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NELSON SENIOR MATHS

FOR THE AUSTRALIAN CURRICULUM | **SPECIALIST 11**

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ABOUT THIS SERIES

There are 6 books for this series currently. These cover the subjects General Mathematics, Mathematical Methods and Specialist Mathematics. An additional 2 books are to follow, which will cover the Essential Mathematics course.

These books have all been written specifically, from scratch, by a national author team for the Senior Australian Curriculum.

There are worked solutions available for purchase, which accompany each of the Student Books.

There are also ExamView question banks which can be purchased separately. These will contain a large number of multiple choice questions covering each course: General, Methods and Specialist. An ExamView test generator is also provided, so that teachers can construct their own tests for each topic.

Accompanying each printed textbook is a digital textbook called the NelsonNetBook and a NelsonNet website.

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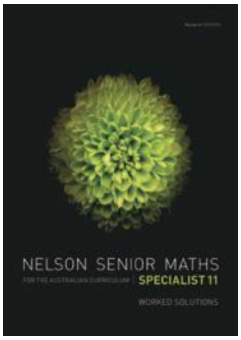

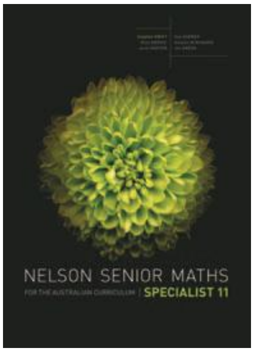

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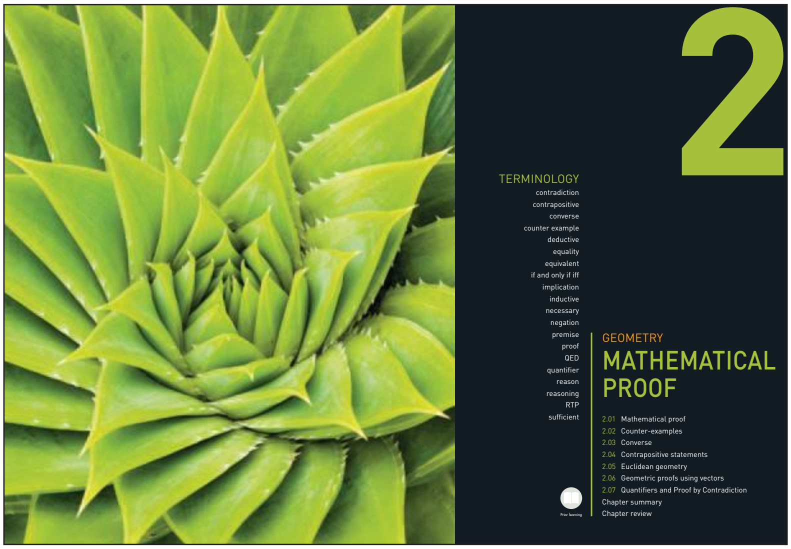
ABOUT THIS BOOK


2

TERMINOLOGY
contradiction
contrapositive
converse
counter example
deductive
equality
equivalent
if and only if iff
implication
inductive
necessary
negation
premise
proof
QED
quantifier
reason
reasoning
RTP
sufficient

GEOMETRY
MATHEMATICAL PROOF

2.01 Mathematical proof
2.02 Counter-examples
2.03 Converse
2.04 Contrapositive statements
2.05 Euclidean geometry
2.06 Geometric proofs using vectors
2.07 Quantifiers and Proof by Contradiction
Chapter summary
Chapter review

A clear outline of chapter contents is provided.

Links to curriculum content descriptions are included.

A Prior learning worksheet is provided, enabling teachers to assess pre-requisite understanding.

Example 1

Decide whether these proofs are deductive, inductive or by contradiction.

a All sheep are black.
This is a sheep.
Therefore it is black.

b In triangle A, the angles add up to 180°.
In triangle B, the angles add up to 180°.
Therefore, in all triangles the angles add up to 180°.

c Assume that pentagons are squares.
Squares have 4 sides.
Therefore pentagons have 4 sides.
But pentagons have 5 sides, so the assumption cannot be true.
Therefore pentagons are not squares.

Solution

a Proof **a** begins with some statements and a conclusion is drawn.	Proof **a** is deductive.
b Proof **b** begins with some specific statements and a general conclusion is drawn.	Proof **b** is inductive.
c Proof **c** begins with an assumption and a contradiction is shown.	Proof **c** is by contradiction.

Examples are clearly set out with reasoning (in black) and writing (in blue) in separate columns.

Examples are sequenced in a logical order.

There are generally three examples leading to an exercise.

Examples show solutions and steps which guide students through the use of the TI-Nspire CX and the CASIO ClassPad CAS calculators.

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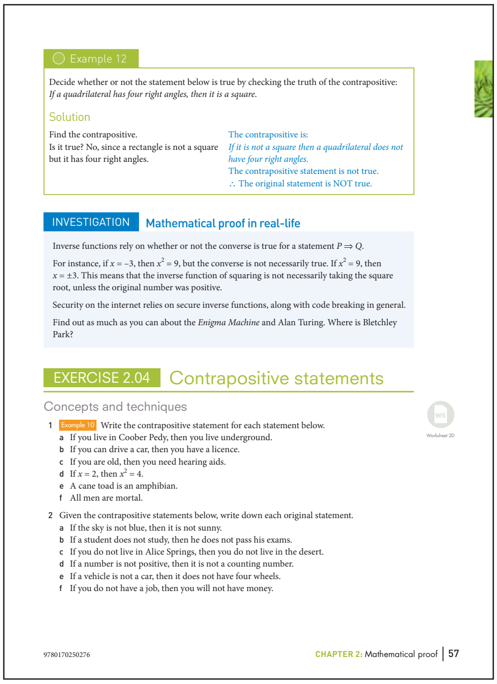

Example 12

Decide whether or not the statement below is true by checking the truth of the contrapositive:
If a quadrilateral has four right angles, then it is a square.

Solution

Find the contrapositive. The contrapositive is:
If it is not a square then a quadrilateral does not have four right angles.

Is it true? No, since a rectangle is not a square but it has four right angles. The contrapositive statement is not true.
∴ The original statement is NOT true.

INVESTIGATION Mathematical proof in real-life

Inverse functions rely on whether or not the converse is true for a statement $P \Rightarrow Q$.

For instance, if $x = -3$, then $x^2 = 9$, but the converse is not necessarily true. If $x^2 = 9$, then $x = \pm 3$. This means that the inverse function of squaring is not necessarily taking the square root, unless the original number was positive.

Security on the internet relies on secure inverse functions, along with code breaking in general.

Find out as much as you can about the *Enigma Machine* and Alan Turing. Where is Bletchley Park?

EXERCISE 2.04 Contrapositive statements

Concepts and techniques

WS Worksheet 2D

1 Example 10 Write the contrapositive statement for each statement below.
a If you live in Coober Pedy, then you live underground.
b If you can drive a car, then you have a licence.
c If you are old, then you need hearing aids.
d If $x = 2$, then $x^2 = 4$.
e A cane toad is an amphibian.
f All men are mortal.

2 Given the contrapositive statements below, write down each original statement.
a If the sky is not blue, then it is not sunny.
b If a student does not study, then he does not pass his exams.
c If you do not live in Alice Springs, then you do not live in the desert.
d If a number is not positive, then it is not a counting number.
e If a vehicle is not a car, then it does not have four wheels.
f If you do not have a job, then you will not have money.

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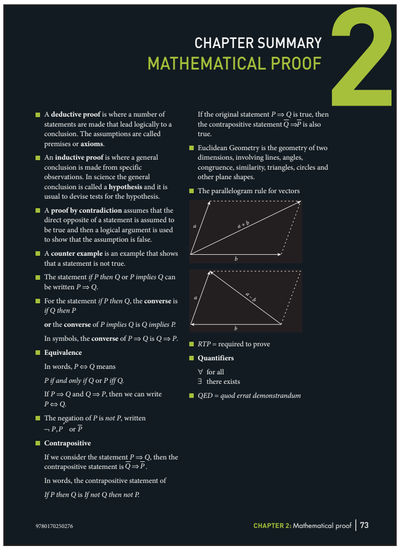
Exercise questions are clearly divided into two categories: concepts and techniques and reasoning and communication.

Where appropriate, worksheets are provided for additional practice and consolidation of key concepts.

Each chapter contains at least one investigation, providing students with the opportunity to apply their understanding to a practical application.



CHAPTER SUMMARY
MATHEMATICAL PROOF
2

- A **deductive proof** is where a number of statements are made that lead logically to a conclusion. The assumptions are called premises or **axioms**.
- An **inductive proof** is where a general conclusion is made from specific observations. In science the general conclusion is called a **hypothesis** and it is usual to devise tests for the hypothesis.
- A **proof by contradiction** assumes that the direct opposite of a statement is assumed to be true and then a logical argument is used to show that the assumption is false.
- A **counter example** is an example that shows that a statement is not true.
- The statement *if P then Q* or *P implies Q* can be written $P \Rightarrow Q$.
- For the statement *if P then Q*, the **converse** is *if Q then P*

  or the **converse** of *P implies Q* is *Q implies P*.

  In symbols, the **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- **Equivalence**

  In words, $P \Leftrightarrow Q$ means

  P if and only if Q or *P iff Q*.

  If $P \Rightarrow Q$ and $Q \Rightarrow P$, then we can write $P \Leftrightarrow Q$.
- The negation of P is *not P*, written $\neg P, P'$ or $\overline{P}$
- **Contrapositive**

  If we consider the statement $P \Rightarrow Q$, then the contrapositive statement is $\overline{Q} \Rightarrow \overline{P}$.

  In words, the contrapositive statement of

  If P then Q is *If not Q then not P*.

  If the original statement $P \Rightarrow Q$ is true, then the contrapositive statement $\overline{Q} \Rightarrow \overline{P}$ is also true.
- Euclidean Geometry is the geometry of two dimensions, involving lines, angles, congruence, similarity, triangles, circles and other plane shapes.
- The parallelogram rule for vectors

- *RTP* = required to prove
- **Quantifiers**

  ∀ for all
  ∃ there exists
- *QED = quod errat demonstrandum*

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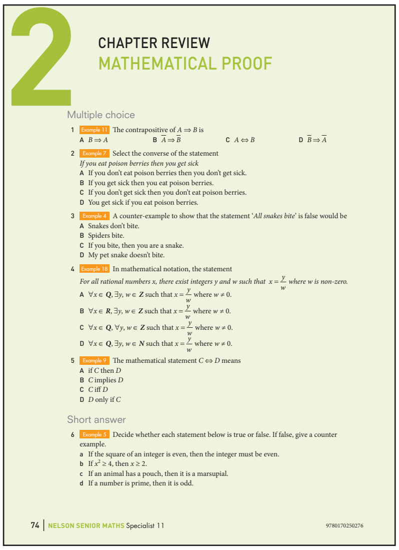

2
CHAPTER REVIEW
MATHEMATICAL PROOF

Multiple choice

1 Example 11 The contrapositive of $A \Rightarrow B$ is
A $B \Rightarrow A$ B $\overline{A} \Rightarrow \overline{B}$ C $A \Leftrightarrow B$ D $\overline{B} \Rightarrow \overline{A}$

2 Example 7 Select the converse of the statement
If you eat poison berries then you get sick
A If you don't eat poison berries then you don't get sick.
B If you get sick then you eat poison berries.
C If you don't get sick then you don't eat poison berries.
D You get sick if you eat poison berries.

3 Example 4 A counter-example to show that the statement '*All snakes bite*' is false would be
A Snakes don't bite.
B Spiders bite.
C If you bite, then you are a snake.
D My pet snake doesn't bite.

4 Example 18 In mathematical notation, the statement
For all rational numbers x, there exist integers y and w such that $x = \frac{y}{w}$ *where w is non-zero.*
A $\forall x \in \boldsymbol{Q}, \exists y, w \in \boldsymbol{Z}$ such that $x = \frac{y}{w}$ where $w \neq 0$.
B $\forall x \in \boldsymbol{R}, \exists y, w \in \boldsymbol{Z}$ such that $x = \frac{y}{w}$ where $w \neq 0$.
C $\forall x \in \boldsymbol{Q}, \forall y, w \in \boldsymbol{Z}$ such that $x = \frac{y}{w}$ where $w \neq 0$.
D $\forall x \in \boldsymbol{Q}, \exists y, w \in \boldsymbol{N}$ such that $x = \frac{y}{w}$ where $w \neq 0$.

5 Example 9 The mathematical statement $C \Leftrightarrow D$ means
A if C then D
B C implies D
C C iff D
D D only if C

Short answer

6 Example 5 Decide whether each statement below is true or false. If false, give a counter example.
a If the square of an integer is even, then the integer must be even.
b If $x^2 \geq 4$, then $x \geq 2$.
c If an animal has a pouch, then it is a marsupial.
d If a number is prime, then it is odd.

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Each chapter concludes with a detailed chapter summary and a comprehensive review of concepts.

At the end of every three chapters a comprehensive mixed revision set is provided.

ICONS IN THE TEXTBOOK

Example 1	Link from question to worked example
Prior learning	Prior learning
CAS	CAS exercise and example
Worksheets	Worksheet
Weblink	Weblink
Spreadsheet	Interactive spreadsheet
Practice quiz	Practice quiz

GREEK ALPHABET

GREEK ALPHABET

Α,	α	alpha	Ι,	ι	iota	Ρ,	ρ	rho
Β,	β	beta	Κ,	κ	kappa	Σ,	σ	sigma
Γ,	γ	gamma	Λ,	λ	lambda	Τ,	τ	tau
Δ,	δ	delta	Μ,	μ	mu	Υ,	υ	upsilon
Ε,	ε	epsilon	Ν,	ν	nu	Φ,	ϕ	phi
Ζ,	ζ	zeta	Ξ,	ξ	xi	Χ,	χ	chi
Η,	η	eta	Ο,	ο	omicron	Ψ,	ψ	psi
Θ,	θ	theta	Π,	π	pi	Ω,	ω	omega

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ABOUT THE AUTHORS

Lead author: Stephen Swift
Stephen Swift was the lead author of the Nelson Think Maths series. He has also written as a member of an author team on successful senior mathematics textbooks. Stephen started teaching Mathematics, Science and Computing in 1973 and has taught at all levels from Year 7 to 12 in several states, in urban and country schools until retiring in 2010 from the role of Mathematics Head of Department at Wellington Point State High School in Brisbane.

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Ross has worked as a classroom teacher and Mathematics Head of Department at a number of regional and metropolitan secondary schools. He has taught Mathematics at Years 8 to 12. Ross brings a wide range of experience from education and other sectors to the writing of mathematics student books.

Jim Green
Jim started his teaching career in Sydney over 30 years ago. He moved to Lismore in Northern NSW in 1985. He has been a Head of Mathematics for over 20 years. Jim has been involved in various mathematics associations for most of his teaching career. Jim has been a writer for Mathematics competitions, NSW School Certificate Examinations and HSC Trial Examination Papers. Jim has written articles for teaching journals and has presented at local, state and national professional development conferences. Jim is very interested in the use of technology to teach Mathematics.

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Sue Garner is assistant Head of VCE Mathematics at Ballarat Grammar in Victoria, where she has taught for twenty years, specialising in both Mathematical Methods (CAS) and Specialist Mathematics. Sue has been involved in the assessment of senior examinations in both of these subject areas. Sue has taught in secondary schools in Melbourne Ballarat and the United Kingdom. She has lectured at the University of Ballarat, La Trobe University and the University of Melbourne and has published extensively.

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Allason is the Head of Mathematics at Mount Scopus Memorial College. She has a Masters in Mathematical Modelling and Data Analysis and has been a co-author of several mathematics texts. Allason is currently the President of the MAV and is also a member of the AAMT council.

The **Digital Resources Team** includes Margaret Denham, Issam Kanaan, Jason Warwick and Roger Walter.

SYMBOLS AND ABBREVIATIONS

$\approx$	approximately equal to
$=$	equal to
$\equiv$	identically equal to, congruent
$\neq$	not equal to
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\pm$	plus or minus
$\sqrt{}$	square root
$*$	a general operation
Σ	summation
$\Rightarrow$	if … then, implies, so $P \Rightarrow Q$ means if P then Q or P implies Q
$\Leftrightarrow$, iff	if and only if, so $P \Leftrightarrow Q$ means P if and only if Q
$\overline{\mathbf{Q}}$, $\neg Q$, Q'	not Q
$\parallel\mid$, ~	similar
$\parallel$	parallel
$\perp$	perpendicular
RTP	required to prove
QED	quod erat demonstrandum, indicating a proof is complete
$\forall$	for all
$\exists$	there exists
LHS	left-hand side
RHS	right-hand side
$\boldsymbol{N}$, $\mathbb{N}$	the set of natural numbers, {1, 2, 3, 4, ….}
$\boldsymbol{Z}$, $\mathbb{Z}$ or $\boldsymbol{J}$	the set of integers, {… -3, -2, -1, 0, 1, 2, 3, …}
$\boldsymbol{Q}$, $\mathbb{Q}$	the set of rational numbers
$\boldsymbol{Q}'$ or $\overline{\mathbf{Q}}$	the set of irrational numbers
$\boldsymbol{Q}^+$	the set of positive rational numbers
$\boldsymbol{R}$, $\mathbb{R}$	the set of real numbers
$\boldsymbol{R}/0$ or $\boldsymbol{R}/\{0\}$	means the real numbers except 0
$\boldsymbol{C}$, $\leq$	the set of complex numbers
$\in$	is a member of
$\notin$	is not a member of
$\mathbf{p}$, $\underset{\sim}{p}$	the vector $\mathbf{p}$
$\lvert\mathbf{p}\rvert$, p	the magnitude (norm) of the vector $\mathbf{p}$
$\mathbf{AB}$	displacement vector from point A to point B
$\mathbf{OA}$, $\mathbf{a}$, $\overrightarrow{OA}$, $\overline{OA}$	the position vector of the point A
(r, θ), $\begin{pmatrix} r \\ \theta \end{pmatrix}$	polar form of the vector with magnitude r and direction θ
(x, y), $\begin{pmatrix} x \\ y \end{pmatrix}$	component (Cartesian) form of vector
$-\mathbf{a}$	additive inverse of the vector $\mathbf{a}$
$\hat{\mathbf{p}}$	unit vector in the direction of the vector $\mathbf{p}$
$\mathbf{i}$, $\mathbf{j}$	unit vectors in the x and y directions
$\Delta\mathbf{v}$	change in the vector $\mathbf{v}$
$\mathbf{a} \cdot \mathbf{b}$	the scalar product (dot product) of the vectors $\mathbf{a}$ and $\mathbf{b}$
080°	bearing of 80°
°	degree
$\angle$	angle
$\triangle$	triangle
SSS	side, side, side test of triangle congruency
SAS	side, angle, side test of triangle congruency
AAS	angle, angle, side test of triangle congruency
RHS	right angle, hypotenuse, side test of triangle congruency
!	factorial, so $5! = 5 \times 4 \times 3 \times 2 \times 1$
t_n	the nth term of a sequence

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^nP_r	the number of permutations of r things from n things
$\binom{n}{r}$, ^nC_r	the number of combinations of r objects from n objects
$Pa_{n,r}$	element r in row n of Pascal's triangle
$A = \{2, 4, 6, 8\}$	the set A consisting of the numbers 2, 4, 6 and 8
$A = \{x: 0 < x < 10$ and x an even number$\}$	the set A of elements x such that $0 < x < 10$ and x is an even number
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\cap$	intersection
$\cup$	union
$\{\}, \varnothing$	null set (empty set)
U	universal set
$\overline{A}$, A'	the complement of the set or event A
$\lvert A\rvert$	the cardinality of the set A; for finite sets, the number of elements
$n(A)$	for finite sets, the number of elements in the set A
$P(A)$	the probability of the event A
φ	the golden ratio
$P(n)$	a proposition associated with the number n
$\mathbf{A}$, $\underset{\sim}{A}$	the matrix $\mathbf{A}$

$\mathbf{A} = (a_{ij})_{2\times 3} = \begin{pmatrix} 3 & -2 & 5 \\ -4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 5 \\ -4 & 1 & 0 \end{pmatrix}$
the 2×3 matrix A with elements a_{ij}

$-\mathbf{A}$	the additive inverse of the matrix $\mathbf{A}$
$\mathbf{I}_n$	the $n \times n$ identity matrix
δ_{ij}	the Kronecker delta function
$\mathbf{A}^{-1}$	the multiplicative inverse of the (square) matrix $\mathbf{A}$
det $\mathbf{A}$, $\lvert\mathbf{A}\rvert$, A	the determinant of the matrix A
$\mathbf{T}: (x, y) \rightarrow (x', y')$	the transformation that changes the point (x, y) to (x', y')
$\mathbf{D}$	a dilation
λ_1, λ_2	the dilation factors in the x and y directions for a dilation
$\mathbf{R}_\alpha$	a rotation around the origin through and angle a
$\mathbf{M}$	a reflection across a line through the origin
$\mathbf{S} \circ \mathbf{T}$	the composition of the transformations $\mathbf{S}$ and $\mathbf{T}$
$\mathbf{T}^{-1}$	the inverse of the transformation $\mathbf{T}$
π (pi)	about 3.14159…
$\sin(\theta)$	the sine of the angle θ
$\cos(\theta)$	the cosine of the angle θ
$\tan(\theta)$	the tangent of the angle θ
$\cot(\theta)$	the cotangent of the angle θ
$\sec(\theta)$	the secant of the angle θ
$\operatorname{cosec}(\theta)$	the cosecant of the angle θ
$x \rightarrow \infty$	x becomes large and positive (x approaches infinity)
$x \rightarrow -\infty$	x becomes large and negative (x approaches negative infinity)
$\arcsin(a)$, $\sin^{-1}(a)$	the angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ such that $\sin(\theta) = a$
$\arccos(a)$, $\cos^{-1}(a)$	the angle $0 \leq \theta \leq \pi$ such that $\cos(\theta) = a$
$\arctan(a)$, $\tan^{-1}(a)$	the angle $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ such that $\tan(\theta) = a$
i	the imaginary number $(\sqrt{-1})$
$\mathrm{Re}(z)$	the real part of the complex number z
$\mathrm{Im}(z)$	the imaginary part of the complex number z
$\bar{z}$	the complex conjugate of z
$Arg(z)$	the argument of z
$\lvert z\rvert$, mod(z)	the modulus of the complex number z

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1

TERMINOLOGY

displacement
ground speed
head-to-tail
initial
linear combination
magnitude
norm
parallelogram rule
polar form
position vector
resultant
scalar
terminal point
triangle rule
unit vector
vector

VECTORS IN THE PLANE

BASIC VECTORS

Prior learning

REPRESENTING VECTORS IN THE PLANE BY DIRECTED LINE SEGMENTS

- examine examples of vectors including displacement and velocity (ACMSM010)
- define and use the magnitude and direction of a vector (ACMSM011)
- represent a scalar multiple of a vector (ACMSM012)
- use the triangle rule to find the sum and difference of two vectors (ACMSM013)

ALGEBRA OF VECTORS IN THE PLANE

- use ordered pair notation and column vector notation to represent a vector (ACMSM014)
- define and use unit vectors and the perpendicular unit vectors *i* and *j* (ACMSM015)
- express a vector in component form using the unit vectors *i* and *j* (ACMSM016)
- examine and use addition and subtraction of vectors in component form (ACMSM017)
- define and use multiplication by a scalar of a vector in component form (ACMSM018) AC

1.01 TWO-DIMENSIONAL VECTORS

Vectors are used to model important aspects of the physical world. They are used to predict what will happen under different circumstances and to determine the actions that are required to achieve desired outcomes. Physics relies heavily on geometric vectors to model a wide range of phenomena, including forces, velocity, electromagnetic fields and applications, and so on. Engineers use vectors to work out how to construct cars, bridges, buildings and other objects in a safe and durable way.

Many quantities, such as mass, distance, volume and speed, are specified by a single magnitude. Quantities of this type are called **scalar** quantities. Other quantities, such as displacement, velocity and acceleration, need to have both magnitude and direction to complete their specification. These are known as **vector** quantities.

IMPORTANT

A **geometric vector** (or just **vector**) is a quantity with both **magnitude** (size) and **direction**. Both magnitude and direction must be stated to specify a geometric vector.

A vector is shown using a lower-case bold letter such as **a**. Vectors are sometimes shown by putting an arrow or wavy line over the top of the letter ($\vec{a}$ or $\tilde{a}$). In handwriting it can be shown by putting a wavy line under the lower-case letter, as in $\underset{\sim}{a}$.

The **magnitude** of a vector **p** is written as $|\mathbf{p}|$ or p. The magnitude is also called the **norm**.

A vector is shown as a **directed line segment**. The length of the segment represents the magnitude of the vector and an arrow is used to show its direction.

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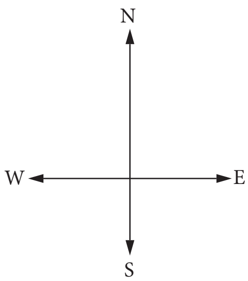
Example 1

Use directed line segments to represent the following displacements on the same set of axes.

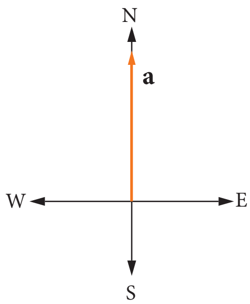
a 4 m north **b** 8 m east **c** 4 m south **d** 12 m north-west

Solution

Draw north–south and east–west lines to act as a reference when drawing the vectors.
We need to choose a suitable scale.
A scale of 1 cm = 2 m will do.



a Using the scale, **a** will be 2 cm on the diagram. Draw **a** as a 2 cm arrow in the direction of north.

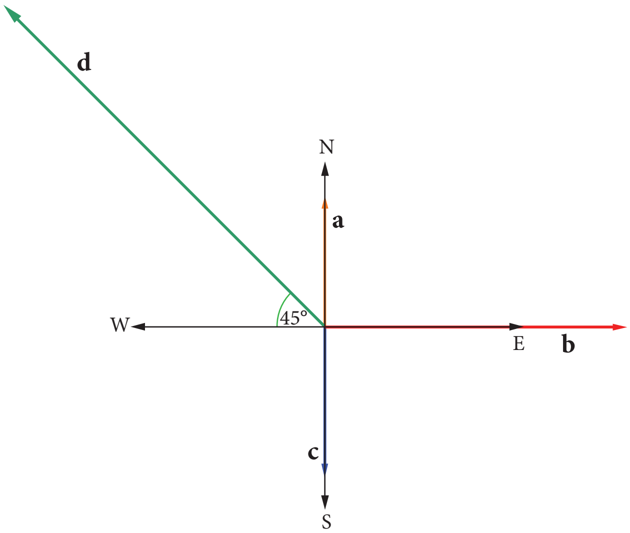


Use the same procedure for the other vectors.

b This is a 4 cm arrow in the direction of east.

c This is a 2 cm arrow in the direction of south.

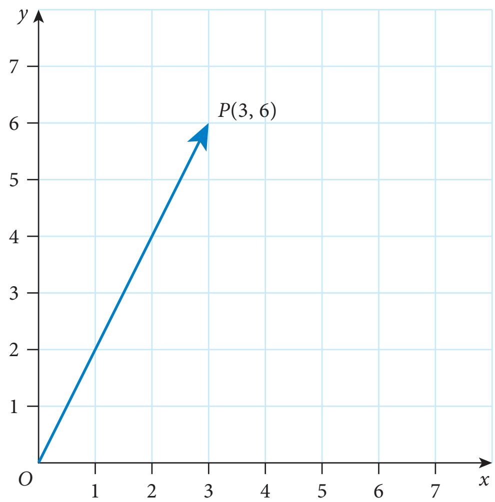
d This is a 6 cm arrow in the direction of north-west.



It is more efficient and easier to show the vectors with a common origin.

WS
Representing vectors

A vector that refers to a fixed point or origin is called a **position vector**. The diagram on the right shows the position vector of $P(3, 6)$ with respect to the origin $O(0, 0)$ of the Cartesian plane. The position vector of P is written as **OP** or $\overrightarrow{OP}$. O is the **initial point** and P is the **terminal point** of the vector.

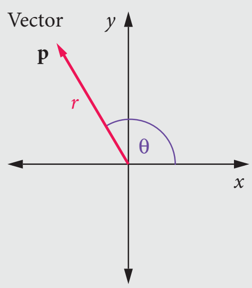


Vectors may be represented numerically by specifying the magnitude and direction.

IMPORTANT

Two-dimensional vectors are represented in polar form by an ordered pair or column vector. The first number is the magnitude of the vector and the second number gives the direction of the vector as an angle *anticlockwise* from the positive direction of the x-axis.

The two-dimensional vector **p** shown in the diagram can be written as the ordered pair (r, θ) or the column vector $\begin{bmatrix} r \\ \theta \end{bmatrix}$.



Example 2

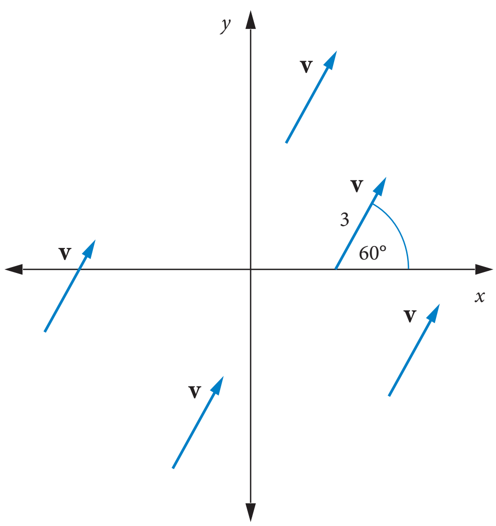
A vector **v** of magnitude 3 is at 60° to the x-axis. Show some representations of the vector.

Solution

It doesn't matter where **v** is placed. As in the diagram, the vector must have an angle of 60° to the x-axis and length 3. It has the polar representation

$\mathbf{v} = (3, 60°)$ or $\mathbf{v} = \begin{bmatrix} 3 \\ 60° \end{bmatrix}$.

Notice that the vector still has the same magnitude and direction wherever it is drawn in the plane. It is the *same vector*, although it may be drawn in different positions.



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One of the most important vector quantities in scientific applications is **displacement**. The displacement from one position to another is the *change of position*. When you look at displacement, it doesn't matter how the change has occurred. It is a geometric vector because a change of position has both magnitude and direction.

WS

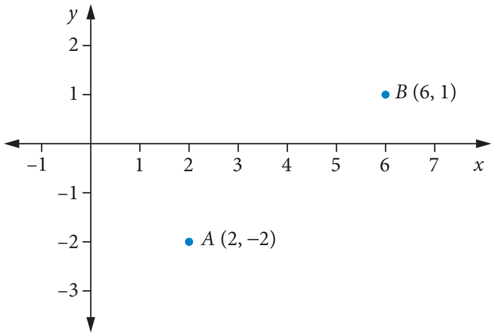
Vectors in polar form

Example 3

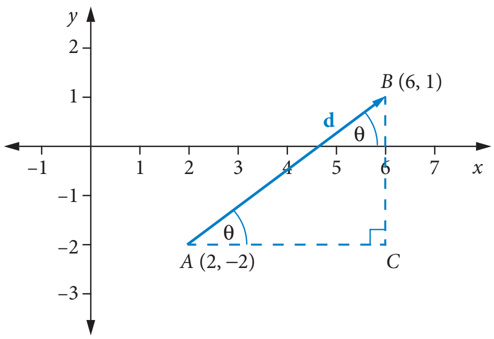
Find the displacement vector, **d**, from $A(2, -2)$ to $B(6, 1)$.

Solution

Draw a set of axes and plot $A(2, -2)$ and $B(6, 1)$.



The displacement from A to B can be shown as an arrow from A to B.
Label the arrow **d**.
You need to calculate the distance from A to B to find the magnitude of **d**, and the angle θ to find the direction.



Use the distance formula.
Substitute in the known values.
Evaluate.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6-2)^2 + (1 - -2)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Calculate θ using $\triangle ABC$ and the tan ratio.

$$\tan(\theta) = \frac{BC}{AC}$$

Substitute in the known values.

$$= \frac{3}{4}$$

Evaluate.

$$= 0.75$$

Use $\tan^{-1}$ to calculate θ.

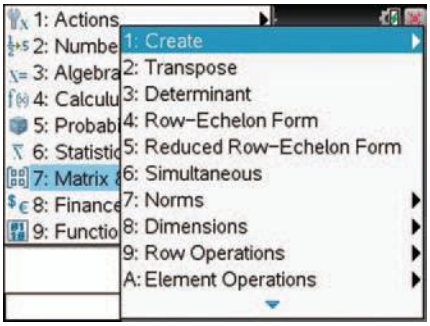
$$\tan^{-1}(0.75) = 36.8698\ldots°$$

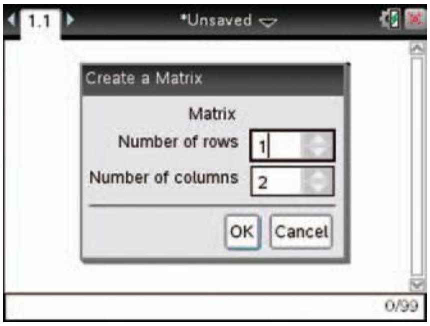
Round the angle and write **d** in polar form.

$$\mathbf{d} \approx (5, 36.9°)$$

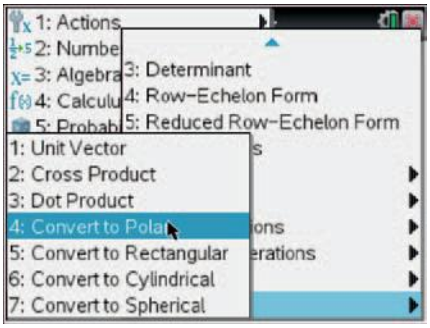
TI-Nspire CAS

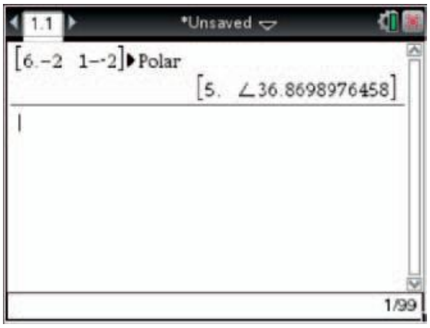
You can do this on your CAS calculator by entering the coordinates as a subtraction *with the coordinates of B first*, and asking for the result in polar form.
Make sure that your calculator angle is set to degrees in your document by pressing doc▾, 7:Settings & Status and 2: Document settings.
Then use menu, 7:Matrix & Vector, 1:Create and 1: Matrix. Choose 1 row and 2 columns for coordinates or 2 rows and 1 column for a column vector.





Subtract the x-coordinates in the first box, shown as ⬚, and the y-coordinates in the second box.
Enter one of the coordinates with a decimal point to make the answer approximate. In the screen shown, the 6 was entered as a decimal. Make sure that you use the (–) button for the negative sign of –2 and the – button for subtraction.
Press menu, 7:Matrix & Vector, C: Vector and 4: Convert to Polar to change the answer to polar coordinates.

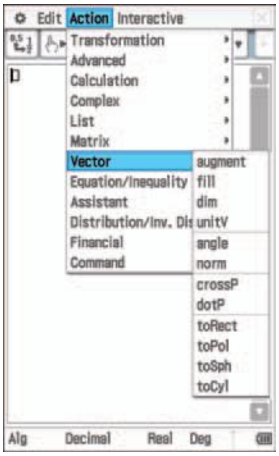




Round the angle and write **d** in polar form. **d** ≈ (5, 36.9°)

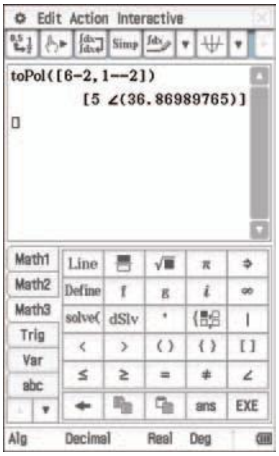
ClassPad

Tap $\overset{\text{Main}}{\sqrt{\alpha}}$ and clear the screen by tapping **Edit** and then **Clear All** and **OK**.
Make sure the screen is set to degrees (**Deg**) and **Decimal** at the bottom.
Tap the **Action** menu at the top, then tap **Vector** and then **toPol**. (Convert to Polar.)

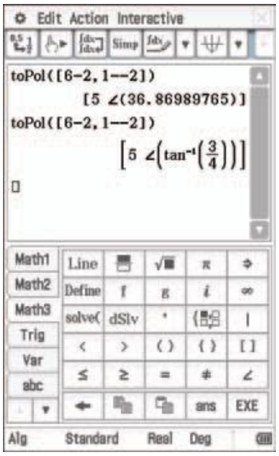


Vectors are entered using square brackets (press `Keyboard` then tap `Math3` to see a button for [ ]) with a comma `,` between the elements.

Enter the vector of the difference of points, subtracting A from B ($B - A$), first the x and then the y coordinates, separated by a comma. Press `EXE`.



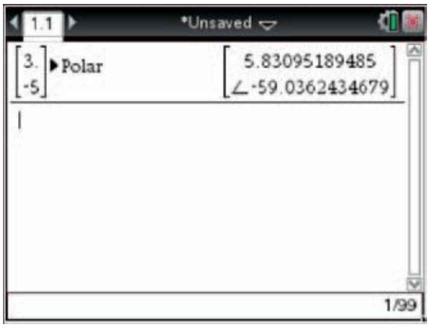
An exact answer can also be found by setting the calculator to **Standard**.



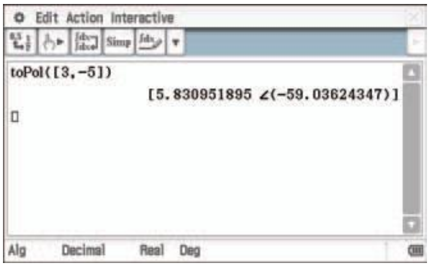
Round the angle and write **d** in polar form. **d** ≈ (5, 36.9°)

You can use your CAS calculator to find the polar form of a position vector by entering the vector as in Example **3** and asking for the polar form. For example, the position vector for the point (3, –5) would be done as follows.

TI-Nspire CAS



ClassPad



WS Displacement vectors

The position vector is approximately (5.8, –59.0°), or (5.8, 301.0°) with a positive angle.

Since a displacement vector may show a change from one position to another position, it is also symbolised by writing the first and second positions in order. The displacement from A to B is shown typographically as **AB**, or $\overline{AB}$ or $\overrightarrow{AB}$, and in handwriting as $\underset{\sim}{AB}$, or $\overrightarrow{AB}$.

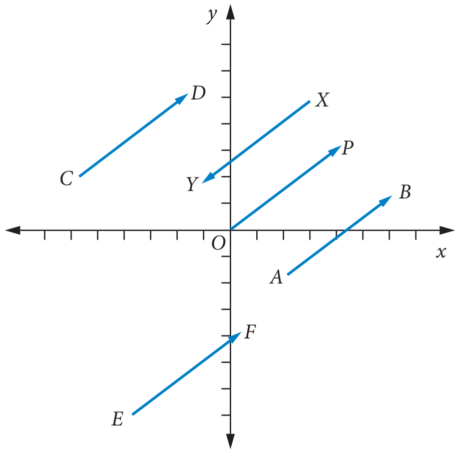
Even though a displacement from A to B can be written using A and B, it is still the same as another vector of the same magnitude and direction in another position.

In the diagram on the right, **AB**, **CD**, **EF** and **OP** are all equal vectors. We could write

$$\mathbf{AB} = \mathbf{CD} = \mathbf{EF} = \mathbf{OP}$$

Because **XY** has the same magnitude as **OP**, but is in the opposite direction, we write

$$\mathbf{XY} = -\mathbf{OP} = -\mathbf{AB} = -\mathbf{CD} = -\mathbf{EF}$$



EXERCISE 1.01 Two-dimensional vectors

Concepts and techniques

1 Which of the following is not a vector quantity?
A displacement B speed C acceleration D force E velocity

2 What is the magnitude of the position vector of $P(5, 0)$?
A 0 B 1 C 3 D 2.5 E 5

3 What is the direction of the position vector of $Q(0, -3)$?

A $-180°$ **B** $0°$ **C** $90°$ **D** $180°$ **E** $270°$

4 A vessel is travelling south-west of a fixed point at $5\ \text{km h}^{-1}$. Which of the following polar forms best describes the velocity of the vessel?

A $(5, 45°)$ **B** $(5, 225°)$ **C** $(5\ \text{km h}^{-1}, 270°)$ **D** $(5, 135°)$ **E** $(5, 215°)$

5 Example 1 Use directed line segments to represent the following displacements on the same set of axes.

a 5 km west **b** 7 km north **c** 10 km south-east **d** 8 m south-west

6 Sketch the position vectors of each of the following points on the same set of axes.

a $(4, 7)$ **b** $(-3, -5)$ **c** $(4, -4)$ **d** $(-2, 6)$

7 Example 2 A vector **h** has a magnitude of 6 and is in the direction 30° south of west. Show at least five different representations of **h**.

8 Calculate the norm of the position vector for each of the following points in exact form.

a $(3, 4)$ **b** $(-5, 12)$ **c** $(24, -7)$ **d** $(-1, -1)$
e $(-2, 7)$ **f** $(3, -6)$ **g** $(a, 4)$ **h** $(6, b)$

9 Calculate the direction (in degrees correct to 1 decimal place) of the position vectors of the points in question **8** for parts **a** to **f**.

10 Sketch each of the following vectors.

a $(4, 120°)$ **b** $(5, 210°)$ **c** $(3, -40°)$ **d** $(5, 300°)$ **e** $(4, 45°)$

11 Express each of the position vectors below in polar form.

a $(3, -4)$ **b** $(-5, 5)$ **c** $(-7, -10)$ **d** $(4, 12)$ **e** $(-10, 16)$

12 Find the polar form of the position vectors for each of the following points, correct to one decimal place.

a $(8, 15)$ **b** $(12, 5)$ **c** $(-9, -12)$ **d** $(-7, -12)$ **e** $(8, -8)$

13 Example 3 Find the displacement vector (in polar form) for each of the following movements.

a $A(3, 4)$ to $B(5, 11)$ **b** $R(-2, 1)$ to $B(-7, -2)$ **c** $M(-3, -8)$ to $B(-9, 1)$

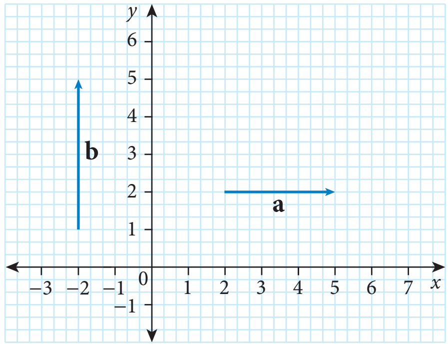
Reasoning and communication

14 A vector **u** has a magnitude of 8 and is in the direction 60° north of west.

a Describe the vector −**u** in words.
b Sketch **u** and −**u** on the same set of axes so that their initial points coincide.
c Sketch **u** and −**u** on a different set of axes so that the initial point of **u** coincides with the terminal point of −**u**.
d If **u** is a displacement vector, describe the displacement vector you get by doing **u** and then −**u**.

1.02 ADDITION OF VECTORS

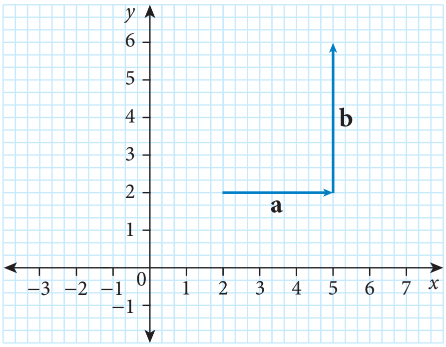
You saw in the previous section that a vector can be represented in any position on the plane. This means that we can slide or translate a vector without changing its value because its magnitude and direction have been unaltered. This is useful when we want to find the outcome of adding vectors.

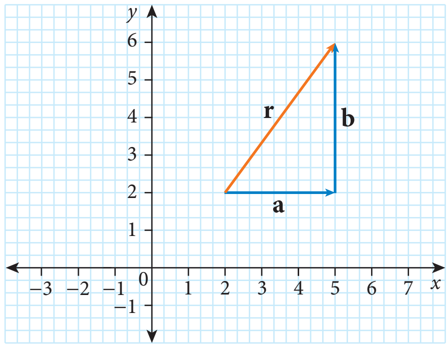


Consider the situation where there are two displacements, **a** and **b**, as shown on the right.

To find the displacement that is the same as **a** followed by **b**, slide **b** across so that its tail is on the head (tip) of **a**.

Now, the overall result of adding displacement **a** to **b** is the vector from the head of **a** to the tail of **b**. This is shown on the diagram as **r**.





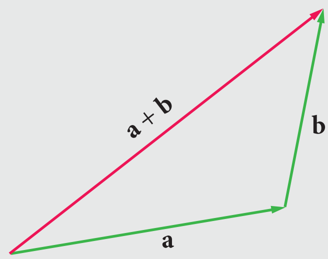
IMPORTANT

WS

Triangle addition of vectors

Triangle rule for vector addition

If two vectors **a** and **b** are placed head-to-tail to form two sides of a triangle, as shown in this diagram, then **a** + **b** is defined by the triangle of vectors. The sum **a** + **b** is also called the **resultant** of **a** and **b**.



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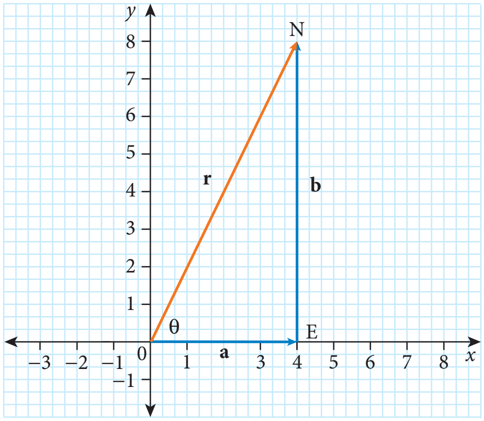
This is also called the **head-to-tail** (or **tip-to-tail**) method of vector addition.

Example 4

A boat travels due east for 4 km and then due north for 8 km. Calculate the resultant displacement of the boat.

Solution

Sketch a diagram showing the displacements **a** and **b**.
Mark in the resultant displacement, **r**.
Mark in θ as the angle made between **r** and the positive direction of the *x*-axis.



Calculate the magnitude of **r**.

$$|\mathbf{r}| = \sqrt{4^2 + 8^2}$$
$$= 4\sqrt{5}$$

Calculate the direction of **r**.

$$\theta = \tan^{-1}\left(\frac{8}{4}\right)$$
$$\approx 63.4°$$

Change to a bearing by finding the angle clockwise from north (*y*-axis).

Bearing ≈ 026.6°

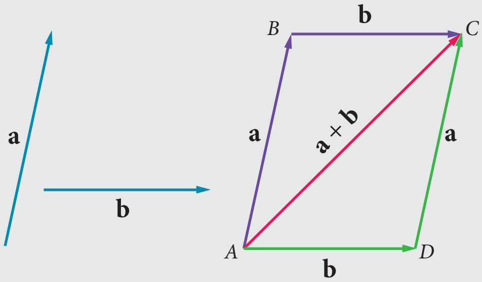
Convert the direction of **r** to a bearing and state the result with magnitude and direction.

The resultant displacement is $4\sqrt{5}$ km at a bearing of about 026.6° (or N 26.6° E).

IMPORTANT

Parallelogram rule for vector addition

Consider vectors **a** and **b** as shown on the right. If we form the *parallelogram ABCD* using **a** and **b** as the sides, the diagonal *AC* is the resultant of **a** + **b**.



WS Parallelogram rule for vector addition

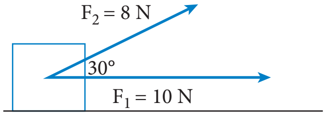
The parallelogram rule demonstrates that vector addition is commutative, as **a** + **b** = **b** + **a**.

Basic trigonometry is often used in the calculation of resultant vectors.

Example 5

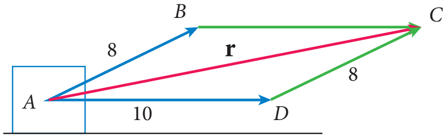
Two forces given in newtons (N) are applied to a block as shown in the diagram on the right.

Calculate the resultant force on the block.



Solution

Complete the parallelogram of vectors and label the vertices A, B, C and D.
Draw in the diagonal of the parallelogram and label it **r** (the resultant vector).



Use the properties of a parallelogram to determine the internal angles.

$\angle BAD = 30°$ (Given)
$\angle BCD = \angle BAD = 30°$ (Opposite angles of a parallelogram)
$\angle ADC = 150°$ (Co-interior angles on parallel lines)
$\angle ADC = \angle ABD = 150°$ (Opposite angles of a parallelogram)

Use the cosine rule in $\angle ADC$ to find the magnitude of $\mathbf{r}$ ($|\mathbf{r}| = r$).

$$AC^2 = AD^2 + DC^2 - 2\ AD \times DC \cos(\angle ADC)$$

Substitute in the known values and evaluate.

$$\begin{aligned} r^2 &= 10^2 + 8^2 - 2 \times 10 \times 8 \cos(150°) \\ &= 302.56\ldots \end{aligned}$$

Take the square root of both sides.

$$r \approx 17.4$$

Use the sine rule in $\angle ADC$ to find $\angle CAD$.

$$\frac{AC}{\sin(\angle ADC)} = \frac{CD}{\sin(\angle CAD)}$$

Substitute in the known values.

$$\frac{17.4}{\sin(150°)} = \frac{8}{\sin(\angle CAD)}$$

Rearrange and evaluate.

$$\begin{aligned} \sin(\angle CAD) &= \frac{8 \times \sin(150°)}{17.4} \\ &= 0.2299\ldots \end{aligned}$$

Calculate $\angle CAD$.

$$\begin{aligned} \angle CAD &= \sin^{-1}(0.2299\ldots) \\ &\approx 13.3° \end{aligned}$$

State the result.

The resultant force on the block is about 17.4 N acting at 13.3° above the horizontal.

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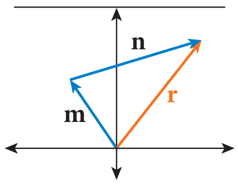
Example 6

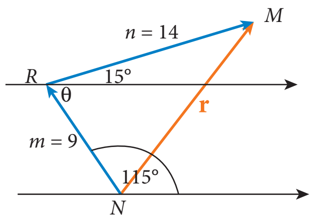
Vectors **m** and **n** are (9, 115°) and (14, 15°). Find the resultant **r** = **m** + **n**.

Solution

Position **m** with its tail at the origin.
Position **n** with its tail on the head of **m**.
Draw in the resultant vector, **r**.



Find the angle between **m** and **n**.
$\theta = 65°$ (Co-interior with 115°)
$\angle NRM = 65° + 15° = 80°$



Calculate the magnitude of **r** using the cosine rule.

$$r^2 = m^2 + n^2 - 2nm\cos(R)$$
$$= 14^2 + 9^2 - 2 \times 14 \times 9 \times \cos(80°)$$
$$= 233.2406\ldots$$

Find the square root.

$$r = \sqrt{233.2406\ldots}$$
$$= 15.2722...$$

Find $\angle RNM$ using the sine rule.

$$\frac{\sin(N)}{n} = \frac{\sin(R)}{r}$$

Use the exact value on your calculator.

$$\frac{\sin(N)}{14} = \frac{\sin(80°)}{15.2722\ldots}$$

Rearrange.

$$\sin(N) = \frac{14\sin(80°)}{15.2722\ldots}$$

Calculate.

$$= 0.9027\ldots$$

Calculate $\angle RNM$.

$$N = \sin^{-1}(0.9027\ldots)$$
$$= 64.5246\ldots$$

Find the direction of **r**.

Direction of $\boldsymbol{r} = 115° - 64.5246\ldots°$
$\approx 50.5°$

State the result.

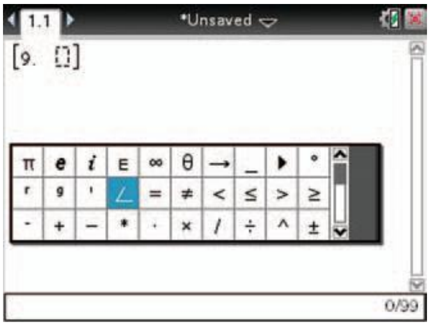
r ≈ (15.3, 50.5°)

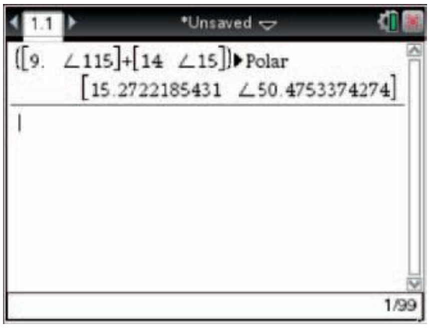
TI-Nspire CAS

Use menu, 7:Matrix & Vector, 1:Create and 1: Matrix for each vector.

The angle symbol is necessary. It is in the symbols menu. You get this by pressing ctrl and 📖.

Make sure that you ask for the answer in polar form using menu, 7:Matrix & Vector, C: Vector and 4: Convert to Polar.





Round and state the result.

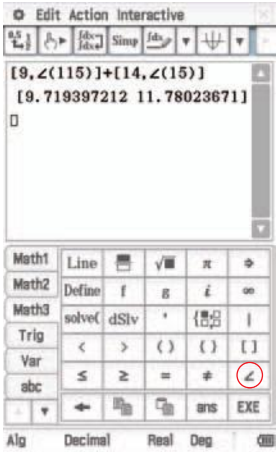
r ≈ (15.3, 50.5°)

ClassPad

Tap Main $\sqrt{\alpha}$ and clear the screen by tapping **Edit** and then **Clear All** and **OK**.

Make sure the screen is set to degrees (**Deg**) and **Decimal** at the bottom.

Press Keyboard then tap Math3 to see a button for [ ]. Vectors are entered using these square brackets with the magnitude first and then the direction, separated by ,.



The angle symbol is necessary. This is visible in the lower right part of the screen.

After tapping the symbol, enter the angle and close the brackets with).

Press EXE to obtain the result.

Round and state the result.

r ≈ (15.3, 50.5°)

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EXERCISE 1.02 Addition of vectors

Concepts and techniques

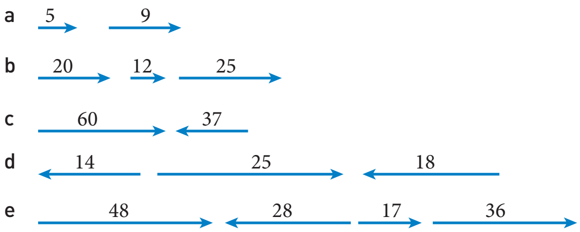
1 A boat travels due east and then a different distance due north. Which of the following best describes the direction of its resultant displacement?

A north B east C north-east D north of east E south of west

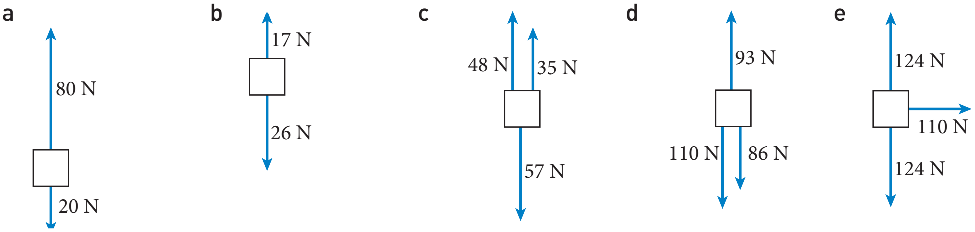
2 A mass is suspended vertically by a string. The force due to gravity acting on the mass is 170 N. The force exerted by the string is 170 N vertically upwards. The resultant force exerted on the mass is:

A 170 N vertically up B 0 C 340 N vertically down
D 170 N vertically down E 340 N vertically up

3 Calculate the resultant vector in each of the following situations.



4 In each of the following situations a mass is shown with a number of forces acting on it. Calculate the resultant force in each case.



5 Example 6 Given **w** = (6, 220°), **x** = (10, 60°), **y** = (8, 100°) and **z** = (9, –50°), work out the following resultant vectors, correct to one decimal place.

a **w** + **y** b **y** + **z** c **x** + **z** d **w** + **z** e **y** + **x**

6 Given **a** = (7, 160°), **b** = (15, 40°), **c** = (18, –120°) and **d** = (11, –50°), work out the following resultant vectors, correct to one decimal place.

a **a** + **b** b **c** + **b** c **d** + **a** d **a** + **c** e **c** + **d**

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Reasoning and communication

7 Example 4 A ship leaves harbour and sails 12 km north to port A. From here, the ship travels 24 km due east to port B before sailing 11 km south-west to port C. Calculate the ship's resultant displacement.

8 A large rock partly buried in the ground is lifted using crowbars on opposite sides. One crowbar is exerting a vertical force of 400 N while the other is exerting a force of 300 N at an angle of 35° to the vertical. What is the total force acting and what is its direction?

9 Example 5 Sam walks from her home along a footpath to an intersection 1 km away in a north-west direction. She then walks 800 m along a road in the direction 60° north of east. Calculate Sam's resultant displacement.

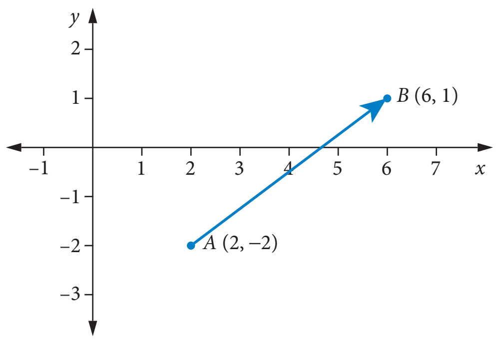
10 A boat travels 5 km NE and then 7 km at a bearing of 120°. Find its distance and direction from its starting point.

11 A bush walker walks from a base station to a viewing platform 24 km away in the direction north-east. Then the bush walker walks a further 16 km due east to a hut. Calculate the resultant displacement.

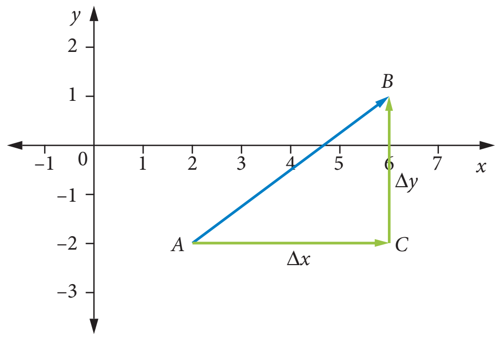
12 A bird flies from her nest looking for food. She flies at a velocity of 20 ms^{-1} in the direction of 45° south of east with respect to the air. There is a wind blowing at a constant velocity of 12 ms^{-1} in the direction 40° south of west. Calculate the resultant velocity of the bird.

1.03 COMPONENT AND POLAR FORMS OF VECTORS

Consider the displacement **AB** from $A(2, -2)$ to $B(6, 1)$ as shown on the right.



This displacement could have occurred in many different ways. It could have occurred by a movement from A to C and then from C to B, as shown in the diagram on the right. This would be an x change of $6 - 2 = 4$ and a y change of $1 - (-2) = 3$.



You can show the displacement **AB** as the x change and the y change. This gives the ordered pair (4, 3). This can be done for any vector and is called the **component form** of the vector. So, 4 is the ***x*-component** and 3 is the ***y*-component** of the vector **AB**.

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IMPORTANT

A two-dimensional geometric vector is represented in **component form** by an ordered pair, column matrix or row matrix. The first number is the x-component and the second is the y-component of the vector. They form a right-angled triangle with the vector as the hypotenuse.

Thus a two-dimensional vector **p** may be written as the row vector (x, y) or the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

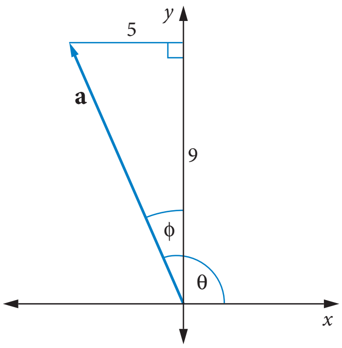
Being able to express a vector in both component and polar form is a useful skill. Trigonometry is used to convert vectors between these two representations.

Example 7

Change the vector $\mathbf{a} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$ to polar form.

Solution

Draw **a** on the Cartesian plane using the x and y components. Draw a right-angled triangle to show the relationship between the two forms of the vector. To obtain the correct directions, care must be taken to allow for the signs of the components.



We need to calculate the magnitude a and direction θ of **a**.

Use Pythagoras' theorem to find a.

$$a^2 = 5^2 + 9^2$$
$$= 106$$
$$a \approx 10.30$$

Use the diagram to find the tan ratio.

$$\tan(\phi) = \frac{5}{9}$$
$$= 0.555\ldots$$

Use $\tan^{-1}$ to find ϕ.

$$\phi = \tan^{-1}(0.555\ldots)$$
$$\approx 29.1°$$

Use the diagram to find θ.

$$\theta \approx 90° + 29.1°$$
$$= 119.1°$$

Write **a** in polar form.

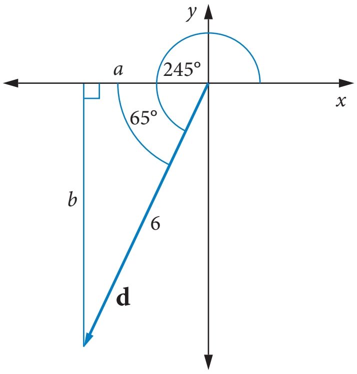
$$\mathbf{a} \approx (10.30, 119.1°)$$

Example 8

Write the vector **d**, of magnitude 6 in the direction 245°, in component form in the x and y directions.

Solution

Use the information to draw the vector. The vector is in the third quadrant.



Calculate a using the cos ratio.

$$a = 6\cos(65°) \approx 2.54$$

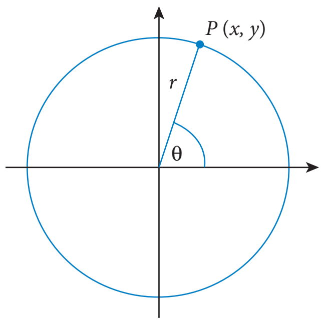
Calculate b using the sin ratio.

$$b = 6\sin(65°) \approx 5.44$$

Write **d**, taking signs into account.

$$\mathbf{d} \approx (-2.54, -5.44)$$

For any angle θ, sin (θ), cos (θ) and tan (θ) are defined in terms of the coordinates of the point $P(x, y)$ at an angle θ from the x-axis on a circle of radius r, as shown in the diagram on the right.



$\sin(\theta) = \frac{y}{r}$, $\cos(\theta) = \frac{x}{r}$ and $\tan(\theta) = \frac{y}{x}$

Also $r = \sqrt{x^2 + y^2}$.

By rearranging these definitions, you get the following rules for the conversion of the polar and component forms of vectors.

IMPORTANT

For a vector **v** with polar form (r, θ) and component form (x, y):

$x = r\cos(\theta)$ $\quad$ $y = r\sin(\theta)$

$\tan(\theta) = \frac{y}{x}$ $\quad$ $r^2 = x^2 + y^2$

and $|\mathbf{v}| = \sqrt{x^2 + y^2}$.

When you are making vector calculations with angles, you should set the angle measure on your calculator to degrees.

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TECHNOLOGY

You can use a CAS calculator to change between the component and polar forms of a vector, using similar procedures to those shown in the previous sections.

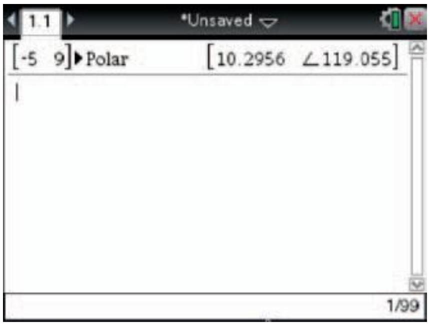
TI-Nspire CAS

Change $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$ to polar form.

In Document settings, check that the angle setting is in degrees and change the Calculation Mode to Approximate.

You can use menu, 7: Matrix & Vector and 1: Create to enter the vector or use the keyboard short cut
ctrl ((−) 5 , 9

Then move the cursor to the right and change to polar form using menu, 7: Matrix & Vector, C: Vector and 4: Convert to Polar.

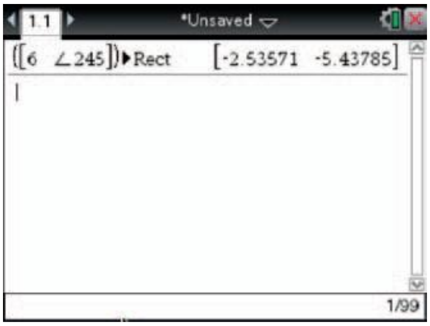


$(-5, 9) \approx (10.2956, 119.055°)$

To change (6, 245°) to component form, enter the vector as before, but use the angle symbol in front of the angle. You can get this from the catalogue or by using ctrl menu and 7:Symbols.

Move the cursor to the right and use menu, 7: Matrix & Vector, C:Vector and 5:Convert to Rectangular.

You can just press enter to get rectangular coordinates because this is the default vector format.



$(6, 245°) \approx (-2.53571, -5.43785)$

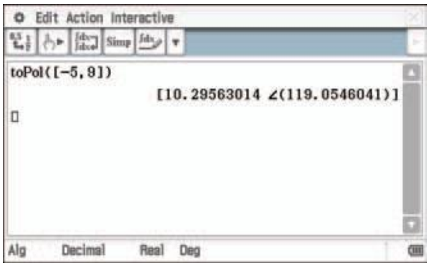
ClassPad

Change $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$ to polar form.

Use **toPol**. First make sure that the calculator is set to **Decimal** and **Deg**.

Use the Main $\sqrt{\alpha}$ application. Tap **Action**, then **Vector**, and then **toPol**.

Enter the brackets using the [] key (located on the screen after you press Keyboard and tap Math3) then enter the vector, separating the numbers with a comma and tap **OK**.

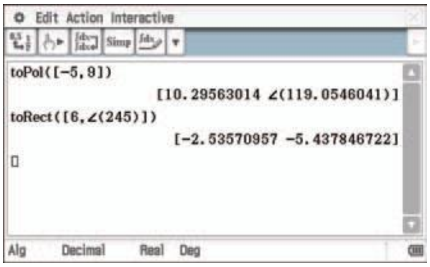


$(-5, 9) \approx (10.2956, 119.055°)$

To change (6, 245°) to component form, use **toRect**. Tap **Action**, then **Vector**, and then **toRect**.

Enter the vector as before, but use the angle symbol in front of the angle and close the open (with) after entering the value. Vectors are entered using square brackets with the magnitude first and then the direction, separated by a comma , .

Tap **OK** to perform the calculation.



$(6, 245°) \approx (-2.53571, -5.43785)$

EXERCISE 1.03 Component and polar forms of vectors

Concepts and techniques

WS Component and polar forms of vectors

1 Calculate the direction of each of the following vectors, correct to 1 decimal place.

a (2, 7) b (1, 9) c (3, 6) d (5, 8)
e (−2, 9) f (6, −4) g (−8, −7) h (12, −5)

2 Calculate the magnitude of each of the following vectors, correct to 2 decimal places if necessary.

a (−3, 4) b (12, −5) c (−6, −2) d (11, 4)
e (7, −9) f (13, 8) g (−7, 12) h (13, −2)

3 Example 7 Change each of the following vectors to polar form, correct to one decimal place.

a (3, −4) b $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$ c (−7, −10) d $\begin{bmatrix} 4 \\ 12 \end{bmatrix}$

e (−10, 16) f $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$ g (12, 5) h $\begin{bmatrix} -9 \\ -12 \end{bmatrix}$

i (−7, −12) i $\begin{bmatrix} 8 \\ -8 \end{bmatrix}$

4 CAS Change each of the following vectors to polar form, correct to 2 decimal places.

a (9, 12) b (11, 17) c (−9, 15) d (18, −13)
e (−20, −17) f (16.4, 8.7) g (6.37, −12.8) h (−3.91, 11.62)

5 Example 8 Change each of the following vectors to component form, correct to 2 decimal places.

a (5, 30°) b (10, 300°) c (24, 90°) d (16, 135°)
e (28, −120°) f (70, 270°) g (35, 0°) h (22, 180°)

6 CAS Change each of the following vectors to component form, correct to 2 decimal places.

a (6, 60°) b (12, 120°) c (17, 52°) d (23, 166°)
e (14, −132°) f (15, 287°) g (24, 221°) h (43, −17°)
i (4, 200°) i (8, −60°)

7 For each of the following pairs of vectors, which has the larger magnitude?

a (1, 5) or (2, 2) b (4, 7) or (3, 9) c (−2, 6) or (3, 8)
d (7, 7) or (4, −8) e (−3, −10) or (−6, 8) f (10, 4) or (6, −11)

8 If A(5, −6) and B(−2, 2) are the head and tail respectively of **AB**, express **AB** in polar form.

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1.04 MULTIPLICATION BY SCALARS

If you wanted to make a new vector twice as big as an old one, the direction wouldn't change. You just double the magnitude. In component form, this doubles both of the components.

IMPORTANT

Multiplication by a scalar

For a vector $\mathbf{a} = (x, y)$ and a constant c, the **scalar multiple** $c\mathbf{a}$ is given by $c\mathbf{a} = (cx, cy)$.

If the vector is in polar form, i.e. $\mathbf{a} = (r, \theta)$, then for $c \geq 0$, $c\mathbf{a} = (cr, \theta)$.

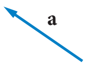
For $c < 0$, the vector is in the opposite direction, so $c\mathbf{a} = (-cr, 180° + \theta)$.

The negative sign in the last part just makes the magnitude positive.

The product $(-1)\mathbf{a}$ is normally written as $-\mathbf{a}$.

Example 9

Vector **a** is shown on the right.



Draw vectors representing:

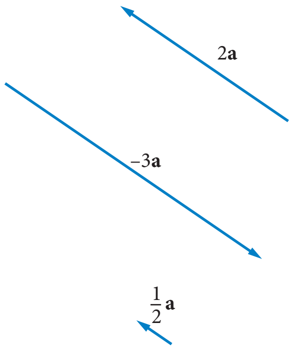
a $2\mathbf{a}$ b $-3\mathbf{a}$ c $\frac{1}{2}\mathbf{a}$

Solution

a $2\mathbf{a}$ will be in the same direction but will be twice as long as **a**.

b $-3\mathbf{a}$ will be in the opposite direction and will be three times as long as **a**.

c $\frac{1}{2}\mathbf{a}$ will be in the same direction but will be half as long as **a**.



Example 10

a Given $\mathbf{f} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, find $-3\mathbf{f}$.

b Given $\mathbf{g} = (5, 223°)$, find $-4\mathbf{g}$.

Solution

a Multiply each component of **f** by −3.

$$-3\mathbf{f} = \begin{bmatrix} -3 \times (-6) \\ -3 \times 3 \end{bmatrix} = \begin{bmatrix} 18 \\ -9 \end{bmatrix}$$

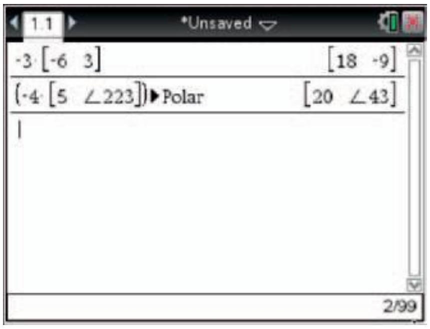
b Multiply the norm by 4 and add 180° to the angle.

$$-4\mathbf{g} = (4 \times 5, 223° + 180°)$$

Convert the angle to a value in the domain $0 \leq \theta < 360°$.

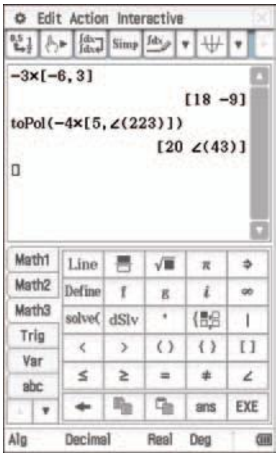
$$= (20, 403°)$$
$$= (20, 43°)$$

You can do scalar multiplication with CAS calculators.



TI-Nspire CAS

Make sure your Document Settings is in degrees. Enter the vector as you did before, but put the multiplication in front and make sure the answer is given in polar form for part **b**.



ClassPad

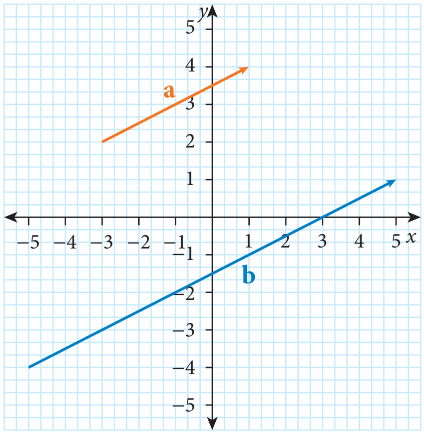
When in polar form, you must use **toPol** so the answer remains in polar form. The calculator must be set to degrees.

In Polar form, the magnitude is given as a positive number and the angle is usually between 0° and 360°.

EXERCISE 1.04 Multiplication by scalars

Concepts and techniques

Scalar multiplication



1 The diagram on the right shows two vectors **a** and **b**.
Which of the following best describes the relationship between the two vectors?

A $\mathbf{a} = 2.5\mathbf{b}$ B $\mathbf{a} = \frac{2}{5}\mathbf{b}$ C $\mathbf{a} = \frac{1}{2}\mathbf{b}$

D $\mathbf{b} = -2\mathbf{a}$ E $\mathbf{b} = 2\mathbf{a}$

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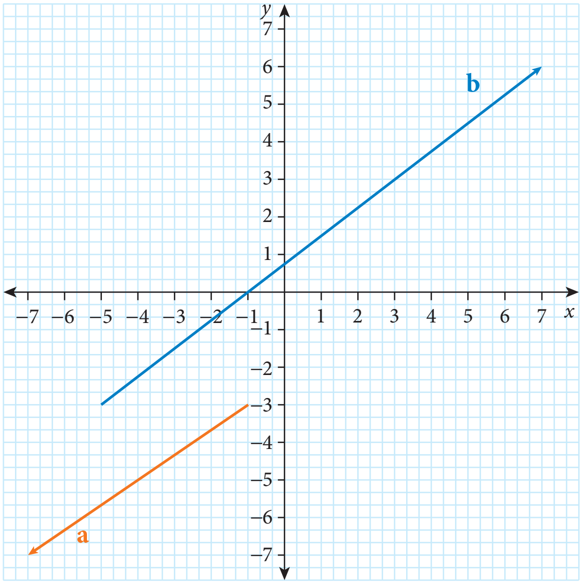
2 The diagram on the right shows two vectors **a** and **b**.
Which of the following best describes the relationship between the two vectors?

A $\mathbf{a} = 2\mathbf{b}$ B $\mathbf{b} = -2\mathbf{a}$
C $\mathbf{a} = 3\mathbf{b}$ D $\mathbf{b} = 3\mathbf{a}$
E $\mathbf{a} = -\frac{1}{3}\mathbf{b}$



3 Example 9 If $\mathbf{b} = (6, 2)$, draw vectors representing:

a $2\mathbf{b}$ b $-\mathbf{b}$ c $3\mathbf{b}$ d $\frac{1}{2}\mathbf{b}$ e $-2\frac{1}{2}\mathbf{b}$

4 Example 10 If $\mathbf{d} = \begin{bmatrix} 18 \\ -12 \end{bmatrix}$, find:

a $3\mathbf{d}$ b $-2\mathbf{d}$ c $\frac{1}{3}\mathbf{d}$ d $6.4\mathbf{d}$
e $\frac{1}{2}\mathbf{d}$ f $-2.5\mathbf{d}$ g $\frac{2}{3}\mathbf{d}$ h $-4\mathbf{d}$

5 If $\mathbf{v} = (6, 47°)$, find

a $3\mathbf{v}$ b $-4\mathbf{v}$ c $2.5\mathbf{v}$ d $-\mathbf{v}$
e $-5\mathbf{v}$ f $10\mathbf{v}$ g $-7\mathbf{v}$ h $\frac{1}{2}\mathbf{v}$

6 If $\mathbf{a} = (2, 7)$, $\mathbf{b} = (-4, 9)$ and $\mathbf{c} = (3, -12)$, calculate:

a $4\mathbf{a}$ b $-3\mathbf{c}$ c $3.5\mathbf{b}$ d $9.7\mathbf{a}$
e $-\frac{1}{3}\mathbf{b}$ f $\frac{1}{2}\mathbf{c}$ g $\frac{3}{4}\mathbf{b}$ h $-2\mathbf{a}$

7 If $\mathbf{a} = (4, 195°)$, $\mathbf{b} = (5, 69°)$ and $\mathbf{c} = (2, 304°)$, calculate:

a $3\mathbf{a}$ b $-2\mathbf{c}$ c $1.5\mathbf{b}$ d $-4\mathbf{a}$
e $-\frac{1}{2}\mathbf{b}$ f $\frac{1}{3}\mathbf{c}$ g $-4\mathbf{b}$ h $-5\mathbf{a}$

1.05 UNIT VECTORS

You have used two different forms of geometric vectors in two dimensions—the polar form and the component form. The component form is conveniently expressed as a row or column vector. It is also useful to express it in terms of **unit vectors**.

IMPORTANT

A **unit vector** has magnitude 1. The unit vector in the same direction as a given vector $\mathbf{p}$ is symbolised by placing a circumflex (ˆ) over the vector symbol, written as $\hat{\mathbf{p}}$.

A unit vector may be derived from any vector $\mathbf{p}$ by multiplying by $\frac{1}{|\mathbf{p}|}$.

Thus $\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$.

The unit vectors in the x and y directions are given the special symbols $\mathbf{i}$ and $\mathbf{j}$, so $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. In polar form i = (1, 0°) and j = (1, 90°).

Example 11

If $\mathbf{p} = (3, -4)$, find $\hat{\mathbf{p}}$.

Solution

Find the magnitude of $\mathbf{p}$. $|\mathbf{p}| = \sqrt{3^2 + (-4)^2}$

Evaluate. $= 5$

Write the rule for the unit vector. $\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$

Substitute in the values. $= \frac{1}{5}(3, -4)$

Evaluate. $\hat{\mathbf{p}} = \left(\frac{3}{5}, -\frac{4}{5}\right)$

WS Unit vectors

A **linear combination** of vectors is a sum of scalar multiples of the vectors. Thus a linear combination of vectors $\mathbf{p}$ and $\mathbf{q}$ is of the form $a\mathbf{p} + b\mathbf{q}$, where a and b are real numbers. Any vector in two dimensions may be written as a linear combination of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Example 12

Write the following vectors in terms of $\mathbf{i}$ and $\mathbf{j}$.

a $\mathbf{f} = (5, -3)$ b $\mathbf{g} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ c $\mathbf{h} = (5, 300°)$

Solution

a Write $\mathbf{f}$. $\mathbf{f} = (5, -3)$

Express as a sum of vectors. $= (5, 0) + (0, -3)$

Write as multiples of unit vectors. $= 5(1, 0) + [-3(0, 1)]$

State the result. $\mathbf{f} = 5\mathbf{i} + (-3\mathbf{j})$

$= 5\mathbf{i} - 3\mathbf{j}$

b Write **g**.

$$\mathbf{g} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

Express as a sum of vectors.

$$= \begin{bmatrix} -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Write as multiples of unit vectors.

$$= -6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

State the result.

$$\mathbf{g} = -6\mathbf{i} + 0\mathbf{j}$$

c Write **h**.

$$\mathbf{h} = (5, 300°)$$

Express in component form.

$$= (5\cos(300°), 5\sin(300°))$$

Evaluate and round off.

$$\approx (2.5, -\frac{5\sqrt{3}}{2})$$

Express as a sum of vectors.

$$= (2.5, 0) + (0, -\frac{5\sqrt{3}}{2})$$

Write as multiples of unit vectors.

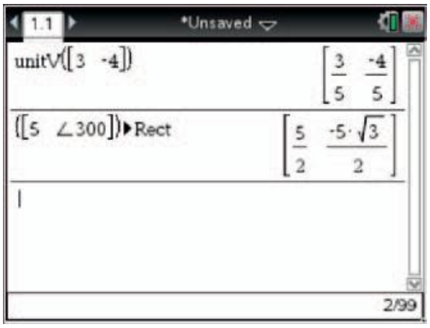
$$= 2.5(1, 0) + [-\frac{5\sqrt{3}}{2}(0, 1)]$$

State the result.

$$\mathbf{h} = \frac{5}{2}\mathbf{i} - \frac{5\sqrt{3}}{2}\mathbf{j}$$

TI-Nspire CAS

You can calculate a unit vector by typing unitV() or using menu, 7: Matrix & Vector, C: Vector and 1: Unit Vector.
To change a vector to component form, type the vector and press enter or use menu, 7: Matrix & Vector, C: Vector and 5: Convert to rectangular.



$$\mathbf{h} = \frac{5}{2}\mathbf{i} - \frac{5\sqrt{3}}{2}\mathbf{j}$$

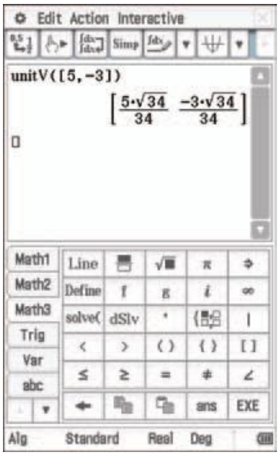
ClassPad

A unit vector can be calculated using **unitV**.

In the Main $\sqrt{\alpha}$ application, tap **Action**, then **Vector** and then **unitV**.

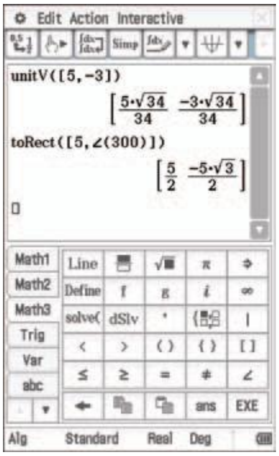
For exact answers (surds, fractions), set the calculator to **Standard**, otherwise **Decimal**.

After you have entered the vector, tap **OK**.



To write a polar vector in terms of **i** and **j**, calculate it as a column vector.

Tap **Action**, then **Vector** and then **toRect** and proceed as before.



$\mathbf{h} = \frac{5}{2}\mathbf{i} - \frac{5\sqrt{3}}{2}\mathbf{j}$

EXERCISE 1.05 Unit vectors

Concepts and techniques

1 Which of the following is a unit vector?

A $(2, 90°)$ B $(1, 131°)$ C $\left(\frac{1}{2}, 180°\right)$ D $\left(1.5, \frac{\pi}{2}\right)$ E $(2, 0°)$

2 Which of the following is not a unit vector?

A $(0, 1)$ B $(1, 1)$ C $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ D $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ E $(1, 0)$

3 Example 11 Calculate the unit vector for each of the following vectors.

a $(2, 3)$ b $(4, -5)$ c $(-2, -6)$ d $(3.7, -8.2)$

e $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ f $\begin{bmatrix} -5 \\ -12 \end{bmatrix}$ g $\begin{bmatrix} 24 \\ -7 \end{bmatrix}$ h $\begin{bmatrix} 8.22 \\ 4.69 \end{bmatrix}$

i $(10, 86°)$ i $(12, -165°)$ k $\left(5, \frac{2\pi}{3}\right)$ l $\left(11, \frac{7\pi}{5}\right)$

4 Example 12 Write the following vectors as linear combinations of **i** and **j**.

a $(5, -3)$ b $(-6, 4)$ c $(-4, -7)$ d $(6, 5)$ e $(-3.2, -9.4)$

f $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ g $\begin{bmatrix} -2 \\ -8 \end{bmatrix}$ h $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ i $\begin{bmatrix} 7.59 \\ 3.68 \end{bmatrix}$ i $\begin{bmatrix} 0.07 \\ 0.19 \end{bmatrix}$

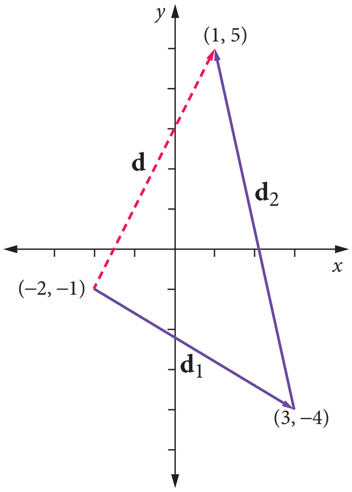
k $(7, -74°)$ l $(9, 125°)$ m $\left(11, \frac{11\pi}{6}\right)$ n $\left(16, \frac{4\pi}{3}\right)$ o $(4.8, -119°)$

Reasoning and communication

5 Find a unit vector (in component form), that is in the opposite direction to $\mathbf{m} = 3\mathbf{i} - 4\mathbf{j}$.

6 Find a vector (in component form) of magnitude 6 that has the same direction as $\mathbf{g} = 4\mathbf{i} - 7\mathbf{j}$.

1.06 USING COMPONENTS



Consider displacements from (–2, –1) to (3, –4) and then from (3, –4) to (1, 5), as shown in the diagram on the right. The overall displacement is from (–2, –1) to (1, 5). As you have previously seen, the resultant displacement is formed as the third side of the triangle made by the first two displacements, with the vectors placed head-to-tail.

You can find the resultant displacement without using a diagram and the triangle of vectors.

In component form, the displacements are $\mathbf{d}_1 = (5, -3)$ and $\mathbf{d}_2 = (-2, 9)$ and the resultant displacement is $\mathbf{d} = (3, 6)$.

You can simply add the components of $\mathbf{d}_1$ and $\mathbf{d}_2$ as shown below.

$$\begin{aligned}\mathbf{d}_1 + \mathbf{d}_2 &= (5, -3) + (-2, 9)\\ &= (5 + (-2), -3 + 9)\\ &= (3, 6)\\ &= \mathbf{d}\end{aligned}$$

When displacements are given in polar form, we must either draw a diagram and use the triangle of vectors or convert vectors to component form before adding.

IMPORTANT

The **sum** (or **resultant**) of two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ is given by

$$\mathbf{a} + \mathbf{b} = (x_1 + x_2, y_1 + y_2)$$

In terms of the unit vectors **i** and **j**, if $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$, then

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= x_1\mathbf{i} + x_2\mathbf{i} + y_1\mathbf{j} + y_2\mathbf{j}\\ &= (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}\end{aligned}$$

Example 13

Find the sum of the vectors:

a (3, –7) and (–11, 8) **b** $\begin{bmatrix}7\\3\end{bmatrix}$ and $\begin{bmatrix}5\\-4\end{bmatrix}$ **c** $5\mathbf{i} + 7\mathbf{j}$ and $-9\mathbf{i} - 4\mathbf{j}$

Solution

a The vectors are in component form, so add the components.

$$\begin{aligned}(3, -7) + (-11, 8) &= (3 + (-11), -7 + 8)\\ &= (-8, 1)\end{aligned}$$

b The vectors are in component form, so add the components.

$$\begin{aligned}\begin{bmatrix}7\\3\end{bmatrix} + \begin{bmatrix}5\\-4\end{bmatrix} &= \begin{bmatrix}7+5\\3+(-4)\end{bmatrix}\\ &= \begin{bmatrix}12\\-1\end{bmatrix}\end{aligned}$$

c The vectors are expressed in terms of **i** and **j**, so add the **i** and **j** parts separately.

$$\begin{aligned}5\mathbf{i} + 7\mathbf{j} + (-9\mathbf{i} - 4\mathbf{j}) &= (5 + -9)\mathbf{i} + [7 + (-4)]\mathbf{j}\\ &= -4\mathbf{i} + 3\mathbf{j}\end{aligned}$$

You looked at scalar multiples of vectors in the previous section.

IMPORTANT

The difference of vectors

The product $(-1)\mathbf{a}$ is normally written as $-\mathbf{a}$.

The difference of vectors is found by addition of the negative.

Thus $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

Example 14

Given that $\mathbf{a} = (6, -4)$ and $\mathbf{b} = (-5, 2)$, find:

a $\mathbf{a} - \mathbf{b}$ b $2\mathbf{a} - 3\mathbf{b}$

Solution

a Use the definition of subtraction. $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

Substitute in the known values. $= (6, -4) + [-(-5, 2)]$

Evaluate. $= (6 + 5, -4 - 2)$
$= (11, -6)$

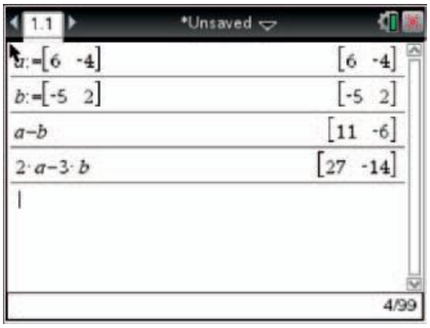
b Use the definition of scalar multiplication. $2\mathbf{a} - 3\mathbf{b} = 2(6, -4) - 3(-5, 2)$

Evaluate. $= (12, -8) - (-15, 6)$

Use the definition of subtraction and evaluate. $= (12, -8) + [-(-15, 6)]$
$= (27, -14)$

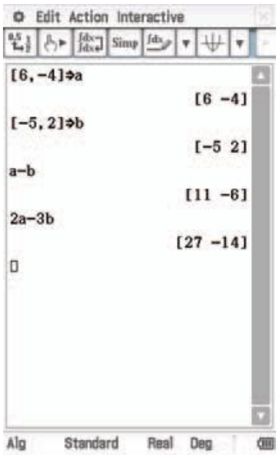
TI-Nspire CAS

You can name vectors and add, subtract, and multiply them by scalars just as you would do for pronumerals.
To define a vector, use the := symbol (ctrl) or use Define.
The screen on the right first defines vectors **a** and **b**, and then calculates **a** – **b** and 2**a** – 3**b**.



ClassPad

You can name vectors and add, subtract, and multiply them by scalars just as you would do for pronumerals.
To define a vector as **a**, use ⇒ a.
Press Keyboard then tap Math3 for the keyboard with the square brackets and ⇒, and tap abc for the letters keyboard.
The screen on the right first defines vectors **a** and **b**, and then calculates **a** – **b** and 2**a** – 3**b**.



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EXERCISE 1.06 Using components

WS Addition of vectors using components

Concepts and techniques

1 Example 13 Find the sum of the following pairs of vectors.

a $(-4, 3)$ and $(-9, 10)$
b $(7, -9)$ and $(11, -5)$
c $(-13, 6)$ and $(8, -7)$
d $(3.8, -4.5)$ and $(6.2, -7.3)$
e $(-1.07, 0.35)$ and $(5.24, -4.57)$
f $\left(3\frac{1}{2}, -2\frac{4}{5}\right)$ and $\left(-1\frac{3}{4}, 5\frac{2}{3}\right)$

2 Find the sum of the following pairs of vectors.

a $\begin{bmatrix} 14 \\ -11 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ -6 \end{bmatrix}$
b $\begin{bmatrix} -5 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ 7 \end{bmatrix}$
c $\begin{bmatrix} 13 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ -8 \end{bmatrix}$
d $\begin{bmatrix} -3.6 \\ -8.7 \end{bmatrix}$ and $\begin{bmatrix} 6.9 \\ -4.6 \end{bmatrix}$
e $\begin{bmatrix} 2.07 \\ -4.66 \end{bmatrix}$ and $\begin{bmatrix} -0.97 \\ -1.78 \end{bmatrix}$
f $\begin{bmatrix} 2\frac{1}{2} \\ -3\frac{4}{9} \end{bmatrix}$ and $\begin{bmatrix} 3\frac{3}{4} \\ -2\frac{1}{3} \end{bmatrix}$

3 Find the sum of the vectors below.

a $5\mathbf{i} - 6\mathbf{j}$ and $2\mathbf{i} - 7\mathbf{j}$
b $-8\mathbf{i} + 7\mathbf{j}$ and $-9\mathbf{i} + 3\mathbf{j}$
c $-11\mathbf{i} - 4\mathbf{j}$ and $6\mathbf{i} + 9\mathbf{j}$
d $5.8\mathbf{i} - 6.7\mathbf{j}$ and $-9.2\mathbf{i} - 5.3\mathbf{j}$
e $-2.14\mathbf{i} + 1.79\mathbf{j}$ and $-6.09\mathbf{i} + 3.36\mathbf{j}$
f $-4\frac{2}{5}\mathbf{i} + 3\frac{3}{8}\mathbf{j}$ and $-5\frac{7}{10}\mathbf{i} + 6\frac{1}{4}\mathbf{j}$

4 Example 14 Given that $\mathbf{a} = (1, 4)$, $\mathbf{b} = (-7, 8)$, $\mathbf{c} = (2, 4)$, $\mathbf{d} = (5, -2)$ and $\mathbf{e} = (-6, 1)$, find:

a $\mathbf{c} + \mathbf{e}$
b $\mathbf{b} + \mathbf{c} + \mathbf{d}$
c $5\mathbf{a}$
d $-2\mathbf{e}$
e $3\mathbf{a} + 2\mathbf{e}$
f $4\mathbf{b} - 2\mathbf{a}$
g $\mathbf{a} - \mathbf{c} - \mathbf{e}$
h $7\mathbf{c} - 7\mathbf{a} + \mathbf{d} + 2\mathbf{e}$
i $7\mathbf{a} + 5\mathbf{b} + \mathbf{e}$
j $2\mathbf{d} - 7\mathbf{c}$

5 Change the following to component form, find the result, and then change back to polar form.

a $(6, 20°) + (9, 55°)$
b $(25, 120°) + (16, 80°)$
c $(7, 30°) - (9, 100°)$
d $(105, 300°) - (95, 60°)$
e $(6, 70°) - (6, 10°)$

Reasoning and communication

6 Find the displacement vector for each of the following movements.

a $A(3, 4)$ to $B(5, 11)$
b $R(-2, 1)$ to $B(-7, -2)$
c $M(-3, -8)$ to $B(-9, 1)$
d $Q(1, 8)$ to $P(1, -10)$
e $B(-2, -5)$ to $X(6, 9)$
f $F(-1, 4)$ to $G(7, -7)$

7 If $\mathbf{a} = (x_1, y_1)$ and $c < 0$, prove that $|c\mathbf{a}| = -c|\mathbf{a}|$.

8 If $\mathbf{a} = (x_1, y_1)$, $\mathbf{b} = (x_2, y_2)$ and c is a scalar, prove that $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$.

1.07 VECTOR PROPERTIES

In the previous sections of this chapter you discovered a number of vector properties. In this section, some of the more important properties will be formalised.

IMPORTANT

Vector properties for addition and multiplication by a scalar

The following properties are defined for vectors **a**, **b** and **c** and scalars r and s.

1 Vector addition is **associative**.

$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

2 Vector addition is **commutative**.

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

3 There is an **additive identity**.

The additive identity has all elements equal to zero and is called the **zero vector**, written as **0**.

$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

4 There is an additive **inverse**.

The additive inverse of $\mathbf{a} = (x_1, y_1)$ is $-\mathbf{a} = (-x_1, -y_1)$.

In polar form, $\mathbf{a} = (r, \theta)$ has the additive inverse $-\mathbf{a} = (r, \theta + 180°)$.

$\mathbf{a} + (-\mathbf{a}) = -\mathbf{a} + \mathbf{a} = \mathbf{0}$

5 **Cancellation laws** hold for multiplication of a vector by a scalar.

$r\mathbf{a} = r\mathbf{b} \Leftrightarrow \mathbf{a} = \mathbf{b}\ (r \neq 0)$

$r\mathbf{a} = s\mathbf{a} \Leftrightarrow r = s\ (\mathbf{a} \neq \mathbf{0})$

6 **Distributive laws** hold for multiplication over addition.

$(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$

$r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$

7 An **associative law** holds for multiplication and multiplication by a scalar.

$(rs)\mathbf{a} = r(s\mathbf{a})$

The properties outlined above can be demonstrated through the use of particular vectors and scalars.

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Example 15

Demonstrate that vector addition is commutative for vectors $\mathbf{u} = (5, -1)$ and $\mathbf{v} = (-6, 8)$.

Solution

Write down what needs to be demonstrated.	Need to show: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Calculate $\mathbf{u} + \mathbf{v}$.	$\mathbf{u} + \mathbf{v} = (5, -1) + (-6, 8)$ $= (5 - 6, -1 + 8)$ $= (-1, 7)$
Calculate $\mathbf{u} + \mathbf{v}$.	$\mathbf{v} + \mathbf{u} = (-6, 8) + (5, -1)$ $= (-6 + 5, 8 - 1)$ $= (-1, 7)$
You have shown that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.	Vector addition is commutative for vectors $\mathbf{u}$ and $\mathbf{v}$.

The proofs of the previously described vector properties follow the same approach as outlined in Example **15**, but use general rather than particular values.

WS Vector properties

Example 16

Prove the distributive law for multiplication of a scalar over vector addition.

Solution

Write what is required to be proved (RTP).	RTP: $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$
Identify the vectors and the scalar quantity.	Let $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ and let r be a scalar quantity.
Substitute the values in the LHS of the RTP statement.	$r(\mathbf{a} + \mathbf{b}) = r(x_1 + x_2, y_1 + y_2)$
Expand the brackets	$= (rx_1 + rx_2, ry_1 + ry_2)$
Regroup like terms.	$= (rx_1, ry_1) + (rx_2, ry_2)$
Factorise.	$= r(x_1, y_1) + r(x_2, y_2)$
Substitute in the defined terms.	$= r\mathbf{a} + r\mathbf{b}$ **QED**

EXERCISE 1.07 Vector properties

Concepts and techniques

1 If $\mathbf{c}$ and $\mathbf{d}$ are vectors, which of the following statements is not necessarily true?

A $\mathbf{c} + \mathbf{d} = \mathbf{d} + \mathbf{c}$ B $\mathbf{d} + \mathbf{0} = \mathbf{d}$ C $\mathbf{c} + (-\mathbf{c}) = \mathbf{0}$

D $\mathbf{c} - \mathbf{d} = \mathbf{d} - \mathbf{c}$ E $\mathbf{0} + \mathbf{c} = \mathbf{c}$

2 If $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are vectors, and m and n are scalar quantities, which of the following statements is not necessarily true?

A $(n + m)\mathbf{r} = n\mathbf{r} + m\mathbf{r}$ B $\mathbf{r}\mathbf{p} + \mathbf{r}\mathbf{q} = \mathbf{r}(\mathbf{p} + \mathbf{q})$ C $m\mathbf{q} + n\mathbf{q} = (m + n)\mathbf{q}$

D $m(\mathbf{p} + \mathbf{q}) = m\mathbf{p} + m\mathbf{q}$ E $(mn)\mathbf{r} = m(n\mathbf{r})$

3 Find the additive inverse of each of the following vectors.

a $\mathbf{m} = (-5, 8)$ b $\mathbf{d} = (-11, -4)$ c $\mathbf{r} = (0, -6)$

d $\mathbf{q} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ e $\mathbf{g} = \begin{bmatrix} 14 \\ 3 \end{bmatrix}$ f $\mathbf{h} = \begin{bmatrix} -12 \\ 11 \end{bmatrix}$

g $\mathbf{a} = 4\mathbf{i} - 9\mathbf{j}$ h $\mathbf{p} = \frac{2}{3}\mathbf{i} + \frac{4}{5}\mathbf{j}$ i $\mathbf{n} = -0.24\mathbf{i} - 1.06\mathbf{j}$

4 Example 15 Demonstrate that vector addition is commutative for each pair of vectors below.

a $\mathbf{m} = (-3, -5)$ and $\mathbf{p} = (2, 9)$ b $\mathbf{q} = (10, -7)$ and $\mathbf{d} = (-8, 4)$

c $\mathbf{f} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ d $\mathbf{u} = \begin{bmatrix} 15 \\ -6 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} -7 \\ 11 \end{bmatrix}$

e $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$ and $\mathbf{n} = -5\mathbf{i} + 7\mathbf{j}$ f $\mathbf{e} = -11\mathbf{i} - 6\mathbf{j}$ and $\mathbf{n} = 13\mathbf{i} + 8\mathbf{j}$

5 Demonstrate that vector addition is associative for each set of vectors below.

a $\mathbf{a} = (-2, 6)$, $\mathbf{b} = (1, -7)$ and $\mathbf{c} = (-8, 0)$

b $\mathbf{p} = (-1, 9)$, $\mathbf{q} = (-4, -6)$ and $\mathbf{r} = (5, -3)$

c $\mathbf{t} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -8 \\ 10 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

d $\mathbf{k} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$, $\mathbf{m} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$

e $\mathbf{w} = -\mathbf{i} - 3\mathbf{j}$, $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j}$ and $\mathbf{s} = 9\mathbf{i} - 6\mathbf{j}$

f $\mathbf{e} = -4\mathbf{i} + 11\mathbf{j}$, $\mathbf{d} = -\mathbf{i} - 3\mathbf{j}$ and $\mathbf{f} = 8\mathbf{i} - 4\mathbf{j}$

6 Demonstrate that the distributive law for multiplication of a vector over scalar addition holds true for each grouping of vector and scalars shown below.

a $\mathbf{h} = (1, -2)$, $w = 3$ and $g = 2$ b $\mathbf{d} = (-12, 18)$, $a = \frac{1}{2}$ and $b = \frac{2}{3}$

c $\mathbf{a} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, $m = 4$ and $n = 5$ d $\mathbf{b} = \begin{bmatrix} -16 \\ -8 \end{bmatrix}$, $r = 0.5$ and $s = 0.25$

e $\mathbf{q} = 5\mathbf{i} - \mathbf{j}$, $c = 2$ and $d = 6$ f $\mathbf{p} = -12\mathbf{i} - 36\mathbf{j}$, $k = \frac{1}{3}$ and $v = \frac{3}{4}$

Reasoning and communication

7 Example 16 Prove that vector addition is associative.

8 Prove that vector addition is commutative.

9 Given that $\mathbf{p}$ and $\mathbf{q}$ are vectors and k is a scalar quantity, prove that $k(\mathbf{p} + \mathbf{q}) = k\mathbf{p} + k\mathbf{q}$.

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1.08 APPLICATIONS OF VECTORS

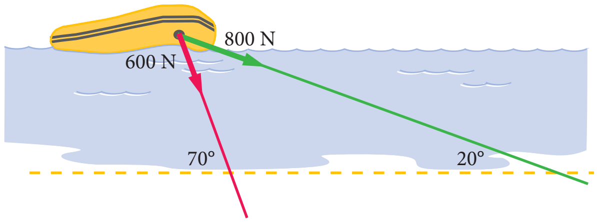
Many situations, such as displacement, velocity, force and acceleration, involve vector quantities. The techniques dealt with in this chapter can be used to solve real world problems involving vectors.

Example 17

Two ropes are tied to the prow of a rubber boat and are pulling the boat towards the beach. The first rope is being pulled by two people at an angle of 20° to the shoreline, with a force of 800 N (newtons). The second rope is being pulled in the same general direction by a lifeguard, with a force of 600 N at an angle of 70° to the shoreline. What is the total force acting on the boat and in what direction is it acting?

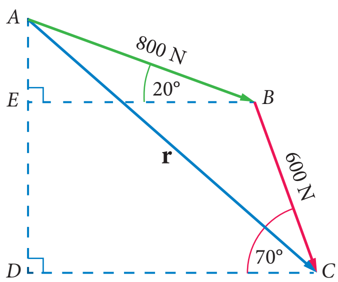
Solution

Draw a diagram to show the forces.



Draw the forces head-to-tail to view the triangle of forces *ABC* at a larger scale, and complete the triangle with the resultant.

Since *EBCD* is a quadrilateral, $\angle EBC = 110°$, so $\angle ABC = 130°$.



Use the cosine rule to find *r*, stating the rule first.

$$\begin{aligned} r^2 &= AB^2 + BC^2 - 2 \times AB \times BC \cos(\angle ABC) \\ &= 800^2 + 600^2 - 2 \times 800 \times 600 \cos(130°) \\ &= 1\,617\,076.105\ldots \end{aligned}$$

Keep the exact answer in your calculator.

$$r = 1271.643\ldots \text{ N}$$

Use the sine rule to find $\angle BCA$.

$$\frac{\sin(\angle BCA)}{AB} = \frac{\sin(\angle ABC)}{AC}$$

Rearrange.

$$\sin(\angle BCA) = \frac{AB \times \sin(\angle ABC)}{AC}$$

Use the exact answer from your calculator.

$$\begin{aligned} &= \frac{800 \times \sin(130°)}{1271.643\ldots} \\ &= 0.481\,9924\ldots \end{aligned}$$

Use $\sin^{-1}$ to find $\angle BCA$.

$\angle BCA = 28.811...°$

Calculate the angle of the resultant with the shoreline.

$$\text{Resultant angle} = 70° - 28.811...° = 41.188...°$$

State the answer in an appropriate form.

The total force acting on the boat is about 1272 N, at an angle of 41.2° to the shoreline.

The change in a quantity from x_1 to x_2, can be shown as $\Delta x = x_2 - x_1$. This also applies to vector quantities, but you need to use vector addition. So, for velocity, $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_2 + (-\mathbf{v}_1)$.

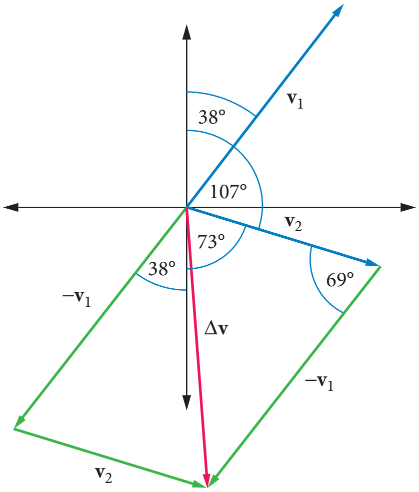
Example 18

The velocity of a boat changes from 25 knots at N 38° E to 15 knots at a bearing of 107°. What is the change in velocity?

iStockphoto/Ian Hamilton

Solution

Sketch a diagram to represent the situation.
Let $\mathbf{v}_1$ be the velocity 25 knots at N 38° E.
Let $\mathbf{v}_2$ be the velocity 15 knots at 107°.
Put $-\mathbf{v}_1$ on the diagram.
Rearrange $-\mathbf{v}_1$ and $\mathbf{v}_2$ on the diagram so that the change of velocity can be found.
Find the angle between the vectors.
The obtuse angle in the parallelogram is 111° (38° + 73°), so the acute angle is 69°.



Express the change of velocity in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_2 + -\mathbf{v}_1$$

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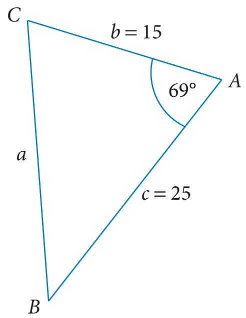
We could proceed using either the component or polar form of the vectors.
We will use the polar form.
Draw a diagram to show the information of interest with a, b and c for the magnitudes of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\Delta\mathbf{v}$.



Use the cosine rule to find a.
Substitute in the known values.

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$= 15^2 + 25^2 - 2 \times 15 \times 25 \times \cos(69°)$$

Evaluate and retain the unrounded result.

$$a = 24.108\ldots$$

Use the sine rule to find C.

$$\frac{\sin(C)}{c} = \frac{\sin(A)}{a}$$

Substitute in the known values.

$$\frac{\sin(C)}{25} = \frac{\sin(69°)}{24.108\ldots}$$

Rearrange and evaluate.

$$\sin(C) = 0.9680\ldots$$

Use $\sin^{-1}$ to find C.

$$C \approx 75.5°$$

Calculate the bearing of the resultant.

$$\text{Bearing} \approx 107° + 75.5°$$

Evaluate.

$$= 182.5°$$

If you refer back to the original diagram, you can see that the sketch was slightly incorrect because it shows the bearing of $\Delta\mathbf{v}$ as less than 180° (107° + 73°). So you can see that as long as the sketch is roughly correct to start with, a slight error of this nature doesn't matter.

State the result.

The change in velocity is about 24.1 knots at a bearing of 182.5°.

EXERCISE 1.08 Application of vectors

Reasoning and communication

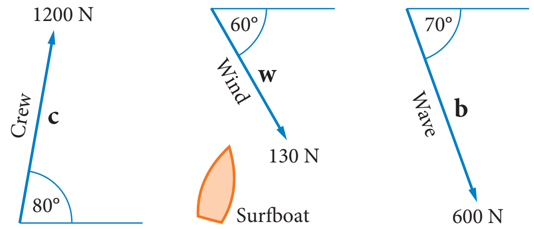
WS Vector applications

1 Example 17 Two winches are being used to pull out a bogged car. One winch is on the driver's side at an angle of 32° to the forward direction and the larger winch is on the passenger's side at an angle of 39° to the forward direction. The first is exerting a force of 6000 N and the other is exerting a force of 9000 N. What is the total force on the car, and in what direction does it act?

2 An orienteer runs 5 km at a bearing of 125° and then 3 km at a bearing of 205°. Find her distance and bearing from the starting point.

3 A boat is being pulled along a canal by two people using ropes. One person is on the left bank pulling at an angle of 30° to the bank with a force of 400 N. The other person is on the right bank pulling at an angle of 45° to the bank. The boat is moving straight along the canal. With what force is the second person pulling?

4 A surfboat at a lifesavers' regatta is putting out from the beach. The directions of the forces acting on the boat are shown below, with the crew exerting a force of 1200 N, the wind a force of 130 N and a breaking wave a force of 600 N. Find the direction and magnitude of the resultant force on the surfboat.



Newspix/Scott Fletcher

5 Example 18 A car travelling at 20 m/s south changes velocity to 18 m/s east. Find the change in velocity.

6 Police called to an emergency are travelling directly west along a main road and slow down to 60 km h^{-1} as they approach an intersection, where they turn right (by 90°). What is the change of velocity of the car?

7 An aircraft travelling at 140 knots at a bearing of 197° changes direction to a bearing of 116° at the same speed to approach the runway from the seaward side. Find the change in velocity.

8 What is the change of velocity when a cyclist in a road race changes velocity from 15 m/s NW to 12 m/s N?

9 Find the change of velocity when an object travelling at v m/s changes direction by θ° without changing speed.

123rf/dziewul

CHAPTER SUMMARY
BASIC VECTORS

1

- A **scalar** quantity is a **magnitude** expressed by a single number, but a **geometric vector** has both a magnitude (**norm**) and a **direction**. A vector is usually denoted by a lower case bold letter, such as $\mathbf{a}$, and may be shown by a **directed line segment** whose length represents the magnitude $|\mathbf{a}| = a$.
- A **displacement** vector showing a change of position from A to B may be written as $\mathbf{AB}$ or $\overrightarrow{AB}$. The **position vector** of a point P is the vector $\mathbf{p} = \mathbf{OP}$. O is the **initial point** and P is the **terminal point** of the vector.
- The **polar form** of a **two-dimensional vector** $\mathbf{v} = (r, \theta)$ shows the magnitude r and the direction θ of the vector. θ is the angle in the positive direction (anticlockwise) from the x-axis. The **component form** (x, y) or $\begin{bmatrix} x \\ y \end{bmatrix}$ shows the vector as **components** in the x- and y-directions.

  The polar and component forms of the vector $\mathbf{v}$ are related by:

  $$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

  $$\tan(\theta) = \frac{y}{x} \qquad r^2 = x^2 + y^2$$

  $$|\mathbf{v}| = \sqrt{x^2 + y^2}.$$
- The sum $\mathbf{a} + \mathbf{b}$ is also called the **resultant** of $\mathbf{a}$ and $\mathbf{b}$. The resultant vector may be found geometrically using the **triangle of vectors**. This is also called the head-to-tail method. Addition of vectors can also be done geometrically using a **parallelogram** of vectors. For a vector $\mathbf{a} = (x_1, y_1)$ and a constant c, the **scalar multiple** $c\mathbf{a}$ is defined as $c\mathbf{a} = (cx_1, cy_1)$. If the vector is in polar form, i.e. $\mathbf{a} = (r, \theta)$, the scalar multiple $c\mathbf{a}$ is (cr, θ) for $c \ge 0$, and $(-cr, 180° + \theta)$ for $c < 0$. The product $(-1)\mathbf{a}$ is normally written as $-\mathbf{a}$.
- A **unit vector** has magnitude 1. The unit vector in the same direction as a given vector $\mathbf{p}$ is symbolised as $\hat{\mathbf{p}}$, where $\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$.
- The unit vectors in the x- and y-directions are $\mathbf{i}$ and $\mathbf{j}$, so $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

  $\mathbf{i} = (1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

  Any vector can be shown as a linear combination of $\mathbf{i}$ and $\mathbf{j}$, so $\mathbf{v} = r\mathbf{i} + s\mathbf{j}$ for some $r, s \in \mathbf{R}$.
- The **sum** (or **resultant**) of two vectors $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b} = (x_2, y_2)$ is defined by

  $$\mathbf{a} + \mathbf{b} = (x_1 + x_2, y_1 + y_2).$$
- Addition of vectors is **associative** and **commutative**, there is an **identity** vector $\mathbf{0} = (0, 0)$ and any vector $\mathbf{a}$ has an additive inverse $-\mathbf{a}$.

  Cancellation, commutative, associative and **distributive laws** hold for vector addition and scalar multiplication.

  $$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

  $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

  $r\mathbf{a} = r\mathbf{b} \rightarrow \mathbf{a} = \mathbf{b}$ $(r \ne 0)$ and
  $r\mathbf{a} = s\mathbf{a} \rightarrow r = s$ $(\mathbf{a} \ne \mathbf{0})$

  $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + s\mathbf{a}$ and
  $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$
  $r(s\mathbf{a}) = (rs)\mathbf{a}$

  The change in a vector from $\mathbf{v}_1$ to $\mathbf{v}_2$ can be written as $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_2 + (-\mathbf{v}_1)$.

1

CHAPTER REVIEW
BASIC VECTORS

Multiple choice

1 Example 4 What is the direction of the position vector of $A(-2, -2)$?

A $-45°$ B $45°$ C $135°$ D $225°$ E $315°$

2 Examples 1, 2 An aircraft is travelling north-west at 128 km h^{-1}. Which of the following polar forms best describes the velocity of the vessel?

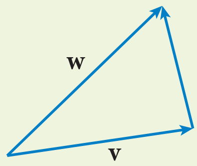
A $(128, 135°)$ B $(128, 45°)$ C $(128, 315°)$

D $(128 \text{ km h}^{-1}, 225°)$ E $(128, -135°)$

3 Example 3 A ball falls vertically downwards 11 m and rebounds to a height of 7 m. Which of the following best describes the resultant displacement of the ball?

A 18 m B 4 m down C 7 m up D 4 m E 4 m up

4 Example 4 This diagram shows vectors **w**, **v** and a third vector. Which of the following best describes the third vector?



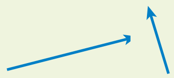
A $\mathbf{w} + \mathbf{v}$ B $\mathbf{v} + (-\mathbf{w})$ C $-\mathbf{w} + \mathbf{v}$

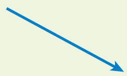
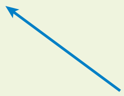
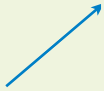
D $\mathbf{w} - \mathbf{v}$ E $\mathbf{v} + \mathbf{w}$

5 Example 10 If $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j}$, which of the following vectors is opposite in direction to **b**?

A $2\mathbf{i} + 5\mathbf{j}$ B $\begin{bmatrix} -4 \\ 10 \end{bmatrix}$ C $-\mathbf{w} + \mathbf{v}$ D $\mathbf{w} - \mathbf{v}$ E $\mathbf{v} + \mathbf{w}$

6 Example 4 This diagram shows vectors **a** and **b**. Which of the following best represents the resultant of **a** and **b**?



A B C  D  E 

7 Example 11 Which of the following is a unit vector?

A $\left(\frac{1}{2}, 90°\right)$ B $(0, 0)$ C $(1.5, 90°)$ D $(1, 248°)$ E $(0.5, 180°)$

8 Example 11 Which of the following is not a unit vector?

A $(1, 1)$ B $\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ C $(1, 0)$ D $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ E $(-1, 0)$

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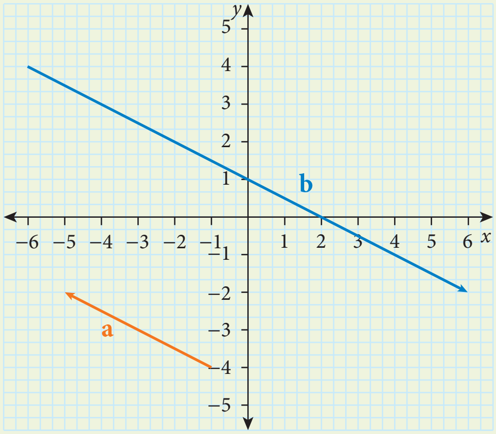
9 Example 9 The diagram on the right shows two vectors **a** and **b**.
Which of the following best describes the relationship between the two vectors?

A $\mathbf{a} = -2\mathbf{b}$ B $\mathbf{b} = 2\mathbf{a}$
C $\mathbf{a} = -\frac{1}{3}\mathbf{b}$ D $\mathbf{b} = 3\mathbf{a}$
E $\mathbf{a} = \frac{1}{2}\mathbf{b}$



Short answer

10 Examples 1, 2 Use directed line segments to represent the following displacements on the same set of axes.

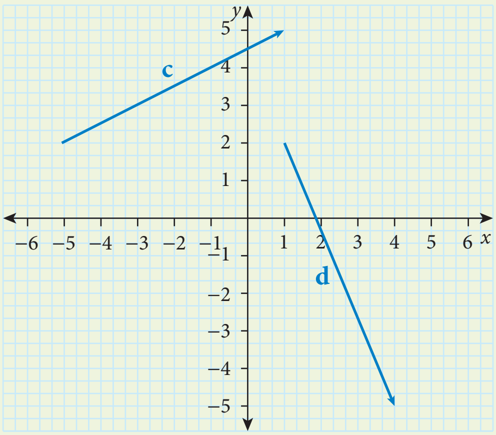
a 5 units north-east b 6 units in the direction 150°

11 Example 3 Find the displacement vectors for each of the following movements.

a $D(3, -4)$ to $E(1, 2)$ b $T(-3, -5)$ to $S(5, 0)$
c $G(1, 1)$ to $J(-3, 5)$ d $R(2, 4)$ to $K(0, 8)$
e $Y(-6, 3)$ to $Z(1, -9)$

12 Example 4 Vectors **c** and **d** are shown on the right. Calculate the resultant of each of the following.

a $\mathbf{c} + \mathbf{d}$ b $\mathbf{d} + \mathbf{c}$
c $\mathbf{d} + -\mathbf{d}$ d $\mathbf{c} + -\mathbf{d}$
e $\mathbf{c} + \mathbf{c} + \mathbf{d}$



13 Example 7 Convert the following vectors to polar form.

a (5, 8) b (−3, 4) c (−6, −10) d (9, −3) e (−4, 0)

14 Example 8 Convert the following vectors to component form.

a (3, 30°) b (8, 150°) c (4, 300°) d (7, 45°) e (9, 230°)

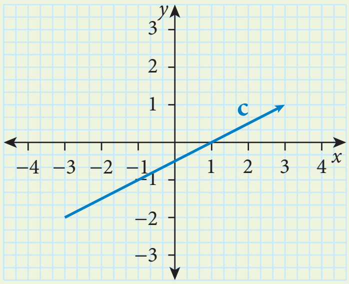
15 Example 7 Convert the following vectors to polar form.

a $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ b $\begin{bmatrix} -2 \\ -6 \end{bmatrix}$ c $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ d $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ e $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$

16 Example 7 Find the norm of each of the following vectors.

a (24, 10) b (−3, −4) c $\begin{bmatrix}-8\\15\end{bmatrix}$ d $\begin{bmatrix}5\\9\end{bmatrix}$ e (10, 12)

17 Example 9 Vector **c** is shown on the right. Draw vectors representing $-2\mathbf{c}$ and $\frac{1}{3}\mathbf{c}$.



18 Example 10 If $\mathbf{a} = \begin{bmatrix}12\\-9\end{bmatrix}$, find $-6\mathbf{a}$.

19 Example 11 If $\mathbf{m} = (7, -24)$ find $\hat{\mathbf{m}}$.

20 Example 12 Write the following vectors in terms of **i** and **j**.

a (1, −3) b (4, 5) c (−2, 5) d $\begin{bmatrix}4\\-3\end{bmatrix}$

e $\begin{bmatrix}-2\\-5\end{bmatrix}$ f (6, 56°) g (8, 200°) h (3, 180°)

i (10, 142°) i (5, 120°)

21 Example 13 Find the resultant of each of the following pairs of vectors.

a $4\mathbf{i} - 7\mathbf{j}$ and $8\mathbf{i} + 2\mathbf{j}$ b (3, −8) and (−9, 6) c $\begin{bmatrix}-1\\10\end{bmatrix}$ and $\begin{bmatrix}6\\-6\end{bmatrix}$

22 Example 14 Given $\mathbf{a} = (3, 5)$, $\mathbf{b} = (-2, -3)$ and $\mathbf{c} = (-1, 4)$, find:

a $\mathbf{a} + \mathbf{c}$ b $\mathbf{b} - \mathbf{a}$ c $3\mathbf{b} + 2\mathbf{c}$ d $5\mathbf{a} - 3\mathbf{b}$ e $\mathbf{a} + \mathbf{b} + \mathbf{c}$

23 Example 14 Given $\mathbf{x} = \begin{bmatrix}-2\\3\end{bmatrix}$, $\mathbf{y} = \begin{bmatrix}4\\-1\end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix}1\\3\end{bmatrix}$, find:

a $\mathbf{x} - \mathbf{y}$ b $2\mathbf{x} + 3\mathbf{y}$ c $\mathbf{y} + 3\mathbf{z}$ d $\mathbf{z} - 2\mathbf{x}$ e $\mathbf{x} + \mathbf{z}$

24 Example 6 Given $\mathbf{p} = (5, 85°)$, $\mathbf{q} = (7, 146°)$ and $\mathbf{r} = (6, 212°)$, find the following without changing to component form.

a $\mathbf{p} + \mathbf{q}$ b $\mathbf{q} + \mathbf{r}$ c $\mathbf{r} + \mathbf{p}$ d $\mathbf{p} - \mathbf{r}$ e $\mathbf{r} - \mathbf{q}$

25 Example 14 Given $\mathbf{f} = (5, 150°)$, $\mathbf{g} = (8, 60°)$ and $\mathbf{h} = (9, 225°)$, find the following by changing to component form and expressing the answer in polar form.

a $\mathbf{f} - \mathbf{g}$ b $\mathbf{g} - \mathbf{h}$ c $\mathbf{g} + \mathbf{h}$ d $\mathbf{f} + \mathbf{g}$ e $\mathbf{h} + \mathbf{g} + \mathbf{f}$

26 Example 15 Demonstrate that vector addition is associative for $\mathbf{r} = (2, -5)$, $\mathbf{g} = (-2, -7)$ and $\mathbf{k} = (3, 2)$.

27 Example 16 If $\mathbf{m} = (x_1, y_1)$ and $\mathbf{n} = (x_2, y_2)$, prove that $\mathbf{m} + \mathbf{n} = \mathbf{n} + \mathbf{m}$.

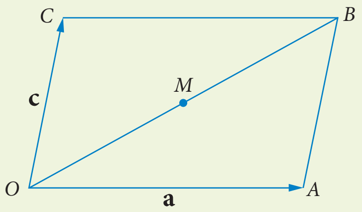
Application

28 *OABC* is a parallelogram and *M* is the midpoint of **OB**. Taking *O* as the origin, the position vectors of *A* and *C* are **a** and **c** respectively. Express each of the following in terms of **a** and **c**.

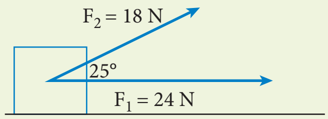
a **BC** b **AC**

c **OB** d **OM**

e **AM** f **MC**



29 Two forces are applied to a block as shown in the diagram on the right. Calculate the resultant force acting on the block.



30 A police officer travels 3 city blocks south and then 2 blocks east. If each block is 300 m long, find the officer's total displacement as a vector in polar form.

31 A ship is being towed by two tugs. One tug is pulling with a force of 30 000 N in the direction N 23° E, while the other is pulling with a force of 35 000 N in the direction N 46° W. What is the resultant force on the ship?

32 A snooker ball approaching the cushion at an angle of 60° to the cushion has been given spin that makes it bounce off at an angle of 45°. If the ball also changes speed from 0.3 m/s to 0.2 m/s, find its change of velocity.

Practice quiz

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2

TERMINOLOGY

contradiction
contrapositive
converse
counterexample
deductive
equality
equivalent
if and only if
implication
inductive
necessary
negation
premise
proof
QED
quantifier
reason
reasoning
RTP
sufficient

GEOMETRY

MATHEMATICAL PROOF

Prior learning

THE NATURE OF PROOF

- use implication, converse, equivalence, negation, contrapositive (ACMSM024)
- use proof by contradiction (ACMSM025)
- use the symbols for implication ($\Rightarrow$), equivalence ($\Leftrightarrow$), and equality ($=$) (ACMSM026)
- use the quantifiers 'for all' and 'there exists' (ACMSM027)
- use examples and counterexamples. (ACMSM028)

GEOMETRIC PROOFS USING VECTORS IN THE PLANE INCLUDING

- The diagonals of a parallelogram meet at right angles if and only if it is a rhombus (ACMSM039)
- Midpoints of the sides of a quadrilateral join to form a parallelogram (ACMSM040) AC

2.01 MATHEMATICAL PROOF

A proof in Mathematics shows something to be true for all cases. This is different from an example where something is true for a particular case. There are different types of proof: deductive, inductive and proof by contradiction. They are based on *reason*, a way of thinking attributed most often to the Classical Greek mathematicians and philosophers like Euclid, Socrates, Plato and Aristotle.

In Mathematics **deductive proof** and **proof by contradiction** are often used, but in Science **inductive proof** is common.

IMPORTANT

A **deductive proof** is where a number of statements are made that lead logically to a conclusion. The assumptions are called **premises** or **axioms**.

An **inductive proof** is where a general conclusion is made from specific observations. In Science the general conclusion is called a **hypothesis** and it is usual to devise tests for the hypothesis.

A **proof by contradiction** assumes that the direct opposite of a statement is true and then a logical argument is used to show that the assumption is false.

Example 1

Decide whether these proofs are deductive, inductive or by contradiction.

a All sheep are black.
This is a sheep.
Therefore it is black.

b In triangle A, the angles add up to 180°.
In triangle B, the angles add up to 180°.
Therefore, in all triangles the angles add up to 180°.

c Assume that pentagons are squares.
Squares have 4 sides.
Therefore pentagons have 4 sides.
But pentagons have 5 sides, so the assumption cannot be true.
Therefore pentagons are not squares.

Solution

a Proof **a** begins with some statements and a conclusion is drawn. — Proof **a** is deductive.

b Proof **b** begins with some specific statements and a general conclusion is drawn. — Proof **b** is inductive.

c Proof **c** begins with an assumption and a contradiction is shown. — Proof **c** is by contradiction.

Some true statements work both ways, forwards and backwards. These are ‘if and only if’ or ‘necessary and sufficient’ statements.

Some true statements work only one way. These are ‘necessary’ or ‘sufficient’ but not both. Statements that work only one way are often written as ‘If … then …’

Example 2

Decide whether the following statements are 'if and only if' or 'necessary but not sufficient'.

a A square is a plane shape with 4 equal sides.

b In a right-angled triangle with hypotenuse h and other sides a and b, $h^2 = a^2 + b^2$.

Solution

a Check whether the statement can be true the other way: A plane shape with 4 equal sides is a square. This is not always true, but it is necessary that to be a square, it must have 4 equal sides.

For a plane shape to be a square it is necessary that it has 4 sides, but having four sides is not sufficient to make it a square, so it is necessary but not sufficient.

b The statement is true in reverse. If $h^2 = a^2 + b^2$, then the triangle must be right-angled.

If a triangle is right-angled, then it is true that $h^2 = a^2 + b^2$. It is also true that only if three of the sides of a triangle are related by $h^2 = a^2 + b^2$ can it be right-angled. The statement works in both directions, so it is 'if and only if' or 'necessary and sufficient'.

Arguments can be false or incorrect in different ways. It is possible to begin with correct premises and to draw a false conclusion. You can also start with incorrect premises and to draw a correct conclusion.

Example 3

Explain why each proof below is incorrect.

a All girls have red hair.
This is a girl.
$\therefore$ She has red hair.

b Lions like to eat meat.
Sally likes to eat meat.
$\therefore$ Sally is a lion.

Solution

a Even though the conclusion is valid, the original premise that all girls have red hair is false.

False premise

b The two premises may be correct but Sally may not be a lion.

The conclusion is invalid because the statements are not reversible. It is not necessarily true that anything that eats meat is a lion.

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EXERCISE 2.01 Mathematical proof

Concepts and techniques

Proof by induction

Proof by induction

Necessary and sufficient

1 Example 1 Decide whether each proof is deductive, inductive or by contradiction.

a $1 = 1^2$

$1 + 3 = 2^2$

$1 + 3 + 5 = 3^2$

$\vdots$

$\therefore\ 1 + 3 + 5 + \cdots + (2n - 1) = n^2$

b All marsupials have pouches.
A wallaby is a marsupial.
$\therefore$ A wallaby has a pouch.

c All rectangles have diagonals that are the same length.
$ABCD$ is a rectangle.
$\therefore\ AC = BD$

d If a is rational and b is irrational, then $a + b$ is irrational.
Assume that $a + b$ is rational.

Then $a + b$ can be written in the form $a+b=\dfrac{c}{d}$, where c and d are integers.

Since a is rational, it can also be written in the form $a=\dfrac{g}{h}$, where g and h are integers.

Clearly, $b=(a+b)-a$

$$\begin{aligned} &= \frac{c}{d}-\frac{g}{h} \\ &= \frac{ch-gd}{dh} \end{aligned}$$

But $(ch - gd)$ and dh are integers, so b is rational.
Therefore $a + b$ cannot be rational, so it must be irrational.

2 Write a valid conclusion for the following deductive argument.
All quadrilaterals have an interior angle sum of 360°.
$PQRS$ is a kite.
A kite is a quadrilateral.
$\therefore$ ______________________________

3 What general conclusion can be drawn from this inductive argument?
Sea temperatures rose in 2010.
Sea temperatures rose in 2011.
Sea temperatures rose in 2012.
$\therefore$ ______________________________

4 Example 2 Decide whether each statement is 'if and only if' or only 'necessary' but not 'sufficient'.

a All mammals are animals.

b All rhombuses have 4 equal sides.

c Congruent triangles are similar.

d All quadrilaterals with equal diagonals that bisect each other at right angles are squares.

5 Example 3 Determine whether the following proofs are true or false, giving reasons.

a A cactus is a prickly plant.
This plant is prickly.
∴ It is a cactus.

b All plane shapes with 2 pairs of adjacent sides equal are rectangles.
WXYZ has 2 pairs of adjacent sides equal.
∴ *WXYZ* is a rectangle.

6 Decide what type of argument this is below (deductive or inductive) and whether or not it is valid.
Mary has 5 children.
All of Mary's children are boys.
Mary is having another baby.
∴ It will be a boy.

7 Complete the proof so that it is valid.
If a number is divisible by 4, then it is composite.
This number is ______________________
∴ It is ______________________________.

Reasoning and communication

8 Johnny gave the following argument in a test.
Squares have two equal diagonals.
This shape has two equal diagonals.
∴ It is a square.
Explain why this argument is invalid and alter it to make a valid argument.

9 Consider the following argument.
All the people in the class have blond hair.
All the people in the class scored well in the maths test.
∴ People with blond hair score well in maths tests.
Decide whether or not the argument is valid. If not, can any valid conclusion be drawn from the statements?

10 Explain why this very common argument is wrong.
Mandy didn't study for the test.
Mandy did well in the test.
Dennis studied for the test.
Dennis did badly.
∴ Studying makes no difference so people shouldn't study for tests.

2.02 COUNTEREXAMPLES

A rigorous proof is required to show that a statement is true for all cases. It is much easier to show that a statement is false. All you need to do is find one case where the statement is false. This is called a **counterexample**.

IMPORTANT

A **counterexample** is an example that shows that a statement is not true for all cases.

Example 4

Provide a counterexample to show that the following statement is false.

All mammals have four legs.

Solution

Can you think of a mammal that doesn't have four legs?

Counterexample:
A blue whale is a mammal, but it doesn't have four legs.
Therefore the statement is false.

Example 5

Sophie gave the following argument to her teacher.

Statement: If a number is prime, then it is odd.

Decide whether or not she is correct.

Solution

Are all prime numbers odd?

The number 2 is prime but it is not odd.
This is a counterexample, therefore Sophie is incorrect.

Example 6

Jack was given the statement:

All rectangles have two pairs of opposite sides that are equal in length.

He said that this was not true because rhombuses have two pairs of opposite sides equal in length, but a rhombus is not a rectangle. Has he provided a counterexample?

Solution

The statement is about rectangles. Is it possible to find a rectangle where the opposite sides are not equal?

Jack is mistaken. He has given a counterexample to the converse statement: All quadrilaterals with two pairs of opposite sides equal in length are rectangles.

In Example **6**, Jack has assumed that the statement was 'necessary and sufficient' (two-way) when it was only 'necessary'. It is necessarily true that 'all rectangles have pairs of opposite sides equal in length', but this is 'not sufficient' to show that a figure is a rectangle.

EXERCISE 2.02 Counterexamples

Concepts and techniques

WS Counterexamples

1 Example 4 Find a counterexample for each statement below to demonstrate that the statement is false.

a If $x^2 = 9$, then $x = 3$.

b The statement $x^2 - 3x + 2 = 0$ is true for $x = 1, 2, 3, \ldots$

c All lines that never meet are parallel.

d If an animal has eight legs, then it is a spider.

2 Example 5 Decide whether or not the statements below are true or false and provide counterexamples for false ones.

a If a quadrilateral has diagonals that are equal in length, then it is a square.

b If $y^2 \geq 4$, then $y \geq 2$.

c $a^2 + b^2 \leq (a + b)^2$ for all $a, b \in \boldsymbol{R}$.

d If $nm = np$, then $m = p$.

e All triangles with three corresponding angles equal are congruent.

