)$. The starting point will depend on the information given.

The following diagrams illustrate the velocity and acceleration of a particle undergoing SHM, including the extreme values.



Note that to the left of O, x is negative, so $-n^2x$ (the acceleration) is positive.

The equation $\ddot{x} = -n^2x$ describes the motion of a particle under the influence of a force that is directed towards the origin O and is proportional to the distance of the particle from O. The force (and hence the acceleration) is zero at O, where the speed is greatest. The force (and hence the acceleration) is greatest at the extreme points, where the speed is zero.

This type of motion occurs in real physical situations (either approximately or exactly) where a particle oscillates about an equilibrium position. For example:

- the to-and-fro motion of a pendulum bob (approximately SHM)
- the up-and-down motion of a mass attached to a spring
- the bobbing motion of a buoy floating on water.

The equations of motion for Simple Harmonic Motion about the point $x = c$ may be written as:

Displacement:	$x = a\sin(nt + \alpha) + c$	$x = a\cos(nt + \alpha) + c$
Velocity:	$\dot{x} = an\cos(nt + \alpha)$	$\dot{x} = -an\sin(nt + \alpha)$
Acceleration:	$\ddot{x} = -an^2\sin(nt + \alpha)$	$\ddot{x} = -an^2\cos(nt + \alpha)$
Or:	$\ddot{x} = -n^2(x - c)$	$\ddot{x} = -n^2(x - c)$

Newton's first law of motion

- A body remains at rest or in uniform motion in a straight line unless it is acted on by a non-zero resultant force.

Newton's second law of motion

- The rate of change of momentum is proportional to the applied force and occurs in the direction of the force.

If a body of mass m is acted on by a non-zero resultant force F, then:

$F \propto$ rate of change of momentum

$$\propto \frac{d}{dt}(mv)$$

$$\propto m\frac{dv}{dt} \qquad \text{if } m \text{ is constant}$$

$$\therefore\ F \propto ma \quad \text{where } a = \frac{dv}{dt}$$

i.e. $F = kma$ where k is a constant.

$k = 1$ when we define 1 standard unit of force, called a newton (N), as the amount that produces an acceleration of $1\,\text{m}\,\text{s}^{-2}$ in a body of mass 1 kg, hence: $F = ma$.

Alternative definition for Newton's second law of motion

The acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body.

Newton's third law of motion

- When two objects exert force on each other, the forces are equal in magnitude but opposite in direction. In other words:

  For every action there is an equal but opposite reaction.

Dynamics of a particle

We will consider all bodies as particles, so that all external forces acting on a body are regarded as acting through a single point in the body and producing only a translational effect (i.e. no rotation).

Recall that acceleration may be expressed in any of the forms: $\dfrac{dv}{dt}$, $\dfrac{d^2x}{dt^2}$, $v\dfrac{dv}{dx}$, $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$.

The form to use in a particular problem will depend on the form of the equation that defines acceleration or force:

- Given $a = f(t)$, use $\dfrac{dv}{dt}$ or $\dfrac{d^2x}{dt^2}$
- Given $a = g(x)$, use $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$
- Given $a = h(v)$, use

$$\begin{cases} \dfrac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\dfrac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v \end{cases}$$

Derivatives with respect to time are often written using dots above the dependent variable, e.g.

$\dot{x} = \dfrac{dx}{dt}, \ddot{x} = \dfrac{d^2x}{dt^2}, \dot{v} = \dfrac{dv}{dt}$

Note also that for constant m we have $F = m\dfrac{dv}{dt}$

and so: $\dfrac{dv}{dt} = \dfrac{F}{m}$

The derivative on the left-hand side is the acceleration, which forms the basis for the solution of a differential equation.

Resisted motion

Whenever a body moves through a medium (such as air, water, oil etc.), it is subjected to a resistance that acts in the opposite direction to the motion.

In general, the faster the body moves, the greater the resistance. Air resistance is typically proportional to some power of the speed, so that air resistance $= kv^n$. We will consider this for $n = 1$ and $n = 2$.

The effect of resistance is always retardation:

- **Resistance** is force acting in the opposite direction to the motion.
- **Retardation** is acceleration acting in the opposite direction to the motion.

In problems involving resisted motion, define the positive direction to be the direction in which motion is actually occurring and take the origin O as the point from which the motion begins.

Terminal velocity

As t increases, the gravitational force acting on an object P causes it to accelerate downwards. However, as the speed increases, the resistance also increases until eventually the downwards weight force and the resistance force are equal in magnitude but opposite in direction. At this stage the resultant force is zero: according to Newton's first law of motion, P will now move in a straight line at constant speed, i.e. the terminal velocity.

- The terminal velocity occurs when the acceleration is equal to zero.

Summary of equations for projectile motion at an angle θ to the horizontal

A particle projected from the ground with an initial velocity V at an angle θ to the horizontal where $u_x = V\cos\theta$, $u_y = V\sin\theta$. If projected from above the ground, then this initial height of projection needs to be added to the equation for y. k is the constant of proportionality for any resistance.

No resistance

$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -g$

$\dot{x} = V\cos\theta$ $\qquad$ $\dot{y} = V\sin\theta - gt$

$x = V\cos\theta\, t$ $\qquad$ $y = V\sin\theta\, t - \dfrac{1}{2}gt^2$

Resistance proportional to the velocity

$\ddot{x} = -k\dot{x}$ $\qquad$ $\ddot{y} = -g - k\dot{y}$

$\dot{x} = V\cos\theta e^{-kt}$ $\qquad$ $\dot{y} = \dfrac{1}{k}\left((g + kV\sin\theta)e^{-kt} - g\right)$

$x = \dfrac{V\cos\theta}{k}\left(1 - e^{-kt}\right)$ $\qquad$ $y = \dfrac{\left(g + kV\sin\theta\right)}{k^2}\left(1 - e^{-kt}\right) - \dfrac{gt}{k}$

Resistance proportional to the square of the velocity, mathematical model

$\ddot{x} = -k(\dot{x})^2$ $\qquad$ $\ddot{y} = -g - k(\dot{y})^2$

$\dot{x} = \dfrac{V\cos\theta}{1 + k\cos\theta t}$ $\qquad$ $\dot{y} = \dfrac{\sqrt{gk}\,V\sin\theta - 10\tan\left(\sqrt{gk}\,t\right)}{\sqrt{gk} + kV\sin\theta\tan\left(\sqrt{gk}\,t\right)}$

$x = \dfrac{1}{k}\log_e\left(1 + kV\cos\theta t\right)$ $\qquad$ $y = \dfrac{1}{k}\log_e\left|\cos(\sqrt{gkt}) + \sqrt{\dfrac{k}{g}}u\sin\theta\sin\left(\sqrt{gkt}\right)\right|$

Mathematics Extension 2 Course Outcomes

Objective **Students:** • develop efficient strategies to solve complex problems using pattern recognition, generalisation, proof and modelling techniques
Year 12 Mathematics Extension 2 outcomes A student:
MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

Objective **Students:** • develop their knowledge, skills and understanding to model and solve complex and interconnected problems in the areas of proof, vectors and mechanics, calculus and complex numbers
Year 12 Mathematics Extension 2 outcomes A student:
MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings
MEX12-3 uses vectors to model and solve problems in two and three dimensions
MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems
MEX12-5 applies techniques of integration to structured and unstructured problems
MEX12-6 uses mechanics to model and solve practical problems

Mathematics Extension 2 Course Outcomes

Objective
Students: • develop their problem-solving and reasoning skills to create appropriate mathematical models in a variety of forms and apply these to difficult unstructured problems
Year 12 Mathematics Extension 2 outcomes A student:
MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

Objective
Students: • use mathematics as an effective means of communication and justification in complex situations
Year 12 Mathematics Extension 2 outcomes A student:
MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

ANSWERS

CHAPTER 1

EXERCISE 1.1

1 C

2 (a) $x = \pm 3i$ (b) $x = \pm 5i$ (c) $x = -1 \pm 4i$ (d) $x = 1 \pm 2i$

(e) $x = 2 \pm 4i$ (f) $x = \frac{1}{2} \pm \frac{5}{2}i$ (g) $x = \frac{1 \pm i\sqrt{31}}{2}$

(h) $x^2 - x + 4 = 0,\ x = \frac{1 \pm i\sqrt{15}}{2}$

3 (a) $-i$ (b) 1 (c) -1 (d) $-i$ (e) 1

4 (a) $\frac{5}{29} + \frac{2i}{29}$ (b) $5 + 2i$ (c) 29 (d) $29 - 20i$ (e) -16

(f) $\frac{13}{17} + \frac{1}{17}i$

5 (a) $10 + 3i$ (b) $6 - 2i$ (c) $23 - 14i$ (d) 58 (e) $-21 - 20i$

(f) i (g) 7 (h) $\frac{2}{13} - \frac{3}{13}i$ (i) $\frac{17}{25} + \frac{44}{25}i$ (j) $\frac{19}{29} + \frac{16}{29}i$

(k) $-\frac{7}{5} - \frac{17}{10}i$ (l) $-\frac{48}{25} - \frac{247}{50}i$

6 (a) $x = 3, y = -2$ (b) $x = 6, y = 2$

7 (a) $-15 + 8i$ (b) $9 + 7i$ (c) $9 + 7i$ (d) $-\frac{3}{29} + \frac{22}{29}i$

8 (a) $(z + 3i)(z - 3i)$ (b) $(z + 6i)(z - 6i)$

(c) $(z - 3 + 4i)(z - 3 - 4i)$ (d) $(2z + 3 + 2\sqrt{2}i)(2z + 3 - 2\sqrt{2}i)$

(e) $(z + 1 + 5i)(z + 1 - 5i)$ (f) $(z - 3 + \sqrt{11}i)(z - 3 - \sqrt{11}i)$

(g) $2\left(z + \frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{7}}{2}i\right)$

(h) $(z + 10)(z - 5 + 5\sqrt{3}i)(z - 5 - 5\sqrt{3}i)$

9 (a) $z = 8 - 4i$ (b) $z = 2i$

10 (a) $\pm(1 + 3i)$ (b) $z = i,\ -\frac{1}{2} - \frac{1}{2}i$

(c) Sum of roots $= -\frac{1}{2} - \frac{1}{2}i + i = -\frac{1}{2} + \frac{1}{2}i = -\frac{b}{a}$

Product of roots $= i\left(-\frac{1}{2} - \frac{1}{2}i\right) = \frac{1}{2} - \frac{1}{2}i = \frac{c}{a}$

11 (a) $\pm(1 - 3i)$ (b) $\pm(2\sqrt{2} - 2\sqrt{2}i)$ (c) $\pm\left(\frac{5\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

12 (a) LHS $= (\sqrt{3} - i)^3 - (\sqrt{3} - i)(\sqrt{3} - i)^2 + 9(\sqrt{3} - i) - 9\sqrt{3} + 9i$
$= 0 =$ RHS

(b) $z^3 - (\sqrt{3} - i)z^2 + 9z - 9\sqrt{3} + 9i = 0$

$z^2(z - (\sqrt{3} - i)) + 9(z - \sqrt{3} + i) = 0$

$(z - \sqrt{3} + i)(z^2 + 9) = 0$

$\therefore z = 3i$ and $z = -3i$ are the other solutions.

(c) Sum of roots $= \sqrt{3} - i + 3i - 3i = \sqrt{3} - i = -\frac{b}{a}$

Sum of products of pairs of roots

$= 3i(\sqrt{3} - i) - 3i(\sqrt{3} - i) + 3i(-3i) = 9 = \frac{c}{a}$

Product of roots $= 3i(-3i)(\sqrt{3} - i) = 9(\sqrt{3} - i) = -\frac{d}{a}$

13 (a) $z = 2 - i, 1 - i$ (b) $z = 1 - 2i, 2i$ (c) $z = 2 + i, i$

(d) $z = 3 + 2i, -i$

14 Let $(x + iy)^2 = a + ib$ where x, y are real. Then $x^2 - y^2 = a$ [1] and $2xy = b$ [2].

Substitute [2] into [1]: $x^4 - 4a^2x^2 - b^2 = 0$

Solving this as a quadratic in x^2: $x^2 = \frac{4a^2 \pm \sqrt{16a^4 + 4b^2}}{2}$

Note that $16a^4 + 4b^2 \geq 0$ (as a, b are real) and that $16a^4 + 4b^2 = 0$ only when $a = b = 0$.

Also, as $\sqrt{16a^4 + 4b^2} \geq 4a^2$, there will be one positive value of x^2 and one negative value of x^2. But x is real, so the negative value of x^2 is disregarded and there are exactly two values of x, each with a unique value of y (from [2]).

$\therefore$ There are two square roots of every complex number (except $0 + 0i$).

15 (a) $z_1 + \overline{z_1} = 2x_1 = 2 \times \text{Re}(z_1)$

(b) $z_1 - \overline{z_1} = 2iy_1 = 2i \times \text{Im}(z_1)$

(c) LHS $= \overline{x_1 + iy_1 + x_2 + iy_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$
$= (x_1 + x_2) - i(y_1 + y_2)$

RHS $= \overline{z_1} + \overline{z_2} = x_1 - iy_1 + x_2 - iy_2 =$ LHS

(d) LHS $= \overline{x_1 + iy_1 - x_2 - iy_2} = \overline{(x_1 - x_2) + i(y_1 - y_2)}$
$= (x_1 - x_2) - i(y_1 - y_2)$

RHS $= \overline{z_1} - \overline{z_2} = x_1 - iy_1 - (x_2 - iy_2) =$ LHS

(e) LHS $= \overline{z_1 \times z_2} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$
$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$

RHS $= \overline{z_1} \times \overline{z_2} = (x_1 - iy_1)(x_2 - iy_2) =$ LHS

16 (a) $(z + 5)\left(z - \frac{5}{2} - \frac{5\sqrt{3}}{2}i\right)\left(z - \frac{5}{2} + \frac{5\sqrt{3}}{2}i\right)$

(b) w satisfies $z^2 - 5z + 25 = 0$ (from linear factors)

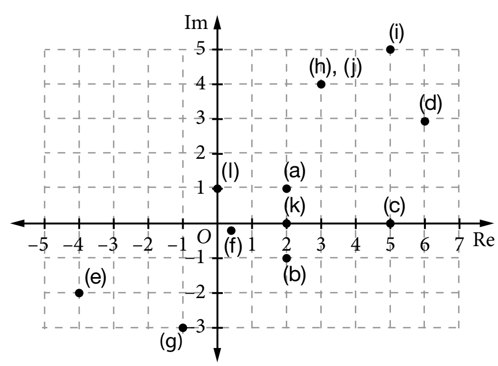
$\therefore w^2 = 5w - 25$

(c) $(5w - 25)^3 = (w^2)^3 = (w^3)^2 = (-125)^2 = 15\,625$

(*Note:* $w^3 = -125$ because w is a root of $z^3 + 125 = 0$)

EXERCISE 1.2

1



2 A **3** D

4 (a) $2\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$ (b) $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(c) $6\sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$ (d) $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(e) $4(\cos\pi + i\sin\pi)$ (f) $2\sqrt{3}\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)$

(g) $4\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)$ (h) $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

5 (a) $2 + 2\sqrt{3}i$ (b) $4\sqrt{2} - 4\sqrt{2}i$

(c) $-3\sqrt{2} + 3\sqrt{2}i$ (d) $-1 - \sqrt{3}\,i$

6 (a) $16\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

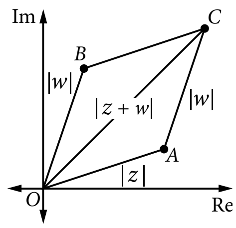
(b) $15\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, $\frac{5}{3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(c) $2\left(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}\right)$, $\cos\pi + i\sin\pi$

7 (a) LHS $= \sqrt{x^2 + y^2}$, RHS $= |x - iy| = \sqrt{x^2 + y^2}$

(b) LHS $= (x + iy)(x - iy) = x^2 + y^2 =$ RHS

(c) LHS $= z + \frac{z\overline{z}}{z} = z + \overline{z} = 2x = 2\text{Re}(z) =$ RHS

8 

In parallelogram $OACB$:

$OA = |z|, OB = AC = |w|, OC = |z + w|$

In ΔOAC: $OC < OA + AC$ (each side of a triangle is less than the sum of the other two sides)

$\therefore |z + w| < |z| + |w|$

If O, B and A are collinear then $|z + w| = |z| + |w|$

$\therefore |z + w| \le |z| + |w|$

9 (a) $-32i$ (b) 16 (c) -64 (d) $-4 + 4i$

(e) $-8 - 8\sqrt{3}\,i$ (f) $-\dfrac{\sqrt{3}}{2048} - \dfrac{1}{2048}i$ (g) $\dfrac{1}{512}$

(h) $128 + 128i$ (i) $\dfrac{1}{2} - \dfrac{1}{2}i$ (j) -4

10 (a) $-7 - 24i$ (b) $-4 - 3i$ (c) $\dfrac{3}{25} + \dfrac{4}{25}i$

11 $\text{LHS} = \dfrac{r(\cos\theta + i\sin\theta)}{r^2(\cos 2\theta + i\sin 2\theta) + r^2}$

$= \dfrac{r(\cos\theta + i\sin\theta)}{r^2(2\cos^2\theta - 1 + 2i\sin\theta\cos\theta + 1)}$

$= \dfrac{\cos\theta + i\sin\theta}{2r\cos\theta(\cos\theta + i\sin\theta)} = \dfrac{1}{2r\cos\theta}$, which is real.

12 (a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{3}$

13 (a) *Step 1* Prove true for $n = 1$: $z = \text{cis}\,\theta$, $z^1 = \text{cis}\,1\theta = z$

Step 2 Assume true for $n = k$, i.e. assume that $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Now when $n = k + 1$:

$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k + 1)\theta + i\sin(k + 1)\theta$

Step 3 Conclusion

True for $n = k + 1$ if true for $n = k$. True for $n = 1$.

$\therefore$ True for all positive integers $n \ge 1$.

(b) $(\cos\theta + i\sin\theta)^{-n} = \dfrac{1}{(\cos\theta + i\sin\theta)^n}$

$= \dfrac{1}{(\cos n\theta + i\sin n\theta)} \times \dfrac{(\cos n\theta - i\sin n\theta)}{(\cos n\theta - i\sin n\theta)} = \dfrac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$

$= \cos(n\theta) - i\sin(n\theta) = \cos(-n\theta) + i\sin(-n\theta)$

14 $z = r(\cos\theta + i\sin\theta)$

$\therefore z^n = r^n(\cos n\theta + i\sin n\theta)$

$\therefore \overline{z^n} = r^n(\cos n\theta - i\sin n\theta)$

$\overline{z} = r(\cos(-\theta) + i\sin(-\theta))$

$\therefore (\overline{z})^n = r^n(\cos(-n\theta) + i\sin(-n\theta))$

$= r^n(\cos n\theta - i\sin n\theta)$

$\therefore \overline{z^n} = (\overline{z})^n$

15 (a) $z^n + (\overline{z})^n = z^n + \overline{z^n} = 2\,\text{Re}(z^n)$

(b) $1 + \sqrt{3}\,i = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

$\therefore \left(1 + \sqrt{3}\,i\right)^{10} = 2^{10}\left(\cos\dfrac{10\pi}{3} + i\sin\dfrac{10\pi}{3}\right)$

$= 2^{10}\left(\cos\dfrac{-2\pi}{3} + i\sin\dfrac{-2\pi}{3}\right)$

$\therefore \left(1 + \sqrt{3}\,i\right)^{10} + \left(1 - \sqrt{3}\,i\right)^{10} = 2 \times 2^{10}\cos\dfrac{-2\pi}{3} = -1024$

16 z is a root of $P(x) = 0$ $\therefore az^3 + bz^2 + cz + d = 0$

$\overline{az^3 + bz^2 + cz + d} = \overline{0}$ (taking conjugates of both sides)

$\overline{az^3} + \overline{bz^2} + \overline{cz} + \overline{d} = 0$ (conjugate of a sum is the sum of the conjugates; conjugate of 0 is 0)

$\overline{a} \times \overline{z^3} + \overline{b} \times \overline{z^2} + \overline{c} \times \overline{z} + \overline{d} = 0$ (conjugate of a product is the product of the conjugates)

$a \times (\overline{z})^3 + b \times (\overline{z})^2 + c \times \overline{z} + d = 0$ (conjugate of a real is itself; conjugate of a power is the power of the conjugate)

$\therefore P(\overline{z}) = 0$ i.e. $\overline{z}$ is also a root

EXERCISE 1.3

1 (a) $e^{\frac{i\pi}{3}} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

(b) $e^{\frac{i\pi}{2}} = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = 0 + i = i$

(c) $e^{\frac{5\pi i}{6}} = \cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

(d) $e^{\frac{i\pi}{4}} = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

(e) $e^{\frac{-i\pi}{2}} = \cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right) = 0 - i = -i$

(f) $e^{\frac{-2\pi i}{3}} = \cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right) = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

(g) $e^{1-\frac{i\pi}{2}} = e \times e^{\frac{-i\pi}{2}} = e\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right) = e(0 - i) = -ei$

(h) $e^{2+\frac{i\pi}{3}} = e^2 \times e^{\frac{i\pi}{3}} = e^2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right) = e^2\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)$

2 (a) $3e^{1.5i}$

(b) $-\sqrt{3} + i = 2\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right) = 2e^{\frac{5i\pi}{6}}$

(c) $3 + 2i = \sqrt{13}\left(\dfrac{3}{\sqrt{13}} + \dfrac{2}{\sqrt{13}}i\right) = \sqrt{13}(\cos\theta + i\sin\theta)$

where $\theta = \tan^{-1}\left(\dfrac{2}{3}\right) = 0.59$

$= \sqrt{13}e^{0.59i}$

(d) $4(\cos 2 - i\sin 2) = 4(\cos(-2) + i\sin(-2)) = 4e^{-2i}$

(e) $2 - 2i = 2\sqrt{2}\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$

$= 2\sqrt{2}e^{\frac{-i\pi}{4}}$

(f) $4\left(-\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right) = 4\left(\cos\dfrac{4\pi}{5} + i\sin\dfrac{4\pi}{5}\right)4e^{\frac{4i\pi}{5}}$

(g) $-2 - 2\sqrt{3}\,i = 4\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right) = 4\left(\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)\right)$

$= 4e^{\frac{-2i\pi}{3}}$

(h)

$\left(1 + \sqrt{2}\right) + \left(1 - \sqrt{2}\right)i = \sqrt{6}\left(\dfrac{1 + \sqrt{2}}{\sqrt{6}} + \dfrac{1 - \sqrt{2}}{\sqrt{6}}i\right) = \sqrt{6}(\cos\theta + i\sin\theta)$

where $\theta = \tan^{-1}\left(\dfrac{1 - \sqrt{2}}{1 + \sqrt{2}}\right)$

$= \tan^{-1}\left(\dfrac{2\sqrt{2} - 3}{1}\right) = -0.17$

$= \sqrt{6}e^{-0.17i}$

3 C. $\cos^2\dfrac{\pi}{8} = \dfrac{1}{2}\left(1 + \dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\left(1 + \dfrac{\sqrt{2}}{2}\right) = \dfrac{1}{4}\left(2 + \sqrt{2}\right)$.

$\cos\dfrac{\pi}{8} = \dfrac{1}{2}\sqrt{2 + \sqrt{2}}$. $\sin\dfrac{\pi}{8} = \dfrac{1}{2}\sqrt{2 - \sqrt{2}}$

4 (a) $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$

(b) $e^{i\theta}+e^{-i\theta}=\cos\theta+i\sin\theta+\cos\theta-i\sin\theta=2\cos\theta.$

$\cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$

$e^{i\theta}-e^{-i\theta}=\cos\theta+i\sin\theta-(\cos\theta-i\sin\theta)=2i\sin\theta.$

$\sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$

5 (a) $z=\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}i\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2e^{\frac{i\pi}{6}}$

(b)(i) $z^2=2^2e^{\frac{2i\pi}{6}}=4e^{\frac{i\pi}{3}}=4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2+2\sqrt{3}i$

(ii) $z^3=2^3e^{\frac{3i\pi}{6}}=8e^{\frac{i\pi}{2}}=8\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=8i$

(iii) $z^4=2^4e^{\frac{4i\pi}{6}}=16e^{\frac{2i\pi}{3}}=16\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=-8+8\sqrt{3}i$

(iv) $\sqrt{z}=\sqrt{2}e^{\frac{i\pi}{12}}=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)=1.37+0.37i$

(v) $z^{-1}=\frac{1}{2}e^{\frac{-i\pi}{6}}=\frac{1}{2}\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)=\frac{\sqrt{3}}{4}-\frac{1}{4}i$

6 (a) $z=1-\sqrt{3}i=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2e^{\frac{-i\pi}{3}}$

(b) (i) $z^2=2^2e^{\frac{-2i\pi}{3}}=4e^{\frac{-2i\pi}{3}}=4\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)$

$=-2-2\sqrt{3}i$

(ii) $z^3=2^3e^{\frac{-3i\pi}{3}}=8e^{-i\pi}=8(\cos(-\pi)+i\sin(-\pi))=-8+0i$

(iii) $z^5=2^5e^{\frac{-5i\pi}{3}}=32e^{\frac{i\pi}{3}}=32\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=16+16\sqrt{3}i$

(iv) $\sqrt{z}=\sqrt{2}e^{\frac{-i\pi}{6}}=\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)=\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i$

(v) $\frac{1}{z}=\frac{1}{2}e^{\frac{i\pi}{3}}=\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\frac{1}{4}+\frac{\sqrt{3}}{4}i$

(vi) $\frac{1}{z^3}=\frac{1}{2^3}e^{\frac{3i\pi}{3}}=\frac{1}{8}e^{i\pi}=\frac{1}{8}(\cos\pi+i\sin\pi)=-\frac{1}{8}+0i$

7 (a) Cosine and sine are both negative in the third quadrant.

$z=\sqrt{2}\left(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)=\sqrt{2}e^{\frac{-3i\pi}{4}}$

or $z=\sqrt{2}\left(-\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\cos\frac{5\pi}{4}-i\sin\frac{5\pi}{4}\right)=\sqrt{2}e^{\frac{5\pi i}{4}}$

(b) (i) $z^2=\left(\sqrt{2}\right)^2e^{\frac{-6i\pi}{4}}=2e^{\frac{-3i\pi}{2}}=2e^{\frac{i\pi}{2}}=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

$=0+2i$

(ii) $z^3=\left(\sqrt{2}\right)^3e^{\frac{-9i\pi}{4}}=2\sqrt{2}e^{\frac{-i\pi}{4}}$

$=2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)=2-2i$

(iii) $z^4=\left(\sqrt{2}\right)^4e^{\frac{-12i\pi}{4}}=4e^{-3i\pi}=4e^{-i\pi}$

$=4(\cos(-\pi)+i\sin(-\pi))=-4+0i$

(iv) $\sqrt{z}=\left(\sqrt{2}\right)^{\frac{1}{2}}e^{\frac{-3i\pi}{8}}=2^{\frac{1}{4}}e^{\frac{-3i\pi}{8}}$

$=2^{\frac{1}{4}}\left(\cos\left(-\frac{3\pi}{8}\right)+i\sin\left(-\frac{3\pi}{8}\right)\right)=0.46-1.10i$

(v) $\frac{1}{z}=\frac{1}{\sqrt{2}}e^{\frac{3i\pi}{4}}=\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-\frac{1}{2}+\frac{1}{2}i$

(vi) $\frac{1}{z^2}=\left(\sqrt{2}\right)^{-2}e^{\frac{6i\pi}{4}}=\frac{1}{2}e^{\frac{3i\pi}{2}}=\frac{1}{2}e^{\frac{-i\pi}{2}}$

$=\frac{1}{2}\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)=0-\frac{1}{2}i$

8 (a) $e^{2i\theta}=\cos 2\theta+i\sin 2\theta,$

$\left(e^{i\theta}\right)^2=(\cos\theta+i\sin\theta)^2=\cos^2\theta-\sin^2\theta+2i\sin\theta\cos\theta$

(b) $\cos 2\theta=\cos^2\theta-\sin^2\theta$; $\sin 2\theta=2\sin\theta\cos\theta$

9 (a) $e^{3i\theta}=\cos 3\theta+i\sin 3\theta$

$\left(e^{i\theta}\right)^3=(\cos\theta+i\sin\theta)^3$

$=\cos^3\theta+3i\cos^2\theta\sin\theta+3i^2\cos\theta\sin^2\theta+i^3\sin^3\theta$

$=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$

$=\cos^3\theta-3\cos\theta\sin^2\theta+\left(3\cos^2\theta\sin\theta-\sin^3\theta\right)i$

(b) $\cos 3\theta=\cos^3\theta-3\cos\theta\,(1-\cos^2\theta)=4\cos^3\theta-3\cos\theta$

(c) $\sin 3\theta=3\sin\theta\,(1-\sin^2\theta)-\sin^3\theta=3\sin\theta-4\sin^3\theta$

10 $\frac{1-3i}{1+2i}=\frac{1-3i}{1+2i}\times\frac{1-2i}{1-2i}=\frac{-5-5i}{5}=-1-i=\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)$

$=\sqrt{2}e^{\frac{-3i\pi}{4}}.\quad r=\sqrt{2},\quad\theta=-\frac{3\pi}{4}$

11 $\left(\sqrt{3}-i\right)\left(-\sqrt{3}+i\right)=-3+2\sqrt{3}i-i^2=-2+2\sqrt{3}i=4\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

$=4e^{\frac{2i\pi}{3}}.\quad r=4,\quad\theta=\frac{2\pi}{3}$

12 (a) $z_1\times z_2=2e^{\frac{-i\pi}{6}}\times 3e^{\frac{2i\pi}{3}}=6e^{\frac{-i\pi}{6}+\frac{2i\pi}{3}}=6e^{\frac{i\pi}{2}}\ (=6i)$

$=6\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

(b) $z_2\times z_3=3e^{\frac{2i\pi}{3}}\times e^{\frac{i\pi}{4}}=3e^{\frac{2i\pi}{3}+\frac{i\pi}{4}}=3e^{\frac{11i\pi}{12}}=3\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$

(c) ${z_1}^2\times z_2=2^2e^{\frac{-2i\pi}{6}}\times 3e^{\frac{2i\pi}{3}}=12e^{\frac{i\pi}{3}}=12\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

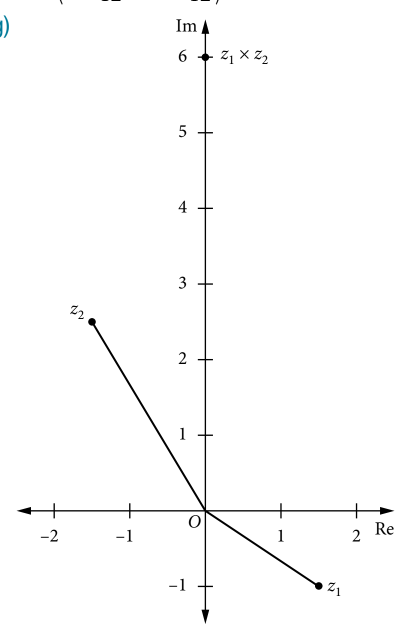
(d) $\frac{z_1}{z_2}=\frac{2e^{\frac{-i\pi}{6}}}{3e^{\frac{2i\pi}{3}}}=\frac{2}{3}e^{\frac{-i\pi}{6}-\frac{2i\pi}{3}}=\frac{2}{3}e^{\frac{-5i\pi}{6}}=6\left(\cos\frac{-5i\pi}{6}+i\sin\frac{-5i\pi}{6}\right)$

(e) $\frac{z_2}{z_3}=\frac{3e^{\frac{2i\pi}{3}}}{e^{\frac{i\pi}{4}}}=3e^{\frac{2i\pi}{3}-\frac{i\pi}{4}}=3e^{\frac{5i\pi}{12}}=3\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$

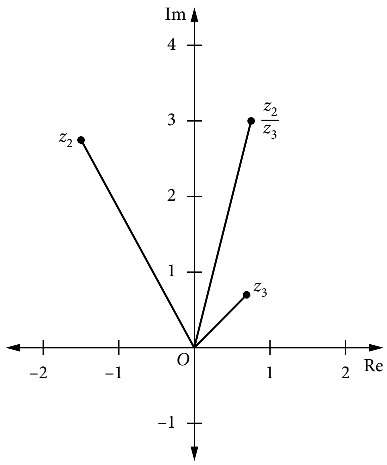
(f) $\frac{{z_1}^2\times z_2}{z_3}=\frac{2^2e^{\frac{-2i\pi}{6}}\times 3e^{\frac{2i\pi}{3}}}{e^{\frac{i\pi}{4}}}=12e^{\frac{i\pi}{3}-\frac{i\pi}{4}}=12e^{\frac{i\pi}{12}}$

$=12\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

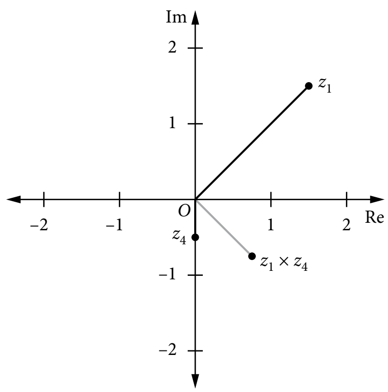
(g)



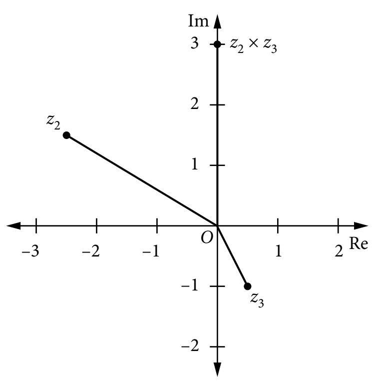
(h)



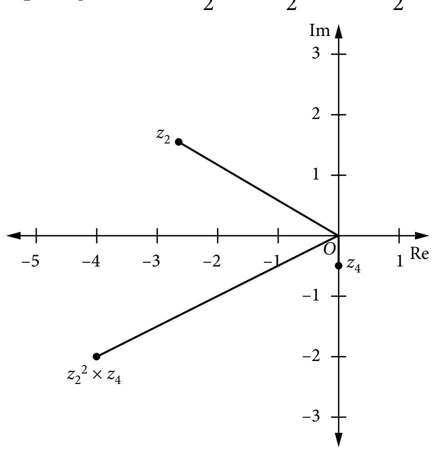
13 (a) $z_1 \times z_4 = 2e^{\frac{i\pi}{4}} \times \frac{1}{2}e^{\frac{-i\pi}{2}} = e^{\frac{i\pi}{4}+\frac{-i\pi}{2}} = e^{\frac{-i\pi}{4}}$



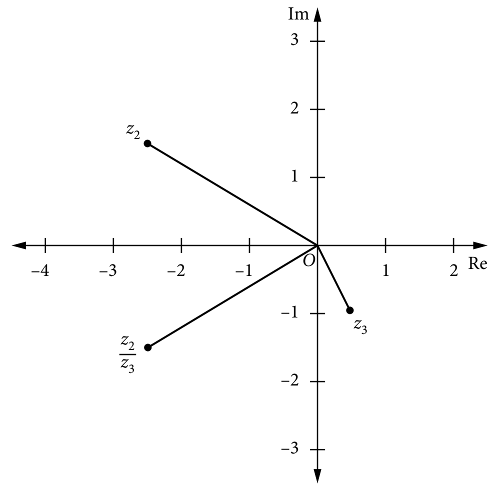
(b) $z_2 \times z_3 = 3e^{\frac{5i\pi}{6}} \times e^{\frac{-i\pi}{3}} = 3e^{\frac{5i\pi}{6}+\frac{-i\pi}{3}} = 3e^{\frac{3i\pi}{6}} = 3e^{\frac{i\pi}{2}} \; (=3i)$



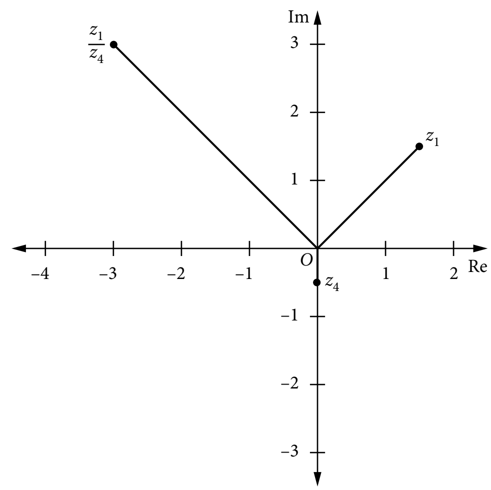
(c) $z_2{}^2 \times z_4 = 3^2 e^{\frac{10i\pi}{6}} \times \frac{1}{2}e^{\frac{-i\pi}{2}} = \frac{9}{2}e^{\frac{5i\pi}{3}-\frac{i\pi}{2}} = \frac{9}{2}e^{\frac{7i\pi}{6}} = \frac{9}{2}e^{\frac{-5i\pi}{6}}$



(d) $\frac{z_2}{z_3} = \frac{3e^{\frac{5i\pi}{6}}}{e^{\frac{-i\pi}{3}}} = 3e^{\frac{5i\pi}{6}+\frac{i\pi}{3}} = 3e^{\frac{7i\pi}{6}} = 3e^{\frac{-5i\pi}{6}}$

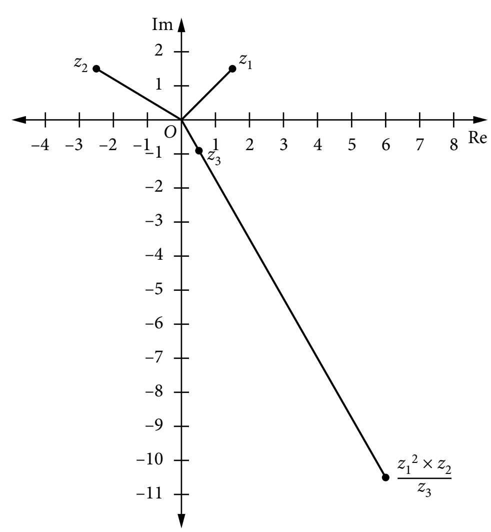


(e) $\frac{z_1}{z_4} = \frac{2e^{\frac{i\pi}{4}}}{\frac{1}{2}e^{\frac{-i\pi}{2}}} = 4e^{\frac{i\pi}{4}+\frac{i\pi}{2}} = 4e^{\frac{3i\pi}{4}}$

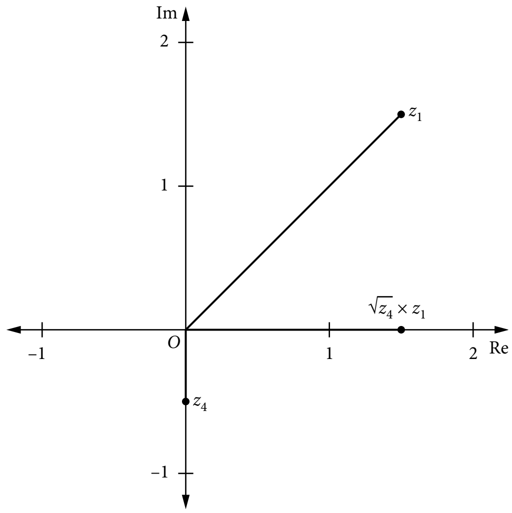


(f) $\frac{z_1{}^2 \times z_2}{z_3} = \frac{2^2 e^{\frac{2i\pi}{4}} \times 3e^{\frac{5i\pi}{6}}}{e^{\frac{-i\pi}{3}}} = 12e^{\frac{2i\pi}{4}+\frac{5i\pi}{6}+\frac{i\pi}{3}}$

$= 12e^{\frac{i\pi}{2}+\frac{7i\pi}{6}} = 12e^{\frac{10i\pi}{6}} = 12e^{\frac{-i\pi}{3}}$

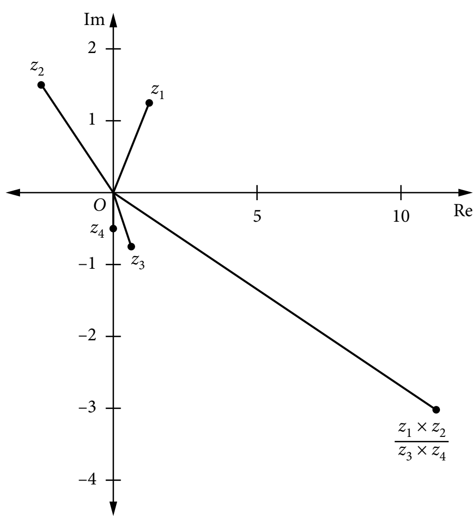


(g) $\sqrt{z_4}\times z_1 = \sqrt{\frac{1}{2}e^{\frac{-i\pi}{2}}}\times 2e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}}e^{\frac{-i\pi}{4}}\times 2e^{\frac{i\pi}{4}} = \sqrt{2}e^{0}\left(=\sqrt{2}\right)$

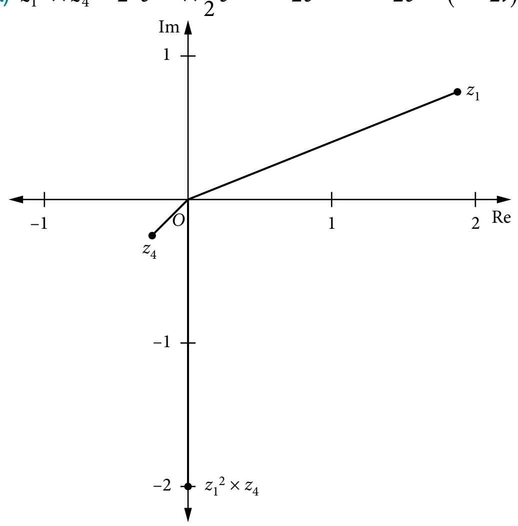


(h) $\frac{z_1\times z_2}{z_3\times z_4} = \frac{2e^{\frac{i\pi}{4}}\times 3e^{\frac{5i\pi}{6}}}{e^{\frac{-i\pi}{3}}\times\frac{1}{2}e^{\frac{-i\pi}{2}}} = \frac{6e^{\frac{i\pi}{4}+\frac{5i\pi}{6}}}{\frac{1}{2}e^{\frac{-i\pi}{3}+\frac{-i\pi}{2}}} = \frac{12e^{\frac{13i\pi}{12}}}{e^{\frac{-5i\pi}{6}}}$

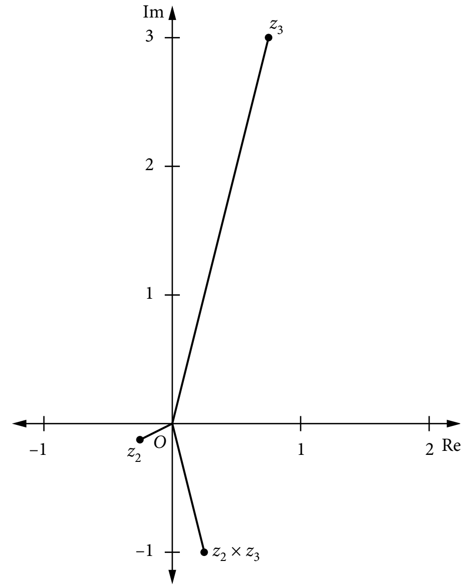
$= 12e^{\frac{13i\pi}{12}+\frac{5i\pi}{6}} = 12e^{\frac{23i\pi}{12}} = 12e^{\frac{-i\pi}{12}}$



14 (a) $z_1^{\,2}\times z_4 = 2^2 e^{\frac{2i\pi}{8}}\times\frac{1}{2}e^{\frac{-3i\pi}{4}} = 2e^{\frac{i\pi}{4}-\frac{3i\pi}{4}} = 2e^{\frac{-i\pi}{2}}\,(=-2i)$

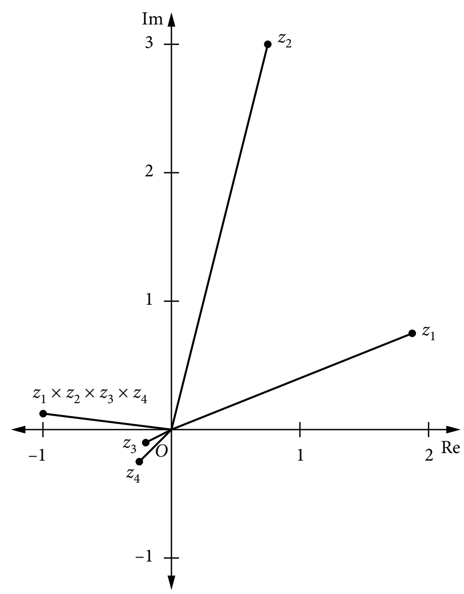


(b) $z_2\times z_3 = 3e^{\frac{5i\pi}{12}}\times\frac{1}{3}e^{\frac{-5i\pi}{6}} = e^{\frac{5i\pi}{12}-\frac{5i\pi}{6}} = e^{\frac{-5i\pi}{12}}$

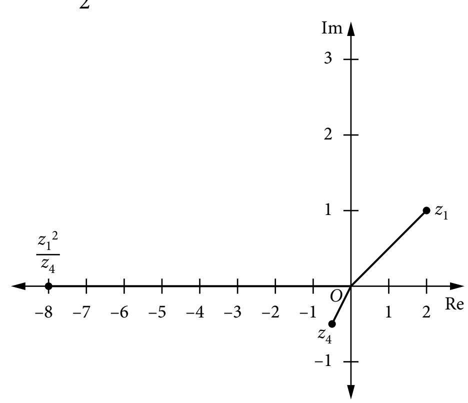


(c) $z_1\times z_2\times z_3\times z_4 = 2e^{\frac{i\pi}{8}}\times e^{\frac{-5i\pi}{12}}\times\frac{1}{2}e^{\frac{-3i\pi}{4}}$

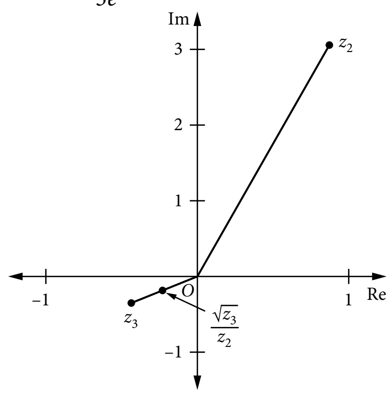
$= e^{\frac{i\pi}{8}-\frac{5i\pi}{12}-\frac{3i\pi}{4}} = e^{\frac{-25i\pi}{24}} = e^{\frac{23i\pi}{24}}$



(d) $\frac{z_1^{\,2}}{z_4} = \frac{2^2 e^{\frac{2i\pi}{8}}}{\frac{1}{2}e^{\frac{-3i\pi}{4}}} = 8e^{\frac{i\pi}{4}+\frac{3i\pi}{4}} = 8e^{i\pi}\,(=-8)$



(e) $\dfrac{\sqrt{z_3}}{z_2} = \dfrac{\frac{1}{\sqrt{3}}e^{\frac{-5i\pi}{12}}}{3e^{\frac{5i\pi}{12}}} = \dfrac{1}{3\sqrt{3}}e^{\frac{-5i\pi}{12}-\frac{5i\pi}{12}} = \dfrac{\sqrt{3}}{9}e^{\frac{-5i\pi}{6}}$



15 $z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos\theta + i\sin\theta)^n$

$= r^n e^{ni\theta}$

$= r^n(\cos n\theta + i\sin n\theta)$

16 (a) C is $2+4i$, D is $-1+3i$; or C is $-2-4i$, D is $-3-i$

(b) $-\dfrac{\pi}{4}, \dfrac{\pi}{4}$ (respectively)

17 $\dfrac{1}{2}(1+2i)z$ or $\dfrac{1}{2}(1-2i)z$

EXERCISE 1.4

1 (a) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

$(\cos\theta + i\sin\theta)^4 = (8\cos^4\theta - 8\cos^2\theta + 1) + i$
$(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

(i) Equate the real parts of the expansion.

(ii) Equate the imaginary parts of the expansion.

(b) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}$

(c) (i) $\pm\cos\dfrac{\pi}{8}, \pm\cos\dfrac{3\pi}{8}$ (ii) $\pm\cos\dfrac{\pi}{12}, \pm\cos\dfrac{5\pi}{12}$

(iii) $\pm\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}$ (iv) $\tan\dfrac{\pi}{16}, \tan\dfrac{5\pi}{16}, -\tan\dfrac{3\pi}{16}, -\tan\dfrac{7\pi}{16}$

2 (a) $\cos\dfrac{\pi}{9}, -\cos\dfrac{2\pi}{9}, -\cos\dfrac{4\pi}{9}$

(b) Product of roots [of part (a)] $= -\dfrac{d}{a} = \dfrac{1}{8}$

3 (a) (i) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

(ii) $z^n - z^{-n} = \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= 2i\sin n\theta$

(b) LHS $= z^3 - 3z^2 \times \dfrac{1}{z} + 3z \times \dfrac{1}{z^2} - \dfrac{1}{z^3}$

$= z^3 - 3z + \dfrac{3}{z} - \dfrac{1}{z^3} = (z^3 - z^{-3}) - 3(z - z^{-1}) =$ RHS

(c) $(2i\sin\theta)^3 = 2i\sin 3\theta - 3 \times 2i\sin\theta$

$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$4\sin^3\theta = -\sin 3\theta + 3\sin\theta$

$\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$

4 (a) $w^3 - 2w^2 - w + 2 = \left(z + \dfrac{1}{z}\right)^3 - 2\left(z + \dfrac{1}{z}\right)^2 - \left(z + \dfrac{1}{z}\right) + 2$

$= z^3 + 3z + \dfrac{3}{z} + \dfrac{1}{z^3} - 2z^2 - 2 - \dfrac{2}{z^2} - z - \dfrac{1}{z} + 2$

$= \left(z^3 + \dfrac{1}{z^3}\right) - 2\left(z^2 + \dfrac{1}{z^2}\right) + 2\left(z + \dfrac{1}{z}\right) - 2$

(b) $0, \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}$

5 $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\cos 2\theta = 2\cos^2\theta - 1$;

$4\cos^3\theta - 3\cos\theta = 2\cos^2\theta - 1$

$4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0$

$\therefore 4x^3 - 2x^2 - 3x + 1 = 0$ where $x = \cos\theta$

$\cos 3\theta = \cos 2\theta$:

$3\theta = 2\theta + 2k\pi$ or $3\theta = -2\theta + 2k\pi$,

$\theta = 2k\pi$ or $5\theta = 2k\pi$, $\theta = \dfrac{2k\pi}{5}$

Cubic equation $\therefore$ only 3 roots: $x = \cos 0, \cos\dfrac{2\pi}{5}, \cos\dfrac{4\pi}{5}$

i.e. $x = 1, \cos\dfrac{2\pi}{5}, \cos\dfrac{4\pi}{5}$

By division, equation becomes: $(x-1)(4x^2 + 2x - 1) = 0$

$x = \dfrac{-2 \pm \sqrt{20}}{8} = \dfrac{-1 \pm \sqrt{5}}{4}, 1 \therefore \cos\dfrac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}$

6 (a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(b) $\cos 4\theta = 0$

$4\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$

$\theta = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}$

$x = \cos\theta$: $x = \cos\dfrac{\pi}{8}, \cos\dfrac{3\pi}{8}, \cos\dfrac{5\pi}{8}, \cos\dfrac{7\pi}{8}$

(c) Sum of roots: $\cos\dfrac{\pi}{8} + \cos\dfrac{3\pi}{8} + \cos\dfrac{5\pi}{8} + \cos\dfrac{7\pi}{8} = 0$

(d) Product of roots: $\cos\dfrac{\pi}{8}\cos\dfrac{3\pi}{8}\cos\dfrac{5\pi}{8}\cos\dfrac{7\pi}{8} = \dfrac{1}{8}$

7 (a) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

(b) $\tan 5\theta = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$

$= \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$

(c) $\tan 5\theta = 0$:

$5\theta = 2k\pi$ or $\pi + 2k\pi$, hence $\theta = \dfrac{2k\pi}{5}$ or $\theta = \dfrac{\pi}{5} + \dfrac{2k\pi}{5}$

i.e. $\theta = \ldots -\dfrac{4\pi}{5}, -\dfrac{2\pi}{5}, 0, \dfrac{2\pi}{5}, \dfrac{4\pi}{5}, \ldots$

or $\theta = \ldots -\dfrac{3\pi}{5}, -\dfrac{\pi}{5}, \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \pi, \ldots$

$t = \tan\theta$, so $t = \tan 0 = \tan\dfrac{10\pi}{5} = \tan\pi = \ldots,$

$t = \tan\dfrac{2\pi}{5} = \tan\dfrac{-3\pi}{5} = \ldots, t = \tan\dfrac{4\pi}{5} = \tan\dfrac{-\pi}{5} = \ldots,$

$t = \tan\dfrac{\pi}{5} = \tan\dfrac{-4\pi}{5} = \ldots, t = \tan\dfrac{3\pi}{5} = \tan\dfrac{-2\pi}{5} = \ldots$

$t(t^4 - 10t^3 + 5) = 0$: the roots are $t = 0$

and the roots of $t^4 - 10t^3 + 5 = 0$:

$t = \tan\dfrac{\pi}{5}, \tan\dfrac{2\pi}{5}, \tan\dfrac{3\pi}{5}, \tan\dfrac{4\pi}{5}$

Product of roots: $\tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5}\tan\dfrac{3\pi}{5}\tan\dfrac{4\pi}{5} = 5$

EXERCISE 1.5

1 (a) $P(i) = -i - 2i + 3i = 0$

(b) $P(3) = 9 - 15 + 3i + 6 - 3i = 0$

(c) $P(-2+i) = 2(3-4i)(-2+i) + 3(3-4i) - (-12+i) - 17 - 9i$

$= 2(-2+11i) + 9 - 12i + 12 - i - 17 - 9i = 0$

(d) $P\left(3 - \sqrt{2}i\right) = 2\left(7 - 6\sqrt{2}i\right)^2 - 12\left(3 - \sqrt{2}i\right)\left(7 - 6\sqrt{2}i\right)$
$+ 23\left(7 - 6\sqrt{2}i\right) - 18 + 6\sqrt{2}i + 11$

$= 2\left(-23 - 84\sqrt{2}i\right) - 12\left(9 - 25\sqrt{2}i\right) + 154 - 132\sqrt{2}i$

$= -46 - 168\sqrt{2}i - 108 + 300\sqrt{2}i + 154 - 132\sqrt{2}i = 0$

2 C **3** $a = 6$ **4** $a = -2, b = -2$

5 $a = 4, b = -4$

6 (a) $\left(z+1-\sqrt{2}i\right)\left(z+1+\sqrt{2}i\right)$ (b) $2\left(z-\frac{1}{2}-\frac{i}{2}\right)\left(z-\frac{1}{2}+\frac{i}{2}\right)$

(c) $(2z-3)(z-i)(z+i)$ (d) $(z+3)(z-i)(z+i)$

7 (a) $(z-1)(2z^2-2z+1)$

(b) $2(z-1)\left(z-\frac{1}{2}-\frac{i}{2}\right)\left(z-\frac{1}{2}+\frac{i}{2}\right)$

8 (a) $(z-2)(z+2)(z^2+4)$

(b) $(z-2)(z+2)(z-2i)(z+2i)$

9 (a) $(z^2-2)(z^2+3)$ (b) $\left(z-\sqrt{2}\right)\left(z+\sqrt{2}\right)\left(z^2+3\right)$

(c) $\left(z-\sqrt{2}\right)\left(z+\sqrt{2}\right)\left(z-\sqrt{3}i\right)\left(z+\sqrt{3}i\right)$

10 (a) $(z-2)(z^2-2z+5)$ (b) $(z-2)(z-1-2i)(z-1+2i)$

11 (a) $(z+2)(z^2-2z+4)$ (b) $(z+2)\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)$

12 D

13 (a) $(z-1)(z+1)(z^2+z+1)(z^2-z+1)$

(b) $(z-1)(z+1)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

$\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

14 (a) $(z-1)(z+1)(z^2+1)(z+3)$

(b) $(z-1)(z+1)(z-i)(z+i)(z+3)$

15 $(z-1)(z+1)(4z-i)$

16 (a) $(z+3)(z^2-z+1)$

(b) $(z+3)\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

17 $2(z-2)\left(z-\frac{3}{4}-\frac{\sqrt{23}}{4}i\right)\left(z-\frac{3}{4}+\frac{\sqrt{23}}{4}i\right)$

EXERCISE 1.6

1 (a) $(z^2+4z+5)(z-\sqrt{2})(z+\sqrt{2})$

(b) $(z+2-i)(z+2+i)(z-\sqrt{2})(z+\sqrt{2})$

2 (a) $z=2+2i,\ 2-2i$

(b) $P(z)=(z+3)(z^2-z+1),\ z=-3,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$

(c) $P(z)=(z^3+8)(z^3-1)$

$=(z+2)(z^2-2z+4)(z-1)(z^2+z+1)$

$z=1,\ -2,\ 1\pm\sqrt{3}i,\ -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

(d) $z=\pm i$

3 (a) $(z-1)(z^2+2z+5)=0,\ z=1$

(b) $z=1,\ -1+2i,\ -1-2i$

4 $(z-1)(z+1)(z+3)(z^2+1)=0,\ z=\pm1,\ -3$

5 B

6 $P(1+i)=(1+i)^3+a(1+i)+b$

$2i-2+ai+a+b=0$

$a=-2,\ b=4$

7 $(z-1)(z+2)(z^2+z+2)=0$

$z=1,\ -2,\ -\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$

8 $-1,\ -\frac{2}{3},\ 3$

9 $(z^2-2z+5)(z^2+z+3)$

$z=1\pm2i,\ -\frac{1}{2}\pm\frac{\sqrt{11}}{2}i$

10 $(z-1)(z^2-2z+5)=0,\ z=1$

11 (a) $(z-1)(z+4)(z^2+z+1)=0$

$z=1,\ -4,\ -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

(b) $(z-2)^2(z^2+2z+4)^2=0$

$z=2,\ -1\pm\sqrt{3}i$

12 (a) (i) $z^2-(3+i)z+2+2i=0$ (ii) $z^3-4z^2+6z-4=0$

(b) (i) $z^2-(3+\sqrt{3}-i)z+2\sqrt{3}+2-\sqrt{3}i-i=0$

(ii) $z^4-6z^3+11z^2-2z-10=0$

(c) (i) $z-3-\sqrt{2}i=0$ (ii) $z^2-6z+11=0$

(d) (i) $z^2-(4-2i)z+3-4i=0$

(ii) $z^4-8z^3+26z^2-40z+25=0$

(e) (i) $z-3-2i=0$ (ii) $z^2-6z+13=0$

(f) (i) $z^2-(2+2\sqrt{3}i)z-2+2\sqrt{3}i=0$

(ii) $z^4-4z^3+12z^2-16z+16=0$

13 $P(ai)=a^4+2a^3i-7a^2-4ai+10=0$

Imaginary parts: $2a^3-4a=0$

$a=\pm\sqrt{2}\ (z=\pm\sqrt{2}i,\ 1\pm2i)$

14 $a=-2,\ b=-1$

15 (a) $-1,\ 1\pm i$ (b) $3-i$

16 $-k^3i-(2+i)k^2+(2+2i)ki+4=0$

Imaginary parts (divided by $-k$): $k^2+k-2=0$

$k=1,\ -2\quad(z=i,\ -2i,\ -2)$

17 (a) $P(z)=z^4+4z^3+5z^2+4z+4$

$P'(z)=4z^3+12z^2+10z+4$

The factors of 4 are $\pm1,\ \pm2,\ \pm4$. If $z>0$, then $P'(z)>0$.

$P'(-1)=2,\ P'(-2)=0,\ P'(-4)=-100$

Hence $z=-2$ is the double real root.

$P(z)=(z+2)^2(z^2+bz+c)$

$4c=4,\ c=1$

$P(z)=(z+2)^2(z^2+bz+1)$

$(z^2+4z+4)(z^2+bz+1)=z^4+4z^3+5z^2+4z+4$

$4z+4bz=4z$ gives $b=0$

$P(z)=(z+2)^2(z^2+1)$

$z=-2,\ \pm i$

(b) $P(z)=z^4+2z^3-2z^2-6z+5$

$P'(z)=4z^3+6z^2-4x^2-6$

The common factors of 5 and 6 are ±1.

$P'(1)=0,\ P'(-1)=-8$

Hence $z=1$ is the double real root.