3 Example 6 In each case below, determine whether or not the counterexample shows that the statement is false.

a Statement: $x^3 + 11x = 6x^2 + 6$ for $x = 1, 2, 3, \ldots$

Counterexample: For $x = 4$, LHS $= 4^3 + 11(4) = 108$

RHS $= 6(4)^2 + 6 = 102$

LHS $\neq$ RHS

The statement is not true.

b Statement: All fish live in the sea.

Counterexample: Turtles live in the sea.

The statement is not true.

Reasoning and communication

4 Decide whether the statement below is true or false. If it is false, find a counterexample.

If $a > b$, then $\frac{1}{a} < \frac{1}{b}$.

5 Is it always true that $\frac{1}{n} > \frac{1}{n+1}$?

6 Is it always true that two straight lines must be either parallel or intersecting?

7 Are the intersections of the perpendicular bisectors of triangles always inside the triangle?

8 A circle can always be drawn through the three vertices of a triangle. Is this true for quadrilaterals?

9 Do the diagonals of a trapezium always intersect inside the trapezium?

10 Do the diagonals of a quadrilateral always intersect inside the quadrilateral?

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2.03 CONVERSE

Using mathematical symbolism, we can write *if P then Q* as $P \Rightarrow Q$. The notation $P \Rightarrow Q$ can also be read as *P implies Q*.

To find the converse of a statement, reverse the implication, so *if P then Q* becomes *if Q then P*, or $P \Rightarrow Q$ becomes $Q \Rightarrow P$, or $P \Leftarrow Q$.

IMPORTANT

For the statement *if P then Q*, the **converse** is *if Q then P*

or the **converse** of *P implies Q* is *Q implies P*.

In symbols, the **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.

The converse of a true statement may or may not be true, and vice versa. The converse may be true but the original statement may or may not be true.

Example 7

Write the converse of each statement.

a If a cat is grey, then it is Burmese.

b A triangle has two equal angles $\Rightarrow$ it is isosceles.

Solution

a In the original statement, P = grey cat, Q = Burmese cat.
In the converse, Q = Burmese cat, P = grey cat.

If a cat is Burmese, then it is grey.

b $P \Rightarrow Q$ becomes $Q \Rightarrow P$.

An isosceles triangle $\Rightarrow$ A triangle with two equal angles.

Example 8

Given the true statements below, write the converse of each and decide whether or not the converse is also true.

a If $x = 3$, then $x^2 = 9$.

b A set of points equidistant from a fixed point $\Rightarrow$ the points are on the circumference of a circle.

c If you are a police constable, then you wear a uniform.

Solution

a We know that if $x^2 = 9$, then $x = -3$ is also a solution.

Converse: If $x^2 = 9$, then $x = 3$.
The converse is not true because $(-3)^2 = 9$.

b	There is only one type of shape that obeys the rule.	Converse: Points on the circumference of a circle $\Rightarrow$ A set of points equidistant from a fixed point. The converse is true.
c	Police constables are not the only people to wear a uniform.	Converse: If you wear a uniform, you are a police constable. The converse is not true as there are other people who wear uniforms.

If a statement and its converse are both true, then this is called an **equivalence**.

IMPORTANT

Equivalence

If $P \Rightarrow Q$ and $Q \Rightarrow P$, then we can write $P \Leftrightarrow Q$.

In words, $P \Leftrightarrow Q$ means P *if and only if* Q or P *iff* Q.

Example 9

Decide whether equivalence holds for the statements below. If so, write $A \Leftrightarrow B$.

a A: The triangle is right-angled.
B: The sides a, b, c of the triangle are such that $a^2 = b^2 + c^2$ (a is the longest side).

b A: You get older.
B: You have more wrinkles.

Solution

a	$A \Rightarrow B$: If the triangle is right-angled, then the sides a, b, c of the triangle are such that $a^2 = b^2 + c^2$ (a is the longest side). $B \Rightarrow A$: If the sides a, b, c of the triangle are such that $a^2 = b^2 + c^2$ (a is the longest side), then the triangle is right-angled.	$A \Rightarrow B$ is true. $B \Rightarrow A$ is true. $\therefore A \Leftrightarrow B$ or A *iff* B.
b	$A \Rightarrow B$: If you get older, you have more wrinkles. $B \Rightarrow A$: If you have more wrinkles, you get older.	$A \Rightarrow B$ is true. $B \Rightarrow A$ is not true as you may have had more sun exposure leading to more wrinkles, rather than age. This is not an equivalence.

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EXERCISE 2.03 Converse

Concepts and techniques

WS Converse

1 Example 7 Write the converse of each statement below.

a If you live in Queensland, then you surf at the beach.
b If you are an interesting person, then you like maths.
c If you are the Premier, then you went to school in Darwin.
d If an animal is a lion, then it has big teeth.

2 Example 8 For each statement below, find its converse and decide whether or not it is true.

a If you do a lot of exercise, then you sweat.
b If you study hard, then your Mathematics improves.
c If a boy has a P-plate, then he has learnt to drive.
d If an animal is a koala, then it lives in a tree.

3 Example 9 For each true statement below, find its converse and decide whether or not it is an equivalence. If so, write it in the form *P iff Q*.

a If $x = 5$, then $3x = 15$.
b If a quadrilateral has two pairs of opposite sides equal, then it is a parallelogram.
c If $a > b$, then $a^2 > b^2$.
d If you have big feet, then you will have big shoes.

Reasoning and communication

4 Bib and Bob were having an argument. Bib stated: If students study more maths, then they get better jobs.
Bob stated: If students don't study more maths, then they don't get better jobs.
Explain to Bob why his statement is not the converse of Bib's.

5 Shane said that the opposite of being male was being female. Sarah said this was not true because the converse of male was not female. Who was right?

2.04 CONTRAPOSITIVE STATEMENTS

The contrapositive statement to *if P then Q* relies on the negation of *P*, or *not P*, written P', $\neg P$ or $\overline{P}$, and the negation of *Q*, or *not Q*, written Q', $\neg Q$ or $\overline{Q}$.

IMPORTANT

Contrapositive

If we consider the statement $P \Rightarrow Q$, then the contrapositive statement is $\overline{Q} \Rightarrow \overline{P}$.

In words, the contrapositive statement of *If P then Q* is *If not Q then not P*.

If the original statement $P \Rightarrow Q$ is true, then the contrapositive statement $\overline{Q} \Rightarrow \overline{P}$ is also true.

For instance, consider the statement:

If you have been awarded a university degree, then you have studied university subjects.

The contrapositive statement is:

If you have not studied university subjects, then you have not been awarded a university degree.

Example 10

Write the contrapositive of the following statements.

a If it is raining, then the road is wet.

b If a triangle has two equal angles, then it is an isosceles triangle.

Solution

a P = it is raining, Q = the road is wet
$\overline{P}$ = it is not raining, $\overline{Q}$ = the road is not wet.
Contrapositive: $\overline{Q} \Rightarrow \overline{P}$

The contrapositive is
If the road is not wet, then it is not raining.

b P = a triangle has two equal angles
Q = a triangle is an isosceles triangle
$\overline{P}$ = a triangle does not have two equal angles
$\overline{Q}$ = a triangle is not an isosceles triangle
Contrapositive: $\overline{Q} \Rightarrow \overline{P}$

The contrapositive is
If it is not an isosceles triangle, then a triangle does not have two equal angles.

You can verify that if $P \Rightarrow Q$ is true, then the contrapositive statement $\overline{Q} \Rightarrow \overline{P}$ is also true.

Example 11

Consider the following statement: If you are mortal, then you will die.

Identify P and Q and verify that $P \Rightarrow Q$ is true and that the contrapositive $\overline{Q} \Rightarrow \overline{P}$ is also true.

Solution

P = the first part	P = you are mortal.
Q = the second part	Q = you will die.
Is $P \Rightarrow Q$ true?	The statement is true.
$\overline{P}$ = the negation of P	Contrapositive:
$\overline{Q}$ = the negation of Q	*If you will not die, then you are not mortal.*
Contrapositive: $\overline{Q} \Rightarrow \overline{P}$	This statement is true.

Checking whether or not the contrapositive statement is true is a useful way of determining whether or not an original statement is true. A statement and its contrapositive are either both true or both false.

Example 12

Decide whether or not the statement below is true by checking the truth of the contrapositive.
If a quadrilateral has four right angles, then it is a square.

Solution

Find the contrapositive.

The contrapositive is:
If it is not a square, then a quadrilateral does not have four right angles.

Is it true? No, since a rectangle is not a square but it has four right angles.

The contrapositive statement is not true.
∴ The original statement is NOT true.

INVESTIGATION Mathematical proof in real-life

Inverse functions rely on whether or not the converse is true for a statement $P \Rightarrow Q$.

For instance, if $x = -3$, then $x^2 = 9$, but the converse is not necessarily true. If $x^2 = 9$, then $x = \pm 3$. This means that the inverse function of squaring is not necessarily taking the square root, unless the original number was positive.

Security on the internet relies on secure inverse functions, along with code-breaking in general.

Find out as much as you can about the *Enigma Machine* and Alan Turing. Where is Bletchley Park?

EXERCISE 2.04 Contrapositive statements

WS Contrapositive

Concepts and techniques

1 Example 10 Write the contrapositive statement for each statement below.
- a If you live in Coober Pedy, then you live underground.
- b If you can drive a car, then you have a licence.
- c If you are old, then you need a hearing aid.
- d If $x = 2$, then $x^2 = 4$.
- e A cane toad is an amphibian.
- f All men are mortal.

2 Given the contrapositive statements below, write down each original statement.
- a If the sky is not blue, then it is not sunny.
- b If a student does not study, then he does not pass his exams.
- c If you do not live in Alice Springs, then you do not live in the desert.
- d If a number is not positive, then it is not a counting number.
- e If a vehicle is not a car, then it does not have four wheels.
- f If you do not have a job, then you will not have money.

3 Example 11 Write the contrapositive of each true statement below, then verify that its contrapositive is also true.

a If $x \geq 3$, then $x^2 \geq 9$.

b If a quadrilateral is a rectangle, then it has diagonals equal in length.

c People who live in Tasmania live in Australia.

d All koalas are marsupials.

4 Example 12 Write the contrapositive of each statement below, then state whether the statement and its contrapositive are both true or both false.

a If $x \geq -3$, then $x^2 \geq 9$.

b If an animal is a bird, then it can fly.

c If you have been to university, then you will get a good job.

d If a number is rational, then it is real.

Reasoning and communication

5 Write the contrapositive of each statement below, then determine whether the statement is true or false.

a If two triangles have matching equal angles, then they are similar shapes.

b A quadrilateral with diagonals that meet at right angles is a rhombus.

c All elephants have a trunk.

6 Mickey was trying to work out whether the following statement was true by stating the contrapositive. He decided it was true. Can you explain where he went wrong?
Statement: If a quadrilateral has four equal sides, then it is a square.
Contrapositive: If a quadrilateral does not have four equal sides, then it is not a square.

7 If $P \Rightarrow Q$ is a true statement, which of the following is always true?

A $Q \Rightarrow P$ B $\neg P \Rightarrow \neg Q$ C $P \Leftrightarrow Q$ D $\neg Q \Rightarrow \neg P$

2.05 EUCLIDEAN GEOMETRY

In earlier years you may have studied geometry involving angles, lines, polygons, etc. This is called Euclidean geometry after the Greek mathematician Euclid (325 BC – 265 BC) who wrote about it in his now famous book *The Elements*. Euclidean geometry is the geometry of the plane, or two dimensions. There are other geometries, called *non-Euclidean geometries*, such as parabolic and hyperbolic geometries, which go beyond the plane. Euclidean geometry relies on *deductive* reasoning.

Recall the following theorems and definitions.

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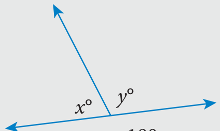
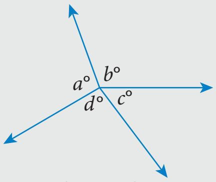
IMPORTANT

Angles on a straight line are supplementary (add to 180°)



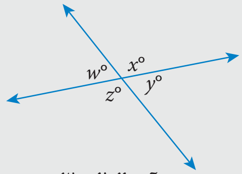
$x + y = 180$

Angles at a point (in a revolution) add to 360°



$a + b + c + d = 360$

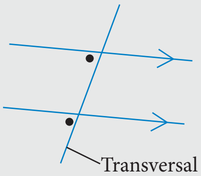
Vertically opposite angles are equal



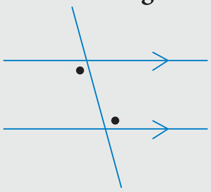
$w = y, x = z$

When parallel lines are crossed by a *transversal,* special pairs of angles are formed.

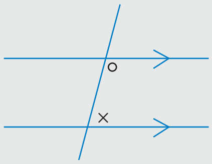
Corresponding angles are equal



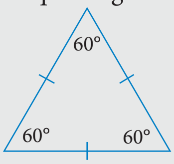
Alternate angles are equal



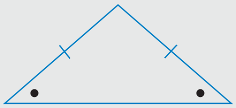
Co-interior angles are supplementary (add to 180°)



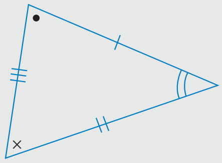
Equilateral triangle
3 equal sides
3 equal angles of size 60°



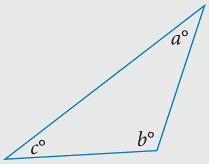
Isosceles triangle
2 equal sides 2 equal angles, opposite the equal sides



Scalene triangle
No equal sides
No equal angles

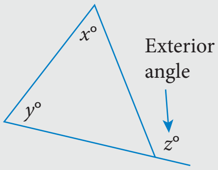


Angle sum of a triangle
Angles add up to 180°



$a + b + c = 180$

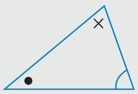
Exterior angle of a triangle
equal to the sum of the two interior opposite angles



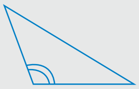
$z = x + y$

Classifying triangles by angles

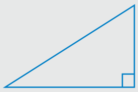
Acute-angled triangle



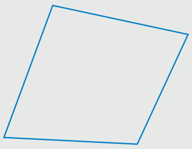
Obtuse-angled triangle



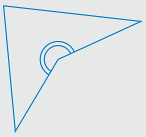
Right-angled triangle



Convex quadrilateral

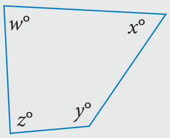


Non-convex quadrilateral



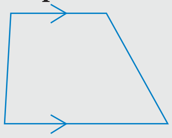
One angle greater than 180°

Angle sum of a quadrilateral



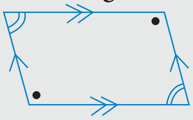
$w + x + y + z = 360$

Trapezium



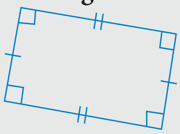
One pair of parallel sides

Parallelogram



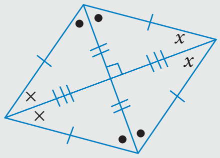
2 pairs of parallel sides, opposite angles equal

Rectangle



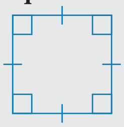
4 right angles

Rhombus



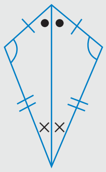
4 equal sides

Square



4 equal sides,
4 right angles

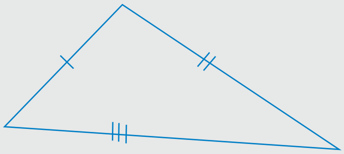
Kite

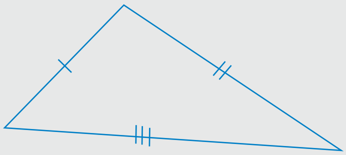


2 pairs of equal adjacent sides

Side, Side, Side (SSS)

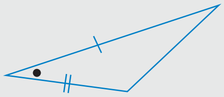
If three sides are equal in both triangles, then the two triangles are congruent.

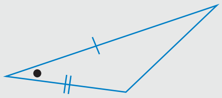




Side, Angle, Side (SAS)

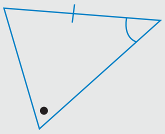
If two sides and the *included* angle are equal in both triangles, then the triangles are congruent.

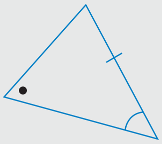




Angle, Angle, Side (AAS)

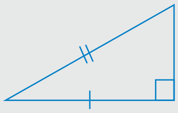
If two angles and the corresponding side are equal in both triangles, then the two triangles are congruent.

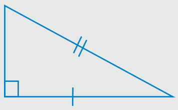




Right angle, Hypotenuse, Side (RHS)

In a right-angled triangle, if the hypotenuse and a second side are equal in both triangles, then the two triangles are congruent.





Similar figures

- have the same shape
- matching angles are equal
- matching sides are in the same ratio
- the symbol ||| means 'is similar to'
- vertices are named in matching order.

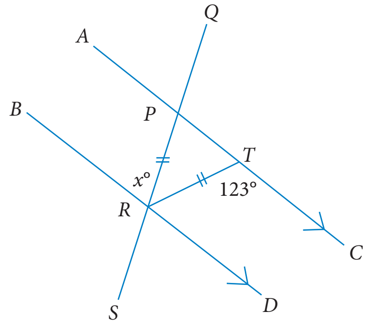
Tests for similar triangles

- two angles equal (equiangular)
- two sides about an included angle in proportion
- three matching sides in proportion.

Example 13

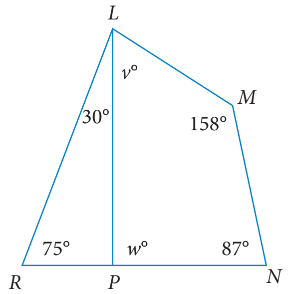
Find the value of each pronumeral.

a



QS is a line.

b



$RNML$ is a quadrilateral.

Solution

a $\triangle PTR$ is isosceles and $\angle CTR$ is the exterior angle of $\triangle PTR$. Find the base angles of $\triangle PTR$.

$\angle PTR = 180° - 123°$
$= 57°$ ($\angle$s on straight line AC)
$\therefore \angle TPR = 57°$ (base $\angle$s of $\triangle PTR$ equal)

$\angle TPR$ and $\angle PRB$ are alternate.

$\therefore x = 57$ (alternate $\angle$s equal, $AC \parallel BD$)

b First find w using $\triangle PRL$.

$w = 30 + 75$
$= 105$ (exterior $\angle$ of $\triangle PRL$ = sum of two interior opposite $\angle$s)

Now find v using the quadrilateral.

$v + 158 + 87 + 105 = 360$
$v = 10$
($\angle$ sum of quadrilateral $PLMN$)

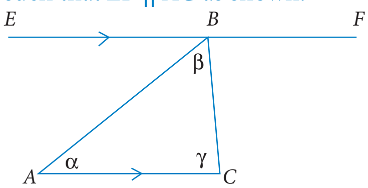
Example 14

Prove that the angle sum of a triangle is 180°.

Solution

Draw a diagram with the information you can use. Label the angles.

Construct $\triangle ABC$ and EF passing through B such that $EF \parallel AC$ as shown.



State what has to be proved.
RTP = required to prove

RTP
$\alpha + \beta + \gamma = 180°$.

Use the given information to deduce further information, giving reasons for each step. Use abbreviations and mathematical symbols where appropriate. Conclude the proof.

Proof
$\angle EBA = \alpha$ (alternate $\angle$s, $EF \parallel AC$)
$\angle FBC = \gamma$ (alternate $\angle$s, $EF \parallel AC$)
$\therefore \alpha + \beta + \gamma = 180°$ ($\angle$s on line EF)
$\therefore$ the angle sum of a triangle is 180°.

Example 15

Prove that any line between two sides of a triangle that is parallel to the third side is bisected by the line from the midpoint of the third side to the opposite vertex.

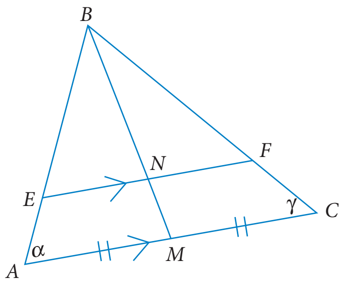
Solution

State what has to be proved.

RTP
Any line between two sides of a triangle that is parallel to the third side is bisected by the line from the midpoint of the third side to the opposite vertex.

Start by drawing a diagram and naming points.

Proof
Draw a diagram with triangle ABC and choose E on AB. Draw a line through E parallel to AC and name its intersection with BC point F. Draw a line from B to M, the midpoint of AC. Label the intersection with EF as N.



To prove the bisection, you need to show that $EN = NF$. You will use similar triangles. Name the base angles for convenience.

Name $\angle BAC = \alpha$ and $\angle BCA = \gamma$.

First prove that $\triangle BAM \parallel\!| \triangle BEN$. Use the parallel lines. Do the first angle.

$\angle BAM = \angle BEN$ (corresponding angles)

Do the next angle.

$\angle BMA = \angle BNE$ (corresponding angles)

Do the third angle.

$\angle ABM = \angle EBN$ (common angle)

State the similarity.

Thus $\triangle BAM \parallel\!| \triangle BEN$ (equiangular)

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The proof of similarity between the triangles on the other side is identical, apart from the names.	Similarly, $\triangle BCM \parallel\!\mid \triangle BFN$
Use similarity of $\triangle BAM$ and $\triangle BEN$.	$\dfrac{AM}{EN} = \dfrac{BM}{BN}$ ($\triangle BAM \parallel\!\mid \triangle BEN$)
Use similarity of $\triangle BCM$ and $\triangle BFN$.	$\dfrac{CM}{FN} = \dfrac{BM}{BN}$ ($\triangle BCM \parallel\!\mid \triangle BFN$)
Use the equalities to obtain the desired result.	Thus $\dfrac{AM}{EN} = \dfrac{CM}{FN}$ (both equal to $\dfrac{BM}{BN}$).
Use the fact that M is the midpoint.	But $AM = CM$
Draw the conclusion.	Thus $EN = FN$
Say you have done it!	**QED**

Example **15** proves that any line parallel to the side of a triangle will be bisected by the corresponding median. This result is a further deduction arising from the axiom that parallel lines never meet (Playfair's axiom).

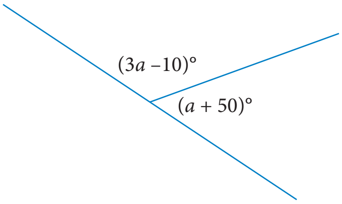
Congruence proofs and similarity proofs are also part of Euclidean geometry.

EXERCISE 2.05 Euclidean geometry

Concepts and techniques

1 Example 13 Find the value of the pronumerals. Give reasons.

a



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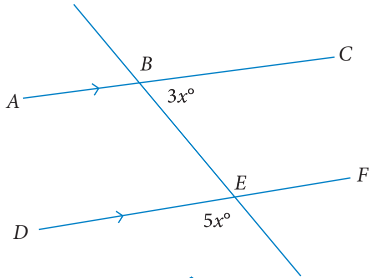

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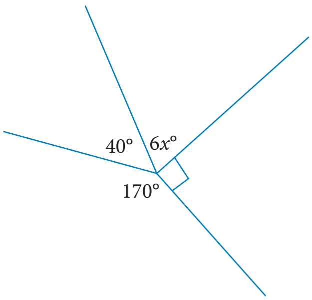
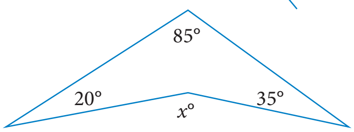


d



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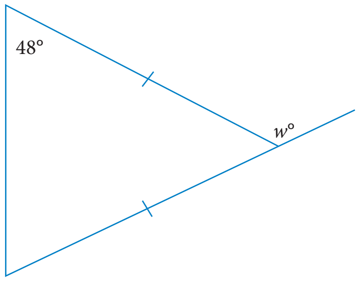
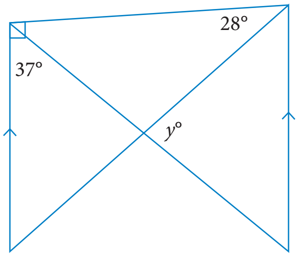
e



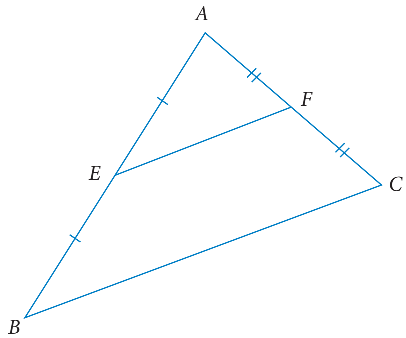
f



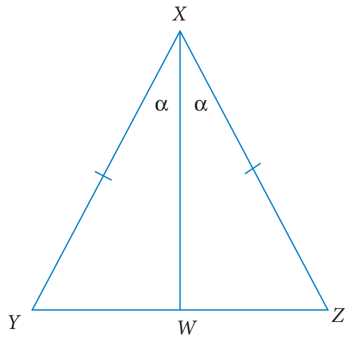
2 Example 15 In the diagram, E and F are the midpoints of AB and AC in $\triangle ABC$. Prove that

a $BC = 2EF$

b $BC \parallel EF$

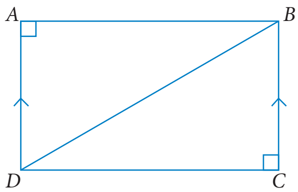


3 In the diagram, $XY = XZ$. XW bisects $\angle YXZ$. Prove that XW bisects YZ.



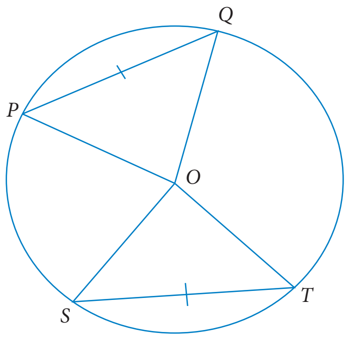
4 Recall the four triangle congruence proofs: SSS, SAS, AAS and RHS. Select the appropriate test and hence prove the following.

a



RTP: $\triangle ADB \equiv \triangle CBD$

b



RTP: $\triangle PQO \equiv \triangle TSO$

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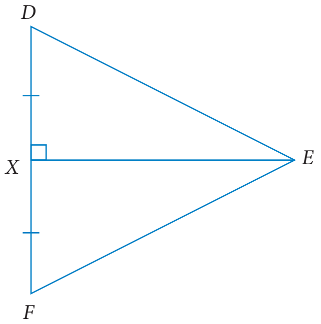
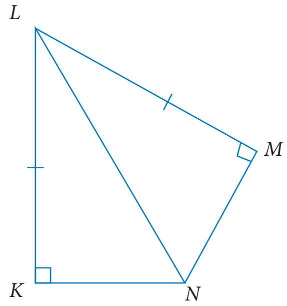
c



RTP: $\triangle DXE \equiv \triangle FXE$

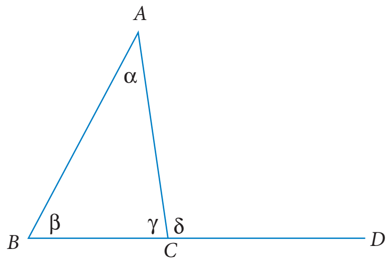
d



RTP: $\triangle LKN \equiv \triangle LMN$

Reasoning and communication

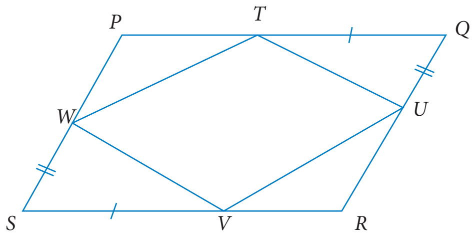
5 Example 14 Prove that the exterior angle of a triangle is equal to the sum of the two interior opposite angles. Use the diagram shown.



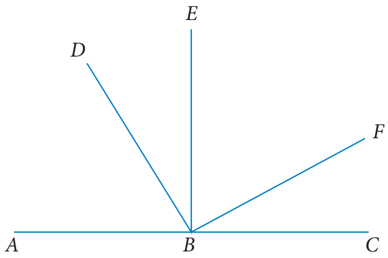
6 In the diagram, $PQRS$ is a parallelogram. T, U, V and W are points lying on the sides of the parallelogram as shown. $QU = WS$ and $TQ = VS$. Prove

a $\triangle WSV \equiv \triangle UQT$

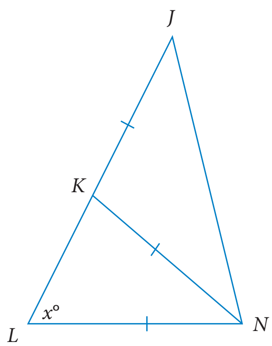
b $TUVW$ is a parallelogram.



7 In the diagram, $EB \perp AC$ and $DB \perp BF$. Prove that $\angle ABF + \angle DBC = 270°$.



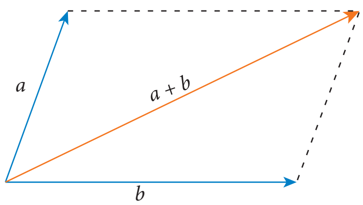
8 In the diagram, $JN = JL$, $JK = KN$ and $KN = LN$. Find the value of the pronumeral.

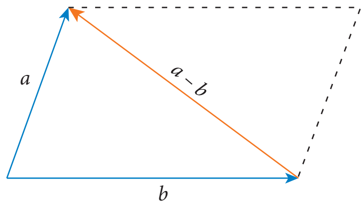


2.06 GEOMETRIC PROOFS USING VECTORS

You can use rules for vectors, including the parallelogram rule, to prove many of geometric properties of plane shapes.

Recall the parallelogram rule for vectors:





Example 16

Use vectors to prove that the diagonals of a parallelogram bisect each other.

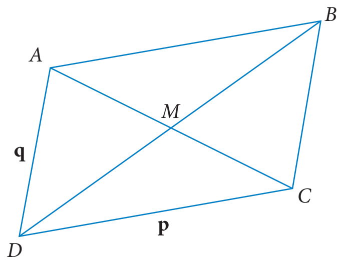
Solution

State what has to be proved.

RTP
The diagonals of a parallelogram bisect each other.

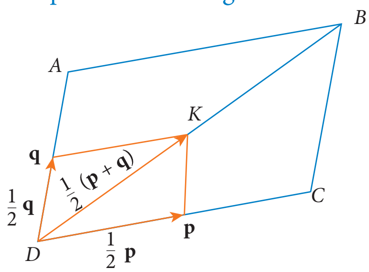
Start by drawing a diagram and naming points.

Proof
Draw a parallelogram with vertices of A, B, C, D and corresponding vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. The diagonals AC and BD meet at M ($\mathbf{m}$).



Express the lengths of the sides as vectors.
We need to consider the two separate diagonals, DB and AC.
We need to show that the midpoints of DB and AC, K and N respectively, coincide with the point M.

Let $\overrightarrow{DC} = \mathbf{p}$ and $\overrightarrow{DA} = \mathbf{q}$. Let $K(\mathbf{k})$ and $N(\mathbf{n})$ be the midpoints of the diagonals DB and AC respectively.



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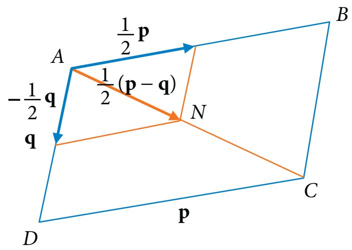


We need to find the vector $\mathbf{k}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. Use the parallelogram rule.	Consider the midpoint $K(\mathbf{k})$ of DB. $\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{DA}$ $= \mathbf{p} + \mathbf{q}$ So $\frac{1}{2}\overrightarrow{DB} = \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\overrightarrow{DA}$ $= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$ $= \frac{1}{2}(\mathbf{p}+\mathbf{q})$ $\therefore$ If $K(\mathbf{k})$ is the midpoint of DB, then $\mathbf{k} = \mathbf{d} + \frac{1}{2}(\mathbf{p}+\mathbf{q})$.
Now we need to find the vector $\mathbf{n}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. Use the parallelogram rule.	Now consider the midpoint N ($\mathbf{n}$)of AC. $\overrightarrow{AC} = \overrightarrow{DC} - \overrightarrow{DA}$ $= \mathbf{p} - \mathbf{q}$ So $\frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{DC} - \frac{1}{2}\overrightarrow{DA}$ $= \frac{1}{2}\mathbf{p} - \frac{1}{2}\mathbf{q}$ $= \frac{1}{2}(\mathbf{p}-\mathbf{q})$ $\therefore$ If $N(\mathbf{n})$ is the midpoint of AC, then $\mathbf{n} = \mathbf{a} + \frac{1}{2}(\mathbf{p}-\mathbf{q})$ $= (\mathbf{q}+\mathbf{d}) + \frac{1}{2}(\mathbf{p}-\mathbf{q})$ since $\mathbf{q} = \mathbf{a} - \mathbf{d}$ $= \mathbf{d} + \frac{1}{2}(\mathbf{p}+\mathbf{q})$ $= \mathbf{k}$
We have shown that the points K and N can be represented by the same vector, so they must be the same point.	$\therefore$ K and N are the same point, i.e., they both coincide with the point M.
Conclude the proof.	$\therefore$ The diagonals of a parallelogram bisect each other.
Say you have done it.	**QED**

Example 17

Use vectors to prove that if the diagonals of a quadrilateral bisect each other, then it must be a parallelogram.

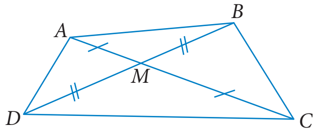
Solution

State what is required.

RTP
If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Begin the proof with a diagram and label the points and vectors.

Proof
Draw the quadrilateral $ABCD$ and label the intersection of the diagonals M. Call the position vectors of the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{m}$.



State the assumptions in useable form.

$AM = MC$ so $\mathbf{AM} = \mathbf{MC}$
$DM = MB$ so $\mathbf{MD} = \mathbf{BM}$

Write $\mathbf{AD}$ and $\mathbf{BC}$ as sums of known vectors.

$\mathbf{AD} = \mathbf{AM} + \mathbf{MD}$
$\mathbf{BC} = \mathbf{BM} + \mathbf{MC}$

Use the assumptions.

But $\mathbf{AM} = \mathbf{MC}$ and $\mathbf{MD} = \mathbf{BM}$

$$\begin{aligned}\text{So } \mathbf{AD} &= \mathbf{AM} + \mathbf{MD}\\ &= \mathbf{BM} + \mathbf{MC}\\ &= \mathbf{BC}\end{aligned}$$

State the meaning of the equality.

Since $\mathbf{AD} = \mathbf{BC}$, AD and BC are parallel.

There is no need to repeat the same steps with different letters.

Similarly, AB and DC are parallel.

State the overall conclusion.

Since both pairs of opposite sides are parallel, $ABCD$ is a parallelogram.

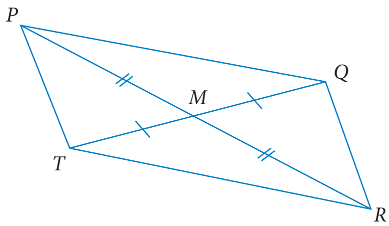
Say you have finished.

QED

EXERCISE 2.06 Geometric proofs using vectors

Concepts and techniques

1 Examples 16 , 17 Use vectors to prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
[Hint: In the diagram, M is the midpoint of PR and QT. Use vectors to prove that $PQRT$ is a parallelogram.]



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2 In order to prove the following statement:
The diagonals of a parallelogram meet at right angles if and only if it is a rhombus
you will need to prove two statements.

1 If the diagonals of a parallelogram meet at right angles, then it is a rhombus.

2 If it is a rhombus, then ____.

a Complete the statement above.

b Prove the first statement using vector geometry.

c Prove the second statement using vector geometry.

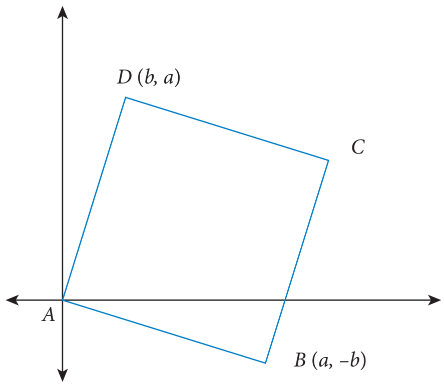
d What property of a rhombus does this prove?

3 Use vectors to prove that the diagonals of a square are equal in length.
Consider the diagram of the square with vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ represented by the coordinates $(a, -b)$ and (b, a) as shown.

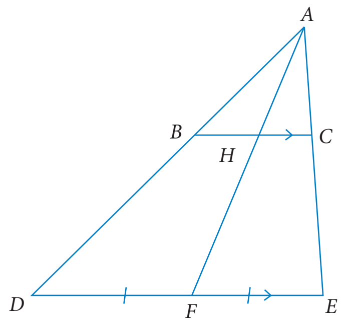
a Find the vector $\overrightarrow{AC}$ in terms of a and b.

b Find the vector $\overrightarrow{DB}$ in terms of a and b.

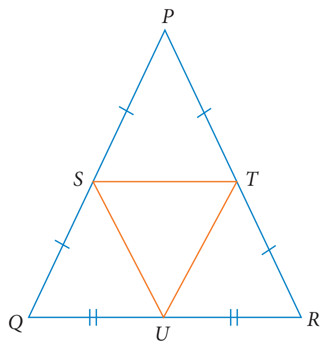
c Explain why AC and DB are equal in length.



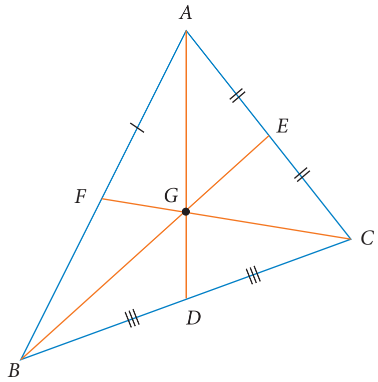
4 Use vectors to show that any line between two sides of a triangle that is parallel to the third side is bisected by the line from the midpoint of the third side to the opposite vertex.
Consider the diagram on the right.
RTP: $BH = HC$



5 Use vectors to show that the midpoints of the sides of an isosceles triangle form another isosceles triangle.
Consider the diagram on the right.
RTP: $\triangle STU$ is isosceles.



6 Use vectors to show that the medians of any triangle are concurrent.
To prove this theorem, you will need to use two definitions.
The **median** of a triangle is a line that joins the midpoint of a side to the opposite vertex.
Three or more lines are **concurrent** if they all meet at the same point.
Consider the diagram on the right.
RTP: FC, BE and DA intersect at the point G.
[Hint: Find the point of intersection G of FC and BE and show that AD passes through G.]



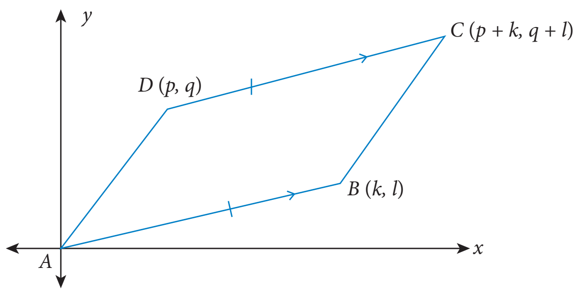
Reasoning and communication

7 In order to prove the following statement:
The line joins the midpoints of two sides of a triangle iff it is parallel to the third side and half its length
you will need to prove two statements.
1 If the line joins the midpoints of two sides of a triangle, then it is parallel to the third side and half its length.
2 If a line is parallel to one side of a triangle and half its length, then it ____________.
a Complete the statement above.
b Prove the first statement using vector geometry.
c Prove the second statement using vector geometry.

8 Use vector geometry to prove that the midpoints of the sides of any quadrilateral join to form a parallelogram.

9 Use vector geometry to show that the opposite sides of a parallelogram are equal in length (Hint: Place the origin \and axes so that so that the vertex A of the parallelogram $ABCD$ is at the origin and the side AD is along the x-axis). Write the vectors $\mathbf{AB} = (p, q)$ and $\mathbf{AD} = (0, t)$. Use vector geometry to show that if the opposite sides of a quadrilateral are equal in length and parallel, then it is a parallelogram.

10 Consider the diagram below, showing the quadrilateral $ABCD$ represented by the coordinates shown. $DC \parallel AB$ and $DC = AB$.
a Prove that $AD \parallel BC$.
b Prove that $AD = BC$.
c Hence explain why $ABCD$ is a parallelogram.



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2.07 QUANTIFIERS AND PROOF BY CONTRADICTION

You can use mathematical symbols to write mathematics in a succinct and unambiguous form. There are some mathematical symbols called *quantifiers* that are commonly used in mathematics proofs. Two of these are shown on the right.

IMPORTANT

Quantifiers

$\forall$	for all
$\exists$	there exists

Example 18

Use mathematical symbols to convert the following word sentences into mathematical statements.

a For all rational numbers n, there exist integers p and q such that $n = \frac{p}{q}$.

b For all positive real numbers x there exists a real number y such that $2^y = x$.

Solution

a Use the quantifiers to replace 'for all' and 'there exists'. Use set notation to describe rational numbers or integers.

$\forall n \in \boldsymbol{Q}, \exists p, q \in \boldsymbol{Z}$ such that $n = \frac{p}{q}$.

b We need to specify that x is real and positive.

$\forall x \in \boldsymbol{R}$ such that $x > 0$,
$\exists y \in \boldsymbol{R}$ such that $2^y = x$.

In Section 2.01 you saw some examples of proof by contradiction. The next example is a well-known proof by contradiction using algebraic notation. Note the use of mathematical symbols.

Example 19

RTP: $\sqrt{2}$ is irrational.

Solution

Proof by contradiction begins by assuming the opposite of what we are trying to prove. It is also necessary to define the variables carefully.

Proof

Assume $\sqrt{2}$ is rational, i.e., assume that $\exists p, q \in \boldsymbol{Z}$ such that $\sqrt{2} = \frac{p}{q}$, where $q \neq 0$ and p, q have no common factors.

Squaring both sides, $2 = \frac{p^2}{q^2}$.

Rearranging.

$\Rightarrow p^2 = 2q^2$

If the square of an integer is even, then the integer must be even.

$\Rightarrow p^2$ is even
$\Rightarrow p$ is even

Equate both equations.	$\Rightarrow p = 2k$ for some $k \in \mathbf{Z}$ $\Rightarrow p^2 = 4k^2 = 2q^2$ $\Rightarrow q^2 = 2k^2$ $\Rightarrow q^2$ is even $\Rightarrow q$ is even
We have proved that the original assumption is flawed.	But this means that both p and q have a common factor of 2. CONTRADICTION. $\therefore \sqrt{2}$ is irrational. **QED**

Note: The abbreviation *QED* stands for *quod errat demonstrandum*, which can be translated from the Latin as *demonstrated as required.* It is commonly put at the end of a proof to show that the proof is complete.

EXERCISE 2.07 Quantifiers and Proof by Contradiction

WS Quantifiers

Concepts and techniques

1 Example 18 Use mathematical symbols, including quantifiers, to write each worded statement below in mathematical notation.

a For all real numbers x, the square of x is positive or zero.

b There exists a rational number w such that w is greater than 2 but is less than or equal to 7.

c For all adult human beings, there exists a perfect match.

d For all real numbers a and b, there exists a real number c such that c is between a and b.

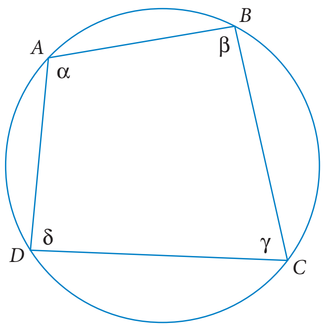
e For all real numbers x such that $x \geq 0$, there exists a real number y, where $y \geq 0$ such that $y = \sqrt{x}$.

2 Example 19 Follow the steps in Example **19** in order to use proof by contradiction to prove that:

a $\sqrt{3}$ is irrational

b $\sqrt{5}$ is irrational.

3 Use proof by contradiction to prove the following theorem.
If the four vertices of a quadrilateral lie on a circle, then the opposite angles are supplementary.
Hint: Begin with the diagram and statement:
Proof:
Assume that A, B, C and D lie on a circle of centre O.
Assume that $\alpha + \gamma \neq 180°$.



Reasoning and communication

Using Proof by Contradiction, prove each of the theorems below.

4 The bisector of the angle between the equal sides of an isosceles triangle intersects the third side at right angles.

5 The opposite sides of a parallelogram are equal in length.

6 The base angles of an isosceles triangle are equal.

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CHAPTER SUMMARY
MATHEMATICAL PROOF

2

- A **deductive proof** is where a number of statements are made that lead logically to a conclusion. The assumptions are called premises or **axioms**.
- An **inductive proof** is where a general conclusion is made from specific observations. In science, the general conclusion is called a **hypothesis** and it is usual to devise tests for the hypothesis.
- A **proof by contradiction** assumes that the direct opposite of a statement is true and then a logical argument is used to show that the assumption is false.
- A **counterexample** is an example that shows that a statement is not true.
- The statement *if P then Q* or *P implies Q* can be written $P \Rightarrow Q$.
- For the statement *if P then Q*, the **converse** is *if Q then P*

  or the **converse** of *P implies Q* is *Q implies P.*

  In symbols, the **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- **Equivalence**

  In words, $P \Leftrightarrow Q$ means

  P if and only if Q or *P iff Q.*

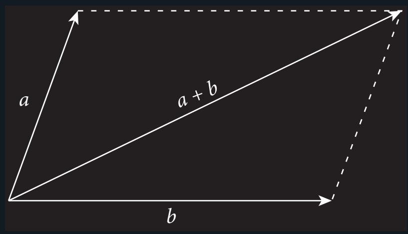
  If $P \Rightarrow Q$ and $Q \Rightarrow P$, then we can write $P \Leftrightarrow Q$.
- The negation of P is *not P*, written $\neg P$, P' or $\overline{P}$
- **Contrapositive**

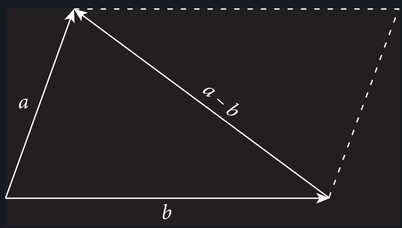
  If we consider the statement $P \Rightarrow Q$, then the contrapositive statement is $\overline{Q} \Rightarrow \overline{P}$.

  In words, the contrapositive statement of

  If P then Q is *If not Q then not P.*

  If the original statement $P \Rightarrow Q$ is true, then the contrapositive statement $\overline{Q} \Rightarrow \overline{P}$ is also true.
- Euclidean geometry is the geometry of two dimensions, involving lines, angles, congruence, similarity, triangles, circles and other plane shapes.
- The parallelogram rule for vectors





- *RTP* = required to prove
- **Quantifiers**

  $\forall$ for all
  $\exists$ there exists
- *QED* = *quod errat demonstrandum*

2

CHAPTER REVIEW
MATHEMATICAL PROOF

Multiple choice

1 Example 11 The contrapositive of $A \Rightarrow B$ is

A $B \Rightarrow A$ B $\overline{A} \Rightarrow \overline{B}$ C $A \Leftrightarrow B$ D $\overline{B} \Rightarrow \overline{A}$

2 Example 7 Select the converse of the statement

If you eat poison berries, then you get sick.

A If you don't eat poison berries, then you don't get sick.
B If you get sick, then you eat poison berries.
C If you don't get sick, then you don't eat poison berries.
D You get sick if you eat poison berries.

3 Example 4 A counterexample to show that the statement '*All snakes bite*' is false would be

A snakes don't bite
B spiders bite
C if you bite, then you are a snake
D my pet snake doesn't bite

4 Example 18 Choose the mathematical notation for the statement

For all rational numbers x, there exist integers y and w such that $x = \frac{y}{w}$ *where w is non-zero.*

A $\forall x \in \boldsymbol{Q}, \exists y, w \in \boldsymbol{Z}$ such that $x = \frac{y}{w}$ where $w \neq 0$.

B $\forall x \in \boldsymbol{R}, \exists y, w \in \boldsymbol{Z}$ such that $x = \frac{y}{w}$ where $w \neq 0$.

C $\forall x \in \boldsymbol{Q}, \forall y, w \in \boldsymbol{Z}$ such that $x = \frac{y}{w}$ where $w \neq 0$.

D $\forall x \in \boldsymbol{Q}, \exists y, w \in \boldsymbol{N}$ such that $x = \frac{y}{w}$ where $w \neq 0$.

5 Example 9 The mathematical statement $C \Leftrightarrow D$ means

A if C then D
B C implies D
C C iff D
D D only if C

Short answer

6 Example 5 Decide whether each statement below is true or false. If false, give a counterexample.

a If the square of an integer is even, then the integer must be even.
b If $x^2 \geq 4$, then $x \geq 2$.
c If an animal has a pouch, then it is a marsupial.
d If a number is prime, then it is odd.

7 Example 1 Complete the deductive argument below.
People who practise the piano every day improve their playing.
Elsie practises the piano every day.
∴ ____________________.

8 Example 1 Complete the inductive argument below.
There were floods in Queensland in 2010.
There were floods in Queensland in 2012.
There were floods in Queensland in 2014.
∴ ______________________________.

9 Example 7 State the converse of each statement below.
a If you are a politician, then you tell the truth.
b If a number is positive, then the square of the number is positive.
c If a student has failed an exam, then he/she must resit the exam.
d If an animal is a fish, then it can swim.

10 Example 11 State the contrapositive of each statement in question **9**.

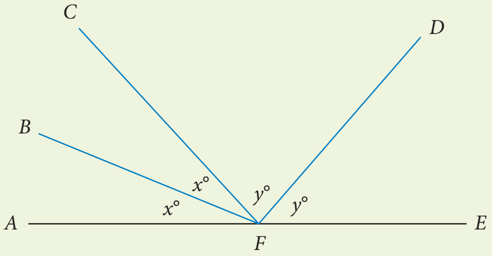
11 Example 9 Decide which of the following true statements are equivalences. Rewrite them using *iff*.
a If you are royal, then you are often photographed.
b If a number is composite, then it has more than two factors.
c If a quadrilateral has diagonals that bisect each other at right angles, then it is a rhombus.
d If an animal is a giraffe, then it eats plants.

Application

12 Consider the contrapositive to determine whether each statement is true.
a If a quadrilateral is a kite, then it has two pairs of adjacent sides equal.
b If $\exists a, b \in \boldsymbol{R}$ such that $a > b$, then $a^2 > b^2$.
c If a triangle has two equal angles, then it has two equal sides.
d If a vehicle is a bike, then it has wheels.

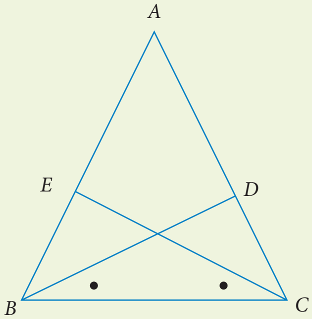
13 Use Euclidean geometry to prove each of the following.

a



Prove that $\angle BFD = 90°$

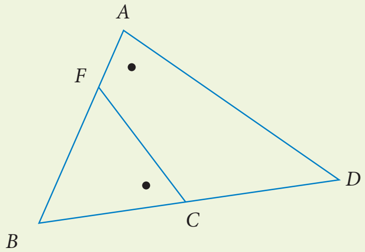
b



$AB \perp EC$, $AC \perp DB$

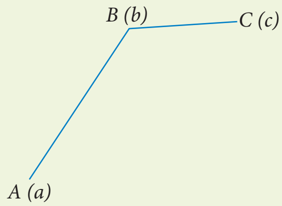
Prove $\triangle BEC \equiv \triangle CDB$

c



Prove $\triangle BAD \,|||\, \triangle BCF$

14 The points A, B and C are represented by the vectors **a**, **b** and **c** as shown. $ABCD$ forms a parallelogram.



Use vectors to find:

a the fourth vertex D (**d**) of the parallelogram $ABCD$, in terms of **a**, **b** and **c**

b the midpoint M (**m**) of AC.

15 Use proof by contradiction to prove that $\sqrt{7}$ is irrational.

16 Explain what is wrong with the following argument.
Ali said "If you go swimming, then you get wet."
Ben argued "No, you can get wet having a shower."

17 Decide whether the statement is 'if and only if', or 'necessary but not sufficient'.
A trapezium has a pair of parallel opposite sides.

18 Consider the statement:

If $x < 2$, *then* $\dfrac{1}{x^2} > \dfrac{1}{4}$.

Zeni tested this with $x = 1$ and decided that the statement was true.
Can you find a better counterexample to prove that the statement is false?

19 Write the converse of the true statement below and decide whether or not the converse is also true.
If a median of a triangle is perpendicular to the side it intersects, then the triangle is isosceles.

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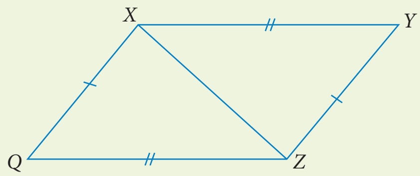
20 Write the contrapositive statement for:

If $0 < a < 3$, *then* $\frac{1}{a} > \frac{1}{3}$.

Is the statement true or false?

21 Consider the quadrilateral $XYZQ$ as shown.



Prove that

a $\triangle QXZ \equiv \triangle YZX$

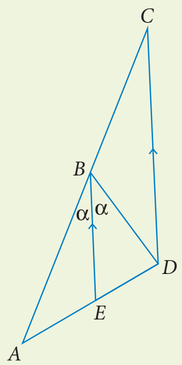
b $XY \parallel QZ$

c $XQ \parallel YZ$

d Explain why $XYZQ$ is a parallelogram.

22 Consider the diagram on the right where A, B, C are collinear, BE bisects $\angle ABD$ and $BE \parallel CD$.

Prove that $\triangle DBC$ is isosceles.



Practice quiz

9780170250276

1 MIXED REVISION
CHAPTERS 1 • 2 • 3

Multiple choice

1 The direction of the vector (–3, 7) is about:
A –113° B –67° C 67° D 113° E 247°

2 What is the contrapositive to the statement?
'*People with an X and a Y sex chromosome are male*'?
A *People with two X chromosomes are female*
B *People who do not have an X and a Y chromosome are female*
C *Males have an X and a Y chromosome*
D *Females do not have an X and a Y chromosome*
E *People who are not male do not have an X and a Y chromosome*

3 A bag contains 12 identical beads, of which 6 are green, 2 are black and 4 are red. How many ways can you put all the beads in a line?
A 576 B 11 880 C 13 860 D 27 720 E 479 001 600

4 If $\mathbf{m} = 3\mathbf{i} - 4\mathbf{j}$, which of the following vectors is in the opposite direction to $\mathbf{m}$?
A $1.5\mathbf{i} - 2\mathbf{j}$ B (–9, 12) C $6\mathbf{i} + 8\mathbf{j}$ D (–4, 3) E $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

5 The mathematical statement $C \Rightarrow B$ means
A B is a necessary and sufficient condition for C
B C is a necessary condition for B
C C is a sufficient condition for B
D B if and only if C
E If B is true, then C must also be true

6 What is the value of ${}^{10}P_3$?
A 30 B 120 C 720 D 1000 E 604 800

7 A golf ball is thrown from a height of 1.7 m at a brick wall 3.5 m from the thrower. The ball rebounds and is caught at a height of 1.5 m by the thrower who has remained stationary. The displacement of the ball is:
A 0 B 20 cm C 1.5 m D 1.7 m E 3.5 m

8 Which of the following is a counter example to the statement?
'*The time is earlier in all countries to the west of Jamaica*'?
A Honduras is to Jamaica's west and its time is 1 hour earlier than Jamaica's
B Jamaica's time is 3 hours later than Vancouver's (in Canada), which is to its west
C Australia is to the west of Jamaica and its time is 15 hours later than Jamaica's
D Britain is to Jamaica's east and its time is 4 hours later than Jamaica's
E Jamaica is west of Latvia and Latvia's time is 8 hours later than Jamaica's

9 A product key has 2 letters followed by 3 digits. How many different product keys are possible?
A 260 B 39 000 C 58 500 D 676 000 E 936 000

Short answer

1 Convert the vector (3, –8) to polar form.

2 Give counter examples for each of the following statements.
   a All mammals have four legs.
   b The square of a number is always larger than the number.
   c A quadrilateral with a right-angle must be a rectangle.

3 A customer at an organic produce shop is buying ripe peaches, apricots and mangoes. Her basket has three compartments that can hold 6 peaches, 10 apricots and 4 mangoes without squashing them. Her children have to pick individual fruit out for her, and she puts them in the basket or tells the child to take it back.
   a How many items *must* she put in to ensure at least one compartment is full?
   b If the fruit are put into the basket in random order, what is the probability that the first 3 she puts in will be apricots?

4 Find the resultant of $\mathbf{a} = (8, -7)$ and $\mathbf{b} = -6\mathbf{i} + 11\mathbf{j}$.

5 Give examples to show the difference between inductive and deductive logic.

6 How many four letter words can be made from the letters of SIGNATURE if one of the centre letters must be a vowel?

Application

1 A boat travels 75 km north-east and then 48 km in the direction N 60° W. Find the displacement of the boat as a vector in polar form.

2 The velocity of a boat changes from 32 knots at N 25° E to 24 knots at a bearing of 113°. Calculate the change in velocity.

3 Joeline asserted that the formula $x^2 + x + 11$ always gives a prime number. Prove that she is wrong.

4 Prove that the diagonals of a rhombus intersect at right angles.

5 Of the 23 students in a class, 2 are called Elizabeth, 2 are called Michael and 3 are called Ben. What is the probability that if the students leave the classroom in random order, the Elizabeths, Michaels and Bens are first out the door?

6 In Chess competitions, players score 1 point for a win, $\frac{1}{2}$ a point for a draw and no points for a loss. The Doeberl Cup is played over the Easter long weekend in Canberra and the Major division has 7 rounds. Like most Chess competitions, players participate in every round and are paired using the Swiss system. Commonly, the winner has a final score of $5\frac{1}{2}$. This means that it is *possible* for someone to win the tournament after losing their first game! What is the probability that in a year where the winner did have a score of $5\frac{1}{2}$, the winner lost his/her first game?

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3

TERMINOLOGY

addition principle
circular permutation
combinatorics
factorial
multiplication principle
mutually exclusive
permutation
pigeonhole principle
recursive definition

COMBINATORICS

COUNTING METHODS AND PERMUTATIONS

Prior learning

PERMUTATIONS (ORDERED ARRANGEMENTS)

- solve problems involving permutations (ACMSM001)
- use the multiplication principle (ACMSM002)
- use factorial notation (ACMSM003)
- solve problems involving permutations and restrictions with or without repeated objects. (ACMSM004)

THE PIGEON-HOLE PRINCIPLE

- solve problems and prove results using the pigeon-hole principle. (ACMSM006) AC

3.01 THE MULTIPLICATION PRINCIPLE

Counting is fundamental to mathematics, science, technology and our daily lives. **Combinatorics** is the mathematics of systematically counting *finite* collections. The methods of counting are most important for finite sets that produce many things to be counted.

Consider a restaurant menu that has 5 different starters, 6 different main courses and 4 different desserts. How many different three-course meals could you choose? Obviously, for each starter you could have 6 different main courses, so that makes $5 \times 6 = 30$ different combinations of starter and main course. For every one of these combinations, you could have 4 different desserts, so there would be $30 \times 4 = 120$ different three-course meals. This is an example of the **multiplication principle**.

IMPORTANT

The **multiplication principle** says that for a choice made in two stages with a ways for one part and b ways for the second part, there are $a \times b$ choices altogether.

For a choice made in n stages: if there are a_1 ways for the first part, a_2 ways for the second part, a_3 ways for the third part, and so on, then there are $a_1 a_2 a_3 \ldots a_n$ choices altogether.

For an extended list like that above, we usually say 'a_i ways for the ith part'.

You can show a multiple product like a multiple sum, but you use Π instead of Σ. So the product notation $\prod_{i=1}^{3} a_i$ means $a_1 \times a_2 \times a_3$.

Example 1

A personal identification number (PIN) has 4 digits. How many PINs are possible?

Solution

There are 4 digits in the PIN.	Number of choices $= a_1 \times a_2 \times a_3 \times a_4$
Each part has 10 possible choices.	$= 10 \times 10 \times 10 \times 10 = 10\,000$
Write the answer.	There are 10 000 possible PINs.

When you choose coloured markers for a board game like monopoly, each person has one less marker to pick from than the person before. You usually throw a die to pick who goes first and then go around the rest of the group.

Example 2

The markers in a board game are green, red, black, white, blue and purple. In how many different ways can 4 players pick their colours?

Solution

There are four people.	Number of choices $= a_1 \times a_2 \times a_3 \times a_4$
The first person picks from 6 colours, the next from 5; so $a_1 = 6$, $a_2 = 5$, and so on.	$= 6 \times 5 \times 4 \times 3$
Work out the answer.	$= 360$
Write the answer.	The players can pick their colours in 360 ways.

You often get the pattern of numbers shown in Example **2**. This kind of pattern is so common that there is a special notation for products that go all the way down to 1. An exclamation mark is used on the end, for example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

> **IMPORTANT**
>
> The **factorial** sequence is written as $n!$ and defined as either:
>
> $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$,
>
> or **recursively** as $0! = 1$ and $n! = n \times (n-1)!$

The second method of working out the factorial sequence is a **recursive** definition. You saw in junior maths that the sequence 3, 5, 7, 9, 11, … can be given as $t_n = 2n + 1$ or recursively as $t_1 = 3$ and $t_n = t_{n-1} + 2$.

If you work downwards from $4! = 24$, then $3! = \frac{4!}{4} = \frac{24}{4} = 6$, $2! = \frac{3!}{3} = \frac{6}{3} = 2$, $1! = \frac{2!}{2} = \frac{2}{2} = 1$ and $0! = \frac{1!}{1} = \frac{1}{1} = 1$. You can also think of 0! as the number of ways of selecting no colours in Example **2**. There is only one way to pick none because there is only one way to leave out all of the colours.

WS Factorial notation

INVESTIGATION Growth rates

Some sequences of numbers increase by adding the same amount to each successive term. For example, the sequence 3, 8, 13, … increases by 5 each term. This is called **arithmetic growth**.

Another kind of growth uses powers. For example, triangular numbers like 1, 3, 6, 10, 15, … show the number of dots in triangular arrangements with sides of 1, 2, 3, … The number of dots go up according to the rule $a_n = 0.5n^2 + 0.5n$. This is called **polynomial growth** because the term is a polynomial involving n. This is 'faster' than increasing by a fixed amount each time.

In a geometric sequence like 3, 15, 75, … the general term is $3 \times 5^{n-1}$. The n is in the exponent, so this kind of growth is called **exponential growth**. Think of some examples of exponential growth.

Compare arithmetic growth and exponential growth: which is faster?

Now think of factorial growth: $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, …

Compare **factorial growth** with exponential growth, for any power.

Consider powers of 5, compared to factorial growth: 1, 5, 25, 125, 625, …

Which gets really big first?

Choose another power, like 15. Does $n!$ ever get bigger than 15^n?

Can you demonstrate that factorial growth always (eventually) passes exponential growth, exponential always passes polynomial growth and polynomial growth always passes arithmetic growth?

Example 3

A coin is tossed 6 times.

a How many different sequences of heads and tails are there?

b What is the probability of guessing the exact sequence of heads and tails?

Solution

a How many H/T choices are there?

Each toss has 2 possible outcomes: heads or tails.

$$\text{Number of choices} = a_1 \times a_2 \times a_3 \times a_4 \times a_5 \times a_6$$
$$= 2 \times 2 \times 2 \times 2 \times 2 \times 2$$
$$= 64$$

b Only one of the sequences is correct.

$$\text{Probability} = \frac{1}{64} = 0.015625$$

Write the answer.

There is a 1.5625% probability of guessing the exact sequence.

EXERCISE 3.01 The multiplication principle

1 $5! =$

A 15 **B** 20 **C** 25 **D** 120 **E** 240

2 $50! =$

A $50 \times 49 \times 48$ **B** $2450 \times 48!$ **C** $2 \times 25!$

D $5! \times 10!$ **E** 5×10^{10}

3 Example 1 A password has 4 letters, with repetition allowed. How many combinations are possible?

4 A motorcycle numberplate is made up of any 2 letters followed by any 2 numbers. How many numberplates of this type are available?

5 A password has 5 letters followed by 4 numbers. If any letter of the alphabet or number (including repetitions) can be used, how many different passwords can be formed? Leave your answer in index form.

6 A witness saw most of the numberplate on a getaway car except for the first letter and the last number. How many different cars do the police need to check in order to find this car, assuming the numberplate has 3 letters and 3 numbers?

7 A certain brand of computer has a serial number made up of 10 letters followed by 15 numbers. How many computers with this type of serial number can be made? Leave your answer in index form.

8 Australia uses 4 digit postcodes and Victoria's start with 3. How many different postcodes are available in Victoria?

9 A country town has telephone numbers starting with 63 followed by any 6 other numbers from 0 to 9. How many telephone numbers are possible in this town?

10 Yasmin has 12 tops, 5 pairs of jeans and 5 pairs of shoes in her wardrobe. If she randomly chooses a top, pair of jeans and shoes, how many combinations are possible?

11 A car manufacturer produces cars in 8 different colours, with either manual or automatic gear transmission, and 4 different types of wheels. How many different combinations can it produce?

12 A company manufactures 20 000 000 computer chips. If it uses a serial number on each one that consists of 10 letters, will there be enough combinations for all these chips?

13 A manufacturer of computer parts puts a serial number on each part, consisting of 3 letters, 4 numbers, then 4 letters. The number of parts sold is estimated to be 5 million. Will there be enough combinations of this serial number to cope with these sales?

14 Example 2 Kate has chosen a pale blue, a pale green and a pale orange dress for her 3 bridesmaids. If the three colours are randomly given to each bridesmaid, how many different possibilities are there?

15 In a computer car race game, the cars that come first, second and third are randomly awarded. If there are 20 cars, how many possible combinations of first, second and third are there?

16 Jacquie only has 4 chocolates left and decides to randomly choose which of her 6 friends will receive one. How many possible ways are there in which can she give the chocolates away?

17 Three prizes are given away at a concert by taping them underneath random seats. If there are 200 people in the audience, in how many ways can these prizes be won?

18 There are 7 clients at a hairdressing salon. If 3 free haircuts are randomly given away, in how many ways could this be done?

19 A flock of 28 pelicans is fed 6 fish carcasses. If each carcass is given to a different pelican, in how many ways can this happen?

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20 Example 3 A PIN has 4 numbers. If I forget my PIN, I am allowed 3 tries to get it right. Find the probability that I get it within the 3 tries.

21 A restaurant offers 7 main courses and 4 desserts, as well as 3 different types of coffee.
- a How many different combinations of main course, dessert and coffee are possible?
- b Find the probability that I randomly pick the combination voted as the best by customers.

22 A telephone number in a capital city can start with a 9 and has 8 digits altogether.
- a How many telephone numbers are possible?
- b If I forget the last 3 digits of my friend's telephone number, how many numbers would I have to try for the correct number?

23 A password consists of 2 letters followed by 5 numbers. What is the probability that I randomly guess the correct password?

24 A bridal shop carries 12 different types of bridal dresses, 18 types of veils and 24 different types of shoes. If Kate chooses a combination of dress, veil and shoes at random, what is the probability that she chooses the same combination as her friend Jane?

25 A set of cards is numbered 1 to 100 and 2 are chosen at random.
- a How many different arrangements of ordered pairs are possible?
- b W hat is the probability that a particular ordered pair is chosen?

26 Each of 10 cards has a letter written on it from A to J. If 3 cards are selected at random, find the probability that they spell out CAB.

Reasoning and communication

27 A city has a population of 3 500 000. How many digits should its telephone numbers have so that every person can have one?

28 Three-pin locks have three 'pins' inside the lock that are pushed into the correct position by the key to open the lock. The pins have different possible lengths that require different keys. The pins are normally made to be of different lengths, so that the key is not straight.
- a A cheap lock has five possible pin lengths. How many different keys are possible?
- b A better three-pin lock has ten possible pin lengths. How many different keys are possible?
- c Good locks have five pins with ten possible pin lengths. How many different keys are possible for these locks?

29 The genetic code of a DNA molecule consists of arrangements of four nucleotides with the bases adenine, cytosine, guanine and thymine. The bases, referred to by their initial letters A, C, G and T, are arranged in long chains but are often broken up into three-letter 'words' such as AAT for legibility. How many such words are possible?

30 Ordinary Victorian numberplates consist of three letters followed by three numbers. How many different numberplates are possible with this arrangement? Suggest why some other states changed from two letters and four numbers to this arrangement.

31 A flute manufacturer makes flutes from different materials in different sections of the flute. Different sections can be put together for different prices to suit the customer's needs. The *head joint* may be made entirely from alloy, be silver-plated, have a silver lip-plate or be solid silver. The *body* may be alloy, silver-plated or solid silver, and the *foot joint* may be alloy, silver-plated or solid silver. In addition, the foot joint may be replaced by a B-flat foot joint. How many different combinations of parts are possible?

32 In horseracing, a quadrella requires a punter to select the winners of four particular races or 'legs' of the quadrella. A punter makes the following selections for each leg of the quadrella.

1st leg	2nd leg	3rd leg	4th leg
Ripabuy	Roy's Joy	Fab	Cunning As
Run Barry Run	Did We Win	Gadzoox	Notta Brass Razoo
	Pedal To The Metal	Just A Con	

How many possible quadrellas can be formed from these selections?

33 A lab rat is outside a feeding enclosure with five doors.

a In how many ways can the lab rat enter and leave the enclosure via a door?

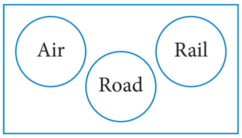
b In how many ways can the lab rat enter the enclosure through one door and leave via a different door?

3.02 THE ADDITION PRINCIPLE

Consider travelling from Melbourne to Alice Springs. You can fly to the Alice via Darwin, Sydney or Adelaide, or direct from Melbourne, so there are 4 ways to fly. There are 3 main ways to travel by road and 1 way by train.

The total number of ways of travelling from Melbourne to Alice Springs is 4 + 3 + 1 = 8.

This is an example of the **addition principle**. The three transport modes are **mutually exclusive** because there is no overlap; you cannot use two methods of transport at once.



IMPORTANT

The **addition principle** says that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.

Example 4

Dimitri keeps his t-shirts in a drawer and his collared shirts on hangers in his wardrobe. On Thursday morning he already has 5 shirts in the wash. He has 8 t-shirts left in his drawers and 4 collared shirts left in his wardrobe.

a How many shirts of either kind can he choose from to wear on Thursday?

b How many t-shirts and collared shirts has he got altogether?

Solution

a Use the addition principle on clean shirts.

$$\begin{aligned}\text{Number of clean shirts} &= 8 + 4\\ &= 12\end{aligned}$$

b Add the numbers of clean and dirty shirts.

$$\begin{aligned}\text{Number of shirts} &= 12 + 5\\ &= 17\end{aligned}$$

The addition principle is often used with the multiplication principle. In some cases the multiplication part is trivial, but you can use them together to solve some quite complex problems.

Example 5

A drawer has single socks that are identical except for colour. Three are black and the other four are white. In how many ways can you choose a matching pair?

Solution

Choose a black pair. Number of black pairs $= 3 \times 2$
$= 6$

Choose a white pair. Number of white pairs $= 4 \times 3$
$= 12$

Add the mutually exclusive pairs. There are $6 + 12 = 18$ ways to choose a matching pair.

The addition principle is used when you can divide a set into mutually exclusive sets. The number of elements in a union of exclusive sets is the total of the number of elements in each set. For example, the set of 5-letter words that contain B and the set of 5-letter words that do not contain the letter B are mutually exclusive sets. The sets of dogs, cats, birds, fish, and so on are mutually exclusive.

Example 6

How many 4-letter 'words' can be made from the letters A, B, C, D, E, F and G if the letter B must be used and no repetition of letters is allowed?

Solution

Imagine the B in the first position.

B			

6 letters could be second.

B	6		

There are 5 left for the third place.

B	6	5	

There are 4 left for the last place.

B	6	5	4

Write the number of 'words' with B first. Number with B first $= 6 \times 5 \times 4$
$= 120$

Do the same for B in the second position.

6	**B**	5	4

Write the number with B in the second position. Number with B 2nd $= 6 \times 5 \times 4$
$= 120$

Do the same for B in the third position.	6 | 5 | **B** | 4
Write the number with B in the third position.	Number with B 3rd $= 6 \times 5 \times 4$ $= 120$
Do it for B in the last position.	6 | 5 | 4 | **B**
Write the number with B in the last position.	Number with B last $= 6 \times 5 \times 4$ $= 120$
Write the necessary condition.	B first, B 2nd, B third and B last are mutually exclusive.
Use the addition principle.	Total number of words containing B $= 120 + 120 + 120 + 120$ $= 480$
Write the answer.	The number of 4-letter 'words' containing B that can be made without repetition from the letters A, B, C, D, E, F and G is 480.

You could also do Example **6** by saying that B could go in any of the 4 places, and then there would be choices of 6, 5 and 4 letters for each of the other places, so the total number would be $4 \times 6 \times 5 \times 4 = 480$.

EXERCISE 3.02 The addition principle

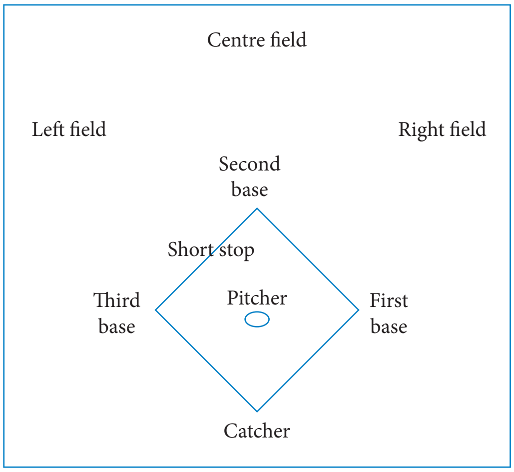
1 Example 4 An Indian restaurant has 6 chicken dishes, 8 beef dishes, 5 pork dishes, 7 lamb dishes and 10 vegetarian dishes, including 6 Vegan dishes. How many different dishes are there?

2 Meg has done her research and knows that the tablet deals from three manufacturers have enough memory and cloud access for her intended use. The first maker has only one tablet in her price range, the second has four and the third has only two. How many tablets can she pick from?

3 On a holiday in north Queensland, Jack had to decide between 3 tours of the Atherton tableland, 4 reef tours, a hot-air balloon trip and a trip on a steam train for his last day before flying home to Melbourne. How many choices did he have?

4 Example 5 Marnie has 3 red, 4 green and 2 blue tops, that she can match with her 2 red, 3 green and 2 blue skirts. The tops and skirts are all different. How many ensembles with the same colour top and skirt can she make?

5 A baseball team has 9 players. Four of the players are not very good as pitcher, short stop or catcher, so are never put in these positions. Two are good pitchers, two are good short stops, but only one is a good catcher. One of the good pitchers or short stops is always put on first base. The other fielding positions are second base, third base, left field, centre field and right field. In how many ways can they arrange the team for fielding?



6 A CD rack has three spare slots. Max has five reggae CDs and four heavy metal CDs.
  a In how many ways can Max fill the three slots with CDs from the same music style?
  b In how many ways can Max fill the three slots if any CD may be used?

7 Example 6 How many different 4-letter 'words' can you make from the letters of the word PARKING if no repetition is allowed and:
  a you can use any letters
  b you must use A
  c you must use I
  d you must use exactly one vowel
  e you must use both vowels
  f you must use at least one vowel.

8 How many 'words' can be made from the letters of the word CATERING according to each of the following rules?
  a Any four letters may be used, without repetition.
  b Any four letters may be used, with repetition permitted.
  c One vowel and three consonants must be used in four-letter words, without repetition.
  d Two vowels and two consonants must be used in four-letter words, without repetition.
  e Three vowels and one consonant must be used in four-letter words, without repetition.
  f At least one vowel and at least one consonant must be used, without repetition, in four-letter words.
  g At least two vowels and at least two consonants must be used, without repetition, in six-letter words.

9 a How many arrangements of five different letters can be made using each of the letters in the word BEGAN?
  b How many of the arrangements in part **a** begin with either A or N?

10 Each of the four digits 0, 4, 5 and 9 are used to form integers, which cannot start with 0.
  a How many numbers can be formed?
  b How many of the numbers in part **a** end in 0?
  c How many end in 4?
  d How many are even?

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e How many of the numbers are odd?
f How many of the numbers are divisible by 5?

11 How many 'words' can be made from the letters of the word MATHEMATICS, according to each of the following rules?
a Four-letter words are to be made, without repetition.
b Five-letter words are to be made, without repetition.
c Five-letter words with one or two vowels are to be made, without repetition.
d Four-letter words containing the letter M are to be made, without repetition.
e Five-letter words containing the letter M are to be made, without repetition.

12 How many different arrangements can be made with the five letters E, F, G, H and I if there are three different letters in each arrangement, provided that, if an arrangement starts with a consonant, the second letter must be a vowel?

13 How many different four digit numbers can be formed with the digits 1, 4, 6, 8:
a if no digit occurs more than once in any number?
b if there are no restrictions?

Reasoning and communication

14 A theme park has two major attractions: Underwater World and Future Land. People can go between Underwater World and Future Land by water ferry in three different ways, and by light rail in two ways.
a In how many ways can people go between Underwater World and Future Land by either water ferry or light rail?
b In how many ways can people go by water ferry and return by light rail?
c In how many ways can people go by one of the two forms of transport and return by the other?

Alamy/David Wall

15 A restaurant has breads, starters, main courses, desserts and cheese courses. There are 3 breads, 5 starters, 6 mains, 4 desserts and 2 cheese platters. How many three course meals including a course before the main course, a main course and dessert or cheese can you have?

3.03 THE PIGEONHOLE PRINCIPLE

The **pigeonhole principle** has many uses. Think of post office boxes.

If the post office has more letters than boxes, at least one of the boxes must get more than one letter.

If you count off boxes and objects in matching pairs, then you have some objects left over, so they must go into boxes that are already occupied. Dirichlet first stated the principle in 1834, so it is sometimes called Dirichlet's box principle.

Shutterstock.com/Steve Wood

IMPORTANT

The pigeonhole principle

If m objects are placed in n boxes, where $m > n$, then at least one box must contain more than one object.

Example 7

If you have 20 married couples, how many people must you pick to make sure you have at least one couple?

Solution

Set up your boxes.	Make 20 boxes, one for each couple.
Put people into the boxes.	You *can* choose one from each couple to put in the boxes.
Choose another person.	This person must be married to one of the first 20.
Write the answer.	21 people are needed to make sure you have at least one couple.

When you use the pigeonhole principle, you work through the worst possible case to find the minimum requirement to make sure that something happens.

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Example 8

A basket of fruit contains apples, bananas and oranges. What is the smallest number of fruit that you should have so that there will be at least 4 apples or 5 bananas or 2 oranges in the basket?

Solution

Make a box for each fruit.	Make an apple box, a banana box and an orange box.
Work through the worst case.	Put in 3 apples, 4 bananas and 1 orange = 8 fruit
Put one more fruit in.	Whatever it is, one box will be full.
Write the answer.	9 pieces of fruit will make sure there are at least 4 apples or 5 bananas or 2 oranges.

Notice that in Example **8** you need

$(4 - 1) + (5 - 1) + (2 - 1) + 1 = 4 + 5 + 2 - 3 + 1 = 9$ fruit

to be certain of getting the required number of one fruit. This is one more than the sum of the numbers of fruit minus the number of types. It illustrates a stronger form of the pigeonhole principle.

IMPORTANT

The pigeonhole principle (strong form)

For positive integers $m_1, m_2, m_3, \ldots, m_n$ associated with n boxes numbered 1 to n, if $m_1 + m_2 + m_3 + \cdots + m_n - n + 1$ objects are placed in the boxes, then at least one box must contain at least m_i objects.

Proof

If you assume that the strong form is not true, then the ith box would contain a maximum of $m_i - 1$ objects, so the total number of objects would be a maximum of

$$(m_1 - 1) + (m_2 - 1) + (m_3 - 1) + \cdots + (m_n - 1) = m_1 + m_2 + m_3 + \cdots + m_n - n$$

which is less than the total number of objects. This is a contradiction, so it must be true. **QED**

In simple applications of the strong principle, the size of each box will often be the same. It then becomes 'if $nm - n + 1$ objects are placed in n boxes, then at least one box contains at least m objects'.

Example 9

In a group of forty people, what is the largest number you can be certain will have their birthday on the same day of the week?

Solution

The worst case is when there are (nearly) the same numbers for each day.	Assume there are m people with birthdays on at least one of the days.
Write down m_i and n.	$n = 7$ and $m_i = m$ for $i = 1$ to 7.

Use the strong version.	Number $= m_1 + m_2 + \cdots + m_7 - n + 1$
Substitute in the values.	$40 = 7m - 7 + 1$
Solve for m.	$m = 6\frac{4}{7}$
Use the fact that m is an integer.	The integer part of $m = 6$.
Write the answer.	There must be at least 6 people with their birthday on the same day of the week.

You could do Example **9** by distributing the people evenly between the days of the week to use 35 people, so the next person from the 40 people must make 6 on at least one day. This is really the same thing as using the principle.

EXERCISE 3.03 The pigeonhole principle

Concepts and techniques

1 Example 7 From a standard pack of cards, what is the smallest number needed to be sure to get:
   a two of the same suit
   b two of the same rank (two 10s, two kings, etc.)

2 What is the smallest number of socks you need to pick from a drawer with 10 single green and 10 single red socks to be certain of getting a matching pair?

3 How many people must you have at a party to be sure that at least two people will have the same birthday?

4 How many words from the first chapter in a book do you need to read to be sure of having two words starting with the same letter?

5 Example 8 A box contains mixed spare USB cables. How many cables must be in the box to be certain of having at least 6 standard cables or 5 standard to mini cables or 3 standard to micro cables or 2 standard to printer cables or 4 standard to female USB standard extension cords?

6 A citrus tart can use the juice and pulp of lemons, oranges, limes or finger limes. It requires one of the following: 4 lemons, 3 oranges, 5 limes or 8 finger limes. How many lemons, oranges, limes and finger limes need to be in a mixed bag to be certain of having enough of one kind for the recipe?

7 When David goes on holiday he likes to read a series of books from one author. Some authors take longer to read than others. He thinks he would be able to read ten of the 'Number one detective agency' books but only half as many of the 'Sunday philosophy club' series. In the same time he would be able to read six 'Rebus' novels but only three of the 'Smiley' books. If he borrowed books from a friend who had lots of books from these series, how many would he need to borrow to be certain of having enough of at least one series?

8 Example 9 There are over 33 000 undergraduate students at Monash University and there is at least one from each of the 196 countries in the world. What is the largest number of students that you can be certain must be from the same country?

9 People can have green, blue, brown, hazel or grey eyes. How many people do you need in a group to be certain of having at least 4 with eyes of the same colour?

Reasoning and communication

10 Show that from any 101 different integers from the numbers 1 to 200, at least one is double another.

11 Show that from any 135 different integers from the numbers 1 to 200, at least 1 must be triple another.

12 Show that at least one integer must be greater than or equal to the average of a group of integers.

13 Show that in any group of 6 people there must be three people who are either mutual friends or mutual strangers.

14 Show that there are at least two people with the same number of friends in any group (more than 1), assuming that if you are friends with someone, they are also friends with you.

3.04 SIMPLE PERMUTATIONS

Eat, ate, eta and tea all mean something different. 'The cup split, spilt tea and was thrown away' would be wrong if you misspelt *split* and *spilt*. In a group of letters that spell words, the order is very important. When you are arranging groups of any type, it is important to be clear whether the order is important or not.

In how many different orders could we arrange the letters of the word *spilt*?

Think of the letters filling up five boxes in a row.

There are 5 to pick from for the first box. You could pick p.

p				

If p is gone, there are 4 left (silt). You could pick l.

p	l			

If p and l are gone, there are 3 left (sit). You could pick i.

p	l	i		

If p, l and i are gone, you could pick s from the 2 left (st).

p	l	i	s	

If p, l, i and s are gone, there is only one left. You must pick t.

p	l	i	s	t

Write the number of choices for each box.

5	4	3	2	1

Actually, it doesn't make any difference what letters you choose, the number of choices *always* goes 5, 4, 3, 2 and 1.

By the multiplication principle, the total number of orders $= 5 \times 4 \times 3 \times 2 \times 1 = 5!$

This can be generalised as follows.

IMPORTANT

The number of ways of arranging n **objects in an ordered list** is given by

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Example 10

Eight school ID cards were dropped on the floor. In how many possible orders can the cards be picked up?

Solution

This is just an ordered list. You can visualise the boxes or use the rule.

8	7	6	5	4	3	2	1

Number of arrangements = 8!

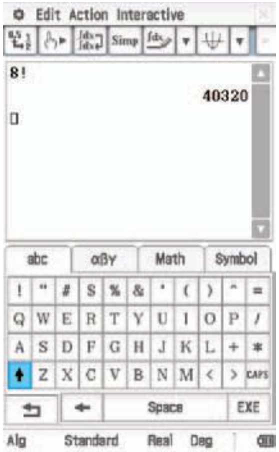
TI-Nspire CAS

Use a calculator page.

Use the ?!▸ button on the right to get !

ClassPad

Simply enter 8! and tap EXE or press EXE. The factorial is found by pressing Keyboard and tapping abc, then 🡅. ! is shown in the top left of the keyboard.



Write the answer.

There are 40 320 orders in which the ID cards can be picked up.

There will be occasions when you will want to choose only some of a group of objects in order, rather than making a list of all of them.

IMPORTANT

A **permutation** is an arrangement of some or all of a set of symbols or objects in a particular order.

The order of objects in a permutation is important. For the three letters A, B and C there are $3 \times 2 \times 1 = 6$ different possible orders, so there are 6 permutations of the three letters A, B and C. They are ABC, ACB, BAC, BCA, CAB, and CBA.

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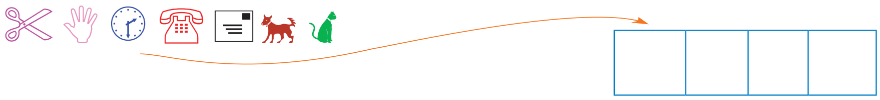
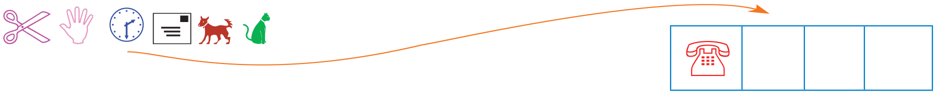
In how many ways can four objects from a set of 7 objects be arranged in a row?

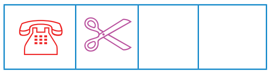
We can think of the objects filling up four boxes placed in a row.



The first one chosen can be any of the 7 objects.

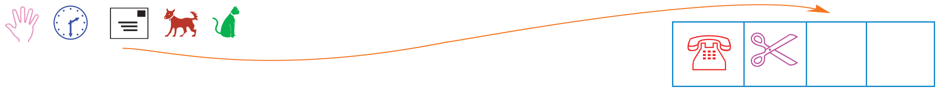
But now there are only 6 left.

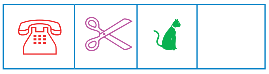




The second one chosen can be any of the 6 left.

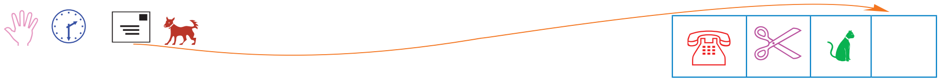
But now there are only 5 left.

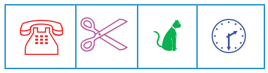




The third one chosen can be any of the 5 left.

But now there are only 4 left.





The last one can be any of the remaining 4.

Write the number of choices each time in the boxes. $\boxed{7}\times\boxed{6}\times\boxed{5}\times\boxed{4}$

By the multiplication principle, the total number of choices is $7 \times 6 \times 5 \times 4 = 840$.

Notice that $7\times6\times5\times4 = \dfrac{7\times6\times5\times4\times3\times2\times1}{3\times2\times1} = \dfrac{7!}{3!} = \dfrac{7!}{(7-4)!}$.

You can obviously generalise this to give the following rule.

IMPORTANT

The number of permutations of r objects from n objects is written as ${}^nP_r$, which is calculated using the formula ${}^nP_r = \dfrac{n!}{(n-r)!} = n\times(n-1)\times(n-2)\times\cdots\times(n-r+1)$.

You may prefer to use boxes instead of the formula to calculate permutations, at least for small numbers of items. For example, if you had to select a committee of three people from Anne, Bob, Cathy, David and Elizabeth. You can also use your CAS calculator.

Example 11

How many different arrangements of 5 objects in a straight line can you make from 20 objects?

Solution

You can use 5 boxes.

20	19	18	17	16

Write the formula. $^nP_r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$

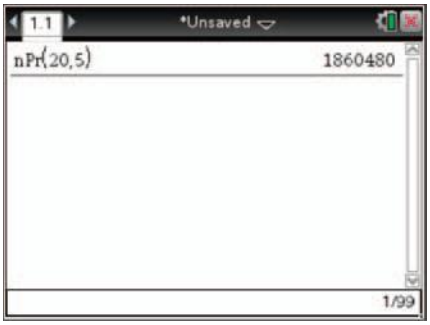
Work out the last number. $n - r + 1 = 20 - 5 + 1 = 16$

Substitute in the values. $^{20}P_5 = 20 \times 19 \times 18 \times 17 \times 16$

Calculate the answer. $= 1\,860\,480$

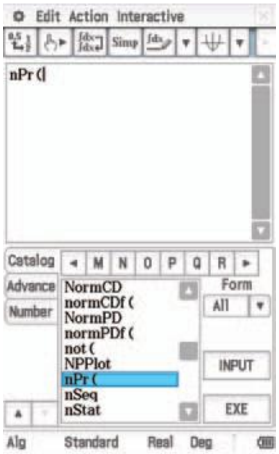
TI-Nspire CAS

Use a Calculator page.
Use menu, 5: Probability and 2: Permutations to get nPr(). Type in the 20, 5 and press enter.

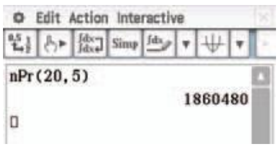


ClassPad

For ^nP_r press Keyboard and tap ▼ to access the list of functions. Tap **N** and scroll down to tap **nPr(**. Tap ▲ to return to the previous screen.



Enter the values of n and r after the bracket, separated by a comma , and close the bracket with). Tap EXE or press EXE.



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Example 12

An anagram is an arrangement of given letters in a particular order. How many anagrams of COMFREY are there?

Solution

An anagram is a permutation of letters. Number of anagrams $= {}^7P_7$

Calculate the answer. $= 7!$

$= 5040$

WS Permutation calculations

EXERCISE 3.04 Simple permutations

WS Permutations

Concepts and techniques

1 ${}^nP_0 =$ **A** 0 **B** 1 **C** n **D** n^2 **E** $n^2 - n$

2 ${}^nP_n =$ **A** 0 **B** 1 **C** $n!$ **D** n^2 **E** $n^2 - n$

3 ${}^nP_1 =$ **A** 0 **B** 1 **C** n **D** n^2 **E** $n^2 - n$

4 ${}^nP_2 =$ **A** 0 **B** 1 **C** n **D** n^2 **E** $n^2 - n$

5 Example 10 In how many ways can each of the following numbers of different objects be placed in an ordered list?
a 5 **b** 9 **c** 10 **d** 4 **e** 12

6 A family has 7 people living in the house: Mum and Dad, Grandpa and Nan, Peter, Mary and Tom. In how many different orders can their individual photos be lined up on the hall table?

7 The alphabet is normally given in the order a, b, c, … How many different orders are possible? Give your answer in an appropriate notation.

8 Example 11 Calculate the following values.
a ${}^3P_2$ **b** ${}^5P_3$ **c** ${}^7P_2$ **d** ${}^9P_8$ **e** ${}^8P_4$
f ${}^5P_5$ **g** ${}^{11}P_2$ **h** ${}^{10}P_5$ **i** ${}^{85}P_3$ **j** ${}^{14}P_5$

9 Find n if:
a ${}^nP_4 = 56 \times {}^nP_2$ **b** $12 \times {}^nP_3 = {}^{3n}P_2$ **c** $8 \times {}^nP_2 = 7 \times {}^{n+1}P_2$
d $9 \times {}^{n+1}P_3 = 10 \times {}^nP_3$ **e** ${}^{n+1}P_4 = 18 \times {}^nP_2$ **f** $56 \times {}^nP_2 = {}^nP_4$

10 Four cars enter a carpark and four parking spaces are available. In how many ways can the cars occupy the parking spaces?

11 Example 12 In how many ways can the letters of the word FIGURES be arranged?

12 A school orchestra has to play six pieces at a concert. The conductor is trying to decide the order of the pieces. How many choices does she have?

13 How many four-digit odd numbers can be formed with the digits 1, 4, 5 and 7:
a if repetitions are not allowed? **b** if repetitions are allowed?

14 A combination lock is opened by turning the dial to three *different* numbers in succession. How many different combinations are possible if the numbers run from:

a 0 to 9? **b** 0 to 19? **c** 0 to 29?

Reasoning and communication

15 **a** How many permutations of the letters of the word ABLE are there?

b How many permutations of the letters of the word VIABLE are there?

c What is the probability that a random permutation of the word VIABLE begins with VI?

16 Find the probability that an anagram of the word COMPUTER will start with the letter C.

17 Show that if an extra item is added to a list of n items, then the number of possible orders increases by $(n-1)!\,n^2$.

3.05 RESTRICTED PERMUTATIONS

Permutations may be restricted in some way, and you may have to use the addition and/or multiplication principles with permutations to solve problems.

Example 13

How many different whole numbers can be made from the digits 0, 1, 2, 3 and 4?

Solution

Count the 1-digit numbers.	No. of 1-digit numbers = 5
The 2-digit numbers cannot start with 0.	No. of 2-digit numbers $= 4 \times {}^4P_1 = 4 \times 4 = 16$
The 3-digit numbers cannot start with 0.	No. of 3-digit numbers $= 4 \times {}^4P_2 = 4 \times 6 = 24$
The 4-digit numbers cannot start with 0.	No. of 4-digit numbers $= 4 \times {}^4P_3 = 4 \times 4 = 16$
The 5-digit numbers cannot start with 0.	No. of 5-digit numbers $= 4 \times {}^4P_4 = 4 \times 1 = 4$
The 1, 2, 3, 4 and 5 digit numbers are mutually exclusive, so add the numbers.	Total number $= 5 + 16 + 24 + 16 + 4 = 65$
Write the answer.	65 whole numbers can be made.

In some cases, you need to think of permutations as ‘blocks’ collapsed into a single box to solve a problem. Then you use the multiplication principle.

Example 14

Seven people, including three friends, are seated in a row at random. In how many ways can the three friends be seated together?

Solution

Show the row with the friends together.

		F_1	F_2	F_3		

Collapse them to a single box to make a row of 5.

		F		

Write the number of permutations of the new row.

Row permutations = 5!

Show the block of 3 friends.

F_1	F_2	F_3

Write the number of permutations of the friends.

Friend permutations = 3!

Use the multiplication principle.

Total arrangements = $5! \times 3!$

Calculate the answer.

$= 120 \times 6$

$= 720$

Write the answer.

There are 720 ways in which the three friends can be seated together.

Permutations that are arrangements in a circle are called **circular permutations**. They are restricted compared with straight-line permutations because they have no starting point. For example, if there are six friends sitting at a round table, then their relative positions do not depend on where you start looking.

IMPORTANT

The number of **circular permutations** of n different objects or symbols is $(n - 1)!$

Proof

Suppose there are X circular permutations.

Consider one such permutation. It can be unbent into n straight permutations by cutting it before each of the objects or symbols.

Thus the number of straight permutations = nX.

But the number of straight permutations is $n!$

Thus $nX = n!$, so $X = \frac{n!}{n} = (n - 1)!$ **QED**

You can also show that the number of circular permutations is $(n - 1)!$ by using the fact that since the symbols are in a circle, it doesn't matter which one you start with.

Example 15

In a barn dance there are 8 boys in one circle. How many possible arrangements of the boys are there?

Solution

Use the formula.	Number of circular permutations $= (8 - 1)!$ $= 7!$
Calculate the result.	$= 5040$
Write the answer.	There are 5040 arrangements of the 8 boys.

EXERCISE 3.05 Restricted permutations

Concepts and techniques

Circular calculations

1 The number of circular permutations of 5 objects is:
A 5 B 24 C 25 D 120 E 720

2 Example 13 How many four-digit numbers can be made using the digits 0, 3, 5, 8, 2 and 7 if:
a the same digit cannot be used twice?
b repetition is allowed?

3 Given the digits 0, 2, 4, 6 and 7 and not repeating any:
a how many 5-digit numbers can be made?
b how many numbers can be made?
c how many numbers can be made that are under 3000?

4 Example 14 How many arrangements of the letters of the word FASHION are possible if:
a the vowels must all be together?
b the vowels are not all together?
c the letter N must be last?

5 In how many ways can four different black and five different red discs be placed in a row if they are placed alternately?

6 A group of 5 boys and 5 girls line up outside a cinema. Find the number of ways that can they be arranged for each of the following cases.
a With no restriction
b If a particular girl stands in line first.
c If they alternate between boys and girls.

7 Example 15 A group of friends go into a restaurant and are seated around a circular table. Find the number of arrangements that are possible if there are:
a 4 friends b 7 friends c 8 friends d 10 friends e 11 friends

8 A merry-go-round proprietor has ten different ‘horses’ to place on the outer edge of the merry-go-round. How many different placements can be made?

9 A group of 7 people sit around a table. Write the number of ways in which they can be arranged for each of the following cases.
   a With no restrictions
   b If 2 people want to sit together
   c If 2 people cannot sit together
   d If 3 people want to sit together

10 Find the probability that if 10 people sit around a table, 2 particular people will be seated together at random.

11 Six women and six men attend a dinner at a restaurant that has round dining tables. In how many ways can the people be seated so that no two women sit together?

Reasoning and communication

12 A string of different beads looks the same when turned over. Find the number of different arrangements possible with each of the following numbers of beads.
   a 10 b 12 c 9 d 11 e 13

13 A singer has arrangements of 20 different songs from her repertoire that are suitable for a particular backing group. Only three of them can be used as the opening number, and there are another four that she would prefer to choose from for the finale. In a six-song bracket, how many different programs are possible if the middle four songs are chosen from the remaining 13 arrangements?

14 The flags of a number of countries consist of three vertical stripes in different colours. For example, France's flag, from the flagpole side, is blue, white and red. If the colours red, blue, green, white, yellow and black may be used, how many flags with three vertical stripes are possible, given that:
   a the three colours must be different?
   b the centre colour must be different from the end colour(s)?

15 A bookshelf is to hold 5 mathematics books, 8 novels and 7 cookbooks.
   a In how many different ways could the books be arranged? (Leave your answer in factorial notation.)
   b If the books are grouped in categories, in how many ways can they be arranged? (Answer in factorial notation.)

16 A minibus has 6 forward-facing and 2 backward-facing seats. If 8 people use the bus, in how many ways can they be seated:
   a with no restrictions?
   b if one particular person must sit in a forward-facing seat?
   c if 2 particular people must sit in forward-facing seats?

17 There are ten contestants in an archery competition. Four of the contestants are women. Prizes are awarded to the top four competitors. If at least one woman finishes in the first four places, in how many ways can the top four places be filled?

Getty Images/Amwell

3.06 PERMUTATIONS WITH REPETITION

You will often have to deal with a permutation where some of the objects are identical. Think about anagrams of word BABY. If you write the first B as B_1 and the second as B_2, the word is B_1AB_2Y. Without the subscripts, you cannot tell the difference between this and B_2AB_1Y. In the same way, AB_1B_2Y and AB_2B_1Y are the same. Every permutation has two versions where the Bs are swapped. Instead of having 4! permutations there will actually only be $\frac{4!}{2}$ permutations.

What about the word NEEDLE? You would not be able to tell the difference between $NE_1E_2DLE_3$, $NE_1E_3DLE_2$, $NE_2E_1DLE_3$, $NE_2E_3DLE_1$, $NE_3E_1DLE_2$ and $NE_3E_2DLE_1$. This time the 3 Es would make $3! = 6$ versions of every permutation, so the number of permutations of the word NEEDLE would be $\frac{6!}{3!}$

The word COOEE would have two versions of the positions of the Os and two versions of the positions of the Es, so the number of permutations would be $\frac{5!}{2\times 2}$.

The permutations of the letters A, A, A, B, B, C, D, E would have $3! \times 2!$ versions of the anagram EBABACAD. These correspond to the possible orders of the 3 As and the 2 Bs. There would actually be only $\frac{8!}{3!\times 2!}$ permutations, instead of 8!.

IMPORTANT

The number of permutations of n symbols, of which a are identical, is given by $\frac{n!}{a!}$.

If there are $a, b, c, \ldots$ identical symbols, the number of permutations of n symbols is $\frac{n!}{a!b!c!\cdots}$, where $a + b + c + \cdots \leq n$.

Example 16

In how many ways can the letters of the word SUCCESSES be arranged?

Solution

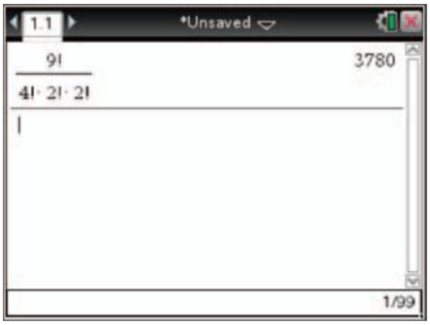
Write the number of symbols. — There are 9 letters, including 4 Ss, 2 Cs and 2 Es.

Use the formula. — Number of permutations $= \frac{9!}{4!\times 2!\times 2!}$

TI-Nspire CAS

Use the Calculator page.

Press t to get the fraction template and to get !.



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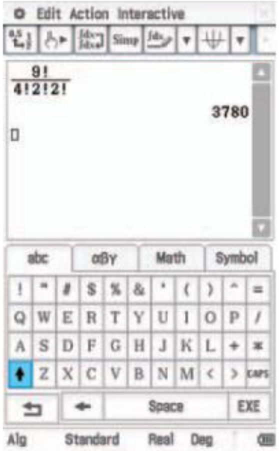
ClassPad

First set up a fraction using ▭.(First press Keyboard then tap Math1).
Then type in the factorials from the abc keyboard under the **abc** tab. There is no need to enter the multiplication signs. Press EXE to complete the calculation.



Write the answer.

There are 3780 arrangements of SUCCESSES.

If you select only some of a group including separate symbols, then there is no simple formula for determining the number of permutations.

EXERCISE 3.06 Permutations with repetition

Concepts and techniques

Permutations with repetitions

1 Example 16 How many anagrams are there for each of the following words?

a CENTIPEDE   b ALGEBRA   c TELEVISION
d ANTARCTICA   e DONOR   f BASKETBALL
g GREENSLEEVES   h HOLLOWLY   i ABRACADABRA
j PRESTIDIGITATION

2 a How many 5-digit numbers can be made using each of the digits 3, 4, 4, 5 and 6?
b How many of the numbers are greater than 4000?
c How many of the numbers are less than 5000?
d If one is chosen at random, find the probability that it is less than 4000.

3 How many 5-digit numbers can be made from the digits 0, 3, 4, 4, 7?

4 How many 6-digit numbers can be made from the digits 0, 3, 4, 4, 4, 9?

5 In how many ways can 10 objects be arranged if 4 of them are identical?

6 In how many ways can 8 objects be arranged if there are 3 pairs of identical objects among them?

7 In how many ways can 20 objects be arranged if there are 2 sets of 4 identical objects among them?

Reasoning and communication

8 How many different bracelets of 6 beads can be made from 10 different beads? Remember that a bracelet can be flipped over.

9 A competition involves 20 competitors and has three prizes. In how many ways can the prizes by awarded:

a if no competitor can win more than one prize?

b if any competitor can win all the prizes?

c if no competitor can win more than two prizes?

10 One artist has 5 paintings, one has 6 and the other has 4 paintings in one room of an exhibition in a local gallery. There are doors in each of the walls of the room where they are exhibited, so the exhibition space is symmetrical around the room. In how many different ways can the artists' works be placed around the walls?

11 A coin is tossed six times. Find:

a the number of ways it can fall

b the number of ways you could get two heads, by finding the number of permutations of HHTTTT

c the probability of obtaining exactly two heads from six tosses.

12 Find the probability of obtaining exactly:

a 2 heads from 4 tosses of a coin

b 2 heads from 8 tosses of a coin

c 3 heads from 5 tosses of a coin

d 6 heads from 8 tosses of a coin

e 10 heads from 20 tosses of a coin.

3.07 APPLICATIONS OF COUNTING AND PERMUTATIONS

Probability problems are mostly calculated by counting the number of ways that things might happen. When solving them, you need to work out the kind of counting involved, or whether multiple methods are needed.

Example 17

The Melbourne Cup is a handicap race with 24 horses running. The handicapper adjusts the weight that each horse carries to give each horse an equal chance of winning. A trifecta is a bet on the first three places in the correct order. Assuming that the handicapper has done the job perfectly, what is the probability of winning the trifecta on the Melbourne Cup?

Solution

The order matters, so this is a permutation.	Number of possible trifectas $= {}^{24}P_3$
Work out the number.	$= 24 \times 23 \times 22$ $= 12\,144$
Find the probability.	$P(\text{Trifecta}) = \frac{1}{12\,144}$ $\approx 0.000\,082$ or 0.0082%
Write the answer.	The probability of winning the trifecta is about 0.0082%.

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When doing multiple calculations, it is often easier to do all the evaluations at the end.

Example 18

Four couples go to a play, but their tickets have been mixed up, although the eight people are still sitting together. What is the probability that, according to the tickets, all four couples are sitting with their partners?

Solution

Find the orders of the whole 8. Possible orders $= {}^8P_8 = 8!$

Find the couples' orders. Possible orders of couples = 4!

Find the orders of each couple. Possible orders of a couple = 2!

Use the multiplication principle. Possible orders with couples together = 4! × 2! × 2! × 2! × 2!

Find the probability. $\text{Probability(couples together)} = \dfrac{4! \times 2 \times 2 \times 2 \times 2}{8!}$

Simplify. $= \dfrac{4 \times 3 \times 2 \times 1 \times 2 \times 2 \times 2 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

Cancel and calculate the answer. $= \dfrac{1}{7 \times 3 \times 5}$

$= \dfrac{1}{105}$

$\approx 0.009\,524$

Write the answer. The probability of the couples being together is just under 1%.

EXERCISE 3.07 Applications of counting and permutations

Concepts and techniques

1 Example 17 A horserace meeting at a country racecourse had only 8 runners in the first race. What is the probability of winning the trifecta for this race?

2 A quadrella is a bet to pick the first four horses in a race in the correct order. What is the probability of winning a quadrella for a race with 15 runners?

3 A country race meeting only has 6 runners in the first race, 5 in the next two races and 7 in the last two races of the 5 races on the day.
What is the probability of winning a trifecta on:
a the first race? b the last race?
An accumulator bet is one such that the winnings from one race are automatically bet on the next race. To win an accumulator bet you have to pick first place correctly on each of the races you bet on. What is the probability of winning an accumulator on
c the first 3 races? d all of the races?

4 Example 18 What is the probability that a group of four friends will be sitting together in a row of 10 seats on an English excursion if the seats in the row are assigned at random, but all four friends are in that row?

5 Trucks taking material from construction sites to a disposal centre might be carrying soil, vegetation, rocks or old concrete. Trucks arrive at the disposal site effectively in random order. In a line of a dozen trucks waiting to enter the site, 3 are carrying soil, 3 are carrying vegetation, 3 are carrying rocks and 3 are carrying old concrete. What is the probability that the trucks carrying the same material are together in the line?

Reasoning and communication

6 At a café there are 4 different starter courses, 7 different main courses and 5 different desserts. If the dishes are equally popular, what is the probability that the first three-course orders on Monday and Tuesday are exactly the same?

7 A number is chosen at random from those that can be made using some or all of the digits 3, 4, 5, 6 and 7 without repetition. What is the probability that it is between 400 and 4000??

8 What is the probability that a random 5-letter 'word' made from the letters of the word WICKEDLY contains a LIE? That is, contains the sequence LIE?

9 In a 'baby' competition for charity, the names of 10 teachers and photographs of them as babies are shown and students pay 50c to match the photos to the names. What is the probability of guessing correctly?

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CHAPTER SUMMARY
COUNTING METHODS AND PERMUTATIONS

3

- **Combinatorics** is the mathematics of systematically counting *finite* collections.
- The **multiplication principle** states that, for a choice made in 2 stages, if there are a ways for one part and b ways for the second part, then there are $a \times b$ choices altogether. If a choice is made in n stages, each of which has a_i possibilities, then there are $a_1 a_2 a_3 \cdots a_n$ possible choices altogether.
- A **recursive** definition of a sequence finds each term from the previous one(s).
- The factorial sequence, shown as $n!$, is given by

  $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$, or $0! = 1$ and $n! = n \times (n-1)!$
- The nth term of a sequence with arithmetic growth is a fixed amount greater than the previous term.
- **Mutually exclusive** sets have no elements in common.
- The **addition principle** states that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.
- The **pigeonhole principle** states that if m objects are placed in n boxes, where $m > n$, then at least one box must contain more than one object.
- The **strong form** of the pigeonhole principle states that for positive integers $m_1, m_2, m_3, \ldots, m_n$ associated with n boxes numbered 1 to n, if $m_1 + m_2 + m_3 + \cdots + m_n - n + 1$ objects are placed in the boxes, then for at least one i, the ith box contains at least m_i objects.
- The number of ways of arranging n **objects in an ordered list** is given by

  $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$
- A **permutation** is a selection of some or all of a set of symbols or objects in a particular order.
- The number of permutations of r objects from n distinct objects is written as ${}^nP_r$ and is given by

  $${}^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1).$$
- **Circular permutations** are ordered arrangements in a circle. The number of circular permutations of n different objects or symbols is $(n-1)!$
- The number of permutations of n symbols, of which a are identical, is given by $\frac{n!}{a!}$. If there are $a, b, c, \ldots$ identical symbols, the number of permutations of n symbols is $\frac{n!}{a!b!c!\cdots}$, where $a + b + c + \cdots \leq n$.

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3

CHAPTER REVIEW
COUNTING METHODS AND PERMUTATIONS

Multiple choice

1 Example 10 $6! =$

A 36 B 120 C 216 D 360 E 720

2 Example 11 The number of permutations of 4 symbols from 8 different symbols is

A 24 B 64 C 70 D 420 E 1680

3 Example 15 The number of ways of arranging 6 people in a circle is

A 30 B 120 C 144 D 720 E 5040

4 Example 16 The number of permutations of 6 symbols with 2 identical pairs is

A 24 B 30 C 64 D 180 E 720

5 Example 2 A new car comes in 8 different colours with 5 choices of interior trim, 3 styles of wheels and 4 different seat options. How many different cars are possible?

A 20 B 40 C 280 D 480 E 560

Short answer

6 Example 1 An activation code for some software consists of alternating letters and numbers, with 4 letters and 3 numbers. How many different activation codes are possible?

7 Example 2 A game has 8 different coloured markers so that each player can have a different coloured marker to move around the board. In how many different ways can 6 players choose their markers?

8 Example 3 What is the probability of the first 7 babies born at the Royal Women's Hospital in Melbourne in December being boys, assuming that girls and boys are equally likely?

9 Example 5 Liza has 2 red tops, 3 pairs of red slacks, 3 blue tops, 4 pairs of blue slacks and 2 black tops. None of the items of clothing are identical. She can pair the black tops with any slacks, but otherwise needs to match the colours. How many outfits can she make up?

10 Example 6 How many 5-letter 'words' can be made from the letters of the word COSTUMING if you must use at least one vowel?

11 Example 7 A drawer contains unpaired red, blue, green, white and black socks that are identical except for their colours. How many socks must you take out to be sure of getting a matching pair?

12 Example 8 A family reunion of over 200 people has members from 4 different generations present. How many people must be in a group to be certain that there are at least 2 great-grandparents or 3 grandparents or 3 parents or 4 children?

13 Example 9 A train carriage has 60 people in it. What is the largest number of people that you can be certain will get out at one of the remaining seven stations before the end of the line?

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14 Example 10 In how many different orders can 9 people in a queue be arranged?

15 Example 11 In how many ways can 5 people lined up at a buffet each choose a piece of fruit from a tray with 10 different pieces left?

16 Example 12 In how many ways can the letters of the word NUMBER be arranged?

17 Example 13 How many different whole numbers greater than 9 can be formed from the digits 0, 2, 3, 4, and 5 without repetition?

18 Example 14 In how many ways can 7 books be arranged on a shelf so that the four by the same author are together?

19 Example 16 How many anagrams of the word MACADAM are there?

20 Example 17 What is the probability of winning a trifecta in a race with 17 runners if three horses are picked at random?

Application

21 The menu at a memorial dinner has choices of 3 different entrées, 4 different main courses and 2 different sweets. How many different three-course meals can be chosen?

22 A necklace is made from 8 natural pearls, so they are all slightly different. It also has a gold bead. The necklace is put on over the head, as it has no clasp. In how many different ways could the beads be arranged?

23 A home theatre store sells speakers from 8 different manufacturers, with a total of 92 different speakers altogether. Show that there must be at least 12 speakers of the same brand.

Practice quiz

24 How many three-letter 'words' can be made with the letters of the word THREAD? The rules are that H can only go at the start or with the T to make TH, there must be at least one vowel and one consonant, and T cannot go next to D.

25 How many numbers larger than 6000 can be made from some or all of the digits 8, 6, 4, 2, 1 and 0, without repetition?

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4

TERMINOLOGY

airspeed
component
course made good
dot product
ground speed
heading
orthogonal component
parallel component
projection
resolution
resolve
scalar product
speed made good
speed through the water
true course
wind
work

Prior learning

VECTORS IN THE PLANE

APPLICATIONS OF VECTORS

ALGEBRA OF VECTORS IN THE PLANE

- define and use scalar (dot) product (ACMSM019)
- apply the scalar product to vectors expressed in component form (ACMSM020)
- examine properties of parallel and perpendicular vectors and determine if two vectors are parallel or perpendicular (ACMSM021)
- define and use projections of vectors (ACMSM022)
- solve problems involving displacement, force and velocity involving the above concepts. (ACMSM023)

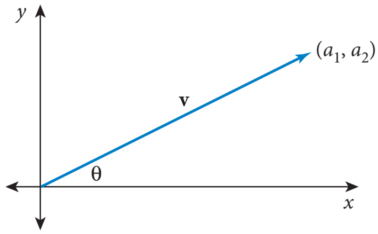
GEOMETRIC PROOFS USING VECTORS IN THE PLANE

- the diagonals of a parallelogram meet at right angles if and only if it is a rhombus (ACMSM039)
- the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides. (ACMSM041) AC

4.01 THE SCALAR PRODUCT

In Chapter 1, you saw that a vector is a quantity with both magnitude and direction.

The vector $\mathbf{v}$ shown on the right has a magnitude v, where $v = |\mathbf{v}| = \sqrt{a_1^2 + a_2^2}$, and makes an angle of θ with the positive direction of the x-axis.



The polar form of $\mathbf{v}$ is (r, θ).

The component form of $\mathbf{v}$ is (a_1, a_2) or $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.

The unit vector form of $\mathbf{v}$ is $a_1\mathbf{i} + a_2\mathbf{j}$, where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$.

You already know how to add two vectors and how to multiply a vector by a scalar.

The **scalar product** or **dot product** of two vectors produces a scalar (real number) answer.

IMPORTANT

For two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of θ between them, the **scalar product** or **dot product** is given by $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$

Example 1

Find the scalar product of vectors with magnitudes 5 and 6 at an angle of 60° to each other.

Solution

Write the formula. $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$

Substitute in the values. $= 5 \times 6 \times \cos(60°)$

Evaluate. $= 15$

The dot product can be positive or negative.

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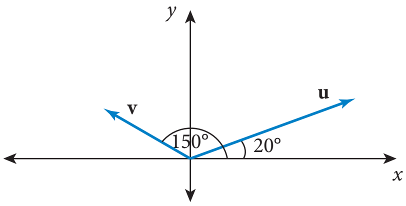
Example 2

If **u** = (4, 20°) and **v** = (2, 150°), calculate **u** · **v** correct to 2 decimal places.

Solution

Sketch a diagram.



Find the angle between the vectors.	$\theta = 150° - 20° = 130°$
Write the formula.	$\mathbf{u} \cdot \mathbf{v} = \lvert\mathbf{u}\rvert \times \lvert\mathbf{v}\rvert \cos(\theta)$
Substitute in the values.	$= 4 \times 2 \times \cos(130°)$
Evaluate and round to 2 decimal places.	≈ -5.14

You can use your CAS calculator to find the dot product of vectors.

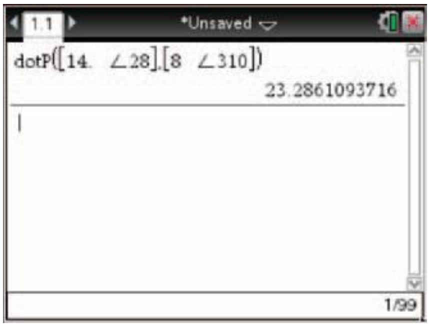
Example 3

Use your CAS calculator to find (14, 28°) · (8, 310°), correct to 6 significant figures.

Solution

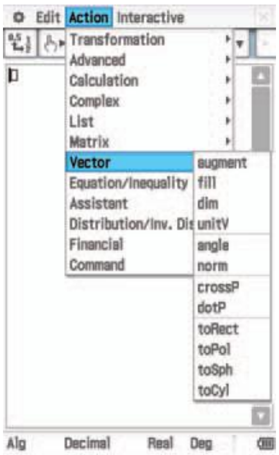
TI-Nspire CAS

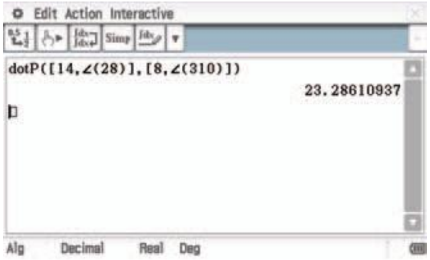
Use a calculator page.
Make sure that your calculator is set to degrees and the calculation mode is set to approximate (or put a decimal point in one of the numbers).
Use menu, 7: Matrix & Vector, C: Vectors and 3: Dot Product to get dotP(). Enter the vectors separated by a comma.



ClassPad

Use the Main $\sqrt{\alpha}$ menu.
First set the calculator to degrees (**Deg**) and **Decimal**.
Tap **Action**, then **Vector** and then **dotP**.
Enter the two vectors, separated by a comma.
Tap EXE or press the EXE key. Remember to put each vector in square brackets.
The vectors can be entered in either polar or component form.
To get the angle sign, press Keyboard, tap Math3.





Write the answer to the specified accuracy. $(14, 28°) \cdot (8, 310°) \approx 23.2861$

EXERCISE 4.01 The scalar product

Concepts and techniques

1 Example 1 The angle between vectors **a** and **b** is θ. Find **a** · **b** (correct to 2 decimal places if necessary) for each of the following.

a $|\mathbf{a}| = 2$, $|\mathbf{b}| = 7$ and $\theta = 30°$
b $|\mathbf{a}| = 1$, $|\mathbf{b}| = 11$ and $\theta = 60°$
c $|\mathbf{a}| = 5$, $|\mathbf{b}| = 8$ and $\theta = 45°$
d $|\mathbf{a}| = 7$, $|\mathbf{b}| = 3$ and $\theta = 115°$
e $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$ and $\theta = 90°$
f $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$ and $\theta = 0°$
g $|\mathbf{a}| = 4$, $|\mathbf{b}| = 9$ and $\theta = 184.6°$
h $|\mathbf{a}| = 12$, $|\mathbf{b}| = 8$ and $\theta = 311.6°$

2 The angle between vectors **v** and **w** is θ. Find **v** · **w** (correct to 2 decimal places if necessary) for each of the following.

a $|\mathbf{v}| = 3$, $|\mathbf{w}| = 2$ and $\theta = \frac{\pi}{3}$
b $|\mathbf{v}| = 4$, $|\mathbf{w}| = 7$ and $\theta = \frac{\pi}{6}$
c $|\mathbf{v}| = 8$, $|\mathbf{w}| = 5$ and $\theta = \frac{\pi}{4}$
d $|\mathbf{v}| = 7$, $|\mathbf{w}| = 9$ and $\theta = \frac{3\pi}{4}$
e $|\mathbf{v}| = 9$, $|\mathbf{w}| = 3$ and $\theta = \frac{\pi}{2}$
f $|\mathbf{v}| = 9$, $|\mathbf{w}| = 3$ and $\theta = 0$
g $|\mathbf{v}| = 11$, $|\mathbf{w}| = 7$ and $\theta = \frac{8\pi}{5}$
h $|\mathbf{v}| = 8$, $|\mathbf{w}| = 15$ and $\theta = \frac{15\pi}{7}$

3 Example 2 Calculate each of the following (correct to 2 decimal places).

a $(18, 354°) \cdot (14, 157°)$
b $(10, 279°) \cdot (17, 152°)$
c $(11, 65°) \cdot (17, 345°)$
d $(9, 256°) \cdot (9, 260°)$
e $(10, 307°) \cdot (14, 219°)$
f $(4, 202°) \cdot (19, 241°)$
g $(12, 314°) \cdot (7, 140°)$
h $(8, 236°) \cdot (8, 40°)$

4 Example 3 CAS Find each of the following (correct to 2 decimal places).

a $(16, 53°) \cdot (18, 259°)$
b $(12, 204°) \cdot (13, 359°)$
c $(14, 359°) \cdot (24, 313°)$
d $(7, 195°) \cdot (4, 35°)$
e $(10, 125°) \cdot (22, 31°)$
f $(13, 137°) \cdot (13, 134°)$
g $(14, 19°) \cdot (4, 154°)$
h $(6, 290°) \cdot (10, 48°)$

Reasoning and communication

5 Change each of the following to polar form and find their dot product, correct to 2 decimal places if necessary.

a $(1, 4)$ and $(4, 3)$
b $(6, 2)$ and $(8, 4)$
c $(-5, 3)$ and $(2, 6)$
d $(4, -3)$ and $(6, 2)$
e $(-2, -7)$ and $(-3, 5)$
f $(-4, 6)$ and $(3, -8)$

4.02 THE COMPONENT FORM

As you have seen, the component form of vectors is easier to use for addition or subtraction of vectors. This is also true for the dot product.

IMPORTANT

Scalar product of vectors in component form

For two vectors $\mathbf{v}_1 = (a_1, a_2)$ and $\mathbf{v}_2 = (b_1, b_2)$, the **scalar product** or **dot product** is given by:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (a_1, a_2) \cdot (b_1, b_2) = a_1 \times b_1 + a_2 \times b_2$$

Use of either form of the dot product will give the same answer.

Example 4

If $\mathbf{a} = (1, -1)$ and $\mathbf{b} = (3, 4)$ calculate $\mathbf{a} \cdot \mathbf{b}$ using:

a the component form **b** the polar form.

c What do you find?

Solution

a Write the rule. $(a_1, a_2) \cdot (b_1, b_2) = a_1 \times b_1 + a_2 \times b_2$

Substitute in the values. $(1, -1) \cdot (3, 4) = 1 \times 3 + (-1) \times 4$

Calculate the answer. $= -1$

b Change $(1, -1)$ to polar form.
$$\tan(\theta) = \frac{-1}{1} = -1$$

Find θ. $\theta = -45°$ or $315°$

Find the magnitude.
$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2} = 1.41\ldots$$

Change $(3, 4)$ to polar form.
$$\tan(\theta) = \frac{4}{3} = 1\frac{1}{3} = 1.33\ldots$$

Find θ. $\theta = 53.13\ldots°$

Find the magnitude.
$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Write the rule for the polar form. $(1.41\ldots, -45°) \cdot (5, 53.13\ldots°)$

The angle between the vectors is $98.13\ldots°$. $= 1.41\ldots \times 5 \times \cos(\theta)$

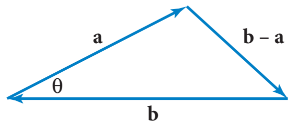
Find the answer, using the extended values in your calculator. $= 1.41\ldots \times 5 \times \cos(98.13\ldots°) = -1$

c Write your findings. The answers are the same.

You can use the cosine rule [$c^2 = a^2 + b^2 - 2ab\cos(C)$] to prove that the polar form and the component form of the dot product are *always* the same.

Consider two vectors $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and their vector difference $\mathbf{b} - \mathbf{a}$.

This is shown by the triangle of vectors to the right.



In this case, $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, $c = |\mathbf{b} - \mathbf{a}|$ and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Now $\mathbf{b} - \mathbf{a} = (b_1 - a_1, b_2 - a_2)$, so using $r = \sqrt{a_1^2 + a_2^2}$ gives

$$a^2 = a_1^2 + a_2^2,\ b^2 = b_1^2 + b_2^2 \text{ and } c^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2.$$

Substitute all these into the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

This gives

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2|\mathbf{a}| \times |\mathbf{b}| \cos(\theta)$$

Multiply out the brackets to get the following.

$$a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2|\mathbf{a}| \times |\mathbf{b}| \cos(\theta)$$

So $$-2a_1b_1 - 2a_2b_2 = -2|\mathbf{a}| \times |\mathbf{b}| \cos(\theta)$$

Divide by -2 to get $$a_1b_1 + a_2b_2 = |\mathbf{a}| \times |\mathbf{b}| \cos(\theta)$$ **QED**

Example 5

Calculate $\mathbf{q} \cdot \mathbf{r}$ when:

a $\mathbf{q} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{r} = \mathbf{i} + 6\mathbf{j}$ b $\mathbf{q} = (2, 4)$ and $\mathbf{r} = (2, 30°)$.

Solution

a Write $\mathbf{q}$ and $\mathbf{r}$ in component form.

$\mathbf{q} = (2, -3)$
$\mathbf{r} = (1, 6)$

Apply the rule in component form. $\mathbf{q} \cdot \mathbf{r} = 2 \times 1 + (-3) \times 6$

Evaluate. $= -16$

b Convert $\mathbf{r}$ from polar to component form.

$\mathbf{r} = (2\cos(30°), 2\sin(30°))$
$= (\sqrt{3}, 1)$

Apply the rule in component form. $\mathbf{q} \cdot \mathbf{r} = 2 \times \sqrt{3} + 4 \times 1$

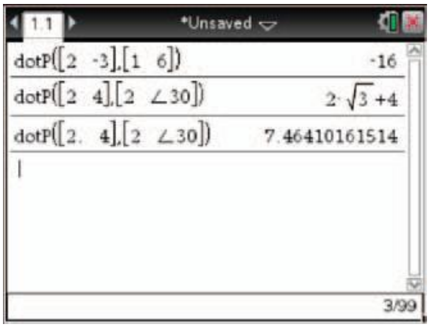
Simplify. $= 4 + 2\sqrt{3}$

TI-Nspire CAS

Use a Calculator page.

You can do both parts without changing the form of the vectors.

If your calculation mode is set on auto, put in a decimal point to get the approximate answer.



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ClassPad

Use the Main $\sqrt{\alpha}$ application.

First set the calculator to degrees (**Deg**) and **Standard**.

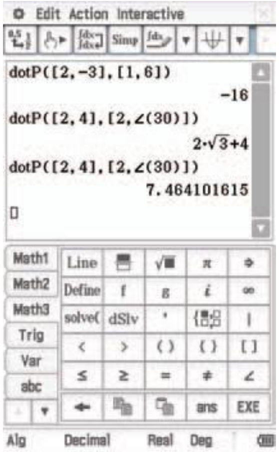
Tap **Action**, then **Vector** and then **dotP**.

Enter the two vectors, separated by a comma.

Tap EXE or press the EXE key.

Remember to put each vector in square brackets.

The third result was obtained by setting the calculator to **Decimal**.



The formula for scalar products provides a convenient way of calculating the angle between two vectors.

State the definition. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\times|\mathbf{b}|\cos(\theta)$

Rearrange. $\cos(\theta) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

IMPORTANT

Angle between two vectors

For vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, the angle θ between **a** and **b** is given by

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \text{ or } \cos(\theta) = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2+a_2^2}\times\sqrt{b_1^2+b_2^2}}$$

Example 6

Calculate the angle between $\mathbf{g} = (2, 6)$ and $\mathbf{h} = (7, 1)$.

Solution

Calculate the magnitude of the vectors.

$$|\mathbf{g}| = \sqrt{2^2+6^2} = 2\sqrt{10}$$

$$|\mathbf{h}| = \sqrt{7^2+1^2} = 5\sqrt{2}$$

Calculate the dot product.

$$\mathbf{g}\cdot\mathbf{h} = 2\times 7 + 6\times 1 = 20$$

Write the formula.

$$\cos(\theta) = \frac{\mathbf{g}\cdot\mathbf{h}}{|\mathbf{g}||\mathbf{h}|}$$

Substitute in the required values.

$$= \frac{20}{2\sqrt{10}\times 5\sqrt{2}}$$

Evaluate.

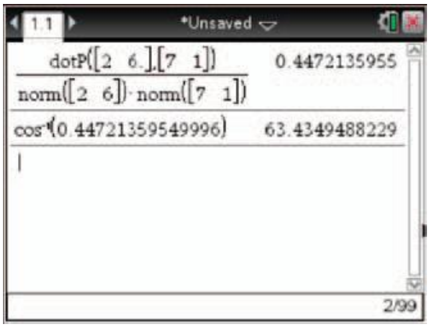
$$= \frac{20}{20\sqrt{5}}$$
$$= \frac{1}{\sqrt{5}}$$

Use $\cos^{-1}$ to calculate θ.

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$
$$\approx 63.43°$$

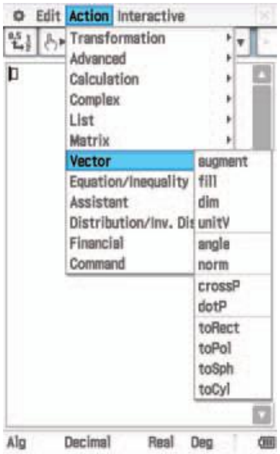
TI-Nspire CAS

Use a Calculator page.
Use the fraction template or brackets and division.
You can use the norm() function to find magnitudes. You can type it in or use [menu], 7: Matrix & Vector, 7: Norms and 1: Norm.
Remember to press [TRIG] to get $\cos^{-1}$.
Press [ctrl] (–) to get the previous answer in the last part.

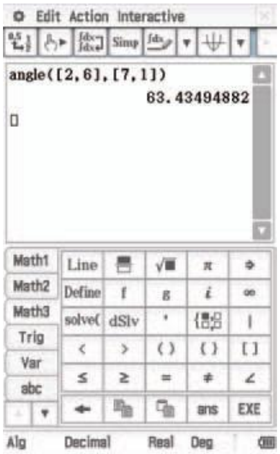


ClassPad

Use the Main $\sqrt{\alpha}$ application.
First set the calculator to degrees (**Deg**) and **Standard**.
Tap **Action**, then **Vector** and then **angle**.



Enter the two vectors, separated by a comma.
Tap [EXE] or press the [EXE] key.



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EXERCISE 4.02 The component form

Concepts and techniques

Component form of the dot product

1 Example 4 In each of the following cases, calculate $\mathbf{a} \cdot \mathbf{b}$ using both the component and polar forms.

a $\mathbf{a} = (2, 0)$ and $\mathbf{b} = (5, 12)$ b $\mathbf{a} = (3, 0)$ and $\mathbf{b} = (4, 3)$

c $\mathbf{a} = (-1, 0)$ and $\mathbf{b} = (-5, 7)$ d $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (8, 2)$

e $\mathbf{a} = (0, -3)$ and $\mathbf{b} = (5, -6)$

2 Find the following scalar products.

a $(9, 120°) \cdot (16, 48°)$ b $(3, 185°) \cdot (16, 200°)$ c $(19, 300°) \cdot (6, 240°)$

3 Example 5 Calculate the scalar product of each of the following pairs of vectors.

a $5\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$ b $2\mathbf{i} + 7\mathbf{j}$ and $8\mathbf{i} + \mathbf{j}$ c $6\mathbf{i} + 2\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$

d $-3\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{i} + 7\mathbf{j}$ e $4\mathbf{i} - 2\mathbf{j}$ and $3\mathbf{i} - 3\mathbf{j}$ f $-9\mathbf{i} + 2\mathbf{j}$ and $-3\mathbf{i} - 11\mathbf{j}$

4 Calculate the dot products of each of the following pairs of vectors.

a $(1, 5)$ and $(5, 0)$ b $(3, 7)$ and $(0, 2)$ c $(-2, 6)$ and $(4, 2)$

d $(7, -3)$ and $(5, 5)$ e $(6, -2)$ and $(3, 8)$ f $(6, -1)$ and $(-3, -7)$

5 CAS Find the following scalar products.

a $\begin{bmatrix} 3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix}$ b $\begin{bmatrix} -2 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ c $\begin{bmatrix} -6 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -6 \end{bmatrix}$

d $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ e $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -9 \end{bmatrix}$ f $\begin{bmatrix} -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

6 Calculate the scalar product of the following pairs of vectors, correct to two decimal places if necessary.

a $(4, 3)$ and $(6, 60°)$ b $(-1, 5)$ and $(4, 45°)$ c $\left(7, \frac{\pi}{6}\right)$ and $(3, -2)$

d $(5, 5)$ and $\left(3, \frac{7\pi}{8}\right)$ e $(6, 115°)$ and $(-4, 4)$ f $(5, -2)$ and $(3, 98°)$

7 Example 6 Calculate the angles between the following pairs of vectors, correct to two decimal places if necessary.

a $(3, 4)$ and $(5, 12)$ b $(1, 3)$ and $(3, 6)$ c $(5, -4)$ and $(-2, 1)$

d $(1, 5)$ and $(4, -6)$ e $(-3, 5)$ and $(4, -2)$ f $(-2, 5)$ and $(3, -2)$

8 CAS Calculate the angles between the following pairs of vectors, correct to two decimal places if necessary.

a $(2, 6)$ and $(-3, 5)$ b $(-4, 5)$ and $(5, 3)$ c $(2, -8)$ and $(-3, 7)$

d $(4, 9)$ and $(3, -2)$ e $(-5, 3)$ and $(2, -4)$ f $(-6, -8)$ and $(-5, -2)$

Reasoning and communication

9 Use the scalar product to show that $(-4, 0)$ and $(0, 5)$ are perpendicular.

10 Consider the scalar product of unit vectors with directions A and B. In polar form, the vectors are $(1, A)$ and $(1, B)$. Use the scalar product in component form to find an expression for the cosine of the angle between the vectors. Use $\cos(A - B)$.

4.03 PROPERTIES OF THE SCALAR PRODUCT

The scalar product of two vectors has a number of very important properties. You may have noticed some of these in the previous sections.

IMPORTANT

Properties of the scalar product

1. The dot product may be calculated from either polar or component forms of vectors as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \cos(\theta)$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ or

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

2. The scalar product is a real number, *not a vector*.
3. The scalar product is commutative.

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

4. The scalar product is distributive over vector addition.

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

5. $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$ for all real m and vectors $\mathbf{a}$ and $\mathbf{b}$.
6. $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

You can verify properties of the scalar product for particular vectors.

Example 7

Show that the scalar product is commutative for the vectors $\mathbf{p} = (2, 7)$ and $\mathbf{q} = (3, -8)$.

Solution

Calculate $\mathbf{p} \cdot \mathbf{q}$.	$\mathbf{p} \cdot \mathbf{q} = 2 \times 3 + 7 \times (-8) = -50$
Calculate $\mathbf{q} \cdot \mathbf{p}$.	$\mathbf{q} \cdot \mathbf{p} = 3 \times 2 + (-8) \times 7 = -50$
State the result.	$\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$

To prove the properties of scalar products you need to show that they are true of all vectors.

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Example 8

Prove that the scalar product is distributive over vector addition.

Solution

Write down what is required to be proved.	**RTP** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
	Proof
Identify the vectors.	Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and $\mathbf{c} = (c_1, c_2)$
Write the LHS of the equation in component form.	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (a_1, a_2) \cdot (b_1 + c_1, b_2 + c_2)$
Apply the rule for dot products.	$= a_1(b_1 + c_1) + a_2(b_2 + c_2)$ $= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2$
Write the RHS of the equation in component form.	$\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = (a_1, a_2) \cdot (b_1, b_2) + (a_1, a_2) \cdot (c_1, c_2)$
Apply the rule for dot products.	$= a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2$ $= \mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
	QED

EXERCISE 4.03 Properties of the scalar product

Concepts and techniques

WS Properties of the dot product

1 Example 7 Use the following pairs of vectors to demonstrate that the scalar product is commutative (i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$).

a (2, 5) and (1, 4)
b (–2, 3) and (4, 5)
c (6, 7) and (–3, –4)
d (–3, 4) and (9, –8)

2 Use the following sets of vectors to show that the scalar product is distributive over vector addition (i.e. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$).

a (3, 7), (2, 5) and (4, 1)
b (2, 3.5), (4.1, 2) and (1.2, 3.4)
c (6.1, –1.8), (5, 2) and (–2, –3)
d (4.6, 9.2), (–5.5, –4.6) and (2, 0)

3 Use the following pairs of vectors to demonstrate that $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$, where m is a real number and **a** and **b** are vectors.

a (2, 5), (3, 4) and $m = 3$
b (1, 6), (8, 3) and $m = 5$
c (4, 1), (0, –6) and $m = –2$
d (–5, 9), (7, 2) and $m = 2.5$

4 Use the following vectors to show that $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

a (1, 2)
b (–3, 4)
c (4, –7)
d (–3, –6)

Reasoning and communication

5 Example 8 Prove that $\mathbf{q} \cdot \mathbf{r} = \mathbf{r} \cdot \mathbf{q}$, where $\mathbf{q} = (q_1, q_2)$ and $\mathbf{r} = (r_1, r_2)$.

6 Prove that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$.

7 Prove that $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$ for all real m and vectors $\mathbf{a}$ and $\mathbf{b}$.

8 Prove that $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

9 Use the polar form of the dot product to show that $\mathbf{a} \cdot (-\mathbf{b}) = -(\mathbf{a} \cdot \mathbf{b})$

4.04 PARALLEL AND PERPENDICULAR VECTORS

INVESTIGATION Parallel and perpendicular vectors

1 Make up some vectors that are parallel.

2 Find the scalar products of parallel vectors.

3 What do you find?

4 Choose some vectors that are perpendicular.

5 Find the scalar products of perpendicular vectors.

6 What do you find?

Consider the scalar product of two vectors **a** and **b**.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

If $\mathbf{a} \cdot \mathbf{b} = 0$, then either $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$ or $\cos(\theta) = 0$.

So, if **a** and **b** are nonzero, $\theta = 90°$ because $\cos(90°) = 0$.

Example 9

Show that the vectors (2, 7) and (−7, 2) are perpendicular.

Solution

Calculate the dot product.

$$(2, 7) \cdot (-7, 2) = 2 \times (-7) + (-7) \times 2$$
$$= 0$$

State the result.

(2, 7) and (−7, 2) are both nonzero, so $\cos(\theta) = 0$.
Therefore they are perpendicular.

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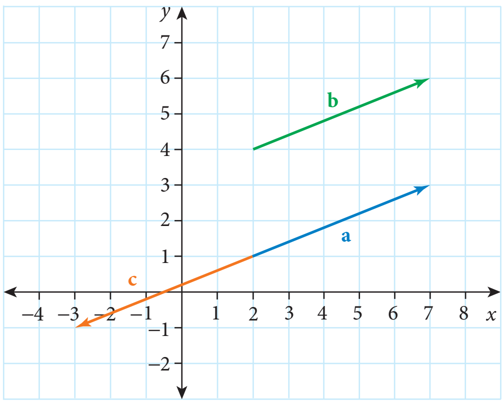
The diagram on the right shows three vectors: **a**, **b** and **c**.

The three vectors are parallel. Vectors **a** and **b** are in the same direction so the angle between them is 0°. Vectors **a** and **c** are in the opposite direction, so the angle between them is 180°.

Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$:

- if $\theta = 0°$, then $\cos(\theta) = 1$, so $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$
- if $\theta = 180°$, then $\cos(\theta) = -1$, so $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$



IMPORTANT

Perpendicular and parallel vectors

If **a** and **b** are perpendicular (or **orthogonal**), then $\mathbf{a} \cdot \mathbf{b} = 0$

If **a** and **b** are parallel, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$ or $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$

For the unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$:

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$$

The dot products of the unit vectors **i** and **j** are as shown because any vector is parallel to itself and **i** and **j** are perpendicular. Sometimes vectors in the opposite direction are called antiparallel. This is the case where $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

Example 10

Determine if the vectors $\mathbf{c} = (5, -4)$ and $\mathbf{d} = (-5, 4)$ are perpendicular, parallel or neither.

Solution

Calculate the magnitude of each vector.	$\lvert\mathbf{c}\rvert = \sqrt{5^2 + (-4)^2} = \sqrt{41}$ $\lvert\mathbf{d}\rvert = \sqrt{(-5)^2 + 4^2} = \sqrt{41}$
Calculate the dot product.	$\mathbf{c} \cdot \mathbf{d} = (5, -4) \cdot (-5, 4)$ $= 5 \times (-5) + (-4) \times 4$ $= -41$
Calculate $\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert$.	$\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert = \sqrt{41} \times \sqrt{41} = 41$
Compare $\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert$ with the dot product.	$\lvert\mathbf{c}\rvert\lvert\mathbf{d}\rvert = -\mathbf{c} \cdot \mathbf{d}$
State the result.	**c** and **d** are parallel but in opposite directions (antiparallel).

EXERCISE 4.04 Parallel and perpendicular vectors

Concepts and techniques

WS Parallel and perpendicular vectors

1 The vector $\mathbf{a} = 3\mathbf{i}$. Which of the following are orthogonal to $\mathbf{a}$ (at right angles)?

A $-3\mathbf{i}$ B $-3\mathbf{j}$ C $3\mathbf{i} + 3\mathbf{j}$ D $-3\mathbf{i} - 3\mathbf{j}$ E 0

2 Example 9 Show that the following pairs of vectors are perpendicular.

a (2, 0) and (0, 6)
b (4, –2) and (2, 4)
c (–5, 5) and (5, 5)
d (–12, 0) and (0, 16)
e $(3\sqrt{5}, 2\sqrt{3})$ and $(-2\sqrt{3}, 3\sqrt{5})$
f $\left(\frac{2}{3}, \frac{-4}{5}\right)$ and $\left(\frac{4}{5}, \frac{2}{3}\right)$

3 Show that the following pairs of vectors are perpendicular.

a $\mathbf{i} - 3\mathbf{j}$ and $3\mathbf{i} + \mathbf{j}$
b $7\mathbf{i}$ and $4\mathbf{j}$
c $9\mathbf{i} + 6\mathbf{j}$ and $6\mathbf{i} - 9\mathbf{j}$
d $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$
e $\begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \end{bmatrix}$ and $\begin{bmatrix} -8 \\ \frac{16}{3} \end{bmatrix}$
f $\begin{bmatrix} 5\sqrt{6} \\ -3\sqrt{10} \end{bmatrix}$ and $\begin{bmatrix} 3\sqrt{3} \\ 3\sqrt{5} \end{bmatrix}$

4 Use the scalar product to show that the following pairs of vectors are parallel or antiparallel.

a (3, 5) and (6, 10)
b (–4, 2) and (8, –4)
c (7, 3) and $(9\frac{1}{3}, 4)$
d (–5, 2) and (7, –2.8)
e $(4\sqrt{6}, 2\sqrt{3})$ and $(-12\sqrt{2}, -6)$
f $\left(\frac{2}{3}, \frac{-4}{5}\right)$ and $\left(\frac{4}{5}, -\frac{24}{25}\right)$

5 Use the scalar product to show that the following pairs of vectors are parallel or antiparallel.

a $2\mathbf{i} + 6\mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$
b $4\mathbf{i} - 5\mathbf{j}$ and $-4.8\mathbf{i} + 6\mathbf{j}$
c $4\mathbf{i} + 9\mathbf{j}$ and $2\frac{2}{3}\mathbf{i} + 6\mathbf{j}$
d $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 14 \end{bmatrix}$
e $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2\frac{2}{3} \end{bmatrix}$
f $\begin{bmatrix} 3\sqrt{6} \\ 4\sqrt{15} \end{bmatrix}$ and $\begin{bmatrix} 18\sqrt{2} \\ 24\sqrt{5} \end{bmatrix}$

6 Example 10 Determine if the following pairs of vectors are perpendicular, parallel or neither.

a (5, –4) and (–4, 5)
b (8, –3) and (–12, 4.5)
c (4, –3) and (–6, 5)
d (–6, 4) and (–15, 10)
e $\left(\frac{3}{4}, \frac{5}{8}\right)$ and (–5, 6)
f $\left(3\sqrt{10}, 2\sqrt{3}\right)$ and $\left(4\sqrt{2}, 5\sqrt{6}\right)$

Reasoning and communication

7 Vectors $\mathbf{p} = (a, -3)$ and $\mathbf{q} = (8, 6)$ are perpendicular. Calculate the possible values of a.

8 Vectors $\mathbf{r} = (4, b)$ and $\mathbf{t} = (3, 6)$ are parallel. Calculate the possible values of b.

9 The vector $\mathbf{m}$ is fixed and its magnitude is 4 while the vector $\mathbf{n}$ is free to rotate and has a magnitude of 5. What are the maximum and minimum values of the dot product $\mathbf{m} \cdot \mathbf{n}$ as $\mathbf{n}$ rotates through all possible positions? What positions of $\mathbf{m}$ and $\mathbf{n}$ lead to these values?

10 Consider the points $A(-1, 6)$, $B(-3, -2)$ and $C(7, 3)$. Calculate the angle between **BA** and **BC**.

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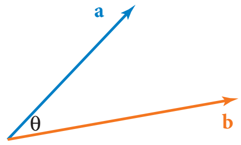
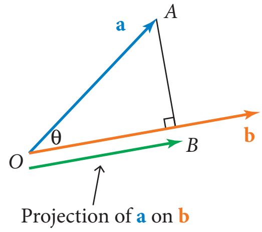
4.05 PROJECTION OF A VECTOR

Consider the two vectors **a** and **b** shown here. The angle between the vectors is θ.



If a line is drawn from **a** perpendicular to **b** as shown by AB here, the part of **b** indicated by the green arrow (OB) is called the **projection** of **a** on **b**. It is also called the projection of **a** in the direction of **b**.

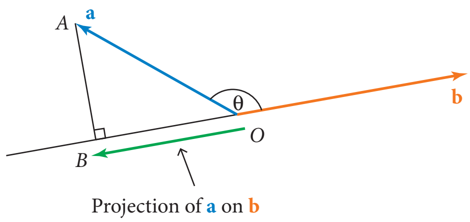


If the angle between the two vectors is obtuse, then the projection is in the opposite direction to vector **b** and will be negative.

From the right-angled triangle OAB and using trigonometry:

$$\cos(\theta) = \frac{OB}{|\mathbf{a}|}$$

or $$OB = |\mathbf{a}|\cos(\theta)$$

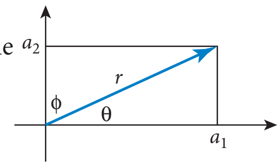


So, if the value of θ is known, the projection p of **a** on **b** is given by:

$$p = |\mathbf{a}|\cos(\theta)$$

Since a projection is in a particular direction, it is actually a vector, but the magnitude is generally referred to as the projection. The phrase **vector projection** may be used to mean the vector with magnitude $|\mathbf{a}|\cos(\theta)$ in the direction of **b**, namely, $|\mathbf{a}|\cos(\theta)\hat{\mathbf{b}}$.

As can be seen from the diagram on the right, the component form of a vector is actually the projection onto the x-axis and the projection onto the y-axis.



$\mathbf{a} = \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, where $a_1 = r\cos(\theta)$ and $a_2 = r\sin(\theta)$

But $\sin(\theta) = \cos(90° - \theta) = \cos(\phi)$ and $r = |\mathbf{a}|$, so

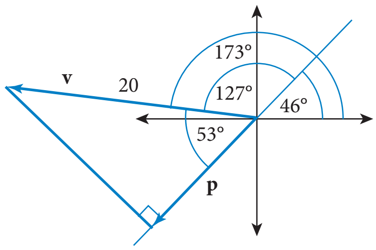
$a_1 = |\mathbf{a}|\cos(\theta)$ and $a_2 = |\mathbf{a}|\cos(\phi)$, the projections on the x and y axes (or on **i** and **j**).

Example 11

Find the projection in the direction 46° of a vector of magnitude 20 in the direction 173°.

Solution

Sketch a diagram showing the vector in the direction 173° and the direction of the projection at 46°.
Indicate the projection p and the vector $\mathbf{v}$ on the diagram.



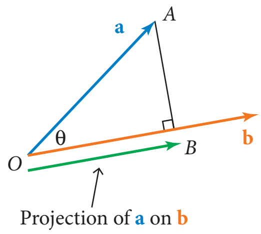
Calculate the projection. $p = 20\cos(127°)$

Evaluate. ≈ -12.0

The negative sign indicates that the projection is in the opposite direction. It could also have been obtained using $-20\cos(53°)$.

State the result. The required projection is -12.0.

Consider again the diagram used before Example **11**.



The projection of $\mathbf{a}$ on $\mathbf{b}$ is given by $OB = |\mathbf{a}|\cos(\theta)$

But $\quad \cos(\theta) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

So $\quad OB = |\mathbf{a}|\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

You can write this as $OB = \mathbf{a}\cdot\dfrac{\mathbf{b}}{|\mathbf{b}|}$

But $\hat{\mathbf{b}} = \dfrac{1}{|\mathbf{b}|}\mathbf{b}$

So $OB = \mathbf{a}\cdot\hat{\mathbf{b}}$

IMPORTANT

The magnitude of a **projection**, $\mathbf{p}$, of $\mathbf{a}$ on $\mathbf{b}$ is the scalar product of $\mathbf{a}$ and a unit vector in the direction of $\mathbf{b}$.

$$p = \mathbf{a}\cdot\hat{\mathbf{b}}$$

where $\hat{\mathbf{b}}$ is a unit vector in the direction of $\mathbf{b}$.

Alternatively: $p = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

The **vector projection** is $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = |\mathbf{a}|\cos(\theta)\hat{\mathbf{b}}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

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Example 12

Find the projection of $\mathbf{m} = (5, 7)$ on $\mathbf{n} = (6, 2)$.

Solution

Find a unit vector in the direction of $\mathbf{n}$.

$$\begin{aligned}\hat{\mathbf{n}} &= \frac{1}{|\mathbf{n}|}\mathbf{n}\\ &= \frac{1}{\sqrt{6^2+2^2}}(6, 2)\\ &= \frac{1}{2\sqrt{10}}(6, 2)\\ &= \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\end{aligned}$$

Use the formula for the projection of $\mathbf{m}$ on $\mathbf{n}$.

$$p = \mathbf{m}\cdot\hat{\mathbf{n}}$$

Substitute in the known values.

$$\begin{aligned}&= (5, 7)\cdot\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\\ &= \frac{15}{\sqrt{10}} + \frac{7}{\sqrt{10}}\end{aligned}$$

Evaluate.

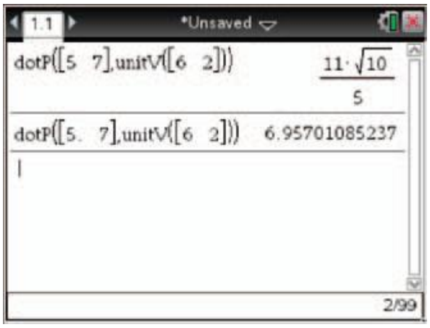
$$= \frac{22}{\sqrt{10}}$$

Rationalise the denominator.

$$\begin{aligned}&= \frac{11\sqrt{10}}{5}\\ &\approx 6.96\end{aligned}$$

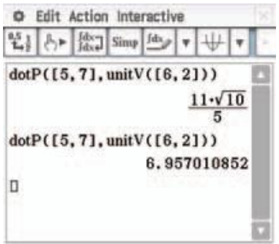
TI-Nspire CAS

You can find the projection using the dot product and unit vector functions.
You type unitV() or use menu, 7: Matrix & Vector, C: Vector and 1: Unit Vector.



ClassPad

Use the Main $\sqrt{\alpha}$ application. Set the calculator to **Standard** for an exact result (which may be a surd or a fraction) and **Decimal** for an approximate answer. Mathematicians generally prefer exact answers.



Take the dot product of $\mathbf{m}$ and $\hat{\mathbf{n}}$.

Tap **Action** and then **Vector** to get **dotP** and **unitV**.

EXERCISE 4.05 Projection of a vector

Concepts and techniques

WS Projections

1 The projection of $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j}$ on the positive direction of the x-axis is:
A -2 B 0 C 2 D 4 E $2\sqrt{5}$

2 Example 11 The projection of $\mathbf{m} = (4, 60°)$ on $\mathbf{n} = (11, 240°)$ is:
A -11 B -4 C 0 D 4 E 11

3 The projection of $\mathbf{p} = (5, 120°)$ on $\mathbf{q} = (7, 30°)$ is:
A -7 B -5 C 0 D 5 E 7

4 Example 12 The projection of $2\mathbf{i} + 4\mathbf{j}$ on $-\mathbf{i} + \frac{1}{2}\mathbf{j}$ is:
A -1 B 0 C $\frac{1}{2}$ D 2 E 4

5 Find the projection in the direction 60° of a vector of magnitude 18 and direction 86°.

6 Find the projection in the direction $\frac{2\pi}{3}$ of a vector of magnitude 16 in the direction 82°.

7 Find the projection in the direction 206° of the vector (8, 100°).

8 Find the projection of (32, 178°) in the direction 200°.

9 What is the projection of (16, 35°) in the direction 305°?

10 What is the projection of (19, 116°) in the direction 116°?

11 Find the projection of $\mathbf{a} = (1, 5)$ on $\mathbf{b} = (3, 4)$.

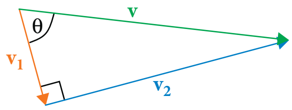
12 Find the projection of $-10\mathbf{i} - 2\mathbf{j}$ on $5\mathbf{i} + 12\mathbf{j}$.

13 Find the projection of $\mathbf{m} = (-3, 5)$ on $\mathbf{n} = (-7, 2)$.

14 Find the projection of $\mathbf{u} = (4, -3)$ on $\mathbf{v} = (-6, -2)$.

15 Find the projection of $-9\mathbf{i} - 3\mathbf{j}$ on $-\mathbf{i} - 2\mathbf{j}$.

4.06 RESOLUTION OF FORCES



Practical problems with vectors often involve combining several vectors. As you have seen it is easier to combine vectors in component form than in polar form. The natural direction to use may not be in the directions of **i** or **j**. In this case we change the vectors to a sum of projections that are perpendicular to each other.

In the diagram above right, the vector $\mathbf{v}$ is the sum of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular to each other. If the angle between $\mathbf{v}$ and $\mathbf{v}_1$ is θ, then $v_1 = v \cos(\theta)$ and $v_2 = v \sin(\theta)$.

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IMPORTANT

Orthogonal components of a vector are projections of the vector in perpendicular directions. Writing a given vector as a sum of orthogonal components is called **resolving** the vector into orthogonal components.

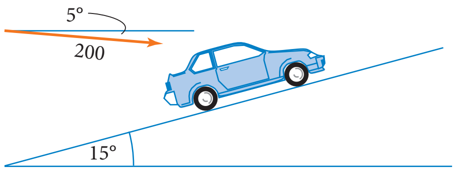
Example 13

Some people are pushing a car up a hill with a slope of 15°. One person who is pushing is applying a force of 200 N at an angle 5° downwards from the horizontal. Resolve this force into components parallel and perpendicular to the road.

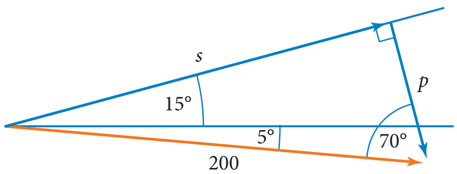
Alamy/Juice Images

Solution

Sketch a diagram to represent the given information.



Draw another diagram to show the orthogonal components. From the diagram, you can see that the angles between the force of 200 N and its orthogonal components s and p are 20° and 70° respectively.



Calculate the projection s. $s = 200\cos(20°)$

Evaluate. ≈ 187.9

Calculate the other component, p. $p = 200\sin(20°)$

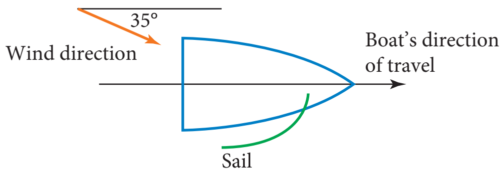
Evaluate. ≈ 68.4

State the result. The components of the 200 N force parallel and perpendicular to the road are about 187.9 N and 68.4 N respectively.

In some cases, you will not know the actual force involved, but will want to find part of a force in a particular direction.

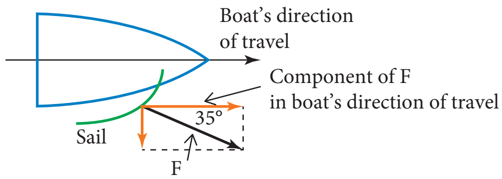
Example 14

A boat is sailing in a particular direction with the wind direction 35° to the south of the direction of travel, as shown in the diagram below. The wind exerts a force **F** on the boat's sail. Calculate the percentage of the force of the wind that is in the direction of travel of the boat.



Solution

Redraw the diagram showing **F** and its components.



Find the forward component.

$$F_d = F\cos(35°)$$
$$= 0.8191\ldots F$$

State the result.

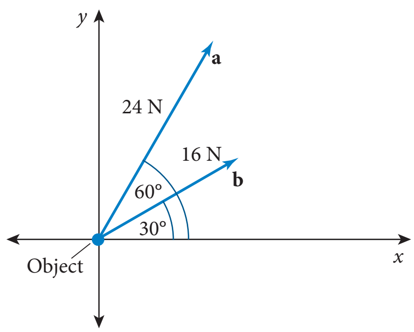
The component of the force of the wind in the direction of travel of the boat is about 82% of the force exerted by the wind.

Example 15

One person applies a force of 24 newtons to an object at an angle of 60° upwards from the horizontal. Another person applies a force of 16 N in the same plane at an angle of 30° upwards from the horizontal. What is the resultant force acting on the object?

Solution

Sketch a diagram, making x the horizontal direction and y the vertical direction.
Let the 24 N force be **a**. Show **a** acting at 60° from the horizontal.
Show the force of 16 N as **b**, acting at 30° from the horizontal.
In order to add the forces, each can be resolved into its x and y components using $x = r\cos(\theta)$ and $y = r\sin(\theta)$.



Resolve **a**.

$\mathbf{a} = (24\cos(60°), 24\sin(60°))$

Evaluate using $\cos(60°) = \frac{1}{2}$ and $\sin(60°) = \frac{\sqrt{3}}{2}$.

$= (12, 12\sqrt{3})$

Resolve **b**.

$\mathbf{b} = (16\cos(30°), 16\sin(30°))$

Evaluate using $\cos(30°) = \frac{\sqrt{3}}{2}$ and $\sin(30°) = \frac{1}{2}$.

$= (8\sqrt{3}, 8)$

Find the resultant, **r**.

$\mathbf{r} = (12, 12\sqrt{3}) + (8\sqrt{3}, 8)$

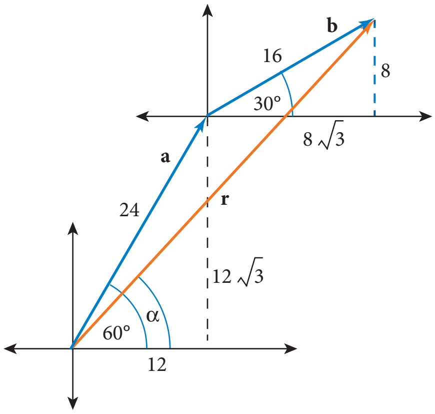
Add the x and y components separately.

$= (12 + 8\sqrt{3}, 12\sqrt{3} + 8)$

Evaluate and store the answers.

$= (25.86\ldots, 28.78\ldots)$

Use a diagram to show the addition of the vectors. Find the magnitude and direction of **r**.



Find $r = |\mathbf{r}|$.
Substitute the stored answers.

$$r^2 = x^2 + y^2$$
$$= 25.86\ldots^2 + 28.78\ldots^2$$
$$r = 38.69\ldots$$

Find the direction of **r**.

$$\tan(\alpha) = \frac{28.78\cdots}{25.86\cdots}$$

Use $\tan^{-1}$ to find α.

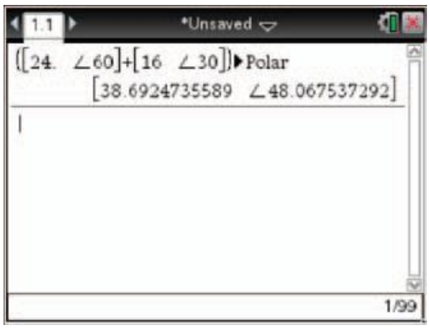
$$\alpha = 48.06\ldots°$$

Round and state the result.

The resultant force is approximately 38.7 N at 48.1° to the horizontal.

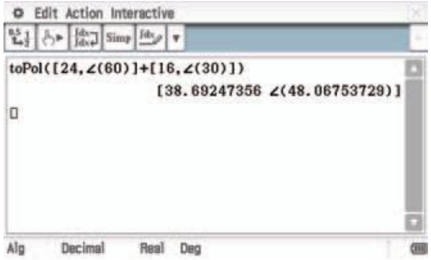
TI-Nspire CAS

Use a Calculator page.
Write the addition of the vectors and express the answer in polar form.
Use a decimal point to force approximate calculation.
If you type the vectors make sure you use square brackets.



ClassPad

Use the Main $\sqrt{\alpha}$ application and add the vectors.
The answer must be in polar form, so start with **toPol**.
Then enter the vector sum,
toPol([24,∠(60)]+[16,∠(30)])
Tap **Action**, then **Vector** for **toPol**.



EXERCISE 4.06 Resolution of forces

Concepts and techniques

WS Resolution of forces

1 Example 14 An aeroplane is flying toward the north-east. Which of the following wind velocity vectors increases the plane's speed the most?

A $\mathbf{w}_1 = 4\mathbf{i} + \mathbf{j}$ B $\mathbf{w}_2 = -\mathbf{i} + 2\mathbf{j}$ C $\mathbf{w}_3 = \mathbf{i} - 8\mathbf{j}$
D $\mathbf{w}_4 = -10\mathbf{i} - 2\mathbf{j}$ E $\mathbf{w}_5 = -5\mathbf{i} - 2\mathbf{j}$

2 For the aeroplane described in question **1**, which of the following wind velocity vectors slows down the plane the most?

A $\mathbf{w}_1 = 4\mathbf{i} + \mathbf{j}$ B $\mathbf{w}_2 = -\mathbf{i} + 2\mathbf{j}$ C $\mathbf{w}_3 = \mathbf{i} - 8\mathbf{j}$
D $\mathbf{w}_4 = -10\mathbf{i} - 2\mathbf{j}$ E $\mathbf{w}_5 = -5\mathbf{i} - 2\mathbf{j}$

3 Example 13 Given vector $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$ and force vector **F** below, calculate:

i the component of **F** parallel to **v**
ii the component of **F** perpendicular to **v**.

a $\mathbf{F} = 4\mathbf{i} + \mathbf{j}$ b $\mathbf{F} = 0.5\mathbf{i} - 0.2\mathbf{j}$ c $\mathbf{F} = 16\mathbf{i} + 12\mathbf{j}$
d $\mathbf{F} = -0.4\mathbf{i} - 0.3\mathbf{j}$ e $\mathbf{F} = -5\mathbf{i} - 3\mathbf{j}$ f $\mathbf{F} = -8\mathbf{i} - 6\mathbf{j}$

4 The force on an object is give by $\mathbf{F} = -15\mathbf{j}$. For each vector **v** below, find:

i the component of **F** parallel to **v**
ii the component of **F** perpendicular to **v**

a $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$ b $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$ c $\mathbf{v} = 6\mathbf{j}$ d $\mathbf{v} = 4\mathbf{i}$

Reasoning and communication

5 Example 15 Three horizontal forces of 300 N, 400 N and 500 N are being exerted on an object. The 300 N and 500 N forces are exerted either side of the 400 N force at angles of 30° and 50° respectively. Find the total acting force and the direction of that force.

6 The winning team of a tug-of-war competition offers to take on two opponents. The teams working together have their ropes at an angle of 70° to each other and are exerting about the same force. The winning team is exerting a force of 1200 N but is easily overcome by the other two teams. The combined force of the other teams is actually 400 N more than the winning team. What forces were exerted by each of the two other teams?

iStockphoto/omgimages

7 A 100 m sprint is run on an athletics track in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$. The wind velocity is $\mathbf{w} = 6\mathbf{i} + \mathbf{j}$ km/h. How much assistance (in km/h) do the athletes get from the wind in the direction of the race?

8 A car going up a hill inclined at 7° has a weight of 9000 N acting vertically downwards. Resolve the weight into components parallel and perpendicular to the slope.

9 In a physics experiment, a small shot put ball that is rolling down a plank sloped at 12° to the horizontal has a weight of 6 N. The total force acting on the ball is worked out from its acceleration to be 0.7 N acting down the slope. Resolve the weight into components parallel and perpendicular to the slope. The force acting down the slope is reduced by the force of friction acting back up the slope. What must the force of friction be?

10 A hang-glider is flying at a constant speed from a cliff-top down towards a long beach. The angle of flight is 5° down from the horizontal. The weight of the glider and its rider is 1100 N altogether. Find the drag acting on the hang-glider back along the flight path, assuming that the drag force of the air on the wing acts perpendicularly to the path.

iStockphoto/iFelino

11 Before a boat on a trailer can be hitched to a car off the sand, is has to be pulled up the beach. The slope of the sand is 5° upwards and the person at the front is pulling upwards with a force of 400 N at an angle of 15° to the horizontal. Another person is pushing horizontally at the back with a force of 350 N. What is the total force moving the boat and trailer up the beach?

4.07 APPLICATIONS OF THE SCALAR PRODUCT

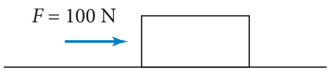
Projections and the dot product are used extensively in Physics to deal with vectors such as force, momentum, displacement and field strength.

The work done by a force is defined in physics as the product of the force and the displacement in the direction of the force. The displacement in the direction of the force is actually the projection of the displacement in the direction of the force. This means that $W = \mathbf{F} \cdot \mathbf{s}$, where W is the work done, $\mathbf{F}$ is the force and $\mathbf{s}$ is the displacement.

The SI units of force, displacement and energy (work) are the newton (N), metre and joule (J) respectively.

Example 16

Calculate the work done by a horizontal force of 100 N that moves an object 20 m horizontally.



Solution

Use the formula for W for displacement in the same direction as the force.	$W = F \times s$
Substitute in the given values.	$= 100 \text{ N} \times 20 \text{ m}$
Evaluate.	$= 2000 \text{ J}$

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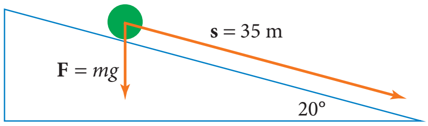
The energy gained by an object is the work done by the forces acting on the object.

Example 17

How much energy is gained by a mass of 25 kg when it accelerates 35 m down a slope at 20° to the horizontal? (Note that the force exerted by gravity near the Earth is given by $\mathbf{F} = 9.81m$ newtons downwards, where m is the mass of the object in g).

Solution

Draw a diagram.



Find the vectors.

Gravitational force $= 25 \times 9.81$ N vertically
$= 245.25$ N vertically

Displacement $= 35$ m at 20° to the horizontal
$= 35$ m at 70° to the vertical

Find the dot product.

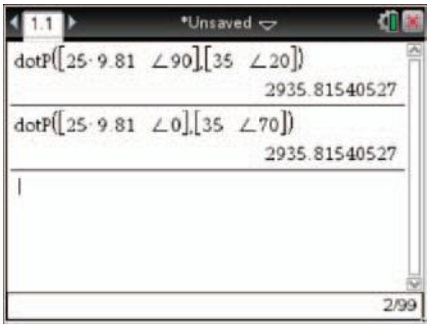
Work done $= \mathbf{F} \cdot \mathbf{s}$
$= 245.25 \times 35 \times \cos(70°)$
$= 2935.815\ldots$ J

Write the answer.

The mass gains about 2936 J of energy.

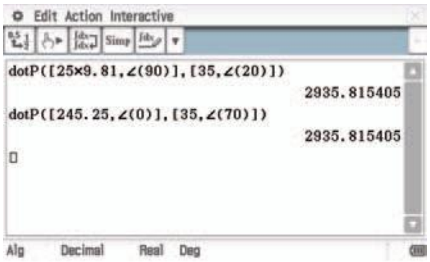
TI-Nspire CAS

Use a Calculator page.
Use the dot product, specifying the directions of the force and displacement relative to one or the other of the horizontal and vertical directions.



ClassPad

Use the Main $\sqrt{\alpha}$ application and **dotP**.
Set the calculator to **Decimal** and **Deg**.
You can specify the forces at angles of 90° and 20° or at angles of 0° and 70°.
You can specify the first force as either 25×9.81 or 245.25.
The screen on the right shows dotP([25×9.81, ∠(90)],[35, ∠(20)]) and dotP([245.25, ∠(0)],[35, ∠(70)])



The scalar product can be used to prove geometric results.

Example 18

Show that the diagonals of a rhombus intersect at right angles.

Solution

State the equalities of a rhombus *ABCD*.	$\mathbf{AB} = \mathbf{DC}$, $\mathbf{BC} = \mathbf{AD}$ and $\lvert\mathbf{AB}\rvert^2 = \lvert\mathbf{DC}\rvert^2 = \lvert\mathbf{BC}\rvert^2 = \lvert\mathbf{AD}\rvert^2$
Find the diagonal's dot product.	$\mathbf{AC}\cdot\mathbf{BD} = (\mathbf{AB}+\mathbf{BC})\cdot(\mathbf{BA}+\mathbf{AD})$
Expand.	$= \mathbf{AB}\cdot\mathbf{BA} + \mathbf{AB}\cdot\mathbf{AD} + \mathbf{BC}\cdot\mathbf{BA} + \mathbf{BC}\cdot\mathbf{AD}$
Use equalities.	$= \mathbf{AB}\cdot\mathbf{BA} + \mathbf{AB}\cdot\mathbf{BC} + \mathbf{BC}\cdot\mathbf{BA} + \mathbf{BC}\cdot\mathbf{BC}$
Use commutativity and simplify.	$= \mathbf{BC}\cdot(\mathbf{AB}+\mathbf{BA}) + \lvert\mathbf{BC}\rvert^2 - \lvert\mathbf{AB}\rvert^2$
But $\mathbf{AB}+\mathbf{BA} = 0$ and $\lvert\mathbf{BC}\rvert^2 - \lvert\mathbf{AB}\rvert^2 = 0$.	$= 0$, so the diagonals are perpendicular.

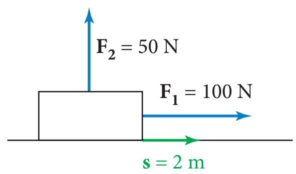
EXERCISE 4.07 Applications of the scalar product

Concepts and techniques

WS Applications of the dot product

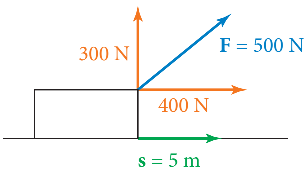
You should use your CAS calculator wherever appropriate in this exercise.

1 Example 16 An object is subjected to two forces as shown in the diagram on the right. If the object has a displacement of 2 m, then the work done by the forces is:



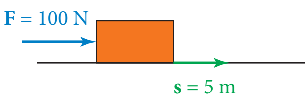
A 0 B 100 J C 200 J
D 224 J E 300 J

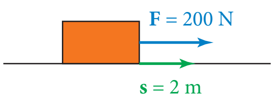
2 An object is subjected to a force of 500 N, as shown in the diagram on the right. The components of the force parallel and perpendicular to the displacement ($\mathbf{s}$) are 400 N and 300 N respectively. The work done by $\mathbf{F}$ is:

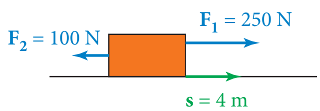


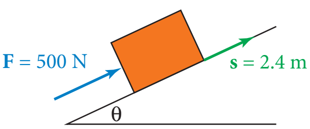
A 1500 J B 2000 J C 2500 J
D 3500 J E 4500 J

3 Calculate the work done in each of the following situations.

a


b


c


d


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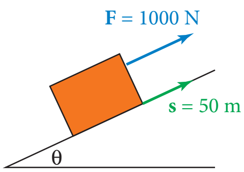

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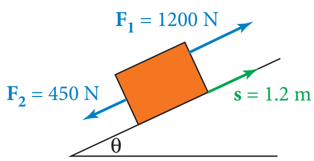

4 Example 17 A ball is rolling down a slope at an angle of 34° to the horizontal. The weight of the ball is 40 N and this acts vertically downwards.

a What is the component of the weight that acts parallel to the direction of motion of the ball?

b How much work is done when the ball travels 1.8 m down the slope?

5 An object undergoes a displacement of $\mathbf{s} = 3\mathbf{i} + 4\mathbf{j}$ metres as a result of a force $\mathbf{F}$ newtons. Calculate the work done in each of the following cases.

a $\mathbf{F} = 4\mathbf{i} + \mathbf{j}$ b $\mathbf{F} = 2\mathbf{i} - 4\mathbf{j}$ c $\mathbf{F} = 9\mathbf{i} + 12\mathbf{j}$

d $\mathbf{F} = -3\mathbf{i} - 5\mathbf{j}$ e $\mathbf{F} = -4\mathbf{i} + 3\mathbf{j}$ f $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j}$

Reasoning and communication

6 A force of 500 N is exerted on an object. The force acts in a direction 45° above the horizontal and results in the object being displaced 20 m horizontally. Calculate the work done by the force.

7 A 36 kg object falls freely under the influence of gravity. Under gravity, $F = mg$, where m is in kilograms and $g = 9.81$ ms^{-2}. Calculate the work done when the object falls 28 m.

8 A force of 12 N causes an object to move 12 m at an angle of 60° to the direction of the force. What amount of energy is transferred?

9 Two forces of 30 N and 50 N act on an object. The 50 N force is at an angle of 35° to the 30 N force. The object moves 8 m at an angle of 15° to the 30 N force, away from the 50 N force.

a Find the total acting force.

b Find the energy transferred by the:

i total force ii 30 N force iii 50 N force.

c Compare the sizes of the answers in part **b** and interpret this information mathematically.

10 Three people are pushing parallel to the slope on the back of a car facing uphill at an angle of 5° to the horizontal. The weight of the car is about 8600 N.

a Resolve the weight into parallel and perpendicular components to the slope to find the average force they must each exert to overcome the weight pushing the car back down the hill.

b If the people stop pushing and let the car roll down the hill, calculate the work done when the car travels 120 m down the slope.

11 A boat and trailer are winched up an 8 m ramp inclined at 40° to the horizontal. The force of gravity on the boat and trailer is 1500 N. What work is done against gravity?

12 A yacht is sailing on a bearing of 240° with a force of 6000 N exerted on its sails by a south-easterly wind. What work is done on the yacht when it sails a distance of 1 nautical mile (n.m.)? (1 n.m. = 1852 m)

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13 A downhill skier loses 15% of the energy gained going down a 50 m long slope at an angle of 60° to the horizontal, due to friction and air resistance. What energy is gained (from gravity) if the skier's total mass (with skis, etc.) is 78 kg and the vertical force of gravity is $9.81 \times m$?

14 Example 18 Use the scalar product to prove that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus.

15 Use the scalar product to prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.
Hint: Use $AC^2 = \mathbf{AC} \cdot \mathbf{AC}$ and express the scalar product in terms of all the sides.

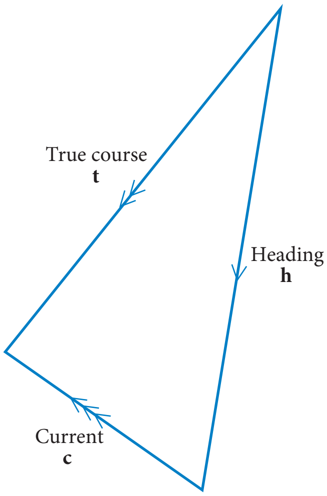
4.08 APPLICATION OF VECTORS TO NAVIGATION

Vectors are particularly useful when constructing problems invovling navigation models. When steering a boat, the effect of the current must be taken into account. Similarly, the effect of the wind must be accounted for when determining the direction in which a plane must fly to reach its destination. In *marine navigation*, the direction in which a boat is steered is called the **heading**. The **speed through the water** is the speed of the boat relative to the water. The **speed made good** and **course made good** are the true speed and course of the boat when the effects of the current are taken into account. Taken together, the speed and course made good constitute a single vector of the true velocity.


iStockphoto/Gord Horne

A boat is carried with the current, so the speed and course made good is the vector sum of the boat's heading and speed through the water and the current. This is illustrated in the vector triangle on the right. It is standard practice to use one arrowhead for the vector showing the course steered and the boat's speed through the water (heading), two arrowheads for the vector showing the course and speed made good (true course), and three arrowheads for the current vector.

Using $\mathbf{h}$ for the heading, $\mathbf{c}$ for the current and $\mathbf{t}$ for the course made good, you get $\mathbf{t} = \mathbf{h} + \mathbf{c}$.



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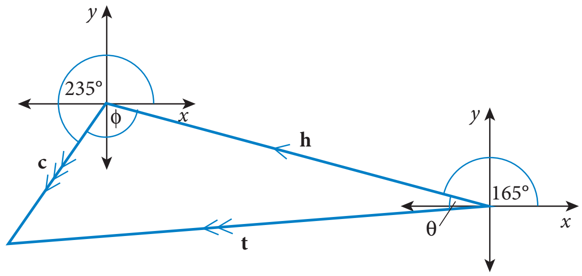
Example 19

A yacht making 7 knots through the water is headed on a bearing of 285°. There is a current of 3 knots on a bearing of 215°. What is the speed and course made good?

Solution

Start by making a rough sketch. Take east as the **i** direction and north as the **j** direction. The heading **h** is on a bearing of 285°. This means that **h** is at an angle of 165° to the **i** direction. The current **c** is at an angle of 235° to the **i** direction.



First calculate the magnitude of **t**.

Write **h** in terms of **i** and **j**. Evaluate and round off.

$$\begin{aligned}\mathbf{h} &= 7\cos(165°)\,\mathbf{i} + 7\sin(165°)\,\mathbf{j}\\ &\approx -6.761\mathbf{i} + 1.812\mathbf{j}\end{aligned}$$

Write **c** in terms of **i** and **j**. Evaluate and round off.

$$\begin{aligned}\mathbf{c} &= 3\cos(235°)\,\mathbf{i} + 3\sin(235°)\,\mathbf{j}\\ &\approx -1.721\mathbf{i} - 2.457\mathbf{j}\end{aligned}$$

Find **t**, using vector addition.

$$\mathbf{t} = \mathbf{h} + \mathbf{c}$$

Substitute for **h** and **c**. Evaluate.

$$\begin{aligned}&\approx -6.761\mathbf{i} + 1.812\mathbf{j} + (-1.721)\mathbf{i} - 2.457\mathbf{j}\\ &= -8.482\mathbf{i} - 0.645\mathbf{j}\end{aligned}$$

Find the magnitude of **t**. Evaluate and round off.

$$\begin{aligned}|\mathbf{t}| &= \sqrt{(-8.482)^2 + (-0.645)^2}\\ &\approx 8.506\end{aligned}$$

Now calculate the direction of **t**.

Use the diagram to find ϕ.

$$\begin{aligned}\phi &= 360° - 235° - (180° - 165°)\\ &= 110°\end{aligned}$$

Use the sine rule to find θ.

$$\frac{8.506}{\sin(110°)} = \frac{3}{\sin(\theta)}$$

Rearrange and evaluate.

$$\theta \approx 19.4°$$

Calculate the bearing of **t**.

$$\begin{aligned}\text{Bearing of } \mathbf{t} &\approx 360° - (165° - 90°) - 19.4°\\ &= 265.6°\end{aligned}$$

State the result.

The speed and course made good is about 8.5 knots on a bearing of 265.6°.

For Example **18**, you can use a CAS calculator to add the vectors, but you need to change the bearings measured clockwise from north into angles measured anticlockwise from the x-axis. When you get your answer, you need to change the angle back to a bearing.

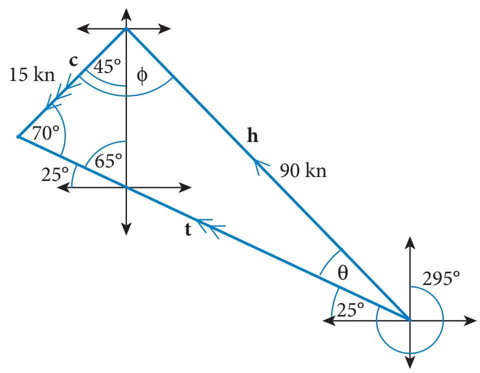
In *air navigation*, the equivalent of speed through the water is **airspeed**, shown with one arrowhead. The course made good is referred to as the **true course**, and the speed made good is called the **ground speed**, shown with two arrowheads. The air current is the **wind** and this is shown with three arrowheads. In air navigation, you normally know the wind direction and speed from meteorological data, and choose your heading in order to fly in a particular direction. In other words, **t** and **c** are known, and you wish to calculate **h**.

Example 20

A pilot calculating a flight plan for a single-engine Cessna wants to fly on a true course of 295°. The wind is a 15-knot north-easterly and she plans to fly at an airspeed of 90 knots. What heading should she make and what will be the ground speed?

Solution

Make a rough sketch.
A north-easterly wind blows from NE, so it blows towards SW.
You could solve this problem using vector components, but it is easier to use the vector triangle.
Use basic geometry to find the remaining angles.



Use the sine rule to find θ.

$$\frac{15}{\sin(\theta)} = \frac{90}{\sin(70°)}$$

Rearrange and evaluate.

$$\theta = 9.010...°$$

Calculate the bearing of **h**.

$$\text{Bearing of } \mathbf{h} = 295° + 9.010...° = 304.010...°$$

Calculate the third angle of the triangle.

$$\text{Third angle} = 180° - 70° - 9.010...° = 100.989...°$$

Use the sine rule to find t.

$$\frac{t}{\sin(100.989...°)} = \frac{90}{\sin(70°)}$$

Rearrange and evaluate.

$$t = 94.019...$$

State the result.

The pilot's heading will be about 304° and the ground speed is about 94 knots.

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EXERCISE 4.08 Application of vectors to navigation

Concepts and techniques

You should use your CAS calculator wherever appropriate in this exercise.

WS Aerial and nautical applications

1 In marine navigation, what is the direction in which a boat is steered called?

A bearing B course C heading
D direction E course made good

2 In air navigation, what is the *speed made good* of an aircraft called?

A ground speed B airspeed C true speed
D wind speed E speed through the air

Reasoning and communication

3 A boat travels 5 km NE and then 7 km at a bearing of 120°. Find its distance and direction from its starting point.

4 A strong swimmer who can swim at 6 km/h for a short period is caught in a rip 20 m from the beach. The rip is moving at 10 km/h away from the beach. Knowing safety procedures, the swimmer tries to swim across the rip, He reaches the edge of the rip after 3 minutes. How far from the shore is he?

5 An aircraft travelling at 140 knots at a bearing of 197° changes its direction to a bearing of 116° at the same speed to approach the runway from the seaward side. Find the change in velocity.

6 Example 19 Find the speed and course made good for a yacht sailing at 7 knots at 135° with a current of 2 knots at 060°.

7 Find the speed and course made good for a boat sailing at 12 knots at 200° with a current of 4 knots at 340°.

8 Example 20 The owner of a cruiser wants to sail on a true course of 120°. The current is very strong, being 5 knots towards the NW. The cruiser can make a speed of 15 knots through the water. At what heading should the owner steer the cruiser, and what will be the true speed?

9 A pilot wants to fly on a true course of 320°. The airspeed is 130 knots and wind is a steady westerly of 15 knots. Find the heading he should take and the speed over the ground of the aircraft.

10 Wind can be much stronger at high altitudes. A passenger airliner with a cruising speed of 560 km/h needs to travel 1500 km south-west at cruising height before beginning the descent for landing. At this height, the wind is an 80 km/h northerly. The plane reaches cruising height at 10:05 a.m. When will the plane commence the descent for landing?

11 A whale is moving up the east coast of Australia in a direction that is essentially at a bearing of 25°. The whale moves at a leisurely pace of about 8 knots through the water and the current up the east coast is about 2 knots northwards. What is the speed that would be observed by a whale spotter at Point Lookout?

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12 A dinghy with an outboard motor has a speed through the water of 8 knots. It is moving directly across the mouth of a creek where the incoming tide has a speed of 5 knots. Find the true speed and direction of the dinghy.

13 Find the ground speed and true course for a light plane heading 210° at an airspeed of 120 knots if the wind is a southerly at 20 knots.

14 A pilot wants to fly 600 km at a bearing of 015° in a plane capable of 95 knots. The wind is predicted to be a south-easterly at 25 knots. If she takes off at 10:30 a.m., what heading should she take, and what is her estimated time of arrival? (1 knot = 1.852 km h^{-1})

CHAPTER SUMMARY
APPLICATIONS OF VECTORS

4

- The **dot product** of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of θ between them is given by:

  $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$
- The scalar product of two vectors $\mathbf{v}_1 = (a_1, a_2)$ and $\mathbf{v}_2 = (b_1, b_2)$, is given by:

  $\mathbf{v}_1 \cdot \mathbf{v}_2 = (a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2$
- The angle between two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ is given by:

  $\cos(\theta) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}$
- The properties of the scalar product include:

  1 the scalar product is a real number, *not a vector*

  2 the scalar product is commutative, namely

  $$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

  3 the scalar product is distributive over vector addition

  $$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

  4 $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b})$, for m real and vectors **a** and **b**

  5 $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$
- If **a** and **b** are perpendicular (or **orthogonal**), then $\mathbf{a} \cdot \mathbf{b} = 0$ and vice-versa for non-zero **a** and **b**.
- If **a** and **b** are parallel, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$ or $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.
- For the unit vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$:

  $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$

  $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$
- The **projection** of **a** on **b** is a vector **p** in the direction of **b** with magnitude given by:

  $p = |\mathbf{a}| \cos(\theta)$ where θ is the angle between **a** and **b**

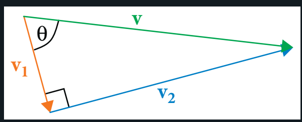
  or $p = \mathbf{a} \cdot \hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is a unit vector in the direction of **b**

  or $p = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$

  The vector projection of **a** on **b** is given by

  $\mathbf{p} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = |\mathbf{a}|\cos(\theta)\hat{\mathbf{b}}$, where θ is the angle between **a** and **b**
- **Orthogonal components** of a vector are projections of the vector in perpendicular directions. Writing a given vector as a sum of orthogonal components is called **resolving** the vector into orthogonal components.

  If the angle between a vector **v** and one of the directions in which you want to resolve it is θ, then the orthogonal components are given by $v_1 = v\cos(\theta)$ and $v_2 = v\sin(\theta)$.



- The work done by a force in moving an object is:

  $W = \mathbf{F} \cdot \mathbf{s}$ where **F** is the force and **s** is the displacement. If the force is in the direction of the displacement then $W = Fs$.

- The velocity of a boat through water is called the **speed through the water** and is called the **heading**. The **speed** and **course made good** (**true course**) is the boat's velocity relative to the land (Earth).
- For air navigation, speed through the air is called the **airspeed** and the speed relative to the Earth is called the **ground speed**.

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CHAPTER REVIEW
APPLICATIONS OF VECTORS

4

Multiple choice

1 Example 5 If $\mathbf{p} = (0, 3)$ and $\mathbf{q} = (0, -3)$, then $\mathbf{p} \cdot \mathbf{q}$ is:

A -9 B -6 C 0 D 6 E 9

2 Example 12 The projection of $\mathbf{b} = -3\mathbf{i} + \mathbf{j}$ on the negative direction of the x-axis is:

A 0 B 1 C 3 D $\sqrt{10}$ E 10

3 Example 11 The projection of $\mathbf{u} = (5, 30°)$ in the direction 210° is:

A -10 B -5 C 0 D 5 E 10

4 Example 11 The projection of $\mathbf{a} = (3, 160°)$ on $\mathbf{b} = (8, 70°)$ is:

A -8 B -3 C 0 D 3 E 8

5 Example 12 The projection of $4\mathbf{i} + 2\mathbf{j}$ on $\frac{1}{2}\mathbf{i} - \mathbf{j}$ is:

A -1 B 0 C $\frac{1}{2}$ D 2 E 4

6 Example 13 A plane is flying toward the south-east. Which of the following wind velocity vectors increases the plane's speed the most?

A $\mathbf{w}_1 = 6\mathbf{i} - 3\mathbf{j}$ B $\mathbf{w}_2 = 6\mathbf{j}$ C $\mathbf{w}_3 = 6\mathbf{i} + \mathbf{j}$

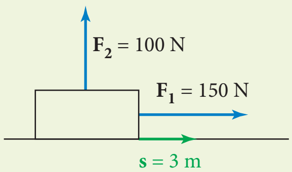
D $\mathbf{w}_4 = -6\mathbf{i} - 6\mathbf{j}$ E $\mathbf{w}_5 = -6\mathbf{i} + 6\mathbf{j}$

7 Example 13 For the plane described in question **6**, which of the following wind velocity vectors slows down the plane the most?

A $\mathbf{w}_1 = 6\mathbf{i} - 3\mathbf{j}$ B $\mathbf{w}_2 = 6\mathbf{j}$ C $\mathbf{w}_3 = 6\mathbf{i} + \mathbf{j}$

D $\mathbf{w}_4 = -6\mathbf{i} - 6\mathbf{j}$ E $\mathbf{w}_5 = -6\mathbf{i} + 6\mathbf{j}$

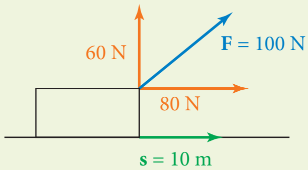
8 Example 16 An object is subjected to two forces, as shown in the diagram on the right. If the object has a displacement of 3 m, then the work done by the forces is:



A 0 B 200 J C 250 J

D 450 J E 750 J

9 Example 17 An object is subjected to a force of 100 N, as shown in the diagram on the right. The components of the force parallel and perpendicular to the displacement (**s**) are 80 N and 60 N respectively. The work done by **F** is:



A 600 J B 800 J C 1000 J

D 1400 J E 2400 J

Short answer

10 Example 1 The angle between vectors **a** and **b** is θ. Find $\mathbf{a} \cdot \mathbf{b}$ (correct to 2 decimal places if necessary) when:

a $|\mathbf{a}| = 3$, $|\mathbf{b}| = 7$ and $\theta = 12°$

b $|\mathbf{a}| = 9$, $|\mathbf{b}| = 5$ and $\theta = 71°$

11 Example 4 Use the geometric definition to calculate the dot product of each of the following pairs of vectors.

a $3\mathbf{i} - 5\mathbf{j}$ and $-4\mathbf{i} + 4\mathbf{j}$

b $-3\mathbf{i} - 4\mathbf{j}$ and $2\mathbf{i} + 5\mathbf{j}$

12 Example 3 The dot product of $\mathbf{d} = (1, 7)$ and $\mathbf{e} = (6, 2)$ is 20. Calculate the angle between **d** and **e**.

13 If $\mathbf{m} = (8, 32°)$ and $\mathbf{n} = (12, 325°)$, calculate the value of $\mathbf{m} \cdot \mathbf{n}$.

14 Example 4 Calculate the dot product of (1, 5) and (3, 7) using both the algebraic and geometric definitions of the dot product.

15 Example 4 Calculate the scalar product of each of the following pairs of vectors.

a $-3\mathbf{i} + 10\mathbf{j}$ and $5\mathbf{i} - 4\mathbf{j}$

b $(-2, -3)$ and $(7, 9)$

c $\begin{bmatrix} 21 \\ 12 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ 11 \end{bmatrix}$

16 Example 5 Calculate the scalar product of the following pair of vectors, correct to two decimal places if necessary.

a $(5, 2)$ and $(7, 80°)$

b $\left(12, \frac{\pi}{3}\right)$ and $(4, -7)$

17 Example 6 Calculate the angle between the following pairs of vectors.

a $(1, 2)$ and $(6, 8)$

b $(3, -4)$ and $(-2, 3)$

18 Example 7 Use the pair of vectors $(3, -7)$ and $(4, 2)$ to demonstrate that the scalar product is commutative (i.e. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$).

19 Example 9 Show that the following pairs of vectors are perpendicular.

a $(3, 0)$ and $(0, -6)$

b $(5, 2)$ and $(2, -5)$

20 Example 10 Show that the following pairs of vectors are parallel.

a $(9, 12)$ and $\left(1, \frac{4}{3}\right)$

b $(-5, -2)$ and $\left(1, \frac{2}{5}\right)$

21 Example 10 Determine if the following pairs of vectors are perpendicular, parallel or neither.

a $(3, 2)$ and $(-2, 3)$

b $(5, -1)$ and $(2, 3)$

c $\left(-4, \frac{3}{2}\right)$ and $(8, -3)$

22 Example 11 Find the projection in the direction 210° of a vector of magnitude 50 and direction 280°.

23 Example 12 Find the projection of $\mathbf{p} = (2, 7)$ on $\mathbf{q} = (5, 12)$.

24 Example 15 Given vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and force vector $\mathbf{F} = -2\mathbf{i} + 4\mathbf{j}$, calculate:

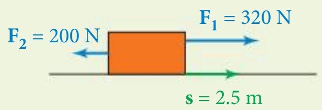
a the component of **F** parallel to **v**

b the component of **F** perpendicular to **v**

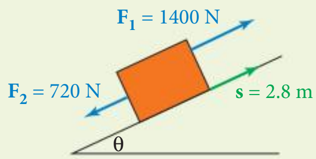
25 Example 13 A force of 200 N acts in the direction N 25° E. Resolve the force into components in the north and east directions.

26 Example 16 Calculate the work done in each of the following situations.

a



b



Application

27 Prove that $|\mathbf{m}| = \sqrt{\mathbf{m} \cdot \mathbf{m}}$, where $\mathbf{m} = (m_1, m_2)$.

28 Consider the points $A(-6, 7)$, $B(-2, 3)$ and $C(3, 4)$. Calculate the angle between **BA** and **BC**.

29 Two ice-skaters are helping a beginner to get started. The ice-skater on the left is exerting a force of 15 N at an angle of 30° from the forward direction and to the left. The skater on the right is exerting a force of 20 N at an angle of 40° from the forward direction and to the right. What is the total force acting on the beginner?

30 A car going up a hill inclined at 10° has a weight of 8500 N acting vertically downwards. Resolve the weight into components parallel and perpendicular to the slope.

31 Four people are attempting to pull a tree stump out of the ground. The first person is pulling at an angle of 30° to the horizontal with a force of 200 N. The second person is 60° clockwise around the stump from the first and is pulling at an angle of 56° to the horizontal with a force of 155 N. The third person is 75° anticlockwise from the first and is pulling with a force of 300 N at an angle of 48° to the horizontal. The fourth person is directly opposite the first and is pulling with a force of 300 N at an angle of 45° to the horizontal. Find the total force acting on the stump, and the lifting force.

32 Use the scalar product to find the amount of work that is done when a force of 200 N moves its point of application 20 m at an angle of 36° to the force.

Practice quiz

33 Find the heading needed and speed made good for a true course of 135° in a boat capable of 8 knots in a current of 2 knots in the direction 045°.

34 Find the heading required to fly a plane with an airspeed of 105 knots on a true course of 225° with a southwesterly wind of 10 knots. Find the ground speed.

35 Use the scalar product to prove the diagonals of a kite are perpendicular.

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5

TERMINOLOGY

addition principle
combination
factorial
inclusion-exclusion principle
multiplication principle
Pascal's triangle
permutation

COMBINATORICS

COUNTING METHODS AND COMBINATIONS

Prior learning

THE INCLUSION-EXCLUSION PRINCIPLE FOR THE UNION OF TWO SETS AND THREE SETS

- determine and use the formulas for finding the number of elements in the union of two and the union of three sets. (ACMSM005)

COMBINATIONS (UNORDERED SELECTIONS)

- solve problems involving combinations (ACMSM007)
- use the notation $\binom{n}{r}$ or ${}^nC_r$ (ACMSM008)
- derive and use simple identities associated with Pascal's triangle. (ACMSM009)

5.01 SIMPLE COMBINATIONS

You studied permutations in Chapter 3. You can think of a permutation as an *ordered* list. Now imagine a child's box of different coloured pencils. To colour the sky in a picture, it doesn't matter where the blue pencil is in the box. All that matters is that there *is* a blue one in the set of pencils. So the order of the colours in a list of the pencils is irrelevant. A list like this, where the order is unimportant, is called a **combination**. You may have already done some work with combinations in Maths Methods.

IMPORTANT

A **combination** is unordered.

A combination of a set of symbols is a selection of none, some or all of the symbols.

The number of combinations of r objects from n distinct objects is given by

$$
\begin{aligned}
{}^nC_r = \binom{n}{r} &= \frac{n!}{r!(n-r)!} \\
&= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}
\end{aligned}
$$

Proof

If you have a combination of r objects, then there are $r!$ permutations of them.

Thus ${}^nP_r = {}^nC_r \times r!$ so ${}^nC_r = \frac{{}^nP_r}{r!}$, and substituting ${}^nP_r = \frac{n!}{(n-r)!}$, you get the formula. **QED**

There are usually far fewer combinations than permutations. For example, if you picked the top 3 students from 5 to enter a competition, there would be 10 possible combinations of 3 students. However, if the order mattered, there would be 60 possible permutations.

Example 1

Calculate the value of ${}^{8}C_3$.

Solution

Write the formula. $\quad {}^{n}C_r = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$

Find the last term of the numerator. $\quad n - r + 1 = 8 - 3 + 1 = 6$

Substitute values. $\quad {}^{8}C_3 = \dfrac{8\times7\times6}{1\times2\times3}$

Cancel where possible and evaluate. $\quad = \dfrac{8\times7\times\cancel{6}^{\cancel{3}\,1}}{1\times{}_1\cancel{2}\times{}_1\cancel{3}} = 56$

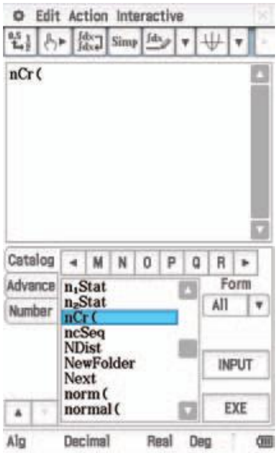
Write the answer. $\quad {}^{8}C_3 = 56$

TI-Nspire CAS

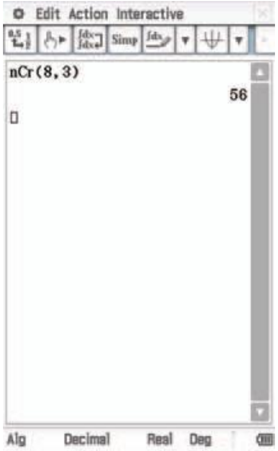
Use a Calculator page.
Type nCr() or use menu, 5: Probability and 3: Combinations. Insert n = 8 and r = 3 separated by a comma and press enter.

ClassPad

Use the Main $\sqrt{\alpha}$ application.
Press Keyboard and tap ▼.
Tap ▶ as necessary to scroll along and tap **N**.
Tap **nCr(** .



Enter the values of n, then r, separated by a comma.
Close the brackets and tap EXE or press EXE.



Example 2

How many different committees of four people can be chosen from a group of 15?

Solution

Write the formula. $${}^{n}C_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Substitute the values in. $${}^{15}C_4 = \frac{15\times 14\times 13\times 12}{1\times 2\times 3\times 4} = 1365$$

Write the answer. There are 1365 different committees.

Example 3

WS
Combination calculations

Find the value of n if ${}^{n+1}C_6 = {}^{n}C_5$.

Solution

Write the formula. $$\frac{(n+1)!}{6!(n+1-6)!} = \frac{n!}{5!(n-5)!}$$

Expand the formula. $$\frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1\times 2\times 3\times 4\times 5\times 6} = \frac{n(n-1)(n-2)(n-3)(n-4)}{1\times 2\times 3\times 4\times 5}$$

Cancel where possible. $$\frac{(n+1)}{6} = \frac{1}{1}$$

Solve for n. $n = 5$

EXERCISE 5.01 Simple combinations

Concepts and techniques

Combinations

1 Example 1 Calculate the values in **a**, **b** and **c** without using your calculator. Simplify the expressions in **d** and **e**.

a $^{6}C_{2}$ b $^{8}C_{8}$ c $^{25}C_{1}$ d $^{n}C_{3}$ e $^{n+1}C_{2}$

2 CAS Work out the values of the following.

a $^{8}C_{3}$ b $^{10}C_{6}$ c $^{12}C_{4}$ d $^{13}C_{5}$ e $^{13}C_{8}$

3 Example 2 Find the number of different ways that a committee of 6 people can be made randomly from a group of:

a 8 people b 9 people c 11 people
d 15 people e 20 people.

4 Example 3 Find n if:

a $^{n}C_{2} = 6$ b $^{n+1}C_{3} = {^{n}C_{2}}$ c $^{n}C_{2} = 45$
d $^{n+1}C_{7} = {^{n}C_{6}}$ e $^{n}C_{2} = 66$ f $^{n}C_{5} = {^{n}C_{11}}$

Reasoning and communication

5 How many different selections of five cards can be made from a deck of 52 different cards?

6 Ten people are introduced and shake hands with each other.

a How many handshakes are there?
b How many handshakes are there for n people?

7 How many triangles can be drawn using five given points on the circumference of a circle?

8 A short test has ten questions, all of which must be answered Yes or No. In how many sets of answers can a person get half the questions wrong?

9 A coin is tossed 20 times. How many different arrangements are there for tossing 5 heads?

10 A set of 10 different coloured marbles are placed in a bag and 6 are selected at random. In how many different ways can this happen?

11 A bag contains 12 different lollies with blue wrappers and 15 different lollies with red wrappers. If I take a handful of 6 lollies out of the bag, how many different handfuls are possible?

12 A car has a radio with five push-buttons that can be tuned to different stations. In a city that has 12 radio stations, how many different selections of those stations can be chosen for the push-buttons if they are placed in order of the wavelength?

13 A simple Lotto game uses the numbers 1–40, with six numbers being chosen. How many different ways can the numbers be chosen?

14 Show that $\binom{10}{4} = \binom{9}{4} + \binom{9}{3}$

15 Show that

a $\binom{13}{7} = \binom{13}{6}$ b $\binom{n}{r} = \binom{n}{n-r}$

5.02 USING COMBINATIONS

Sometimes you have to reduce the number of combinations to allow for restrictions.

Example 4

A committee of five people is to be chosen at random from a class of 20.

a How many different committees could be chosen?

b How many different committees that include Johanna and Esrif could be chosen?

Solution

a Use the formula.

Number of committees $= {}^{20}C_5$
$= 15\,540$

b Committees with both will have 3 people from the other 18.

Number including Esrif and Johanna $= {}^{18}C_3$
$= 816$

You will often need to use the multiplication principle with combinations to solve a problem. You should remember this from your earlier work in Chapter 3.

IMPORTANT

The **multiplication principle** says that for a choice made in two stages with a ways for one part and b ways for the second part, there are $a \times b$ choices altogether.

For a choice made in n stages: if there are a_1 ways for the first part, a_2 ways for the second part, a_3 ways for the third part, and so on, then there are $a_1 a_2 a_3 \ldots a_n$ choices altogether.

Example 5

A team of 6 men and 5 women is chosen at random from 10 men and 9 women. How many different possible teams include Kaye, Stella and Peter?

Solution

The male combinations including Peter have 5 other men.

Male combinations with Peter $= {}^{9}C_5$

Female combinations including Kaye and Stella have 3 others.

Female combinations with Kaye and Stella $= {}^{7}C_3$

Use the multiplication rule.

Number of teams with all three $= {}^{9}C_5 \times {}^{7}C_3$

Use your calculator.

$= 126 \times 35$
$= 4410$

Write the answer.

There are 4410 possible teams that include Kaye, Stella and Peter.

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You may need to use the addition principle with combinations to solve a problem.

IMPORTANT

The **addition principle** says that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.

Remember that mutually exclusive sets have no intersection; they have no elements in common with one another.

Example 6

The executive committee, comprising the president, secretary, treasurer and three other members of a club are elected at the annual general meeting, which is attended by a total of 22 people, including three members of the former executive committee. How many possible committees could be elected from those attending that include at least one of the former executive?

Solution

State what has to be found.	The number of combinations that include 1, 2 or 3 of the former executive committee members is needed.
How many others have to be elected with one of the former executive committee members?	To include exactly one particular member of the former executive committee members, 5 out of the other 19 people need to be elected. Number including a particular one $= {}^{19}C_5 = 11\,628$
There are 3 possible former members.	Number of combinations including one only $= 3 \times 11\,628 = 34\,884$
How many combinations are there with two of the former executive committee members?	To include exactly 2 particular members of the former executive, 4 more are to be elected. Number with two $= {}^{19}C_4 = 3876$
There are 3 ways to choose 2 of the former 3.	Number of combinations including two $= 3 \times 3876 = 11\,628$
There is only one way to include all three.	Number of combinations including three $= {}^{19}C_3 = 969$
Use the addition principle.	Number of possible committees including at least one of the former executive committee $= 34\,884 + 11\,628 + 969$ $= 47\,481$

INVESTIGATION Poker hands

In the card game of poker, players try to get special combinations of 5 cards. The highest possible combination is Ace-King-Queen-Jack-10 of the same suit. This is called a royal flush.

How many possible poker hands are there? How many are royal flushes? What is the probability of getting a royal flush?

The other combinations are as follows.

- Straight flush: 5 cards in a row of the same suit such as 3-4-5-6-7 of spades
- Four of a kind: 4 cards all the same such as 10-10-10-10-Q
- Full house: 3 cards the same and 2 others the same such as 7-7-7-A-A
- Flush: 5 cards of the same suit such as 3, 6, 8, J, Q of clubs
- Straight: 5 cards in a row, such as 7-8-9-10-J
- Three of a kind: 3 cards the same, such as J-J-J-A-3
- Two pair: Two lots of 2 cards the same such as Q-Q-6-6-9
- A pair: 2 cards the same, such as K-K-Q-7-6

Work in groups to determine the number and probabilities of each combination. Compare the probabilities to the order of the hands from highest to lowest. What do you find?

EXERCISE 5.02 Using combinations

Reasoning and communication

1 Example 4 An amateur theatrical company is forming a new committee for the following year. Since nobody really wants to do the committee work, it is decided to select the five-member committee at random from the 20 full members of the company. It is generally agreed that no committee should include both Jodie and Michael, as they do not work well together.
  a How many possible committees would there be, regardless of who is on the committee?
  b How many possible committees would include both of them?

2 Example 5 A company board has ten directors. Six of the directors are in favour of a merger proposal while four are opposed to it. A subcommittee of three directors is to be chosen to investigate the merger proposal. How many subcommittees can be selected so that exactly two directors are in favour of the proposal?

3 Example 6 A family has 19 movies on DVD, including all three of the 'Back to the Future' movies. How many selections of 5 movies would include at least one of the 'Back to the Future' DVDs?

4 The school team has 10 players available from the netball squad for the next game. A netball team has 7 players.
  a How many possible teams are there?
  b How many of the possible teams would include Sonja or Renee or both?

5 Of the players in the squad for the premier school cricket team, 5 are primarily fast bowlers, 2 are primarily spin bowlers, 7 are primarily batsmen and there is one all-rounder. How many different teams of 11 could be chosen that included the all-rounder and at least one spin bowler?

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6 A pet shop owner wishes to display five animals in the shop window. There are six cats and three dogs available for display. How many combinations of five display animals can be chosen so as to include:

a one dog?

b at least one dog?

Alamy/John Norman

7 A committee of 6 people is to be selected randomly from a group of 11 men and 12 women. Find the number of possible committees if:

a there is no restriction on who is on the committee

b all committee members are to be male

c all members are to be female

d there are to be 3 men and 3 women

e a particular woman is included

f a particular man is not included

g there are to be 4 women and 2 men.

8 A horserace has 15 competing horses. At the TAB, a quinella pays out on the horses that come in first and second, in either order. Ryan decides to bet on all possible combinations of quinellas. If it costs him $1 a bet, how much does he pay?

9 A group of 25 students consists of 11 who play a musical instrument and 14 who don't. Find the number of different arrangements possible if a group of 9 students is selected at random:

a with no restriction

b who all play musical instruments

c where 5 play musical instruments

d where 2 don't play musical instruments.

10 A set of cards consists of 8 yellow and 7 red cards.

a If 10 cards are selected at random, find the number of different combinations possible.

b If 8 cards are selected, find the number of combinations of:

i 4 yellow cards ii 6 yellow cards iii 7 yellow cards iv 5 red cards.

11 Ten cards are randomly selected from a set of 52 playing cards. Find the number of combinations selected if:

a there are no restrictions (answer in scientific notation, correct to 3 significant figures)

b they are all hearts

c there are 7 hearts

d they are all red cards

e there are 4 aces.

12 An animal refuge has 17 dogs and 21 cats. If a nursing home orders 12 animals at random, find the number of ways that the order would have

a 7 dogs b 9 dogs c 10 dogs d 4 cats e 6 cats.

13 There are 8 white, 9 red and 5 blue marbles in a bag and 7 are drawn out at random. Find the number of selections possible:

a with no restriction

b if all marbles are red

c if there are 3 white and 2 red marbles

d if there are 4 red and 1 blue marble

e if there are 4 white and 2 blue marbles.

14 Out of a group of 25 students, 7 walk to school, 12 catch a train and 6 catch a bus. If 6 students are selected, find the number of combinations if:

a all walk to school

b no one catches a bus

c 3 walk to school and 1 catches a bus

d 1 walks to school and 4 catch a train

e 3 catch a train and 1 catches a bus.

15 At a karaoke night, a group of 14 friends decide that 4 of them will sing a song together. Of the friends, 5 have previously sung this song before. In how many ways can they do this if they select:

a friends who have all sung the song previously

b 2 of the friends who sang the song previously

c none of the friends who sang the song previously?

16 A cricket team of 11 players is to be chosen from a squad of 15 players. How many different teams can be chosen that include the following?

a Sam and David

b Sam, David and Paresh

c Sam, David, Paresh and Akmal

d Sam or David

5.03 PASCAL'S TRIANGLE

Pascal's triangle is named after Blaise Pascal (1623–1662), although it is also called Yang Hui's triangle in China, Tartaglia's triangle in Italy and Khayyam's triangle in Iran after earlier mathematicians who studied its properties.

The first 7 rows are shown on the right. The rows are numbered from 0 down.

The numbers are staggered in each successive row. You can work them out by adding the numbers diagonally above each position, as shown by the arrows. If there is no number above, it is taken as 0. Work out the next few rows of the triangle for yourself.

1	Row 0
1 1	Row 1
1 2 1	Row 2
1 3 3 1	Row 3
1 4 6 4 1	Row 4
1 5 10 10 5 1	Row 5
1 6 15 20 15 6 1	Row 6
	Row 7?
	Row 8?
	Row 9?

The numbers in a row are counted across from 0, so element number 3 in row 6 is 20. However the third element in row 6 is 15. This can be quite confusing, so you always need to be clear whether you are referring to a row or element by its number or its position.

The symbol Pa is sometimes used to refer to numbers in Pascal's triangle. Using this notation, $Pa_{4,\,2} = 6$ is the centre element in the row that begins 1, 4, …

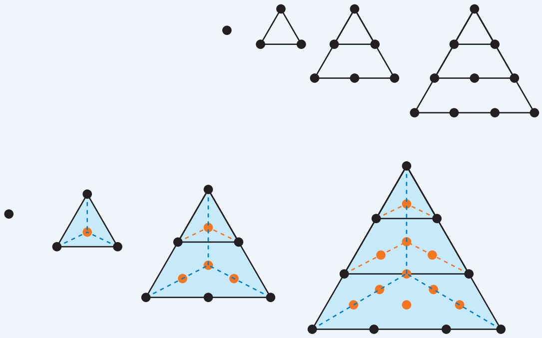
INVESTIGATION Patterns in Pascal's triangle

Look at the diagonals of Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Notice that the first two diagonals are just one and the counting numbers.

Compare the numbers in the diagonals with the number of dots in each of the following sequences of shapes.



Is there a shape for the next diagonal? Can you draw it?

Now look at the digits in the rows as numbers: 1, 11, 121, 1331, …

How is 121 related to 11?

How is 1331 related to 121?

Does the pattern continue?

Can you make it so for rows with double digit numbers? How?

The numbers in Pascal's triangle are actually the same as the numbers for combinations.

IMPORTANT

The numbers in Pascal's triangle are the combinatorial numbers given by ^nC_r.

For example, counting from 0, the third number in row 6 is 20.

It is also the case that $^6C_3 = 20$.

Apart from the ends, any number in any row of Pascal's triangle is the sum of the two diagonally above. Looking at rows 4 and 5 of Pascal's triangle, the first 10 in row 5 is the sum of the 4 and 6 above it.

$$\begin{array}{ccccccccccc} & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

It is true that ${}^4C_1 = 4$, ${}^4C_2 = 6$, ${}^5C_2 = 10$, and ${}^4C_1 + {}^4C_2 = {}^5C_2$ $(4 + 6 = 10)$, but is this always going to work for the combinations? For that to be true, it would have to be true that ${}^{n-1}C_{m-1} + {}^{n-1}C_m = {}^nC_m$, no matter what the values of n and m were (with $m < n$).

In the case above, $n = 5$ and $m = 2$.

What about $n = 8$ and $m = 6$? Does ${}^7C_5 + {}^7C_6 = {}^8C_6$, corresponding to the third last number on row of Pascal's triangle?

${}^7C_5 = 21$, ${}^7C_6 = 7$ and ${}^8C_6 = 28$, so it does work in this case as well.

The proof that the numbers in Pascal's triangle are the combinatorial numbers, ${}^nC_r$ is shown below.

Proof

${}^0C_0 = \dfrac{0!}{0! \times 0!} = 1$, the very top row of Pascal's triangle.

Consider the other rows. For each row:

${}^nC_0 = 1$, the number on the left of row n.

${}^nC_n = 1$, the number of the right of row n of Pascal's triangle.

Thus the numbers at the ends of each row are correct.

Now consider a number somewhere in the middle.

$$\begin{aligned}
{}^{n-1}C_{m-1} + {}^{n-1}C_m &= \frac{(n-1)!}{(m-1)![(n-1)-(m-1)]!} + \frac{(n-1)!}{m![(n-1)-m]!} \\
&= \frac{(n-1)!}{(m-1)![n-1-m+1]!} + \frac{(n-1)!}{m![n-1-m]!} \\
&= \frac{(n-1)!}{(m-1)!(n-m)!} + \frac{(n-1)!}{m![n-m-1]!} \\
&= \frac{(n-1)!}{(m-1)!(n-m)(n-m-1)!} + \frac{(n-1)!}{m(m-1)![n-m-1]!} \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!}\left[\frac{1}{(n-m)} + \frac{1}{m}\right] \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!}\left[\frac{m+n-m}{(n-m) \times m}\right] \\
&= \frac{(n-1)!}{(m-1)!(n-m-1)!} \times \frac{n}{m \times (n-m)} \\
&= \frac{n!}{m!(n-m)!} = {}^nC_m
\end{aligned}$$

Therefore ${}^{n-1}C_{m-1} + {}^{n-1}C_m = {}^nC_m$

Since it works for the first rows, this shows that combinations follow the same rule as for the numbers in Pascal's triangle, so they must be the same numbers. **QED**

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Example 7

Prove that the second number in row n of Pascal's triangle is always n for $n > 0$.

Solution

Write the equivalent combinatorial formula, remembering that the second number is $\text{Pa}_{n,\,1}$.	**Proof** $\text{Pa}_{n,\,1} = {}^nC_1$
Substitute in the formula.	$= \dfrac{n!}{1!(n-1)!}$
Write $n! = n(n-1)!$	$= \dfrac{n(n-1)!}{1\times(n-1)!}$
Simplify.	$= n$ **QED**

In Maths Methods, you may have seen the expansion

$$\begin{aligned}(x+y)^5 &= {}^5C_5x^5 + {}^5C_4x^4y + {}^5C_3x^3y^2 + {}^5C_2x^2y^3 + {}^5C_1xy^4 + {}^5C_0y^5 \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\end{aligned}$$

In Maths Methods, you were *given* the fact that the coefficients of $(x+y)^n$ are the values ${}^nC_r$.

However, this was not properly proved.

IMPORTANT

The binomial coefficients, the coefficients in the expansion of $(x+y)^n$, are given by ${}^nC_r$.

Proof

Consider the x^2y^3 products in the expansion of

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

Each x^2y^3 product is obtained by multiplying the x terms from 2 of the 5 brackets and y terms from the rest of the brackets, so you are choosing 2 xs from 5 possible xs and there are ${}^5C_2$ ways to do this.

Now consider the general case.

Consider x^ry^{n-r} products in the expansion of

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y)$$

Each product is obtained by multiplying x terms from r of the brackets and y terms from the rest of the brackets.

The number of products is precisely the number of ways of choosing r of the brackets to get the x terms. This is ${}^nC_r$.

Hence $(x+y)^n = {}^nC_0x^n + {}^nC_1xy^{n-1} + {}^nC_2x^2y^{n-2} + \cdots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$ **QED**

A similar method can be applied to other expansions.

Example 8

Use combinations to expand $(x+y+z)^4$.

Solution

Write out the problem.	$(x+y+z)^4$
Write out the power.	$=(x+y+z)(x+y+z)(x+y+z)(x+y+z)$
Write the possible terms.	$=a_1x^4+a_2x^3y+a_3x^3z+a_4x^2y^2+a_5x^2yz+a_6x^2z^2$ $+a_7xy^3+a_8xy^2z+a_9xyz^2+a_{10}xz^3+a_{11}y^4$ $+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+a_{15}z^4$ where $a_1, a_2, \ldots$ are constants.
Write the constants.	$a_1={}^4C_4$, the number of combinations of 4 xs (leaving 0 ys and 0 zs) from 4 brackets. Similarly $a_{11}=a_{15}={}^4C_4={}^4C_0$, the number of combinations of 4 ys or 4 zs from 4 brackets. $a_2={}^4C_3$, the number of combinations of 3 xs (leaving 1 y) from 4 brackets. Similarly $a_3=a_7=a_{10}=a_{12}=a_{14}={}^4C_1={}^4C_3$ the number of combinations of 3 of one symbol (leaving 1 of another) from the 4 brackets. $a_4=a_6=a_{13}={}^4C_2$, the number of combinations of 2 of one symbol (leaving 2 of another) from 4 brackets. Finally, $a_5=a_8=a_9={}^4C_2\times{}^2C_1$, the number of combinations of 2 of one symbol from 4 brackets and 1 of another symbol from the remaining 2 brackets (leaving 1 of the third symbol).
Calculate the values.	$a_1=a_{11}=a_{15}={}^4C_0={}^4C_4=1$ $a_2=a_3=a_7=a_{10}=a_{12}=a_{14}={}^4C_1={}^4C_3=4$ $a_4=a_6=a_{13}={}^4C_2=6$ $a_5=a_8=a_9={}^4C_2\times{}^2C_1=6\times2=12$
Write the answer.	$(x+y+z)^4=x^4+4x^3y+4x^3z+6x^2y^2+12x^2yz+6x^2z^2$ $+4xy^3+12xy^2z+12xyz^2+4xz^3$ $+y^4+4y^3z+6y^2z^2+4yz^3+z^4$

The combination numbers, which are the numbers in Pascal's triangle, have many other interesting properties.

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EXERCISE 5.03 Pascal's triangle

Reasoning and communication

1 Example 7 Show that the third number of any row of Pascal's triangle is always $\frac{1}{2}n(n-1)$ for $n > 1$.

2 The Fibonacci numbers are obtained by starting with the numbers 1 and 1 and adding every two numbers to get the next one. Thus they are 1, 1, 2, 3, 5, … as $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on. Show that the 'shallow diagonals' of Pascal's triangle sum to the Fibonacci numbers.

$$\begin{array}{c}
1,\quad 1,\quad 2,\quad 3,\quad 5,\quad 8,\quad 13 \\
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}$$

3 Show that the numbers in any row of Pascal's triangle are symmetrical.

4 Show that for any row the rth number is $\frac{n-r}{r+1}$ times the $(r-1)$th number, that is, ${}^nC_{r+1} = \frac{n-r}{r+1} \times {}^nC_r$, assuming that ${}^nC_0 = 1$ and $n > 0$.

5 Show that the sum of row n of Pascal's triangle is 2^n. (Hint: use the binomial expansion).

6 Example 8 Use combinations to expand $(x + y + z)^3$.

7 When the numbers in the diagonal part (1, 4, 10, 20) are added, they give 35, which is on the next row in the opposite diagonal direction. Show that this is always true for a hockey stick pattern like this.

$$\begin{array}{c}
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}$$

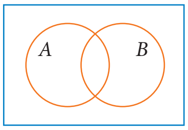
8 Show that the products of the numbers in each triangle in the 'Star of David' patterns as shown below are the same.

$$\begin{array}{c}
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1
\end{array}$$

In the examples, $6 \times 5 \times 20 = 4 \times 15 \times 10$ and $21 \times 70 \times 84 = 35 \times 126 \times 28$.

5.04 THE INCLUSION-EXCLUSION PRINCIPLE

How can you count arrangements that can be divided into different parts that are *not* mutually exclusive? You need to allow for arrangements that are counted twice.

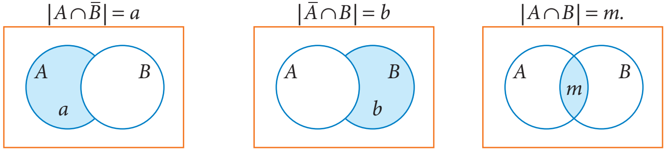


Consider the number of elements in $A \cup B$, that is $|A \cup B|$ or $n(A \cup B)$.

The Venn diagram shows there could be elements in both A and B.

Suppose that there are a elements in A that are not in B, b elements in B that are not in A, and m elements in $A \cap B$. This is shown below.

$|A \cap \bar{B}| = a$ $\quad$ $|\bar{A} \cap B| = b$ $\quad$ $|A \cap B| = m.$



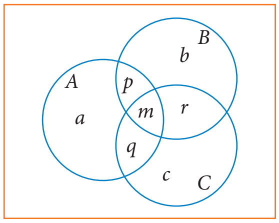
It is clear that $|A \cup B| = a + b + m = (a + m) + (b + m) - m = |A| + |B| - |A \cap B|$.

This is the simplest case of the **inclusion-exclusion principle**.

The number of elements in the union of two sets *includes* the number in each set, and *excludes* the number counted twice because they are in both sets (the intersection of the two).

For example, if there are 12 blue-eyed people and 10 people with fair hair in a class, and 5 of those with blue eyes have fair hair, then there must be $12 + 10 - 5 = 17$ people with either blue eyes, fair hair or both.

What about three sets?



Suppose there are a elements in A that are not in B or C ,

b elements in B that are not in A or C,

c elements in C that are not in A or B,

p elements in $A \cap B$ that are not in C,

q elements in $A \cap C$ that are not in B,

r elements in $B \cap C$ that are not in A,

and m elements in $A \cap B \cap C$.

Clearly

$|A \cup B \cup C|$

$$= a + b + c + p + q + r + m$$

$$= (a + p + m + q) + (b + p + m + r) + (c + q + m + r) - (p + m) - (q + m) - (r + m) + m$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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This is the inclusion-exclusion principle for three sets.

The number of elements in the union of three sets *includes* the number in each set, and twice *excludes* the number counted three times because they are in the intersections of pairs of sets and *includes* the number added and subtracted three times because they are in all three sets.

If 10 students take geography, 9 study history, 7 do art, 6 take geography and history, 5 study geography and art, 4 do history and art and 2 take all three, then there must be $10 + 9 + 7 - 6 - 5 - 4 + 2 = 13$ that study geography, art, history or a combination of some of these subjects. You should satisfy yourself that this example of the principle is true using a Venn diagram.

IMPORTANT

Inclusion-exclusion principle

The number of elements in a union of sets is:

- for 2 sets: $|A \cup B| = |A| + |B| - |A \cap B|$
- for 3 sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

The pattern of inclusions and exclusions is clear and extends to any number of sets.

While the pattern for the inclusion-exclusion principle can be extended to any number of sets, you are only required to use it for unions involving 2 or 3 sets.

Example 9

How many counting numbers up to 100 are divisible by 5 or 6?

Solution

Count the numbers divisible by 5.	No. divisible by $5 = 100 \div 5 = 20$
Count the numbers divisible by 6.	No. divisible by $6 = 100 \div 3 = 16\frac{2}{3}$, so 16
Count those divisible by 5 and 6.	No. divisible by $30 = 100 \div 30 = 3\frac{1}{3}$, so 3
Use inclusion-exclusion.	No. divisible by 5 or $6 = 20 + 16 - 3$
Calculate the result.	$= 33$
Write the answer.	33 counting numbers up to 100 are divisible by 5 or 6.

In Example **9**, if the question had asked for 'numbers under 100 divisible by 5 or 6', then 100 itself, which is divisible by 5, would be excluded so the number would be 32.

Example 10

How many positive whole numbers under 1000 are divisible by 2, 3 or 5?

Solution

Count the numbers divisible by 2.	No. divisible by 2 $= 999 \div 2$ $= 499.5$, so 499
Count the numbers divisible by 3.	No. divisible by 3 $= 999 \div 3 = 333$
Count the numbers divisible by 5.	No. divisible by 5 $= 999 \div 5$ $= 199.8$, so 199
Count those divisible by 2 and 3.	No. divisible by 6 $= 999 \div 6$ $= 166\frac{1}{2}$, so 166
Count those divisible by 2 and 5.	No. divisible by 10 $= 999 \div 10 = 99.9$, so 99
Count those divisible by 3 and 5.	No. divisible by 15 $= 999 \div 15$ $= 66.6$, so 66
Count those divisible by 2, 3 and 5.	No. divisible by 30 $= 999 \div 30$ $= 33.3$, so 33
Use inclusion-exclusion.	No. divisible by 2, 3 or 5 is $499 + 333 + 199 - 166 - 99 - 66 + 33$
Calculate the result.	$= 733$

EXERCISE 5.04 The inclusion-exclusion principle

WS The inclusion-exclusion principle

Concepts and techniques

1 Example 9 How many counting numbers up to and including 100 are multiples of 2 or 3?

2 How many natural numbers under 2000 are multiples of 5 or 7?

3 How many natural numbers up to and including 1500 are multiples of 3 or 5?

4 Example 10 How many whole positive numbers under 1000 are multiples of 2, 3 or 7?

5 How many three-digit counting numbers (numbers from 100 to 999) are multiples of 2 or 3?

6 How many three-digit numbers are multiples of 2, 3 or 4?

7 How many three-digit numbers are multiples of 2, 3 or 7?

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8 How many four-digit numbers are multiples of 2, 3 or 5?

9 How many numbers from 300 to 700 inclusive are multiples of 3 or 5?

10 How many numbers from 200 to 800 inclusive are multiples of 2, 3 or 5?

11 How many numbers from 600 to 2400 inclusive are multiples of 3, 5 or 7?

Reasoning and communication

12 17 of the houses in a short street have both fluorescent tubes and fluorescent bulbs. 10 have fluorescent tubes, fluorescent bulbs and some incandescent bulbs. 21 have fluorescent tubes and incandescent bulbs and 19 have incandescent and fluorescent bulbs. 30 of the houses have incandescent bulbs, 30 have fluorescent tubes and 30 have fluorescent bulbs. All the houses have fluorescent or incandescent lights. How many houses are in the street?

5.05 SIMPLE APPLICATIONS TO PROBABILITY

The inclusion-exclusion principle may be used in combination with other counting methods.

Example 11

The first buses going to the school swimming carnival are carrying competitors. The first bus has 45 students entered in the 50 m events and 35 entered in other events. 28 students are entered in both the 50 m and other events. The fourth (last) teacher got on the bus, counted the number on board, and said 'That's it, no more legally allowed on this bus!"

a What is the legal carrying capacity of the bus?

b What is the probability that a randomly selected student is entered in a 50 m event?

Solution

a Use the inclusion-exclusion principle.

$$\begin{aligned}\text{Number of students} &= 45 + 35 - 28 \\ &= 52\end{aligned}$$

Add the teachers.

$$\begin{aligned}\text{Legal carrying capacity} &= 52 + 4 \\ &= 56\end{aligned}$$

b Use the rule for probability.

$$\begin{aligned}P(50\text{ m event}) &= \frac{45}{52} \\ &\approx 0.8654\end{aligned}$$

Example 12

Out of 50 people, 39 like vanilla ice-cream, 33 like strawberry ice-cream and 32 like chocolate ice-cream. 27 like vanilla and strawberry, 23 like vanilla and chocolate and 22 like chocolate and strawberry. What is the probability that someone chosen at random will like all three flavours, given they all like at least one flavour?

Solution

Choose letters for the sets of people who like each flavour.	Let V = people who like vanilla S = people who like strawberry and C = people who like chocolate
Write the inclusion-exclusion principle.	$\lvert V\cup S\cup C\rvert = \lvert V\rvert + \lvert S\rvert + \lvert C\rvert - \lvert V\cap S\rvert - \lvert V\cap C\rvert - \lvert S\cap C\rvert + \lvert V\cap S\cap C\rvert$
Substitute in the values.	$50 = 39 + 33 + 32 - 27 - 23 - 22 + \lvert V\cap S\cap C\rvert$
Simplify.	$50 = 32 + \lvert V\cap S\cap C\rvert$
Solve.	$\lvert V\cap S\cap C\rvert = 18$
Find the probability.	$P(\text{all flavours}) = \frac{18}{50} = 0.36$
Write the answer.	The probability that the person will like all three flavours is 0.36.

EXERCISE 5.05 Simple applications to probability

Reasoning and communication

1 Example 11 A 'project' home builder offers the following interior floor coverings in the living areas of its standard designs: carpets from a range of 8 different patterns, or tiles from a range of 6. In homes built in the last 6 months, 28 clients chose carpet, 32 chose to use tiles and 16 chose carpet in some rooms and tiles in others.
   a How many houses were built?
   b What is the probability of randomly choosing one of the houses with both carpet and tiles?

2 Example 12 A buffet supper offers pasta salad, rice salad and garden salad with a choice of meats. Sixty of the people present took at least some salad, with 5 taking some of all three salads. 28 people took rice salad and 32 people took pasta salad. 15 took both garden and rice salad, 10 took both pasta and rice salad and 20 took both pasta and garden salad. What is the probability that a person chosen at random from those that took salad had some garden salad?

3 A child likes to play the 'guess my number' game with her older relatives. One is a Year 11 maths student, who tries to improve his chances by saying that she is only allowed to pick numbers from 1 to 100 inclusive that do not divide by 2, 3, 4 or 5. What is the probability of randomly guessing one of these numbers?

4 Three people went into the art gallery together on a rainy day and all three had black umbrellas. Because it was a major exhibition, and because of the rain, there were lots of umbrellas, coats, hats and bags in the cloakroom. The cloakroom attendant assumed that since they came in

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together, they were together, and put all three umbrellas together in the last 'box' space in the cloakroom. When they left, they were given one each at random.

a What is the probability that they all got the correct umbrella?

b What is the probability that they all got the wrong umbrella?

5 A waiter on his first day became nervous and could not remember who ordered what at a table of four people who had all ordered different dishes. Instead of asking, he tried to bluff his way out by putting the meals down at random.

a What is the probability that he got them all right?

b What is the probability that he got them all wrong?

6 An assistant in an antiques store was dusting and accidentally knocked the price cards off four possum (colonial) dressers. The dressers were priced at \$1200, \$1300, \$2700 and \$2800. Not wanting to make a fuss, the assistant put the cards back at random. The assistant then sold one of the dressers (now priced at \$1200) to a customer before the owner came back from lunch.

a What is the probability that all the prices went back correctly?

b What is the probability that \$1200 was the right price?

c What is the probability that all the prices were wrong?

d What is the probability that the customer got a huge bargain?

5.06 GENERAL USE OF COUNTING METHODS

You have now studied various counting methods and related topics: the multiplication principle, the addition principle, the pigeonhole principle, permutations with and without repetition, combinations, Pascal's triangle and the inclusion-exclusion principle.

You need to be able to choose the most appropriate methods to solve problems and may have to use several methods together.

You should remember the following rules for permutations.

IMPORTANT

The number of ways of arranging n **objects in an ordered list** is given by

$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

The number of permutations of r objects from n different objects is written as ${}^nP_r$, which is calculated using the formula

$${}^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1).$$

The number of **circular permutations** of n different objects or symbols is $(n-1)!$.

The number of permutations of n symbols, of which a are identical, is given by $\frac{n!}{a!}$.

If there are $a, b, c, \ldots$ identical symbols, the number of permutations of n symbols is $\frac{n!}{a!b!c!\cdots}$, where $a+b+c+\cdots \leq n$.

Example 13

How many different 4-letter 'words' can be made from the letters of ALMANAC if, in making a word, each of the seven letters can be used only once in each 'word'?

Solution

Count the combinations of four different letters.	Number of combinations of different letters $= {}^5C_4$ $= 5$
Count their permutations.	Permutations of four different symbols $= 4!$ $= 24$
Use the multiplication principle.	Number of words with four different letters $= 5 \times 24$
Count the combinations with two As.	Number of combinations with two As $= {}^4C_2$ $= 6$
Count their permutations.	Permutations of 4 symbols with 2 the same $= \dfrac{4!}{2!}$ $= 12$
Use the multiplication principle.	Number of words with two As $= 6 \times 12$
Count the combinations with three As	Number of combinations with three As $= 4$
Count their permutations.	Permutations of 4 symbols with 3 the same $= \dfrac{4!}{3!}$ $= 4$
Use the multiplication principle.	Number of words with three As $= 4 \times 4$
Use the addition principle.	Total number of words $= 5 \times 24 + 6 \times 12 + 4 \times 4$
Calculate the answer.	$= 208$
State the answer.	208 different 'words' can be made from the letters of ALMANAC using each letter only once.

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Example 14

How many different 5-digit whole numbers can be made from the digits 2, 3, 4, 4, 4, 5, 7, 7, 9 if each of the 9 digits can be used only once in each number?

Solution

Count the combinations with no repeats.	Combinations with no repeats $= {}^6C_5 = 6$
Count their permutations.	Permutations of 5 symbols $= 5! = 120$
Count the numbers with no repeated digits.	Number with no repeats $= 6 \times 120 = 720$
Count the combinations with two 4s.	Combinations with two 4s $= {}^5C_3 = 10$
Count their permutations.	Permutations of 5 symbols with 2 the same $= \frac{5!}{2!} = 60$
Count the numbers with two 4s.	Number with two 4s $= 10 \times 60 = 600$
It is the same for the two 7s.	Number with two 7s $= 10 \times 60 = 600$
Count the combinations with three 4s.	Combinations with three 4s $= {}^5C_2 = 10$
Count their permutations.	Permutations of 5 symbols with 3 the same $= \frac{5!}{3!} = 20$
Count the numbers with three 4s.	Number with three 4s $= 10 \times 20 = 200$
Count the combinations with two 4s and two 7s.	There are 4 combinations with two 4s and two 7s
Count their permutations.	Permutations of two 4s and two 7s $= \frac{5!}{2! \times 2!} = 30$
Count the numbers with two 4s and two 7s.	Number with two 4s and two 7s $= 4 \times 30 = 120$
Count the combinations with three 4s and two 7s.	There is 1 combination with three 4s and two 7s.
Count the permutations.	Permutations of three 4s and two 7s $= \frac{5!}{2! \times 3!} = 10$
Count the numbers with three 4s and two 7s.	Number with three 4s and two 7s $= 1 \times 10 = 10$
Use the addition principle.	Total number $= 720 + 600 + 600 + 200 + 120 + 10 = 2250$
State the answer.	There are 2250 possible 5-digit numbers.

Sometimes the inclusion-exclusion principle, or a variation of it, can be used to solve a problem more efficiently than by using only using combinations or permutations.

Example 15

Peter, Freda and Elena go to buy tickets for a concert. They travel to the box office separately, so their positions in the queue are essentially random. There are 10 people in the queue.

a In how many different ways could Peter and Freda be together in the queue?

b In how many ways could all three friends be together in the queue?

c In how many ways could they all be separated?

Solution

a Consider Peter and Freda as one unit.	The number of permutations of 9 items = 9!
In how many orders can Peter and Freda be?	The number of permutations of 2 items = 2!
Use the multiplication principle.	Ways Peter and Freda are together $= 9! \times 2!$ $= 725\,760$
Write the answer.	There are 725 760 ways Peter and Freda could be together in the queue.
b Consider the three friends as a group.	The number of permutations of 8 items = 8!
In how many orders can the three be?	The number of permutations of 3 items = 3!
Use the multiplication principle.	Ways the three are together $= 8! \times 3!$ $= 241\,920$
Write the answer.	There are 241 920 ways that all three friends could be together in the queue.
c Find the number of possible queues.	Number of queues = 10! $= 3\,628\,800$
State the ways that they can be together.	All three can be together or two can be together.
State the overlap.	The number of any two together includes the three together in some orders.
Peter and Freda together includes EFP, EPF, FPE and PFE, but not FEP or PEF.	In this case, two together includes three together, but only in four orders, not all six.
State the actual intersection, considering the three as one unit.	The number in the intersection of Peter and Freda together with all three together $= 4 \times 8!$ $= 161\,280$

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Two together can be Peter and Freda, Peter and Elena or Elena and Freda.	Each of the three ways that two friends can be together has the same number.
Use the inclusion exclusion principle.	Number of ways some of the friends can be together $= 3 \times 725\ 760 - 3 \times 161\ 280 + 241\ 920$
Calculate the number.	$= 1\ 935\ 360$
Find the number of ways they are separated.	Number of ways all separate $= 3\ 628\ 800 - 1\ 935\ 360$ $= 1\ 693\ 440$
State the answer.	There are 1 693 440 ways that the 3 friends can be separated in the queue of 10 people.

EXERCISE 5.06 General use of counting methods

Reasoning and communication

1 Example 13 How many different 4-letter 'words' can be made from the letters of SELECTED if, in making a word, each of the 8 letters can be used only once in each 'word'?

2 Example 14 How many numbers of each of the following lengths can be made from the digits 1, 2, 2, 3, 4, 5, 5 if each of the 7 digits can be used only once in each number?

a 2 digits
b 3 digits
c 4 digits
d 5 digits

3 Example 15 Seven people, including three friends, are seated in row at random. In how many ways can each of the three friends be seated apart from each other?

4 How many different 4-letter 'words' can be made from the letters of ENTERPRISES if, in making a word, each of the 11 letters can be used only once in each 'word'?

5 How many different 5-letter 'words' can be made from the letters of SYNTHESIS if, in making a word, each of the 9 letters can be used only once in each 'word'?

6 Four friends at an amusement park are in the line to get on a 'terror ride'. They are all in the first 20 in the queue, so they will all get on the next ride.

a In how many ways could John and Mary be together?
b In how many ways could John, Mary and Massoud be together?
c In how many ways could all four be together?
d In how many ways could they all be separated?

7 A restaurant has round tables that seat 8 people and square tables that seat 4 people. In how many ways could a party of 12 people be seated using one 4-person and one 8-person table?

5.07 GENERAL APPLICATIONS TO PROBABILITY

Probability problems are mostly calculated by counting the number of ways that things might happen. When solving them, you need to work out the kind of counting involved: addition, multiplication, pigeonhole or inclusion-exclusion principles, permutations or combinations or multiple methods.

Example 16

In melding card games like Canasta and Gin Rummy, players attempt to make three or four of a kind or to make runs of three or more cards of the same suit. What is the probability that the first three cards a player receives is a run of three cards of the same suit (with Ace high)?

Solution

State the melds in one suit.	The spade melds are 2-3-4, 3-4-5, …., 10-J-Q, J-Q-K, Q-K-A
Count them.	There are 11 possible spade melds of 3 cards.
There are 4 suits.	Number of possible melds altogether $= 4 \times 11$ $= 44$
Count the possible 3-card hands.	Number of possible 3-card hands $= {}^{52}C_3$ $= 22\ 100$
Work out the probability.	Probability of a 3-card run $= \dfrac{44}{22\ 100}$ $= \dfrac{11}{5525}$ $\approx 0.001\,99$
Round and write the answer.	There is about a 0.2% probability of a 3-card run in the first three cards.

When doing multiple calculations, it is often easier to evaluate everything at the end.

Example 17

Four marbles are taken at random from a bag containing 5 red, 3 green and 4 yellow marbles. What is the probability that two marbles are red and the other two are yellow?

Solution

The order is not important.	The number of ways of taking 4 marbles $= {}^{12}C_4$
You want 2 red and 2 yellow.	Use the multiplication principle, so $n(\text{2R and 2Y}) = {}^5C_2 \times {}^4C_2$

Write the probability.

$$P(\text{2R and 2Y}) = \frac{{}^5C_2 \times {}^4C_2}{{}^{12}C_4}$$

$$= {}^5C_2 \times {}^4C_2 \div {}^{12}C_4$$

Substitute formulas.

$$= \frac{5\times4}{1\times2} \times \frac{4\times3}{1\times2} \div \frac{12\times11\times10\times9}{1\times2\times3\times4}$$

Multiply by the reciprocal.

$$= \frac{5\times4}{1\times2} \times \frac{4\times3}{1\times2} \times \frac{1\times2\times3\times4}{12\times11\times10\times9}$$

Cancel and evaluate.

$$= \frac{4}{33}$$

$$\approx 0.12$$

Write the answer.

The probability of getting 2 red and 2 yellow marbles is $\frac{4}{33}$, or about 12%.

EXERCISE 5.07 General applications to probability

Reasoning and communication

1 Example 16 In the card game pontoon, the highest hand (a pontoon) consists of an ace and a court card (J, Q, K) or 10. Each player is dealt two cards to start with. What is the probability of getting a pontoon?

2 Example 17 A box of spare parts has 18 pistons with a tolerance of 0.05 mm in the diameter. There are 8 oversized pistons and 10 undersized pistons in the box. Four pistons are taken from the box at random to rebuild an engine. What is the probability that they are:

a all undersized?

b all oversized?

c half undersized and half oversized?

3 A full hand in poker has a pair and three of a kind, such as 2 fives and 3 kings. What is the probability of being dealt a full house from a normal pack?

4 A flush in poker has 5 cards of the same suit. What is the probability of being dealt a flush from each of the following packs?

a A normal 52 card pack

b A euchre pack (7s up) of 32 cards

c A 6-handed 500 pack that includes 11s, 12s and red 13s in addition to the normal 52 cards.

5 In a busy office, three letters are prepared with envelopes addressed, ready for posting to three different people. Unfortunately, the letters and envelopes are dropped on the floor and mixed up. The person who is asked to place the letters in the envelopes is very careless and doesn't pay attention to the address on the envelopes when the letters are placed in them. What is the probability that no one gets the right letter?

6 25 students were selected at random from a group of Year 11 students who received a grade of A, B or C on a test. There were 30 who got As, 50 who got Bs and 80 who got Cs.

a What is the probability that exactly 15 of the students selected had the same grade?

b What is the probability that at least 10 students had the same grade?

c What is the smallest number n such that the probability of 10 students getting the same grade when n are selected is 1?

7 A band has four guitarists, each of whom has their own guitar. The guitars all look similar and it is common for the guitarists to pick up someone else's guitar by mistake.

a What is the probability of exactly three picking up the wrong guitar?

b What is the probability of at least three picking up the wrong guitar?

c What is the probability that no one picks up the right guitar?

8 Max's mother is colour blind. In his wardrobe he has 3 green shirts, 2 red shirts, 1 blue shirt, 2 black pairs of pants and 3 cream pairs of pants. His mother picks some clothes for him to wear while he is in the shower, but Max is particular about what he wears and will only wear outfits where the shirt and pants colours 'go with' each other. In Max's view, black pants only go with blue or red shirts, and cream pants only go with red or green shirts. What is the probability that Max's mother has picked an acceptable outfit?

9 Ten friends were all on holidays in different parts of the world. They all sent postcards to five different friends from the ten, essentially choosing the 5 at random. What is the probability that at least two people sent cards to each other?

10 To enter a building, workers must enter a code into a touchpad at the entrance. The code has 6 digits. Helen remembers 5 of the digits for her code, but can't remember the order. She knows that the digits are all different. What is the probability of her getting the correct code in the three goes she is allowed?

11 On the first day of the school year, a teacher assigns desks in his classroom to students at random as they enter the classroom. The desks are set out together in groups of 3. There are 24 seats and 24 students in one class.

a What is the probability that three friends are seated together?

b What is the probability that they are seated apart (at different sets of 3 desks)?

12 Some students arrive at the basketball court and decide to make up two teams to play a practice game. Three of the players are much better than the others and a team where all three are on the same side is too strong. The teams are chosen at random using 'eeny, meeny, miny, moe'. The last person must be the umpire. What is the probability of one team being too strong if there are

a 7 students? b 9 students? c 11 students? d 15 students?

(*Note:* The odd one out must umpire!)

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CHAPTER SUMMARY
PERMUTATIONS AND COMBINATIONS

5

- **Combinatorics** is the mathematics of systematically counting finite collections
- A **combination** is an unordered selection. A combination of a set of symbols is a selection of none, some or all of the symbols.
- The number of combinations of r objects from n distinct objects is given by

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$
$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

- The **addition principle** states that for choices made in mutually exclusive ways, the total number of choices is the sum of the choices made in each way.
- The **multiplication principle** states that, for a choice made in 2 stages, if there are a ways for one part and b ways for the second part, then there are $a \times b$ choices altogether. If a choice is made in n stages, each of which have a_i possibilities, then there are $a_1 a_2 a_3 \ldots a_n$ possible choices altogether.
- Pascal's triangle is an arrangement of staggered numbers in rows such that each row begins and ends with 1 and the other numbers are the sums of the two numbers immediately diagonally above them.

						1							Row 0
					1		1						Row 1
				1		2		1					Row 2
			1		3		3		1				Row 3
		1		4		6		4		1			Row 4
	1		5		10		10		5		1		Row 5
1		6		15		20		15		6		1	Row 6

- The numbers in Pascal's triangle are the combinatorial numbers given by ^nC_r and are also the binomial coefficients, the coefficients in the expansion of $(x+y)^n$.
- The **inclusion-exclusion principle** states that the number of elements in a union of sets is given by:

  $|A \cup B| = |A| + |B| - |A \cap B|$ for 2 sets

  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$
  $- |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  for 3 sets

  The pattern of inclusions and exclusions extends to any number of sets.
- The number of ways of arranging n **objects in an ordered list** is given by $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$
- A **permutation** is an ordered arrangement. A permutation of a set of symbols is a selection of none, some or all of the symbols in a particular order.
- The number of permutations of r objects from n distinct objects is written as ^nP_r and is given by

$$^nP_r = \frac{n!}{(n-r)!}$$
$$= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

- **Circular permutations** are ordered arrangements in a circle. The number of circular permutations of n different objects or symbols is $(n-1)!$.
- The number of permutations of n symbols, of which a are identical, is given by $\frac{n!}{a!}$.

  If there are $a, b, c, \ldots$ identical symbols, the number of permutations of n symbols is $\frac{n!}{a!b!c!\cdots}$, where $a + b + c + \ldots \leq n$.

5 CHAPTER REVIEW
COUNTING METHODS AND COMBINATIONS

Multiple choice

1 Example 1 The number of combinations of 4 symbols from 8 different symbols is:
A 24 B 64 C 70 D 420 E 1680

2 Example 2 The number of committees of 3 that can be chosen from 10 people is:
A 30 B 120 C 270 D 720 E 810

3 Example 8 The coefficient of x^3yz in the expansion of $(x + y + z)^5$ is:
A 10 B 12 C 15 D 20 E 50

4 Example 9 The number of natural numbers up to 200 divisible by 3 or 5 is:
A 92 B 93 C 106 D 107 E 108

5 Example 16 The probability that the first 4 cards dealt from a normal pack make two (different) pairs like 4 K K 4 is:
A about 0.000 044 3 B about 0.003 53 C about 0.004 61
D about 0.006 41 E about 0.010 37

Short answer

6 Example 4 How many different committees of four people including Mohammed and Julius could be chosen from a group of fifteen people?

7 Example 5 A netball team of 7 players is to be chosen from a squad of twelve players. How many different teams can be chosen that include Kylie or Simone, but not both?

8 Example 6 A class of 26 students with 14 girls and 12 boys has three representatives on the student council.
a In how many ways can the three be chosen with no restrictions?
b In how many ways can they be chosen if at least one person of each sex must be included?

9 Example 10 How many positive whole numbers under 1000 are divisible by 5, 7 or 9?

10 Example 11 35 of the cows in a dairy herd have horns and 9 of these are a uniform colour. Altogether, there are 72 cows in the herd that have either horns or a uniform colour. There are 220 cows altogether in the herd. What is the probability that a cow chosen at random for testing is of uniform colour?

11 Example 12 In a group of 200 people, 70 have fair skin and the rest have olive or dark complexions. 80 have light-coloured hair (fair or red) and the rest have dark hair. There are 90 with light-coloured eyes (blue, grey, green or hazel) and the rest have dark-coloured eyes. Of those with fair skin 30 have light-coloured hair and 30 have light-coloured eyes. 50 have light-coloured eyes and hair. Of this 50, 20 also have fair skin. How many people in the group have at least one of fair skin or light-coloured eyes or hair?

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12 Example 13 How many different 4-letter 'words' can be made from the letters of ENVELOPE if, in making a word, each of the 8 letters can be used only once in each 'word'?

13 Example 17 A box of printer cartridges has eight new cartridges and 10 remanufactured cartridges. What is the probability that when six are taken out at random, equal numbers of new and remanufactured cartridges are selected?

Application

14 Prove that ${}^nC_r = \frac{n}{r}\left({}^{n-1}C_{r-1}\right)$.

15 Prove that if the first number after the 1 in a row of Pascal's triangle is prime, then it divides all the other numbers, apart from the 1s at either end.

16 In the Australian version of the card game Euchre, the cards 7, 8, 9, 10, J, Q, K and A of each suit are used, making 32 cards altogether. One suit is chosen as trumps, and the Jack of the same colour is included in the trump suit as the second highest card (the left Bower). How many hands of 5 cards are possible that include exactly

a 4 trumps
b 3 trumps
c no trumps?

Practice quiz

17 Three couples go to a concert together and they all sit together. In how many ways can everyone be seated so that no couple is together?

MIXED REVISION CHAPTERS 4 • 5 • 6

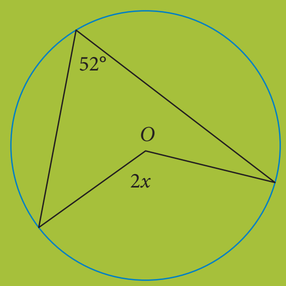
2

Multiple choice

1 The projection of $\mathbf{q} = 5\mathbf{i} + 3\mathbf{j}$ on the positive direction of the x-axis is:
A -5 B 0 C 3 D 5 E $3\sqrt{5}$

2 How many different committees of 4 people can you select from 15 people?
A 1365 B 2730 C 32 760 D 50 625 E 360 360

3 In the diagram, O is the centre of the circle.

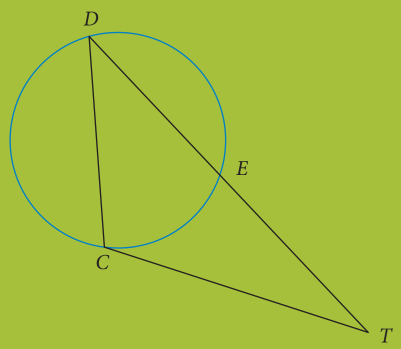


What is the value of x?
A 26° B 52° C 76° D 104° E 208°

4 The projection of $\mathbf{a} = (4, 155°)$ on $\mathbf{b} = (2, 335°)$ is:
A -4 B -2 C 2 D 4 E 6

5 What is the value of ${}^{10}C_4$?
A 210 B 400 C 5040 D 10 000 E 151 200

6 In the diagram, CT is a tangent. Choose the correct statement.



A $CT^2 = DE \times ET$
B $CT^2 = CD \times ET$
C $CT^2 = DE \times DT$
D $CT^2 = CD \times DE$
E $CT^2 = DT \times ET$

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7 An aeroplane is flying toward the northwest. Which of the following wind velocity vectors increases the plane's speed the most?

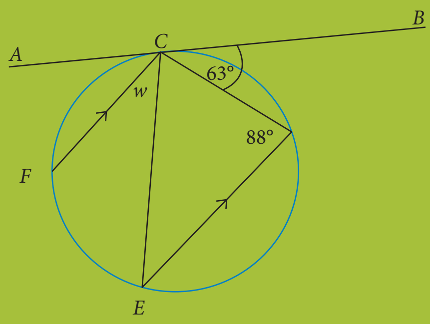
A $\mathbf{w}_1 = -\mathbf{i} - 5\mathbf{j}$ B $\mathbf{w}_2 = -5\mathbf{j}$ C $\mathbf{w}_3 = \mathbf{i} + 5\mathbf{j}$

D $\mathbf{w}_4 = 5\mathbf{i} + \mathbf{j}$ E $\mathbf{w}_5 = -\mathbf{i} + 5\mathbf{j}$

8 What is the probability of selecting 3 brass screws at random from a box containing 12 brass and 8 zinc-plated screws of the same size?

A $\frac{1}{1140}$ B $\frac{1}{220}$ C $\frac{14}{285}$ D $\frac{14}{55}$ E $\frac{11}{57}$

9 In the diagram, *ACB* is a tangent. The value of *w* is:



A 25° B 63° C 88° D 92° E none of the above

Short answer

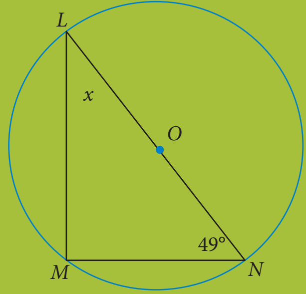
1 Calculate the dot products of each of the following.

a (–2, 4) and (3, 5)

b (5, 84°) and (7, 122°), correct to two decimal places

2 Of the little cakes in a baker's shop, 70 have cream, 80 have custard and 90 have a dusting of icing sugar. 25 have both custard and cream, 35 have cream and icing sugar and 50 have custard and icing sugar. There are 150 little cakes for sale altogether and all have cream, custard or cream or icing sugar. Do any have all three of custard, cream and icing sugar, and if so, how many?

3 Find the value of *x* giving reasons. *O* is the centre of the circle.



4 Find the projection of $\mathbf{p} = (1, 5)$ on $\mathbf{q} = (6, 3)$.

5 At an international sports event, 18 of the athletes from North America, 25 from Europe, 8 from Australia, 13 from Africa, 15 from South America and 20 from Asia are in the dining hall. What is the probability that three athletes chosen at random are all from the Americas?

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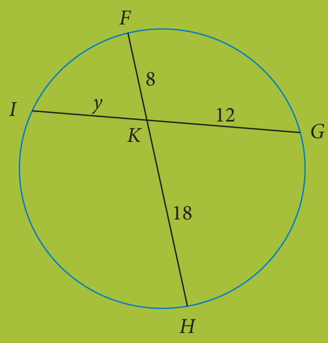
6 Find the value of y in the diagram below. Explain your reasoning.



Application

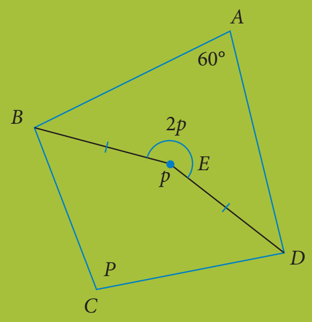
1 Consider the points $A(-5, -1)$, $B(2, -3)$ and $C(2, 6)$, which are the end points of vectors **BA** and **BC**. Use vectors to calculate the angle between **BA** and **BC**.

2 The wind is blowing in the direction N 20° E and a yacht is moving on a bearing of 340° at a speed of 8 m s^{-1}. If the wind is exerting a force of 5000 N on the sails, use the scalar product to find the energy transferred in one minute. Remember that the work done (using $\mathbf{F} \cdot \mathbf{s}$), is the energy transferred.

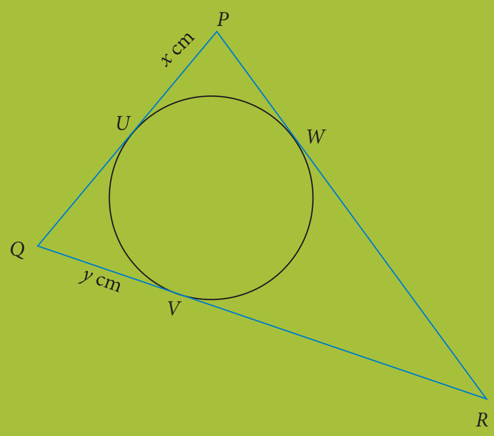
3 A jar contains a mixture of 10 sherbies, 12 milk chews and 8 chocolate eclairs. What is the probability that in a handful of 6 lollies taken out at random, there are equal numbers of each?

4 How many numbers from 2000 to 6000 inclusive are divisible by 4, 5 or 6?

5 Consider the quadrilateral $ABCD$, where E is an internal point such that $EB = ED$.
Prove that $ABCD$ is a cyclic quadrilateral.



6 Consider the diagram where PQ, QR and RP are tangents to the circle, with points of contact U, V and W respectively. $PU = x$ cm and $QV = y$ cm. $VR = 20$ cm and $x + y = 16$.
Find the perimeter of $\triangle PQR$.



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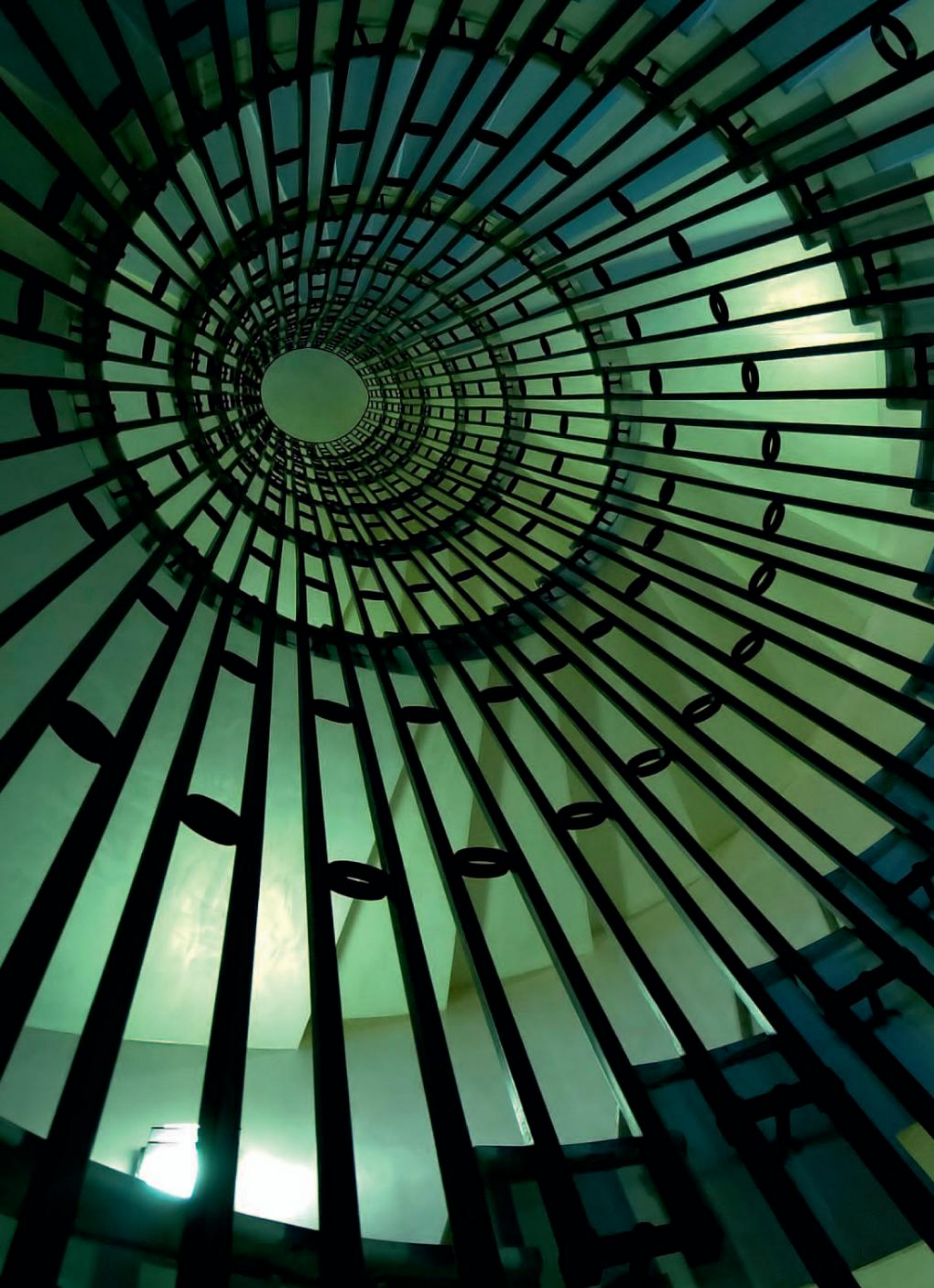


Shutterstock.com/Vlade Shestakov

6

TERMINOLOGY

alternate segment
arc
concyclic
cyclic
intercepts
proof
secant
subtend
tangent

GEOMETRY

CIRCLE GEOMETRY

Prior learning

CIRCLE PROPERTIES AND THEIR PROOFS INCLUDING THE FOLLOWING THEOREMS

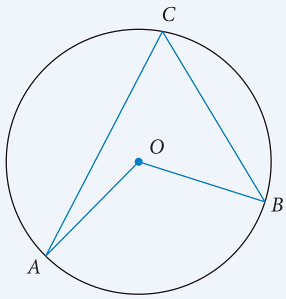
- An angle in a semicircle is a right angle (ACMSM029)
- The angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc (ACMSM030)
- Angles at the circumference of a circle subtended by the same arc are equal (ACMSM031)
- The opposite angles of a cyclic quadrilateral are supplementary (ACMSM032)
- Chords of equal length subtend equal angles at the centre and conversely chords subtending equal angles at the centre of a circle have the same length (ACMSM033)
- The alternate segment theorem (ACMSM034)
- When two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord (ACMSM035)
- When a secant (meeting the circle at A and B) and a tangent (meeting the circle at T) are drawn to a circle from an external point M, the square of length of the tangent equals the product of the lengths to the circle on the secant. ($AM \times BM = TM^2$) (ACMSM036)
- Suitable converses of some of the above results (ACMSM037)
- Solve problems finding unknown angles and lengths and prove further results using the results listed above. (ACMSM038)

6.01 ANGLES AT THE CENTRE OF CIRCLES

There are many theorems in Euclidean geometry that relate specifically to circles. The centre of a circle is usually labelled as O.

INVESTIGATION The angle at the centre of a circle

Consider the circle with points A, B and C on the circumference, with lines AO, BO, AC and BC as shown. $\angle ACB$ is on the circumference and $\angle AOB$ is at the centre, as shown in the diagram below.



Construct the segment CO and extend it to point D as shown on the top right.

For convenience, write $\angle CAO = \alpha$ and $\angle CBO = \beta$.

What kind of triangle is $\triangle COB$?

How do you know?

What is $\angle BCO$ equal to?

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How is $\angle DOB$ related to $\angle BCO$ and $\angle CBO$?

What is $\angle DOB$ equal to?

What is $\angle DOA$ equal to?

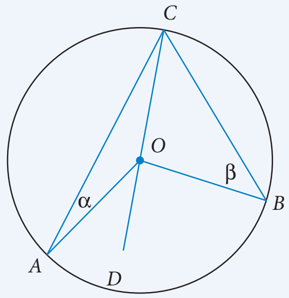
How do you know?

What is $\angle AOB$ equal to?

What is $\angle ACB$ equal to?

What does this prove about $\angle AOB$ and $\angle ACB$?

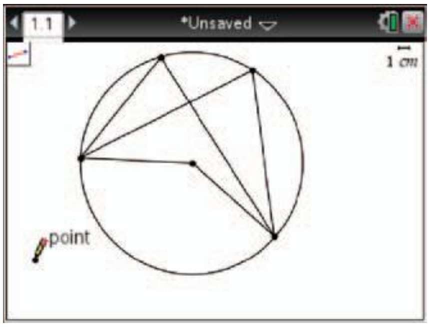
What does this prove about angles subtended at the centre and circumference of a circle by the same arc?



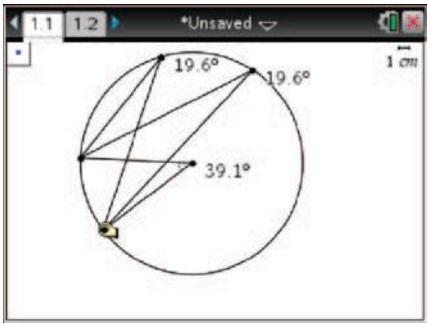
Circle properties can be explored with CAS calculators.

TI-Nspire CAS

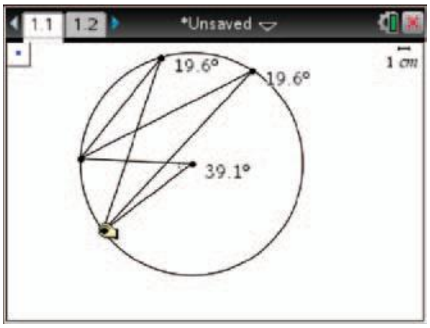
Use a Geometry page.
Use menu, 5: Shapes and 1: Circle to draw a circle.
Use the 'point' to place the centre and move out to place the circumference.
Then use menu, 4: Points & Lines and 5: Segment to draw the lines shown on the screenshot on the right.



Now use menu, 6: Measurement and 4: Angle to measure some of the angles. Move the 'point' to a point on one ray of the angle (even one of the ends of the segment), the centre point of the angle and a point on the other ray of the angle. Notice in the screenshot that the angles have been rounded so they may not be exactly as shown.



Now use menu, 4: Points & Lines and 1: Point. Grab a point on the circle. Move it around the circumference. What do you observe?



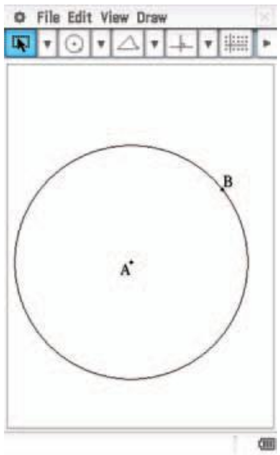
ClassPad

Use the **Geometry** menu.

Use the top menu bar to choose the circle drawing tool .

To draw a circle, tap a point in the middle of the screen for the centre and a point further out that will be on the circumference of the circle.

From the menu where you found choose the line segment tool to draw the angle subtended at the centre and two angles subtended at the circumference (including one at point *B*) from the same arc.



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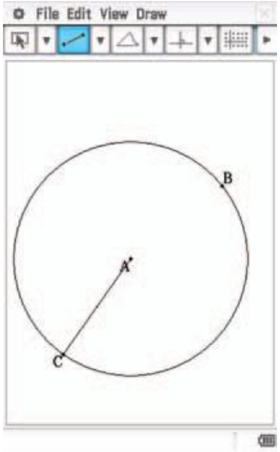
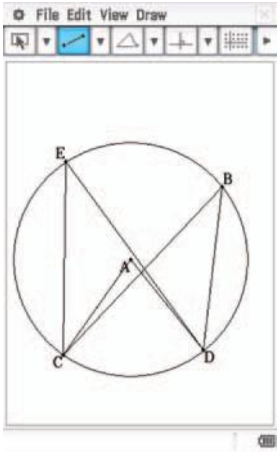
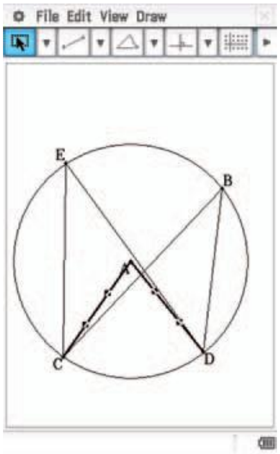
To draw a line from the circumference to the centre, first tap a point on the circumference, then tap point A.