$P(z)=(z-1)^2(z^2+bz+c)$

$c=5$

$P(z)=(z-1)^2(z^2+bz+5)$

$(z^2-2z+1)(z^2+bz+5)=z^4+2z^3-2z^2-6z+5$

$-10z+bz=-6z$ gives $b=4$

$P(z)=(z-1)^2(z^2+4z+5)$

$z=1,\ z=\dfrac{-4\pm\sqrt{16-20}}{2}=-2\pm i$

18 $z^2=1\pm2\sqrt{2}i=(\sqrt{2}\pm i)^2;\ z=\pm(\sqrt{2}+i),\ \pm(\sqrt{2}-i)$

EXERCISE 1.7

1 (a) $\sqrt{2i}$

$(x+iy)^2=2i$

$x^2-y^2+2xyi=0+2i$

$x^2-y^2=0,\ 2xy=2$

$(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=0+2^2=4$

$x^2+y^2=2$

$x^2-y^2=0$

$2x^2=2$

$x^2=1,\ y=\frac{1}{x}$

$x=1,\ y=1.\ x=-1,\ y=-1$

$\sqrt{2i}=1+i$ or $-1-i$

(b) $\sqrt{3+4i}$

$(x+iy)^2=3+4i$

$x^2-y^2+2xyi=3+4i$

$x^2-y^2=3,\ 2xy=4$

$(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=3^2+4^2=25$

$x^2+y^2=5$

$x^2-y^2=3$

$2x^2=8$

$x^2 = 4,\ y = \frac{2}{x}$
$x = 2, y = 1.\ x = -2, y = -1$
$\sqrt{3+4i} = 2+i$ or $-2-i$

(c) $\sqrt{5-12i}$
$(x+iy)^2 = 5-12i$
$x^2 - y^2 + 2xyi = 5 - 12i$
$x^2 - y^2 = 5,\ 2xy = -12$
$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = 5^2 + 12^2 = 169$
$x^2 + y^2 = 13$
$x^2 - y^2 = 5$
$2x^2 = 18$
$x^2 = 9,\ y = -\frac{6}{x}$
$x = 3, y = -2.\ x = -3, y = 2$
$\sqrt{5-12i} = 3-2i$ or $-3+2i$

(d) $\sqrt{-8+15i}$
$(x+iy)^2 = -8+15i$
$x^2 - y^2 + 2xyi = -8 + 15i$
$x^2 - y^2 = -8,\ 2xy = 15$
$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = 8^2 + 15^2 = 289$
$x^2 + y^2 = 17$
$x^2 - y^2 = -8$
$2x^2 = 9$
$x^2 = \frac{18}{4},\ y = \frac{15}{2x}$
$x = \frac{3\sqrt{2}}{2},\ y = \frac{5\sqrt{2}}{2}.\ x = -\frac{3\sqrt{2}}{2},\ y = -\frac{5\sqrt{2}}{2}$
$\sqrt{-8+15i} = \frac{3\sqrt{2}}{3} + \frac{5\sqrt{2}}{2}$ or $-\frac{3\sqrt{2}}{3} - \frac{5\sqrt{2}}{2}$

(e) $\sqrt{-3-4i}$
$(x+iy)^2 = -3-4i$
$x^2 - y^2 + 2xyi = -3 - 4i$
$x^2 - y^2 = -3,\ 2xy = -4$
$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = 3^2 + 14^2 = 25$
$x^2 + y^2 = 5$
$x^2 - y^2 = -3$
$2x^2 = 2$
$x^2 = 1,\ y = -\frac{2}{x}$
$x = 1, y = -2.\ x = -1, y = 2$
$\sqrt{-3-4i} = 1-2i$ or $-1+2i$

(f) $\sqrt{1+i}$
$(x+iy)^2 = 1+i$
$x^2 - y^2 + 2xyi = 1 + i$
$x^2 - y^2 = 1,\ 2xy = 1$
$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = 1^2 + 1^2 = 2$
$x^2 + y^2 = \sqrt{2}$
$x^2 - y^2 = 1$
$2x^2 = \sqrt{2} + 1$
$x^2 = \frac{2\left(\sqrt{2}+1\right)}{4},\ y = \frac{1}{2x}$
$x = \frac{\sqrt{2\left(\sqrt{2}+1\right)}}{2},\ y = \frac{1}{2} \times \frac{2}{\sqrt{2\left(\sqrt{2}+1\right)}} = \frac{\sqrt{2\left(\sqrt{2}-1\right)}}{2}.$
$x = -\frac{\sqrt{2\left(\sqrt{2}+1\right)}}{2},\ y = -\frac{\sqrt{2\left(\sqrt{2}-1\right)}}{2}.$

$\sqrt{1+i} = \frac{\sqrt{2\left(\sqrt{2}+1\right)}}{2} + \frac{\sqrt{2\left(\sqrt{2}-1\right)}}{2}i$
or $-\frac{\sqrt{2\left(\sqrt{2}+1\right)}}{2} - \frac{\sqrt{2\left(\sqrt{2}-1\right)}}{2}i$

2 (a) $x^2 + 2x + 2i = 0$
$x^2 + 2x + 1 = 1 - 2i$
$(x+1)^2 = 1 - 2i$
Let $x + 1 = \alpha + \beta i$
$(\alpha + \beta i)^2 = 1 - 2i$
$\alpha^2 - \beta^2 + 2\alpha\beta i = 1 - 2i$
$\alpha^2 - \beta^2 = 1,\ 2\alpha\beta = -2$
$\left(\alpha^2+\beta^2\right)^2 = \left(\alpha^2-\beta^2\right)^2 + \left(2\alpha\beta\right)^2 = 1^2 + (-2)^2 = 5$
$\alpha^2 + \beta^2 = \sqrt{5}$
$\alpha^2 - \beta^2 = 1$
$2\alpha^2 = \sqrt{5} + 1$
$\alpha^2 = \frac{2\left(\sqrt{5}+1\right)}{4},\ \beta = -\frac{1}{\alpha}$
$\alpha = \frac{\sqrt{2\left(\sqrt{5}+1\right)}}{2},\ \beta = -\frac{2}{\sqrt{2\left(\sqrt{5}+1\right)}} = -\frac{\sqrt{2\left(\sqrt{5}-1\right)}}{2}.$
$\alpha = -\frac{\sqrt{2\left(\sqrt{5}+1\right)}}{2},\ \beta = \frac{\sqrt{2\left(\sqrt{5}-1\right)}}{2}$
$x + 1 = \frac{\sqrt{2\left(\sqrt{5}+1\right)}}{2} - \frac{\sqrt{2\left(\sqrt{5}-1\right)}}{2}i,$
$x + 1 = -\frac{\sqrt{2\left(\sqrt{5}+1\right)}}{2} + \frac{\sqrt{2\left(\sqrt{5}-1\right)}}{2}i$
$x = \frac{\sqrt{2\left(\sqrt{5}+1\right)}}{2} - 1 - \frac{\sqrt{2\left(\sqrt{5}-1\right)}}{2}i$
$x = -\frac{\sqrt{2\left(\sqrt{5}+1\right)}}{2} - 1 + \frac{\sqrt{2\left(\sqrt{5}-1\right)}}{2}i$

(b) $x^2 - 4x + 2 - i = 0$
$x^2 - 4x + 4 = 4 - 2 + i$
$(x-2)^2 = 2 + i$
Let $x - 2 = \alpha + \beta i$
$(\alpha + \beta i)^2 = 2 + i$
$\alpha^2 - \beta^2 + 2\alpha\beta i = 2 + i$
$\alpha^2 - \beta^2 = 2,\ 2\alpha\beta = 1$
$\left(\alpha^2+\beta^2\right)^2 = \left(\alpha^2-\beta^2\right)^2 + \left(2\alpha\beta\right)^2 = 2^2 + 1^2 = 5$
$\alpha^2 + \beta^2 = \sqrt{5}$
$\alpha^2 - \beta^2 = 2$
$2\alpha^2 = \sqrt{5} + 2$
$\alpha^2 = \frac{2\left(\sqrt{5}+2\right)}{4},\ \beta = \frac{1}{2\alpha}$
$\alpha = \frac{\sqrt{2\left(\sqrt{5}+2\right)}}{2},\ \beta = \frac{1}{2\alpha} = \frac{1}{2} \times \frac{2}{\sqrt{2\left(\sqrt{5}-2\right)}} = \frac{\sqrt{2\left(\sqrt{5}-2\right)}}{2}.$
$\alpha = -\frac{\sqrt{2\left(\sqrt{5}-2\right)}}{2},\ \beta = -\frac{\sqrt{2\left(\sqrt{5}+2\right)}}{2}$
$x - 2 = \frac{\sqrt{2\left(\sqrt{5}-2\right)}}{2} + \frac{\sqrt{2\left(\sqrt{5}+2\right)}}{2}i,$
$x = 2 + \frac{\sqrt{2\left(\sqrt{5}-2\right)}}{2} + \frac{\sqrt{2\left(\sqrt{5}+2\right)}}{2}i$

$$x-2=-\frac{\sqrt{2\left(\sqrt{5}-2\right)}}{2}-\frac{\sqrt{2\left(\sqrt{5}+2\right)}}{2}i,$$

$$x=2-\frac{\sqrt{2\left(\sqrt{5}-2\right)}}{2}-\frac{\sqrt{2\left(\sqrt{5}+2\right)}}{2}i$$

(c) $x^2+2(2+i)x+3=0$

$x^2+2(2+i)x+(2+i)^2=(2+i)^2-3$

$(x+2+i)^2=4i$

Let $x+2+i=\alpha+\beta i$

$(\alpha+\beta i)^2=4i$

$\alpha^2-\beta^2+2\alpha\beta i=4i$

$\alpha^2-\beta^2=0,\ 2\alpha\beta=4$

$$\left(\alpha^2+\beta^2\right)^2=\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2=0^2+4^2=16$$

$\alpha^2+\beta^2=4$

$\alpha^2-\beta^2=0$

$2\alpha^2=4$

$\alpha^2=\sqrt{2},\ \beta=\dfrac{2}{\alpha}$

$\alpha=\sqrt{2},\ \beta=\dfrac{2}{\sqrt{2}}=\sqrt{2}.\ \ \alpha=-\sqrt{2},\ \beta=-\sqrt{2}.$

$x+2+i=\sqrt{2}+\sqrt{2}i,$

$x=\left(\sqrt{2}-2\right)+\left(\sqrt{2}-1\right)i$

$x+2+i=-\sqrt{2}-\sqrt{2}i,$

$x=-\left(\sqrt{2}+2\right)-\left(\sqrt{2}+1\right)i$

(d) $x^2+(3+4i)x-4=0$

$$x^2+(3-4i)x+\left(\frac{3-4i}{2}\right)^2=\left(\frac{3-4i}{2}\right)^2+4$$

$$\left(x+\frac{3-4i}{2}\right)^2=\frac{9-24i}{4}$$

Let $x+\dfrac{3-4i}{2}=\alpha+\beta i$

$(\alpha+\beta i)^2=\dfrac{9-24i}{4}$

$\alpha^2-\beta^2+2\alpha\beta i=\dfrac{9}{4}-6i$

$\alpha^2-\beta^2=\dfrac{9}{4},\ 2\alpha\beta=6$

$$\left(\alpha^2+\beta^2\right)^2=\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2=\left(\frac{9}{4}\right)^2+6^2=\frac{657}{16}=\frac{9\times 73}{16}$$

$\alpha^2+\beta^2=\dfrac{3\sqrt{73}}{4}$

$\alpha^2-\beta^2=\dfrac{9}{4}$

$2\alpha^2=\dfrac{3\sqrt{73}+9}{4}$

$\alpha^2=\dfrac{6\left(\sqrt{73}+3\right)}{16},\ \beta=\dfrac{3}{\alpha}$

$$\alpha=\frac{\sqrt{6\left(\sqrt{73}+3\right)}}{4},\ \beta=\frac{3\times 4}{\sqrt{6\left(\sqrt{73}+3\right)}}=\frac{\sqrt{6\left(\sqrt{73}-3\right)}}{4}.$$

$$\alpha=-\frac{\sqrt{6\left(\sqrt{73}+3\right)}}{4},\ \beta=-\frac{\sqrt{6\left(\sqrt{73}-3\right)}}{4}$$

$$x+\frac{3-4i}{2}=\frac{\sqrt{6\left(\sqrt{73}+3\right)}}{4}+\frac{\sqrt{6\left(\sqrt{73}-3\right)}}{4}i,$$

$$x+\frac{3-4i}{2}=-\frac{\sqrt{6\left(\sqrt{73}+3\right)}}{4}-\frac{\sqrt{6\left(\sqrt{73}-3\right)}}{4}i$$

$$x=\frac{\sqrt{6\left(\sqrt{73}+3\right)}-6}{4}+\frac{\sqrt{6\left(\sqrt{73}-3\right)}+8}{4}i$$

$$x=\frac{-6-\sqrt{6\left(\sqrt{73}+3\right)}}{4}+\frac{8-\sqrt{6\left(\sqrt{73}-3\right)}}{4}i$$

(e) $x^2+2(1-3i)x+2+i=0$

$x^2+2(1-3i)x+(1-3i)^2=(1-3i)^2-2-i$

$(x+1-3i)^2=-10-7i$

Let $x+1-3i=\alpha+\beta i$

$(\alpha+\beta i)^2=-10-7i$

$\alpha^2-\beta^2+2\alpha\beta i=-10-7i$

$\alpha^2-\beta^2=-10,\ 2\alpha\beta=-7$

$$\left(\alpha^2+\beta^2\right)^2=\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2=10^2+7^2=149$$

$\alpha^2+\beta^2=\sqrt{149}$

$\alpha^2-\beta^2=-10$

$2\alpha^2=\sqrt{149}-10$

$\alpha^2=\dfrac{2\left(\sqrt{149}-10\right)}{4},\ \beta=\dfrac{-7}{2\alpha}$

$$\alpha=\frac{\sqrt{2\left(\sqrt{149}-10\right)}}{2},\ \beta=\frac{-7}{2}\times\frac{2}{\sqrt{2\left(\sqrt{149}-10\right)}}=\frac{-\sqrt{2\left(\sqrt{149}+10\right)}}{2}.$$

$$\alpha=\frac{-\sqrt{2\left(\sqrt{149}-10\right)}}{2},\ \beta=\frac{\sqrt{2\left(\sqrt{149}+10\right)}}{2}$$

$$x+1-3i=\frac{\sqrt{2\left(\sqrt{149}-10\right)}}{2}-\frac{\sqrt{2\left(\sqrt{149}+10\right)}}{2}i$$

$$x=\frac{\sqrt{2\left(\sqrt{149}-10\right)}-2}{2}-\frac{\sqrt{2\left(\sqrt{149}+10\right)}-3}{2}i,$$

$$x+1-3i=\frac{-\sqrt{2\left(\sqrt{149}-10\right)}}{2}+\frac{\sqrt{2\left(\sqrt{149}+10\right)}}{2}i$$

$$x=\frac{-\sqrt{2\left(\sqrt{149}-10\right)}-2}{2}+\frac{\sqrt{2\left(\sqrt{149}+10\right)}+6}{2}i$$

(f) $ix^2-4x+3=0$

$x^2+4ix-3i=0$

$x^2+4ix+(2i)^2=(2i)^2+3i$

$(x+2i)^2=-4+3i$

Let $x+2i=\alpha+\beta i$

$(\alpha+\beta i)^2=-4+3i$

$\alpha^2-\beta^2+2\alpha\beta i=-4+3i$

$\alpha^2-\beta^2=-4,\ 2\alpha\beta=3$

$$\left(\alpha^2+\beta^2\right)^2=\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2=4^2+3^2=25$$

$\alpha^2+\beta^2=5$

$\alpha^2-\beta^2=-4$

$2\alpha^2=1$

$\alpha^2=\dfrac{2}{4},\ \beta=\dfrac{3}{2\alpha}$

$\alpha=\dfrac{\sqrt{2}}{2},\ \beta=\dfrac{3\sqrt{2}}{2}.\ \ \alpha=-\dfrac{\sqrt{2}}{2},\ \beta=-\dfrac{3\sqrt{2}}{2}.$

$$x+2i=\frac{\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i,\ x=\frac{\sqrt{2}}{2}+\frac{3\sqrt{2}-4}{2}i$$

$$x+2i=-\frac{\sqrt{2}}{2}-\frac{3\sqrt{2}}{2}i,\ x=-\frac{\sqrt{2}}{2}-\frac{3\sqrt{2}+4}{2}i$$

(g) $ix^2+2ix+3=0$

$x^2+2x-3i=0$

$x^2+2x+1=1+3i$

$(x+1)^2=1+3i$

Let $x+1=\alpha+\beta i$

$(\alpha+\beta i)^2=1+3i$

$\alpha^2-\beta^2+2\alpha\beta i=1+3i$

$\alpha^2-\beta^2=1,\ 2\alpha\beta=3$

$$\left(\alpha^2+\beta^2\right)^2=\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2=1^2+3^2=10$$

$\alpha^2+\beta^2=\sqrt{10}$

$\alpha^2-\beta^2=1$

$2\alpha^2=\sqrt{10}+1$

$$\alpha^2=\frac{2\left(\sqrt{10}+1\right)}{4},\ \beta=\frac{3}{2\alpha}$$

$$\alpha=\frac{\sqrt{2\left(\sqrt{10}+1\right)}}{2},\ \beta=\frac{3}{2}\times\frac{2}{\sqrt{2\left(\sqrt{10}+1\right)}}=\frac{\sqrt{2\left(\sqrt{10}-1\right)}}{2}.$$

$$\alpha=\frac{-\sqrt{2\left(\sqrt{10}+1\right)}}{2},\ \beta=\frac{-\sqrt{2\left(\sqrt{10}-1\right)}}{2}.$$

$$x+1=\frac{\sqrt{2\left(\sqrt{10}+1\right)}}{2}+\frac{\sqrt{2\left(\sqrt{10}-1\right)}}{2}i,$$

$$x=\frac{\sqrt{2\left(\sqrt{10}+1\right)}-2}{2}+\frac{\sqrt{2\left(\sqrt{10}-1\right)}}{2}i$$

$$x+1=\frac{-\sqrt{2\left(\sqrt{10}+1\right)}}{2}-\frac{\sqrt{2\left(\sqrt{10}-1\right)}}{2}i,$$

$$x=-\frac{\sqrt{2\left(\sqrt{10}+1\right)}+2}{2}-\frac{\sqrt{2\left(\sqrt{10}-1\right)}}{2}i$$

(h) $(2-i)x^2+2x+1=0$

$5x^2+2(2+i)x+2=0$

$$x^2+\frac{2(2+i)}{5}x+\frac{2+i}{5}=0$$

$$x^2+\frac{2(2+i)}{5}x+\left(\frac{2+i}{5}\right)^2=\left(\frac{2+i}{5}\right)^2-\frac{2+i}{5}$$

$$\left(x+\left(\frac{2+i}{5}\right)\right)^2=-\frac{7+i}{25}$$

Let $x+\frac{2+i}{5}=\alpha+i\beta$

$$(\alpha+i\beta)^2=-\frac{7}{25}-\frac{1}{25}$$

$$\alpha^2-\beta^2+2\alpha\beta i=-\frac{7}{25}-\frac{1}{25}i$$

$$\alpha^2-\beta^2=-\frac{7}{25},\ 2\alpha\beta=-\frac{1}{25}$$

$$\left(\alpha^2+\beta^2\right)^2=\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2=\left(\frac{7}{25}\right)^2+\left(\frac{1}{25}\right)^2=\frac{2}{25}$$

$$\alpha^2+\beta^2=\frac{\sqrt{2}}{5}$$

$$\alpha^2-\beta^2=\frac{7}{25}$$

$$2\alpha^2=\frac{5\sqrt{2}-7}{25}$$

$$\alpha^2=\frac{2\left(5\sqrt{2}-7\right)}{100}$$

$$\alpha=\frac{\sqrt{2\left(5\sqrt{2}-7\right)}}{10}$$

$$\beta=\frac{-\sqrt{2\left(5\sqrt{2}+7\right)}}{10}$$

$$\alpha=\frac{-\sqrt{2\left(5\sqrt{2}-7\right)}}{10},\ \beta=\frac{\sqrt{2\left(5\sqrt{2}+7\right)}}{10}$$

$$x+\frac{(2+i)}{5}=\frac{\sqrt{2\left(5\sqrt{2}-7\right)}}{10}-\frac{\sqrt{2\left(5\sqrt{2}+7\right)}}{10}i$$

$$x=\frac{\sqrt{2\left(5\sqrt{2}-7\right)}-4}{10}-\frac{\sqrt{2\left(5\sqrt{2}+7\right)}+2}{10}i$$

$$x+\frac{(2+i)}{5}=\frac{-\sqrt{2\left(5\sqrt{2}-7\right)}}{10}+\frac{\sqrt{2\left(5\sqrt{2}+7\right)}}{10}i$$

$$x=\frac{\sqrt{2\left(5\sqrt{2}-7\right)}+4}{10}+\frac{\sqrt{2\left(5\sqrt{2}+7\right)}-2}{10}i$$

3 (a) Expression $(x-3)(x-1-i)(x-1+i)$

$=(x-3)(x^2-2x+2)$

$=x^3-5x^2+8x-6$

(b) $x=3,\ 1+i,\ 1-i$

EXERCISE 1.8

1 D

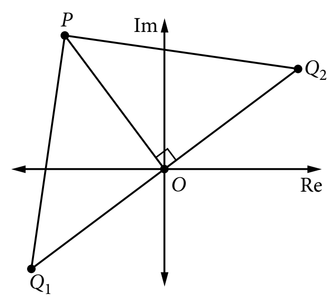
2 (a) For Q_1: $w=(-3+4i)\times i$

$=-4-3i$

For Q_2: $\overrightarrow{OQ_2}=\overrightarrow{Q_1O}$

$w=-(-4-3i)$

$=4+3i$



(b) For Q_1: $\overrightarrow{PO}$ represents $3-4i$

$\overrightarrow{PQ_1}=\overrightarrow{PO}$ rotated anticlockwise $\frac{\pi}{2}$

$\therefore\ \overrightarrow{PQ_1}$ represents $(3-4i)\times i=4+3i$

Then $\overrightarrow{OQ_1}=\overrightarrow{OP}+\overrightarrow{PQ_1}=-3+4i+4+3i=1+7i$

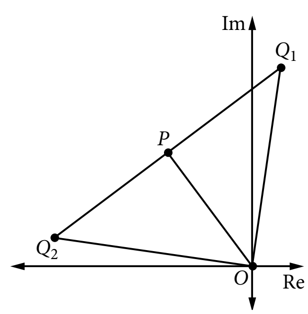
i.e. $w=1+7i$

For Q_2: $\overrightarrow{PQ_2}=\overrightarrow{PO}$ rotated anticlockwise $\frac{3\pi}{2}$

$\therefore\overrightarrow{PQ_2}$ represents $(3-4i)\times i^3=-4-3i$

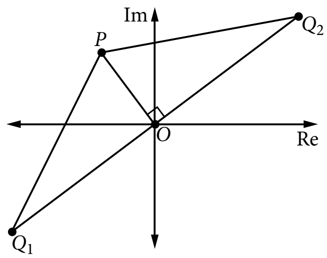
Then $\overrightarrow{OQ_2}=\overrightarrow{OP}+\overrightarrow{PQ_2}=-3+4i+(-4-3i)=-7+i$

i.e. $w=-7+i$



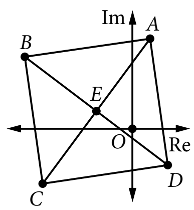
(c) For Q_1: $w = (-3+4i)\times 2i$
$= -8-6i$

For Q_2: $\overrightarrow{OQ_2} = \overrightarrow{Q_1O}$
$w = -(-8-6i)$
$= 8+6i$



3 For B: $\overrightarrow{EB} = \overrightarrow{EA}$ rotated anticlockwise $\frac{\pi}{2}$
$= i(3+4i) = -4+3i$

Then $\overrightarrow{OB} = \overrightarrow{OE} + \overrightarrow{EB} = -2+i+-4+3i = -6+4i$
i.e. B represents $-6+4i$.
For C: $\overrightarrow{EC} = \overrightarrow{AE} = -3-4i$
Then $\overrightarrow{OC} = \overrightarrow{OE} + \overrightarrow{EC}$
$= -2+i+-3-4i = -5-3i$
i.e. C represents $-5-3i$.
For D: $\overrightarrow{ED} = \overrightarrow{BE} = 4-3i$
Then $\overrightarrow{OD} = \overrightarrow{OE} + \overrightarrow{ED}$
$= -2+i+4-3i = 2-2i$
i.e. D represents $2-2i$.



4 (a) $|z_1| = 10$ and $|z_2| = 15$ $\therefore |z_1+z_2| \le |z_1| + |z_2| \le 25$

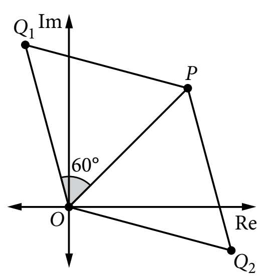
(b) $|z_1+z_2| = 25$ when $z_2 = kz_1$ (real k) $\therefore |z_2| = k\times|z_1|$,
$15 = 10k$, $k = 1.5$
$\therefore z_2 = 1.5(6+8i) = 9+12i$

5 For Q_1: $\overrightarrow{OQ_1} = \overrightarrow{OP}$ rotated anticlockwise $\frac{\pi}{3}$

$= (1+i)\times \text{cis}\frac{\pi}{3} = (1+i)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i$

For Q_2: $\overrightarrow{OQ_2} = \overrightarrow{OP}$ rotated anticlockwise $\frac{5\pi}{3}$

$= (1+i)\times \text{cis}\frac{5\pi}{3} = (1+i)\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i$



6 (a) $OA = |z_1| = 2$, $OB = |z_2| = 2$
$\therefore$ $OACB$ is a rhombus (parallelogram with adjacent sides equal)

(b) $z_3 = z_1 + z_2 = \sqrt{3}+i+(-\sqrt{3}+i) = 2i$

(c) $OC = 2$ $\therefore$ equilateral triangle $\therefore \theta = \frac{\pi}{3}$

7 (a) C is the midpoint of AB

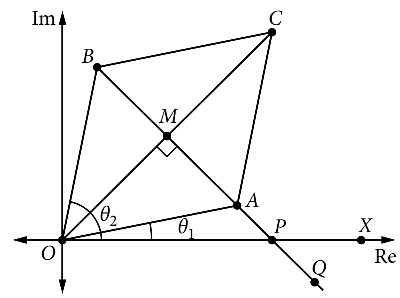
$\therefore \alpha = \frac{1}{2}(wz+\overline{w}z) = \frac{1}{2}(w+\overline{w})z = \frac{1}{2}\times 2\text{Re}(w)\times z$

$= \cos\frac{3\pi}{4}\times z = -\frac{1}{\sqrt{2}}z$

(b) Diagonals of a parallelogram bisect each other, so C is also the midpoint of OD.

$\therefore \overrightarrow{OD} = 2\times\overrightarrow{OC} = 2\times -\frac{1}{\sqrt{2}}z = -\sqrt{2}z$ i.e. D represents $-\sqrt{2}z$

8 $|z_1| = |z_2|$ $\therefore OA = OB$ $\therefore OACB$ is a rhombus.
$\angle AOB = \theta_2 - \theta_1$ $\therefore \angle AOC = \frac{1}{2}(\theta_2-\theta_1)$
$\arg(z_1+z_2) = \angle XOC = \angle XOA + \angle AOC$
$= \theta_1 + \frac{1}{2}(\theta_2-\theta_1) = \frac{1}{2}(\theta_1+\theta_2)$
$\arg(z_1-z_2) = -(\angle XPQ) = -(\angle APO)$
$= -(\pi - (\frac{\pi}{2} + \angle XOM))$ (angle sum of ΔOPM)
$\therefore \arg(z_1-z_2) = \frac{\theta_1+\theta_2-\pi}{2}$



9 (a) $\overrightarrow{OQ} = \overrightarrow{OP}$ rotated anticlockwise $\frac{\pi}{2}$
$\therefore w = z\times i$
$w^2 = -z^2$
$w^2 + z^2 = 0$

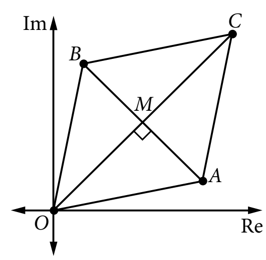
(b) R is $z+w$. E is midpoint of OR.
$\therefore$ E represents $\frac{1}{2}(z+w) = \frac{1}{2}(1+i)z$

10 (a) $\overrightarrow{OE} = \overrightarrow{AB} = w_2 - w_1$ $\therefore$ E is $w_2 - w_1$

(b) $\overrightarrow{OF} = \overrightarrow{OE}$ rotated anticlockwise $\frac{\pi}{2}$
$\therefore$ F is $(w_2-w_1)\times i$

(c) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \overrightarrow{OF} = w_1 + (w_2-w_1)i$

11 $\frac{z_1+z_2}{z_1-z_2} = 2i$ $\therefore OC = 2\times AB$ and $OC \perp AB$



(a) $OACB$ is a rhombus (diagonals perpendicular)
$\therefore OA = OB$ $\therefore |z_1| = |z_2|$

(b) $\angle AOC = \frac{\alpha}{2}$ (diagonals of rhombus bisect angles)
In ΔAOM: $\tan\frac{\alpha}{2} = \frac{AM}{OM} = \frac{1}{2}$

(c) $z_1 + z_2 = 2i(z_1 - z_2)$
$z_2 + 2iz_2 = 2iz_1 - z_1$
$z_2(1+2i) = z_1(-1+2i)$
$z_2 = \frac{-1+2i}{1+2i}z_1 = \frac{1}{5}(3+4i)z_1$

12 (a) $|z| = 1$, $\arg z = \frac{\pi}{2}$, $|w| = 1$, $\arg w = \frac{\pi}{4}$

(b) rhombus (equal sides $|z| = |w|$)

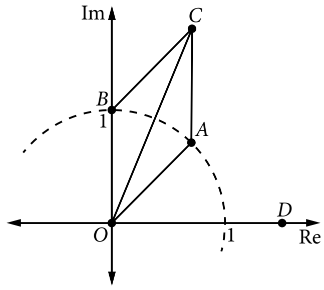
(c) $\arg(z+w) = \frac{3\pi}{8}$ (diagonals of rhombus bisect angles)

$z + w = \frac{\sqrt{2}}{2} + \frac{(2+\sqrt{2})}{2}i$

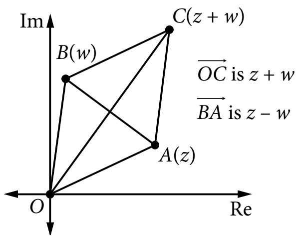
But $z+w = r\,\text{cis}\frac{3\pi}{8}$, $OC = r$

$\therefore\ r\cos\frac{3\pi}{8} = \frac{\sqrt{2}}{2}$ [1] and $r\sin\frac{3\pi}{8} = \frac{(2+\sqrt{2})}{2}$ [2]

[2] ÷ [1]: $\tan\frac{3\pi}{8} = \sqrt{2}+1$



13 (a) Values of z and w may vary. Example:



(b) (i) $|z| = |w|$ $\therefore\ OA = OB$ $\therefore$ rhombus

$\therefore$ diagonals are perpendicular $\therefore\ \frac{z+w}{z-w} = ki$

(ii) $|z+w| = |z-w|$ $\therefore\ OC = BA$

$\therefore$ rectangle (parallelogram with equal diagonals)

$\therefore\ \frac{z}{w} = ki$

14 (a) $w_2 - w_1 = i(w_3 - w_1)$

$\therefore\ |w_2 - w_1| = |i(w_3 - w_1)| = |w_3 - w_1|$

$\therefore$ two sides equal, also perpendicular (multiplication by i)

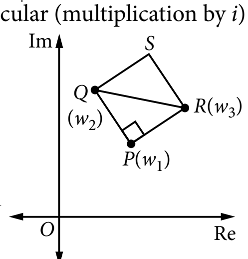
$\therefore$ right-angled isosceles triangle

(b) $\overrightarrow{PR} = w_3 - w_1$, $\overrightarrow{PQ} = w_2 - w_1$,

$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{PQ} = w_3 + w_2 - 2w_1$

(c) $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS}$

$= w_1 + w_3 + w_2 - 2w_1 = w_3 + w_2 - w_1$



15 (a) $z_1 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, $z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(b) rhombus (parallelogram with adjacent sides equal)

(c) $\arg(z_1 + z_2) = \frac{1}{2}\left(\frac{3\pi}{4} + \frac{\pi}{6}\right) = \frac{11\pi}{24}$

(d) $z_1 + z_2 = (\sqrt{3} - \sqrt{2}) + (\sqrt{2} + 1)i$ in Cartesian form

$= r\left(\cos\frac{11\pi}{24} + i\sin\frac{11\pi}{24}\right)$ in mod–arg form ($r = OR$)

So $r\cos\frac{11\pi}{24} = \sqrt{3} - \sqrt{2}$ [1]

and $r\sin\frac{11\pi}{24} = \sqrt{2} + 1$ [2]

[2] ÷ [1] gives the result.

16 (a) LHS $= |z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)}$ (using the property $z\overline{z} = |z|^2$)

$= (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) + (z_1 - z_2)(\overline{z}_1 - \overline{z}_2)$ (The conjugate of a sum or difference is the sum or difference of conjugates.)

$= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2 + z_1\overline{z}_1 - z_1\overline{z}_2 - z_2\overline{z}_1 + z_2\overline{z}_2$

$= 2(z_1\overline{z}_1 + z_2\overline{z}_2)$

$= 2(|z_1|^2 + |z_2|^2) =$ RHS

(b) In a parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of the (four) sides.

17 (a) midpoint of AC

(b) Parallelogram. $\frac{1}{2}(\beta + \delta)$ is midpoint of BD;

when $\alpha + \gamma = \beta + \delta$, the midpoints are the same point, i.e. the diagonals bisect each other $\therefore$ parallelogram.

18 $z_1 - z_2 + z_3 - z_4 = 0$ i.e. $z_1 + z_3 = z_2 + z_4$ $\therefore$ parallelogram

$z_1 - iz_2 - z_3 + iz_4 = 0$ i.e. $z_1 - z_3 = i(z_2 - z_4)$ i.e. diagonal AC is equal to and perpendicular to diagonal BD $\therefore\ ABCD$ is a square.

19 Let $z_1 = r_1\,\text{cis}\,\theta_1$, $z_2 = r_2\,\text{cis}\,\theta_2$, $z_3 = r_3\,\text{cis}\,\theta_3$

Then $z_1z_2 = r_1r_2\,\text{cis}(\theta_1 + \theta_2)$ and $z_3^{\,2} = r_3^{\,2}\,\text{cis}\,2\theta_3$

If $z_1z_2 = z_3^{\,2}$ then $r_1r_2 = r_3^{\,2}$, i.e. $\frac{r_1}{r_3} = \frac{r_3}{r_2}$ $\therefore$ geometric series

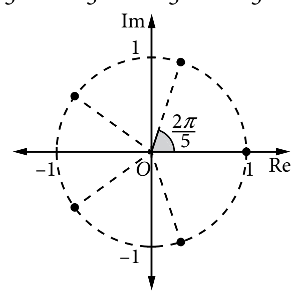
Also: $\theta_1 + \theta_2 = 2\theta_3$, i.e. $\arg z_3$ is the average of $\arg z_1$ and $\arg z_2$

$\therefore\ OZ_3$ bisects $\angle Z_1OZ_2$.

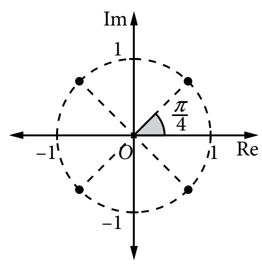
EXERCISE 1.9

1 B

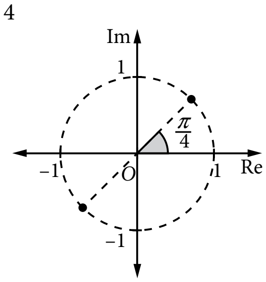
2 (a) $\text{cis}\,0$, $\text{cis}\frac{2\pi}{5}$, $\text{cis}\frac{-2\pi}{5}$, $\text{cis}\frac{4\pi}{5}$, $\text{cis}\frac{-4\pi}{5}$



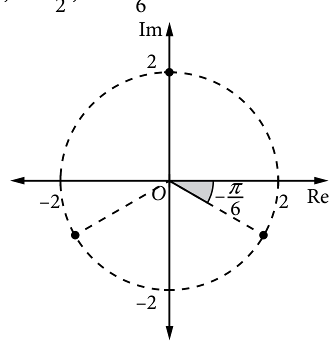
(b) $\text{cis}\frac{\pi}{4}$, $\text{cis}\frac{-\pi}{4}$, $\text{cis}\frac{3\pi}{4}$, $\text{cis}\frac{-3\pi}{4}$



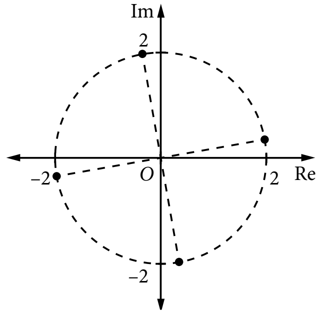
(c) $\text{cis}\frac{\pi}{4}$, $\text{cis}\frac{-3\pi}{4}$



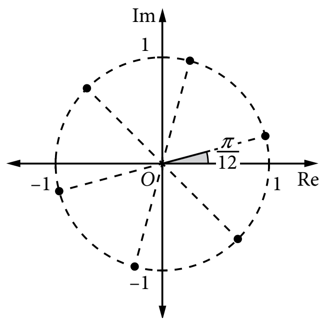
(d) $2\,\text{cis}\frac{-\pi}{6}$, $2\,\text{cis}\frac{\pi}{2}$, $2\,\text{cis}\frac{-5\pi}{6}$



(e) $2\operatorname{cis}\frac{\pi}{24}, 2\operatorname{cis}\frac{13\pi}{24}, 2\operatorname{cis}\frac{-23\pi}{24}, 2\operatorname{cis}\frac{-11\pi}{24}$



(f) $\operatorname{cis}\frac{\pi}{12}, \operatorname{cis}\frac{5\pi}{12}, \operatorname{cis}\frac{3\pi}{4}, \operatorname{cis}\frac{-11\pi}{12}, \operatorname{cis}\frac{-7\pi}{12}, \operatorname{cis}\frac{-\pi}{4}$



3 (a) $2, 2\operatorname{cis}\frac{2\pi}{5}, 2\operatorname{cis}\frac{-2\pi}{5}, 2\operatorname{cis}\frac{4\pi}{5}, 2\operatorname{cis}\frac{-4\pi}{5}$

(b) $2\operatorname{cis}\frac{\pi}{4}, 2\operatorname{cis}\frac{-\pi}{4}, 2\operatorname{cis}\frac{3\pi}{4}, 2\operatorname{cis}\frac{-3\pi}{4}$

(c) $-1, \operatorname{cis}\frac{3\pi}{5}, \operatorname{cis}\frac{-3\pi}{5}, \operatorname{cis}\frac{\pi}{5}, \operatorname{cis}\frac{-\pi}{5}$

(d) $\sqrt{3}\operatorname{cis}\frac{\pi}{18}, \sqrt{3}\operatorname{cis}\frac{7\pi}{18}, \sqrt{3}\operatorname{cis}\frac{13\pi}{18}, \sqrt{3}\operatorname{cis}\frac{-5\pi}{18},$

$\sqrt{3}\operatorname{cis}\frac{-11\pi}{18}, \sqrt{3}\operatorname{cis}\frac{-17\pi}{18}$

4 (a) $-2, 1-\sqrt{3}i$ (b) $z^3=-8$

5 (a) $w_1=\operatorname{cis}\frac{2\pi}{3}, w_2=\operatorname{cis}\frac{-2\pi}{3} \quad \therefore w_1=\overline{w_2}$

Also: $(w_2)^2=\operatorname{cis}\frac{-4\pi}{3}=\operatorname{cis}\frac{2\pi}{3}=w_1$

(b) Roots of $z^3-1=0$ are $1, w_1, w_2$

Sum of roots $=-\frac{b}{a}$: $1+w_1+w_2=0 \quad \therefore w_1+w_2=-1$

(c) Product of roots $=-\frac{d}{a}$: $w_1w_2=1$

6 (a) $w^3-1=0$

$(w-1)(w^2+w+1)=0$

$\therefore w^2+w+1=0$ as $w\neq 1$ (w is non-real)

(b) $(1-w)(1-w^2)=1-w-w^2+w^3$

$=1-(w+w^2)+1$

$=1-(-1)+1=3$

(c) -1 (d) 3 (e) 1 (f) 27

7 (a) w is a root, so $w^5=1$

$\therefore (w^2)^5=(w^5)^2=1$, i.e. w^2 is also a root of $z^5=1$

Similarly: $(w^{-1})^5=(w^5)^{-1}=1$ and $(w^{-2})^5=(w^5)^{-2}=1$

i.e. w^{-1} and w^{-2} are also roots of $z^5=1$

(b) Sum of roots $=-\frac{b}{a} \quad \therefore w^2+w+1+w^{-1}+w^{-2}=0$

(c) Let $w=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$:

$$\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+1+\cos\frac{-2\pi}{5}+i\sin\frac{-2\pi}{5}+\cos\frac{-4\pi}{5}+i\sin\frac{-4\pi}{5}=0$$

But $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$

$$\therefore \cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+1+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}+\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}=0$$

$\therefore \cos\frac{4\pi}{5}+\cos\frac{2\pi}{5}=-\frac{1}{2}$ (equating real parts)

But $\cos\frac{4\pi}{5}=-\cos\frac{\pi}{5} \quad \therefore \cos\frac{2\pi}{5}-\cos\frac{\pi}{5}=-\frac{1}{2}$

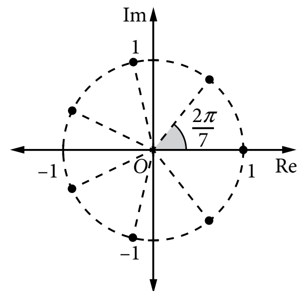
(d) Using $\cos 2\theta=2\cos^2\theta-1$: $4\cos^2\frac{\pi}{5}-2\cos\frac{\pi}{5}-1=0$

$\therefore \cos\frac{\pi}{5}=\frac{2\pm\sqrt{20}}{8}=\frac{1\pm\sqrt{5}}{4}$

But $\frac{\pi}{5}$ is an acute angle, so it must be the positive option:

$\therefore \cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}$

8 (a) $1, \operatorname{cis}\frac{2\pi}{7}, \operatorname{cis}\frac{-2\pi}{7}, \operatorname{cis}\frac{4\pi}{7}, \operatorname{cis}\frac{-4\pi}{7}, \operatorname{cis}\frac{6\pi}{7}, \operatorname{cis}\frac{-6\pi}{7}$



(b) $w^7-1=0$

$(w-1)(w^6+w^5+w^4+w^3+w^2+w+1)=0$

$w\neq 1$ as w is non-real

$\therefore w^6+w^5+w^4+w^3+w^2+w+1=0$

(c) Let $\alpha=w+w^2+w^4$ and $\beta=w^3+w^5+w^6$

$\therefore \alpha+\beta=w^6+w^5+w^4+w^3+w^2+w=-1$

$\alpha\beta=(w+w^2+w^4)(w^3+w^5+w^6)=2$

Equation with roots α and β is $z^2-(\alpha+\beta)z+\alpha\beta=0$

$\therefore z^2+z+2=0$

(d) Sum of roots $=-\frac{b}{a}$:

$1+\operatorname{cis}\frac{2\pi}{7}+\operatorname{cis}\frac{-2\pi}{7}+\operatorname{cis}\frac{4\pi}{7}+\operatorname{cis}\frac{-4\pi}{7}+\operatorname{cis}\frac{6\pi}{7}+\operatorname{cis}\frac{-6\pi}{7}=0$

$\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$

$\therefore 1+2\cos\frac{2\pi}{7}+2\cos\frac{4\pi}{7}+2\cos\frac{6\pi}{7}=0$ (equating real parts)

But $\cos\frac{6\pi}{7}=-\cos\frac{\pi}{7} \quad \therefore \cos\frac{\pi}{7}=\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\frac{1}{2}$

9 (a) $w^6-1=0$

$(w^2-1)(w^4+w^2+1)=0$ (difference of two cubes)

$\therefore w^2-1=0$ or $w^4+w^2+1=0$

But w is non-real $\therefore w^4+w^2+1=0$

(b) LHS $=w^{-2}\times w^6=w^4$ (as $w^6=1$)

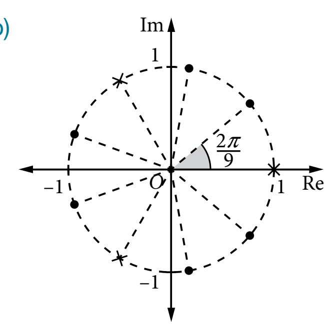
RHS $=1+w^2+w^4+w^4=0+w^4=$ LHS

10 (a) $z^9-1=(z^3-1)(z^6+z^3+1)$ (difference of two cubes)

$\therefore$ roots of $z^9-1=0$ include

roots of $z^3-1=0$, and roots of $z^6+z^3+1=0$

(b)



(c) Roots of $z^6+z^3+1=0$ are

$\operatorname{cis}\frac{2\pi}{9}, \operatorname{cis}\frac{-2\pi}{9}, \operatorname{cis}\frac{4\pi}{9}, \operatorname{cis}\frac{-4\pi}{9}, \operatorname{cis}\frac{8\pi}{9}, \operatorname{cis}\frac{-8\pi}{9}.$

As $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$:

Sum of roots $=2\left(\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}\right)$

(d) Sum of roots $= -\frac{b}{a} = 0$

$\therefore \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$

$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$ (as $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$)

(e) The result follows from using the expansion $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\times\text{Re}(\alpha)\times z + |\alpha|^2$ with each of the three conjugate pairs.

(f) If expanded there are 27 terms, although many are like terms in z^2 or z. LHS term in z^2 has coefficient 0. Examining RHS terms in z^2 and result of part (e):

$4\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} - 4\cos\frac{4\pi}{9}\cos\frac{\pi}{9} - 4\cos\frac{\pi}{9}\cos\frac{2\pi}{9} + 3 = 0$

$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} - \cos\frac{4\pi}{9}\cos\frac{\pi}{9} - \cos\frac{\pi}{9}\cos\frac{2\pi}{9} = -\frac{3}{4}$

11 (a) $z^5 - 1 = 0 \quad \therefore (z-1)(z^4+z^3+z^2+z+1) = 0$

$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a root of $z^5 - 1 = 0$,

hence also a root of $z^4+z^3+z^2+z+1 = 0$.

(b) The roots are the non-real roots of $z^5 - 1 = 0$

i.e. $\text{cis}\frac{2\pi}{5}$, $\text{cis}\frac{-2\pi}{5}$, $\text{cis}\frac{4\pi}{5}$, $\text{cis}\frac{-4\pi}{5}$

(c) Sum of roots $= -\frac{b}{a}$:

$\text{cis}\frac{2\pi}{5} + \text{cis}\frac{-2\pi}{5} + \text{cis}\frac{4\pi}{5} + \text{cis}\frac{-4\pi}{5} = -1$

But $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$:

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

(d) $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

$\cos\frac{2\pi}{5} + 2\cos^2\frac{2\pi}{5} - 1 = -\frac{1}{2}$

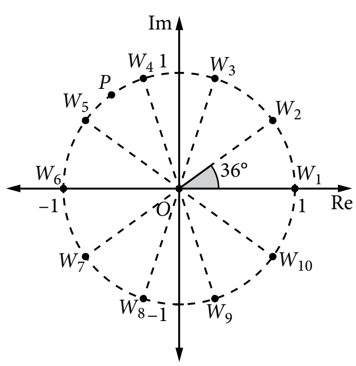
$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$

$\therefore \cos\frac{2\pi}{5} = \frac{-2\pm\sqrt{20}}{8} = \frac{-1\pm\sqrt{5}}{4}$

But $\frac{2\pi}{5}$ is an acute angle, so it must be the positive option:

$\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$

12 (a)



(b) $w_1, w_2, \ldots w_{10}$ are roots of $z^{10} - 1 = 0$

Coefficients of the equation are real, so roots occur as conjugate pairs.

Sum of roots $= -\frac{b}{a} = 0 \quad \therefore \sum_{i=1}^{10} w_i = \sum_{i=1}^{10} \overline{w_i} = 0$

(c) Distance $PW_i = |z - w_i|$

$\therefore [PW_i]^2 = |z-w_i|^2 = (z-w_i)\times\overline{(z-w_i)}$

$= (z-w_i)\times(\bar{z}-\overline{w_i}) = z\bar{z} - z\overline{w_i} - \bar{z}w_i + w_i\overline{w_i}$

$= 2 - z\overline{w_i} - \bar{z}w_i$

(as $z\bar{z} = |z|^2 = 1$ and $w\bar{w} = |w|^2 = 1$)

(d) $\sum_{i=1}^{10}[PW_i]^2 = \sum_{i=1}^{10}(2 - z\overline{w_i} - \bar{z}w_i)$

$= \sum_{i=1}^{10} 2 - \sum_{i=1}^{10} z\overline{w_i} - \sum_{i=1}^{10}\bar{z}w_i$

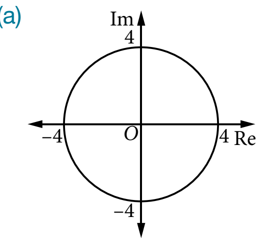
$= 20 - z\sum_{i=1}^{10}\overline{w_i} - \bar{z}\sum_{i=1}^{10} w_i$

$= 20 - z\times 0 - \bar{z}\times 0 = 20$

EXERCISE 1.10

1 B **2** C

3 (a)



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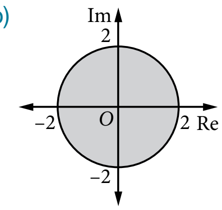

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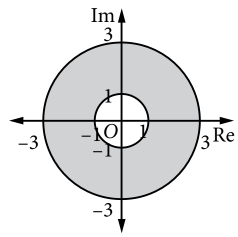

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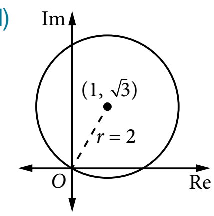

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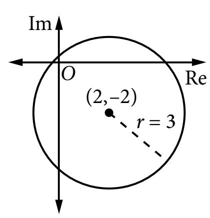

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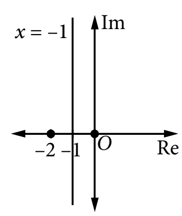

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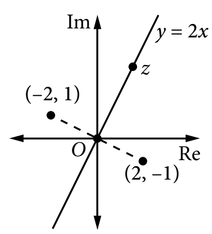

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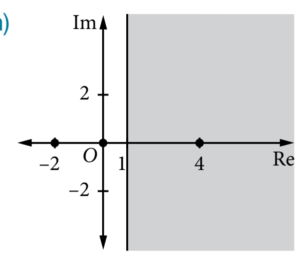
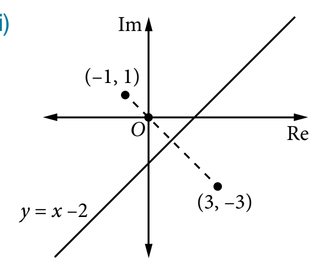
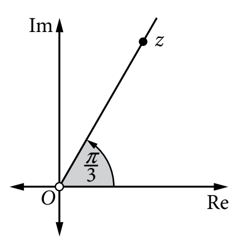


(i)



4 (a)



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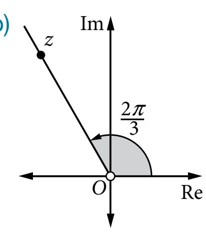

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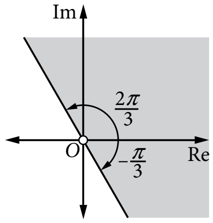

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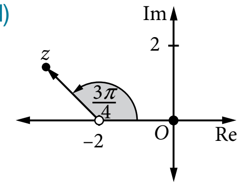

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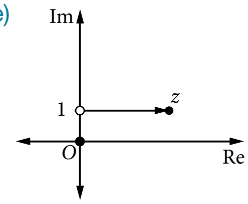

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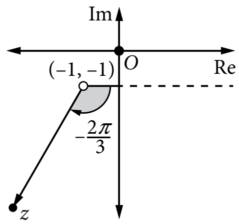

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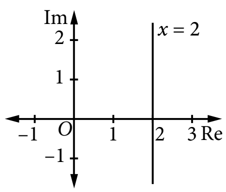

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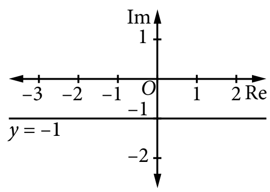

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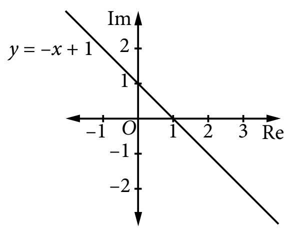

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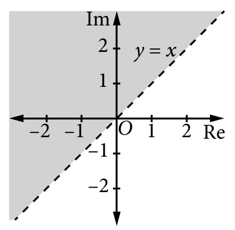

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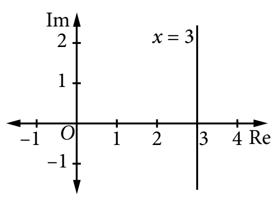

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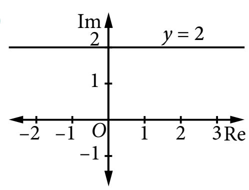

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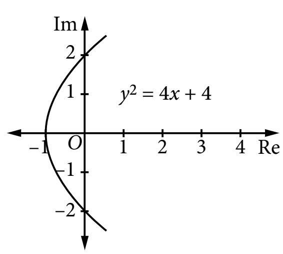

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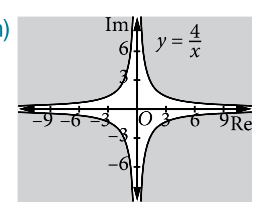

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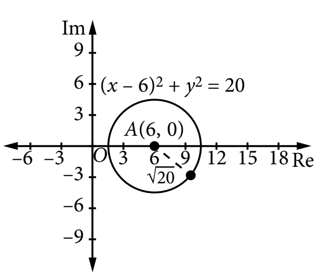

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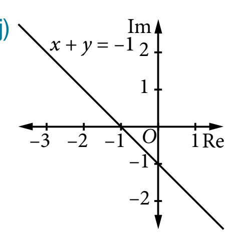
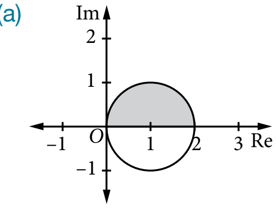


6 (a)



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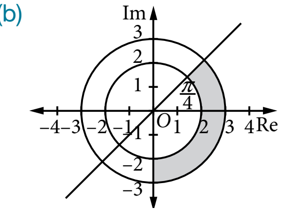

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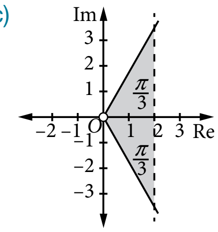

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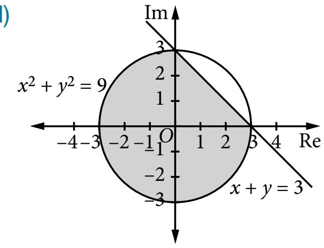

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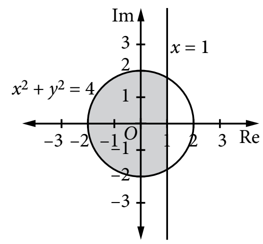


CHAPTER REVIEW 1

1 (a) $2i$ (b) $-3 + 4i$ (c) 25

(d) $-\frac{11}{25} - \frac{2}{25}i$ (e) $\pm(1 - 2i)$

2 (a) $\pm 5i$ (b) $-1 \pm i$

(c) $\frac{5 \pm 3i}{4}$ (d) $\frac{11}{10} + \frac{3}{10}i$

3 (a) $p = 1, q = 1$ (b) $p = -1, q = 2$ or $p = -\frac{4}{3}, q = \frac{3}{2}$

4 (a) $3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (b) $\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$

5 (a) $-2\sqrt{3} + 2i$ (b) $-3\sqrt{2} - 3\sqrt{6}\,i$

6 (a) (i) $8\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$, $-8i$

(ii) $\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$, $\frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i$

(b) Equate the real parts: $\sqrt{2}\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2}$

$\therefore \cos\frac{7\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}$

7 $\operatorname{cis}\frac{3\pi}{10}$

8 (a) $3 \le |z + 3 - 3i| \le 3\sqrt{2}$ (b) $|z - 4i| = |z + 2 - 2i|$

(c) $2 \le |z| \le 3$ and $-\frac{\pi}{4} \le \arg z \le \frac{\pi}{3}$

(d) $\arg(z - 1 + i) = \frac{3\pi}{4}$ (e) $\arg\left(\frac{z + 2 - 2i}{z - 1 + i}\right) = \pi$

9 (a) $\operatorname{cis}\frac{\pi}{4}$, $\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $\operatorname{cis}\frac{3\pi}{4}$, $\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

(b) $\operatorname{cis}0$, $\operatorname{cis}\frac{\pi}{3}$, $\operatorname{cis}\left(-\frac{\pi}{3}\right)$, $\operatorname{cis}\frac{2\pi}{3}$, $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$, $\operatorname{cis}\pi$

(c) $\operatorname{cis}\frac{\pi}{3}$, $\operatorname{cis}\left(-\frac{\pi}{3}\right)$, $\operatorname{cis}\frac{2\pi}{3}$, $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$, $\operatorname{cis}\pi$

10 $\left(1 + \sqrt{2}i\right)^3 = -5 + \sqrt{2}i$

11 (a) Let A, B, C represent z, z^2, z^4 respectively.
Let D represent $z^2 + z^4$.
On an Argand diagram $OBDC$ is a rhombus, $OB = OC = 1$, so diagonal OD bisects $\angle BOC$.
$\arg(z^2) = 2\theta$ and $\arg(z^4) = 4\theta$, so $\arg(z^2 + z^4) = 3\theta$

(b) In ΔOBD: $OD = 2\cos\theta$, i.e. $\left|z^2 + z^4\right| = 2\cos\theta$
$\therefore z^2 + z^4 = 2\cos\theta\,(\cos 3\theta + i\sin 3\theta)$

(c) $\pm\dfrac{\pi}{6}$

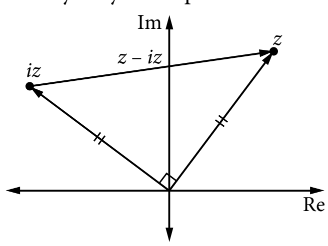
12 $\dfrac{\pi}{3} \le \arg z \le \dfrac{2\pi}{3}$

13 (a) $(-w)^{12} = w^{12} = i$

(b) Roots of $z^{12} = i$ are on the circumference of a circle of radius 1.
$\left|z_1 + z_2\right| = \left|z_1 - (-z_2)\right| =$ distance from $-z_2$ to z_1,
i.e. the distance between two points on the circumference of the circle.
As z_1 and z_2 are two distinct roots, z_1 and $-z_2$ are not endpoints of a diameter $\therefore \left|z_1 + z_2\right| < 2$

14 (a) $\arg iz = \arg i + \arg z = \dfrac{\pi}{2} + \alpha$

(b) Values of z may vary. Example:



(c) Pythagoras' theorem shows the result.

(d) Area $= \dfrac{1}{2} \times |z| \times |iz| = \dfrac{1}{2} \times |z| \times 1 \times |z| = \dfrac{1}{2} \times |z|^2 = \dfrac{1}{2} z\overline{z}$

(e) Interval from iz to z is a diameter (subtends right angle at O).
Length of diameter is $\sqrt{10}$, area of circle is $\dfrac{5\pi}{2}$.
Area of triangle is $\dfrac{1}{2}(1 + 2i)(1 - 2i) = \dfrac{5}{2}$
$\therefore$ Area required $= \dfrac{5}{2}(\pi - 1)$

15 C

16 (a) midpoint of AC

(b) $\alpha + \gamma = \beta + \delta \;\therefore\; \dfrac{1}{2}(\alpha + \gamma) = \dfrac{1}{2}(\beta + \delta)$
$\therefore$ midpoints of AC and BD are coincident,
i.e. diagonals bisect each other $\therefore$ parallelogram

17 (a) $z + \dfrac{1}{z} = k,\; x + iy + \dfrac{1}{x + iy} = k + 0i$

$$x + iy + \frac{1}{x + iy} \times \frac{x - iy}{x - iy} = k + 0i$$

$$x + iy + \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = k + 0i$$

Equating imaginary parts: $y - \dfrac{y}{x^2 + y^2} = 0$
$$y(x^2 + y^2) - y = 0$$
$$y(x^2 + y^2 - 1) = 0$$
$$\therefore y = 0 \text{ or } x^2 + y^2 = 1$$

(b) If $y = 0$ then $z + \dfrac{1}{z} = k$ becomes $x + \dfrac{1}{x} = k$
$\therefore x^2 - kx + 1 = 0$
This equation must have real roots (or else there are no solutions to the original equation).
$\therefore \Delta \ge 0$, i.e. $k^2 - 4 \ge 0$, hence $|k| \ge 2$

(c) If $x^2 + y^2 = 1$ then $|z| = 1$, $\left|\dfrac{1}{z}\right| = 1$
Using the triangle inequality: $\left|z + \dfrac{1}{z}\right| \le |z| + \left|\dfrac{1}{z}\right|$
$\therefore |k| \le 2$

18 (a) $x - 2y - 4 = 0$

(b) $z + \overline{z} = 2x \;\therefore\; x = \dfrac{z + \overline{z}}{2}$

$z - \overline{z} = 2yi \;\therefore\; y = \dfrac{z - \overline{z}}{2i} = \dfrac{-i(z - \overline{z})}{2}$

Substituting into $x - 2y - 4 = 0$:
$$\frac{z + \overline{z}}{2} - 2 \times \frac{-i(z - \overline{z})}{2} - 4 = 0$$
$$z + \overline{z} + 2i(z - \overline{z}) - 8 = 0$$
$$(1 + 2i)z + (1 - 2i)\overline{z} - 8 = 0$$

19 (a) $w^5 - 1 = 0$
$(w - 1)(w^4 + w^3 + w^2 + w + 1) = 0$
But $w \ne 1 \;\therefore\; 1 + w + w^2 + w^3 + w^4 = 0$

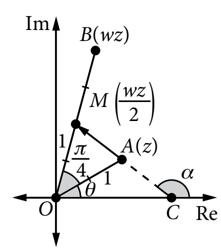
(b) $(1 - w)(1 - w^2)(1 - w^3)(1 - w^4)$
$= (1 - w)(1 - w^4) \times (1 - w^2)(1 - w^3)$
$= (1 - w - w^4 + w^5) \times (1 - w^2 - w^3 + w^5)$
$= (2 - w - w^4) \times (2 - w^2 - w^3)$
$= 4 - 2w^2 - 2w^3 - 2w + w^3 + w^4 - 2w^4 + w^6 + w^7$
$= 4 - 2(w + w^2 + w^3 + w^4) + (w + w^2 + w^3 + w^4)$
$= 4 - 2(-1) + (-1) = 5$

(c) Sum and product of roots:
$z_1 + z_2 = w + w^2 + w^3 + w^4 = -1$
$z_1 z_2 = (w + w^4)(w^2 + w^3)$
$= w^3 + w^4 + w^6 + w^7$
$= w^3 + w^4 + w + w^2 = -1$
$\therefore$ Quadratic equation: $z^2 - (z_1 + z_2)z + (z_1 z_2) = 0$
$$z^2 + z - 1 = 0$$

20 (a) $2\operatorname{cis}\dfrac{\pi}{3}$, $2\operatorname{cis}\pi$, $2\operatorname{cis}\left(-\dfrac{\pi}{3}\right)$

(b) Without loss of generality,
let $w_1 = 2\operatorname{cis}\dfrac{\pi}{3}$ and $w_2 = 2\operatorname{cis}\left(-\dfrac{\pi}{3}\right)$.
Then $w_1^{6n} + w_2^{6n}$
$= 2^{6n}\left(\cos\dfrac{6n\pi}{3} + i\sin\dfrac{6n\pi}{3}\right) + 2^{6n}\left(\cos\left(-\dfrac{6n\pi}{3}\right) + i\sin\left(-\dfrac{6n\pi}{3}\right)\right)$
$= 2^{6n}(\cos 2n\pi + i\sin 2n\pi + \cos(-2n\pi) + i\sin(-2n\pi))$
$= 2^{6n}(1 + i \times 0 + 1 + i \times 0) = 2^{6n+1}$

21



(a) M is midpoint of OB
$\therefore M$ represents $\dfrac{1}{2}wz$,
$\overrightarrow{AM} = \dfrac{1}{2}wz - z$

(b) Cosine rule in ΔOAM: $AM = \left|\dfrac{1}{2}wz - z\right| = \sqrt{2 - \sqrt{2}}$

(c) $\angle OMA = \dfrac{3\pi}{8}$ (angle sum of isosceles triangle)
$\arg\left(\dfrac{1}{2}wz - z\right) = \alpha = \dfrac{\pi}{4} + \theta + \dfrac{3\pi}{8} = \dfrac{5\pi}{8} + \theta$ (exterior angle of ΔOMC)

22 (a) LHS $= 1 - \cos\theta - i\sin\theta + \cos\theta - \cos^2\theta - i\sin\theta\cos\theta + i\sin\theta - i\sin\theta\cos\theta + \sin^2\theta$
$= 1 - (\cos^2\theta - \sin^2\theta) - 2i\sin\theta\cos\theta$
$= 1 - (\cos 2\theta + i\sin 2\theta) =$ RHS

(b) *Step 1* Prove true for $n = 1$: (See part (a))
Step 2 Assume true for $n = k$, i.e. assume that:
$$1 + \operatorname{cis}\theta + \operatorname{cis}2\theta + \ldots + \operatorname{cis}k\theta = \frac{1 - \operatorname{cis}(k+1)\theta}{1 - \operatorname{cis}\theta}$$

Now when $n = k + 1$, need to prove that:

$1 + \text{cis}\,\theta + \text{cis}\,2\theta + \ldots + \text{cis}(k+1)\theta = \dfrac{1 - \text{cis}(k+2)\theta}{1 - \text{cis}\theta}$

$\text{LHS} = 1 + \text{cis}\,\theta + \text{cis}\,2\theta + \ldots + \text{cis}\,k\theta + \text{cis}(k+1)\theta$

$= \dfrac{1 - \text{cis}(k+1)\theta}{1 - \text{cis}\theta} + \text{cis}(k+1)\theta$

$= \dfrac{1 - \text{cis}(k+1)\theta}{1 - \text{cis}\theta} + \dfrac{\text{cis}(k+1)\theta \times (1 - \text{cis}\theta)}{1 - \text{cis}\theta}$

$= \dfrac{1 - \text{cis}(k+1)\theta + \text{cis}(k+1)\theta - \text{cis}(k+1)\theta \times \text{cis}\theta}{1 - \text{cis}\theta}$

$= \dfrac{1 - \text{cis}(k+2)\theta}{1 - \text{cis}\theta} = \text{RHS}$

Step 3 Conclusion

True for $n = k + 1$ if true for $n = k$. True for $n = 1$.