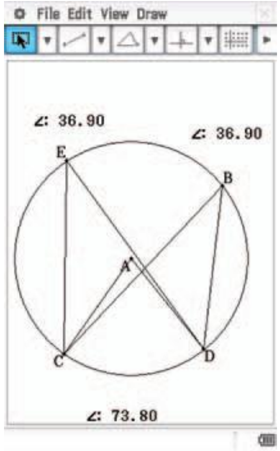




Tap the selection arrow
Select lines AC and AD for $\angle CAD$ by tapping them.



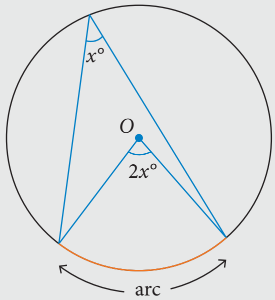
Tap the top right arrow ▸.
Touch and drag the angle measurement below the circle.
Tap the arrow ◂ and repeat the sequence above for ∩ *CED* and ∩ *CBD*, but drag the measurements to the top left and top right of the circle.
Note: Before selecting the next angle, you may need to tap a blank area of screen to remove all previous selections.



IMPORTANT

Theorem 1: Angle at the centre of a circle

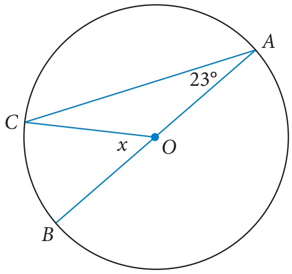
The **angle at the centre** subtended by an arc of a circle is twice the **angle at the circumference** subtended by the same arc.



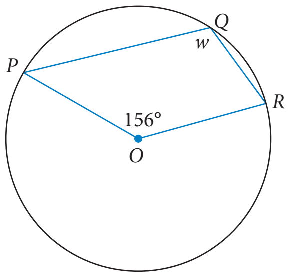
Example 1

Find the value of the unknowns.

a



b



Solution

a $\angle CAB$ and $\angle COB$ are both standing on arc BC.

$x = 2 \times 23°$ (Angle at centre)

Write the answer.

$\therefore\ x = 46°$

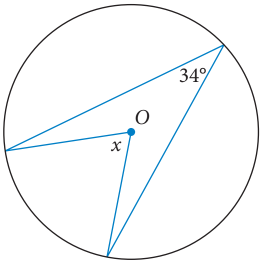
b Find the reflex $\angle PQR$.	Reflex $\angle POR = 360° - 156°$ (Revolution) $= 204°$
$\angle PQR$ and reflex $\angle POR$ are both on the major arc PR.	$2w = 204°$ (Angle at centre)
Write the answer.	$w = 102°$

EXERCISE 6.01 Angles at the centre of circles

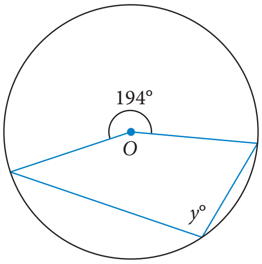
Concepts and techniques

1 Example 1 Find the value of the pronumerals (O is the centre of each circle) in the circles below.

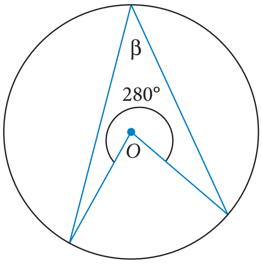
a



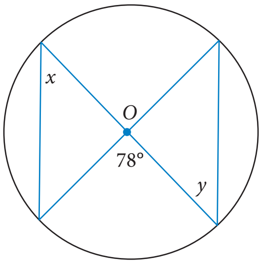
b



c

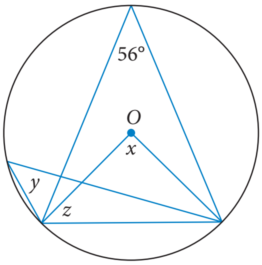


d

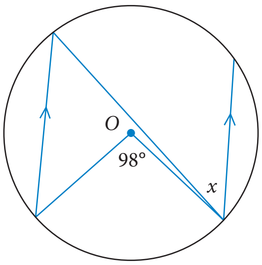


2 Find the values of all pronumerals (O is the centre of each circle) in the following circles.

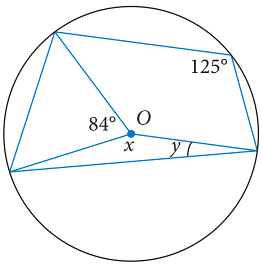
a



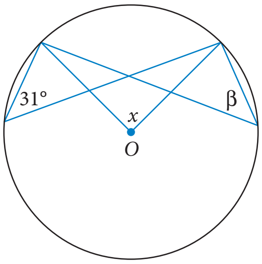
b



c

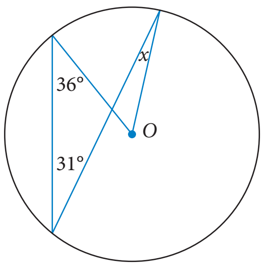


d

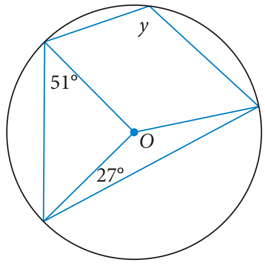


3 Find the values of all pronumerals (O is the centre of each circle) in the following circles.

a

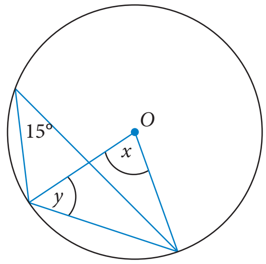


b

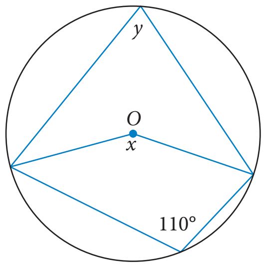


Reasoning and communication

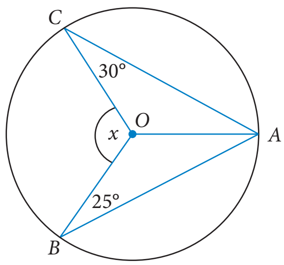
4 Find the value of x and y, giving reasons.



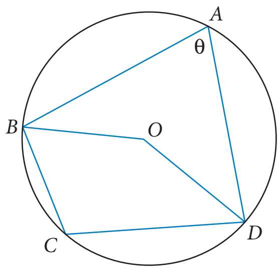
5 Find the value of x and y, giving reasons.



6 Find the value of x. Give reasons for each step in your calculation.



7 The circle has centre O and $\angle DAB = \theta$.
Show that $\angle DAB$ and $\angle BCD$ are supplementary.



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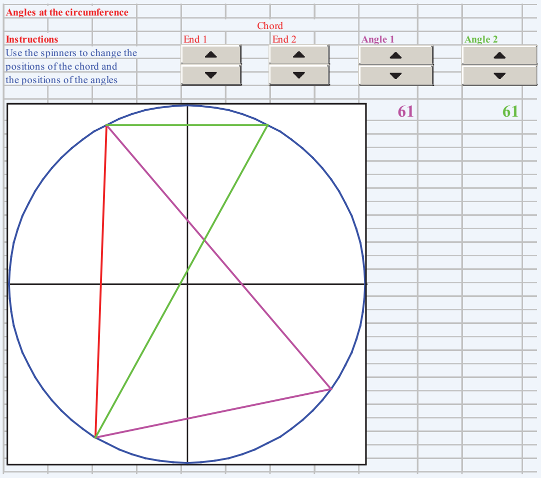
6.02 ANGLES AT THE CIRCUMFERENCE OF CIRCLES

TECHNOLOGY Angles at the circumference

Angles at the circumference

The spreadsheet 'Angles at the circumference' is available on NelsonNet. You can use the spinners to change the positions of the chords and angles. Use the spreadsheet to investigate their relationship.



Consider the situation in the diagram on the right.

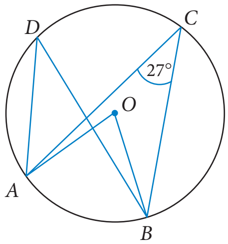
What is $\angle AOB$?

What does that make $\angle ADB$?

What is the relationship between $\angle AOB$ and $\angle ADB$?

Is the relationship still true if you erase the centre and the lines AO and BO?

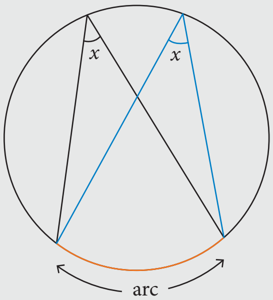
Would this relationship be true for other angles at the circumference standing on the same arc?



IMPORTANT

Theorem 2: Angles on the same arc

Angles at the **circumference** of a circle subtended by the **same arc** are **equal**.



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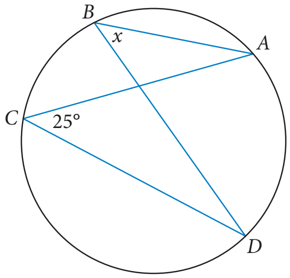
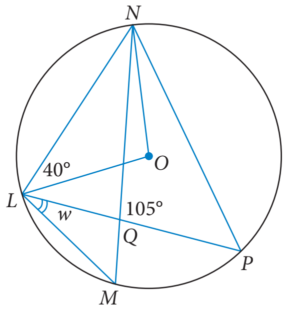
Example 2

Find the value of the unknowns in the circles below.

a



b



Solution

a $\angle DCA$ and $\angle DBA$ are both standing on arc AD. $x = 25°$ (Same arc)

b $ON = OL$ are both radii. $\angle LNO = 40°$ (Isosceles $\triangle$)

$\therefore\ \angle LON = 100°$ ($\triangle LNO$ angle sum)

$\angle LPN$ and $\angle LON$ are both on arc LN. $\therefore\ \angle LPN = 50°$ (Angle at centre)

Use $\triangle QPN$ to find $\angle MNP$. $\therefore\ \angle MNP = 25°$ ($\triangle QPN$ angle sum)

$\angle MNP$ and w are both on arc PM. $\therefore\ w = 25°$ (Angle at centre)

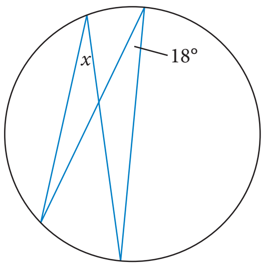
There is no unique solution to examples like the ones shown above. You should aim for the most efficient solution. The solution requiring the least number of steps is the most 'elegant'.

EXERCISE 6.02 Angles at the circumference of circles

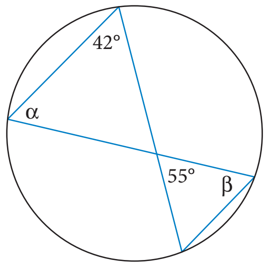
Concepts and techniques

1 Example 2 Find the values of all pronumerals (O is the centre of each circle) in the circles below.

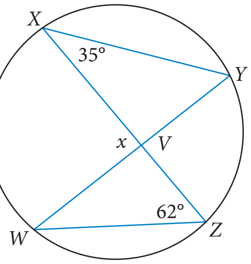
a



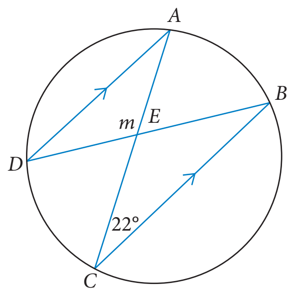
b



c



d



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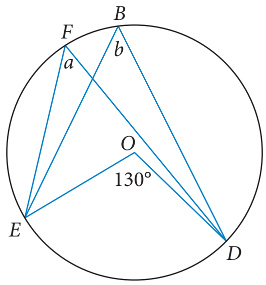
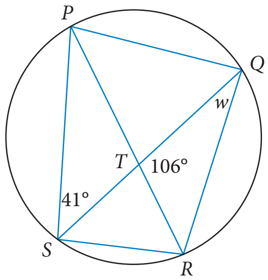
2 Find the value of the pronumerals in the circles below, giving reasons.

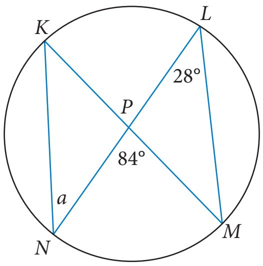
a



b



c



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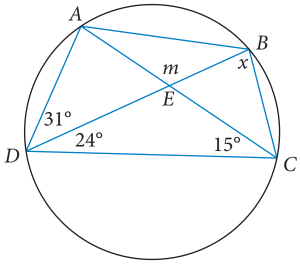
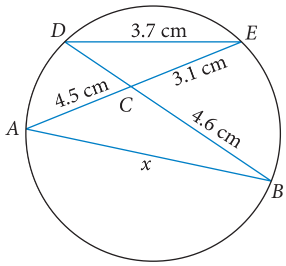


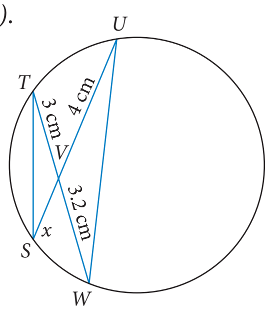
Reasoning and communication

3 a Prove that $\triangle ABC \,|||\, \triangle DEC$.

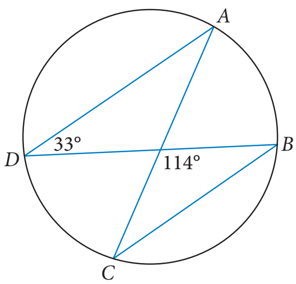
b Hence find the value of x, correct to 1 decimal place.



4 Prove that $\triangle STV$ and $\triangle WUV$ are similar. Hence find the length x (VS).



5 Show that $AD \parallel BC$ in the circle below.



6 (Proof of Theorem 2)

a Construct a circle with arc AB and construct $\angle ACB$ and $\angle ADB$ on the circumference of the circle such that C and D are not on the arc AB.

b Mark the centre and draw in the lines AO and BO.

c What can you say about $\angle AOB$ and $\angle ACB$?

d What can you say about $\angle AOB$ and $\angle ADB$?

e What does this prove about $\angle ACB$ and $\angle ADB$?

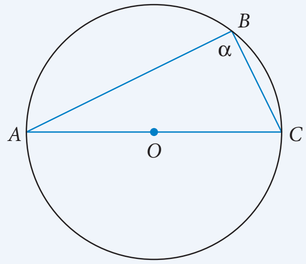
f Set out a formal proof that angles subtended by the same arc on the circumference of a circle are equal.

6.03 SEMICIRCLE ANGLE

The angle in a semicircle is a special case of Theorem 1, where the angle at the centre is 180°.

INVESTIGATION The angle in a semicircle

Consider the circle with diameter AC and point B on the circumference as shown.
Let $\angle ABC = \alpha$.



We know that AOC is a straight line.

What is the size of $\angle AOC$?

Now using Theorem 1, what can you say about $\angle AOC$ and $\angle ABC$?

Therefore what is the size of $\angle ABC$?

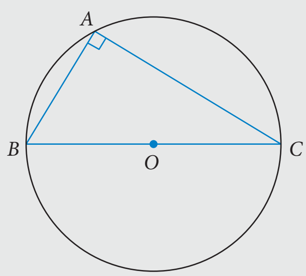
What does this prove about α, the angle at the circumference subtended by the semicircle?

IMPORTANT

Theorem 3: Semicircle angle

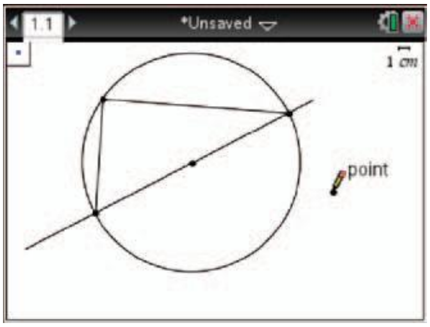
An **angle at the circumference subtended by a diameter** is a **right angle**.

Conversely, a chord that subtends a right angle at the circumference is a diameter.

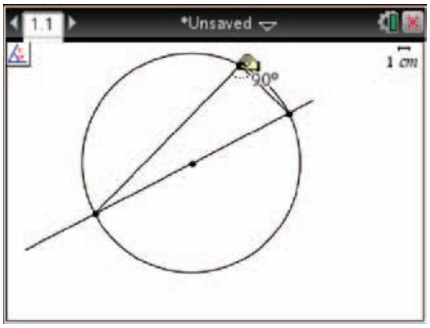


TI-Nspire CAS

Use a Geometry page.
Use menu, 5: Shapes and 1: Circle to draw a circle.
Use menu, 4: Points & Lines and 4: Line to draw a line from the circumference through the centre.
Use menu, 4: Points & Lines and 3: Intersection Point to select the line and a point on the circumference to extend to the other side of the circle.
Use Line Segment to draw the other lines shown on the screenshot.



Use 6: Measurement to measure the angle subtended by the diameter (see page 185).
Now grab the point and move it around the circumference. What do you observe?

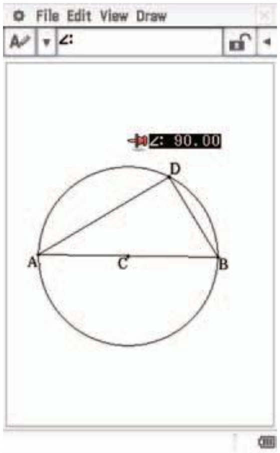


ClassPad

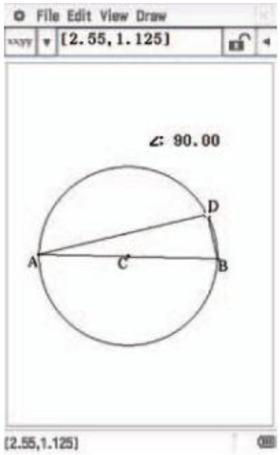
Use the **Geometry** application.
Use the line segment tool then tap two points on the screen. This will draw a line between points A and B that will become the diameter of a circle.
Tap the selection arrow and select *AB*.

Tap **Draw**, then **Construct**, then **Midpoint**. Point C will appear.

Select the circle construction tool, tap *C* to select it as the centre and then tap either *A* or *B*.
Use the line segment tool to draw *AD* and *BD* so *D* is on the circumference.
Tap the selection arrow and select ∩ *ADB* by tapping *AD* and *BD*.
Tap the top right arrow.
Drag the measurement of ∠ *ADB* down onto the screen.



Tap a blank part of the screen and then tap *D*. Hold the stylus point on *D* then move the point by dragging it. Try moving the other points.

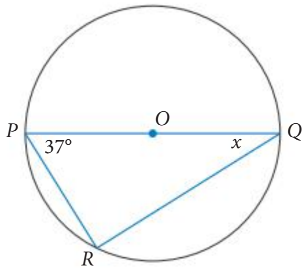


Angles in semicircles and subtended by equal chords

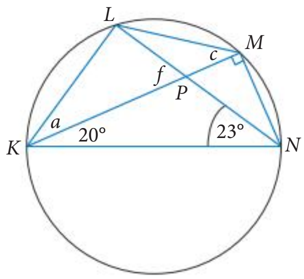
Example 3

Find the value of the unknowns in the circles below.

a



b



Solution

a *PQ* is a diameter.	$\angle PRQ = 90°$ (Semicircle angle)
Use the angles in the triangle.	$x = 53°$ ($\triangle PRQ$ angle sum)
b Use the converse of the theorem.	*KN* is a diameter (Semicircle angle converse)
Now use the theorem normally.	$\therefore \angle KLN = 90°$ (Semicircle angle)
	$\therefore a + 20° + 23° + 90° = 180°$ ($\triangle KLN$ angle sum)
	$\therefore a = 47°$
23° and *c* are both on the arc *KL*.	$c = 23°$ (Same arc)
Use the exterior angle rule for *f*.	$\therefore f = 43°$ (Exterior angle of $\triangle PKN$)

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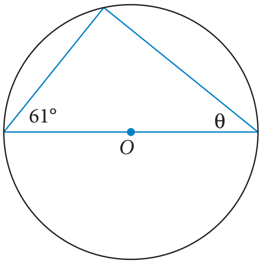
EXERCISE 6.03 Semicircle angle

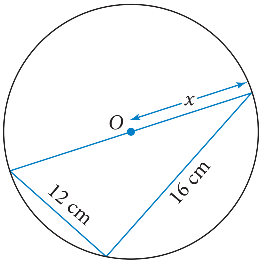
Concepts and techniques

1 Example 3 Find the values of all pronumerals (O is the centre of each circle) for the following circles.

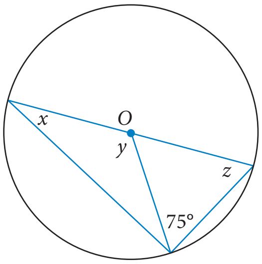
a



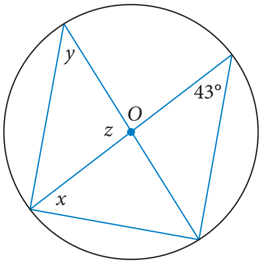
b



c

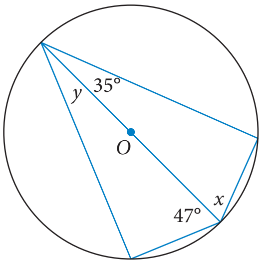


d

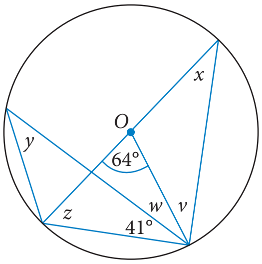


2 Find the values of all pronumerals (O is the centre of each circle) for the following circles.

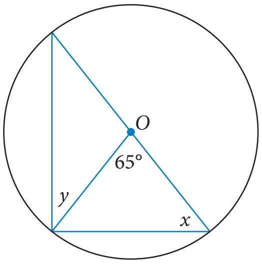
a



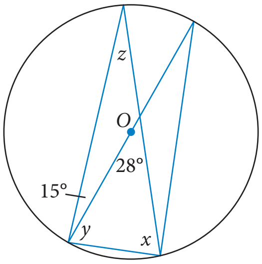
b



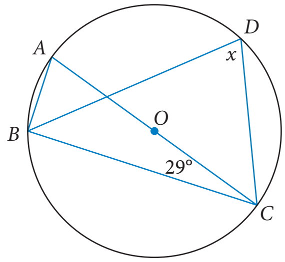
c



d



3 Evaluate x, giving reasons for each step in your working out.



Reasoning and communication

4 (Proof of the converse of Theorem 3, i.e., that the vertices of a right-angled triangle lie on a circle with the hypotenuse as the diameter.)

a Construct a diagram showing a right-angled triangle PRQ, with the right angle at R and the midpoint of PQ at S.
Draw in the point X such that $PRQX$ is a rectangle.

b Connect S and X.

c Given that $PRQX$ is a rectangle, what can you say about the lengths of PQ and RX?

d What is the significance of the point S in relation to PQ and RX?

e Therefore, what can you say about the lengths SR, SQ, SP and SX?

f Therefore, S is the centre of the circle passing through which points?

g Set out a formal proof that the vertices of a right-angled triangle lie on a circle with the hypotenuse as the diameter.

5 Alternate proof of an angle in a semicircle.

a Construct a diagram showing a circle with O as the centre, AB as the diameter and a point C on the circumference such at ABC forms a triangle. Let $\angle ACO = \alpha$ and $\angle BCO = \beta$.

b What sort of triangles are ACO and BCO?

c Which angle is equal to $\angle ACO$?

d Which angle is equal to $\angle BCO$?

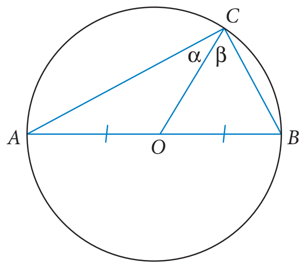
e What is the size of $\angle AOC$ in terms of α?

f What is the size of $\angle BOC$ in terms of β?

g What is the size of $\angle AOB$ in terms of α and β?

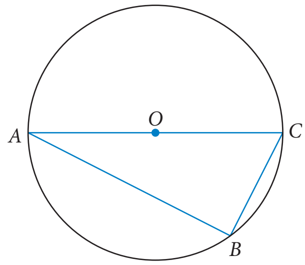
h Complete the statement: $2\alpha + 2\beta =$ __________°
$\therefore\ \alpha + \beta =$ __________°

i Set out a formal proof that the angle in a semicircle is 90°.



6 $AB = 6$ cm and $BC = 3$ cm. O is the centre of the circle.

Show that the radius of the circle is $\dfrac{3\sqrt{5}}{2}$ cm.



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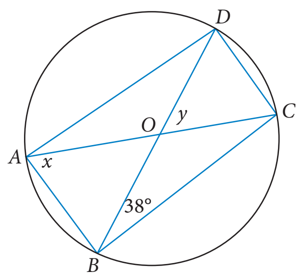
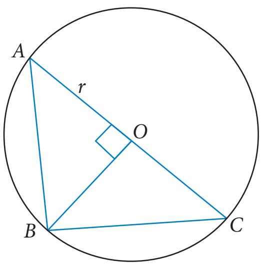
7 The circle has centre O.
  a Evaluate x and y.
  b Show that $AD = BC$.



8 A circle has centre O and radius r as shown.
  a Show that triangles AOB and ABC are similar.
  b Show that $BC = \sqrt{2}r$.

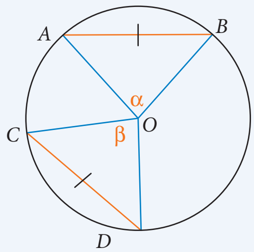


6.04 ARCS AND CHORDS

It is possible to relate equal chords or equal arcs to the angle subtended at the centre of a circle.

INVESTIGATION Equal chords subtending equal angles

Consider a circle, centre O, with chords AB and CD as shown.
Let $\angle AOB = \alpha$ and $\angle COD = \beta$.



Which congruence test would you use to prove that $\triangle AOB$ and $\triangle COD$ are congruent?

What can you say about $\angle AOB$ and $\angle COD$?

What does this prove about the angles subtended at the centre of a circle by equal chords?

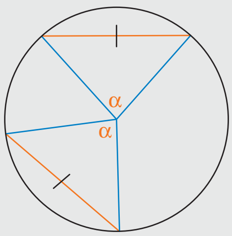
Is the converse true?

IMPORTANT

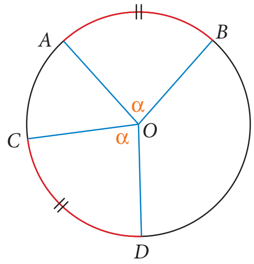
Theorem 4: Centre angles on equal chords

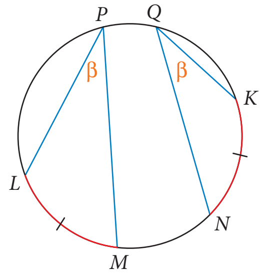
Equal chords of a circle subtend equal angles at the centre.

Conversely, equal angles at the centre subtend equal chords.



It follows that the angles subtended at the centre or at the circumference by equal arcs are also equal. Conversely, equal angles at the centre subtend equal arcs.

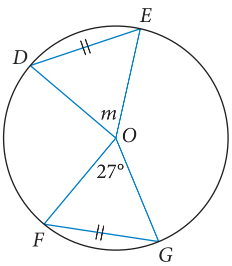


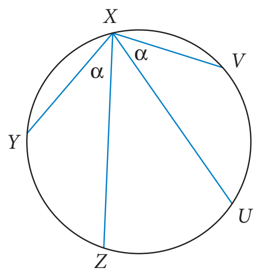


Example 4

a Find the value of m.

b In the diagram, $YZ = 15$ cm. Find the length of UV.





Solution

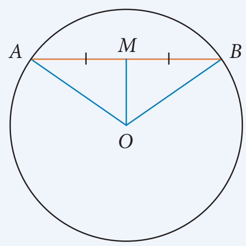
a $DE = FG$ $\quad$ $m = 27°$ (Equal chords)

b $\angle YXZ = \angle UXV$ $\quad$ $UV = 15$ cm (Equal chords/arcs converse)

Another theorem relates the perpendicular bisector of a chord to the centre of the circle.

INVESTIGATION The bisector of a chord passing through the centre of a circle

Consider a circle, centre O, with chord AB. Let M be the midpoint of AB.



Join AO, BO and OM.

Which congruence test would you use to prove that $\triangle AMO$ and $\triangle BMO$ are congruent?

Why are $\angle AMO$ and $\angle BMO$ equal?

What is the sum of $\angle AMO$ and $\angle BMO$?

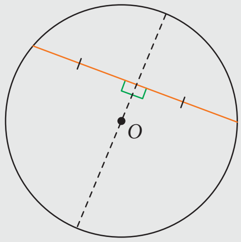
Therefore, what is the size of $\angle AMO$?

What does this say about the angle between the bisector of a chord and the chord?

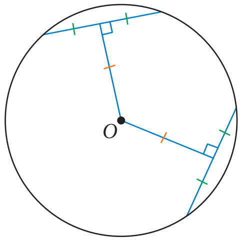
IMPORTANT

Theorem 5: Chord bisector

A radius that bisects a chord is perpendicular to the chord.



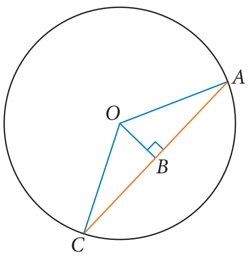
The following results (corollaries) follow from the previous theorem.



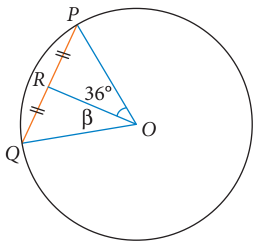
1. The perpendicular bisector of a chord passes through the centre of a circle.
2. The line passing through the centre of a circle perpendicular to a chord bisects the chord.
3. The perpendicular bisector of a chord bisects the angle at the centre of a circle subtended by the chord.
4. Equal chords are equidistant from the centre of a circle.

Example 5

a $AC = 16$ cm and $OB = 5$. Find the radius.



b Find the value of β.



Solution

a $OB \perp AC$, so B is the midpoint of AC. — $AB = 8$ cm (Chord bisector corollary)

Use Pythagoras' theorem. — $OA^2 = OB^2 + AB^2$

Substitute values. — $= 5^2 + 8^2 = 89$

Write the answer. — Radius $= \sqrt{89}$

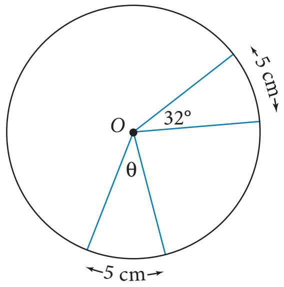
b $PR = QR$, so OR is the perpendicular bisector of PQ. — $\beta = 36°$ (Chord bisector corollary)

EXERCISE 6.04 Arcs and chords

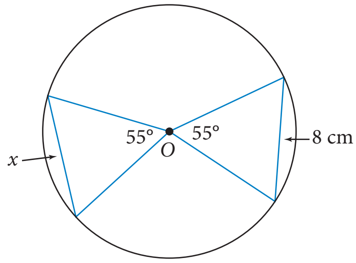
Concepts and techniques

1 Example 4 Find the values of all pronumerals (O is the centre of each circle) in the following circles.

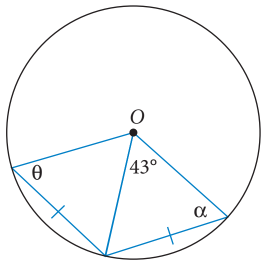
a



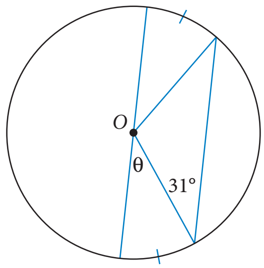
b



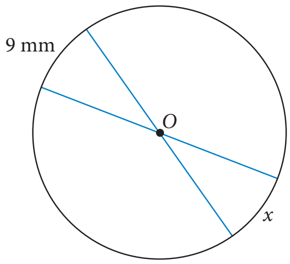
c



d



e



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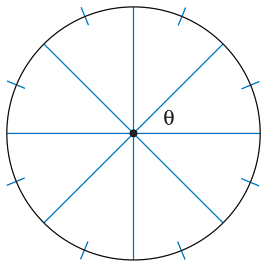
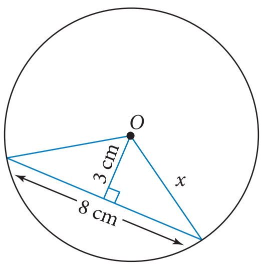


2 Example 5 Find the values of all pronumerals (O is the centre of each circle) in the following circles.

a



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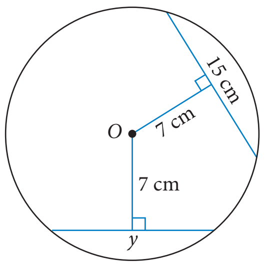

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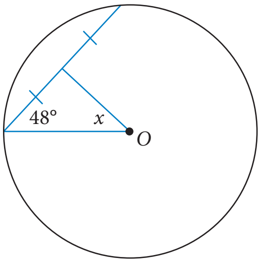

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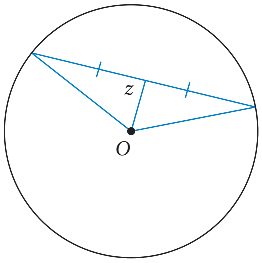

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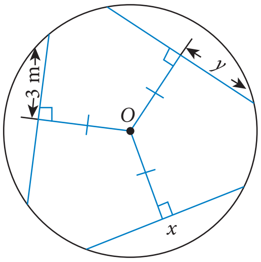

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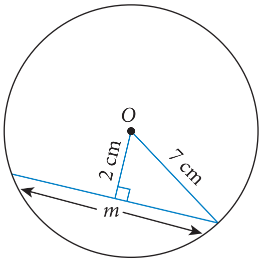
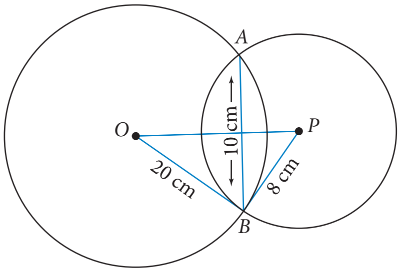


3 Find the exact radius of a circle with a chord that is 8 cm long and 5 cm from the centre.

4 A circle with radius 89 mm has a chord drawn 52 mm from the centre. How long, to the nearest millimetre, is the chord?

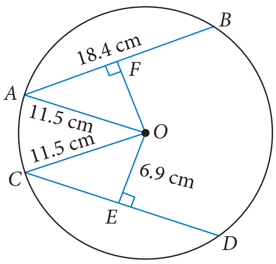
Reasoning and communication



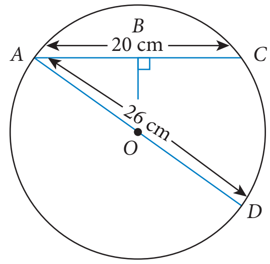
5 In this circle, O and P are the centres of intersecting circles with radii 20 cm and 8 cm respectively. AB is the common chord through the intersections of the circles.

Prove that $\triangle OAP \equiv \triangle OBP$ and hence find the distance OP correct to 1 decimal place.

6 In the circle below, show that $AB = CD$.



7 In the circle on the right, $AC = 20$ cm and $AD = 26$ cm. Find OB, correct to 1 decimal place.



8 [Proof that the perpendicular bisector of a chord bisects the angle at the centre of a circle subtended by the chord.]

a Construct a chord AB on a circle with centre O. Join AO and BO. Join the midpoint M of AB to O.

b Use a congruence test to prove that $\triangle AMO \equiv \triangle BMO$.

c What can you say about $\angle AOM$ and $\angle BOM$?

d Set out a proof that shows that the perpendicular bisector of a chord bisects the angle at the centre of the circle subtended by the chord.

9 [Proof that equal chords are equidistant from the centre of a circle.]

a Construct two equal non-intersecting chords in a circle. Label them AB and CD. The centre of the circle is O. Construct AO and CO.

b Join the midpoint F of AB to the centre O. Join the midpoint G of CD to the centre O.

c Use a congruence test to prove that $\triangle AFO \equiv \triangle CGO$.

d What can you say about the lengths FO and GO?

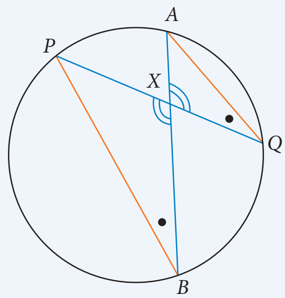
e Set out a formal proof that equal chords are equidistant from the centre of a circle.

6.05 INTERSECTING CHORDS

If two chords intersect, we can relate the lengths of the resulting intercepts.

INVESTIGATION The intervals of intersecting chords

Consider the circle with chords AB and PQ intersecting at X as shown. Join PB and AQ.



Which similarity test would you use to prove that the resulting triangles $\triangle PBX$ and $\triangle AQX$ are similar?

What can you say about the ratios of the matching sides $\dfrac{PB}{AQ}, \dfrac{BX}{QX}$ and $\dfrac{PX}{AX}$?

Why does $AX \cdot BX = PX \cdot QX$?

What does this say about the products of the lengths of the intervals on two intersecting chords?

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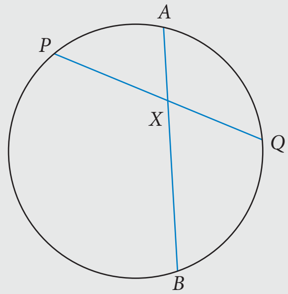
IMPORTANT

Theorem 6: Intersecting chords

When two chords of a circle intersect, the **product of the lengths** of the intervals on one chord **equals** the **product of the lengths** of the intervals on the other chord.

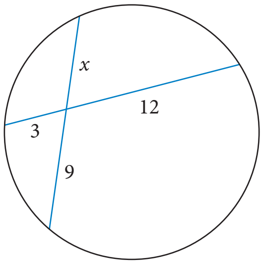
In the circle, $AX \cdot BX = PX \cdot QX$



Example 6

Find the value of the unknowns in the following circles.

a



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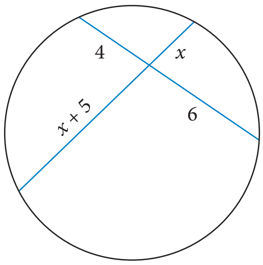


Solution

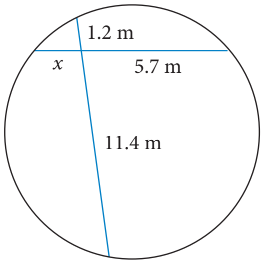
a Use the theorem.	$x \times 9 = 3 \times 12$ (Intersecting chords)
Solve the equation.	$9x = 36$ $x = 4$
b Use the theorem.	$x(x + 5) = 4 \times 6$ (Intersecting chords)
Expand brackets.	$x^2 + 5x = 24$
Factorise.	$x^2 + 5x - 24 = 0$
Find the values of x.	$(x - 3)(x + 8) = 0$ $x = 3$ or $x = -8$
A length cannot be negative.	So $x \neq -8$
Write the answer.	$x = 3$.

EXERCISE 6.05 Intersecting chords

Concepts and techniques

1 Example 6 Find the values of all pronumerals (O is the centre of each circle) in the following circles.

a



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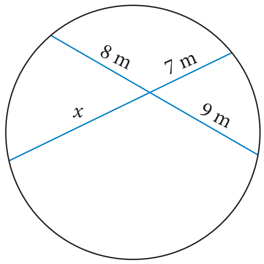

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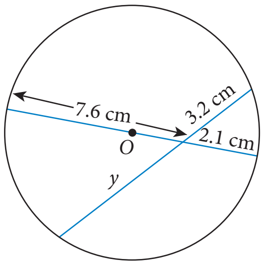

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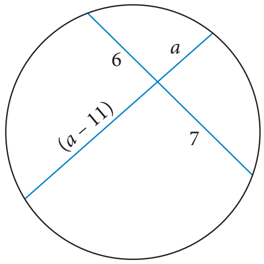
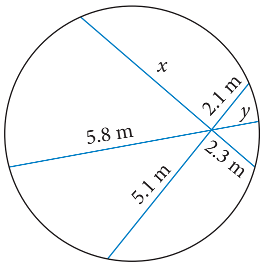


2 Find unknowns in the circles below correct to 1 decimal place if necessary.

a



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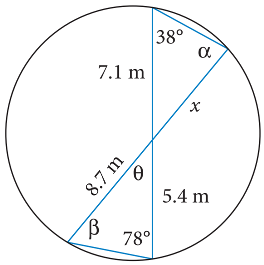
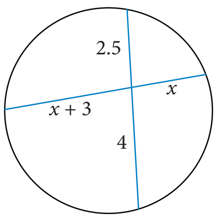


3 Find the pronumerals in the circles below.

a



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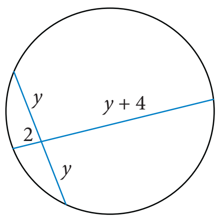

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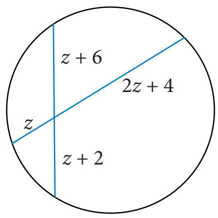
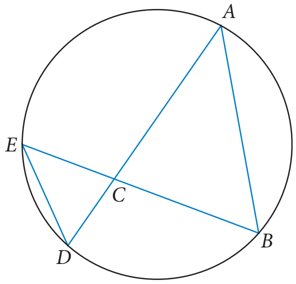


Reasoning and communication

4 Theorem proof

a Prove that triangles ABC and EDC are similar.

b Show that $AC \cdot CD = BC \cdot CE$.

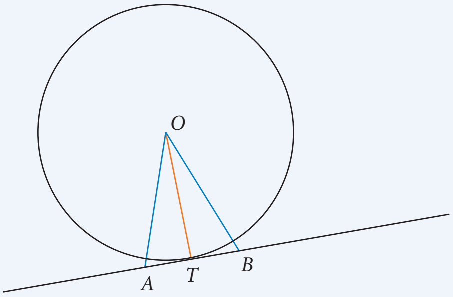


6.06 TANGENTS AND SECANTS

There are a number of theorems relating a tangent to a circle.

INVESTIGATION The relationship between the radius and a tangent at the point of contact

Consider a circle where T is the point of contact of a tangent with the circle. A and B are any other points on the tangent as shown. Join line segments AO, BO and TO.



TO is a radius because T lies on the circle.

Why are AO and BO longer than the radius TO?

By definition, what is the shortest distance from a point to a line?

What is the shortest distance from O to the tangent?

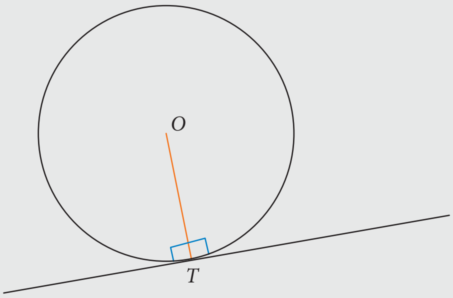
Why is TO perpendicular to the tangent?

What does this say about the angle between a tangent and a radius at the point of contact?

IMPORTANT

Theorem 7: Tangent and radius

A **tangent** drawn to a circle is **perpendicular** to the **radius** at the point of contact.

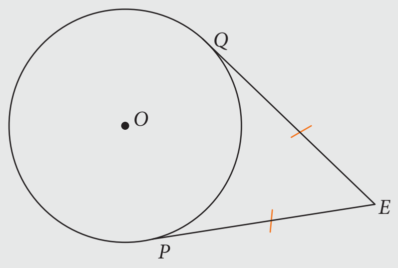


The next theorem relates the two tangents that can be drawn to a circle from an external point. You will prove it in Exercise **6.06**.

IMPORTANT

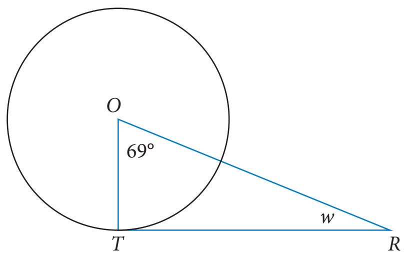
Theorem 8: External tangents

Tangents drawn to a circle from an **external** point are **equal** in length.



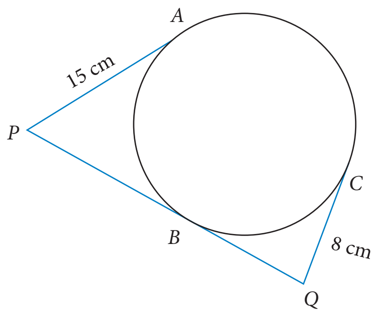
Example 7

a TR is a tangent.



Find the value of w.

b The points A, B and C lie on the circle.



Find the length of PQ.

Solution

a Find a second angle.

$\angle OTR = 90°$ (Tangent and radius)

$w + 69° = 90°$ ($\triangle OTR$ angle sum)

$w = 21°$

b Use the tangents.

$PB = 15$ cm (External tangents)

Use the other tangents.

$QB = 8$ cm (External tangents)

$\therefore PQ = 15 + 8$

$= 23$ cm

The next theorem is known as *the alternate segment* theorem.

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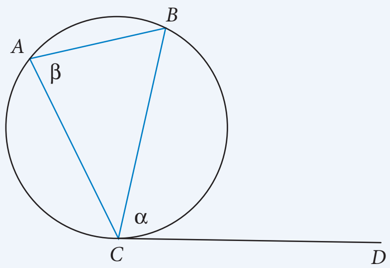
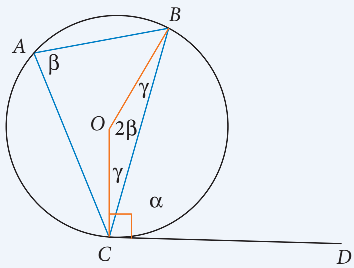
INVESTIGATION The alternate segment theorem

Consider the circle with chord CB and tangent CD meeting at C as shown.





Let $\angle BCD = \alpha$, $\angle BAC = \beta$. Join OC and OB. Let $\angle OCB = \gamma$.

What is the relationship between $\angle COB$ and $\angle BAC$?

What type of triangle is COB?

Explain why $\angle OBC = \gamma$.

What can you say about $\angle OCD$?

Explain why $\beta + \gamma = 90°$.

Explain why $\alpha + \gamma = 90°$

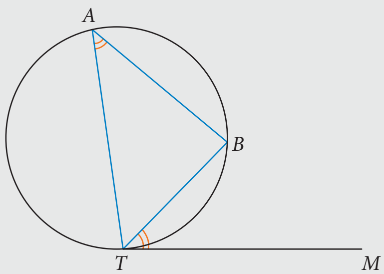
What does this say about α and β?

IMPORTANT

Theorem 9: Alternate segment

The angle between the tangent and a chord is equal to the angle in the alternate segment.

In the diagram, $\angle TAB = \angle BTM$.



A further theorem relates a tangent to a secant. You will prove this in Exercise **6.06**.

WS

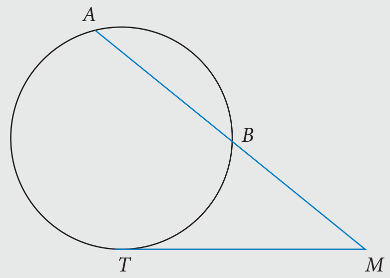
Intersecting chords, radii, tangents and secants

IMPORTANT

Theorem 10: Tangent and secant

When a **secant** and a **tangent** are drawn to a circle from an **external** point, the **square** of the length of the **tangent** equals the **product** of the **lengths** to the circle on the **secant**.

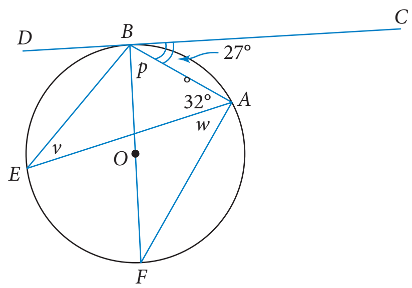
In the diagram, $AM \cdot BM = TM^2$



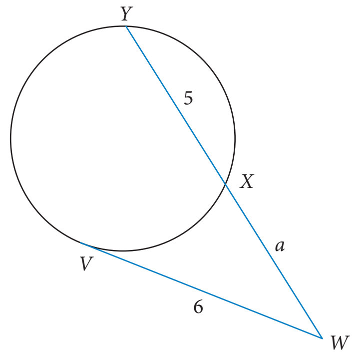
Example 8

Find the values of the unknowns.

a DC is a tangent.



b VW is a tangent.



Solution

a BF is a diameter. — $\angle BAF = 90°$ (Semicircle angle)

Use the right angle. — $w = 58°$ (Complementary)

BO is a radius. — $\angle CBO = 90°$ (Tangent and radius)

$p = 63°$ (Complementary)

Look at BC and $\triangle BEA$. — $v = 27°$ (Alternate segment)

b Use the theorem. — $a(a + 5) = 6^2$ (Tangent and secant)

Expand brackets. — $a^2 + 5a = 36$

Write in standard form. — $a^2 + 5a - 36 = 0$

Factorise. — $(a - 4)(a + 9) = 0$

$a = 4$ or $a = -9$

a is a length. — But $a \geq 0$, since a is a length, so $a \neq -9$.

Write the answer. — $a = 4$.

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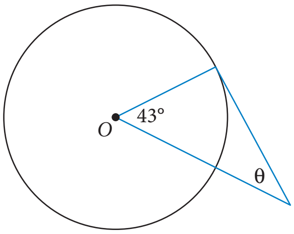
EXERCISE 6.06 Tangents and secants

Concepts and techniques

1 Example 7 Find the values of the pronumerals in the following circles.

a



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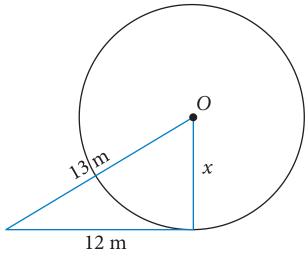

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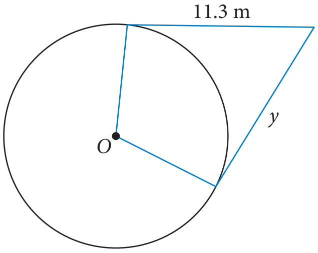

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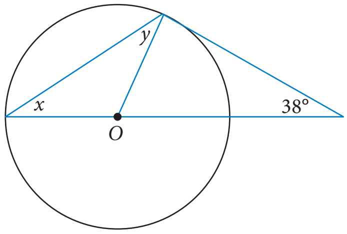

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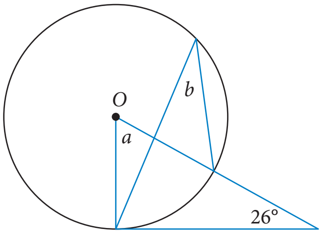

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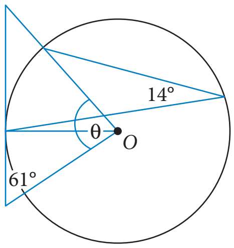

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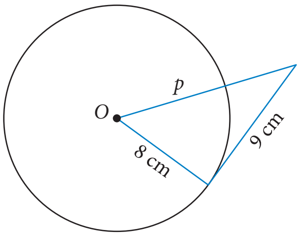

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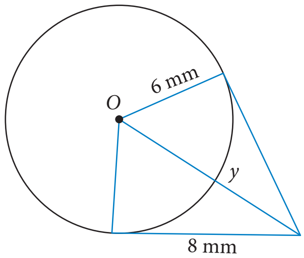

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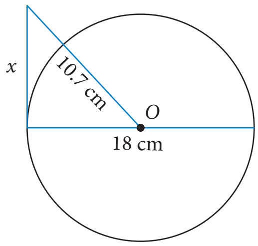

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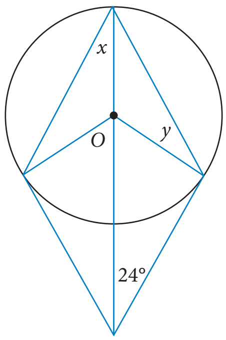
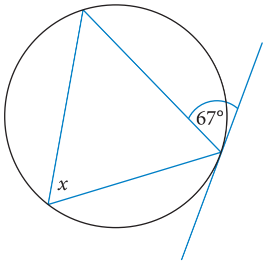


2 Example 8 Find the values of all pronumerals (O is the centre of each circle; all external lines are tangents) in the following circles.

a



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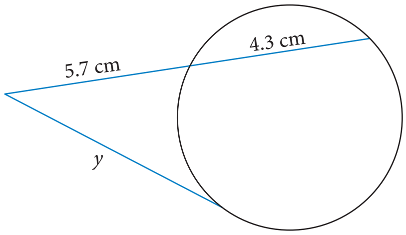

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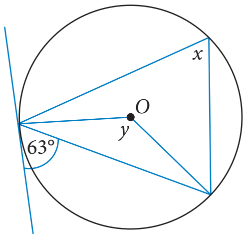

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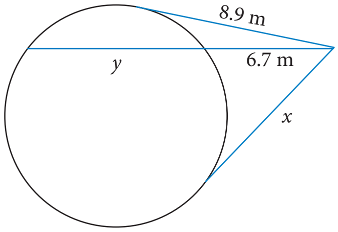

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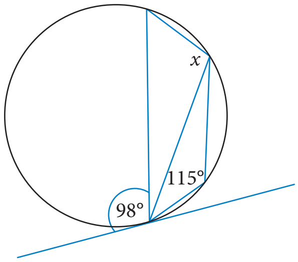

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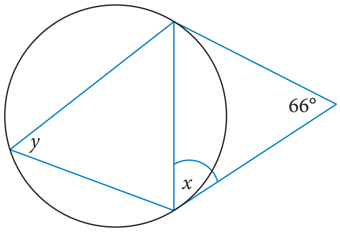

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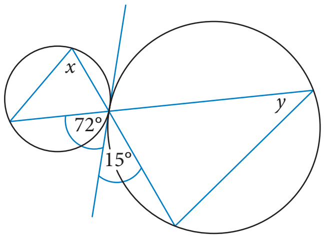

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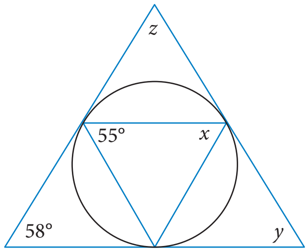
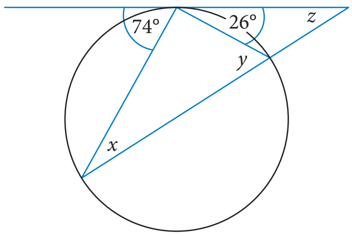


3 Find the values of all pronumerals in the following circles.

a



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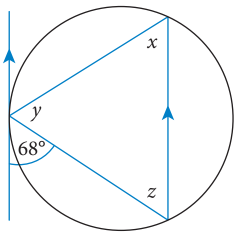

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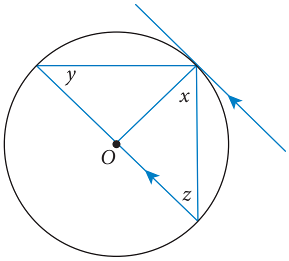

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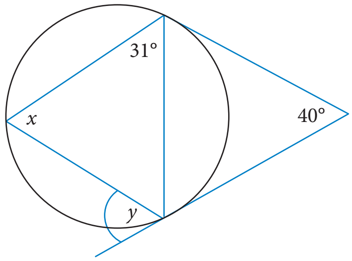

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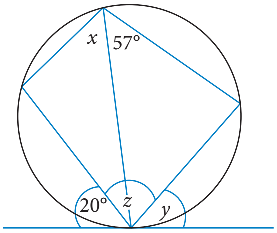

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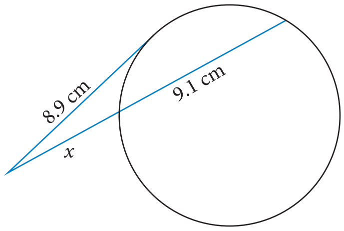

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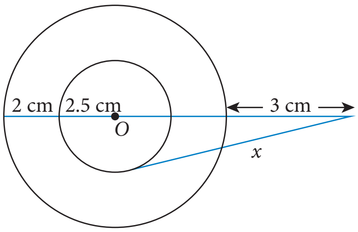

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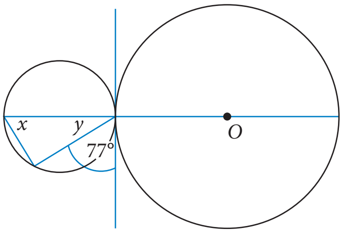

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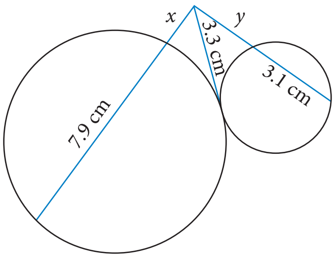

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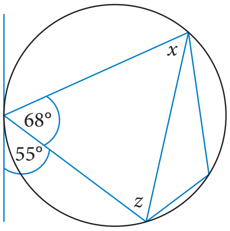
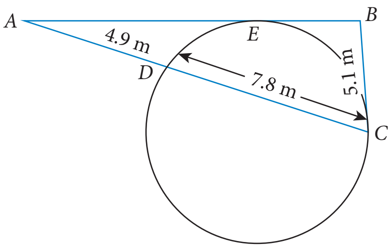


Reasoning and communication

4 Find AB, given $AD = 4.9$ m, $BC = 5.1$ m and $CD = 7.8$ m.



5 [Proof of Theorem 8]

a Construct a circle with tangents EQ and EP, where E is an external point and Q and P are the points of contact of the tangents with the circle.

b Join OP, OQ and OE.

c Use a congruence test to prove that $\triangle POE$ and $\triangle QOE$ are congruent.

d Explain why PE and QE are equal in length.

e Set out a proof that shows that tangents drawn to a circle from an external point are equal in length.

6 [Proof of Theorem 10]

a Construct a circle with tangent TM meeting the circle at T. Mark a point A on the circumference and join A to M, meeting the circle at B.

b Join AT and TB.

c Use a similarity proof to show that $\triangle TAM \parallel\mid \triangle BTM$.

d Which test did you use?

e Write a statement about the matching sides.

f Explain why $\dfrac{AM}{TM} = \dfrac{TM}{BM}$.

g Is it true that $AM \cdot BM = TM^2$?

h Set out a proof that shows that the square of the length of the tangent to a circle is equal to the product of the lengths to the circle on the secant.

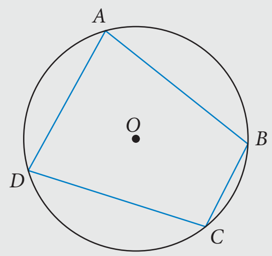
6.07 FIGURES IN CIRCLES

You saw in Section **6.03** that the angle in a semicircle is a right angle. This theorem relates to a right-angled triangle inscribed in a circle. There are also theorems related to quadrilaterals inscribed in circles. These are called *cyclic quadrilaterals.*

IMPORTANT

A **cyclic quadrilateral** is a quadrilateral whose four vertices lie on the circumference of a circle.

In the diagram, $ABCD$ is a cyclic quadrilateral.



INVESTIGATION Opposite angles in a cyclic quadrilateral

Consider the circle with cyclic quadrilateral *ABCD* as shown.
Join *OB* and *OD*.

Let $\angle DAB = \alpha$ and $\angle DCB = \beta$.

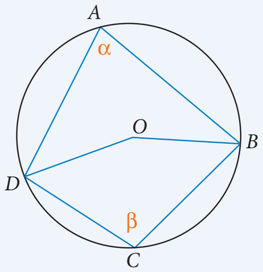
What is the size of obtuse $\angle DOB$ standing on arc *DCB*?

What is the size of reflex $\angle DOB$ standing on arc *DAB*?

Explain why $2\alpha + 2\beta = 360°$.

What does this say about the sum $\alpha + \beta$?

What does this say about the opposite angles in a cyclic quadrilateral?

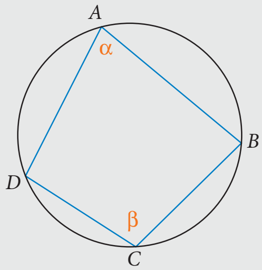


IMPORTANT

Theorem 11: Opposite angles of a cyclic quadrilateral

The **opposite angles** of a cyclic quadrilateral are **supplementary**.

In the diagram, $\alpha + \beta = 180°$.

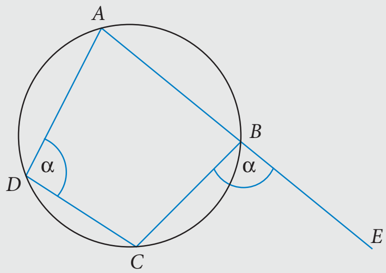


The next theorem will be proved in Exercise **6.07**.

IMPORTANT

Theorem 12: Exterior angle of a cyclic quadrilateral

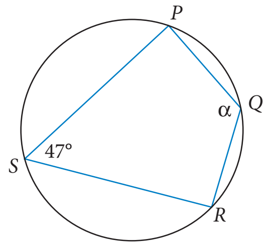
The **exterior angle** of a cyclic quadrilateral is **equal** to the **interior opposite angle**.



Example 9

Find the value of the unknown angles in the circles below.

a



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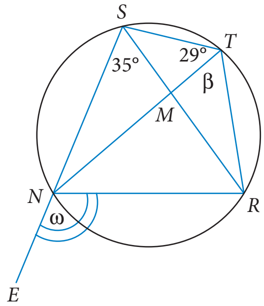


Solution

a *PQRS* is a cyclic quadrilateral.

$\alpha = 180° - 47°$ (Opposite angles of cyclic quadrilateral)

$= 133°$

b β and 35° are both on *NR*.

$\beta = 35°$ (Same chord)

Find the opposite angle.

$\angle STR = 35° + 29° = 64°$

Find the exterior angle.

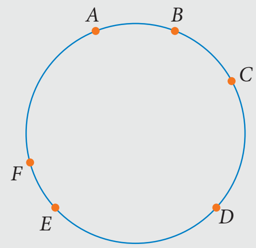
$\omega = 64°$ (Exterior angle of cyclic quadrilateral)

It is possible to draw a circle through any three non-collinear points. Special conditions must apply, however, for four or more non-collinear points to lie on a circle. These conditions relate to the converses of Theorems 11 and 12. The converses of Theorems 11 and 12 are also true.

IMPORTANT

Four or more points are called ***concyclic*** if they all lie on the same circle.

In the diagram, *A*, *B*, *C*, *D*, *E*, *F* are concyclic.



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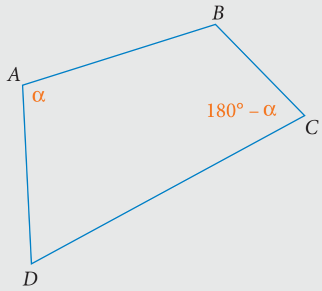
IMPORTANT

Converse of Theorem 11: Opposite angles of a cyclic quadrilateral

If the opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.

In the diagram, *ABCD* is a cyclic quadrilateral.
The four vertices *A*, *B*, *C*, *D* are concyclic.

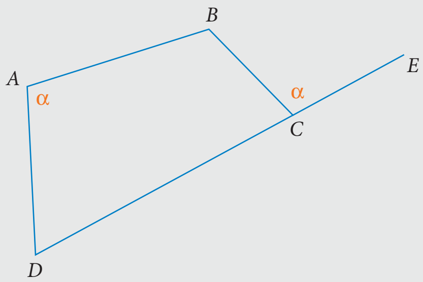


IMPORTANT

Converse of Theorem 12: Exterior angle of a cyclic quadrilateral

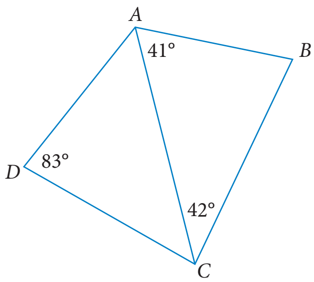
If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.

In the diagram, *ABCD* is a cyclic quadrilateral.



Example 10

Prove that the points *A*, *B*, *C* and *D* form a cyclic quadrilateral.



Solution

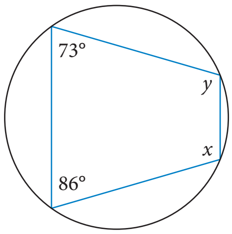
Find $\angle ABC$.	$\angle ABC = 180° - (41 + 42)°$ ($\triangle ABC$ angle sum) $= 97°$
Show the opposite angles are supplementary.	Now $\angle ADC + \angle ABC = 83° + 97°$ $= 180°$
Write the conclusion.	*ABCD* is a cyclic quadrilateral as its opposite angles are supplementary.

EXERCISE 6.07 Figures in circles

Concepts and techniques

1 Example 9 Find the values of all pronumerals in the following circles.

a



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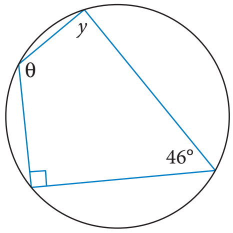

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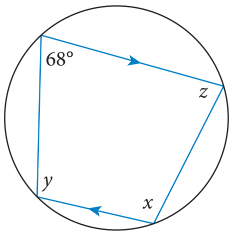

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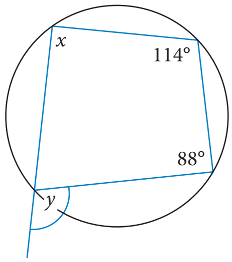

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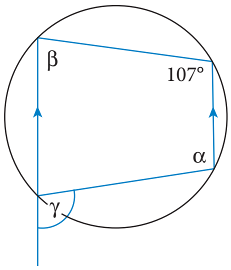

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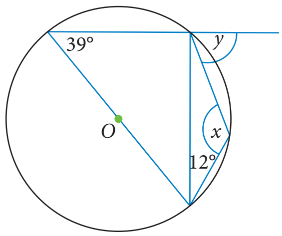

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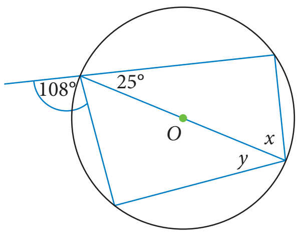

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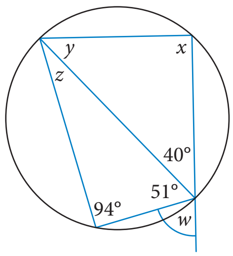

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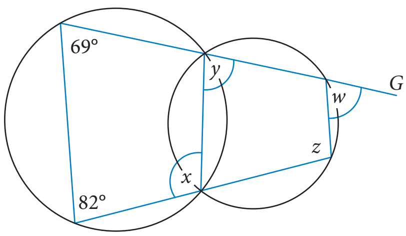

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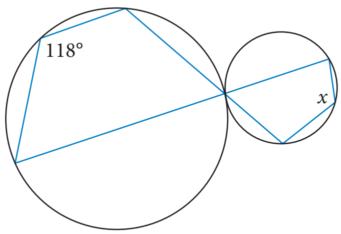

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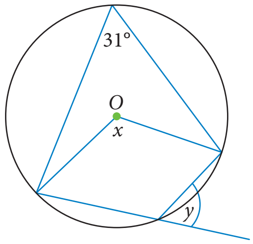
2 Find the values of all pronumerals in the following circles.

a



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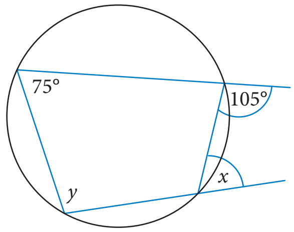

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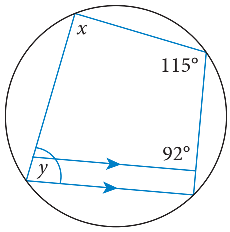

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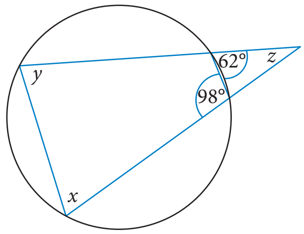

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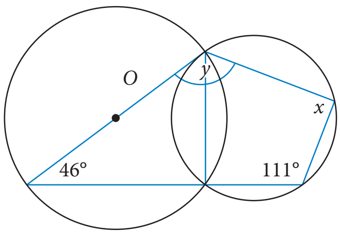

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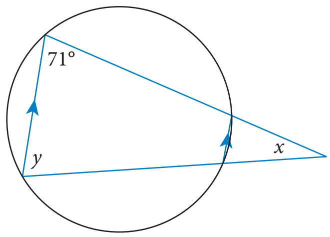

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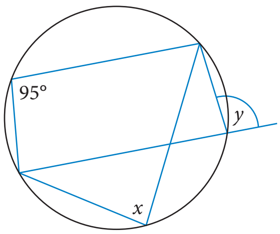

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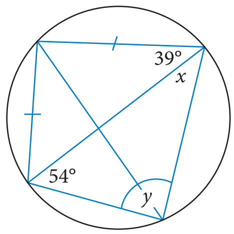

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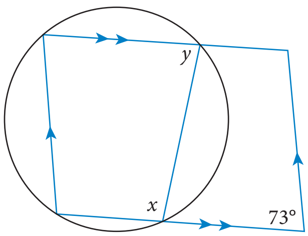

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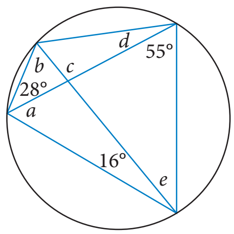
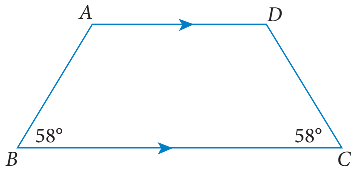


Reasoning and communication

3 Example 10 Show that $ABCD$ is a cyclic quadrilateral.

a



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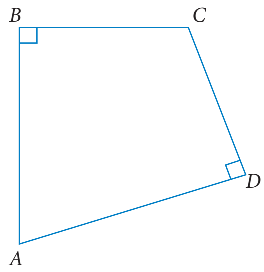

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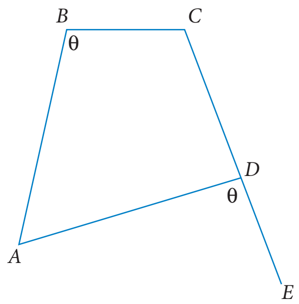

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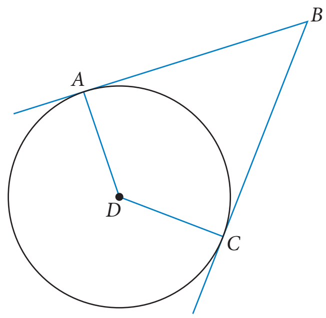


4 [Proof of Theorem 12]

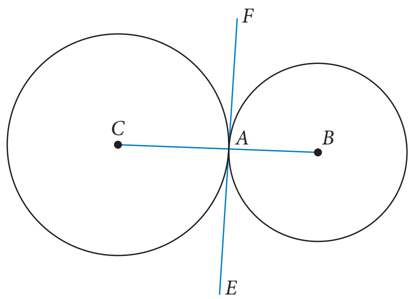
- a Construct a circle with cyclic quadrilateral $ABCD$. Produce AB to E.
- b What angle is supplementary with $\angle ADC$?
- c What angle is supplementary with $\angle CBE$?
- d What can you say about $\angle ADC$ and $\angle CBE$?
- e Set out a formal proof to show that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

6.08 MIXED CIRCLE PROBLEMS

You can use several theorems to find unknown angles and lengths.

Example 11

Prove that the centres and point of contact of two circles that just touch are collinear.



Prove that C, A, B are collinear.

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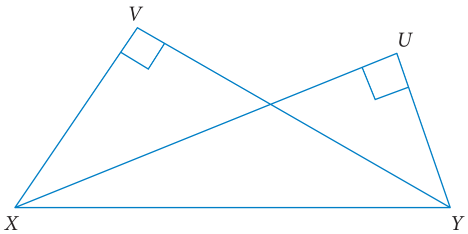
Solution

Find part of $\angle CAB$.	$\angle FAC = 90°$ (Tangent and radius)
Find the other part.	$\angle FAB = 90°$ (Tangent and radius)
Find the whole angle.	$\angle CAB = \angle FAB + \angle FAC = 180°$
Use $\angle CAB$.	CAB is a straight line
Write the conclusion.	C, A, B are collinear. QED

Example 12

Prove that X, Y, U, V are concyclic.



Solution

Consider XUY.	XY is a diameter on a circle through X, Y and U (Semicircle converse)
Consider XVY.	XY is a diameter on a circle through X, Y and V (Semicircle converse)
Use the common diameter.	The circles must be the same.
Write the conclusion.	X, Y, U, V are concyclic. QED

INVESTIGATION Circle geometry in real-life

National Library of Australia (nla.pic-an7650469)

In 1957, Jorn Utzon won the design competition for Sydney's new opera house which was to be built on Bennelong Point. He designed shells of 'undefined geometry' and it became known as one of the greatest engineering and architectural challenges to resolve the shells into a shape that could be built.

The breakthrough finally came with what is now known as *Utzon's Sphere.*

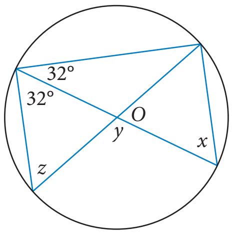
Find out about the design of the Sydney Opera House and the spherical shells that signify it. Can you build it?

EXERCISE 6.08 Mixed circle problems

Concepts and techniques

1 Find the values of all pronumerals (O is the centre of each circle) in the circles below.

a



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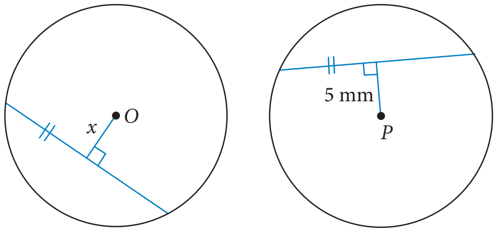
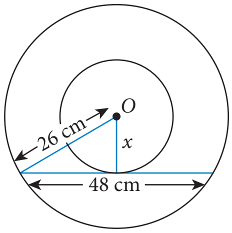


2 Find the values of all pronumerals (all external lines are tangents to the circles) in the following diagrams.

a



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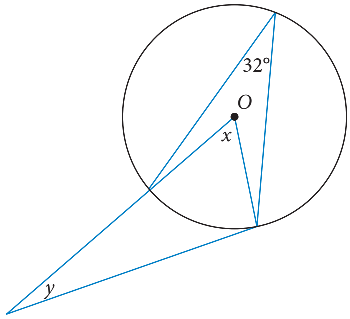

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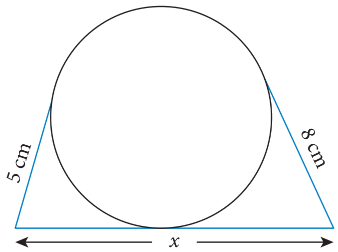

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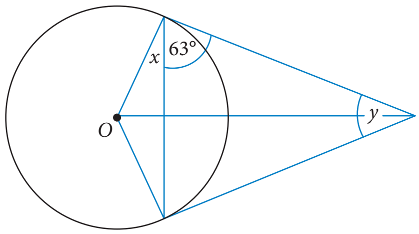

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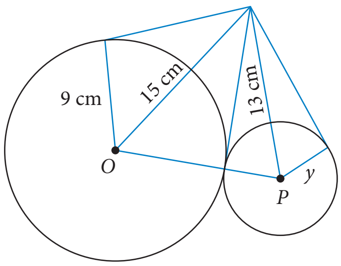

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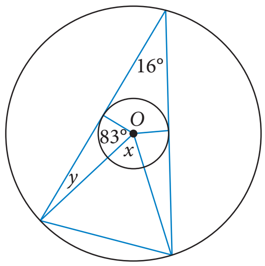

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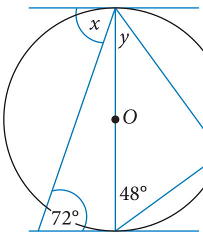

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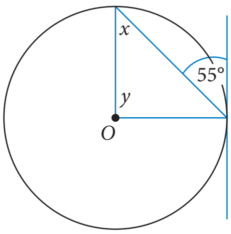

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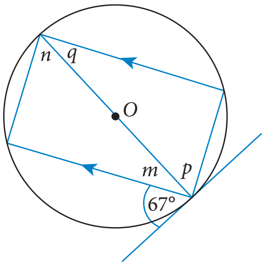

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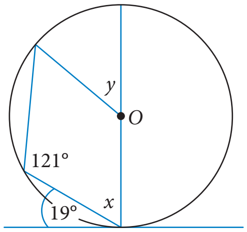
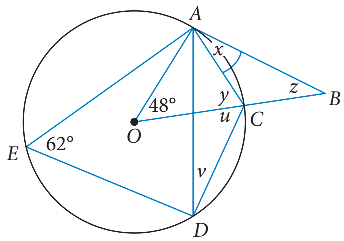
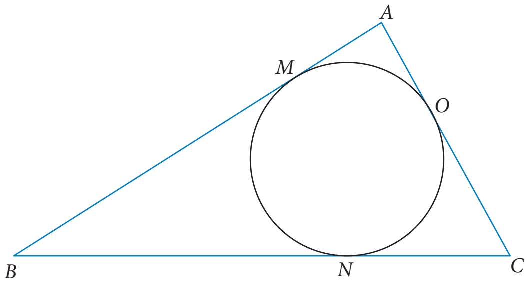


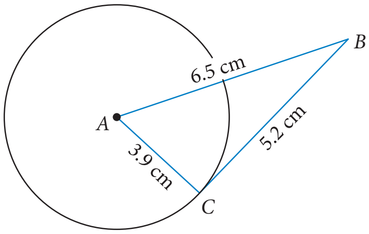
3 Find the values of all pronumerals, giving reasons for each step of your working (*O* is the centre of the circle, *AB* is a tangent).



4 *AB*, *BC* and *AC* are tangents, with *AB* = 24 cm, *BC* = 27 cm and *BM* = 15 cm. Find the length of *AC*.

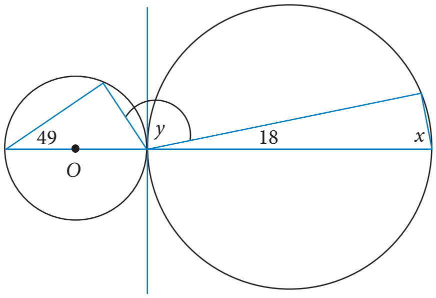


5 Example 11 *AB* = 6.5 m, *AC* = 3.9 m and *BC* = 5.2 m. Prove that *A* lies on a diameter of the circle, given that *BC* is a tangent to the circle.

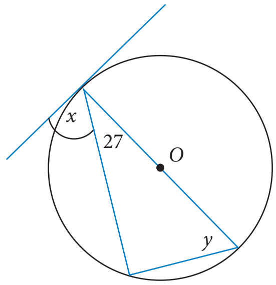


6 Find the values of all pronumerals (all external lines are tangents to the circles) in the circles below.

a

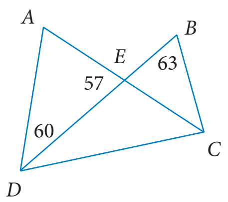


b



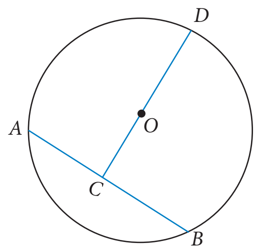
Reasoning and communication

7 Example 12 Prove that A, B, C, D are concyclic.

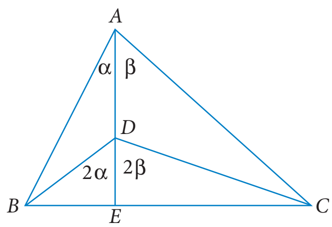


8 A circle with centre O has radius r and chord $AB = x$ with $\angle ACD = 90°$.

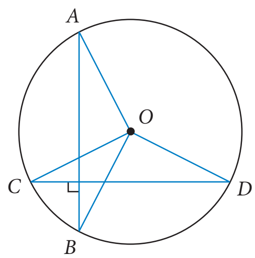
Show that $CD = \dfrac{2r + \sqrt{4r^2 - x^2}}{2}$.



9 In the triangles below, $\angle BDE = 2\angle BAD$ and $\angle CDE = 2\angle CAD$. Prove that a circle can be drawn through A, B and C with centre D.



10 Two chords AB and CD intersect at 90°. Prove, for obtuse $\angle AOD$, that $\angle AOD + \angle COB = 180°$, where O is the centre of the circle.



11 Prove that if a pair of opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.

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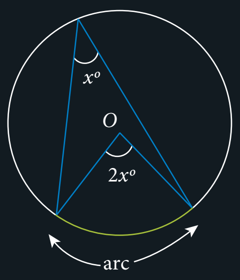
CHAPTER SUMMARY
CIRCLE GEOMETRY

6

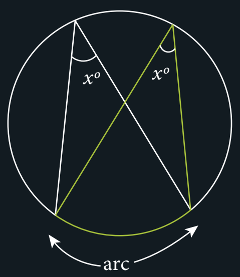
Theorem 1: Angles at the centre of a circle

- The angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc.



Theorem 2: Angles on the same arc

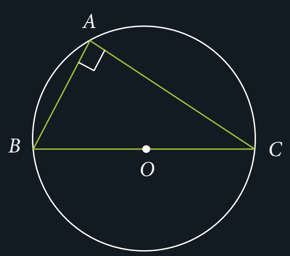
- Angles at the circumference of a circle subtended by the same arc are equal.



Theorem 3: Semicircle angle

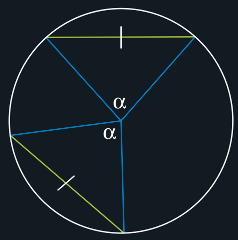
- An angle at the circumference subtended by a diameter is a right angle.

  Conversely, a chord that subtends a right angle at the circumference is a diameter.



Theorem 4: Centre angles on equal chords

- Equal chords of a circle subtend equal angles at the centre. Conversely, equal angles at the centre subtend equal chords.

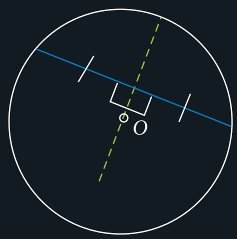


Corollary

Angles subtended at the centre or at the circumference by equal arcs are equal; conversely, equal angles at the centre subtend equal arcs.

Theorem 5: Chord bisector

- A radius that bisects a chord is perpendicular to the chord.

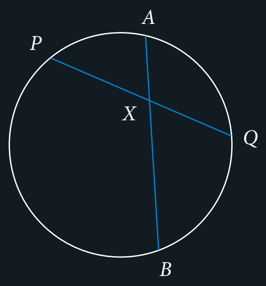


Corollaries are:

1. The perpendicular bisector of a chord passes through the centre of a circle.
2. The line passing through the centre of a circle perpendicular to a chord bisects the chord.
3. The perpendicular bisector of a chord bisects the angle at the centre of a circle subtended by the chord.
4. Equal chords are equidistant from the centre of a circle.

Theorem 6: Intersecting chords

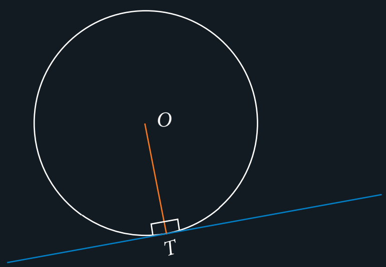
- When two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord.



In the circle, $AX \cdot BX = PX \cdot QX$

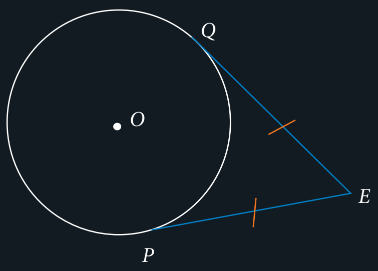
Theorem 7: Tangent and radius

- A tangent drawn to a circle is perpendicular to the radius at the point of contact.



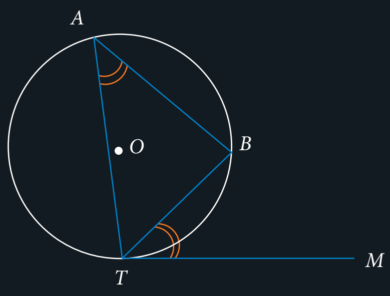
Theorem 8: External tangents

- Tangents drawn to a circle from an external point are equal in length.



Theorem 9: Alternate segment

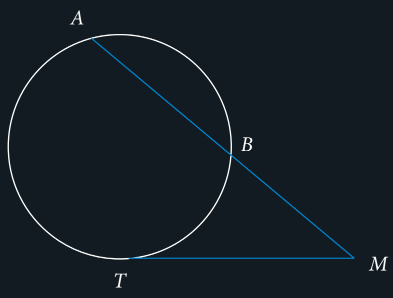
- The angle between the tangent and a chord is equal to the angle in the alternate segment.



In the diagram, $\angle TAB = \angle BTM$

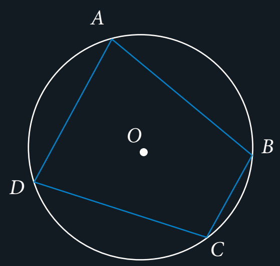
Theorem 10: Tangent and secant

- When a secant and a tangent are drawn to a circle from an external point, the square of the length of the tangent equals the product of the lengths to the circle on the secant.



In the diagram, $AM \cdot BM = TM^2$

- A **cyclic quadrilateral** is a quadrilateral whose four vertices lie on the circumference of a circle.



In the diagram, $ABCD$ is a cyclic quadrilateral.

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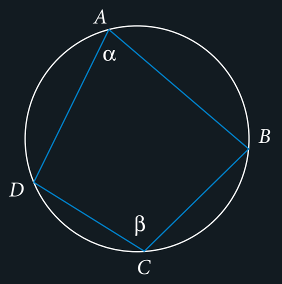
Theorem 11: Opposite angles in a cyclic quadrilateral

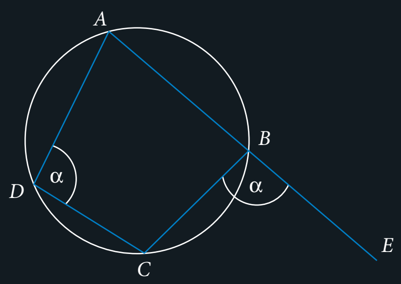
- The opposite angles of a cyclic quadrilateral are supplementary.



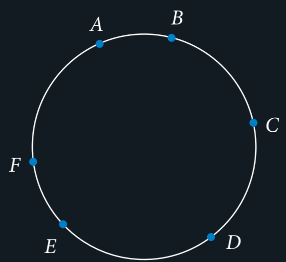
In the diagram, $\alpha + \beta = 180°$

Theorem 12: Exterior angle of a cyclic quadrilateral

- The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



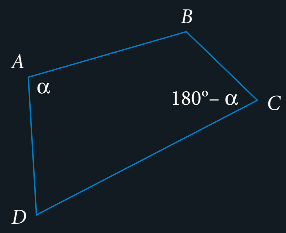
- Four or more points are called **concyclic** if they all lie on the same circle.



In the diagram, A, B, C, D, E, F are concyclic.

Converse of Theorem 11

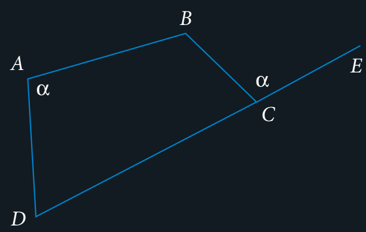
- If the opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.



In the diagram, $ABCD$ is a cyclic quadrilateral. The four vertices A, B, C, D are concyclic.

Converse of Theorem 12

- If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.



In the diagram, $ABCD$ is a cyclic quadrilateral.

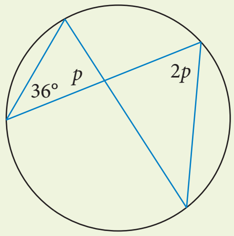
6

CHAPTER REVIEW
CIRCLE GEOMETRY

Multiple choice

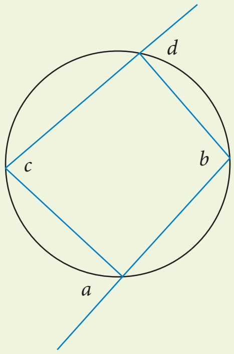
1 Example 2 In the diagram, $p = ...$

A $18°$ B $36°$
C $48°$ D $60°$
E $72°$



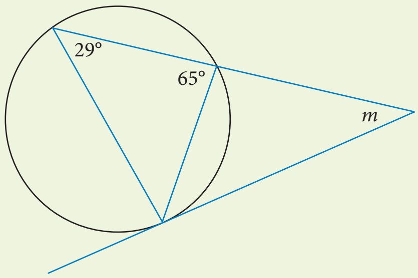
2 Example 9 In the diagram which statement is always true?

A $d = b$ B $b = c$
C $b = c$ D $d + c = 180°$
E $b + c = 180°$



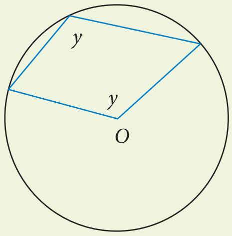
3 Example 7 In the diagram $m = ...$

A $29°$ B $36°$
C $61°$ D $86°$
E $94°$



4 Example 1 In the diagram $y = ...$

A $90°$ B $100°$
C $110°$ D $120°$
E $150°$



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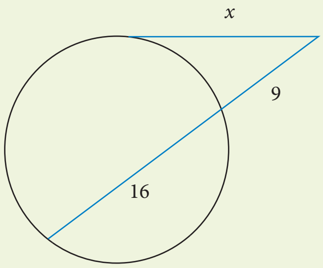
5 Example 8 In the diagram $x = \ldots$

A 5
B 9
C 10
D 12
E 15



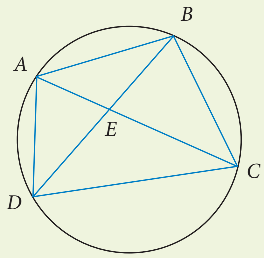
6 Example 2 In the diagram, consider the statements

1 $\triangle AED \mid\mid\mid \triangle BEC$

2 $\triangle ACD \mid\mid\mid \triangle BDC$

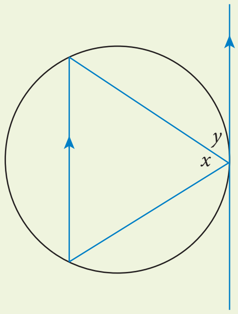
Which of the following are true?

A both 1 and 2
B neither 1 nor 2
C only 1
D only 2
E It is impossible to determine



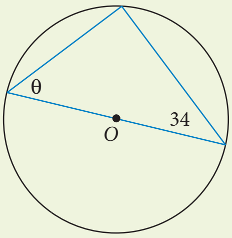
7 Example 8 In the diagram,

A $x = y$
B $x = 2y$
C $x = \dfrac{180° - y}{2}$
D $x = 180° - 2y$
E $x = 120° - 2y$

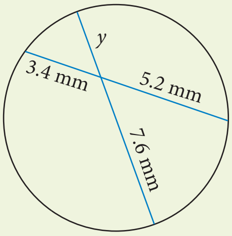


Short answer

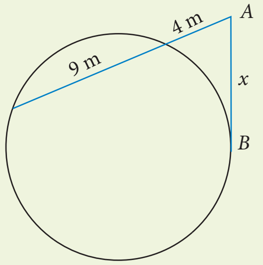
8 Example 3 O is the centre of the circle. Evaluate θ.



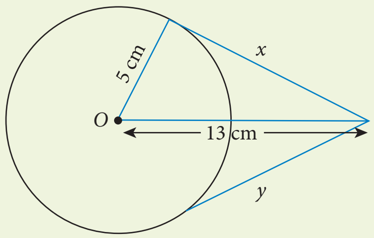
9 Example 6 Evaluate y, correct to 1 decimal place.



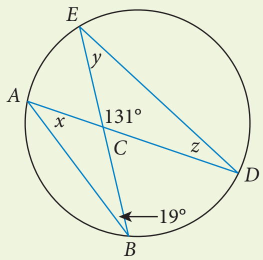
10 Example 8 AB is a tangent to the circle. Find the value of x correct to 1 decimal place.



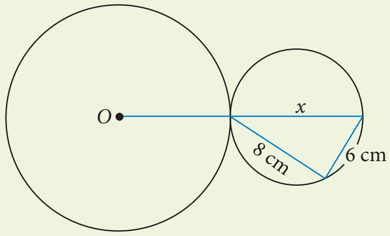
11 Example 7 O is the centre of the circle. Find the length of tangents x and y.



12 Example 2 Evaluate x, y and z, giving reasons for each step of your working.



13 Example 3 O is the centre of the larger circle. Find the value of x.



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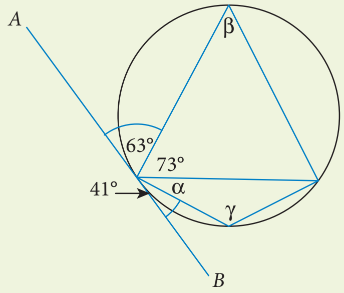
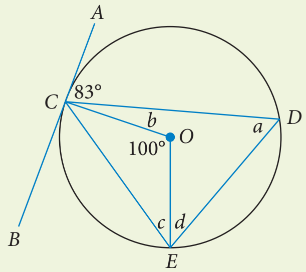
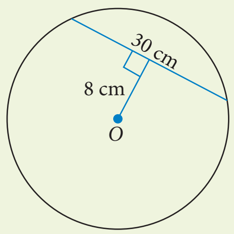
14 Examples 8,9 AB is a tangent to the circle. Evaluate α, β and γ.



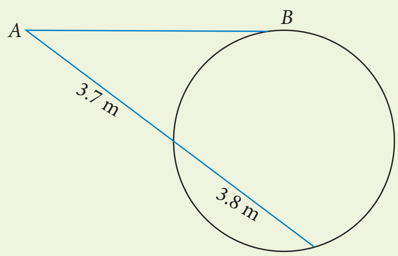
15 Examples 1, 7 O is the centre of the circle, and AB is a tangent. Evaluate a, b, c and d, giving reasons for each step of your working.



16 Example 5 Find the length of the radius of the circle. O is the centre.



17 Example 8 Find the length of tangent AB.



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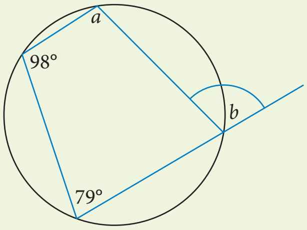
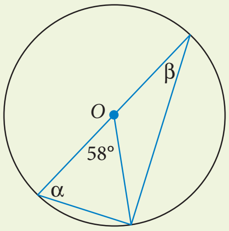
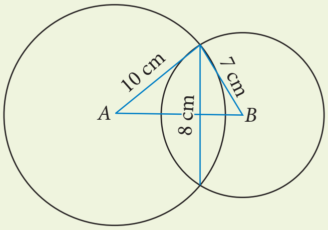
18 Example 9 Evaluate a and b.



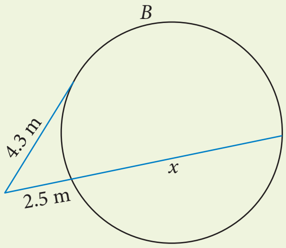
19 Examples 1, 3 O is the centre of the circle. Find the value of α and β.



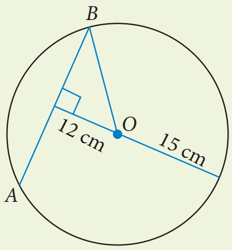
20 Example 11 Calculate the length of AB, correct to 3 significant figures, given that A and B are the centres of the circles.



21 Example 8 Find the value of x correct to 1 decimal place.



22 Example 5 Find the length of AB.



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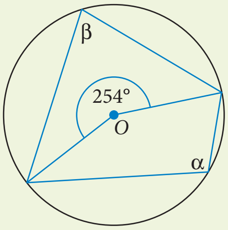
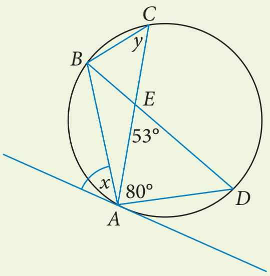
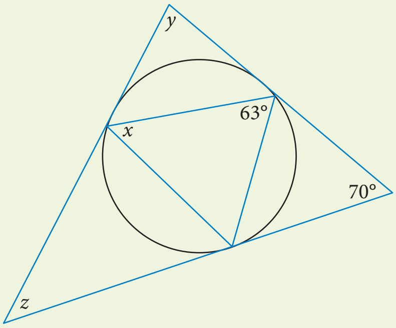
23 Example 9 Evaluate α and β.



24 Examples 2, 8 Evaluate x and y, giving reasons for your working.

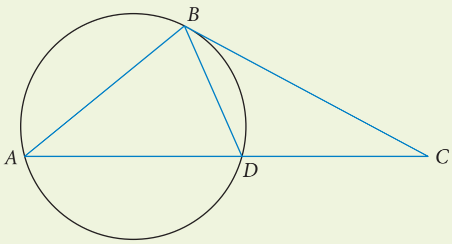


25 Example 7 Evaluate x, y and z.



Application

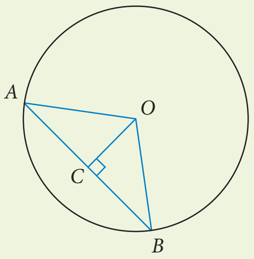
26 Prove that $\triangle BCD$ is similar to $\triangle ABC$.



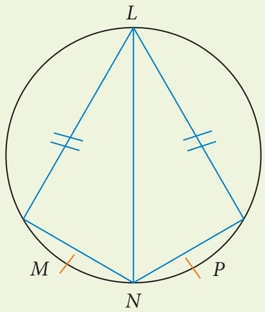
27 O is the centre of the circle.

a Prove that $\triangle OAC$ and $\triangle OBC$ are congruent.

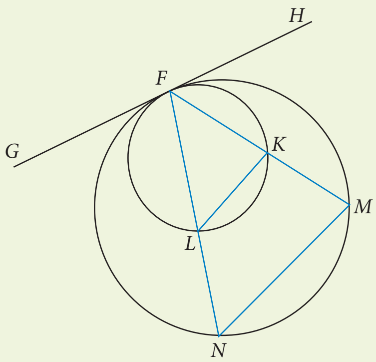
b Show that OC bisects AB.



28 In the circle, arc MN = arc PN, $LM = LP$. Prove that the quadrilateral $LMNP$ is a kite.

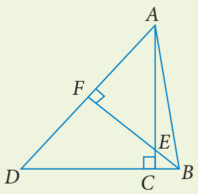


29 GFH is a common tangent to both circles. Prove that $LK \parallel NM$.



Practice quiz

30 In the triangle DAB, AC and BF are altitudes. Prove that D, C, E, F are concyclic.



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7

TERMINOLOGY

abundant number
amicable number
closure law
composite number
continued fraction
contradiction
contraposition
counter example
deficient number
double primes
integer
irrational number
mathematical induction
Mersenne prime
natural number
perfect number
proper divisors
rational number
real number
recurring decimal
surd
terminating decimal
transcendental number
twin primes

REAL AND COMPLEX NUMBERS

REAL NUMBERS AND PROOFS

Prior learning

PROOFS INVOLVING NUMBERS

- prove simple results involving numbers. (ACMSM061)

RATIONAL AND IRRATIONAL NUMBERS

- express rational numbers as terminating or eventually recurring decimals and vice versa (ACMSM062)
- prove irrationality by contradiction for numbers such as $\sqrt{2}$ and $\log_2(5)$. (ACMSM063)

AN INTRODUCTION TO PROOF BY MATHEMATICAL INDUCTION

- understand the nature of inductive proof including the 'initial statement' and inductive step (ACMSM064)
- prove results for sums such as $1+4+9+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for any positive integer n (ACMSM065)
- prove divisibility results, such as $3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n. (ACMSM066)

7.01 INTEGERS AND SUBSETS

Numbers were first used for counting. These counting numbers are now referred to as the **natural numbers**. They have the symbol $\boldsymbol{N}$. So the set is $\boldsymbol{N} = \{1, 2, 3, 4, 5, ...\}$. They are also called the positive **integers**, $\boldsymbol{Z}^+$. The set of integers, $\boldsymbol{Z}$ (or $\boldsymbol{J}$), contains the natural numbers, zero and the negative integers, $\boldsymbol{Z}^-$. These integers make the set $\boldsymbol{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

The integers are divided into **even** and **odd numbers**. An even number is an integer that is divisible by 2. It is of the form $n = 2k$, where k is an integer.

The even numbers = $\{..., -4, -2, 0, 2, 4, ...\} = \{n: n = 2k, k \in \boldsymbol{Z}\}$.

An odd number is an integer that is not divisible by 2. The odd numbers are of the form $n = 2k + 1$, where k is an integer. When you divide an odd number by 2 you always get a remainder of 1.

The odd numbers = $\{..., -3, -1, 1, 3, ...\} = \{n: n = 2k + 1, k \in \boldsymbol{Z}\}$. Some people prefer the equivalent $\{n: n = 2k - 1, k \in \boldsymbol{Z}\}$.

You can see patterns when operating with even and odd numbers. Some are shown in the table below.

Addition and Subtraction	Multiplication
even number ± even number = even number example: $2 + 6 = 8$	even number × even number = even number example: $4 \times 12 = 48$
odd number ± odd number = even number example: $3 + 9 = 12$	odd number × odd number = odd number example: $9 \times 11 = 99$
even number ± odd number = odd number example: $12 - 3 = 9$	even number × odd number = even number example: $10 \times 9 = 90$

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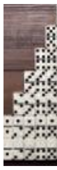
Example 1

a Show that when an odd number is subtracted from an even number, the result is an odd number.

b Show that when two odd numbers are multiplied together, the result is an odd number.

Solution

a	Write an odd number.	Let $x = 2a + 1$, where $a \in \mathbf{Z}$
	Write an even number.	Let $y = 2b$, where $b \in \mathbf{Z}$
	Subtract the numbers.	$y - x$
	Substitute.	$= 2b - (2a + 1)$
	Remove the brackets.	$= 2b - 2a - 1$
	Factorise $2b - 2a$.	$= 2(b - a) - 1$
	State the nature of $b - a$.	$b - a$ is an integer as a and b are integers
	State the nature of $2(b - a)$.	$2(b - a)$ is even as it is $2c$, where $c \in \mathbf{Z}$
	Write the conclusion.	Hence $2(b - a) - 1$ is odd. **QED**
b	Represent one of the odd numbers as $2a + 1$ and the other as $2b + 1$, where a and b are integers.	
	Multiply the two numbers together.	$(2a + 1)(2b + 1)$
	Expand the brackets.	$= 4ab + 2a + 2b + 1$
	Factorise $4ab + 2a + 2b$.	$= 2(2ab + a + b) + 1$
	Show that the result is odd.	$2ab + a + b$ is an integer as a and b are integers. $2(2ab + a + b)$ is even. Hence $2(2ab + a + b) + 1$ is odd. **QED**



A **perfect number** is a positive integer that is equal to the sum of its positive **proper divisors**. Proper divisors are the factors of the number, excluding the number itself. 6 and 28 are perfect numbers because $1 + 2 + 3 = 6$ and $1 + 2 + 4 + 7 + 14 = 28$. The Greek mathematician Nicomachus described **deficient numbers** as numbers where the sum of the proper divisors is less than the number; for example, 8 is deficient as $1 + 2 + 4 < 8$. He called numbers where the sum of the proper divisors is greater than the number **abundant numbers**. 12 is the first abundant number. The sum of its proper factors is $1 + 2 + 3 + 4 + 6 > 12$.

Amicable numbers are pairs of numbers where each one is the sum of the proper divisors of the other. The first pair of amicable numbers is 220 and 284.

The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110.

$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$

The proper divisors of 284 are 1, 2, 4, 71 and 142.

$1 + 2 + 4 + 71 + 142 = 220$

Their divisors add to each other, so they *are* amicable numbers.

Proper divisors are also called **proper factors**. The smallest prime number is 2 and it is the only even prime number. There are 25 prime numbers that are less than 100: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97.

IMPORTANT

A **prime number** is a positive integer that has exactly two factors, one and itself. A **composite number** is a positive integer that has more than two factors.

Example 2

a Show that 496 is a perfect number.

b Write 496 in the form $2^{p-1}(2^p - 1)$, where p is a prime number.

c Is $2^6(2^7 - 1)$ a perfect number?

Solution

a List the proper divisors of 496.	1, 2, 4, 8, 16, 31, 62, 124, 248
Add the proper divisors.	$1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$
Write the conclusion.	Hence 496 is a perfect number.
b Test some prime numbers. Try $p = 5$. Note that $2^2(2^3 - 1) = 28$ and $2^1(2^2 - 1) = 6$.	$2^4(2^5 - 1) = 496$
c List the proper divisors of $2^6(2^7 - 1)$.	$2^6(2^7 - 1) = 8128$ 1, 2, 4, 8, 16, 32, 64, 127, 254, 508, 1016, 2032, 4064
Add the proper divisors.	$1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508$ $+ 1016 + 2032 + 4064 = 8128$
Write the answer.	$2^6(2^7 - 1)$ is a perfect number.

Notice in Example **2** that the proper factors of 496 are 1, 2, …, 2^4 and 31, 2×31, …, $2^3 \times 31$.

To check if a number is prime, you only need to check if the number is divisible by the prime numbers less than or equal to its square root. For example, to check if 59 is prime, you only need to check the prime numbers up to $\sqrt{59} \approx 7.68$. Since 59 is not divisible by 2, 3, 5 or 7, it must be prime.

Proof

Suppose that the number's square root is a whole number. It follows immediately that the number is not prime. Consider cases where the square root is not a whole number.

Suppose there was a number whose only factors were bigger than the square root.

Let the number be n and the factor larger than the square root be f.

Since f is a factor, $fp = n$ for some positive integer p.

It follows that $p = \frac{n}{f}$.

But $f > \sqrt{n}$, so $\frac{1}{f} < \frac{1}{\sqrt{n}}$ and thus $p = \frac{n}{f} < \frac{n}{\sqrt{n}} = \sqrt{n}$.

Thus, the number has a factor less than $\sqrt{n}$.

This contradicts the assumption. Thus there is no number whose only factors are bigger than its square root.

We've also shown that any factor larger than $\sqrt{n}$ has a co-factor less than $\sqrt{n}$.

Thus, it is sufficient to check for factors that are less than or equal to the square root. **QED**

As an example, consider $n = 103$, $\sqrt{103} \approx 10.1$. You only need to check if 103 is divisible by 2, 3, 5 and 7. 103 is not divisible by 2 as the last digit is not even, nor 3 as the sum of the digits, which is 4, is not divisible by 3, nor 5 as the last digit is neither 0 or 5 and not 7. Hence 103 is prime.

The **Sieve of Eratosthenes** is a method for finding prime numbers up to a particular number. The table below shows how it works for 50. Circle the lowest prime number, 2 and cross out all the multiples of that number, 4, 6, 8, … Select the next lowest prime number, 3 and cross out all the multiples of that number that have not already been crossed out, which means you can start at $3^2 = 9$ as the smaller multiples have been crossed out. Continue the process until you have checked the prime numbers less than or equal to $\sqrt{n}$. In this case $\sqrt{50} \approx 7.07$. So cross out the multiples of the prime numbers up to and including 7. The rest of the unmarked numbers should be prime.

	2	3	$\cancel{4}$	5	$\cancel{6}$	7	$\cancel{8}$	$\cancel{9}$	$\cancel{10}$
11	$\cancel{12}$	13	$\cancel{14}$	$\cancel{15}$	$\cancel{16}$	17	$\cancel{18}$	19	$\cancel{20}$
$\cancel{21}$	$\cancel{22}$	23	$\cancel{24}$	$\cancel{25}$	$\cancel{26}$	$\cancel{27}$	$\cancel{28}$	29	$\cancel{30}$
31	$\cancel{32}$	$\cancel{33}$	$\cancel{34}$	$\cancel{35}$	$\cancel{36}$	37	$\cancel{38}$	$\cancel{39}$	$\cancel{40}$
41	$\cancel{42}$	43	$\cancel{44}$	$\cancel{45}$	$\cancel{46}$	47	$\cancel{48}$	$\cancel{49}$	$\cancel{50}$

Hence 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47 are prime numbers.

There are many special types of prime numbers. **Double or twin primes**, for example, differ by two. Three and five are twin primes and five and seven are twin primes. Another type of prime number is a **Mersenne prime**. A Mersenne prime is a (prime) number that can be written in the form $2^p - 1$, where p is prime. For example, $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$ and $2^7 - 1 = 127$ are Mersenne primes, but $2^4 - 1 = 15$ is not as 4 is not a prime number.

Example 3

Is $2^{11} - 1 = 2047$ a Mersenne prime?

Solution

Find $\sqrt{2047}$. $\sqrt{2047} \approx 45$

Write the prime numbers less than $\sqrt{2047}$. Check 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 and 43

Perform the divisibility tests. 2047 is not divisible by 2 as the last digit is not even, nor 3 as the sum of the digits, 13, is not divisible by 3. It is not divisible by 5 as the last digit is not 0 or 5.

Check the rest on a calculator.

For example, $\frac{2047}{7} = 292.4\ldots$

It is not divisible by 7, 11, 13, 17 and 19 either.

$\frac{2047}{23} = 89$, so it is divisible by 23.

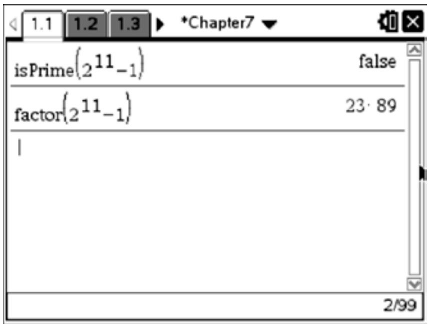
Write the answer.

2047 is not prime. Hence $2^{11} - 1$ is not a Mersenne prime.

You can easily check if a number is prime using technology.

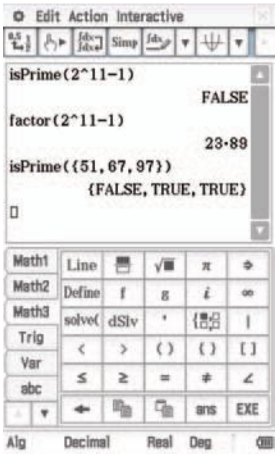
TI-Nspire CAS

Insert a calculator page, and type the commands directly or press [catalog] and select **isPrime** (. Complete the command and press [enter].
Use [menu] and select 3: Algebra and 2: Factor to find the factors.
isPrime tells you if a number is prime.



ClassPad

Use the Main $\sqrt{\alpha}$ application. Navigate to the list of functions (press [Keyboard] and tap [▾]) and find **isPrime(** in the list. The command **isPrime** tells you if a number is prime or not.
The command **factor** gives you the number as a product using its prime factors.
isPrime can be used to test more than one number at a time by entering the values inside of a pair of braces with each value separated from the previous one by a comma.



INVESTIGATION Playing with integers

Sieve of Eratosthenes

1 Finding primes

a Answer the following questions by downloading and using the Sieve of Eratosthenes at the site given using the web link.

Note: To run the demonstration you need to have Mathematica or install the free Mathematica player from Wolfram.

i How many prime numbers are less than 315?

ii What was the largest prime that needed to be tested?

iii What is the largest prime that needs to be tested to determine the number of primes less than 960?

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b Use a suitable CAS calculator, Mathematica or the website to answer the following questions.

- **i** List the primes less than 100.
- **ii** Find the 1000th smallest prime number.
- **iii** How many prime numbers are there between 7000 and 8000?
- **iv** Is 10 347 a prime number?

WolframAlpha

2 Perfect numbers and Mersenne primes

View the demonstration at the website.

Perfect number generator

a **i** How many deficient numbers are less than 6?

- **ii** How many abundant numbers are less than 28?
- **iii** Is 496 a perfect number?

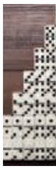
b Using a suitable CAS, Mathematica or the website:

- **i** show that 496 and 8128 are perfect numbers
- **ii** write 496 and 8128 as products of their prime factors

WolframAlpha

c Show, by summing the factors of a perfect number, that if a perfect number can be written in the form $2^{p-1} \times b$, where b is prime, then $b = 2^p - 1$.

A Mersenne prime is a prime number which can be written in the form $2^p - 1$, where p is a prime number.



d Use a suitable CAS application such as Mathematica to complete the following table.

p	Is $2^p - 1$ prime?	Is $2^{p-1}(2^p - 1)$ a Perfect number?
2	Yes	Yes
3		
5		
7		
11		
13		
17		
19		
23		
29		
31		

e **i** Is a perfect number ever a prime number? Explain.

- **ii** What conclusion can you draw from the table?

f How many perfect numbers were found by October 2009?

EXERCISE 7.01 Integers and subsets

Concepts and techniques

WS Proper divisors

1 -3 belongs to which one of the following sets?
A counting numbers B Z^+ C N D odd numbers
E even numbers

2 Example 1 a Show that when two odd numbers are added together, the result is an even number.
b Show that when an odd number is multiplied by an even number, the result is an even number.

3 Which one of the following is not a proper divisor of 312?
A 1 B 13 C 24 D 104 E 312

4 List the numbers less than 20 which are abundant.

5 Example 2 a Show that 33 550 336 is a perfect number.
b Write 33 550 336 in the form $2^{p-1}(2^p - 1)$, where p is a prime number.

6 Show that 1184 and 1210 are amicable numbers.

7 Is the set of prime numbers a subset of the set of odd numbers? Explain.

8 Use the Sieve of Eratosthenes to find the prime numbers less than 100.

9 Determine whether each of the following numbers is prime.
a 247 b 631

10 Example 3 Is $2^{13} - 1$ a Mersenne prime?

11 If (a, b) represents twin prime pairs, how many pairs exist if a and b are less than 50?

Reasoning and communication

12 a $2^{17} - 1$ is a prime number. What should it be multiplied by to get a perfect number?
b Explain why $2^a(2^{23} - 1)$, where a is a positive integer, cannot represent a perfect number.

7.02 SIMPLE PROOFS INVOLVING INTEGERS

A mathematical proof is a logical argument which demonstrates that something is true. There are several different types of proofs. **Direct proofs** were used in the previous section to show that, for example, if you add an even and an odd integer, the result is an odd integer. Other types of proof include proof by **counter example**, **contraposition** and **contradiction**. You have already seen a proof by contradiction. This was used in the previous section to show that if a number is prime, you only need to check if the number is divisible by the prime numbers less than or equal to its square root.

Closure laws are statements about operations. A set is closed under an operation if the result of the operation on members of the set always belongs to the set.

IMPORTANT

Closure law of addition

The sum of any two integers results in another integer.

In set notation, $a, b \in \mathbf{Z} \Rightarrow c = a + b \in \mathbf{Z}$.

Closure law of multiplication

The product of any two integers results in another integer.

In set notation, $a, b \in \mathbf{Z} \Rightarrow c = a \times b \in \mathbf{Z}$.

A formal definition of closure using mathematical symbols is given below. The symbol $*$ is often used to denote an operation using two numbers, called a binary operation. Remember that $\Leftrightarrow$ means 'if and only if' and $\forall$ means 'for all'.

To prove whether a set is closed or not, you can do a direct proof or find a counter example to show that it is not closed. Both are shown in Example **4**.

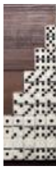
A **counterexample** is where you give an example to show that a statement is false. It is only necessary to give one example. For example, the statement 'all months of the year have 31 days' is false because the counterexample September has 30 days.

IMPORTANT

A set S is **closed** under an operation $*$ if and only if, for all a and b in S, $a * b$ is also in S.

In set notation:

S is closed under $* \Leftrightarrow a * b \in S \; \forall \; a, b \in S$.



Example 4

Consider the set $T = \{1, 3, 5, 7, 9, ...\}$.

a Give a counterexample to show that each of the following statements is false.

  i The set is closed under addition.

  ii The set is closed under subtraction.

b Using a direct proof, show a positive odd number is never produced by addition of two members of the set.

Solution

a **i** Give a counterexample.	$1 + 3 = 4$
Write the conclusion.	Since 4 is not an odd number, the set T is not closed under addition. **QED**
ii Give a counterexample.	$1 - 3 = -2$
Write the conclusion.	Since -2 is not an odd number, the set T is not closed under subtraction. **QED**

b Choose odd numbers. Let $x = 2a + 1$ and $y = 2b + 1$

Add the numbers. $x + y$

Substitute expressions. $= 2a + 1 + 2b + 1$

Put it in the form of an even integer. $= 2a + 2b + 2$

$= 2(a + b + 1)$

Write the conclusion. $2(a + b + 1)$ is even so addition of two numbers from the set T never gives a member of the set. **QED**

Another type of proof is proof by contraposition. Some examples are:

- saying that 'if you wear Ugg boots you will have warm feet' is the same as saying 'if you do not have warm feet you cannot be wearing Ugg boots'
- saying that 'a dog is an animal with four legs' is the same as saying that 'an animal which doesn't have four legs cannot be a dog'

IMPORTANT

Contraposition

The statement 'If p then q' is equivalent to 'If not q then not p'

You can prove that that 'if a number is divisible by 6, then it is divisible by 3' by proving the contrapositive statement 'if a number is not divisible by 3, then it is not divisible by 6'.

IMPORTANT

The phrase 'if and only if', shown by the symbol $\Leftrightarrow$ means that both the first statement and its converse are true.

It means that the statement is two-way.

Example 5

a Prove that if n^2 is odd, then n is odd, where $n \in \mathbb{Z}$.

b Prove that n^2 is odd if and only if n is odd, where $n \in \mathbb{Z}$.

Solution

a Write a contrapositive statement. If n is even, then n^2 is even, where $n \in \mathbb{Z}$.

Assume that n is an even integer. Let $n = 2k$, where k is an integer.

Square both sides of the equation. $n^2 = 4k^2 = 2(2k^2)$

Write the answer. Since k is an integer, $2k^2$ is an integer and $2(2k^2)$ is an even integer. That is, it cannot be odd.

Hence if n^2 is odd, then n is odd, where $n \in \mathbb{Z}$. **QED**

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b One half has been proven.	Only the converse is needed.
Assume that n is odd.	Let $n = 2k + 1$, where $k \in \mathbf{Z}$, be an odd integer.
Square n.	$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
What kind of number is n^2?	But $2(2k^2 + 2k) + 1$ is odd, so n^2 is odd
State the result.	From part **a**, if n^2 is odd then n is odd, and from the above, if n is odd then n^2 is odd, so n^2 is odd if and only if n is odd, where $n \in \mathbf{Z}$. **QED**

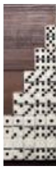
To prove a statement A by contradiction, assume that A is false or A' is true. Show that A' leads to B and B contradicts A'. Hence A' is false and A is true. The proof in Example **6** is due to Euclid who lived around 300 BC.

Example 6

Show that there are infinitely many prime numbers.

Solution

Write the contradictory statement.	There are a finite number of prime numbers.
Use mathematical notation.	Let the n primes be $p_1, p_2, p_3, \cdots, p_n$.
Choose a particular number a which is bigger than any of the primes.	$a = p_1 \times p_2 \times p_3 \times \cdots \times p_n + 1$ a cannot be prime as it is bigger than all the primes.
Write the steps which show that there is a contradiction.	If a is not prime, it must be divisible by one of the primes. However, when you divide a by any of the primes, there is always a remainder of 1. Hence a must be prime or have new prime factors.
State the answer.	This is a contradiction. So the original assumption is incorrect and there are infinitely many primes. **QED**



EXERCISE 7.02 Simple proofs involving integers

Concepts and techniques

1 Example 4 Consider the set $O = \{1, 3, 5, 7, 9, ...\}$.
   a Provide a counter example to show that the set is not closed under division.
   b Use a direct proof to show that the set is closed under multiplication.

2 a Is the set of negative integers closed under
   i addition? ii subtraction? iii multiplication? iv division?
   b Give a counter example for the ones that are not closed.

3 Is the set of integers closed under division? If not, give a counter example.

4 Suppose that $A = \{-1, 0, 1\}$.
   a Is A closed under addition?
   b Is A closed under multiplication?

5 Give counterexamples to show that each of the following are false.
   a All prime numbers are odd.
   b When you square an integer, the result is always a positive integer.

6 Show that the sum of two consecutive odd integers is divisible by 4.

7 Prove that every odd integer is the difference of two perfect squares.

8 Show that if p and q are twin primes, then 1 more than their product is divisible by their average.

Reasoning and communication

Use proof by contraposition to prove questions **9** to **12**.

9 Example 5 a Prove that if n^2 is even, then n is even, where $n \in \mathbf{Z}$.
   b Prove that n^2 is even if and only if n is even, where $n \in \mathbf{Z}$.

10 a If for two integers p and q, $p + q$ is even, then p and q have the same parity (both odd or both even).
   b Prove that the sum of two integers, p and q, is even if and only if p and q have the same parity.

11 If for two integers p and q, pq is even, then both of them will not be odd.

12 If for two integers p and q, pq is odd, then p and q must be odd.

Use proof by contradiction to prove questions **13** and **14**.

13 Example 6 A Diophantine equation is one where the solutions are required to be integers. Prove by contradiction that there are no positive integer solutions to the Diophantine equation $a^2 - b^2 = 1$.

14 Prove by contradiction that there are an infinite number of even integers.

15 Prove that the only prime of the form $n^3 - 1$ is 7.

16 Given $x \in \mathbf{Z}$, prove that if $x^2 - 2x + 5$ is odd, then x is even.

7.03 DECIMAL REPRESENTATION

Terminating decimals are decimals which have a finite number of digits after the decimal place like 0.25. **Recurring** decimals are decimals that have digits in a repeating pattern after the decimal place; for example, $\frac{1}{7} = 0.142\,857\,142\,857\,142\,857... = 0.\overline{142\,857}$ or $0.\dot{1}42\,85\dot{7}$. The sequence of digits that is repeated is 142857. You put a **vinculum** (line or bar) above these numbers to show that they are repeated or a dot above the first and last digits of the repeating part.

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Terminating and recurring decimals can all be expressed in the form $\frac{a}{b}$, where a and b are integers and b is not equal to zero. $\frac{a}{b}$ is called a **fraction**. We call a the **numerator** and b the **denominator**. The line above the denominator is also called a vinculum. Fractions are said to be written in simplest form when $x = \frac{c}{d}$, and c and d have no common factors except 1, and d does not equal zero.

In this section you will be converting terminating and recurring decimals to fractions and vice versa.

The following example shows this.

Example 7

a Convert the following terminating decimals to fractions.

i 9.9999 **ii** −435.078 65

b Convert the following fractions to terminating decimals.

i $-\frac{6666}{10\ 000}$ **ii** $\frac{237}{8}$

Solution

a i There are four digits after the decimal place.
So multiply 9.9999 by 10 000 and divide by 10 000.

Write the answer.

$$9.9999 = \frac{9.9999 \times 10\ 000}{10\ 000} = \frac{99\ 999}{10\ 000}$$

ii There are five digits after the decimal place.
So multiply −435.078 65 by 100 000 and divide by 100 000.

$$-435.078\ 65 \times \frac{100\ 000}{100\ 000}$$

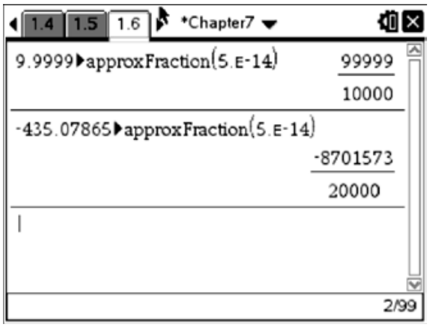
Write the answer.
Answers are normally written in simplest form.

$$= -\frac{43\ 507\ 865}{100\ 000}$$

$$= -\frac{8\ 701\ 573}{20\ 000}$$

TI-Nspire CAS

Insert a calculator page. Type the number and select menu 2: Number and 2: Approximate to Fraction.



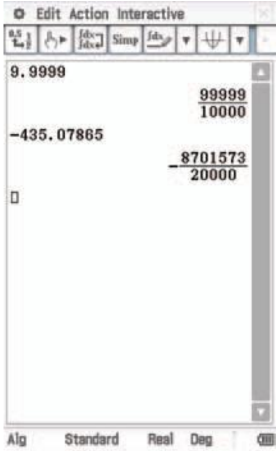
ClassPad

Use the Main $\sqrt{\alpha}$ application.

Set the calculator to **Standard**.

Simply enter the decimal and press EXE and the calculator will convert it to a fraction, expressed in simplest form.

When entering negative numbers, use the (−) key.

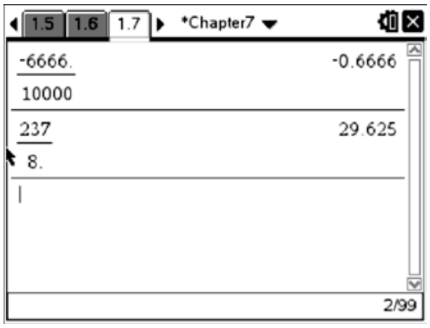


b **i** Divide –6666 by 10 000. –0.6666

ii Divide 237 by 8. 29.625

TI-Nspire CAS

Insert a calculator page. Press (−) and type the number. Press ctrl and enter to get the decimal equivalent.



ClassPad

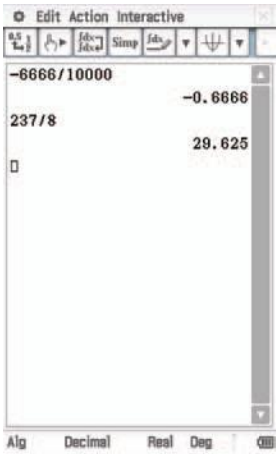
Use the Main $\sqrt{\alpha}$ application.

Set the calculator to **Decimal**.

Enter the expression and press EXE.

If you want the answer as a fraction, just set the calculator to Standard.

When entering negative numbers, use the (−) key.



The next example shows how to convert a fraction to a recurring decimal. You keep dividing until a pattern emerges. The pattern will be a finite sequence of digits that keep recurring.

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Example 8

Convert each of the following to a recurring decimal.

a $\dfrac{239}{11}$ **b** $\dfrac{239}{12}$

Solution

a Divide 239 by 11.
Stop when a pattern emerges.

$$\begin{array}{r} 2\,1.\;7\;2\;7\;2\ldots \\ 11\overline{\big)\,23^{1}9.^{8}0^{3}0^{8}0^{3}0} \end{array}$$

There are two recurring digits.
Put a line above the recurring digits.

$21.\overline{72}$

b Divide 239 by 12.
Stop when a pattern emerges.

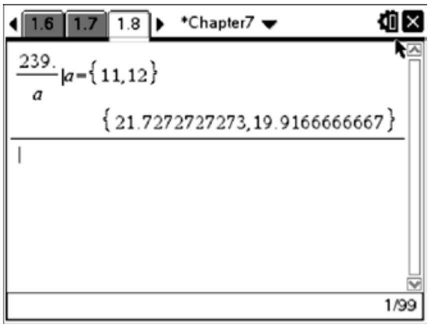
$$\begin{array}{r} 1\;9.\;\;9\;1\;6\;6\ldots \\ 12\overline{\big)\,23^{11}9.^{11}0^{2}0^{8}0^{8}0} \end{array}$$

There is one recurring digit.
Put a bar above the recurring digit.

$19.91\overline{6}$

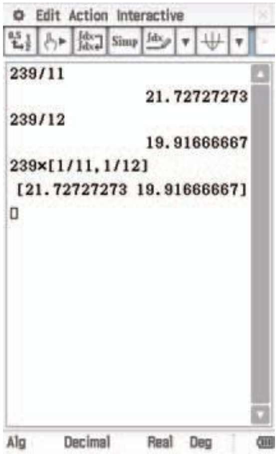
TI-Nspire CAS

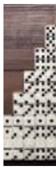
Insert a calculator page. Type the number and press ctrl, = and select |. Press ctrl,), and type the remaining numbers.
You might need to change the Document Settings to get a 12-digit display.



ClassPad

Main $\sqrt{\alpha}$ application.
Set the calculator to **Decimal**.
Enter the expression and press EXE.





There are a number of ways to convert recurring decimals to fractions. The method shown in Example **12** is summarised in the box below.

IMPORTANT

Converting a recurring decimal to a fraction

1 Let x represent the number.

2 Multiply the number x by 10^n where n is the number of recurring digits.

3 Subtract **1** from **2**.

4 Rearrange the equation to write x as a fraction.

5 Write your answer in simplest form.

Example 9

Express the following recurring decimals in rational form.

a $0.\overline{9}$ **b** $0.\overline{36}$ **c** $2.7\overline{451}$

Solution

a Write the original number as x, showing the recurring pattern. $x = 0.9999\ldots$

Since one digit recurs, multiply x by 10. $10x = 9.9999\ldots$

Subtract x from $10\,x$.

$$\begin{aligned} 10x &= 9.9999\ldots \\ -\ x &= 0.9999\ldots \\ \hline 9x &= 9 \end{aligned}$$

Divide by 9. $x = \frac{9}{9} = 1$

b Write the original number as x, showing the recurring pattern. $x = 0.36\ 36\ 36\ 36\ 36\ \ldots$

Since two digits recur, multiply by 100. $100x = 36.36\ 36\ 36\ 36\ \ldots$

Subtract x from $100x$.

$$\begin{aligned} 100x &= 36.36\ 36\ 36\ 36\ \ldots \\ -\ x &= 0.36\ 36\ 36\ 36\ \ldots \\ \hline 99x &= 36 \end{aligned}$$

Divide by 99 and simplify.

$$\begin{aligned} x &= \frac{36}{99} \\ &= \frac{4}{11} \end{aligned}$$

c Write the original number as x, showing the recurring pattern. $x = 2.7\ 451\ 451\ 451\ 451\ \ldots$

Since three digits recur, multiply by 1000. $1000x = 2745.145\ 145\ 145\ 1\ldots$

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Subtract x from $1000x$.

$$
\begin{aligned}
1000x &= 2745.1\ 451\ 451\ 451\ 451\ \ldots \\
\underline{-x} &= \underline{2.7\ 451\ 451\ 451\ 451\ \ldots} \\
999x &= 27\ 42.4
\end{aligned}
$$

Multiply by 10.

$$9990x = 27\ 424$$

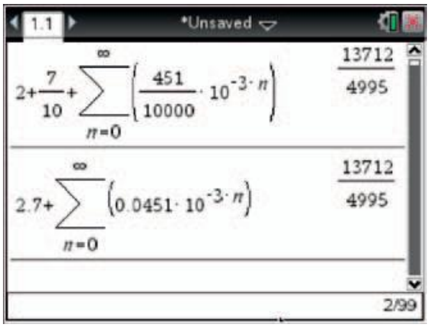
Divide by 9990 and simplify.

$$
\begin{aligned}
x &= \frac{27\ 424}{9990} \\
&= \frac{13\ 712}{4995}
\end{aligned}
$$

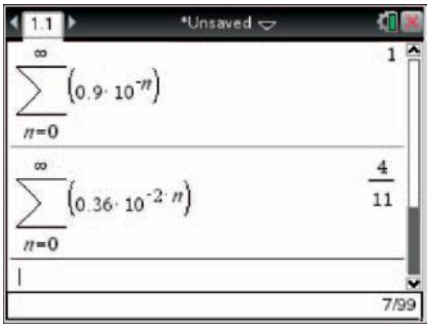
TI-Nspire CAS

$2.7\overline{451} = 2.745\ 145\ 145\ 145\ \ldots$

You can enter this as shown on the top line, using the [templates] key to get the fraction and Σ templates and the [π▸] key to get ∞. Otherwise, change the Calculation Mode ([doc▾], 7: Settings and Status, 2: Document settings) to Exact and enter as decimals as shown.



You can also do this for parts **a** and **b** but the whole expression is repeating for these, so you don't need to put anything before Σ.
0.999… repeats one digit only so you use 10^{-1n}, while 0.3636… repeats 2 digits so you use 10^{-2n}.

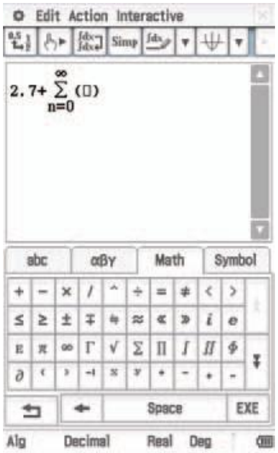


ClassPad

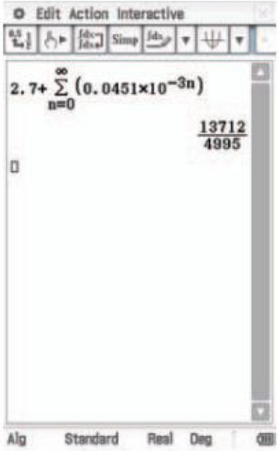
2.745 145 145 145 1…

Use the Main $\sqrt{\alpha}$ application and set the calculator to **Standard**.
Enter the non-recurring part, if it exists.
In this case it is 2.7.
Press [+] and tap the symbol [Σ□], found after pressing [Keyboard] and tapping [Math2].

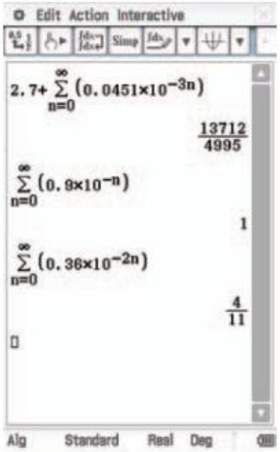
Fill it in to make $\sum_{n=0}^{\infty}(\square)$.



In the bracket, insert the exact value of the first recurring part, 0.0451.
There are 3 repeating digits, so multiply by 10 to the power $-3n$.
Tap EXE or press EXE.



This can also be done for parts **a** and **b**, 0.9999... and 0.3636...
In these cases, the whole expression is repeating.



The result in Example **12a**, $\overline{0.9} = 1$ is significant. This means that numbers such as 3.999… and 4 are equivalent. It is also very clear that every rational number can be expressed in many different ways; for example, $-2\frac{2}{5} = -2.4 = -2.400... = -2.399... = -\frac{12}{5}$.