$\therefore$ True for all positive integers $n \geq 1$.

(c) Let $\theta = \dfrac{\pi}{36}$ and apply part (b) with $n = 71$:

$1 + \text{cis}\dfrac{\pi}{36} + \text{cis}\dfrac{2\pi}{36} + \ldots + \text{cis}\dfrac{71\pi}{36} = \dfrac{1 - \text{cis}\frac{72\pi}{36}}{1 - \text{cis}\frac{\pi}{36}}$

$= \dfrac{1 - \text{cis}\,2\pi}{1 - \text{cis}\frac{\pi}{36}} = \dfrac{1 - 1}{1 - \text{cis}\frac{\pi}{36}} = 0$

(d) The sum of the 72 roots of $z^{72} = 1$ is 0.

23 (a) $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2} \quad \therefore \cos\left(2 \times \dfrac{\pi}{12}\right) = \dfrac{\sqrt{3}}{2}$

$2\cos^2\dfrac{\pi}{12} - 1 = \dfrac{\sqrt{3}}{2},\ \cos^2\dfrac{\pi}{12} = \dfrac{2+\sqrt{3}}{4},$

$\cos\dfrac{\pi}{12} = \pm\dfrac{\sqrt{2+\sqrt{3}}}{2}$

But $\dfrac{\pi}{12}$ is acute: $\therefore \cos\dfrac{\pi}{12} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$

(b) $\sin^2\dfrac{\pi}{12} = 1 - \cos^2\dfrac{\pi}{12} = 1 - \dfrac{2+\sqrt{3}}{4} = \dfrac{2-\sqrt{3}}{4}$

$\therefore \sin\dfrac{\pi}{12} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

(c) $\left(\dfrac{\sqrt{2+\sqrt{3}}}{2} + \dfrac{\sqrt{2-\sqrt{3}}}{2}i\right)^4 = \left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right)^4$

$= \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

24 (a) $x = \tan\dfrac{\pi}{9},\ -\tan\dfrac{2\pi}{9},\ \tan\dfrac{4\pi}{9}$

(b) Sum of roots [of part (a)] $= -\dfrac{b}{a} = 3\sqrt{3}$

25 $\left(\dfrac{z}{z-i}\right)^4 = w^4 = -4$ has roots $\sqrt[4]{4}\left(\cos\dfrac{\pi + 2k\pi}{4} + i\sin\dfrac{\pi + 2k\pi}{4}\right)$

$\therefore w = 1 + i,\ 1 - i,\ -1 + i,\ -1 - i$:

For $\dfrac{z}{z-i} = 1 + i$

$z = z + iz - i + 1$

$iz = -1 + i$

$z = 1 + i$

For $\dfrac{z}{z-i} = 1 - i$

$z = z - iz - i - 1$

$iz = -1 - i$

$z = -1 + i$

For $\dfrac{z}{z-i} = -1 + i$

$z = -z + iz + i + 1$

$z(2 - i) = 1 + i$

$z = \dfrac{1 + 3i}{5}$

For $\dfrac{z}{z-i} = -1 - i$

$z = -z - iz + i - 1$

$z(2 + i) = -1 + i$

$z = \dfrac{-1 + 3i}{5}$

26 $(2x + 1)(4x^2 + 2x - 1) = 0$: $x = \dfrac{-1 \pm \sqrt{5}}{4}$

27 (a) $x^2 - y^2 = 5,\ xy = 6$: $\pm(3 + 2i)$

(b) $z = \dfrac{-3 \pm \sqrt{9 - 4(1 - 3i)}}{2}$

$= \dfrac{-3 \pm \sqrt{5 + 12i}}{2} = \dfrac{-3 \pm (3 + 2i)}{2} = i,\ -3 - i$

28 $P'(x) = 3x^2 + 2ax + b$

$P'(1) = 3 + 2a + b = 0$

$P(1) = a + b + 3 = 0$

$a = 0,\ b = -3$

29 (a) Coefficients real $\therefore$ other zeros are conjugates

$(a - ib), (a - 2ib)$: $a = 1,\ b = 2$

(b) $(x^2 - 2x + 5)(x^2 - 2x + 17)$

30 $P'(x) = 3ax^2 + b$:

$8a + 2b + c = 0,\ 12a + b = 0,\ -8a - 2b + c = 20$

$a = \dfrac{5}{8},\ b = -\dfrac{15}{2},\ c = 10$

CHAPTER 2

Many of the proofs below are not complete: in most cases only the outline of the proof of $S(k + 1)$ is given. For more complete solutions, see the *Student Worked Solutions* book.

EXERCISE 2.1

1 (a) Either p is odd or q is odd.

(b) $x \leq 5$ and $x \geq -5$ or $-5 \leq x \leq 5$

(c) x is divisible by neither 7 nor 8

(d) Neither x nor y are zero or both x and y are non-zero.

2 (a) Doubling any integer and adding 3 results in an odd number. This is true as doubling any integer results in an even number, and the sum of any even number and any odd number is odd.

(b) There is a real number that is equal to its reciprocal. This is true as the number 1 has this property.

(c) The square of a real number is always greater than zero. This is false since the square of zero is zero.

(d) There is a real number whose square is -1. This is false.

(e) The product of any integer and the next is divisible by 3. This is false (since, for example, 4×5 is not divisible by 3).

(f) The difference between any two real numbers is positive. This is false since $3 - 5$ is negative, for example.

(g) For all real numbers, there is a real number that can be added to it to obtain an answer of zero. This is true (since adding the negative of any number to itself results in an answer of zero).

(h) There is a real number with the property that multiplying it by any other real number (y) gives an answer of y. This is true, as the number 1 has this property.

3 (a) $\forall$ integers n, $n^2 > n$. This is false as $0^2 = 0$.

(b) $\exists$ a real number x, such that $5x = 0$. ($\exists\, x \in \mathbb{R}$ such that $5x = 0$.) This is true as the number 0 has this property.

(c) $\forall$ integers n, $n + n + 1$ is odd. This is true.

(d) $\exists$ a real number x such that $x = x^2$. ($\exists\, x \in \mathbb{R}$ such that $x = x^2$) This is true as both 0 and 1 have this property.

(e) $\forall$ real numbers x and y, $x^2 + y^2 < xy$. ($\forall\, x, y \in \mathbb{R},\ x^2 + y^2 < xy$) This is false (for example, when $x = -1$ and $y = 1$).

(f) $\exists$ a real number x such that for all real numbers y, $\dfrac{y}{x} = y$. ($\exists\, x \in \mathbb{R}$ such that $\forall\, y \in \mathbb{R}$, $\dfrac{y}{x} = y$) This is true. The number 1 has this property.

(g) $\forall$ non-zero integers n, $\exists$ an integer m such that n is divisible by m. This is true as every integer is divisible by itself (and 1).

4 (a) $\exists$ a real number x such that $x^2 \le 0$. Negation is true (since $0^2 \le 0$).
(b) $\forall$ real numbers x, $x^2 \ne x$. Original is true (since $1^2 = 1$).
(c) $\exists$ a positive integer n such that $10n \le n$. Original is true since 10 times any positive number is greater than the number.
(d) $\exists$ a real number x such that x is neither positive nor negative. Negation is true (as the number 0 has this property).
(e) $\forall$ integers n, either $n = 0$ or $n^2 \ge 1$. Negation is true as squaring any integer gives a positive integer, except for the integer 0.
(f) $\exists$ an integer n such that either $(-1)n \ne 1$ and $(-1)n \ne -1$. Original is true as raising -1 to any integer power always results in 1 or -1.

5 (a) $x > 3 \Rightarrow x^2 > 9$
(b) n is divisible by 9 $\Rightarrow$ n is divisible by 3
(c) $n > 5 \Rightarrow n > 4$
(d) $p > 3 \Rightarrow 7p$ is positive
(e) $2q$ is a perfect square $\Rightarrow$ q is even
(f) m is a multiple of 6 $\Rightarrow$ m is divisible by 3
(g) $x < -2 \Rightarrow x^2 > 2$
(h) n even and $n > 2 \Rightarrow n$ is not prime

6 (a) Original: This is true, since a number divisible by 20 must be divisible by any factor of 20.
Converse: If n is divisible by 5, then n is divisible by 20. This is false. For example, 10 is divisible by 5 but not by 20.
Contrapositive: If n is not divisible by 5, then n is not divisible by 20. This must be true as the original statement is true.
Negation: There exists an integer n with the property that n is divisible by 20 and n is not divisible by 5. This must be false as the original statement was true.
(b) Original: This is true since the square of any integer is divisible by all factors of the integer.
Converse: If n^2 is divisible by 3, then n is divisible by 3. This is actually true as the square of any integer must have the same prime factors as the integer.
Contrapositive: If n^2 is not divisible by 3, then n is not divisible by 3. This must be true as the original statement is true.
Negation: There exists an integer n with the property that n divisible by 3 but n^2 is not divisible by 3. This must be false as the original statement was true.
(c) Original: This is true, as the second inequality can be obtained by multiplying both sides of the first by 10.
Converse: If $10x > 70$, then $x > 7$. This is true as the first inequality can be obtained by dividing both sides of the second by 10.
Contrapositive: If $10x \le 70$, then $x \le 7$. This must be true as the original statement is true.
Negation: There exists a real number x with the property that $x > 7$ and $10x \le 70$. This must be false as the original statement was true.
(d) Original: This is true. (It is the null-factor law.)
Converse: If $x = 0$ or $y = 0$, then $xy = 0$. This is true.
Contrapositive: If $x \ne 0$ and $y \ne 0$, then $xy \ne 0$. This must be true as the original statement is true.
Negation: There exist real numbers x and y with the property that $xy = 0$ and $x \ne 0$ and $y \ne 0$. This must be false as the original statement was true.
(e) Original: This is false. For example, 20 is divisible and its final digit is not 5.
Converse: If the final digit of n is 5, then n is divisible by 5. This is true.
Contrapositive: If the final digit of n is not 5, then n is not divisible by 5. This must be false as the original statement is false.
Negation: There exists an integer n that is divisible by 5, and with the final digit of n not 5. This must be true as the original statement was false.
(f) Original: This is clearly true.
Converse: If $xy = 16$, then $x = 4$ and $y = 4$. This is false. For example, $8 \times 2 = 16$.
Contrapositive: If $xy \ne 16$, then $x \ne 4$ or $y \ne 4$. This must be true as the original statement is true.
Negation: There exists real numbers x and y with the property that $x = 4$ and $y = 4$, and $xy \ne 16$. This must be false as the original statement was true.
(g) Original: This is true as any number divisible by 24 must contain factors of 2 and 3.
Converse: If n is even and n is divisible by 3, then n is divisible by 24. This is false. For example, 12 is even and divisible by 3 but not by 24.
Contrapositive: If n is odd or n is not divisible by 3, then n is not divisible by 24. This must be true as the original statement is true.
Negation: There exists an integer n with the property that n is divisible by 24, and n is either odd or not divisible by 3. This must be false as the original statement was true.

7 (a) n is even $\Leftrightarrow n^2$ is even
(b) $x + y = 0 \Leftrightarrow x = -y$
(c) n is even and divisible by 3 $\Leftrightarrow$ n is divisible by 6

8 (a) Examples: 7, 14, 28. Counterexample: 21
(b) Examples: 2, 3, 4. Counterexample: 0.5
(c) Examples: 2, 3, 5. Counterexample: 11
(d) Examples: 60, 120, 180. Counterexample: 30
(e) Examples: $x = 10, y = 0$; $x = 10, y = 1$; $x = 10, y = 2$. Counterexample: $x = 4, y = 100$
(f) Examples: $x = 3, y = 3$; $x = 3, y = 4$; $x = 3, y = 5$. Counterexample: $x = 0.1, y = 10$

9 D. Alternative A, in everyday language, says that for all real numbers x, there exists a real number y such that $xy = 6$. But this is not true for $x = 0$.
Alternative B, in everyday language, says that there exists a real number x, such that for all real numbers y, $xy = 6$. There is no single number (for x) that multiplies with every real number to give a result of 6.
Alternative C, in everyday language, says that there exists a real number x, such that for all real numbers y, $x + y = 6$. There is no single number (for x) that adds to every real number to give a result of 6.
Alternative D, in everyday language, says that for all real numbers x, there exists a real number y such that $x + y = 6$. This is definitely true. If you start with any real number, you'll be able to find a real number to add to it that gives a result of 6.

10 B. The original statement, in everyday language, says that for all real numbers x, there exists a real number y such that $x + y = 6$. If this is not the case, it would mean that there must exist a real number x such that for all real numbers y, $x + y \ne 6$. Thus, B is the correct alternative.
Notice that alternative B is different from alternative C. Alternative C says that the result of adding any two real numbers is never 6. This is different from saying there is at least one special number, such that when any number is added to it, the result is never 6.

11 $\exists$ a real number x such that $x > 0$ and $x < 10$, but $x < 0$ or $x > 10$. It is clear that the original statement is true.

12 (a) Starting with the number 6, you obtain the sequence: $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. Since the sequence reaches the number 1, the conjecture is verified for this case.

(b) Starting with the number 13, you obtain the sequence: $13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. Since the sequence reaches the number 1, the conjecture is verified for this case.

(c) Starting with the number 7, you obtain the sequence: $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. Since the sequence reaches the number 1, the conjecture is verified for this case.

EXERCISE 2.2

1 (a) Let p and q be odd integers.
Let $p = 2k + 1$ and $q = 2l + 1$ for integers k and l.

$$\begin{aligned} p + q &= 2k + 1 + 2l + 1 \\ &= 2k + 2l + 2 \\ &= 2(k + l + 1) \end{aligned}$$

Since $(k + l + 1)$ must be an integer then it follows that $p + q$ is even.

(b) Let p be an odd integer and q be an even integer.
Let $p = 2k + 1$ and $q = 2l$ for integers k and l.

$$\begin{aligned} p + q &= 2k + 1 + 2l \\ &= 2(k + l) + 1 \end{aligned}$$

Since $(k + l)$ must be an integer then it follows that $p + q$ is odd.

(c) Let p and q be odd integers.
Let $p = 2k + 1$ and $q = 2l + 1$ for integers k and l.

$$\begin{aligned} pq &= (2k + 1)(2l + 1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

Since $(2kl + k + l)$ must be an integer then it follows that pq is odd.

(d) Let p and q be two consecutive odd integers.
Let $p = 2k + 1$ and $q = 2k + 3$ for some integer k.

$$\begin{aligned} p + q &= 2k + 1 + 2k + 3 \\ &= 4k + 4 \\ &= 4(k + 1) \end{aligned}$$

Since $(k + 1)$ must be an integer then it follows that $p + q$ is divisible by 4.
(Could also have used $2k - 1$ and $2k + 1$.)

(e) Let n, $n + 1$, $n + 2$, $n + 3$, $n + 4$ be five consecutive integers.
Sum of squares $= n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2 + (n + 4)^2$
$= n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + 9 + n^2 + 8n + 16$
$= 5n^2 + 20n + 30$
$= 5(n^2 + 4n + 6)$
Since $n^2 + 4n + 6$ must be an integer then the sum of the squares is divisible by 5.
(Could also have used $(n - 2)$, $(n - 1)$, n, $(n + 1)$, $(n + 2)$.)

(f) Let a and b be rational numbers, i.e. let $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where p, q, r, s are integers with q, s non-zero.

$$\begin{aligned} ab &= \frac{p}{q} \times \frac{r}{s} \\ &= \frac{pr}{qs} \end{aligned}$$

Since pr and qs must both be integers, and qs cannot be zero (since neither q nor s is 0), it follows that ab is rational.

(g) Let a and b be rational numbers, i.e. let $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where p, q, r, s are integers with q, s non-zero.

$$\begin{aligned} a + b &= \frac{p}{q} + \frac{r}{s} \\ &= \frac{ps + qr}{qs} \end{aligned}$$

Since $ps + qr$ and qs must both be integers, and qs cannot be zero (since neither q nor s is 0), it follows that $a + b$ is rational.

(h) If n is odd, then $n = 2k + 1$ for some integer k.

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since $2k^2 + 2k$ must be an integer it follows that n^2 is odd.

(i) If n is divisible by 7 then, $n = 7k$ for some integer k.

$$\begin{aligned} n^2 &= (7k)^2 \\ &= 7 \times 7k^2 \end{aligned}$$

Since $7k^2$ must be an integer then it follows that n^2 is divisible by 7.

(j) If $m + n$ and $n + p$ are even, then $m + n = 2k$ and $n + p = 2l$ for integers k and l.
Hence $n = 2k - m$
Substitute in $n + p = 2l$: $2k - m + p = 2l$

$$\begin{aligned} -m + p &= 2l - 2k \\ m + p &= 2l - 2k + 2m \\ m + p &= 2(l - k + m) \end{aligned}$$

Since $(l - k + m)$ must be an integer then it follows that $m + p$ is even.

2 (a) The contrapositive statement is: if n is odd, then $3n + 2$ is odd.
Suppose that n is odd. Then $n = 2k + 1$ for some integer k.

$$\begin{aligned} 3n + 2 &= 3(2k + 1) + 2 \\ &= 6k + 3 + 2 \\ &= 6k + 5 \\ &= 2(3k + 2) + 1 \end{aligned}$$

Since $(3k + 2)$ must be an integer then it follows that $3n + 2$ is odd.

(b) The contrapositive statement is: if neither a nor b are even then at least one of a and b is even.
Suppose that neither a nor b are even. This means that both are odd so then $a = 2k + 1$ and $b = 2l + 1$ for integers k and l.

$$\begin{aligned} ab &= (2k + 1)(2l + 1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

Since $(2kl + k + l)$ must be an integer then it follows that ab is odd.

(c) The contrapositive statement is: if n is odd, then $n^3 + 5$ is even.
Suppose that n is odd, so $n = 2k + 1$ for some integer k.

$$\begin{aligned} n^3 + 5 &= (2k + 1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \end{aligned}$$

Since $(4k^3 + 6k^2 + 3k + 3)$ must be an integer then it follows that $n^3 + 5$ is even.

(d) The contrapositive statement is: if $\sqrt{x}$ is rational, then x is rational.
Suppose that $\sqrt{x}$ is rational, then $\sqrt{x} = \frac{p}{q}$ where p and q are integers and $q \neq 0$.
Square both sides: $x = \frac{p^2}{q^2}$.
Since p^2 and q^2 are integers with $q^2 \neq 0$ then x is rational.

(e) The contrapositive statement is: if $\frac{1}{x}$ is rational, then x is rational.

Suppose that $\frac{1}{x}$ is rational, then $\frac{1}{x} = \frac{p}{q}$ where p and q are integers and $q \neq 0$. Also, $p \neq 0$ as $\frac{1}{x} \neq 0$.

Thus $x = \frac{q}{p}$ and since p and q are integers with $p \neq 0$, it follows that x is rational.

3 (a) Suppose, for a contradiction that $\sqrt{3}$ is rational, then $\sqrt{3} = \frac{p}{q}$ for integers p and q, with p and q having no common factors other than 1.

$3 = \frac{p^2}{q^2}$

$p^2 = 3q^2$ and thus p^2 is divisible by 3.
If p^2 is divisible by 3 then p is divisible by 3 so write $p = 3m$ for some integer m.
$9m^2 = 3q^2$
$3m^2 = q^2$
Hence q^2 is also divisible by 3 so q is divisible by 3.
Since p and q are both divisible by 3 then you have a contradiction since p and q have no common factors other than 1.
It follows that $\sqrt{3}$ is an irrational number.

(b) Suppose, for a contradiction, that $\sqrt{5}$ is rational, then $\sqrt{5} = \frac{p}{q}$ for integers p and q, with p and q having no common factors other than 1.

$5 = \frac{p^2}{q^2}$

$p^2 = 5q^2$ and thus p^2 is divisible by 5.
If p^2 is divisible by 5 then p is divisible by 5 so write $p = 5m$ for some integer m.
$25m^2 = 5q^2$
$5m^2 = q^2$
Hence q^2 is also divisible by 5 so q is divisible by 5.
Since p and q are both divisible by 5 then you have a contradiction since p and q have no common factors other than 1.
It follows that $\sqrt{5}$ is an irrational number.

(c) Let a be a rational number and b an irrational number. Suppose, for a contradiction, that $a + b$ is rational. Since a and $a + b$ are both rational, $a = \frac{p}{q}$ and $a + b = \frac{r}{s}$ where p, q, r, s are integers with $q \neq 0$ and $s \neq 0$.

$$a + b = \frac{r}{s}$$
$$\frac{p}{q} + b = \frac{r}{s}$$
$$b = \frac{r}{s} - \frac{p}{q}$$
$$= \frac{rq - ps}{sq}$$

This is a contradiction as b is irrational. Hence, $a + b$ must be irrational.

(d) Let a be a non-zero rational number and b an irrational number. Suppose, for a contradiction, that ab is rational. Since a and ab are both rational, $a = \frac{p}{q}$ and $ab = \frac{r}{s}$ where p, q, r, s are integers with $q \neq 0$ and $s \neq 0$. Note also that $p \neq 0$ since a is non-zero.

$$ab = \frac{r}{s}$$
$$\frac{p}{q}b = \frac{r}{s}$$
$$b = \frac{rq}{sp}$$

This is a contradiction as b is irrational. Hence, ab must be irrational.

(e) Suppose, for a contradiction, that there exists integers a and b such that $18a + 6b = 1$.
$6(2a + b) = 1$
$3a + b = \frac{1}{6}$

This is a contradiction as $3a + b$ is an integer. Therefore, there cannot be integers a and b such that $18a + 6b = 1$.

4 (a) First, suppose that $n + 9$ is even. Then $n + 9 = 2k$ for some integer k.
$n + 6 = n + 9 - 3$
$= 2k - 3$
$= 2(k - 2) + 1$
Thus, $n + 6$ is odd.
Conversely, suppose that $n + 6$ is odd. Then $n + 6 = 2k + 1$ for some integer k.
$n + 9 = n + 6 + 3$
$= 2k + 1 + 3$
$= 2k + 4$
$= 2(k + 20$
Thus $n + 9$ is even.

(b) First, suppose that $n - 3$ is odd. Then $n - 3 = 2k + 1$ for some integer k.
$n + 2 = n - 3 + 5$
$= 2k + 1 + 5$
$= 2k + 6$
$= 2(k + 3)$
Thus, $n + 2$ is even.
Conversely, suppose that $n + 2$ is even. Then $n + 2 = 2k$ for some integer k.
$n - 3 = n + 2 - 5$
$= 2k - 5$
$= 2(k - 3) + 1$
Thus, $n - 3$ is odd.

(c) First, suppose that n is even. Then $n = 2k$ for some integer k.
$13n + 4 = 13 \times 2k + 4$
$= 26k + 4$
$= 2(13k + 2)$
Thus, $13n + 4$ is even.
Conversely, suppose that $13n + 4$ is even. Suppose, for a contradiction, that n is odd. Then $n = 2k + 1$ for some integer k.
$13n + 4 = 13(2k + 1) + 4$
$= 26k + 17$
$= 2(13k + 8) + 1$
Thus, $13n + 4$ is odd—a contradiction. Hence, if $13n + 4$ is even, n must be even.

(d) First, suppose that n is odd. Then $n = 2k + 1$ for some integer k.

$7n + 6 = 7(2k + 1) + 6$
$= 14k + 13$
$= 2(7k + 6) + 1$

Thus, $7n + 6$ is odd.

Conversely, suppose that $7n + 6$ is odd. Suppose, for a contradiction, that n is even. Then $n = 2k$ for some integer k.

$7n + 6 = 7 \times 2k + 6$
$= 14k + 6$
$= 2(7k + 3)$

Thus, $7n + 6$ is even – a contradiction. Hence, if $7n + 6$ is odd, n must be odd.

(e) First, suppose that n is even. Then $n = 2k$ for some integer k.

$n^2 = (2k)^2$
$= 2 \times 2k^2$

Thus, n^2 is even.

To prove the converse, suppose that n^2 is even. Suppose, for a contradiction, that n is not even. Then this would mean that n is odd, meaning $n = 2k + 1$ for some integer k.

$n^2 = (2k + 1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$

But this would mean that n^2 is odd—a contradiction. Hence, if n^2 is even, n must be even.

5 Alternative A is the converse of the given statement, which is not logically equivalent to it (despite being a true statement in this case).

Alternative B is not the contrapositive, as the statement 'at least one of the integers is odd' is not the negation of the statement 'at least one of the two integers must be even', i.e. if you have one odd integer and one even integer, then both statements 'at least one odd' and 'at least one even' are true.

C. A contrapositive proof must start by assuming that it is not the case that at least one of the two integers is even—in other words, assuming that both integers are odd. Hence, C is the correct alternative.

Alternative D is how a proof by contradiction would proceed.

6 Suppose, for a contradiction, that this number is rational. Then $1 - 5\sqrt{2} = \frac{p}{q}$ for integers p, q with $q \neq 0$.

$-5\sqrt{2} = \frac{p}{q} - 1$

$\sqrt{2} = -\frac{p}{5q} + \frac{1}{5}$

$\sqrt{2} = \frac{-p + q}{5q}$

This is a contradiction as $\sqrt{2}$ is irrational, and thus cannot be expressed as the ratio of two integers. Therefore, the number $1 - 5\sqrt{2}$ must be irrational.

7 Suppose, for a contradiction, that both a and b are greater than $\sqrt{c}$.

Then $a \times b > \sqrt{c} \times \sqrt{c}$
$ab > c$

This is a contradiction as $ab = c$. Thus, it must be the case that at least one of a and b is less than or equal to $\sqrt{c}$.

8 This is most readily proved by proving the contrapositive statement: if a is not odd, then $a^2 - 2a + 7$ is not even.

If a is not odd, then it is even, and can be expressed as $a = 2k$ for some integer k.

$a^2 - 2a + 7$
$= 4k^2 - 4k + 7$
$= 2(2k^2 - 2k + 3) + 1$

Thus, $a^2 - 2a + 7$ is odd (not even), as required.

9 Let a, b, c, d be the digits, in order, of a four-digit number, N.

The actual number is therefore $N = 1000a + 100b + 10c + d$.

First, suppose that N is divisible by 9.

Thus, $1000a + 100b + 10c + d = 9k$ for some integer k.

Then $999a + 99b + 9c + a + b + c + d = 9k$
$a + b + c + d = 9k - 999a - 99b - 9c$
$a + b + c + d = 9(k - 111a - 11b - c)$

Thus, the sum of the digits is divisible by 9.

Conversely, suppose the sum of the digits is divisible by 9. Then $a + b + c + d = 9k$ for some integer k.

Then $N = 1000a + 100b + 10c + d$
$= 999a + 99b + 9c + a + b + c + d$
$= 999a + 99b + 9c + 9k$
$= 9(111a + 11b + c + k)$

Thus, N is divisible by 9.

10 (a) Squaring a real number produces a non-negative real number.

Hence, $(a - b)^2 \geq 0$
$a^2 - 2ab + b^2 \geq 0$
$a^2 + b^2 \geq 2ab$

(b) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 1 + \frac{a}{b} + \frac{b}{a} + 1$

$= 2 + \frac{a^2 + b^2}{ab}$

From part (a), $a^2 + b^2 \geq 2ab$, and so provided that ab is positive (which it is in this case, since a and b are positive), it follows that $\frac{a^2 + b^2}{ab} \geq \frac{2ab}{ab}$

$\frac{a^2 + b^2}{ab} \geq \frac{2\cancel{ab}}{\cancel{ab}}$

$\frac{a^2 + b^2}{ab} \geq 2$

Combining this inequality with the earlier one yields the result that $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

(c) From part (a): $a^2 + b^2 \geq 2ab$

$a^2 + 2ab + b^2 \geq 4ab$

$(a+b)^2 \geq 4ab$

$a + b \geq \sqrt{4ab}$ (provided a, b positive)

$a + b \geq 2\sqrt{ab}$

$\frac{a+b}{2} \geq \sqrt{ab}$

11 Let n be an odd integer. Then $n = 2k + 1$ for some integer k.

$n = 2k + 1$
$= k^2 + 2k + 1 - k^2$
$= (k + 1)^2 - k^2$

12 For a contradiction, suppose there exist integers a and b such that $a^2 - 4b - 3 = 0$, and so $a^2 = 4b + 3$.

From this equation, a^2 must be odd, and so a must be odd.

Hence, $a = 2k + 1$ for some integer k.

Then $(2k + 1)^2 = 4b + 3$
$4k^2 + 4k + 1 = 4b + 3$
$4k^2 + 4k - 4b = 2$
$4(k^2 + k - b) = 2$
$k^2 + k - b = \frac{1}{2}$

The left-hand side of this equation is an integer, while the right-hand side is not. This is a contradiction. Therefore, there cannot possibly exist integers a and b such that $a^2 - 4b - 3 = 0$. The stated result follows.

13 If $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+2} + 3^{3k} = 5A$ where A is an integer.

Therefore, $2^{k+2} = 5A - 3^{3k}$

$2^{k+3} = 2(5A - 33^k)$

$2^{k+3} = 10A - 2 \times 3^{3k}$

Thus, $2^{k+3} + 33^{k+3} = 10A - 2 \times 3^{3k} + 3^{3k+3}$

$= 10A - 2 \times 3^{3k} + 3^3 \times 3^{3k}$

$= 10A - 2 \times 3^{3k} + 27 \times 3^{3k}$

$= 10A + 25 \times 3^{3k}$

$= 5(2A + 5 \times 3^{3k})$

As the expression in the brackets is an integer, it follows that $2^{k+3} + 3^{3k+3}$ is divisible by 5.

14 Suppose, for a contradiction, that $\sqrt{6} + \sqrt{10}$ is rational. Then $\sqrt{6} + \sqrt{10} = \frac{p}{q}$ for integers p, q with $q \neq 0$.

$$\left(\sqrt{6} + \sqrt{10}\right)^2 = \left(\frac{p}{q}\right)^2$$

$$6 + 2\sqrt{6}\sqrt{10} + 10 = \frac{p^2}{q^2}$$

$$16 + \sqrt{60} = \frac{p^2}{q^2}$$

$$\sqrt{60} = \frac{p^2}{q^2} - 16$$

$$= \frac{p^2 - 16q^2}{q^2}$$

This is a contradiction, as $\sqrt{60}$ is irrational (since 60 is not a perfect square). The result follows.

15 Suppose, for a contradiction, that $r = \frac{p}{q}$ is a rational solution to the equation, where p, q are integers with no common factor other than 1 and with $q \neq 0$.

Then $\left(\frac{p}{q}\right)^3 + \frac{p}{q} + 1 = 0$

$$\frac{p^3}{q^3} + \frac{p}{q} + 1 = 0$$

$$p^3 + pq^2 + q^3 = 0$$

If p is even and q is odd, then p^3 must be even, pq^2 must be even, and q^3 must be odd; this would mean the left-hand side of the equation must be an odd number—a contradiction (since the right-hand side is 0).

If p is odd and q is even, then p^3 must be odd, pq^2 must be even, and q^3 must be even; this would mean again that the left-hand side of the equation must be an odd number—a contradiction (since the right-hand side is 0).

Finally, if both p and q are odd, then p^3 must be odd, pq^2 must be odd, and q^3 must be odd; this would mean again that the left-hand side of the equation must be an odd number—a contradiction (since the right-hand side is 0).

(Note that both p and q cannot be even as it was assumed that p and q have no common factor other than 1.)

Thus, in all cases, a contradiction is obtained. The result follows.

EXERCISE 2.3

1 $(a-b)^2 \geq 0$, $a^2 + b^2 \geq 2ab$

$\therefore ab(a^2 + b^2) \geq 2a^2b^2$ as both a, b are positive.

2 $x < y$ and x is positive $\quad \therefore x^2 < xy \quad$ [1]

Also $x < y$ and y is positive $\therefore xy < y^2 \quad$ [2]

Link [1] and [2]: $x^2 < y^2$

3 (a) $\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 \geq 0 \quad \therefore \frac{x}{y} + \frac{y}{x} \geq 2$

(b) $(x-y)^2 \geq 0$, $x^2 - xy + y^2 \geq xy$

(c) $x^3 + y^3 = (x+y)(x^2 - xy + y^2) \geq (x+y)xy = xyz\left(\frac{x}{z} + \frac{y}{z}\right)$

(d) $y^3 + z^3 \geq xyz\left(\frac{y}{x} + \frac{z}{x}\right)$, $x^3 + z^3 \geq xyz\left(\frac{x}{y} + \frac{z}{y}\right)$

(e) Add the results of parts (c) and (d):

$$2(x^3 + y^3 + z^3) \geq xyz\left(\frac{x}{z} + \frac{y}{z} + \frac{y}{x} + \frac{z}{x} + \frac{x}{y} + \frac{z}{y}\right)$$

$$\geq xyz(2 + 2 + 2) = 6xyz \text{ (using part (a))}$$

$\therefore x^3 + y^3 + z^3 \geq 3xyz$

(f) (i) Let $a = x^3$, $b = y^3$, $c = z^3$ and substitute into part (e):

$\therefore a + b + c \geq 3\sqrt[3]{abc}$

(ii) Similarly $a + b + d \geq 3\sqrt[3]{abd}$, $a + c + d \geq 3\sqrt[3]{acd}$, $b + c + d \geq 3\sqrt[3]{bcd}$: Multiplying gives the result required.

4 (a) $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$, $c^2 + a^2 \geq 2ca$

Adding gives the result required.

(b) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq 3(ab + bc + ca)$

5 Let $a = xy$, $b = yz$, $c = zx$ and substitute into the given result.

6 (a) $(\sqrt{a} - \sqrt{b})^2 \geq 0 \quad \therefore a + b \geq 2\sqrt{ab} \quad \therefore \frac{a+b}{2} \geq \sqrt{ab}$

Similarly: $\frac{c+d}{2} \geq \sqrt{cd}$

Add these results: $\frac{a+b+c+d}{2} \geq \sqrt{ab} + \sqrt{cd}$

$\therefore \frac{a+b+c+d}{4} \geq \frac{\sqrt{ab} + \sqrt{cd}}{2} \quad$ [1]

Using the result $\frac{A+B}{2} \geq \sqrt{AB}$ with $A = \sqrt{ab}$ and $B = \sqrt{cd}$:

$\therefore \frac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt{\sqrt{ab} \times \sqrt{cd}} = \sqrt[4]{abcd} \quad$ [2]

Link [1] and [2] to obtain the required result.

(b) Let $d = \frac{a+b+c}{3}$; from part (a):

$$\frac{a+b+c+\frac{a+b+c}{3}}{4} \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}$$

$$\frac{a+b+c}{3} \geq \sqrt[4]{abc} \times \sqrt[4]{\frac{a+b+c}{3}}$$

$$\therefore \left(\frac{a+b+c}{3}\right)^4 \geq abc \times \left(\frac{a+b+c}{3}\right) \quad \therefore \left(\frac{a+b+c}{3}\right)^3 \geq abc$$

7 (a) $lb \leq \left(\frac{l+b}{2}\right)^2$, question 6(a)

(b) $lbh \leq \left(\frac{l+b+h}{3}\right)^3$, question 6(b)

8 (a) Using $a^2 + b^2 \geq 2ab \quad$ [1]

Let $a = \frac{1}{\sqrt{x}}$, $b = \frac{1}{\sqrt{y}}$ to obtain $\frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}} \quad$ [2]

Let $a = \sqrt{x}$, $b = \sqrt{y}$ and substitute into [1]: $x + y \geq 2\sqrt{xy} \quad$ [3]

Taking reciprocals of both sides of [3]: $\frac{1}{x+y} \leq \frac{1}{2\sqrt{xy}}$

$\therefore \frac{1}{2\sqrt{xy}} \geq \frac{1}{x+y}$, $\frac{2}{\sqrt{xy}} \geq \frac{4}{x+y} \quad$ [4]

Link [2] and [4]: $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$

(b) From [1], let $a = \frac{1}{x}$, $b = \frac{1}{y}$: $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{2}{xy} \quad$ [5]

From [3], squaring both sides: $(x+y)^2 \geq 4xy$

$\therefore \frac{1}{(x+y)^2} \leq \frac{1}{4xy}$ or $\frac{1}{4xy} \geq \frac{1}{(x+y)^2}$

$\therefore \frac{2}{xy} \geq \frac{8}{(x+y)^2} \quad$ [6]

Link [5] and [6] to get the required result.

9 (a) Assume that $4a^2 - 6ab + 4b^2 < a^2 + b^2$:

$\therefore 3(a-b)^2 < 0$ which is a contradiction as a, b are real

$\therefore$ Assumption is false, so: $4a^2 - 6ab + 4b^2 \geq a^2 + b^2$

(b) $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$
Now $(a-b)^4 \geq 0$, so: $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0$
$\therefore\ a^4 + b^4 \geq 4a^3b - 6a^2b^2 + 4ab^3 = ab(4a^2 - 6ab + 4b^2) \geq ab$
$(a^2 + b^2) = a^3b + ab^3$

10 Assume that $\frac{a^2+b^2}{2} \leq \left(\frac{a+b}{2}\right)^2$: $\frac{a^2+b^2}{2} \leq \frac{a^2+2ab+b^2}{4}$
i.e. $a^2 - 2ab + b^2 \leq 0$ or $(a-b)^2 \leq 0$
which is a contradiction as a, b are real and unequal.
$\therefore$ Assumption is false, so: $\frac{a^2+b^2}{2} > \left(\frac{a+b}{2}\right)^2$

11 (a) $a + b \geq 2\sqrt{ab}$: similarly $b + c \geq 2\sqrt{bc}$ and $c + a \geq 2\sqrt{ca}$
Multiply to get the required result.
(b) $A + B + C \geq 3\sqrt[3]{ABC}$
Let $A = bc(b+c)$, $B = ca(c+a)$, $C = ab(a+b)$:
$bc(b+c) + ca(c+a) + ab(a+b) \geq$
$3\sqrt[3]{bc(b+c)ca(c+a)ab(a+b)}$
$= 3\sqrt[3]{a^2b^2c^2(b+c)(c+a)(a+b)} \geq 3\sqrt[3]{a^2b^2c^2\,8abc}$
using (a), and the result is then obtained.
(c) From $a + b \geq 2\sqrt{ab}$, let $a = b^2c^2$ and
$b = c^2a^2$: $b^2c^2 + c^2a^2 \geq 2abc^2$
Similarly: $c^2a^2 + a^2b^2 \geq 2a^2bc$ and $a^2b^2 + b^2c^2 \geq 2ab^2c$
Add these results to obtain the required result.
(d) From $a + b \geq 2\sqrt{ab}$, let $a = ab$ and $b = xy$:
$ab + xy \geq 2\sqrt{abxy}$
Similarly: $ax + by \geq 2\sqrt{axby}$ Multiply to get the result.
(e) From $a + b \geq 2\sqrt{ab}$, let $a = a^2$ and $b = x^2$:
$a^2 + x^2 \geq 2ax,\ ax \leq \frac{a^2+x^2}{2}$
Similarly: $by \leq \frac{b^2+y^2}{2}$
Adding: $ax + by \leq \frac{a^2+x^2+b^2+y^2}{2} = 1$
(as $a^2 + b^2 = 1$ and $x^2 + y^2 = 1$)

12 (a) (i) From $a + b + c \geq 3\sqrt[3]{abc}$, let $a = x^2y$, $b = y^2z$, $c = z^2x$:
$x^2y + y^2z + z^2x \geq 3xyz$
(ii) Let $a = xy^2$, $b = yz^2$, $c = zx^2$:
$xy^2 + yz^2 + zx^2 \geq 3xyz$
(b) $(x+y+z)^3$
$= x^3 + y^3 + z^3 + 3(x^2y + y^2z + z^2x) + 3(xy^2 + yz^2 + zx^2) + 6xyz \geq$
$3xyz + 9xyz + 9xyz + 6xyz = 27xyz$

13 (a) $(ad - bc)^2 \geq 0 \quad \therefore\ 2abcd \leq a^2d^2 + b^2c^2$
Add $a^2b^2 + c^2d^2$ to both sides:
$(ab + cd)^2 \leq a^2d^2 + b^2c^2 + a^2b^2 + c^2d^2 = (a^2 + c^2)(b^2 + d^2)$
Take the square root of both sides to obtain the result.
(b) Assume that $(a + 5b)(a + 2b) < 9b(a + b)$:
$a^2 + 7ab + 10b^2 < 9ab + 9b^2$
i.e. $a^2 - 2ab + b^2 < 0$ or $(a-b)^2 \leq 0$
which is a contradiction as a, b are real and unequal.
$\therefore$ Assumption is false, so: $(a + 5b)(a + 2b) \geq 9b(a + b)$
(c) $\frac{a^2}{b^2}+\frac{b^2}{a^2}+6-\frac{4a}{b}-\frac{4b}{a} = \frac{a^2b^2}{a^2b^2}\times\left(\frac{a^2}{b^2}-\frac{4a}{b}+3+\frac{b^2}{a^2}-\frac{4b}{a}+3\right)$
$=\frac{a^2(a^2-4ab+3b^2)+b^2(b^2-4ab+3a^2)}{a^2b^2}$
$=\frac{a^2(a-3b)(a-b)+b^2(b-3a)(b-a)}{a^2b^2}$
$=\frac{(a-b)\left[(a^3-3a^2b)-(b^3-3ab^2)\right]}{a^2b^2}$
$=\frac{(a-b)(a-b)^3}{a^2b^2}=\frac{(a-b)^4}{a^2b^2}\geq 0$ as a, b are real

(d) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = 1 + \frac{a}{b} + \frac{b}{a} + 1 \geq 4$
(using the result $\frac{a}{b} + \frac{b}{a} \geq 2$, which has been proved in earlier questions)

14 $\sqrt{s(s-a)(s-b)(s-c)} \leq \frac{s(s-a)+(s-b)(s-c)}{2}$
$\leq \frac{s^2 - as + s^2 - bs - cs + bc}{2}$
$\leq \frac{2s^2 - s(a+b+c) + bc}{2} = \frac{bc}{2}$
i.e. $A \leq \frac{bc}{2}$. Similarly: $A \leq \frac{ca}{2}$ and $A \leq \frac{ab}{2}$
Adding: $3A \leq \frac{ab+bc+ca}{2}$ hence $A \leq \frac{ab+bc+ca}{6}$
But: $ab + bc + ca \leq a^2 + b^2 + c^2\ \therefore\ A \leq \frac{a^2+b^2+c^2}{6}$

15 Let $f(x) = x^p - px + (p-1)$ where integer $p > 1$. Note that $f(x)$ is a continuous function.
Now $f'(x) = px^{p-1} - p$ and $f''(x) = p(p-1)x^{p-2}$
Stationary points where $f'(x) = 0$,
i.e. where $p = 0$ (not possible as $p > 1$) or $x = 1$.
Also: $f''(1) = p(p-1)$ (which is positive as $p > 1$); and $f(1) = 0$
$\therefore$ The only stationary point on the continuous function is a minimum at $(1, 0)$.
So $f(x) \geq 0$ for all x, i.e. $x^p - px + (p-1) \geq 0$
$\therefore\ x^p + (p-1) \geq px$
Equality holds when $f(x) = 0$:
$(x-1)\left(\sum_{i=0}^{p-1} x^i - p\right) = 0$
$\therefore\ x = 1$ or $\sum_{i=0}^{p-1} x^i = p$

16 Let $f(x) = e^x - e^a - e^a x + e^a a$. Note that $f(x)$ is a continuous function.
Now $f'(x) = e^x - e^a$ and $f''(x) = e^x$
Stationary points where $f'(x) = 0$, i.e. where $x = a$.
Also: $f''(a) = e^a$ (which is positive); and $f(a) = 0$
$\therefore$ The only stationary point on the continuous function is a minimum at $(a, 0)$.
So $f(x) \geq 0$ for all x, i.e. $e^x - e^a - e^a x + e^a a \geq 0$
$\therefore\ e^x - e^a \geq e^a(x - a)$

17 (a) $g(0) = \sin 0 - 0 = 0$
$g'(x) = \cos x - 1$: $g'(0) = \cos 0 - 1 = 0$
(b) $-1 \leq \cos x \leq 1$: $-2 \leq \cos x - 1 \leq 0$
i.e. $-2 \leq g'(x) \leq 0$
(c) $g'(x)$ is non-positive $\therefore$ $g(x)$ is non-increasing,
i.e. at every point $g(x)$ is either decreasing or stationary.
From part (a), there is a stationary point at $(0, 0)$, but the function decreases as x increases, i.e. $g(x) \leq 0$ for $x > 0$.
(d) $g(x) \leq 0$: $\sin x - x < 0$, $\sin x < x$ (for $x > 0$)

18 (a) $\frac{d^2y}{dx^2} = e^x$ is positive for all x, so the graph is concave up for all x.
(b) R is above the curve, so the y-coordinate of R is greater than the y-coordinate of the point on the curve directly below R.
(c) $e^a + e^b > 2e^{\frac{a+b}{2}}$; similarly $e^c + e^d > 2e^{\frac{c+d}{2}}$
Also $\frac{a+b}{2} > \frac{c+d}{2}$ so $e^{\frac{a+b}{2}} + e^{\frac{c+d}{2}} > 2e^{\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2}} = 2e^{\frac{a+b+c+d}{4}}$
$\therefore\ e^a + e^b + e^c + e^d > 2e^{\frac{a+b}{2}} + 2e^{\frac{a+b}{2}}$
$= 2\left(e^{\frac{a+b}{2}} + e^{\frac{a+b}{2}}\right) > 4e^{\frac{a+b+c+d}{4}}$

19 (a) Expression $\left(\frac{1}{x}+\frac{1}{y}\right)^2-\frac{4}{xy}$

$$=\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}-\frac{4}{xy}$$

$$=\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}$$

$$=\left(\frac{1}{x}-\frac{1}{y}\right)^2$$

$$\geq 0$$

Hence $\left(\frac{1}{x}+\frac{1}{y}\right)^2 \geq \frac{4}{xy}$

Taking the positive square root of both sides gives $\frac{1}{x}+\frac{1}{y}\geq\frac{2}{\sqrt{xy}}$.

(b) Expression $\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^2-\frac{4}{x^2y^2}$

$$=\frac{1}{x^4}+\frac{2}{x^2y^2}+\frac{1}{y^4}-\frac{4}{x^2y^2}$$

$$=\frac{1}{x^4}-\frac{2}{x^2y^2}+\frac{1}{y^4}$$

$$=\left(\frac{1}{x^2}-\frac{1}{y^2}\right)^2$$

$$\geq 0$$

Hence $\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^2 \geq \frac{4}{x^2y^2}$

Taking the positive square root of both sides gives $\frac{1}{x^2}+\frac{1}{y^2}\geq\frac{2}{xy}$.

20 $1\leq x\leq 4$

$1\leq\sqrt{x}\leq 2$

$2\leq 1+\sqrt{x}\leq 3$

$\frac{1}{3}\leq\frac{1}{1+\sqrt{x}}\leq\frac{1}{2}$

21 (a) (i) $f'(x)=1-e^{x-1}$ and $f''(x)=-e^{x-1}$

Stationary points where $f'(x)=0$, i.e. where $x=1$

Also: $f'(0)>0$ and $f'(2)<0$

i.e. $f'(x)$ changes from +ve to −ve at $x=1$

$\therefore$ Maximum turning point at $(1,0)$

(ii) $f(x)$ is a continuous function with a maximum value of 0

$\therefore f(x)\leq 0$ for all x, i.e. $x-e^{x-1}\leq 0$, $x\leq e^{x-1}$

(iii) $\frac{x_1x_2\ldots x_n}{X^n}=\frac{x_1}{X}\times\frac{x_2}{X}\times\ldots\times\frac{x_n}{X}$

$$\leq e^{\frac{x_1}{X}-1}\times e^{\frac{x_2}{X}-1}\times\ldots\times e^{\frac{x_n}{X}-1}$$

$$=e^{\frac{x_1+x_2+\ldots+x_n}{X}-n}=e^{n-n}=1$$

$\therefore \frac{x_1x_2\ldots x_n}{X^n}\leq 1$

Hence $X^n\geq x_1x_2\ldots x_n$ and so: $X\geq\sqrt[n]{x_1x_2\ldots x_n}$

(b) (i) $\frac{d^2y}{dx^2}=-\frac{1}{x^2}$ is negative for all x in the domain $(x>0)$, so the curve is concave down.

(ii) $\frac{\log_e x_1+\log_e x_2+\ldots+\log_e x_n}{n}\leq\log_e\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)$

(iii) $\log_e(x_1x_2\ldots x_n)\leq n\log_e\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)$

$$\log_e(x_1x_2\ldots x_n)\leq\log_e\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^n$$

$$(x_1x_2\ldots x_n)\leq\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^n$$

$$\therefore\ \frac{x_1+x_2+\ldots+x_n}{n}\geq\sqrt[n]{x_1x_2\ldots x_n}$$

(c) Let $x_1=1, x_2=2, \ldots x_n=n$.

LHS is simplified as the sum of the terms of an arithmetic series to give $\frac{n(n+1)}{2n}\geq\sqrt[n]{n!}$ and hence the result.

EXERCISE 2.4

1 $n=1$: LHS $=\frac{1}{1\times 2}=\frac{1}{2}$; RHS $=\frac{1}{1+1}=\frac{1}{2}=$ LHS

Result is true for $n=1$

Assume the result is true for $n=k$, i.e. assume that

$$\sum_{r=1}^{k}\frac{1}{r(r+1)}=\frac{k}{k+1}.$$

Prove the result is true for $n=k+1$, i.e. prove that

$$\sum_{r=1}^{k+1}\frac{1}{r(r+1)}=\frac{k+1}{k+2}.$$

$$\text{LHS}=\sum_{r=1}^{k+1}\frac{1}{r(r+1)}$$

$$=\sum_{r=1}^{k}\frac{1}{r(r+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}$$

$$=\text{RHS}$$

The result is true for $n=k+1$ if it is true for $n=k$. But the result is true for $n=1$, hence it is true for $n=1+1$ and by the principle of mathematical induction it is true for all $n\geq 1$.

2 $n=2$: Expression $2^3-2=6$, which is a multiple of 6.

Result is true for $n=2$.

Assume the result is true for $n=k$, i.e. assume that $k^3-k=6M$, where M is a positive integer.

Prove the result is true for $n=k+1$, i.e. prove that $(k+1)^3-(k+1)$ is a multiple of 6.

$(k+1)^3-(k+1)$

$=k^3+3k^2+3k+1-k-1$

$=k^3-k+3k^2+3k$

$=6M+3k(k+1)$

$=6\left(M+\frac{k(k+1)}{2}\right)$

If k is odd, $(k+1)$ is even and thus $\frac{k(k+1)}{2}$ is always a positive integer.

Hence $=6\left(M+\frac{k(k+1)}{2}\right)$ is a multiple of 6.

The result is true for $n=k+1$ if it is true for $n=k$. But the result is true for $n=2$, hence it is true for $n=2+1$ and by the principle of mathematical induction it is true for all $n\geq 2$.

3 $n = 2$: Expression is $2^2 + 4 = 8$, which is divisible by 8.
Result is true for $n = 2$.
Assume the result is true for $n = 2k$, i.e. assume that $4k^2 + 4k = 8M$, where M is a positive integer.
Prove the result is true for $n = 2k + 2$, i.e. prove that $4(k + 1)^2 + 4(k + 1)$ is divisible by 8.

$4(k + 1)^2 + 4(k + 1)$
$= 4(k + 1)(k + 1 + 1)$
$= 4(k + 1)(k + 2)$
$= 4(k^2 + 3k + 2)$
$= 4k^2 + 4k + 8k + 8$
$= 8M + 8(k + 1)$
$= 8(M + k + 1)$, which is divisible by 8.

The result is true for $n = 2k + 2$ if it is true for $n = 2k$. But the result is true for $n = 2$, hence it is true for $n = 2 + 2$ and by the principle of mathematical induction it is true for all even integers, $n \geq 2$.

4 $n = 1$: Expression $3^4 - 1 = 80$, which is divisible by 80.
Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that $3^{4k} - 1 = 80M$, where M is a positive integer.
Prove the result is true for $n = k + 1$, i.e. prove that $3^{4(k+1)} - 1$ is divisible by 80.

$3^{4(k+1)} - 1$
$= 3^4 \times 3^{4k} - 1$
$= 81 \times 3^{4k} - 81 + 80$
$= 81(3^{4k} - 1) + 80$
$= 81 \times 80M + 80$
$= 80(81M + 1)$, which is divisible by 80.

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

5 $n = 1$: LHS $= 2 - 1 = 1$; RHS$= 1^2 = 1 =$ LHS
Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that

$$\sum_{r=1}^{k}(2r-1) = k^2$$

Prove the result is true for $n = k + 1$, i.e. prove that

$$\sum_{r=1}^{k+1}(2r-1) = (k+1)^2$$

$$\text{LHS} = \sum_{r=1}^{k+1}(2r-1)$$
$$= \sum_{r=1}^{k}(2r-1) + (2k+1)$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$
$$= \text{RHS}$$

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

6 $n = 1$: LHS $= \dfrac{4}{1 \times 2 \times 3} = \dfrac{2}{3}$; RHS $= 1 - \dfrac{2}{2 \times 3} = \dfrac{2}{3} =$ LHS
Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that

$$\sum_{r=1}^{k}\frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(k+1)(k+2)}$$

Prove the result is true for $n = k + 1$, i.e. prove that

$$\sum_{r=1}^{k+1}\frac{4}{r(r+1)(r+2)} = 1 - \frac{2}{(k+2)(k+3)}$$

$$\text{LHS} = \sum_{r=1}^{k+1}\frac{4}{r(r+1)(r+2)}$$
$$= \sum_{r=1}^{k}\frac{4}{r(r+1)(r+2)} + \frac{4}{(k+1)(k+2)(k+3)}$$
$$= 1 - \frac{2}{(k+1)(k+2)} + \frac{4}{(k+1)(k+2)(k+3)}$$
$$= 1 - \frac{2(k+3)-4}{(k+1)(k+2)(k+3)}$$
$$= 1 - \frac{2k+2}{(k+1)(k+2)(k+3)}$$
$$= 1 - \frac{2}{(k+2)(k+3)}$$
$$= \text{RHS}$$

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

7 $n = 2$: LHS $= \quad 1 - \dfrac{1}{2^2} \quad = \dfrac{3}{4}$; RHS $= \dfrac{2+1}{2 \times 2} = \dfrac{3}{4} =$ LHS
Result is true for $n = 2$.
Assume the result is true for $n = k$, i.e. assume that

$$1 - \frac{1}{2^2} \quad 1 - \frac{1}{3^2} \quad \ldots \quad 1 - \frac{1}{k^2} \quad = \frac{k+1}{2k}$$

Prove the result is true for $n = k + 1$, i.e. prove that

$$1 - \frac{1}{2^2} \quad 1 - \frac{1}{3^2} \quad \ldots \quad 1 - \frac{1}{(k+1)^2} \quad = \frac{k+2}{2(k+1)}$$

$$\text{LHS} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right)$$
$$= \frac{(k+1)}{2k}\left(1 - \frac{1}{(k+1)^2}\right) = \frac{(k+1)}{2k} \times \frac{k(k+2)}{(k+1)^2}$$
$$= \frac{k+2}{2(k+1)} = \text{RHS}$$

8 $n = 1$: LHS $= (-1)^{1-1}(1)^2 = 1$; RHS $= (-1)^{1-1} \quad \dfrac{1(1+1)}{2} \quad = 1 =$ LHS
Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1}k^2 = (-1)^{k-1} \quad \frac{k(k+1)}{2}$$

Prove the result is true for $n = k + 1$, i.e. prove that

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2$$
$$= (-1)^k \quad \frac{(k+1)(k+2)}{2}$$

$$\text{LHS} = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2$$
$$= (-1)^{k-1}\left[\frac{k(k+1)}{2}\right] + (-1)^k(k+1)^2$$
$$= (-1)^k \times \frac{(k+1)}{2} \times [-k + 2(k+1)]$$
$$= (-1)^k \times \frac{(k+1)(k+2)}{2} = \text{RHS}$$

9 $n = 2$: LHS $= \log\left(\frac{1+1}{1}\right) = \log 2$; RHS $= \log(1+1) = \log 2 = $ LHS

Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that

$$\log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log\left(\frac{k+1}{k}\right) = \log(k+1)$$

Prove the result is true for $n = k+1$, i.e. prove that

$$\log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log\left(\frac{k+1}{k}\right) + \log\left(\frac{k+2}{k+1}\right) = \log(k+2)$$

$$\text{LHS} = \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \ldots + \log\left(\frac{k+1}{k}\right) + \log\left(\frac{k+2}{k+1}\right)$$
$$= \log(k+1) + \log\left(\frac{k+2}{k+1}\right)$$
$$= \log\left((k+1) \times \frac{(k+2)}{(k+1)}\right) = \log(k+2) = \text{RHS}$$

10 $n = 1$: Expression $x - y$, which is divisible by $(x - y)$.
Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that $x^k - y^k$ is divisible by $(x - y)$, i.e. that $x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + \ldots + xy^{k-2} + y^{k-1})$.
Prove the result is true for $n = k + 1$, i.e. prove that $x^{k+1} - y^{k+1}$ is divisible by $(x - y)$.
$x^{k+1} - y^{k+1}$
$= x^{k+1} - x^k y + x^k y - y^{k+1}$
$= x^k(x - y) + y(x^k - y^k)$
$= x^k(x - y) + y(x - y)(x^{k-1} + x^{k-2}y + \ldots + xy^{k-2} + y^{k-1})$
$= (x - y)(x^k + y(x^{k-1} + x^{k-2}y + \ldots + xy^{k-2} + y^{k-1}))$,
which is divisible by $(x - y)$.
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

11 $n = 1$: Expression is $x^1 - 1$, which is divisible by $(x - 1)$.
Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that $x^k - 1 = (x - 1)(x^{k-1} + x^{k-2} + \ldots + x + 1)$.
Prove the result is true for $n = k + 1$, i.e. prove that $x^{k+1} - 1$ is divisible by $(x - 1)$.
Consider $\frac{x^{k+1} - 1}{x - 1}$

$$\frac{x^{k+1} - 1}{x - 1}$$
$$= x^k + \frac{x^k - 1}{x - 1}$$
$$= x^k + \frac{(x-1)(x^{k-1} + x^{k-2} + \ldots + x + 1)}{x - 1}$$
$$= x^k + x^{k-1} + x^{k-2} + \ldots + x + 1$$

which has no remainder, hence it is divisible.
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

12 $n = 1$: LHS $= 2\log 2$; RHS $= \log\frac{2^1}{1} = \log 2 = $ LHS
Result is true for $n = 1$.
Assume the result is true for $n = k$, i.e. assume that

$$\sum_{r=1}^{k} r\log\left(\frac{r+1}{r}\right) = \log\frac{(k+1)^k}{k!}.$$

Prove the result is true for $n = k + 1$, i.e. prove that

$$\sum_{r=1}^{k+1} r\log\left(\frac{r+1}{r}\right) = \log\frac{(k+2)^{k+1}}{(k+1)!}$$

$$\text{LHS} = \sum_{r=1}^{k+1} r\log\left(\frac{r+1}{r}\right)$$
$$= \sum_{r=1}^{k} r\log\left(\frac{r+1}{r}\right) + (k+1)\log\left(\frac{k+2}{k+1}\right)$$
$$= \log\frac{(k+1)^k}{k!} + (k+1)\log\left(\frac{k+2}{k+1}\right)$$
$$= k\log(k+1) - \log k! + (k+1)\log(k+2) - (k+1)\log(k+1)$$
$$= (k+1)\log(k+2) - \log(k+1) - \log k!$$
$$= \log(k+2)^{k+1} - \log((k+1)k!)$$
$$= \log(k+2)^{k+1} - \log(k+1)!$$
$$= \log\frac{(k+2)^{k+1}}{(k+1)!}$$
$$= \text{RHS}$$

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

EXERCISE 2.5

1 $S(k)$ is $k^2 - 11k + 30 \geq 0$
For $S(k + 1)$: $(k + 1)^2 - 11(k + 1) + 30$
$= k^2 + 2k + 1 - 11k - 11 + 30$
$= (k^2 - 11k + 30) + (2k - 10) \geq 0$
as $k^2 - 11k + 30 \geq 0$ and $k \geq 6$

2 $S(k)$ is $k^2 + 5k - 14 > 0$
For $S(k + 1)$: $(k + 1)^2 + 5(k + 1) - 14$
$= k^2 + 2k + 1 + 5k + 5 - 14$
$= (k^2 + 5k - 14) + (2k + 6) > 0$
as $k^2 + 5k - 14 > 0$ and $k > 0$

3 $S(k)$ is $12^k - 7^k - 5^k > 0$
For $S(k + 1)$: $12^{k+1} - 7^{k+1} - 5^{k+1}$
$= 12 \times (12^k - 7^k - 5^k) + 5 \times 7^k + 7 \times 5^k > 0$

4 $S(k)$ is $10^k - 2^k - 3^k - 5^k \geq 0$
For $S(k + 1)$: $10^{k+1} - 2^{k+1} - 3^{k+1} - 5^{k+1}$
$= 10 \times (10^k - 2^k - 3^k - 5^k) + 8 \times 2^k + 7 \times 3^k + 5 \times 5^k \geq 0$

5 $S(k)$ is $(1 + x)^k - 1 - kx \geq 0$
For $S(k + 1)$: $(1 + x)^{k+1} - 1 - (k + 1)x$
$= (1 + x)[\,(1 + x)^k - 1 - kx] + kx^2 \geq 0$

6 Let $S(n)$ be the statement that $2n - n^2 > 0$.
Prove that $S(5)$ is true.
LHS $= 2^5 - 5^2$
$= 32 - 25$
$= 7 > 0$
Hence $S(5)$ is true.
Assume that $S(k)$ is true, i.e. assume that $2k - k^2 > 0$.
Prove that $S(k + 1)$ is true if $S(k)$ is true, i.e. $2k^{+1} - (k + 1)^2 > 0$.
LHS $= 2k^{+1} - (k + 1)^2$
$= 2 \times 2k - (k^2 + 2k + 1)$
$= 2 \times 2k - 2k^2 + k^2 - 2k - 1)$
$= 2(2k - k^2) + (k^2 - 2k - 1)$
$> 0 + (k^2 - 2k - 1)$
In the worked example it was shown that $k^2 - 2k - 1 \geq 0$ for $k \geq 3$, so LHS > 0.
Hence S$(k + 1)$ is true if $S(k)$ is true.
But $S(5)$ is true so by the principle of mathematical induction $S(n)$ is true for all $n > 4$.

7 (a) $F(n) = n^3 + 20n$: $F(n+2) = (n+2)^3 + 20(n+2)$

$F(n+2) - F(n) = (n+2)^3 + 20(n+2) - (n^3 + 20n)$
$= (n+2)^3 - n^3 + 40$
$= (n+2-n)((n+2)^2 + n(n+2) + n^2) + 40$
$= 2(n^2 + 4n + 4 + n^2 + 2n + n^2) + 40$
$= 2(3n^2 + 6n + 4) + 40$
$= 6n^2 + 12n + 48$
$= 6(n^2 + 2n + 8)$

(b) $n = 2$: Expression is $2^3 + 40 = 48$, which is divisible by 48.
Hence result is true for $n = 2$.
Assume the result is true for $n = k$, k even, i.e. assume that $k^3 + 20k = 48M$ where M is an integer.
Prove the result is true for $n = k + 2$, i.e. prove that $(k+2)^3 + 20(k+2)$ is divisible by 48 when k is even.
$F(k+2) - F(k) = 6(k^2 + 2k + 8)$
Now prove that $n^2 + 2n + 8$ is divisible by 8 when n is even.
$n = 2$: Expression is $4 + 4 + 8 = 16$, which is divisible by 8.
Hence result is true for $n = 2$.
Assume $k^2 + 2k + 8 = 8N$, where N is an integer.
Prove that $(k+2)^2 + 2(k+2) + 8$ is divisible by 8 when k is even.
$(k+2)^2 + 2(k+2) + 8$
$= k^2 + 4k + 4 + 2k + 4 + 8$
$= k^2 + 2k + 8 + 4k + 8$
$= 8N + 4(k+2)$
Since k is even then $k + 2$ has a factor of 2 and hence the expression is divisible by 8.
Hence $n^2 + 2n + 8$ is divisible by 8 when n is even.
Hence $F(n+2) - F(n)$ is divisible by $6 \times 8 = 48$ when n is even.
Hence $F(n+2)$ and $F(n)$ are each divisible by 48 when n is even.
Hence prove $n^3 + 20n$ is divisible by 48 if n is even.

8 (a) (i) $S(1)$: LHS $= \frac{d}{dx}(x) = 1$; RHS $= 1 \times x^0 = 1 =$ LHS
Hence result is true for $n = 1$.

(ii) Now $x^{k+1} = x \times x^k$.
$S(k)$ assume that $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$
$S(k+1)$ prove that $\frac{d}{dx}\left(x^{k+1}\right) = (k+1)x^k$

$$\begin{aligned}\text{LHS} &= \frac{d}{dx}\left(x^{k+1}\right)\\ &= \frac{d}{dx}\left(x \times x^k\right)\\ &= 1 \times x^k + x \times kx^{k-1}\\ &= x^k + kx^k\\ &= (k+1)x^k\\ &= \text{RHS}\end{aligned}$$

(b) The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

9 The odd numbers are 1, 3, 5, 7, …, $(2n - 1)$ for n a positive integer.
$n = 1$; Expression $2(1) - 1 = 1$. Hence the result is true for $n = 1$.
The first odd number from the formula is 1.
Assume that the kth odd number is $(2k - 1)$.
Prove that the $(k + 1)$th odd number is $(2k + 1)$.
Odd numbers go up by 2 so the odd number after $(2k - 1)$ is $(2k - 1) + 2$, which simplifies to $2k + 1$.
Hence the $(k + 1)$th odd number is $(2k + 1)$ if the kth odd number is $(2k - 1)$.
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

10 $n = r = 1$: LHS $= 1 + 1 = 2$; RHS $= \frac{1+1}{1} = 2 =$ LHS
Result is true when $n = r = 1$.
Assume the result is true for $n = r = k$, i.e. assume that

$$1 + n + \frac{n(n+1)}{2!} + \ldots + \frac{n(n+1)(n+2)\ldots(n+k-1)}{k!} = \frac{(n+1)(n+2)\ldots(n+k)}{k!}.$$

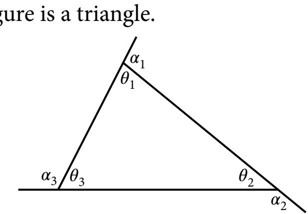
Prove the result is true for $n = r = k + 1$, i.e. prove that

$$1 + n + \frac{n(n+1)}{2!} + \ldots + \frac{n(n+1)(n+2)\ldots(n+k-1)}{k!} + \frac{n(n+1)(n+2)\ldots(n+k)}{(k+1)!} = \frac{(n+1)(n+2)\ldots(n+k)(n+k+1)}{(k+1)!}.$$

$$\begin{aligned}\text{LHS} &= 1 + n + \frac{n(n+1)}{2!} + \ldots + \frac{n(n+1)(n+2)\ldots(n+k-1)}{k!} + \frac{n(n+1)(n+2)\ldots(n+k)}{(k+1)!}\\ &= \frac{(n+1)(n+2)\ldots(n+k)}{k!} + \frac{n(n+1)(n+2)\ldots(n+k)}{(k+1)!}\\ &= \frac{(n+1)(n+2)\ldots(n+k)((k+1)+n)}{(k+1)!}\\ &= \frac{(n+1)(n+2)\ldots(n+k)(n+k+1)}{(k+1)!}\\ &= \text{RHS}\end{aligned}$$

The result is true for $n = r = k + 1$ if it is true for $n = r = k$.
But the result is true for $n = r = 1$, hence by the principle of mathematical induction it is true for all $n = r \geq 1$.

11 $n = 3$: The figure is a triangle.



The α_i are the exterior angles of the triangle, the θ_i are the interior angles of the triangle.
$\theta_1 + \theta_2 + \theta_3 = 180°$, the sum of the angles of the triangle.
$\alpha_1 + \theta_1 = 180°$, a straight angle.
Similarly, $\alpha_2 + \theta_2 = \alpha_3 + \theta_3 = 180°$
So $\alpha_1 + \theta_1 + \alpha_2 + \theta_2 + \alpha_3 + \theta_3 = 3 \times 180°$
$\alpha_1 + \alpha_2 + \alpha_3 = 540° - (\theta_1 + \theta_2 + \theta_3)$
$\alpha_1 + \alpha_2 + \alpha_3 = 540° - 180°$
$\alpha_1 + \alpha_2 + \alpha_3 = 360°$
Hence the result is true when $n = 3$.
The variable in this question is really the angle sum of the polygon involved since the sum of the exterior angles is a constant.
When $n = 3$, the sum of each interior angle and its corresponding exterior angle was $3 \times 180°$.
Thus what really has to be proved is that in an n-sided polygon the sum of each interior angle and its corresponding exterior angle is $n \times 180°$.
Assume that the sum of each interior angle and its corresponding exterior angle of a k-sided polygon is $k \times 180°$.
Prove that the sum of each interior angle and its corresponding exterior angle of a $(k + 1)$ sided polygon is $(k+1) \times 180°$.
In the $(k + 1)$ sided polygon, draw a diagonal to create a k-sided polygon and a triangle.
This triangle gives another interior angle, of the original $(k + 1)$ sided polygon, and its exterior angle, whose sum is 180°.

Hence the sum of each interior angle and its corresponding exterior angle of a $(k+1)$ sided polygon $= k \times 180° + 1 \times 180° = (k+1) \times 180°$.
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 3$, hence by the principle of mathematical induction it is true for all $n \geq 3$.
The angle sum of an n-sided polygon is $(n - 2) \times 180°$ for all integers $n \geq 3$.
Hence the sum of the exterior angles of an n-sided polygon $= n \times 180° - (n-2) \times 180 = 2 \times 180° = 360°$.

12 $S(1)$: LHS$= 2^2 = 4$; RHS $= \dfrac{1\times3\times8}{6} = 4 =$ LHS
Hence $S(1)$ is true.
Assume that $S(k)$ is true, i.e. assume that
$$(k+1)^2+(k+2)^2+(k+3)^2+\ldots+(2k)^2=\frac{k(2k+1)(7k+1)}{6}.$$
Prove that $S(k+1)$ is true, i.e. prove that
$$(k+2)^2+(k+3)^2+(k+4)^2+\ldots+(2k+2)^2 = \frac{(k+1)(2k+3)(7k+8)}{6}.$$
For $S(k+1)$
$$\begin{aligned}\text{LHS} &= (k+2)^2+(k+3)^2+(k+4)^2+\ldots+(2k)^2+(2k+1)^2 \\ &\quad +(2k+2)^2 \\ &= (k+2)^2+(k+3)^2+(k+4)^2+\ldots+4(k+1)^2 \\ &= (k+1)^2+(k+2)^2+(k+3)^2+(k+4)^2+\ldots+(2k)^2 \\ &\quad +(2k+1)^2+3(k+1)^2 \\ &= \frac{k(2k+1)(7k+1)}{6}+(2k+1)^2+3(k+1)^2 \\ &= \frac{2k+1}{6}\big(k(7k+1)+6(2k+1)\big)+3(k+1)^2 \\ &= \frac{2k+1}{6}\left(7k^2+k+12k+6\right)+3(k+1)^2 \\ &= \frac{2k+1}{6}\left(7k^2+13k+6\right)+3(k+1)^2 \\ &= \frac{(2k+1)(7k+6)(k+1)}{6}+3(k+1)^2 \\ &= \frac{k+1}{6}\left(14k^2+19k+6+18k+18\right) \\ &= \frac{k+1}{6}\left(14k^2+37k+24\right) \\ &= \frac{(k+1)(2k+3)(7k+8)}{6}\end{aligned}$$
Hence $S(k+1)$ is true if $S(k)$ is true.
But $S(1)$ is true so by the principle of mathematical induction $S(n)$ is true for all positive integers n.

13 $n = 1$: LHS $= \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{-1}{x^2}$; RHS $= \dfrac{-1}{x^{1+1}} = \dfrac{-1}{x^2} =$ LHS
Result is true for $n = 1$
Assume that the result is true for $n = k$, i.e. that $\dfrac{d}{dx}\left(\dfrac{1}{x^k}\right)=\dfrac{-k}{x^{k+1}}$.
Prove that the result is true for $n = k + 1$, i.e. that
$$\frac{d}{dx}\left(\frac{1}{x^{k+1}}\right)=\frac{-(k+1)}{x^{k+2}}$$
$$\begin{aligned}\text{LHS} &= \frac{d}{dx}\left(\frac{1}{x^{k+1}}\right) \\ &= \frac{d}{dx}\left(\frac{1}{x^k}\times\frac{1}{x}\right) \\ &= \frac{-k}{x^{k+1}}\times\frac{1}{x}+\frac{1}{x^k}\times\frac{-1}{x^2} \\ &= \frac{-k}{x^{k+2}}-\frac{1}{x^{k+2}} \\ &= \frac{-(k+1)}{x^{k+2}} \\ &= \text{RHS}\end{aligned}$$
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence by the principle of mathematical induction it is true for all $n \geq 1$.

14 $n = 1$: LHS $= \dfrac{d}{dx}(x) = 1$; RHS $= 1! = 1 =$ LHS
Result is true for $n = 1$.
Assume that the result is true for $n = k$, i.e. that $\dfrac{d^k}{dx^k}\left(x^k\right) = k!$.
Prove that the result is true for $n = k + 1$, i.e. that
$$\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\right)=(k+1)!.$$
$$\begin{aligned}\text{LHS} &= \frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\right) \\ &= \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(x^{k+1}\right)\right] \\ &= \frac{d}{dx}\big((k+1)k(k-1)\ldots\times2\times x\big) \\ &= (k+1)! \\ &= \text{RHS}\end{aligned}$$
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence by the principle of mathematical induction it is true for all $n \geq 1$.

15 $(x+a)^n=\displaystyle\sum_{r=0}^{n}{}^nC_r x^r a^{n-r}$
$n = 1$: LHS $= x + a$; RHS $= {}^1C_0x^0a^1 + {}^1C_1x^1a^0 = x + a =$ LHS
Hence result is true when $n = 1$.
Assume the result is true for $n = k$, i.e. assume
$$(x+a)^k=x^k+kx^{k-1}a+\frac{k(k-1)}{2!}x^{k-2}a^2+\ldots+kxa^{k-1}+a^k.$$
Prove the result is true for $n = k + 1$, i.e. prove
$$(x+a)^{k+1}=x^{k+1}+(k+1)x^ka+\frac{(k+1)(k)}{2!}x^{k-1}a^2+\ldots$$
$$+(k+1)xa^k+a^{k+1}.$$
$$\begin{aligned}\text{LHS} &= (x+a)(x+a)^k \\ &= (x+a)\left(x^k+kx^{k-1}a+\frac{k(k-1)}{2!}x^{k-2}a^2+\ldots+kxa^{k-1}+a^k\right) \\ &= x^{k+1}+kx^ka+\frac{k(k-1)}{2!}x^{k-1}a^2+\ldots+kx^2a^{k-1}+xa^k \\ &\quad +x^ka+kx^{k-1}a^2+\frac{k(k-1)}{2!}x^{k-2}a^3+\ldots \\ &\quad +\frac{k(k-1)}{2!}x^2a^{k-1}+kxa^k+a^{k+1} \\ &= x^{k+1}+(k+1)x^ka+\left(k+\frac{k(k-1)}{2!}\right)x^{k-1}a^2+\ldots \\ &\quad +\left(k+\frac{k(k-1)}{2!}\right)x^2a^{k-1}+(k+1)xa^k+a^{k+1} \\ &= x^{k+1}+(k+1)x^ka+\frac{k(k+1)}{2!}x^{k-1}a^2+\ldots+\frac{k(k+1)}{2!}x^2a^{k-1} \\ &\quad +(k+1)xa^k+a^{k+1} \\ &= \text{RHS}\end{aligned}$$
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n \geq 1$.

16 (a) $n = 1$, no handshakes
$n = 2$, 1 handshake. $1 = \dfrac{2\times1}{2}$
$n = 3$, say A, B, C so handshakes: AB, AC, BC so $2 + 1 = 3 = \dfrac{3\times2}{2}$
$n = 4$, say A, B, C, D so handshakes: AB, AC, AD, BC, BD, CD so $3 + 2 + 1 = 6 = \dfrac{4\times3}{3}$
For n people, $(n-1) + (n-2) + (n-3) + \ldots + 3 + 2 + 1 = \dfrac{n(n-1)}{2}$

(b) Prove that for n people there are $\frac{n(n-1)}{2}$ handshakes, $n > 1$.

$n = 2$: $\frac{2 \times 1}{2} = 1$

Assume the result is true for $n = k$, i.e. assume that there are $\frac{k(k-1)}{2}$.

Prove the result is true for $n = k + 1$, i.e. prove that there are $\frac{(k+1)k}{2}$.

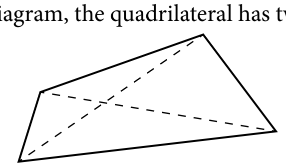
When there are $k + 1$ people, each of the k people originally there now have another person whose hand they can shake, this means another k handshakes.

So with $(k + 1)$ people there are $\frac{k(k-1)}{2} + k$ handshakes,

i.e. $\frac{k^2 - k + 2k}{2} = \frac{k^2 + k}{2} = \frac{(k+1)k}{2}$ handshakes.

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, hence it is true for $n = 1 + 1$ and by the principle of mathematical induction it is true for all $n > 1$.

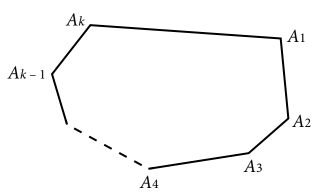
17 $S(4)$: From the diagram, the quadrilateral has two diagonals.



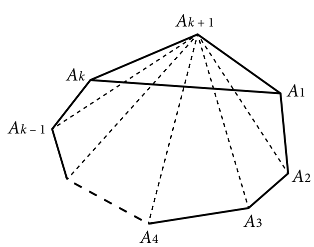
$S(4) = \frac{4(4-3)}{2} = 2$. Hence $S(4)$ is true.

$S(k)$ is that there are $\frac{k(k-3)}{2}$ diagonals.

$S(k + 1)$ is that there are $\frac{(k+1)(k-2)}{2}$ diagonals.



Polygon with k vertices



Polygon with $(k + 1)$ vertices

$S(k + 1)$: The polygon with $(k + 1)$ vertices has all the diagonals of the polygon with k vertices plus the $(k - 2)$ diagonals drawn (dotted lines) plus the diagonal joining A_kA_1 (which was a side in the first polygon).

Number of diagonals =

$\frac{k(k-3)}{2} + (k-2) + 1 = \frac{k(k-3) + 2(k-1)}{2}$

$= \frac{k^2 - k - 2}{2} = \frac{(k+1)(k-2)}{2}$.

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 4$, hence it is true for $n = 4 + 1$ and by the principle of mathematical induction it is true for all $n \geq 4$.

18 (a) $(A^n - B^n)(A - B)$

(b) If $A > B$ then $A^n > B^n$ as A, B both positive

$\therefore (A^n - B^n)(A - B) > 0$

If $A < B$ then $A^n < B^n$ as A, B both positive

$\therefore (A^n - B^n)(A - B) > 0$

If $A = B$ then $(A^n - B^n)(A - B) = 0$

$\therefore (A^n - B^n)(A - B) \geq 0$

$\therefore A^{n+1} - A^nB + B^{n+1} - B^nA \geq 0$

$\therefore A^{n+1} + B^{n+1} \geq A^nB + B^nA$

(c) $S(k)$ is $\left(\frac{A+B}{2}\right)^k - \frac{A^k + B^k}{2} \leq 0$

For $S(k + 1)$:

$\left(\frac{A+B}{2}\right)^{k+1} - \frac{A^{k+1} + B^{k+1}}{2} = \left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^k - \frac{A^{k+1} + B^{k+1}}{2}$

$= \left(\frac{A+B}{2}\right)\left[\left(\frac{A+B}{2}\right)^k - \frac{A^k + B^k}{2}\right] + \frac{AB^k + BA^k - A^{k+1} - B^{k+1}}{4} \leq 0$

using $S(k)$ and the result from part (b).

19 To generate the greatest number of regions, each new line has to intersect each of the existing lines inside the circle. So, if there are two existing lines (giving four regions), then the third line would cut the other two lines inside the circle, producing three extra regions for a total of seven regions.

$S(k)$ is 'k straight lines gives a maximum of $\frac{1}{2}(k^2 + k + 2)$ regions'.

For $S(k + 1)$: next line produces maximum of $(k + 1)$ new regions

$\therefore$ number of regions $= \frac{1}{2}(k^2 + k + 2) + k + 1 = \frac{1}{2}(k^2 + 3k + 4)$

$= \frac{1}{2}((k+1)^2 + (k + 1) + 2)$

20 $S(k)$ is $\sin(x + k\pi) = (-1)^k \sin x$

For $S(k + 1)$: LHS $= \sin(x + (k + 1) \times \pi) = \sin((x + k\pi) + \pi)$

$= -\sin(x + k\pi)$ as $\sin(a + \pi) = -\sin\alpha$

$\therefore$ LHS $= -1 \times (-1)^k \sin x = (-1)^{k+1} \sin x =$ RHS

21 (a)

\$1	(impossible)
\$2	(impossible)
\$3	1 × \$3
\$4	(impossible)
\$5	1 × \$5
\$6	2 × \$3
\$7	(impossible)
\$8	1 × \$3, 1 × \$5
\$9	3 × \$3
\$10	2 × \$5
\$11	2 × \$3, 1 × \$5
\$12	4 × \$3
\$13	1 × \$3, 2 × \$5
\$14	3 × \$3, 1 × \$5
\$15	3 × \$5

(b) 'Using only \$3 and \$5 coins, it is possible to make \$$n$ for all integers $n \geq 8$.'

For $S(k + 1)$, there are 3 cases to consider.

Case 1: k is a multiple of 3, e.g. 9 or 12.

To generate $k + 1$, replace 3 × \$3 by 2 × \$5

Case 2: k is a multiple of 5, e.g. 10.

To generate $k + 1$, replace 1 × \$5 by 2 × \$3

Case 3: $k = 3p + 5q$ where p, q are positive integers.

To generate $k + 1$, replace by $3(p + 2) + 5(q - 1)$

22 (a) $n = 1$: $p_1 = 1$ and $q_1 = 1$ $\quad\therefore$ true for $n = 1$.

Assume true for $n = k$, i.e. assume that p_k and q_k are unique positive integers such that: $(1 + \sqrt{3})^k = p_k + q_k\sqrt{3}$

For $n = k + 1$:

$(1 + \sqrt{3})^{k+1} = (1 + \sqrt{3})^k \times (1 + \sqrt{3})$

$= (p_k + q_k\sqrt{3}) \times (1 + \sqrt{3})$

$= (p_k + 3q_k) + (p_k + q_k)\sqrt{3}$

Hence $p_{k+1} = p_k + 3q_k$ is a unique positive integer and $q_{k+1} = p_k + q_k$ is also a unique positive integer.

(b) $n = 1$: $p_1^2 - 3q_1^2 = 1 - 3 = -2$ $\therefore$ true for $n = 1$.
Assume true for $n = k$, i.e. assume that $(p_k)^2 - 3(q_k)^2 = (-2)^k$.
For $n = k + 1$:
$(p_{k+1})^2 - 3(q_{k+1})^2 = (p_k + 3q_k)^2 - 3(p_k + q_k)^2$
$= (p_k)^2 + 6p_kq_k + 9(q_k)^2 - 3(p_k)^2 - 6p_kq_k - 3(q_k)^2$
$= -2(p_k)^2 + 6(q_k)^2 = -2(p_k^2 - 3q_k^2) = (-2)^{k+1}$ as required.

23 (a) $k^p + pk^{p-1} + \frac{p(p-1)k^{p-2}}{2!} + \frac{p(p-1)(p-2)k^{p-3}}{3!} + \ldots + \frac{p(p-1)(p-2)k^3}{3!} + \frac{p(p-1)k^2}{2!} + pk + 1$

(b) k^p, 1

(c) Verify for $n = 1$. Assume true for $n = k$, i.e. assume that $k^p - k$ is a multiple of p. For $n = k + 1$: $(k + 1)^p - (k + 1)$
$= k^p + pk^{p-1} + \frac{p(p-1)k^{p-2}}{2!} + \frac{p(p-1)(p-2)k^{p-3}}{3!} + \ldots + \frac{p(p-1)(p-2)k^3}{3!} + \frac{p(p-1)k^2}{2!} + pk + 1 + k - 1$
$= (k^p - k) + pk^{p-1} + \frac{p(p-1)k^{p-2}}{2!} + \frac{p(p-1)(p-2)k^{p-3}}{3!} + \ldots + \frac{p(p-1)(p-2)k^3}{3!} + \frac{p(p-1)k^2}{2!} + pk$
which is divisible by p, as required.

24 (a) $\tan\left(\theta + \frac{\pi}{2}\right) = \cot\left(\frac{\pi}{2} - \left(\theta + \frac{\pi}{2}\right)\right) = \cot(-\theta) = -\cot\theta$

(b) Verify for $n = 1$. Assume true for $n = k$, i.e. assume that $\tan\left[(2k+1)\frac{\pi}{4}\right] = (-1)^k$.
For $n = k + 1$: $\tan\left[(2(k+1)+1)\frac{\pi}{4}\right] = \tan\left[(2k+1)\frac{\pi}{4} + \frac{\pi}{2}\right]$
$= -\cot(2k+1)\frac{\pi}{4} = \frac{-1}{(-1)^k} = (-1)^{k+1}$ as required.

25 (a) Assume that $2k + 3 \le 2\sqrt{(k+1)(k+2)}$ for some $k > 0$.
Square both sides, which are known to be positive:
$4k^2 + 12k + 9 \le 4k^2 + 12k + 8$
which produces the contradiction $1 \le 0$
$\therefore$ Assumption is false, so required result is true.

(b) Verify for $n = 1$. Assume true for $n = k$.
Prove true for $n = k + 1$ as follows:
$\text{LHS} > 2\left(\sqrt{k+1} - 1\right) + \frac{1}{\sqrt{k+1}} = \frac{2(k+1) - 2\sqrt{k+1} + 1}{\sqrt{k+1}}$
$= \frac{2k+3}{\sqrt{k+1}} - 2$
$> \frac{2\sqrt{(k+1)(k+2)}}{\sqrt{k+1}} - 2 = 2\left(\sqrt{k+2} - 1\right)$ as required.

(c) Incorrect; $n = 10^{20}$ is one counterexample.

EXERCISE 2.6

The following are only brief outline proofs. For more detailed solutions of the odd-numbered questions, see *New Senior Mathematics Extension 2 for Year 12 Student Worked Solutions*, for even-numbered questions, see Reader+ Worked Solutions.

1 $n = 1$: $u_1 = 1$, $u_2 = 3 \times 1 + 4 = 7$
$S(1)$: $u_1 = 3^1 - 2 = 1$, which is true.
$u_2 = 3^2 - 2 = 7$, which is true.
Result is true when $n = 1$.
$S(k)$: Assume true for $n = k$, i.e. assume that $u_k = 3^k - 2$.
$S(k + 1)$: Prove true for $n = k + 1$, i.e. prove that $u_{k+1} = 3^{k+1} - 2$.
Now $u_{k+1} = 3u_k + 4$
$= 3 \times (3^k - 2) + 4$
$= 3^{k+1} - 6 + 4$
$= 3^{k+1} - 2$
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 4$, hence it is true for $n = 4 + 1$ and by the principle of mathematical induction it is true for all $n \ge 1$.

2 $n = 1$: $u_1 = 6$; $u_2 = 6 + 2^1 + 4 = 12$
$S(1)$: $u_1 = 2^1 + 4 = 6$; $u_2 = 2^2 + 8 = 12$
Result is true when $n = 1$
$S(k)$: Assume true for $n = k$, i.e. assume that $u_k = 2^k + 4k$
$S(k + 1)$: Prove true for $n = k + 1$, i.e. prove that $u_{k+1} = 2^{k+1} + 4(k + 1)$
Now $u_{k+1} = u_k + 2^k + 4$
$= 2^k + 4k + 2^k + 4$
$= 2 \times 2^k + 4k + 4$
$= 2^{k+1} + 4(k + 1)$
The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 4$, hence it is true for $n = 4 + 1$ and by the principle of mathematical induction it is true for all $n \ge 1$.

3 Verify for $n = 1$ and $n = 2$. Assume true for $n = 1, 2, \ldots k$.
Prove true for $n = k + 1$ as follows:
$u_{k+1} = 2u_k - u_{k-1} = 2(2k - 1) - [2(k - 1) - 1]$
$= 4k - 2 - 2k + 2 + 1$
$= 2k + 1 = 2(k + 1) - 1$ as required.

4 Verify for $n = 1$ and $n = 2$. Assume true for $n = 1, 2, \ldots k$.
Prove true for $n = k + 1$ as follows:
$u_{k+1} = 5u_k - 6u_{k-1} = 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1})$
$= 5 \times 2^k + 5 \times 3^k - 3 \times 2^k - 2 \times 3^k$
$= 2 \times 2^k + 3 \times 3^k = 2^{k+1} + 3^{k+1}$ as required.

5 Verify for $n = 1$ and $n = 2$. Assume true for $n = 1, 2, \ldots k$.
Prove true for $n = k + 1$ as follows:
$u_{k+1} = 11u_k - 28u_{k-1} = 11(7^k - 4^k) - 28(7^{k-1} - 4^{k-1})$
$= 11 \times 7^k - 11 \times 4^k - 4 \times 7^k + 7 \times 4^k = 7 \times 7^k - 4 \times 4^k$
$= 7^{k+1} - 4^{k+1}$ as required.

6 Verify for $n = 1$ and $n = 2$. Assume true for $n = 1, 2, \ldots k$.
Prove true for $n = k + 1$ as follows:
$u_{k+1} = 10u_k - 25u_{k-1} = 10(k - 1) \times 5^k - 25(k - 2) \times 5^{k-1}$
$= (2k - 2) \times 5^{k+1} - (k - 2) \times 5^{k+1} = k \times 5^{k+1}$ as required

7 Verify for $n = 1$ and $n = 2$. Assume true for $n = 1, 2, \ldots k$.
Prove true for $n = k + 1$ as follows:
$u_{k+1} = u_k + u_{k-1} < \left(\frac{5}{3}\right)^k + \left(\frac{5}{3}\right)^{k-1} = \left(\frac{5}{3}\right)^{k-1}\left[\left(\frac{5}{3}\right) + 1\right] = \left(\frac{5}{3}\right)^{k-1} \times \frac{8}{3}$
$< \left(\frac{5}{3}\right)^{k-1} \times \frac{25}{9} = \left(\frac{5}{3}\right)^{k+1}$ as required

8 $n = 2$: $a_0 = 1$, $a_1 = 6$, $a_2 = 6a_1 - 9a_0 = 6 \times 6 - 9 = 27$
$S(2)$: $a_2 = 3^2 + 2 \times 3^2 = 9 + 18 = 27$
Result is true when $n = 2$
$S(k)$: Assume true for $n = k - 1$, i.e. assume that $u_{k-1} = 3^{k-1} + (k - 1)3^{k-1}$
Assume true for $n = k$, i.e. assume that $u_k = 3^k + k3^k$
$S(k + 1)$: Prove true for $n = k + 1$, i.e. prove that $u_{k+1} = 3^{k+1} + (k + 1)3^{k+1}$
Now $u_{k+1} = 6u_k - 9u_{k-1}$
$= 6(3^k + k3^k) - 9(3^k - 1 + (k - 1)3^k - 1)$
$= 2 \times 3^{k+1} + 2k3^{k+1} - 3^{k+1} - (k - 1)\, 3^{k+1}$
$= 3^{k+1} + 2k3^{k+1} - k3^{k+1} + 3^{k+1}$
$= 3^{k+1} + k3^{k+1} + 3^{k+1}$
$= 3^{k+1} + (k + 1)3^{k+1}$
The result is true for $n = k + 1$ if it is true for $n = k$ and $n = k - 1$. But the result is true for $n = 2$ and $n = 1$, and by the principle of mathematical induction it is true for all $n \ge 2$.

9 (a) Assume that $(4k + 3)\sqrt{k} > (4k + 1)\sqrt{k+1}$ for some $k \ge 0$.
Square both sides:
$16k^3 + 24k^2 + 9k > 16k^3 + 24k^2 + 9k + 1$
which produces the contradiction $0 > 1$
$\therefore$ Assumption is false, so required result is true

(b) Verify for $n = 1$. Assume true for $n = k$. For $n = k + 1$:

$$\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{k}+\sqrt{k+1} \le \frac{(4k+3)\sqrt{k}}{6}+\sqrt{k+1}$$

$$\le \frac{(4k+1)\sqrt{k+1}}{6}+\frac{6\sqrt{k+1}}{6}=\frac{(4k+7)\sqrt{k+1}}{6} \text{ as required}$$

10 $4=\frac{3}{u_1}$: $u_1=\frac{3}{4}$

$n=1$: $u_1=\frac{3^2-3}{3^2-1}=\frac{3}{4}$ $\quad\therefore$ true for $n = 1$. Assume true for $n = k$.

$n=k+1$: $4=u_k+\frac{3}{u_{k+1}}$ to find:

$$u_{k+1}=\frac{3}{4-u_k}=\frac{3}{4-\frac{3^{k+1}-3}{3^{k+1}-1}}=\frac{3}{\left(4-\frac{3^{k+1}-3}{3^{k+1}-1}\right)}\times\frac{(3^{k+1}-1)}{(3^{k+1}-1)}$$

$$=\frac{3^{k+2}-3}{4(3^{k+1}-1)-(3^{k+1}-3)}=\frac{3^{k+2}-3}{3^{k+2}-1} \text{ as required}$$

11 $n=1$: $u_1=0$, $u_2=(1-x)\times 0+x=x$

$S(1)$: $u_1=\frac{1}{x}(x-1+1-x)=0,\ u_2=\frac{1}{x}\left(2x-1+(1-x)^2\right)$

$$=\frac{1}{x}\left(2x-1+1-2x+x^2\right)=x$$

Result is true when $n = 1, 2$.

$S(k)$: Assume true for $n = k$, i.e. assume that

$u_k=\frac{1}{x}\left(kx-1+(1-x)^k\right)$.

$S(k + 1)$: Prove true for $n = k + 1$, i.e. prove that

$u_{k+1}=\frac{1}{x}\left((k+1)x-1+(1-x)^{k+1}\right)$

Now $u_{k+1}=(1-x)u_k+kx$

$$=\frac{1-x}{x}\left(kx-1+(1-x)^k\right)+kx$$

$$=\frac{1}{x}\left(kx-kx^2-1+x+(1-x)^{k+1}\right)+kx$$

$$=\frac{1}{x}\left(kx-1+x+(1-x)^{k+1}\right)-kx+kx$$

$$=\frac{1}{x}\left((k+1)x-1+(1-x)^{k+1}\right)$$

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$ and, by the principle of mathematical induction, it is true for all $n \ge 1$.

CHAPTER REVIEW 2

1 (a) $n = 1$: LHS = 1; RHS $=\frac{1}{3}(4-1)=1=$ LHS

Result is true for $n = 1$.

Assume that $\sum_{r=1}^{k}(2r-1)^2=\frac{k}{3}\left(4k^2-1\right)$.

Prove that $\sum_{r=1}^{k+1}(2r-1)^2=\frac{(k+1)}{3}\left(4(k+1)^2-1\right)$.

$$\text{LHS}=\sum_{r=1}^{k+1}(2r-1)^2$$

$$=\frac{k}{3}\left(4k^2-1\right)+(2k+1)^2$$

$$=\frac{1}{3}\left(k(2k+1)(2k-1)+3(2k+1)^2\right)$$

$$=\frac{(2k+1)}{3}\left(2k^2-k+6k+3\right)$$

$$=\frac{(2k+1)}{3}((2k+3)(k+1))$$

$$=\frac{(k+1)(2k+1)(2k+3)}{3}$$

$$\text{RHS}=\frac{(k+1)}{3}\left(4(k+1)^2-1\right)$$

$$=\frac{(k+1)}{3}\left(4k^2+8k+4-1\right)$$

$$=\frac{(k+1)}{3}\left(4k^2+8k+3\right)$$

$$=\frac{(k+1)(2k+1)(2k+3)}{3}=\text{LHS}$$

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, and by the principle of mathematical induction it is true for all $n \ge 1$.

(b) $n = 1$: LHS $=\frac{1}{1\times 4}=\frac{1}{4}$; RHS $=\frac{1}{3+1}=\frac{1}{4}=$ LHS

Result is true for $n = 1$.

Assume that $\sum_{r=1}^{k}\frac{1}{(3r-2)(3r+1)}=\frac{k}{3k+1}$.

Prove that $\sum_{r=1}^{k+1}\frac{1}{(3r-2)(3r+1)}=\frac{k+1}{3k+4}$

$$\text{LHS}=\sum_{r=1}^{k+1}\frac{1}{(3r-2)(3r+1)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{k+1}{3k+4}=\text{RHS}$$

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$, and by the principle of mathematical induction it is true for all $n \ge 1$.

(c) $n = 1$: LHS = 2; RHS = $1 \times 2! = 2 =$ LHS

Result is true for $n = 1$.

Assume that $\sum_{r=1}^{k}\left(r^2+1\right)r!=k\times(k+1)!$.

Prove that $\sum_{r=1}^{k+1}\left(r^2+1\right)r!=(k+1)\times(k+2)!$.

$$\text{LHS}=\sum_{r=1}^{k+1}\left(r^2+1\right)r!$$

$$=k\times(k+1)!+\left((k+1)^2+1\right)(k+1)!$$

$$=(k+1)!\left[k+k^2+2k+1+1\right]$$

$$=(k+1)!\left[k^2+3k+2\right]$$

$$=(k+1)!(k+1)(k+2)$$

$$=(k+1)\times(k+2)!=\text{RHS}$$

The result is true for $n = k + 1$ if it is true for $n = k$. But the result is true for $n = 1$ and, by the principle of mathematical induction, it is true for all $n \ge 1$.

(d) $n = 1$: LHS = 1 + 3 = 4; RHS $=\frac{1\times 2^3}{2}=4=$ LHS

Result is true for $n = 1$.

Assume that $\sum_{r=1}^{k}\left(r^3+3r^5\right)=\frac{k^3(k+1)^3}{2}$.

Prove that $\sum_{r=1}^{k+1}\left(r^3+3r^5\right)=\frac{(k+1)^3(k+2)^3}{2}$.

$$\text{LHS}=\sum_{r=1}^{k+1}\left(r^3+3r^5\right)$$
$$=\frac{k^3(k+1)^3}{2}+(k+1)^3+3(k+1)^5$$
$$=\frac{(k+1)^3}{2}\left[k^3+2+6(k+1)^2\right]$$
$$=\frac{(k+1)^3}{2}\left[k^3+2+6k^2+12k+6\right]$$
$$=\frac{(k+1)^3}{2}\left[k^3+6k^2+12k+8\right]$$
$$=\frac{(k+1)^3(k+2)^3}{2}=\text{RHS}$$

The result is true for $n=k+1$ if it is true for $n=k$. But the result is true for $n=1$, and by the principle of mathematical induction, it is true for all $n\geq 1$.

2 (a) Expression $=\dfrac{r+2-r}{r(r+1)(r+2)}=\dfrac{2}{r(r+1)(r+2)}$.

(b) $$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right)$$
$$=\frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{12}\right)+\ldots+\left(\frac{1}{(n-1)n}-\frac{1}{n(n+1)}\right)+\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right)\right]$$
$$=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right)$$
$$=\frac{n^2+3n+2-2}{4(n+1)(n+2)}$$
$$=\frac{n(n+3)}{4(n+1)(n+2)}$$

(c) $n=1$: LHS $=\frac{1}{6}$; RHS $=\dfrac{1\times 4}{4\times 2\times 3}=\dfrac{1}{6}=$ LHS

Result is true for $n=1$.

Assume that $\displaystyle\sum_{r=1}^{k}\frac{1}{r(r+1)(r+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$.

Prove that $\displaystyle\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$.

$$\text{LHS}=\sum_{r=1}^{k+1}\frac{1}{r(r+1)(r+2)}$$
$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$
$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$
$$=\frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$$
$$=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$
$$\text{RHS}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$
$$=\frac{k^2+5k+4}{4(k+2)(k+3)}$$
$$=\frac{(k+1)\left(k^2+5k+4\right)}{4(k+1)(k+2)(k+3)}$$
$$=\frac{k^3+5k^2+4k+k^2+5k+4}{4(k+1)(k+2)(k+3)}$$
$$=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}=\text{LHS}$$

3 (a) 3, 9, 21, 45, 93

(b) $S(k)$ is $T_k=3(2^k-1)$

For $S(k+1)$: LHS $=T_{k+1}$
$=2\times T_k+3=6(2^k-1)+3$
$=6\times 2^k-3=3(2^{k+1}-1)=$ RHS

4 (a) $S(1)$: LHS $=u_1+1$; RHS $=2^0(u_1+1)=u_1+1)=$ LHS

Hence $S(1)$ is true.

$S(k)$: $u_k+1=2^{k-1}(u_1+1)$

$S(k+1)$: $u_{k+1}+1=2^k(u_1+1)$

Now $u_{k+1}=2u_k+1$
$=2(2^{k-1}(u_1+1)-1)+1$
$=2^k(u_1+1)-2+1$
$=u_{k+1}+1=2^k(u_1+1)$

Hence $S(k+1)$ is true if $S(k)$ is true. Since $S(1)$ is true then by the principle of mathematical induction the result is true for all positive integral values of n.

(b) $u_r=2^{r-1}(1+1)-1=2^r-1$

$$\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}2^r-1=\sum_{r=1}^{n}2^r-n=\frac{2\left(2^n-1\right)}{2-1}-n=2^{n+1}-n-2$$

5 $n=1$: $u_2=(1+x)u_1-1x=(1+x)\times 0-x=-x$

$S(1)$: $u_1=\dfrac{1}{x}\left[1+x-(1+x)^1\right]=0$

$S(2)$: $u_2=\dfrac{1}{x}\left[1+2x-(1+x)^2\right]=\dfrac{1}{x}\left[1+2x-\left(1+2x+x^2\right)\right]=-x$

Hence $S(1)$ and $S(2)$ are true.

$S(k)$: Assume that $u_k=\dfrac{1}{x}\left[1+kx-(1+x)^k\right]$.

$S(k+1)$: Prove that $u_{k+1}=\dfrac{1}{x}\left[1+(k+1)x-(1+x)^{k+1}\right]$.

Now:

$$u_{k+1}=(1+x)u_k-kx$$
$$=(1+x)\times\frac{1}{x}\left[1+kx-(1+x)^k\right]-kx$$
$$=\frac{1}{x}\left[(1+x)\left(1+kx-(1+x)^k\right)\right]-kx$$
$$=\frac{1}{x}\left[(1+x)+kx+kx^2-(1+x)^{k+1}\right]-kx$$
$$=\frac{1}{x}\left[1+(k+1)x+kx^2-(1+x)^{k+1}-kx^2\right]$$
$$=\frac{1}{x}\left[1+(k+1)x-(1+x)^{k+1}\right]$$

Hence $S(k+1)$ is true if $S(k)$ is true. Since $S(1)$ is true then by the principle of mathematical induction the result is true for all positive integral values of n.

6 $S(k)$ is $(x+y)^k-x^k-y^k>0$

For $S(k+1)$: $(x+y)^{k+1}-x^{k+1}-y^{k+1}$
$=(x+y)(x+y)^k-x\times x^k-y\times y^k$
$=(x+y)[(x+y)^k-x^k-y^k]+y\times x^k+x\times y^k>0$

using $S(k)$ and $x>0$, $y>0$

7 (a) $$\text{LHS}=\frac{\sin 2kx}{2\sin x}+\cos(2k+1)x$$
$$=\frac{\sin 2kx}{2\sin x}+\cos 2kx\cos x-\sin 2kx\sin x$$
$$=\frac{\sin 2kx+2\sin x(\cos 2kx\cos x-\sin 2kx\sin x)}{2\sin x}$$
$$=\frac{\sin 2kx+\sin 2x\cos 2kx-2\sin 2kx\sin^2 x}{2\sin x}$$
$$=\frac{\sin 2x\cos 2kx-\sin 2kx(2\sin^2 x-1)}{2\sin x}$$
$$=\frac{\sin 2x\cos 2kx+\sin 2kx\cos 2kx}{2\sin x}$$
$$=\frac{\sin\left(2(k+1)x\right)}{2\sin x}=\text{RHS}$$

(b) For $S(k+1)$:

$$\text{LHS} = \frac{\sin 2kx}{2\sin x} + \cos(2k+1)x = \frac{\sin(2(k+1)x)}{2\sin x} = \text{RHS}$$

using the result of part (a).

8 $n = 1$: LHS = 1 and RHS = 1

$\therefore$ True for $n = 1$. Assume true for $n = k$, i.e. assume that:

$$(x_1 + x_2 + x_3 + \ldots + x_k)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_k}\right) \geq k^2$$

Prove true for $n = k+1$ if true for $n = k$.

$$n = k+1: \quad \text{LHS} = \left[(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}\right] \times \left[\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_k}\right) + \frac{1}{x_{k+1}}\right]$$

$$= (x_1 + x_2 + x_3 + \ldots + x_k) \times \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_k}\right) + (x_1 + x_2 + x_3 + \ldots + x_k) \times \frac{1}{x_{k+1}} + x_{k+1} \times \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_k}\right) + x_{k+1} \times \frac{1}{x_{k+1}}$$

$$\geq k^2 + \frac{x_1}{x_{k+1}} + \frac{x_2}{x_{k+1}} + \ldots + \frac{x_k}{x_{k+1}} + \frac{x_{k+1}}{x_1} + \frac{x_{k+1}}{x_2} + \ldots + \frac{x_{k+1}}{x_k} + 1$$

$\geq k^2 + 2k + 1$ (using $a + \frac{1}{a} \geq 2$)

$= (k+1)^2$

$\therefore$ True for $n = k+1$ if true for $n = k$.

True for $n = 1$, $\therefore$ by induction true for all integers $n \geq 1$.

9 (a) Assume that $x + \sqrt{x} < \sqrt{x(x+1)}$ for some $x \geq 0$. Square both sides (which are known to be positive):

$x^2 + 2x\sqrt{x} + x < x^2 + x$

$2x\sqrt{x} < 0$ which is a contradiction.

$\therefore x + \sqrt{x} \geq \sqrt{x(x+1)}$

(b) Verify for $n = 1$ and $n = 2$. Assume true for $n = 1, 2, \ldots k$.

$n = k+1$:

$$u_{k+1} = u_k + ku_{k-1} \geq \sqrt{k!} + k\sqrt{(k-1)!} = \sqrt{(k-1)!} \times (\sqrt{k} + k)$$
$$\geq \sqrt{(k-1)!} \times \sqrt{k(k+1)} = \sqrt{(k+1)!}$$

$\therefore$ True for $n = k+1$ if true for $n = 1, 2, \ldots k$.

True for $n = 1$, $n = 2$, $\therefore$ by induction, true for all integers $n \geq 1$.

CHAPTER 3

EXERCISE 3.1

1 (a) (i) 5 (ii) $\frac{1}{5}\left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right)$ (b) (i) 13 (ii) $\frac{1}{13}\left(-5\underset{\sim}{i} + 12\underset{\sim}{j}\right)$

(c) (i) 25 (ii) $\frac{1}{25}\left(-7\underset{\sim}{i} - 24\underset{\sim}{j}\right)$ (d) (i) 5 (ii) $\frac{1}{5}\left(3\underset{\sim}{i} - 4\underset{\sim}{i}\right)$

2 (a) (i) $-\underset{\sim}{i} + 4\underset{\sim}{j}$ (ii) $\sqrt{17}$ (iii) $\frac{\sqrt{17}}{17}\left(-\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

(b) (i) $6\underset{\sim}{i} + 8\underset{\sim}{j}$ (ii) 10 (iii) $\frac{1}{5}\left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

(c) (i) $2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ (ii) 3 (iii) $\frac{1}{3}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

(d) (i) $-3\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$ (ii) $5\sqrt{2}$ (iii) $\frac{\sqrt{2}}{10}\left(-3\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}\right)$

(e) (i) $4\underset{\sim}{i}$ (ii) 4 (iii) $\underset{\sim}{i}$

(f) (i) $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ (ii) $\sqrt{3}$ (iii) $\frac{\sqrt{3}}{3}\left(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$

3 (a) $2\underset{\sim}{i} + 6\underset{\sim}{j} - \underset{\sim}{k}$ (b) $6\underset{\sim}{j} - 4\underset{\sim}{k}$ (c) $42\underset{\sim}{i} - 6\underset{\sim}{j} - 5\underset{\sim}{k}$

(d) $-12\underset{\sim}{i} - 9\underset{\sim}{j} + 15\underset{\sim}{k}$ (e) $62\underset{\sim}{i} + 3\underset{\sim}{k}$ (f) $-10\underset{\sim}{i} + 27\underset{\sim}{j} - 42\underset{\sim}{k}$

4 $9, \frac{1}{9}\left(\underset{\sim}{i} - 4\underset{\sim}{j} + 8\underset{\sim}{k}\right)$

5 $\hat{\underset{\sim}{a}} = \frac{1}{13}\left(12\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}\right), \hat{\underset{\sim}{b}} = \frac{\sqrt{2}}{10}\left(4\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k}\right),$

$\hat{\underset{\sim}{c}} = \frac{1}{7}\left(-6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

6 $\overrightarrow{PQ} = \underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}, \overrightarrow{RS} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k} = 2\left(\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right) = 2\overrightarrow{PQ}$

$\therefore \overrightarrow{PQ} \parallel \overrightarrow{RS}$.

7 (a) $5\underset{\sim}{i} + 6\underset{\sim}{j} + 8\underset{\sim}{k}$ (b) $-7\underset{\sim}{i} + 35\underset{\sim}{j} + 42\underset{\sim}{k}$ (c) $-7\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$

(d) $8\underset{\sim}{i} - 5\underset{\sim}{j} - 5\underset{\sim}{k}$ (e) $r = 3, s = 2, t = -1$

8 $m = -3, n = 2$

9 $\overrightarrow{RP} = (4, 4, -7), \left|\overrightarrow{RP}\right| = \sqrt{16 + 16 + 49} = 9$

$\overrightarrow{RQ} = (1, 2, -2), \left|\overrightarrow{RQ}\right| = \sqrt{1 + 4 + 4} = 3 \therefore \left|\overrightarrow{RP}\right| = 3\left|\overrightarrow{RQ}\right|$

10 $p = \frac{1}{6}, q = -\frac{11}{12}$

11 (a) 1 (b) 4 (c) 3 (d) 5 (e) $\sqrt{10}$ (f) $\sqrt{17}$

EXERCISE 3.2

1 (a) (i) 1 (ii) 81°52′ (b) (i) 4 (ii) 43°5′

(c) (i) 24 (ii) 47°28′ (d) (i) −2 (ii) 98°27′

2 (a) $\overrightarrow{AB} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}, \overrightarrow{CD} = 5\underset{\sim}{i} - 10\underset{\sim}{j} - 10\underset{\sim}{k}$ (b) 143°18′

(c) $\overrightarrow{BC} = -4\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}, \overrightarrow{AD} = 2\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}$ (d) 142°34′

(e) −3 (f) $-\frac{45\sqrt{14}}{14}$

3 (a) $\underset{\sim}{b} + \underset{\sim}{c} = -2\underset{\sim}{i} - \underset{\sim}{j} - 5\underset{\sim}{k}$,

$$\underset{\sim}{a} \bullet \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right) \bullet \left(-2\underset{\sim}{i} - \underset{\sim}{j} - 5\underset{\sim}{k}\right) = -20$$

$$\underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c} = \left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right) \bullet \left(4\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\right)$$
$$+ \left(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right) \bullet \left(-6\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}\right)$$
$$= 8 - 3 - 3 - 12 + 2 - 12 = -20$$

Hence $\underset{\sim}{a} \bullet \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{a} \bullet \underset{\sim}{c}$

(b) $-100\underset{\sim}{i} + 70\underset{\sim}{j} + 14\underset{\sim}{k}$

4 (a) 1 (b) 29 (c) 3 (d) $\frac{\sqrt{3}}{3}\left(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$

(e) (i) $\frac{1}{3}\left(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$ (ii) $\frac{1}{3}\left(5\underset{\sim}{i} + 13\underset{\sim}{j} - 8\underset{\sim}{k}\right)$

5 (a) 8 (b) 11 (c) $\frac{3\sqrt{29}}{29}$

6 (a) $\frac{\sqrt{5}}{5}\left(2\underset{\sim}{j} + \underset{\sim}{k}\right)$ (b) $\pm\frac{1}{7}\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

7 (a) 0 (b) $3\underset{\sim}{a} \infty \underset{\sim}{b}$

8 (a) (i) $-\frac{42\sqrt{13}}{13}$ (ii) $-\frac{7}{5}\left(\underset{\sim}{i} + 2\underset{\sim}{j}\right)$

(b) (i) $-\frac{3\sqrt{14}}{14}$ (ii) $-\frac{1}{7}\left(4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

(c) (i) $\frac{3\sqrt{29}}{29}$ (ii) $\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ (d) (i) $\frac{2\sqrt{5}}{5}$ (ii) $\underset{\sim}{i} - \underset{\sim}{j}$

9 (a) $a \bullet b = -5,\ a \bullet a = 9,\ b \bullet b = 27$ (b) $-\frac{5\sqrt{3}}{27}$

(c) $\frac{5}{27}(i + j - 5k),\ -\frac{5}{9}\left(2i - 2j + k\right)$

(d) $\pm\frac{1}{\sqrt{218}}\left(11i + 9j - 4k\right)$ or $\pm\frac{\sqrt{218}}{218}\left(11i + 9j - 4k\right)$

10 (a) $a \bullet b = 0,\ a \bullet a = 49,\ b \bullet b = 49$ (b) 0 (c) 0

11 (a) $\frac{1}{3}\left(i + 2j - 2k\right)$ (b) $\frac{1}{7}\left(2i + 3j - 6k\right)$ (c) $\frac{\sqrt{5}}{5}\left(j + 2k\right)$

(d) (i) $\frac{20}{49}\left(2i + 3j - 6k\right)$ (ii) $\frac{1}{49}\left(9i + 38j + 22k\right)$

12 $a \bullet b = |a||b|\cos\theta$

If $a \bullet b = |a||b|$ then $\cos\theta = 1$ and $\theta = 0°$. Thus, the vectors a and b are parallel.

13 $p = -1$

14 (a) $\frac{\sqrt{2}}{2}\left(i + j\right)$ (b) $2i + 2j,\ -i + j - 4k$

15 (a) $\sqrt{76} = 2\sqrt{19}$ (b) $-\frac{13\sqrt{19}}{19}$ (c) 13 (d) $5\frac{2}{13}$

16 $-\frac{1}{\sqrt{21}},\ \frac{22}{\sqrt{21}},\ \frac{21}{\sqrt{21}}$

The projection of the sum of the vectors equals the sum of the projections of each vector.

17 (a) $b - a$ (b) $-a$ (c) $-b$ (d) $-(a + b)$ or $-a - b$ (e) $2(b - a)$

(f) $\overrightarrow{BD} \bullet \overrightarrow{FC} = -(a + b) \bullet 2(b - a)$

$= -2(a \bullet b - a \bullet a + b \bullet b - b \bullet a)$

$= -2\left(|b|^2 - |a|^2\right)$

$= 0$ since $|a| = |b|$ since the hexagon is regular.

Hence $\overrightarrow{BD}$ and $\overrightarrow{FC}$ are perpendicular.

18 $c = -\frac{4}{9}\left(i + 2j - 2k\right),\ d = \frac{1}{9}\left(22i - j + 10k\right)$

19 (a) $\sqrt{14}$ (b) $-2i - 2j - 2k$

20 (a) $\frac{\sqrt{3}}{3}$ (b) $\pm\frac{\sqrt{2}}{2}\left(j - k\right)$

21 (a) $2j + 3k$ (b) $\frac{1}{3}\left(i + 2j + 2k\right)$ (c) 8 (d) $\frac{8}{3}$

22 (a) $\overrightarrow{AC} = 3i + 2j,\ \overrightarrow{DB} = 3i - 2j$

(b) 113°. The acute angle is 67°.

23 (a) $\frac{1}{2}\left(i + j + \sqrt{2}k\right)$ (b) 60° (c) $\frac{1}{2}\left(i + j + \sqrt{2}k\right)$

24 120°

EXERCISE 3.3

1 (a) $\overrightarrow{AC} = (6, 12),\ \overrightarrow{BC} = (2, 4)$ (b) $\overrightarrow{AC} = (6, 12) = 3(2, 4) = 3\overrightarrow{BC}$

(c) Since $\overrightarrow{AC} = 3\overrightarrow{BC}$ then $\overrightarrow{AC} \parallel \overrightarrow{BC}$.

As the vectors share a common point, then A, B and C are collinear.

2 (a) $-2i + 4j$

(b) $\left|\overrightarrow{OA}\right| = 5, \left|\overrightarrow{OB}\right| = 5, \left|\overrightarrow{AB}\right| = 2\sqrt{5}$. Since $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| \neq \left|\overrightarrow{AB}\right|$ then ΔOAB is isosceles.

(c) $4i + 2j$

3 (a) $\overrightarrow{AB} = i + 3j,\ \overrightarrow{CB} = 6i$ (b) $\overrightarrow{OB} = 7i + 3j,\ \overrightarrow{CA} = 5i - 3j$

(c) $\overrightarrow{ON} = \frac{1}{2}\left(7i + 3j\right),\ \overrightarrow{OM} = \frac{1}{2}\left(7i + 3j\right)$

M and N are the same point, hence the diagonals bisect each other.

(d) $\overrightarrow{CP} = 2i - 3j,\ \overrightarrow{BP} = -4i - 3j$

4 (a) $\overrightarrow{AB} = -2i + 4j,\ \overrightarrow{OC} = 7i + 2j$

(b) $\overrightarrow{DC} = 4i$. $\overrightarrow{OA} = 8i = 2\overrightarrow{DC}$ hence DC is parallel to OA and equal to half its length.

5 (a) $\overrightarrow{AB} = i + 2j,\ \overrightarrow{BC} = 2i - j,\ \overrightarrow{AC} = 3i + j$

(b) $\left|\overrightarrow{AB}\right| = \sqrt{5}, \left|\overrightarrow{BC}\right| = \sqrt{5}, \left|\overrightarrow{AC}\right| = \sqrt{10}$.

$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 = 10 = \left|\overrightarrow{AC}\right|^2$.

The triangle is right-angled at B, so $\angle ABC$ is a right angle.

(c) $3i - j$

6 (a) $\overrightarrow{OP} = 5i + j,\ \overrightarrow{OQ} = 3i + 5j,\ \overrightarrow{PQ} = -2i + 4j$

(b) $\overrightarrow{AC} = -4i + 8j = 2\left(-2i + 4j\right) = 2\overrightarrow{PQ}$. Hence $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}$.

$\overrightarrow{PQ} \parallel \overrightarrow{AC}$ and half its length.

7 $\overrightarrow{OA} = i + 2j,\ \overrightarrow{OB} = 5i + 2j,\ \overrightarrow{OC} = 4i - j,\ \overrightarrow{OD} = 2i - j$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 3i - 3j$

$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = -3i - 3j$

$\overrightarrow{AC} \bullet \overrightarrow{BD} = 3\left(i - j\right) \bullet 3\left(-i - j\right) = 9(-1 + 1) = 0.$

Therefore, the diagonals intersect at right angles.

8 (a) $\overrightarrow{OM} = i + \frac{1}{2}j,\ \overrightarrow{ON} = \frac{1}{2}i + j$ (b) 36°52′

9 (a) $m = \sqrt{3}$

(b) $\overrightarrow{OA} = \sqrt{3}i,\ \overrightarrow{OC} = i + j + k,\ \overrightarrow{AC} = \left(1 - \sqrt{3}\right)i + j + k$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \left(1 + \sqrt{3}\right)i + j + k$

$\overrightarrow{AC} \bullet \overrightarrow{OB} = \left(\left(1 - \sqrt{3}\right)i + j + k\right) \bullet \left(\left(1 + \sqrt{3}\right)i + j + k\right)$

$= -2 + 1 + 1 = 0$

Hence the diagonals of $OABC$ are perpendicular.

10 (a) $\overrightarrow{PQ} = -mi + \sqrt{3}mj$ (b) $\overrightarrow{OM} = \frac{1}{2}\left(3mi + \sqrt{3}mj\right)$

(c) $\overrightarrow{OM} \bullet \overrightarrow{PQ} = \left(\frac{1}{2}\left(3mi + \sqrt{3}mj\right)\right) \bullet \left(-mi + \sqrt{3}mj\right)$

$= \frac{1}{2}\left(-3m^2 - 3\sqrt{3}m^2 + 3\sqrt{3}m^2 + 3m^2\right) = 0$

Hence $\overrightarrow{OM} \perp \overrightarrow{PQ}$.