EXERCISE 7.03 Decimal representation

Concepts and techniques

Decimal representation

1 Example 7 a Convert the following terminating decimals to fractions.
   i 3.245 ii −521.078574
   b Convert the following fractions to terminating decimals.
   i $-\frac{5659}{8}$ ii $\frac{31129}{500}$

2 Which one of the following can be written as a terminating decimal?
   A $\frac{189\,767}{3}$ B $\frac{189\,767}{7}$ C $\frac{189\,767}{8}$ D $\frac{189\,767}{9}$ E $\frac{189\,767}{11}$

3 Example 8 Convert each of the following to a recurring decimal.
   a $\frac{564}{11}$ b $\frac{239}{13}$ c $\frac{22}{7}$

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4 Example 9 Convert each of the following recurring decimals to fractions.

a $0.\overline{1}$ b $0.58\overline{3}$ c $3.1\overline{28}$ d $1.\overline{714\,285}$ e $1.4\overline{9}$

Reasoning and communication

5 Recurring decimals can be expressed as the sum of an infinite geometric series. $S_\infty = \frac{a}{1-r}$, where a is the first term and r is the common ratio, i.e., the second term divided by the first term. Given $\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots$,

a write the sum as a recurring decimal

b write the sum as a fraction using $S_\infty = \frac{a}{1-r}$

6 Consider $\frac{1}{7}, \frac{1}{17}$ and $\frac{1}{19}$.

a How many recurring digits does each of them have?

b Write a general rule connecting the numbers and the number of recurring digits.

c Give a counter example to show that your rule does not work for all $\frac{1}{p}$, where p is a prime number.

7 Consider the numbers $\frac{a}{7}$, where $a = \{1, 2, 3, 4, 5, 6\}$.

a Write them as recurring decimals.

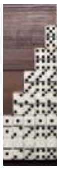
b Describe the pattern that occurs.

8 $\frac{1}{7} = 0.142857142857...$ The repeated digits are 142857. Notice that 142 + 857 = 999, which is a string of 9s.

a Show that the pattern occurs for $\frac{1}{13}$.

b Find another example where the pattern works for $\frac{1}{p}$, where p is prime.

c Find a counter example to show that p does not have to be prime for the pattern to work.



7.04 RATIONAL NUMBERS

We call the set of **rational numbers** $\mathbf{Q}$. It includes any number that can be written in the form $\frac{a}{b}$, where a and b are both integers and b is not equal to zero. This means that the set of integers $\mathbf{Z}$, terminating decimals and recurring decimals make up the set of rational numbers. Such examples are:

- 2, as it can be written as $\frac{2}{1}$
- 0.875, as it can be written as $\frac{7}{8}$
- $0.\overline{3}$, as it can be written as $\frac{1}{3}$

IMPORTANT

A rational number is a number x that can be expressed in the form $x = \frac{a}{b}$, where $a, b \in \mathbf{Z}$ and $b \neq 0$.

In set notation: $\mathbf{Q} = \left\{x = \frac{a}{b} : a, b \in \mathbf{Z}, b \neq 0\right\}$.

Any rational number can be written in the form $x = \frac{c}{d}$, where c and d have no common factors except 1. d is usually taken to be positive, so for negative x, c is negative.

Example 10

Show that each of the following numbers are rational by writing them in the form $\frac{a}{b}$, where a and b are integers and b is not equal to zero.

a -3 **b** 0.75 **c** 2.17×10^4

Solution

a Write -3 as a fraction.
There are two equivalent forms. These are also equivalent to $-\frac{3}{1}$.

$$-3 = \frac{-3}{1} \text{ or } \frac{3}{-1}$$

b Write 0.75 as a fraction.
Divide 75 by 100.
$\frac{3}{4}$ is the simplest form.

$$0.75 = \frac{75}{100} = \frac{3}{4}$$

c Multiply 2.17 by 10 000.
Write the answer as a fraction.

$$2.17 \times 10^4 = 21\,700$$
$$= \frac{21\,700}{1}$$

Example **11** is an example of a direct proof using rational numbers, showing that they are closed under addition.

Example 11

Prove that if x and y are rational numbers, then $x + y$ is rational.

Solution

Represent x and y as a fraction.

Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c and d are integers and b and d are not equal to zero.

Sum x and y.

$$\frac{a}{b} + \frac{c}{d}$$

Write the sum as a single fraction.

$$= \frac{ad + bc}{bd}$$

Explain why ad, bc and bd are all integers.

According to the Closure law of multiplication, ad, bc and bd are all integers. $bd \neq 0$.

Explain why $ad + bc$ is an integer.

According to the Closure law of addition, $ad + bc$ is an integer.

State the conclusion.

Hence $\frac{ad + bc}{bd}$ is rational and can be written in the form $\frac{e}{f}$ where $e, f \in \mathbf{Z}, f \neq 0$.

If x and y are rational numbers, then $x + y$ is rational. **QED**

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Example **12** is a proof using contradiction with rational numbers.

Example 12

Prove that there are no rational number solutions to the equation $x^3 + x + 1 = 0$ using contradiction.

Solution

Write the contradictory statement.

There is a rational number solution to $x^3 + x + 1 = 0$.

Substitute $\frac{a}{b}$ into the equation, where $a, b \in \boldsymbol{Z}$, $b \neq 0$ and a and b have no common factors except 1, and b is positive.

$$\left(\frac{a}{b}\right)^3 + \frac{a}{b} + 1 = 0$$

Multiply both sides of the equation by b^3.

$a^3 + ab^2 + b^3 = 0$

Consider the different cases.

Zero is even.

1. a is odd and b is even.

a^3 is odd, ab^2 is even, b^3 is even.
LHS is odd and RHS is even: impossible.

2. a is even and b is odd.

a^3 is even, ab^2 is even, b^3 is odd.
LHS is odd and RHS is even: impossible.

3. a is odd and b is odd.

a^3 is odd, ab^2 is odd, b^3 is odd.
LHS is odd and RHS is even: impossible.

4. a is even and b is even.

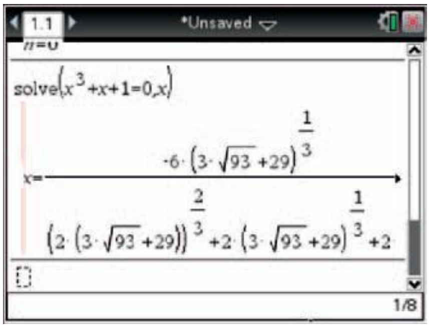
a and b would have a common factor of 2, which is a contradiction.

Write the answer.

There is no rational number solution to the equation $x^3 + x + 1 = 0$. **QED**

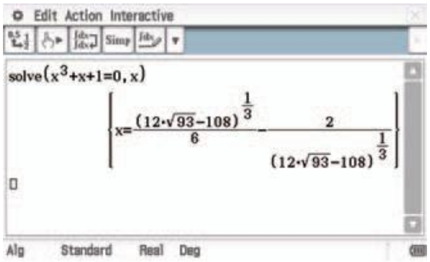
TI-Nspire CAS

Use **solve** to work out the solution to the equation with the calculator set to Exact calculation mode. A rational answer will be shown as a fraction or whole number. In this case the solution involves surds.



ClassPad

Use the Main $\sqrt{\alpha}$ application and set the calculator to **Standard**. A rational answer will be shown as a fraction or whole number. In this case the solution involves surds.



EXERCISE 7.04 Rational numbers

Concepts and techniques

1 Example 10 Show that each of the following numbers are rational by writing them in the form $\frac{a}{b}$, where a and b are integers and b is not equal to zero.

a $-\frac{2}{3}$ b 0.375 c 5.67×10^3

d 8.2×10^{-2} e 0 f $2\frac{3}{4}$

2 A recurring decimal belongs to which one of the following sets of numbers?

A $\boldsymbol{Z}^-$ B $\boldsymbol{Z}^+$ C $\boldsymbol{Z}$ D $\boldsymbol{Q}$ E $\boldsymbol{N}$

3 Simplify each of the following and show that the result is a rational number.

a $\frac{2}{3}+\frac{5}{7}$ b $2\frac{1}{2}-3\frac{2}{5}$ c $\frac{2}{3}\times\frac{5}{7}$ d $2\frac{1}{2}\times 3\frac{2}{5}$ e $\frac{2}{3}\div\frac{5}{7}$

Use a direct proof to prove questions **4** to **6**.

4 Example 11 Show that the product of two rational numbers is rational.

5 Show that the difference between two rational numbers is rational.

6 a Show that if a rational number is divided by a rational number, then the result is a rational number.

b Are the rational numbers closed under division?

7 Consider $A = \left\{\ldots, \frac{1}{n}, \ldots \frac{1}{8}, \frac{1}{4}, \frac{1}{2}\right\}$, where n is a positive even integer.

a Use a counter example to show that A is not closed under

i addition ii subtraction iii division

b Use a direct proof to show that A is closed under multiplication.

Reasoning and communication

8 Use proof by contraposition to show that if $x^2 - 2x + 5$ is not rational, then x is not rational.

Use proof by contradiction to do questions **9** and **10**.

9 Example 12 Show that there are no rational solutions to the equation $x^3 + 2x + 2 = 0$.

10 Show that there are no rational solutions to the equation $x^4 + x^2 + x - 2 = 0$.

11 a Find an equation of the form $ax^5 + bx^4 + cx^3 + dx^2 + ex + 1 = 0$ which has no rational solutions, where $a, b, c, d, e \in \boldsymbol{Z}$.

b Sketch the graph of $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + 1$ for your values of a, b, c, d and e. Give the coordinates of any axis intercepts, correct to two decimal places.

c Use proof by contradiction to show that $ax^5 + bx^4 + cx^3 + dx^2 + ex + 1 = 0$ has no rational solutions using your values of a, b, c, d and e.

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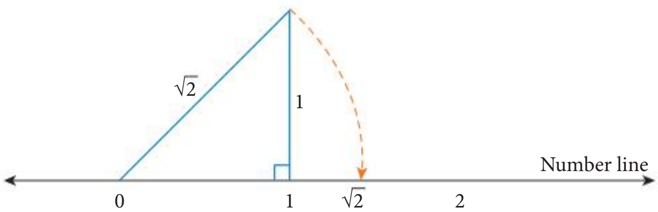
7.05 IRRATIONAL NUMBERS

Real numbers that cannot be written in the form $\frac{a}{b}$, where $a, b \in \boldsymbol{Z}, b \neq 0$, are called **irrational numbers**. Examples of such numbers are $\sqrt{2}$, $-3\sqrt{5}$, $\log_2(5)$, e and π. $\sqrt{4}$ is not an irrational number as it can be written as 2. Irrational numbers cannot be written as either terminating or recurring decimals. This means that decimal representations of these numbers will be approximations and not exact values; for example, $\sqrt{2} \approx 1.414\ 213\ 562$ and $e \approx 2.\ 718\ 281\ 828$. The ancient Babylonians used to approximate π as $\frac{25}{8}$. π has also been approximated as $\frac{22}{7}$, $\frac{333}{106}$ and 3.1416. However, it is actually an irrational number.

Irrational numbers that must be expressed in the form $\sqrt[a]{n}$, where $n, a \in \boldsymbol{N}$, are called **surds**. Even though it is not possible to find an exact decimal value for a surd, each surd corresponds to some exact value. This can be shown by the construction on the right.



Irrational numbers that are not surds or combinations of surds are called **transcendental numbers**; for example, e and π. These numbers can also be represented on a number line.

IMPORTANT

Irrational numbers

An irrational number cannot be expressed in the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers.

You can also choose $b > 0$ instead because a positive or negative is enough.

Expressed as decimals, irrational numbers do not recur and do not terminate. However, they do have definite locations on the number line. In set notation: $\boldsymbol{Q}' = \{x \in \boldsymbol{R}: x \notin \boldsymbol{Q}\}$.

In Examples **13** and **14**, proof by contradiction is used to show that $\sqrt{2}$ and $\log_2(5)$are irrational. In both cases, you use the fact that any number in fractional form can always be written in its simplest form by cancelling down any factors other than 1.

Example 13

WS Rational and irrational numbers

Use proof by contradiction to show that $\sqrt{2}$ is an irrational number.

Solution

Write the contradictory statement.	Assume that $\sqrt{2}$ is a rational number.
What does this mean?	Then $\sqrt{2}$ can be written as a ratio of integers.
Write $\sqrt{2}$ as a rational number in simplest form.	Write $\sqrt{2} = \frac{a}{b}, a, b \in \boldsymbol{Z}, b \neq 0$, where a and b only have 1 as a common factor.

Square both sides of the equation.	$2=\frac{a^2}{b^2}$
Rearrange the equation.	$a^2 = 2b^2$
What does $a^2 = 2b^2$ mean?	a^2 must be even, so a must be even.
Write a as double another integer.	$a = 2k$, where $k \in \mathbb{Z}$
Substitute $a = 2k$ into the equation.	$(2k)^2 = 2b^2$
Simplify and make b^2 the subject.	$2b^2 = 4k^2$ $b^2 = 2k^2$
What does $b^2 = 2k^2$ mean?	b^2 must be even, so b is even.
State the critical point.	a and b are even, so they have a common factor of 2.
Write the contradiction.	This contradicts the assumption that $\sqrt{2}$ is rational, so it must be false.
Write the conclusion.	$\sqrt{2}$ is irrational. **QED**

Example 14

Use proof by contradiction to show that $\log_2(5)$ is irrational.

Solution

Write the contradictory statement.	Assume that $\log_2(5)$ is a rational number.
Write $\log_2(5)$ as a rational number in simplest form.	Write $\log_2(5)=\frac{a}{b}$, $a, b \in \mathbb{Z}^+$, where a and b only have 1 as a common factor.
Put in index form.	$2^{\frac{a}{b}}=5$
Raise both sides of the equation to the power of b.	$\left(2^{\frac{a}{b}}\right)^b=5^b$
Tidy up the equation.	$2^a = 5^b$
State the critical point.	For any positive integer a, 2^a is even For any positive integer b, 5^b is odd A positive integer cannot be both even and odd.
Write the contradiction.	This contradicts the assumption that $\log_2(5)$ is a rational number, so it must be false.
Write the conclusion.	$\log_2(5)$ is irrational. **QED**

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INVESTIGATION Infinite continued fractions for irrational numbers

This investigation is designed for a computer-based CAS such as Mathematica.

Irrational numbers can be written as Infinite continued fractions. These are fractions in the form $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$, where $a_0 \in \boldsymbol{Z}$ and a_1, a_2, ... are positive integers.

a $\sqrt{2}$ can be written as an infinite continued fraction, $1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}$.

Write the decimal equivalent for each of the following finite continued fractions and comment on the result.

$$1 + \frac{1}{2},\ 1 + \cfrac{1}{2 + \cfrac{1}{2}},\ 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}},\ 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}$$

The decimal equivalent of other irrational numbers can be constructed in the same way.

b You can construct the decimal 0.01001000100001… using an infinite continued fraction.

- **i** Find the finite continued fractions for 0.01 and 0.01001 by finding values for a_0, a_1, a_2 and a_3.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = 0.01,\quad a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}} = 0.01001$$

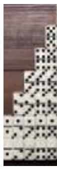
- **ii** Using an appropriate CAS, find the continued fraction for 0.010010001. The command for Mathematica is ContinuedFraction[0.010010001]. This cannot presently be done on CAS calculators.

c Find infinite continued fractions for e and π. The command for Mathematica is ContinuedFraction[Pi,20]. Comment on the results.

d Watch the Mathematica demonstration at the website.

http://demonstrations.wolfram.com/ContinuedFractions/

What conclusion can you draw about square root functions as opposed to cube roots or higher? Give some examples to support your conclusion.



WolframAlpha

Infinite continued fractions can be used to construct decimals like 0.01001000100001… as shown in the investigation. However, there is a much simpler way to construct this decimal, as shown in Example **15**.

Example 15

Construct the decimal 0.01001000100001… by adding fractions of the form $\frac{1}{10^{2+a}}$, where $a \in \mathbf{Z}^+ \cup \{0\}$.

Solution

Write 0.01 as a fraction. $\quad \frac{1}{100} = \frac{1}{10^2}$

Write 0.000 01 as a fraction. $\quad \frac{1}{100\,000} = \frac{1}{10^5}$

Write 0.000 000 001 as a fraction. $\quad \frac{1}{1\,000\,000\,000} = \frac{1}{10^9}$

Write 0.01001000100001… as the sum of the fractions by observing the pattern. $\quad \frac{1}{10^2} + \frac{1}{10^5} + \frac{1}{10^9} + \frac{1}{10^{14}} + \dots \frac{1}{10^{\frac{1}{2}n^2+\frac{3}{2}n}} + \dots$

$n \in \mathbf{N}$

EXERCISE 7.05 Irrational numbers

Concepts and techniques

1 Which one of the following is not an irrational number?

A $2\sqrt{6}$ B π C e D $-\sqrt{99}$ E $2\sqrt{16}$

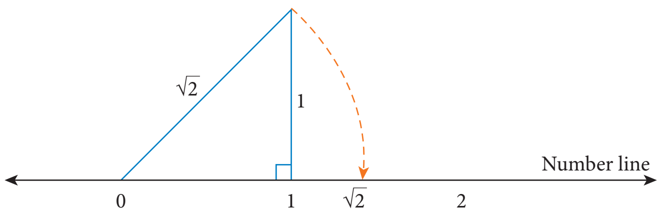
2 Give a counter example to show that irrational numbers are not closed under

a addition b subtraction c multiplication d division.

3 Using the result $1^2 + 2^2 = 5$, show $\sqrt{5}$ on the number line below.

4 Does $\pi = \frac{22}{7}$? Explain.

5 Besides π and e, give two other examples of transcendental numbers.



Reasoning and communication

6 Example 13 Use proof by contradiction to show that $\sqrt{5}$ is irrational.

7 Example 14 Use proof by contradiction to show that $\log_2(3)$ is irrational.

8 Using simultaneous equations or finite differences, show that the sequence 2, 5, 9, 14, … in Example **15** can be written as $\frac{1}{2}n^2 + \frac{3}{2}n$.

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9 Example 15 Construct the decimal 0.01000100001000001… by adding fractions of the form $\frac{1}{10^{2+a}}$, where $a \in \boldsymbol{Z}^{+} \cup \{0\}$.

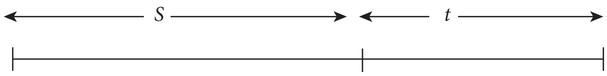
10 Construct the decimal 0.01002000400008… by adding fractions of the form $\frac{2^{b}}{10^{2+a}}$, where a, $b \in \boldsymbol{Z}^{+} \cup \{0\}$.

11 Give the conditions for which the solutions to the quadratic equation $ax^2 + bx + c = 0$, where a, b, $c \in \boldsymbol{Z}$, are irrational.

12 Give an example of a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$ which only has irrational solutions.

13 a Solve $x^2 - x - 1 = 0$.

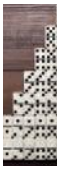
b Use the result $\frac{s+t}{s} = \frac{s}{t} = 1.618... = \varphi =$ **Golden Ratio**, where $s > t$, to answer the following questions.



i Find the Golden Ratio in surd form. Hint: let $t = 1$.

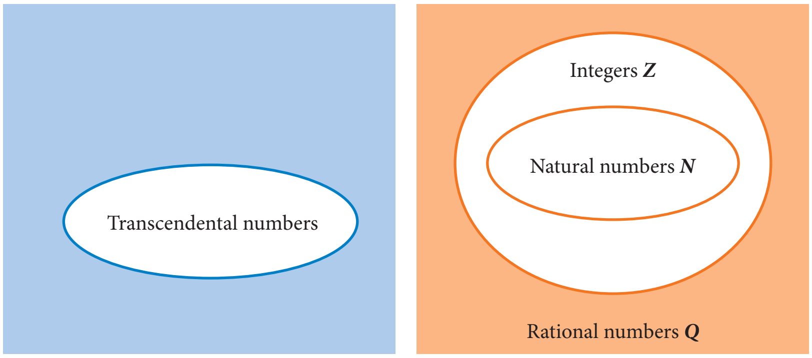
ii Prove by contradiction that the Golden Ratio is irrational.

7.06 REAL NUMBERS



The inclusion of irrational numbers with rational numbers gives the set of **real numbers**, $\boldsymbol{R}$. The diagram below shows the relationship between the various sets of numbers we have discussed in this chapter.

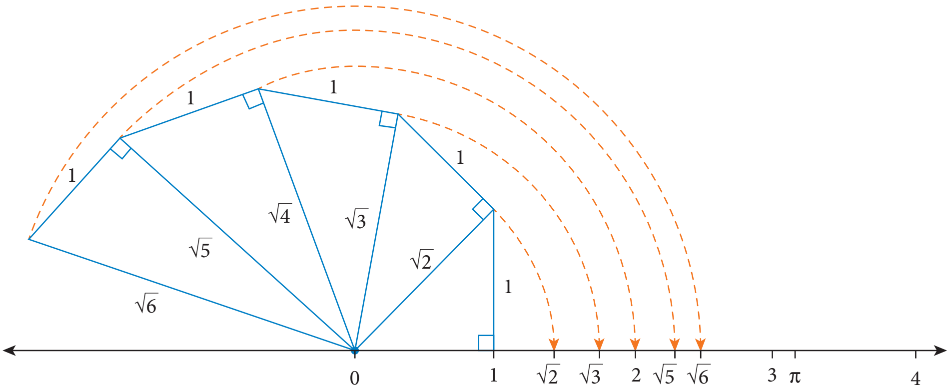
Real numbers ($\boldsymbol{R}$)



The set of real numbers can be thought of as the universal set, where

$$\boldsymbol{N} \subset \boldsymbol{Z} \subset \boldsymbol{Q} \subset \boldsymbol{R} \text{ and } \{\text{irrational numbers, } \boldsymbol{Q}'\} \subset \boldsymbol{R}.$$

The set of real numbers can be represented on a number line as shown in the diagram below.



Examples **16** and **17** are proofs by contradiction which involve real numbers.

Example 16

Use proof by contradiction to show that for all real numbers x and y, if x is rational and y is irrational, then $x + y$ is irrational.

Solution

Write the contradictory statement.	Assume that there exists a rational number x and an irrational number y such that $x + y$ is rational.
Write x as a rational number.	$x = \frac{a}{b}$, where $a, b \in \mathbf{Z}$, $b \neq 0$
Write $x + y$ as rational number.	$\frac{a}{b} + y = \frac{p}{q}$, where $a, b, p, q \in \mathbf{Z}$, and $b, q \neq 0$
Make y the subject.	$y = \frac{p}{q} - \frac{a}{b}$
Show that $y = \frac{p}{q} - \frac{a}{b}$ is rational.	$\frac{p}{q} - \frac{a}{b} = \frac{bp - aq}{bq} = \frac{m}{n}$, where $m, n \in \mathbf{Z}$, $n \neq 0$
State the vital point.	Since $\frac{p}{q} - \frac{a}{b}$ is rational, y is rational.
State the contradiction.	But this is a contradiction, as y is irrational.
State the effect of the contradiction.	Thus, there is no rational number x and irrational number y such that $x + y$ is rational.
State the conclusion.	For all real numbers x and y, if x is rational and y is irrational, then $x + y$ is irrational. **QED**

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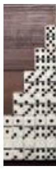
Example 17

Prove $\dfrac{1}{x+y} \neq \dfrac{1}{x} + \dfrac{1}{y}$ by contradiction for all non-zero real numbers x and y.

Solution

Write the contradictory statement.	Assume that there exist $x, y \in \boldsymbol{R}$, where $x, y \neq 0$, such that $\dfrac{1}{x+y} = \dfrac{1}{x} + \dfrac{1}{y}$.
Write the RHS as a single fraction.	Then $\dfrac{1}{x+y} = \dfrac{y+x}{xy}$.
Cross multiply.	So $(y+x)^2 = xy$
Expand the LHS.	$y^2 + 2xy + x^2 = xy$
Simplify.	So $x^2 + xy + y^2 = 0$
Use the quadratic formula.	This gives $x = \dfrac{-y \pm \sqrt{y^2 - 4 \times 1 \times y^2}}{2}$
Simplify.	So $x = \dfrac{-y + \sqrt{-3y^2}}{2}$ or $x = \dfrac{-y - \sqrt{-3y^2}}{2}$
State the vital point.	But for $y \in \boldsymbol{R}$ $\sqrt{-3y^2}$ is not real, so $x \notin \boldsymbol{R}$.
State the contradiction.	But this contradicts the assumption that $x \in \boldsymbol{R}$.
State the effect of the contradiction.	Thus there are no $x, y \in \boldsymbol{R}$, where $x, y \neq 0$, such that $\dfrac{1}{x+y} = \dfrac{1}{x} + \dfrac{1}{y}$.
Write the answer.	For all nonzero real numbers x and y, $\dfrac{1}{x+y} \neq \dfrac{1}{x} + \dfrac{1}{y}$. **QED**

Quadratic equations, $y = ax^2 + bx + c$, where $a, b, c \in \boldsymbol{R}$, have real roots if the discriminant, $b^2 - 4ac$ is greater than or equal to zero.



Example 18

For what values of c will the solutions of the quadratic equation $x^2 + x + c = 0$ be:

a real **b** integers?

Solution

a Solve $x^2 + x + c = 0$ for x using the quadratic formula.

$$x = \frac{-1 \pm \sqrt{1-4c}}{2}$$

For real solutions the discriminant has to be greater than or equal to zero.

$1 - 4c \geq 0$

Make c the subject.

$-4c \geq -1$

$c \leq \frac{1}{4}$

b Write the solutions.

$$x = \frac{-1 \pm \sqrt{1-4c}}{2}$$

Choose integer solutions.

Put $\frac{-1+\sqrt{1-4c}}{2} = m$ and $\frac{-1-\sqrt{1-4c}}{2} = n$; $n, m \in \mathbf{Z}$.

Choose one value and square it.

This gives $1 - 4c = (2m + 1)^2$.

Rearrange to make c the subject.

$$c = \frac{1-(2m+1)^2}{4} = \frac{-4m^2 - 4m}{4} = -m^2 - m = -m(m+1)$$

Comment for other value.

c has the same form for n.

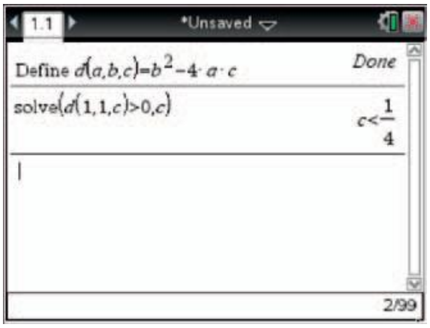
Write the answer.

For integer solutions, c must be of the form $-m\,(m + 1)$, where $m \in \mathbf{Z}$.

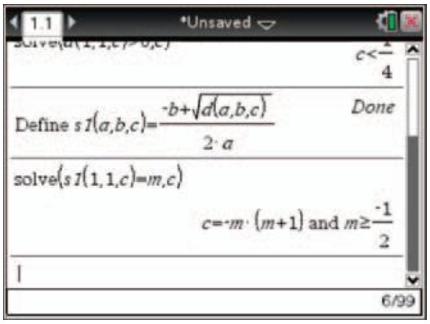
a **TI-Nspire CAS**

Define the function $d(a, b, c)$ as $d(a, b, c) = b^2 - 4ac$ for the discriminant. Make sure that you put a multiplication between a and c.

Now solve $d(1, 1, c) > 0$ for c.



b Choose one of the solutions, say $s1 = (-b+\sqrt{d(a,b,c)})/(2a)$ and solve $s1=m$ for c.



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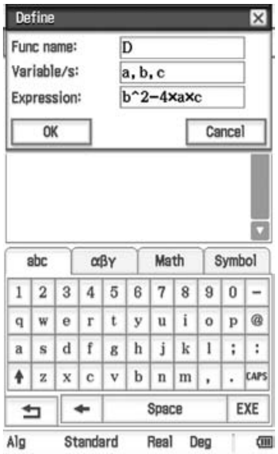
a **ClassPad**

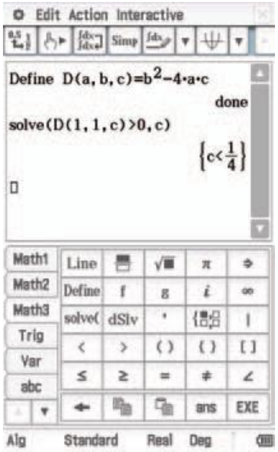
Use the $\sqrt{\alpha}$ (Main) application and set the calculator to **Standard**.

Define $D(a, b, c) = b^2 - 4ac$.

(The **Define** function is found at the bottom of the **Interactive** menu.) Fill in the screen so that it matches the one on the right and tap **OK**.



In this case, $a = 1$, $b = 1$ and c is c.
Solve for $D > 0$, looking for values of c.



b Define the two solutions as $S1$ and $S2$ (see right screen).

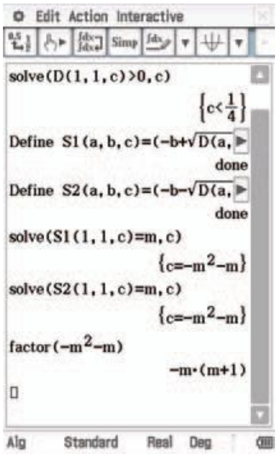
$S1 = (-b+\sqrt{D(a,b,c)})/(2a)$

$\boxed{\text{EXE}}$

$S2 = (-b-\sqrt{D(a,b,c)})/(2a)$

$\boxed{\text{EXE}}$

Solve $S1=m$ for c and also $S2=m$ for c.



Write the solutions.

a $c \le \frac{1}{4}$

b $c = -m(m+1)$ for $m \in \mathbb{Z}$

EXERCISE 7.06 Real numbers

Concepts and techniques

1 a If $x = 2+\sqrt{3}$ and $y = 2-\sqrt{3}$, then find each of the following.

i $x+y$ ii $x-y$ iii xy iv $\frac{x}{y}$

b State which subset of the real number system each of the results in part **a** belongs to.

2 Is the set of real numbers closed under

a addition?   b subtraction?   c multiplication?   d division?

3 If $x = 3$ and $y = 6$, find each of the following.

a $\frac{1}{x+y}$ b $\frac{1}{x}+\frac{1}{y}$ c $\frac{1}{x-y}$ d $\frac{1}{x}-\frac{1}{y}$

Reasoning and communication

For questions **4** to **7**, use proof by contradiction to show that each of the statements is true.

4 Example 16 For all real numbers x and y, if x is rational and y is irrational, then $x \times y$ is irrational.

5 For all real numbers x and y, if x is rational and y is irrational, then $x - y$ is irrational.

6 Example 17 For all nonzero real numbers x and y, $\frac{1}{x-y} \neq \frac{1}{x} - \frac{1}{y}$.

7 a In a right-angled triangle, the length of the hypotenuse is less than the sum of the lengths of the other two sides.

b Give an example of where the solutions to $a^2 + b^2 = c^2$ are integer values only.

8 Example 18 Consider the quadratic equation $x^2 + x + c = 0$. For what values of c will the solutions be rational?

9 Give a counter example to show that if the solutions to the equation

$$x^3 + bx^2 + cx + d = (x - e)(x - f)(x - g) = 0$$

are irrational, then d is irrational is a false statement.

7.07 THE PRINCIPLE OF MATHEMATICAL INDUCTION

Mathematical induction is a technique which is used to prove whether a generalisation is true or not.

Proof by induction follows a set of formal steps.

> **IMPORTANT**
>
> **The principle of mathematical induction**
>
> Let there be associated with each positive integer n, a proposition $P(n)$.
>
> If **1** $P(1)$ is true, and
>
> **2** for all k, if $P(k)$ is true, it follows that $P(k + 1)$ is true,
>
> then $P(n)$ is true for all positive integers n.

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IMPORTANT

To prove something by mathematical induction, you should follow the steps below.

Step 1: Under **RTP** (required to prove) state what has to be proved.

Step 2: Prove that the statement is true for $n = 1$.

Step 3: Assume that the statement is true for $n = k$, where $k \in \boldsymbol{Z}^+$.

Step 4: Show that it necessarily follows that the statement is true for $n = k + 1$.

Step 5: Make a formal statement that 'By the principle of mathematical induction …' and put in the statement, followed by **QED**.

Example **19** is a simple example of proving results for sums. Example **20** is more complicated.

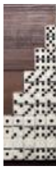
Example 19

Use the method of mathematical induction to prove that

$1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2$ for all $n \in \boldsymbol{Z}^+$.

Solution

State what has to be proved.	**RTP** $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2$ for all $n \in \boldsymbol{Z}^+$.
Show that it is true for $n = 1$.	LHS = 1 RHS = $1^2 = 1$
State the first part.	Since the LHS = RHS, it is true for $n = 1$.
Assume that it is true for $n = k$, where $k \in \boldsymbol{Z}^+$.	Assume that $1 + 3 + 5 + 7 + 9 + \cdots + (2k - 1) = k^2$ for $k \in \boldsymbol{Z}^+$
Add the $(k + 1)$th term to both sides of the equation.	$1 + 3 + 5 + 7 + 9 + \cdots + (2k - 1) + [2(k + 1) - 1]$ $= k^2 + [2(k + 1) - 1]$
Expand the RHS and simplify.	$= k^2 + 2k + 2 - 1$ $= k^2 + 2k + 1$
Factorise so that the RHS is in the same form as the original statement.	$= (k + 1)^2$
Summarise the statement for $k + 1$.	Thus $1 + 3 + 5 + 7 + 9 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$ so it is true for $k + 1$.
Make a formal statement for all values of n. Replace $k + 1$ with n.	By the principle of mathematical induction, $1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2$ for all $n \in \boldsymbol{Z}^+$. **QED**



Example 20

Use the method of mathematical induction to prove that

$$1+4+9+\cdots+n^2=\frac{n(n+1)(2n+1)}{6} \text{ for all } n\in \boldsymbol{Z}^+.$$

Solution

State what has to be proved.	**RTP** $1+4+9+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$ for all $n\in \boldsymbol{Z}^+$.
Show that it is true for $n = 1$.	LHS $= 1^2 = 1$ RHS $=\frac{1\times 2\times 3}{6}=\frac{6}{6}=1$
State the first part.	Since the LHS = RHS, it is true for $n = 1$.
Assume that it is true for $n = k$, where $k\in \boldsymbol{Z}^+$.	Assume that $1+4+9+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$ for $k\in \boldsymbol{Z}^+$
Add the $(k + 1)$th term to both sides of the equation.	$1+4+9+\cdots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$
Simplify the RHS by using the LCD of 6.	RHS $=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$
Use $k + 1$ as a common factor.	$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$
Expand $[k(2k + 1) + 6(k + 1)]$.	$=\frac{(k+1)(2k^2+7k+6)}{6}$
Factorise $(2k^2 + 7k + 6)$.	$=\frac{(k+1)(k+2)(2k+3)}{6}$
Summarise the statement for $k + 1$.	$=\frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$ so it is true for $k + 1$.
Make a formal statement.	By the principle of mathematical induction,
Replace $k + 1$ with n.	$1+4+9+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$ for all $n\in \boldsymbol{Z}^+$. **QED**

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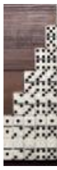
Example **21** involves proving a divisibility result using mathematical induction.

Example 21

Use the method of mathematical induction to prove that $3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n.

Solution

State what has to be proved.	**RTP** $3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n.
Show that it is true for $n = 1$.	$3^{2n+4} - 2^{2n} = 3^6 - 2^4 = 725$
State the first part.	725 is divisible by 5.
Assume that it is true for $n = k$, where $k \in \boldsymbol{Z}^+$.	Write $3^{2k+4} - 2^{2k} = 5M$, where $M \in \boldsymbol{Z}^+$.
Write the expression for $k + 1$.	$3^{2(k+1)+4} - 2^{2(k+1)}$
Simplify	$= 3^{2k+4+2} - 2^{2k+2}$
Write the expression in terms of $3^{2k+4} - 2^{2k}$.	$= 3^{2k+4} \times 9 - 2^{2k} \times 4$ $= 3^{2k+4} \times 5 + 3^{2k+4} \times 4 - 2^{2k} \times 4$ $= 4(3^{2k+4} - 2^{2k}) + 5 \times 3^{2k+4}$
Substitute $5M$ for $3^{2k+4} - 2^{2k}$.	$= 4 \times 5M + 5 \times 3^{2k+4}$
Factorise, since 5 is a common factor.	$= 5(4M + 3^{2k+4})$, which is divisible by 5
Summarise the statement for $k + 1$.	$3^{2(k+1)+4} - 2^{2(k+1)}$ is divisible by 5 so it is true for $k + 1$.
Make a formal statement. Replace $k + 1$ with n.	By the principle of mathematical induction, $3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n. **QED**



EXERCISE 7.07 The principle of mathematical induction

Reasoning and communication

1 Example 19 Use mathematical induction to prove each of the following for $n \in \mathbf{Z}^+$.

a $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

b $1 + 4 + 7 + 10 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$ for all $n \geq 1$.

c $2 + 4 + 6 + \cdots + 2n = n(n + 1)$ for all $n \geq 1$.

2 Example 20 Show that for every positive integer n,

a $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$

b $2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$

c $1^3 + 3^3 + 5^3 + \cdots$ to n terms $= n^2(2n^2 - 1)$

3 Use mathematical induction to prove each of the following, for all positive integers $n \geq 1$.

a $\frac{1}{3\times5} + \frac{1}{5\times7} + \frac{1}{7\times9} + \cdots$ to n terms $= \frac{n}{3(2n+3)}$

b $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right) = \frac{1}{n}$

4 Example 21 Prove the following, where $n \in \mathbf{Z}^+$.

a $3^{2n} - 1$ is divisible by 8.

b $4^n - 1$ is divisible by 3.

c $3^n - 1$ is divisible by 2.

d $n^7 - n$ is divisible by 7.

e $n(n + 1)(n + 2)$ is divisible by 6.

5 Prove the following inequalities using the method of mathematical induction.

a $2^n > n$ for all $n \in \mathbf{Z}^+$.

b $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} > \frac{13}{24}$, for integers $n > 1$.

6 If $r \neq 1$, show that, for any positive integer n, $a + ar + ar^2 + \cdots + ar^n = \frac{a(r^n - 1)}{r-1}$.

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7

CHAPTER SUMMARY

REAL NUMBERS AND PROOFS

- The **real** number system, $\mathbf{R}$, consists of the **rational** numbers, $\mathbf{Q}$ and the **irrational** numbers, $\mathbf{Q}'$.
- Any rational number can be written in the form $\frac{a}{b}$, where $a, b \in \mathbf{Z}$ and $b \neq 0$. They can be written as **terminating** decimals or **recurring** decimals.
- An **integer** is a positive or negative whole number. $\mathbf{Z} = \{..., -2, -1, 0, 1, 2, ...\}$.
- The counting numbers or **natural numbers**, $\mathbf{N}$ is the set of positive integers. $\mathbf{N} = \{1, 2, 3, ...\}$.
- The set of integers can be divided into **even** numbers, $\{..., -4, -2, 0, 2, 4, ...\}$ and **odd** numbers $\{..., -3, -1, 1, 3, ...\}$. Even numbers can be written in the form $2k$ and odd numbers in the form $2k + 1$, where k is an integer.
- The set of irrational numbers consists of **surds** and **transcendental numbers**. They cannot be written in the form $\frac{a}{b}$, where $a, b \in$ and $b \neq 0$. They do not have digits that repeat when represented as a decimal.
- Surds are irrational numbers that can be expressed in the form $\sqrt[a]{n}$, where $n, a \in \mathbf{N}$.
- **Transcendental numbers** are irrational numbers that cannot be expressed as a combination of surds; for example, e and π.
- A **perfect** number is a positive integer that is equal to the sum of its positive **proper divisors**. Proper divisors are the factors of the number, excluding the number itself.
- A **deficient** number is a positive integer where the sum of its proper divisors is less than the number itself.
- An **abundant** number is a positive integer where the sum of its proper divisors is greater than the number itself.
- **Amicable** numbers are pairs of positive integers where each one is the sum of the proper divisors of the other.
- A **prime number** is a positive integer that has exactly two factors, one and itself. A **composite** number is a positive integer that has more than two factors.
- The **Sieve of Eratosthenes** is a method for finding prime numbers up to a particular number, n. Only the prime numbers less than or equal to $\sqrt{n}$ need to be checked.
- **Double or twin** primes differ by two. A **Mersenne** prime is a prime number that can be written in the form $2^p - 1$, where p is prime.
- A set is **closed** under an operation $*$ if and only if, for all a and b in S, $a * b$ is also in S.
- A **direct** proof involves giving logical arguments to show that a statement is true.
- A **counterexample** shows that a statement is false by giving one instance where it is not true.
- **Contraposition** means that 'if p then q' is the same as 'if not q then not p'.
- In a proof by **contradiction** the statement you are trying to prove, say A, is assumed to be false. This is then shown to lead to something that contradicts the assumption. This means that the assumption is false, so A is true.

- The principle of mathematical induction states that for a statement $P(n)$, if $P(1)$ is true and for all k, if $P(k)$ is true then it follows that $P(k + 1)$ is true then $P(n)$ is true for all positive integers n.
- To prove something by **mathematical induction,** perform the following steps.
  - Step 1: Under **RTP** (required to prove) state what has to be proved.
  - Step 2: Prove that the statement is true for $n = 1$.
  - Step 3: Assume that the statement is true for $n = k$, where $k \in \boldsymbol{Z}^+$.
  - Step 4: Show that it necessarily follows that the statement is true for $n = k + 1$.
  - Step 5: Make a formal statement that 'By the principle of mathematical induction …' and put in the statement, followed by **QED**.

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CHAPTER REVIEW

REAL NUMBERS AND PROOFS

7

Multiple choice

1 Example 1 Which one of the following sets of natural numbers does not contain an even number?

A composite
B prime
C numbers of the form $n^2 + n + 1$, where n is natural
D numbers of the form $n^2 + 2n + 3$, where n is natural
E perfect squares

2 Example 16 Which one of the following is incorrect?

A $\boldsymbol{Q}' \subset \boldsymbol{R}$ B $\boldsymbol{N} \subset \boldsymbol{Z}^+ \subset \boldsymbol{Q} \subset \boldsymbol{R}$ C $\boldsymbol{N} \subset \boldsymbol{Z} \subset \boldsymbol{Q} \subset \boldsymbol{R}$
D $\boldsymbol{Q} \subset \boldsymbol{Q}' \subset \boldsymbol{R}$ E $\boldsymbol{N} \subset \boldsymbol{Z}^+ \subset \boldsymbol{R}^+$

3 Example 7, 8 If a is the last digit of the number $219a$, then $\frac{219a}{175}$ can be written as a terminating decimal if a equals

A 1 only B 1 and 8 only C 8 only D 2 and 8 only E 0 and 1 only

4 Example 4 The set of integers is not closed under

A multiplication B squaring C division D subtraction E addition

5 Example 9 $\frac{1}{2}$ does not equal

A 0.500… B $0.4\overline{9}$ C 0.5 D 0.499 E $\frac{5}{10}$

Short answer

6 Example 1 a Show that when three odd numbers are added together, the result is an odd number.
b Show that when three odd numbers are multiplied together, the result is an odd number.

7 Example 2 Show that 100 is not a perfect number.

8 Example 3 a Write 511 in the form $2^a - 1$, where a is an integer, and explain why it is not a Mersenne prime.
b By testing prime numbers less than $\sqrt{511}$, show that 511 is not a prime number.

9 Example 4 Give counter examples to show that each of the following statements is false.
a When an even number is divided by an even number, the result is an even number.
b The set of composite numbers is closed under subtraction.

10 Example 5 Prove by contraposition that 2 is the only even prime number.

11 Example 6 Prove by contradiction that there is no greatest integer.

12 Example 7 a Convert −65.009 856 into a fraction.

b Convert $\frac{844}{2500}$ to a terminating decimal.

13 Example 8 Convert each of the following into a recurring decimal.

a $\frac{6543}{7}$ b $\frac{1}{81}$

14 Example 9 Express $3.82\overline{34}$ in rational form.

15 Example 10 Show that each of the following numbers are rational by writing them in the form $\frac{a}{b}$, where a and b are integers and b is not equal to zero.

a $-\frac{5}{6}$ b 0.875 c 7.23×10^{-2} d 0.24×10^{4} e $5\frac{4}{5}$

16 Example 11 a Show that the set of rational numbers is closed under subtraction.

b Give a counter example to show that the rational numbers are not closed under division.

17 Example 12 Prove that there are no rational solutions to the equation $x^3 + 2x - 1 = 0$ using contradiction.

18 Example 13 Prove that $\sqrt{3}$ is irrational.

19 Example 14 Prove $\log_2(7)$ is irrational.

20 Example 14 Prove by contradiction that the negative of any irrational number is irrational.

21 Example 15 Construct the decimal 0.010 000 100 000 001… by adding fractions of the form $\frac{1}{10^{2+a}}$, where $a \in \mathbf{Z}^+ \cup \{0\}$.

22 Examples 16, 17 Prove by contradiction that for every nonzero real number there exists a unique reciprocal.

23 Example 17 Disprove each of the following by giving a counter example.

a If a and b are rational, then a^b is rational.

b If a and b are irrational, then ab is irrational.

24 Example 18 Consider $x^4 + x^2 + c = 0$. For what values of c will there be real solutions?

25 Examples 19 and 20 Use mathematical induction to prove that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \text{ for } n \in \mathbf{N}.$$

26 Example 21 Use mathematical induction to prove that for all positive integers n, the sum of any three consecutive integers is divisible by 3.

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Application

Practice quiz

27 **a** **i** Write out the factors of 496.

ii Find the sum of the five smallest factors and comment on the result.

iii Find the sums of the six smallest factors and comment on the result.

iv Repeat the process until there are no more factors to sum.

v What type of number is 496?

b **i** Write 496 in the form $2^{p-1}(2^p - 1)$.

ii What does 2^{p-1} equal?

iii What does $2^p - 1$ equal?

c Use a direct proof to show that $2^p - 1$ is equal to the sum of the factors of 2^{p-1}, where p is a positive integer.

d **i** Write each of the following as the product of two factors, one of which is $x - 1$.
$x^2 - 1$, $x^3 - 1$, $x^4 - 1$, $x^5 - 1$ and $x^n - 1$, where n is a positive integer greater than 1.

ii Hence use proof by contraposition to show that if $2^p - 1$ is prime, then p is prime.

iii Give a counter example to show that if p is prime, then $2^p - 1$ is prime is a false statement.

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8

TERMINOLOGY

addition of matrices
column matrix
determinant
dimension
element
identity matrix
inverse
invertible
leading diagonal
matrix (matrices)
multiplication of matrices
non-singular
order
row matrix
scalar
singular
square matrix
zero matrix

Prior learning

MATRICES

MATRIX ARITHMETIC

MATRIX ARITHMETIC

- understand the matrix definition and notation (ACMSM051)
- define and use addition and subtraction of matrices, scalar multiplication, matrix multiplication, multiplicative identity and inverse (ACMSM052)
- calculate the determinant and inverse of 2 × 2 matrices and solve matrix equations of the form AX = B, where A is a 2 × 2 matrix and X and B are column vectors. (ACMSM053) AC

8.01 MATRICES

Matrices are one of the most important structures in mathematics. There is evidence of the study of matrices in the form of 'magic squares' in Chinese literature dating back to 650 BCE. The earliest mathematical application of matrices appears to be in solving linear equations, where they proved to be particularly useful in solving simultaneous equations. The study of matrices was developed by Arab mathematicians and spread to India when the Arabs conquered parts of the Indian subcontinent in the 7th century. The study of matrices was then applied to additional branches of mathematics as well as astronomy. The term 'matrix' first appeared in texts in 1848, after which time a number of famous mathematicians worked on matrix theory.

Matrices can be used to encrypt numerical data and they have a wide range of applications in computer graphics. The power of matrices lies in the fact that they provide a concise way to write and process information that would otherwise be unmanageable due to their volume and complexity.

IMPORTANT

A **matrix** (the plural is **matrices**) is a rectangular array of numbers enclosed in large parentheses () or brackets [ ]. In typing, a capital letter in boldface is used as the symbol for a matrix. In handwriting, you can underline the capital letter with a wavy line.

For example, $\mathbf{B} = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & 0 & 5 & 6 \end{bmatrix}$ could be written as $\underset{\sim}{B}$.

Each number in the matrix is called an **element** of the matrix.

The **size** (**dimension** or **order**) of a matrix is always given with the *rows first*.

Matrix **B** above is a 2 × 4 matrix.

A great deal of information is shown as tables of numbers. For some tables, there will be blank cells where it does not make sense to have an entry, but in most cases every cell will have a number, even if it is zero.

For example, the following table shows the wins, draws and losses in the 2011–2012 Tri-Series Cricket held in Australia between Australia, Sri Lanka and India.

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Team	Wins	Draws	Losses	Points
Sri Lanka	4	1	3	19
Australia	4	0	4	19
India	3	1	4	15

The match data is shown by the numbers in the rows and columns. You can see that there are 3 rows and 4 columns of data; they form a rectangular array of numbers.

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IMPORTANT

Elements of a matrix are referred to using the lower case letter of the matrix name and subscripts to indicate the row and column of the element. A general matrix may be represented by

$$\begin{array}{ccc} & n \text{ columns} & \\ \begin{array}{c} m \\ \text{rows} \end{array} & \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots \\ c_{21} & c_{22} & c_{23} & \cdots \\ c_{31} & c_{32} & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \searrow \end{bmatrix} & \begin{array}{c} i \text{ changes} \\ \downarrow \end{array} \\ & \text{j changes} \rightarrow & \end{array}$$

This means that matrix **C** has elements c_{ij} and is of size $m \times n$.

Example 1

a Write the matrix **C** for the cricket results shown in the table at the top of the page.
b What is the value of c_{23}?
c What is the size of the matrix?

Solution

a The headings and team names are omitted. There are 3 rows and 4 columns of data.

$$\begin{bmatrix} 4 & 1 & 3 & 19 \\ 4 & 0 & 4 & 19 \\ 3 & 1 & 4 & 15 \end{bmatrix}$$

b c_{23} is the value in the 2nd row and 3rd column. $c_{23} = 4$

c Size = no. of rows × no. of columns. The size of the matrix is 3×4

If a matrix **A** has the same number of rows as columns, then we say that it is a **square matrix**.

e.g. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

In a square matrix the elements a_{ii}, with $i = 1, 2, 3, \ldots$, are called **diagonal elements**.

A **diagonal matrix** is a square matrix with all the non-diagonal elements zero.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

A matrix with one row is called a **row matrix**.

e.g. $\begin{bmatrix} 2 & 1 & -3 & 0 \end{bmatrix}$

A matrix with one column is called a **column matrix**.

e.g. $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$

EXERCISE 8.01 Matrices

Concepts and techniques

1 Example 1 Which of the following represent matrices?

a $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

b $\begin{bmatrix} 2 & & \\ & 1 & \\ & & 3 \end{bmatrix}$

c $\begin{matrix} 1 & 4 \\ 3 & 2 \end{matrix}$

d $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & -1 \\ & -3 & -1 \end{bmatrix}$

e $\begin{bmatrix} \sqrt{3} & \frac{5}{6} \\ -6 & 2 \end{bmatrix}$

f $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2 $\mathbf{X} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 1 & -5 & 2 & -2 \\ -2 & 3 & 1 & -2 \end{bmatrix}$. If possible, write down each of the following.

a x_{12} b x_{21} c x_{34} d x_{22} e x_{32}

3 $\mathbf{R} = \begin{bmatrix} 0 & 4 & 7 & 2 & 3 \\ -1 & 3 & 5 & 0 & 8 \\ 8 & 6 & 1 & -2 & 10 \\ 3 & -9 & 0 & 3 & 4 \end{bmatrix}$

Express the relationship between each pair of elements below in simple forms like $2r_{41} = r_{32}$.

a r_{31} and r_{25}
b r_{14} and r_{45}
c r_{32} and r_{15}
d r_{42} and r_{44}
e r_{25} and r_{34}
f r_{35} and r_{34}

4 Name each of the following matrices stating their size, type and any other distinguishing features.

a $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
b $\begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$
c $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
d $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
e $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix}$

Reasoning and communication

5 A manufacturer builds desktop computers with a variety of central processor speeds and hard drive capacities. The result of a stock inventory is shown in the table below.

Hard Drive	1.86 GHz	2.0 GHz	3.8 GHz	2.33 GHz	2.67 GHz	2.8 GHz	3.0 GHz	3.2 GHz	3.5 GHz
160 GB	3	2	4	1	5	0	3	6	1
250 GB	1	8	1	7	1	8	0	5	8
320 GB	4	0	5	6	4	5	9	1	7
500 GB	7	9	2	0	1	1	10	9	1

a Write the information in the table as a 4×9 matrix **M**.
b Write the value of m_{32}.
c Write the value of m_{21}.
d Write the value of m_{27}.
e What is the order of **M**?

6 On a particular trading day, the exchange rates for Australian dollars (AUD), United States dollars (USD), Great Britain pounds (GBP) and Japanese yen (JPY) were as follows.

1 AUD = 1.05664 USD, 0.65720 GBP, 94.3892 JPY
1 USD = 0.94642 AUD, 0.62192 GBP, 89.3362 JPY
1 GBP = 1.52148 AUD, 1.60769 USD, 143.625 JPY
1 JPY = 0.01060 AUD, 0.01120 USD, 0.00696 GBP

a What is the equivalence of 1 AUD in USD?
b Put the information in a 4×4 matrix **E** in the order AUD, USD, GPB and JPY.
c What is the value of e As e_{24}?
d What is the value of e_{32}?
e In this matrix, what is the relationship between e_{ij} and e_{ji}, where $i \neq j$?

7 The following table indicates extreme temperatures and rainfall levels for each state in Australia as at 1 January 2013.

Capital	Max	Year	Min	Year	Rainfall	Year
South Australia	50.7	1960	−8.2	1976	1852.6	1917
Western Australian	50.5	1998	−7.2	2008	2380.6	2000
New South Wales	49.7	1939	−23.0	1994	4539.7	1950
Queensland	49.5	1972	−10.6	1965	12461.0	2000
Victoria	48.8	2009	−11.7	1970	3738.5	1956
Northern Territory	48.3	1960	−7.5	1976	2953.2	2000
Tasmania	42.2	2009	−13.0	1983	4504.1	1948

Bureau of Meteorology

a What would be the size of the matrix **W** formed by this information?
b What is the value of w_{11}?
c What is the value of w_{76}?
d What is the maximum value for $(c_{i1} - c_{i3})$, and what does this represent?

8 An ecologist interested in the predation of species in a freshwater pool studied the diets of water beetles, mosquito larvae, fingerlings and adult fish. She found that mosquito larvae and fingerlings both ate microscopic algae and microscopic animals in the water, but that fingerlings also ate approximately 3 mosquito larvae each day. Water beetles ate about 5 mosquito larvae and 2 fingerlings each day, and adult fish ate about 6 fingerlings and 4 water beetles each day.
a Write the information as the 4×4 matrix **P**, showing the predation of species on each other, with the animals in the order: mosquito larvae, fingerlings, water beetles and adult fish.
b What is the significance of the row of zeros for mosquito larvae?
c What is the meaning of p_{33}?

9 Construct the matrix (d_{ij}) if $1 \le i \le 3$, $1 \le j \le 4$, $d_{ij} = 1$ if $i = j$, $d_{ij} = 2$ if $i > j$ and $d_{ij} = 0$ if $i < j$.

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8.02 SCALAR MULTIPLICATION OF MATRICES

Inflation has the effect of increasing prices over time. The retail prices of certain 'milk' varieties were recorded in a particular year and are shown in the table below.

Milk type	1 L bottle	2 L bottle
Whole milk	\$1.85	\$3.09
Low fat	\$1.94	\$3.46
High calcium	\$2.04	\$3.95
Soy milk	\$2.35	\$4.55

In matrix form this is: $\mathbf{M} = \begin{bmatrix} 1.85 & 3.09 \\ 1.94 & 3.46 \\ 2.04 & 3.95 \\ 2.35 & 4.55 \end{bmatrix}$

Inflation is currently at approximately 3% per annum, meaning that prices of products such as milk can expect to increase by 3% each year.

So to increase each element in the matrix **M** by 3%, we obtain a new matrix **N** by multiplying by 1.03 as follows:

$$\mathbf{N} = 1.03 \times \begin{bmatrix} 1.85 & 3.09 \\ 1.94 & 3.46 \\ 2.04 & 3.95 \\ 2.35 & 4.55 \end{bmatrix} = \begin{bmatrix} 1.03 \times 1.85 & 1.03 \times 3.09 \\ 1.03 \times 1.94 & 1.03 \times 3.46 \\ 1.03 \times 2.04 & 1.03 \times 3.95 \\ 1.03 \times 2.35 & 1.03 \times 4.55 \end{bmatrix} \approx \begin{bmatrix} 1.91 & 3.18 \\ 2.00 & 3.56 \\ 2.10 & 4.07 \\ 2.42 & 4.69 \end{bmatrix}$$

Multiplication of a matrix by a number is called multiplication by a **scalar**.

IMPORTANT

Given a matrix **A** and a constant c, the matrix $c\mathbf{A}$ is defined by $c\mathbf{A} = (ca_{ij})$.

The new matrix is referred to as a **scalar multiple** of the original matrix.

Example 2

Given that

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & -3 & -5 \\ -2 & 5 & 4 & 2 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 6 & 3 \\ 4 & -2 \end{bmatrix}, \text{ find}$$

a $3\mathbf{A}$ **b** $-\mathbf{B}$ **c** $\frac{1}{2}\mathbf{C}$

Solution

a Multiply every element by 3.

$$3\mathbf{A} = 3\begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 9 \\ -3 & 3 \\ 9 & 0 \end{bmatrix}$$

b Multiply every element by -1.

$$-1\mathbf{B} = -1 \times \begin{bmatrix} 1 & 0 & -3 & -5 \\ -2 & 5 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 & 5 \\ 2 & -5 & -4 & -2 \end{bmatrix}$$

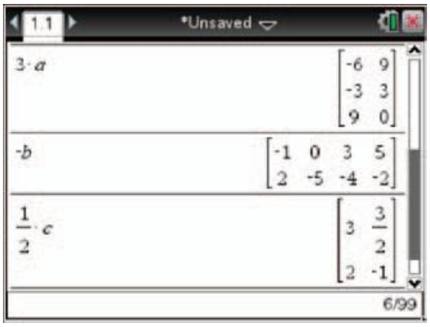
c Multiply every element by $\frac{1}{2}$.

$$\frac{1}{2}\mathbf{C} = \frac{1}{2} \times \begin{bmatrix} 6 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 1\frac{1}{2} \\ 2 & -1 \end{bmatrix}$$

TI-Nspire CAS

From the menu, choose 1: Actions 1: Define. Press A and use menu, 7:Matrix & Vector, 1:Create and 1: Matrix. Select 3 rows and 2 columns and fill to make **A**.

When you've made all three, find 3**A**, –**B** and use and choose for $\frac{1}{2}$**C.**



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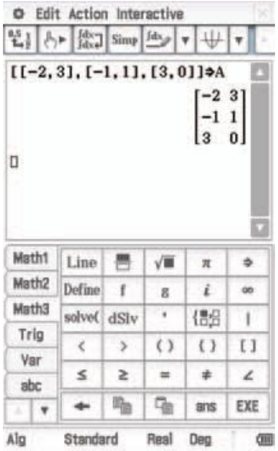
ClassPad

Use the $\overset{\text{Main}}{\sqrt{\alpha}}$ application. .

Start with square brackets obtainable from the keyboard shown after you press Keyboard then tap Math3.

Enter each row in square brackets, with the numbers separated by commas.

Finish with ⇒ A.

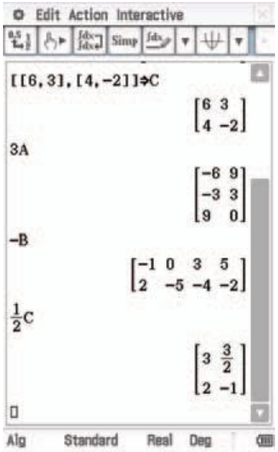


Follow a similar process to enter **B** and **C**.

To multiply by 3, type 3**A** then tap EXE or press EXE.

Repeat for –**B** and $\frac{1}{2}$**C**

For –**B**, use the (–) key.



EXERCISE 8.02 Scalar multiplication of matrices

Concepts and techniques

1 Example 2 Given that $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}$, find:

a 3**A** b 0.4**B** c 5**B** d −4**A** e **B** ÷ 3

2 Given that $\mathbf{X} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 5 & 0 & -1 \\ 0 & 5 & -2 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & -3 & 1 \\ 4 & 2 & 4 \\ -4 & -3 & -2 \end{bmatrix}$, find:

a −2**X** b 0.5**Y** c −**Y** d 3**X**

3 Given that $\mathbf{M} = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 5 & -4 & 3 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 1 & 4 & 3 & -5 \\ -3 & 2 & -1 & -6 \\ 0 & 5 & -2 & 3 \end{bmatrix}$, find:

a $5\mathbf{M}$ b $-3\mathbf{N}$ c $4\mathbf{M}$ d $-5\mathbf{N}$

Reasoning and communication

4 The wholesale prices ($) of some paint products are shown in the following matrix. The first column represents the prices of 100 mL tubes, the second the prices of 250 mL tubes and the third the prices of 400 mL tubes. The first row is for *black*, the second for *red*, the third for *yellow* and the fourth for *white*.

$$\mathbf{W} = \begin{bmatrix} 2.40 & 4.90 & 8.40 \\ 4.20 & 7.90 & 11.80 \\ 3.80 & 8.40 & 11.00 \\ 5.90 & 11.80 & 14.40 \end{bmatrix}$$

a The retailer applies a mark-up of 70%. Calculate the matrix for the retail prices.
b A GST of 10% has to be added to get the final price for the customer. What is the matrix for the prices actually paid by the customer?

5 A local cattle farmer sells the following head of cattle at the local saleyard each Monday for 5 weeks. Matrix **Y** represents the number of yearlings sold and matrix **C** represents the number of calves sold. The rows represent the different Mondays that he went to the saleyards.

$$\mathbf{Y} = \begin{bmatrix} 20 \\ 16 \\ 25 \\ 23 \\ 19 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 4 \\ 1 \\ 7 \\ 2 \\ 3 \end{bmatrix}$$

a If yearlings sold each week for $350, write the matrix to show the income for the five weeks from the sale of the yearlings.
b If calves sold for $60 each week, write a matrix to show the income for the five weeks from the sale of the calves.
c What was the total income from all sales over the five weeks?

8.03 ADDITION AND SUBTRACTION OF MATRICES

The sales of mobile telephone plans for January for three Telstra outlets are shown below.

Telstra outlet	Cap plan	Phone plan	Member plan
Queen Street Mall, Brisbane	150	28	35
Castletown Shopping Centre, Townsville	89	15	22
Pacific Fair Shopping Centre, Gold Coast	125	12	43

Telstra website

This information can be represented as the matrix $\mathbf{J} = \begin{bmatrix} 150 & 28 & 35 \\ 89 & 15 & 22 \\ 125 & 12 & 43 \end{bmatrix}$.

If the sales figures for February are represented as $\mathbf{F} = \begin{bmatrix} 182 & 33 & 47 \\ 112 & 23 & 17 \\ 138 & 36 & 46 \end{bmatrix}$, how can **J** and **F** be combined to show the sales figures for the various locations for the January/February period?

The January sales for the Queen Street Mall outlet are shown in the first row of **J** and the February sales for the same store are shown in the first row of **F**. This means that it only makes sense to add corresponding rows of **J** and **F** to find the totals. As the first columns of **J** and **F** show the number of cap plans sold in each month, it makes sense to add corresponding elements within each row.

In doing this, the total sales by Telstra (**T**) can be shown as follows.

$$\mathbf{T} = \begin{bmatrix} 150 & 28 & 35 \\ 89 & 15 & 22 \\ 125 & 12 & 43 \end{bmatrix} + \begin{bmatrix} 182 & 33 & 47 \\ 112 & 23 & 17 \\ 138 & 36 & 46 \end{bmatrix} = \begin{bmatrix} 150+182 & 28+33 & 35+47 \\ 89+112 & 15+23 & 22+17 \\ 125+138 & 12+36 & 43+46 \end{bmatrix} = \begin{bmatrix} 332 & 61 & 82 \\ 201 & 38 & 39 \\ 263 & 48 & 89 \end{bmatrix}$$

IMPORTANT

The process of adding matrices is known as **matrix addition**.

Given two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, we define the sum $\mathbf{A} + \mathbf{B}$ as the $m \times n$ matrix $\mathbf{C}$ such that $c_{ij} = a_{ij} + b_{ij}$ for all i, j.

Corresponding elements are added. That is, $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.

Example 3

Find $\mathbf{A} + \mathbf{B}$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 9 & 8 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

Solution

Add corresponding elements, i.e., $a_{11} + b_{11}$, $a_{12} + b_{12}$, etc

$$\begin{bmatrix} 0 & 1 & 2 \\ 9 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0+6 & 1+5 & 2+4 \\ 9+3 & 8+4 & 7+5 \end{bmatrix}$$

Write $\mathbf{A} + \mathbf{B}$

$$= \begin{bmatrix} 6 & 6 & 6 \\ 12 & 12 & 12 \end{bmatrix}$$

If two matrices are not the same size, they cannot be added.

Work out $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -3 \\ -2 & 5 & 4 \end{bmatrix}$. What do you find?

Work out $\begin{bmatrix} -3 & -4 & 3 \\ 2 & -5 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -3 \\ -2 & 5 & 4 \end{bmatrix}$. What do you find?

IMPORTANT

The identity for matrix addition is the **zero matrix** $\mathbf{O} = (o_{ij})$, where $o_{ij} = 0$ for all i, j.

If required, the size of the zero matrix is included as a subscript, as in $\mathbf{O}_{4\times 5}$.

The inverse for matrix addition is the matrix such that every element is the opposite sign of the given matrix. This is actually $(-1)\mathbf{A}$, normally written as $-\mathbf{A}$.

The subtraction of matrices is considered as addition of the **additive inverse**.

$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = \mathbf{A} + (-\mathbf{B})$

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Example 4

Given $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & -3 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 6 & -4 \\ 2 & -3 \\ 3 & 2 \end{bmatrix}$, find:

a $\mathbf{A} - \mathbf{C}$ **b** $\mathbf{B} - \mathbf{A} - \mathbf{C}$ **c** $2\mathbf{A} - \mathbf{C}$

Solution

a Treat $\mathbf{A} - \mathbf{C}$ as $\mathbf{A} + (-\mathbf{C})$

$$\mathbf{A} - \mathbf{C} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 7 \\ -3 & 4 \\ 0 & -2 \end{bmatrix}$$

b Treat $\mathbf{B} - \mathbf{A} - \mathbf{C}$ as $\mathbf{B} + (-\mathbf{A}) + (-\mathbf{C})$

$$\mathbf{B} - \mathbf{A} - \mathbf{C} = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 1 & -1 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -1 & 0 \\ -5 & -5 \end{bmatrix}$$

c Treat $2\mathbf{A} - \mathbf{C}$ as $2\mathbf{A} + (-\mathbf{C})$

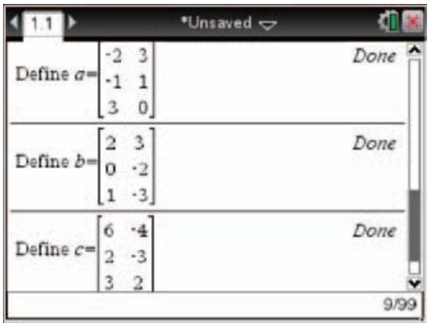
$$2\mathbf{A} - \mathbf{C} = 2 \times \begin{bmatrix} -2 & 3 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 6 \\ -2 & 2 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ -2 & 3 \\ -3 & -2 \end{bmatrix}$$

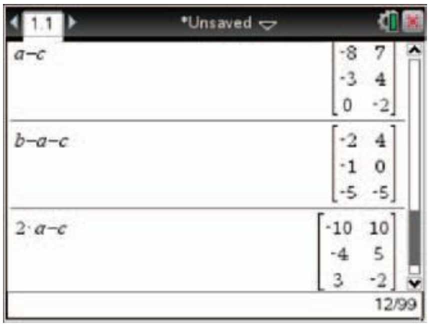
$$= \begin{bmatrix} -10 & 10 \\ -4 & 5 \\ 3 & -2 \end{bmatrix}$$

TI-Nspire CAS

Define each matrix (see page 288).

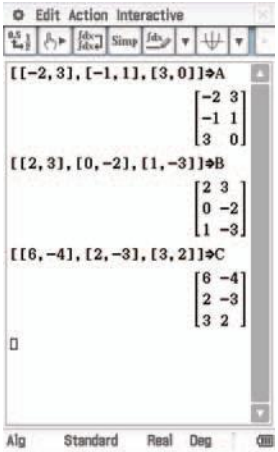


Now calculate **A** – **C**, **B** – **A** – **C** and 2**A** – **C**.

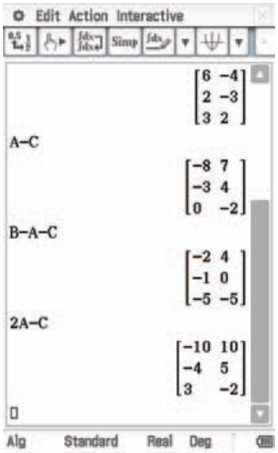


ClassPad

Define **A**, **B** and **C** as before.



Enter each matrix operation, followed by EXE.



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IMPORTANT

Laws for addition and scalar multiplication of matrices

The following laws hold for any matrices **A**, **B** and **C** and scalars r and s.

1. Matrix addition is associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
2. Matrix addition is commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
3. The additive identity **O** has all elements equal to zero and is called the **zero matrix**.
4. The additive inverse of $\mathbf{A} = (a_{ij})$ is $-\mathbf{A} = (-a_{ij})$.
5. Cancellation laws hold for scalar multiplication of a matrix:
   $$r\mathbf{A} = r\mathbf{B} \Leftrightarrow \mathbf{A} = \mathbf{B}\ (r \neq 0)$$
   $$r\mathbf{A} = s\mathbf{A} \Leftrightarrow r = s\ (\mathbf{A} \neq \mathbf{O})$$
6. Distributive laws hold for:
   Scalar multiplication over addition: $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$
   Scalar multiplication over matrix addition: $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$
7. An associative law holds for multiplication and scalar multiplication:
   $$(rs)\mathbf{A} = r(s\mathbf{A})$$

EXERCISE 8.03 Addition and subtraction of matrices

Concepts and techniques

WS
Addition and subtraction of matrices

1 Example 3 Given that

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -6 & 4 \\ 8 & -10 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 5 & -7 \\ -3 & 1 \end{bmatrix}, \text{ find:}$$

a $\mathbf{C} + \mathbf{A}$　b $\mathbf{B} + \mathbf{D}$　c $\mathbf{C} + \mathbf{B}$　d $5\mathbf{A} + \mathbf{B}$　e $3\mathbf{B} + 2\mathbf{D}$

2 Example 4 Given that

$$\mathbf{X} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 5 & 0 & -1 \\ 0 & 5 & -2 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & -3 & 1 \\ 4 & 2 & 4 \\ -4 & -3 & -2 \end{bmatrix} \text{ and } \mathbf{Z} = \begin{bmatrix} 1 & 5 & 8 \\ 0 & 2 & 4 \\ 0 & 6 & 2 \\ 7 & 0 & 0 \end{bmatrix}, \text{ find:}$$

a $\mathbf{Z} + \mathbf{X}$　b $\mathbf{X} + \mathbf{Z}$　c $\mathbf{Y} + \mathbf{Z}$
d $4\mathbf{Z} + \mathbf{X}$　e $2\mathbf{Z} + 3\mathbf{X} + 2\mathbf{Y}$

3 Given that

$$\mathbf{L} = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 5 & -4 & 3 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 1 & 4 & 3 & -5 \\ -3 & 2 & -1 & -6 \\ 0 & 5 & -2 & 3 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} 6 & -5 & 4 & -3 \\ -1 & 8 & -9 & 2 \\ 4 & 7 & -2 & 5 \end{bmatrix}, \text{ find:}$$

a $\mathbf{L} + \mathbf{M}$　b $\mathbf{N} + \mathbf{M}$　c $4\mathbf{L} + 3\mathbf{N}$
d $2\mathbf{L} + 5\mathbf{N} + \mathbf{M}$　e $3\mathbf{N} - 6\mathbf{M} - 5\mathbf{L}$

4 Given that

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 9 & -3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -6 & 4 \\ 8 & -10 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 5 & -7 \\ -3 & 1 \end{bmatrix}, \text{ find:}$$

a $\mathbf{C} - \mathbf{A}$ b $\mathbf{B} - \mathbf{D}$ c $\mathbf{C} - \mathbf{B}$ d $5\mathbf{A} - \mathbf{B}$ e $3\mathbf{B} - 2\mathbf{D}$

5 Given that

$$\mathbf{X} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 3 & 6 \\ 5 & 0 & -1 \\ 0 & 5 & -2 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} -1 & -4 & -3 \\ 2 & -3 & 1 \\ 4 & 2 & 4 \\ -4 & -3 & -2 \end{bmatrix} \text{ and } \mathbf{Z} = \begin{bmatrix} 1 & 5 & 8 \\ 0 & 2 & 4 \\ 0 & 6 & 2 \\ 7 & 0 & 0 \end{bmatrix}, \text{ find:}$$

a $\mathbf{Z} - \mathbf{X}$ b $\mathbf{X} - \mathbf{Z}$ c $\mathbf{Y} - \mathbf{Z}$ d $4\mathbf{Z} - \mathbf{X}$ e $2\mathbf{Z} - 3\mathbf{X} + 2\mathbf{Y}$

6 Given that

$$\mathbf{L} = \begin{bmatrix} 3 & 0 & -1 & 2 \\ 5 & -4 & 3 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 1 & 4 & 3 & -5 \\ -3 & 2 & -1 & -6 \\ 0 & 5 & -2 & 3 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} 6 & -5 & 4 & -3 \\ -1 & 8 & -9 & 2 \\ 4 & 7 & -2 & 5 \end{bmatrix}, \text{ find:}$$

a $\mathbf{L} - \mathbf{M}$ b $\mathbf{N} - \mathbf{M}$ c $4\mathbf{L} - 3\mathbf{N}$
d $2\mathbf{L} + 5\mathbf{M} - \mathbf{N}$ e $3\mathbf{N} - 6\mathbf{M} - 5\mathbf{L}$

Reasoning and communication

7 A bank has three branches in Melbourne: City, Essendon and St Kilda. There are four types of accounts: Low Fee, Business, Savings and Student/Pensioner. The numbers of each type of account at the beginning of a year were as shown in the table below.

	Low Fee	Business	Savings	Stud/Pens
City	300	400	200	150
Essendon	400	220	300	200
St Kilda	250	150	400	150

The numbers of closures of accounts in that year were:

	Low Fee	Business	Savings	Stud/Pens
City	30	70	20	20
Essendon	20	30	40	30
St Kilda	40	10	50	10

The numbers of new accounts opened in that year were:

	Low Fee	Business	Savings	Stud/Pens
City	50	100	40	15
Essendon	15	50	30	10
St Kilda	30	25	20	40

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Use 3×4 matrices to show:

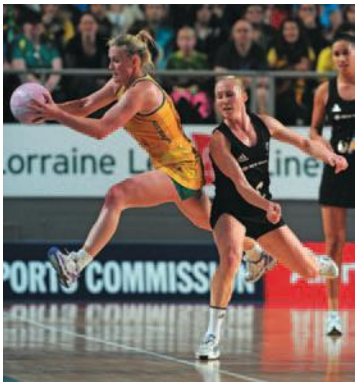
a the number of accounts at the beginning of the year

b the number of closures in the year

c the number of accounts opened in the year

d an evaluated matrix expression for the number of accounts at the end of the year.

8 The results of the first five rounds, middle five rounds and last five rounds of a netball competition are shown by the following three matrices. Each row represents the results of a team. The first column is the number of games won, the second is drawn games and the third is games lost.


Newspix/George Salpigtidis

$$\mathbf{F} = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{L} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

a Write a matrix expression for the total results.

b Evaluate your expression to obtain the final results.

c Which team (1st to 6th) is the top team?

8.04 MATRIX MULTIPLICATION

Matrices provide a useful way to write numerical information that is categorised simultaneously in two ways. The purchase of items from suppliers at different prices and the performances of bank branches with different types of accounts are two of the applications you have already seen. In many cases, the main categories are further divided into subcategories, leading to the use of related matrices.

A supplier of bulk meat makes up different packs so that hotels, hostels and caterers may order different mixes of meat to suit their requirements. The quantities, in kilograms, of different meats in each type of pack are shown in the table below.

Pack	Sausages	Mince	Chops	Stewing Steak	BBQ Steak	Prime Steak
BBQ	20	10	15	0	25	0
Basic	15	15	10	20	5	5
Quality	5	20	20	10	0	15
Deluxe	5	15	15	5	0	30

This can be written as the matrix:

$$\mathbf{B} = \begin{bmatrix} 20 & 10 & 15 & 0 & 25 & 0 \\ 15 & 15 & 10 & 20 & 5 & 5 \\ 5 & 20 & 20 & 10 & 0 & 15 \\ 5 & 15 & 15 & 5 & 0 & 30 \end{bmatrix}$$


Shutterstock.com/Kitch Bain

Over a period of 3 months, a caterer specialising in home parties orders the following packs.

Month	BBQ	Basic	Quality	Deluxe
July	2	3	1	0
August	3	2	2	1
September	4	1	1	3

This can be written as the matrix:

$$\mathbf{C} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 1 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

The two matrices can be combined to find the total quantity of each type of meat ordered each month. We need to combine the packs for each month with the quantities of meat in each pack. That is, we need to combine the matrices in the order

$$\mathbf{C} \times \mathbf{B} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 1 \\ 4 & 1 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 20 & 10 & 15 & 0 & 25 & 0 \\ 15 & 15 & 10 & 20 & 5 & 5 \\ 5 & 20 & 20 & 10 & 0 & 15 \\ 5 & 15 & 15 & 5 & 0 & 30 \end{bmatrix}$$

To obtain the quantity of sausages ordered in July, we work out:

$$2 \times 20 + 3 \times 15 + 1 \times 5 + 0 \times 5 = 90 \text{ kg}$$

To obtain the quantity of mince ordered in July, we work out:

$$2 \times 10 + 3 \times 15 + 1 \times 20 + 0 \times 15 = 85 \text{ kg}$$

For each type of meat in July, we multiply the packs ordered in July by the composition of each pack, and add the results.

Similarly, for each type of meat in August, we multiply the elements of the August row by the corresponding elements of the meat column.

Placing each month's meats in a row, and each type of meat in a column, we obtain:

$$\mathbf{A} = \begin{bmatrix} 90 & 85 & 80 & 70 & 65 & 30 \\ 105 & 115 & 120 & 65 & 85 & 70 \\ 115 & 120 & 135 & 45 & 105 & 110 \end{bmatrix}$$

The matrix gives the total amount of each meat each month, and could be written in table form as follows.

Month	Sausages	Mince	Chops	Stewing Steak	BBQ Steak	Prime Steak
July	90	85	80	70	65	30
August	105	115	120	65	85	70
September	115	120	135	45	105	110

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Although the operation we have performed on **C** and **B** involves both multiplication and addition, we call it **matrix multiplication**. We write either **A** = **CB** or **A** = **C** × **B**.

IMPORTANT

The **product** of two matrices **A** and **B** exists if and only if **A** has the same number of columns as **B** has rows. Matrices that fulfill this condition are called **conformable matrices**. If **A** is an $n \times m$ matrix (a_{ij}) and **B** is an $m \times p$ matrix (b_{ij}), then the product matrix **C** is an $n \times p$ matrix (c_{ij}) such that c_{ij} is the sum of the products of elements in the ith row of **A** and the corresponding elements of the jth column of **B**.

That is, $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$

Example 5

Find the product **AB** of matrices $\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}$.

Solution

Call the product **P** for convenience.

$$\mathbf{P} = \mathbf{AB} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}$$

Multiply the first *row by* the first *column*.

$$p_{11} = 2 \times 1 + 3 \times 8 + (-1) \times 6 = 20$$

Multiply the second *row by* the first *column*.

$$p_{12} = 4 \times 1 + 2 \times 8 + 2 \times 6 = 32$$

Put the numbers in their correct places and write the product of **A** and **B**.

$$\mathbf{AB} = \begin{bmatrix} 20 \\ 32 \end{bmatrix}$$

Example 6

Find the product **FE** of the matrices $\mathbf{E} = \begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 0 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{F} = \begin{bmatrix} -1 & 0 & -2 & -3 \\ 1 & 2 & 2 & 1 \\ -3 & 4 & -4 & 0 \\ 2 & -5 & -1 & 3 \end{bmatrix}$.

Solution

F has to be first. Call the product **P** for convenience.

$$\begin{aligned} \mathbf{P} &= \mathbf{FE} \\ &= \begin{bmatrix} -1 & 0 & -2 & -3 \\ 1 & 2 & 2 & 1 \\ -3 & 4 & -4 & 0 \\ 2 & -5 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 0 & 3 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

Multiply the first *row by* the first *column*.

$$\begin{aligned} p_{11} &= (-1) \times 1 + 0 \times (-2) + (-2) \times 0 + (-3) \times 4 \\ &= -13 \end{aligned}$$

Multiply the first *row by* the second *column*.

$$\begin{aligned} p_{12} &= (-1) \times 4 + 0 \times 1 + (-2) \times 3 + (-3) \times 2 \\ &= -16 \end{aligned}$$

Put these answers in place.

$$\begin{aligned} \mathbf{P} &= \mathbf{FE} \\ &= \begin{bmatrix} -13 & 16 \end{bmatrix} \end{aligned}$$

Now do the second row.

$$\begin{aligned} p_{21} &= 1 \times 1 + 2 \times (-2) + 2 \times 0 + 1 \times 4 \\ &= 1 \\ p_{22} &= 1 \times 4 + 2 \times 1 + 2 \times 3 + 1 \times 2 \\ &= 14 \end{aligned}$$

Put the answers in place.

$$\begin{aligned} \mathbf{P} &= \mathbf{FE} \\ &= \begin{bmatrix} -13 & 16 \\ 1 & 14 \end{bmatrix} \end{aligned}$$

Continue with the other two rows.

$$\begin{aligned} p_{31} &= (-3) \times 1 + 4 \times (-2) + (-4) \times 0 + 0 \times 4 \\ &= -11 \\ p_{32} &= (-3) \times 4 + 4 \times 1 + (-4) \times 3 + 0 \times 2 \\ &= -20 \\ p_{41} &= 2 \times 1 + (-5) \times (-2) + (-1) \times 0 + 3 \times 4 \\ &= 24 \\ p_{42} &= 2 \times 4 + (-5) \times 1 + (-1) \times 3 + 3 \times 2 \\ &= 6 \end{aligned}$$

Write the product of **F** and **E**.

$$\mathbf{FE} = \begin{bmatrix} -13 & -16 \\ 1 & 14 \\ -11 & -20 \\ 24 & 6 \end{bmatrix}$$

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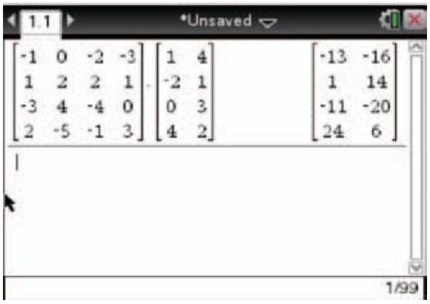
TI-Nspire CAS

From the [menu], choose 7:Matrix & Vector, 1:Create and 1: Matrix. Select 4 rows and 4 columns for **F** and fill the matrix with elements. Now multiply [×] by **E**, creating a 4×2 matrix.



ClassPad

Matrices can be entered directly using the [Math2] keyboard.

Press [Keyboard] then tap [Math2].

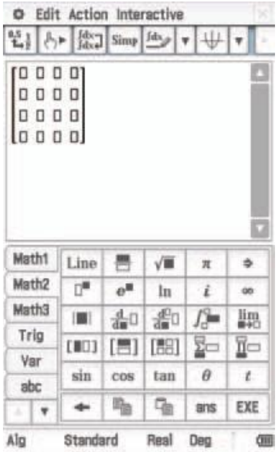
Note the effect of the following keys.

[1×2 matrix key]: Starts a 1×2 matrix; adds a column.

[2×1 matrix key]: Starts a 2×1 matrix; adds a row.

[2×2 matrix key]: Starts a 2×2 matrix; adds a row and a column.

The easiest way to get a 4×4 matrix is to tap [2×2 matrix key] three times.

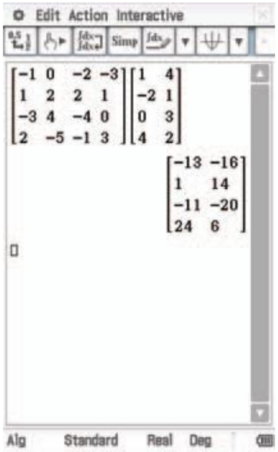


Enter the numbers in the matrix by tapping each square and typing the number. You can also use the arrows to move around the elements of the matrix.

To start the second matrix, tap [1×2 matrix key], then tap [2×1 matrix key] three times to add three more rows.

There is no need for a multiplication sign.

Enter the numbers and press [EXE].



You could use Define() in Example **6** so that once you had set up **F** and **E**, they could be used again. You would only do this if you had a number of operations to perform with the same matrices.

To multiply matrices, the number of columns in the first matrix *must* equal the number of rows in the second matrix.

If **C** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, then **CB** can be calculated and it is an $m \times p$ matrix. However, if we try to multiply **BC**, we find that the number of columns and rows are not equal as required and so the multiplication is not defined.

Thus, in general, matrix multiplication is not commutative, $\mathbf{AB} \neq \mathbf{BA}$. However, it is associative so that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, although the general proof of associativity is quite difficult. For matrices of small sizes we can show that associativity is true, using the properties of real numbers.

Example 7

Prove that products of matrices are associative, where the first matrix is 1×2, the second is 2×2 and the third is 2×1.

Solution

State what has to be proved.	**RTP** $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 1×2, 2×2 and 2×1 respectively.
Set up general matrices for a proof.	**Proof** Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$
Start with the LHS.	$\text{LHS} = \mathbf{A}(\mathbf{BC})$ $= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}\right)$
Work out **BC**.	$= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}\begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} \\ b_{21}c_{11} + b_{22}c_{21} \end{bmatrix}$
Now multiply by **A**.	$= [a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21}]$
Rearrange.	$= [a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21}]$
Factorise.	$= [(a_{11}b_{11} + a_{12}b_{21})c_{11} + (a_{11}b_{12} + a_{12}b_{22})c_{21}]$
Express as a product.	$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$
Express the first matrix as a product.	$= \left(\begin{bmatrix} a_{11} & a_{11} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$
Write in symbols.	$= \mathbf{AB}(\mathbf{C}) = \text{RHS}$
Complete proof.	Since LHS = RHS, $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ **QED**

EXERCISE 8.04 Matrix multiplication

Concepts and techniques

WS Multiplying matrices

1 Example 5 Calculate the following.

a $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$

c $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$

d $\begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}$

e $\begin{bmatrix} -9 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

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2 Example 6 Calculate the following.

a $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 0 & 4 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 5 & -1 \\ -2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ -1 & 2 \\ 4 & -3 \end{bmatrix}$

d $\begin{bmatrix} 2 & 3 & 1 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 0 & -5 \\ 1 & -2 \\ -4 & 3 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 5 & -3 \\ 2 & 3 & 4 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 4 \\ 4 & -3 & 2 \end{bmatrix}$

3 Calculate $\begin{bmatrix} 3 & 6 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 10 & 5 \end{bmatrix}$ and comment on the result.

4 Calculate $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 11 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 3 \\ 1 & 11 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ and comment on the result.

Reasoning and communication

5 Example 7 Prove that $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ for 2×2 matrices **B** and **C** and a 1×2 matrix **A**. What is the name of this law?

6 Prove that matrix multiplication is associative for products of 2×2 matrices in general.

7 A kitchenware chain has four stores ordering stock from a central warehouse. The orders for cutlery sets for a catalogue special are as follows.

Setting	Store A	Store B	Store C	Store D
6 place	12	15	10	12
8 place	10	8	10	12
10 place	8	4	10	4

a Write the orders as a 4×3 matrix.

b The prices of the sets are \$117.80, \$155.00 and \$190.00 respectively. Write the prices as a column matrix.

c Multiply the matrices to obtain the invoice amounts for each store.

8 The costs of trucking small parcels between Brisbane, Mt Isa, Rockhampton, Sydney and Toowoomba are shown by the following matrix, where the rows give the costs from each place in alphabetical order.

$$\mathbf{C} = \begin{bmatrix} 0 & 10.80 & 9.80 & 6.40 & 5.00 \\ 10.80 & 0 & 7.20 & 16.40 & 15.40 \\ 9.80 & 7.20 & 0 & 16.20 & 14.40 \\ 6.40 & 16.40 & 16.20 & 0 & 11.00 \\ 5.00 & 15.40 & 14.40 & 11.00 & 0 \end{bmatrix}$$

a Why does the diagonal have zeros?

b The handling costs for parcel delivery are a flat \$4. Write a matrix **H** to give the handling costs and calculate the matrix **T** to give the total delivery costs for small parcels.

c Medium-sized parcels cost $1\frac{1}{2}$ times as much for trucking but the handling costs are the same. Write a matrix expression for the delivery costs for medium-sized parcels.

d Large parcels cost twice as much for handling and three times as much for trucking as small parcels. Write a matrix expression for the delivery costs of large parcels.

9 The results of the first five rounds, middle five rounds and last five rounds of a netball competition are shown in the following three matrices. Each row represents the results of a team. The first column is the number of games won, the second is the number of drawn games and the third is the number of games lost.

$$\mathbf{F} = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & 0 & 2 \\ 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix}, \mathbf{L} = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

Team scores are calculated by allotting 3 points for a win, 1 point for a draw and 0 for a loss.

a Write the points as a column matrix **P**.

b Use matrix multiplication to find the team points for each group of five rounds.

c Calculate $(\mathbf{F} + \mathbf{M} + \mathbf{L})\mathbf{P}$ and $\mathbf{FP} + \mathbf{MP} + \mathbf{LP}$ and comment on your results.

8.05 IDENTITIES AND INVERSES

You need to understand multiplicative identity and inverse matrices to solve matrix equations.

IMPORTANT

The **identity matrix**, $\mathbf{I}_n$ is such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all $n \times n$ matrices **A**.

Since the identity and matrices must commute, an identity can exist only for square matrices.

Example 8

Show that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity for 2×2 matrices.

Solution

Specifically state what has to be proved.

RTP

$$\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{A} = \mathbf{A} \text{ for all } 2 \times 2 \text{ matrices } \mathbf{A}.$$

Choose a general matrix.

Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Left multiply.

$$\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \times 1 + a_{12} \times 0 & a_{11} \times 0 + a_{12} \times 1 \\ a_{21} \times 1 + a_{22} \times 0 & a_{21} \times 0 + a_{22} \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{A}$$

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Right multiply.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
	$= \begin{bmatrix} 1 \times a_{11} 1 + 0 \times a_{12} & 0 \times a_{11} + 1 \times a_{12} \\ 1 \times a_{21} + 0 \times a_{22} & 0 \times a_{21} + 1 \times a_{22} \end{bmatrix}$
	$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathbf{A}$
Write the conclusion.	Thus $\mathbf{A}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{A} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$. QED

The function δ_{ij} is called the **Kronecker delta** function and is used in many areas of mathematics.

In the real numbers, 1 is the only number that can be squared to give itself and 0 is the only number that can be squared to give 0. This is not true for matrices.

Since there is a multiplicative identity, it is possible that an inverse will also exist.

IMPORTANT

The **identity matrix**, $\mathbf{I}_n$ has elements of 1 along the **leading diagonal** and zeros for the other elements.
$\mathbf{I}_n = (\delta_{ij})$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

IMPORTANT

If $\mathbf{A}$ is a square matrix, then its **inverse matrix** is denoted by $\mathbf{A}^{-1}$.

$\mathbf{A}^{-1}$ is the matrix such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the **identity matrix**.

A matrix that has an inverse is called **invertible** or **non-singular**.

A matrix for which no inverse exists is said to be **singular**.

A matrix that squares to give itself ($\mathbf{A}^2 = \mathbf{A}$) is called **idempotent**.

A matrix that squares to give the zero matrix ($\mathbf{A}^2 = \mathbf{0}$) is called **nilpotent**.

Example 9

Find the inverse matrix for $\mathbf{X} = \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$.

Solution

Write a general matrix for the inverse.

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{I}$.

Work out the matrix product.

$$\mathbf{AX} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a \times (-4) + b \times 1 & a \times 2 + b \times 0 \\ c \times (-4) + d \times 1 & c \times 2 + d \times 0 \end{bmatrix}$$
$$= \begin{bmatrix} b - 4a & 2a \\ d - 4c & 2c \end{bmatrix}$$

Now write the equality.

$$\begin{bmatrix} b - 4a & 2a \\ d - 4c & 2c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Write the equation from the equality.

$b - 4a = 1$, $2a = 0$, $d - 4c = 0$ and $2c = 1$

Solve for a and c.

$a = 0$ and $c = \frac{1}{2}$

Find b and d.

$b = 1$ and $d = 2$

Write $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}$$

Check the value of $\mathbf{AX}$.

$$\mathbf{AX} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \times (-4) + 1 \times 1 & 0 \times 2 + 1 \times 0 \\ \frac{1}{2} \times (-4) + 2 \times 1 & \frac{1}{2} \times 2 + 2 \times 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now work out $\mathbf{XA}$.

$$\mathbf{XA} = \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -4 \times 0 + 2 \times \frac{1}{2} & -4 \times 1 + 2 \times 2 \\ 1 \times 0 + 0 \times \frac{1}{2} & 1 \times 1 + 0 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Write the result.

$\mathbf{AX} = \mathbf{XA} = \mathbf{I}$, so $\mathbf{X}^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 2 \end{bmatrix}$

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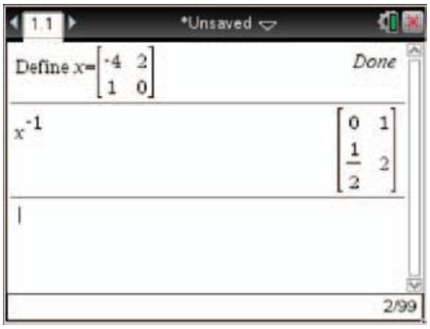
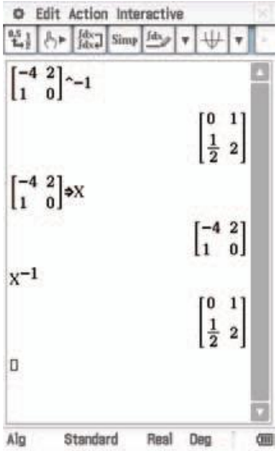
TI-Nspire CAS

Define the matrix **X** (see page 288).
Now use [^] to find the inverse of the matrix (–1).



ClassPad

Find the inverse of the matrix (–1) using either [^] or tapping [▫■] in the [Math1] or [Math2] keyboard.
It doesn't have to be defined as **X**.



IMPORTANT

For a 2 × 2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse is the matrix $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The quantity $ad - bc$ is called the **determinant**, shown symbolically as det **A**, |**A**| or *A*.

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, which is a real number.

The inverse of a matrix can be written as $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided det $\mathbf{A} \neq 0$.

A **singular** matrix is one with no inverse.

A matrix is singular if its determinant is zero.

Example 10

For the matrix $\mathbf{M} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, find each of the following using the formulas for 2 × 2 matrices.

a det **M**

b $\mathbf{M}^{-1}$

Solution

a Write the formula. det **M** = $ad - bc$

Substitute values and simplify. $= 6 \times 3 - (-1) \times 2 = 20$

Write the answer. $= 20$

b Write the formula.

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Substitute values and simplify.

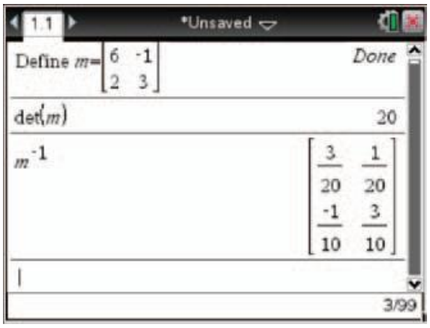
$$= \frac{1}{20}\begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$$

Write the answer.

$$= \begin{bmatrix} \frac{3}{20} & \frac{1}{20} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$$

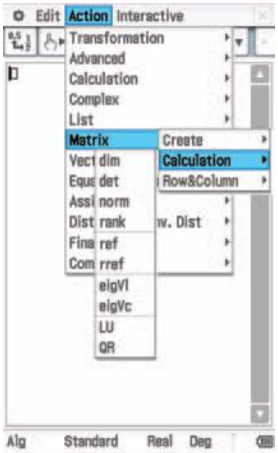
TI-Nspire CAS

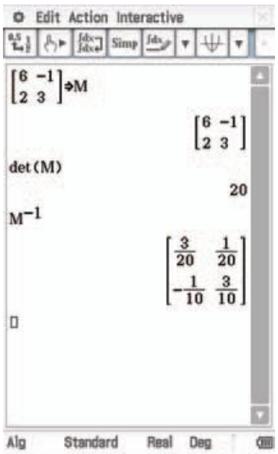
Enter the matrix as **M**.
Use [menu] 7: Matrices & Vectors and 3: Determinant to find the determinant.
Find $\mathbf{M}^{-1}$.



ClassPad

Define the matrix as **M**.
det can be typed in or found in the menu obtained by tapping **Action** then **Matrix** then **Calculation**.
Find $\mathbf{M}^{-1}$ or M^–1.





IMPORTANT

Laws for matrix multiplication

The following laws hold for any matrices **A, B** and **C** and scalars r and s.

1 Matrix multiplication is associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
2 Matrix multiplication is *not commutative*: in general $\mathbf{AB} \neq \mathbf{BA}$
3 The multiplicative identity **I** is a square matrix with 1s in the leading diagonal and 0 everywhere else. For $n \times n$ matrices, $\mathbf{I}_n = (\delta_{ij})$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.
4 Matrices that have a multiplicative inverse are said to be **invertible** or **non-singular** matrices. Matrices without multiplicative inverses are called **singular** matrices.
5 Cancellation laws hold for invertible matrices.
If **A** is invertible, then $\mathbf{AB} = \mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$ and $\mathbf{BA} = \mathbf{CA} \Rightarrow \mathbf{B} = \mathbf{C}$
6 Distributive laws hold for matrix multiplication over matrix addition.
Left distributive law: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
Right distributive law: $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
7 Matrix multiplication over addition: $\mathbf{A}(r + s) = r\mathbf{A} + r\mathbf{B}$
8 The associative law holds for matrix multiplication and scalar multiplication:
$r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B}$

There are left and right distributive laws because matrix multiplication is not commutative.

EXERCISE 8.05 Identities and inverses

Concepts and techniques

WS
Inverse and determinant of a matrix

1 Example 9 Find the inverses of the following matrices using the method of Example **9**.

a $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ b $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$ c $\mathbf{A} = \begin{bmatrix} 3 & -3 \\ -1 & 0 \end{bmatrix}$

2 Find the inverses of the following matrices.

a $\mathbf{A} = \begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix}$ b $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & \frac{1}{3} \end{bmatrix}$ c $\mathbf{A} = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$

3 State whether each of the following is nilpotent.

a $\begin{bmatrix} -4 & 2 \\ -8 & 4 \end{bmatrix}$ b $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ c $\begin{bmatrix} 12 & -9 \\ 16 & -12 \end{bmatrix}$

4 Which of the following matrices are idempotent?

a $\begin{bmatrix} 5 & 4 \\ -5 & -4 \end{bmatrix}$ b $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ c $\begin{bmatrix} 9 & -4 \\ 18 & -8 \end{bmatrix}$

5 Example 10 Use the formula to find the determinants of the following matrices.

a $\begin{bmatrix} -12 & 2 \\ 1 & -3 \end{bmatrix}$ b $\begin{bmatrix} 2 & 6 \\ -6 & 11 \end{bmatrix}$ c $\begin{bmatrix} -7 & 3 \\ -3 & 3 \end{bmatrix}$

d $\begin{bmatrix} -15 & 14 \\ 3 & 0 \end{bmatrix}$ e $\begin{bmatrix} 3 & 0 \\ -11 & 7 \end{bmatrix}$

6 Use your CAS calculator to find the determinant of each of the following.

a $\begin{bmatrix} 7 & -1 \\ -10 & 9 \end{bmatrix}$ b $\begin{bmatrix} 13 & 6 \\ 1 & -13 \end{bmatrix}$ c $\begin{bmatrix} -14 & -15 \\ 11 & 0 \end{bmatrix}$

d $\begin{bmatrix} 4 & -8 \\ 11 & 6 \end{bmatrix}$ e $\begin{bmatrix} -1 & 4 \\ -14 & 13 \end{bmatrix}$

7 Use the formula to find the inverses of the following, if they exist.

a $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ b $\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ c $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

d $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ e $\begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}$

8 Find the inverse of $\mathbf{B} = \begin{bmatrix} -1 & -1 & -1 \\ 4 & 5 & 0 \\ 0 & 1 & -3 \end{bmatrix}$ using your CAS calculator.

Reasoning and communication

9 Find a matrix of order 2 that is both nilpotent and idempotent.

10 Given that $\mathbf{X} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} -5 & -3 \\ 4 & 3 \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that:

a $\mathbf{XM} = \mathbf{X}$ but $\mathbf{MX} \neq \mathbf{X}$ b $\mathbf{NX} = \mathbf{X}$ but $\mathbf{XN} \neq \mathbf{X}$ c $\mathbf{XI} = \mathbf{X}$ and $\mathbf{IX} = \mathbf{X}$

11 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (assume $ad - bc \neq 0$).

a Show that $\mathbf{IA} = \mathbf{A}$.   b Show that $\mathbf{AI} = \mathbf{A}$.   c Show that $\mathbf{A} \times \mathbf{0} = \mathbf{0}$.

d Show that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.   e Show that $\mathbf{AA}^{-1} = \mathbf{I}$.

12 Prove that the inverse of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

13 A pizza shop makes pizzas in three sizes: small, medium and large. In one particular week the sales were recorded for each day and each size of pizza. This data is shown in the table.

	Number of pizzas sold						
Size	**Sun**	**Mon**	**Tue**	**Wed**	**Thu**	**Fri**	**Sat**
Small	25	20	50	25	30	50	50
Medium	35	35	80	35	40	70	75
Large	45	40	100	40	40	90	85

a Write the information in the table as a 3×7 matrix, $\mathbf{P}$.

b Find a matrix, $\mathbf{A}$, such that when $\mathbf{P}$ is multiplied by $\mathbf{A}$, the resulting matrix will give the number of each pizza size sold that week.

c Write the matrix $\mathbf{PA}$.

d Find a matrix, $\mathbf{B}$, such that when $\mathbf{B}$ is multiplied by $\mathbf{P}$, the resulting matrix will give the number of pizzas sold on each day of the week.

e Write the matrix $\mathbf{BP}$.

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The cost of a small pizza is \$7.50, a medium pizza is \$11.00 and a large pizza is \$15.00.

f Write a matrix, $\mathbf{C}$, that represents the cost of the different-sized pizzas.

g Use an appropriate matrix method to find the income from each type of pizza sold that week.

8.06 MATRIX EQUATIONS

You can manipulate matrix expressions and equations in almost the same way as ordinary algebraic expressions and equations. The two exceptions are that matrix multiplication is not generally commutative, and you cannot divide matrices. Where you would normally divide, you must multiply by the inverse, provided that it exists.

You will often find that it is necessary to insert the identity matrix into an equation or expression to factorise it.

While it is correct to write $ab - a = a(b - 1)$ for real numbers, $\mathbf{AB} - \mathbf{A} \neq \mathbf{A}(\mathbf{B} - 1)$ because subtracting the number 1 from the matrix $\mathbf{B}$ does not make sense.

You must write $\mathbf{A} = \mathbf{AI}$ so that $\mathbf{AB} - \mathbf{AI} = \mathbf{A}(\mathbf{B} - \mathbf{I})$.

Notice that it would not be correct to use $\mathbf{A} = \mathbf{IA}$, because then $\mathbf{A}$ would not be on the left side as it is in $\mathbf{AB}$.

Example 11

Factorise the following matrix expressions where possible.

a $\mathbf{AB} + \mathbf{B}$

b $\mathbf{BA} - \mathbf{B}$

c $\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I}$

d $\mathbf{X}^2 - 5\mathbf{XY} - 6\mathbf{Y}^2$

Solution

a Write the matrix expression. $\mathbf{AB} + \mathbf{B}$

Insert the identity on the left to match $\mathbf{A}$. $= \mathbf{AB} + \mathbf{IB}$

Use the right distributive law. $= (\mathbf{A} + \mathbf{I})\mathbf{B}$

b Write the matrix expression. $\mathbf{BA} - \mathbf{B}$

Insert the identity on the right to match $\mathbf{A}$. $= \mathbf{BA} - \mathbf{BI}$

Use the left distributive law. $= \mathbf{B}(\mathbf{A} - \mathbf{I})$

c Use the decomposition method as you would for $x^2 - 3x - 10$.

$-10 = -5 \times (+2)$

$-3 = -5 + (+2)$

Write the matrix expression. $\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I}$

Break up the terms to produce the common factor $(\mathbf{A} - 5\mathbf{I})$. $= \mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{A} - 10\mathbf{I}$

Insert the identity on the left and right. $= \mathbf{A} \times \mathbf{A} - 5\mathbf{I} \times \mathbf{A} + \mathbf{A} \times 2\mathbf{I} - 5\mathbf{I} \times 2\mathbf{I}$

Use the right distributive law twice. $= (\mathbf{A} - 5\mathbf{I})\mathbf{A} + (\mathbf{A} - 5\mathbf{I})2\mathbf{I}$

Use the left distributive law. $= (\mathbf{A} - 5\mathbf{I})(\mathbf{A} + 2\mathbf{I})$

d Use the decomposition method as you would for $x^2 - 5xy - 6y^2$.

$-6 = -6 \times (+1)$

$-5 = -6 + (+1)$

Write the matrix expression. $\mathbf{X}^2 - 5\mathbf{XY} - 6\mathbf{Y}^2$

Break up the terms to produce $\mathbf{X} - 6\mathbf{Y}$. $= \mathbf{X}^2 - 6\mathbf{XY} + \mathbf{XY} - 6\mathbf{Y}^2$

Write each pair of terms so it can be factorised..	$= \mathbf{X}^2 - \mathbf{X} \times 6\mathbf{Y} + 1\mathbf{XY} - 6\mathbf{Y}^2$
Apply the left and right distributive laws. No further factorisation can be done because the $\mathbf{X} - 6\mathbf{Y}$ terms are on different sides.	$= \mathbf{X}(\mathbf{X} - 6\mathbf{Y}) + (\mathbf{X} - 6\mathbf{Y})\mathbf{Y}$

As you can see in Example **11** part **d**, you can factorise $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$ or $\mathbf{BA} + \mathbf{CA} = (\mathbf{B} + \mathbf{C})\,\mathbf{A}$, but they are not the same and you cannot factorise $\mathbf{AB} + \mathbf{CA}$.

Example 12

Solve the following equations for **X** and **Y**.

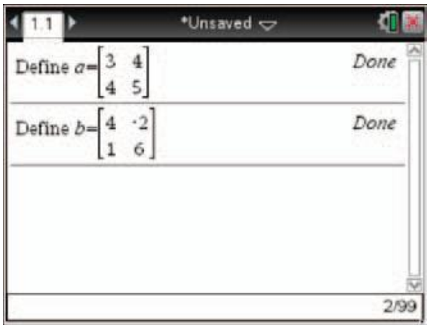
a $\mathbf{X}\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$

Solution

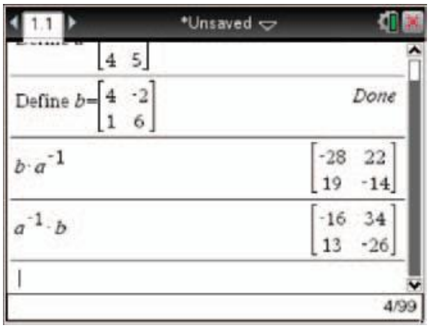
a Write the matrix equation.	$\mathbf{X}\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$
To find **X**, the inverse of $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ must exist.	For $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$, $\det \mathbf{A} = -1$
Since $\det \mathbf{A} \neq 0$, the inverse exists.	$\mathbf{A} = \frac{1}{-1}\begin{bmatrix} 5 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$
Multiply the left side of both sides of the equation by $\mathbf{A}^{-1}$. Let $\mathbf{B} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$.	$\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$
Simplify $\mathbf{AA}^{-1}$.	$\mathbf{XI} = \mathbf{BA}^{-1}$
Simplify **XI**.	$\mathbf{X} = \mathbf{BA}^{-1}$
Substitute the matrices.	$\mathbf{X} = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$
Multiply.	$= \begin{bmatrix} -28 & 22 \\ 19 & -14 \end{bmatrix}$
b Write the matrix equation.	$\mathbf{AY} = \mathbf{B}$
Multiply the right side of both sides of the equation by $\mathbf{A}^{-1}$.	$\mathbf{A}^{-1}\mathbf{AY} = \mathbf{A}^{-1}\mathbf{B}$
Simplify $\mathbf{A}^{-1}\mathbf{A}$.	$\mathbf{IY} = \mathbf{A}^{-1}\mathbf{B}$
Simplify **IY**.	$\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}$
Substitute the matrices.	$\mathbf{Y} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$
Multiply.	$= \begin{bmatrix} -16 & 34 \\ 13 & -26 \end{bmatrix}$

TI-Nspire CAS

Enter the matrices **A** and **B**.

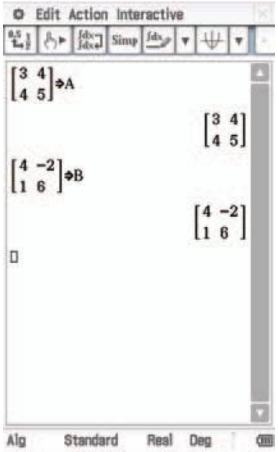


Calculate $\mathbf{BA}^{-1}$ and $\mathbf{A}^{-1}\mathbf{B}$.

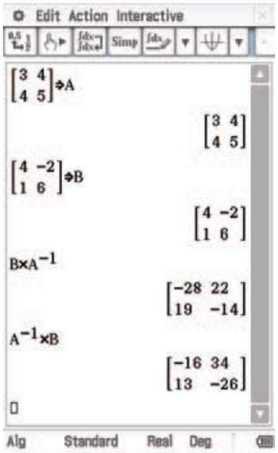


ClassPad

Enter the matrices **A** and **B**.



Calculate $\mathbf{B} \times \mathbf{A}^{-1}$ and $\mathbf{A}^{-1} \times \mathbf{B}$.
The multiplication sign is essential in this case.



The order of calculations is very important because matrix multiplication is not commutative ($\mathbf{AB} \neq \mathbf{BA}$).

Example 13

Solve the following equation using your CAS calculator as needed.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 2\mathbf{X}\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix}$$

Solution

Write the equation.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 2\mathbf{X}\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix}$$

Take **X** terms to the LHS and the others to the RHS.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 2\mathbf{X}\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix}$$

Reorder the second term.

$$\mathbf{X}\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - \mathbf{X}\times 2\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix} = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix}$$

Factorise.

$$\mathbf{X}\left(\begin{bmatrix}2 & 2\\4 & 3\end{bmatrix} - 2\begin{bmatrix}-2 & 3\\3 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 5\\-2 & -4\end{bmatrix} + 3\begin{bmatrix}-1 & -4\\3 & 2\end{bmatrix}$$

Write in simplified form.

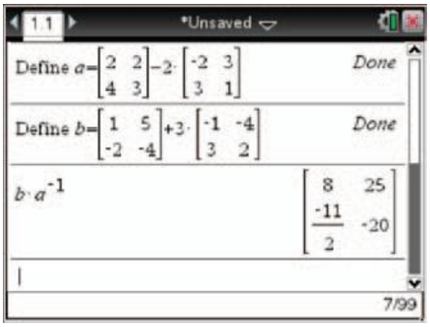
$$\mathbf{XA} = \mathbf{B}$$

Perform the necessary operation.

$$\mathbf{X} = \mathbf{BA}^{-1}$$

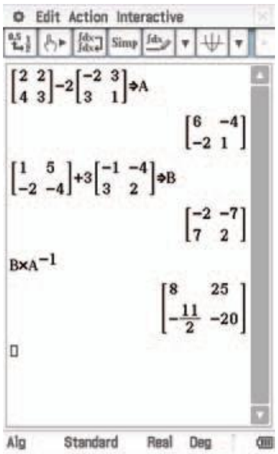
TI-Nspire CAS

Perform the matrix operations and call the results **A** and **B**.
Calculate $\mathbf{BA}^{-1}$.



ClassPad

Perform the matrix operations and call the results **A** and **B**.
Calculate $\mathbf{BA}^{-1}$.



Write the answer.

$$\mathbf{X} = \begin{bmatrix}8 & 25\\-5\frac{1}{2} & -20\end{bmatrix}$$

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EXERCISE 8.06 Matrix equations

Concepts and techniques

1 Example 11 Factorise each of the following matrix expressions, if possible.

a $\mathbf{M}^2 + 4\mathbf{M}$ b $7\mathbf{GX} - \mathbf{X}$ c $\mathbf{AB} - 3\mathbf{A}$

d $\mathbf{H}^2\mathbf{G} + 4\mathbf{HG}^2$ e $\mathbf{X}^2 - 2\mathbf{XY} + \mathbf{Y}^2$

2 Example 12 Solve the following.

a $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 4 & 3 \\ -1 & -4 \end{bmatrix}$ b $\mathbf{X}\begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 3 & -2 \end{bmatrix}$ c $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 17 \\ 9 \end{bmatrix}$

d $\begin{bmatrix} 4 & 1 \\ 3 & -2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ e $\begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 5 & -2 \\ 7 & -5 \end{bmatrix}$

3 Solve the following.

a $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 2 & 4 \\ -6 & -8 \end{bmatrix}$ b $\mathbf{X}\begin{bmatrix} -4 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ 10 & -4 \end{bmatrix}$ c $\begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

d $\mathbf{X}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 5 \\ -5 & 3 \end{bmatrix}$ e $\begin{bmatrix} -5 & 2 \\ -10 & 3 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 2 & -2 & 5 \\ 3 & 0 & -3 \end{bmatrix}$

Reasoning and communication

4 Example 13 Solve the following, using your CAS calculator as needed.

a $2\mathbf{X} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}\mathbf{X} - \begin{bmatrix} 2 & 12 \\ 20 & 12 \end{bmatrix} = \begin{bmatrix} -14 & -6 \\ 22 & -3 \end{bmatrix}$ b $\begin{bmatrix} 11 & 6 \\ 3 & 5 \end{bmatrix}\mathbf{X} - 3\mathbf{X} + \begin{bmatrix} 4 & 6 \\ 4 & 3 \end{bmatrix} = 4\mathbf{I}$

c $2\mathbf{X} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}\mathbf{X} - 4\mathbf{X} = \mathbf{X} + \begin{bmatrix} -14 & -6 \\ 22 & -3 \end{bmatrix}$ d $5\mathbf{X} - \mathbf{X}\begin{bmatrix} -1 & 5 \\ 2 & 1 \end{bmatrix} + 7\mathbf{I} = \mathbf{0}$

e $\mathbf{X}\begin{bmatrix} 8 & -2 & 6 \\ -5 & 5 & 4 \\ -4 & 2 & 5 \end{bmatrix} - 2\mathbf{X} + 3\mathbf{I} = \begin{bmatrix} -9 & 3 & 1 \\ 2 & 6 & 2 \\ 3 & 4 & -3 \end{bmatrix}$

8 CHAPTER SUMMARY
MATRIX ARITHMETIC

- A **matrix** (plural = matrices) is a rectangular array of **elements** shown in rows and columns enclosed by brackets, such as $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$. The **size** (or **dimension** or **order**) of a matrix is stated with the number of rows first, so **A** is a 2×3 matrix.
- A **square** matrix has the same number of rows and columns, while a row matrix has only 1 row and a column matrix has only 1 column.
- A general matrix may be written as $\mathbf{C} = c_{ij}$ or $(c_{ij})_{m \times n}$ if the size needs to be shown.
- **Matrix addition** is only defined for matrices of the same size. For the $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, $\mathbf{A} + \mathbf{B} = \mathbf{C}$ such that $c_{ij} = a_{ij} + b_{ij}$ for all i, j; i.e. $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.
- Multiplying a matrix by a constant (c) produces a **scalar multiple** of the matrix, so $c\mathbf{A} = (ca_{ij})$. The product $(-1)\mathbf{A}$ is normally written as $-\mathbf{A}$, so matrix subtraction can be shown as $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = \mathbf{A} + (-\mathbf{B})$.
- **A** and **B** are **equal** if and only if they are the same size and $a_{ij} = b_{ij}$, for all i, j.
- The laws for matrix addition and scalar multiplication are:

  Associativity:
  $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$, $(rs)\mathbf{A} = r(s\mathbf{A})$

  Commutivity: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

  Additive identity: $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$

  Additive inverse: $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$, where $\mathbf{A} = (a_{ij})$ and $-\mathbf{A} = (-a_{ij})$

  Cancellation: $r\mathbf{A} = r\mathbf{B} \Leftrightarrow \mathbf{A} = \mathbf{B}$ $(r \neq 0)$, $r\mathbf{A} = s\mathbf{A} \Leftrightarrow r = s$ $(\mathbf{A} \neq \mathbf{0})$

  Distributivity: $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$, $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$

- The product of two matrices **A** and **B** exists if and only **A** and **B** are **conformable**. This means that **A** has the same number of columns as **B** has rows. If **A** is an $n \times m$ matrix (a_{ij}) and **B** is an $m \times p$ matrix (b_{ij}) and $\mathbf{AB} = \mathbf{C}$, then **C** is an $n \times p$ matrix (c_{ij}) such that $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$.
- An **idempotent matrix A** is such that $\mathbf{A}^2 = \mathbf{A}$. A **nilpotent matrix B** of order n is such that $\mathbf{B}^2 = \mathbf{0}_n$, where $\mathbf{0}_n$ is the null matrix of size $n \times n$.
- **Identity matrices** (**I**) are diagonal matrices with 1s in the diagonal and 0s everywhere else. $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.
- The multiplicative inverse of a 2×2 matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The quantity $ad - bc$ is a number called the **determinant** of the matrix.
- Matrix multiplication is not commutative.

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- The laws of matrix multiplication are:

  Associativity: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

  Identity: $\mathbf{I}$ is a square matrix with 1s in the leading diagonal and 0 everywhere else. For $n \times n$ matrices $\mathbf{I}_n = (\delta_{ij})$ where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

  Inverse: matrices that have a multiplicative inverse are called **invertible** or **non-singular** matrices. Matrices without multiplicative inverses are called **singular** matrices.

  Cancellation: If $\mathbf{A}$ is invertible, then $\mathbf{AB} = \mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$ and $\mathbf{BA} = \mathbf{CA} \Rightarrow \mathbf{B} = \mathbf{C}$

  Left distributive law: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

  Right distributive law: $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$

  Distributivity over addition: $\mathbf{A}(r + s) = r\mathbf{A} + s\mathbf{A}$

  Associativity with scalar multiplication: $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B}$

- Matrix algebra differs from the algebra of real numbers because matrix multiplication is not commutative and there is no division. It is often necessary to introduce the identity to complete factorisations in the solution of matrix equations.

8

CHAPTER REVIEW
MATRIX ARITHMETIC

Multiple choice

1 Example 1 What is the size of the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 0 \\ -1 & 3 \end{bmatrix}$?

A 2×3 B 3×2 C 2×2 D 1×3 E 6

2 Example 2 Given that $\mathbf{X} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$, which matrix represents $-2\mathbf{X}$?

A $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ B $\begin{bmatrix} 0 & -2 & 4 \\ -2 & 0 & -2 \\ 4 & -2 & 0 \end{bmatrix}$ C $\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}$

D $\begin{bmatrix} 0 & 2 & -2 \\ 2 & 0 & 2 \\ -2 & 2 & 0 \end{bmatrix}$ E $\begin{bmatrix} 0 & 2 & -4 \\ 2 & 0 & 2 \\ -4 & 2 & 0 \end{bmatrix}$

3 Example 3 If $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 5 \end{bmatrix}$, choose the correct answer for $3\mathbf{A} + 2\mathbf{B}$.

A It cannot be done. B $\begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}$ C $\begin{bmatrix} 6 & 3 \end{bmatrix}$

D $\begin{bmatrix} 15 & 4 \\ 3 & 0 \end{bmatrix}$ E $\begin{bmatrix} 6 & 3 \\ 1 & 0 \end{bmatrix}$

4 Example 4 If $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$, then $3\mathbf{A} - 2\mathbf{B}$ is equal to:

A $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ B $\begin{bmatrix} 8 & -3 \\ 3 & 2 \end{bmatrix}$ C $\begin{bmatrix} 7 & 0 \\ 3 & 4 \end{bmatrix}$

D $\begin{bmatrix} 11 & -12 \\ 3 & 4 \end{bmatrix}$ E $\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$

5 Example 10 What is the inverse of $\begin{bmatrix} 6 & -11 \\ -2 & 4 \end{bmatrix}$?

A $\begin{bmatrix} 3 & -1 \\ -5\frac{1}{2} & 2 \end{bmatrix}$ B $\begin{bmatrix} 2 & 5\frac{1}{2} \\ 1 & 3 \end{bmatrix}$ C $\begin{bmatrix} 2 & 1 \\ 5\frac{1}{2} & 3 \end{bmatrix}$

D $\begin{bmatrix} \frac{2}{23} & -\frac{11}{46} \\ -\frac{1}{23} & \frac{3}{23} \end{bmatrix}$ E $\begin{bmatrix} -\frac{2}{23} & \frac{11}{46} \\ \frac{1}{23} & -\frac{3}{23} \end{bmatrix}$

Short answer

6 Example 5 If $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$, find **BA**.

7 Example 6 If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$, find **AB**.

8 Example 9 Find the inverse of $\begin{bmatrix} -6 & 2 \\ -5 & 2 \end{bmatrix}$ from the definition (without using a formula).

9 Example 10 For each of the following matrices, find the determinant and inverse, if they exist.

a $\begin{bmatrix} 3 & 3 \\ 8 & 9 \end{bmatrix}$

b $\begin{bmatrix} -4 & 10 \\ -2 & 5 \end{bmatrix}$

10 Example 11 Factorise $\mathbf{P}^2\mathbf{Q} + \mathbf{P}\mathbf{Q}^2$ using matrix algebra, if possible.

11 Example 12 Solve the matrix equation $\begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix}\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

12 Example 13 Solve the matrix equation $\begin{bmatrix} 5 & 2 \\ 8 & 4 \end{bmatrix}\mathbf{C} + 3\begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} = 2\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, using your CAS calculator as needed.

Application

13 Show that matrices **A**, **B** and **C**, each with dimensions of 2 × 2, are associative under addition, i.e. **A** + (**B** + **C**) = (**A** + **B**) + **C**.

14 A smallgoods manufacturer produces three types of salami – Italian, Hungarian and Austrian – in mild and hot versions.
The prices in dollars of each type are shown in the table below.

Type of salami	Italian	Hungarian	Austrian
Mild	12.50	15.00	17.50
Hot	13.20	16.40	18.00

The orders from three delicatessens are as follows.

Orders	MI	HI	MH	HH	MA	HA
George's	3	1	2	3	1	0
Frank's	2	2	3	1	3	2
Maria's	4	5	0	1	0	2

Practice quiz

a Write the prices as a 1 × 6 matrix.
b Write the orders as a 6 × 3 matrix.
c Use matrix multiplication to find the price of each deli's order.

3 MIXED REVISION
CHAPTERS 7 • 8 • 9

Multiple choice

1 Which one of the following is not a prime number?

A 1 B 2 C 137 D 211 E 503

2 Matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$ and matrix $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$. Matrix $\mathbf{A} + \mathbf{B}$ = ?

A $\begin{bmatrix} 1 & 3 & 5 & 1 \\ 4 & 0 & 3 & 2 \end{bmatrix}$ B $\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 5 & 1 \\ 3 & 2 \end{bmatrix}$ C $\begin{bmatrix} 6 & 4 \\ 7 & 2 \end{bmatrix}$ D $\begin{bmatrix} 14 & 7 \\ 20 & 4 \end{bmatrix}$ E $\begin{bmatrix} 9 & 15 \\ 11 & 9 \end{bmatrix}$

3 $\cot\left(\frac{5\pi}{6}\right) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C $\sqrt{3}$ D $-\sqrt{3}$ E -1

4 $3.\overline{142857}$ is

A equal to π B an irrational number C an integer

D equal to $\frac{22}{7}$ E a transcendental number.

5 The matrix **A** has order $2 \times m$, matrix **B** has order $4 \times n$ and matrix **AB** has order 2×3. The values of m and n are:

A $m = 3, n = 3$ B $m = 2, n = 3$ C $m = 4, n = 3$

D $m = 2, n = 4$ E $m = 3, n = 4$

6 $\sin\left(\frac{\pi}{12}\right)$ can be expressed as

A $2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$ B $\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

C $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$ D $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

E $\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

7 Which one of the following sets is **not** closed under multiplication?

A Real numbers B Irrational numbers C $\{-1, 0, 1\}$

D Integers E Positive integers

8 For the matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $\mathbf{A}^3$ = ?

A $\begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix}$ B $\begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$ C $\begin{bmatrix} 35 & 152 \\ 189 & 152 \end{bmatrix}$

D $\begin{bmatrix} 8 & 27 \\ 64 & 125 \end{bmatrix}$ E $\begin{bmatrix} 116 & 153 \\ 204 & 269 \end{bmatrix}$

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9 Expressed as a difference, $\sin(x)\sin(5x) =$

A $\frac{1}{2}[\cos(4x)-\cos(6x)]$ B $\frac{1}{2}[\cos(4x)+\cos(6x)]$ C $\frac{1}{2}[\cos(x)-\cos(5x)]$

D $\frac{1}{2}[\sin(4x)-\sin(6x)]$ E $\frac{1}{2}[\cos(6x)-\cos(4x)]$

Short answer

1 a Show that the equation $4x + 6y = 1987$ has no integer solutions.
b Find a counter example to show that the statement below is false.
The equation $3x + 5y = 40$ has no positive integer solutions.

2 Show that the matrix equation below has no solution. Justify your reasoning.

$$\begin{bmatrix}12 & 28\\15 & 35\end{bmatrix}\mathbf{X}=\begin{bmatrix}2 & 5\\4 & 10\end{bmatrix}$$

3 Select a suitable double angle formula to express $\sin(x)\sin(2x)$ in terms of $\cos(x)$ only.

4 Construct the decimal 0.05005000500005… by adding fractions of the form $\frac{1}{2\times10^a}$, where $a \in \boldsymbol{Z}^+$.

5 Find the solution to the equation $\begin{bmatrix}4 & 2\\6 & 5\end{bmatrix}\mathbf{X}=\begin{bmatrix}2\\9\end{bmatrix}$

6 a Expand the expression $\cos(x + 2x)$ using compound angle and double angle formulas.
b Hence show that $\cos(x + 2x) = \cos(x)[1 - 4\sin^2(x)]$

Application

1 Prove by contradiction that $x^3 + x - 7 = 0$ has no rational solutions.

2 a Show that the following statement is false.
If $2n + 7$ is odd, then n is odd, where $n \in \boldsymbol{Z}$.
b Show that for all $n \in \boldsymbol{Z}$, $2n + 7$ is odd.
c Show by mathematical induction that $\frac{1}{7\times9}+\frac{1}{9\times11}+\frac{1}{11\times13}+\cdots$ to n terms $=\frac{n}{7(2n+7)}$, where $n \in \boldsymbol{N}$.
d For what values of n will the reciprocal of the sum in part **c** be an odd integer?

3 $\mathbf{C}=\begin{bmatrix}5 & 3\\7 & 4\end{bmatrix}$. Find the inverse matrix $\mathbf{C}^{-1}$.

4 Solve for $\mathbf{X}$.

$$2\mathbf{X}+\begin{bmatrix}5 & 2\\3 & 1\end{bmatrix}\mathbf{X}-\begin{bmatrix}2 & 12\\20 & 12\end{bmatrix}=\begin{bmatrix}-14 & -6\\22 & -3\end{bmatrix}$$

5 Prove that $[\operatorname{cosec}(x) + \cot(x)][\operatorname{cosec}(x) - \cot(x)] = 1$.

6 Prove that $\frac{1+\cot(\beta)}{\operatorname{cosec}(\beta)}-\cos(\beta)=\frac{\sec(\beta)}{\tan(\beta)+\cot(\beta)}$.

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9

TRIGONOMETRY

TRIGONOMETRIC IDENTITIES

TERMINOLOGY

addition theorems

angle sum and difference identities

compound angle formula

cosecant

cosine

cotangent

double angle formula

frequency

half-angle formula

products to sums and differences

Pythagorean identity

ratio

reciprocal function

reference angle

secant

sine

sum and difference formula

sums to products

tangent

trigonometric identity

unit circle

x-coordinate

y-coordinate

Prior learning

COMPOUND ANGLES

- prove and apply the angle sum, difference and double angle identities. (ACMSM044)

THE RECIPROCAL TRIGONOMETRIC FUNCTIONS, SECANT, COSECANT AND COTANGENT

- define the reciprocal trigonometric functions, sketch their graphs, and graph simple transformations of them. (ACMSM045)

TRIGONOMETRIC IDENTITIES

- prove and apply the Pythagorean identities (ACMSM046)
- prove and apply the identities for products of sines and cosines expressed as sums and differences (ACMSM047)
- convert sums $a \cos x + b \sin x$ to $R \cos(x \pm a)$ or $R \sin(x \pm a)$ and apply these to sketch graphs, solve equations of the form $a \cos x + b \sin x = c$ and solve problems (ACMSM048)
- prove and apply other trigonometric identities such as $\cos 3x = 4 \cos^3 x - 3 \cos x$. (ACMSM049) AC

9.01 RECIPROCAL TRIGONOMETRIC FUNCTIONS

Trigonometry is used in many fields of science, construction and engineering. Trigonometric functions are periodic functions that repeat themselves in a 'cycle'. Many functions that we see all around us are periodic. Periodic functions are found in the way waves move across water, light moves through space, sound moves through air and earthquakes travel around the Earth.

Basic trigonometry

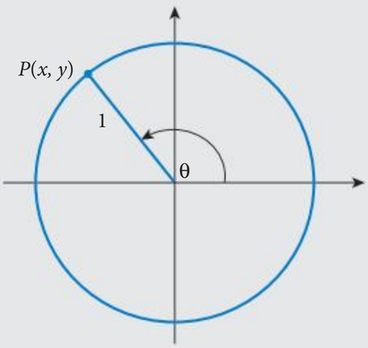
Previous work in trigonometric functions has dealt with the trigonometric ratios of sine, cosine and tangent.

Reciprocal trigonometric functions introduce three more functions: **secant**, **cosecant** and **cotangent**. They are the reciprocals of cosine, sine and tangent respectively.

IMPORTANT

Trigonometric functions

Angles are measured anticlockwise from the positive x-axis. For a point $P(x, y)$ at an angle of θ on the **unit circle**, we define each trigonometric **ratio** as below.



The **sine** of θ is the y-coordinate of the point P, so $\sin(\theta) = y$.

The **cosine** of θ is the x-coordinate of the point P, so $\cos(\theta) = x$.

The **tangent** of θ is the gradient of the line segment OP, so $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$.

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The **secant** of θ is the reciprocal of the cosine, so sec (θ) = $\frac{1}{x} = \frac{1}{\cos(\theta)}$.

The **cosecant** of θ is the reciprocal of the sine, so cosec (θ) = $\frac{1}{y} = \frac{1}{\sin(\theta)}$.

The **cotangent** of θ is the reciprocal of the tangent, so cot (θ) = $\frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)}$.

The abbreviations of sine, cosine, tangent, secant, cosecant and cotangent are sin, cos, tan, sec, cosec and cot respectively.

The trigonometric ratios of angles are always abbreviated to the letters shown above.

IMPORTANT

Reciprocal trigonometric relationships are:

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

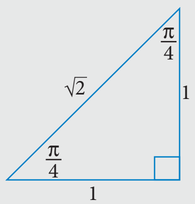
$$\cot(x) = \frac{1}{\tan(x)}$$

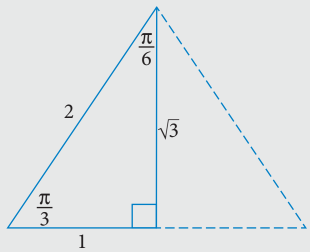
You should remember the exact values for the trigonometric functions of sin, cos and tan.

IMPORTANT

The exact values of the trig ratios for the angles $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°) and $\frac{\pi}{3}$ (60°), are shown in the table and **standard triangles** below.

Angle	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$





Example 1

State the exact values for the reciprocal trig functions: cot (30°), sec (45°) and cosec (60°).

Solution

Write each reciprocal function in terms of the original functions tan, cos or sin.

$$\cot(30°) = \frac{1}{\tan(30°)}$$
$$\sec(45°) = \frac{1}{\cos(45°)}$$
$$\operatorname{cosec}(60°) = \frac{1}{\sin(60°)}$$

Substitute the exact values for tan (30°), cos (45°) and sin (60°).

$$\cot(30°) = \frac{1}{\frac{1}{\sqrt{3}}}$$
$$\sec(45°) = \frac{1}{\frac{1}{\sqrt{2}}}$$
$$\operatorname{cosec}(60°) = \frac{1}{\frac{\sqrt{3}}{2}}$$

Simplify each fraction and write the answers.

$$\cot(30°) = \sqrt{3}$$
$$\sec(45°) = \sqrt{2}$$
$$\operatorname{cosec}(60°) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

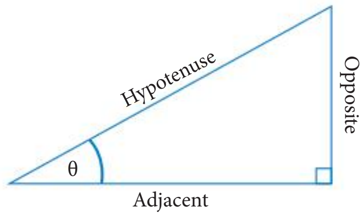
Degrees and radians can both be used to display symmetry.

Trigonometric ratios of triangle sides

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \operatorname{cosec}(\theta) = \frac{\text{Hypotenuse}}{\text{Opposite}}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \sec(\theta) = \frac{\text{Hypotenuse}}{\text{Adjacent}}$$

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} \qquad \cot(\theta) = \frac{\text{Adjacent}}{\text{Opposite}}$$



Remember that cos (45) means that the 45 is a huge angle in radians, but cos (45°) means the angle is in degrees.