11 $\overrightarrow{OP} = i + j,\ \overrightarrow{OQ} = 5i + 4j,\ \overrightarrow{OR} = 2i + 8j,\ \overrightarrow{OS} = -2i + 5j.$

$\overrightarrow{PQ} = 4i + 3j,\ \overrightarrow{SR} = 4i + 3j$. $\overrightarrow{PQ} = \overrightarrow{SR}$ so $\overrightarrow{PQ} \parallel \overrightarrow{SR}$.

$\overrightarrow{PS} = -3i + 4j,\ \overrightarrow{QR} = -3i + 4j$. $\overrightarrow{PS} = \overrightarrow{QR}$ so $\overrightarrow{PS} \parallel \overrightarrow{QR}$. Hence $PQRS$ is a parallelogram (both pairs of opposite sides parallel).

$\overrightarrow{PQ} \bullet \overrightarrow{QR} = \left(4i + 3j\right) \bullet \left(-3i + 4j\right) = -12 + 12 = 0$ so $\overrightarrow{PQ} \perp \overrightarrow{QR}$ and $PQRS$ is a rectangle (parallelogram with one angle a right angle).

$\left|\overrightarrow{PQ}\right| = 5, \left|\overrightarrow{QR}\right| = 5$, so $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{QR}\right|$. Hence $PQRS$ is a square (rectangle with a pair of adjacent sides equal).

12 $\overrightarrow{OA} = 2i + j + 2k,\ \overrightarrow{OB} = 3i - j,\ \overrightarrow{OC} = i - 2j - 2k.$

$\overrightarrow{AC} = -i - 3j - 4k$

$\overrightarrow{AC} \bullet \overrightarrow{OB} = \left(-i - 3j - 4k\right) \bullet \left(3i - j\right) = -3 + 3 + 0 = 0.$

Hence $\overrightarrow{AC} \perp \overrightarrow{OB}$.

$\left|\overrightarrow{AO}\right| = 3,\ \overrightarrow{AB} = i - 2j - 2k, \left|\overrightarrow{AB}\right| = 3$, so ΔOAB is isosceles with $AO = AB$ so the line from the vertex A perpendicular to the side OB bisects OB.

Hence AC bisects OB at right angles.

13 (a) $\overrightarrow{OP} = 4i + j,\ \overrightarrow{OQ} = 3i - 3j + 4k,\ \overrightarrow{OR} = i - 2j + k.$

$\overrightarrow{PQ} = -i - 4j + 4k,\ \overrightarrow{QR} = -2i + j - 3k,\ \overrightarrow{RP} = 3i + 3j - k.$

$\overrightarrow{QR} \bullet \overrightarrow{RP} = \left(-2i + j - 3k\right) \bullet \left(3i + 3j - k\right) = -6 + 3 + 3 = 0$

so $\overrightarrow{QR} \perp \overrightarrow{RP}$ and ΔPQR is right-angled at R.

(b) $\left|\overrightarrow{QR}\right| = \sqrt{14}, \left|\overrightarrow{RP}\right| = \sqrt{19}$. Area $\Delta PQR\ \frac{1}{2} \times \sqrt{14} \times \sqrt{19} = \frac{\sqrt{266}}{2}$

14 (a) $\overrightarrow{AB} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$, $\overrightarrow{BC} = -8\underset{\sim}{i} - 2\underset{\sim}{j}$, $\overrightarrow{AC} - 5\underset{\sim}{i} + 3\underset{\sim}{j}$

(b) $\overrightarrow{AB} \bullet \overrightarrow{AC} = \left(3\underset{\sim}{i} + 5\underset{\sim}{j}\right) \bullet \left(-5\underset{\sim}{i} + 3\underset{\sim}{j}\right) = -15 + 15 = 0.$

Hence ABC is right-angled.

15 Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$. Thus, $\overrightarrow{OP} = \frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{OQ} = \frac{1}{2}\underset{\sim}{b}$ and

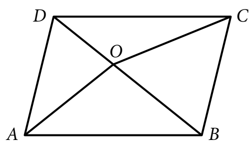
$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$.

Now $\overrightarrow{PQ} = \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a} = \frac{1}{2}\left(\underset{\sim}{b} - \underset{\sim}{a}\right) = \frac{1}{2}\overrightarrow{AB}$

Hence $\overrightarrow{PQ} \parallel \overrightarrow{AB}$ and is half its length.

16 $\overrightarrow{AB} = 3\left(\underset{\sim}{i} + \underset{\sim}{j}\right)$

17 $\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{AB} + \overrightarrow{OC}$



$\overrightarrow{AB} = \overrightarrow{DC}$

$\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{DC} + \overrightarrow{OC}$

$= \overrightarrow{DC} - \overrightarrow{CO}$

$= \overrightarrow{OD}$

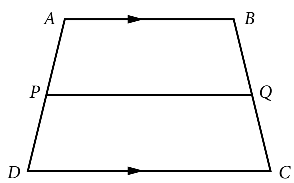
$\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{CO} = \overrightarrow{AB} + \overrightarrow{CO}$

$\overrightarrow{AB} = \overrightarrow{DC}$

$\overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{DC} + \overrightarrow{CO}$

$= \overrightarrow{DO}$

18 (a) $\overrightarrow{PQ} = \overrightarrow{PD} + \overrightarrow{DC} + \overrightarrow{CQ}$



$= \frac{1}{2}\overrightarrow{AD} + \overrightarrow{DC} + \frac{1}{2}\overrightarrow{CB}$

$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AB} + \overrightarrow{BQ}$

$= \frac{1}{2}\overrightarrow{DA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$

(b) $2\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AD} + \overrightarrow{DC} + \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{DA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$

$= \frac{1}{2}\overrightarrow{AD} - \frac{1}{2}\overrightarrow{AD} + \overrightarrow{DC} + \frac{1}{2}\overrightarrow{CB} - \frac{1}{2}\overrightarrow{CB} + \overrightarrow{AB}$

$= \overrightarrow{AB} + \overrightarrow{DC}$

$\overrightarrow{PQ} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{DC}\right)$

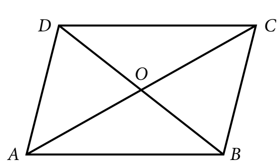
Since $AB \parallel DC$ then $\overrightarrow{AB} = k\overrightarrow{DC}$

So $\overrightarrow{PQ} = \frac{1}{2}\left(k\overrightarrow{DC} + \overrightarrow{DC}\right) = \frac{k+1}{2}\overrightarrow{DC}$

Hence $PQ \parallel DC$ and $PQ \parallel AB$.

Since $\overrightarrow{PQ} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{DC}\right)$ then $\left|\overrightarrow{PQ}\right| = \frac{1}{2}\left(\left|\overrightarrow{AB}\right| + \left|\overrightarrow{DC}\right|\right)$

19 Let $\overrightarrow{AB} = \underset{\sim}{u} = \overrightarrow{DC}$, $\overrightarrow{AD} = \underset{\sim}{v} = \overrightarrow{BC}$



$\overrightarrow{DB} = \overrightarrow{AB} - \overrightarrow{AD} = \underset{\sim}{u} - \underset{\sim}{v}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underset{\sim}{u} + \underset{\sim}{v}$

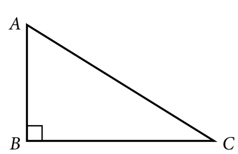
$DB \perp AC$: $\overrightarrow{DB} \bullet \overrightarrow{AC} = 0$

$\left(\underset{\sim}{u} - \underset{\sim}{v}\right) \bullet \left(\underset{\sim}{u} + \underset{\sim}{v}\right) = 0$

$\underset{\sim}{u} \bullet \underset{\sim}{u} + \underset{\sim}{u} \bullet \underset{\sim}{v} - \underset{\sim}{v} \bullet \underset{\sim}{u} - \underset{\sim}{v} \bullet \underset{\sim}{v} = 0$

$\underset{\sim}{u} \bullet \underset{\sim}{u} = \underset{\sim}{v} \bullet \underset{\sim}{v}$

20 $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$, $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$



Now $AB \perp BC$: $\left|\overrightarrow{AB}\right| \bullet \left|\overrightarrow{BC}\right| = 0$

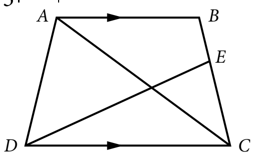
$\left(\underset{\sim}{b} - \underset{\sim}{a}\right) \bullet \left(\underset{\sim}{c} - \underset{\sim}{b}\right) = 0$

$\underset{\sim}{b} \bullet \underset{\sim}{c} - \underset{\sim}{b} \bullet \underset{\sim}{b} - \underset{\sim}{a} \bullet \underset{\sim}{c} + \underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

$\underset{\sim}{b} \bullet \underset{\sim}{b} = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{c} - \underset{\sim}{a} \bullet \underset{\sim}{c}$

$\underset{\sim}{b}^2 = \underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{c} - \underset{\sim}{a} \bullet \underset{\sim}{c}$

21 $\left|\overrightarrow{AB}\right| = x, \left|\overrightarrow{DC}\right| = 2x, \left|\overrightarrow{DA}\right| = y, \left|\overrightarrow{BE}\right| = \frac{1}{3}\left|\overrightarrow{BC}\right|$



Let $\overrightarrow{AB} = \underset{\sim}{b}$, $\overrightarrow{DA} = \underset{\sim}{a}$, $\overrightarrow{DC} = 2\underset{\sim}{b}$

Now $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = 2\underset{\sim}{b} - \underset{\sim}{a}$

and $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = \underset{\sim}{a} + \underset{\sim}{b}$

so $\overrightarrow{BC} = \overrightarrow{DC} - \overrightarrow{DB} = 2\underset{\sim}{b} - \underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{b} - \underset{\sim}{a}$

Hence $\overrightarrow{BE} = \frac{1}{3}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

So $\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE} = \underset{\sim}{a} + \underset{\sim}{b} + \frac{1}{3}\left(\underset{\sim}{b} - \underset{\sim}{a}\right) = \frac{2}{3}\left(2\underset{\sim}{b} + \underset{\sim}{a}\right)$

$\overrightarrow{AC} \bullet \overrightarrow{DE} = \left(2\underset{\sim}{b} - \underset{\sim}{a}\right) \bullet \frac{2}{3}\left(2\underset{\sim}{b} + \underset{\sim}{a}\right)$

$= \frac{2}{3}\left(4\underset{\sim}{b} \bullet \underset{\sim}{b} + 2\underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{a} \bullet 2\underset{\sim}{b} - \underset{\sim}{a} \bullet \underset{\sim}{a}\right)$

$= \frac{2}{3}\left(4\left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{DA}\right|^2\right)$

$= \frac{2}{3}\left(4x^2 - y^2\right)$

22 $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$, $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$

$\overrightarrow{OC} \bullet \overrightarrow{AB} = 0$

$\underset{\sim}{c} \bullet \left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 0$

$\underset{\sim}{c} \bullet \underset{\sim}{b} - \underset{\sim}{c} \bullet \underset{\sim}{a} = 0$

$\underset{\sim}{c} \bullet \underset{\sim}{b} = \underset{\sim}{c} \bullet \underset{\sim}{a}$

$\overrightarrow{OB} \bullet \overrightarrow{AC} = 0$

$\underset{\sim}{b} \bullet \left(\underset{\sim}{c} - \underset{\sim}{a}\right) = 0$

$\underset{\sim}{b} \bullet \underset{\sim}{c} - \underset{\sim}{b} \bullet \underset{\sim}{a} = 0$

$\underset{\sim}{b} \bullet \underset{\sim}{c} = \underset{\sim}{b} \bullet \underset{\sim}{a}$

Consider $\overrightarrow{OA} \bullet \overrightarrow{BC} = \underset{\sim}{a} \bullet \left(\underset{\sim}{c} - \underset{\sim}{b}\right)$

$= \underset{\sim}{a} \bullet \underset{\sim}{c} - \underset{\sim}{a} \bullet \underset{\sim}{b}$

$= \underset{\sim}{c} \bullet \underset{\sim}{b} - \underset{\sim}{b} \bullet \underset{\sim}{c}$ from previous two parts.

$= 0$

Hence $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$.

23 (a) $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 6\underset{\sim}{k}$

$\underset{\sim}{a} \bullet \underset{\sim}{b} = 6 - 12 + 6 = 0$

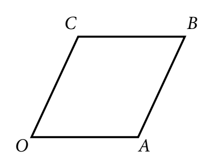
The diagonals of the parallelogram are perpendicular, so the shape is a rhombus.

(b) $\left|\underset{\sim}{a}\right| = \sqrt{26}, \left|\underset{\sim}{b}\right| = 7$

$\frac{5\sqrt{3}}{2}$, 72°8′, 107°52′

24 $5\sqrt{3}$

25 Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{b}$ where $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$ since $OABC$ is a rhombus.



$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{AC} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{OB} \bullet \overrightarrow{AC} = \left(\underset{\sim}{a} + \underset{\sim}{b}\right) \bullet \left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

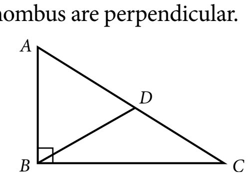
$= \underset{\sim}{a} \bullet \underset{\sim}{b} - \underset{\sim}{a} \bullet \underset{\sim}{a} + \underset{\sim}{b} \bullet \underset{\sim}{b} - \underset{\sim}{a} \bullet \underset{\sim}{b}$

$= \left|\underset{\sim}{b}\right|^2 - \left|\underset{\sim}{a}\right|^2$

$= 0$

Hence $OC \perp AC$, so the diagonals of a rhombus are perpendicular.

26 D is the midpoint of AC.



Let $\overrightarrow{BA} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$

$\overrightarrow{BA} \bullet \overrightarrow{BC} = 0$

$\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

$\overrightarrow{AC} = \underset{\sim}{b} - a$ so $\overrightarrow{AD} = \overrightarrow{DC} = \frac{1}{2}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$\overrightarrow{BD} = \underset{\sim}{a} + \frac{1}{2}\left(b - \underset{\sim}{a}\right) = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

$\left|\overrightarrow{AC}\right| = \left|\underset{\sim}{b} - \underset{\sim}{a}\right|, \left|\overrightarrow{AB}\right| = \left|\underset{\sim}{a}\right|, \left|\overrightarrow{BC}\right| = \left|\underset{\sim}{b}\right|$

Now:

$\left|\overrightarrow{AD}\right| = \left|\overrightarrow{DC}\right| = \frac{1}{2}\left|\left(\underset{\sim}{b} - \underset{\sim}{a}\right)\right| = \frac{1}{2}\sqrt{\left|\underset{\sim}{b} - \underset{\sim}{a}\right|^2} = \frac{1}{2}\sqrt{\left|\underset{\sim}{b}\right|^2 - 2\underset{\sim}{b} \bullet \underset{\sim}{a} + \left|\underset{\sim}{a}\right|^2}$

$= \frac{1}{2}\sqrt{\left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a}\right|^2}$

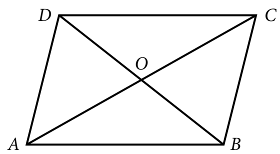
and $\left|\overrightarrow{BD}\right| = \frac{1}{2}\left|\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\right| = \frac{1}{2}\sqrt{\left|\underset{\sim}{b}\right|^2 + 2\underset{\sim}{b} \bullet \underset{\sim}{a} + \left|\underset{\sim}{a}\right|^2} = \frac{1}{2}\sqrt{\left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a}\right|^2}$

Hence D is equidistant from the three vertices of ΔABC.

27 Given $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BD}\right|$, $ABCD$ is a parallelogram.

Let $\overrightarrow{AD} = \underset{\sim}{a} = \overrightarrow{BC}$, $\overrightarrow{AB} = \underset{\sim}{b} = \overrightarrow{DC}$.

Now $\overrightarrow{AC} = \underset{\sim}{b} + a$ and $\overrightarrow{BD} = \underset{\sim}{a} - \underset{\sim}{b}$



Hence $\left|\underset{\sim}{b} + \underset{\sim}{a}\right| = \left|\underset{\sim}{a} - \underset{\sim}{b}\right|$

$\left|\underset{\sim}{b} + \underset{\sim}{a}\right|^2 = \left|\underset{\sim}{a} - \underset{\sim}{b}\right|^2$

$\left|\underset{\sim}{b}\right|^2 + 2\underset{\sim}{b} \bullet \underset{\sim}{a} + \left|\underset{\sim}{a}\right|^2 = \left|\underset{\sim}{a}\right|^2 - 2\underset{\sim}{a} \bullet \underset{\sim}{b} + \left|\underset{\sim}{b}\right|^2$

$4\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

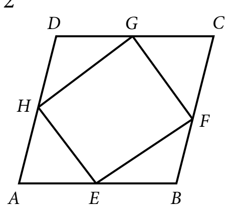
$\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

Thus, $\overrightarrow{AD} \perp \overrightarrow{AB}$ and the parallelogram $ABCD$ is a rectangle.

28 $ABCD$ is a rhombus. E, F, G and H and the midpoints of the sides AB, BC, CD and DA respectively.

Let $\overrightarrow{AD} = \overrightarrow{BC} = \underset{\sim}{a}$, $\overrightarrow{AB} = \overrightarrow{DC} = \underset{\sim}{b}$, $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$

$\overrightarrow{AE} = \overrightarrow{EB} = \frac{1}{2}\underset{\sim}{b} = \overrightarrow{DG} = \overrightarrow{GC}$, $\overrightarrow{AH} = \overrightarrow{HD} = \frac{1}{2}\underset{\sim}{a} = \overrightarrow{BF} = \overrightarrow{FC}$



$\overrightarrow{EH} = \overrightarrow{EA} + \overrightarrow{AH} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$

$\overrightarrow{HG} = \overrightarrow{HD} + \overrightarrow{DG} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

$\overrightarrow{FG} = \overrightarrow{FC} + \overrightarrow{CG} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

Hence $\overrightarrow{EH} = \overrightarrow{FG} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$ and $\overrightarrow{EF} = \overrightarrow{HG} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$, so $EFGH$ is a parallelogram (both pairs of opposite sides parallel).

$\overrightarrow{EH} \bullet \overrightarrow{EF} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right) \bullet \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$

$= \frac{1}{4}\left(\underset{\sim}{a} \bullet b + \underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{b} \bullet \underset{\sim}{b} - \underset{\sim}{b} \bullet \underset{\sim}{a}\right)$

$= \frac{1}{4}\left(\left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{b}\right|^2\right)$

$= 0$

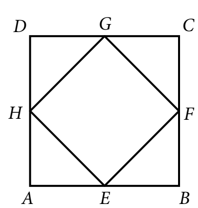
Hence $\overrightarrow{EH} \perp \overrightarrow{EF}$ and thus, $EFGH$ is a rectangle (parallelogram with one angle 90°).

29 $ABCD$ is a square. E, F, G and H and the midpoints of the sides AB, BC, CD and DA respectively.

Prove that $EFGH$ is a square.

Let $\overrightarrow{AD} = \overrightarrow{BC} = \underset{\sim}{a}$, $\overrightarrow{AB} = \overrightarrow{DC} = \underset{\sim}{b}$, $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$

$\overrightarrow{AE} = \overrightarrow{EB} = \frac{1}{2}\underset{\sim}{b} = \overrightarrow{DG} = \overrightarrow{GC}$, $\overrightarrow{AH} = \overrightarrow{HD} = \frac{1}{2}\underset{\sim}{a} = \overrightarrow{BF} = \overrightarrow{FC}$



$\overrightarrow{EH} = \overrightarrow{EA} + \overrightarrow{AH} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$

$\overrightarrow{HG} = \overrightarrow{HD} + \overrightarrow{DG} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

$\overrightarrow{FG} = \overrightarrow{FC} + \overrightarrow{CG} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

Hence $\overrightarrow{EH} = \overrightarrow{FG} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$ and $\overrightarrow{EF} = \overrightarrow{HG} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$, so $EFGH$ is a parallelogram (both pairs of opposite sides parallel).

Consider $\overrightarrow{EH} \bullet \overrightarrow{EF} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right) \bullet \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$

$= \frac{1}{4}\left(\underset{\sim}{a} \bullet b + \underset{\sim}{a} \bullet \underset{\sim}{a} - \underset{\sim}{b} \bullet \underset{\sim}{b} - \underset{\sim}{b} \bullet \underset{\sim}{a}\right)$

$= \frac{1}{4}\left(\left|\underset{\sim}{a}\right|^2 - \left|\underset{\sim}{b}\right|^2\right)$

$= 0$

Hence $\overrightarrow{EH} \perp \overrightarrow{EF}$ and thus, $EFGH$ is a rectangle (parallelogram with one angle 90°).

$\left|\overrightarrow{EH}\right| = \frac{1}{2}\left|\underset{\sim}{a} - \underset{\sim}{b}\right| = \frac{1}{2}\sqrt{\left(\underset{\sim}{a} - \underset{\sim}{b}\right)^2} = \frac{1}{2}\sqrt{\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 - 2\underset{\sim}{a} \bullet \underset{\sim}{b}}$

$= \frac{1}{2}\sqrt{2\left|\underset{\sim}{a}\right|^2} = \frac{\sqrt{2}}{2}\left|\underset{\sim}{a}\right|$

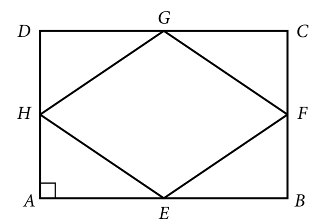
$\left|\overrightarrow{EF}\right| = \frac{1}{2}\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \frac{1}{2}\sqrt{\left(\underset{\sim}{a} + \underset{\sim}{b}\right)^2} = \frac{1}{2}\sqrt{\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + 2\underset{\sim}{a} \bullet \underset{\sim}{b}}$

$= \frac{1}{2}\sqrt{2\left|\underset{\sim}{a}\right|^2} = \frac{\sqrt{2}}{2}\left|\underset{\sim}{a}\right|$

Hence $\left|\overrightarrow{EH}\right| = \left|\overrightarrow{EF}\right|$, so $EFGH$ is a square (rectangle with a pair of adjacent sides equal).

30 Let $\overrightarrow{AD} = \overrightarrow{BC} = \underset{\sim}{a}$, $\overrightarrow{AB} = \overrightarrow{DC} = \underset{\sim}{b}$

$\overrightarrow{AE} = \overrightarrow{EB} = \frac{1}{2}\underset{\sim}{b} = \overrightarrow{DG} = \overrightarrow{GC}$, $\overrightarrow{AH} = \overrightarrow{HD} = \frac{1}{2}\underset{\sim}{a} = \overrightarrow{BF} = \overrightarrow{FC}$



$\overrightarrow{EH} = \overrightarrow{EA} + \overrightarrow{AH} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$

$\overrightarrow{HG} = \overrightarrow{HD} + \overrightarrow{DG} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

$\overrightarrow{FG} = \overrightarrow{FC} + \overrightarrow{CG} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$

Hence $\overrightarrow{EH} = \overrightarrow{FG} = \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right)$ and $\overrightarrow{EF} = \overrightarrow{HG} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$, so $EFGH$ is a parallelogram (both pairs of opposite sides parallel).

Now $AD \perp AB$ so $\overrightarrow{AD} \bullet \overrightarrow{AB} = 0$ hence $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$.

$\left|\overrightarrow{EH}\right| = \frac{1}{2}\left|\underset{\sim}{a} - \underset{\sim}{b}\right| = \frac{1}{2}\sqrt{\left(\underset{\sim}{a} - \underset{\sim}{b}\right)^2} = \frac{1}{2}\sqrt{\underset{\sim}{a} \bullet \underset{\sim}{b} - 2\underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{b}}$

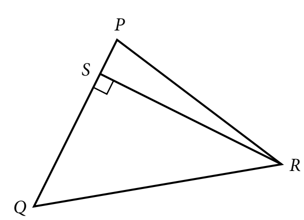
$= \frac{1}{2}\sqrt{\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2}$

$\left|\overrightarrow{EF}\right| = \frac{1}{2}\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \frac{1}{2}\sqrt{\left(\underset{\sim}{a} + \underset{\sim}{b}\right)^2} = \frac{1}{2}\sqrt{\underset{\sim}{a} \bullet \underset{\sim}{b} + 2\underset{\sim}{a} \bullet \underset{\sim}{b} + \underset{\sim}{b} \bullet \underset{\sim}{b}}$

$= \frac{1}{2}\sqrt{\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2}$

Thus, $\left|\overrightarrow{EH}\right| = \left|\overrightarrow{EF}\right|$, so the parallelogram $EFGH$ has a pair of adjacent sides equal, so it is a rhombus.

31 $RS \perp PQ$



In ΔSQR, $\cos Q = \frac{\left|\overrightarrow{SQ}\right|}{\left|\overrightarrow{QR}\right|}$

$\left|\overrightarrow{SQ}\right| = \left|\overrightarrow{QR}\right|\cos Q$

In ΔSPR, $\cos P = \frac{\left|\overrightarrow{PS}\right|}{\left|\overrightarrow{PR}\right|}$

$\left|\overrightarrow{PS}\right| = \left|\overrightarrow{PR}\right|\cos P$

$\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{PS}\right| + \left|\overrightarrow{SQ}\right| = \left|\overrightarrow{PR}\right|\cos P + \left|\overrightarrow{QR}\right|\cos Q$

$\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ}$

$\overrightarrow{PQ} \bullet \overrightarrow{PQ} = \overrightarrow{PR} \bullet \overrightarrow{PQ} + \overrightarrow{RQ} \bullet \overrightarrow{PQ}$

$\left|\overrightarrow{PQ}\right|^2 = \left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|\cos P + \left|\overrightarrow{RQ}\right|\left|\overrightarrow{PQ}\right|\cos Q$

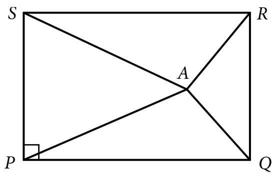
Divide both sides by $\left|\overrightarrow{PQ}\right|$:

$\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{PR}\right|\cos P + \left|\overrightarrow{RQ}\right|\cos Q$

Write this in terms of lengths.

$PQ = PR\cos P + RQ\cos Q$

32



Let $\overrightarrow{PQ} = \overrightarrow{SR} = \underset{\sim}{a}$, $\overrightarrow{PS} = \overrightarrow{QR} = \underset{\sim}{b}$.

Since $PQRS$ is a rectangle, $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$.

Now $\overrightarrow{AQ} = \underset{\sim}{a} + \overrightarrow{AP}$, $\overrightarrow{AS} = \underset{\sim}{b} + \overrightarrow{AP}$

$$\begin{aligned}\left|\overrightarrow{AS}\right|^2 + \left|\overrightarrow{AQ}\right|^2 &= \left|\underset{\sim}{b} + \overrightarrow{AP}\right|^2 + \left|\underset{\sim}{a} + \overrightarrow{AP}\right|^2 \\ &= \left(\underset{\sim}{b} + \overrightarrow{AP}\right)\bullet\left(\underset{\sim}{b} + \overrightarrow{AP}\right) + \left(\underset{\sim}{a} + \overrightarrow{AP}\right)\bullet\left(\underset{\sim}{a} + \overrightarrow{AP}\right) \\ &= \left|\underset{\sim}{b}\right|^2 + 2\underset{\sim}{b}\bullet\overrightarrow{AP} + \left|\overrightarrow{AP}\right|^2 + \left|\underset{\sim}{a}\right|^2 + 2\underset{\sim}{a}\bullet\overrightarrow{AP} + \left|\overrightarrow{AP}\right|^2 \\ &= \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + 2\overrightarrow{AP}\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + 2\left|\overrightarrow{AP}\right|^2\end{aligned}$$

Now $\overrightarrow{AR} = \overrightarrow{AQ} + \underset{\sim}{b}$.

$$\begin{aligned}\left|\overrightarrow{AP}\right|^2 + \left|\overrightarrow{AR}\right|^2 &= \left|\overrightarrow{AP}\right|^2 + \left|\underset{\sim}{a} + \underset{\sim}{b} + \overrightarrow{AP}\right|^2 \\ &= \left|\overrightarrow{AP}\right|^2 + \left|\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + \overrightarrow{AP}\right|^2 \\ &= \left|\overrightarrow{AP}\right|^2 + \left(\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + \overrightarrow{AP}\right)\bullet\left(\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + \overrightarrow{AP}\right) \\ &= \left|\overrightarrow{AP}\right|^2 + \left(\underset{\sim}{a} + \underset{\sim}{b}\right)\bullet\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + 2\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\bullet\overrightarrow{AP} + \left|\overrightarrow{AP}\right|^2 \\ &= 2\left|\overrightarrow{AP}\right|^2 + \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + 2\underset{\sim}{a}\bullet\underset{\sim}{b} + 2\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\bullet\overrightarrow{AP} \\ &= \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + 2\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\bullet\overrightarrow{AP} + 2\left|\overrightarrow{AP}\right|^2 \\ &= \left|\overrightarrow{AS}\right|^2 + \left|\overrightarrow{AQ}\right|^2\end{aligned}$$

EXERCISE 3.4

1 (a) $AB = 3$, $BC = 3$, $AC = \sqrt{26}$. Isosceles.

(b) $AB = \sqrt{66}$, $BC = \sqrt{245}$, $AC = \sqrt{155}$. None of these.

(c) $AB = \sqrt{14}$, $BC = \sqrt{59}$, $AC = \sqrt{73}$. ΔABC is right-angled at B.

(d) $AB = \sqrt{26}$, $BC = \sqrt{26}$, $AC = \sqrt{26}$. Equilateral.

2 (a) Not collinear (b) Not collinear

(c) Collinear (d) Collinear

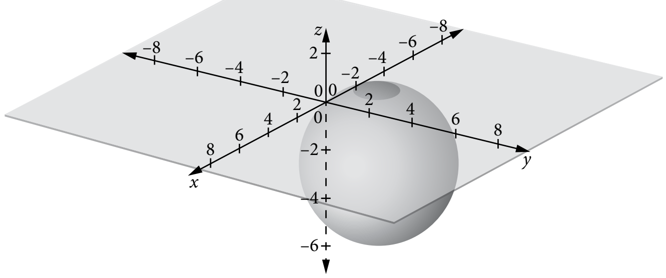
3 (a) $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$

(b) $(x+1)^2 + y^2 + (z-5)^2 = 3$

(c) $(x-3)^2 + (y+2)^2 + (z-4)^2 = 2.25$

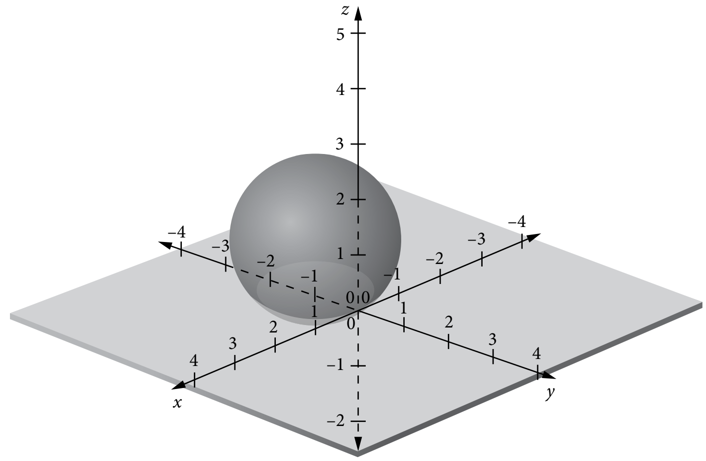
(d) $x^2 + (y+1)^2 + (z-2)^2 = 18$

4 (a) Completing the square gives $(x+2)^2 + (y-1)^2 + (z+3)^2 = 10$.
This is a circle with centre $(-2, 1, -3)$ and radius $\sqrt{10}$.



(b) Completing the square gives $(x-4)^2 + (y+2)^2 + z^2 = 14$.
This is a circle with centre $(4, -2, 0)$ and radius $\sqrt{14}$.

(c) Completing the square gives $x^2 + (y+1)^2 + (z-1)^2 = 2$.
This is a circle with centre $(0, -1, 1)$ and radius $\sqrt{2}$.



(d) Completing the square gives $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 + (z+1)^2 = \frac{3}{2}$.

This is a circle with centre $\left(-\frac{1}{2}, 1\frac{1}{2}, -1\right)$ and radius $\frac{\sqrt{6}}{2}$.

5 $(x-3)^2 + y^2 + (z-7)^2 = 14$

6 $(x-4)^2 + (y-5)^2 + (z+2)^2 = 38$

7 (a) $\overrightarrow{AB} \bullet \overrightarrow{AC} = -2 + 4 - 2 = 0$, $\angle BAC = 90°$ (b) $3\sqrt{2}$

8 $\overrightarrow{DE} = (2,3,4)$, $\overrightarrow{EF} = (-1,3,-9)$, $\overrightarrow{DF} = (1,6,-5)$.

$\overrightarrow{DE} \bullet \overrightarrow{DF} = (2,3,4)\bullet(1,6,-5) = 2 + 18 - 20 = 0$, $\therefore \angle EDF = 90°$.

$\left|\overrightarrow{DE}\right| = \sqrt{29}$, $\left|\overrightarrow{EF}\right| = \sqrt{91}$, $\left|\overrightarrow{DF}\right| = \sqrt{62}$.

$\left|\overrightarrow{DE}\right|^2 + \left|\overrightarrow{DF}\right|^2 = 29 + 62 = 91 = \left|\overrightarrow{EF}\right|^2$,

$\therefore$ DEF is right-angled at D.

9 (a) $(x-2)^2 + (y-2)^2 + (z-3)^2 = 4$

(b) $(x-2)^2 + (y-2)^2 + (z-3)^2 = 9$

10 (a) $z = \frac{7}{8}$

(b) $x^2 + y^2 = \frac{5\sqrt{39}}{8}^2$, $z = \frac{7}{8}$ Centre $\left(0, 0, \frac{7}{8}\right)$, radius $\frac{5\sqrt{39}}{8}$

11 (a) $y = \frac{11}{4}$

(b) $x^2 + z^2 = \frac{135}{16}$, $y = \frac{11}{4}$ Centre $\left(0, \frac{11}{4}, 0\right)$, radius $\frac{3\sqrt{15}}{4}$

12 (a) $x = -\frac{9}{8}$

(b) $y^2 + z^2 = \frac{735}{64}$, $x = -\frac{9}{8}$ Centre $\left(-\frac{9}{8}, 0, 0\right)$, radius $\frac{\sqrt{735}}{8}$

EXERCISE 3.5

1 (a) (i) $y = \frac{x}{2} + 2$ or $x - 2y + 4 = 0, x \ge 0$ (ii) $\underset{\sim}{r} = 2\underset{\sim}{i} + (t+2)\underset{\sim}{j}, t \ge 0$

(b) (i) $y = \frac{1}{x}, x > 0$ (ii) $\underset{\sim}{r} = t\underset{\sim}{i} + \frac{1}{t}\underset{\sim}{j}, t > 0$

(c) (i) $x^2 + y^2 = 4$ (ii) $\underset{\sim}{r} = 2\cos\theta\underset{\sim}{i} + 2\sin\theta\underset{\sim}{j}, 0 \le \theta \le 2\pi$

(d) (i) $y = (x-3)^2 - 5$ or $y = x^2 - 6x + 4, x \ge 3$

(ii) $\underset{\sim}{r} = (t+3)\underset{\sim}{i} + (t^2 - 5)\underset{\sim}{j}, t \ge 0$

(e) (i) $y = 1 - x^{\frac{2}{3}}, -1 \le x \le 1$ (ii) $\underset{\sim}{r} = u^3\underset{\sim}{i} + \left(1 - u^2\right)\underset{\sim}{j}, -1 \le u \le 1$

(f) (i) $(x-y)^2 = 2(x+y), x > 0$ (ii) $\underset{\sim}{r} = \left(2+s\right)\underset{\sim}{i} + \left(s^2 - s\right)\underset{\sim}{j}, s > 0$

(g) (i) $2y^2 = x + 1, -1 \le x \le 1$ (ii) $\underset{\sim}{r} = \cos 2\theta\underset{\sim}{i} + \cos\theta\underset{\sim}{j}, \le \theta \le 2\pi$

(h) (i) $\left(\frac{x}{2}\right)^2 + \frac{y}{\sqrt{3}}^2 = 1$ or $3x^2 + 4y^2 = 12, -2 \le x \le 2$

(ii) $\underset{\sim}{r} = 2\cos t\underset{\sim}{i} + \sqrt{3}\sin t\underset{\sim}{j}, t \ge 0$

2 (a) (i) $x^2+y^2=1$ (ii) $r=\frac{2t}{1+t^2}i+\frac{1-t^2}{1+t^2}j,\ t\in R$

(b) (i) $\sin\phi=\frac{x}{a},\ \cos\phi=\frac{y}{b},\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(ii) $r=a\sin\phi\, i+b\cos\phi\, j,\ \phi\in R$

(c) (i) $y=\pm 2x\sqrt{1-x^2}$ (ii) $r=\sin t\, i+\sin 2t\, j,\ 0\le t\le 2\pi$

(d) (i) $(x-2)^2+(1-y)^2=1$

(ii) $r=(2+\sin\theta)i+(1-\cos\theta)j,\ \theta\ge 0$

(e) (i) $x^2-y^2=4$ (ii) $r=\left(u+\frac{1}{u}\right)i+\left(u-\frac{1}{u}\right)j,\ u\ne 0$

(f) (i) $\frac{x}{3}+\frac{y}{4}=1$ or $4x+3y-12=0,\ 0\le x\le 3,\ 0\le y\le 4$

(ii) $r=3\cos^2 t\, i+4\sin^2 t\, j,\ t\ge 0$

(g) (i) $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$ or $9x^2+4y^2=36$

(ii) $r=\frac{4t}{1+t^2}i+\frac{3(1-t^2)}{1+t^2}j,\ t\in R$

3 (a) (i) $x=2t,\ y=t^2,\ t\ge 0$ (ii) $y=\frac{x^2}{4},\ x\ge 0$

(b) (i) $x=\cos 2\theta,\ y=\sin 2\theta,\ \theta\in R$ (ii) $x^2+y^2=1$

(c) (i) $x=4\cos^2\theta,\ y=3\sin^2\theta,\ \theta\in R$

(ii) $3x+4y=12,\ 0\le x\le 4,\ 0\le y\le 3$

(d) (i) $x=t,\ y=4at^2,\ t\ge 0$ (ii) $y=4ax^2,\ x\ge 0$

(e) (i) $x=2-\sin\theta,\ y=1+\cos\theta,\ 0\le\theta\le 2\pi$

(ii) $(x-2)^2+(y-1)^2=1$

(f) (i) $x=t+\frac{1}{t},\ y=t-\frac{1}{t},\ t\ne 0$ (ii) $x^2-y^2=4,\ x+y\ne 0$

(g) (i) $x=u+3,\ y=u^2-5,\ u\in R$

(ii) $y=(x-3)^2-5$ or $y=x^2-6x+4$

(h) (i) $x=2\sin t,\ y=4\cos t,\ t\in R$ (ii) $\frac{x^2}{4}+\frac{y^2}{16}=1$

(i) (i) $x=t-\frac{1}{t},\ y=t^2+\frac{1}{t^2},\ t\ne 0$ (ii) $y=x^2+2$

4 $x=a\cos t,\ y=a\sin t$

$\frac{x}{a}=\cos t,\ \frac{y}{a}=\sin t$

$\sin^2 t+\cos^2 t=1$

$\frac{y^2}{a^2}+\frac{x^2}{a^2}=1$

$x^2+y^2=a^2$

This is the equation of a circle centre (0, 0), radius = a.

5 (a), (b) $x=4\cos 3t,\ y=4\sin 3t$

$\frac{x}{4}=\cos 3t,\ \frac{y}{4}=\sin 3t$

$\sin^2 3t+\cos^2 3t=1$ so $\frac{x^2}{16}+\frac{y^2}{16}=1$

$x^2+y^2=16$ is a circle, centre (0, 0), radius = 4

(c) $|r|=4$

EXERCISE 3.6

1 (a) $r=(1+\lambda)i+(2+\lambda)j$ (b) (i) (1, 2) (ii) (0, 1) (iii) (3, 4)

2 (a) $r=(-1+\lambda)i+(4-\lambda)j+(6+\lambda)k$

(b) (i) $\left(-\frac{1}{2},\ 3\frac{1}{2},\ 6\frac{1}{2}\right)$ (ii) (5, –2, 12) (iii) (–4, 7, 3)

3 (a) $r=(4+4\lambda)i+(2+4\lambda)j$

(b) (i) (12, 10) (ii) (20, 18) (iii) (36, 34)

4 (a) $r=(2-5\lambda)i+(3-5\lambda)j+(4+3\lambda)k$

(b) (i) (7, 8, 1) (ii) (2, 3, 4) (iii) (–3, –2, 7)

5 $r=(4+2\lambda)i+12j+(-3-3\lambda)k$

6 $r=(3+3\lambda)i+(5-\lambda)j+(7-2\lambda)k$

EXERCISE 3.7

1 D 2 B 3 A 4 C

5 (a) $a=9$ (b) $a=-12$

6 (a) $x=-1+2\lambda,\ y=2-\lambda,\ z=1+3\lambda$

(b) (1, 2, 3) does not lie on the line.

(3, 0, 7) lies on the line.

(c) $\lambda=3$, Answers will vary. Two examples are (5, –1, 10) ($\lambda=3$) and (–3, 3, –2) ($\lambda=-1$).

7 y–z plane: (0, 6, 2)

x–z plane: (3, 0, 5)

x–y plane: (–2, 10, 0)

8 (a) C does not lie on AB. (b) C lies on AB. (c) C lies on AB.

9 (a) $b_1=i-3j+k,\ b_2=4i-12j+4k,\ b_2=4(i-3j+k)=4b_1$.

Hence the lines are parallel.

(b) L_1 is $r_1=(2+\lambda)i+(1-3\lambda)j+(-1+\lambda)k$.

When $x=1,\ 2+\lambda=1,\ \lambda=-1$.

For $\lambda=-1,\ y=1-3\lambda=4,\ z=-1+\lambda=-2$.

$\therefore$ (1, 4, –2) lies on L_1.

L_2 is $r_2=(-2+4\lambda)i+(3-12\lambda)j+(2+4\lambda)k$.

When $x=6,\ -2+4\lambda=6,\ \lambda=2.\ y=-21,\ z=10$.

For $\lambda=-2,\ y=3-24=-21,\ z=2+4\lambda=10$.

$\therefore$ (6, –21, 10) lies on L_2.

10 $b_1=i-3j+4k,\ b_2=-2i+2j+2k$.

$b_1\bullet b_2=(i-3j+4k)\bullet(-2i+2j+2k)$

$=-2-6+8$

$=0$

Hence the lines are perpendicular.

11 (a) $r=(2+\lambda)i+(1-2\lambda)j+(-2+3\lambda)k$

(b) $r=(1-\lambda)i+(-2+4\lambda)j+(1-3\lambda)k$

(c) (i) Not parallel (ii) Not perpendicular (iii) Do not intersect.

12 C

CHAPTER REVIEW 3

1 (a) (i) $-i+4j+2k$ (ii) $\sqrt{21}$ (iii) $\frac{\sqrt{21}}{21}(-i+4j+2k)$

(b) (i) $3i+6j+8k$ (ii) $\sqrt{109}$ (iii) $\frac{\sqrt{109}}{109}(3i+6j+8k)$

(c) (i) $-2i+2j-k$ (ii) 3 (iii) $\frac{1}{3}(-2i+2j-k)$

2 (a) 86°51′ (b) 133°19′ (c) 131°51′

3 (a) Not collinear (b) Collinear

4 $\hat{a}=\frac{\sqrt{29}}{29}(2i+3j-4k),\ \hat{b}=\frac{\sqrt{2}}{10}(3i-5j-4k),$

$\hat{c}=\frac{1}{7}(2i+6j+3k)$

5 (a) $-17i+2j+19k$ (b) –17 (c) –36

6 43°46′

7 Any multiple of $4i+4j+3k$.

8 (a) Completing the square gives $(x+7)^2+(y-6)^2+(z+1)^2=81$.

This is a sphere with centre (–7, 6, –1) and radius 9.

(b) Completing the square gives $(x-3)^2+y^2+(z+1)^2=4$.

This is a sphere with centre (3, 0, –1) and radius 2.

9 (a) (i) $y=\frac{x^2}{4}$ (ii) $r=2ti+t^2 j$

(b) (i) $x^2-y^2=1$

For $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, the only restriction will be $|x|\ge 1$.

(ii) $r=\sec\theta\, i+\tan\theta\, j$

10 (a) (i) $x=4t^2,\ y=2t$ (ii) $x=y^2$

(b) (i) $x=\operatorname{cosec} 2\theta,\ y=\sec 2\theta$ (ii) $y^2=\frac{x^2}{x^2-1}$

11 $r=(2+2\lambda)i+(19-3\lambda)j-31k$

12 $r=(4-2\lambda)i+(3+2\lambda)j+(6-3\lambda)k$

13 $\overrightarrow{AB}=(4,\ 4,\ 2),\ \overrightarrow{CD}=(3,\ -5,\ 4)$

$\overrightarrow{AB}\bullet\overrightarrow{CD}=(4,\ 4,\ 2)\bullet(3,\ -5,\ 4)=12-20+8=0$

$\therefore AB\perp CD$

14 $\underset{\sim}{c} = -\frac{2}{9}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right), \underset{\sim}{d} = \frac{1}{9}\left(13\underset{\sim}{i} - 16\underset{\sim}{j} + 5\underset{\sim}{k}\right)$

15 $p = \frac{1}{6}, q = -\frac{11}{12}$

16 (a) $OA = (1, 1, 0), |OA| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$OB = (1, 0, 1), |OB| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$OC = (0, 1, 1), |OC| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$AB = (0, -1, 1), |AB| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$AC = (-1, 0, 1), |AC| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$BC = (-1, 1, 0), |AB| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

All lengths are equal, so all faces will be equilateral triangles and the shape must be a regular tetrahedron.

(b) 60° (c) 90°

17 (a) $m = 0$ (b) $m = 3$

CHAPTER 4

EXERCISE 4.1

1 (a) $\frac{x}{2} - \frac{1}{4}\sin 2x + C$ (b) $\frac{x}{2} + \frac{1}{8}\sin 4x + C$

(c) $\frac{x}{2} - \frac{1}{2}\sin x + C$ (d) $\frac{x}{2} + \frac{1}{12}\sin 6x + C$

(e) $\frac{x}{2} - \frac{1}{12}\sin 6x + C$ (f) $\frac{x}{2} - \frac{1}{16}\sin 8x + C$

(g) $\frac{x}{2} + \frac{1}{6}\sin 3x + C$ (h) $\frac{x}{2} + \frac{1}{10}\sin 5x + C$

2 D

3 (a) $\frac{1}{3}\sin^3 x + C$ (b) $\frac{1}{2}\tan^2 x + C$ (c) $-\frac{1}{4}\cos^4 x + C$

(d) $\frac{1}{5}\sin^5 x + C$ (e) $-\frac{2}{3}\cos^3 x + C$ (f) $\frac{x}{2} + \frac{1}{4}\cos 2x + C$

(g) $-\frac{1}{5}\cos^5 x + C$ (h) $\sec x + C$ (i) $-\operatorname{cosec} x + C$

4 (a) $\int \sec^2 x\,dx = \tan x + C$ (b) $2\tan\frac{x}{2} + C$

(c) $\frac{1}{3}\tan 3x + C$ (d) $\frac{1}{2}\tan x + C$ (e) $-\frac{x}{2} - \frac{1}{8}\sin 4x + C$

(f) $\frac{1}{2}\sin^2 x + C$ (g) $x + \sin x + C$ (h) $\frac{x}{2} + \frac{1}{4}\sin 2x + C$

(i) $-\frac{2}{3}\cos^3 x + \cos x + C$ (j) $\tan\frac{x}{2} + C$

5 (a) $\frac{x}{2} - \frac{1}{8}\sin 4x + C$ (b) $\frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C$

(c) $\frac{x}{8} - \frac{1}{64}\sin 8x + C$ (d) $\frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

(e) $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

(f) $\frac{1}{12}\sin 6x + \frac{1}{4}\sin 2x + C$

(g) $\sin x - \frac{1}{2}\sin^3 x + \frac{1}{5}\sin^5 x + C$

(h) $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

(i) $\frac{1}{3}\tan^3 x - \tan x + x + C$

6 $\int \cos^4 x(1 - \cos^2 x)\sin x\,dx = \int(\cos^4 x - \cos^6 x)\sin x\,dx$
$= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

7 (a) $\frac{1}{2}\int_0^{\frac{\pi}{4}}(1 - \cos 2x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$

(b) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1 + \cos 2x)\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$
$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{12}$

(c) $\frac{1}{16}$ (d) $\left[\frac{\sin^4 x}{4}\right]_{-\pi}^{\pi} = \frac{1}{4}(1 - 1) = 0$

(e) $\frac{1}{2}$ (f) $\frac{1}{2}$ (g) $\frac{4}{3}$ (h) $\frac{\pi}{2}$ (i) $\frac{9\sqrt{3} - 4\sqrt{2}}{160}$

(j) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin\theta}{\cos^2\theta}\,d\theta = \left[\frac{1}{\cos\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2 - \frac{2}{\sqrt{3}} = \frac{6 - 2\sqrt{3}}{3}$

EXERCISE 4.2

1 (a) $\sin^{-1}\frac{x}{3} + C$ (b) $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$ (c) $\cos^{-1}\frac{x}{4} + C$

2 (a) $\sin^{-1}\frac{x}{\sqrt{3}} + C$ (b) $\frac{1}{3}\cos^{-1}\frac{3x}{4} + C$ (c) $\frac{1}{2}\cos^{-1}\frac{2x}{5} + C$

3 B

4 (a) $\tan^{-1}(x + 2) + C$ (b) $\tan^{-1}(x - 3) + C$

(c) $\sin^{-1}\left(\frac{x - 1}{2}\right) + C$ (d) $\sin^{-1}\left(\frac{x - 1}{4}\right) + C$

(e) $-\frac{1}{3}(16 - x^2)^{\frac{3}{2}} + C$ (f) $\frac{9}{2}\sin^{-1}\frac{x}{3} + \frac{x}{2}\sqrt{9 - x^2} + C$

(g) $\frac{1}{2}\sin^{-1} x - \frac{x}{2}\sqrt{1 - x^2} + C$ (h) $\sin^{-1}\frac{x}{2} - \sqrt{4 - x^2} + C$

(i) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x + 2}{\sqrt{2}}\right) + C$

5 (a) $\frac{\pi}{18}$ (b) $\frac{\pi}{12}$ (c) $\frac{1}{3}$ (d) $\frac{3\sqrt{3}}{8}$

(e) $\int_{-2}^{-1}\frac{dx}{1 + (x + 2)^2} = \left[\tan^{-1}(x + 2)\right]_{-2}^{-1} = \frac{\pi}{4}$

(f) $\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{dx}{\sqrt{1 - (x - 1)^2}} = \left[\sin^{-1}(x - 1)\right]_{\frac{1}{2}}^{\frac{3}{2}}$
$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3}$

(g) $\frac{25\pi}{8}$ (h) $\frac{1}{\sqrt{3}}$ (i) $\frac{\pi\sqrt{3}}{6}$

(j) $\int_{-1}^{1}\frac{dx}{(x - 1)^2 + 4} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x - 1}{2}\right)\right]_{-1}^{1} = \frac{1}{2}\left(0 - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{8}$

(k) $\frac{\pi}{4}$ (l) $\int_{-1}^{0}\frac{dx}{(x + 1)^2 + 3} = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x + 1}{\sqrt{3}}\right)\right]_{-1}^{0} = \frac{\pi}{6\sqrt{3}}$

(m) $\int_{-\frac{7}{4}}^{\frac{3}{4}}\frac{dx}{\sqrt{\frac{25}{4} - \left(x + \frac{1}{2}\right)^2}} = \left[\sin^{-1}\frac{x + \frac{1}{2}}{\frac{5}{2}}\right]_{-\frac{7}{4}}^{\frac{3}{4}}$
$= \left[\sin^{-1}\left(\frac{2x + 1}{5}\right)\right]_{-\frac{7}{4}}^{\frac{3}{4}}$
$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3}$

(n) $\frac{\pi a^2}{4}$ (o) $\left[\tan^{-1} t\right]_{\tan x}^{\cot x} = \tan^{-1}(\cot x) - \tan^{-1}(\tan x)$
$= \left(\frac{\pi}{2} - x\right) - x = \frac{\pi}{2} - 2x$

6 C

EXERCISE 4.3

1 (a) $\log_e(x + 1) + C, x > -1$ (b) $\log_e(2x + 1) + C, x > -\frac{1}{2}$

(c) $-\frac{1}{3}\log_e(8 - x^3) + C, x < 2$ (d) $-\frac{1}{4}\log_e(1 - 2x) + C, x < \frac{1}{2}$

(e) $\frac{1}{2}\log_e(x^2 + 2x + 3) + C$, x real (f) $2x + 5\log_e x + C, x > 0$

(g) $\log_e(\cos x + \sin x) + C, 2n\pi - \frac{\pi}{4} < x < 2n\pi + \frac{3\pi}{4}$

(h) $\frac{1}{2}\log_e(1 + x^2) + C$, x real (i) $-\frac{1}{2(1 + x^2)} + C$, x real

(j) $\sqrt{1 + x^2} + C$, x real (k) $\frac{1}{2}\log_e(2x + 5) + C, x > -\frac{5}{2}$

(l) $-\frac{1}{2(2x+5)}+C,\ x\neq -\frac{5}{2}$

(m) $\int\left(1+\frac{2}{2x-1}\right)dx = x+\log_e(2x-1)+C,\ x>\frac{1}{2}$

(n) $\log_e(\sec x)+C,\ \left(2n-\frac{1}{2}\right)\pi < x < \left(2n+\frac{1}{2}\right)\pi$

(o) $\log_e(1+\sin x)^2+C,\ x\neq\left(2n-\frac{1}{2}\right)\pi$

(p) $x+\frac{1}{2}\log_e(x^2+1)+C,\ x$ real

(q) $\frac{x^2}{2}-3x-5\log_e(x-2)+C,\ x>2$

(r) $\int\left(x^2-x+1-\frac{1}{x+1}\right)dx$

$= \frac{x^3}{3}-\frac{x^2}{2}+x-\log_e(x+1)+C,\ x>-1$

(s) $\frac{1}{2}\log_e(x^2+6x-7)+C,\ x\neq -7, 1$

(t) $\log_e(\sin x)+C,\ 2n\pi<x<(2n+1)\pi$

2 A

3 (a) $\ln 3$ (b) $\frac{3}{4}\ln\left(\frac{7}{3}\right)$ (c) $\ln\left(\frac{13}{7}\right)$

(d) $1+\log_e 3$ (e) $3+\frac{1}{2}\log_e 10$ (f) $\frac{13}{12}-\frac{1}{16}\log_e 5$

(g) $\frac{17}{2}+11\log_e 2$ (h) $\frac{3}{2}\log_e 5$ (i) $\frac{7}{2}+\log_e 2$

(j) $\frac{7}{8}\log_e 5-\frac{5}{2}$ (k) $\frac{1}{2}\log_e 3$ (l) $\log_e 2$

(m) $\left[\sqrt{1+x^2}\right]_2^3=\sqrt{10}-\sqrt{5}$ (n) 1 (o) $\frac{9}{4}$

(p) $\log_e 2$ (q) $\int_1^2\left(x^2-\frac{2}{x}+\frac{1}{x^4}\right)dx=\left[\frac{x^3}{3}-2\log_e|x|-\frac{1}{3x^3}\right]_1^2$

$= \frac{21}{8}-2\log_e 2$

(r) $\log_e 2$ (s) $\log_e\left(\frac{1+e}{2}\right)$ (t) $\log_e 2$

4 (a) $\frac{1}{x+\sqrt{x^2-a^2}}\times\left(1+\frac{x}{\sqrt{x^2-a^2}}\right)$

$= \frac{1}{x+\sqrt{x^2-a^2}}\times\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}}=\frac{1}{\sqrt{x^2-a^2}}$

(b) $\log_e\left|x+\sqrt{x^2-a^2}\right|+C$

5 (a) $\frac{1}{x+\sqrt{x^2+a^2}}\times\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)$

$= \frac{1}{x+\sqrt{x^2+a^2}}\times\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}=\frac{1}{\sqrt{x^2+a^2}}$

(b) $\log_e\left|x+\sqrt{x^2+a^2}\right|+C$

6 (a) $\log_e\left|x+\sqrt{x^2-1}\right|+C$ (b) $\log_e\left|x+\sqrt{x^2+1}\right|+C$

(c) $\log_e\left|x-2+\sqrt{x^2-4x+3}\right|+C$

(d) $\log_e\left|x+3+\sqrt{x^2+6x+13}\right|+C$

(e) $\log_e\left|x-\frac{5}{2}+\sqrt{x^2-5x+7}\right|+C$

(f) $\log_e\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$

7 (a) $\log_e\left(\frac{4+2\sqrt{3}}{3+\sqrt{5}}\right)$ (b) $\log_e\left(\frac{1+\sqrt{10}}{3}\right)$

(c) $\log_e\left(\frac{5+\sqrt{26}}{3+\sqrt{10}}\right)$ (d) $\log_e\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)$

EXERCISE 4.4

1 A

2 (a) $\log_e|2-\cos x|+C$ (b) $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{5}}\right)+C$

(c) $\frac{-2}{1+\tan\frac{x}{2}}+C$

3 (a) $\log_e\left(2+\sqrt{3}\right)$ (b) $\log_e\frac{3}{2}$ (c) $\frac{\pi}{3\sqrt{3}}$

4 (a) $\frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right)+C$ (b) $\frac{4}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{\theta}{2}\right)-\theta+C$

(c) $2\tan^{-1}\left(\tan\frac{\theta}{2}\right)-\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{\theta}{2}-1}{\sqrt{3}}\right)+C$

(d) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}\tan\frac{x}{2}\right)+C$ (e) $\log_e|\sin(x)+1|+C$

(f) $\tan\frac{x}{2}+C$ (g) $C-\log_e\left|1-\tan^2\frac{x}{2}\right|$

5 (a) $\int\frac{1}{1+\frac{2t}{1+t^2}}\times\frac{dt}{1+t^2}=\int\frac{dt}{1+t^2+2t}=\int\frac{dt}{(1+t)^2}$

$= -\frac{1}{t+1}+C=C-\frac{1}{\tan x+1}$

(b) $\int\frac{2t}{1-t^2}\times\frac{1}{1+\frac{1-t^2}{1+t^2}}\times\frac{dt}{1+t^2}=\int\frac{t}{1-t^2}dt$

$= C-\frac{1}{2}\log_e\left|1-t^2\right|=C-\log_e\left|1-\tan^2 x\right|$

6 (a) $x=\frac{\pi}{3},\ t=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$. $x=\frac{2\pi}{3},\ t=\tan\frac{\pi}{3}=\sqrt{3}$

$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{dx}{\sin x}=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1+t^2}{2t}\times\frac{2dt}{1+t^2}$

$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{dt}{t}$

$= \left[\log_e t\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

$= \log_e\sqrt{3}-\log_e\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{1}{2}\log_e 3+\frac{1}{2}\log_e 3=\log_e 3$

(b) $x=0,\ t=0$. $x=\frac{\pi}{2},\ t=\tan\frac{\pi}{4}=1$

$\int_0^{2\pi}\frac{dx}{5+4\cos x}=\int_0^1\frac{1}{5+\frac{4(1-t^2)}{1+t^2}}\times\frac{2dt}{1+t^2}$

$= 2\int_0^1\frac{dt}{5+5t^2+9-4t^2}$

$= 2\int_0^1\frac{dt}{9+t^2}$

$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1$

$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

(c) $x=0, t=0. \quad x=\frac{\pi}{3}, t=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$

$$\int_0^{\frac{\pi}{3}}\frac{\tan x}{1+\cos x}dx=\int_0^{\frac{1}{\sqrt{3}}}\frac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}}\times\frac{2dt}{1+t^2}$$

$$=4\int_0^{\frac{1}{\sqrt{3}}}\frac{t}{1+t^2+1-t^2}dt$$

$$=4\int_0^{\frac{1}{\sqrt{3}}}\frac{t}{2(1-t^2)}dt$$

$$=-\left[\log_e(1-t^2)\right]_0^{\frac{1}{\sqrt{3}}}$$

$$=-\log_e\left(1-\frac{1}{3}\right)-0=\log_e\left(\frac{3}{2}\right)$$

(d) $x=0, t=0.\ x=\frac{\pi}{2}, t=\tan\frac{\pi}{4}=1$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{3-\cos x-2\sin x}=\int_0^1\frac{1}{3-\frac{1-t^2}{1+t^2}-\frac{4t}{1+t^2}}\times\frac{2dt}{1+t^2}$$

$$=2\int_0^1\frac{dt}{3+3t^2-1+t^2-4t}$$

$$=2\int_0^1\frac{dt}{4t^2-4t+2}$$

$$=2\int_0^1\frac{dt}{(2t-1)^2+1}$$

$$=\left[\tan^{-1}\left(\frac{2t-1}{1}\right)\right]_0^1$$

$$=\tan^{-1}1-\tan^{-1}(-1)$$

$$=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$$

7 $$\int\frac{d\theta}{1+\cos\theta+\sin\theta}=\int\frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\times\frac{2dt}{1+t^2}$$

$$=\int\frac{2dt}{1+t^2+1-t^2+2t}$$

$$=\int\frac{dt}{1+t}$$

$$=\ln|1+t|+C$$

$$=\ln\left|1+\tan\frac{\theta}{2}\right|+C$$

8 $x=0, t=0. \quad x=\frac{\pi}{3}, t=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$

$$\int_0^{\frac{\pi}{3}}\frac{dx}{1+\cos x-\sin x}=\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}\times\frac{2dt}{1+t^2}$$

$$=\int_0^{\frac{1}{\sqrt{3}}}\frac{2dt}{1+t^2+1-t^2-2t}$$

$$=\int_0^{\frac{1}{\sqrt{3}}}\frac{2dt}{2-2t}$$

$$=\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{1-t}$$

$$=\left[-\log_e(1-t)\right]_0^{\frac{1}{\sqrt{3}}}$$

$$=-\log_e\left(1-\frac{1}{\sqrt{3}}\right)+0$$

$$=\log_e\left(\frac{\sqrt{3}}{\sqrt{3}-1}\right)=\log_e\left(\frac{3+\sqrt{3}}{2}\right)$$

9 $t=\tan\theta, d\theta=\frac{dt}{1+t^2}.\ \theta=0, t=0.\ \theta=\frac{\pi}{4}, t=1$

$$\int_0^{\frac{\pi}{4}}\frac{d\theta}{2+\sin 2\theta}=\int_0^1\frac{1}{2+\frac{2t}{1+t^2}}\times\frac{dt}{1+t^2}$$

$$=\frac{1}{2}\int_0^1\frac{dt}{1+t+t^2}$$

$$=\frac{1}{2}\int_0^1\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\frac{1}{2}\times\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$=\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$

$$=\frac{1}{\sqrt{3}}\left(\tan^{-1}(\sqrt{3})-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$=\frac{1}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi}{6\sqrt{3}}=\frac{\sqrt{3}\pi}{18}$$

10 $x=\frac{\pi}{3}, t=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}. \quad x=\frac{\pi}{2}, t=\tan\frac{\pi}{4}=1$

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{dx}{12\sin x-5\cos x+13}$$

$$=\int_{\frac{1}{\sqrt{3}}}^1\frac{1}{12\times\frac{2t}{1+t^2}-5\times\left(\frac{1-t^2}{1+t^2}\right)+13}\times\frac{2dt}{1+t^2}$$

$$=\int_{\frac{1}{\sqrt{3}}}^1\frac{2dt}{24t-5+5t^2+13+13t^2}$$

$$=\int_{\frac{1}{\sqrt{3}}}^1\frac{2dt}{18t^2+24t+8}$$

$$=\int_{\frac{1}{\sqrt{3}}}^1\frac{dt}{9t^2+12t+4}$$

$$=\int_{\frac{1}{\sqrt{3}}}^1\frac{dt}{(3t+2)^2}$$

$$=\frac{-1}{3}\left[\frac{1}{3t+2}\right]_{\frac{1}{\sqrt{3}}}^1$$

$$=-\frac{1}{3}\left(\frac{1}{5}-\frac{1}{\sqrt{3}+2}\right)$$

$$=-\frac{1}{3}\left(\frac{1}{5}-\frac{\sqrt{3}-2}{-1}\right)$$

$$=\frac{9-5\sqrt{3}}{15}$$

CHAPTER REVIEW 4

1 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{8}$ (c) 0 (d) 1

2 (a) $\frac{x}{\sqrt{1-x^2}}+C$ (b) $\frac{x^2}{2}+\tan^{-1}x-\frac{1}{2}\log_e(x^2+1)+C$

(c) $\frac{x^2}{2}-2x+\frac{1}{x+1}+3\log_e|x+1|+C$

3 (a) $\frac{1}{6}$ (b) $\log_e 2$ (c) $\frac{1}{2}\log_e\frac{1+2e}{3}$

(d) $\frac{\pi+2}{64}$ (e) $6\frac{1}{3}$

4 (a) $\frac{\pi\sqrt{3}}{6}$ (b) $\frac{3\pi+8}{32}$ (c) $\sqrt{3}-\frac{\pi}{3}$

5 $\frac{81\pi}{16}$

6 (a) $\log_e|x|-2x^2+C$ (b) $x-\frac{\cos 2x}{2}+C$

(c) $\frac{1}{3}\sin^3 x+C$ (d) $\sec x+C$

(e) $\tan x-x+C$ (f) $\frac{4x-\sin 4x}{32}+C$

(g) $\frac{x}{2}+\frac{\sin 2x}{4}+C$ (h) $\frac{3x}{8}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C$

7 (a) $\frac{1}{12}$ (b) $\frac{1}{8}\log_e\frac{17}{5}$ (c) $\frac{\pi}{4}$

(d) 1 (e) $\frac{\pi}{6}$ (f) $\frac{\pi}{4}$

8 (a) $-e^{\cos x}+C$ (b) $\frac{2}{5}(x+2)(x-3)^{\frac{3}{2}}+C$

(c) $\frac{2}{5}(x-5)x^{\frac{3}{2}}+C$ (d) $-\frac{1}{32(x^2-2)^2}+C$

(e) $-\frac{1}{3}\cos^3 x+C$ (f) $\log_e(\sin x)+C$

(g) $-2\sqrt{1+\cos\theta}+C$

9 (a) $-\frac{\operatorname{cosec} 2\theta}{2}+C$ (b) $-\frac{1}{2}e^{-x^2}+C$

(c) $\log_e(x^2+1)+C$ (d) $\frac{1}{2}\sin(x^2)+C$

(e) $\frac{1}{3}\tan^3 x+C$

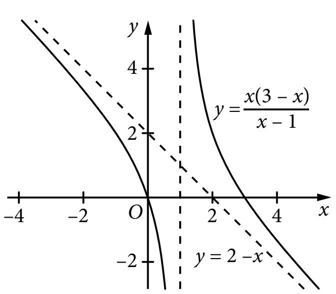
10 (a) $1-\frac{\pi}{4}$ (b) 2 (c) $1-\frac{\pi}{4}$

(d) $\left[\log_e\left(\frac{x+1}{x+4}\right)\right]_1^2=\log_e\frac{5}{4}$ (e) $\left[\log_e(x+1)(x+4)\right]_1^2=\log_e\frac{9}{5}$

11 $x=-\frac{\pi}{2}, t=-1. \quad x=\frac{\pi}{2}, t=1.$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{2+\cos x}=\int_{-1}^{1}\frac{1}{2+\frac{1-t^2}{1+t^2}}\times\frac{2dt}{1+t^2}$$
$$=\int_{-1}^{1}\frac{2dt}{2+2t^2+1-t^2}$$
$$=\int_{-1}^{1}\frac{2dt}{t^2+3}$$
$$=\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_{-1}^{1}$$
$$=\frac{2}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right)$$
$$=\frac{2}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{6}\right)=\frac{2\sqrt{3}\pi}{9}$$

12 (a)



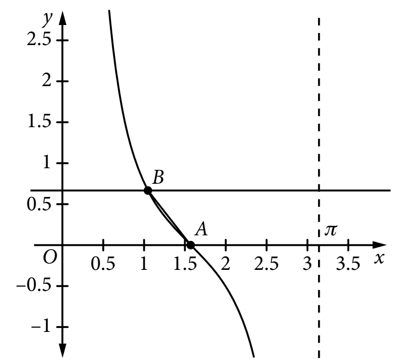
(b) Area $=\int_{-3}^{0}\left(2-x-\frac{x(3-x)}{x-1}\right)dx=\int_{-3}^{0}\frac{2}{1-x}dx$

$=\left[-2\log_e(1-x)\right]_{-3}^{0}=2\log_e 4\approx 2.773$ units^2

13 $x=0, t=0. \quad x=\frac{\pi}{2}, t=1$

$$\int_0^{\frac{\pi}{2}}\frac{1}{3\sin x+4\cos x+5}dx=\int_0^1\frac{1}{\frac{6t}{1+t^2}+\frac{4(1-t^2)}{1+t^2}+5}\times\frac{2dt}{1+t^2}$$
$$=\int_0^1\frac{2dt}{6t+4-4t^2+5+5t^2}$$
$$=\int_0^1\frac{2dt}{t^2+6t+9}$$
$$=\int_0^1\frac{2dt}{(t+3)^2}$$
$$=-2\left[\frac{1}{t+3}\right]_0^1$$
$$=-2\left(\frac{1}{4}-\frac{1}{3}\right)=\frac{1}{6}$$

14



$A\left(\frac{\pi}{2},0\right)$. For B: $\frac{2}{3}=\frac{\cos x}{\sin^2 x}$, $2\sin^2 x=3\cos x$,

$2\cos^2 x+3\cos x-2=0$, $(2\cos x-1)(\cos x+2)=0$

$\therefore x=\frac{\pi}{3}$: $B\left(\frac{\pi}{3},\frac{2}{3}\right)$

Area $=\frac{1}{2}\times\left(\frac{\pi}{2}-\frac{\pi}{3}\right)\times\frac{2}{3}-\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}dx$

$=\frac{\pi}{18}-\left[\frac{-1}{\sin x}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}=\frac{\pi}{18}+\left(1-\frac{2}{\sqrt{3}}\right)=\frac{\pi+18-12\sqrt{3}}{18}$ units^2

15 Area $=\int_{-1}^{1}\left(-1+\frac{2}{1+x^2}\right)dx=\left[-x+2\tan^{-1}x\right]_{-1}^{1}$

$=\left(-1+\frac{2\pi}{4}\right)-\left(1-\frac{2\pi}{4}\right)=\pi-2$ units^2

16 $\frac{x^2}{4}+\frac{y^2}{9}=1$, $y=\frac{3}{2}\sqrt{4-x^2}$:

Area $=4\int_0^2\frac{3}{2}\sqrt{4-x^2}\,dx=6\pi$ units^2

17 (a) $x=0, t=0. \quad x=\frac{\pi}{2}, t=1$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sin x}=\int_0^1\frac{1}{1+\frac{2t}{1+t^2}}\times\frac{2dt}{1+t^2}$$
$$=\int_0^1\frac{2dt}{1+t^2+2t}$$
$$=\int_0^1\frac{2dt}{(t+1)^2}$$
$$=\left[\frac{-2}{t+1}\right]_0^1$$
$$=-1+2=1$$

(b) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}dx = \int_0^{\frac{\pi}{2}} \frac{1+\sin x-1}{1+\sin x}dx$

$= \int_0^{\frac{\pi}{2}} \left(1-\frac{1}{1+\sin x}\right)dx$

$= [t]_0^{\frac{\pi}{2}} - 1$

$= \frac{\pi-2}{2}$

CHAPTER 5

EXERCISE 5.1

1 D

2 (a) $\frac{1}{x-1}-\frac{1}{x+3}$ (b) $\frac{1}{x+2}+\frac{1}{x-3}$

(c) $\frac{1}{x}+\frac{2}{x+1}$ (d) $\frac{1}{x}-\frac{2}{x-1}+\frac{3}{x+2}$

3 (a) $\frac{1}{x-1}-\frac{1}{x+2}$ (b) $\frac{1}{x-2}+\frac{2}{x+3}$

(c) $\frac{3}{x-2}-\frac{2}{x+2}$ (d) $\frac{1}{x}+\frac{2}{x-3}-\frac{3}{x+4}$

4 C

5 (a) $1-\frac{1}{3(x+1)}+\frac{4}{3(x-2)}$ (b) $1-\frac{3}{x-2}+\frac{9}{x-4}$

(c) $1-\frac{3}{2(x+3)}+\frac{3}{2(x-3)}$ (d) $\frac{4}{7(x-4)}+\frac{3}{7(x+3)}$

6 (a) $\frac{4}{x+3}-\frac{5}{2x-1}$ (b) $\frac{3}{x+3}+\frac{2}{x-2}$

(c) $\frac{3}{3x+1}+\frac{4}{x+8}$ (d) $\frac{2}{x+2}+\frac{3}{x+1}-\frac{4}{x-3}$

(e) $\frac{5}{x-3}+\frac{4}{x+3}$ (f) $\frac{8}{2x+3}-\frac{5}{x+2}$

(g) $2+\frac{4}{x-2}-\frac{3}{x+2}$ (h) $1+\frac{1}{x}-\frac{1}{5(x-1)}-\frac{8}{5(2x+3)}$

7 $\frac{4}{x+1}+\frac{3}{x+2}-\frac{2}{x+3}$

EXERCISE 5.2

1 C

2 (a) $\frac{1}{x^2+1}-\frac{1}{x^2+4}$ (b) $\frac{1}{x^2+1}-\frac{1}{x^2+9}$

(c) $\frac{4}{x^2+4}-\frac{1}{x^2+1}$ (d) $\frac{3}{x^2+1}-\frac{1}{x^2+2}$

3 (a) $\frac{3x-2}{x^2+1}-\frac{2}{x-2}$ (b) $\frac{2x+1}{x^2+x+1}-\frac{1}{x+2}$

(c) $\frac{2}{x-1}-\frac{3}{x+1}+\frac{3x+2}{x^2+1}$ (d) $\frac{4}{x}-\frac{x-1}{x^2+x+1}$

4 B

5 (a) $\frac{1}{x+2}+\frac{2}{x^2-2x+5}$ (b) $\frac{1}{2(1-x)}+\frac{3}{2(1+x)}+\frac{3x-1}{1+x^2}$

(c) $x-1+\frac{2}{x+1}-\frac{3x-1}{x^2+1}$ (d) $x+\frac{1}{x-2}+\frac{2x+2}{x^2+2x+4}$

(e) $\frac{2}{3(x-1)}+\frac{x-1}{3(x^2+x+1)}$ (f) $\frac{2}{2x+1}-\frac{x-1}{x^2+1}$

6 (a) $a=4, b=1$ (b) $a=3, b=-4$

(c) $a=3, b=4$ (d) $a=2, b=5$

7 (a) $\text{RHS} = \frac{3}{2x+3}-\frac{4}{x-2}+\frac{2}{(x-2)^2}$

$= \frac{3(x-2)^2-4(2x+3)(x-2)+2(2x+3)}{(2x+3)(x-2)^2}$

$= \frac{3x^2-12x+12-8x^2+4x+24+4x+6}{(2x+3)(x-2)^2}$

$= \frac{-5x^2-4x+42}{(2x+3)(x-2)^2} = \text{LHS}$

(b) $\text{RHS} = \frac{2}{x-2}-\frac{3}{2x+1}-\frac{4}{(2x+1)^2}$

$= \frac{2(2x+1)^2-3(x-2)(2x+1)-4(x-2)}{(x-2)(2x+1)^2}$

$= \frac{8x^2+8x+2-6x^2+9x+6-4x+8}{(x-2)(2x+1)^2}$

$= \frac{2x^2+13x+16}{(x-2)(2x+1)^2} = \text{LHS}$

8 (a) $a=-3, b=5, c=2$ (b) $a=-1, b=-1, c=2$

EXERCISE 5.3

1 (a) $\log_e\left|\frac{x-3}{x-1}\right|+C$ (b) $\log_e\left|\frac{x-2}{x+1}\right|+C$

(c) $\frac{1}{6}\log_e\left|\frac{x-4}{x+2}\right|+C$ (d) $\log_e\left|x^2-1\right|+C$

(e) $\log_e\left|(x-1)^2(x+1)\right|+C$ (f) $\log_e\left|(x-3)(x-2)\right|+C$

2 D

3 (a) $\log_e|x+1|-\frac{1}{2}\log_e(x^2+1)+C$ (b) $\log_e|x+2|-\tan^{-1}\frac{x}{2}+C$

(c) $\log_e|x-1|-\tan^{-1}x+C$ (d) $\log_e|x+3|+\frac{1}{3}\tan^{-1}\frac{x}{3}+C$

(e) $\log_e|x-2|+\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$

(f) $\log_e|x-5|-\frac{1}{2}\log_e\left(x^2+3\right)+C$

4 (a) $\tan^{-1}x-\frac{1}{3}\tan^{-1}\frac{x}{3}+C$ (b) $4\tan^{-1}\frac{x}{4}-\tan^{-1}x+C$

(c) $\tan^{-1}\frac{x}{2}-\tan^{-1}x+C$ (d) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$

(e) $\frac{1}{2}\log_e\left(\frac{x^2+4}{x^2+5}\right)+C$ (f) $\frac{1}{2}\log_e(x^2+9)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$

5 (a) $\log_e\left|\frac{(x+2)^5}{(x-1)^3}\right|+C$ (b) $\log_e\left|(x-3)^5(x+3)^4\right|+C$

(c) $\log_e\left|(x+1)^3\sqrt{x^2+1}\right|-3\tan^{-1}x+C$

(d) $2x+\log_e\left|\frac{(x-2)^4}{(x+2)^3}\right|+C$ (e) $\log_e(x-1)^2+3\tan^{-1}x+C$

(f) $\frac{15}{4}\log_e|x-1|+\frac{9}{4}\log_e|x+3|-2\log_e|x+1|+C$

(g) $\log_e\left|\frac{(x+3)^4}{(2x-1)^{\frac{5}{2}}}\right|+C$ (h) $\log_e\left|\frac{(x-3)^5}{\sqrt{2x-1}}\right|+C$

(i) $\log_e\left(\frac{e^x+1}{e^x+2}\right)+C$

6 (a) $\frac{1}{6}\left(5\log_e\frac{8}{7}+\log_e 2\right)$ (b) $\left[\log_e\left(\frac{x}{x+1}\right)\right]_3^4 = \log_e\frac{16}{15}$

(c) $\log_e 3-\frac{1}{2}\log_e 5$ (d) $\frac{1}{2}\log_e 3$ (e) $\frac{1}{4}\log_e 3$ (f) $\log_e\frac{72}{25}$

7 (a) $1+\log_e\frac{12}{5}$ (b) $\frac{1}{2}\left(\tan^{-1}3-\frac{\pi}{6\sqrt{3}}-\frac{\pi}{4}\right)$ (c) $4\log_e\frac{7}{6}+\log_e 2$

8 $\frac{1}{2}+\log_e\frac{3}{4}$

9 (a) $\text{RHS} = \frac{x^2+1+(1-x)(x+1)}{(x+1)(x^2+1)} = \frac{x^2+1+x+1-x^2-x}{x^3+x^2+x+1}$

$= \frac{2}{x^3+x^2+x+1} = \text{LHS}$

(b) $\int_1^2\left(\frac{1}{x+1}+\frac{1}{x^2+1}-\frac{x}{x^2+1}\right)dx$

$= \left[\log_e|x+1|+\tan^{-1}x-\frac{1}{2}\log_e(x^2+1)\right]_1^2$

$= \frac{1}{2}\log_e\frac{9}{10}-\frac{\pi}{4}+\tan^{-1}2$

10 (a) $1 \equiv (a+b)x^2 + (b+c)x + c$: $\quad a=1, b=-1, c=1$

(b) $\log_e\left|\frac{x+1}{x}\right| - \frac{1}{x} + C$

11 (a) $4x-7 \equiv a(x-2)+b$: $\quad a=4,\ b=1$

(b) $4\log_e|x-2| - \frac{1}{x-2} + C$

12 (a) $a=-3, b=5, c=2$ (b) $\log_e\left|\frac{(x-2)^5}{(x+3)^3}\right| - \frac{2}{x-2} + C$

13 (a) $a=0, b=4, c=3$ (b) $\log_e\left|(x-2)^3\right| - \frac{4}{x+1} + C$

14 (a) $a=1, b=-1, c=-1$ (b) $\left[\log_e\frac{x}{x+1} + \frac{1}{x+1}\right]_1^2 = \log_e\frac{4}{3} - \frac{1}{6}$

15 $\int_0^1\left(\frac{2x}{4-x^2} + \frac{2x-4}{x^2-4x+5} + \frac{1}{(x-2)^2+1}\right)dx$

$= \left[-\log_e(4-x^2) + \log_e(x^2-4x+5) + \tan^{-1}(x-2)\right]_0^1$

$= \log_e\frac{8}{15} - \frac{\pi}{4} + \tan^{-1}2$

16 (a) $\text{RHS} = \frac{x^2 - x(x-3) - 3(x-3)}{9(x-3)x^2}$

$= \frac{x^2 - x^2 + 3x - 3x + 9}{9(x-3)x^2} = \frac{1}{(x-3)x^2} = \text{LHS}$

(b) $\text{Area} = \int_4^6\left(\frac{1}{9(x-3)} - \frac{1}{9x} - \frac{1}{3x^2}\right)dx$

$= \left[\frac{1}{9}\log_e\left(\frac{x-3}{x}\right) + \frac{1}{3x}\right]_4^6$

$= \frac{1}{9}\log_e 2 - \frac{1}{36} \approx 0.049 \text{ units}^2$

17 $\text{Area} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{8}{2x+3} - \frac{5}{x+2}\right)dx$

$= \left[4\log_e(2x+3) - 5\log_e(x+2)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= 4\log_e 2 + 5\log_e\frac{3}{5} \approx 0.218 \text{ units}^2$

18 (a) $a=2, b=1, c=-1$

(b) $\int\left(\frac{2}{(x-2)^2} + \frac{1}{x-2} - \frac{1}{x+3}\right)dx = \log_e\left|\frac{x-2}{x+3}\right| - \frac{2}{x-2} + C$

19 $\int_{-1}^1\left(\frac{2}{4+(x-1)^2} - \frac{2x-2}{x^2-2x+5} - \frac{2x}{2-x^2}\right)dx$

$= \left[\tan^{-1}\left(\frac{x-1}{2}\right) - \log_e\left|x^2-2x+5\right| + \log_e\left|2-x^2\right|\right]_{-1}^1$

$= 0 - \log_e 4 + \log_e 1 - \left(\tan^{-1}(-1) - \log_e 8 + \log_e 1\right)$

$= -2\log_e 2 + \frac{\pi}{4} + 3\log_e 2 = \log_e 2 + \frac{\pi}{4}$

20 (a) $a=1, b=2, c=-1$

(b) $\int\left(\frac{1}{x} + \frac{2}{x^2} - \frac{1}{x+1}\right)dx = \log_e|x| - \frac{2}{x} - \log_e|x+1| + C$

$= \log_e\left|\frac{x}{x+1}\right| - \frac{2}{x} + C$

21 $\int\frac{dx}{\sin 2x - \cos 2x} = \int\frac{1}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \times \frac{dt}{1+t^2} = \int\frac{dt}{2t-1+t^2}$

$= \int\frac{dt}{(t^2+2t+1)-2} = \int\frac{dt}{(t+1)^2-2}$

$= \int\frac{dt}{(t+1-\sqrt{2})(t+1+\sqrt{2})}$

$= \frac{1}{2\sqrt{2}}\int\left(\frac{1}{(t+1-\sqrt{2})} - \frac{1}{(t+1+\sqrt{2})}\right)dt$

$= \frac{1}{2\sqrt{2}}\left(\log_e\left|t+1-\sqrt{2}\right| - \log_e\left|t+1+\sqrt{2}\right|\right) + C$

$= \frac{1}{2\sqrt{2}}\log_e\left|\frac{t+1-\sqrt{2}}{t+1+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}}\log_e\left|\frac{\tan x+1-\sqrt{2}}{\tan x+1+\sqrt{2}}\right| + C$

22 $\int\frac{dx}{1+3\sin x} = \frac{1}{4\sqrt{2}}\log_e\left(\frac{\tan\frac{x}{2}+3-2\sqrt{2}}{\tan\frac{x}{2}+3+2\sqrt{2}}\right) + C$

23 $\int\frac{\left(1+\sin^2 x\right)}{1+\cos^2 x}dx = \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) - x + C$

EXERCISE 5.4

1 C

2 (a) $\frac{-4}{15}(4-x)^{\frac{5}{2}} - \frac{2}{3}(4-x)^{\frac{3}{2}} + C, x \le 4$

(b) $\frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2} + C$ (c) $-x\cos x + \sin x + C$

(d) $\frac{1}{4}\sin 2x - \frac{x}{2}\cos 2x + C$ (e) $\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$

(f) $x\sin^{-1}2x + \frac{1}{2}\sqrt{1-4x^2} + C, |x| \ge -1$

(g) $\frac{x}{2}\sin(2x+1) + \frac{1}{4}\cos(2x+1) + C$

(h) $x\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2} + C, |x| \le \frac{1}{2}$

(i) $\left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + C, |x| < 1$

3 (a) $\frac{2}{15}(x+1)^{\frac{3}{2}}(3x-2) + C, x \ge -1$

(b) $-x^2\cos x + 2x\sin x + 2\cos x + C$

(c) $\left(\frac{x^2}{2} - \frac{1}{4}\right)\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + C, |x| \le 1$

(d) $x^2\sin x + 2x\cos x - 2\sin x + C$

(e) $-(x+1)e^{-x} + C$ (f) $\frac{x^2}{2}\log_e|x| - \frac{x^2}{4} + C$

4 (a) $-\frac{e^{-x}}{2}(\sin x + \cos x) + C$ (b) $\frac{e^x}{2}(\sin x + \cos x) + C$

(c) $\frac{e^{-x}}{2}(\sin x - \cos x) + C$

(d) $\frac{x}{2}\left[\sin\left(\log_e|x|\right) - \cos\left(\log_e|x|\right)\right] + C$

(e) $\frac{x}{2}\left[\sin\left(\log_e|x|\right) + \cos\left(\log_e|x|\right)\right] + C$

(f) $x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2) + C$

5 (a) $\frac{\pi}{2} - 1$ (b) 2 (c) $\frac{\pi}{8}$

(d) $\frac{5\pi}{12} - \frac{\sqrt{3}-1}{2}$ (e) 0 (f) $\frac{1}{4} - \frac{\sqrt{2}(4-\pi)}{32}$

6 (a) $\frac{\sqrt{3}}{4} - \frac{\pi}{12}$ (b) $\frac{1}{2}\left(e^{\frac{\pi}{2}} + 1\right)$ (c) $\frac{2}{e} - \frac{3}{e^2}$

(d) 1 (e) $2\log_e 2 - \frac{3}{4}$ (f) $\frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)$

(g) $5\log_e 2 - \frac{7}{9}$ (h) $\frac{5\sqrt{3}\pi}{18} - \frac{1}{2}\log_e 3$ (i) $-\frac{2}{5}\left(e^{-\pi} + e^{-\frac{\pi}{2}}\right)$

7 B

8 $\text{Area} = \int_1^a \log_e x\,dx = \left[x\log_e x - x\right]_1^a = a\log_e a - a + 1$

EXERCISE 5.5

1 (a) $\frac{d}{dx}\left(\sin x\cos^{n-1}x\right)$

$= \cos x\cos^{n-1}x + \sin x \times (n-1)\cos^{n-2}x \times (-\sin x)$

$\frac{d}{dx}\left(\sin x\cos^{n-1}x\right) = \cos^n x - (n-1)\sin^2 x\cos^{n-2}x$

$\cos^n x = \frac{d}{dx}\left(\sin x\cos^{n-1}x\right) + (n-1)\sin^2 x\cos^{n-2}x$

$\int\cos^n x\,dx = \sin x\cos^{n-1}x + (n-1)\int\sin^2 x\cos^{n-2}x\,dx$

From here the recurrence relation is found as in Example 25 (see page 148).

(b) (i) $I_6 = \left[\frac{1}{6}\sin x\cos^5 x\right]_0^{\frac{\pi}{2}} + \frac{5}{6}I_4 = \frac{5}{6}I_4,$

$I_4 = \left[\frac{1}{4}\sin x\cos^3 x\right]_0^{\frac{\pi}{2}} + \frac{3}{4}I_2 = \frac{3}{4}I_2$

$I_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2x)\,dx$

$= \left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2}+0=0\right) = \frac{\pi}{4}$

$\therefore I_6 = \frac{5}{6}\times\frac{3}{4}\times\frac{\pi}{4} = \frac{5\pi}{32}$

(ii) $\int_0^{\frac{\pi}{2}}\sin^6 x\,dx = \int_0^{\frac{\pi}{2}}(1-\cos^2 x)^3\,dx$

$= \int_0^{\frac{\pi}{2}}(1-3\cos^2 x+3\cos^4 x-\cos^6 x)\,dx$

$I_6 = \int_0^{\frac{\pi}{2}}\cos^6 x\,dx$:

$\int_0^{\frac{\pi}{2}}\sin^6 x\,dx = \int_0^{\frac{\pi}{2}}dx - 3I_2 + 3I_4 - I_6$

$= \frac{\pi}{2} - 3\times\frac{\pi}{4} + 3\times\frac{3}{4}\times\frac{\pi}{4} - \frac{5\pi}{32} = \frac{5\pi}{32}$

(iii) $\int_0^{\frac{\pi}{2}}\cos^4 x\sin^2 x\,dx = \int_0^{\frac{\pi}{2}}\cos^4 x(1-\cos^2 x)\,dx$

$= I_4 - I_6 = \frac{3\pi}{16} - \frac{5\pi}{32} = \frac{\pi}{32}$

(iv) $\int_0^{\frac{\pi}{4}}\sec^4 x\,dx = \int_0^{\frac{\pi}{4}}\sec^2 x\sec^2 x\,dx$

$= \left[\sec^2 x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\tan x\times 2\sec x\frac{\sin x}{\cos^2 x}\,dx$

$= 2 - 2\int_0^{\frac{\pi}{4}}\tan^2 x\sec^2 x\,dx$

$= 2 - 2\int_0^{\frac{\pi}{4}}(\sec^4 x - \sec^2 x)\,dx$

$= 2 + 2[\tan x]_0^{\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}}\sec^4 x\,dx$

$\therefore 3\int_0^{\frac{\pi}{4}}\sec^4 x\,dx = 4,\ \int_0^{\frac{\pi}{4}}\sec^4 x\,dx = \frac{4}{3}$

2 $\frac{d}{dx}\left(x^n e^x\right) = nx^{n-1}e^x + x^n e^x$ hence by integration:

$x^n e^x = \int nx^{n-1}e^x\,dx + \int x^n e^x\,dx$

$\therefore \int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

3 (a) $\frac{d}{dx}\left(x^n\log_e x\right) = nx^{n-1}\log_e x - \frac{x^n}{x} = nx^{n-1}\log_e x + x^{n-1}$

(b) (i) $\int_1^2 x^2\log_e x\,dx = \left[\frac{1}{3}x^3\log_e x\right]_1^2 - \frac{1}{3}\int_1^2 x^2\,dx$

$= \frac{8}{3}\log_e 2 - \left[\frac{x^3}{9}\right]_1^2 = \frac{8}{3}\log_e 2 - \frac{7}{9} \approx 1.071$

(ii) $\int_1^2 x^3\log_e x\,dx = \left[\frac{1}{4}x^4\log_e x\right]_1^2 - \frac{1}{4}\int_1^2 x^3\,dx$

$= 4\log_e 2 - \left[\frac{x^4}{16}\right]_1^2 = 4\log_e 2 - \frac{15}{16} \approx 1.835$

4 (a) $I_n = \int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$= \left[\cos^{n-1}x\sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}}\sin^2 x\cos^{n-2}x\,dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}}(1-\cos^2 x)\cos^{n-2}x\,dx$

$= (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x\,dx - (n-1)\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n;\ nI_n = (n-1)I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

(b) $I_5 = \frac{4}{5}I_3,\ I_3 = \frac{2}{3}I_1,\ I_1 = \int_0^{\frac{\pi}{2}}\cos x\,dx = [\sin x]_0^{\frac{\pi}{2}} = 1$

$I_5 = \frac{4}{5}\times\frac{2}{3}\times 1 = \frac{8}{15}$

5 (a) $\int\sin^n x\,dx = \int\sin^{n-1}x\sin x\,dx$

$= \sin^{n-1}x(-\cos x) - (n-1)\int\sin^{n-2}x\cos x(-\cos x)\,dx$

$= -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x(1-\sin^2 x)\,dx$

$= -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x\,dx - (n-1)\int\sin^n x\,dx$

$\therefore n\int\sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int\sin^{n-2}x\,dx$

$\int\sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \left(\frac{n-1}{n}\right)\int\sin^{n-2}x\,dx$

(b) $\int_0^{\frac{\pi}{6}}\sin^4 x\,dx = -\frac{1}{4}\left[\sin^3 x\cos x\right]_0^{\frac{\pi}{6}} + \frac{3}{4}\int_0^{\frac{\pi}{6}}\sin^2 x\,dx$

$= -\frac{1}{4}\left(\frac{\sqrt{3}}{16}\right) + \frac{3}{8}\int_0^{\frac{\pi}{6}}(1-\cos 2x)\,dx$

$= -\frac{\sqrt{3}}{64} + \frac{3}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$

$= -\frac{\sqrt{3}}{64} + \frac{3}{8}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{16} - \frac{7\sqrt{3}}{64}$

6 (a) $I_n = \int\sec^n x\,dx = \int\sec^{n-2}x\sec^2 x\,dx$

$= \sec^{n-2}x\tan x - (n-2)\int\tan x\sec^{n-3}x\frac{\sin x}{\cos^2 x}\,dx$

$= \sec^{n-2}x\tan x - (n-2)\int\sec^{n-2}x\tan^2 x\,dx$

$= \sec^{n-2}x\tan x - (n-2)\int\sec^n\,dx + (n-2)\int\sec^{n-2}x\,dx$

$\therefore I_n = \sec^{n-2}x\tan x - (n-2)I_n + (n-2)I_{n-2}$

$(n-1)I_n = \sec^{n-2}x\tan x + (n-2)I_{n-2}$

$I_n = \frac{1}{n-1}(\sec^{n-2}x\tan x) + \frac{n-2}{n-1}I_{n-2}$

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sec^4 x\,dx = \frac{1}{3}\left[\sec^2 x\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sec^2 x\,dx$

$= \frac{1}{3}\left(4\sqrt{3} - \frac{4\sqrt{3}}{9}\right) + \frac{2}{3}[\tan x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{32\sqrt{3}}{27} + \frac{2}{3}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \frac{44\sqrt{3}}{27}$

7 (a) $I_n = \int_0^{\frac{\pi}{4}}\tan^{2n}x\,dx = \int_0^{\frac{\pi}{4}}\tan^{2n-2}x\tan^2 x\,dx$

$= \int_0^{\frac{\pi}{4}}\tan^{2n-2}x(\sec^2 x - 1)\,dx$

$= \int_0^{\frac{\pi}{4}}\tan^{2n-2}x\sec^2 x\,dx - \int_0^{\frac{\pi}{4}}\tan^{2(n-1)}x\,dx$

$= \left[\frac{1}{2n-1}\tan^{2n-1}x\right]_0^{\frac{\pi}{4}} - I_{2n-2} = \frac{1}{2n-1} - I_{n-1}$

(b) $I_2 = \int_0^{\frac{\pi}{4}}\tan^4 x\,dx = \frac{1}{3} - \int_0^{\frac{\pi}{4}}\tan^2 x\,dx$

$= \frac{1}{3} - \int_0^{\frac{\pi}{4}}(\sec^2 x - 1)\,dx$

$= \frac{1}{3} - [\tan x - x]_0^{\frac{\pi}{4}} = \frac{1}{3} - \left(1 - \frac{\pi}{4} - 0\right) = \frac{\pi}{4} - \frac{2}{3}$

8 (a) $I_n = \int_0^1 x^{2n-1}e^{x^2}\,dx = \frac{1}{2}\int_0^1 x^{2n-2}2xe^{x^2}\,dx$

$= \frac{1}{2}\left[x^{2n-2}e^{x^2}\right]_0^1 - \frac{2n-2}{2}\int_0^1 x^{2n-3}e^{x^2}\,dx = \frac{e}{2} - (n-1)I_{n-1}$

(b) $I_2 = \int_0^1 x^3 e^{x^2}\,dx = \frac{e}{2} - \int_0^1 x e^{x^2}\,dx$

$= \frac{e}{2} - \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{e}{2} - \frac{1}{2}(e-1) = \frac{1}{2}$

9 (a) $I_n = \int x^5 (\log_e x)^n\,dx$

$= \frac{x^6}{6}(\log_e x)^n - \frac{1}{6}\int x^6 n(\log_e x)^{n-1}\frac{1}{x}\,dx$

$= \frac{x^6}{6}(\log_e x)^n - \frac{n}{6}\int x^5(\log_e x)^{n-1}\,dx$

$\therefore I_n = \frac{x^6}{6}(\log_e x)^n - \frac{n}{6}I_{n-1}$

(b) $I_2 = \int x^5(\log_e x)^2\,dx = \frac{x^6}{6}(\log_e x)^2 - \frac{2}{6}I_1$

$I_1 = \frac{x^6}{6}(\log_e x) - \frac{1}{6}\int x^5\,dx$

$= \frac{x^6\log_e x}{6} - \frac{1}{6}\times\frac{x^6}{6} = \frac{x^6 \log_e x}{6} - \frac{x^6}{36}$

$\therefore I_2 = \frac{x^6}{6}(\log_e x)^2 - \frac{1}{3}\left(\frac{x^6\log_e x}{6} - \frac{x^6}{36}\right)$

$= \frac{x^6}{6}(\log_e x)^2 - \frac{x^6\log_e x}{18} + \frac{x^6}{108} + C$

$= \frac{x^6}{108}\left(18(\log_e x)^2 - 6\log_e x + 1\right) + C$

10 (a) $I = \int e^{ax}\sin x\,dx = -\cos x e^{ax} - \int(-\cos x)ae^{ax}\,dx$

$= -\cos x e^{ax} + a\int \cos x e^{ax}\,dx$

$H = \int\cos x e^{ax}\,dx = \sin x e^{ax} - a\int \sin x e^{ax}\,dx = \sin x e^{ax} - aI$

$\therefore I = -\cos x e^{ax} + a(\sin x e^{ax} - aI)$

$(a^2+1)I = -\cos x e^{ax} + a\sin x e^{ax}$

$\therefore \int e^{ax}\sin x\,dx = \frac{1}{a^2+1}e^{ax}(a\sin x - \cos x)$

(b) (i) $\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$

(ii) $\int e^{3x}\sin x\,dx = \frac{e^{3x}}{10}(3\sin x - \cos x)+C$

11 $I_n = \int_0^x \frac{t^n}{1+t}dt,\ I_{n-1} = \int_0^x \frac{t^{n-1}}{1+t}dt$

$\therefore I_n + I_{n-1} = \int_0^x \frac{t^n}{1+t}dt + \int_0^x\frac{t^{n-1}}{1+t}dt = \int_0^x\left(\frac{t^n}{1+t}+\frac{t^{n-1}}{1+t}\right)dt$

$= \int_0^x t^{n-1}\left(\frac{t}{1+t}+\frac{1}{1+t}\right)dt = \int_0^x t^{n-1}\,dt = \left[\frac{t^n}{n}\right]_0^x = \frac{x^n}{n}$

EXERCISE 5.6

1 B

2 $\int_0^{\frac12}\sqrt{\frac{1+x}{1-x}}\,dx = \int_0^{\frac12}\sqrt{\frac{(1+x)^2}{1-x^2}}\,dx = \int_0^{\frac12}\frac{1+x}{\sqrt{1-x^2}}\,dx$

$= \int_0^{\frac12}\left(\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}\right)dx$

$= \left[\sin^{-1}x - \sqrt{1-x^2}\right]_0^{\frac12}$

$= \sin^{-1}\frac12 - \frac{\sqrt3}{2} - (0-1) = 1 - \frac{\sqrt3}{2}+\frac{\pi}{6}$

3 (a) $-x^2+4x-3 = 1-(x^2-4x+4) = 1-(x-2)^2$

(b) $\int_1^3\sqrt{(x-1)(3-x)}\,dx = \int_1^3\sqrt{1-(x-2)^2}\,dx$

$x-2=\sin\theta,\ dx=\cos\theta\,d\theta$

Limits $x=1:\ \theta=-\frac{\pi}{2}\quad x=3:\ \theta=\frac{\pi}{2}$

$\int_{-\frac\pi2}^{\frac\pi2}\sqrt{1-\sin^2\theta}\cos\theta\,d\theta = \int_{-\frac\pi2}^{\frac\pi2}\cos^2\theta\,d\theta = 2\int_0^{\frac\pi2}\cos^2\theta\,d\theta$

$= \int_0^{\frac\pi2}(1+\cos2\theta)\,d\theta = \left[\theta + \frac{\sin2\theta}{2}\right]_0^{\frac\pi2}$

$= \left(\frac\pi2+\frac12\sin\pi - 0\right) = \frac\pi2$

4 (a) $\int\sec x\,dx = \int\left(\frac{1+t^2}{1-t^2}\times\frac{2}{1+t^2}\right)dt = \int\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt$

$= \log_e\left|\frac{1+t}{1-t}\right| + C = \log_e\left|\frac{1+\tan\frac x2}{1-\tan\frac x2}\right| + C$

(b) $\int \sec x\,dx = \int\frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dt = \log_e|\sec x+\tan x|+C$

5 (a) $\sec x + C$

(b) $u = \frac\pi2 - x,\ du=-dx$:

$\int \operatorname{cosec}x\cot x\,dx = \int\operatorname{cosec}\left(\frac\pi2-u\right)\cot\left(\frac\pi2-u\right)(-du)$

$= -\int\sec u\tan u\,du = -\sec u + C$

$= -\sec\left(\frac\pi2 - x\right)+C = -\operatorname{cosec}x + C$

6 $x^2+y^2=r^2$: $2x+2y\frac{dy}{dx}=0,\ \frac{dy}{dx}=\frac{-x}{y}$

$C = 4\int_0^r\sqrt{1+\left(\frac{-x}{y}\right)^2}\,dx = 4\int_0^r\sqrt{\frac{y^2+x^2}{y^2}}\,dx = 4\int_0^r\frac{r}{\sqrt{r^2-x^2}}\,dx$

$= 4r\left[\sin^{-1}\left(\frac xr\right)\right]_0^r = 4r\times\left(\frac\pi2 - 0\right) = 2\pi r$

7 (a) $I = \int_2^1\frac{\cos^2\left(\frac\pi6(3-u)\right)}{(3-u)u}(-du) = \int_1^2\frac{\cos^2\left(\frac\pi2-\frac\pi6u\right)}{u(3-u)}du$

$= \int_1^2\frac{\sin^2\left(\frac\pi6 u\right)}{u(3-u)}du$

(b) $2I = \int_1^2\frac{\cos^2\left(\frac\pi6x\right)}{x(3-x)}du + \int_1^2\frac{\sin^2\left(\frac\pi6x\right)}{x(3-x)}dx = \int_1^2\frac{dx}{x(3-x)}$

$=\frac13\int_1^2\left(\frac1x+\frac1{3-x}\right)dx = \frac13\left[\log_e\left|\frac{x}{3-x}\right|\right]_1^2$

$=\frac13\left(\log_e2 - \log_e\left(\frac12\right)\right) = \frac23\log_e 2$

$\therefore I = \frac13\log_e 2$

8 (a) $\frac{d}{dx}\left(x^2\tan^{-1}x\right) = 2x\tan^{-1}x + \frac{x^2}{x^2+1}$

(b) $\int 2x\tan^{-1}x\,dx = \int\left(\frac{d}{dx}\left(x^2\tan^{-1}x\right) - \frac{x^2}{x^2+1}\right)dx$

$= x^2\tan^{-1}x - \int\left(1-\frac{1}{x^2+1}\right)dx$

$= x^2\tan^{-1}x - x + \tan^{-1}x + C$

9 (a) $f(x)=x^3$ is odd, $g(x)=(1-x^2)^2$ is even.

Hence $f(x)\times g(x)$ is odd: $\int_{-1}^1 x^3(1-x^2)^2\,dx = 0$

(b) $f(x)=x^2$ is even, $g(x)=(1-x^2)^3$ is even (as $1-x^2$ is even).

Hence $f(x)\times g(x)$ is even: $\int_{-1}^1 x^2(1-x^2)^3\,dx$

$= 2\int_0^1 x^2(1-x^2)^3\,dx$

10 (a) Let $x = a + b - u$: $dx = -du$

Limits are $x = a$: $u = b$ $x = b$: $u = a$

$$\int_a^b f(x)\,dx = \int_b^a f(a+b-u)(-du)$$
$$= \int_a^b f(a+b-u)\,du = \int_a^b f(a+b-x)\,dx$$

(b) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\left(\frac{\pi}{2}-x\right)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x\,dx$$

11 (a) Let $u = \pi - x$: $du = -dx$

Limits are $x = 0$: $u = \pi$ $x = \pi$: $u = 0$

$$\int_0^{\pi} x\sin x\,dx = \int_{\pi}^{0} (u-\pi)\sin(u-\pi)(-du)$$
$$= \int_0^{\pi} (u-\pi)\sin(u-\pi)\,du$$
$$= \int_0^{\pi} (u-\pi)\sin(\pi+u)\,du$$
$$= -\int_0^{\pi} (u-\pi)\sin u\,du$$
$$= \int_0^{\pi} (\pi-u)\sin u\,du$$

But the variable in the integrand is arbitrary, so:

$$\int_0^{\pi} x\sin x\,dx = \int_0^{\pi} (\pi-x)\sin x\,dx$$
$$\int_0^{\pi} x\sin x\,dx = \int_0^{\pi} \pi\sin x\,dx - \int_0^{\pi} x\sin x\,dx$$
$$2\int_0^{\pi} x\sin x\,dx = \pi\int_0^{\pi} \sin x\,dx$$
$$\int_0^{\pi} x\sin x\,dx = \frac{\pi}{2}\int_0^{\pi} \sin x\,dx$$

(b) $$\int_0^{\pi} x\sin x\,dx = \frac{\pi}{2}[-\cos x]_0^{\pi} = -\frac{\pi}{2}(-1-1) = \pi$$

12 (a) Take reciprocals of each term in $1 < u < 1 + x$, remembering to reverse the inequality signs:

$$1 > \frac{1}{u} > \frac{1}{1+x}$$

Hence: $$\frac{1}{1+x} < \frac{1}{u} < 1$$

(b) $$\int_1^{1+x} \frac{1}{1+x}\,du < \int_1^{1+x} \frac{du}{u} < \int_1^{1+x} du$$
$$\left[\frac{u}{1+x}\right]_1^{1+x} < [\log_e u]_1^{1+x} < [u]_1^{1+x}$$
$$1 - \frac{1}{1+x} < \log_e(1+x) < 1 + x - 1$$
$$\frac{x}{1+x} < \log_e(1+x) < x$$

EXERCISE 5.7

1 $y = \tan^{-1} x$ is an odd function. Hence

$\int_{-1}^{0} \tan^{-1} x\,dx = -\int_0^1 \tan^{-1} x\,dx$ so that $\int_{-1}^{1} \tan^{-1} x\,dx = 0$.

2 Area $= \int_1^2 \frac{1}{x(3-x)}dx = \frac{1}{3}\left(\ln 2 - \ln\frac{1}{2}\right) = \frac{2}{3}\ln 2$ units^2

3 Area $= \int_4^5 \frac{x^2-6x+8}{x^2-3x}dx = 1 + \frac{8}{3}\ln\frac{4}{5} - \frac{1}{3}\ln 2 = 0.174$ units^2

4 $1 - x^2 = 0 \Rightarrow x = \pm 1$

Area $= \int_{-1}^{1} \frac{1-x^2}{1+x^2}dx = \pi - 2$ units^2

5 Volume $= \pi\int_{-1}^{1}\left(\frac{1}{\sqrt{4-x^2}}\right)^2 dx = \frac{\pi}{4}\left(\ln 3 - \ln\frac{1}{3}\right) = \frac{\pi}{2}\ln 3$ units^3

6 Volume $= \pi\int_2^3\left(\frac{x}{\sqrt{x^2-1}}\right)^2 dx = \pi\left(3 + \frac{1}{2}\ln\frac{1}{2} - \left(2 + \frac{1}{2}\ln\frac{1}{3}\right)\right)$

$= \frac{\pi}{2}(2 + \ln 1.5)$ units^3

7 (a) $\int_1^t \ln x\,dx = t\ln t - t + 1$

(b) Area of triangle $= \frac{1}{2}\times(t-1)\times\ln t$

$= \frac{(t-1)\ln t}{2}$

Area under curve > Area of triangle

$t\ln t - t + 1 > \frac{(t-1)\ln t}{2}$

$2t\ln t - 2t + 2 > t\ln t - \ln t$

$t\ln t + \ln t > 2(t-1)$

$\ln t(t+1) > 2(t-1)$

$\ln t > 2\left(\frac{t-1}{t+1}\right), t > 1$

8 (a) $\frac{d}{dx}\left(\sin^{-1} x - \sqrt{1-x^2}\right) = \frac{1+x}{\sqrt{1-x^2}}$

(b) Now $\frac{d}{dx}\left(\sin^{-1} x - \sqrt{1-x^2}\right) = \frac{1+x}{\sqrt{(1-x)(1+x)}} = \sqrt{\frac{1+x}{1-x}}$

$$\int_0^{\theta}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} dx = \int_0^{\theta} \frac{d}{dx}\left(\sin^{-1} x - \sqrt{1-x^2}\right)dx$$
$$= \left[\sin^{-1} x - \sqrt{1-x^2}\right]_0^{\theta}$$
$$= \sin^{-1}\theta - \sqrt{1-\theta^2} - (0-1)$$
$$= \sin^{-1}\theta + 1 - \sqrt{1-\theta^2}$$

9 $V = \pi\int_0^{\pi} \sin^2 x\,dx = \frac{\pi^2}{2}$ units^3

10 $V = \pi\int_2^4 \frac{(x-2)^2(4-x)}{x}dx = \pi\left(16\log_e 2 - 10\frac{2}{3}\right) \approx 2.38$ units^3

11 (a) $V = \pi\int_0^{\frac{\pi}{4}}\left(\cos^2 x - \sin^2 x\right)dx = \pi\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{2}$ units^3

(b) $V = \pi\int_0^{\frac{1}{\sqrt{2}}}\left(\sin^{-1} y\right)^2 dy + \pi\int_{\frac{1}{\sqrt{2}}}^{1}\left(\cos^{-1} y\right)^2 dy$

$\approx 0.42 + 0.28 \approx 0.70$ units^3

12 $V = 4\pi\int_{-a}^{a}(b-x)\sqrt{a^2-x^2}\,dx = 2\pi^2 a^2 b$ units^3

13 (a) $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi^2}{2}$ units^3

(b) $V = \pi\times\left(\frac{\pi}{2}\right)^2\times 1 - \pi\int_0^1\left(\sin^{-1} x\right)^2 dx \approx 6.283$ units^3

(c) $V = 2\pi\int_0^{\frac{\pi}{2}} x\sin x\,dx = 2\pi$ units^3

14 $y = -x^2\cos x + 6\cos x + 4x\sin x + C$

$x = 0, y = 6$: $6 = 6 + C, C = 0$

$y = (6 - x^2)\cos x + 4x\sin x$

15 $y = -\frac{e^{-x}}{2}(\cos x + \sin x)$

16 $y = 2 - \log_e\left(x^2+1\right)$

17 $y = \frac{e^{-x}\cos x}{2}$

18 $V = 2\pi\int_0^1\left(1-e^{-x}\right)(1-x)\,dx = \pi\left(1-\frac{2}{e}\right)$ units^3

19 (a) $V = \pi\int_{-a}^{a} y^2\,dx = \dfrac{\pi b^2}{a^2}\int_{-a}^{a}\left(a^2 - x^2\right)dx$

$$= \frac{\pi b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{\pi b^2}{a^2}\left(a^3 - \frac{a^3}{3}\right) = \frac{4\pi ab^2}{3}$$

(b) $V = 2\pi\int_{-a}^{a} 2y(a-x)\,dx = 2\pi^2a^2b$ units^3

20 $y = 4x - x\log_e\left(x^2+1\right) - 2\tan^{-1}x + 1$

21 $y = \dfrac{1}{x^2+1} + 1$

22 (a) Integrate by parts: $I_n = \left[x\left(\log_e x\right)^n\right]_1^2 - n\int_1^2\left(\log_e x\right)^{n-1}dx$

$$= 2\left(\log_e x\right)^n - nI_{n-1}$$

(b) $\dfrac{dy}{dx} = \dfrac{n\left(\log_e x\right)^{n-1}}{x}$:

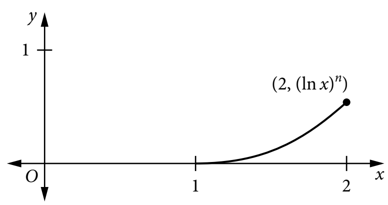
As $n \geq 2$, for $1 < x \leq 2$ all factors are positive

$\therefore \dfrac{dy}{dx} > 0$. Hence the function increases for $1 < x \leq 2$.

$\dfrac{d^2y}{dx^2} = \dfrac{n\left(\log_e x\right)^{n-2}\left(n-1-\log_e x\right)}{x^2}$:

By a similar argument $\dfrac{d^2y}{dx^2} > 0$ for $1 < x \leq 2$.

Hence the function is concave up for $1 < x \leq 2$.



(c) Area under the curve < Area of triangle

From part (a): $2\left(\log_e 2\right)^n - nI_{n-1} < \dfrac{\left(\log_e 2\right)^n}{2}$

which yields the result.

(d) $I_1 = 2\log_e 2 - 1$; from part (c) with $n = 2$, the result follows.

23 (a) $64 \equiv (2x+a)(x^2-4x+4) - (2x-b)(x^2+4x+4)$

$x = 2$: $64 = -(4-b)\times 16 \Rightarrow b = 8$

$x = -2$: $64 = (-4+a)\times 16 \Rightarrow a = 8$

(or use $x = 0$: $64 = 4a + 4b \Rightarrow a + b = 16$, $a = 16 - b$)

(b) $\displaystyle\int_0^m \frac{64}{x^4-8x^2+16}dx = \int_0^m\left(\frac{2x+8}{x^2+4x+4} - \frac{2x-8}{x^2-4x+4}\right)dx$

$$= \int_0^m\left(\frac{2x+4}{x^2+4x+4} + \frac{4}{x^2+4x+4} - \frac{2x-4}{x^2-4x+4} + \frac{4}{x^2-4x+4}\right)dx$$

$$= \left[\log_e\left(x^2+4x+4\right) - \log_e\left(x^2-4x+4\right)\right]_0^m$$

$$+ 4\int_0^m\left(\frac{1}{(x+2)^2} + \frac{1}{(x-2)^2}\right)dx$$

$$= \log_e\left(\frac{m^2+4m+4}{m^2-4m+4}\right) - 0 + 4\left[\frac{-1}{x+2} + \frac{-1}{x-2}\right]_0^m$$

$$= \log_e\left(\frac{m+2}{m-2}\right)^2 - 4\left(\frac{1}{m+2} + \frac{1}{m-2} - \left(\frac{1}{2} - \frac{1}{2}\right)\right)$$

$$= 2\log_e\left(\frac{m+2}{m-2}\right) - \frac{8m}{m^2-4}$$

24 (a) $36 \equiv (2x+a)(x^2-3x+3) - (2x-b)(x^2+3x+3)$

$x = 0$: $36 = 3a + 3b \Rightarrow a + b = 12 \Rightarrow a = 12 - b$

$x = 1$: $36 = (2+a)\times 1 - (2-b)\times 7 \Rightarrow a + 7b = 48$

Substitute $a = 12 - b$: $12 - b + 7b = 48 \Rightarrow b = 6, a = 6$

(b) $\displaystyle\int_0^p \frac{36}{x^4-3x^2+9} = \int_0^p\left(\frac{2x+6}{x^2+3x+3} - \frac{2x-6}{x^2-3x+3}\right)dx$

$$= \int_0^p\left(\frac{2x+3}{x^2+3x+3} + \frac{3}{x^2+3x+3} - \frac{2x-3}{x^2-3x+3} + \frac{3}{x^2-3x+3}\right)dx$$

$$= \left[\log_e\left(x^2+3x+3\right) - \log_e\left(x^2-3x+3\right)\right]_0^p$$

$$+ \int_0^p\left(\frac{3}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{3}{\left(x-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)dx$$

$$= \log_e\left(\frac{p^2+3p+3}{p^2-3p+3}\right) - 0$$

$$+ \left[\frac{3\times 2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{3\times 2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^p$$

$$= \log_e\left(\frac{p^2+3p+3}{p^2-3p+3}\right) + 2\sqrt{3}\left[\tan^{-1}\left(\frac{2x+3}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right)\right]_0^p$$

$$= \log_e\left(\frac{p^2+3p+3}{p^2-3p+3}\right)$$

$$+ 2\sqrt{3}\left(\tan^{-1}\left(\frac{2p+3}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2p-3}{\sqrt{3}}\right) - \left(\tan^{-1}\sqrt{3} + \tan^{-1}\left(-\sqrt{3}\right)\right)\right)$$

$$= \log_e\left(\frac{p^2+3p+3}{p^2-3p+3}\right) + 2\sqrt{3}\left(\tan^{-1}\left(\frac{2p+3}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2p-3}{\sqrt{3}}\right)\right)$$

(c) $\displaystyle\lim_{p\to\infty}\int_0^p \frac{36}{x^4-3x^2+9}dx = \lim_{p\to\infty}\left[\log_e\left(\frac{p^2+3p+3}{p^2-3p+3}\right)\right.$

$$\left. + 2\sqrt{3}\left(\tan^{-1}x\left(\frac{2p+3}{\sqrt{3}}\right) + \tan^{-1}x\left(\frac{2p-3}{\sqrt{3}}\right)\right)\right]$$

$$\lim_{p\to\infty}\left[\log_e\left(\frac{p^2+3p+3}{p^2-3p+3}\right)\right] = \log_e 1 = 0$$

$$\lim_{p\to\infty}\left[\left(\tan^{-1}x\left(\frac{2p+3}{\sqrt{3}}\right) + \tan^{-1}x\left(\frac{2p-3}{\sqrt{3}}\right)\right)\right] = \left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \pi$$

Hence $\displaystyle\lim_{p\to\infty}\int_0^p \frac{36}{x^4-3x^2+9}dx = 2\sqrt{3}\pi$

25 $V = 2\pi\int_1^e xy\,dx = \dfrac{2\pi}{9}\left(2e^3+1\right)$ units^3

26 $V = \int_0^{\frac{\pi}{2}} y\sqrt{1-y^2}\,dx = \dfrac{1}{2}\left[\sin^2 x\right]_0^{\frac{\pi}{2}} = \dfrac{1}{2}$ units^3

CHAPTER REVIEW 5

1 (a) $\dfrac{(2\sqrt{3}-1)\pi}{12} + \dfrac{1-\sqrt{3}}{2}$

(b) $\left[\dfrac{5}{2}\log_e\left|x^2-3x+2\right| + \dfrac{1}{2}\log_e\left|\dfrac{x-2}{x-1}\right|\right]_3^4 = \log_e 18$

2 (a) $x(\log_e|2x| - 1) + C$ (b) $\dfrac{1}{2}\log_e\left|\dfrac{(x-1)^3}{x+1}\right| + C$

3 $\log_e\dfrac{9}{8}$

4 (a) $\dfrac{(4\sqrt{3}-3)\pi}{12} - \dfrac{1}{2}\log_e 2$

(b) $2\log_e 5$

5 $-\operatorname{cosec} x$

(a) $-2\left[\log_e\left|\operatorname{cosec}\dfrac{\theta}{2}+\cot\dfrac{\theta}{2}\right|\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2\log_e\left(\dfrac{\sqrt{3}+2}{\sqrt{2}+1}\right)$

(b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec u\,du = \left[\log_e(\sec u+\tan u)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \log_e(2+\sqrt{3}) - \log_e\left(\dfrac{2}{\sqrt{3}}+\dfrac{1}{\sqrt{3}}\right)$

$= \log_e\left(\dfrac{2+\sqrt{3}}{\sqrt{3}}\right)$

6 (a) $\dfrac{d}{dx}\left(\dfrac{\sin x}{\cos^2 x}+\dfrac{1}{2}\log_e(1+\sin x)-\dfrac{1}{2}\log_e(1-\sin x)\right)$

$= \dfrac{\cos x\cos^2 x - \sin x\times 2\cos x(-\sin x)}{\cos^4 x}$

$+ \dfrac{\cos x}{2(1+\sin x)} - \dfrac{-\cos x}{2(1-\sin x)}$

$= \dfrac{\cos^2 x+2\sin^2 x}{\cos^3 x} + \dfrac{\cos x}{2}\left(\dfrac{1-\sin x+1+\sin x}{1-\sin^2 x}\right)$

$= \dfrac{\cos^2 x+2\sin^2 x}{\cos^3 x} + \dfrac{\cos x}{2}\times\dfrac{2}{\cos^2 x}$

$= \dfrac{2(\cos^2 x+\sin^2 x)}{\cos^3 x} = 2\sec^3 x$

(b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^3\theta\,d\theta = \frac{1}{2}\left[\frac{\sin\theta}{\cos^2\theta}+\log_e\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \dfrac{1}{2}\left(\dfrac{\sqrt{3}}{2}\times 4+\log_e\sqrt{\dfrac{2+\sqrt{3}}{2-\sqrt{3}}}-\left(\dfrac{1}{2}\times\dfrac{4}{3}+\log_e\sqrt{\dfrac{2+1}{2-1}}\right)\right)$

$= \dfrac{1}{2}\left(2\sqrt{3}+\log_e\sqrt{\dfrac{(2+\sqrt{3})^2}{4-3}}-\dfrac{2}{3}-\log_e\sqrt{3}\right)$

$= \dfrac{1}{2}\left(2\sqrt{3}-\dfrac{2}{3}+\log_e\left(\dfrac{2+\sqrt{3}}{\sqrt{3}}\right)\right)$

7 (a) $-7+8x-x^2 = b^2-a^2+2ax-x^2$: $a=4$, $b=3$

(b) $\displaystyle\int_1^7 \sqrt{(x-1)(7-x)}\,dx = \int_1^7\sqrt{9-(x-4)^2}\,dx$

$x-4 = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$x=1,\ \theta=-\dfrac{\pi}{2};\ \ x=7,\ \theta=\dfrac{\pi}{2}$

$\displaystyle\int_1^7\sqrt{9-(x-4)^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta$

$\displaystyle = 9\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$

$\displaystyle = 9\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

$\displaystyle = \frac{9}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$

$\displaystyle = \frac{9}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{9\pi}{2}$

8 $(a+c)x^2+(a+b)x+b \equiv 1$

$a+c=0,\ a+b=0$: $b=1,\ a=-1,\ c=1$

9 (a) $\dfrac{2}{x-1}+\dfrac{3}{x-2}-\dfrac{4}{x-3}$ (b) $\dfrac{10}{x-2}-\dfrac{9}{x-1}$

(c) $x+1+\dfrac{3}{x-3}-\dfrac{2}{x-2}$ (d) $\dfrac{4}{2x-3}-\dfrac{3}{x-3}+\dfrac{5}{2x+1}$