Degrees and radians

You know the exact values for some angles less than 90° (less than $\frac{\pi}{2}$). Values in other quadrants for secant, cosecant and cotangent follow the same patterns of symmetry that sine, cosine and tangent follow.

Whenever sine, cosine or tangents are positive or negative, so are their reciprocal pairs. This is summarised using the familiar CAST (or ASTC) diagram for a circle of radius *r*.

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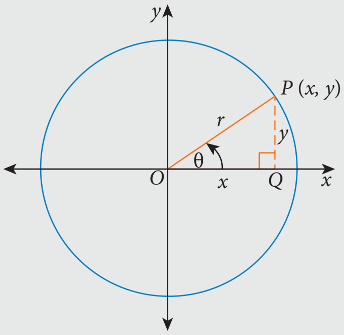
IMPORTANT

For any point $P(x, y)$ on a circle with radius r and centre at the origin:

$\sin(\theta)=\frac{y}{r}$, $\cos(\theta)=\frac{x}{r}$ and $\tan(\theta)=\frac{y}{x}$

$\text{cosec}(\theta)=\frac{r}{y}$, $\sec(\theta)=\frac{r}{x}$ and $\cot(\theta)=\frac{x}{y}$



The acute angle, θ in this case, is always identified as the angle in the 1st quadrant that is drawn between the terminal side of θ and the x-axis. This angle is known as the **reference angle**.

You know that in the 2nd quadrant:

$\sin(\pi - \theta) = \sin(\theta)$

$\cos(\pi - \theta) = -\cos(\theta)$

$\tan(\pi - \theta) = -\tan(\theta)$

It follows that for reciprocal functions in the 2nd quadrant:

$\text{cosec}(\pi - \theta) = \text{cosec}(\theta)$

$\sec(\pi - \theta) = -\sec(\theta)$

$\cot(\pi - \theta) = -\cot(\theta)$

IMPORTANT

Symmetry of the unit circle for reciprocal functions

2nd quadrant

$\text{cosec}(\pi - \theta) = \text{cosec}(\theta)$

$\sec(\pi - \theta) = -\sec(\theta)$

$\cot(\pi - \theta) = -\cot(\theta)$

3rd quadrant

$\text{cosec}(\pi + \theta) = -\text{cosec}(\theta)$

$\sec(\pi + \theta) = -\sec(\theta)$

$\cot(\pi + \theta) = \cot(\theta)$

4th quadrant

$\text{cosec}(2\pi - \theta) = -\text{cosec}(\theta) = \text{cosec}(-\theta)$

$\sec(2\pi - \theta) = \sec(\theta) = \sec(-\theta)$

$\cot(2\pi - \theta) = -\cot(\theta) = \cot(-\theta)$

IMPORTANT

Signs of the trig ratios

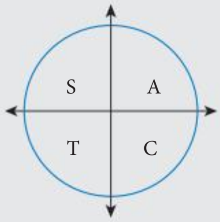
1st quadrant: All ratios are positive (**A**)

2nd quadrant: Sin only is positive (**S**)

3rd quadrant: Tan only is positive (**T**)

4th quadrant: Cos only is positive (**C**)

The above can be remembered by using the mnemonic '**CAST**' or '**A**ll **S**cience **T**eachers are **C**urious'.



Example 2

Simplify the expression $\cot\left(2\pi - \frac{\pi}{3}\right)$.

Solution

cot is the reciprocal of tan.

$$\cot\left(2\pi - \frac{\pi}{3}\right) = \frac{1}{\tan\left(2\pi - \frac{\pi}{3}\right)}$$

Note that $\cot\left(2\pi - \frac{\pi}{3}\right)$ places the angle in the fourth quadrant. tan and cot are both negative in the fourth quadrant.

$$\cot\left(2\pi - \frac{\pi}{3}\right) = -\cot\left(\frac{\pi}{3}\right)$$

Evaluate $\cot\left(\frac{\pi}{3}\right)$.

$$-\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\tan\left(\frac{\pi}{3}\right)}$$

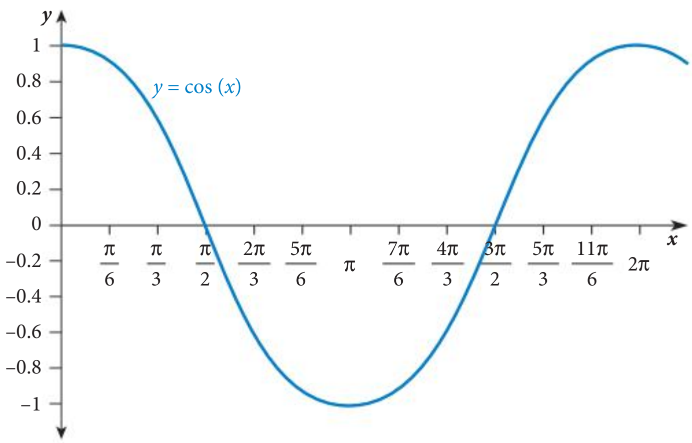
Write the answer.

$$\cot\left(2\pi - \frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$

WS Solving periodic functions

The graphs of secant, cosecant and cotangent can be developed from the graphs of cosine, sine and tangent.

The graph of $y = \cos(x)$ for $0 \le x \le 2\pi$ looks like:



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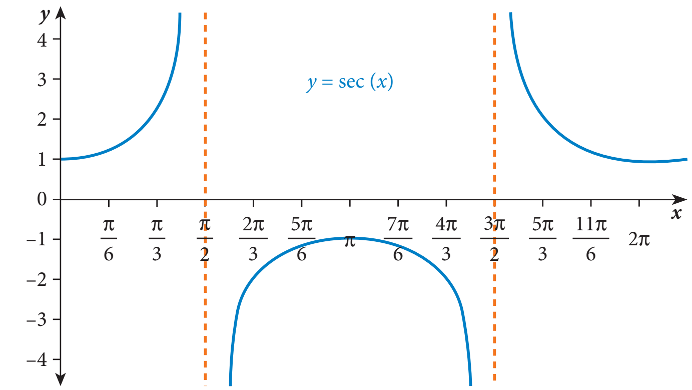
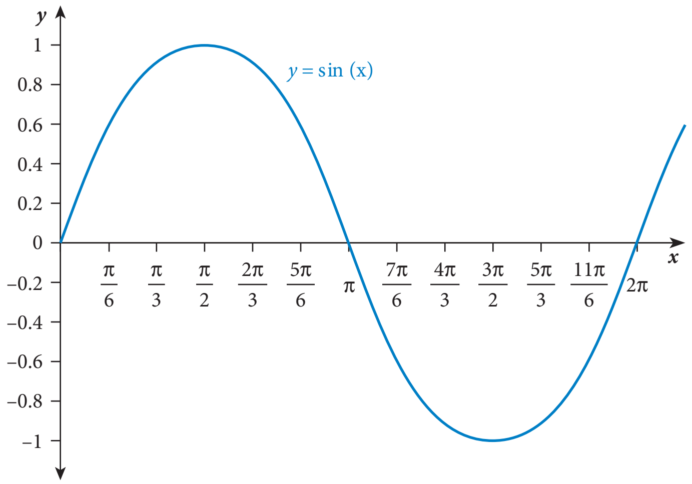
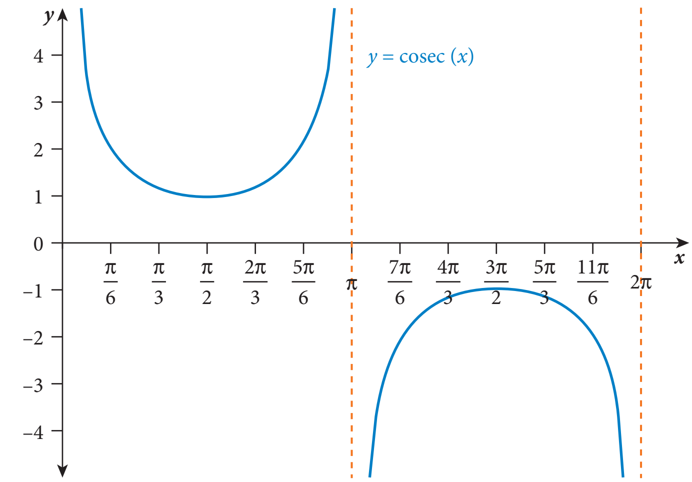
The graph of $y = \sec(x)$ for $0 \le x \le 2\pi$ looks like:



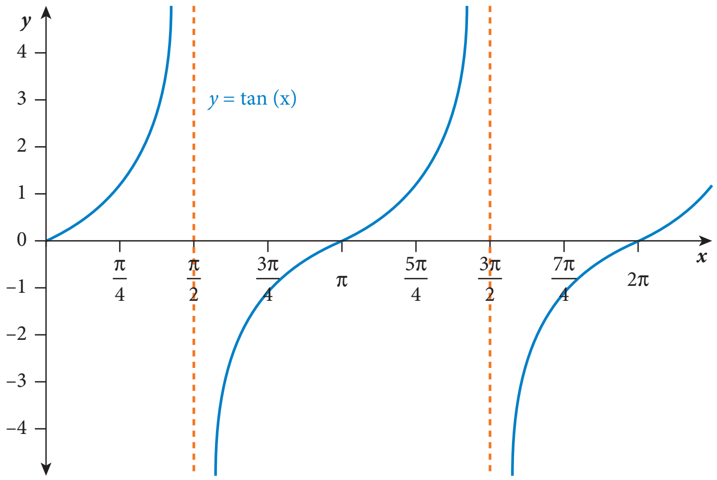
The graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ looks like:



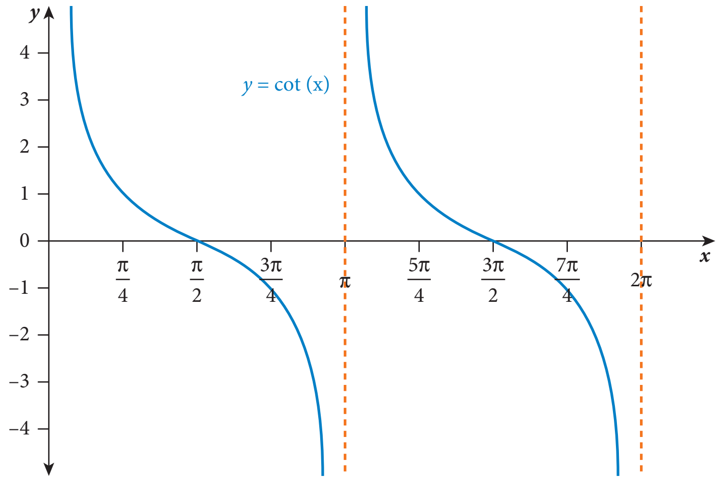
The graph of $y = \operatorname{cosec}(x)$ for $0 < x < 2\pi$ looks like:



The graph of $y = \tan(x)$ for $0 \le x \le 2\pi$ looks like:



The graph of $y = \cot(x)$ for $0 < x < 2\pi$ looks like:



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IMPORTANT

Graphs of reciprocal functions

In general, if $y = f(x)$, then its reciprocal $y = \frac{1}{f(x)}$ follows the rules:

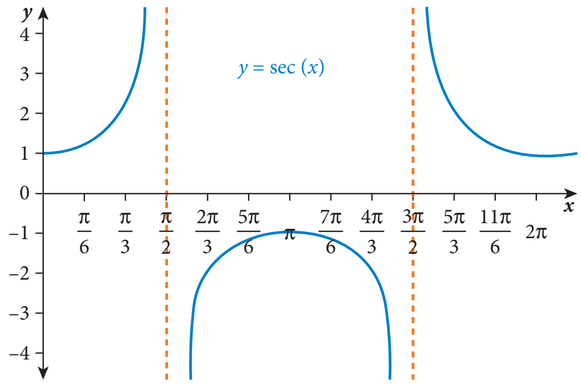
- if there are any x-intercepts in $y = f(x)$, then a vertical asymptote will be found at that point for the graph of $y = \frac{1}{f(x)}$ as you cannot divide by zero.
- if the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect, they will do so at the points $y = \pm 1$
- if the y values in $y = f(x)$ approach infinity as x approaches infinity, then the y values in $y = \frac{1}{f(x)}$ will approach zero as x approaches infinity.
- if $y \to 0$ in $y = \frac{1}{f(x)}$ as $x \to \infty$, then $y \to \infty$ in $y = f(x)$ as $x \to \infty$
- the sign of the reciprocal function will be the same as the sign of the function

Example 3

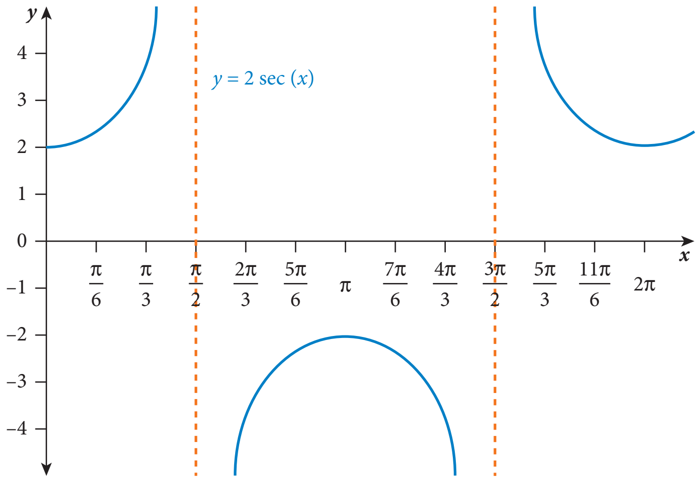
Sketch the graph of $y = 2 \sec (x) + 1$ for $0 \le x \le 2\pi$.

Solution

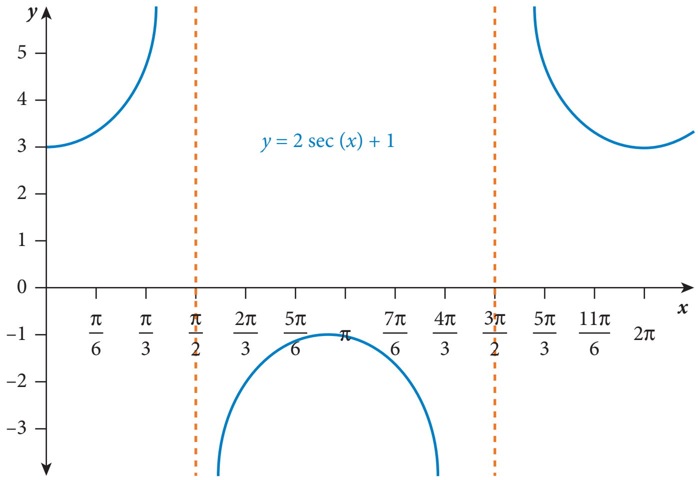
sec (x) is the reciprocal of cos (x) so its asymptotes are where cos $(x) = 0$. This is at $\frac{\pi}{2}, \frac{3\pi}{2}$ and so on. Sketch the graph of sec (x) for $0 \le x \le 2\pi$.



Now draw a graph of $y = 2 \sec(x)$ for $0 \le x \le 2\pi$, by multiplying all y values by 2.

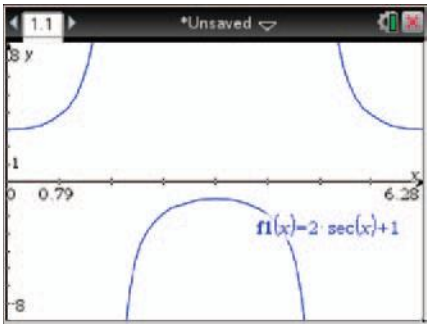


Now draw a graph of $y = 2 \sec(x) + 1$ for $0 \le x \le 2\pi$, by shifting the previous graph upwards 1 unit.



TI-Nspire CAS

Insert a Graph page and make the Window settings $0 \le x \le 2$ with scale $\pi/4$ and $-8 \le y \le 8$ with scale 1. Press **µ** and put **f1**(x) = $2 \sec(x) + 1$.



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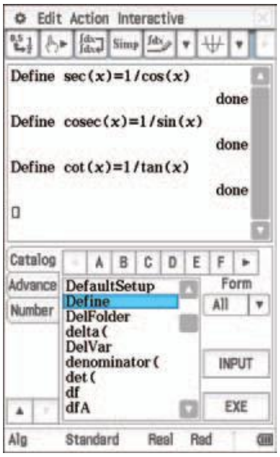
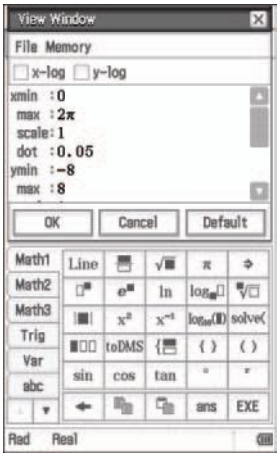
ClassPad

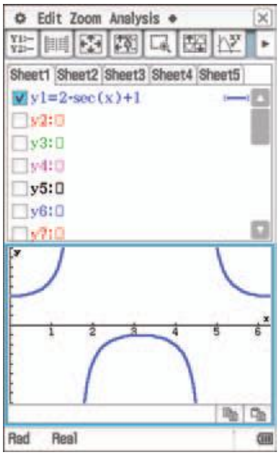
The ClassPad does not have specific defined functions for sec, cosec or cot, so the reciprocals of sin, cos and tan must be used. It is probably easiest to define these three functions using the Main $\sqrt{\alpha}$ application and **Define**, which is found in the Catalog menu (after pressing Keyboard and tapping ▼). You will only need to do this once.
Make sure the calculator is set to radians (**Rad**).



Use the **Graph&Table** application and set y1=2sec(x)+1
Tap for **View Window** to set $0 \leq x \leq 2\pi$ and $-8 \leq y \leq 8$. Tap **OK**.



Tap to draw the graph.



EXERCISE 9.01 Reciprocal trigonometric functions

Concepts and techniques

1 Example 1 $\cot\left(\frac{\pi}{3}\right) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C 1 D $\sqrt{2}$ E $\sqrt{3}$

2 $\operatorname{cosec}\left(\frac{\pi}{4}\right) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C 1 D $\sqrt{2}$ E $\sqrt{3}$

3 $\sec\left(\frac{\pi}{6}\right) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C $\sqrt{3}$ D $\frac{2}{\sqrt{3}}$ E $\frac{\sqrt{3}}{2}$

4 $\operatorname{cosec}(90°) =$

A 0 B $\frac{2}{\sqrt{3}}$ C $\frac{1}{\sqrt{2}}$ D 1 E undefined

5 $\cot(45°) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C 1 D $\sqrt{2}$ E $\sqrt{3}$

6 Example 2 Evaluate:

a $\cot\left(\pi - \frac{\pi}{4}\right)$ b $\sec\left(\pi - \frac{\pi}{4}\right)$ c $\operatorname{cosec}\left(\pi + \frac{\pi}{3}\right)$ d $\sec\left(2\pi - \frac{\pi}{3}\right)$

e $\cot\left(\pi + \frac{\pi}{6}\right)$ f $\sec\left(\pi + \frac{\pi}{4}\right)$ g $\sec\left(2\pi + \frac{\pi}{4}\right)$ h $\cot\left(2\pi + \frac{\pi}{6}\right)$

7 Write a simplified expression for:

a $\sec(2\pi + \theta)$ b $\sec(\pi - \theta)$ c $\operatorname{cosec}(2\pi - \theta)$

d $\cot(\pi + \theta)$ e $\operatorname{cosec}(\pi + \theta)$ f $\cot(2\pi - \theta)$

8 Example 3 For $0 < x < 2\pi$, sketch the graph of $y = 3\operatorname{cosec}(x) - 1$

9 For $0 < x < 2\pi$, sketch the graph of $y = -\sec\left(x - \frac{\pi}{2}\right) + 1$

10 For $0 < x < 2\pi$, sketch the graph of $y = 4\cot(2x) + 1$

11 For $-\pi \le x \le \pi$, sketch the graph of $y = -2\sec\left(2x + \frac{\pi}{2}\right)$

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Reasoning and communication

12 Evaluate and compare $\sec\left(\frac{\pi}{4}\right)$ and $\text{cosec}\left(\frac{\pi}{4}\right)$.

13 Evaluate and compare $\sec\left(\frac{\pi}{3}\right)$ and $\text{cosec}\left(\frac{\pi}{6}\right)$.

14 Evaluate:

a $\sec\left(\frac{\pi}{6}\right)$

b $\cot\left(\frac{\pi}{4}\right)$

c $\text{cosec}\left(-\frac{\pi}{3}\right)$

d $\text{cosec}\left(\frac{2\pi}{3}\right)$

e $\sec(0)$

f $\cot(0)$

g $\cot\left(-\frac{4\pi}{3}\right)$

h $\sec\left(-\frac{7\pi}{4}\right)$

i $\text{cosec}\left(\frac{3\pi}{2}\right)$

i $\sec\left(\frac{\pi}{3}\right)$

15 Use the graph of $y = \tan(x)$ to sketch $y = \cot(x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

16 Use the graph of $y = \cos(3x)$ to sketch the graph of $y = \sec(3x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

17 Use the graph of $y = \sin(4x)$ to sketch the graph of $y = \text{cosec}(4x)$ from 0 to π.

18 Use the graph of $y = \tan(2x)$ to sketch the graph of $y = \cot(2x)$ from $-\pi$ to π.

19 Use the graph of $y=\cos\left(x-\frac{\pi}{4}\right)$ to sketch the graph of $y=\sec\left(x-\frac{\pi}{4}\right)$ from $-\pi$ to π.

20 Use the graph of $y=\sin\left(x+\frac{\pi}{6}\right)$ to sketch the graph of $y=\text{cosec}\left(x+\frac{\pi}{6}\right)$ from 0 to 2π.

21 Use the graph of $y=\tan\left(x+\frac{\pi}{4}\right)$ to sketch the graph of $y=\cot\left(x+\frac{\pi}{4}\right)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

22 Sketch the graph of $y=\text{cosec}\left(x+\frac{\pi}{4}\right)$ from 0 to 2π.

23 Sketch the graph of $y=\sec\left(3x+\frac{\pi}{4}\right)$ from 0 to π.

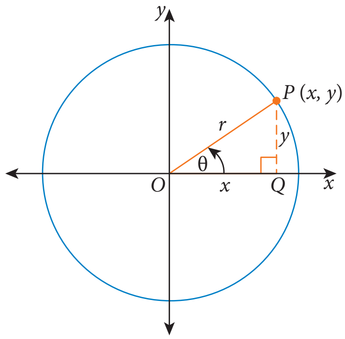
24 Sketch the graph of $y=\cot\left(2x-\frac{\pi}{4}\right)$ from 0 to π.

25 Sketch the graph of $y=\sec\left(2x-\frac{\pi}{3}\right)$ from 0 to π.

9.02 THE PYTHAGOREAN IDENTITIES

The earliest applications of trigonometry were in the fields of astronomy, navigation and surveying. Trigonometry was used to calculate distances that could not be measured directly. You may have investigated how to measure the height of tall structures like the school flagpole, weather vane, or wind turbine in your own previous work on trigonometry.

You should already be familiar with Pythagoras' theorem in a right angled triangle.



Pythagorean identities are so called because they follow the rules for Pythagoras' theorem.

For any point $P(x, y)$ on a circle of radius r with centre at the origin:

$$\sin(\theta) = \frac{y}{r}$$

$$\cos(\theta) = \frac{x}{r}$$

$$\tan(\theta) = \frac{y}{x}$$

Using Pythagoras' theorem, we get $r^2 = x^2 + y^2$.

Example 4

Prove the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$

Solution

Using Pythagoras' theorem, rewrite this relationship in terms of the lengths in the first quadrant of the unit circle.	Using the equation $c^2 = a^2 + b^2$, we get the relationship $r^2 = x^2 + y^2$
Relate the lengths r, x and y to trig functions.	We know that $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$
Rearrange the trig function equations.	$\cos(\theta) = \frac{x}{r} \Rightarrow x = r\cos(\theta)$ and $\sin(\theta) = \frac{y}{r} \Rightarrow y = r\sin(\theta)$
Substitute using $r^2 = x^2 + y^2$.	We know that $r^2 = x^2 + y^2$, giving $r^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta)$ So for any r: $1 = \cos^2(\theta) + \sin^2(\theta)$ **QED**

This proves the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

Rearrangement and substitution of A for θ gives the following.

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IMPORTANT

The basic Pythagorean identity

$\sin^2(A) + \cos^2(A) = 1$

$\cos^2(A) = 1 - \sin^2(A)$

$\sin^2(A) = 1 - \cos^2(A)$

where A is any angle, measured in either degrees or radians.

Example 5

Prove the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$

Solution

Write the basic Pythagorean identity.	$\sin^2(\theta) + \cos^2(\theta) = 1$
Divide by $\cos^2(\theta)$.	$\dfrac{\sin^2(\theta)}{\cos^2(\theta)} + \dfrac{\cos^2(\theta)}{\cos^2(\theta)} = \dfrac{1}{\cos^2(\theta)}$
Simplify and rearrange.	$\left[\dfrac{\sin(\theta)}{\cos(\theta)}\right]^2 + 1 = \left[\dfrac{1}{\cos(\theta)}\right]^2$
Write the basic functions.	$\tan^2(\theta) + 1 = \sec^2(\theta)$ **QED**

WS Using Pythagoras' theorem

You can prove the other identity in the same way.

IMPORTANT

Pythagorean identities

$\cos^2(A) + \sin^2(A) = 1$

$\tan^2(A) + 1 = \sec^2(A)$

$\cot^2(A) + 1 = \mathrm{cosec}^2(A)$

where A is any angle, measured in either degrees or radians.

The triple bar symbol '$\equiv$' is often used in identities, meaning that the statement is true for every value of the unknown. For example, you could write $\cos^2(\theta) + \sin^2(\theta) \equiv 1$.

IMPORTANT

You can use the following hints in proving trigonometric identities.

1 Try starting with the most complicated-looking side of the identity.

2 Fractional expressions are generally more complicated than sums and differences. Products are the simplest.

3 If the expressions involve squares, try using the Pythagorean identities.

4 If you cannot see how to proceed, try changing all the functions to sin and cos.

5 If you are getting nowhere, try starting from the other side of the identity.

Example 6

Prove the identity $\text{cosec}^2(x) = \sec(x)\,\text{cosec}(x)\cot(x)$

Solution

The RHS looks more complicated.

$$\text{RHS} = \sec(x)\,\text{cosec}(x)\cot(x)$$

Change to simpler trig functions.

$$= \frac{1}{\cos(x)} \times \frac{1}{\sin(x)} \times \frac{1}{\tan(x)}$$

Use $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

$$= \frac{1}{\cos(x)} \times \frac{1}{\sin(x)} \times \frac{1}{\frac{\sin(x)}{\cos(x)}}$$

Simplify and cancel.

$$= \frac{1}{\cos(x)} \times \frac{1}{\sin(x)} \times \frac{\cos(x)}{\sin(x)}$$

$$= \frac{1}{\sin(x)} \times \frac{1}{\sin(x)}$$

Simplify.

$$= \left[\frac{1}{\sin(x)}\right]^2$$

$$= \text{cosec}^2(x)$$

$$= \text{LHS}$$

QED

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EXERCISE 9.02 The Pythagorean identities

Concepts and techniques

1 Example 4 Prove that $\sin^2(A) + \cos^2(A) \equiv 1$ from the definitions.

2 Example 5 Prove that $\tan(A) \equiv \dfrac{\sin(A)}{\cos(A)}$ from the definitions.

3 Example 6 Prove that $1 + \cot^2(A) \equiv \text{cosec}^2(A)$.

4 Prove that $\cot(A) \equiv \dfrac{\cos(A)}{\sin(A)}$ from the definitions.

5 Show that $\tan\left(\dfrac{2\pi}{3}\right) = \dfrac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)}$.

Reasoning and communication

6 Prove that $[1 - \sin(\theta)][1 + \sin(\theta)] = \cos^2(\theta)$.

7 Prove that $\sin^4(\theta) - \cos^4(\theta) = 1 - 2\cos^2(\theta)$.

8 Prove that $[\cot(x) + \text{cosec}(x)]^2 = \dfrac{1 + \cos(x)}{1 - \cos(x)}$.

9 Prove that $\sec(y) + \tan(y) = \dfrac{1 + \sin(y)}{\cos(y)}$.

10 Prove that $2\,\text{cosec}^2(a) = \dfrac{1}{1 - \cos(a)} + \dfrac{1}{1 + \cos(a)}$.

11 Prove that $\dfrac{\sin(\beta) + \cos(\beta)}{\cos(\beta)\sin(\beta)} = \sin(\beta)[1 + \tan(\beta)] + \cos(\beta)[1 + \cot(\beta)]$.

12 Prove that $1 + \sin(\theta)\cos(\theta) = \dfrac{\sin(\theta)\cos(\theta)[\cos(\theta)\cot(\theta) - \sin(\theta)\tan(\theta)]}{\cos(\theta) - \sin(\theta)}$.

9.03 ANGLE SUM AND DIFFERENCE IDENTITIES

You can simplify problems that include more complex angles using the **angle sum and difference identities**. These are also commonly called '**sum and difference formulas**' or '**compound angle formulas**'.

We usually use x, y, etc. for the variables because they represent real numbers.

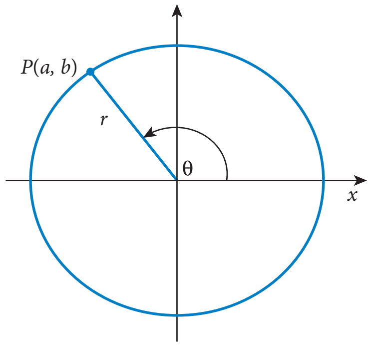
The identity

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

is proved first and the others are developed from it.

Create a circle diagram, with radius r, where the angle $(x - y)$ can be represented.

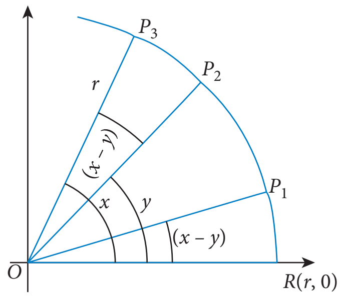
Choose a point $p(a, b)$ at a radius of r on the ray OP, drawn at an angle of θ with the x-axis.



We know that $\sin(\theta) = \frac{b}{r}$ and $\cos(\theta) = \frac{a}{r}$ so $b = r\sin(\theta)$ and $a = r\cos(\theta)$.
Point p at radius r on the ray OP drawn at an angle of θ with the x-axis can be written as $P(r\cos(\theta), r\sin(\theta))$.

Focus on just the first quadrant of the circle, plotting points on the arc that are subtended by the angle $(x - y)$.

Consider points P_1, P_2 and P_3 drawn at radius r in rays OP_1, OP_2 and OP_3 at angles $(x - y)$, y, x with the x-axis.



These points can be written as $P_1(r\cos(x - y), r\sin(x - y))$, $P_2(r\cos(y), r\sin(y))$ and $P_3(r\cos(x), r\sin(x))$.
$R(r, 0)$ is on the x-axis at radius r.

The angle subtended by the chord P_2P_3 at the centre of the circle is the angle x. This is because the chord P_2P_3 subtends the angle $(x + y) - y$, which equals x.

The chords OP_1, OP_2 and OP_3 are of equal length as they are radii of the circle with radius r.
Since the chords P_2P_3 and RP_1 subtend the same angle, x, at the centre of the circle, they must be equal in length.
Thus $|P_2P_3| = |RP_1|$

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Continue the proof by using the distance formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.	Using the distance formula, we get $\sqrt{[r\cos(x)-r\cos(y)]^2+[r\sin(x)-r\sin(y)]^2}$ $=\sqrt{[r-r\cos(x-y)]^2+[0-r\sin(x-y)]^2}$
Square both sides.	$[r\cos(x)-r\cos(y)]^2+[r\sin(x)-r\sin(y)]^2$ $=[r-r\cos(x-y)]^2+[0-r\sin(x-y)]^2$
Take out the common factor of r^2 and then divide both sides by r^2.	$[\cos(x)-\cos(y)]^2+[\sin(x)-\sin(y)]^2$ $=[1-\cos(x-y)]^2+[-\sin(x-y)]^2$
Multiply out the brackets and simplify, using the Pythagorean identity. Collect and rearrange the terms.	$\cos^2(x)-2\cos(x)\cos(y)+\cos^2(y)+\sin^2(x)-2\sin(x)\sin(y)+\sin^2(y)$ $=1-2\cos(x-y)+\cos^2(x-y)+\sin^2(x-y)$ This gives $2-2\cos(x)\cos(y)-2\sin(x)\sin(y)$ $=2-2\cos(x-y)$ leading to $-\cos(x)\cos(y)-\sin(x)\sin(y)=-\cos(x-y)$
Simplify to get the result required.	$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$ **QED**

The identity $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$ can also be written as $\cos(A-B)\equiv\cos(A)\cos(B)+\sin(A)\sin(B)$.

IMPORTANT

Angle sum and difference identities

The following identities apply for all A and B.

$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$

$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$

$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$

$\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$

$$\tan(A+B)=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$\tan(A-B)=\frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

You will notice that the sum and difference formulas for tangent look quite different from the sine and cosine formulas. It is quite common for tangent expressions and graphs to be quite different from those of sine and cosine. This is because it is a quotient.

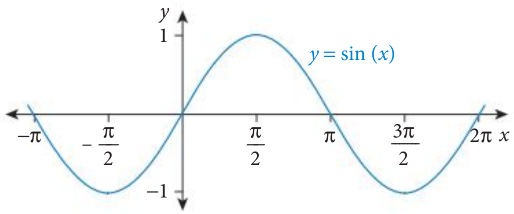
Example 7

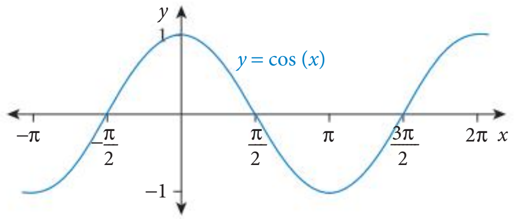
Expand $\cos(3x + 2y)$.

Solution

Write the expression.	$\cos(3x + 2y)$
Use $\cos(A + B)$	$= \cos(3x)\cos(2y) - \sin(3x)\sin(2y)$
Write the result.	$\cos(3x + 2y) = \cos(3x)\cos(2y) - \sin(3x)\sin(2y)$

You may have already learnt about **complementary** trig functions.





The graphs of $y = \sin(x)$ and $y = \cos(x)$ look very similar. In fact, in the graph of $y = \cos(x)$, if you use a **phase shift** of $\frac{\pi}{2}$ in the positive direction of the x-axis, you will notice that the cosine graph looks exactly like the sine graph.

WS Simplifying periodic functions

There are complementary relationships between sine and cosine, tangent and cotangent, secant and cosecant.

Note the prefix 'co'.

Important

Complementary function identities

The following identities apply for all values of θ.

$$\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin(\theta)$$

$$\tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot(\theta)$$

Also

$$\sin\left(\frac{3\pi}{2} \pm \theta\right) = -\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp\cot(\theta)$$

Example 8

Use the expansion of cos $(y - x)$ to simplify $\cos\left(\frac{\pi}{2} - x\right)$.

Solution

Write the theorem for cos $(y - x)$.

$\cos(y - x) = \cos(y)\cos(x) + \sin(y)\sin(x)$

Let $y = \frac{\pi}{2}$.

$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2}\right)\cos(x) + \sin\left(\frac{\pi}{2}\right)\sin(x)$

Evaluate.

$= 0 \times \cos(x) + 1 \times \sin(x)$

Write the result.

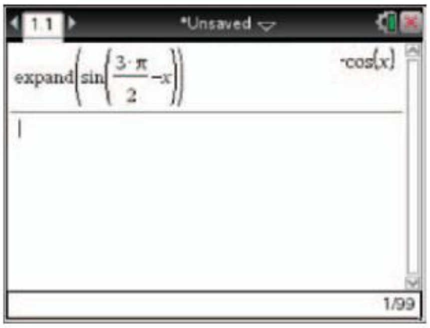
$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$

Complementary angle properties

TI-Nspire CAS

Make sure that you have the Angle in radians and the Calculation Mode in Auto or Exact.

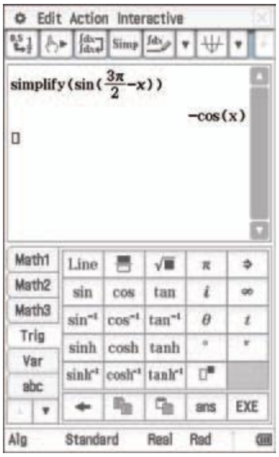
Use the expand() instruction, either from the catalogue (k)m or from b, 3: Algebra and 3: Expand.



ClassPad

From the Main $\sqrt{\alpha}$ application you can use **simplify** (first tap **Action** then **Transformation**) to simplify these expressions.

Make sure that the calculator is set to radians (**Rad**).



You can use the sum and difference formulas to prove identities in a similar way to those already encountered.

Example 9

Prove that $\cot(x)+\cot(y)=\dfrac{\sin(x+y)}{\sin(x)\sin(y)}$

Solution

The right-hand side looks more complicated, so start with it.

$$\text{RHS}=\frac{\sin(x+y)}{\sin(x)\sin(y)}$$

Use the theorem for sin $(x + y)$.

$$=\frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\sin(x)\sin(y)}$$

Express this as separate fractions.

$$=\frac{\sin(x)\cos(y)}{\sin(x)\sin(y)}+\frac{\cos(x)\sin(y)}{\sin(x)\sin(y)}$$

$$=\frac{\cos(y)}{\sin(y)}+\frac{\cos(x)}{\sin(x)}$$

Use the definition for cot and rearrange.

$= \cot(y) + \cot(x)$

$= \cot(x) + \cot(y)$

$= \text{LHS}$ **QED**

EXERCISE 9.03 Angle sum and difference identities

Concepts and techniques

1 Example 7 Expand the following.

a $\sin(2x + y)$ b $\cos(4x - 3y)$ c $\sin(3p - 5q)$

d $\tan(2e - f)$ e $\cos(3\alpha + 2\beta)$ f $\sin(4m + n)$

g $\tan(w + 3x)$ h $\sin(2D - 5E)$ i $\cos(P - 4Q)$

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2 Example 8 Simplify the following.

a $\sin(\pi - x)$ b $\cos(\pi + x)$ c $\tan\left(\frac{\pi}{2} - x\right)$

d $\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$ e $\cot\left(\frac{3\pi}{2} - x\right)$ f $\sec\left(x + \frac{\pi}{2}\right)$

g $\sin(2\pi - x)$ h $\cos\left(\frac{3\pi}{2} - x\right)$ i $\tan(x - \pi)$

3 Use the expansion of $\cos(x - y)$ and $\sin(x - y)$ to prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ and $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$.

4 Uses expansions of $\sin\left(\frac{\pi}{2} + x\right)$ and $\cos\left(\frac{\pi}{2} + x\right)$ to prove that $\tan\left(\frac{\pi}{2} + x\right) = -\cot(x)$.

Reasoning and communication

5 Prove that $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

6 Prove that $\tan(x - y) = \dfrac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$.

7 Example 9 Prove that $\cot(x + y) = \dfrac{\cot(x)\cot(y) - 1}{\cot(y) + \cot(x)}$.

8 Prove that $\tan(x) + \tan(y) = \dfrac{\sec(x)\sec(y)}{\operatorname{cosec}(x + y)}$.

9 Prove that $\sin(x + y) + \cos(x - y) = [\sin(x) + \cos(x)][\sin(y) + \cos(y)]$.

10 Prove that $\sin(x + y)\sin(x - y) = \sin^2(x) - \sin^2(y)$.

11 Prove that $\dfrac{\cot(x) - \cot(y)}{1 - \cot(x)\cot(y)} = \dfrac{\sin(x - y)}{\cos(x + y)}$.

12 Prove that $\tan(x - y) + \tan(y - z) + \tan(z - x) = \tan(x - y)\tan(y - z)\tan(z - x)$.

9.04 DOUBLE ANGLE FORMULAS

Double angle formulas can be developed from sum and difference identities.

Example 10

Show that $\sin(2x) = 2\sin(x)\cos(x)$

Solution

Express $2x$ as $x + x$.	$\sin(2x) = \sin(x + x)$
Use $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.	$\sin(x + x) = \sin(x)\cos(x) + \cos(x)\sin(x)$
Collect terms.	$\sin(x + x) = 2\sin(x)\cos(x)$
Write the result.	$\sin(2x) = 2\sin(x)\cos(x)$

You can develop each of the double angle formulas in a similar way.

IMPORTANT

Double angle formulas

The following identities apply for all A.

$\sin(2A) = 2\sin(A)\cos(A)$

$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$

$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

You have three choices for the double angle formula for $\cos(2A)$. You choose whichever is more suitable for the problem.

You can use the double angle formulas for multiple angles.

Example 11

Use the double angle formula for tan to find an expression for $\tan(6x)$.

Solution

Express $6x$ as $3x + 3x$.

$$\tan(6x) = \tan(3x + 3x)$$

Use the identity $\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ to find an expression for $\tan(6x)$.

$$\tan(3x+3x) = \frac{\tan(3x)+\tan(3x)}{1-\tan(3x)\tan(3x)}$$

Collect terms.

$$\tan(3x+3x) = \frac{2\tan(3x)}{1-\tan^2(3x)}$$

Write the result.

$$\tan(6x) = \frac{2\tan(3x)}{1-\tan^2(3x)}$$

It is important for you to remember that it is the **angle** that is doubled and not the rest of the expression.

You could continue to simplify questions similar to Example **11**, using a combination of double angle formulas and sum and difference identities.

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Example 12

Find an expression for $\sin(3x)$ in terms of $\sin(x)$ only.

Solution

Express $3x$ as $2x + x$.	$\sin(3x) = \sin(2x + x)$
Use $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$= \sin(2x)\cos(x) + \cos(2x)\sin(x)$
Use the $\sin(2x)$ and $\cos(2x)$ formulas.	$= 2\sin(x)\cos(x)\cos(x) + [1 - 2\sin^2(x)]\sin(x)$
Remove the brackets.	$= 2\sin(x)\cos^2(x) + \sin(x) - 2\sin^3(x)$
Use $\sin^2(x) + \cos^2(x) = 1$	$= 2\sin(x)[1 - \sin^2(x)] + \sin(x) - 2\sin^3(x)$
Remove the brackets.	$= 2\sin(x) - 2\sin^3(x) + \sin(x) - 2\sin^3(x)$
Collect like terms.	$= 3\sin(x) - 4\sin^3(x)$
Write the result.	$\sin(3x) = 3\sin(x) - 4\sin^3(x)$

WS Advanced exponential and periodic functions 1

You can also use double angle formulas to prove trig identities as you did in Section **9.03**.

EXERCISE 9.04 Double angle formulas

Concepts and techniques

1 Example 10 Show that $\cos(2A) = 1 - 2\sin^2(A)$

2 Show that $2\cos^2(A) - 1 = 1 - 2\sin^2(A)$

3 Show that $\cos^2(y) - \sin^2(y) = 1 - 2\sin^2(y)$

4 Example 11 Show that $\sin(8A) = 2\sin(4A)\cos(4A)$.

5 Show that $\cos(5A) = 1 - 2\sin^2\left(\frac{5A}{2}\right)$

6 Use a suitable double angle formula to expand:

a $\sin(4x)$ **b** $\cos(6A)$ **c** $\tan(4y)$ **d** $\cos(y)$ **e** $\tan(4x)$

7 Example 12 Use sum and difference identities and double angle formulas, where suitable, to continue to expand the expressions in question **6a**, **b**, **c**.

Reasoning and communication

8 Prove that $\dfrac{1}{\cos(\theta)+\sin(\theta)} + \dfrac{1}{\cos(\theta)-\sin(\theta)} = 2\cos(\theta)\sec(2\theta)$

9 Prove that $\tan(4x) - \tan(x) = \dfrac{\sec(4x)\sec(x)}{\operatorname{cosec}(3x)}$.

10 Show that $\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$.

11 Show that $\cot(2x) = \dfrac{\cot^2(x) - 1}{2\cot(x)}$.

12 Prove that cot (θ) – tan (θ) = 2 cot (2θ).

13 Prove that $\tan(2x) = \dfrac{1}{1 - \tan(x)} - \dfrac{1}{1 + \tan(x)}$.

14 Prove that tan (x) = cot (x) – 2 cot (2x).

9.05 FINDING AND USING EXACT VALUES

You can simplify many expressions that include angles with exact trigonometric ratios using the sum and difference identities and the double angle identities. They also allow you to calculate the exact values of other angles, such as $\frac{\pi}{12}$.

Example 13

Find the exact value of cos (15°).

Solution

Think of 2 values, using 30°, 45° or 60° that will combine, using + or – to get 15°.	$60° - 45° = 15°$
Use the difference formula for cos.	$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
Use this difference expression in cos (15°).	$\cos(60° - 45°) = \cos(60°)\cos(45°) + \sin(60°)\sin(45°)$
Use exact values to evaluate the RHS of the expression.	$= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$ $= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$ $= \frac{\sqrt{2} + \sqrt{6}}{4}$
Simplify the LHS.	$\cos(60° - 45°) = \cos(15°)$
Write the answer.	$\cos(15°) = \frac{\sqrt{2} + \sqrt{6}}{4}$

You can use any of the other sum and difference identities to simplify angle expressions that can be expressed in terms of the three exact-value angles $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ (30°, 45°, 60°).

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Example 14

Find the exact value of $\sin\left(\frac{\pi}{12}\right)$.

Solution

Think of 2 values, using $\frac{\pi}{6}$, $\frac{\pi}{4}$ or $\frac{\pi}{3}$ that will combine, using + or – to get $\frac{\pi}{12}$.

$$\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$$

Use the difference formula for sin.

$$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$$

Use this difference expression in $\sin\left(\frac{\pi}{12}\right)$.

$$\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

Use exact values to evaluate the RHS of the expression.

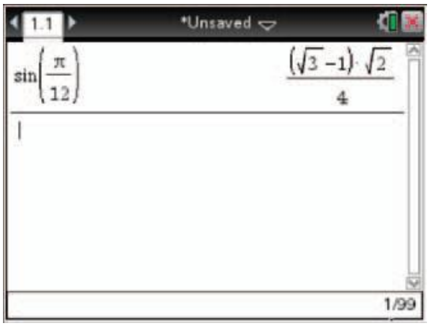
$$\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$
$$=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}-\frac{1}{2}\times\frac{\sqrt{2}}{2}$$
$$=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$$
$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

Simplify the LHS.

$$\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{12}\right)$$

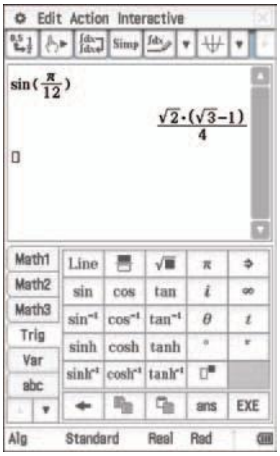
TI-Nspire CAS

Make sure that your calculator is set on radians and is in auto or exact Calculation Mode.

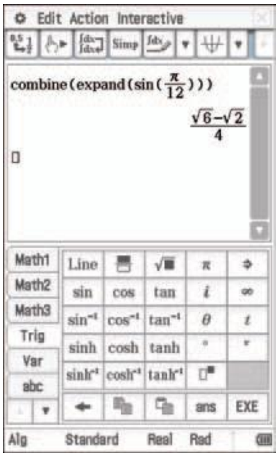


ClassPad

You can find exact values in one step.
Use the Main $\sqrt{\alpha}$ application and make sure that the calculator is set to **Standard**.
Sin, cos and tan and and fractions [fraction key] can be found on the [Trig] menu that is available after pressing [Keyboard].



You can use the command **expand** to get rid of the brackets and **combine** to present the answer with a common denominator. (Both commands are on the menu accessed by tapping **Action** then **Transformation**.)



Write the answer.

$$\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$$

Double angle formulas can be used to find the exact values of trigonometric ratios that have angles other than $\frac{\pi}{6}, \frac{\pi}{4}$ or $\frac{\pi}{3}$.

Example 15

Find the exact value of $\sin\left(\frac{3\pi}{8}\right)$ using a double angle formula.

Solution

Express $\frac{3\pi}{8}$ in terms of an angle with an exact trig value.

$$2\times\frac{3\pi}{8}=\frac{3\pi}{4}$$

Write the cos $(2x)$ rule with sin (x).

$$\cos(2x) = 1 - 2\sin^2(x)$$

Use the angle expression.

$$\cos\left(\frac{3\pi}{4}\right) = 1 - 2\sin^2\left(\frac{3\pi}{8}\right)$$

Rearrange.

$$2\sin^2\left(\frac{3\pi}{8}\right) = 1-\cos\left(\frac{3\pi}{4}\right)$$

Use the value $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$$2\sin^2\left(\frac{3\pi}{8}\right) = 1 + \frac{\sqrt{2}}{2}$$

Simplify the RHS.

$$2\sin^2\left(\frac{3\pi}{8}\right) = \frac{2+\sqrt{2}}{2}$$

Divide by 2.

$$\sin^2\left(\frac{3\pi}{8}\right) = \frac{2+\sqrt{2}}{4}$$

Find the square root.

$$\sin\left(\frac{3\pi}{8}\right) = \pm\frac{\sqrt{2+\sqrt{2}}}{2}$$

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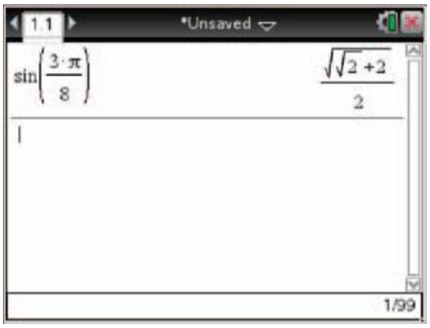
In the first quadrant sin (x) is positive.

$$\sin\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

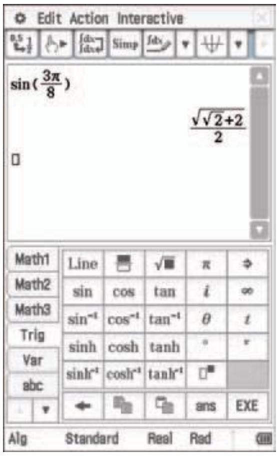
TI-Nspire CAS

You can check this on your CAS calculator. Make sure that it is set to radians and auto or exact Calculation Mode.



ClassPad

Make sure that your calculator is set on **Standard** calculation mode and radians (**Rad**).



EXERCISE 9.05 Finding and using exact values

Concepts and techniques

1 Example 13 Find the exact values of each of the following using angle sum and difference identities. Check your answers with your CAS calculator.

a tan (15°) b cos (75°) c $\tan\left(\frac{7\pi}{12}\right)$

d $\sin\left(\frac{5\pi}{12}\right)$ e $\cot\left(\frac{11\pi}{12}\right)$ f sin (105°)

g cos (195°) h $\sin\left(\frac{7\pi}{12}\right)$ i sin (195°)

2 Example 14 Find the exact value of tan (75°), using angle sum and difference identities and a combination of the angles 45° and 30°.

3 Example 15 Find exact values of the following using a suitable double angle formula.

a $\sin\left(\frac{\pi}{12}\right)$ b $\cos\left(\frac{\pi}{8}\right)$ c $\tan\left(\frac{5\pi}{12}\right)$

d $\sin\left(\frac{\pi}{8}\right)$ e $\cos\left(\frac{5\pi}{12}\right)$ f $\tan\left(\frac{\pi}{8}\right)$

9.06 PRODUCTS TO SUMS

To simplify calculations, sometimes you will need to change products involving trigonometric functions into sums. The rules for these conversions can be proved using the addition theorems.

We first show that $\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Example 16

Show that $\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Solution

Simplify the RHS using sum and difference formulas.	**Proof** $\frac{1}{2}[\cos(A-B)+\cos(A+B)]$ $=\frac{1}{2}[\cos(A)\cos(B)+\sin(A)\sin(B)+\cos(A)\cos(B)-\sin(A)\sin(B)]$
Collect terms and simplify.	$=\frac{1}{2}\times 2\cos(A)\cos(B)$ $=\cos(A)\cos(B)$
Write the result.	$\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$ **QED**

Sum and difference identities similar to the one shown above yield the following rules for conversion from **products to sums and differences**.

IMPORTANT

Products as sums and differences

$$\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

$$\sin(A)\sin(B)=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\sin(A)\cos(B)=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$\cos(A)\sin(B)=\frac{1}{2}[\sin(A+B)-\sin(A-B)]$$

You will find these formulas useful when rearranging exact angle expressions.

Example 17

Given that sin (50°) ≈ 0.76604, use an appropriate products as sums and differences formula to evaluate the expression cos (40°) sin (10°). Give your answer correct to 3 decimal places.

Solution

Use the formula

$\cos(A)\sin(B) = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$.

$$\cos(40°)\sin(10°)$$
$$= \frac{1}{2}[\sin(40°+10°) - \sin(40°-10°)]$$

Collect terms and evaluate.

$$= \frac{1}{2}[\sin(50°) - \sin(30°)]$$
$$= \frac{1}{2}(0.76604 - 0.5)$$
$$= \frac{1}{2} \times 0.26604$$

Write the result.

$$= 0.133$$

Example 18

Express sin (4x) sin (7x) as a sum or a difference of trigonometric functions.

Solution

Use

$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.

$$\sin(4x)\sin(7x)$$
$$= \frac{1}{2}[\cos(4x-7x) - \cos(4x+7x)]$$

Collect terms and simplify.

$$= \frac{1}{2}[\cos(-3x) - \cos(11x)]$$

Remember that cos (–θ) = cos (θ).

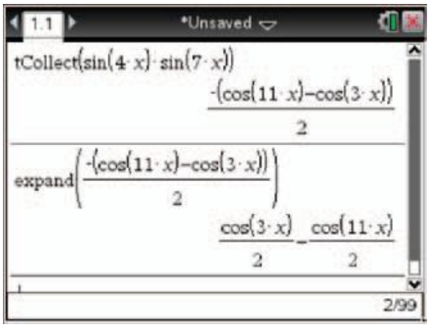
$$= \frac{1}{2}[\cos(3x) - \cos(11x)]$$

Write the result.

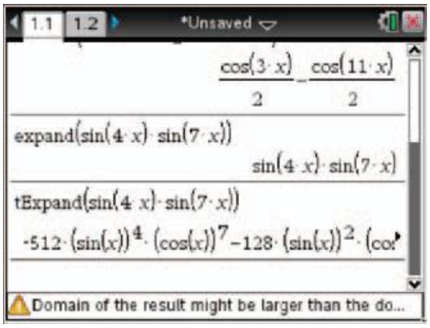
$$\sin(4x)\sin(7x) = \frac{1}{2}[\cos(3x) - \cos(11x)]$$

TI-Nspire CAS

Use tCollect() from the catalogue (k).
You can expand the terms using b, 3: Algebra, 3: Expand and using /v to insert the answer. The calculator will automatically insert the expression from the answer.

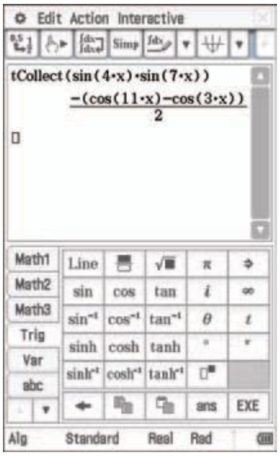


You cannot use expand() on its own or texpand() because you will then get an expression that has only powers of sin (x) and cos (x).



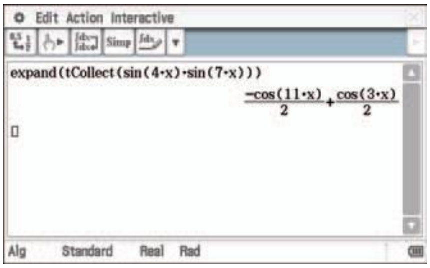
ClassPad

Use the Main $\sqrt{\alpha}$ application.
Tap **Interactive**, then **Transformation** and **tCollect**, type **tCollect** or choose it from the catalogue.

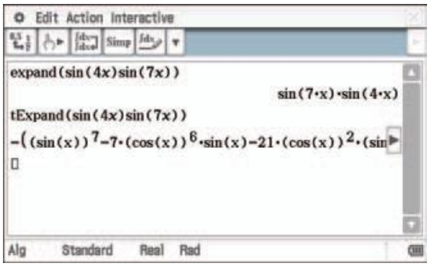


It is important that the 4***x*** and the 7***x*** are in brackets.
Note that the terms would need to be expanded.

This expansion can be done on the calculator by expand(tCollect(sin(4***x***)sin(7***x***))).



You cannot use **expand** or **tExpand** (also on the menu found by tapping **Action** then **Transformation**) on their own, as the screen on the right shows.
tExpand will use the double angle and the angle sum formulas to expand sin (4x) and sin (7x) until everything is expressed in terms of sin (x) and cos (x).



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EXERCISE 9.06 Products to sums

Concepts and techniques

1 Example 17 Given that $\sin(40°) \approx 0.6428$ and $\cos(40°) \approx 0.7660$, use an appropriate Products as sums and differences formula to evaluate the following expressions, giving your answers correct to 3 decimal places.

a $\cos(10°)\cos(50°)$ b $\sin(10°)\sin(50°)$
c $\sin(10°)\cos(50°)$ d $\cos(10°)\sin(50°)$

2 Example 18 Express the following as sums or differences of trigonometric functions.

a $2\sin(5x)\cos(7x)$ b $8\sin(3x)\sin(9x)$ c $6\cos(4x)\cos(6x)$
d $4\cos(5x)\sin(6x)$ e $\sin(6x)\cos(10x)$ f $3\cos(10x)\cos(14x)$
g $5\sin(2x)\sin(3x)$ h $7\cos(9x)\cos(11x)$ i $\cos(8x)\sin(x)$

3 Express each of the following as sums or differences of trigonometric functions and hence find their exact values.

a $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$ b $\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$

c $\sin\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{4}\right)$ d $2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)$

Reasoning and communication

4 Example 16 Show that $\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

5 Show that $2\cos(A)\sin(B) = \sin(A+B) - \sin(A-B)$

6 Show that $\sin(g)\sin(f) = \frac{1}{2}[\cos(f-g)-\cos(f+g)]$

7 Show that $\sin(c+d) = 2\sin(c)\cos(d) - \sin(c-d)$

8 Show that $\sin(11x)\cos(7x) - \sin(8x)\cos(4x) = \sin(3x)\cos(15x)$.

9 Show that $4\sin(3x)\sin(5x)\sin(8x) = \sin(6x) + \sin(10x) - \sin(16x)$.

10 Waves breaking on a beach may have travelled thousands of kilometres from where they were formed by wind blowing on the water. The waves can actually consist of smaller waves of different frequencies, amplitudes and wavelengths because they may have arisen in different parts of the ocean. Suppose that the waves arriving at Kirra, Qld. on a particular day actually consist of waves that can be modelled as $\sin(5t)$, $\sin(6t)$, $\sin(7t)$ and $\sin(8t)$. Show that every sixth wave is larger.


Shutterstock.com/Joshua Rainey Photography

9.07 SUMS TO PRODUCTS

You have examined the shapes of the trigonometric functions. In practice, the functions rarely occur singly, so we need to be able to combine the functions to obtain the overall shape of a combined graph.

In AC circuits, the effects of inductors and capacitors are such that the voltages in circuits involving these elements consist of two parts that are out of phase by 90°. We know $\sin(\theta + 90°) = \cos(\theta)$, so this effectively means that there is a combination of the sine and cosine functions involved in the voltages of AC circuits.

However, the coefficients of the functions are different, since they are determined by the sizes of the circuit elements, the applied voltages and other properties of the circuit.

There is a practical need to be able to express a function of the form $a \sin(x) + b \cos(x)$ as a single function. You can do this by using the addition theorems in reverse.

Example 19

Express $3 \sin(x) + 5 \cos(x)$ in the form $A \sin(x + \theta)$ and sketch the graph for $x = 0$ to 2π.

Solution

Use the addition theorem.

$$A \sin(x + \theta) = A[\sin(x) \cos(\theta) + \cos(x) \sin(\theta)] = A \cos(\theta) \sin(x) + A \sin(\theta) \cos(x)$$

Let $A \sin(x + \theta) = 3 \sin(x) + 5 \cos(x)$.

$$3 \sin(x) + 5 \cos(x) = A \cos(\theta) \sin(x) + A \sin(\theta) \cos(x)$$

Compare the LHS and RHS.

$$A \cos(\theta) = 3 \text{ and } A \sin(\theta) = 5.$$

We know that $A^2 = A^2 \times 1$, so we can use $\cos^2(\theta) + \sin^2(\theta) = 1$ as follows.

$$A^2 = A^2[\cos^2(\theta) + \sin^2(\theta)]$$

Expand and factorise.

$$= [A \cos(\theta)]^2 + [A \sin(\theta)]^2$$

Substitute values.

$$= 3^2 + 5^2 = 34$$

Take square roots to evaluate A.

$$A = \sqrt{34} \approx 5.83$$

Use the definition for $\tan(\theta)$.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{A \sin(\theta)}{A \cos(\theta)} = \frac{5}{3} \approx 1.67$$

Write the result.

$$3 \sin(x) + 5 \cos(x) = \sqrt{34} \sin\left[x + \tan^{-1}\left(\frac{5}{3}\right)\right]$$

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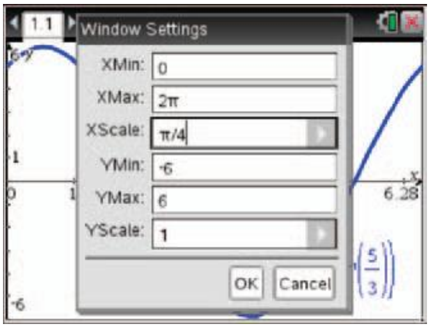
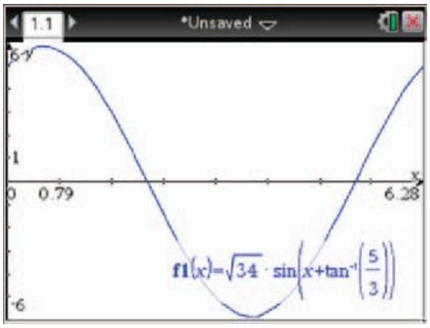
You can sketch this graph using a CAS calculator.

TI-Nspire CAS

Insert a graph and change the window settings to $0 \le x \le 2\pi$ and $-6 \le y \le 6$ using b, 4: Window/Zoom and 1: Window settings.

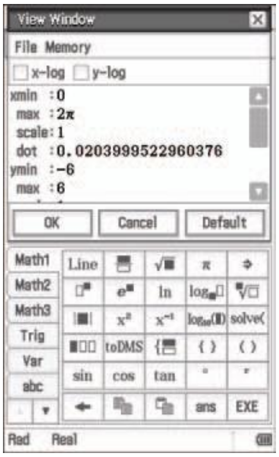


Enter the function as f1(x) = $\sqrt{34}\sin\left[x + \tan^{-1}\left(\frac{5}{3}\right)\right]$.



ClassPad

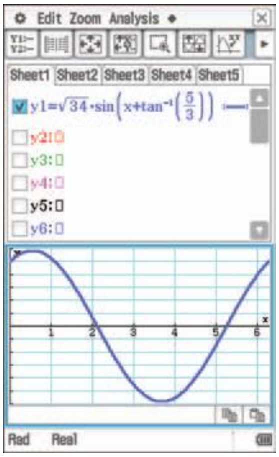
Use the **Graph&Table** application.
Set the calculator to radian measure (**Rad**).
Tap and adjust the viewing window for $0 \le x \le 2\pi$ and $-6 \le y \le 6$, using a scale of 1.



Enter y1= $\sqrt{34}\sin(x+\tan^{-1}(\frac{5}{3}))$

Use the [Trig] menu to access the fraction and trigonometric functions. Press [EXE] to finish entering the function.

Tap [graph icon] to draw the graph.



You can use the change of variables $A = x + y$ and $B = x - y$ to express sums as products.

First, we show that

$$\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right].$$

Write the change of variables. $A = x + y$ and $B = x - y$

Solve as simultaneous equations for x and y. $x=\frac{1}{2}(A+B)$ and $y=\frac{1}{2}(A-B)$

Write the formula for $2\sin(x)\cos(y)$. $2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$

Substitute for x, y, $(x + y)$ and $(x - y)$. $2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]=\sin(A)+\sin(B)$

Reverse to get the result. $\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$

QED

The same change of variables can be used with the other formulas to give the following.

IMPORTANT

Sums to products

The following identities apply for all values of A and B.

$$\sin(A)+\sin(B)=2\sin\left[\tfrac{1}{2}(A+B)\right]\cos\left[\tfrac{1}{2}(A-B)\right]$$

$$\sin(A)-\sin(B)=2\cos\left[\tfrac{1}{2}(A+B)\right]\sin\left[\tfrac{1}{2}(A-B)\right]$$

$$\cos(A)+\cos(B)=2\cos\left[\tfrac{1}{2}(A+B)\right]\cos\left[\tfrac{1}{2}(A-B)\right]$$

$$\cos(A)-\cos(B)=-2\sin\left[\tfrac{1}{2}(A+B)\right]\sin\left[\tfrac{1}{2}(A-B)\right]$$

Note: the *last* formula has a *negative* sign.

The sums to products rules can be used to analyse wave interference.

Example 20

A sound wave can be modelled as a sine or cosine function specifying the displacement of air molecules as a function of time. For a **frequency** of f Hz (hertz), the displacement can be written as $A\sin(2\pi ft)$, where t is in seconds and A is the maximum displacement. Find the resulting sound when two guitar strings are plucked at frequencies of 256 Hz and 258 Hz with similar loudness.

Solution

Add these together to obtain the resulting sound.	$A\sin(2\pi f_1 t) + A\sin(2\pi f_2 t)$
Factorise.	$= A[\sin(2\pi f_1 t) + \sin(2\pi f_2 t)]$
Use the sum-to-product rule.	$= A\left\{2\sin\left[\tfrac{1}{2}(2\pi f_1 t + 2\pi f_2 t)\right]\cos\left[\tfrac{1}{2}(2\pi f_1 t - 2\pi f_2 t)\right]\right\}$
Factorise.	$= A\left\{2\sin\left[2\pi\left(\dfrac{f_1+f_2}{2}\right)t\right]\cos\left[2\pi\left(\dfrac{f_1-f_2}{2}\right)t\right]\right\}$
	$= 2A\sin\left[2\pi\left(\dfrac{f_1+f_2}{2}\right)t\right]\cos\left[2\pi\left(\dfrac{f_1-f_2}{2}\right)t\right]$
Substitute for f_1 and f_2 and simplify.	$= 2A\sin(2\pi\times 257t)\cos(2\pi\times 1t)$
Rearrange terms.	$= 2A\cos(2\pi\times 1t)\sin(2\pi\times 257t)$
Let $f_3 = 257$ and $A' = 2A\cos(2\pi f_4 t)$ where $f_4 = 1$.	$= A'\sin(2\pi f_3 t)$
Compare the form of the last expression with the original expressions and write a conclusion.	The sound will have a frequency of 257 Hz, but the loudness will vary from loud to soft with a frequency of 1 Hz. This variation is called 'beating' and can be heard clearly when tuning the guitar.

You can use the sums to products formulas to simplify expressions containing exact values.

Example 21

Simplify the expression cos (75°) + cos (–15°) and find its exact value.

Solution

Write the expression.

$\cos(75°) + \cos(-15°)$

Express cos (–15°) in terms of cos (15°).

$= \cos(75°) + \cos(15°)$

Use $\cos(A) + \cos(B) = 2\cos(A + B)\cos(A - B)$.

$= 2\cos\left[\frac{1}{2}(75° + 15°)\right]\cos\left[\frac{1}{2}(75° - 15°)\right]$

Simplify.

$= 2\cos(45°)\cos(30°)$

Substitute exact values.

$= 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}$

Simplify.

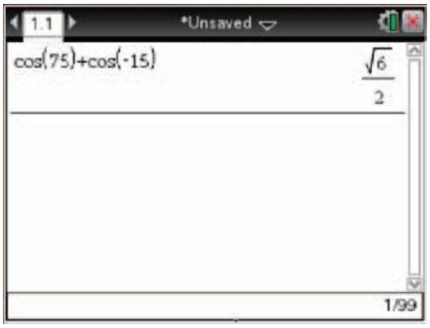
$= \frac{\sqrt{6}}{2}$

Write the result.

$\cos(75°) + \cos(-15°) = \frac{\sqrt{6}}{2}$

TI-Nspire CAS

Make sure that your calculator is set on degrees.

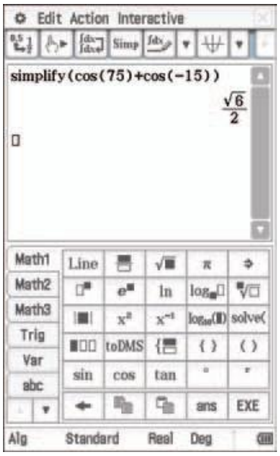


ClassPad

This question can be done using **simplify**.

Make sure that your calculator is set on **Standard** and **Deg**.

Use the Main $\sqrt{\alpha}$ application and tap **Action** then **Transformation** then **simplify**.



You can also use the sums to products formulas to prove trigonometric identities.

EXERCISE 9.07 Sums to products

Concepts and techniques

You should use a CAS calculator to sketch the graphs in questions **1** to **6**.

1 Example 19 **a** Express $6 \sin(x) + 8 \cos(x)$ in each of the following forms and sketch the graphs for $x = 0$ to 2π.

i $A \sin(x + \theta)$ **ii** $A \cos(x + \theta)$ **iii** $A \sin(x - \theta)$ **iv** $A \cos(x - \theta)$

b Comment on the graphs obtained.

2 Express $\sin(x)+\sqrt{3}\cos(x)$ in the form $A \sin(x + \theta)$ and sketch the graph for $x = 0$ to 2π.

3 Express $\sin(x) + \cos(x)$ in the form $A \cos(x - \theta)$ and sketch the graph for $x = 0$ to 2π.

4 Express $\sqrt{3}\sin(x)-\cos(x)$ in the form $A \cos(x + \theta)$ and sketch the graph for $x = 0$ to 2π.

5 Express $12 \sin(x) + 5 \cos(x)$ in the form $A \sin(x + \theta)$ and sketch the graph for $x = 0$ to 2π.

6 Express $8 \sin(x) - 15 \cos(x)$ in the form $A \sin(x - \theta)$ and sketch the graph for $x = 0$ to 2π.

7 Express each of the following as a product.

a $\sin(3x) + \sin(7x)$ **b** $\sin(8x) - \sin(2x)$ **c** $\cos(5x) + \cos(9x)$

d $\cos(11x) - \cos(7x)$ **e** $\sin(4x) + \sin(9x)$ **f** $\sin(2x) - \sin(5x)$

g $\cos(x) + \cos(5x)$ **h** $\cos(3x) - \cos(11x)$ **i** $\sin(x) + \sin(2x)$

8 Example 21 Use the formula $\cos(A)+\cos(B)=2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$ to simplify the expression $\cos(120°) + \cos(60°)$.

9 Use the formula $\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$ to simplify the expression $\sin(75°) + \sin(15°)$.

10 Use the formula $\sin(A)-\sin(B)=2\cos\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$ to simplify the expression $\sin(135°) - \sin(-45°)$.

Reasoning and communication

11 Show that $2 \cos(x) \cos(y) = \cos(x + y) + \cos(x - y)$.

12 Use the expansions of $\sin(2x)$, $\sin(x + y)$ and $\cos(x - y)$ to show that

$$\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right].$$

(*Hint*: Write $A = 2x$ and $B = 2y$.)

13 Use the expansions of $\cos(2x)$, $\cos(x + y)$ and $\cos(x - y)$ to show that

$$\cos(A)+\cos(B)=2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right].$$

14 Example 20 A 6th (lowest, E) string on a guitar is in tune at 82.4 Hz and is being tuned upward. When it is pushed against the 5th fret, it should give the same note (110 Hz) as the next string, A. In fact, when both strings are plucked, there is a 'beat' with intervals of 0.5 s. When the A

string is tightened, the 'beat' changes to every second. The correct note can be modelled as a sine function, with the formula $y = a\sin\left(\frac{110t}{2\pi}\right)$, where a is the loudness and t is time. Show that this is consistent with making the guitar closer to being in tune using a sums to products formula.

15 a Express $\sin(x + h) - \sin(x)$ as a product.

b Find an expression for $\frac{\sin(x+h)-\sin(x)}{h}$.

c Given that $\lim_{z\to 0}\frac{\sin(z)}{z}=1$, find $\frac{d}{dx}\sin(x)$ from the definition of the derivative.

9.08 TRIGONOMETRIC IDENTITIES

The use of trigonometric functions extends far beyond the solution of geometric problems involving triangles. In fact, their most important application is in the modelling of periodic phenomena. These applications include light, sound, electrical circuits, electronics, shock absorbers and even the way a door with a damper closes. While it is obvious that AC electrical theory uses trigonometric functions, it is not so obvious that advanced DC current theory also uses them. However, to model what happens when a switch is turned on or off, trigonometric functions are vital in the analysis of transient currents that can easily damage sensitive components in a circuit.

When you prove identities, start with the side that is more complicated.

Example 22

Prove the identity $2\sin^2(\theta)\cos^2(\theta)=\frac{1}{2}\sin^2(2\theta)$

Solution

Start with the RHS.	$\frac{1}{2}\sin^2(2\theta)$
Use the double angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.	$=\frac{1}{2}[\sin(2\theta)]^2$ $=\frac{1}{2}[2\sin(\theta)\cos(\theta)]^2$
Simplify.	$=\frac{1}{2}\times 4\sin^2(\theta)\cos^2(\theta)$ $= 2\sin^2(\theta)\cos^2(\theta)$
Compare with the LHS.	$=$ LHS
Write the result.	$2\sin^2(\theta)\cos^2(\theta)=\frac{1}{2}\sin^2(2\theta)$ **QED**

Sometimes it is easier to simplify both the LHS and the RHS.

Example 23

Prove the identity $\dfrac{1}{\operatorname{cosec}(x)+\cot(x)}=\dfrac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$

Solution

Start with the LHS by simplifying the fraction.

$$\frac{1}{\operatorname{cosec}(x)+\cot(x)}$$

Use the reciprocal identities $\operatorname{cosec}(x)=\dfrac{1}{\sin(x)}$ and $\cot(x)=\dfrac{1}{\tan(x)}$.

$$=\frac{1}{\frac{1}{\sin(x)}+\frac{1}{\tan(x)}}$$

Simplify using the identity $\tan(x)=\dfrac{\sin(x)}{\cos(x)}$.

$$=\frac{1}{\frac{1}{\sin(x)}+\frac{\cos(x)}{\sin(x)}}$$

Express and simplify using a common denominator.

$$=\frac{\sin(x)}{1+\cos(x)}$$

Write the result for the LHS.

$$\frac{1}{\operatorname{cosec}(x)+\cot(x)}=\frac{\sin(x)}{1+\cos(x)}$$

Now simplify the RHS.
Use the double angle formulas $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = 1 - 2\sin^2(x)$.

$$\frac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$$

$$=\frac{\frac{1}{2}\left(1-\left[1-2\sin^2(x)\right]\right)}{\frac{\sin(x)}{\cos(x)}\cos(x)+\frac{1}{2}\times 2\sin(x)\cos(x)}$$

Cancel and collect terms.

$$=\frac{\sin^2(x)}{\sin(x)+\sin(x)\cos(x)}$$

Take out $\sin(x)$ as a common factor and cancel.

$$=\frac{\sin^2(x)}{\sin(x)[1+\cos(x)]}$$

$$=\frac{\sin(x)}{1+\cos(x)}$$

$= \text{LHS}$

Write the result.

$$\frac{1}{\operatorname{cosec}(x)+\cot(x)}=\frac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$$

You can simplify trigonometric expressions in terms of either sine or cosine only.

Example 24

Rewrite the expression 3 sin $(2x)$ cos (x) + sin (x) cos $(2x)$ in terms of sin (x) only.

Calculating exact values of trigonometric functions

Solution

Use the double angle formulas
$\sin(2x) = 2\sin(x)\cos(x)$
and
$\cos(2x) = 1 - 2\sin^2(x)$
to simplify the expression.

$3\sin(2x)\cos(x) + \sin(x)\cos(2x)$
$= 3 \times 2\sin(x)\cos(x)\cos(x) + \sin(x) \times [1 - 2\sin^2(x)]$

Collect terms.

$= 6\sin(x)\cos^2(x) + \sin(x) - 2\sin^3(x)$

Use the rearrangement of the Pythagorean identity
$1 - \sin^2(x) = \cos^2(x)$

$= 6\sin(x)[1 - \sin^2(x)] + \sin(x) - 2\sin^3(x)$

Expand the brackets and collect terms.

$= 6\sin(x) - 6\sin^3(x) + \sin(x) - 2\sin^3(x)$
$= 7\sin(x) - 8\sin^3(x)$

Write the result.

$3\sin(2x)\cos(x) + \sin(x)\cos(2x) = 7\sin(x) - 8\sin^3(x)$

EXERCISE 9.08 Trigonometric identities

Reasoning and communication

1 Example 22 Prove the identity $\frac{1}{8}\sin^2(4x) = \frac{1}{2}\sin^2(2x)\cos^2(2x)$

2 Show that $2\sin^2(6x) = 8[\sin^2(3x) - \sin^4(3x)]$

3 Prove the following identities.

a $2\sin(x) - 2\sin^2(x) - \cos^2(x) = -[\sin(x) - 1]^2$

b $\frac{1}{2}\left[\cos(x) + \cos\left(\frac{\pi}{3}\right)\right] = \cos\left(\frac{x}{2} + \frac{\pi}{6}\right)\cos\left(\frac{x}{2} - \frac{\pi}{6}\right)$

4 Example 23 Prove the following identities.

a $\dfrac{-1}{\operatorname{cosec}(x) + \cot(x)} = \dfrac{\frac{1}{2}[1 - \cos(2x)]}{\tan(x)\cos(x) + \frac{1}{2}\sin(2x)}$

b $\frac{1}{4}[\sin(x + y) + \sin(x - y)]^2 = [\sin(x) - \sin(x)\sin(y)]^2 - 2\sin^2(x)\sin^2(y) + 2\sin^2(x)\sin(y)$

c $\dfrac{1}{\sin(x) + \operatorname{cosec}(x)} = \dfrac{\frac{1}{2}\sec(x)\sin(2x)}{1 - \cos^2(x) + \operatorname{cosec}^2(x) - \cot^2(x)}$

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5 Example 24 Rewrite the expression $\frac{1}{3}\sin(3x)+\cos(2x)$ in terms of sin only.

6 Rewrite the expression $-2\sin^2(x)\cos(2x)$ in terms of $\cos(x)$ only.

7 Prove that $\sec^2(x) = \sec(x)\operatorname{cosec}(x)\tan(x)$.

8 Prove that $\frac{1}{\operatorname{cosec}(x)+1}+\frac{1}{\operatorname{cosec}(x)-1}=2\tan(x)\sec(x)$.

9 Show that $1 + \cos(2x) = 2\cos^2(x)$.

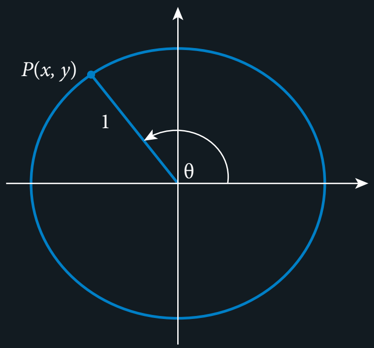
10 Prove that $\cot(x)+\cot(y)=\frac{\operatorname{cosec}(x)\operatorname{cosec}(y)}{\operatorname{cosec}(x+y)}$.

9 CHAPTER SUMMARY

TRIGONOMETRIC IDENTITIES

- Trigonometric functions. For a point $P(x, y)$ at an angle θ on the unit circle,

$\sin(\theta) = y$, $\cos(\theta) = x$ and $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$.



- Compound angles are found in expressions that involve more than one angle, such as $\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)$ or $\cos(2\theta)$.
- An identity is a formula that is true for every value.
- Reciprocal trigonometric relationships are

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)},\ \sec(x) = \frac{1}{\cos(x)} \text{ and}$$

$$\cot(x) = \frac{1}{\tan(x)}.$$

- The symmetry of the unit circle gives:

2nd quadrant

- $\operatorname{cosec}(\pi - \theta) = \operatorname{cosec}(\theta)$
- $\sec(\pi - \theta) = -\sec(\theta)$
- $\cot(\pi - \theta) = -\cot(\theta)$

3rd quadrant

- $\operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}(\theta)$
- $\sec(\pi + \theta) = -\sec(\theta)$
- $\cot(\pi + \theta) = \cot(\theta)$

4th quadrant

- $\operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec}(\theta) = \operatorname{cosec}(-\theta)$
- $\sec(2\pi - \theta) = \sec(\theta) = \sec(-\theta)$
- $\cot(2\pi - \theta) = -\cot(\theta) = \cot(-\theta)$

- The Pythagorean identities are:

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$

Angle sum and difference identities are:

- $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
- $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
- $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
- $\tan(A + B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$
- $\tan(A - B) = \dfrac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$

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Complementary function identities are:

- $\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos(\theta)$
- $\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin(\theta)$
- $\tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot(\theta)$
- $\sin\left(\frac{3\pi}{2} \pm \theta\right) = -\cos(\theta)$
- $\cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm\sin(\theta)$
- $\tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp\cot(\theta)$

Double angle formulas are:

- $\sin(2A) = 2\sin(A)\cos(A)$
- $\cos(2A) = \cos^2(A) - \sin^2(A)$
  $= 2\cos^2(A) - 1$
  $= 1 - 2\sin^2(A)$
- $\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

Products as sums and differences formulas are:

- $\cos(A)\cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
- $\sin(A)\sin(B) = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- $\sin(A)\cos(B) = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
- $\cos(A)\sin(B) = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

Sums to products formulas are:

- $\sin(A) + \sin(B) = 2\sin\left[\frac{1}{2}(A + B)\right]\cos\left[\frac{1}{2}(A - B)\right]$
- $\sin(A) - \sin(B) = 2\cos\left[\frac{1}{2}(A + B)\right]\sin\left[\frac{1}{2}(A - B)\right]$
- $\cos(A) + \cos(B) = 2\cos\left[\frac{1}{2}(A + B)\right]\cos\left[\frac{1}{2}(A - B)\right]$
- $\cos(A) - \cos(B) = -2\sin\left[\frac{1}{2}(A + B)\right]\sin\left[\frac{1}{2}(A - B)\right]$

9

CHAPTER REVIEW
TRIGONOMETRIC IDENTITIES

Multiple choice

1 Example 1 $\sin(60°) =$

A $\frac{2}{\sqrt{3}}$ B $\frac{3}{\sqrt{3}}$ C $\frac{\sqrt{3}}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{1}{2}$

2 Example 1 $\operatorname{cosec}\left(\frac{\pi}{3}\right) =$

A $\frac{2}{\sqrt{3}}$ B $\frac{3}{\sqrt{3}}$ C $\frac{\sqrt{3}}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{1}{2}$

3 Example 2 $\sin\left(\pi - \frac{\pi}{6}\right) =$

A $\frac{2}{\sqrt{3}}$ B $\frac{3}{\sqrt{3}}$ C $\frac{\sqrt{3}}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{1}{2}$

4 Example 2 $\tan(360° - 30°) =$

A $\tan(30°)$ B $-\tan(60°)$ C $-\tan(30°)$ D $\cot(30°)$ E $\tan(60°)$

5 Example 13 $\cos(75°) =$

A $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ B $\frac{\sqrt{6}}{4}$ C $\frac{3\sqrt{2}}{4}$ D $\frac{\sqrt{6}+\sqrt{2}}{4}$ E $\frac{\sqrt{3}}{4} - \frac{\sqrt{2}}{4}$

6 Example 15 $\sin\left(\frac{5\pi}{8}\right) =$

A $\frac{\sqrt{2+\sqrt{2}}}{4}$ B $\frac{\sqrt{2-\sqrt{2}}}{2}$ C $\frac{2+\sqrt{2}}{4}$ D $\frac{\sqrt{2-\sqrt{2}}}{4}$ E $\frac{\sqrt{2+\sqrt{2}}}{2}$

7 Example 18 $\cos(5x)\cos(9x) =$

A $\frac{1}{2}[\cos(7x) + \cos(2x)]$
B $\frac{1}{2}[\sin(14x) + \sin(4x)]$
C $2[\cos(7x) - \cos(2x)]$
D $\frac{1}{2}[\sin(7x) + \sin(2x)]$
E $\frac{1}{2}[\cos(14x) + \cos(4x)]$

8 Example 21 $\sin(12x) - \sin(8x) =$

A $\frac{1}{2}\sin(20x)\cos(4x)$
B $2\cos(10x)\sin(2x)$
C $2\sin(10x)\cos(2x)$
D $2\sin(20x)\cos(4x)$
E $\frac{1}{2}[\cos(2x)\cos(10x)]$

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Short answer

9 Example 2 Write a simplified expression for:

a $\sec(2\pi - \theta)$ b $\operatorname{cosec}(\pi - \theta)$ c $\cot(2\pi - \theta)$
d $\sec(\pi + \theta)$ e $\cot(\pi + \theta)$ f $\operatorname{cosec}(2\pi - \theta)$

10 Example 8 Write a simplified expression for:

a $\sec\left(\frac{\pi}{2}-\theta\right)$ b $\operatorname{cosec}\left(\frac{3\pi}{2}-\theta\right)$ c $\cot\left(\frac{\pi}{2}-\theta\right)$
d $\sec\left(\frac{3\pi}{2}+\theta\right)$ e $\cot\left(\frac{3\pi}{2}-\theta\right)$ f $\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)$

11 Example 3 Sketch the graph of $y = 3\operatorname{cosec}(x)$ for $0 \le x \le 2\pi$.

12 Example 3 Sketch the graph of $y = 2\sec(x) - 1$ for $0 \le x \le 2\pi$.

13 Example 5 Prove the identity $\sec^2(x) - 1 = \tan^3(x)\cot(x)$

14 Example 14 Find the exact value of $\cos\left(\frac{\pi}{12}\right)$.

15 Example 11 Find the exact value of $\tan\left(\frac{\pi}{12}\right)$.

16 Example 15 Find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

Application

17 Find an expression for $\sin(4x)$ in terms of $\cos(x)$ and $\sin(x)$ only.

18 Show that $\sin(A)\sin(B)=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$.

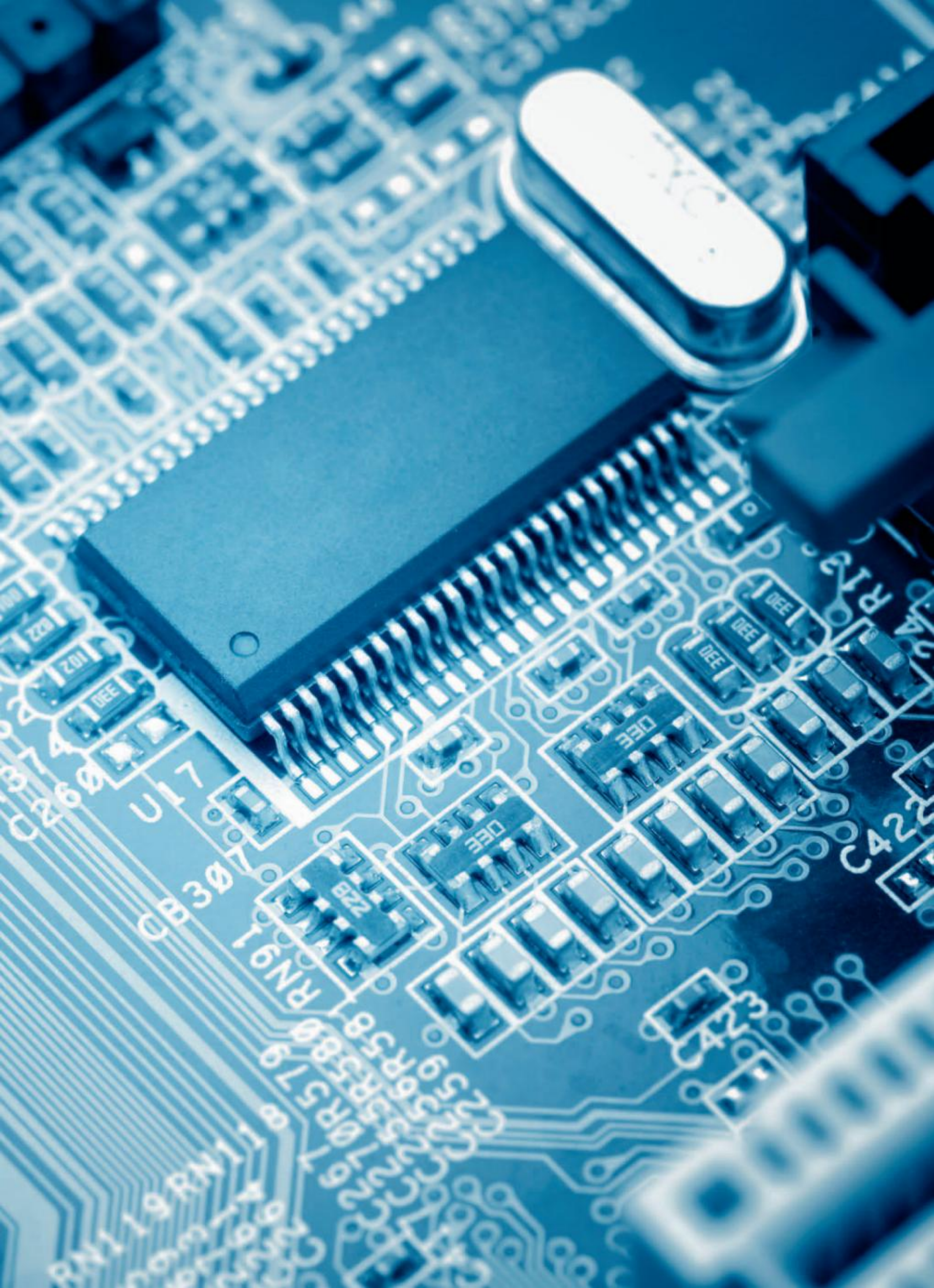
19 Show that $\dfrac{\cos(x)+\sec(x)}{\cot(2x)}=\dfrac{2\tan(x)\cdot[\cos^2(x)+1]}{\cos(x)\cdot[1-\tan^2(x)]}$

20 Show that $\dfrac{1-\cos(x)}{8}=[4\sin(x)\cos^3(x)-2\sin(x)\cos(x)]^2$

21 Show that $\cos(x)\tan(x)\sin(3x) = \sin^2(x)\cos(2x) + \sin(x)\cos(x)\sin(2x)$

22 Rewrite the expression $3\sin(2x)\sin(x)\cos(x) + \cos(x)\cos(2x)$ in terms of $\cos(x)$ only.

Practice quiz


U17
C260
CB307
RN91
C422

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10

TERMINOLOGY

Argand diagram
closure
complex
complex conjugate
imaginary
inverse
modulus
real
realising the denominator

REAL AND COMPLEX NUMBERS

COMPLEX NUMBERS

Prior learning

COMPLEX NUMBERS

- define the imaginary number i as the root of the equation $x^2 = -1$ (ACMSM067)
- use complex numbers in the form $a + bi$ where a and b are the real and imaginary parts (ACMSM068)
- determine and use complex conjugates (ACMSM069)
- perform complex-number arithmetic: addition, subtraction, multiplication and division. (ACMSM070)

THE COMPLEX PLANE

- consider complex numbers as points in a plane with real and imaginary parts as Cartesian coordinates (ACMSM071)
- examine addition of complex numbers as vector addition in the complex plane (ACMSM072)
- understand and use location of complex conjugates in the complex plane (ACMSM073)

ROOTS OF EQUATIONS

- use the general solution of real quadratic equations (ACMSM074)
- determine complex conjugate solutions of real quadratic equations (ACMSM075)
- determine linear factors of real quadratic polynomials. (ACMSM076) AC

10.01 IMAGINARY NUMBERS

You have seen before that some quadratic equations do not have a solution in the set of real numbers. For instance, $x^2 = -4$ does not have any real solutions because there is no real number x that gives a negative answer when it is squared. If, however, we make such a number and give it a name, then we can solve these equations and many more.

Unlike real numbers, this number cannot be plotted on the real number line. Although it is imaginary in this sense, the mathematics of i has turned out to be incredibly useful in mathematics, electronics, engineering and technology. Most of modern number theory relies on imaginary numbers for proof. For example, the proof of Fermat's last theorem, that '2 is the only power for which there are distinct positive integer solutions to $a^n + b^n = c^n$', relies heavily on complex and imaginary numbers. The design of electronic circuits would be virtually impossible without imaginary numbers.

> **IMPORTANT**
>
> The **imaginary number** i is defined as $i = \sqrt{-1}$.

Example 1

Use the fact that $i = \sqrt{-1}$ to write the following in terms of i.

a $\sqrt{-4}$ b $\sqrt{-9}$ c $\sqrt{-75}$

Solution

a Write −4 as a product and use the surd rules.

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)}\\ &= \sqrt{4} \times \sqrt{-1}\\ &= 2 \times i\\ &= 2i\end{aligned}$$

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b Write –9 as a product.

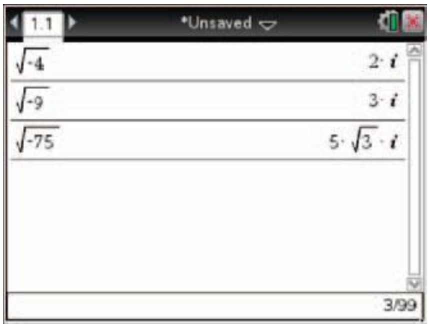
$$\sqrt{-9} = \sqrt{9 \times (-1)}$$
$$= \sqrt{9} \times \sqrt{-1}$$
$$= 3i$$

c Write –75 as a product, then use surd rules to simplify.

$$\sqrt{-75} = \sqrt{25 \times 3 \times (-1)}$$
$$= 5 \times \sqrt{3} \times i$$
$$= 5i\sqrt{3}$$

TI-Nspire CAS

Change the Real or Complex setting to Rectangular using doc▾, 7: Settings & Status, 2: comment settings. Then do the calculations shown in the screen shot.

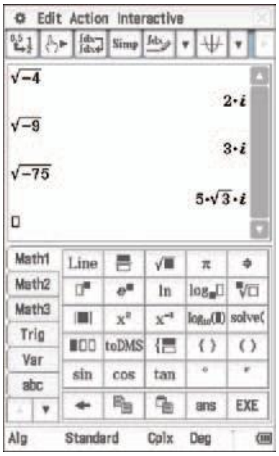


ClassPad

Use the Main $\sqrt{\alpha}$ application.

Set the calculator to **Cplx** (complex).

If necessary, tap **Real** at the bottom.



It is much clearer to write $5i\sqrt{3}$ than $5\sqrt{3}i$, because it is clear that the square root applies only to the 3. It is now possible to solve quadratic equations that involve the square root of a negative number.

Example 2

Solve the following equations.

a $x^2 = -16$ **b** $p^2 = -54$ **c** $7z^2 + 1 = 0$

Solution

a Take the square root of both sides, noting that there are two possibilities for x.

$$x^2 = -16$$
$$x = \pm\sqrt{-16}$$
$$x = \pm 4i$$

b Simplify the surd.

$$p^2 = -54$$
$$p = \pm\sqrt{9 \times 6 \times (-1)}$$
$$p = \pm 3i\sqrt{6}$$

c Make z the subject of the equation first, then assign the negative sign to the numerator.

$$7z^2 + 1 = 0$$
$$z^2 = -\frac{1}{7}$$
$$z = \pm\sqrt{\frac{-1}{7}}$$
$$= \pm\frac{i}{\sqrt{7}}$$

Let us now look at powers of the imaginary number i.

Example 3

Evaluate each of the following.

a i^2 b i^3 c i^4 d i^5 e i^{50}.

Solution

a Square $\sqrt{-1}$.

$$i^2 = \left(\sqrt{-1}\right)^2$$
$$= -1$$

b Use the fact that $i^2 = -1$.

$$i^3 = i^2 \times i$$
$$= -1 \times i$$
$$= -i$$

c Use the fact that $i^2 = -1$ twice.

$$i^4 = i^2 \times i^2$$
$$= -1 \times (-1)$$
$$= 1$$

d Use the fact that $i^4 = 1$.

$$i^5 = i^4 \times i$$
$$= 1 \times i$$
$$= i$$

e Write the power to get a multiple of 4.
Use the index laws.
Use fact that $i^4 = 1$.

$$i^{50} = i^{48} \times i^2$$
$$= (i^4)^{12} \times (-1)$$
$$= 1^{12} \times (-1)$$
$$= -1$$

IMPORTANT

The powers of i make a pattern that repeats each four numbers: $i, -1, -i, 1, i, -1, -i, 1, \ldots$

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EXERCISE 10.01 Imaginary numbers

Concepts and techniques

1 Example 1 Evaluate each of the following using imaginary numbers.

a $\sqrt{-25}$ b $\sqrt{-36}$ c $\sqrt{-20}$ d $\sqrt{-\frac{8}{9}}$

2 Example 2 Solve the following equations using imaginary numbers.

a $x^2 = -1$ b $x^2 = -9$ c $x^2 = -49$ d $x^2 = -\frac{1}{4}$

3 Example 3 Evaluate each power of i.

a i^6 b i^{11} c i^{28} d i^{97}

4 Simplify each expression.

a $i \times (-i)$ b $-i \times (-i)$ c $(-i)^7$ d $5i \times 9i$

5 Expand and simplify the following.

a $(1 + i)(1 - i)$ b $3i(2i + 5)$

6 Evaluate: $1 + i - i^2 + i^3 - i^4 + i^5$

7 By first writing $x^2 + 16 = x^2 - 16i^2$, factorise $x^2 + 16$.

Reasoning and communication

8 Evaluate the series:

a $i + i^2 + i^3 + i^4 + \cdots + i^{100}$ b $\sum_{n=0}^{n=253} i^n$

9 By first writing $1 = i^4$, evaluate each of the following.

a $\frac{1}{i}$ b $\frac{1}{i^3}$ c $\frac{1}{i^4}$ d i^{-5} e i^{-10}

10 Factorise each expression below.

a $x^2 + 4$ b $x^2 + 81$

10.02 COMPLEX NUMBERS

You saw in the previous section that imaginary numbers arise when solving a quadratic equation like $x^2 = -4$. Consider the solution to the equation $x^2 - 3x + 4 = 0$. Using the quadratic formula, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{+3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{3 \pm \sqrt{-7}}{2} \end{aligned}$$

$x = \dfrac{3 \pm i\sqrt{7}}{2}$, where $i = \sqrt{-1}$

The solutions can be written in the form $x = \frac{3}{2} + i\frac{\sqrt{7}}{2}$ or $x = \frac{3}{2} - i\frac{\sqrt{7}}{2}$. These solutions are part real and part imaginary. We call these **complex numbers**.

IMPORTANT

A **complex number** is a number that can be written in the form $a + ib$, where a and b are real numbers.

A complex number is often denoted by the letter z, so $z = a + ib$.

The set of complex numbers is denoted by $\boldsymbol{C}$.

The **real part** of $z = a + ib$ is denoted by $\operatorname{Re}(z)$, where $\operatorname{Re}(z) = a$ and the **imaginary part** of $z = a + ib$ is denoted by $\operatorname{Im}(z) = b$.

If $\operatorname{Re}(z) = 0$, then we say that z is *purely imaginary.*

If $\operatorname{Im}(z) = 0$, then we say that z is *purely real* or just *real.*

Example 4

Write down $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ for each of the following complex numbers.

a $z = 5 - 4i$ **b** $z = i\sqrt{2} + 9$

c $z = 2x + y + (x - y)i$ **d** $z = \dfrac{x - iy}{x^2 + y^2}$

Solution

a For $z = 5 - 4i$, the real part is the number 5. The coefficient of i is -4.

$\operatorname{Re}(z) = 5$
$\operatorname{Im}(z) = -4$

b $z = i\sqrt{2} - 9$ is the same as $z = -9 + i\sqrt{2}$.

$\operatorname{Re}(z) = -9$
$\operatorname{Im}(z) = \sqrt{2}$

c For $z = 2x + y + (x - y)i$, all the terms without an i attached form the real part. Those with an i attached form the imaginary part.

$\operatorname{Re}(z) = 2x + y$
$\operatorname{Im}(z) = x - y$

d Write $z = \dfrac{x - iy}{x^2 + y^2}$ as $z = \dfrac{x}{x^2 + y^2} + i\left(\dfrac{-y}{x^2 + y^2}\right)$.

$\operatorname{Re}(z) = \dfrac{x}{x^2 + y^2}$

$\operatorname{Im}(z) = \dfrac{-y}{x^2 + y^2}$

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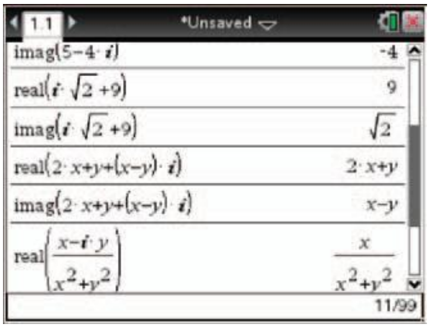
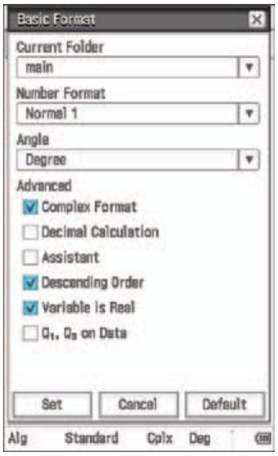
TI-Nspire CAS

Make sure that in Document Settings, Real or Complex is set to Rectangular. Then use menu, 2: Number, 9: Complex Number Tools and 2: Real Part or 3: Imaginary Part to perform the operations for parts **a**, **b**, **c** and **d**.
Use the π▸ button to get i.

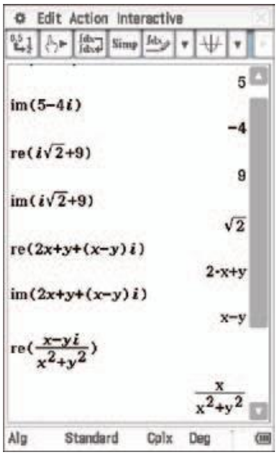


ClassPad

First set up the properties to define variables such as x and y to be real.
Tap ⚙ and then select **Basic Format**.
Make sure that **Variable** is **Real** is ticked.
Complex Format should also be ticked.
Tap Set.



Use the Main $\sqrt{\alpha}$ application.
Set the calculator to **Cplx** (complex) at the bottom, not **Real**.
Use **re** to find the real part and **im** for the imaginary part. (These functions are available on the list obtained by pressing Keyboard then tapping ▼.)
Use i on the Math2 keyboard.



The fact that complex numbers have a real and an imaginary part means that the set of real numbers is a subset of the set of complex numbers ($\boldsymbol{R} \subset \boldsymbol{C}$).

It is now possible to solve any quadratic equation. The solutions could be real, imaginary or complex.

Example 5

Solve each quadratic equation, expressing your answers as complex numbers in the form $a + ib$, where a and b are real.

a $x^2 - 2x + 3 = 0$ b $3x^2 + 10x + 15 = 0$

Solution

a Use the quadratic formula.

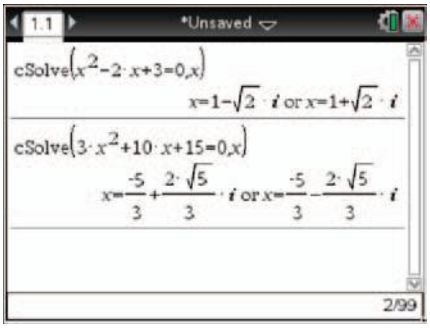
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{+2 \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{-8}}{2}$$
$$x = \frac{2 \pm i2\sqrt{2}}{2}$$
$$= 1 \pm i\sqrt{2}$$

b Use the quadratic formula and simplify as far as possible.

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(3)(15)}}{2(3)}$$
$$= \frac{-10 \pm \sqrt{-80}}{6}$$
$$= \frac{-10 \pm i4\sqrt{5}}{6}$$
$$= \frac{-5 \pm 2i\sqrt{5}}{3}$$
$$= -\frac{5}{3} \pm \frac{2\sqrt{5}}{3}i$$

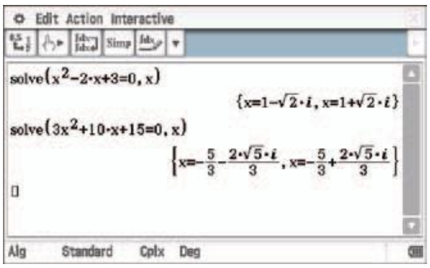
TI-Nspire CAS

Make sure that Real or Complex is set to Rectangular. Then use cSolve() to find roots that include complex cases.



ClassPad

Use the Main $\sqrt{\alpha}$ application.
Set the calculator to **Cplx** (complex).
If necessary, tap **Real** at the bottom.
Use **solve** in the normal manner.



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EXERCISE 10.02 Complex numbers

Concepts and techniques

1 Example 4 Categorise each of the following as real, complex, or purely imaginary.

a $5 + 2i$ b $i\sqrt{5}$ c $2i^2$ d $\dfrac{4-\sqrt{3}}{2}$

e $\sqrt{-8}$ f $i^2 + i^4$ g $i - \sqrt{2}$ h $\dfrac{3i - i\sqrt{2}}{i}$

2 Example 4 State Re(z) and Im(z) for each complex number.

a $z = -2 - 4i$ b $z = \dfrac{7+3i}{4}$

c $z = i\sqrt{2} - 6$ d $\dfrac{x^2 - iy^2}{x^2 + y^2}$ where x and y are real.

3 Example 4 State the complex number w if Re(w) and Im(w) are the values stated.

a $\text{Re}(w) = 5, \text{Im}(w) = -4$ b $\text{Re}(w) = \sqrt{2}, \text{Im}(w) = \sqrt{7}$

c $\text{Re}(w) = -\frac{1}{2}, \text{Im}(w) = \dfrac{\sqrt{3}}{2}$ d $\text{Re}(w) = -2\sqrt{2}, \text{Im}(w) = -5\sqrt{2}$

e $\text{Re}(w) = \dfrac{y}{x^2+y^2}, \text{Im}(w) = \dfrac{x}{x^2+y^2}$ where x and y are real.

4 Example 5 Solve the quadratic equations below, expressing your answers as complex numbers in the form $a + ib$, where a and b are real.

a $x^2 + 49 = 0$ b $z^2 + 1 = 0$ c $w^2 + 25 = 0$

d $x^2 + 4x + 5 = 0$ e $z^2 - z + 2 = 0$ f $w^2 - 2w + 2 = 0$

g $x^2 - 2x + 4 = 0$ h $z^2 + 10 = 6z$ i $w^2 = 4w - 8$

Reasoning and communication

5 Example 4 State Re (z) and Im (z) for each complex number (x and y are real).

a $z = (x + 3) + i(y - 2)$ b $z = x - 3y + (2x + y)i$

c $z = 5i - x + 4 - yi$ d $z = 4x - y - i(3x - 2y)$

e $\dfrac{x^2 + 2xyi + y^2}{x^2 + y^2}$, where x and y are real.

6 Example 5 Solve each quadratic equation, expressing your answers as complex numbers in the form $a + ib$, where a and b are real.

a $2z^2 + 3z + 4 = 0$ b $3z^2 + 6zi = 2$ c $z^2 + a^2 + b^2 = 2az$

10.03 COMPLEX CONJUGATES

Recall that conjugate surds are surds that make a difference of two squares when multiplied. For instance, the conjugate of $2-\sqrt{5}$ is $2+\sqrt{5}$ and the product $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)$ forms a difference of two squares. Note the answer is *rational*, since $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)=-1$.

In the examples above, you may have noticed that the solutions to the quadratic equations are similar to conjugate surds. For instance, the solutions to $x^2 - 2x + 3 = 0$ are $1-i\sqrt{2}$ or $1+i\sqrt{2}$. These are called **complex conjugates**.

IMPORTANT

For a complex number z, where $z = a + ib$ (where a and b are real numbers), the **complex conjugate** of z is denoted by $\bar{z}$ and $\bar{z} = a - ib$.

In a similar way to surds, the product of complex conjugates forms a difference of two squares. For instance, the conjugate of $1-i\sqrt{2}$ is $1+i\sqrt{2}$ and the product $\left(1-i\sqrt{2}\right)\left(1+i\sqrt{2}\right)$ is a difference of two squares. The answer is *real*, since

$$\begin{aligned}\left(1-i\sqrt{2}\right)\left(1+i\sqrt{2}\right)&=1^2-(i\sqrt{2})^2\\&=1-i^2(\sqrt{2})^2\\&=1-(-1)\times 2\\&=3\end{aligned}$$

IMPORTANT

For a complex number z and its conjugate $\bar{z}$, where $z = a + ib$ (where a and b are real numbers), the product $z\bar{z}$ is real.

Proof

Let $z = a + ib$, then $\bar{z} = a - ib$, where a and b are real. Then

$$\begin{aligned}z\bar{z}&=(a+ib)(a-ib)\\&=a^2-i^2b^2\\&=a^2+b^2\end{aligned}$$

Since a and b are real, then $a^2 + b^2$ is also real.

$\therefore z\bar{z}$ is always real.

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Example 6

Write the complex conjugate of each complex number.

a $z = u - vi$ **b** $x + iy - w$ **c** $z = 3 + i(5 - x)$ **d** $\dfrac{ix - y}{x^2 + y^2}$

Solution

a Change the sign of Im(z). $\bar{z} = u + vi$

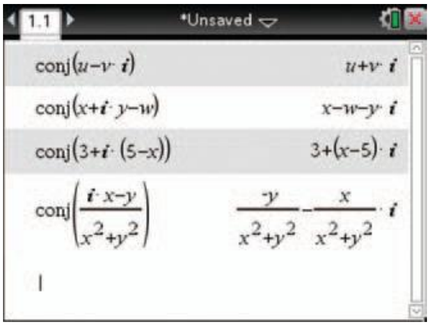
b First group the real parts and imaginary parts separately, i.e. $(x - w) + (iy)$. $x - w - iy$

c Change the sign of Im(z). $\bar{z} = 3 - i(5 - x)$

d Note that $\dfrac{ix - y}{x^2 + y^2}$ can be written as $\dfrac{-y + ix}{x^2 + y^2}$. $\dfrac{-y - ix}{x^2 + y^2}$

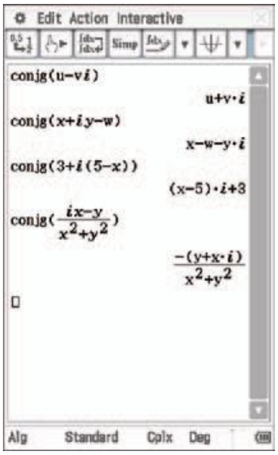
TI-Nspire CAS

Make sure Real or Complex is set to Rectangular. Then use Complex Conjugate from the Complex Number Tools menu to find the conjugates.



ClassPad

Tap ⚙ then **Basic Format**. Check that this screen has **Complex Format** and **Variable is Real** set, then tap **Set**. (see page 377)
Then use **conjg** to find the conjugates. This only works properly with numbers, but can be used to find the conjugate of a complex expression.



Note that $\dfrac{ix - y}{x^2 + y^2} = \left(\dfrac{-y}{x^2 + y^2}\right) + \left(\dfrac{x}{x^2 + y^2}\right)i$ and $\dfrac{-ix - y}{x^2 + y^2} = \left(\dfrac{-y}{x^2 + y^2}\right) - \left(\dfrac{x}{x^2 + y^2}\right)i$, which shows the conjugates more clearly.

Example 7

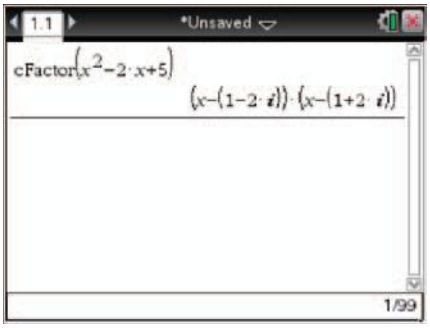
Show that $x^2 - 2x + 5 = (x - 1)^2 - 4i^2$ and hence factorise $x^2 - 2x + 5$.

Solution

Start with the RHS.	$(x - 1)^2 - 4i^2$
Multiply out the brackets.	$= x^2 - 2x + 1 - 4 \times (-1)$
Simplify.	$= x^2 - 2x + 1 + 4$
Add the constants.	$= x^2 - 2x + 5 = \text{LHS}$
$(x - 1)^2 - 4i^2$ is a difference of two squares.	$(x - 1)^2 - 4i^2 = (x - 1 - 2i)(x - 1 + 2i)$ $\therefore\ x^2 - 2x + 5 = (x - 1 - 2i)(x - 1 + 2i)$.

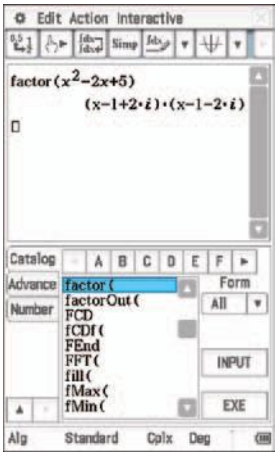
TI-Nspire CAS

Make sure that Real or Complex is set to Rectangular. Then use cFactor() from Complex in the Algebra menu.



ClassPad

Use the Main $\sqrt{\alpha}$ application.
Set the calculator to **Cplx** (complex).
Use **factor** in the normal manner.



From Example 7, you could solve $x^2 - 2x + 5 = 0$ using the null factor law to get $x = 1 + 2i$ or $x = 1 - 2i$.

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Example 8

A quadratic equation in the form $x^2 + bx + c = 0$ has complex roots $7 \pm 3i$. Find b and c.

Solution

Write the factorised form of the quadratic equation used to get the roots.

Let the two roots be $\alpha = 7 + 3i$ and $\beta = 7 - 3i$ in the equation $(x - \alpha)(x - \beta) = 0$

Substitute the values of α and β.

$(x - 7 - 3i)\ (x - 7 + 3i) = 0$

Multiply out the brackets.

$x^2 - 7x + 3ix - 7x + 49 - 21i - 3ix + 21i - 9i^2 = 0$

Simplify.

$x^2 - 14x + 49 - 9 \times (-1) = 0$

Add the constants.

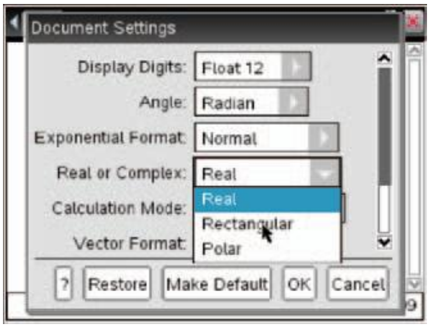
$x^2 - 14x + 58 = 0$

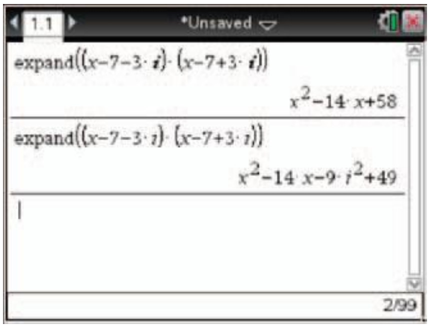
Write the answers.

$b = -14$ and $c = 58$

TI-Nspire CAS

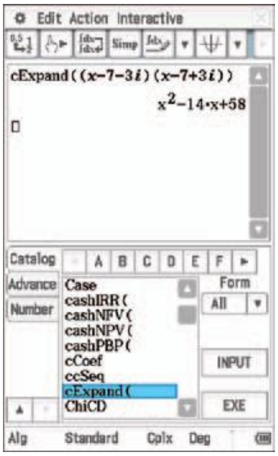
Change your Documents Settings using [doc▾], 7: Settings & Status and 2: Document Settings. Change Real or Complex to Rectangular. Then use [menu], 3: Algebra and 3: Expand to enter the expression on the LHS of the equation. i is in the [π▸] menu.
Note that you must use the $\boldsymbol{i}$ from the [π▸] menu. If you type I from the keyboard it will be treated as a variable.





ClassPad

Use **cExpand** for complex expressions.
Note that you must use the [i] from the [Math2] or [Math3] or [Trig] menu.



EXERCISE 10.03 Complex conjugates

Concepts and techniques

WS Complex conjugates

1 Example 6 State the conjugate $\overline{z}$ for each complex number below.

a $z = -1 - i$ b $z = \dfrac{3+i}{2}$ c $z = i\sqrt{3} - 1$

d $x - y + (2x - 3y)i$, where x and y are real e $\dfrac{x^2 + iy^2}{x^2 + y^2}$, where x and y are real

2 Show that the answer to each of the following products is real.

a $(2 + i)(2 - i)$ b $\dfrac{5+2i}{2} \times \dfrac{5-2i}{2}$ c $(i\sqrt{5} - 3)\left(-i\sqrt{5} - 3\right)$

3 Example 7 By first writing $x^2 + 2x + 3$ in the form $x^2 + 2x + 1 - 2i^2$, factorise $x^2 + 2x + 3$.

4 Use the technique shown in question **5** to factorise each of the quadratic expressions.

a $x^2 + 4x + 5$ b $x^2 - 6x + 13$ c $x^2 - 2x + 2$

5 Use the quadratic formula to solve the following equations.

a $x^2 + x + 4 = 0$ b $x^2 - 3x + 3 = 0$

c $x^2 + 6x + 10 = 0$ d $2x^2 + x + 4 = 0$

6 Example 8 Find a quadratic equation for which the two given complex numbers are roots.

a $-1 + i, -1 - i$ b $1 - 3i, 1 + 3i$ c $\sqrt{3} - i, \sqrt{3} + i$

Reasoning and communication

7 Given each of the following quadratic equations with real and imaginary coefficients, use the quadratic formula to find the roots.

a $x^2 + ix + 1 = 0$ b $x^2 - 2ix - 1 = 0$ c $ix^2 - x + 2i = 0$

8 a Look at each of the equations in question **5**. Were the coefficients real? Were the solutions to each equation complex conjugate pairs? Complete the statement:

If the coefficients of a quadratic equation are real, then the roots …

b Look at each of the equations in question **7**. Were the coefficients real? Were the solutions to each equation complex conjugate pairs? Complete the statement:

If the coefficients of a quadratic equation are not all real, then the roots …

9 Prove that for any complex number $z = x + yi$

a $z + \overline{z} = 2\mathrm{Re}(z)$ b $z - \overline{z} = 2\mathrm{Im}(z)$

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10.04 OPERATIONS WITH COMPLEX NUMBERS

Just like real numbers, complex numbers can be added, subtracted, multiplied and divided. When dealing with surds, we group or equate the *rational* and *irrational* parts. When dealing with complex numbers, we group or equate the *real* and *imaginary* parts.

IMPORTANT

For complex numbers $a + ib$ and $c + id$ (where a, b, c and d are real numbers), $a + ib = c + id$ if and only if $a = c$ AND $b = d$.

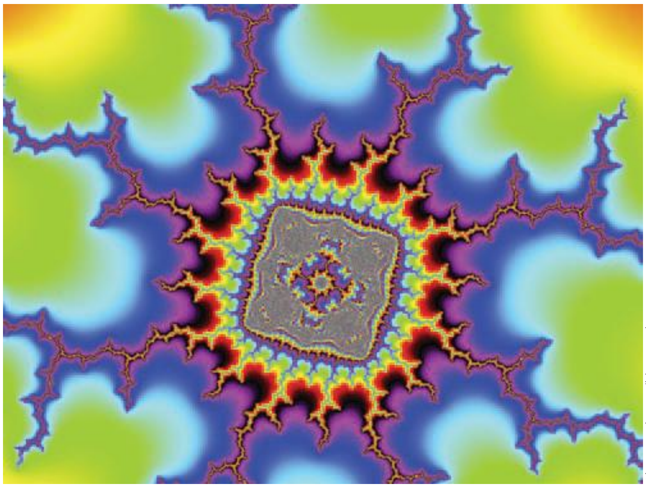
Example 9

Find the value of real numbers x and y if $x - y + i(x + 3y) = 3 + 11i$

Solution

Write the equation.	$x - y + i(x + 3y) = 3 + 11i$	
Equate the real parts and the imaginary parts to make two equations in x and y.	$x - y = 3$ $x + 3y = 11$	[1] [2]
Solve simultaneously.	$4y = 8$ $y = 2$	[2] − [1]
Substitute in one of the equations.	$x - 2 = 3$ $x = 5$	
Write the answers.	$x = 5$ and $y = 2$	

You could do Example **9** on your CAS calculator by solving the equation for x and then solving for y, but it is quicker to do it without your calculator.



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Example 10

Simplify each of the following. Express your answer in the form $a + bi$, where a and b are real.

a $(3 + 4i) + (5 - 2i)$ **b** $(1 - 5i) - (2 - 4i)$

c $(3 + 2i)(4 - 3i)$ **d** $6(2 + i) - 2i(5 + 7i)$

Solution

a Group the real parts and the imaginary parts.

$$\begin{aligned}(3 + 4i) + (5 - 2i) &= (3 + 5) + i(4 - 2)\\ &= 8 + 2i\end{aligned}$$

b Group the real parts and the imaginary parts.

$$\begin{aligned}(1 - 5i) - (2 - 4i) &= (1 - 2) + i(-5 + 4)\\ &= -1 - i\end{aligned}$$

c Expand like a binomial expansion.
Recall that $i^2 = -1$.

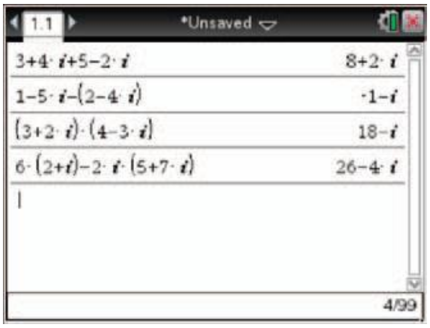
$$\begin{aligned}(3 + 2i)(4 - 3i) &= 12 - 9i + 8i - 6i^2\\ &= 12 - i + 6\\ &= 18 - i\end{aligned}$$

d Remove the brackets and add like terms.

$$\begin{aligned}6(2 + i) - 2i(5 + 7i) &= 12 + 6i - 10i - 14i^2\\ &= 12 - 4i + 14\\ &= 26 - 4i\end{aligned}$$

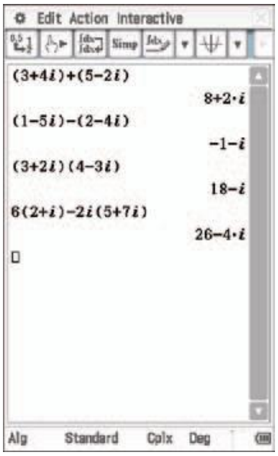
TI-Nspire CAS

You can do these on your calculator by entering the expressions as calculations. Your calculator will automatically express them in the form $a + bi$ when set in Rectangular mode (see page 377).



ClassPad

With numeric expressions, just enter the expression and press $\boxed{\text{EXE}}$.



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When dividing two complex numbers, we multiply by the complex conjugate of the denominator. This is called **realising the denominator**. It is similar to surds, where we rationalise the denominator.

> **IMPORTANT**
>
> To **realise a denominator** of a complex number, you multiply the number by 1 in the form $\frac{\bar{z}}{\bar{z}}$.

Example 11

Simplify by realising the denominator. Give your answer in the form $a + bi$, where a and b are real.

a $\dfrac{3+2i}{4-3i}$ **b** $\dfrac{x-iy}{u+iv}$, where x, y, u and $v \in \boldsymbol{R}$

Solution

a The complex conjugate of $4 - 3i$ is $4 + 3i$.
Recall from the previous section that $(a + bi)(a - bi) = a^2 + b^2$.

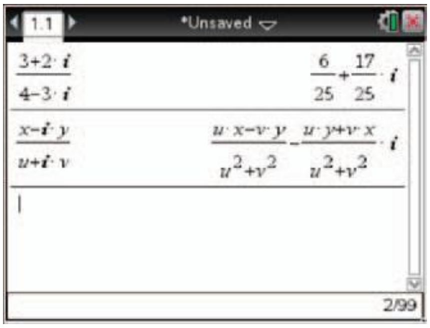
$$\frac{3+2i}{4-3i} \times \frac{4+3i}{4+3i} = \frac{12+9i+8i+6i^2}{4^2+3^2}$$
$$= \frac{12+17i-6}{25}$$
$$= \frac{6+17i}{25}$$
$$= \frac{6}{25} + \frac{17}{25}i$$

b Multiply by the complex conjugate, then group the real and imaginary parts.

$$\frac{x-iy}{u+iv} \times \frac{u-iv}{u-iv} = \frac{xu-xvi-uyi+vyi^2}{u^2+v^2}$$
$$= \frac{(ux-vy)+i(-vx-uy)}{u^2+v^2}$$
$$= \frac{(ux-vy)}{u^2+v^2} + \frac{(-vx-uy)}{u^2+v^2}i$$

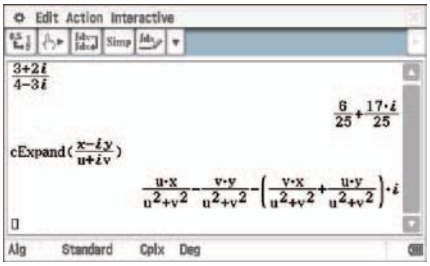
TI-Nspire CAS

Do the calculations in Rectangular mode.



ClassPad

With numeric expressions, just enter the expression and press EXE.
With algebraic expressions, use **cExpand**.



EXERCISE 10.04 Operations with complex numbers

WS Complex number operations

Concepts and techniques

1 Example 9 Find the value of a and b in each of the following, where a and b are real.

a $a + 2bi = 7 - 8i$

b $(a + 4i) - (b + bi) = 3 - 2i$

c $2a + b + 3ai - 2bi = 1 + 5i$

2 Example 10 Simplify each of the following expressions, leaving your answer in the form $a + ib$, where a and b are real.

a $(1 - 3i) + (2 + i)$

b $(6 + 5i) - (3 + 4i)$

c $(3 - 2i)(1 + 9i)$

d $2(3 + 2i) + 4i(i - 7)$

3 If $z = 2 + 3i$ and $w = 5 - 2i$, find the following in $a + ib$ form.

a $4z + w$

b $z - w$

c z^2

d zw

e $\overline{z}\,\overline{w}$

f $(z + w)^2$

g $\overline{w}^2$

h $3z - wi$

4 Example 11 Simplify by realising the denominator. Give your answer in the form $a + ib$, where a and b are real.

a $\dfrac{1}{1+i}$

b $\dfrac{2-i}{2+i}$

c $\dfrac{3+4i}{2-3i}$

d $\dfrac{i}{i-4}$

e $\dfrac{2-i}{5i}$

5 If $u = -1 + 2i$ and $v = 2 - 3i$, find the following in $a + ib$ form.

a $\dfrac{1}{u}$

b $\dfrac{u}{v}$

c $\dfrac{3}{u^2}$

d $\dfrac{1}{iv}$

e $\dfrac{u-1}{v+i}$

6 Simplify each of the following.

a $(2 - i)^2 + 3(1 + 2i)(1 - 2i)$

b $(3 + 5i)(5 - 3i) - (3 - 2i)^2$

c $\dfrac{1+2i}{2+i} + \dfrac{1-i}{1-2i}$

d $\dfrac{1+3i}{(2-i)^2}$

7 Find the value of X and Y where $X, Y \in \boldsymbol{R}$, if $\dfrac{5-4i}{4+3i} = X + iY$.

Reasoning and communication

8 If $z = x + iy$ where $X, Y \in \boldsymbol{R}$, express the following in the form $X + Yi$, where $X, Y \in \boldsymbol{R}$.

a $\dfrac{1}{z}$

b $\dfrac{1}{\overline{z}}$

c $\dfrac{1}{z^2}$

d $\dfrac{1}{iz}$

e $\dfrac{z-i}{z+i}$

9 Show that $z = 1 + i$ is a solution of the equation $z^2 - 2z + 2 = 0$.

10 Show that $z = \dfrac{-1+i\sqrt{3}}{2}$ is a solution of the equation $z^2 + z + 1 = 0$.

11 Show that $w = \dfrac{-1-i\sqrt{3}}{2}$ is a solution of the equation $w^3 = 1$.

12 a Show that $\left(x - 1 - i\sqrt{3}\right)\left(x - 1 + i\sqrt{3}\right) = x^2 - 2x + 4$.

b Show that $\left(x + 1 - i\sqrt{3}\right)\left(x - 1 + i\sqrt{3}\right) = x^2 + 2 + 2i\sqrt{3}$.

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10.05 THE COMPLEX PLANE

It is possible to represent the complex number $z = x + yi$ (where $x, y \in \boldsymbol{R}$) geometrically on a two-dimensional plane similar to the Cartesian plane, with a horizontal axis denoted by x or $\text{Re}(z)$ and vertical axis denoted by y or $\text{Im}(z)$. This plane is called an **Argand diagram** or **Argand plane**, after the French mathematician Jean-Robert Argand (1768–1822).

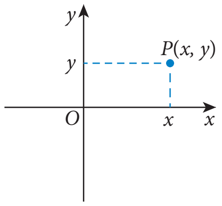


Figure 1

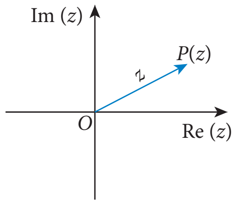


Figure 2

The complex number $z = x + yi$ can be represented by the point $P(x, y)$ (Figure 1) or by the vector $\overrightarrow{OP}$ (Figure 2).

Real numbers are plotted along the x-axis so it is labelled $\text{Re}(z)$, Re or sometimes x.

Purely imaginary numbers are plotted along the y-axis so it is labelled $\text{Im}(z)$, Im, iy or y.

[By convention, the complex number $z = 0 + 0i$ is just written as 0 on the Argand diagram. It is the origin.]

All complex numbers $z = a + bi$ (where $a \neq 0$, $b \neq 0$) correspond to a unique point (a, b) or vector on the Argand diagram.

Example 12

Represent each of the following numbers as a point on an Argand diagram.

a $u = 2 + 3i$ **b** $w = -1 - 4i$ **c** $v = (1 + 2i)(1 - 2i)$ **d** $z = \dfrac{3}{i}$

Solution

a Write u as a point. $u = 2 + 3i$ is shown as $(2, 3)$ or $2 + 3i$

b Write w as a point. $w = -1 - 4i$ is shown as $(-1, -4)$ or $-1 - 4i$

c Change v to the form $a + bi$ first.

$$\begin{aligned} v &= (1 + 2i)(1 - 2i) \\ &= 1 - 2i + 2i - 4i^2 \\ &= 1 + 4 \\ &= 5 + 0i \end{aligned}$$

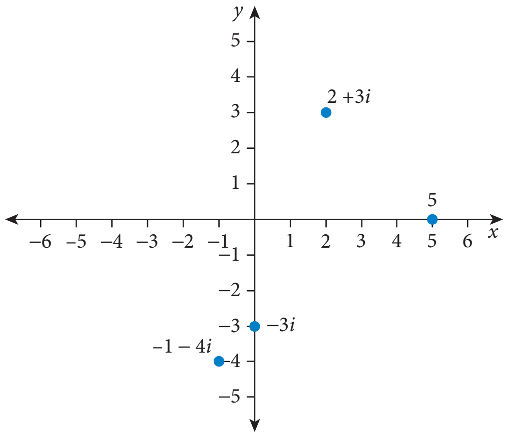
$v = (1 + 2i)(1 - 2i)$ is shown as $(5, 0)$ or 5.

d Change z to the form $a + bi$ first.

$z = \frac{3}{i} = \frac{3}{i} \times \frac{i}{i} = \frac{3i}{-1} = -3i = 0 - 3i$

z is shown as $(0, -3)$ or $-3i$.

Show the numbers on the diagram.



Example 13

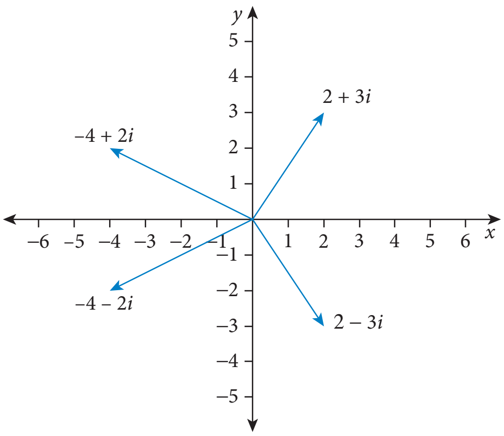
Given that $z = 2 - 3i$ and $w = -4 + 2i$, show $z, \bar{z}, w, \bar{w}$ as vectors on an Argand diagram. What do you notice about the geometric relationship between complex conjugates?

Solution

Write the conjugates.

$\bar{z} = 2 + 3i$ and $\bar{w} = -4 - 2i$

Plot $z = 2 - 3i$, $w = -4 + 2i$, $\bar{z} = 2 + 3i$ and $\bar{w} = -4 - 2i$ with arrows on the ends to show they are vectors.



Comment on the conjugates.

The conjugates are reflections in the x-axis.

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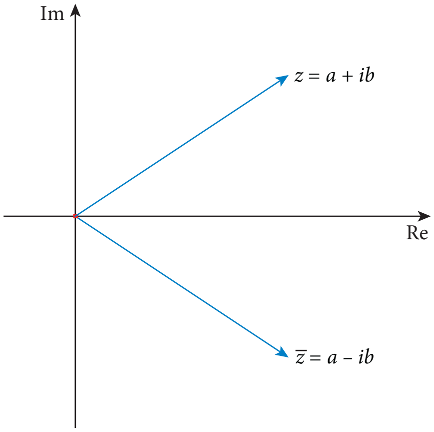
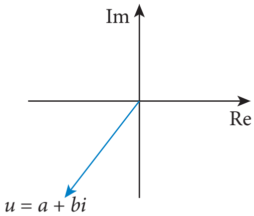


IMPORTANT

On an Argand diagram, if $z = a + bi$, then the complex conjugate $\bar{z} = a - bi$ is the reflection of the vector $z = a + bi$ in the x-axis.