10 (a) $\dfrac{1}{x-3}-\dfrac{2}{x-2}+\dfrac{3}{2x-1}$

(b) $2+\dfrac{4}{x-2}-\dfrac{3}{x+2}$

11 (a) RHS $= \dfrac{(2x+4)(3-x^2)+2x(x^2+2x+2)}{(x^2+2x+2)(3-x^2)}$

$= \dfrac{6x-2x^3+12-4x^2+2x^4+4x^2+4x}{(x^2+2x+2)(3-x^2)}$

$= \dfrac{10x+12}{(x^2+2x+2)(3-x^2)} =$ LHS

(b) LHS $= \dfrac{2x+2+2}{x^2+2x+2} = \dfrac{2x+2}{x^2+2x+2}+\dfrac{2}{x^2+2x+1+1}$

$= \dfrac{2x+2}{x^2+2x+2}+\dfrac{2}{(x+1)^2+1} =$ RHS

12 $a=\dfrac{61}{5}, b=\dfrac{6}{25}, c=-\dfrac{6}{25}$

13 (a) $\dfrac{\pi\sqrt{3}}{18}+\log_e 5$ (b) $\dfrac{\sqrt{2}(\pi-4)}{4}$

14 $\dfrac{a}{x\sqrt{a^2-x^2}}$; $\dfrac{1}{5}\log_e\dfrac{3}{2}$

15 (a) $17\dfrac{1}{15}$ (b) $e-1$ (c) $\dfrac{1}{8}\log_e\dfrac{9}{5}$

(d) $4(\sin 1-\cos 1)\approx 1.205$ (e) $\dfrac{81\pi}{16}$

16 (a) $\dfrac{1}{4}\log_e\left|\dfrac{1+2x}{1-2x}\right|+C$ (b) $-\dfrac{1}{8}\log_e\left|1-4x^2\right|+C$

(c) $-\dfrac{x}{4}+\dfrac{1}{16}\log_e\left|\dfrac{1+2x}{1-2x}\right|+C$ (d) $-\dfrac{1}{4}\sqrt{1-4x^2}+C$

(e) $\dfrac{1}{2}\sin^{-1}2x+C$ (f) $\dfrac{1}{2}\tan^{-1}2x+C$

17 (a) $\dfrac{1}{2}\log_e\left|\tan\dfrac{x}{2}\right|-\dfrac{1}{4}\tan^2\dfrac{x}{2}+C$ (b) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{1}{3}\tan x\right)+C$

(c) $\dfrac{1}{5}\log_e\left(2+\tan\dfrac{\theta}{2}\right)-\dfrac{1}{5}\log_e\left(2\tan\dfrac{\theta}{2}-1\right)+C$

(d) $\dfrac{5}{6}\tan^{-1}\left(2\tan\dfrac{\theta}{2}\right)-\dfrac{\theta}{3}+C$

18 $\displaystyle\int_0^{\frac{\pi}{3}}\frac{1}{4+5\cos x}\,dx = \frac{1}{3}\ln\left(\frac{3\sqrt{3}+1}{3\sqrt{3}-1}\right) = \frac{1}{3}\ln\left(\frac{14+3\sqrt{3}}{13}\right)$

19 $\dfrac{x^2\log_e|2x|}{2}-\dfrac{x^2}{4}+C$

20 (a) $\dfrac{1}{3}\log_e\left|x^3+3x\right|+C$ (b) $-\dfrac{1}{x+1}+C$

(c) $\dfrac{x^3}{3}+\log_e x+C$ (d) $\sqrt{x^2+2x-3}+C$

(e) $\log_e\left|\dfrac{x}{\sqrt{x^2+4}}\right|+\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}+C$

(f) $\dfrac{1}{3}\log_e\left|\dfrac{x-1}{\sqrt{x^2+x+1}}\right|-\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+C$

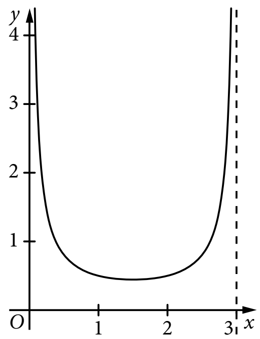
(g) $\dfrac{1}{4}(2x^2-1)\sin^{-1}x+\dfrac{x\sqrt{1-x^2}}{4}+C$

(h) $\dfrac{x^3\tan^{-1}x}{3}-\dfrac{x^2}{6}+\dfrac{1}{6}\log_e(x^2+1)+C$

21 (a) $\dfrac{\sqrt{5}}{10}\log_e\left|\dfrac{x-2-\sqrt{5}}{x-2+\sqrt{5}}\right|+C$ (b) $\dfrac{1}{\sqrt{21}}\tan^{-1}\left(\dfrac{\sqrt{3}(x+1)}{\sqrt{7}}\right)+C$

(c) $\log_e\left|x-2+\sqrt{x^2-4x+1}\right|+C$ (d) $\log_e\left(x+\sqrt{x^2+16}\right)+C$

22 (a)



(b) Area $= \int_1^2 \frac{dx}{x(3-x)}$

$$= \frac{1}{3}\int_1^2 \left(\frac{1}{x} + \frac{1}{3-x}\right) dx$$

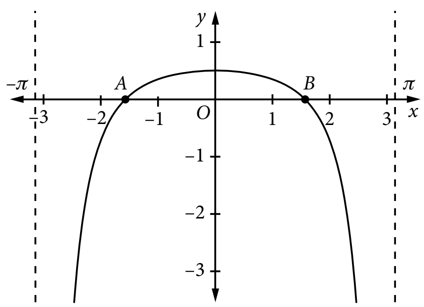
$$= \frac{1}{3}\left[\log_e\left(\frac{x}{3-x}\right)\right]_1^2$$

$$= \frac{1}{3}\log_e 4 \approx 0.462 \text{ units}^2$$

23 Area $= \int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 - \int_0^1 -e^{-x}\,dx$

$$= -\frac{1}{e} - \left[e^{-x}\right]_0^1 = -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e} \approx 0.264 \text{ units}^2$$

24



$$\frac{dy}{dx} = \frac{-\sin x(1+\cos x) - \cos x(-\sin x)}{(1+\cos x)^2} = \frac{-\sin x}{(1+\cos x)^2}$$

Turning point where $\sin x = 0$: $\left(0, \frac{1}{2}\right)$

$A\left(-\frac{\pi}{2}, 0\right)$, $B\left(\frac{\pi}{2}, 0\right)$ $\therefore$ Area $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x}\,dx$

$$\int \frac{\cos x}{1+\cos x}\,dx = \int \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2} = \int \left(-1 + \frac{2}{1+t^2}\right) dt$$

$$= 2\tan^{-1} t - t = x - \tan\frac{x}{2}$$

$$\therefore \text{ Area} = \left[x - \tan\frac{x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi - 2 \text{ units}^2$$

25 Find the particular solution of $\frac{dy}{dx} = \frac{1}{x^2+1} + 1$, given that $y = 1$ when $x = 0$.

$\frac{dy}{dx} = \frac{1}{x^2+1} + 1$: $\quad y = \int \left(\frac{1}{x^2+1} + 1\right) dx$

$$= \tan^{-1} x + x + C$$

$x = 0, y = 1$: $1 = C$

$y = \tan^{-1} x + x + 1$

26 $\frac{dy}{dx} = (2-y)(2+y)$

$$\frac{dx}{dy} = \frac{1}{(2-y)(2+y)}$$

$$= \frac{1}{4}\left(\frac{1}{2-y} + \frac{1}{2+y}\right)$$

$$4x = \int \left(\frac{1}{2-y} + \frac{1}{2+y}\right) dy$$

$$= \left(-\ln|2-y| + \ln|2+y|\right) + C$$

$$4x + C = \ln\left|\frac{2+y}{2-y}\right| + C$$

$$\left|\frac{2+y}{2-y}\right| = e^{4x+C}$$

Let $A = \pm e^C$: $\frac{2+y}{2-y} = Ae^{4x}$

$$2 + y = 2Ae^{4x} - Aye^{4x}$$

$$y\left(1 + Ae^{4x}\right) = 2\left(Ae^{4x} - 1\right)$$

$$y = \frac{2\left(Ae^{4x} - 1\right)}{1 + Ae^{4x}}$$

$$y = \frac{2\left(A - e^{-4x}\right)}{A + e^{-4x}}$$

$y(0) = 1$: $1 = \frac{2(A-1)}{A+1}$

$$A + 1 = 2A - 2$$

$$A = 3$$

$$y = \frac{2\left(3 - e^{-4x}\right)}{3 + e^{-4x}}$$

27 (a) Let $u = a - x$ so $x = a - u$ and $dx = -du$

$x = 0, u = a. \quad x = a, u = 0$

$$\int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du$$

$$= \int_0^a f(a-u)\,du$$

$$= \int_0^a f(a-x)\,dx$$

$$I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$= \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$= \int_0^\pi \frac{(\pi - x)\sin x}{1+(-\cos x)^2}\,dx$$

$$= \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$2I = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx$$

(b) Let $u = \cos x$, $du = -\sin x dx$. $\quad x = 0, u = 1. \quad x = \pi, u = -1$

$$\int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}\,dx = \pi\int_1^{-1} \frac{-du}{1+u^2}$$

$$= \pi\int_{-1}^1 \frac{du}{1+u^2}$$

$$= \pi\left[\tan^{-1} u\right]_{-1}^1$$

$$= \pi\left(\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right)$$

$$= \frac{\pi^2}{2}$$

$$\therefore I = \frac{\pi^2}{4}$$

CHAPTER 6

EXERCISE 6.1

1 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4 + x$, $\frac{1}{2}v^2 = 4x + \frac{1}{2}x^2 + C$

$x = 0, v = 1$: $C = \frac{1}{2}$, $v^2 = 8x + x^2 + 1$

$x = 1$: $v = \sqrt{10}$

2 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 7 - 2x$, $\frac{1}{2}v^2 = 7x - x^2 + C$
$x = 0, v = 0$: $C = 0$, $v^2 = 14x - 2x^2$, $v = \sqrt{14x - 2x^2}$, $0 \le x \le 7$

3 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 4x$, $\frac{1}{2}v^2 = 3x - 2x^2 + C$
$x = 1, v = 0$: $C = -1$, $v^2 = 6x - 4x^2 - 2$
At rest: $-2(2x^2 - 3x + 1) = 0$, $(2x-1)(x-1) = 0$, $x = \frac{1}{2}$

4 C

5 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{6}\sqrt[3]{x}$, $\frac{1}{2}v^2 = -\frac{1}{8}x^{\frac{4}{3}} + C$
$x = 0, v = 2$: $C = 2$, $v^2 = 4 - \frac{1}{4}x^{\frac{4}{3}}$
$v = 0$: $x = 8$ m

6 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \cos x$, $\frac{1}{2}v^2 = \sin x + C$
$x = 0, v = 0$: $C = 0$, $v^2 = 2\sin x$, $v = \pm\sqrt{2\sin x}$

7 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$,
$\frac{1}{2}v^2 = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$
$x = 0, v = 1$: $C = \frac{1}{2}$, $v^2 = x - \frac{1}{2}\sin 2x + 1$,
$\frac{dx}{dt} = \sqrt{x - \frac{1}{2}\sin 2x + 1}$

8 $\frac{1}{2}v^2 = x^2 - x^3 + C$
$x = 0, v = 2$: $C = 2$, $v^2 = 4 + 2x^2 - 2x^3$, $v = \sqrt{4 + 2x^2 - 2x^3}$,
$v > 0$ from initial conditions

9 $\frac{1}{2}v^2 = \frac{1}{2}x^2 - 3x + C$
$x = 1, v = -2$: $C = 4.5$, $v^2 = x^2 - 6x + 9$
$x = 0$: $v^2 = 9$, $v = -3$ $(v < 0)$

10 (a) $\ddot{x} = v\frac{dv}{dx} = (4 + x^2) \times 2x = 2x^3 + 8x$
(b) $\frac{dt}{dx} = \frac{1}{4 + x^2}$, $t = \int \frac{1}{4 + x^2}\,dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$
$t = 0, x = -2$: $C = \frac{\pi}{8}$, $t = \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{\pi}{8}$, $\tan^{-1}\frac{x}{2} = 2\left(t - \frac{\pi}{8}\right)$,
$x = 2\tan 2\left(t - \frac{\pi}{8}\right)$
$t = \frac{\pi}{4}$: $x = 2$ m

11 $t = \frac{1}{2}x^2 + 4x + C$
$t = 0, x = 0$: $C = 0$, $t = \frac{1}{2}x^2 + 4x$
$x = 2$: $t = 10$

12 $t = \int \frac{1}{x + 4}\,dx$, $t = \ln(x + 4) + C$
$t = 0, x = -3$: $C = 0$, $t = \ln(x + 4)$, $x = e^t - 4$

13 $v = 2(4 - x)^2$, $t = \frac{1}{2}\int \frac{1}{(4 - x)^2}\,dx = \frac{1}{2(4 - x)} + C$
$t = 0, x = 0$: $C = -\frac{1}{8}$, $t = \frac{1}{2(4 - x)} - \frac{1}{8}$, $x = \frac{32t}{8t + 1}$

14 $t = \int \frac{1}{(3 - x)^2}\,dx = \frac{1}{3 - x} + C$
$t = 0, x = 2$: $C = -1$, $t = \frac{1}{3 - x} - 1$
(a) $x = \frac{3t + 2}{t + 1}$ (b) $\dot{x} = \frac{1}{(t + 1)^2}$, $\ddot{x} = \frac{-2}{(t + 1)^3} = -2(3 - x)^3$

15 (a) $\ddot{x} = \dot{x} + 0 = x + 3$
(b) $t = \int \frac{1}{x + 3}\,dx = \ln(x + 3) + C$
$t = 0, x = -2$: $C = 0$, $x = e^t - 3$
$t = 1$: $x = e - 3$

16 $\frac{1}{2}v^2 = \frac{k}{x} + C$, $t = 0, v = 0, x = a$:
$C = -\frac{k}{a}$, $\frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{a} = \frac{k(a - x)}{ax}$,
$v^2 = \frac{2k(a - x)}{ax}$, $|v| = \sqrt{\frac{2k(a - x)}{ax}}$

17 (a) $\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + C$, $x = 0, v = 1$: $C = 0$, $v^2 = e^{-2x}$, $v = \pm e^{-x}$
$x = 0, v > 0$: $v = e^{-x}$
(b) $\frac{dx}{dt} = e^{-x}$, $\frac{dt}{dx} = e^x$, $t = e^x + C$
$t = 0, x = 0$: $C = -1$, $t = e^x - 1$, $x = \ln(t + 1)$

18 (a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 32x^3 + 48x^2 + 16x$, $\frac{1}{2}v^2 = 8x^4 + 16x^3 + 8x^2 + C$
$x = -2, v = -8$: $C = 0$, $v^2 = 16x^4 + 32x^3 + 16x^2$,
$v^2 = 16x^2(x^2 + 2x + 1) = 16x^2(x + 1)^2$
(b) $v = \pm 4x(x + 1)$; $x = -2, v < 0$: $v = -4x(x + 1)$
$\frac{dx}{dt} = -4x(1 + x)$, $\frac{dt}{dx} = -\frac{1}{4x(1 + x)}$, $-4t = \int \frac{1}{x(1 + x)}\,dx$
(c) $\log_e\left(1 - \frac{1}{2}\right) = 0 + C$, $t = 0, x = -2$: $C = -\log_e 2$,
$4t = \log_e\left(1 + \frac{1}{x}\right) + \log_e 2 = \log_e 2\left(1 + \frac{1}{x}\right)$, $2\left(1 + \frac{1}{x}\right) = e^{4t}$,
$1 + \frac{1}{x} = \frac{1}{2}e^{4t}$, $x = \frac{2}{e^{4t} - 2}$

19 (a) $\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = 8x \times 4(x^2 + 9) = 32x(x^2 + 9)$
(b) $\frac{dx}{dt} = 4(x^2 + 9)$, $\frac{dt}{dx} = \frac{1}{4} \times \frac{1}{9 + x^2}$, $t = \frac{1}{4} \times \frac{1}{3}\tan^{-1}\frac{x}{3} + C$
$t = 0, x = 0$: $C = 0$, $t = \frac{1}{12}\tan^{-1}\frac{x}{3}$
$x = 3$: $t = \frac{1}{12}\tan^{-1} 1 = \frac{\pi}{48}$ seconds

20 (a) $\frac{d^2x}{dt^2} = 80(0 + 0.4e^{-0.4t}) = 0.4 \times 80e^{-0.4t}$,
$80e^{-0.4t} = 80 - v$, $\frac{d^2x}{dt^2} = 0.4(80 - v) \propto (80 - v)$
(b) $\frac{dx}{dt} = 80(1 - e^{-0.4t})$, $x = 80(t + 2.5e^{-0.4t}) + C$
$t = 0, x = 0, v = 0$: $C = -200$, $x = 80(t + 2.5e^{-0.4t}) - 200$
$t = 5$: $x = 80(5 + 2.5e^{-2}) - 200 = 227$ m
(c) $v = 60$: $e^{-0.4t} = \frac{80 - 60}{80} = 0.25$, $t = 2.5\log_e 4$
$x = 80\left(2.5\log_e 4 + 2.5 \times \frac{1}{4}\right) - 200 = 127$ m

21 $x = 0, \ddot{x} = 3$: $C = 3$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = kx + 3$, $\frac{1}{2}v^2 = \frac{1}{2}kx^2 + 3x$,
$v^2 = kx^2 + 6x$
$\ddot{x} = 0$ when $x = 160$: $k = -\frac{3}{160}$, $v^2 = 6x - \frac{3x^2}{160}$
(a) $x = 160$, $v^2 = 3 \times 160$, $v = 4\sqrt{30}$ m s^{-1}
(b) $x = 80$: $v = 6\sqrt{10}$ m s^{-1}

EXERCISE 6.2

1 $x = 6\cos 4t$ is of the form $x = a\cos(nt + \alpha)$: motion is simple harmonic about the origin, $a = 6$, $n = 4$; period $= \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$.
Particle starts 6 cm to the right of O.
$\dot{x} = -24\sin 4t$, particle is initially at rest.
$\ddot{x} = -96\cos 4t$, $\ddot{x} = -16x$, initial acceleration is 96 cm s^{-2} towards O.

2 (a) $\ddot{x} = -9x$, $n = 3$, $T = \frac{2\pi}{3}$, $v = \sqrt{4 - 9x^2}$, $x = \frac{2}{3}\sin 3t$,
$a = \frac{2}{3}$, $v_{max} = 2$
(b) $\ddot{x} = -9x$, $n = 3$, $T = \frac{2\pi}{3}$, $v^2 = 9\left(\frac{40}{9} - x^2\right)$, $a = \frac{2\sqrt{10}}{3}$,
$v_{max} = 2\sqrt{10}$

3 $v^2 = 5(4-x^2)$, $\frac{1}{2}v^2 = \frac{5}{2}(4-x^2)$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left[\frac{5}{2}(4-x^2)\right]$,

$\ddot{x} = \frac{5}{2}\times(-2x) = -5x$: SHM, $n = \sqrt{5}$, $T = \frac{2\pi}{\sqrt{5}}$, $a = 2$

4 B

5 D. $x = 17\left(\frac{8}{17}\sin 4t + \frac{15}{17}\cos 4t\right)$

$= 17\sin(4t+\alpha)$ where $\tan\alpha = \frac{15}{9}$

$x = 17\times 4\cos(4t+\alpha)$. Maximum velocity is 68.

6 B. $\ddot{x} = 6(6-x)$

7 A. $x = 13\left(\frac{5}{13}\sin 2t + \frac{12}{13}\cos 2t\right) = 13\sin(2t+\alpha)$

where $\tan\alpha = \frac{12}{5}$

$\dot{x} = 13\times 2\cos(2t+\alpha)$. $\ddot{x} = -52\sin(2t+\alpha)$.
Maximum acceleration is 52.

8 (a) $x = 2, \dot{x} = 5$. $x = 5, \dot{x} = 4$.

$v^2 = n^2(a^2 - x^2)$

$25 = n^2(a^2 - 4)$ [1]

$16 = n^2(a^2 - 25)$ [2]

$\frac{25}{16} = \frac{a^2-4}{a^2-25}$ [1]÷[2]

$25a^2 - 625 = 16a^2 - 64$

$9a^2 = 561$

$a^2 = \frac{561}{9}$

$a = \frac{\sqrt{561}}{3}$

(b) $25 = n^2\left(\frac{561}{9} - 4\right)$

$25 = n^2 \times \frac{525}{9}$

$n^2 = \frac{9}{21}$

$n = \frac{3}{\sqrt{21}}$

Period $= \frac{2\pi}{n} = 2\pi\times\frac{\sqrt{21}}{3} = \frac{2\pi\sqrt{21}}{3}$

9 $x = 5\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$, $\dot{x} = -\frac{5\pi}{2}\sin\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$,

$\ddot{x} = -\frac{5\pi^2}{4}\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right) = -\frac{\pi^2}{4}x$: SHM

(a) $T = 2\pi \div \frac{\pi}{2} = 4$ s (b) $a = 5$ m

(c) $\frac{1}{2} = \cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$, $t = \frac{4}{3}$,

$\dot{x} = -\frac{5\pi}{2}\sin\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) = -\frac{5\pi\sqrt{3}}{4}$ m s^{-1}

(d) $\ddot{x} = -\frac{\pi^2}{4}\times\left(-\frac{5}{2}\right) = \frac{5\pi^2}{8}$ m s^{-2}

10 (a) $\frac{1}{2}v^2 = 150 + 50x - \frac{25}{2}x^2$, $\ddot{x} = 50 - 25x = -25(x-2)$:
SHM around $x = 2$
(i) $x = 2$ m
(ii) $T = \frac{2\pi}{5}$, $v^2 = 25(6-x)(2+x) = 25(16-y^2)$
where $y = 2 - x$; $a = 4$ m

(b) $\frac{1}{2}v^2 = 64 - 16x - 8x^2$, $\ddot{x} = -16 - 16x = -16(x+1)$:
SHM around $x = -1$
(i) $x = -1$
(ii) $T = \frac{\pi}{2}$, $v^2 = 16(2-x)(4+x) = 16(9-y^2)$
where $y = x + 1$; $a = 3$ m

(c) $\frac{1}{2}v^2 = 3 + 2x - x^2$, $\ddot{x} = 2 - 2x = -2(x-1)$:
SHM around $x = 1$
(i) $x = 1$
(ii) $T = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$, $v^2 = 2(3-x)(1+x) = 2(4-y^2)$
where $y = 1 - x$; $a = 2$ m

11 (a) $v^2 = 2(3-x)(5+x)$, $x = -5, 3$

(b) $\frac{1}{2}v^2 = 15 - 2x - x^2$, $\ddot{x} = -2 - 2x = -2(x+1)$

(c) $x = -1$, $v^2 = 32$, $v_{max} = 4\sqrt{2}$

12 (a) $\dot{x} = -an\sin(nt+\varepsilon)$
$t = 0, \dot{x} = 0$: $\varepsilon = 0$, $x = a\cos nt$
$t = 0, x = -5$: $-5 = a$, $x = -5\cos nt = 5\cos(\pi - nt)$
or $x = 5\sin\left(nt - \frac{\pi}{2}\right)$

(b) $t = 0, x = 0$: $0 = a\cos\varepsilon$
$\varepsilon = \frac{\pi}{2}$: $x = a\cos\left(nt + \frac{\pi}{2}\right)$, $\dot{x} = -an\sin\left(nt + \frac{\pi}{2}\right)$
$t = 0, v < 0, a > 0$: $x = a\cos\left(nt + \frac{\pi}{2}\right)$

13 $v^2 = 9(16 - x^2)$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left[\frac{9}{2}(16-x^2)\right]$, $\ddot{x} = -9x$:
SHM $x = 4$: $\ddot{x} = -36$ m s^{-2}, maximum speed = 12 m s^{-1}

14 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -16x$, $v^2 = 16(25 - x^2)$, $x = 5\sin(4t - C)$
where $\sin C = -\frac{3}{5}$
$\therefore \cos C = \frac{4}{5}$, $x = 4\sin 4t + 3\cos 4t$
max displacement = 5 m, maximum speed = 20 m s^{-1}

15 (a) $t = 0, x = 0, \dot{x} = -4$, $\frac{2\pi}{n} = \frac{\pi}{2}$, $n = 4$
$x = a\sin 4t$, $\dot{x} = -4a\cos 4t$, $-4 = -4a$, $a = 1$: $x = \sin 4t$
(b) $a = 1$ m

16 $36 = n^2(a^2 - 4)$, $16 = n^2(a^2 - 9)$, $\frac{36}{16} = \frac{a^2-4}{a^2-9}$, $a = \sqrt{13}$, $n = 2$
period = π s, amplitude = $\sqrt{13}$ m

17 $\dot{x} = an\cos nt - bn\sin nt$,
$\ddot{x} = -an^2\sin nt - bn^2\cos nt = -n^2(a\sin nt + b\cos nt) = -n^2x$,
amplitude $= \sqrt{a^2+b^2}$
$\dot{x} = n(a\cos nt - b\sin nt)$, maximum speed $= n\sqrt{a^2+b^2}$

18 $x = 4 - \cos 2t$, $\dot{x} = 2\sin 2t$,
$\ddot{x} = 4\cos 2t = 4(4-x) = -4(x-4)$: SHM
(a) $x = 4$ m (b) $n = 2$, $T = \pi$ s (c) $a = 1$ m

19 $n = \frac{\pi}{4}$, $a = 10$: $v^2 = \frac{\pi^2}{16}(100 - x^2)$; $x = 6$: $v = 2\pi$ cm s^{-1}

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left[\frac{\pi^2}{32}(100-x^2)\right] = -\frac{\pi^2}{16}x$, $\ddot{x} = \frac{3\pi^2}{8}$ m s^{-2}

maximum acceleration when $x = 10$: $\ddot{x}_{max} = \frac{5\pi^2}{8}$ m s^{-2}

20 $\dot{x} = 16\cos 2t - 12\sin 2t$,
$\ddot{x} = -32\sin 2t - 24\cos 2t = -4(8\sin 2t + 6\cos 2t) = -4(x - 10)$:
SHM about $x = 10$, $n = 2$, $T = \pi$ s, $a = 10$ m

21 (a) $x = ut + \frac{1}{2}at^2$, $0.5 = 11u + \frac{1}{2}\times 121a$, $1 = 22u + 121a$

$0 = 18u + \frac{1}{2}\times 324a$, $0 = u + 9a$

$\therefore x = \frac{9t}{77} - \frac{t^2}{154}$: $t = 7, x = 0.5$; $t = 9$, $x = \frac{81}{154}$

(b) $n = \frac{\pi}{6}$, $x = -\sin\frac{\pi t}{6}$: $t = 7$, $x = \frac{1}{2}$; $t = 9$, $x = 1$

22 $a = 2, n = \frac{2\pi}{3}, v^2 = \frac{4\pi^2}{9}(4 - x^2), v_{max} = \frac{4\pi}{3}$ m s^{-1},

$\ddot{x}_{max} = \frac{8\pi^2}{9}$ m s^{-2}

23 (a) $x = 5\cos\frac{\pi t}{6}$ (b) $\frac{5\pi}{6}$ (c) $a = 5,\ f = \frac{1}{12}, T = 12$ s

(d) $\frac{5\pi}{6}$ (e) $\frac{5\pi^2}{36}$

24 $v^2 = 2(25 - x^2)$; $x = 0$: speed $= 5\sqrt{2}$ m s^{-1}

$x = 4$: speed $= 3\sqrt{2}$ m s^{-1}

25 $v^2 = 4(a^2 - x^2)$; $a = 4$ m, $v = 2\sqrt{7}$ m s^{-1}

26 (a) $f = \frac{\sqrt{5}}{2\pi}$ Hz (vibrations per second)

(b) $n = \sqrt{5}$, $a = 10$ m

27 $n = 2, a = 2.5, v^2 = 4(6.25 - x^2)$; $x = 1.5$: $v = 4$ m s^{-1}, $\ddot{x} = 6$ m s^{-2}

28 $a = 10, n = \frac{\pi}{5}, x = 10\sin\frac{\pi t}{5}$; $x = 6$: $t = 1.02$ s

29 $64 = n^2(a^2 - 9), 36 = n^2(a^2 - 16)$: $a = 5, n = 2, T = \pi$ seconds,
$\ddot{x}_{max} = 20$ m s^{-2}

30 (a) 10.5 m (b) $a = 1.5$ m (c) $T = 14$ h

(d) $n = \frac{\pi}{7}$:

$t = 0$ (9 a.m.): $y = 10.5 - 1.5 = 9$ m

$t = 3.5$ (12:30 p.m.): $y = 10.5 - 1.5\cos\frac{\pi}{2} = 10.5$ m

$t = 7$ (4 p.m.): $y = 10.5 - 1.5\cos\pi = 10.5 + 1.5 = 12$ m.

(e) $y = 10$, $10 = 10.5 - 1.5\cos\frac{\pi t}{7}$, $1.5\cos\frac{\pi t}{7} = 0.5$, $\cos\frac{\pi t}{7} = \frac{1}{3}$,

$\frac{\pi t}{7} = 1.231, 5.052$; $t = 2.743$ h, 11.257 h

Between 9 a.m. + 2 h 45 min = 11:45 a.m.

and 9 a.m. + 11 h 15 min = 8:15 p.m.

31 (a) 18°C

(b) $t = 0$ (4 a.m.), $T = 13$: $T = 18 - 5\cos 0 = 13$

$t = 12$ (4 p.m.), $T = 23$: $T = 18 - 5\cos\pi = 18 + 5 = 23$

∴ formula satisfies the given conditions.

(c) (i) 10 a.m and 10 p.m. (from part (a))

(ii) $T = 15$: $15 = 18 - 5\cos\frac{\pi t}{12}$, $5\cos\frac{\pi t}{12} = 3$, $\cos\frac{\pi t}{12} = 0.6$,

$\frac{\pi t}{12} = 0.9273$, $t = 3$ h 33 min = at 7:33 a.m and 12:27 a.m.

(iii) $T = 21$: $21 = 18 - 5\cos\frac{\pi t}{12}$, $5\cos\frac{\pi t}{12} = -3$,

$\cos\frac{\pi t}{12} = -0.6$, $\frac{\pi t}{12} = 2.214$, $t = 8$ h 27 min

= at 12:27 p.m. and 7:33 p.m.

32 $na = 3, n^2a = 8$: $n = \frac{8}{3}, a = \frac{9}{8}$ m, $T = \frac{3\pi}{4}$ s

33 (a) $x = \cos 2t - \sqrt{3}\sin 2t$

$= 2\left(\frac{1}{2}\cos 2t - \frac{\sqrt{3}}{2}\sin 2t\right)$

$= 2(\cos 2t\cos\alpha - \sin 2t\sin\alpha)$ where $\tan\alpha = \sqrt{3}$

$= 2\cos\left(2t + \frac{\pi}{3}\right)$

$R = 2, \alpha = \frac{\pi}{3}$

(b) $x = 2\cos\left(2t + \frac{\pi}{3}\right)$

$\dot{x} = -4\sin\left(2t + \frac{\pi}{3}\right)$

Maximum speed is 4 m s^{-1}.

Occurs when $-4 = -4\sin\left(2t + \frac{\pi}{3}\right)$

$\sin\left(2t + \frac{\pi}{3}\right) = 1$

$2t + \frac{\pi}{3} = \frac{\pi}{2}$

$2t = \frac{\pi}{6}$

$t = \frac{\pi}{12}$

Maximum speed first occurs after $\frac{\pi}{12}$ seconds.

34 (a) $\sqrt{3}\sin 3t - \cos 3t = 2\left(\frac{\sqrt{3}}{2}\sin 3t - \frac{1}{2}\cos 3t\right)$

$= 2\sin(3t - \alpha)$ where $\tan\alpha = \frac{1}{\sqrt{3}}$

$= 2\sin\left(3t - \frac{\pi}{6}\right)$

$R = 2,\ \alpha = \frac{\pi}{6}$

(b) $x = 4 + \sqrt{3}\sin 3t - \cos 3t$

$= 2\sin\left(3t - \frac{\pi}{6}\right) + 4$

$\dot{x} = 6\cos\left(3t - \frac{\pi}{6}\right)$

$\ddot{x} = -18\sin\left(3t - \frac{\pi}{6}\right)$

$= -9 \times 2\sin\left(3t - \frac{\pi}{6}\right)$

$= -9(x - 4)$

Since the acceleration is proportional to, but in the opposite direction to, the displacement from the centre of motion, $x = 4$, the particle is undergoing simple harmonic motion.

(c) $a = 2$, centre of motion is $x = 4$.

(d) Minimum displacement occurs when $\sin\left(3t - \frac{\pi}{6}\right) = -1$

$3t - \frac{\pi}{6} = \frac{3\pi}{2}$

$3t = \frac{10\pi}{6}$

$t = \frac{5\pi}{9}$

Minimum displacement occurs after $\frac{5\pi}{9}$ seconds.

35 (a) $\ddot{x} = -12x, \ddot{x} = -n^2x \Rightarrow n^2 = 12$

$n = 2\sqrt{3}$: Period $= \frac{2\pi}{n} = \frac{2\pi}{2\sqrt{3}} = \frac{\pi\sqrt{3}}{3}$

(b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -12x$

$\frac{1}{2}v^2 = \int_{-4}^{x} -12x\,dx$

$= \left[-6x^2\right]_{-4}^{x}$

$= -6x^2 + 6 \times 16$

$= 6\left(16 - x^2\right)$

$v^2 = 12\left(16 - x^2\right)$

(c) Let $x = 4\cos\left(2\sqrt{3}t + \alpha\right)$

$t = 0, x = -4$: $-4 = 4\cos\alpha$

$\cos\alpha = -1 \Rightarrow \alpha = \pi$

$x = 4\cos\left(2\sqrt{3}t + \pi\right)$

$= -4\cos 2\sqrt{3}t$

36 (a) $x = 4 + \cos^2 t$

$\dot{x} = -2\cos t\sin t$

$= -\sin 2t$

$\ddot{x} = -2\cos 2t$

$= -2\left(2\cos^2 t - 1\right)$

$= -4\cos^2 t + 2$

$= -4(x-4) + 2$

$= 18 - 4x$

(b) $\ddot{x} = 18 - 4x$

$= -4(x - 4.5)$

This is of the form $\ddot{x} = -n^2(x-k)$ so represents SHM about $x = 4.5$.

(c) Centre is $x = 4.5$, $n^2 = 4$ so $n = 2$. Period $= \frac{2\pi}{2} = \pi$

Since $-2\cos 2t = 18 - 4x$, $x = 4.5 - 0.5\cos 2t$, so amplitude = 0.5.

37 (a) $x = 1 + 2\cos\left(2t - \frac{\pi}{4}\right)$

$\dot{x} = -4\sin\left(2t - \frac{\pi}{4}\right)$

$\ddot{x} = -8\cos\left(2t - \frac{\pi}{4}\right)$

But $2\cos\left(2t - \frac{\pi}{4}\right) = x - 1$

$\ddot{x} = -4(x-1)$

(b) Centre of motion is $x = 1$

Maximum speed when $\sin\left(2t - \frac{\pi}{4}\right) = 1$

$2t - \frac{\pi}{4} = \frac{\pi}{2}$

$2t = \frac{3\pi}{4}$

$t = \frac{3\pi}{8}$ seconds

(c) $a = 2$

At rest when $\sin\left(2t - \frac{\pi}{4}\right) = 0$

$2t - \frac{\pi}{4} = 0$

$2t = \frac{\pi}{4}$

$t = \frac{\pi}{8}$ seconds

38 (a) Period = 12.5 hours. $\frac{25}{2} = \frac{2\pi}{n}$, $n = \frac{4\pi}{25}$

The depth halfway between low tide and high tide $= \frac{5-1}{2} =$ 2 m. amplitude = 2

Halfway between 1 and 5 is 3. $c = 3$

$x = 2\cos\frac{4\pi t}{25} + 3$

Check that this fits the given information that low tide is at $t = 0$: $x = 2\cos 0 + 3 = 2 + 3 = 5$. This is high tide.

Hence $x = 3 - 2\cos\frac{4\pi t}{25}$ and $a = -2$

(b) Low tide is at 2 a.m. so the first high tide is at $2 + 6.25$ h = 8.15 a.m.

(c) $x = 3 - 2\cos\frac{4\pi t}{25}$

$\dot{x} = \frac{8\pi t}{25}\sin\frac{4\pi t}{25}$

$\ddot{x} = \frac{32\pi^2}{625}\cos\frac{4\pi t}{25}$

$\ddot{x} = 0$ when $\cos\frac{4\pi t}{25} = 0$

$\frac{4\pi t}{25} = \frac{\pi}{2}$

$t = \frac{25}{8} = 3\frac{1}{8}$ hours after low tide

The second low tide is at 3:30 p.m. (3 + 12.5 h)

Now $\frac{1}{8} \times 60 = 7.5$, so $3\frac{1}{8}$ hours is 3 h 7.5 min.

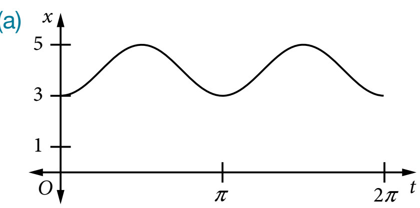
The depth is increasing fastest at 3:30 + 3 h 7.5 min = 6:37:30 p.m.

EXERCISE 6.3

1 (a) $\dot{x} = -24\sin 4t$, $\ddot{x} = -96\cos 4t$ (b) $x = 9$ (c) $-3 \le x \le 9$ (d) $t = \frac{\pi}{6}$

2 D

3 (a)

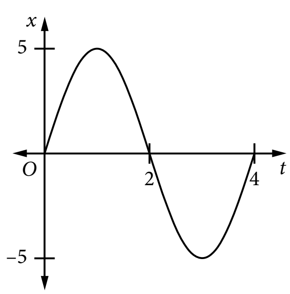


(b) $\dot{x} = 2\sin 2t$: $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(c) (i) $\ddot{x} = 4\cos 2t$ (ii) $\ddot{x} = 4(4 - x)$

4 $a = 2\sin t$, $v = 3 - 2\cos t$, $x = 1 + 3t - 2\sin t$

5 (a)



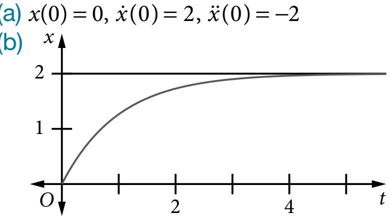
(b) $\dot{x} = \frac{5\pi}{2}\cos\frac{\pi t}{2}$: $t = 1, 3$ seconds

(c) $\frac{5\pi\sqrt{2}}{4}$ cm s^{-1} (d) (i) $\ddot{x} = -\frac{5\pi^2}{4}\sin\frac{\pi t}{2}$ (ii) $\ddot{x} = -\frac{\pi^2 x}{4}$

6 (a) $\dot{x} = 1 + 2\sin t$, $\ddot{x} = 2\cos t$ (b) $t = 0, x = 0$: $\dot{x} = 1$, $\ddot{x} = 2$

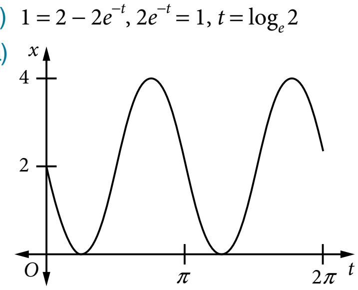
7 (a) $x(0) = 0$, $\dot{x}(0) = 2$, $\ddot{x}(0) = -2$

(b)



(c) $1 = 2 - 2e^{-t}$, $2e^{-t} = 1$, $t = \log_e 2$

8 (a)



(b) $\dot{x} = -4\cos 2t$: $\dot{x} = 0$ at $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$t = \frac{\pi}{4}, x = 0;\ t = \frac{3\pi}{4}, x = 4$

∴ particle oscillates between $x = 0$ and $x = 4$.

(c) $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (d) (i) $\ddot{x} = 8\sin 2t$ (ii) $\ddot{x} = 4(2 - x)$

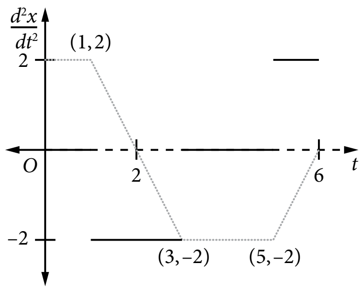
9 (a) $t = 2$; $v = \int_0^2 a\,dt$ is largest value of v

(b) $t = 4$; $v = \int_0^4 a\,dt = 0$

(c) $t = 4$: x is maximum when $v = 0$ and $a < 0$

10 (a) $t = 2$ (b) $t = 4$; when the area below the t-axis equals the area above the axis, the displacement is zero: $x = \int_0^4 \frac{dx}{dt}dt = 0$

(c)



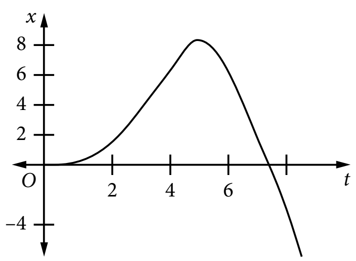
11 distance $\approx \frac{1}{2}(0 + 2(4.6 + 5.7 + 8 + 9.9 + 12.7) + 18.2) = 50\,\text{m}$

12 (a) distance $\approx \frac{2}{2}(0 + 2\times 1 + 4) = 6$ (b) 2 (c) $t > 5$

(d) When the area under the curve to the right of D has a magnitude of 6, at time t_1:

$\int_0^{t_1} \frac{dx}{dt}dt = 0;\ 4(t_1 - 6) = 6,\ t_1 = 7.5$, so when $t \approx 7.5$

(e)



13 (a) 30 m (b) 35 m (c) 45 m **14** D

EXERCISE 6.4

1 C **2** $x = \frac{t^3}{3} + \frac{3t^2}{2} + t + 2$

3 (a) $F = (4t - 12)$ N (b) ± 4 N

4 (a) $F = 20 - t$ (b) $20\,\text{m s}^{-1}$ (c) 5 s

(d) $20\,\text{m s}^{-1}$ (e) $716\frac{2}{3}$ m (f) $15\frac{25}{27}\,\text{m s}^{-1}$

5 $F = 10x - 30$ **6** D **7** $25\,\text{m s}^{-1}$

8 (a) $x = 3 - 3\cos t + 2t$ (b) $v = 4$

(c) $t = \log_e 2$

9 (a) $v = \pm\frac{\sqrt{10x - 3 - 3x^2}}{x}$ (b) $x = 3$ or $\frac{1}{3}$

10 $\log_e 2$ m

11 $\int_0^v v\,dv = -k\int_a^x x^{-2}\,dx,\ \frac{1}{2}v^2 = k\left[\frac{1}{x}\right]_a^x = k\left(\frac{a - x}{ax}\right)$

∴ Speed $= \sqrt{\frac{2k(a - x)}{ax}}$

12 (a) The displacement x in the direction away from O is positive, so acceleration towards O is negative.

(b) $\int_0^v v\,dv = -k\int_b^x x^{-2}\,dx,\ \frac{1}{2}v^2 = k\left[\frac{1}{x}\right]_b^x = k\left(\frac{1}{x} - \frac{1}{b}\right)$

∴ $v^2 = 2k\left(\frac{1}{x} - \frac{1}{b}\right)$

(c) $x = b\cos^2\theta$ ∴ $dx = -2b\cos\theta\sin\theta\,d\theta$

$\int\sqrt{\frac{x}{b - x}}\,dx = -2b\int\cos^2\theta\,d\theta = -b\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$ [1]

As $x = b\cos^2\theta$ ∴ $\cos^2\theta = \frac{x}{b}$, $\cos 2\theta = \frac{2x - b}{b}$

Hence $\sin 2\theta = \frac{2\sqrt{bx - x^2}}{b}$ [2]

and $\theta = \frac{1}{2}\cos^{-1}\left(\frac{2x - b}{b}\right)$ [3]

Substitute [2] and [3] into [1] to complete the proof.

(d) $v = -\sqrt{\frac{2k}{b}}\sqrt{\frac{b - x}{x}}$ (moving towards O)

$\int_0^t dt = -\sqrt{\frac{b}{2k}}\int_b^0\sqrt{\frac{x}{b - x}}\,dx,$

$t = -\sqrt{\frac{b}{2k}}\left[-\frac{b}{2}\cos^{-1}\left(\frac{2x - b}{b}\right) - \sqrt{bx - x^2}\right]_b^0$

which produces the required result.

13 $\int_0^v v\,dv = -k^2\int_a^{\frac{a}{2}}(x + a^4x^{-3})\,dx$ ∴ $\frac{1}{2}v^2 = -k^2\left[\frac{x^2}{2} - \frac{a^4}{2x^2}\right]_a^{\frac{a}{2}}$

which produces the required result.

14 12:47 p.m.

15 (a) $v_1^2 = n^2(a^2 - x_1^2)$ [1] and $v_2^2 = n^2(a^2 - x_2^2)$ [2]

From [1]: $a^2 = x_1^2 + \frac{v_1^2}{n^2}$

Substitute into [2]: $v_2^2 = n^2\left(x_1^2 + \frac{v_1^2}{n^2} - x_2^2\right)$

$v_2^2 = n^2(x_1^2 - x_2^2) + v_1^2,\ \ n = \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}$

∴ $T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

(b) [1] ÷ [2]: $\frac{v_1^2}{v_2^2} = \frac{a^2 - x_1^2}{a^2 - x_2^2}$, $a^2(v_1^2 - v_2^2) = v_1^2x_2^2 - v_2^2x_1^2$

∴ $a = \sqrt{\frac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}}$

16 (a) $\frac{2\pi}{n} = 5T$ ∴ $n = \frac{2\pi}{5T}$

The displacements of the particles are given by:

$x = a\cos\left(\frac{2\pi}{5T}t + \alpha\right)$

For particle 1, $x = a$ when $t = 0$ ∴ $\alpha = 0$: $x_1 = a\cos\frac{2\pi t}{5T}$

For particle 2, $x = a$ when $t = T$ ∴ $a = a\cos\left(\frac{2\pi}{5T}T + \alpha\right)$,

i.e. $\alpha = -\frac{2\pi}{5}$: $x_2 = a\cos\frac{2\pi(t - T)}{5T}$

Particles meet at $x_1 = x_2$: $\cos\frac{2\pi(t - T)}{5T} = \cos\frac{2\pi t}{5T}$

$\frac{2\pi(t - T)}{5T} = \pm\frac{2\pi t}{5T} + 2k\pi$

For $k = 0$: $t = \frac{T}{2}$, which is before particle 2 is in motion.

For $k = 1$: $t = 3T$, i.e. $2T$ after particle 2 starts to move.

(b) $a\cos\frac{\pi}{5}$

17 (a) For particle 1: $x_1 = v_1t\cos\alpha_1$ and $y_1 = v_1t\sin\alpha_1 - \frac{1}{2}gt^2$

For particle 2: $x_2 = v_2t\cos\alpha_2$ and $y_2 = v_2t\sin\alpha_2 - \frac{1}{2}gt^2$

They hit the pole at the same time, so $x_1 = x_2$ at $t = T$:

∴ $v_1\cos\alpha_1T = v_2\cos\alpha_2T$

$T(v_1\cos\alpha_1 - v_2\cos\alpha_2) = 0$

As $T \neq 0$: $v_1\cos\alpha_1 = v_2\cos\alpha_2$

(b) For particle 2: $y_2 = v_2 t\sin\alpha_2 - \frac{1}{2}gt^2$

When $y = 0$: $t\left(v_2\sin\alpha_2 - \frac{1}{2}gt\right) = 0$

So it hits the pole when: $t = \frac{2v_2\sin\alpha_2}{g}$

(c) Distance from O to base of pole:

$OB = v_1\cos\alpha_1 \times \frac{2v_2\sin\alpha_2}{g}$

Let height of pole be h. As $\tan\alpha = \frac{h}{OB}$:

$h = \frac{2v_1v_2\cos\alpha_1\sin\alpha_2\tan\alpha}{g}$

For particle 1: $y_1 = v_1 t\sin\alpha_1 - \frac{1}{2}gt^2$ and $y_1 = h$

when $t = \frac{2v_2\sin\alpha_2}{g}$

Hence: $\frac{2v_1v_2\cos\alpha_1\sin\alpha_2\tan\alpha}{g}$

$= v_1\sin\alpha_1 \times \frac{2v_2\sin\alpha_2}{g} - \frac{g}{2}\times\left(\frac{2v_2\sin\alpha_2}{g}\right)^2$

Multiply both sides by $\frac{g}{2v_2\sin\alpha_2}$:

$v_1\cos\alpha_1\tan\alpha = v_1\sin\alpha_1 - v_2\sin\alpha_2$

But $v_2 = \frac{v_1\cos\alpha_1}{\cos\alpha_2}$ from part (a):

$\therefore v_1\cos\alpha_1\tan\alpha = v_1\sin\alpha_1 - \frac{v_1\cos\alpha_1\sin\alpha_2}{\cos\alpha_2}$

Divide by $v_1\cos\alpha_1$ to obtain the required result.

18 (a) A is $(ut\cos\alpha, ut\sin\alpha - \frac{1}{2}gt^2)$, B is $\left(\frac{u^2}{g}\sin 2\alpha, \frac{2u^2}{g}\sin^2\alpha - ut\sin\alpha\right)$

(b) Gradient $OA = \frac{ut\sin\alpha - \frac{1}{2}gt^2}{ut\cos\alpha} = \frac{2u\sin\alpha - gt}{2u\cos\alpha}$

Gradient $OB = \frac{\frac{2u^2}{g}\sin^2\alpha - ut\sin\alpha}{\frac{u^2}{g}\sin 2\alpha}$

$= \frac{2u^2\sin^2\alpha - gut\sin\alpha}{u^2\sin 2\alpha}$

$= \frac{2u\sin^2\alpha - gt\sin\alpha}{2u\sin\alpha\cos\alpha} = \frac{2u\sin\alpha - gt}{2u\cos\alpha}$

Gradient OA = Gradient OB $\therefore$ O, A and B are in a straight line.

19 (a) At time t during flight, let the slower stone be at C and the faster stone at D where C is $\left(ut\cos\alpha, ut\sin\alpha - \frac{1}{2}gt^2\right)$ and D is $\left(vt\cos\alpha, vt\sin\alpha - \frac{1}{2}gt^2\right)$:

Gradient $CD = \frac{vt\sin\alpha - \frac{1}{2}gt^2 - \left(ut\sin\alpha - \frac{1}{2}gt^2\right)}{vt\cos\alpha - ut\cos\alpha}$

$= \frac{(v-u)t\sin\alpha}{(v-u)t\cos\alpha} = \tan\alpha$

which is independent of t.

(b) In triangle PQZ: $\tan\alpha = \frac{h}{PZ}$ $\therefore PZ = h\cot\alpha$

(c) Slower stone hits ground at P when $t = \frac{2u\sin\alpha}{g}$. At that time, faster stone has:

$\dot{y} = -gt + v\sin\alpha = -2u\sin\alpha + v\sin\alpha = (-2u + v)\sin\alpha$

and $\dot{x} = v\cos\alpha$

So, at Q: $\tan(-\beta) = \frac{(-2u+v)\sin\alpha}{v\cos\alpha}$

$\therefore -v\tan\beta = -2u\tan\alpha + v\tan\alpha$

$2u\tan\alpha = v(\tan\alpha + \tan\beta)$

(d) If $\beta = \frac{\alpha}{2}$, $v\left(\tan\alpha + \tan\frac{\alpha}{2}\right) = 2u\tan\alpha$. Let $t = \tan\frac{\alpha}{2}$:

$\therefore v\left(\frac{2t}{1-t^2} + t\right) = 2u\left(\frac{2t}{1-t^2}\right)$ which gives $u = \frac{v}{4}(3 - t^2)$.

$\alpha > 0$, so $t^2 > 0$: $\therefore 3 - t^2 < 3$

Hence: $u < \frac{3}{4}v$

20 (a) $\left(Vt\cos\alpha, Vt\sin\alpha - \frac{1}{2}gt^2\right)$

(b) $\tan\theta = \frac{Vt\sin\alpha - \frac{1}{2}gt^2}{Vt\cos\alpha} = \frac{2V\sin\alpha - gt}{2V\cos\alpha}$

$\therefore 2V\cos\alpha\tan\theta = 2V\sin\alpha - gt$

$gt = 2V\sin\alpha - 2V\cos\alpha\tan\theta$ [1]

At B: $\dot{x} = V\cos\alpha$ and $\dot{y} = -gt + V\sin\alpha$

$\therefore \tan\beta = \frac{V\sin\alpha - gt}{V\cos\alpha}$, $V\cos\alpha\tan\beta = V\sin\alpha - gt$

$gt = V\sin\alpha - V\cos\alpha\tan\beta$ [2]

Equate [1] and [2]:

$2V\sin\alpha - 2V\cos\alpha\tan\theta = V\sin\alpha - V\cos\alpha\tan\beta$

Divide by $V\cos\alpha$ to obtain the required result.

21 (a) $\left(Vt\cos\alpha, Vt\sin\alpha - \frac{1}{2}gt^2\right)$

(b) $\dot{x} = V\cos\alpha$ and $\dot{y} = -gt + V\sin\alpha$

(c) The particle lands at an angle of 60° below the horizontal, so $\tan(-60°) = \frac{\dot{y}}{\dot{x}}$: $\therefore -\sqrt{3} = \frac{-gt + V\sin\alpha}{V\cos\alpha}$,

$gt = V\sin\alpha + \sqrt{3}V\cos\alpha$ [1]

Time of flight: $t_F = \frac{V(\sin\alpha + \sqrt{3}\cos\alpha)}{g}$

(d) Let P be the point on the horizontal plane through O such that P is vertically below Q. Then:

$\tan 30° = \frac{-\frac{1}{2}gt^2 + Vt\sin\alpha}{Vt\cos\alpha}$

$\therefore gt = 2V\sin\alpha - \frac{2\sqrt{3}}{3}V\cos\alpha$ [2]

Equate [1] and [2]:

$2V\sin\alpha - \frac{2\sqrt{3}}{3}V\cos\alpha = V\sin\alpha + \sqrt{3}V\cos\alpha$,

$\tan\alpha = \frac{5\sqrt{3}}{3}$

(e) In triangle OPQ:

$OQ = \frac{OP}{\cos 30°} = \frac{2}{\sqrt{3}}\times V\cos\alpha \times t_F$

$= \frac{2}{\sqrt{3}}\times V\cos\alpha \times \frac{V(\sin\alpha + \sqrt{3}\cos\alpha)}{g}$ [3]

Now $\tan\alpha = \frac{5\sqrt{3}}{3}$ so: $\sin\alpha = \frac{5\sqrt{3}}{2\sqrt{21}}$ and $\cos\alpha = \frac{3}{2\sqrt{21}}$

Substitute these values into [3] to obtain the required result.

EXERCISE 6.5

1 (a) $\frac{ku}{ud + k}$ (b) $\frac{ud^2 + 2dk}{2ku}$

2 (a) $\ddot{x} = -kv$, $v\frac{dv}{dx} = -kv$, $\frac{dv}{dx} = -k$, $\int_U^V dv = -k\int_0^d dx$,

$V - U = -kd$ $\therefore V = U - kd$

(b) $\frac{dv}{dt} = -kv$, $\frac{dt}{dv} = -\frac{1}{kv}$, $\int_0^T dt = -\frac{1}{k}\int_U^V \frac{dv}{v}$,

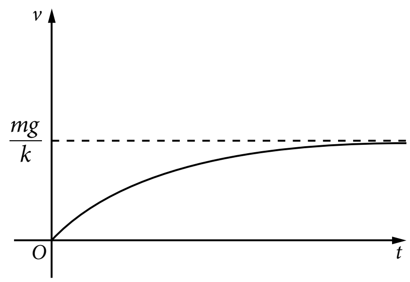
$T = -\frac{1}{k}\log_e\frac{V}{U}$, $\frac{V}{U} = e^{-kT}$ $\therefore U = Ve^{kT}$

(c) From part (a): $k = \frac{U - V}{d}$, substitute into part (b) to obtain result.

3 B

4 (a) $v=\frac{mg}{k}\left(1-e^{-\frac{kt}{m}}\right)$

(b)



(c) Terminal velocity is $\frac{mg}{k}$, $t=\frac{m}{k}\log_e\frac{4}{3}$ (d) $\frac{m^2g}{k^2}\left(\log_e\frac{4}{3}-\frac{1}{4}\right)$

5 (a) Resistance force (b) $49\,\text{m s}^{-1}$ (c) $\frac{245}{e}$ m

6 (a) $\ddot{x}=g-kv^2$ (b) $\sqrt{\frac{g}{k}}$ (c) $\frac{1}{2k}\log_e\left(\frac{g}{g-kv^2}\right)$

(d) $\frac{1}{2k}\log_e\left(\frac{4}{3}\right)$ (e) $\frac{1}{2\sqrt{g}}\log_e 3$

7 (a) $\ddot{x}=g-kv$ (b) $v=\frac{g}{k}(1-e^{-kt})$ (c) $\frac{g}{k}$

(d) $49(1-e^{-1})\,\text{m s}^{-1}$ (e) $5\log_e 2$ seconds

8 (a) $v^2=250g(1-e^{-0.008x})$ (b) $\frac{5\sqrt{10g(e^8-1)}}{e^8}\,\text{m s}^{-1}$

9 (a) $\frac{10g(e-1)}{e}\,\text{m s}^{-1}$ (b) $\frac{g(e^{11}-20+19e)}{2e^{11}}\,\text{m s}^{-1}$ (c) $\frac{g}{2}\,\text{m s}^{-1}$

10 (a) $5\log_e\left(\frac{g+40}{g}\right)$ m (b) $\sqrt{\frac{10}{g}}\tan^{-1}\left(2\sqrt{\frac{10}{g}}\right)$ s

11 (a) $\frac{u}{k}+\frac{g}{k^2}\log_e\left(\frac{g}{g+ku}\right)$ (b) $\frac{1}{k}\log_e\left(\frac{g+ku}{g}\right)$ (c) $\ddot{x}=g-kv$

(d) $\ddot{x}=g-kv$, $v\frac{dv}{dx}=g-kv$, $\frac{dv}{dx}=\frac{g-kv}{v}$,

$\int_0^h dx=\int_0^V\frac{v}{g-kv}dv$, $h=\int_0^V\left(-\frac{1}{k}+\frac{g}{k}\times\frac{1}{g-kv}\right)dv$,

$h=\left[-\frac{v}{k}-\frac{g}{k^2}\log_e(g-kv)\right]_0^V$

$\frac{u}{k}+\frac{g}{k^2}\log_e\left(\frac{g}{g+ku}\right)=-\frac{V}{k}+\frac{g}{k^2}\log_e\left(\frac{g}{g-kV}\right)$

the result follows.

12 (a) $\frac{m}{2k}\log_e\left(1+\frac{ku^2}{mg}\right)$ (b) $V=\frac{u}{\sqrt{1+\frac{ku^2}{mg}}}$

(c) $1+\frac{ku^2}{mg}>1$ as k, m, g all positive $\therefore\ V<u$

13 $\frac{1}{2\sqrt{gk}}\left[2\tan^{-1}\left(U\sqrt{\frac{k}{g}}\right)+\log_e\left(\frac{\sqrt{g}+V\sqrt{k}}{\sqrt{g}-V\sqrt{k}}\right)\right]$

14 (a) Take tan (…) of both sides to obtain the result.