Example 14

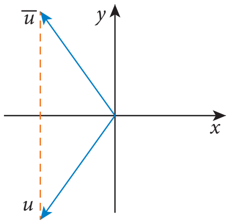
The vector $u = a + bi$ is plotted below on an Argand diagram.



Plot the conjugate vector $\bar{u}$.

Solution

The conjugate is the reflection across the x-axis.



EXERCISE 10.05 The complex plane

Concepts and techniques

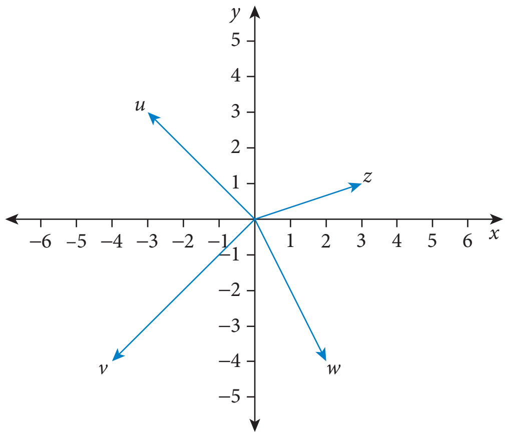
1 Example 12 Given the complex numbers below,

i state the points corresponding to each complex number

ii plot the points corresponding to each complex number on an Argand diagram.

a $a = 1 + 2i$ b $b = 3 - i$ c $c = -3 - 4i$ d $d = -2 + 2i$

2 State the complex number represented by each vector on the Argand diagram below.



3 Example 13 On an Argand diagram, plot the vectors corresponding to each complex number z with its conjugate $\overline{z}$.

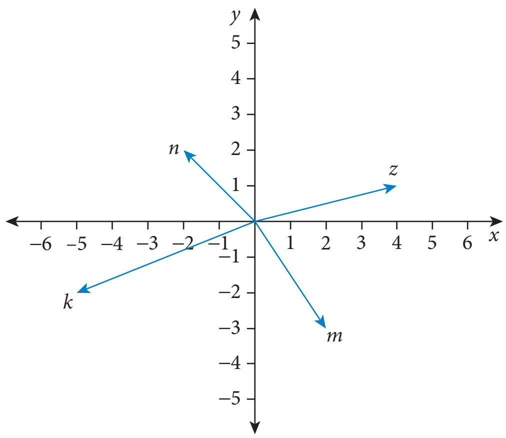
a $z = \sqrt{3} - i$ b $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ c $z = -3 - 2i$

4 Example 14 The vectors corresponding to four complex numbers are shown in the Argand diagram on the right.

a State the complex number representing each vector z, n, k and m.

b State the complex conjugate of each.

c Plot $\overline{z}, \overline{n}, \overline{k}$ and $\overline{m}$ on an Argand diagram.



5 Represent each of the following numbers as points on an Argand plane.

a $z = (1 + i)(2 - i)$ b $w = \frac{1}{-1+i}$ c $v = \left(\sqrt{2} - i\right)^2$

6 If $z = 3 + 2i$,

a calculate the complex numbers

i $w = iz$ ii $v = i^2 z$ iii $u = i^3 z$

b plot the corresponding vectors z, w, v and u.

c How is the position of a vector affected if you multiply a complex number by i ?

Reasoning and communication

7 If $z + \overline{z} = 0$, explain why z lies on the imaginary axis.

8 If z is a vector on the complex plane, explain why $z + \overline{z}$ lies on the real axis, if $z + \overline{z} \neq 0$.

9 If $z = \overline{z}$, explain the position of z on the Argand diagram.

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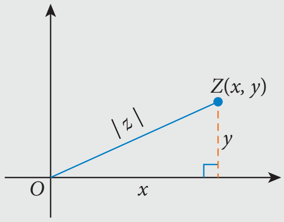
10.06 THE MODULUS OF A COMPLEX NUMBER

For a complex number $z = x + yi$, represented by the point $Z(x, y)$, the length of the vector $\overrightarrow{OZ}$ can be found using Pythagoras' theorem. It is called the **modulus** of z and is denoted by $|z|$. Since it is a length, $|z| \geq 0$.

IMPORTANT

The length of the vector $z = x + yi$ is called the **modulus**, written as **mod** z or $|z|$.

$$|z| = \sqrt{x^2 + y^2} \text{ or } |z| = \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2}$$



Example 15

Find $|z|$ for each of the following complex numbers.

a $z = 1 - i\sqrt{3}$ **b** $z = -2 + 2i$ **c** $z = \dfrac{1+i}{\sqrt{2}}$

Solution

a Identify the real and imaginary parts. Do not include i.

$$|z| = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2}$$
$$= \sqrt{4} = 2$$

b Use the formula.

$$|z| = \sqrt{(-2)^2 + (2)^2}$$
$$= \sqrt{8}$$

Simplify the surd.

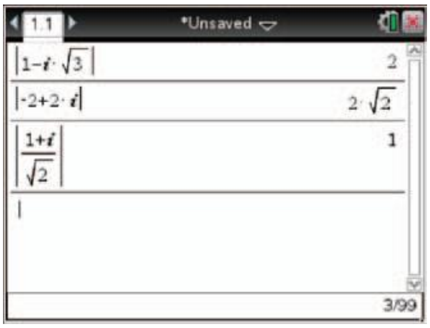
$$= 2\sqrt{2}$$

c Separate into real and imaginary parts.

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$
$$= \sqrt{\frac{1}{2} + \frac{1}{2}}$$
$$= \sqrt{1} = 1$$

TI-Nspire CAS

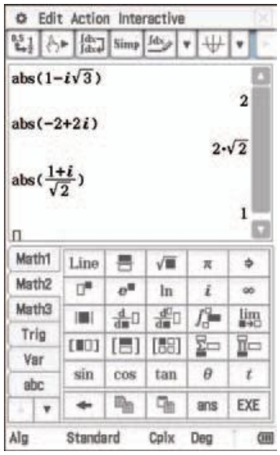
Make sure that Real or Complex is set to Rectangular. Then use Magnitude from the Complex Number Tools.



ClassPad

To find the modulus of a complex number, use the function **abs**.

This is because the modulus of a complex number is an extension of the idea of absolute value for real numbers.



Example 16

Evaluating the modulus of a complex number 1

Evaluating the modulus of a complex number 2

Find in simplest form.

a $\left|x-y+i(x+y)\right|$ **b** $\left|\dfrac{x-yi}{x^2+y^2}\right|$

Solution

a Identify the real and imaginary parts.

$\text{Re}[x-y+i(x+y)]=x-y$

$\text{Im}[x-y+i(x+y)]=x+y$

Use the formula.

$$\begin{aligned}\left|x-y+i(x+y)\right|&=\sqrt{(x-y)^2+(x+y)^2}\\&=\sqrt{2x^2+2y^2}\end{aligned}$$

b Use the formula.

$$\left|\frac{x-yi}{x^2+y^2}\right|=\sqrt{\left(\frac{x}{x^2+y^2}\right)^2+\left(\frac{-y}{x^2+y^2}\right)^2}$$

Simplify.

$$=\sqrt{\frac{x^2}{(x^2+y^2)^2}+\frac{y^2}{(x^2+y^2)^2}}$$

$$=\sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}}$$

Cancel and rationalise the surd.

$$=\sqrt{\frac{1}{x^2+y^2}}$$

$$=\frac{1}{\sqrt{x^2+y^2}}\text{ or }\frac{\sqrt{x^2+y^2}}{x^2+y^2}$$

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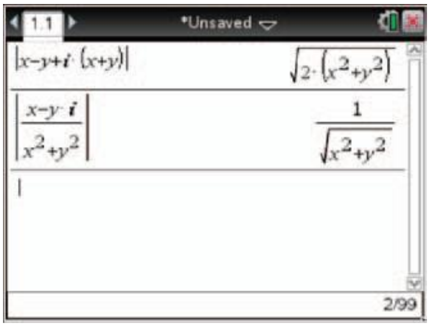
TI-Nspire CAS

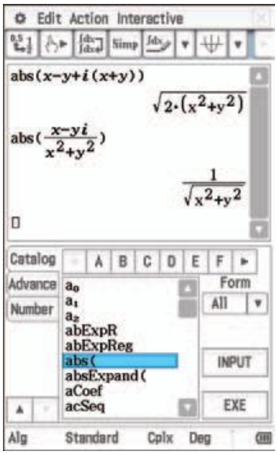
Make sure that Real or Complex is set to Rectangular. You can use Magnitude from the Complex Number Tools with variables.



ClassPad

The function **abs** can also be used with variables, like x and y.

Make sure **Basic Format** (under ⚙) is set to **Variable is Real**. See page 377.



EXERCISE 10.06 The modulus of a complex number

Concepts and techniques

1 Example 15 Find $|z|$ for each of the following complex numbers.

a $z = 3 + 5i$ b $z = -2 + i$ c $z = \sqrt{3} - i$

d $z = 1 + i\sqrt{2}$ e $z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$ f $z = \dfrac{3 - 4i}{5}$

2 If $u = 3 - 2i$, find the value of each of the following.

a $|u|$ b $|u^{-1}|$ c $\dfrac{1}{|u|}$ d $|u^2|$

3 Evaluate the following.

a $\left|\sqrt{3} + i\right|$ b $\left|\sqrt{2} - i\sqrt{7}\right|$ c $\dfrac{1}{|6 + 8i|}$ d $\left|\dfrac{1 - i}{\sqrt{2}}\right|$

4 By first expressing each complex number in the form $a + ib$, find the modulus.

a $5 - i(2 + 3i)$ b $(3 - 2i)(1 + 4i)$ c $2\sqrt{3}(2 + i)$

d $\dfrac{1}{1 + i\sqrt{3}}$ e $\dfrac{1 - i}{1 + i}$ f $i\left(\sqrt{2} - i\right)^2$

5 If $z = -2 + 3i$ and $w = 1 - 4i$, show that each of the following is true.

a $|zw| = |z||w|$

b $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

c $|z + w| \le |z| + |w|$

d $|z - w| \ge |z| - |w|$

6 Example 16 Find in terms of x and y:

a $|x + iy|$

b $\left|\frac{1}{x - iy}\right|$

c $\left|\frac{x + 1 - iy}{x + 1 + iy}\right|$

Reasoning and communication

7 If $z = a + ib$, show that $|z| = |\bar{z}|$. Can you explain geometrically why this is true?

8 If $z = x + yi$, show that $|z|^2 = |z^2|$.

9 If $z = x + yi$, show that $|z|^2 = z\bar{z}$.

10 If $z = x + yi$ and $w = u + vi$, where $u, v, x, y \in \boldsymbol{R}$, prove that

a $|zw| = |z||w|$

b $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

10.07 OPERATIONS IN THE ARGAND PLANE

As discussed in Chapter 3, two vectors can be added or subtracted using the parallelogram rule. Complex numbers can be represented as vectors on the Argand diagram. Using vector geometry, we can add and subtract complex numbers on the Argand diagram.

Example 17

If $z = 1 + 3i$ and $w = 4 + 2i$,

a find $z + w$

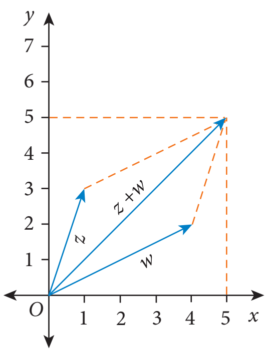
b plot z, w and $z + w$ as vectors on the complex plane.

Solution

a Group the real and imaginary parts.

$$z + w = (1 + 4) + (3 + 2)i$$
$$= 5 + 5i$$

b The vectors z and w form the sides of the parallelogram and $z + w$ is the diagonal.



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Recall that the vector −**z** has the same magnitude as the vector **z** but it is in the opposite direction.

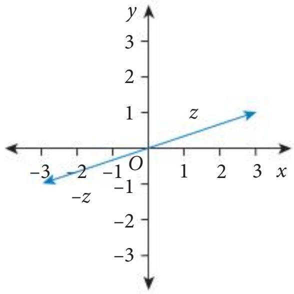
Example 18

Given $z = 3 + i$, plot the following as vectors.

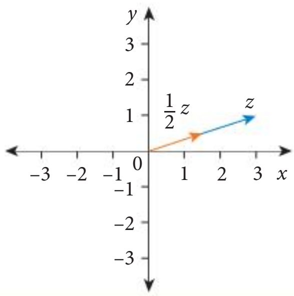
a $-z$ **b** $\frac{1}{2}z$

Solution

a The vector $-z = -3 - i$. It is in the opposite direction.



b $\frac{1}{2}z$ is half the length of z.



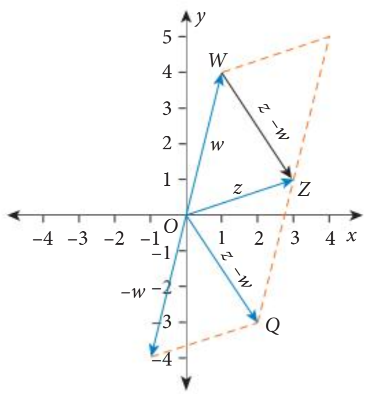
It is possible to subtract two vectors **z** and **w** using −**z** or −**w**, or by using the other diagonal in the parallelogram rule.

Example 19

Given $\mathbf{z} = 3 + i$ and $\mathbf{w} = 1 + 4i$, plot the vector $\mathbf{z} - \mathbf{w}$ and label the corresponding point Q.

Solution

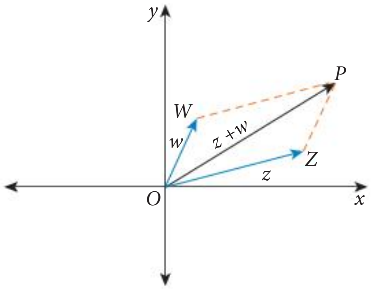
First write $\mathbf{z} - \mathbf{w}$ as $\mathbf{z} + (-\mathbf{w})$.
Use the parallelogram rule with **z** and −**w**.
The vector $\mathbf{z} - \mathbf{w}$ is the diagonal vector from O. The point $Q(2, -3)$ corresponds with the arrowhead.
Alternatively, use the parallelogram rule with **z** and **w**.
The vector joining Z and W with the arrowhead on Z is also the vector $\mathbf{z} - \mathbf{w}$.

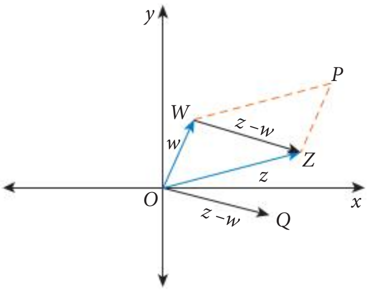


The example above highlights this important point: the vector is movable but the corresponding point is fixed. The point corresponds to the arrowhead of the vector with its tail at the origin because *the vector represents a complex number*.

IMPORTANT

Given the vectors **z** and **w** with corresponding points Z and W respectively, the vectors **z** + **w** and **z** – **w** are the diagonal vectors in the parallelogram $OWPZ$.





The point P corresponds to the vector **z** + **w**.

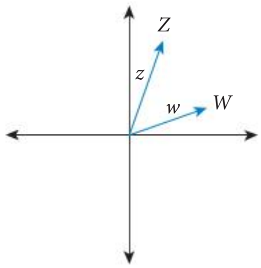
The point Q corresponds to the vector **z** – **w**.

EXERCISE 10.07 Operations in the Argand plane

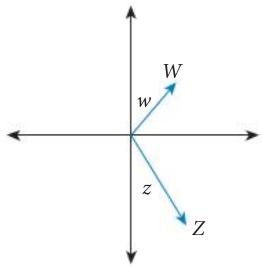
Concepts and techniques

1 Example 17 Copy the diagrams below and use the parallelogram rule to find the vector **z** + **w**. Label the corresponding point P.

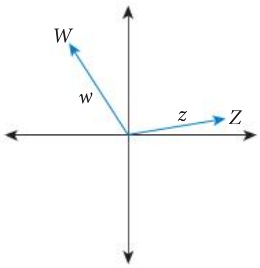
a



b

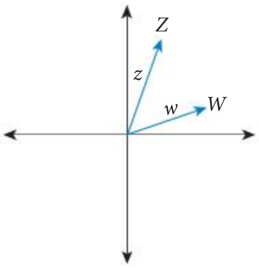


c

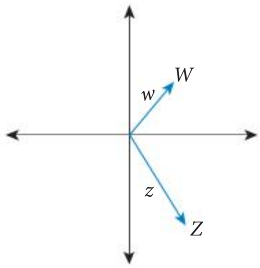


2 Example 18 Copy the diagrams below and find the vectors –**z** and –**w**. Label the corresponding points P and Q.

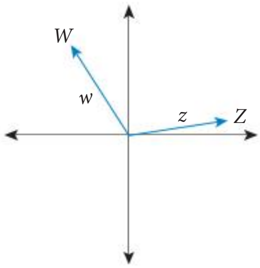
a



b



c



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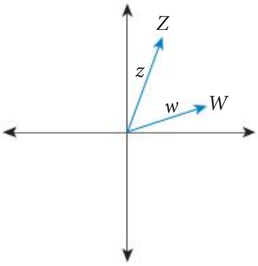
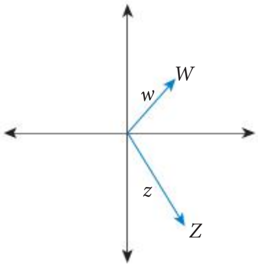
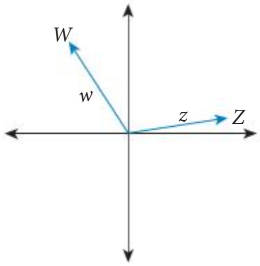
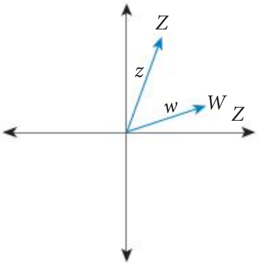
3 Copy the diagrams below and find the vectors $\frac{1}{2}\mathbf{z}$ and $\frac{1}{2}\mathbf{w}$. Label the corresponding points U and V.

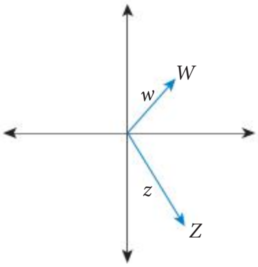
a


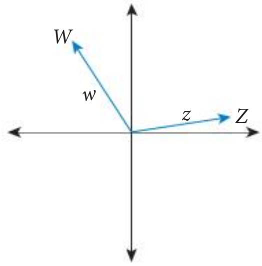
b


c


4 Example 19 Copy the diagrams below and use the parallelogram rule to find the vector $\mathbf{z} - \mathbf{w}$. Label the corresponding point R.

a


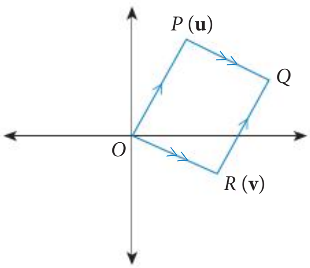
b


c


5 Consider the parallelogram $OPQR$ in the Argand diagram on the right. Let $\overrightarrow{OP} = \mathbf{u}$ and $\overrightarrow{OR} = \mathbf{v}$.
In terms of $\mathbf{u}$ and $\mathbf{v}$ find expressions for

a $\overrightarrow{OQ}$ b $\overrightarrow{PR}$ c $\overrightarrow{QO}$

d $\overrightarrow{RP}$ e M, the midpoint of OQ.



6 Consider the points M and N with corresponding vectors $\mathbf{m} = -2 + i$ and $\mathbf{n} = 3 + 2i$. The points O, M, L, N form a parallelogram $OMLN$. Find the complex number corresponding to the vertex L.

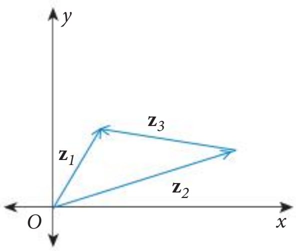
Reasoning and communication

7 Consider the vectors $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ shown in the diagram. Express

a $\mathbf{z}_3$ in terms of $\mathbf{z}_1$ and $\mathbf{z}_2$

b $\mathbf{z}_1$ in terms of $\mathbf{z}_2$ and $\mathbf{z}_3$

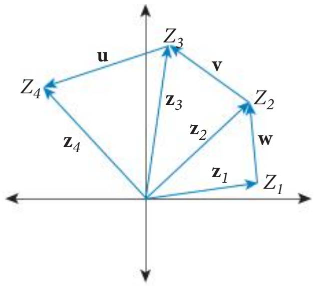
c $\mathbf{z}_2$ in terms of $\mathbf{z}_1$ and $\mathbf{z}_3$



8 Consider any two vectors w_1 and w_2 on an Argand Diagram. Using a diagram, explain why

a $|w_1 + w_2| \le |w_1| + |w_2|$ b $|w_1 - w_2| \ge |w_1| - |w_2|$

9 Consider the points Z_1, Z_2, Z_3, Z_4 corresponding to the complex numbers with vectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ and $\mathbf{z}_4$ as shown in the diagram. Show that $\mathbf{z}_4 - \mathbf{z}_1 = \mathbf{w} + \mathbf{u} + \mathbf{v}$.



10.08 PROPERTIES OF COMPLEX NUMBERS

The complex numbers form a field. The complex numbers complete the development of the number system. They are sufficient for performing all the calculations that we need to be able to do, including in complex problems such as the fluid dynamics used in aerospace engineering and design.

IMPORTANT

For any complex numbers, z, w and $v \in \mathbf{C}$, the following properties hold.

1. Closure under addition: $z + w \in \mathbf{C}$
2. Closure under multiplication: $zw \in \mathbf{C}$
3. The commutative law of addition: $z + w = w + z$
4. The commutative law of multiplication: $zw = wz$
5. The associative law of addition: $(z + w) + v = z + (w + v)$
6. The associative law of multiplication: $(zw)v = z(wv)$
7. The distributive law of multiplication over addition: $z(w + v) = zw + zv$
8. Additive identity: $0 + z = z + 0 = z, \forall z \in \mathbf{C}$
9. Multiplicative identity: $1 \times z = z \times 1 = z$.
10. Additive inverse: $\forall z \in \mathbf{C} \, \exists \, (-z) \in \mathbf{C}$ such that $z + (-z) = (-z) + z = 0$
11. Multiplicative inverse: $\forall z \neq 0 \in \mathbf{C} \, \exists \, z^{-1} \in \mathbf{C}$ such that $zz^{-1} = z^{-1}z = 1$

Example 20

Show that there exists a multiplicative inverse for nonzero numbers in the complex plane.

Solution

We must show that there exists a number z^{-1} that can be written in the form $A + Bi$ such that $z \times z^{-1} = z^{-1} \times z = 1$.	Let $z = a + bi$ The inverse is the reciprocal, $z^{-1} = \dfrac{1}{z}$.

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Write the reciprocal.	$z^{-1} = \dfrac{1}{a+bi}$
Rationalise the denominator.	$= \dfrac{1}{a+bi} \times \dfrac{a-bi}{a-bi}$ $= \dfrac{a-bi}{a^2+b^2}$ $= \dfrac{a}{a^2+b^2} + \dfrac{-b}{a^2+b^2}i$
Check the product zz^{-1}.	$z \times z^{-1} = (a+ib) \times \left(\dfrac{a}{a^2+b^2} + \dfrac{-b}{a^2+b^2}i\right)$
Multiply out the brackets.	$= \dfrac{aa}{a^2+b^2} + \dfrac{-ab}{a^2+b^2}i + \dfrac{aib}{a^2+b^2} + \dfrac{-bb}{a^2+b^2}i^2$
Simplify.	$= \dfrac{a^2}{a^2+b^2} + \dfrac{-b^2i^2}{a^2+b^2}$
Use $i^2 = -1$ and put on a common denominator.	$= \dfrac{a^2+b^2}{a^2+b^2} = 1$
Check the product $z^{-1}z$.	$z^{-1} \times z = \left(\dfrac{a}{a^2+b^2} + \dfrac{-b}{a^2+b^2}i\right) \times (a+ib)$
Multiply out the brackets.	$= \dfrac{aa}{a^2+b^2} + \dfrac{-ab}{a^2+b^2}i + \dfrac{aib}{a^2+b^2} + \dfrac{-bb}{a^2+b^2}i^2$
Simplify.	$= \dfrac{a^2}{a^2+b^2} + \dfrac{-b^2i^2}{a^2+b^2}$
Use $i^2 = -1$ and put on a common denominator.	$= \dfrac{a^2+b^2}{a^2+b^2}$ $= 1$
Write the result.	For any number $z \neq 0 \in \mathbf{C}$ there exists $z^{-1} = \dfrac{1}{z}$ such that $z \times z^{-1} = z^{-1} \times z = 1$. **QED**

There are also other important properties of complex numbers. These are stated below.

IMPORTANT

Properties of complex numbers

1 $|z|^2 = z\bar{z}$

2 $|z|^n = |z^n|$

3 $|z_1 z_2| = |z_1||z_2|$

4 $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

5 $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

6 $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

INVESTIGATION Complex numbers in real life

As mentioned earlier in the chapter, many of the electronic devices you use today such as computers, Smartphones and MP3 players rely on the mathematics of complex numbers. Investigate the following milestones in mathematical and scientific history.

1 Who was the first person to use the idea of imaginary numbers?

2 Who first used the notation $i = \sqrt{-1}$?

3 When was the first electronic device invented? By whom?

4 Give an example of how complex numbers can be utilised in electronics.

Example 21

Prove that for two complex numbers z_1 and z_2 that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

Solution

Express z_1 and z_2 in Cartesian form.

Let $z_1 = a + bi$ and $z_2 = c + di$.

Consider the LHS.

$$\text{LHS} = \left|\frac{z_1}{z_2}\right|$$

Realise the denominator.

$$= \left|\frac{a+bi}{c+di}\right|$$

$$= \left|\frac{a+bi}{c+di} \times \frac{c-di}{c-di}\right|$$

Group the real and imaginary parts and use the formula for the modulus.

$$= \left|\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}\right|$$

$$= \sqrt{\frac{(ac+bd)^2}{(c^2+d^2)^2} + \frac{(bc-ad)^2}{(c^2+d^2)^2}}$$

$$= \frac{\sqrt{a^2c^2 + 2abcd + b^2d^2 + b^2c^2 - 2abcd + a^2d^2}}{c^2+d^2}$$

Re-factorise.

$$= \frac{\sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)}}{c^2+d^2}$$

$$= \frac{\sqrt{(c^2+d^2)(a^2+b^2)}}{c^2+d^2}$$

Cancel.

$$= \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

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Consider the RHS.

$$\text{RHS} = \frac{|z_1|}{|z_2|}$$

$$= \frac{|a + bi|}{|c + di|}$$

Find the modulus of the numerator and denominator.

$$= \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

$$= \sqrt{\frac{a^2 + b^2}{c^2 + d^2}}$$

Write the conclusion.

LHS = RHS

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

QED

EXERCISE 10.08 Properties of complex numbers

Concepts and techniques

WS Complex conjugates and inverses

1 Example 20 Let $z_1 = a + bi$, $z_2 = c + di$ and $z_3 = e + fi$. Prove the properties below for the set of complex numbers $\boldsymbol{C}$.
   a The closure under addition
   b Closure under multiplication
   c The commutative law of addition
   d The commutative law of multiplication
   e The associative law of addition
   f The associative law of multiplication
   g The distributive law of multiplication over addition

2 Let $z = x + yi$, $1 = 1 + 0i$ and $0 = 0 + 0i$. Write down the
   a additive inverse of $z = x + yi$
   b multiplicative inverse of $z = x + yi$
   c the additive identity for $z = x + yi$
   d the multiplicative identity for $z = x + yi$.

3 Example 21 Show that each of the following is true for $z = 3 - 4i$.
   a $|z|^2 = z\overline{z}$ b $|z|^2 = |z^2|$

4 Let $z_1 = 1 - 2i$ and $z_2 = -1 + i$. Show that each property is true.
   a $|z_1 z_2| = |z_1||z_2|$ b $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ c $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ d $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

Reasoning and communication

5 Prove the following properties $\forall z \in C$.

a $|z|^2 = z\bar{z}$

b $|z|^2 = |z^2|$

c $|z_1 z_2| = |z_1||z_2|$

d $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

e $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

f $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

6 Show that $z + \bar{z}$ is real $\forall z \in C$.

7 Show that $z\bar{z}$ is real $\forall z \in C$.

8 Is it true that $\frac{1}{|z|^2} = \frac{1}{|\bar{z}|^2}$ $\forall z \in C$? Give reasons for your answer.

10.09 QUADRATIC EQUATIONS

You have seen before that a quadratic equation written in the form $ax^2 + bx + c = 0$, where a, b and c are real, has two roots that can be real, imaginary or complex. If the two roots are α and β, then we can write

$$\begin{aligned} ax^2 + bx + c &\equiv a(x - \alpha)(x - \beta) \\ &\equiv a(x^2 - \alpha x - \beta x + \alpha\beta) \\ &\equiv a[x^2 - (\alpha + \beta)x + \alpha\beta]. \end{aligned}$$

You can both solve a quadratic equation *and* create a quadratic equation or expression given the roots.

Example 22

Solve each quadratic equation in the complex plane and classify the roots as real, imaginary or complex.

a $x^2 - 6x - 7 = 0$ b $x^2 - 6x + 7 = 0$ c $x^2 - 6x + 10 = 0$

Solution

a Factorise to solve.

$x^2 - 6x - 7 = 0$

$(x - 7)(x + 1) = 0$

$x = 7$ or 1

The roots are real.

b $x^2 - 6x + 7 = 0$ does not factorise.
Use the quadratic formula.

$x^2 - 6x + 7 = 0$

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)} \\ &= \frac{6 \pm \sqrt{8}}{2} \\ &= \frac{6 \pm 2\sqrt{2}}{2} \\ &= 3 \pm \sqrt{2} \end{aligned}$$

The roots are real.

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c $x^2 - 6x + 10 = 0$ does not factorise. Use the quadratic formula OR create a difference of two squares.

$$x^2 - 6x + 10 = 0$$

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)} \\ &= \frac{6 \pm \sqrt{-4}}{2} \\ &= \frac{6 \pm 2i}{2} \\ &= 3 \pm i \end{aligned}$$

OR

$x^2 - 6x + 10 = 0$

$x^2 - 6x + 9 - i^2 = 0$

$(x - 3)^2 - i^2 = 0$

$(x - 3 - i)(x - 3 + i) = 0$

$x - 3 - i = 0$ or $x - 3 + i = 0$

$x = 3 + i$ or $x = 3 - i$

Note that the roots are complex conjugates.

$\therefore\ x = 3 \pm i$

The roots are complex and form a conjugate pair.

IMPORTANT

Theorem:

If a quadratic equation $ax^2 + bx + c = 0$, has real coefficients a, b and c, then, if $\alpha = n + mi$ is a root (n and m are real), then its conjugate $\overline{\alpha} = n - mi$ will also be a root.

The proof of this theorem relies on the four properties of conjugates. Consider any two complex numbers $z_1 = A + Bi$ and $z_2 = C + Di$, where $A, B, C, D \in \boldsymbol{R}$.

1 If $z = 0$, then $\overline{z} = 0$.

2 If $\overline{z_1 + z_2} = 0$, then $\overline{z}_1 + \overline{z}_2 = 0$.

3 If $\overline{kz} = 0$, then $k\overline{z} = 0$, where $k \in \boldsymbol{R}$.

4 If $\overline{z_1 z_2} = 0$, then $\overline{z}_1\,\overline{z}_2 = 0$.

Now we are able to prove the theorem.

Proof

If $\alpha = n + mi$ is a root of $ax^2 + bx + c = 0$, then $\alpha = n + mi$ must satisfy $ax^2 + bx + c = 0$.

That is, $a(\alpha)^2 + b(\alpha) + c = 0$.

Now we must prove that $\bar{\alpha} = n - mi$ must satisfy $ax^2 + bx + c = 0$. Consider

$$a(\alpha)^2 + b(\alpha) + c = 0$$

$$\overline{a(\alpha)^2 + b(\alpha) + c} = 0$$

$$\overline{a(\alpha)^2} + \overline{b(\alpha)} + c = 0$$

$$a\overline{(\alpha)^2} + b\overline{(\alpha)} + c = 0$$

$$a(\bar{\alpha})^2 + b(\bar{\alpha}) + c = 0$$

$\therefore$ both α and $\bar{\alpha}$ satisfy $ax^2 + bx + c = 0$.

This means that complex roots of quadratic equations with *real* coefficients always come in conjugate pairs.

IMPORTANT

If a quadratic equation $ax^2 + bx + c = 0$ has real coefficients a, b and c, then it is called a *real* quadratic equation. The corresponding expression $ax^2 + bx + c$ is called a *real* quadratic expression.

Example 23

Show that $3 - i\sqrt{2}$ is a root of the equation $z^2 - 6z + 11 = 0$ and find the other root.

Solution

Substitute $3 - i\sqrt{2}$ into the RHS.

$$\left(3 - i\sqrt{2}\right)^2 - 6\left(3 - i\sqrt{2}\right) + 11$$

Simplify.

$$= 9 - 6i\sqrt{2} - 2 - 18 + 6i\sqrt{2} + 11$$

$$= 0$$

Write the conclusion.

$\therefore\ 3 - i\sqrt{2}$ is a root.

Complex roots of real quadratic equations come in conjugate pairs.

The other root is $3 + i\sqrt{2}$.

It is easy to form the real quadratic equation or expression whose roots or zeros are α and $\bar{\alpha}$.

IMPORTANT

If a real quadratic equation $ax^2 + bx + c = 0$, or the corresponding real expression $ax^2 + bx + c$ has complex conjugate roots or zeros α and $\bar{\alpha}$, then the equation or expression can be expressed in the form

$x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} = 0$ or $a(x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha})$.

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Example 24

a Form a quadratic equation in x that has roots $\alpha = 3 + i\sqrt{5}$ and $\bar{\alpha} = 3 - i\sqrt{5}$, expressing your answer in the form

i $(x-\alpha)(x-\bar{\alpha}) = 0$ **ii** $x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} = 0$.

b Express the quadratic expression $w^2 + 4w + 7$ as a product of its linear factors.

Solution

a i Write the equation form.	$(x-\alpha)(x-\bar{\alpha}) = 0$
Substitute roots.	$\left[x-\left(3+i\sqrt{5}\right)\right]\left[x-\left(3-i\sqrt{5}\right)\right]=0$
Simplify.	$\left(x-3-i\sqrt{5}\right)\left(x-3+i\sqrt{5}\right)=0$
ii Find the sum of the roots.	$\alpha+\bar{\alpha}=\left(3+i\sqrt{5}\right)+\left(3-i\sqrt{5}\right)$
Simplify.	$= 6$
Find the product of the roots.	$\alpha\bar{\alpha}=\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)$
Simplify.	$= 14$
Write the equation form.	$x^2 - (\alpha + \bar{\alpha})x + \alpha\,\bar{\alpha} = 0$
Substitute sum and product of roots.	$x^2 - 6x + 14 = 0$
b Write the equation.	$w^2 + 4w + 7 = 0$
Complete the square.	$(w^2 + 4w + 4) + 3 = 0$
Write as a difference of squares.	$(w+2)^2-\left(i\sqrt{3}\right)^2=0$
Factorise as a difference of squares.	$\left(w+2-i\sqrt{3}\right)\left(w+2+i\sqrt{3}\right)=0$

EXERCISE 10.09 Quadratic equations

Concepts and techniques

WS Complex numbers

1 Example 22 Find the solutions to the following real quadratic equations.

a $z^2 + 2z + 4 = 0$ **b** $z^2 - 2z - 4 = 0$ **c** $z^2 + 4z + 8 = 0$ **d** $3z^2 + 2z + 8 = 0$

2 Find the complex solutions to the real quadratic equation $(x + 1)^2 + 9 = 0$.

3 Solve $(x + 1)^2 = 1 - 2(x + 3)^2$ over the complex plane.

4 Example 23 Show by substitution that $\beta = -5 - i$ is a root of the real quadratic equation $z^2 + 10z + 26 = 0$. State the other root.

5 Find the unknown real coefficient if $2 - i$ is a solution to the equation $z^2 + mz + 5 = 0$. State the other solution.

6 Example 24 Form quadratic equations for which the following complex conjugates are roots. Give your answer in the form

i $(x-\alpha)(x-\bar{\alpha})=0$ ii $x^2-(\alpha+\bar{\alpha})x+\alpha\bar{\alpha}=0$

a $1 \pm 2i$ b $\sqrt{3} \pm i$ c $-4 \pm i\sqrt{2}$ d $\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$

e $-5 \pm 6i$ f $\frac{\sqrt{3}}{2} \pm \frac{i}{2}$ g $-\frac{3}{5} \pm i\frac{2}{5}$ h $-\frac{1}{\sqrt{3}} \pm i\frac{\sqrt{2}}{\sqrt{3}}$

7 Find the unknown integer coefficients if the roots below are solutions to the equation $kz^2 + nz + p = 0$.

a $2 \pm i$ b $1 \pm i\sqrt{3}$ c $\frac{-1 \pm i\sqrt{15}}{4}$ d $\frac{5}{13} \pm i\frac{12}{13}$

8 Express each of the quadratic expressions as a product of its linear factors.

a $z^2 - 2z + 2$ b $z^2 - 2z + 6$ c $z^2 + 4z + 5$ d $z^2 + z + 1$.

Reasoning and communication

9 The three roots of the equation $z^3 = 1$ are called the roots of unity. By fully factorising the expression $z^3 - 1$, show that the equation $z^3 = 1$ has one real root and two complex roots which are conjugates.

10 If ω and $\bar{\omega}$ are the complex roots of the equation $z^3 = 1$, show that

a $\bar{\omega} = \omega^2$ b $|\omega| = |\bar{\omega}| = |\omega^2| = 1$

11 Plot the solutions to $z^3 = 1$ as vectors on the complex plane. What do you notice?

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10 CHAPTER SUMMARY
COMPLEX NUMBERS

- The **imaginary number *i*** is defined as $i = \sqrt{-1}$.
- The powers of i make a pattern that repeats each four numbers: $i, -1, -i, 1, i, -1, -i, 1, \ldots$
- A complex number is a number that can be written in the form $a + bi$, where a and b are real numbers.
- A complex number is often denoted by the letter z, so $z = a + bi$.
- The real part of $z = a + bi$ is denoted by $\text{Re}(z)$ and $\text{Re}(z) = a$
- The imaginary part of $z = a + bi$ is denoted by $\text{Im}(z)$ and $\text{Im}(z) = b$.
- If $\text{Re}(z) = 0$, then we say that z is *purely imaginary.*
- If $\text{Im}(z) = 0$, then we say that z is *purely real* or just *real.*
- The set of real numbers is a subset of the set of complex numbers.
- For a complex number z, where $z = a + bi$ (where a and b are real numbers), the complex conjugate of z is denoted by $\overline{z}$ and $\overline{z} = a - bi$
- The product $z\overline{z}$ is real.
- When dividing two complex numbers, we multiply by the complex conjugate of the denominator. This is called *realising the denominator.*
- The complex number $z = x + yi$ (where $x, y \in \boldsymbol{R}$) can be represented on the *Argand diagram* with a horizontal axis denoted by x or $\text{Re}(z)$ and a vertical axis denoted by y or $\text{Im}(z)$.
- The complex number $z = x + yi$ can be represented by the point $P(x, y)$ (Figure 1) or by the vector $\overrightarrow{OP}$ (Figure 2).

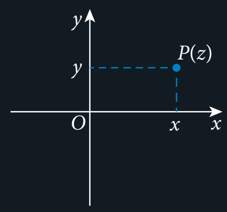


Figure 1

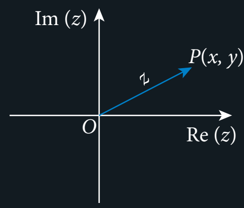
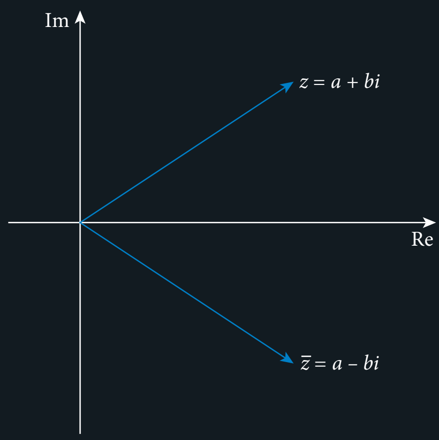
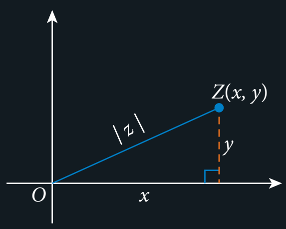


Figure 2

- On the Argand diagram, if $z = a + bi$, then the complex conjugate $\overline{z} = a - bi$ is the reflection of the vector $z = a + bi$ in the x-axis.

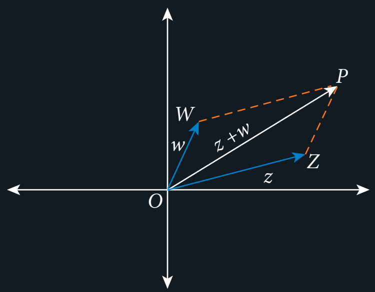


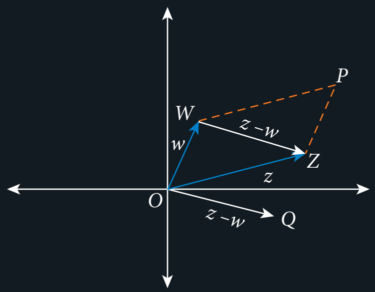
- The length of the vector $z = x + yi$ is called **mod** z or $|z|$.



$|z| = \sqrt{x^2 + y^2}$ or $|z| = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2}$

Given the vectors **z** and **w** with corresponding points Z and W, respectively, the vectors **z** + **w** and **z** – **w** are the diagonal vectors in the parallelogram $OWPZ$.





Point P corresponds with the vector **z** + **w**.

Point Q corresponds with the vector **z** – **w**.

- The complex numbers form a field.
- The following properties hold $\forall z \in \boldsymbol{C}$:
  a $|z|^2 = z\bar{z}$
  b $|z|^2 = |z^2|$
  c $|z_1 z_2| = |z_1||z_2|$
  d $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
  e $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
  f $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
- $z + \bar{z}$ is real $\forall z \in \boldsymbol{C}$.
- $z\bar{z}$ is real $\forall z \in \boldsymbol{C}$.
- If a quadratic equation $ax^2 + bx + c = 0$ has real coefficients a, b and c, then if $\alpha = n + mi$ is a root (n and m are real), then its conjugate $\bar{\alpha} = n - mi$ will also be a root.
- If a quadratic equation $ax^2 + bx + c = 0$ has real coefficients a, b and c, then it is called a *real* quadratic equation. The corresponding expression $ax^2 + bx + c$ is called a *real* quadratic expression.
- If a real quadratic equation has complex conjugate roots α and $\bar{\alpha}$, then the equation can be expressed in the form
  $(x - \alpha)(x - \bar{\alpha}) = 0$ or
  $x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} = 0$.

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CHAPTER REVIEW COMPLEX NUMBERS 10

Multiple choice

1 Example 6, 10 If $z = 1 - 3i$ and $w = 2 - i$, then $\overline{z} + 2w =$

A $2 + 3i$ B $5 + 5i$ C $5 + i$ D $3 + i$ E $5 - 5i$

2 Example 10 If $z = 1 - 3i$ and $w = 2 - i$, then $zw =$

A $-1 - 7i$ B $-1 - 5i$ C $4 - 7i$ D $4 - 5i$ E 5

3 Example 3 The value of i^{107} is

A i B $-i$ C -1 D 1 E $\sqrt{-107}$

4 Example 16 If $z = \sqrt{2} + i\sqrt{2}$, then $\frac{1}{|z|} =$

A 4 B $\frac{1}{4}$ C 2 D $\frac{1}{2}$ E $\frac{1}{2\sqrt{2}}$

5 Example 21 If z and w are complex numbers, then which of the following is NOT true?

A $\overline{z + w} = \overline{z} + \overline{w}$ B $\overline{z \times w} = \overline{z} \times \overline{w}$ C $|zw| = z\overline{w}$

D $|zw| = |z||w|$ E $z\overline{z} = |z|^2$

Short answer

6 Example 1 Write each number in terms of i.

a $\sqrt{-81}$ b $\sqrt{-18}$ c $\sqrt{-48}$ d $\sqrt{-216}$

7 Example 2 Solve the following equations.

a $z^2 = -36$ b $z^2 + 9 = 0$ c $4z^2 + 1 = 0$

8 Example 3 Evaluate each of the following.

a i^3 b i^{200} c i^{49} d i^{590} e $\frac{1}{i^{17}}$

9 Example 4 Write down Re(z) and Im(z) for each complex number.

a $5 - i\sqrt{5}$ b $\frac{1 + i\sqrt{3}}{2}$

c $x + 3yi - 2y + xi$ (assume that x and y are real)

10 Example 5 Solve the following quadratic equations, giving your answers in the form $a + bi$ (assume that a and b are real).

a $x^2 + 4x + 6 = 0$ b $3z^2 - 10z + 12 = 0$ c $w^2 + w + 2 = 0$

11 Example 6 Write the complex conjugate of:

a $-4 + 7i$ b $2 + 5i$

c $2x + y - yi + 3xi$ (assume that x and y are real)

12 Example 7 By first expressing each as a difference of two squares, factorise each quadratic below.

a $x^2 + 4x + 5$ b $z^2 - 10z + 29$ c $w^2 + w + 1$

13 Example 8 Write a quadratic equation in the form $ax^2 + bx + c = 0$ with the following complex roots.

a $3 \pm i$ b $1 \pm i\sqrt{6}$ c $\dfrac{2 \pm i\sqrt{3}}{2}$

14 Example 9 Find the value of real numbers x and y if $2x + y + i(x - 2y) = 4 + 7i$.

15 Example 10 Simplify each expression below, giving your answers in the form $a + bi$ (assume that a and b are real).

a $2 + 8i - 4 - 6i$ b $(4 - i)(7 + 3i)$ c $(3 - 2i)^2$

16 Example 11 Simplify by realising the denominator, giving your answers in the form $a + bi$ (assume that a and b are real).

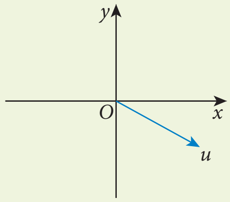
a $\dfrac{1+i}{2-i}$ b $\dfrac{1}{\sqrt{2}+i}$ c $\dfrac{2}{1+i} - \dfrac{2}{1-i}$

17 Example 12 Represent each number as a point on an Argand diagram.

a $z = 2 + i$ b $w = -3 - 2i$ c $v = \dfrac{4}{i}$

18 Example 13 Plot the complex number $z = -1 + 2i$ and its complex conjugate $\bar{z}$ as vectors on an Argand diagram.

19 Example 14 The complex number u is shown as a vector on the Argand diagram to the right. Copy the diagram and plot its conjugate $\bar{u}$.



20 Example 15 For each complex number z find $|z|$.

a $z = -2 + 3i$ b $z = \dfrac{1 + i\sqrt{3}}{4}$ c $z = \dfrac{1}{\sqrt{3} - 2i}$

21 Example 16 Write $|(x - y) - i(2x + y)|$ in simplest form (assume that x and y are real).

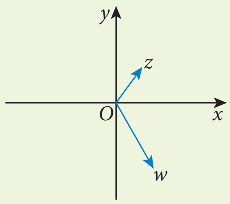
22 Example 17 Plot the complex numbers $z = 3 + i$ and $w = -1 + 2i$ as vectors on an Argand diagram. Hence plot the vector $z + w$.

23 Example 18 Given the complex number $z = 2 - 4i$,

a state the points U and V that represent $\frac{1}{2}z$ and $-z$

b plot U and V on an Argand diagram.

24 Example 19 The complex numbers z and w are shown as vectors on an Argand diagram. Hence plot the point Q that represents the vector $z - w$.



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25 Example 20 Find the multiplicative inverse of each complex number, giving your answer in the form $a + bi$ (assume that a and b are real).

a $1 - i\sqrt{2}$ b $2i^3$ c $\dfrac{1}{3-4i}$

26 Example 21 Prove that for two complex numbers $z = a + bi$ and $w = c + di$ (assume that a, b, c and d are real):

a $\overline{z+w} = \overline{z} + \overline{w}$ b $\left|\overline{zw}\right| = \left|\overline{z}\right|\left|\overline{w}\right|$

27 Example 22 Solve the following quadratic equations, classifying the roots as real, imaginary or complex.

a $x^2 + 4 = 0$ b $3z^2 - 10z + 3 = 0$ c $w^2 - 2w + 2 = 0$

28 Example 23 Show that $4 - i$ is a root of the equation $z^2 - 8z + 17 = 0$. State the other root.

29 Example 24 Form the quadratic equation that has complex conjugate roots $-3 \pm i\sqrt{2}$, giving your answer in the form $(z-\alpha)(z-\overline{\alpha}) = 0$.

30 Example 10, 11 Evaluate the following.

a $(5 - 2i)(6 + 7i)$ b $|6 - 8i|$ c $\dfrac{\sqrt{2}+i}{\sqrt{2}-i}$ d $\overline{3-4i}$

31 Example 4, 9, 15 Consider the complex numbers $u = \sqrt{3} - i$ and $v = -1 - i\sqrt{3}$. Find:

a $u + v$ b $\text{Re}(u + v)$ c $\text{Im}(u + v)$ d $|u + v|$

32 Example 13 If $z = 3 + 2i$, sketch the following as vectors on Argand diagrams.

a z b $\overline{z}$ c $-z$ d $2z$

33 Example 22 Solve each equation in the complex plane.

a $z^2 + 25 = 0$ b $z^2 - 10z + 34 = 0$ c $w^2 + 3w = 0$

d $2u^2 + 8u + 9 = 0$ e $3z^2 + 7z + 5 = 0$ f $w^4 - 5w^2 - 36 = 0$

34 Example 24 If $-2 \pm 2i$ are the zeros of a quadratic $az^2 + bz + c$, find the quadratic.

Application

Practice quiz

35 Find the solutions to $x^2 + 5 = 3x$.

36 By factorising $z^3 - 8$, solve the equation $z^3 = 8$ in the complex plane.

37 The points V, W and U represent the complex numbers $1 + i$, $4 + 2i$ and $-1 + 3i$ respectively on the complex plane. Find the complex number represented by the point X if $VWXU$ is a parallelogram. What is the complex number represented by the midpoint M of VX?

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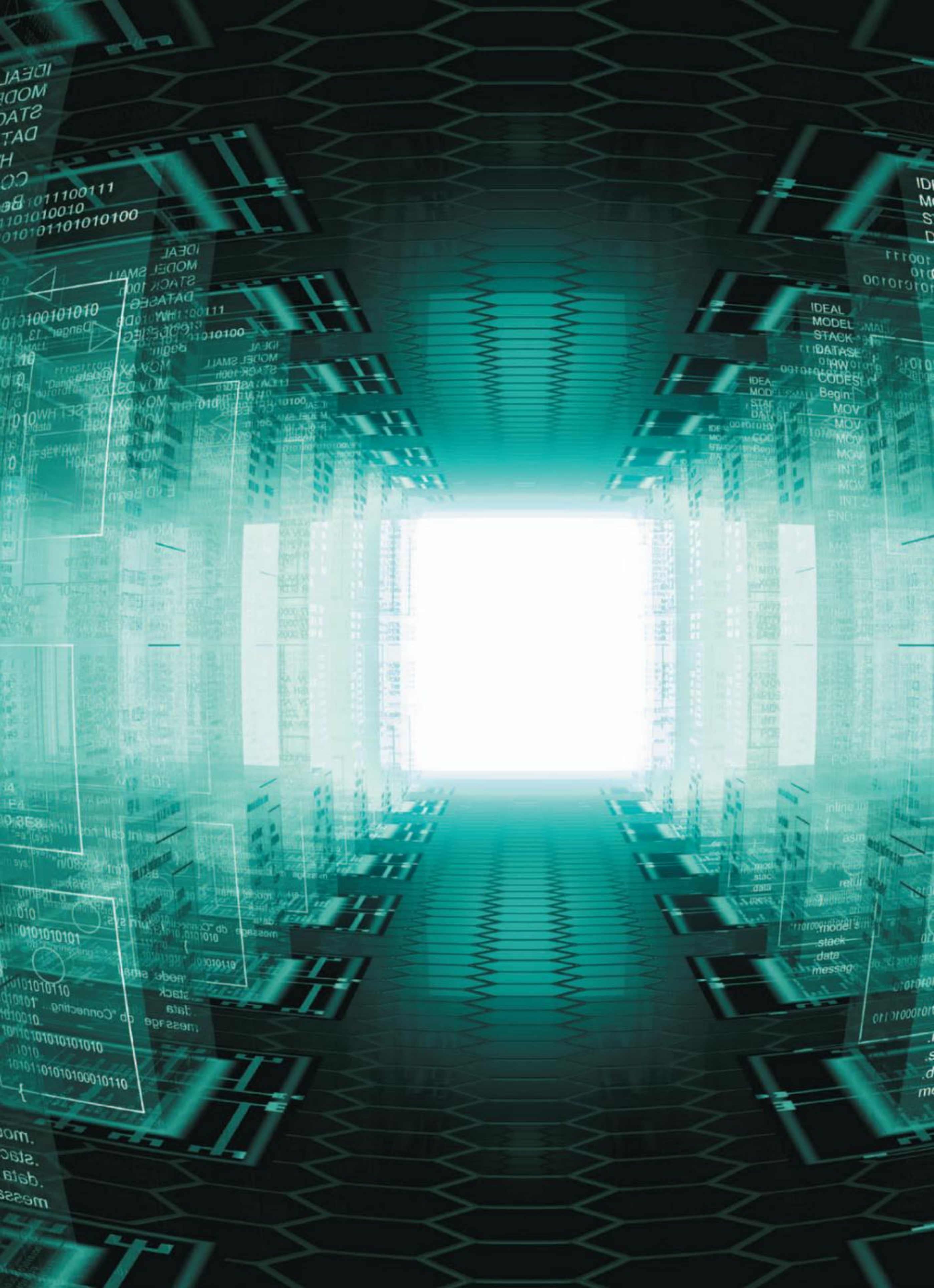

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11

TERMINOLOGY

angle of inclination
composition
dilation
identity
inverse
linear transformation
matrix
reflection
rotation
slope
transformation
translation
vector

MATRICES

TRANSFORMATIONS IN THE PLANE

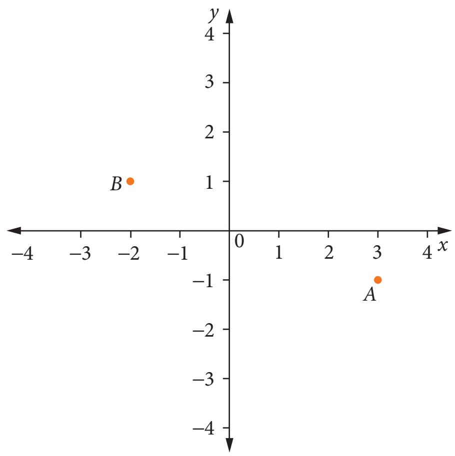
Prior learning

TRANSFORMATIONS IN THE PLANE

- translations and their representation as column vectors (ACMSM054)
- define and use basic linear transformations: dilations of the form $(x, y) \to (\lambda_1 x, \lambda_2 y)$, rotations about the origin and reflection in a line which passes through the origin, and the representations of these transformations by 2×2 matrices (ACMSM055)
- apply these transformations to points in the plane and geometric objects (ACMSM056)
- define and use composition of linear transformations and the corresponding matrix products (ACMSM057)
- define and use inverses of linear transformations and the relationship with the matrix inverse (ACMSM058)
- examine the relationship between the determinant and the effect of a linear transformation on area (ACMSM059)
- establish geometric results by matrix multiplications; for example, show that the combined effect of 2 reflections in lines through the origin is a rotation (ACMSM060) AC

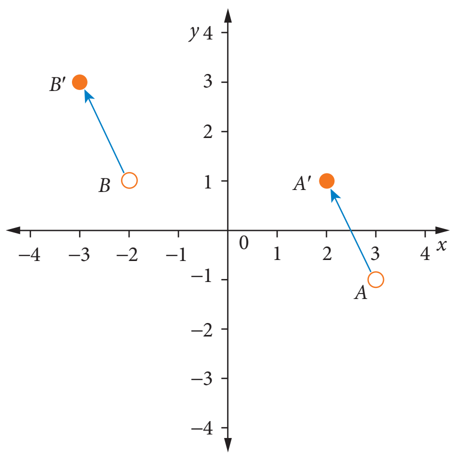
11.01 TRANSLATIONS

In Years 1–10 you examined transformations of geometric objects. You may also have looked at them in terms of changes in the coordinates of drawings in the Cartesian plane. The topic of transformations is very important in mathematics and its applications, such as the design and use of industrial robots.



In this chapter you will look at transformations as matrices. Points in the plane may be shown as column matrices, such as $A\begin{bmatrix}3\\-1\end{bmatrix}$ and $B\begin{bmatrix}-2\\1\end{bmatrix}$, when convenient. Points in the plane will also be written using standard coordinate notation, such as $A(3, -1)$ and $B(-2, 1)$.

In terms of movements, a **translation** is a *slide* of all the points in the plane the same distance in the same direction. The diagram on the right now shows a translation that moves the points in the plane *up* 2 and *left* 1. This is a translation of −2 in the x direction and 3 in the y direction.



The points $A\begin{bmatrix}3\\-1\end{bmatrix}$ and $B\begin{bmatrix}-2\\1\end{bmatrix}$ are translated to the new points $A'\begin{bmatrix}2\\1\end{bmatrix}$ and $B'\begin{bmatrix}-3\\3\end{bmatrix}$.

You can write this as $A\begin{bmatrix}3\\-1\end{bmatrix} \to A'\begin{bmatrix}2\\1\end{bmatrix}$ and $B\begin{bmatrix}-2\\1\end{bmatrix} \to B'\begin{bmatrix}-3\\3\end{bmatrix}$.

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Example 1

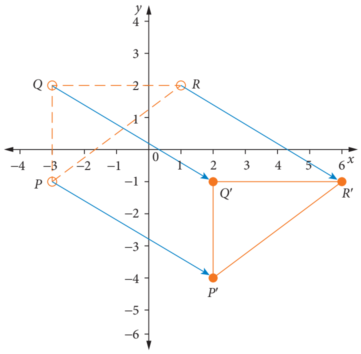
Show a translation of the triangle with vertices $P\begin{bmatrix}-3\\-1\end{bmatrix}$, $Q\begin{bmatrix}-3\\2\end{bmatrix}$ and $R\begin{bmatrix}1\\2\end{bmatrix}$ 3 down and 5 right as:

a a drawing in the plane

b a transformation of column matrices.

Solution

a The x-coordinates of each point increase by 5 and the y-coordinates decrease by 3. Draw the original points and the new points in the plane.



b 3 down and 5 right means that the x-coordinates increase by 5 and the y-coordinates decrease by 3.

$P\begin{bmatrix}-3\\-1\end{bmatrix} \to P'\begin{bmatrix}2\\-4\end{bmatrix}$, $Q\begin{bmatrix}-3\\2\end{bmatrix} \to Q'\begin{bmatrix}2\\-1\end{bmatrix}$

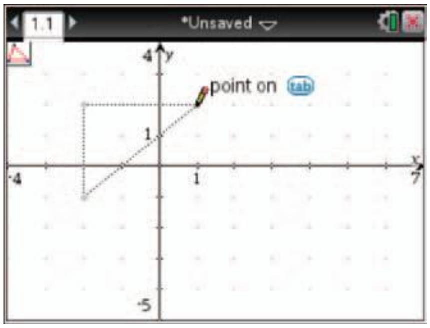
and $R\begin{bmatrix}1\\2\end{bmatrix} \to R'\begin{bmatrix}6\\-1\end{bmatrix}$

TI-Nspire CAS

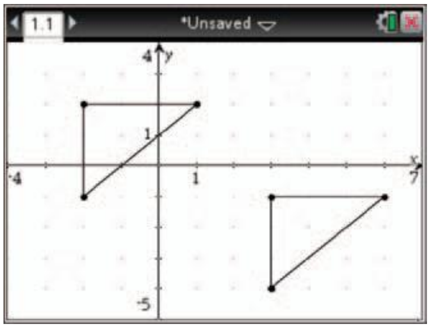
Use a graph page and Window settings of $-4 \le x \le 7$ and $-5 \le y \le 4$ with scales of 1 unit.

Use menu, 2: View, 6: Grid and 2: Dot Grid to show dots.

Use menu, 8: Geometry, 2: Shapes and Grid and 2: Triangle to draw the triangle by clicking the dots.



Now use menu, 8: Geometry, 5: Transformation and 3: Translation to do the translation by clicking the triangle, clicking a corner point and moving it 3 down and 5 right.

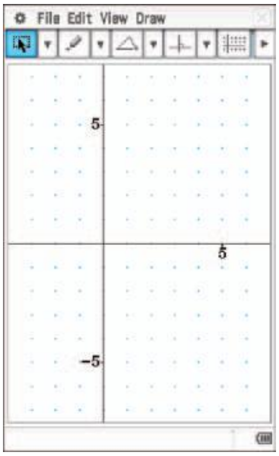


ClassPad

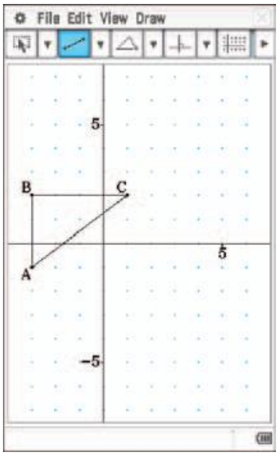
Use the Geometry menu.

Under tap **View Window** and set the fields in the window to make $-4 \le x \le 7$ and ymid = 0. ymid = 0 sets the middle of the screen to $y = 0$. When you have finished, tap **OK**.

Set the screen as shown by tapping . You might have to tap it several times.



Tap **Draw** then **Basic Object** and **Line Segment** (or the symbol) to draw the triangle by doing each side in order. (First tap the first point of the line segment then touch the point again and drag the stylus to the final point.) After these steps the triangle will appear as ΔABC.



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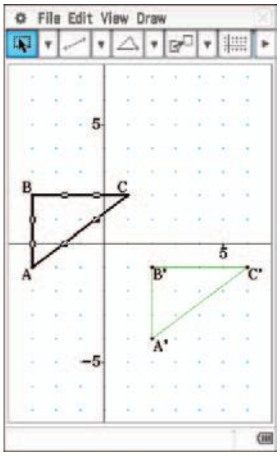
Tap select (icon) and select each side of the triangle. Use **Draw**, **Construct** and **Translation** to do the translation.

Enter $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ for 5 right and 3 down into the column matrix and tap OK.



The ClassPad automatically sets the y-scale to show the shapes accurately, but on the TI-Nspire CAS, if you want to show the shapes accurately, you need to set the x and y scales in a ratio of 2 : 1.

In Example **1**, you can transform the points by adding the column matrix $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ to each of the points.

Thus $P\begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} = P'\begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $Q\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} = Q'\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $R\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} = R'\begin{bmatrix} 6 \\ -1 \end{bmatrix}$.

The matrix $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ represents the translation 5 to the right and 3 down (5 in the x direction and –3 in the y direction).

IMPORTANT

A translation can be modelled as addition of column matrices. The column matrices representing points are translated to the new points by adding the column matrix representing the translation.

Example 2

Use column matrices to find the points that $A(2, 1)$, $B(4, 2)$, $C(5, 1)$, $D(3, -1)$ are transformed to by a translation of 3 left and 2 down.

Solution

Write the points as column matrices.

$A\begin{bmatrix} 2 \\ 1 \end{bmatrix}, B\begin{bmatrix} 4 \\ 2 \end{bmatrix}, C\begin{bmatrix} 5 \\ 1 \end{bmatrix}, D\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Write the translation as a column matrix.

Translation $= \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

Add the translation to each column matrix.

$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = A'\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$B\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = B'\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$C\begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = C'\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$D\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = D'\begin{bmatrix} 0 \\ -3 \end{bmatrix}$

Write the answers.

The points become $A'(-1, -1)$, $B'(1, 0)$, $C'(2, -1)$, $D'(0, -3)$.

EXERCISE 11.01 Translations

Concepts and techniques

1 Example 1 Show each of the following translations as both a drawing in the plane and a transformation of column matrices.
  a The point $P(3, 4)$ is translated 3 down and 4 left.
  b The point $Q(-2, 1)$ is translated 4 right and 3 down.
  c The line segment $A(1, 3)$ $B(-1, -2)$ is translated 2 right and 1 down.
  d The line segment $F(-2, 1)$ $G(1, -2)$ is translated 3 right and 3 up.
  e The triangle $A(1, 2)$ $B(-2, -1)$ $C(2, 0)$ is translated 3 right and 2 down.

2 Example 2 Use matrices to find the points that each of the following are transformed to after a translation of 4 left and 3 up.
  a $A(-1, -5)$ b $B(2, 3)$ c $C(0, 0)$ d $D(1, -3)$ e $E(-3, 6)$

3 Use matrices to find the points to which each of the following are transformed to after a translation of 1 left and 2 down.
  a $A(3, 4)$ b $B(-5, -4)$ c $C(4, -3)$ d $D(6, -3)$ e $E(-4, 8)$

4 Use matrices to find the points that each of the following are transformed to after a translation of 5 right and 6 down.
  a $A(-4, 7)$ b $B(3, -3)$ c $C(-8, 2)$ d $D(-7, -5)$ e $E(-2, -4)$

5 The point $T(2, 4)$ is translated to $T'(-2, 3)$. Write the translation as a column matrix.

6 What translation changes the point $M(-3, 5)$ to $M'(4, 1)$?

7 What translation is necessary to reverse the translation $K(6, 3) \rightarrow K'(2, 9)$?

8 CAS Show a translation of the triangle $A(-2, -3)$ $B(1, 4)$ $C(3, 2)$ 3 left and 4 up.

9 CAS Show a translation of the pentagon $A(2, -1)$ $B(-2, 1)$ $C(0, 3)$ $D(2, 3)$ $D(3, 2)$ 3 down and 4 right.

Reasoning and communication

10 The translation $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is applied to the point $P(1, -2)$, followed by the translation $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
  a What are the coordinates of the final point?
  b What is the combined translation?
  c Generalise this to state the overall effect of the two translations $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$.

11 Consider the combination of two translations, $\mathbf{S} * \mathbf{T}$, as an operation that produces another transformation.
  a Given a translation $\mathbf{T} = \begin{bmatrix} a \\ b \end{bmatrix}$, can you find an *identity* translation $\mathbf{I}$ so that $\mathbf{T} * \mathbf{I} = \mathbf{T}$?
  b Given a translation $\mathbf{T} = \begin{bmatrix} a \\ b \end{bmatrix}$, can you find an *inverse* translation $\mathbf{T}_i$ so that $\mathbf{T} * \mathbf{T}_i$ leaves every point in the same place by undoing the first translation?

12 Show that the order in which you combine two translations does not matter.

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11.02 LINEAR TRANSFORMATIONS

There are many ways in which points and shapes can be transformed. For example, the transformation of points given by $P(x, y) \to P'(x^2, y^3)$ changes every point in the plane to a point that either remains on the y-axis or is on the right of the y-axis. It also changes shapes in different ways in the vertical and horizontal directions.

The transformation $P(x, y) \to P'(xy, x^2 + y^{-1})$ is of 'mind-boggling' complexity, twisting and folding shapes in weird ways.

A **linear transformation** is an important type of transformation. A linear transformation involves only linear functions of x and y and leaves straight lines as straight lines (or shrinks them to a single point), although it may change their position and direction. In computer graphics, linear transformations are used to map 3D objects to 2D images.

A transformation may change a shape into a different one. For example, a sideways stretch would change a square into a rectangle. However, some properties may not change.

IMPORTANT

A **linear transformation of the plane** changes the point $P(x, y)$ to $P'(ax + by, cx + dy)$, where a, b, c and d are constants (real numbers).

Example 3

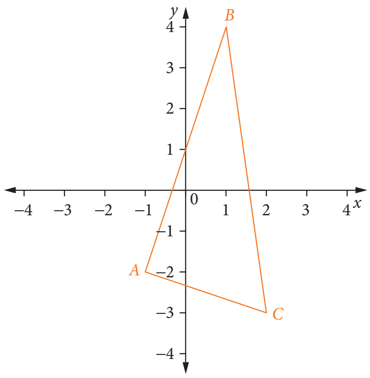
a Show that the triangle $A(-1, -2)$ $B(1, 4)$ $C(2, -3)$ is right-angled.
The triangle is transformed by the linear transformation $(x, y) \to (2x - y, 2x + y)$.

b Find the coordinates of the image $A'B'C'$.

c What is the shape of $A'B'C'$?

Solution

a Draw a diagram.
It looks as if $\angle CAB$ could be a right angle.



Find the vector for the side CA.

$\mathbf{CA} = (-1 - 2, -2 - (-3))$
$= (-3, 1)$

Find the vector for the side AB.

$\mathbf{AB} = (1 - (-1), 4 - (-2))$
$= (2, 6)$

Find $\mathbf{CA} \cdot \mathbf{AB}$.

$(-3, 1) \cdot (2, 6) = -3 \times 2 + 1 \times 6$
$= 0$

What does this mean?

Since $\mathbf{CA} \cdot \mathbf{AB} = 0$, $\mathbf{CA}$ and $\mathbf{AB}$ are perpendicular.

State the result.

ABC is a right-angled triangle.

b Apply $(x, y) \to (2x - y, 2x + y)$ to A.

$A(-1, -2) \to A'(2 \times (-1) - (-2), 2 \times (-1) + (-2))$

Simplify.

$A(-1, -2) \to A'(0, -4)$

Apply $(x, y) \to (2x - y, 2x + y)$ to B.

$B(1, 4) \to B'(2 \times 1 - 4, 2 \times 1 + 4)$

Simplify.

$B(1, 4) \to B'(-2, 6)$

Apply $(x, y) \to (2x - y, 2x + y)$ to C.

$C(2, -3) \to C'(2 \times 2 - (-3), 2 \times 2 + (-3))$

Simplify.

$C(2, -3) \to C'(7, 1)$

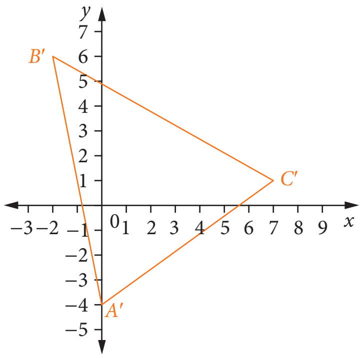
Write the result.

The points are $A'(0, -4)$, $B'(-2, 6)$, $C'(7, 1)$.

c Draw a diagram.

The shape *appears* to be a triangle that is *not* right-angled.

You need to check this.



Find the vector for the side $A'B'$.

$\mathbf{A'B'} = (-2, 6) - (0, -4)$
$= (-2, 10)$

Find the vector for the side $B'C'$.

$\mathbf{B'C'} = (7, 1) - (-2, 6)$
$= (9, -5)$

Find the vector for the side $C'A'$.

$\mathbf{C'A'} = (0, -4) - (7, 1)$
$= (-7, -5)$

Find their magnitudes.

$|\mathbf{A'B'}| = \sqrt{104}$,
$|\mathbf{B'C'}| = \sqrt{106}$,
$|\mathbf{C'A'}| = \sqrt{74}$

Check for a right-angle.

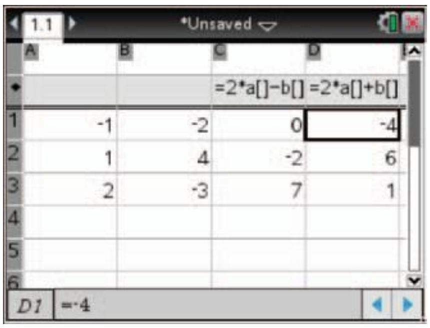
$|\mathbf{B'C'}|^2 = 106 \neq 104 + 74 = |\mathbf{A'B'}|^2 + |\mathbf{C'A'}|^2$

Write your conclusion.

The sides of $\triangle A'B'C'$ are all different and do not make a Pythagorean triple, so it is a scalene triangle.

TI-Nspire CAS

You can work out part **b** using your CAS calculator. Use the Lists & Spreadsheet page and enter the points into columns A and B. Then put =2a-b into column C, choosing column references.
Then put 2a+b into column D.



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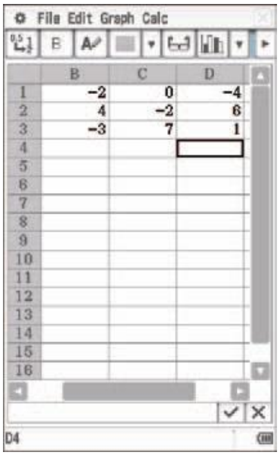
ClassPad

Use the **Spreadsheet** application.
Enter the points into columns A and B. Use one row per point. Enter the x coordinate of a point into column A and the y coordinate into column B.
Tap cell C1, press $\boxed{=}$ 2 $\boxed{\times}$, tap cell A1, press $\boxed{-}$ and tap cell B1 to enter the formula C1=2A1–B1. Press $\boxed{\text{EXE}}$.
Tap C1, **Edit** and **Copy**, then tap C2 and slide down to C3. Tap **Edit** and **Paste** to copy the formula. Follow a similar process to enter the formula D1=2A1+B1 in the D column.



Part **a** of Example **3** could also have been done using gradients or Pythagoras' theorem. It shows that a linear transformation will not always change a geometric figure to an image that is exactly the same shape. However, the triangle *was* changed to another triangle. An obvious question is 'Does a linear transformation change a polygon to another polygon with the same number of sides?'

INVESTIGATION Linear transformations of polygons

In this investigation you will look at some transformations to determine whether polygons are always transformed to polygons with the same number of sides.

1 Consider the transformation $(x, y) \to (x + 2, \sqrt{y^2 - 4})$ and the hexagon $A(3, 2)$ $B(5, 5)$ $C(7, 2)$ $D(7, -2)$ $E(5, -3)$ $F(3, -2)$. Find the shape of the image $A'B'C'D'E'$ under the transformation. Does this show that linear transformations do not always produce an image of the same kind of polygon?

2 Consider the transformation $(x, y) \to (x, x)$. Find the shape of the rectangle $A(3, 2)$ $B(6, 6)$ $C(14, 0)$ $D(11, -4)$ under the transformation. Does this show that linear transformations do not always produce an image of the same kind of polygon?

3 What about the transformation $(x, y) \to (0, 0)$? It obviously shrinks everything to a single point, but is it linear?

4 Can you make up linear transformations that do not transform polygons to the same kind of polygon?

The process of working out the new points under a linear transformation is quite tedious. Consider the transformation in Example **3**, $(x, y) \to (2x - y, 2x + y)$. This can be written in the form of column vectors as $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x - y \\ 2x + y \end{bmatrix}$, treating the points (x, y) as position vectors. You can write this transformation as multiplication by a 2×2 matrix: $\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ 2x + y \end{bmatrix}$.

Remember, matrix multiplication is *row* $\times$ *column*! The same method can be used to write *any* linear transformation $(x, y) \to (ax + by, cx + dy)$ as matrix multiplication.

IMPORTANT

The linear transformation that changes the point $P(x, y) \rightarrow P'(ax + by, cx + dy)$ is modelled by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. The matrix is multiplied by the column vector representing the point $P(x, y)$ to give the column vector representing $P'(x', y')$.

As they can be represented as matrices, we usually use a single capital letter to denote linear transformations, such as $\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Example 4

a Express the linear transformation $\mathbf{T}$: $(x, y) \rightarrow (3x + 2y, 4x - y)$ as a matrix.

b Use the matrix to find the image of the isosceles triangle $A(2, 4)$ $B(3, 7)$ $C(4, 4)$ under the transformation $\mathbf{T}$.

Solution

a Write the rule.

$(x, y) \rightarrow (ax + by, cx + dy)$ is modelled as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Substitute the values.

$(x, y) \rightarrow (3x + 2y, 4x - y)$ is $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$.

Write the answer.

$$\mathbf{T} = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$$

b Apply to **a**.

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$$

Apply to **b**.

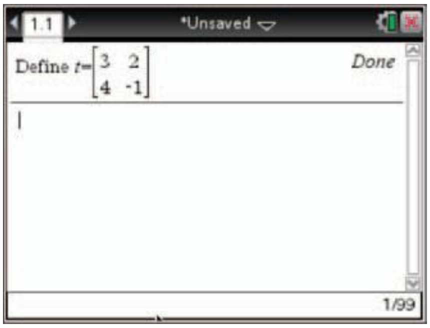
$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 23 \\ 5 \end{bmatrix}$$

Apply to **c**.

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$$

TI-Nspire CAS

You can work this out on your CAS calculator by typing Define (or getting it from the or menu) and then doing menu enter 7: Matrices and Vectors, 1: Create. All variables on the TI-Nspire are changed to lower case.



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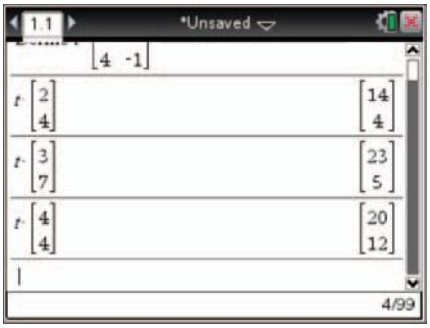
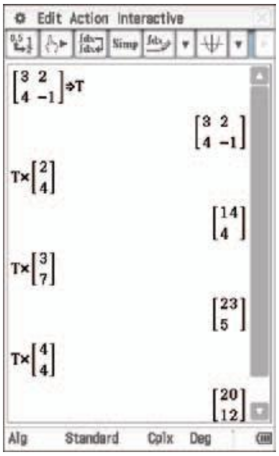
Multiply the matrix by each position vector in succession. Enter each position vector using menu 7: Matrices and Vectors, 1: Create, with 2 rows and 1 column.



ClassPad

Use the Main $\sqrt{\alpha}$ application.
The matrix and vector templates can be found by pressing Keyboard then tapping Math2.
Define the matrix **T** and then do the matrix multiplications.
You must include the multiplication symbol.



Write the answer. $ABC \rightarrow A'(14, 4)\ B'(23, 5)\ C'(20, 12)$

What is the shape of $A'B'C'$ under the transformation **T** in Example **4**?

EXERCISE 11.02 Linear transformations

WS Linear transformations

Concepts and techniques

1 Example 3 **a** Show that the triangle $A(-1, -2)$ $B(-2, 1)$ $C(2, -3)$ is isosceles.
A transformation is given by $(x, y) \rightarrow (-2x - 2y, 3x + y)$.
b Find the coordinates of the image $A'B'C'$ under the transformation.
c What is the shape of $A'B'C'$?

2 **a** Show that $A(4, -3)$ $B(6, 2)$ $C(1, 4)$ $D(-1, -1)$ is a square.
A transformation is given by $(x, y) \rightarrow (2x - 3y, 3y - x)$.
b Find the coordinates of the image $A'B'C'D'$.
c What is the shape of $A'B'C'D'$?

3 **a** Show that $A(4, -3)$ $B(3, -1)$ $C(12, 6)$ $D(13, 4)$ is a parallelogram.
A transformation is given by $(x, y) \rightarrow (3x - y, 4y - 2x)$.
b Find the coordinates of the image $A'B'C'D'$.
c What is the shape of $A'B'C'D'$?

4 Example 4 **a** Express the linear transformation $\mathbf{T}$: $(x, y) \rightarrow (2x - 4y, 4x + 3y)$ as a matrix.
b Use the matrix to find the image of the triangle $A(-1, 3)$ $B(2, 5)$ $C(3, -3)$ under the transformation **T**.

5 a Express the linear transformation $\mathbf{T}: (x, y) \rightarrow (x + y, -x - y)$ as a matrix.
  b Use the matrix to find the image of the triangle $A(1, 2)$ $B(4, 4)$ $C(5, -4)$ under the transformation $\mathbf{T}$.

6 a CAS Express the linear transformation $\mathbf{T}: (x, y) \rightarrow (x - 2y, 2x - y)$ as a matrix.
  b Use the matrix to find the image of the triangle $A(3, 5)$ $B(4, -2)$ $C(1, 1)$ under the transformation $\mathbf{T}$.

7 a CAS Express the linear transformation $\mathbf{T}: (x, y) \rightarrow (3y - 2x, 4x + 3y)$ as a matrix.
  b Use the matrix to find the image of the parallelogram $A(-2, -3)$ $B(-1, -1)$ $C(-3, 2)$ $D(-4, 0)$ under the transformation $\mathbf{T}$.

8 a CAS Express the linear transformation $\mathbf{T}: (x, y) \rightarrow (5x - 2y, 3x + 4y)$ as a matrix.
  b Use the matrix to find the image of $A(1, 5)$ $B(-2, 3)$ $C(5, -4)$ $D(7, 6)$ under the transformation $\mathbf{T}$.

Reasoning and communication

9 $A(2, 3)$ $B(4, 1)$ $C(5, -3)$ $D(3, -1)$ is a parallelogram.
  a Find the images of the parallelogram under each of the transformations $\mathbf{T}_1 = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{T}_2 = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$.
  b Find the shape of each image under each transformation.

10 $A(2, 3)$ $B(3, 6)$ $C(5, 4)$ $D(4, 1)$ is a parallelogram. It is transformed by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
  a Find the image under the transformation.
  b Show that the image is also a parallelogram.

11.03 DILATIONS

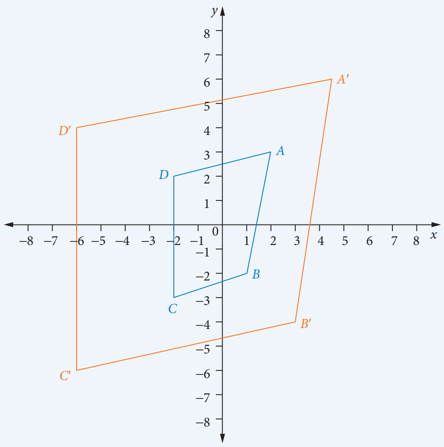
A simple **dilation** stretches or compresses the plane in the x direction and/or the y direction. You have seen the effect of compression or stretching of functions on their graphs. Dilation does the same thing to geometric shapes.

INVESTIGATION Dilations

The dilation below magnifies the quadrilateral by a factor of 3 in the x direction and a factor of 2 in the y direction. The blue one is the original figure and the orange one is the magnified figure.

- What are the coordinates of A and A'?
- What are the coordinates of B and B'?
- How are the coordinates of the points for dilation by a factor of 3 related?
- Complete the transformation for dilation by a factor of 3 below.

  $(x, y) \rightarrow$ (____________, ____________)



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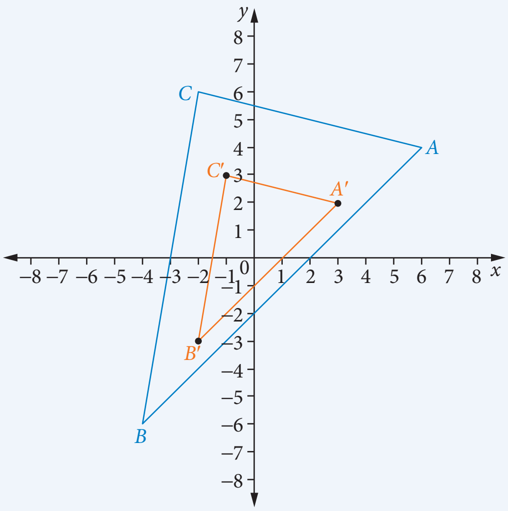
The dilation below reduces the triangle by a factor of 2 in both the x and y directions. The blue triangle is the original figure and the orange triangle is the reduced figure.



- What are the coordinates of A and A'?
- What are the coordinates of B and B'?
- How are the coordinates of the points for dilation by a factor of $\frac{1}{2}$ related?
- Complete the transformation for dilation by a factor of $\frac{1}{2}$ below.

  $(x, y) \rightarrow$ (____________, ____________)
- What is the transformation for dilation by a factor of λ?

  $(x, y) \rightarrow$ (____________, ____________)

From the investigation, the following should be clear.

IMPORTANT

The transformation for **dilation** by a factor of λ_1 in the x direction and λ_2 in the y direction is given by $(x, y) \rightarrow (\lambda_1 x, \lambda_2 y)$.

For $\lambda_1, \lambda_2 > 1$, the dilation is a magnification that makes shapes larger. For $\lambda_1, \lambda_2 < 1$ the dilation is a reduction that makes shapes smaller.

Example 5

What is the transformation for dilation by a factor of 5 in the x direction and 3 in the y direction?

Solution

Write the general dilation.	$(x, y) \rightarrow (\lambda_1 x, \lambda_2 y)$
Substitute the factors $\lambda_1 = 5$ and $\lambda_2 = 3$.	$(x, y) \rightarrow (5x, 3y)$

Example 6

The parallelogram $A(2, -1)$ $B(4, 1)$ $C(1, 3)$ $D(-1, 1)$ is dilated by a factor of 2.5 in the x direction and 1.5 in the y direction.

a Find the image under the transformation.

b Show the object and image on common Cartesian axes.

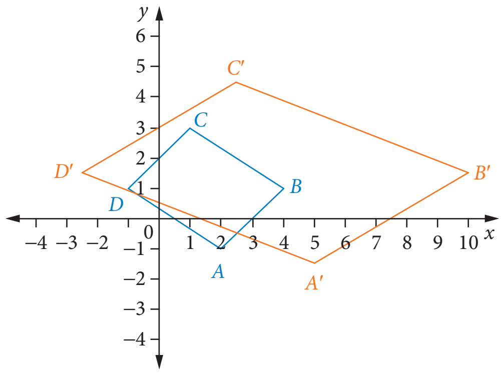
c What is the relationship between the shapes of the object and image?

Solution

a Apply the transformation $(x, y) \rightarrow (2.5x, 1.5y)$ to each point.

$A(2, -1) \rightarrow A'(5, -1.5)$
$B(4, 1) \rightarrow B'(10, 1.5)$
$C(1, 3) \rightarrow C'(2.5, 4.5)$
$D(-1, 1) \rightarrow D'(-2.5, 1.5)$

b Draw the object in blue and its image in orange.



c The object and image are both parallelograms, but the image is larger and distorted.

The object and image are the same kind of shape, but the image is larger and stretched more in the x direction.

A dilation by a factor of λ_1 and λ_2 in the x and y directions respectively can be written as $(x, y) \rightarrow (\lambda_1 x + 0y, 0x + \lambda_2 y)$ so it is a linear transformation. You can use the relationship between linear transformations and matrices to write the matrix for a dilation.

The matrix for the linear transformation

$(x, y) \rightarrow (ax + by, cx + dy)$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Substituting $a = \lambda_1$, $b = c = 0$ and $d = \lambda_2$ gives the following.

IMPORTANT

Dilation by factors of λ_1 and λ_2 in the x and y directions respectively is given by the matrix

$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

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Example 7

a Write the matrix for a dilation by a factor of 0.6 in the x direction and 0.8 in the y direction.
b Use the matrix to find the image of the square $A(2, 3)$ $B(5, 8)$ $C(0, 11)$ $D(-3, 6)$.
c What is the shape of the image $A'B'C'D'$?

Solution

a Write the general matrix. $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Substitute $\lambda_1 = 0.6$ and $\lambda_2 = 0.8$. $\mathbf{D} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}$

b Apply the matrix to the position vector **a**. $\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 2.4 \end{bmatrix}$

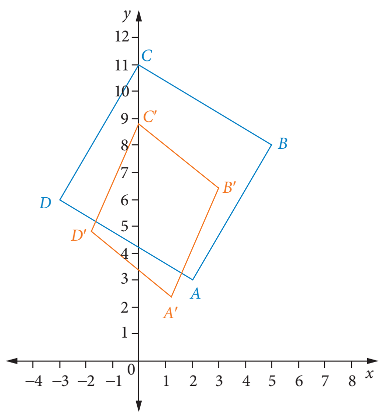
Apply the matrix to the position vector **b**. $\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6.4 \end{bmatrix}$

Apply the matrix to the position vector **c**. $\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 8.8 \end{bmatrix}$

Apply the matrix to the position vector **d**. $\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1.8 \\ 4.8 \end{bmatrix}$

Write the solution. The image is $A'(1.2, 2.4)$ $B'(3, 6.4)$ $C'(0, 8.8)$ $D'(-1.8, 4.8)$.

c Sketch the graph.



The shape *appears* to be a rhombus. It is sufficient to prove that the diagonals bisect each other at right angles.
Find the midpoints of the diagonals and the slopes of the diagonals.

Midpoint of $A'C' = \left(\dfrac{1.2+0}{2}, \dfrac{2.4+8.8}{2}\right) = (0.6, 5.6)$

Midpoint of $B'D' = \left(\dfrac{3+(-1.8)}{2}, \dfrac{6.4+4.8}{2}\right) = (0.6, 5.6)$

Slope of $A'C' = \dfrac{8.8-2.4}{0-1.2} = -\dfrac{16}{3}$

Slope of $B'D' = \dfrac{4.8-6.4}{(-1.8)-3} = \dfrac{1}{3}$

Use the midpoints.

The midpoints are the same so the diagonals bisect each other.

Use the slopes.

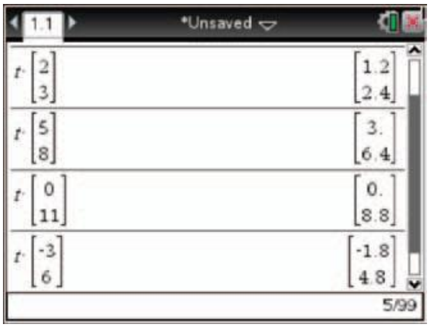
$m_1 m_2 = -\frac{16}{3} \times \frac{1}{3} = -\frac{16}{9} \neq -1$, so the diagonals are not perpendicular.

Make your conclusion.

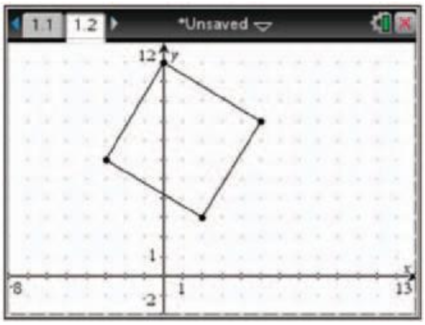
The diagonals bisect each other, so the image $A'B'C'D'$ is a parallelogram but not a rhombus or square.

You can do this on your CAS calculator in two ways. You can define the matrix and apply it in the same way as for general linear transformations (see page s 424–425).

TI-Nspire CAS

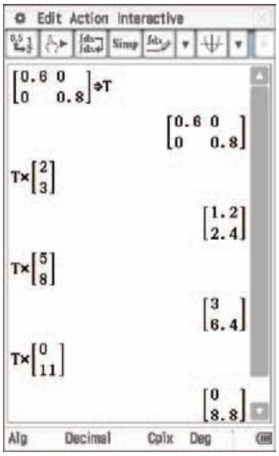


You can also add a Graph page and draw the shapes. Draw the square using 4: Polygon from the Shapes menu (see page 417) with Window settings of $-8 \le x \le 13$ and $-2 \le y \le 12$ for correct proportion.



ClassPad

Ensure that your calculator is set to **Decimal**.



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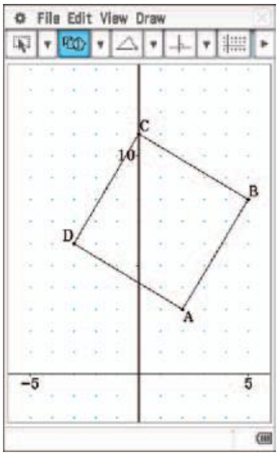
You can also use the Geometry application to draw the shapes.
Set View Window so $-6 \le x \le 6$ and ymid = 6
($y = 6$ marks the middle of the screen.)
The easiest way to draw it is to tap the use the polygon tool ( ).



Notice that unless you carefully set the scales, the graph produced by the TI-Nspire is distorted so it is not a good guide to the real shape.

EXERCISE 11.03 Dilations

Concepts and techniques

1 Example 5 Write the transformations for each of the following dilations.
- a A dilation by a factor of 4 in the x direction and 3 in the y direction.
- b A dilation by a factor of 1.2 in the x direction and 1.2 in the y direction.
- c Magnification by a factor of 3 in the x direction and reduction by a factor of 2 in the y direction.
- d Reduction by a factor of 4 in the x direction and magnification by a factor of 3 in the y direction.

2 Example 7 a Write the matrix for a dilation by a factor of 1.5 in the x direction and 2 in the y direction.
- b Use the matrix to find the image of the triangle $A(3, 4)$ $B(4, 8)$ $C(7, 6)$.
- c What is the shape of the image?

3 a Write the matrix for a dilation by a factor of 0.4 in the x direction and 0.8 in the y direction.
- b Use the matrix to find the image of the rectangle $A(-10, -8)$ $B(-5, 4)$ $C(19, -6)$ $D(14, -18)$.
- c What is the shape of the image?

4 a CAS Write the matrix for a dilation by a factor of 0.7 in the x direction and 0.9 in the y direction.
- b Use the matrix to find the image of the square $A(-10, -8)$ $B(-4, 2)$ $C(6, -4)$ $D(0, -14)$.
- c What is the shape of the image?

5 a CAS Write the matrix for a dilation by a factor of 3 in the x direction and 2 in the y direction.
- b Use the matrix to find the image of the pentagon $A(-2, 1)$ $B(0, 4)$ $C(3, 4)$ $D(3, -1)$, $E(1, -2)$.
- c What is the shape of the image?

Reasoning and communication

6 Example 6 The triangle $A(-3, -6)$ $B(3, 3)$ $C(6, -3)$ is reduced by a factor of 3 in the x direction and 2 in the y direction.
   a Find the image under the transformation.
   b Show the object and image on the same Cartesian axes.
   c What is the relationship between the object and image?

7 The square $A(-3, -5)$ $B(2, 7)$ $C(14, 2)$ $D(9, -10)$ is magnified by a factor of 2 in the x direction and 2.5 in the y direction.
   a Find the image under the transformation.
   b Show the object and image on the same Cartesian axes.
   c What is the relationship between the object and image?

8 The parallelogram $A(-4, 4)$ $B(-2, 8)$ $C(4, 0)$ $D(2, -4)$ is dilated by a factor of 1.5 in both directions.
   a Find the image under the transformation.
   b Show the object and image on the same Cartesian axes.
   c What is the relationship between the object and image?

11.04 ROTATIONS

A simple **rotation** turns all points through the same angle around the origin. The angle may be expressed in either radians or degrees. You need to use trigonometry to find the positions of the points that are rotated. Remember that angles are measured *anticlockwise* from the x-axis, so a rotation of 30° means an anticlockwise rotation of 30° and a rotation of $-\frac{5\pi}{4}$ means a rotation of $\frac{5\pi}{4}$ clockwise. Polar coordinates are easier than Cartesian coordinates to use with rotations. You studied polar coordinates in your vector work. The rules for changing between Cartesian and polar coordinates are given again below.

IMPORTANT

For a vector **v** with polar form (r, θ) and component form (x, y):

$x = r\cos(\theta)$ $y = r\sin(\theta)$

$\tan(\theta) = \frac{y}{x}$ $r^2 = x^2 + y^2$

and $|\mathbf{v}| = r = \sqrt{x^2 + y^2}$.

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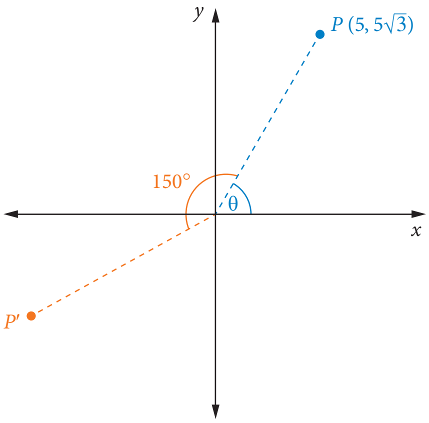
Example 8

Use trigonometry to find the image of the point $P(5, 5\sqrt{3})$ after a rotation of 150° around the origin.

Solution

Sketch a diagram.
Draw the point with an angle θ and the rotation through 150° to the new position P'.



Convert $P(5, 5\sqrt{3})$ to polar coordinates.

$r = \sqrt{5^2 + (5\sqrt{3})^2} = \sqrt{100} = 10$

$\tan(\theta) = \dfrac{5\sqrt{3}}{5} = \sqrt{3}$, so $\theta = 60°$

P is $P(10, 60°)$.

Add 150° to get the angle for P'.

P' is $P'(10, 210°)$.

Convert $P'(10, 210°)$ to Cartesian coordinates.

$x = r\cos(\theta) = 10\cos(210°) = -5\sqrt{3}$

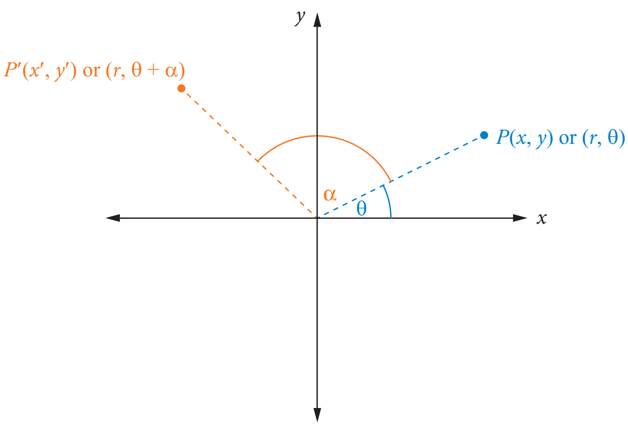
$y = r\sin(\theta) = 10\sin(210°) = -5$

Write the answer.

The image is $P'(-5\sqrt{3}, -5)$.

Calculation of each point as shown in Example **8** is time-consuming. You can use the same approach to find the general form of rotations about the origin.

Consider a rotation of the point (x, y) through an angle of α. You can draw a diagram similar to that shown in Example **8**.



The polar coordinates of P' are $(r, \theta + \alpha)$, so the Cartesian coordinates of P' are given by

$$x' = r \cos(\theta + \alpha) \text{ and } y' = r \sin(\theta + \alpha)$$

Using the identities

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

gives

$$x' = r[\cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha)]$$

and

$$y' = r[\sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha)].$$

Expanding, rearranging the products and using $a + b = b + a$ with y' gives

$$x' = [r\cos(\theta)]\cos(\alpha) - [r\sin(\theta)]\sin(\alpha)$$

and

$$y' = [r\cos(\theta)]\sin(\alpha) + [r\sin(\theta)]\cos(\alpha)$$

But the Cartesian coordinates of P are $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Substituting gives

$$x' = x\cos(\alpha) - y\sin(\alpha) \text{ and } y' = x\sin(\alpha) + y\cos(\alpha)$$

This gives the rule for rotation through an angle α around the origin as follows.

IMPORTANT

The transformation for **rotation** through an angle α around the origin is given by

$$(x, y) \rightarrow (x\cos(\alpha) - y\sin(\alpha), x\sin(\alpha) + y\cos(\alpha))$$

where α is the usual anticlockwise rotation. For a clockwise rotation, α is negative.

Example 9

Use the transformation to find the image of the triangle $A(2, 2)$ $B(5, 6)$ $C(6, -1)$ after rotation through an angle of 125°, correct to 2 decimal places.

Solution

Write the general rotation.	$(x, y) \rightarrow (x\cos(\alpha) - y\sin(\alpha), x\sin(\alpha) + y\cos(\alpha))$
Substitute $\alpha = 125°$.	$(x, y) \rightarrow (x\cos(125°) - y\sin(125°), x\sin(125°) + y\cos(125°))$
Apply to $A(2, 2)$.	$A' = (2\cos(125°) - 2\sin(125°), 2\sin(125°) + 2\cos(125°))$
Simplify and round.	$\approx (-2.79, 0.49)$
Apply to $B(5, 6)$.	$B' = (5\cos(125°) - 6\sin(125°), 5\sin(125°) + 6\cos(125°))$
Simplify and round.	$\approx (-7.78, 0.65)$
Apply to $C(6, -1)$.	$C' = (6\cos(125°) - (-1)\sin(125°), 6\sin(125°) + (-1)\cos(125°))$
Simplify and round.	$\approx (-2.62, 5.49)$
Write the solution.	The image is $A'(-2.79, 0.49)$ $B'(-7.78, 0.65)$ $C'(-2.62, 5.49)$.

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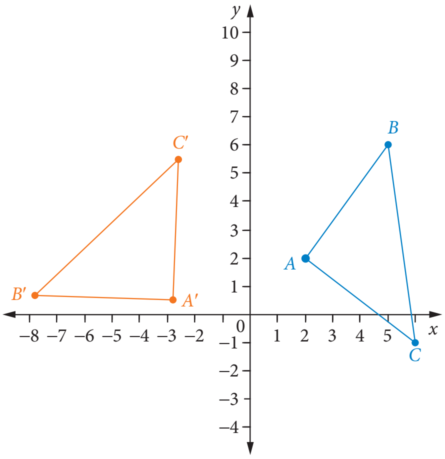
The diagram below shows the object and image from Example **9**. As you would expect, the rotation preserves the shape so that the image is **congruent** to the object.



The matrix for the linear transformation $(x, y) \to (ax + by, cx + dy)$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Substituting $a = d = \cos(\alpha)$, $b = -\sin(\alpha)$ and $c = \sin(\alpha)$ gives the following.

IMPORTANT

Rotation through an angle α around the origin is given by the matrix

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Unless otherwise directed, use exact values for rotations whenever possible.

Example 10

a What is the matrix for a rotation of $-\frac{7\pi}{6}$ around the origin?

b Use the matrix to find the image of the rectangle $A(-2, -3)$ $B(1, 1)$ $C(9, -5)$ $D(6, -9)$ after a rotation of $-\frac{7\pi}{6}$.

Solution

a Write the general formula.

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Substitute values.

$$\mathbf{R}_{-\frac{7\pi}{6}} = \begin{bmatrix} \cos\left(-\frac{7\pi}{6}\right) & -\sin\left(-\frac{7\pi}{6}\right) \\ \sin\left(-\frac{7\pi}{6}\right) & \cos\left(-\frac{7\pi}{6}\right) \end{bmatrix}$$

Simplify.

$$= \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

b Apply the matrix to A.

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{3+2\sqrt{3}}{2} \\ \frac{3\sqrt{3}-2}{2} \end{bmatrix}$$

Apply the matrix to B.

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix}$$

Apply the matrix to C.

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 9 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{5-9\sqrt{3}}{2} \\ \frac{9+5\sqrt{3}}{2} \end{bmatrix}$$

Apply the matrix to D.

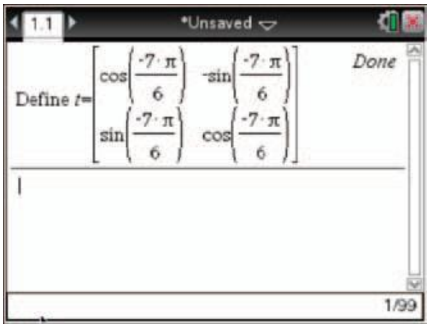
$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 6 \\ -9 \end{bmatrix} = \begin{bmatrix} \frac{9-6\sqrt{3}}{2} \\ \frac{6+9\sqrt{3}}{2} \end{bmatrix}$$

Write the solution.

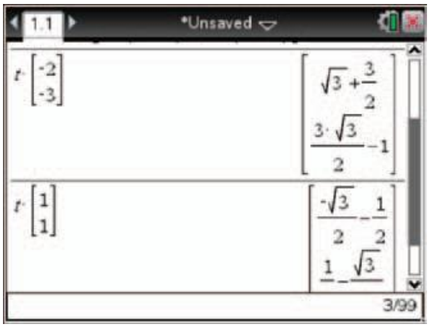
The image is $A'\left(\frac{3+2\sqrt{3}}{2}, \frac{3\sqrt{3}-2}{2}\right)$ $B'\left(-\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$ $C'\left(\frac{5-9\sqrt{3}}{2}, \frac{9+5\sqrt{3}}{2}\right)$ $D'\left(\frac{9-6\sqrt{3}}{2}, \frac{6+9\sqrt{3}}{2}\right)$.

TI-Nspire CAS

You can do this on your CAS calculator in the same way as for general linear transformations (see pages 424–425). Make sure that you set your calculator in radians for this problem.



Calculate each position vector in succession. The calculator separates the rational and irrational parts of the answers.



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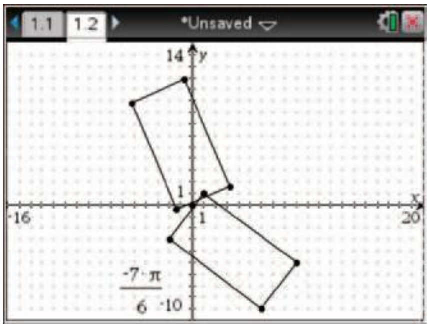
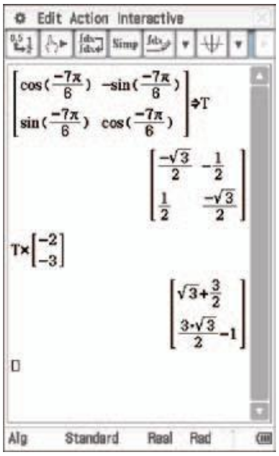
Also, you can add a Graph page and draw the rectangle (see page 417). Make the Window settings $-16 \le x \le 20$ and $-10 \le y \le 20$.
Then use menu, 8: Geometry, 5: Transformation and 4: Rotation to rotate the rectangle by clicking on the origin, clicking on the rectangle and entering the angle.



ClassPad

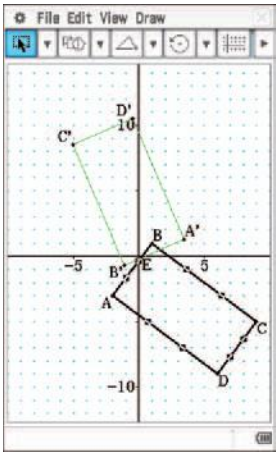
You can define the matrix and calculate the image points as before (see pages 424–425). The multiplication sign is essential.
Only the first point has been calculated in the screen on the right.

The calculator should be set to **Standard** and **Rad**.



You can also use the **Geometry** application. Set the View Window to $-10 \le x \le 10$ and ymid to 1 and tap **OK**. Draw the rectangle (see page 418). Make sure the **Function Angle** and **Measure Angle** are set to radians by tapping and selecting **Geometry Format**.

Now tap the selection tool, select all sides and tap **Draw**, **Construct** and **Rotation**. Tap on the origin as the centre of rotation and enter the angle.



EXERCISE 11.04 Rotations

Concepts and techniques

Rotations

1 Example 8 Use trigonometry to find the image of $P(6, 6)$ after a rotation of 210° around the origin.

2 Use trigonometry to find the image of $P(-2\sqrt{3}, 2)$ after a rotation of $\frac{2\pi}{3}$ around the origin.

3 Example 9 Find the image of the quadrilateral $A(2, 2)$ $B(5, 6)$ $C(6, -1)$ $D(-2, -2)$ after a rotation through an angle of 208°, correct to 2 decimal places.

4 Find the image of the kite $A(3, 7)$ $B(8, 6)$ $C(7, 1)$ $D(-4, -2)$ after a rotation through an angle of 208°, correct to 2 decimal places.

5 Example 10 a What is the matrix for a rotation of 323° around the origin, correct to 2 decimal places?

b Use the matrix to find the image of the triangle $A(4, 5)$ $B(6, 8)$ $C(7, -1)$ after a rotation of 323°, correct to 2 decimal places.

6 a CAS What is the matrix for a rotation of 137° around the origin, correct to 2 decimal places?

b Use the matrix to find the image of the pentagon $A(-4, 2)$ $B(-2, 6)$ $C(4, 8)$ $D(5, 6)$ $E(2, 6)$ after a rotation of 137°, correct to 2 decimal places.

7 a CAS What is the matrix for a rotation of $\frac{3\pi}{4}$ around the origin?

b Use the matrix to find the image of the parallelogram $A(-2, -2)$ $B(1, 3)$ $C(9, 3)$ $D(6, -2)$ after a rotation of $\frac{3\pi}{4}$.

Reasoning and communication

8 a What is the matrix for a rotation of 60°?

b What rotation would restore a figure transformed by the rotation in part **a** to its original position?

c What is the matrix for the restoring rotation?

9 a What is the matrix for a rotation of $\frac{7\pi}{4}$?

b What rotation would restore a figure transformed by the rotation in part **a**?

c What is the matrix for the restoring rotation?

10 The matrix for a rotation is $\mathbf{R}_\theta = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

a What is the angle of rotation for the transformation?

b What is the matrix for the rotation that would restore a figure transformed by this matrix to its original position?

11.05 REFLECTIONS

A simple **reflection** swaps all points across to the opposite side of a line through the origin. The line may be expressed by its inclination angle, gradient or equation. You can use your knowledge of coordinate geometry to find the positions of points that are reflected. Remember that the equation of a straight line with gradient m and y-intercept c is given by $y = mx + c$. The gradient of a line with inclination θ is given by $m = \tan(\theta)$.

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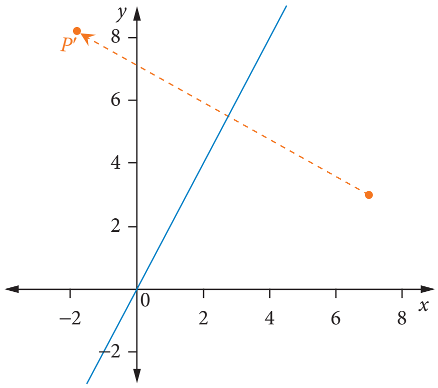
Example 11

Find the image of the point $P(7, 3)$ when it is reflected across the line $y = 2x$.

Solution

Sketch a diagram.
Point P' will be the same distance as P from the line, but on the other side.
The line PP' is perpendicular to the line $y = 2x$ and passes through point P.



Find the slope of PP'. — The slope of $y = 2x$ is 2, so PP' has slope $-\frac{1}{2}$.

Find the equation of PP' using $y - y_1 = m(x - x_1)$ — $y - 3 = -\frac{1}{2}(x - 7)$

$x + 2y - 13 = 0$

Substitute $y = 2x$ to find the intersection. — $x + 2 \times 2x - 13 = 0$

Find x. — $x = 2.6$

Substitute in $y = 2x$ to find y. — $y = 2 \times 2.6 = 5.2$

State the intersection. — $y = 2x$ and PP' intersect at $M(2.6, 5.2)$.

State the relationship of P, M and P'. — M is the midpoint of PP'.

Use the midpoint formula, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. — $M(2.6, 5.2) = \left(\frac{7+x_2}{2}, \frac{3+y_2}{2}\right)$, where (x_2, y_2) are the coordinates of P'.

Find x_2.

$$\frac{7+x_2}{2} = 2.6$$
$$x_2 + 7 = 5.2$$
$$x_2 = -1.8$$

Find y_2.

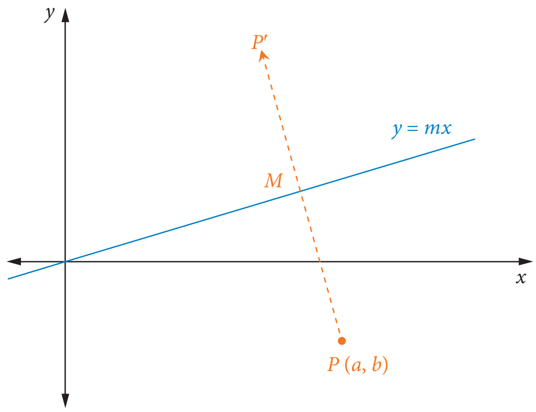
$$\frac{3+y_2}{2} = 5.2$$
$$y_2 + 3 = 10.4$$
$$y_2 = 7.4$$

Write the answer. — The image is $P'(-1.8, 7.4)$.

Repeating the procedure in Example **11** for many points to find the reflection of a geometric shape would take a very, very long time. You can use the same procedure as shown in Example **11** to find the general transformation for reflection in a line through the origin.

Consider the reflection of a point across the line $y = mx$.

Sketch a diagram similar to that in the example.



The slope of the line of reflection is m so the slope of PP' is $-\frac{1}{m}$.

The equation of PP' is given by $y - b = -\frac{1}{m}(x - a)$. Substituting $y = 2x$ gives

$2x - b = -\frac{1}{m}(x - a)$ giving $x = \frac{a+mb}{1+m^2}$.

Substituting in $y = mx$ gives the coordinates of M as $\left(\frac{a+mb}{1+m^2}, \frac{ma+m^2b}{1+m^2}\right)$.

M is the midpoint of $P(a, b)$ and $P'(c, d)$, so $\left(\frac{a+mb}{1+m^2}, \frac{ma+m^2b}{1+m^2}\right) = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

To find c, you solve $\frac{a+mb}{1+m^2} = \frac{a+c}{2}$ to get $c = \left(\frac{1-m^2}{1+m^2}\right)a + \left(\frac{2m}{1+m^2}\right)b$.

To find d, you solve $\frac{ma+m^2b}{1+m^2} = \frac{b+d}{2}$ to get $d = \left(\frac{2m}{1+m^2}\right)a - \left(\frac{1-m^2}{1+m^2}\right)b$

Replacing (a, b) by (x, y) and (c, d) by (x', y') gives

$x' = px + qy$ and $y' = qx - py$ where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$.

If you know the inclination of the line, you can use that directly to find the values of p and q by substituting $m = \tan(\theta)$ in the equations for p and q.

$$p = \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)} = \frac{1-\frac{\sin^2(\theta)}{\cos^2(\theta)}}{1+\frac{\sin^2(\theta)}{\cos^2(\theta)}} = \frac{\cos^2(\theta)-\sin^2(\theta)}{\cos^2(\theta)+\sin^2(\theta)} = \cos(2\theta)$$

and

$$q = \frac{2\tan(\theta)}{1+\tan^2(\theta)} = \frac{\frac{2\sin(\theta)}{\cos(\theta)}}{1+\frac{\sin^2(\theta)}{\cos^2(\theta)}} = \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)+\sin^2(\theta)} = \sin(2\theta)$$

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You can substitute in the general linear transformation matrix to get the matrix for reflection.

IMPORTANT

The transformation for **reflection** across a line with slope m through the origin is given by

$$(x, y) \rightarrow \left(\left(\frac{1-m^2}{1+m^2}\right)x + \left(\frac{2m}{1+m^2}\right)y, \left(\frac{2m}{1+m^2}\right)x - \left(\frac{1-m^2}{1+m^2}\right)y\right)$$

or $(x, y) \rightarrow (px + qy, qx - py)$, where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$.

The transformation for **reflection** across a line of inclination θ through the origin is given by

$$(x, y) \rightarrow (\cos(2\theta)x + \sin(2\theta)y, \sin(2\theta)x - \cos(2\theta)y)$$

The matrix for **reflection** through a line with a slope of m or inclination $\theta = \tan^{-1}(m)$ through the origin is given by the matrix

$$\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix} \text{ or } \mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$

You should use the first method if you are given the slope of the line of reflection and the second if you are given the angle of inclination.

Example 12

Use the transformations given below to find the image of the triangle $A(-5, -5)$ $B(5, 0)$ $C(0, -10)$

a after reflection across a line through the origin with slope -0.5

b after reflection across a line through the origin with inclination $\frac{\pi}{6}$ to the x-axis.

Solution

a Find the value of p.

$$p = \frac{1-m^2}{1+m^2} = \frac{1-0.25}{1+0.25} = 0.6$$

Find the value of q.

$$q = \frac{2m}{1+m^2} = \frac{2\times(-0.5)}{1+0.25} = -0.8$$

Write the matrix for the reflection.

$$\mathbf{M} = \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

Apply the transformation to ABC using $\begin{bmatrix} -5 & 5 & 0 \\ -5 & 0 & -10 \end{bmatrix}$ to represent A, B and C.

$$\begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}\begin{bmatrix} -5 & 5 & 0 \\ -5 & 0 & -10 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 3 & 8 \\ 7 & -4 & 6 \end{bmatrix}$$

Write the answer.

The image is $A'(1, 7)$ $B'(3, -4)$ $C'(8, 6)$.

b Find cos (2θ). $\cos\left(2\times\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

Find sin (2θ). $\sin\left(2\times\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$

Write the matrix for **M**. $\mathbf{M}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$

Apply the transformation to ABC. $\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-5 & 5 & 0\\ -5 & 0 & -10\end{bmatrix}$

Work out the answer for $A'B'C'$. $=\begin{bmatrix}\frac{-5\left(1+\sqrt{3}\right)}{2} & \frac{5}{2} & -5\sqrt{3}\\ \frac{5\left(1-\sqrt{3}\right)}{2} & \frac{5\sqrt{3}}{2} & 5\end{bmatrix}$

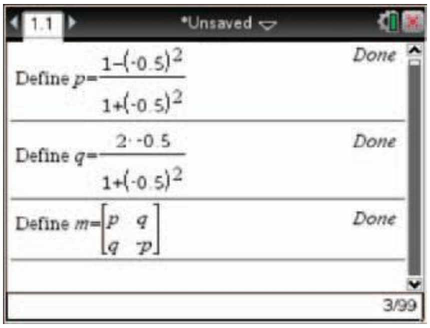
Write the answer. The image is $A'\left(\frac{-5\left(1+\sqrt{3}\right)}{2},\frac{5\left(1-\sqrt{3}\right)}{2}\right)$

$B'\left(\frac{5}{2},\frac{5\sqrt{3}}{2}\right)$ $C'\left(-5\sqrt{3},5\right)$.

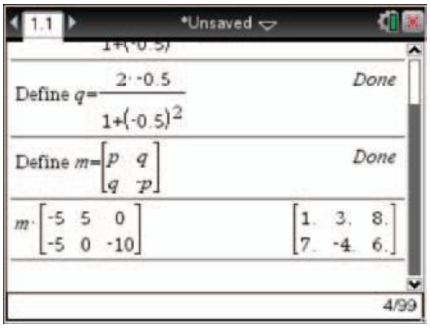
You can use your CAS calculator to perform the matrix calculation.

TI-Nspire CAS

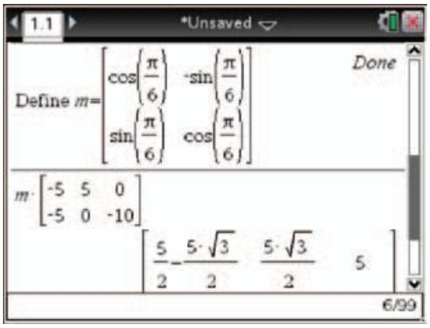
a Define the matrix first.



Then perform the calculations for the triangle ABC.

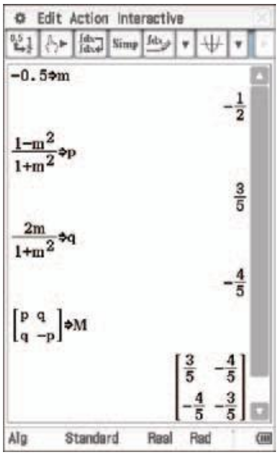


b In this case, make sure that your calculator is set on radians before defining the matrix.

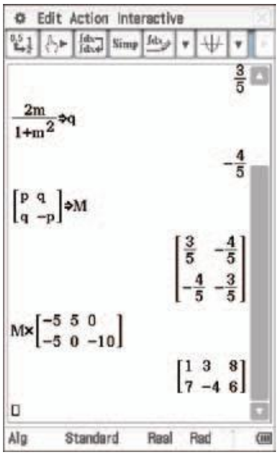


ClassPad

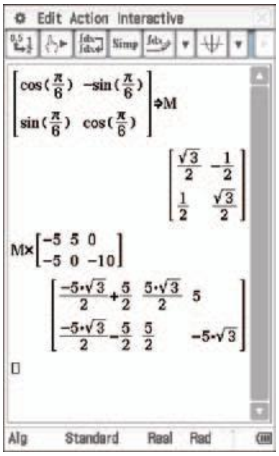
a Use the Main $\sqrt{\alpha}$ application.
Ensure that the calculator is set to Standard.
First define m, then define p and q, and finally define the matrix M.



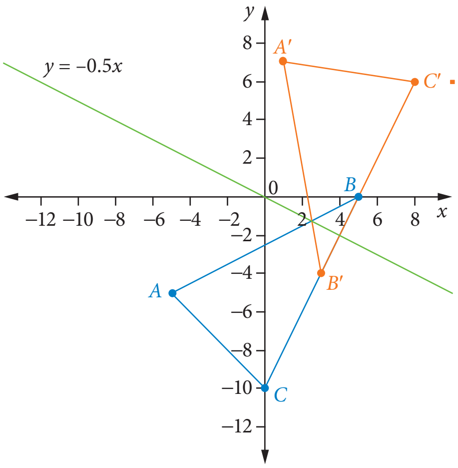
Perform the calculation for ΔABC.
Press Keyboard and tap Math2.
Use [matrix] to open a 2×2 matrix.
Widen it to 2×3 by tapping [column]. Enter the elements of the matrix. Each column corresponds to one point.



b In this case, make sure that your calculator is set to radians.



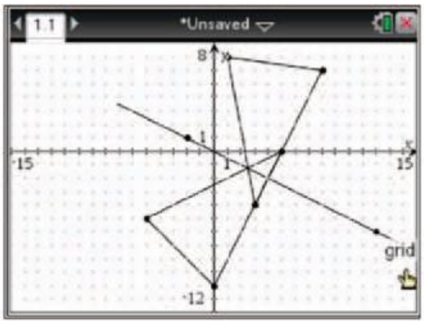
The reflection for Example **12** (part **a**) is shown below. Notice that the shape of the figure is preserved. The image $A'B'C'$ is **congruent** to the object ABC.



You can also use your CAS calculator to draw the triangles and reflect them without using a matrix, but you have to draw the line first.

TI-Nspire CAS

Use menu, 8: Geometry, 5: Transformation and 2: Reflection to reflect the triangle by clicking on it and clicking on the line.



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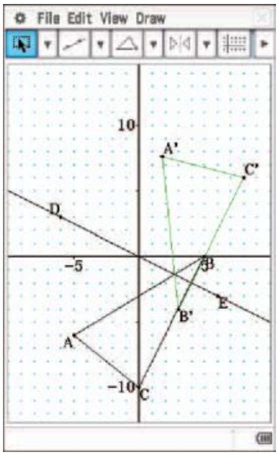
ClassPad

Select the **Geometry** application.

Draw the triangle ABC using the **Line Segment** command.

To draw the line of reflection, select the Line and on the screen tap at one point that is on the line y = –0.5x. Drag the stylus to a second point on the line to draw a line from D to E.

Tap and select each side of the triangle. Tap **Draw**, **Construct**, **Reflection**. Tap the line.



If you recognise the inclination angle from the slope, it is better to use the angle form of the reflection transformation because it is easier to apply.

EXERCISE 11.05 Reflections

Concepts and techniques

WS Reflections

1 Example 11 Use coordinate geometry to find the following.
  a The image of $P(5, 10)$ when it is reflected in the line $y = -2x$
  b The image of $P(3, 7)$ when it is reflected in the line $y = x$
  c The image of $P(-9, 3)$ when it is reflected in the line $y = 3x$
  d The image of $P(20, -15)$ when it is reflected in the line $y = \frac{x}{7}$

2 Example 12 Use a matrix to find the image of $A(-2, 1)$ $B(-1, 3)$ $C(-2, -2)$ after it is reflected across a line through the origin with slope 2.

3 Use a matrix to find the image of $A(5, 5)$ $B(-5, 0)$ $C(-1, 6)$ after it is reflected across a line through the origin with slope $-\frac{1}{3}$.

4 CAS Use a matrix to find the image of $A(-10, 10)$ $B(-5, -5)$ $C(5, -15)$ $D(10, -5)$ after it is reflected across a line through the origin with slope 7.

5 CAS Use a matrix to find the image of $A(4, 5)$ $B(7, 9)$ $C(11, 6)$ $D(7, 1)$ $E(5, 0)$ after it is reflected across a line through the origin with slope –3.

6 Use a matrix to find the image of $A(4, 5)$ $B(6, 9)$ $C(3, 7)$ after it is reflected across a line through the origin with inclination 30°, correct to 2 decimal places.

7 Use a matrix to find the image of $A(-4, 5)$ $B(5, 15)$ $C(0, 20)$ after it is reflected across a line through the origin with inclination $\frac{3\pi}{8}$, correct to 2 decimal places.

8 CAS Use a matrix to find the image of $A(2, -5)$ $B(7, -1)$ $C(15, -7)$ $D(12, -11)$ after it is reflected across a line through the origin with inclination 75°, correct to 2 decimal places.

9 CAS Use a matrix to find the image of $A(1, 5)$ $B(6, 9)$ $C(14, 3)$ $D(11, -1)$ $E(9, 4)$ after it is reflected across a line through the origin with inclination $-\frac{\pi}{15}$, correct to 2 decimal places.

Reasoning and communication

10 Use matrices or other methods to show that any shape is restored to its original position after a reflection in a line through the origin by the same reflection.

11.06 COMPOSITION OF TRANSFORMATIONS

In mathematics, the term **composition** means the application of two processes of the same type, one after the other. Composition is shown by the symbol $\circ$, so $\mathbf{S} \circ \mathbf{T}$ means 'the composition of **S** and **T**'. The order of application is from right to left, so **T** is performed first, then **S** is applied to the result of **T**.

IMPORTANT

The composition of the transformations **S** and **T** is the combined transformation such that if $\mathbf{T}: (x, y) \to (x', y')$ and $\mathbf{S}: (x', y') \to (x'', y'')$, then the composition is

$\mathbf{S} \circ \mathbf{T}: (x, y) \to (x', y') \to (x'', y'')$.

The image of a vector **v** under a transformation **T** is written as **Tv**.

Example 13

The transformation **S** is a translation 4 to the left and 5 up. **T** is a translation 3 to the right and 2 up.

a Write the translations in the form $(x, y) \to (x', y')$ and find the composition $\mathbf{S} \circ \mathbf{T}$.

b Write the translations as column vectors and find the composition $\mathbf{S} \circ \mathbf{T}$.

c Find the image of $A(2, 2)$ $B(6, 5)$ $C(9, 2)$ $D(7, -2)$ under $\mathbf{S} \circ \mathbf{T}$.

d Draw the object and image from part **c** on the same set of axes.

Solution

a **S** decreases x by 4 and increases y by 5.	$\mathbf{S}: (x, y) \to (x - 4, y + 5)$
T increases x by 3 and increases y by 2.	$\mathbf{T}: (x, y) \to (x + 3, y + 2)$
Apply **S** to the image from **T**.	$\mathbf{S}: ((x + 3), (y + 2)) \to ((x + 3) - 4, (y + 2) + 5)$
Simplify.	$((x + 3) - 4, (y + 2) + 5) = (x - 1, y + 7)$
Write the result for $\mathbf{S} \circ \mathbf{T}$.	$\mathbf{S} \circ \mathbf{T}: (x, y) \to (x - 1, y + 7)$
b Write **S** as a column matrix.	$\mathbf{S} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$
Write **T** as a column matrix.	$\mathbf{T} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

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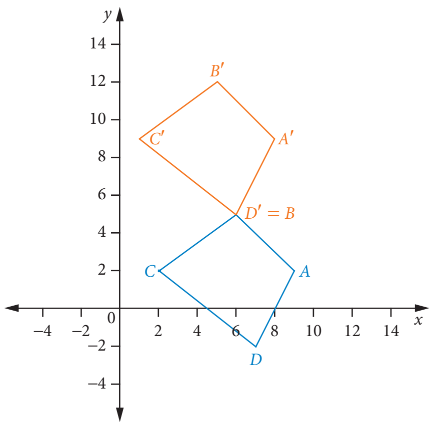
Find $\mathbf{Tv}$ for $\mathbf{v}=\begin{bmatrix}x\\y\end{bmatrix}$ by addition.	$\mathbf{Tv}=\mathbf{T}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3+x\\2+y\end{bmatrix}$
Now find $\mathbf{S}(\mathbf{Tv})$.	$\mathbf{S}(\mathbf{Tv})=\mathbf{S}\begin{bmatrix}3+x\\2+y\end{bmatrix}=\begin{bmatrix}-4\\5\end{bmatrix}+\begin{bmatrix}3+x\\2+y\end{bmatrix}=\begin{bmatrix}-4+3+x\\5+2+y\end{bmatrix}=\begin{bmatrix}-1+x\\7+y\end{bmatrix}$
Write the result for $(\mathbf{S}\circ\mathbf{T})\mathbf{v}$.	$(\mathbf{S}\circ\mathbf{T})\mathbf{v}=\mathbf{S}(\mathbf{Tv})=\begin{bmatrix}-1+x\\7+y\end{bmatrix}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}x\\y\end{bmatrix}$
Write the column matrix for $\mathbf{S}\circ\mathbf{T}$.	$\mathbf{S}\circ\mathbf{T}=\begin{bmatrix}-1\\7\end{bmatrix}$
c Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of A.	$(\mathbf{S}\circ\mathbf{T})\mathbf{a}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}1\\9\end{bmatrix}$
Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of B.	$(\mathbf{S}\circ\mathbf{T})\mathbf{b}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}6\\5\end{bmatrix}=\begin{bmatrix}5\\12\end{bmatrix}$
Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of C.	$(\mathbf{S}\circ\mathbf{T})\mathbf{c}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}9\\2\end{bmatrix}=\begin{bmatrix}8\\9\end{bmatrix}$
Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of D.	$(\mathbf{S}\circ\mathbf{T})\mathbf{d}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}7\\-2\end{bmatrix}=\begin{bmatrix}6\\5\end{bmatrix}$
Write the result.	The image is $A'(1, 9)$ $B'(5, 12)$ $C'(8, 9)$ $D'(6, 5)$.
d Show the object and image on the same axes in different colours.	

What kind of transformation does the composition of two translations produce?

Dilations, rotations and reflections are all linear transformations. It is possible to show that the composition of two linear transformations is also a linear transformation. You can use the product of the matrices to find the effect of composing two linear transformations.

Example 14

The rotation though an angle of 60° is written as $\mathbf{R}_{60°}$ and the dilation by a factor of 2 in the x direction and 4 in the y direction is written as $\mathbf{D}$.

a Use matrices to find the transformation $\mathbf{D} \circ \mathbf{R}_{60°}$

b Use matrices to find the transformation $\mathbf{R}_{60°} \circ \mathbf{D}$.

c Is this composition commutative?

d Find the image of $A(3, 6)$ $B(5, 4)$ $C(8, 5)$ under $\mathbf{R}_{60°} \circ \mathbf{D}$.

e Show ABC and $A'B'C'$ on the same set of axes.

Solution

a Write the matrix for $\mathbf{D}$.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Write the matrix for $\mathbf{R}_{60°}$.

$$\mathbf{R}_{60°} = \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Work out $\mathbf{D} \times \mathbf{R}_{60°}$.

$$\mathbf{D} \circ \mathbf{R}_{60°} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{3} \\ 2\sqrt{3} & 2 \end{bmatrix}$$

b Work out $\mathbf{R}_{60°} \times \mathbf{D}$.

$$\mathbf{R}_{60°} \circ \mathbf{D} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

c You cannot reverse the order.

Since $\mathbf{R}_{60°} \circ \mathbf{D} \neq \mathbf{D} \circ \mathbf{R}_{60°}$, this composition is not commutative.

d Apply $\mathbf{R}_{60°} \circ \mathbf{D}$ to ABC.

$$\begin{bmatrix} 1 & -2\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 & 8 \\ 6 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3-12\sqrt{3} & 5-8\sqrt{3} & 8-10\sqrt{3} \\ 12+3\sqrt{3} & 8+5\sqrt{3} & 10+8\sqrt{3} \end{bmatrix}$$

Write the answer.

The image is $A'(3 - 12\sqrt{3}, 12 + 3\sqrt{3})$ $B'(5 - 8\sqrt{3}, 8 + 5\sqrt{3})$ $C'(8 - 10\sqrt{3}, 10 + 8\sqrt{3})$.

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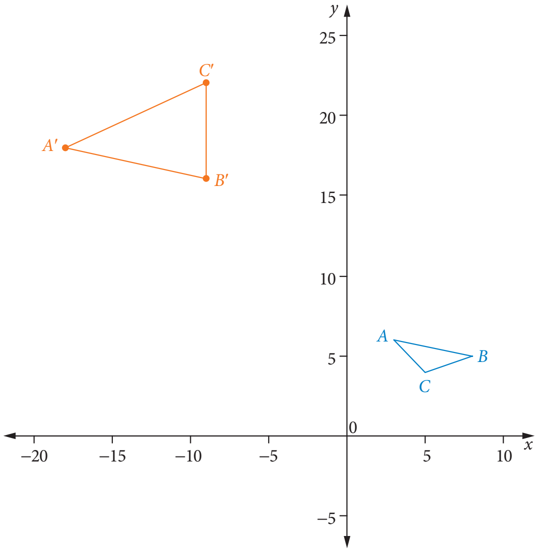
e Use approximations to sketch the triangles on the same axes.



Example 15

The transformation **M** is a reflection in the line through the origin with slope 3, and the transformation $\mathbf{R}_{120°}$ is a rotation through an angle of 120°.

a Use matrices to find the composition $\mathbf{M} \circ \mathbf{R}_{120°}$.

b Use matrices to find the composition $\mathbf{R}_{120°} \circ \mathbf{M}$.

c Is the composition commutative?

d Find the image of $A(2, -2)$ $B(4, 3)$ $C(5, -1)$ under $\mathbf{R}_{120°} \circ \mathbf{M}$.

e Show ABC and $A'B'C'$ on the same set of axes.

Solution

a Find p.

$$p = \frac{1-m^2}{1+m^2} = \frac{1-9}{1+9} = -\frac{4}{5} = -0.8$$

Find q.

$$q = \frac{2m}{1+m^2} = \frac{2\times 3}{1+9} = \frac{3}{5} = 0.6$$

Write **M**.