(b) $v\frac{dv}{dx}=-(v+v^3)$, $\frac{dv}{dx}=-(1+v^2)$, $\frac{dx}{dv}=-\frac{1}{1+v^2}$,

$\int_0^x dx=-\int_Q^v\frac{dv}{1+v^2}$, $x=-[\tan^{-1}v]_Q^v$

$x=\tan^{-1}Q-\tan^{-1}v$, $x=\tan^{-1}\left[\frac{Q-v}{1+Qv}\right]$

(c) $\frac{dv}{dt}=-(v+v^3)$, $\frac{dt}{dv}=-\frac{1}{v+v^3}$, $\int_0^t dt=-\int_Q^v\frac{dv}{v+v^3}$,

$t=-\int_Q^v\left(\frac{1}{v}-\frac{v}{1+v^2}\right)dv$ (using partial fractions)

$t=\frac{1}{2}\left[\log_e\left(\frac{1+v^2}{v^2}\right)\right]_Q^v$, $t=\frac{1}{2}\log_e\left[\frac{Q^2(1+v^2)}{v^2(1+Q^2)}\right]$

(d) $v^2=\frac{Q^2}{e^{2t}(1+Q^2)-Q^2}$ (e) $v\to 0$, $x\to\tan^{-1}Q$

15 (a) Take O as point of projection, take upwards motion as positive. Gravity and resistance both act downwards

$\therefore\ m\ddot{x}=-(mg+0.1v)$

But $m=1$, so: $\ddot{x}=-(g+0.1v)$

(b) $\ddot{x}=-(g+0.1v)$, $v\frac{dv}{dx}=-\frac{(10g+v)}{10}$, $\frac{dx}{dv}=-\frac{10v}{10g+v}$,

$\int_0^h dx=-\int_{V_0}^0\frac{10v}{10g+v}dv$, $h=\int_{V_0}^0\left(10-\frac{100g}{10g+v}\right)dv$

$h=[10v-100g\log_e(10g+v)]_{V_0}^0$

$h=10V_0+100g\log_e\left(\frac{10g}{10g+V_0}\right)$

(c) $\ddot{x}=-(g+0.1v)$, $\frac{dv}{dt}=-\frac{10g+v}{10}$, $\int_0^{T_1}dt=-\int_{V_0}^0\frac{10}{10g+v}dv$

$T_1=10[\log_e(10g+v)]_0^{V_0}$, $T_1=10\log_e\left(\frac{10g+V_0}{10g}\right)$

(d) Take O as maximum height, take downwards motion as positive. Gravity acts downwards, resistance acts upwards

$\therefore\ m\ddot{x}=mg-0.1v$

But $m=1$, so: $\ddot{x}=g-0.1v$

(e) $\ddot{x}=g-0.1v$, $\frac{dv}{dt}=\frac{10g-v}{10}$, $\int_0^{T_2}dt=\int_0^{V_F}\frac{10}{10g-v}dv$

$T_2=-10[\log_e(10g-v)]_0^{V_F}$, $T_2=10\log_e\left(\frac{10g}{10g-V_F}\right)$

(f) $\ddot{x}=g-0.1v$, $v\frac{dv}{dx}=\frac{10g-v}{10}$, $\frac{dx}{dv}=\frac{10v}{10g-v}$

$\int_0^h dx=\int_0^{V_F}\frac{10v}{10g-v}dv$, $h=\int_0^{V_F}\left(-10+\frac{100g}{10g-v}\right)dv$

$h=[-10v-100g\log_e(10g-v)]_0^{V_F}$

$h=-10V_F+100g\log_e\left(\frac{10g}{10g-V_F}\right)$

(g) Equate results of parts (b) and (f):

$10V_0+100g\log_e\left(\frac{10g}{10g+V_0}\right)$

$=-10V_F+100g\log_e\left(\frac{10g}{10g-V_F}\right)$

$V_0+V_F=10g\log_e\left(\frac{10g+V_0}{10g-V_F}\right)$

Total time $T=T_1+T_2$

$=10\log_e\left(\frac{10g+V_0}{10g}\right)+10\log_e\left(\frac{10g}{10g-V_F}\right)$

$=10\log_e\left(\frac{10g+V_0}{10g-V_F}\right)=\frac{V_0+V_F}{g}$

16 (a) For $0\le t\le 10$: $20000\ddot{x}=-(40000+4v^2)$

$\therefore\ \ddot{x}=-\left(2+\frac{v^2}{5000}\right)$

For $t>10$: $20000\ddot{x}=-(40000+4v^2+18000-300v)$

$\therefore\ \ddot{x}=-\left(2.9+\frac{v^2}{5000}-\frac{3v}{200}\right)$

(b) $35.4\,\text{m s}^{-1}$

(c) 473.6 m (d) 23.0 s (e) 708.3 m

(f) $\displaystyle\int_0^{35.4} \frac{dv}{v^2 - 75v + 14500} = \int_0^{35.4} \frac{dv}{v^2 - 75v + 1406.25 + 13093.75}$

$\displaystyle = \int_0^{35.4} \frac{dv}{(v - 37.5)^2 + (114.4279)^2}$

$\displaystyle = \frac{1}{114.4279}\left[\tan^{-1}\left(\frac{v - 37.5}{(114.4279)}\right)\right]_0^{35.4}$

$= 0.00260721$ (to 6 sig. figs)

$$\int_0^{35.4} \frac{v\,dv}{v^2 - 75v + 14500} = \frac{1}{2}\int_0^{35.4} \frac{2v - 75}{v^2 - 75v + 14500}dv + \frac{75}{2}\int_0^{35.4} \frac{dv}{v^2 - 75v + 14500}$$

$\displaystyle = \frac{1}{2}\left[\log_e(v^2 - 75v + 14500)\right]_0^{35.4} + \frac{75}{2} \times 0.00260721$

$= 0.0469319$ (to 6 sig. figs)

EXERCISE 6.6

1 $t = 0, \dot{x} = 45, x = 0, y = 0, \dot{y} = 0.\ t = 10, y = 100.$

(a) $x = 45t$: $t = 10, x = 450$ m

The package hits the ground 450 m relative to the point from where it was dropped.

(b) Taking downwards as positive, travels 100 metres in 10 seconds so $\dot{y} = 10$.

$\dot{x} = 45, \dot{y} = 10, V = \sqrt{45^2 + 10^2} = 46.1$ m s^{-1}. $\tan\theta = \frac{10}{45}$, $\theta = 12.5°$.

Hits the ground with a velocity of 46.1 m s^{-1} at an angle of 12.5° to the horizontal (77.5° to the vertical).

2 $\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$

(a) $|\underset{\sim}{u}| = \sqrt{7^2 + 24^2} = 25$ ms^{-1}. $\theta = \tan^{-1}\frac{24}{7} = 73.7°$

(b) $t = \frac{240}{49} = 4.90$ s

(c) Horizontal distance $= \frac{7 \times 240}{49} = \frac{240}{7} = 34.3$ m

(d) Greatest height when $t = 2.45$ s

Greatest height $= 24 \times 2.45 - 4.9 \times 2.45^2 = 29.4$ m

(e) Find when $\underset{\sim}{v} \bullet \underset{\sim}{u} = 0$: $\left(7\underset{\sim}{i} + (24 - 9.8t)\underset{\sim}{j}\right) \bullet \left(7\underset{\sim}{i} + 24\underset{\sim}{j}\right) = 0$

$49 + 576 - 235.2t = 0$

$t = \frac{625}{235.2} \approx 2.7$ s

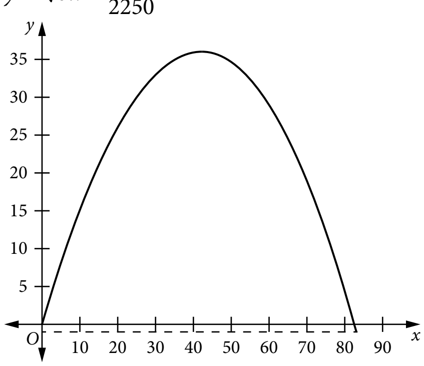
Yes, when $t = 2.7$ seconds.

3 $t = 0, x = 0, y = 1, V = 30$ m s^{-1}, $\alpha = 60°$.

(a) Horizontal distance travelled $= 15 \times 5.34 = 80.1$ m

(b) Maximum height $= 35.4$ m

(c) $y = \sqrt{3}x - \frac{49x^2}{2250}$



Ground level is when $y = -1$.

4 Horizontal distance travelled $\frac{500}{9} \times \frac{1}{2} = 27.8$ m

5 (a) $V = 31.3$ m s^{-1} (b) $V = \sqrt{980} = 31.3$ m s^{-1}

6 $V = 19.6$ m s^{-1}, α.

(a) $\underset{\sim}{a} = -g\underset{\sim}{j}$

$\underset{\sim}{v}(t) = \underset{\sim}{c} - gt\underset{\sim}{j}$

$\underset{\sim}{v}(0) = 19.6\cos\alpha\,\underset{\sim}{i} + 19.6\sin\alpha\,\underset{\sim}{j}$

$\underset{\sim}{v}(t) = 19.6\cos\alpha\,\underset{\sim}{i} + (19.6\sin\alpha - gt)\underset{\sim}{j}$

Greatest height when $19.6\sin\alpha - gt = 0$

$t = \frac{19.6\sin\alpha}{9.8} = 2\sin\alpha$

$\underset{\sim}{r}(t) = 19.6\cos\alpha t\,\underset{\sim}{i} + \left(19.6\sin\alpha t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}: \underset{\sim}{d} = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = 19.6\cos\alpha t\,\underset{\sim}{i} + \left(19.6\sin\alpha t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$

Hits ground when $19.6\sin\alpha t - \frac{1}{2}gt^2 = 0$

$t(19.6\sin\alpha - 4.9t) = 0$

$t = 0, \quad t = \frac{19.6\sin\alpha}{4.9} = 4\sin\alpha$

Hence reaches maximum height halfway through the flight.

(b) $t = 4\sin\alpha$, $x = 19.6\cos\alpha\, t$.

Range $= 19.6\cos\alpha \times 4\sin\alpha = 39.2 \times 2\sin\alpha\cos\alpha = 39.2\sin 2\alpha$

Range is greatest when $\sin 2\alpha = 1$ or $\alpha = 45°$

Maximum range $= 39.2$ m

7 (a) $\underset{\sim}{a} = -g\underset{\sim}{j}$

$\underset{\sim}{v}(t) = \underset{\sim}{c} - gt\underset{\sim}{j}$

$\underset{\sim}{v}(0) = u\cos\alpha\,\underset{\sim}{i} + u\sin\alpha\,\underset{\sim}{j}$

$\underset{\sim}{v}(t) = u\cos\alpha\,\underset{\sim}{i} + (u\sin\alpha - gt)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = u\cos\alpha t\,\underset{\sim}{i} + \left(u\sin\alpha t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}: \underset{\sim}{d} = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = u\cos\alpha t\,\underset{\sim}{i} + \left(u\sin\alpha t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$

$x = u\cos\alpha\, t$ so $t = \frac{x}{u\cos\alpha}$

$y = u\sin\alpha t - \frac{1}{2}gt^2$

$= u\sin\alpha \times \frac{x}{u\cos\alpha} - \frac{1}{2}g \times \left(\frac{x}{u\cos\alpha}\right)^2$

$= x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$

$= x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$

(b) $y = 0$: $x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha = 0$

$x\left(\tan\alpha - \frac{gx}{2u^2}\sec^2\alpha\right) = 0$

$x = 0, x = \frac{2u^2\tan\alpha}{g\sec^2\alpha} = \frac{u^2}{g} \times 2\sin\alpha\cos\alpha = \frac{u^2\sin 2\alpha}{g}$

Range $= \frac{u^2\sin 2\alpha}{g}$

(c) Greatest height when $u\sin\alpha - gt = 0$

$t = \frac{u\sin\alpha}{g}$

Height is given by $y = u\sin\alpha t - \frac{1}{2}gt^2$

Greatest height $= u\sin\alpha \times \frac{u\sin\alpha}{g} - \frac{1}{2}g\left(\frac{u\sin\alpha}{g}\right)^2$

$= \frac{u^2\sin^2\alpha}{g} - \frac{u^2\sin^2\alpha}{2g}$

$= \frac{u^2\sin^2\alpha}{2g}$

(d) Range $= \frac{u^2\sin 2\alpha}{g}$ and is a maximum when $\sin 2\alpha = 1$.

Hence $\alpha = 45°$.

Maximum range $= \frac{u^2\sin 90°}{g} = \frac{u^2}{g}$

8 $V\cos\theta = 6u$, $V\sin\theta = \frac{5u}{2}$.

(a) Time taken to travel from O to P is $\frac{5u}{g}$ seconds.

(b) Range is $\frac{30u^2}{g}$

(c) $V = \frac{13u}{2}$

(d) The minimum speed vertically will be zero when it reaches its highest point. At this time, $t = \frac{5u}{2g}$, the horizontal speed is $6u$.

Hence minimum speed $= \sqrt{36u^2 + 0} = 6u$.

9 Ball is 42.4 m above the ground.

10 (a) $\underset{\sim}{r} = 20t\underset{\sim}{i} + 2t^2\underset{\sim}{j} + (15t - 5t^2)\underset{\sim}{k}$ (b) 18 m (c) $t = \frac{300}{232} \approx 1.29$ s

(d) $t = 1.29$: $v = \sqrt{116 \times 1.29^2 - 300 \times 1.29 + 625} = 20.8$ m s^{-1}

11 (a) $t = \frac{6}{9.8} \approx 0.6$ s

(b) $x = \frac{u_x}{k}\left(1 - e^{-kt}\right)$, $y = \frac{\left(g + ku_y\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$

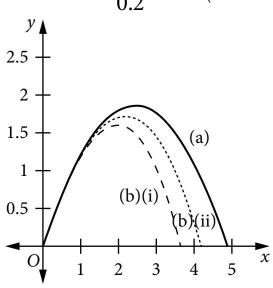
(i) $k = 0.4$: $x = \frac{4}{0.4}\left(1 - e^{-0.4t}\right) = 10\left(1 - e^{-0.4t}\right)$

$y = \frac{(9.8 + 0.4 \times 6)}{0.4^2}\left(1 - e^{0.4t}\right) - \frac{9.8t}{0.4} = 76.25\left(1 - e^{0.4t}\right) - 24.5t$

(ii) $k = 0.2$: $x = \frac{4}{0.2}\left(1 - e^{-0.2t}\right) = 20\left(1 - e^{-0.2t}\right)$

$y = \frac{(9.8 + 0.2 \times 6)}{0.2^2}\left(1 - e^{-0.2t}\right) - \frac{9.8t}{0.2} = 275\left(1 - e^{-0.2t}\right) - 49t$

(c)



(d) The greater the value of k, the lower the greatest height and the shorter the range. The graphs are very similar over the first second.

12 Hits the ground after 4.6 seconds.

13 $x = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$, $y = \frac{(10 + ku\sin\theta)}{k^2}\left(1 - e^{-kt}\right) - \frac{10t}{k}$

(a) $y = \frac{(10 + ku\sin\theta)}{k^2}\left(1 - e^{-kt}\right) - \frac{10t}{k}$

$\dot{y} = \frac{(10 + ku\sin\theta)}{k^2} \times ke^{-kt} - \frac{10}{k}$

$= \frac{(10 + ku\sin\theta)e^{-kt}}{k} - \frac{10}{k}$

$\dot{y} = 0$: $\frac{(10 + ku\sin\theta)e^{-kt}}{k} - \frac{10}{k} = 0$

$(10 + ku\sin\theta)e^{-kt} = 10$

$e^{-kt} = \frac{10}{10 + ku\sin\theta}$

$-kt = \log_e\left(\frac{10}{10 + ku\sin\theta}\right)$

$t = \frac{1}{k}\log_e\left(\frac{10 + ku\sin\theta}{10}\right)$ s

(b) Greatest height is $\frac{u\sin\theta}{k} - \frac{10}{k^2}\log_e\left(\frac{10 + ku\sin\theta}{10}\right)$ m

(c) $k = 0.4$, $t = 0$, $x = 0$, $y = 20$, $u = 30$ m s^{-1}, $\theta = 30°$,

$x = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$,

$y = \frac{(10 + ku\sin\theta)}{k^2}\left(1 - e^{-kt}\right) - \frac{10t}{k}$

$x = \frac{30\cos 30°}{0.4}\left(1 - e^{-0.4t}\right) = 37.5\sqrt{3}\left(1 - e^{-0.4t}\right)$

$y = \frac{(10 + 0.4 \times 30\sin 30°)}{0.4^2}\left(1 - e^{-0.4t}\right) - \frac{10t}{0.4}$

$= 100\left(1 - e^{-0.4t}\right) - 25t$

Greatest height when $t = \frac{1}{0.4}\log_e\left(\frac{10 + 0.4 \times 30\sin 30°}{10}\right)$

$= 2.5\log_e 1.6$

Greatest height is $20 + \frac{30\sin 30°}{0.4}\left(1 - \frac{10}{16}\right) - \frac{10}{0.4^2}\log_e(1.6)$

$= 20 + \frac{1125}{32} - 62.5\log_e 1.6 = 25.8$ m

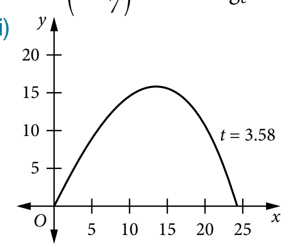
(d) $t = 2.5\log_e 1.6$: $x = \frac{75\sqrt{3}}{2}\left(1 - \frac{10}{16}\right) = \frac{75\sqrt{3}}{2}\left(\frac{6}{16}\right) = \frac{225\sqrt{3}}{16}$ m

≈ 24.4 m

14 (a) $t = 5\log_e 1.4 \approx 1.7$ s

(b) $y = 350\left(1 - \frac{5}{7}\right) - 50 \times 5\log_e 1.4 = 100 - 250\log_e 1.4 \approx 15.9$ m

(c) (i)



(ii) $t = 3.6$: $y = 350\left(1 - e^{-0.2 \times 3.6}\right) - 50 \times 3.6 \approx -0.36 \approx 0$

(d) $x = 50\left(1 - e^{-0.2 \times 3.6}\right) = 50\left(1 - e^{-0.72}\right) \approx 25.7$ m

EXERCISE 6.7

1 $\theta = 45°$, $V = 10\sqrt{2}$, $k = 0.01$, $V\cos\theta = 10$, $V\sin\theta = 10$

(a) $x = 10t$, $y = 10t - 5t^2$ Trajectory: $y = x - \frac{x^2}{20}$

(b) $x = \frac{10}{0.01}\left(1 - e^{-0.01t}\right) = 1000\left(1 - e^{-0.01t}\right)$,

$y = \frac{10 + 0.1}{0.01^2}\left(1 - e^{-0.01t}\right) - \frac{10t}{0.01} = 101000\left(1 - e^{-0.01t}\right) - 1000t$

$\frac{x}{1000} = 1 - e^{-0.01t}$, $e^{-0.01t} = 1 - \frac{x}{1000} = \frac{1000 - x}{1000}$,

$-0.01t = \log_e\left(\frac{1000 - x}{1000}\right)$, $t = 100\log_e\left(\frac{1000}{1000 - x}\right)$

Trajectory is $y = 101x - 100000\log_e\left(\frac{1000}{1000 - x}\right)$

(c) $x = \dfrac{1}{0.01}\log_e\left(1+\dfrac{t}{10}\right) = 100\log_e\left(\dfrac{10+t}{10}\right)$

$y = 100\log_e\left|\cos\left(\sqrt{0.1}\,t\right)+\sqrt{0.1}\sin\left(\sqrt{0.1}\,t\right)\right|$

(d) $\dot{y} = 0: \sqrt{10} - 10\tan\left(\sqrt{0.1}\,t\right) = 0$

$t = 0.97\text{ s}$

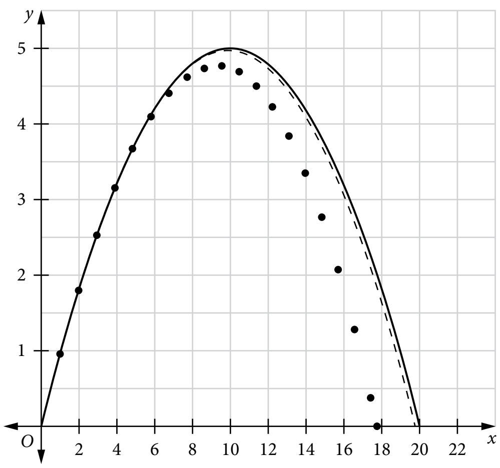
$y = 100\log_e\left|\cos(0.3063)+\sqrt{0.1}\sin(0.3063)\right| = 4.77\text{ m}$

(e) $y = 0: 100\log_e\left|\cos\left(\sqrt{0.1}\,t\right)+\sqrt{0.1}\sin\left(\sqrt{0.1}\,t\right)\right| = 0$

$t = 1.94\text{ s}$

$x = 100\log_e\left(\dfrac{10+1.94}{10}\right) = 17.7\text{ m}$

(f)



part (a) – solid line
part (b) – dashed line
part (c) – dotted line

(g) solid line, 5 m; dashed line, 5 m; dotted line, 4.8 m

(h) 20 m, 19.9 m, 17.8 m

(i) The trajectories with no air resistance and the air resistance proportional to the velocity are very similar. With the air resistance proportional to the square of the velocity, the trajectory is similar until just before the greatest height, which is less, and then falls away faster to give a smaller range.

2 (a) Same as Q**1**(a)

(b) $k = 0.05$

$x = \dfrac{10}{0.05}\left(1-e^{-0.05t}\right) = 200\left(1-e^{-0.05t}\right),$

$y = \dfrac{10+0.5}{0.05^2}\left(1-e^{-0.05t}\right) - \dfrac{10t}{0.05} = 10.5\times 400\left(1-e^{-0.05t}\right)$

$-200t = 4200\left(1-e^{-0.05t}\right) - 200t$

$\dfrac{x}{200} = 1-e^{-0.05t},\ e^{-0.05t} = 1-\dfrac{x}{200} = \dfrac{200-x}{200},$

$-0.05t = \log_e\left(\dfrac{200-x}{200}\right),\ t = 20\log_e\left(\dfrac{200}{200-x}\right)$

Trajectory is $y = 21x - 4000\log_e\left(\dfrac{200}{200-x}\right)$

(c) $x = \dfrac{1}{0.05}\log_e\left(1+\dfrac{t}{2}\right) = 20\log_e\left(\dfrac{2+t}{2}\right)$

$y = 20\log_e\left|\cos\left(\sqrt{0.5}\,t\right)+\sqrt{0.5}\sin\left(\sqrt{0.5}\,t\right)\right|$

(d) $\dot{y} = 0: 5\sqrt{2} - 10\tan\left(\sqrt{0.5}\,t\right) = 0$

$t = 0.87\text{ s}$

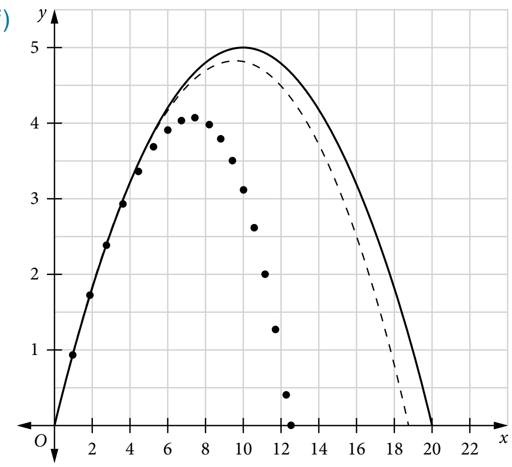
$y = 20\log_e\left|\cos(0.6155)+\sqrt{0.5}\sin(0.6155)\right| = 4.05\text{ m}$

(e) $y = 0: 20\log_e\left|\cos\left(\sqrt{0.5}\,t\right)+\sqrt{0.5}\sin\left(\sqrt{0.5}\,t\right)\right| = 0$

$t = 1.74\text{ s}$

$x = 20\log_e\left(\dfrac{2+1.74}{2}\right) = 12.5\text{ m}$

(f)



part (a) – solid line
part (b) – dashed line
part (c) – dotted line

(g) solid line, 5 m; dashed line, 4.8 m; dotted line, 4.1 m

(h) 20 m, 18.6 m, 12.5 m

(i) The three trajectories are similar to begin with. The greater the air resistance, the sooner the path drops away, and the steeper the descent.

3 (a) Same as Q**1**(a)

(b) $k = 0.1$

$x = \dfrac{10}{0.1}\left(1-e^{-0.1t}\right) = 100\left(1-e^{-0.1t}\right),$

$y = \dfrac{10+1}{0.01^2}\left(1-e^{-0.1t}\right) - \dfrac{10t}{0.1} = 1100\left(1-e^{-0.1t}\right) - 100t$

$\dfrac{x}{100} = 1-e^{-0.1t},\ e^{-0.1t} = 1-\dfrac{x}{100} = \dfrac{100-x}{100},$

$-0.1t = \log_e\left(\dfrac{100-x}{100}\right),\ t = 10\log_e\left(\dfrac{100}{100-x}\right)$

Trajectory is $y = 11x - 1000\log_e\left(\dfrac{100}{100-x}\right)$

(c) $x = \dfrac{1}{0.1}\log_e(1+t) = 10\log_e(1+t)$

$y = 10\log_e\left|\cos(t)+\sin(t)\right|$

(d) $\dot{y} = 0: 10 - 10\tan t = 0$

$t = 0.79\text{ s}$

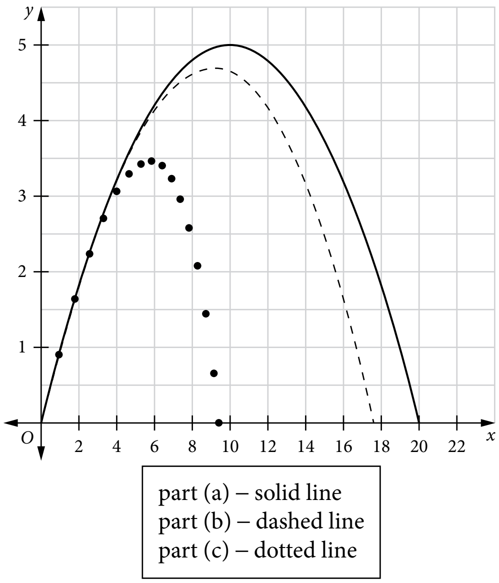
$y = 10\log_e\left|\cos\dfrac{\pi}{4}+\sin\dfrac{\pi}{4}\right| = 3.47\text{ m}$

(e) $y = 0: 10\log_e\left|\cos t + \sin t\right| = 0$

$t = 1.57\text{ s}$

$x = 10\log_e(1+t) = 10\log_e 2.57 = 9.44\text{ m}$

(f)



(g) solid line, 5 m; dashed line, 4.7 m; dotted line, 3.5 m

(h) 20 m, 17.6 m, 9.5 m

(i) The three trajectories are similar to begin with. The greater the air resistance, the sooner the path drops away, and the steeper the descent.
As k increases, the greater effect of air resistance reduces the greatest height and the range. When the air resistance is proportional to the square of the velocity, the descent becomes steeper.

4 $\theta = 30°$, $V = 12$, $k = 0.02$, $V\cos\theta = 6\sqrt{3}$, $V\sin\theta = 6$.

(a) $x = 6\sqrt{3}t$, $y = 6t - 5t^2$

Trajectory is $y = \dfrac{\sqrt{3}x}{3} - \dfrac{5x^2}{108}$

(b) $x = \dfrac{6\sqrt{3}}{0.02}\left(1 - e^{-0.02t}\right) = 300\sqrt{3}\left(1 - e^{-0.02t}\right)$,

$$y = \frac{10 + 0.12}{0.02^2}\left(1 - e^{-0.02t}\right) - \frac{10t}{0.02} = 25300\left(1 - e^{-0.02t}\right) - 500t$$

$$\frac{x}{300\sqrt{3}} = 1 - e^{-0.02t},\ e^{-0.02t} = 1 - \frac{x}{300\sqrt{3}} = \frac{300\sqrt{3} - x}{300\sqrt{3}},$$

$$-0.02t = \log_e\left(\frac{300\sqrt{3} - x}{300\sqrt{3}}\right),$$

$$t = 50\log_e\left(\frac{300\sqrt{3}}{300\sqrt{3} - x}\right)$$

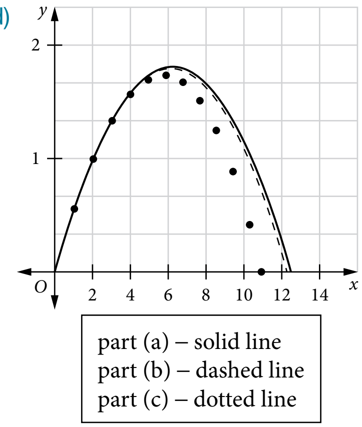
Trajectory is $y = \dfrac{25150x}{150\sqrt{3}} - 500 \times 50\log_e\left(\dfrac{150\sqrt{3}}{150\sqrt{3} - x}\right)$

$$= \frac{503\sqrt{3}x}{9} - 25000\log_e\left(\frac{150\sqrt{3}}{150\sqrt{3} - x}\right)$$

(c) $x = 50\log_e \quad 1 + \dfrac{3\sqrt{3}t}{50}$

$$y = 50\log_e\left|\cos(\sqrt{0.2}\,t) + \frac{3\sqrt{5}}{50}\sin\left(\sqrt{0.2}\,t\right)\right|$$

(d)



(e) solid line, 1.8 m; dashed line, 1.8 m; dotted line, 1.7 m

(f) 12.5 m, 12.3 m, 10.9 m

(g) The three trajectories are similar during the upward motion. The greater the air resistance, the sooner the path drops away, and the steeper the descent.

5 $\theta = 60°$, $t = 0$, $x = 0$, $y = 2$. $V = 10$, $k = 0.005$, $V\cos\theta = 5$, $V\sin\theta = 5\sqrt{3}$

(a) $x = 5t$, $y = 2 + 5\sqrt{3}t - 5t^2$

Trajectory is $y = \sqrt{3}x - 5x^2$

(b) $x = \dfrac{5}{0.005}\left(1 - e^{-0.005t}\right) = 1000\left(1 - e^{-0.005t}\right)$,

$$y = 2 + \frac{10 + 0.025\sqrt{3}}{0.005^2}\left(1 - e^{-0.005t}\right) - \frac{10t}{0.005}$$

$$= 2 + 40000\left(10 + 0.025\sqrt{3}\right)\left(1 - e^{-0.005t}\right) - 2000t$$

$$\frac{x}{1000} = 1 - e^{-0.005t},\ e^{-0.005t} = 1 - \frac{x}{1000} = \frac{1000 - x}{1000},$$

$$-0.005t = \log_e\left(\frac{1000 - x}{1000}\right),\ t = 200\log_e\left(\frac{1000}{1000 - x}\right)$$

Trajectory is

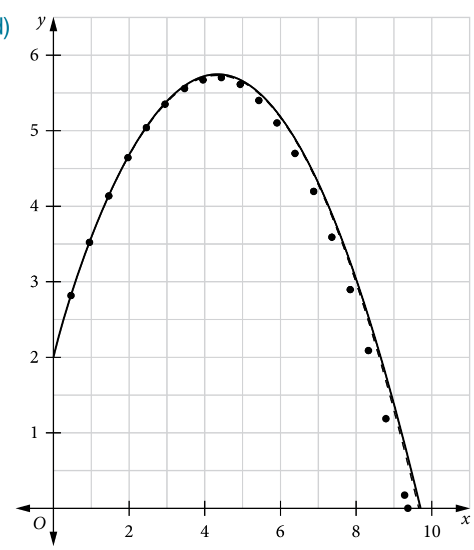
$$y = 2 + 40\left(10 + 0.025\sqrt{3}\right)x - 2000 \times 200\log_e\left(\frac{1000}{1000 - x}\right)$$

$$= 2 + 40\left(10 + 0.025\sqrt{3}\right)x - 400\,000\log_e\left(\frac{1000}{1000 - x}\right)$$

(c) $x = 200\log_e(1 + 0.025t)$

$$y = 2 + 200\log_e\left|\cos\left(\sqrt{0.05}\,t\right) + \frac{\sqrt{15}}{20}\sin\left(\sqrt{0.05}\,t\right)\right|$$

(d)



(e) The greatest height in each case is about 5.7 m (5.75 m in graph (a)).
(f) The range for graphs (a) and (b) is 9.7 m, the range for graph (c) is 9.4 m.
(g) When the resistance is small, the results from the three cases are very similar.

CHAPTER REVIEW 6

1 $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=x+6,\ \frac{1}{2}v^2=\frac{1}{2}x^2+6x+C$
$t=0,\ v=0,\ x=2$: $C=-14,\ v^2=x^2+12x-28$
$x=3$: $v=\sqrt{17}$

2 (a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=x^3+x,\ \frac{1}{2}v^2=\frac{1}{4}x^4+\frac{1}{2}x^2+C$
$x=2,\ v=\frac{5\sqrt{2}}{2}$: $C=\frac{1}{4},\ v^2=\frac{1}{2}(x^4+2x^2+1)$,
$v=\pm\frac{1}{\sqrt{2}}\left(x^2+1\right)$
$v>0$: $v=\frac{1}{\sqrt{2}}\left(x^2+1\right)$
(b) $\frac{dx}{dt}=\frac{x^2+1}{\sqrt{2}},\ \frac{dt}{dx}=\frac{\sqrt{2}}{x^2+1}$: $t=\sqrt{2}\tan^{-1}x+C$
$t=0,\ x=2$: $C=-\sqrt{2}\tan^{-1}2$
$t=\sqrt{2}\tan^{-1}x-\sqrt{2}\tan^{-1}2$
$\frac{t}{\sqrt{2}}=\tan^{-1}x-\tan^{-1}2$
$\tan\left(\frac{t}{\sqrt{2}}\right)=\tan\left(\tan^{-1}x-\tan^{-1}2\right)=\frac{x-2}{1+2x}$
$x=\frac{2+\tan\left(\frac{t}{\sqrt{2}}\right)}{1-2\tan\left(\frac{t}{\sqrt{2}}\right)}$

3 (a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-n^2x,\ \frac{1}{2}v^2=-\frac{1}{2}n^2x^2+C$
$x=a,\ v=0$: $C=\frac{1}{2}a^2x^2,\ \frac{1}{2}v^2=\frac{1}{2}a^2x^2-\frac{1}{2}n^2x^2$,
$v^2=n^2(a^2-x^2)$
(b) $x=0$: $v^2=n^2a^2,\ v=na$
(c) $x=-a$: $\ddot{x}=n^2a$
(d) $t=0,\ x=0,\ x=a\sin nt,\ \dot{x}=an\cos nt,\ \dot{x}=\frac{an}{4}$:
$\frac{an}{4}=an\cos n,\ \cos nt=\frac{1}{4},\ nt=\cos^{-1}\left(\frac{1}{4}\right),\ t=\frac{1}{n}\cos^{-1}\left(\frac{1}{4}\right)$

4 (a) $t=0,\ x=-a;\ a=16,\ T=5$;
$n=\frac{2\pi}{5},\ x=16\sin\left(\frac{2\pi t}{5}-\frac{\pi}{2}\right)$ or $x=16\cos\left(\frac{2\pi t}{5}+\pi\right)$
(b) $x=-8$: $-8=16\sin\left(\frac{2\pi t}{5}-\frac{\pi}{2}\right),\ \sin\left(\frac{2\pi t}{5}-\frac{\pi}{2}\right)=-\frac{1}{2}$,
$\frac{2\pi t}{5}-\frac{\pi}{2}=-\frac{\pi}{6},\ \frac{2\pi t}{5}=\frac{\pi}{3},\ t=\frac{5}{6}$ s

5 (a) $\dot{x}=2\cos 2t+2\sqrt{3}\sin 2t$,
$\ddot{x}=-4\sin 2t+4\sqrt{3}\cos 2t=-4(\sin 2t-\sqrt{3}\cos 2t)$
$=-4(x-3)$
(b) $n=2,\ T=\pi$ s
(c) $\dot{x}=2(\cos 2t+\sqrt{3}\sin 2t)=4\left(\frac{1}{2}\cos 2t+\frac{\sqrt{3}}{2}\sin 2t\right)$
$=4\cos\left(2t-\frac{\pi}{3}\right)$
(d) $\dot{x}=2,\ 4\cos\left(2t-\frac{\pi}{3}\right)=2,\ \cos\left(2t-\frac{\pi}{3}\right)=\frac{1}{2}$,
$2t-\frac{\pi}{3}=-\frac{\pi}{3},\ \frac{\pi}{3},\ \frac{5\pi}{3}$;
$2t=0,\ \frac{2\pi}{3},\ 2\pi;\ t=0,\ \frac{\pi}{3},\ \pi$ s
$\dot{x}=-2,\ 4\cos\left(2t-\frac{\pi}{3}\right)=-2,\ \cos\left(2t-\frac{\pi}{3}\right)=-\frac{1}{2}$,
$2t-\frac{\pi}{3}=\frac{2\pi}{3},\ \frac{4\pi}{3};\ 2t=\pi,\ \frac{5\pi}{3};\ t=\frac{\pi}{2},\ \frac{5\pi}{6}$ s
Hence $t=0,\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \pi$ s

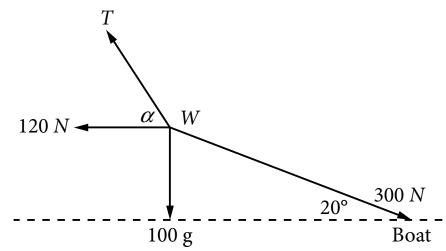
6 (a) 0 (b) $a=\frac{18-2t^2}{\left(9+t^2\right)^2}$
(c) $t=3$ (d) $t=3,\ v=\frac{1}{3}$ m s^{-1}
(e) $x=\int_0^3\frac{2t\,dt}{9+t^2}=\log_e 18-\log_e 9=\log_e 2$

7 (a) 12 s (b) 432 m from O
(c) Particle moves right (i.e. in the positive direction) from O, slowing until momentarily at rest after 12 s, then moves left with increasing speed.

8 (a) $F=-1.6x+50$ (b) $5\sqrt{6}$ m s^{-1}

9 (a) $x=3t-2\sin t+1$ (b) $v=3e^t-3$

10 (a)



(b) The waterskier is moving with constant speed in a straight line. By Newton's first law of motion, either there is no force or the resultant of all forces is zero.
(c) $T=1094.6,\ \alpha=81°$

11 (a) Take O as the centre of the moon.
Take motion away from O as positive.
$\ddot{x}=-\frac{k}{x^2}$ [1]
But when $x=R,\ \ddot{x}=-\frac{g}{6}$
(on the surface, acceleration due to gravity is $-\frac{g}{6}$):
$\therefore -\frac{g}{6}=-\frac{k}{R^2},\ k=\frac{gR^2}{6}$
Substitute into [1]: $\ddot{x}=-\frac{gR^2}{6x^2}\ \therefore\ v\frac{dv}{dx}=-\frac{gR^2}{6}x^{-2}$
$\therefore \int_{v_0}^{v}v\,dv=-\frac{gR^2}{6}\int_R^x x^{-2}\,dx,\ \left[\frac{1}{2}v^2\right]_{v_0}^{v}=\frac{gR^2}{6}\left[x^{-1}\right]_R^x$
$v^2-v_0^2=-\frac{gR^2}{3}\left(\frac{1}{x}-\frac{1}{R}\right),\ v^2=v_0^2+\frac{gR^2}{3}\left(\frac{1}{x}-\frac{1}{R}\right)$
$v=\pm\sqrt{v_0^2+\frac{gR^2}{3}\left(\frac{1}{x}-\frac{1}{R}\right)}$
(b) $v=0$ when $x=2R$: $0=v_0^2+\frac{gR^2}{3}\left(\frac{1}{2R}-\frac{1}{R}\right)$
$v_0^2=\frac{gR^2}{3}\times\frac{1}{2R}=\frac{gR}{6}\ \therefore\ v_0=\sqrt{\frac{gR}{6}}$
(c) If the particle escapes: $x\to\infty\ \therefore\ \frac{1}{x}\to 0$
The equation from part (a) gives:
$v^2\to v_0^2+\frac{gR^2}{6}\left(0-\frac{1}{R}\right),\ v^2\to v_0^2-\frac{gR}{3}$
But $v^2\geq 0$ hence $v_0^2\geq\frac{gR}{3}$
i.e. $v_0\geq\sqrt{\frac{gR}{3}}$: escape velocity is $\sqrt{\frac{gR}{3}}$

12 B

13 (a) $x = Vt\cos\alpha,\ y = Vt\sin\alpha - \frac{1}{2}gt^2$, particle hits ground at

$y = 0$: $t = \frac{2V\sin\alpha}{g}$, $x = \frac{V^2\sin 2\alpha}{g}$

Maximum value of $\sin 2\alpha = 1$ $\therefore$ maximum range $= \frac{V^2}{g}$

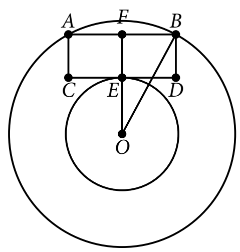
(b) $\alpha = 15°$: $R = \frac{V^2\sin 30°}{g} = \frac{V^2}{2g}$

$\alpha = 60°$: $R = \frac{V^2\sin 120°}{g} = \frac{\sqrt{3}V^2}{2g}$

$\alpha = 45°$: $R = \frac{V^2\sin 90°}{g} = \frac{V^2}{g}$

$\therefore$ spray lands within distances of $\frac{V^2}{2g}$ and $\frac{V^2}{g}$

(c)



$OB = \frac{V^2}{g}$, $OE = \frac{V^2}{2g}$, $EF = 3$, $FB = 3$:

$$\left(\frac{V^2}{g}\right)^2 = \left(\frac{V^2}{2g} + 3\right)^2 + 3^2$$

Let $A = \frac{V^2}{2g}$: $(2A)^2 = (A+3)^2 + 3^2$

$A^2 - 2A - 6 = 0$, $A = \frac{2 \pm \sqrt{28}}{2} = 1 \pm \sqrt{7}$

But $A = \frac{V^2}{2g}$, must be positive $\therefore \frac{V^2}{2g} \geq 1 + \sqrt{7}$

14 (a) $m\ddot{x} = mg - 2v$, $0.5\ddot{x} = 0.5g - 2v$, $\ddot{x} = g - 4v$

(b) $v = \frac{g}{4}(1 - e^{-4t})$

(c) $\frac{g}{4}$ m s^{-1}

(d) $\frac{1}{4}\log_e 2$ s

(e) $\frac{199g}{16}$ m

15 (a) $\underset{\sim}{r} = 3\cos 2t\,\underset{\sim}{i} + 3\sin 2t\,\underset{\sim}{j}$

$x = 3\cos 2t$, $y = 3\sin 2t$

$\cos 2t = \frac{x}{3}$, $\sin 2t = \frac{y}{3}$

But $\sin^2 2t + \cos^2 2t = 1$

So $\frac{x^2}{9} + \frac{y^2}{9} = 1$

Hence $x^2 + y^2 = 1$ is the equation of the path. It is a circle centre (0, 0), radius 3.

(b) $\underset{\sim}{r} = 3\cos 2t\,\underset{\sim}{i} + 3\sin 2t\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}} = -6\sin 2t\,\underset{\sim}{i} + 6\cos 2t\,\underset{\sim}{j}$

Speed $= \left|\dot{\underset{\sim}{r}}\right| = \sqrt{36\sin^2 2t + 36\cos^2 2t} = 6\sqrt{\sin^2 2t + \cos^2 2t} = 6$

Hence particle moves with a constant speed.

(c) $\dot{\underset{\sim}{r}} = -6\sin 2t\,\underset{\sim}{i} + 6\cos 2t\,\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}} = -12\cos 2t\,\underset{\sim}{i} - 12\sin 2t\,\underset{\sim}{j}$

$= -4\left(3\cos 2t\,\underset{\sim}{i} + 3\sin 2t\,\underset{\sim}{j}\right)$

$= -4\underset{\sim}{r}$

$\left|\ddot{\underset{\sim}{r}}\right| = 4\sqrt{9\cos^2 2t + 9\sin^2 2t} = 12$, which is a constant.

Consider

$\ddot{\underset{\sim}{r}} \bullet \dot{\underset{\sim}{r}} = \left(-12\cos 2t\,\underset{\sim}{i} - 12\sin 2t\,\underset{\sim}{j}\right) \bullet \left(-6\sin 2t\,\underset{\sim}{i} + 6\cos 2t\,\underset{\sim}{j}\right)$

$= 72\sin 2t\cos 2t - 72\sin 2t\cos 2t$

$= 0$

Hence the acceleration is perpendicular to the direction of the motion.

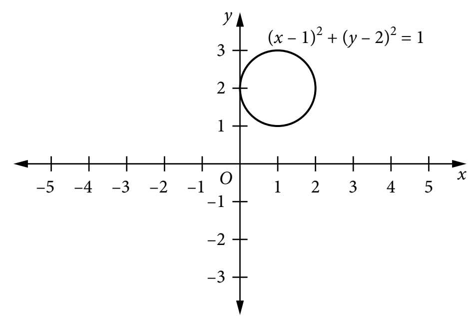
16 (a) $\underset{\sim}{r} = (1 + \sin 4t)\underset{\sim}{i} + (2 - \cos 4t)\underset{\sim}{j}$

$x = 1 + \sin 4t$, $y = 2 - \cos 4t$

$\sin 4t = x - 1$, $\cos 4t = 2 - y$

$\sin^2 4t + \cos^2 4t = 1$

$(x-1)^2 + (y-2)^2 = 1$ is a circle centre (1, 2), radius 1



(b) $\underset{\sim}{r} = (1 + \sin 4t)\underset{\sim}{i} + (2 - \cos 4t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}} = 4\cos 4t\,\underset{\sim}{i} + 4\sin 4t\,\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}} = -16\sin 4t\,\underset{\sim}{i} + 16\cos 4t\,\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}} \bullet \dot{\underset{\sim}{r}} = \left(-16\sin 4t\,\underset{\sim}{i} + 16\cos 4t\,\underset{\sim}{j}\right) \bullet \left(4\cos 4t\,\underset{\sim}{i} + 4\sin 4t\,\underset{\sim}{j}\right)$

$= -64\sin 4t\cos 4t + 64\sin 4t\cos 4t$

$= 0$

Hence the acceleration is perpendicular to the direction of the motion.

17 $\underset{\sim}{r} = 2\cos 3t\,\underset{\sim}{i} + 2\sin 3t\,\underset{\sim}{j} + 3t\,\underset{\sim}{k}$

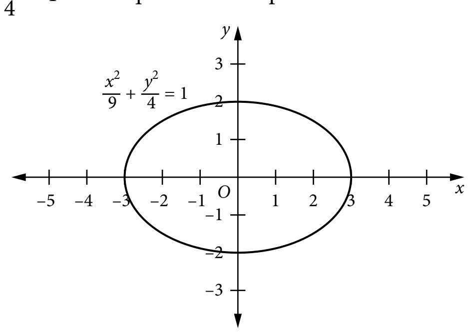
$\dot{\underset{\sim}{r}} = -6\sin 3t\,\underset{\sim}{i} + 6\cos 3t\,\underset{\sim}{j} + 3\underset{\sim}{k}$

$\ddot{\underset{\sim}{r}} = -18\cos 3t\,\underset{\sim}{i} - 18\sin 3t\,\underset{\sim}{j} + 0\underset{\sim}{k}$

$\left|\dot{\underset{\sim}{r}}\right| = \sqrt{36\sin^2 3t + 36\cos^2 3t + 9} = \sqrt{36 + 9} = 3\sqrt{5}$, a constant.

$\left|\ddot{\underset{\sim}{r}}\right| = \sqrt{18^2\cos^2 3t + 18^2\sin^2 3t} = 18$, a constant.

18 (a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is the equation of the path.



(b) $\underset{\sim}{r} = 3\cos t\,\underset{\sim}{i} + 2\sin t\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}} = -3\sin t\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}$

Consider $\underset{\sim}{r} \bullet \dot{\underset{\sim}{r}} = \left(3\cos t\,\underset{\sim}{i} + 2\sin t\,\underset{\sim}{j}\right) \bullet \left(-3\sin t\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}\right)$

$= -9\sin t\cos t + 4\sin t\cos t$

$= -2.5\sin 2t$

Perpendicular when $\underset{\sim}{r} \bullet \dot{\underset{\sim}{r}} = 0$ hence require $\sin 2t = 0$ for $0 \le t \le 2\pi$.

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$t = 0$: $\underset{\sim}{r} = 3\cos 0\,\underset{\sim}{i} + 2\sin 0\,\underset{\sim}{j} = 3\underset{\sim}{i}$

$t = \frac{\pi}{2}$: $\underset{\sim}{r} = 3\cos\frac{\pi}{2}\underset{\sim}{i} + 2\sin\frac{\pi}{2}\underset{\sim}{j} = 2\underset{\sim}{j}$

$t = \pi$: $\underset{\sim}{r} = 3\cos\pi\,\underset{\sim}{i} + 2\sin\pi\,\underset{\sim}{j} = -3\underset{\sim}{i}$

$t = \frac{3\pi}{2}$: $\underset{\sim}{r} = 3\cos\frac{3\pi}{2}\underset{\sim}{i} + 2\sin\frac{3\pi}{2}\underset{\sim}{j} = -2\underset{\sim}{j}$

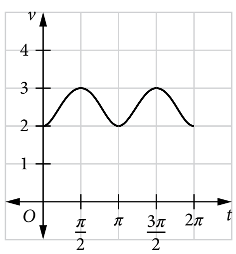
$t = 2\pi$: $\underset{\sim}{r} = 3\cos 2\pi\,\underset{\sim}{i} + 2\sin 2\pi\,\underset{\sim}{j} = 3\underset{\sim}{i}$

(c) $\dot{\underset{\sim}{r}} = -3\sin t\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}$

Speed $= \left|\dot{\underset{\sim}{r}}\right| = \sqrt{9\sin^2 t + 4\cos^2 t}$

Maximum speed when $\cos t = 0$, i.e. $t = \frac{\pi}{2}, \frac{3}{2}$ so maximum speed is 3.

Minimum speed when $\sin t = 0$, i.e. $t = 0, \pi, 2\pi$ so minimum speed is 2.



The maximum and minimum speed can also be read from the graph.

(d) $\underset{\sim}{r} = 3\cos t\,\underset{\sim}{i} + 2\sin t\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}} = -3\sin t\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}$

$\ddot{\underset{\sim}{r}} = -3\cos t\,\underset{\sim}{i} - 2\sin t\,\underset{\sim}{j}$

$= -\underset{\sim}{r}$

Hence the acceleration vector is parallel to the position vector but in the opposite direction. Thus the acceleration vector is always directed towards the origin.

And $\left|\ddot{\underset{\sim}{r}}\right| = \left|\underset{\sim}{r}\right|$ so the magnitude of the acceleration is equal to the particle's distance from the origin.

Or $\left|\ddot{\underset{\sim}{r}}\right| = \sqrt{9\cos^2 t + 4\sin^2 t} = \sqrt{5\cos^2 t + 4}$

(e) Consider $\ddot{\underset{\sim}{r}} \bullet \dot{\underset{\sim}{r}} = \left(-3\cos t\,\underset{\sim}{i} - 2\sin t\,\underset{\sim}{j}\right) \bullet \left(-3\sin t\,\underset{\sim}{i} + 2\cos t\,\underset{\sim}{j}\right)$

$= 9\sin t\cos t - 4\sin t\cos t$

$= 2.5\sin 2t$

Require $\sin 2t = 0$, i.e. $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

GLOSSARY

A

acceleration

The rate of change of velocity with respect to time:
$\frac{d^2x}{dt^2} = \ddot{x} = \dot{v}(t) = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$,
standard units m s^{-2}.

angular acceleration

The rate of change of angular velocity with respect to time: $\frac{d^2\theta}{dt^2} = \ddot{\theta} = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\omega^2\right)$,
standard units radians s^{-2}.

angular displacement

Consider a particle P that moves about the origin O along a curve AP. OA is a fixed arbitrary (reference) line. At time t, $\angle AOP = \theta$, with rotation anticlockwise taken as positive. θ is thus the angular displacement of P about O at time t, in radians.

angular velocity

The rate of change of angular displacement with respect to time: $\omega = \frac{d\theta}{dt} = \dot{\theta}$, standard units radians s^{-1}.

Argand diagram

The representation of complex numbers on a Cartesian plane such that the real axis is x and the imaginary axis is y.

argument and principal argument of a complex number

When a complex number z is represented by a point P in the complex plane, then the argument of z, denoted arg z, is the angle θ that OP (where O denotes the origin) makes with the positive real axis O_x, with the angle measured from O_x.

If the argument is restricted to the interval $(-\pi, \pi]$, this is called the principal argument and is denoted by Arg z.

argument of a complex number

For a complex number $z = x + iy$, the argument is $\arg z = \theta$ such that $x = r\cos\theta$ and $y = r\sin\theta$. On an Argand diagram, this is the angle made with the positive direction of the real axis by the vector from O to the point representing z.

- $\arg 0$ is undefined.
- $\arg(z - z_1)$ is the angle made with the positive direction of the real axis by the vector from the point representing z_1 to the point representing z.

See also *principal argument of a complex number*.

arithmetic mean

The arithmetic mean of the numbers $x_1, x_2, x_3, \ldots, x_n$ is defined to be: $\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$.

asymptote

A line or curve that a function approaches but never reaches.

- *Asymptotes of a hyperbola*

  The asymptotes to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are: $y = \pm\frac{bx}{a}$

C

Cartesian equation

A Cartesian equation is the equation of a relation or a function expressed in terms of the Cartesian coordinates x and y.

A Cartesian equation may sometimes be formed from two parametric equations by eliminating the parameter.

Cartesian form of a complex number

The Cartesian form of a complex number (z) is $z = x + iy$, where x and y are real numbers and i is the imaginary number. Also known as standard or rectangular form.

column vector notation

A vector in two or three dimensions can be represented in column vector notation.

For example, $\underset{\sim}{v} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 6\underset{\sim}{k}$ can be represented as the ordered triple $\underset{\sim}{v} = (4, 5, 6)$, and in column vector notation as $\underset{\sim}{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

complex conjugate

The complex conjugate of the number $z = a + ib$ is given by $\overline{z} = a - ib$, where a and b are real numbers. A complex number and its conjugate are called a conjugate pair.

complex number

Any number z of the form $z = x + iy$, where x and y are real numbers and $i^2 = -1$.

complex plane

A complex plane is a Cartesian plane in which the horizontal axis is the real axis and the vertical axis is the imaginary axis. The complex plane is sometimes called the Argand plane. Geometric plots in the complex plane are known as Argand diagrams.

component form of a vector

The component form of a vector describes the projections of a vector in the x, y and z-directions. It can be expressed as an ordered triple (in three dimensions) or by a linear combination of unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

conjugate hyperbolae

Two hyperbolae are a conjugate pair when the transverse and conjugate axes of one hyperbola are respectively the conjugate and transverse axes of the other.

conjugate of a complex number

For a complex number $z = x + iy$, the conjugate of z is $\bar{z} = x - iy$.

- $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

conjugate root theorem

If a polynomial $P(x)$ has real coefficients and $P(a + bi) = 0$, $b \neq 0$, then $P(a - bi) = 0$. Complex zeros of real polynomials occur in conjugate pairs.

contrapositive

The contrapositive of the statement 'If P, then Q' is 'If not Q, then not P'. The contrapositive is true if and only if the statement itself is also true.

converse

The converse of a statement 'If P, then Q' is 'If Q, then P'.

The statements can be represented as: the converse of $P \quad Q$ is $Q \quad P$ or $P \quad Q$.

The converse of a true statement need not be true.

counterexample

A counterexample is an example that demonstrates that an assertion is not true in general.

D

de Moivre's theorem

If $z = r(\cos\theta + i\sin\theta)$ and n is an integer, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

de Moivre's theorem states that for all integers n: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

In exponential form, when $r = 1$, de Moivre's theorem is simply a statement of the law of indices: $(e^{i\theta})^n = e^{in\theta}$

E

ellipse

(1) A conic with eccentricity $0 < e < 1$. Its equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(2) The locus (or set) of all points P in a plane such that the ratio of the distances from P to a fixed point S (the focus) and to a fixed straight line (the directrix) is a positive constant e (the eccentricity) less than 1.

(3) The locus (or set) of all points P in a plane such that the sum of the distances from P to two fixed points (the foci) is a constant.

Euler's formula

Euler's formula states that for any real number θ: $e^{i\theta} = \cos\theta + i\sin\theta$

even function

A function is even if $f(-x) = f(x)$ for all values of x in the domain. The function is also symmetrical about the y-axis.

exponential form of a complex number

The complex number $z = a + ib$ can be expressed in exponential form as $z = re^{i\theta}$, where r is the modulus of the complex number and θ is the argument expressed in radians.

F

factor theorem

(1) For any polynomial $P(z)$, if $P(a) = 0$, then $(z - a)$ is a factor of $P(z)$.

(1a) For any polynomial $P(z)$, if $P\left(\frac{b}{a}\right) = 0$, then $(az - b)$ is a factor of $P(z)$.

(2) For any polynomial $P(z)$, if $(z - a)$ is a factor of $P(z)$, then $P(a) = 0$.

fundamental theorem of algebra

Every polynomial equation with complex coefficients, $P(z) = 0$, of degree n (where n is a positive integer), has a root that is a complex number.

G

geometric mean

The geometric mean of the positive numbers $x_1, x_2, x_3, \ldots, x_n$, is defined to be: $(x_1 x_2 x_3 \ldots x_n)^{\frac{1}{n}}$

I

identity of polynomials

If two polynomials of degree n are equal for more than n values of the variable, then the polynomials are identically equal, i.e. equal for all values of the variable.

imaginary part of a complex number

For a complex number $z = x + iy$, the imaginary part is $\text{Im}(z) = y$.

implication

To say that P implies Q means that if P is true, then Q is true.

In shorthand it can be written as 'If P, then Q', and in notation form as $P \quad Q$.

implicit differentiation

To differentiate a function without explicitly finding its derivative. For example, the derivative of y^n with respect to x is: $ny^{n-1}\frac{dy}{dx}$

integration by parts

If $u(x)$ and $v(x)$ are differentiable functions of x, then:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \text{ or } \int uv'\,dx = uv - \int vu'\,dx$$

$$\text{or } \int u\,dv = uv - \int v\,du$$

M

modulus of a complex number

For a complex number $z = x + iy$, the modulus is $\text{mod}\, z = |z| = \sqrt{x^2 + y^2} = r$.

- $|z| = r$ is the distance from the origin O to the point (x, y).
- $|z - (a + ib)|$ is magnitude of the vector from (a, b) to the point representing z.

mod–arg form of a complex number

For a complex number $z = x + iy$, the mod–arg form is the expression $z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$, where $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$, $y = r\sin\theta$.

N

negation

If P is a statement, then the statement 'not P' is the negation of P. The negation of P is denoted by $\neg P$ or $\sim P$.

Newton's first law of motion

A body remains at rest or in uniform motion in a straight line unless it is acted on by a non-zero resultant force.

Newton's second law of motion

The rate of change of momentum is proportional to the applied force and occurs in the direction of the force: $F = kma$, where k is a constant.

Newton's third law of motion

When two objects exert force on each other, the forces are equal in magnitude but opposite in direction. 'For every action there is an equal but opposite reaction.'

Newton's laws of motion

Newton's laws of motion consist of three fundamental laws of classical physics:

(1) Unless acted upon by a resultant force, a body remains at rest or in uniform motion in a straight line.

(2) The acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body.

(3) For every action, there is an equal and opposite reaction.

O

odd function

A function is odd if $f(-x) = -f(x)$ for all values of x in the domain. The function has rotational symmetry about the origin.

P

parameter

(1) A parameter is a quantity that defines certain characteristics of a function or system. For example, θ is a parameter in $y = x\cos\theta$.

(2) A parameter can be a characteristic value of a situation; for example, the time taken for a machine to produce a certain product.

parametric equation

Equation in which related variables (e.g. x and y) are expressed in terms of another variable, the 'parameter' (e.g. t or θ), so that $x = f(t)$, $y = g(t)$ or $x = f(\theta)$, $y = g(\theta)$.

- *Parametric equations of a circle*
  $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta < 2\pi$
- *Parametric equations of an ellipse*
  $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta < 2\pi$
- *Parametric equations of a hyperbola*
  $x = a\sec\theta$, $y = b\tan\theta$, $-\pi < \theta \le \pi$, $\theta \ne -\frac{\pi}{2}, \frac{\pi}{2}$
- *Parametric equations of a rectangular hyperbola*
  $x = ct$, $y = \frac{c}{t}$
- *Parametric equations of a parabola*
  $x = 2at$, $y = at^2$

A parametric equation is the equation of a relation or function expressed in terms of independent parameters.

partial fractions

When a function $f(x)$ is the ratio of two polynomials $\frac{R(x)}{B(x)}$ such that degree of $R(x)$ < degree of $B(x)$, this may be be decomposed into partial fractions with denominators that are the factors of $B(x)$, such that:

- $\frac{R(x)}{B(x)} = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \ldots + \frac{c_n}{x - a_n}$
- $\frac{R(x)}{B(x)} = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \ldots + \frac{c_n}{x - a_n} + \frac{dx + e}{x^2 + bx + c}$, degree of $R(x) < n + 2$

perpendicular equation of a straight line

The line perpendicular to the line $ax + by + c = 0$ which passes through (x_1, y_1) has the equation: $b(x - x_1) - a(y - y_1) = 0$

polar form of a complex number

The complex number $z = a + ib$ can be expressed in polar form as:

$z = r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta)$, where r is the modulus of the complex number and θ is its argument expressed in radians. This is also known as modulus–argument form.

polynomial equations and roots

Every polynomial equation $P(z) = 0$ of degree n has n solutions ('roots' of the equation, or 'zeros' of the polynomial) over the field of complex numbers.

Complex roots of a polynomial equation with real coefficients occur in conjugate pairs.

principal argument of a complex number

The argument $\arg z = \theta$ of a complex number z such that $-\pi < \theta \le \pi$.

proof by contradiction

Proof by contradiction is when a mathematical proof assumes the opposite (negation) of the original statement being proven and illustrates through a logical chain of arguments that the opposite is demonstrably false. As the reasoning is correct and the conclusion absurd, the only element that could be wrong was the initial assumption. Therefore, the original statement is true.

R

rational function

A rational function f is the ratio of two polynomials:

$f(x)=\frac{A(x)}{B(x)}$, defined for all values of x except those for which $B(x)=0$.

A rational function is a function of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

real part of a complex number

For a complex number $z=x+iy$, the real part is $\text{Re}(z)=x$.

reciprocal functions

Two functions f and g are a pair of reciprocal functions if: $g(x)=\frac{1}{f(x)}$

recursive formula

A recursive formula defines a sequence in which successive terms are expressed as a function of the preceding terms.

resistance

A force that acts in the opposite direction to a body's motion.

resisted motion

Whenever a body moves through a medium (such as air, water, oil etc.), it is subjected to a resistance that acts in the opposite direction to the motion. If the body falls vertically downwards, the resistance acts upwards; if the body is projected vertically upwards, the resistance acts downwards.

Resisted motion is motion that encounters resisting forces, for example friction and air resistance.

retardation

Acceleration acting in the opposite direction to a body's motion.

roots of unity

A complex number z is an nth root of unity if $z^n = 1$.

The points in the complex plane representing the roots of unity lie on the unit circle and are evenly spaced.

S

smooth

In physical problems, 'smooth' is assumed to mean 'no friction'.

square root of −1

This is the basis of imaginary numbers, denoted by i such that $i^2=-1$. The two square roots of -1 are $\pm i$.

square root of a function

$$\begin{aligned}\sqrt{(f(x))^2} = |f(x)| &= f(x) \quad \text{for} \quad f(x) \geq 0 \\ &= -f(x) \quad \text{for} \quad f(x) < 0\end{aligned}$$

statement

A statement is an assertion that can be true or false but not both.

string

In physical problems, a *light inelastic string* is assumed to have a negligible weight force and constant length, so that it does not stretch.

T

terminal velocity

The maximum velocity of a body falling under the action of gravity and a resistance force, which occurs when the acceleration is equal to zero.

Terminal velocity is the constant velocity that a free-falling object will eventually reach when the resistance of the medium through which the object is falling prevents further acceleration.

t formulae *or t* substitution

Expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of t where $t=\tan\frac{\theta}{2}$.

If $t=\tan\frac{\theta}{2}$, then: $\sin\theta=\frac{2t}{1+t^2}$, $\cos\theta=\frac{1-t^2}{1+t^2}$, $\frac{d\theta}{dt}=\frac{2}{1+t^2}$

triangle inequalities

Where z_1, z_2 are complex numbers and k is real:

- $|z_1+z_2| \leq |z_1|+|z_2|$, with equality when $z_2=kz_1$
- $|z_1-z_2| \geq |z_1|-|z_2|$, with equality when $z_2=kz_1$