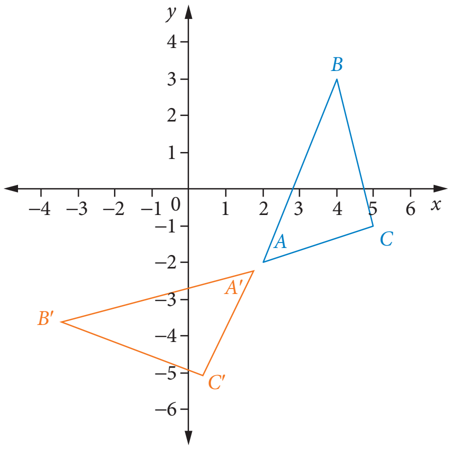
$$\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix} = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

Write $\mathbf{R}_{120°}$.

$$\mathbf{R}_{120°} = \begin{bmatrix} \cos(120°) & -\sin(120°) \\ \sin(120°) & \cos(120°) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

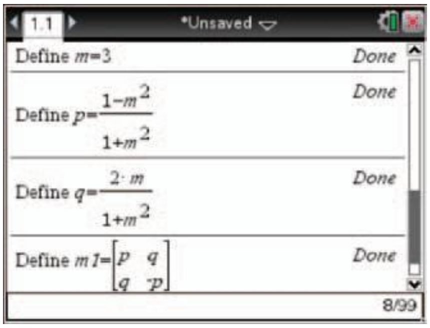
Find $\mathbf{M} \circ \mathbf{R}_{120°}$.	$\mathbf{M} \circ \mathbf{R}_{120°} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ $= \begin{bmatrix} \frac{3\sqrt{3}+4}{10} & \frac{4\sqrt{3}-3}{10} \\ \frac{4\sqrt{3}-3}{10} & \frac{-(3\sqrt{3}+4)}{10} \end{bmatrix}$
b Find $\mathbf{R}_{120°} \circ \mathbf{M}$.	$\mathbf{R}_{120°} \circ \mathbf{M} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ $= \begin{bmatrix} \frac{4-3\sqrt{3}}{10} & \frac{-(4\sqrt{3}+3)}{10} \\ \frac{-(4\sqrt{3}+3)}{10} & \frac{3\sqrt{3}-4}{10} \end{bmatrix}$
c Compare the answers for parts **a** and **b**.	$\mathbf{M} \circ \mathbf{R}_{120°} \neq \mathbf{R}_{120°} \circ \mathbf{M}$ so the composition is not commutative.
d Apply $\mathbf{R}_{120°} \circ \mathbf{M}$ to ABC.	$\begin{bmatrix} \frac{4-3\sqrt{3}}{10} & \frac{-(4\sqrt{3}+3)}{10} \\ \frac{-(4\sqrt{3}+3)}{10} & \frac{3\sqrt{3}-4}{10} \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ -2 & 3 & -1 \end{bmatrix}$ $= \begin{bmatrix} \frac{7+2\sqrt{3}}{5} & \frac{7-24\sqrt{3}}{10} & \frac{23-11\sqrt{3}}{10} \\ \frac{(1-7\sqrt{3})}{5} & \frac{-(24+7\sqrt{3})}{10} & \frac{-(11+23\sqrt{3})}{5} \end{bmatrix}$ $\approx \begin{bmatrix} 1.74... & -3.45... & 0.39... \\ -2.22... & -3.61... & -5.08... \end{bmatrix}$
Write the answer.	The image is $A'\left(\frac{7+\sqrt{3}}{5}, \frac{(1-7\sqrt{3})}{5}\right)$ $B'\left(\frac{7-24\sqrt{3}}{10}, \frac{-(24+7\sqrt{3})}{10}\right)$ $C'\left(\frac{23-11\sqrt{3}}{10}, \frac{-(11+23\sqrt{3})}{5}\right)$.

e Use approximations to sketch the triangles on the same axes.

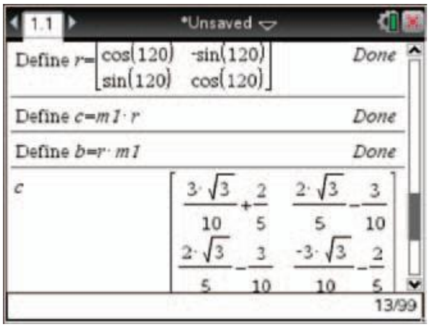


TI-Nspire CAS

Define m, p and q and then the reflection matrix. In the screen shot it is called $m1$ because the variable m has already been used for the slope.

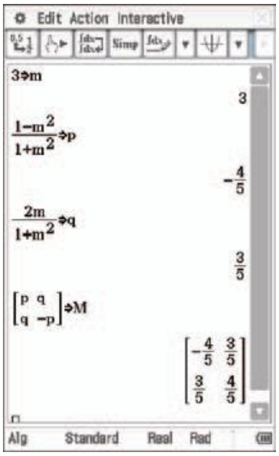


Make sure that your calculator is set on degrees.
Multiply the reflection by the rotation and vice versa, giving them different names. The screen shot shows the result for $\mathbf{M} \circ \mathbf{R}_{120°}$. You can check $\mathbf{R}_{120°} \circ \mathbf{M}$ and apply it to ABC on your own calculator.

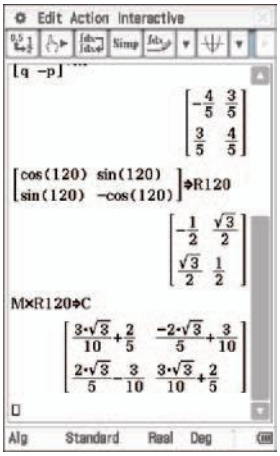


ClassPad

Ensure the calculator is set to **Standard**. Define m, p and q, and then the reflection matrix **M**. The Casio treats M and m as different variables.



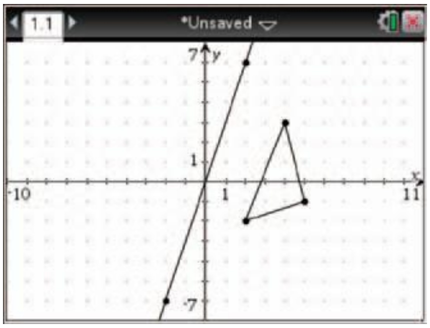
Make sure that your calculator is set on degrees.
Define the rotation matrix as **R**120.
Set **M** ×**R**120 to equal **C** and set **R**120× **M** to equal **B**.
The screen shot shows the result for $\mathbf{M} \circ \mathbf{R}_{120°}$. You can check $\mathbf{R}_{120°} \circ \mathbf{M}$ and apply it to *ABC* on your own calculator.



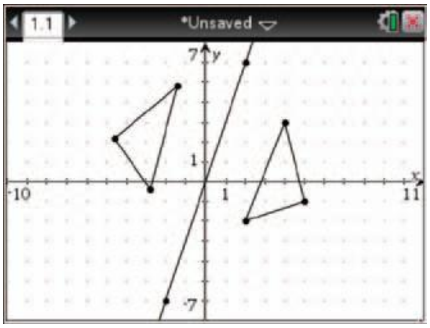
You could work out parts **d** and **c** of Example **15** using the geometry option on your CAS calculator as shown below. Unfortunately, current calculators will not do dilations unless both factors are the same, so you could not work out Example **14** in the same way. However, you could perform it with a general transformation on the CASIO ClassPad.

TI-Nspire CAS

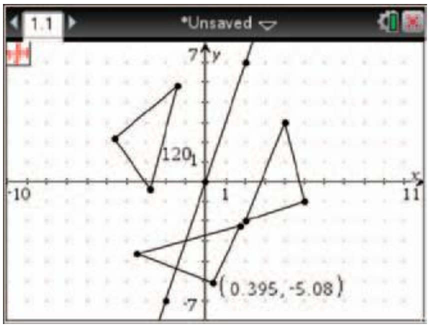
Draw the triangle on a Graph page (see page 417) with Window Settings of $-10 \le x \le 11$ and $-7 \le y \le 7$ for correct proportion. Draw a line through the origin with slope 3.



Use $\boxed{\text{menu}}$, 8: Geometry, 5: Transformation and 2: Reflection to reflect *ABC* across the line by clicking on the triangle and clicking on the line. Use $\boxed{\text{menu}}$, 9: Settings to ensure that the Graphing Angle and Geometry Angle are in degrees.



Now use $\boxed{\text{menu}}$, 8: Geometry, 5: Transformation and 4: Rotation to rotate *ABC* around the origin by clicking on the origin and the reflected triangle and typing in the angle. You can find the coordinates of points using $\boxed{\text{menu}}$, 1: Actions and 8: Coordinates and Equations.



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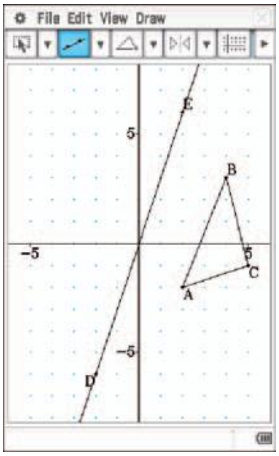
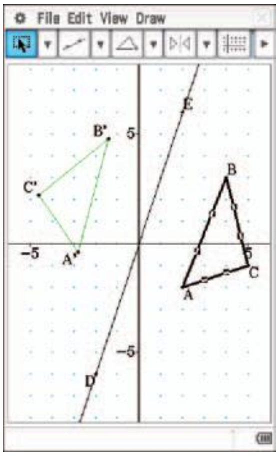
ClassPad

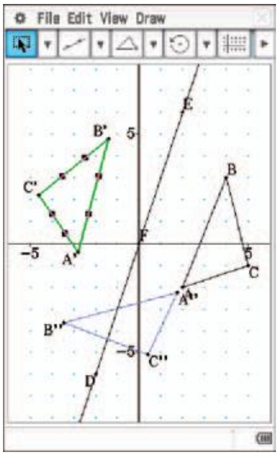
Use the Geometry to draw the triangle and the line of reflection.
Use **View Window** to set $-6 \leq x \leq 6$ and ymid = 0.
Refer to page 418.



Tap and select each side of the triangle.
Tap **Draw**, **Construct** and **Reflection**.
Tap the line of reflection.
The image, triangle $A'B'C'$, is drawn.



Make sure that your calculator is set to degrees by tapping and selecting **Geometry Format**.
Unselect the first triangle and select each side of the second (image) triangle.
Tap **Draw**, **Construct** and **Rotation**.
Tap the origin as the point of rotation.
Enter 120 as the angle of rotation.
The image, triangle $A''B''C''$, is drawn. It represents a reflection followed by a rotation of 120°.



EXERCISE 11.06 Composition of linear transformations

Concepts and techniques

1 Example 13 $\mathbf{T}_1$ is a translation of 3 to the right and 4 up. $\mathbf{T}_2$ is 2 to the left and 7 down and $\mathbf{T}_3$ is 4 to the right and 3 down.
   a Find $\mathbf{T}_1 \circ \mathbf{T}_2$ in the form $(x, y) \to (x', y')$.
   b Find $\mathbf{T}_3 \circ \mathbf{T}_2$ as a column vector.
   c Find the image of $A(-2, -1)$ $B(-1, 5)$ $C(6, 7)$ under the transformation $\mathbf{T}_1 \circ \mathbf{T}_3$.
   d Show ABC and $A'B'C'$ on the same set of axes.

2 $\mathbf{T}_1$ is a translation of 4 to the left and 2 up. $\mathbf{T}_2$ is 5 to the right and 4 down and $\mathbf{T}_3$ is 2 to the right and 4 up.
   a Find $\mathbf{T}_1 \circ \mathbf{T}_2$ in the form $(x, y) \to (x', y')$.
   b Find $\mathbf{T}_3 \circ \mathbf{T}_2$ as a column vector.
   c Find the image of $A(1, -4)$ $B(3, 2)$ $C(7, 5)$ $D(8, -2)$ under the transformation $\mathbf{T}_1 \circ \mathbf{T}_3$.
   d Show ABC and $A'B'C'$ on the same set of axes.

3 Example 14 a Find the composition $\mathbf{R}_{35^\circ} \circ \mathbf{R}_{49^\circ}$ of rotations of 35° and 49°, correct to 4 decimal places.
   b Find the composition $\mathbf{R}_{49^\circ} \circ \mathbf{R}_{35^\circ}$, correct to 4 decimal places.
   c Is this composition commutative?
   d Find the image of $A(3, 4)$ $B(5, 3)$ $C(2, 1)$ under $\mathbf{R}_{49^\circ} \circ \mathbf{R}_{35^\circ}$, correct to 2 decimal places.
   e Draw the object and image from part **d** on the same set of axes.

4 $\mathbf{M}_1$ is a reflection in $y = 3x$ and $\mathbf{M}_2$ is a reflection in $y = -2x$.
   a Find the composition $\mathbf{M}_1 \circ \mathbf{M}_2$.
   b Find the composition $\mathbf{M}_2 \circ \mathbf{M}_1$.
   c Is this composition commutative?
   d Find the image of $A(1, 5)$ $B(2, 3)$ $C(-2, 1)$ $D(-3, 4)$ under $\mathbf{M}_1 \circ \mathbf{M}_2$.
   e Draw the object and image from part **d** on the same set of axes.

5 Example 15 CAS $\mathbf{D}$ is a dilation by a factor of 1.5 in the x direction and 2 in the y direction. $\mathbf{M}$ is a reflection in a line of slope -2.
   a Find the composition $\mathbf{D} \circ \mathbf{M}$.
   b Find the composition $\mathbf{M} \circ \mathbf{D}$.
   c Is this composition commutative?
   d Find the image of $A(-2, 4)$ $B(1, 5)$ $C(3, 2)$ $D(-1, 0)$ under $\mathbf{M} \circ \mathbf{D}$.
   e Draw the object and image from part **d** on the same set of axes.

6 CAS $\mathbf{R}_{80^\circ}$ is a rotation through 80° and $\mathbf{M}$ is a reflection in a 29° line of inclination.
   a Find the composition $\mathbf{R}_{80^\circ} \circ \mathbf{M}$, correct to 4 decimal places.
   b Find the composition $\mathbf{M} \circ \mathbf{R}_{80^\circ}$.
   c Is this composition commutative?
   d Find the image of $A(3, -1)$ $B(5, 1)$ $C(6, -2)$ $D(4, -3)$ under $\mathbf{R}_{80^\circ} \circ \mathbf{M}$.
   e Draw the object and image on the same set of axes.

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7 CAS $\mathbf{M}_1$ is a reflection in $y = x$ and $\mathbf{M}_2$ is a reflection in $y = -x$.
  a Find the composition $\mathbf{M}_1 \circ \mathbf{M}_2$.
  b Find the composition $\mathbf{M}_2 \circ \mathbf{M}_1$.
  c Is this composition commutative?

Reasoning and communication

8 Show that a composition of rotations is always commutative by finding the compositions $\mathbf{R}_\alpha \circ \mathbf{R}_\beta$ and $\mathbf{R}_\beta \circ \mathbf{R}_\alpha$.

9 Show that composition of linear transformations is not always commutative by finding a counter example.

10 A general linear transformation is of the form $\mathbf{T}$: $(x, y) \to (ax + by, cx + dy)$, where a, b, c and d are real numbers. Write another general linear transformation $\mathbf{S}$ and show that the transformation $\mathbf{S} \circ \mathbf{T}$ is also linear.

11.07 INVERSE TRANSFORMATIONS

In the previous sections we covered various transformations on the plane (2D space). Some of these transformations can be reversed by another transformation so that the combined effect is to make no change, i.e., $(x, y) \to (x, y)$. This is like adding 0 or multiplying by 1.

IMPORTANT

If it exists, the element that leaves everything unchanged for an operation is called the **identity** for that operation and is usually written as 1, 0 or $\mathbf{I}$.

For all transformations, the identity is $\mathbf{I}$: $(x, y) \to (x, y)$.

For translations, the identity is none along and none up or down, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The identity matrix representation for linear transformations is $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For rotations, it is a rotation though 0°, $\mathbf{R}_0$, (or 360°) or the equivalent in radians.

For dilations, the identity has a factor of 1 in both the x and y directions.

There is no reflection that is equal to the identity.

A transformation that reverses a particular transformation so that their composition is the identity has a special name.

IMPORTANT

For a transformation $\mathbf{T}$, if there exists a transformation $\mathbf{S}$ such that

$\mathbf{S} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{S} = \mathbf{I}$

then $\mathbf{S}$ is called the **inverse** of $\mathbf{T}$ and written as $\mathbf{T}^{-1}$ so $\mathbf{T}^{-1} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{I}$.

A transformation that has an inverse is called an **invertible** transformation.

You have already seen that some compositions of translations and products of matrices are not commutative, which is the reason why the identity and inverse have to work on *both the left and right*. It is obvious that the inverse of a translation is another translation in the opposite direction. The inverse of a dilation is another dilation with factors that are reciprocals of the factors of the original dilation. The inverse of a rotation is an equal rotation in the opposite direction.

Example 16

Find the inverses of the following transformations and confirm that they are correct by a composition of the transformation and its inverse.

a A translation of 5 up and 3 left.

b A dilation by a factor of 0.5 in the x direction and 0.8 in the y direction.

c A rotation through an angle of $\frac{3\pi}{4}$.

Solution

a Write the translation.

$$\mathbf{T}: (x, y) \rightarrow (x - 3, y + 5)$$

Write the inverse, $\mathbf{T}^{-1}$.

$$\mathbf{T}^{-1}: (x, y) \rightarrow (x + 3, y - 5)$$

Use vectors to do the composition on the left.

$$\begin{aligned}\mathbf{T}^{-1} \circ \mathbf{T} &= \mathbf{T}^{-1} + \mathbf{T}\\ &= \begin{bmatrix} -5 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix}\\ &= \begin{bmatrix} -5+5 \\ 3+(-3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark\end{aligned}$$

Do the composition on the right.

$$\begin{aligned}\mathbf{T} \circ \mathbf{T}^{-1} &= \mathbf{T} + \mathbf{T}^{-1}\\ &= \begin{bmatrix} 5 \\ -3 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \end{bmatrix}\\ &= \begin{bmatrix} 5+(-5) \\ -3+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark\end{aligned}$$

b Write the dilation.

$$\mathbf{D}: (x, y) \rightarrow (0.5x, 0.8y)$$

Use $\frac{1}{0.5} = 2$ and $\frac{1}{0.8} = 1.25$ for $\mathbf{D}^{-1}$.

$$\mathbf{D}^{-1}: (x, y) \rightarrow (2x, 1.25y)$$

Use the matrix representations to check the composition on the left.

$$\begin{aligned}\mathbf{D}^{-1} \circ \mathbf{D} &= \begin{bmatrix} 2 & 0 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}\\ &= \begin{bmatrix} 2\times 0.5 & 0 \\ 0 & 0.8\times 1.25 \end{bmatrix}\\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark\end{aligned}$$

Now repeat it on the right.

$$\begin{aligned}\mathbf{D} \circ \mathbf{D}^{-1} &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1.25 \end{bmatrix}\\ &= \begin{bmatrix} 0.5\times 2 & 0 \\ 0 & 0.8\times 1.25 \end{bmatrix}\\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark\end{aligned}$$

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c The inverse will be a rotation of the same amount in the opposite direction.

Angle of rotation for the inverse $= -\frac{3\pi}{4}$

Write the matrix for the rotation.

$$\mathbf{R} = \begin{bmatrix} \cos\left(\frac{3\pi}{4}\right) & -\sin\left(\frac{3\pi}{4}\right) \\ \sin\left(\frac{3\pi}{4}\right) & \cos\left(\frac{3\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Write the matrix for the inverse.

$$\mathbf{R}^{-1} = \begin{bmatrix} \cos\left(-\frac{3\pi}{4}\right) & -\sin\left(-\frac{3\pi}{4}\right) \\ \sin\left(-\frac{3\pi}{4}\right) & \cos\left(-\frac{3\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Use matrix multiplication to check the composition on the left.

$$\mathbf{R}^{-1} \circ \mathbf{R} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Now repeat it on the right.

$$\mathbf{R} \circ \mathbf{R}^{-1} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

The inverses of translations and rotations are opposite to each other. However, the inverses of reflections are different. How do you reverse a reflection? With a mirror, you use another mirror. The inverse of a reflection is the same reflection. *Reflections are their own inverses.*

Example 17

Use matrices to confirm that the following reflections are their own inverses.

a The reflection in the line with slope –0.6.

b The reflection in the line with an inclination angle of 30° to the x-axis.

Solution

a Find p.

$$p = \frac{1-m^2}{1+m^2} = \frac{1-(-0.6)^2}{1+(-0.6)^2} = \frac{8}{17} \approx 0.4706$$

Find q.

$$q = \frac{2m}{1+m^2} = \frac{2\times -0.6}{1+0.36} = -\frac{15}{17} \approx -0.8824$$

Write the matrix for $\mathbf{M}$.

$$\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix} = \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix}$$

The inverse will be the same.

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix}$$

Use matrix multiplication to check the composition on the left.
Since the matrices are the same it is not necessary to repeat it on the right as well.

$$\mathbf{M} \times \mathbf{M}^{-1} = \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix} \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{64+225}{289} & \frac{-120+120}{289} \\ \frac{120-120}{289} & \frac{64+225}{289} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

b Write the matrix for $\mathbf{M}$.

$$\mathbf{M} = \begin{bmatrix} \cos(60^\circ) & \sin(60^\circ) \\ \sin(60^\circ) & -\cos(60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

The inverse will be the same.

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Use matrix multiplication to check the composition on the left.
Since the matrices are the same it is not necessary to repeat it on the right as well.

$$\mathbf{M} \times \mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}+\frac{3}{4} & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & \frac{1}{4}+\frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

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You can also use your CAS calculator to find the inverse of a square matrix **A** using **A-1** in the same way as for ordinary numbers.

What is the inverse of a composition of two transformations? Is it the composition of the inverses? Does it depend on whether or not the transformations commute?

INVESTIGATION Inverses of compositions

Consider the following pairs of matrices:

$\mathbf{T}_1$ is a rotation of $\frac{\pi}{3}$ and $\mathbf{T}_2$ is a reflection in the line with inclination $-\frac{\pi}{8}$.

$\mathbf{T}_1$ is a reflection in the line with slope 15° and $\mathbf{T}_2$ is a dilation with factor $\frac{2}{3}$ in both directions.

Use matrices or other means to investigate the compositions $\mathbf{T}_1 \circ \mathbf{T}_2$, $\mathbf{T}_2 \circ \mathbf{T}_1$, $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1}$, $(\mathbf{T}_2 \circ \mathbf{T}_1)^{-1}$, $\mathbf{T}_1^{-1} \circ \mathbf{T}_2^{-1}$ and $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$.

What do you find?

Try inverses of compositions of other linear transformations.

Can you make a general rule?

Can you prove it?

From your investigation, you would have found the following.

IMPORTANT

If it exists, the inverse of a composition of linear transformations, $\mathbf{T}_1 \circ \mathbf{T}_2$, is given by $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$.

EXERCISE 11.07 Inverse transformations

Concepts and techniques

1 Example 16 For each of the following translations, write the translation as a vector, find its inverse and verify your results by composing them.
   a 5 up and 3 right
   b 4 down and 5 left
   c 2 right and 6 down
   d 7 left and 5 up

2 Example 17 Find the matrix for dilation with a factor of $\frac{1}{3}$ in the x direction and $\frac{1}{2}$ in the y direction and its inverse and verify your results by composing them.

3 Find the matrix for dilation with a factor of $2\frac{1}{2}$ and its inverse and verify your results by composition.

4 Find the matrix for rotation through an angle of $\frac{4\pi}{3}$ and its inverse and verify your results by composition.

5 Find the matrix for rotation through an angle of 120° and its inverse and verify your results by composition.

6 Find the matrix for reflection through a line with an inclination of 112.5° and its inverse and verify your results by composition.

7 Find the matrix for reflection through a line with slope $-\frac{1}{3}$ and its inverse and verify your results by composition.

8 For each of the following pairs of transformations, find $\mathbf{T}_1 \circ \mathbf{T}_2$, $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1}$, $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$ and $\mathbf{T}_1^{-1} \circ \mathbf{T}_2^{-1}$ using matrices, and comment on the results.

a A rotation through an angle of 30° and a rotation through an angle of 45°.

b A reflection through a line with slope 3 and a reflection through a line with slope $\frac{1}{2}$.

Reasoning and communication

9 $\mathbf{T}_1$ and $\mathbf{T}_2$ are the matrices of invertible linear transformations and $\mathbf{T}_3 = \mathbf{T}_1 \circ \mathbf{T}_2$.

a What are the products of $\mathbf{T}_1$ and its inverse and vice versa?

b What are the products of $\mathbf{T}_2$ and its inverse and vice versa?

c What is the matrix for $\mathbf{T}_1^{-1} \circ \mathbf{T}_3$?

d By what does the matrix for $\mathbf{T}_1^{-1} \circ \mathbf{T}_3$ have to be multiplied on the left to give $\mathbf{I}$?

e Use the results from parts **a** to **d** to prove that $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1} = \mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$ for all invertible linear transformations.

10 Prove that for linear transformations that commute, $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1} = \mathbf{T}_1^{-1} \circ \mathbf{T}_2^{-1}$.

11.08 DETERMINANTS AND GEOMETRY

In Chapter 8, you saw that invertible matrices had a non zero **determinant**. The formula for the determinant of a square matrix of order 2 is shown on the right.

Any linear transformation can be written as a 2×2 matrix, so you can calculate the determinant of its matrix.

Finding the area of a closed plane figure is sometimes important. The simplest way to find the area for many polygons is to divide them into triangles. There are many ways to work out the area of a triangle, including the use of vector properties as shown on the opposite page.

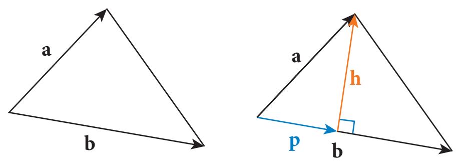
IMPORTANT

The determinant of the matrix

$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$\det \mathbf{A} = |\mathbf{A}| = ad - bc$

Consider the triangle formed by two vectors **a** and **b** as shown below in the diagram on the left.



Construct the orthogonal vectors **p** and **h** as shown on the right. The area of the triangle is then given by $\frac{1}{2}bh$. **p** is the projection of **a** on **b** and $\mathbf{a} = \mathbf{p} + \mathbf{h}$ so $\mathbf{h} = \mathbf{a} - \mathbf{p}$. As you saw in Chapter 4 (see page 128), the projection of **a** on **b** is given by $\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \times \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$.

Thus $\mathbf{h} = \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$ and you can use $|\mathbf{b}|$ and $|\mathbf{h}|$ to find the area of the triangle.

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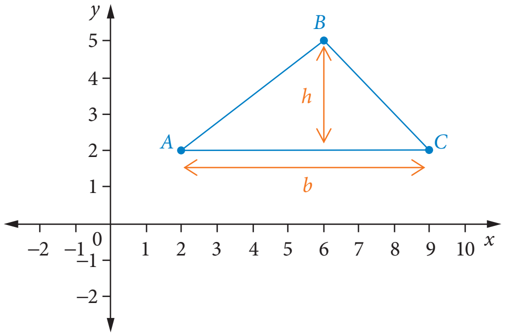
Example 18

a Find the area of the triangle $A(2, 2)$ $B(6, 5)$ $C(9, 2)$.
b Find the matrix for the transformation $\mathbf{T}$: $(x, y) \rightarrow (2x + 3y, -2x + 2y)$.
c Find the image $A'B'C'$ of the triangle under the transformation $\mathbf{T}$.
d Find the area of $A'B'C'$.
e Find the determinant of the matrix of $\mathbf{T}$.
f Comment on your results.

Solution

a Sketch ABC.

To find the area, you can use $A = \frac{1}{2}bh$



Work out the area.

Area of $ABC = \frac{1}{2}bh = \frac{1}{2} \times 7 \times 3 = 10.5$

b Write the matrix for the transformation.

$\mathbf{T} = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}$

c Use the matrix to find $A'B'C'$.

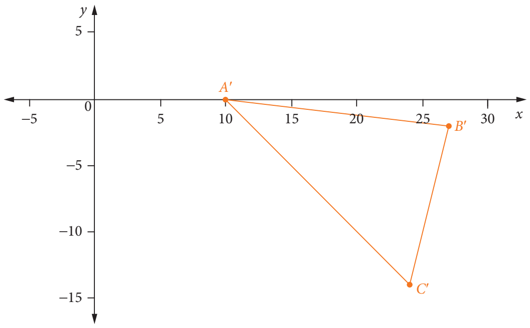
$\begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 & 9 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 27 & 24 \\ 0 & -2 & -14 \end{bmatrix}$

Write the answer.

The image is $A'(10, 0)$ $B'(27, -2)$ $C'(24, -14)$.

d Sketch $A'B'C'$.

No sides are vertical or horizontal, so use vectors or the semiperimeter formula to find the area.

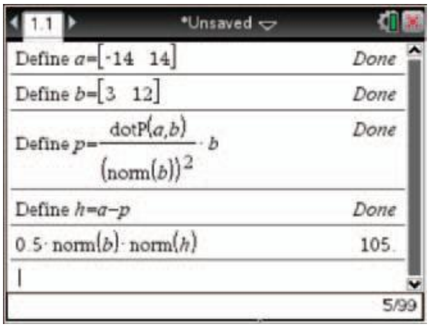


Use the vectors $\mathbf{C'A'}$ and $\mathbf{C'B'}$ to find the area.

$\mathbf{C'A'} = (10 - 24, 0 - (-14)) = (-14, 14)$
$\mathbf{C'B'} = (27 - 24, -2 - (-14)) = (3, 12)$

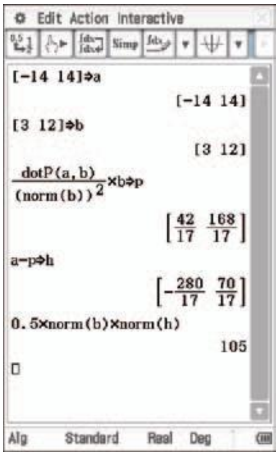
TI-Nspire CAS

Use your calculator to find the projection **p** and hence the height **h** and area of the triangle.



ClassPad

Set the calculator to **Standard** mode. Use your calculator to find the projection **p** and hence the height **h** and area of the triangle.



Write the area. Area of $A'B'C' = 105$

d Find the determinant. det $\mathbf{T} = ad - bc = 2 \times 2 - 3 \times (-2) = 10$

e Compare the areas. $\text{Area}_{ABC} = 10.5$ and $\text{Area}_{A'B'C'} = 105$

f Write a comment. The area of the image is det $\mathbf{T}$ =10 times the area of the object.

From Example **18**, it appears that the area of the object under the linear transformation **T** is multiplied by $|\mathbf{T}|$ to get the area of the image. This can be proved for the general case.

IMPORTANT

The area of the image produced by a linear transformation is $|\mathbf{T}|$ times the area of the object.

Of course, if **T** is negative then you need to change the sign of $|\mathbf{T}|$A because an area is always positive. What is the significance of the negative sign of the determinant for the image compared to the object?

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Example 19

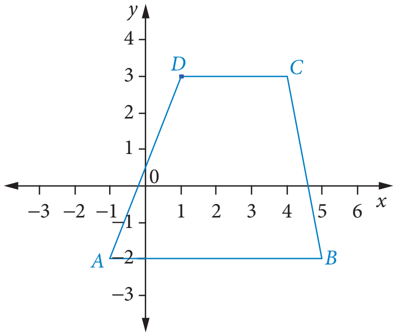
A linear transformation is given by **T**: $(x, y) \rightarrow (3x - y, 2x + 3y)$. What is the area of the new shape of the trapezium $A(-1, -2)$ $B(5, -2)$ $C(4, 3)$ $D(1, 3)$ under the transformation?

Solution

Write the matrix for the transformation. $\mathbf{T} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}$

Find the determinant of **T**. $|\mathbf{T}| = ad - bc = 3 \times 3 - (-1) \times 2 = 11$

Sketch the trapezium.



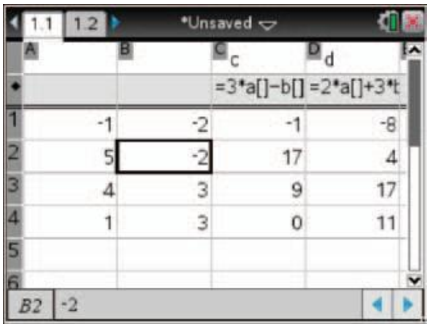
Find the area of the trapezium. $A = \frac{1}{2}(a+b)h = \frac{1}{2}(6+3) \times 5 = 22.5$

Multiply by the determinant. Area of new shape $= 22.5 \times 11 = 247.5$

You can use your CAS calculator to perform this transformation and find the area of the transformed shape. You can also use the det() function on the TI-Nspire CAS or the ClassPad to find the determinant of a matrix.

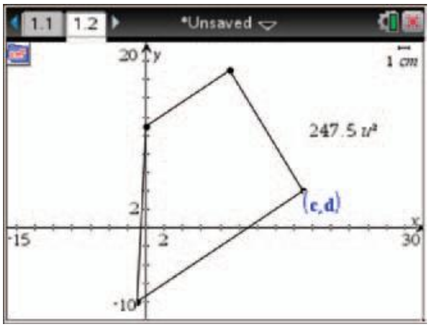
TI-Nspire CAS

Put the points into a Spreadsheet page and apply the transformation (see pages 422–423). Let the variables for columns C and D be c and d, say.



	A	B	C c	D d
◆			=3*a[]−b[]	=2*a[]+3*t
1	-1	-2	-1	-8
2	5	-2	17	4
3	4	3	9	17
4	1	3	0	11
5				

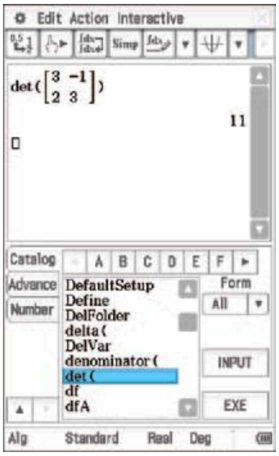
Add a Graph page and make the Window Settings $-15 \le x \le 30$ and $-10 \le y \le 20$ to fit the new shape (see pages 411-412). Now use menu, 3: Graph Entry/Edit, 5: Scatter Plot and put $x \leftarrow c$ and $y \leftarrow d$ in the dialog box. Draw a polygon joining up the points. You can find the area using menu, 8: Geometry, 3: Measurement, 2: Area and clicking on the polygon.



The display digits in 9: Settings have been changed to Float 4.

ClassPad

Use the Main $\sqrt{\alpha}$ application to calculate the determinant.

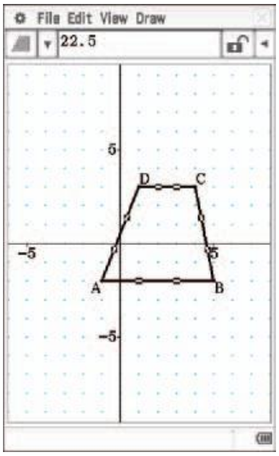


You can do the rest from the **Geometry** application.

First draw the trapezium.

Start with **View Window** set to about $-6 \le x \le 8$ and ymid = 0.

Select tap each side and tap to measure its area. Make sure that the area symbol is displayed in the top left drop down menu.



Area = 22.5

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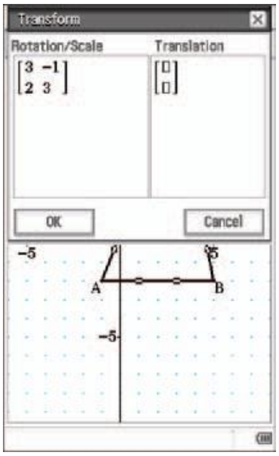
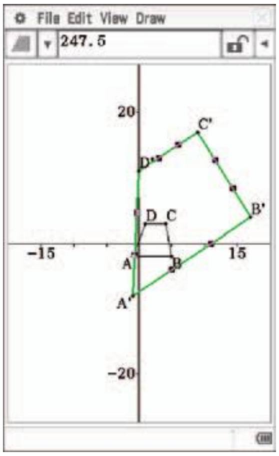
Tap **Draw**, then **Construct**, then **General Transform**. Fill in the 2 × 2 matrix only. Tap **OK**.



Change **View Window** so $-20 \le x \le 20$. The dots disappear.
First tap a blank area of screen if it is not clear whether the small original trapezium is still selected. Select each side of the (larger) image and find its area.



What kind of transformation do you think the combination of two dilations gives? What about two rotations? What about two reflections?

Example 20

a Find the result of applying the reflection in a line of inclination 30°, followed by a reflection in a line of slope 45°.

b What kind of transformation is the composition of the two reflections?

Solution

a Write the matrix for the first reflection.

$$\mathbf{M}_1 = \begin{bmatrix} \cos(60^\circ) & \sin(60^\circ) \\ \sin(60^\circ) & -\cos(60^\circ) \end{bmatrix}$$

Write the matrix for the second reflection.

$$\mathbf{M}_2 = \begin{bmatrix} \cos(90^\circ) & \sin(90^\circ) \\ \sin(90^\circ) & -\cos(90^\circ) \end{bmatrix}$$

Find the matrix for $\mathbf{M}_2 \circ \mathbf{M}_1$.

$$\mathbf{M}_2 \times \mathbf{M}_1 = \begin{bmatrix} \cos(90^\circ) & \sin(90^\circ) \\ \sin(90^\circ) & -\cos(90^\circ) \end{bmatrix}\begin{bmatrix} \cos(60^\circ) & \sin(60^\circ) \\ \sin(60^\circ) & -\cos(60^\circ) \end{bmatrix}$$

Multiply.

$$= \begin{bmatrix} \cos(90^\circ)\cos(60^\circ) & \cos(90^\circ)\sin(60^\circ) \\ +\sin(90^\circ)\sin(60^\circ) & -\sin(90^\circ)\cos(60^\circ) \\ & \\ \sin(90^\circ)\cos(60^\circ) & \cos(90^\circ)\cos(60^\circ) \\ -\cos(90^\circ)\sin(60^\circ) & +\sin(90^\circ)\sin(60^\circ) \end{bmatrix}$$

b Rearrange the top right term of the matrix.

$$= \begin{bmatrix} \cos(90^\circ)\cos(60^\circ) & -[\sin(90^\circ)\cos(60^\circ) \\ +\sin(90^\circ)\sin(60^\circ) & -\cos(90^\circ)\sin(60^\circ)] \\ & \\ \sin(90^\circ)\cos(60^\circ) & \cos(90^\circ)\cos(60^\circ) \\ -\cos(90^\circ)\sin(60^\circ) & +\sin(90^\circ)\sin(60^\circ) \end{bmatrix}$$

Use the trigonometric addition formulas in reverse.

$$= \begin{bmatrix} \cos(90^\circ - 60^\circ) & -\sin(90^\circ - 60^\circ) \\ \sin(90^\circ - 60^\circ) & \cos(90^\circ - 60^\circ) \end{bmatrix}$$

Simplify.

$$= \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix}$$

But this is a rotation.

$$= \mathbf{R}_{30^\circ}$$

Write the answer.

A reflection in a line of inclination 30°, followed by a reflection in a line of slope 45° gives a rotation of 30°.

You can use matrices to prove that the result of Example **21** is generally true.

IMPORTANT

The composition of two reflections in lines through the origin gives a rotation around the origin.

EXERCISE 11.08 Determinants and geometry

WS Matrix multiplications

Concepts and techniques

Your teacher will tell you where you should use your calculator in this exercise.

1 Example 19 Find the area of each shape below under the given transformation.

a The rectangle $A(3, 2)$ $B(5, 2)$ $C(5, -5)$ $D(3, -5)$ under the transformation $\mathbf{T}: (x, y) \to (3x - 2x, 2x + 3y)$.

b The square $A(3, 8)$ $B(-3, 8)$ $C(-3, 2)$ $D(3, 2)$ under the transformation $\mathbf{T}: (x, y) \to (x - 3y, x + 7y)$.

c The kite $A(2, 5)$ $B(-2, 1)$ $C(2, -10)$ $D(6, 1)$ under the transformation $\mathbf{T}: (x, y) \to (2x + y, x + 3y)$.

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2 Example 20 What is result of a reflection in the x-axis followed by a rotation of 90°?

3 What is the rotation of −90° followed by a rotation of 45°?

4 What is the result of a dilation of factors 2 and 3, followed by a dilation of factors 4 and 2, where the factors are given in the order x then y?

5 What is the result of a rotation of 90° followed by a reflection in the y-axis?

Reasoning and communication

6 Example 18 **a** Find the area of the triangle $A(-1, -2)$ $B(2, 2)$ $C(4, 2)$.
  b Find the matrix for the transformation $\mathbf{T}$: $(x, y) \rightarrow (x - 2y, 2x + 3y)$.
  c Find the image $A'B'C'$ of the triangle under the transformation $\mathbf{T}$.
  d Find the area of $A'B'C'$.
  e Find the determinant of the matrix of $\mathbf{T}$.
  f Comment on your results.

7 **a** Find the area of the rectangle $A(3, -3)$ $B(5, 0)$ $C(-1, 4)$ $D(-3, 1)$.
  b Find the matrix for the transformation $\mathbf{T}$: $(x, y) \rightarrow (y - 2x, 2x - 2y)$.
  c What will be the area of the new shape under the transformation $\mathbf{T}$?

8 Find the area of each shape below under the given transformation.
  a The triangle $A(-3, -2)$ $B(-3, 6)$ $C(4, 1)$ under the transformation $\mathbf{T}$: $(x, y) \rightarrow (2y - 3x, x + y)$.
  b The rectangle $A(1, -5)$ $B(5, -2)$ $C(-1, 6)$ $D(-5, 3)$ under the transformation $\mathbf{T}$: $(x, y) \rightarrow (2x + 4y, 3x + 5y)$.
  c The kite $A(3, 10)$ $B(9, 8)$ $C(7, 2)$ $D(-3, 2)$ under the transformation $\mathbf{T}$: $(x, y) \rightarrow (3x + 5y, 3x - y)$.

9 **a** Find the result of applying a reflection in the line through the origin of inclination 40°, followed by a reflection in the line through the origin of slope 15°.
  b What kind of transformation is the composition of the two reflections?

10 **a** Find the result of applying the reflection in a line of slope −2, followed by a reflection in a line of slope 3.
  b What kind of transformation is the composition of the two reflections?

11 Prove that the composition of two dilations is another dilation.

12 Prove that the composition of two reflections across a line through the origin is a rotation.

11 CHAPTER SUMMARY
TRANSFORMATIONS IN THE PLANE

- A **translation** is a *slide* of all the points in the plane the same distance in the same direction.
- A translation can be modelled as addition of column matrices. The column matrices representing points are translated to the new points by adding the column matrix representing the translation.
- A **linear transformation of the plane** changes the point $P(x, y)$ to $P'(ax + by, cx + dy)$, where a, b, c and d are constants (real numbers).
- The linear transformation that changes the point $P(x, y) \to P'(ax + by, cx + dy)$ is modelled by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.
- Linear transformations are usually written as $\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The image of a vector **v** under a transformation **T** is written as **Tv**.
- A simple **dilation** *stretches* or *compresses* the points in the plane in the x and y directions. The dilation by a factor of λ_1 in the x direction and λ_2 in the y direction is given as $(x, y) \to (\lambda_1 x, \lambda_2 y)$.
- Dilation by factors of λ_1 and λ_2 in the x and y directions respectively is modelled by the matrix $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.
- For $\lambda_1, \lambda_2 > 1$, the dilation is a magnification that makes shapes larger. For $\lambda_1, \lambda_2 < 1$ the dilation is a reduction that makes shapes smaller.
- A simple **rotation** *turns* all points through the same angle around the origin. The ***rotation*** through an angle α around the origin is given by $(x, y) \to (x\cos(\alpha) - y\sin(\alpha), x\sin(\alpha) + y\cos(\alpha))$, where α is the usual anticlockwise rotation. For a clockwise rotation, α is negative.
- **Rotation** through an angle α around the origin is modelled by the matrix $\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.
- A simple **reflection** swaps all points across to the opposite side of a line through the origin. The line may be expressed by its inclination angle, gradient or equation. The reflection across a line with slope m through the origin is given by $(x, y) \to$ $\left(\left(\dfrac{1-m^2}{1+m^2}\right)x + \left(\dfrac{2m}{1+m^2}\right)y, \left(\dfrac{2m}{1+m^2}\right)x - \left(\dfrac{1-m^2}{1+m^2}\right)y\right)$ or $(x, y) \to (px + qy, qx - py)$, where $p = \dfrac{1-m^2}{1+m^2}$ and $q = \dfrac{2m}{1+m^2}$. The reflection across a line of inclination θ through the origin is given by $(x, y) \to (\cos(2\theta)x + \sin(2\theta)y, \sin(2\theta)x - \cos(2\theta)y)$
- **Reflection** through a line at a slope of m or inclination $\theta = \tan^{-1}(m)$ through the origin is modelled by the matrix $\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$ where $p = \dfrac{1-m^2}{1+m^2}$ and $q = \dfrac{2m}{1+m^2}$ or $\mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

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- The **composition** of the transformations **S** and **T** is the combined transformation such that if **T**: $(x, y) \to (x', y')$ and **S**: $(x', y') \to (x'', y'')$ then the composition is $\mathbf{S} \circ \mathbf{T}$: $(x, y) \to (x', y') \to (x'', y'')$.
- The composition $\mathbf{S} \circ \mathbf{T}$ is given by the matrix product **ST**. In general, composition of transformations is not commutative.
- If it exists, the element that leaves everything unchanged for an operation is called the **identity** for that operation and is usually written as 1, 0 or **I**. For all transformations, the identity is **I**: $(x, y) \to (x, y)$.
- For translations the identity is none along and none up or down, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- For rotations the identity is a rotation though 0°, $\mathbf{R}_0$, (or 360°) or the equivalents in radians.
- For dilations the identity has both factors equal to 1.
- There is no reflection equal to the identity.
- The matrix representation for **I** for dilations, rotations and reflections is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- For a transformation **T**, if there exists a transformation **S** such that $\mathbf{S} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{S} = \mathbf{I}$ then **S** is called the **inverse** of **T** and written as $\mathbf{T}^{-1}$ so $\mathbf{T}^{-1} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{I}$.
- A translation that has an inverse is called an **invertible** transformation.
- The inverse of a translation $\begin{bmatrix} a \\ b \end{bmatrix}$ is the translation $\begin{bmatrix} -a \\ -b \end{bmatrix}$ in the opposite direction.
- The inverse of a dilation $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is the dilation $\mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}$ with reciprocal factors in the x and y directions.
- The inverse of a rotation $\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ is the rotation $(\mathbf{R}_\alpha)^{-1} = \mathbf{R}_{-\alpha} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$ in the opposite direction.
- Reflections are their own inverses.
- The determinant of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\det \mathbf{A} = |\mathbf{A}| = ad - bc$.
- The area of the image produced by a linear transformation **T** is $|\mathbf{T}|$ times the area of the object.
- If it exists, the inverse of a composition of linear transformations, $\mathbf{T}_1 \circ \mathbf{T}_2$, is given by $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$.

11 CHAPTER REVIEW
TRANSFORMATIONS IN THE PLANE

Multiple choice

1 Example 1 The image of the triangle $A(-2, -3)$ $B(1, 4)$ $C(2, -2)$ under the translation 5 up and 3 left is:

A $A'(3, -6)$ $B'(6, 1)$ $C'(7, 1)$ **B** $A'(3, 0)$ $B'(6, 7)$ $C'(7, 1)$
C $A'(-5, 2)$ $B'(-2, 9)$ $C'(-1, 3)$ **D** $A'(1, 2)$ $B'(4, 9)$ $C'(5, 3)$
E $A'(-7, -6)$ $B'(-4, 1)$ $C'(-3, -5)$

2 Example 4 What is the matrix for $\mathbf{T}: (x, y) \to (2y - 5x, 4x + y)$?

A $\begin{bmatrix} 2 & -5 \\ 4 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 4 \\ -5 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & -5 \\ 4 & 1 \end{bmatrix}$
D $\begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} -5 & 2 \\ 4 & 1 \end{bmatrix}$

3 Example 7 What is the matrix for a dilation by a factor of 2 in the x direction and 3 in the y direction?

A $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$
D $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

4 Example 9 Example 10 What is the image of the point $(2, 6)$ after a rotation of $\frac{2\pi}{3}$ around the origin?

A $(-3\sqrt{3} - 1, \sqrt{3} - 3)$ **B** $(1 - 3\sqrt{3}, 3 - \sqrt{3})$
C $(\sqrt{3} - 1, \sqrt{3} + 1)$ **D** $(1 - 3\sqrt{3}, 3\sqrt{3} + 1)$
E $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

5 Example 12 Example 17 What is the matrix for the inverse of the reflection through the line $y = 2x$?

A $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ **B** $\begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$ **C** $\begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix}$
D $\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$ **E** $\begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}$

6 Example 14 Example 15 Which of the following is true about compositions?

A Dilations and reflections commute **B** No transformations commute
C All transformations commute **D** None of the above
E Two rotations make a reflection

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Short answer

7 Example 2 Use column matrices to find the points $A(-3, 5)$ $B(1, 7)$ $C(4, 3)$ $D(8, -2)$ after they are transformed by a translation of 3 down and 7 right.

8 Example 3 **a** Show that the quadrilateral $A(-4, 3)$ $B(3, 3)$ $C(2, -1)$ $D(-5, -1)$ is a parallelogram.
b Find the image of the parallelogram under the transformation $(x, y) \to (3x + 3y, 2x + 4y)$.
c What is the shape of $A'B'C'D'$?

9 Example 5 Example 6 Example 7 **a** What is the transformation for a dilation **D** of factor 3 in the x direction and 1.5 in the y direction?
b What is the matrix for the dilation **D**?
c Find the image of the right-angled triangle $A(-3, -2)$ $B(3, 6)$ $C(-1, 9)$ under the dilation **D**.
d What is the shape of the image?

10 Example 8 Example 9 **a** What is the matrix for a rotation of 150° around the origin?
b Find the image of the rectangle $A(-1, -2)$ $B(4, -2)$ $C(4, 5)$ $D(-1, 5)$ under the rotation.
c What is the relationship between the object and the image?

11 Example 12 Find the image (correct to two decimal places) of the quadrilateral $A(-3, 4)$ $B(-1, 8)$ $C(2, 7)$ $D(4, 3)$ after reflection across a line through the origin of inclination $\frac{7\pi}{8}$.

12 Example 13 Find the image of the triangle $A(-2, 4)$ $B(7, 2)$ $C(5, 1)$ after application of the transformation $\mathbf{S} \circ \mathbf{T}$, where **S** is the translation 2 right and 4 down and **T** is the translation 5 right and 6 up.

13 Example 14 Example 15 **a** Find the image of the triangle $A(-2, -4)$ $B(1, 5)$ $C(3, 2)$ after the transformation $\mathbf{D} \circ \mathbf{M}$, where **M** is the reflection in the line through the origin with slope $-\frac{1}{3}$ and **D** is the dilation with an x factor of 2 and a y factor of 1.5.
b Use a calculator or other means to determine whether **D** and **M** commute.

14 Example 16 Example 17 Find the inverses of the following transformations.
a A translation 3 left and 4 up.
b A dilation with an x factor of $\frac{1}{2}$ and a y factor of 1.5.
c A rotation of $\frac{4\pi}{3}$.
d A reflection in the line through the origin with an inclination of 25°.

15 Example 19 Find the area of the image of the rectangle $A(-3, -4)$ $B(-3, 5)$ $C(2, 5)$ $D(2, -4)$ after the transformation $(x, y) \to (4x - y, x + y)$.

Application

16 Use vectors or other means to show that translation is associative under composition. That is, for any translations **S**, **R** and **T**, $\mathbf{S} \circ (\mathbf{R} \circ \mathbf{T}) = (\mathbf{S} \circ \mathbf{R}) \circ \mathbf{T}$.

17 **a** Find the area of the triangle $A(2, 6)$ $B(5, 2)$ $C(3, 1)$.
b Show that the image in **a**, after transformation by a dilation with an x factor of 2 and a y factor of 3.5, has an area that is 7 times that of the original triangle.

Practice quiz

18 Use matrices or other methods to prove that the composition of two rotations is another rotation.

4 MIXED REVISION
CHAPTERS 10 • 11 • 12

Multiple choice

1 If z is a complex number, the solutions to $z^2 - 2z + 10 = 0$ are:

A $z = -1 \pm 3i$ B $z = 1 \pm 3i$ C $z = -1 \pm i\sqrt{3}$ D $z = 1 \pm i\sqrt{3}$ E $z = -1 \pm \sqrt{11}$

2 What is the vector for the translation 2 down and 5 to the left?

A $\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$ B $\begin{bmatrix} 2 & -5 \end{bmatrix}$ C $\begin{bmatrix} -5 \\ -2 \end{bmatrix}$ D $\begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}$ E $\begin{bmatrix} -2 \\ -5 \end{bmatrix}$

3 The amplitude and period of the graph of $y = 3 \sin [2(x - \pi)]$ are respectively

A 3, 2 B $-3, \frac{1}{2}$ C $3, \frac{\pi}{2}$ D $3, \pi$ E $3, 2\pi$

4 The value of $\dfrac{1+\sqrt{3}-i\sqrt{3}}{1+i}$ in $a + bi$ form is:

A $-\frac{1}{2} - i\frac{(2\sqrt{3}-1)}{2}$ B $\frac{1}{2} - i\frac{(2\sqrt{3}-1)}{2}$ C $\frac{1}{2} - i\frac{(2\sqrt{3}+1)}{2}$

D $\frac{1}{2} + i\frac{(2\sqrt{3}-1)}{2}$ E $-\frac{1}{2} - i\frac{(2\sqrt{3}+1)}{2}$

5 What is the matrix for a reflection in the line $y = 3x$?

A $\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$ B $\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$ C $\begin{bmatrix} -8 & 10 \\ 10 & 8 \end{bmatrix}$ D $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ E $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

6 $y = 2 \sec (3x)$ can be expressed as

A $y = 2 \cos (3x)$ B $y = \dfrac{2}{\cos (3x)}$ C $y = 2\cos\left(\frac{1}{3}x\right)$

D $y = \dfrac{1}{2\cos(3x)}$ E $y = \frac{1}{2}\cos\left(\frac{1}{3}x\right)$

7 An equation with roots $\dfrac{-1 \pm i\sqrt{3}}{2}$ is:

A $z^2 + z + 1 = 0$ B $z^2 - z + 1 = 0$ C $z^2 - z - 1 = 0$

D $z^2 + z - 1 = 0$ E none of the above

8 What is the area of the image of the rhombus $A(-1, 3)$ $B(1, 7)$ $C(3, 3)$ $D(1, -1)$ under the transformation $(x, y) \rightarrow (3x + y)(5y - 4x)$?

A 16 B 112 C 224 D 304 E 608

9 The solution(s) of the equation $\tan(2x) = -\dfrac{\sqrt{3}}{3}$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is/are

A $-\frac{\pi}{6}, \frac{5\pi}{6}$ B $-\frac{\pi}{3}, \frac{4\pi}{3}$ C $-\frac{\pi}{6}, \frac{5\pi}{6}$ D $\frac{5\pi}{12}$ E $-\frac{\pi}{12}, \frac{5\pi}{12}$

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Short answer

1 Factorise $x^2 - 4x + 5$ over the complex plane.

2 What is the image of $A(-2, 1)$ $B(2, 4)$ $C(7, 2)$ $D(4, -3)$ under a dilation with factors of 3 in the x-direction and 2 in the y-direction?

3 Express $3\cos(\theta) + 2\sin(\theta)$ in the form of $r\cos(\theta - \alpha)$.

4 If $z = 1 + i$ and $w = -2 + 5i$, find in the form $x + yi$:

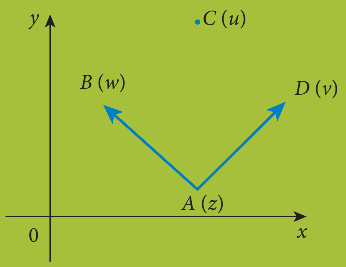
a $z\overline{w}$ b $|z+w|$

5 Find the image of $A(-1, -2)$ $B(2, 3)$ $C(7, 4)$ under a rotation of $-\frac{\pi}{6}$.

6 Sketch the graph of $y = 2\tan\left(\frac{x}{2}\right)$ for $0 < x < 5\pi$.

Application

1 In the diagram, the vertices of a square $ABCD$ are represented by the complex numbers z, w, u and v.



a Find the vector $\overrightarrow{AD}$ in terms of z and v. Find the vector $\overrightarrow{AB}$ in terms of z and w.

b Hence, or otherwise, find the complex number u that represents C in terms of z, w and v.

2 Consider the equation $z^2 + uz + (2 - i) = 0$. If i is a root of the equation, find the complex number u.

3 What is the single transformation that is the result of a reflection in the line through the origin with inclination 50° followed by a rotation of 20°?

4 Use matrices or other methods to prove that dilation is associative under composition. That is, for any dilations **S**, **R** and **T**, $\mathbf{S}\circ(\mathbf{R}\circ\mathbf{T}) = (\mathbf{S}\circ\mathbf{R})\circ\mathbf{T}$.

5 a Express $\sin(x) + 5\cos(x)$ in the form $r\sin(\theta + \alpha)$.

b Find the general solution for the equation $\sin(x) + 5\cos(x) = -\frac{1}{2}$.

c Use a graph to find the solution(s), in degrees correct to 2 decimal places, of $\sin(x) + 5\cos(x) = -\frac{1}{2}$ for $0° \le x \le 360°$.

6 The displacement of a particle is given by $x = 2\sin(3t) - \cos(3t)$, where x is measured in metres and time, t, in seconds.

a Sketch the graph of the displacement to illustrate that the particle follows Simple Harmonic Motion (SHM).

b Find the amplitude and period of the motion.

c Find the greatest distance, correct to 3 decimal places, that the particle moves from its origin.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12

TERMINOLOGY

amplitude
equilibrium
general solution
period
periodic functions
periodic motion
phase shift
reciprocal functions
simple harmonic motion
sum and difference formulas
transformation
translation

TRIGONOMETRY

TRIGONOMETRIC FUNCTIONS AND GRAPHS

Prior learning

THE BASIC TRIGONOMETRIC FUNCTIONS

- find all solutions of $f(a(x - b)) = c$ where f is one of sin, cos or tan (ACMSM042)
- graph functions with rules of the form $y = f(a(x - b))$ where f is one of sin, cos, or tan. (ACMSM043)

THE RECIPROCAL TRIGONOMETRIC FUNCTIONS, SECANT, COSECANT AND COTANGENT

- define the reciprocal trigonometric functions; sketch their graphs and graph simple transformations of them. (ACMSM045)

TRIGONOMETRIC IDENTITIES

- convert sums $a \cos x + b \sin x$ to $R \cos(x \pm \alpha)$ or $R \sin(x \pm \alpha)$ and apply these to sketch graphs, solve equations of the form $a \cos x + b \sin x = c$ and solve problems (ACMSM048)

APPLICATIONS OF TRIGONOMETRIC FUNCTIONS TO MODEL PERIODIC PHENOMENA

- model periodic motion using sine and cosine functions and understand the relevance of the period and amplitude of these functions in the model. (ACMSM050) AC

12.01 PERIOD, AMPLITUDE AND PHASE SHIFT

You already know the shapes of the basic trigonometric functions $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ from earlier work. You also saw how the constants a, b, c and d in $y = a \sin(x)$, $y = \sin(bx)$, $y = \sin(x + c)$ and $y = \sin(x) + d$ change the shape of the graph of $y = \sin(x)$. You have also done the same thing in Maths Methods for $y = \cos(x)$ and $y = \tan(x)$.

IMPORTANT

The **amplitude** of the functions $y = a \sin(x)$ or $y = a \cos(x)$ is a. It is the distance of the peaks and troughs from the average value of the function.

The **period** of a function $y = f(x)$ is the smallest value P such that $f(x) = f(x + P)$ for all x. The period of $y = \sin(bx)$ and $y = \cos(bx)$ is $\frac{2\pi}{b}$. The period of $y = \tan(bx)$ is $\frac{\pi}{b}$.

The **phase shift** of $y = \sin(x + c)$, $y = \cos(x + c)$ or $y = \tan(x + c)$ is c. It is the horizontal translation of the graphs from those of $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$. For positive values of c it is a displacement to the left, and for negative values of c it is to the right.

The value of d in $y = \sin(x) + d$ or $y = \cos(x) + d$ is the **vertical translation** of the graphs compared to those of $y = \sin(x)$ or $y = \cos(x)$. For positive values of d it is up, while for negative values it is down.

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Example 1

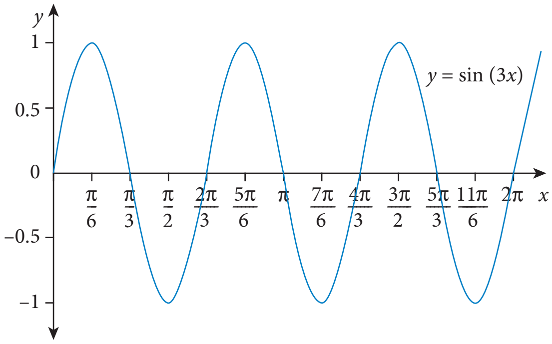
Sketch the graph of $y = \sin(3x)$ for $0 \le x \le 2\pi$.

Solution

Find the period of $y = \sin(3x)$ using $\frac{2\pi}{b}$.

Period $= \frac{2\pi}{3}$

The graph will repeat 3 times over the interval $0 \le x \le 2\pi$, with one cycle every $\frac{2\pi}{3}$ radians. It is compressed horizontally compared to $y = \sin(x)$.



The graphs of $y = -2\sin(x)$ and $y = \frac{1}{3}\cos(x)$ have the amplitudes 2 and $\frac{1}{3}$ respectively.

Example 2

Find the amplitude and period of the graph of $y = 4\cos\left(\frac{1}{2}x\right)$ and sketch its graph for one cycle.

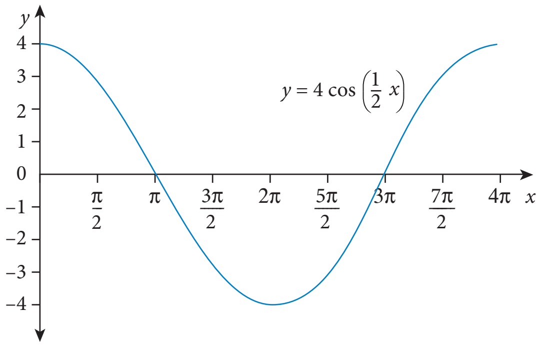
Solution

State the amplitude and period.

Amplitude $= 4$

Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

Sketch the graph for one cycle, with the graphed stretched vertically by a factor of 4 compared to $\cos(x)$ and stretched horizontally by a factor of $\frac{1}{\frac{1}{2}} = 2$.



Example 3

Find the amplitude, period and phase shift of the graph of $y = 6\cos\left(x + \frac{\pi}{4}\right)$ and hence sketch its graph for one cycle.

Solution

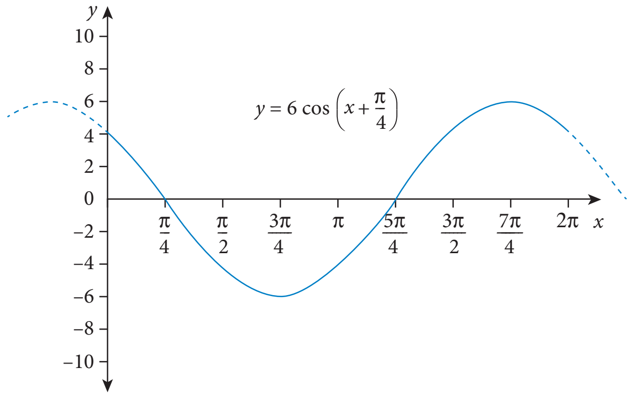
State the amplitude, period and phase shift.

Amplitude = 6

$$\text{Period} = \frac{2\pi}{1} = 2\pi$$

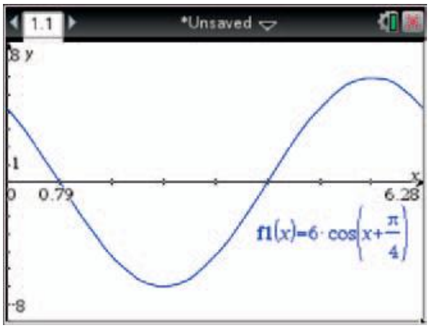
Phase change is $\frac{\pi}{4}$ units to the left.

The graph has the same period as $y = \cos(x)$ but it is shifted $\frac{\pi}{4}$ to the left so that everything occurs sooner. It is stretched vertically by a factor of 6 compared to $\cos(x)$.



TI-Nspire CAS

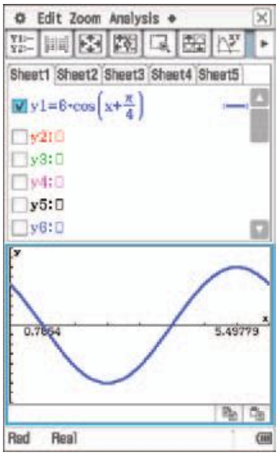
Make sure your calculator is set to radians, including the Graphing Angle in 9: Settings. Make the Window settings $0 \le x \le 2\pi$ with scale $\frac{\pi}{4}$ and $-8 \le y \le 8$ with scale 1.



ClassPad

Use the **Graph** application. Make sure your calculator is set to radians (**Rad**). Enter the function in $y1$ and tap the box to turn it on.

Tap the graph icon to draw the graph. Set the **View Window** to $0 \le x \le 2\pi$ with scale $\frac{\pi}{4}$ and $-8 \le y \le 8$ with scale 1.



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EXERCISE 12.01 Period, amplitude and phase shift

Concepts and techniques

1 Examples 1, 2 Sketch the graphs of the following from $0 \le x \le 2\pi$ and check with a CAS calculator.

a $y = \cos(2x)$ b $y = \tan(2x)$ c $y = -3\sin(x)$ d $y = \frac{1}{2}\sin\left(\frac{2x}{3}\right)$

2 For $-\pi \le x \le \pi$, sketch a graph of each of the following and check with a CAS calculator.

a $y = \sin(2x)$ b $y = \sin\left(\frac{1}{2}x\right)$ c $y = \cos(3x)$ d $y = \sin(2x) + 3$

3 Example 4 For $0 \le x \le 2\pi$, sketch a graph of each of the following and check with a CAS calculator.

a $y = 2\sin(x)$ b $y = \frac{1}{2}\sin(x)$ c $y = 3\cos(x)$ d $y = 2\sin(x) + 1$

4 Sketch a graph of each of the following and check with a CAS calculator, showing a full cycle.

a $y = 3\sin(4x)$ b $y = \frac{1}{2}\cos(3x)$ c $y = 5\sin(2x)$ d $y = \frac{1}{3}\sin(8x) + 4$

5 Find equations of two of the vertical asymptotes and hence sketch the graph of the following for $0 \le x \le \pi$.

a $y = \tan(2x)$ b $y = 2\tan(3x)$ c $y = -\tan(x + \pi)$

6 Example 3 Sketch graphs of each of the following and check with a CAS calculator, showing a full cycle of each.

a $y = \sin\left(x - \frac{\pi}{2}\right)$ b $y = \sin\left(x + \frac{\pi}{2}\right)$ c $y = \cos\left(x - \frac{\pi}{6}\right)$ d $y = \cos\left(x + \frac{\pi}{6}\right)$

e $y = 2\sin\left(x - \frac{\pi}{3}\right)$ f $y = 2\sin\left(x + \frac{\pi}{3}\right)$ g $y = 3\cos\left(x - \frac{\pi}{4}\right)$ h $y = 3\cos\left(x + \frac{\pi}{4}\right)$

Reasoning and communication

7 A large natural harbour has an entrance that is only 900 m wide. The movement of tides keeps the entrance clear, but a channel has to be dredged to the port 14 km from the harbour entrance. At the port, the depth of water in the channel is given by $d = 4.7 + (h - m)\cos(0.5t - 0.21)$, where t is the time in hours after high tide at the harbour entrance, h is the height of the high tide and m is the height of the following low tide. On a particular day, the high tide is given as 2.9 m at 1:37 p.m. and the low tide as 0.5 m.

a Sketch a graph showing the depth of water in the channel from 12 noon on that day until 12 noon on the next day, assuming the model is valid for that time.

b When was the first low tide at the port on that day?

c For what period was the water at a depth of 5 m or more in the channel between successive low tides?

d How long was it between high tides?

e How long was it between high tide at the harbour entrance and high tide at the port?

8 The sales, S, in 100s of units, of a seasonal product are modelled by $S = 54.8 + 32.5\cos\left(\frac{\pi t}{6}\right)$ where t is the time in months ($t = 1$ is January and $t = 12$ is December).

a Draw a graph of the sales for a period of 12 months.

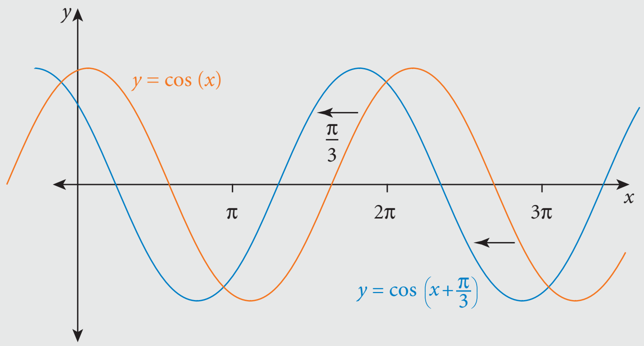
b Use the graph to determine the months for which sales exceed 6800 units.

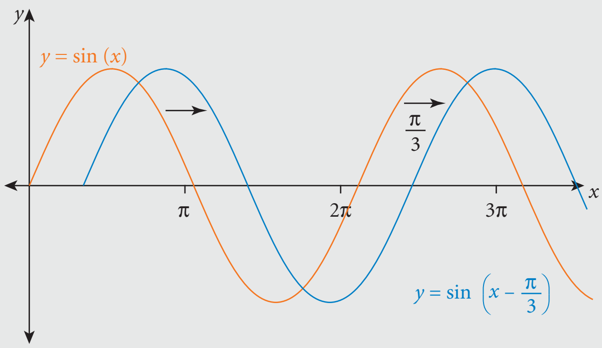
12.02 GENERAL TRIGONOMETRIC FUNCTIONS

You have already sketched graphs of the form $y = f(bx + c)$ where f is one of sine, cosine, or tangent.
$y = f(bx + c)$ can also be expressed as $y = f\left(b\left(x + \frac{c}{b}\right)\right)$.
If you compare $y = \sin(bx + c)$ to $y = f(x + c)$ you can see that the period must be taken into account when finding a phase shift.

IMPORTANT

The graphs of the functions $y = a\sin(bx + c)$, $y = a\cos(bx + c)$ and $y = a\tan(bx + c)$ are related to the graphs of the functions $y = a\sin(x)$, $y = a\cos(x)$ and $y = a\tan(x)$ by a phase shift of $\frac{c}{b}$. If $\frac{c}{b}$ is positive, the shift is to the left. If $\frac{c}{b}$ is negative, it is to the right.





When $\frac{c}{b}$ is positive, everything happens $\frac{c}{b}$ units before it does on the related basic trigonometric graph.

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Example 4

Sketch the graph of $y = 3\sin\left(2\left(x - \frac{\pi}{2}\right)\right)$ for $0 \le x \le \pi$.

Solution

State the amplitude and period.

Amplitude $= 3$

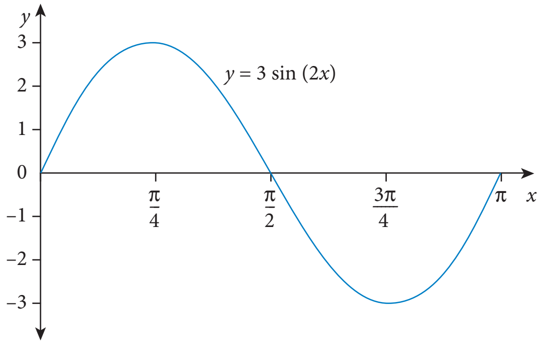
Period $= \frac{2\pi}{2} = \pi$

Using the formula $\frac{c}{b}$, find the horizontal translation.

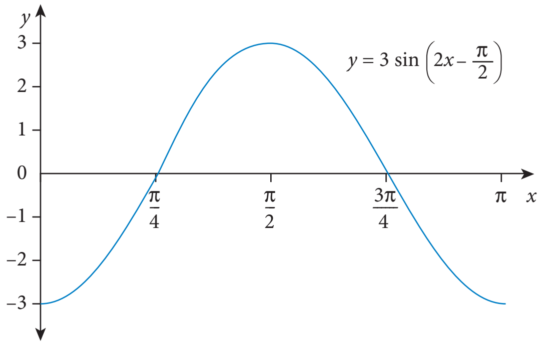
$\frac{c}{b} = -\frac{\pi}{2} \div 2 = -\frac{\pi}{4}$

Since the phase shift is negative, it is to the right.

Sketch the graph of $y = 3\sin(2x)$



Translate $y = 3\sin(2x)$ by $\frac{\pi}{4}$ to the right to sketch the graph of $y = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$.



For $\cos(bx + c)$ you can work the phase shift by asking 'When is $\cos(bx + c)$ the same as $\cos(x)$ for the zeros, maxima and minima?'. For $\cos(0)$, $bx + c = 0$ when $x = -\frac{c}{b}$. You can obviously use the same method with other trigonometric functions.

Example 5

Sketch the graph of $y = 2\cos\left(3x + \frac{\pi}{2}\right)$ for $0 \le x \le \pi$.

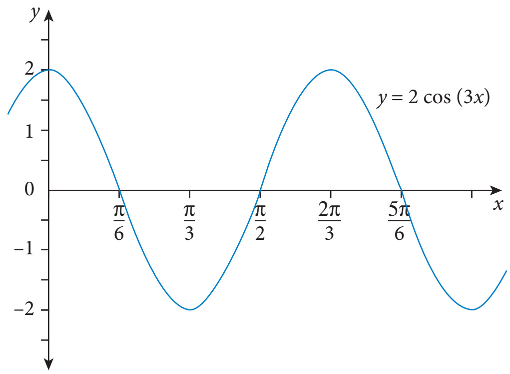
Solution

State the amplitude and period.

Amplitude = 3

Period $= \frac{2\pi}{3}$

Sketch the graph of $y = 2\cos(3x)$.



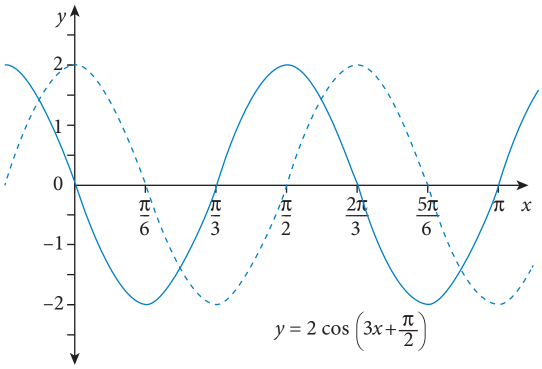
Find the horizontal translation for $y = 2\cos\left(3x + \frac{\pi}{2}\right)$.

$3x + \frac{\pi}{2} = 0$ gives $x = -\frac{\pi}{6}$ so everything moves $\frac{\pi}{6}$ left.

Translate $y = 3\sin(2x)$ by $\frac{\pi}{6}$ to the left to sketch the graph of

$$y = 2\cos\left(3x + \frac{\pi}{2}\right) = 2\cos\left(3\left(x + \frac{\pi}{6}\right)\right)$$

The zeros, maxima and minima all move back $\frac{\pi}{6}$.



Example **5** shows the technique of dotting in the graph of $y = 3\sin(2x)$ before applying the phase shift to draw the desired graph.

INVESTIGATION Musical beats

To tune a guitar, the frets are used to play a note on one string that should be the same as the base note of the next string. When the strings are in tune, the notes will be the same. When the strings are quite out of tune, they sound quite different. When they are close, but not quite the same, the sound will seem to beat (fade and strengthen rhythmically). This can be shown on a CAS calculator.

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1 Start with the Main (Casio) or Calculator (TI) page. Consider a note that should have a frequency of 440 Hz. Define this as c. Define 442 as d.

Now go to the Graph page.

Put in $f1(x)/\text{Y1} = \sin(2\pi cx)$, $f2(x)/\text{Y2} = \sin(2\pi dx)$ and $f3(x)/\text{Y3} = f1(x)/\text{Y1} + f2(x)/\text{Y2}$, and turn $f1(x)/\text{Y1}$ and $f2(x)/\text{Y2}$ off.

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Make the window settings or View Window so that $0 \le x \le 5$ and $-3 \le y \le 3$. Change d to 441 and graph it again.

Change d to 440.2, change the display so that $0 \le x \le 20$. Find what note corresponds to 440 Hz.

2 Try different values of c and d.

3 What do you find?

4 Write a short account of your results.

You can also sketch graphs with vertical translations as well as ones with varying amplitudes, periods and phase shifts.

WS

Sketching periodic functions – amplitude and period

WS

Sketching periodic functions

Example 6

Sketch the graph of $y = \frac{1}{3}\cos\left[4\left(x + \frac{\pi}{8}\right)\right] + 1$ for $0 \le x \le \pi$.

Solution

State the amplitude and period.

Amplitude $= \frac{1}{3}$

Period $= \frac{2\pi}{4} = \frac{\pi}{2}$

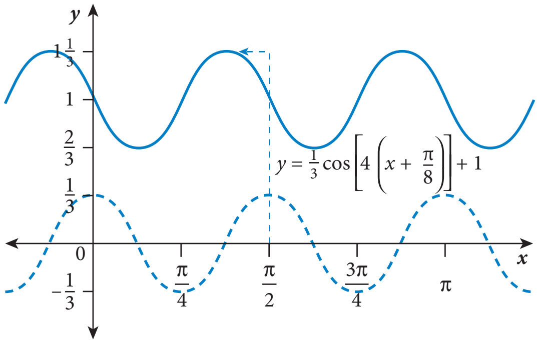
$y = \frac{1}{3}\cos\left[4\left(x + \frac{\pi}{8}\right)\right] + 1$ already has the phase shift worked out.

$4\left(x + \frac{\pi}{8}\right) = 0$ when $x = -\frac{\pi}{8}$, so everything moves $\frac{\pi}{8}$ left from $y = \frac{1}{3}\cos(4x) + 1$.

State the vertical translation.

The vertical translation is 1 unit up.

Sketch the graph of $\frac{1}{3}\cos(4x)$. Then sketch $y = \frac{1}{3}\cos\left[4\left(x + \frac{\pi}{8}\right)\right] + 1$ by moving everything $\frac{\pi}{8}$ left and 1 up.



Sketching periodic functions – phase and vertical shift

IMPORTANT

The graphs of the functions $y = a\sin(bx + c) + d$ and $y = a\cos(bx + c) + d$ are related to the graphs of the functions $y = a\sin(x)$ and $y = a\cos(x)$ by as follows.

Summary of the features of sine and cosine functions

For the functions $y = a\sin(bx + c) + d = a\sin\left[b\left(x + \frac{c}{b}\right)\right] + d$

and $y = a\cos(bx + c) + d = a\cos\left[b\left(x + \frac{c}{b}\right)\right] + d$

- the amplitude is the size of a. If a is negative, the graph is upside down
- the period is $\frac{2\pi}{b}$
- the value $\frac{c}{b}$ is called the **phase shift** and is the horizontal translation
- the average (mean) value is d.

Sketching trigonometric graphs

Graphs of sine and cosine functions may be sketched from the equation by two methods.

Method 1: Identify the translations and changes of scale from the equation.

Method 2: Identify the:

- starting point
- zeros
- end of a cycle
- maxima and minima

Find the values of x that produce the sine or cosine of 0, $\frac{\pi}{2}$, π, $\frac{\pi}{2}$ and 2π and the corresponding values of y.

EXERCISE 12.02 General trigonometric functions

Concepts and techniques

1 Examples 4-6 Sketch graphs of each of the following and check with a CAS calculator, showing a full cycle of each.

a $y=4\sin\left[2\left(x-\frac{\pi}{3}\right)\right]+2$ b $y=2\cos\left[3\left(x+\frac{\pi}{6}\right)\right]-3$ c $y=5\sin\left[4\left(x+\frac{\pi}{8}\right)\right]-2$

2 Sketch graphs of each of the following and check with a CAS calculator, showing a full cycle of each.

a $y=2\sin\left(2x-\frac{\pi}{3}\right)+3$ b $y=3\cos(4x+\pi)-2$ c $y=4-3\sin\left(6x+\frac{3\pi}{2}\right)$

d $y=5\cos\left(2x-\frac{3\pi}{4}\right)-3$ e $y=3-5\cos\left(3x-\frac{\pi}{3}\right)$ f $y=4-3\sin\left(4x+\frac{\pi}{4}\right)$

Reasoning and communication

3 Sketch for $-2\le x\le 2$

a $y=\sin(\pi x)$ b $y=3\cos(2\pi x)$

4 In an unusual meteorological investigation, the temperature, T °C, in a town in central Victoria was found to fluctuate approximately according to the rule $T=25+6\sin(0.1\pi t)$, where t is the number of hours after 10:00 a.m.

a Sketch a graph of the temperature fluctuations for a sufficient number of hours to be able to determine the maximum and minimum temperatures for that day and the next night.

b Use the graph to determine the maximum and minimum temperatures.

c When did they occur?

d At what time was the temperature:

i 27°C? ii 20°C?

e Why was this unusual?

5 In Calcutta, the highest mean monthly temperature is 29.45°C in June and the lowest is 18.3°C in December. Find a model for the temperature throughout the year and graph it.

12.03 APPROXIMATE SOLUTION OF TRIGONOMETRIC EQUATIONS

In previous sections you have sketched the graphs of the form $y=f(a(x-b))+c$ where f is one of sine, cosine or tangent.

In this section you will use these graphs to solve trigonometric equations. In most real situations, exact answers are rare so you need to be able to find approximate answers.

Example 7

Sketch the graph of $y = \tan(2x) + 1$ for $0 \le x \le \pi$ and find solutions to the equation $\tan(2x) + 1 = 0$.

Solution

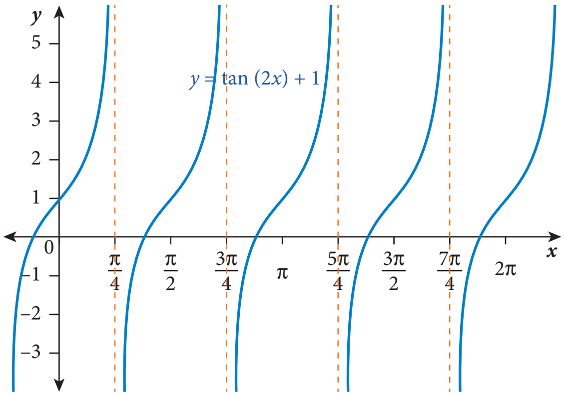
State the period.

Period $= \frac{\pi}{2}$

State the vertical translation.

The graph is translated 1 unit up compared to $y = \tan(2x)$.

Sketch the graph of $y = \tan(2x) + 1$

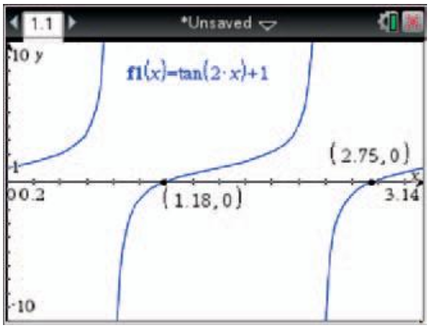


Find the solutions to the equation $\tan(2x) + 1 = 0$.

Solutions to the equation will be the x-intercepts of the graph. Approximate solutions to the equation $\tan(2x) + 1 = 0$ for $0 \le x \le \pi$ are $x = 1.2$, $x = 2.7$

TI-Nspire CAS

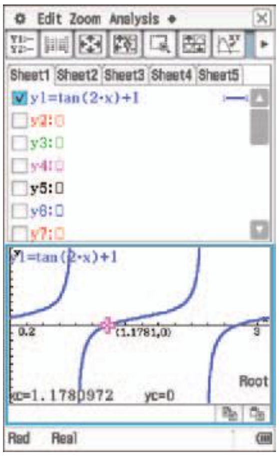
Use a Graph page and type in the function. with Window Settings of $0 \le x \le \pi$ and $-10 \le y \le 10$ (scales 0.2, 1). Use menu, 6: Analyse Graph and 1: Zero to find the zeros.



ClassPad

Use the **Graph** application, enter the function in **y1** and tap the box. Make **View Window** $0 \le x \le \pi$ and $-10 \le y \le 10$ (scales 0.2, 1). Tap to draw the graph.

Tap **Analysis**, **G-Solve** and **Root**.



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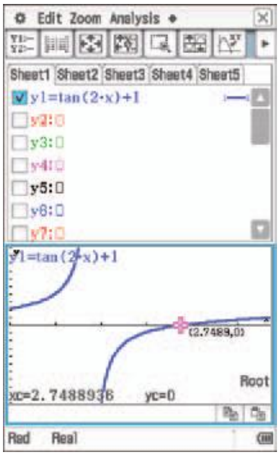
You can obtain the larger root by resetting the lower boundary (such as by setting xmin=2) and repeating the commands given above.



You can find approximate solutions that correspond to the intersections of graphs.

Example 8

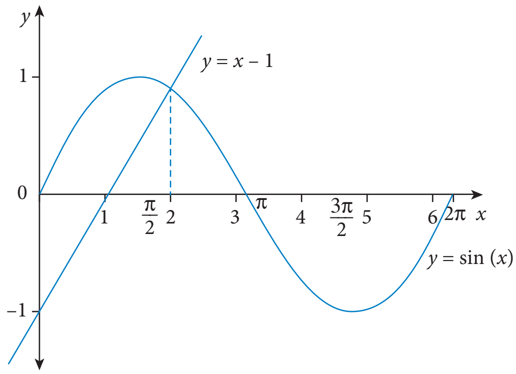
Find approximate solutions to the equation $\sin(x) = x - 1$ by sketching the graphs $y = \sin(x)$ and $y = x - 1$ on a Cartesian plane.

Solution

When sketching the two graphs together, we use $\pi \approx 3.14$, $2\pi \approx 6.28$ and so on to label the x-axis as shown.

To sketch $y = x - 1$, find the gradient and y-intercept or find the x- and y-intercepts.

x-intercept (where $y = 0$) is 1 and y-intercept (where $x = 0$) is -1.

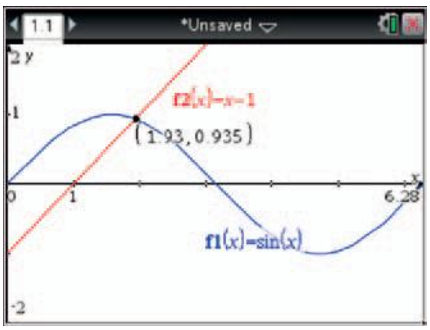


The solution to $\sin(x) = x - 1$ is at the point of intersection of the two graphs.

$x \approx 2$

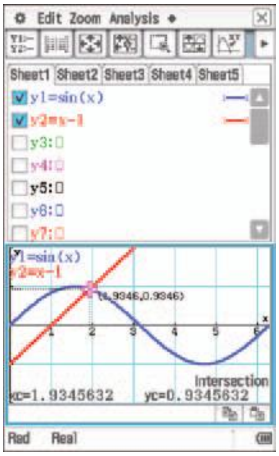
TI-Nspire CAS

Use a Graph page and type in the function. Use Graph Entry/Edit to enter the second function. Set suitable values for the scales and use Analyse Graph to find the Intersection.



ClassPad

Use the **Graph** application and enter the functions in $y1$ and $y2$ and tap the boxes.
Under ⚙ tap **View Window** to set $0 \le x \le 2\pi$ and $-2 \le y \le 2$ and tap to draw the graphs. Tap **Analysis**, **G-Solve** and **Intersection** to find the point.



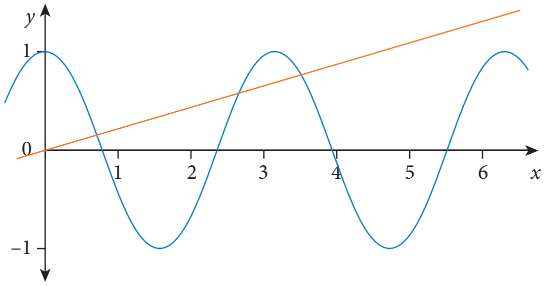
You can often use graphs to find how many solutions an equation has.

Example 9

How many solutions are there for $\cos(2x) = \frac{x}{4}$ in the domain $0 \le x \le 2\pi$?

Solution

Sketch $y = \cos(2x)$ and $y = \frac{x}{4}$ on the same set of axes.
$y = \cos(2x)$ has amplitude 1 and period π.
$y = \frac{x}{4}$ has x-intercept 0 and passes through (4, 1).



There are 3 points of intersection of the graphs.

The equation $\cos(2x) = \frac{x}{4}$ has 3 solutions.

EXERCISE 12.03 Approximate solution of trigonometric equations

Concepts and techniques

1 Example 7 Sketch the graph of $y = 2\sin(x) - 1$ for $0 \le x \le 2\pi$ and hence find approximate solutions to the equation $2\sin(x) - 1 = 0$.

2 Example 8 Solve $\sin(x) = x$ for $0 \le x \le 2\pi$ graphically by sketching $y = \sin(x)$ and $y = x$ on the same number plane.

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3 Solve $\cos(x) = 2x - 3$ for $0 \le x \le 2\pi$ by finding the points of intersection of the graphs $y = \cos(x)$ and $y = 2x - 3$.

4 Solve $\tan(x) = x$ graphically in the domain $0 \le x \le 2\pi$.

5 Example 9 Show graphically that $\sin(x) = \dfrac{x}{2}$ has

a 2 solutions for $0 \le x \le 2\pi$

b 3 solutions for $-\pi \le x \le \pi$

Reasoning and communication

6 The depth of water in a harbour, in metres, can be modelled by $h = 2.8\cos(0.52t)$, where t is the number of hours since high tide. The depth is considered adequate for ships for only 2 hours either side of high tide. What is the minimum depth considered adequate?

12.04 EXACT SOLUTION OF TRIGONOMETRIC EQUATIONS

In this section you will solve trigonometric equations exactly. The basic technique for solving trigonometric equations exactly is to rearrange the original equation to obtain an equation of the type $f(a(x - b)) = c$, where f is the sin, cos or tan function. You then solve the $a(x - b) = f^{-1}(c)$, where f^{-1} gives the value(s) of x that give $\sin(x) = c$, $\cos(x) = c$ or $\tan(x) = c$.

Example 10

Solve the equation $2\sin\left(x-\frac{\pi}{4}\right) = -1$ for $0 \le x \le 2\pi$.

Solution

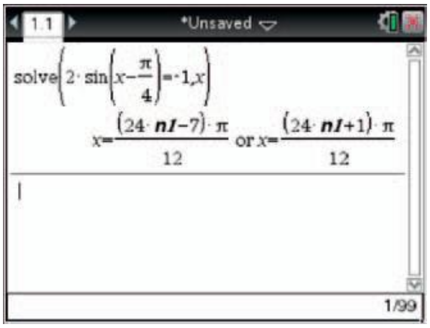
Write the equation.	$2\sin\left(x-\frac{\pi}{4}\right) = -1$
Rearrange to the form sin () =.	$\sin\left(x-\frac{\pi}{4}\right) = -\frac{1}{2}$
Find the value for which $\sin(x) = \frac{1}{2}$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
Use the CAST diagram or another method to find the values for $\sin(x) = -\frac{1}{2}$	$\sin(x) = -\frac{1}{2}$ for $x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$
Write and solve the first simplified equation.	$x-\frac{\pi}{4} = -\frac{5\pi}{6}$, so $x = -\frac{7\pi}{12}$
Do the next one.	$x-\frac{\pi}{4} = -\frac{\pi}{6}$, so $x = \frac{\pi}{12}$
Do the next one.	$x-\frac{\pi}{4} = \frac{7\pi}{6}$, so $x = \frac{17\pi}{12}$
Do the last one.	$x-\frac{\pi}{4} = \frac{11\pi}{6}$, so $x = \frac{25\pi}{12}$

But $-\frac{7\pi}{12}$ and $\frac{25\pi}{12}$ are outside $0 \le x \le 2\pi$.

The solutions are $x = \frac{\pi}{12}$ or $x = \frac{17\pi}{12}$.

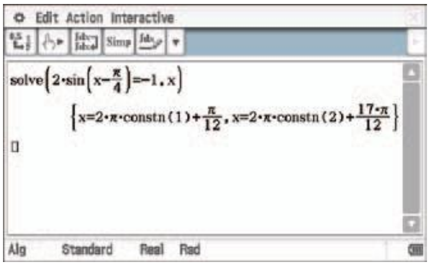
TI-Nspire CAS

Use a Calculator page, and make sure it is set on Radians and Auto or Exact Calculation mode.



ClassPad

Use the Main $\sqrt{\alpha}$ menu and make sure your calculator is set on radians (**Rad**) and **Standard** calculation.



Both calculators have integer constants in their answers. $n1$ in the TI-Nspire means an integer, and constn(1) and constn(2) both mean integers for the CASIO. Only some integer values give answers with $0 \le x \le 2\pi$.

TI-Nspire CAS

Substitute $n1 = 1$ and $n1 = 0$.

$$x = \frac{(24 \times 1 - 7)\pi}{12} = \frac{17\pi}{12}$$

$$x = \frac{(24 \times 0 + 1)\pi}{12} = \frac{\pi}{12}$$

ClassPad

Substitute 0 for both constants.

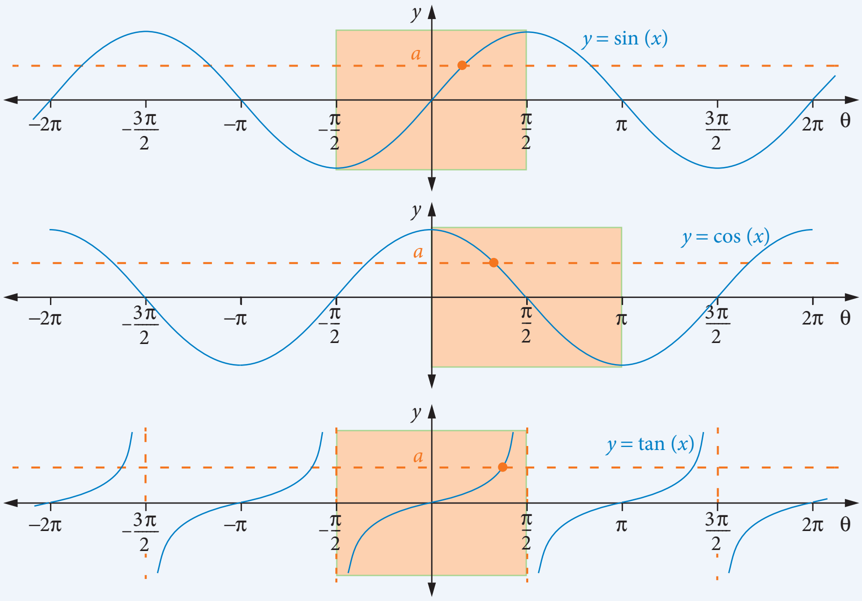
$$x = 2\pi \times 0 + \frac{17\pi}{12} = \frac{17\pi}{12}$$

$$x = 2\pi \times 0 + \frac{\pi}{12} = \frac{\pi}{12}$$

In some trigonometric equations a **general solution** is required because the domain is not restricted and the answers go on and on.

INVESTIGATION General solutions to trigonometric equations

You are familiar with the graphs of $\sin(x)$, $\cos(x)$ and $\tan(x)$ as shown here.



There is exactly one solution to equations of the form $\sin(x) = a$ in the range $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

There is exactly one solution to the equation $\tan(x) = a$ in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

There is exactly one solution to the equation $\cos(x) = a$ in the range $0 \le x \le \pi$.

The functions $\arcsin(a)$, $\arccos(a)$ and $\arctan(a)$ are restricted to these ranges and the calculator functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ show only these values.

Notice that the range for arcsin and arctan are slightly different because tan is not defined for $x = \pm\frac{\pi}{2}$.

For the equation $\tan(x) = a$, if θ is the solution in the domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $x = n\pi + \theta$, where n is an integer. Remember that this means that n can be positive, zero or negative.

This type of solution is known as a general solution to the trigonometric equation. General solutions to $\sin(x) = a$ and $\cos(x) = a$ can also be expressed in terms of n and π, but the expressions are a little more complicated.

Work in groups of two or three to work out the general solutions of $\sin(\theta) = a$ and $\cos(\theta) = a$.

From the investigation above, you and others in your class may have found different ways of writing the general solutions of sine and cosine functions. To check whether they are the same, you could try working out the values for different cases. The formulas given below are one version of the way they can be written.

IMPORTANT

General solutions of trigonometric equations

For the equation $\sin(x) = a$, if θ is the solution in the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $x = n\pi + (-1)^n \times \theta$, where n is an integer.

For the equation $\cos(x) = a$, if θ is the solution in the domain $0 \le \theta \le \pi$, then $x = 2n\pi \pm \theta$, where n is an integer.

For the equation $\tan(x) = a$, if θ is the solution in the domain $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $x = n\pi + \theta$, where n is an integer.

Example 11

Solve $\cos(x) = -\frac{\sqrt{2}}{2}$.

Solution

Find the value for which $\cos(x) = \frac{\sqrt{2}}{2}$. $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Use the CAST diagram to find the solution in the range $0 \le \theta \le \pi$. $\cos\left(\pi - \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ so $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

Use symmetry, the graph of $\cos(x)$ or the formula. The general solution is $x = 2n\pi \pm \frac{3\pi}{4}$ for $n \in \mathbf{Z}$.

Example **10** shows that your CAS calculator will find the general solution unless you restrict the range of solutions. You will usually be asked for answers within a restricted domain. In cases where the trigonometric function is not equal to one of the exact values, you can use your calculator to find approximate answers, or write the exact answer in terms of $\sin^{-1}(a)$, $\cos^{-1}(a)$ or $\tan^{-1}(a)$.

The values of $\sin^{-1}(a)$, $\cos^{-1}(a)$ and $\tan^{-1}(a)$ are always taken to be in the domains $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $0 \le \theta \le \pi$, and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ respectively. These values are sometimes called the **principal values**.

Example 12

Solve $2\tan\left(3x + \frac{\pi}{4}\right) - 7 = 0$ for $-\pi \le x \le 2\pi$, correct to 2 decimal places.

Solution

Write the equation. $2\tan\left(3x + \frac{\pi}{4}\right) - 7 = 0$

Rearrange and simplify. $\tan\left[3\left(x + \frac{\pi}{12}\right)\right] = 3.5$

Find the solution in $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. $3\left(x + \frac{\pi}{12}\right) = \tan^{-1}(3.5) \approx 1.2925$

Write the general solution. $3\left(x + \frac{\pi}{12}\right) = n\pi + \tan^{-1}(3.5)$

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Divide by 3.

$$x+\frac{\pi}{12}=\frac{n\pi+\tan^{-1}(3.5)}{3}$$

Solve for x.

$$x=\frac{n\pi+\tan^{-1}(3.5)}{3}-\frac{\pi}{12}=\frac{(4n-1)\pi+4\tan^{-1}(3.5)}{12}$$

Substitute values of n to find answers in the right range, from $-\pi \approx -3.14$ to $2\pi \approx 6.28$.

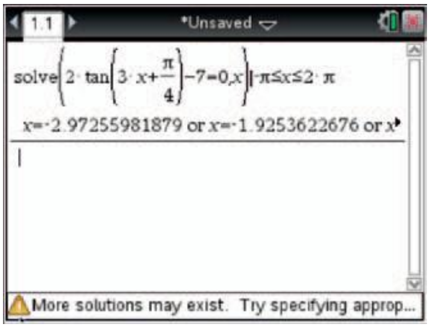
For $n=-4$, $x \approx -4.02$; for $n=-3$, $x \approx -2.97$; for $n=-2$, $x \approx -1.93$; for $n=5$, $x \approx 5.41$; for $n=6$, $x \approx 6.45$

Write the answers.

$x = -2.97, -1.93, -0.88, 0.17, 1.22, 2.26, 3.31, 4.36, 5.41$

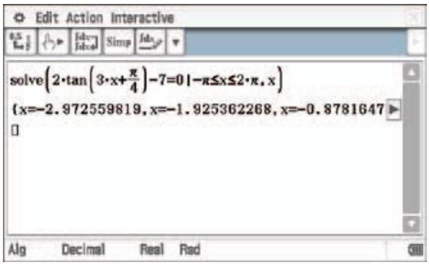
TI-Nspire CAS

Use solve(), but also specify a domain. The | and ≤ signs are in the menu obtained by pressing ctrl =. You can see the rest of the solutions using the arrows.



ClassPad

Use **solve()** with your calculator set on **Decimal**, but also specify a domain. The | and ≤ signs are in the menu obtained by pressing Keyboard then tapping Math3. You can see the rest of the solutions by tapping the ▶ or ◀.



EXERCISE 12.04 Exact solution of trigonometric equations

Concepts and techniques

1 Example 10 Solve the following equations for $0 \le x \le 2\pi$.

a $2\cos\left(x+\frac{\pi}{2}\right)=-1$ b $-\sqrt{3}\tan\left(2x+\frac{\pi}{4}\right)=1$ c $\frac{1}{\sqrt{3}}\sin(3x)=-\frac{1}{2}$

2 Solve each of the following equations for the domain given.

a $2\cos(x)=-2 \quad -\pi \le x \le \pi$
b $\sin(x)-1=0 \quad 0 \le x \le 2\pi$
c $1-\tan(x)=0 \quad -2\pi \le x \le 0$
d $6\cos(x)-3=3 \quad -\pi \le x \le 2\pi$
e $10\cos(x)+5=-5 \quad -\pi \le x \le \pi$
f $\cos(x)+8=8 \quad 0 \le x \le 3\pi$
g $3\tan(x)+3=0 \quad -\pi \le x \le \pi$
h $2\sin(x)-1=0 \quad -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$
i $6-4\cos(x)=4 \quad -\pi \le x \le 2\pi$
j $2\cos(x)-\sqrt{3}=0 \quad \pi \le x \le 3\pi$

3 Solve each of the following for $0° \le \theta \le 360°$.

a $\tan(\theta) = 1$ b $\sin(\theta) = \frac{1}{2}$ c $\cos(\theta) = -\frac{\sqrt{3}}{2}$ d $\sin(\theta) = -\frac{1}{\sqrt{2}}$

e $\cos(\theta) = 0$ f $\tan(\theta) = \sqrt{3}$ g $\cos(\theta) = -\frac{1}{2}$ h $\tan(\theta) = -\frac{1}{\sqrt{3}}$

i $\sin(\theta) = \sin(50°)$ j $\cos(\theta) = \cos(115°)$

4 Example 11 Solve the following equations, giving the general solution.

a $\sin(x) = -\frac{\sqrt{3}}{2}$ b $\cos(x) = \frac{\sqrt{3}}{2}$ c $\tan(x) = \frac{\sqrt{3}}{3}$

5 Example 12 Solve each of the following over the domain $0 \le x \le \pi$, correct to 4 decimal places.

a $3\sin(x) = 1$ b $4\cos(x) + 1 = 0$ c $5\tan(x) - 2 = 0$

d $9\cos(x) = -5$ e $7\sin(x) - 3 = 0$ f $3\tan(x) + 20 = 0$

6 Find all values of x in the domain $0 \le x \le 2\pi$ for which:

a $\sin(2x) = 0.5$ b $\cos(2x) = -1$ c $\sin(2x) = 1$

d $\sin(2x) = -0.5$ e $\sin(x) = \cos(x)$ f $\sin(x) = \sqrt{3}\cos(x)$

Reasoning and communication

7 Find general solutions for each of the following equations.

a $2\cos\left(2\left(x+\frac{\pi}{4}\right)\right)=\sqrt{3}$ b $2\sin\left(3\left(x+\frac{\pi}{3}\right)\right)+1=0$ c $\tan\left(2x-\frac{\pi}{6}\right)-1=0$

d $2\sin\left(2\left(x-\frac{\pi}{2}\right)\right)=-\sqrt{3}$ e $\cos\left(2x+\frac{\pi}{4}\right)-\frac{1}{\sqrt{2}}=0$

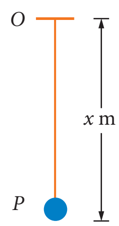
8 Find all solutions of the following equations for $-\pi \le x \le \pi$, correct to 2 decimal places,

a $3\sin\left(2\left(x-\frac{\pi}{3}\right)\right)=-2$ b $2\tan\left(3\left(x+\frac{\pi}{5}\right)\right)=3$ c $4\cos\left(4x+\frac{\pi}{4}\right)=3$

d $5\sin(3x-1)-4=0$ e $2\cos(5x+2)+1.3=0$

9 Find, correct to 2 decimal places, the two values of x closest to 0 for which $5\sin\left(2x-\frac{\pi}{6}\right)-2=0$.

10 A particle P bobs up and down on the end of an elastic string that is fixed at O. Its distance below O is given by the rule $x = 0.34 + 0.04\sin(4t)$ where x is in metres and t is the time in seconds since observation began.



a How far below O is the particle initially?

b How far below O is the particle at $t = 0.6$?

c What are the distance below O of the lowest point in the particle's path, and the value of t when the particle first reaches this point?

12.05 RECIPROCAL TRIGONOMETRIC FUNCTIONS

You have already done some work on the reciprocal trigonometric functions in Chapter 9.

IMPORTANT

Reciprocal functions

$$\operatorname{cosec}(x)=\frac{1}{\sin(x)}$$

$$\sec(x)=\frac{1}{\cos(x)}$$

$$\cot(x)=\frac{1}{\tan(x)}$$

INVESTIGATION Important points in reciprocal functions

1. Use $\operatorname{cosec}(x)=\frac{1}{\sin(x)}$, $\sec(x)=\frac{1}{\cos(x)}$ and $\cot(x)=\frac{1}{\tan(x)}$ to complete the table below.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
cosec (x)					
sec (x)					
cot (x)					

e.g. $$\operatorname{cosec}\left(\frac{\pi}{2}\right)=\frac{1}{\sin\left(\frac{\pi}{2}\right)}$$
$$=\frac{1}{1}$$
$$=1$$

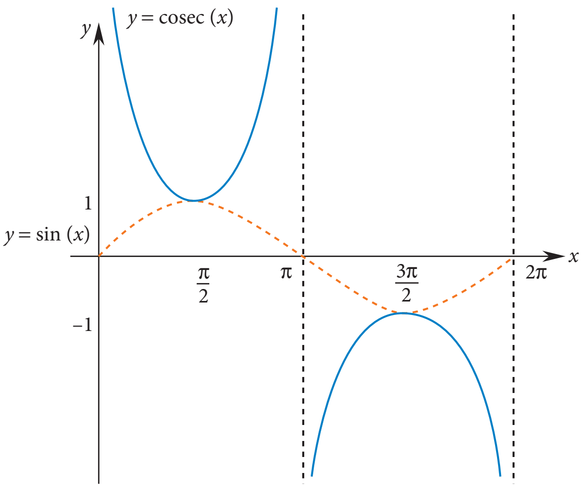
2. Find any asymptotes

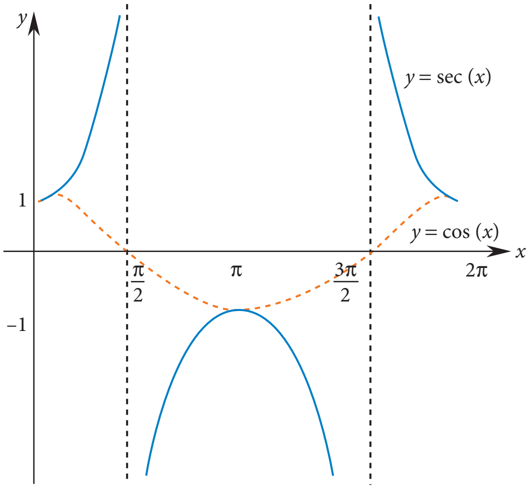
e.g. $$\sec\left(\frac{\pi}{2}\right)=\frac{1}{\cos\left(\frac{\pi}{2}\right)}$$
$$=\frac{1}{0}\quad(\text{undefined})$$

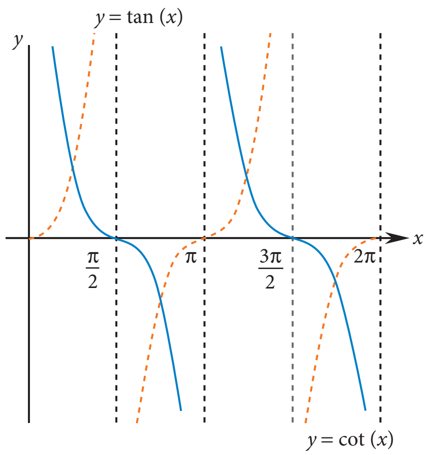
Discover what the values are on either side of the asymptotes.

3. Sketch each graph of the reciprocal trigonometric functions.

The graphs of the reciprocal ratios are drawn below.





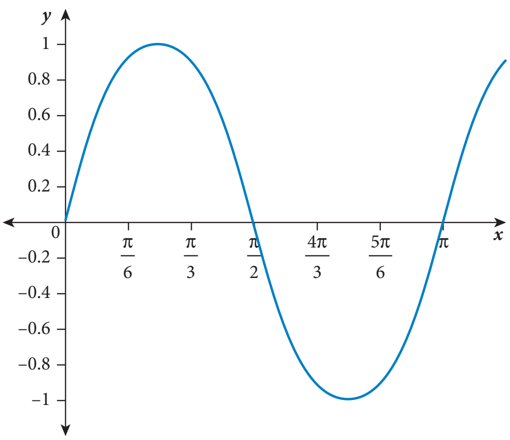


Example 13

Sketch the graph of $y = \sin(2x)$ and hence sketch the graph of $y = \text{cosec}(2x)$, for $0 \le x \le \pi$.

ClassPad

Sketch $y = \sin(2x)$ for $0 \le x \le \pi$.



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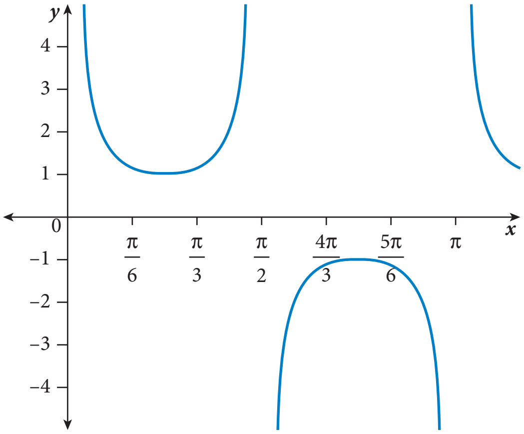
Sketch $y = \text{cosec }(2x)$ as the reciprocal of the graph of $y = \sin(2x)$.



Transformations of the reciprocal functions $y = \text{cosec }(x)$, $y = \sec(x)$ and $y = \cot(x)$ can be graphed using two methods.

IMPORTANT

Sketching reciprocal trigonometric function graphs

Method 1

Sketch the function for sin, cos or tan as appropriate and use the graph to draw the reciprocal function.

Method 2

Sketch the basic graph of $y = \text{cosec }(x)$, $y = \sec(x)$ and $y = \cot(x)$ and transform it using the properties of transformations.

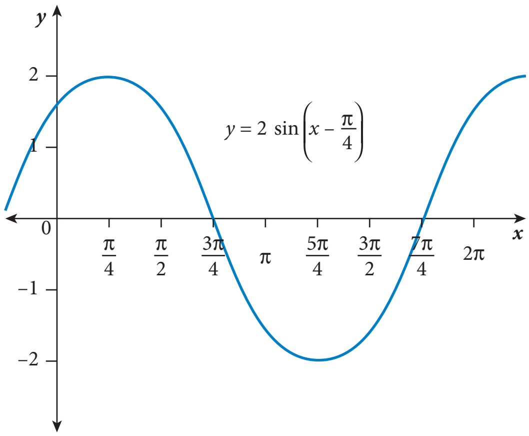
The following is an example showing Method 1.

Example 14

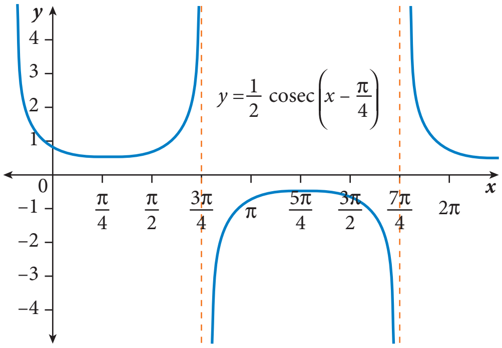
Sketch the graph of $y = 2\sin\left(x - \frac{\pi}{4}\right)$ and hence sketch the graph of $y = \frac{1}{2}\text{cosec}\left(x - \frac{\pi}{4}\right)$, for $0 \le x \le 2\pi$.

Solution

Sketch the graph of $y = 2\sin\left(x - \frac{\pi}{4}\right)$.



Sketch the graph of $y = \frac{1}{2}\operatorname{cosec}\left(x - \frac{\pi}{4}\right)$, as the reciprocal of the graph of $y = 2\sin\left(x - \frac{\pi}{4}\right)$.



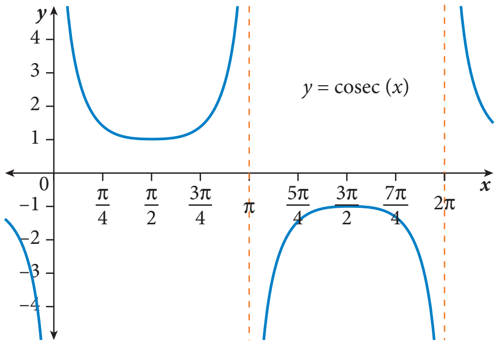
The following is an example showing Method 2.

Example 15

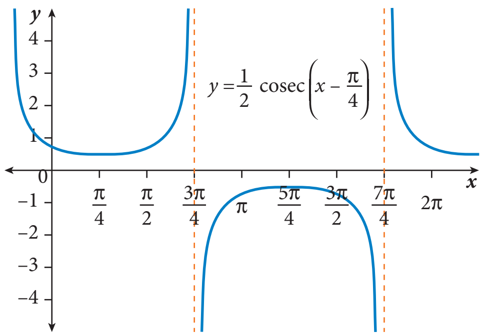
Sketch the graph of $y = \operatorname{cosec}(x)$ and hence sketch the graph of $y = \frac{1}{2}\operatorname{cosec}\left(x - \frac{\pi}{4}\right)$, for $0 \le x \le 2\pi$.

Solution

Sketch the graph of $y = \operatorname{cosec}(x)$.



Sketch the graph of $y = \frac{1}{2}\operatorname{cosec}\left(x - \frac{\pi}{4}\right)$, as the transformation of the graph of $y = \operatorname{cosec}(x)$.



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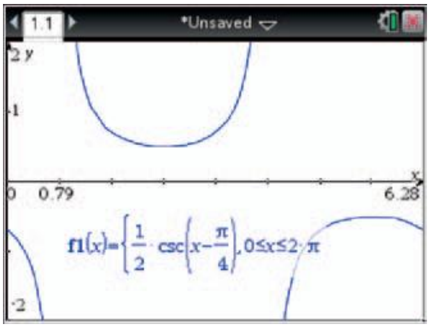
TI-Nspire CAS

Use a Graph page. The cosec function is shown as csc in the [trig] menu. Enter the function and the required range. Change the Window Settings to appropriate values.

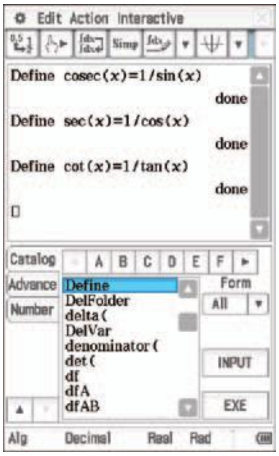


ClassPad

The reciprocal trigonometric functions are not listed on the calculator.

You must first define them, using the Main $\sqrt{\alpha}$ menu.

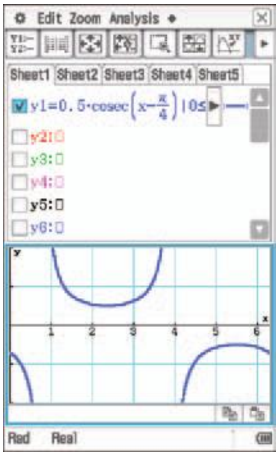
This need only be done once.



The graphs can then be sketched in the normal way.

y1 = 0.5cosec $(x - \frac{\pi}{4})$ | $0 \le x \le 2\pi$

Use **View Window** to set suitable values for x and y.



For the values $0 \le x \le 2\pi$, the y scale was chosen as –2 to 2 on the TI-Nspire to ensure that the graph was shown in approximately correct proportion.

EXERCISE 12.05 Reciprocal trigonometric functions

Concepts and techniques

1 Example 13 Use the graph of $y = \tan(x)$ to sketch $y = \cot(x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

2 Use the graph of $y = \cos(3x)$ to sketch the graph of $y = \sec(3x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

3 Use the graph of $y = \sin(4x)$ to sketch the graph of $y = \operatorname{cosec}(4x)$ from 0 to π.

4 Use the graph of $y = \tan(2x)$ to sketch the graph of $y = \cot(2x)$ from $-\pi$ to π.

5 Example 14 Use the graph of $y=\cos\left(x-\frac{\pi}{4}\right)$ to sketch the graph of $y=\sec\left(x-\frac{\pi}{4}\right)$ from $-\pi$ to π.

6 Use the graph of $y=\sin\left(x+\frac{\pi}{6}\right)$ to sketch the graph of $y=\operatorname{cosec}\left(x+\frac{\pi}{6}\right)$ from 0 to 2π.

7 Use the graph of $y=\tan\left(x+\frac{\pi}{4}\right)$ to sketch the graph of $y=\cot\left(x+\frac{\pi}{4}\right)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

8 Example 15 Sketch the graph of $y=\operatorname{cosec}\left(x+\frac{\pi}{4}\right)$ from 0 to 2π.

9 Sketch the graph of $y=\sec\left(3x+\frac{\pi}{4}\right)$ from 0 to π.

10 Sketch the graph of $y=\cot\left(2x-\frac{\pi}{4}\right)$ from 0 to π.

11 Sketch the graph of $y=\sec\left(2x-\frac{\pi}{3}\right)$ from 0 to π.

12.06 *A* SIN (*X*) + *B* COS (*X*)

You can use the addition theorem for $\sin(x + y)$ to change expressions of the form $a\sin(x) + b\cos(x)$ to expressions of the form $r\sin(x + \alpha)$.

$\sin(x + \alpha) = \sin(x)\cos(\alpha) + \cos(x)\sin(\alpha)$

Multiplying by r gives
$r\sin(x + \alpha) = (r\cos(y))\sin(\alpha) +$
$(r\sin(\alpha))\cos(x)$

IMPORTANT

$a\sin(x) + b\cos(x) =$
$r\sin(x + \alpha)$ where
$r=\sqrt{a^2+b^2}$ and
$\tan(\alpha)=\frac{b}{a}$

Choosing $a = r\cos(\alpha)$ and $b = r\sin(\alpha)$

Now $r=\sqrt{r^2\times 1} = \sqrt{r^2\sin^2(\alpha)+\cos^2(\alpha)} = \sqrt{r^2\sin^2(\alpha)+r^2\cos^2(\alpha)} = \sqrt{a^2+b^2}$

and $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)} = \dfrac{r\sin(\alpha)}{r\cos(\alpha)} = \dfrac{b}{a}$

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Example 16

Write $\sqrt{3}\sin(x)+\cos(x)$ in the form $r\sin(x+\alpha)$.

Solution

Write the rule.

$a\sin(\theta)+b\cos(\theta)=r\sin(\theta+\alpha)$ where $r=\sqrt{a^2+b^2}$ and $\tan(\alpha)=\dfrac{b}{a}$

Write the values for a and b.

$a=\sqrt{3}, b=1$

Calculate r and α.

$$\begin{aligned} r&=\sqrt{a^2+b^2}\\ &=\sqrt{\sqrt{3}^2+1^2}\\ &=\sqrt{3+1}\\ &=\sqrt{4}\\ &=2 \end{aligned}$$

$$\begin{aligned} \tan(\alpha)&=\frac{b}{a}\\ &=\frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \alpha&=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\\ &=30^\circ \end{aligned}$$

Write the answer.

$\sqrt{3}\sin(x)+\cos(x)=2\sin(x+30^\circ)$

Example 17

Write $3\sin(\theta)+2\cos(\theta)$ in the form $r\sin(\theta+\alpha)$.

Solution

Write the rule.

$a\sin(\theta)+b\cos(\theta)=r\sin(\theta+\alpha)$ where $r=\sqrt{a^2+b^2}$ and $\tan(\alpha)=\dfrac{b}{a}$

Write the values for a and b.

$a=3, b=2$

Calculate r and α.

$$\begin{aligned} r&=\sqrt{a^2+b^2}\\ &=\sqrt{3^2+2^2}\\ &=\sqrt{9+4}\\ &=\sqrt{13} \end{aligned}$$

$$\begin{aligned} \tan(\alpha)&=\frac{b}{a}\\ &=\frac{2}{3} \end{aligned}$$

$$\begin{aligned} \alpha&=\tan^{-1}\left(\frac{2}{3}\right)\\ &=33^\circ 41' \end{aligned}$$

Write the answer.

$3\sin(\theta)+2\cos(\theta)=\sqrt{13}\sin(\theta+33^\circ 41')$

Similar results can be found to create the expressions $r\cos(x \pm a)$ or $r\sin(x \pm a)$.

Example 18

Write the expression $3\cos(x) + 4\sin(x)$ in the form of $r\cos(x - \alpha)$.

Solution

Write the expansion of $r\cos(x - \alpha)$.	$r\cos(x-\alpha) = r\cos(x)\cos(\alpha) + r\sin(x)\sin(\alpha)$
Using trigonometric ratios, $\tan(\alpha) = \frac{b}{a}$.	$\tan(\alpha) = \frac{4}{3}$
Use $r = \sqrt{r^2\sin^2(\alpha) + r^2\cos^2(\alpha)}$	$r = \sqrt{b^2 + a^2} = \sqrt{4^2 + 3^2} = 5$
Write the answer.	$3\cos(x) + 4\sin(x) = 5\cos(x - \alpha)$, where $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

EXERCISE 12.06 $a\sin(x) + b\cos(x)$

Concepts and techniques

1 Examples 16, 17 Write each expression in the form $r\sin(\theta + \alpha)$.
  a $2\sin(\theta) + \cos(\theta)$
  b $\sin(\theta) + \sqrt{3}\cos(\theta)$
  c $\sin(\theta) + \cos(\theta)$
  d $5\sin(\theta) + 2\cos(\theta)$
  e $4\sin(\theta) + \cos(\theta)$
  f $3\sin(\theta) + \cos(\theta)$
  g $2\sin(\theta) + 3\cos(\theta)$
  h $4\sin(\theta) + 7\cos(\theta)$
  i $5\sin(\theta) + 4\cos(\theta)$
  j $3\sin(\theta) + 5\cos(\theta)$

2 Write each expression in the form $r\sin(\theta - \alpha)$.
  a $\sin(\theta) - \cos(\theta)$
  b $\sin(\theta) - 2\cos(\theta)$
  c $\sin(\theta) - \sqrt{3}\cos(\theta)$
  d $\sqrt{3}\sin(\theta) - \cos(\theta)$
  e $5\sin(\theta) - 2\cos(\theta)$

3 Example 18 Write the expression $3\cos(\theta) + \sin(\theta)$ in the form $r\cos(\theta - \alpha)$.

4 Write the expression $\cos(\theta) - \sqrt{3}\sin(\theta)$ in the form $r\cos(\theta + \alpha)$.

5 Write the expression $9\sin(\theta) + 2\cos(\theta)$ in the form:
  a $r\sin(\theta + \alpha)$
  b $r\cos(\theta - \alpha)$

12.07 USING $A\cos(X) + B\sin(X)$

You can use expressions of the form $a\cos(x) + b\sin(x)$ to solve equations and sketch graphs.

In Example **18** you saw that $3\cos(x) + 4\sin(x)$ can be expressed as $5\cos(x - \alpha)$, where $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13°$ correct to 2 decimal places.

Thus $3\cos(x) + 4\sin(x) \approx 5\cos(x - 53.13°)$.

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Example 19

Use the fact that $3\cos(x) + 4\sin(x) = 5\cos(x - 53.13°)$ to graph the function $y = 3\cos(x) + 4\sin(x)$ for $0 \le x \le 360°$.

Solution

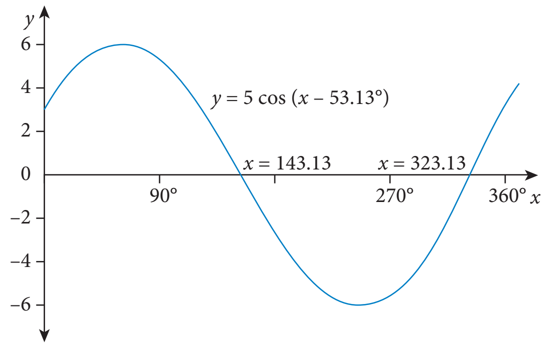
State the period, amplitude and phase shift of the graph of $y = 5\cos(x - 53.13°)$.

$P = \frac{360}{1} = 360°$

Amplitude = 5

Phase shift = 53.13° to the right.

Sketch the graph of $y = 5\cos(x - 53.13°)$.



You can make the equation, $3\cos(x) + 4\sin(x) =$ constant, simpler to solve by using the equivalence $3\cos(x) + 4\sin(x) = 5\cos(x - 53.13°)$.

Example 20

Use the fact that $3\cos(x) + 4\sin(x) = 5\cos(x - 53.13°)$ to solve the equation $3\cos(x) + 4\sin(x) = 0$ for $0 \le x \le 360°$.

Solution

Simplify the equation $5\cos(x - 53.13°) = 0$.

$5\cos(x - 53.13°) = 0$

$\therefore \cos(x - 53.13°) = 0$

Solve the equation $5\cos(x - 53.13°) = 0$ for $0 \le x \le 360°$.

$\cos(x - 53.13°) = 0$

gives $(x - 53.13) = -90, 90, 270, 450, \ldots$

Consider the domain required.

$\therefore x = 90 + 53.13, 270 + 53.13, \ldots$

State solutions.

$x = 143.13°, 323.13°$

The solution in Example **20**, $x = 143.13°, 323.13°$, matches the zeros of the graph, $x = 143.13°, 323.13°$, in Example **19**.

Example 21

Solve $\sqrt{3}\sin(x) + \cos(x) = 1$ for $0° \le x \le 360°$.

Solution

Use the result for $a\sin(x) + b\cos(x)$. Find the values of a and b.	For $\sqrt{3}\sin(x)+\cos(x)$, $a=\sqrt{3}$ and $b=1$.
Calculate r and α.	$r=\sqrt{a^2+b^2}$ $=\sqrt{\left(\sqrt{3}\right)^2+1^2}$ $=2$ $\tan(\alpha)=\frac{b}{a}$ $=\frac{1}{\sqrt{3}}$ $\therefore \alpha = 30°$
Rewrite the equation and the domain.	$\sqrt{3}\sin(x) + \cos(x) = 1$ for $0° \le x \le 360°$ $2\sin(x + 30°) = 1$ for $30° \le x + 30° \le 390°$
Simplify the equation.	$\sin(x+30°)=\frac{1}{2}$
Solve for $x + 30°$.	$x + 30° = 30°, 180° - 30°, 360° + 30°$ $x = 30°, 150°, 390°$
Write the answer.	$x = 0°, 120°, 360°$

EXERCISE 12.07 Using $a\cos(x) + b\sin(x)$

Concepts and techniques

1 Example 19 Express $3\cos(x) + 4\sin(x)$ in the form of $r\sin(x + \alpha)$. Hence graph the function $y = 3\cos(x) + 4\sin(x)$ for $0 \le x \le 360°$.

2 Express $4\sin(x) - 3\cos(x)$ in the form of $r\sin(x - \alpha)$. Hence graph the function $y = 4\sin(x) - 3\cos(x)$ for $0 \le x \le 360°$.

3 Express $2\cos(x) - 5\sin(x)$ in the form of $r\cos(x + \alpha)$. Hence graph the function $y = 2\cos(x) - 5\sin(x)$ for $0 \le x \le 360°$.

4 Examples 20, 21 Solve for $0° \le \theta \le 360°$

a $3\sin(\theta) + 4\cos(\theta) = 0$

b $5\cos(\theta) - 12\sin(\theta) = -3$

c $\sin(\theta) - \sqrt{3}\cos(\theta) = 0$

d $\sin(\theta) + \cos(\theta) = -1$

e $4\sin(\theta) - \cos(\theta) + 3 = 0$

f $\sin(\theta) - \cos(\theta) = 1$

g $\sqrt{2}\cos(\theta)+\sin(\theta)=1$

h $2\sin(\theta)-\cos(\theta)=\frac{\sqrt{5}}{2}$

i $3\cos(\theta) - 5\sin(\theta) + 2 = 0$

j $\sqrt{2}\cos(\theta) + \sin(\theta) + 1 = 0$

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Reasoning and communication

5 Find the general solution of $6\sin(\theta) - 8\cos(\theta) = 5$.

6 Find the general solution of $\sqrt{3}\sin(\theta) + \cos(\theta) = 1$

7 A radio wave follows the path of the equation $h = 9\sin\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{4}\right)$, where h (metres) is the height from a mean level and t (hours) is the time after 9 a.m.

a Simplify the expression $9\sin\left(\frac{\pi t}{4}\right) + \cos\left(\frac{\pi t}{4}\right)$, and show that it can be written in the form $r\sin\left(\frac{\pi t}{4} + \alpha\right)$.

b Using the expression found in part **a**, find the height of the radio waves at 9 a.m.

c Using the expression found in part **a**, find the height of the radio waves at 11 a.m.

d Find the time(s) in a 24-hour period when the height of the radio waves returns to that of 9 a.m.

12.08 MODELLING PERIODIC MOTION

The sine and cosine curves are used in many applications, including the study of waves. There are many different types of waves, including water, light and sound waves. Oscilloscopes display patterns of electrical waves on the screen of a cathode-ray tube.

Simple harmonic motion (such as the movement of a pendulum) is a wave-like or oscillatory motion when graphed against time. In 1581, when he was 17 years old, Galileo noticed a lamp swinging backwards and forwards in Pisa cathedral. He found that the lamp took the same time to swing to and fro, no matter how much weight it had on it. This led him to discover the pendulum.

IMPORTANT

A particle in **simple harmonic motion** (SHM) moves backwards and forwards (**oscillates**) in accordance with an equation of the type $x = a\cos(nt + \varepsilon)$, where a, n, ε are constants, x is its displacement from a central position and t is time. The central position is called the point of **equilibrium**.

A pendulum, a mass attached to a string, a buoy rising and falling with the tide, a violin or guitar string when plucked or a vibrating tuning fork are all examples of objects with simple harmonic motion (SHM).

Example 22

The displacement of a particle in metres over time t seconds is given by $x = 5\cos(2t)$. Describe and graph its motion.

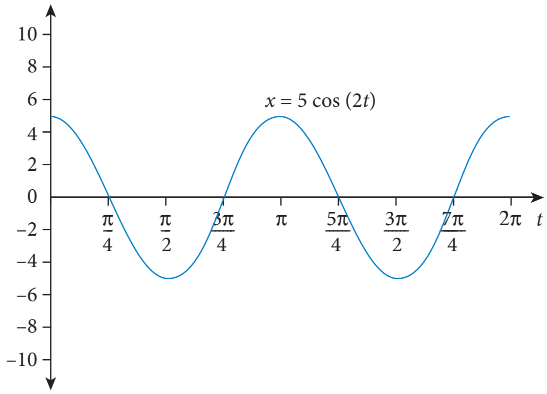
Solution

Find the period and amplitude.

$$\text{Period } \frac{2\pi}{2} = \pi$$

$$\text{Amplitude} = 5$$

Sketch the graph of $x = 5\cos(2t)$ for $0 \le t \le 2\pi$.



Describe the graph.

The particle starts at 5 metres from the origin. It then passes through the origin at $\frac{\pi}{4}$ seconds. The particle then travels to the other side of the origin until it is 5 metres in the other direction, and then back to the origin at $\frac{3\pi}{4}$ seconds. It then oscillates backwards and forwards.

At maximum displacement the particle is at rest.

The equation for SHM can also be in the form $x = a\sin(nt + \varepsilon)$.

The equation $x = a\cos(nt + \varepsilon)$ includes amplitude, period and also **phase shift**.

IMPORTANT

Since the position of a particle is given by $x = a\cos(nt + \varepsilon)$, the **amplitude** of the motion is a and the **period** is $\frac{2\pi}{n}$.

Example 23

A particle moves in a straight line with the equation of displacement given by $x = 3\sqrt{2}\cos\left(4t - \frac{\pi}{4}\right)$, where x is measured in metres and t in seconds. Describe and graph its motion.

Solution

State the period and amplitude.

$\text{Period} = \frac{2\pi}{4} = \frac{\pi}{2}$

$\text{Amplitude} = 3\sqrt{2}$

State the phase shift.

Express $x = 3\sqrt{2}\cos\left(4t - \frac{\pi}{4}\right)$ as

$x = 3\sqrt{2}\cos\left[4\left(t - \frac{\pi}{16}\right)\right]$.

Phase shift is $\frac{\pi}{16}$ in a horizontal positive direction.

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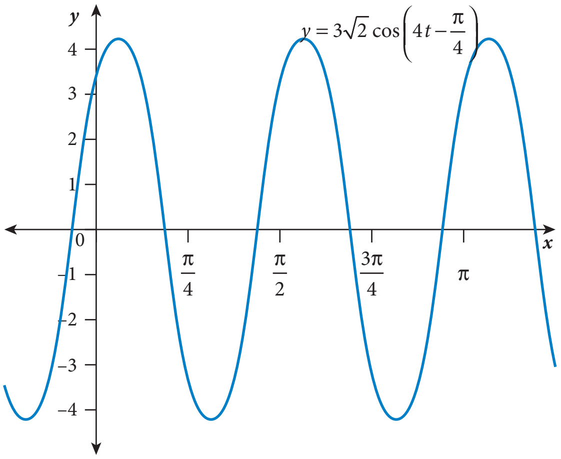
Sketch the graph of $x=3\sqrt{2}\cos\left(4t-\frac{\pi}{4}\right)$ for two cycles.



Describe the graph.

The particle starts at 3 metres from the origin. It then moves further away until it reaches 4.24 metres from the origin. It turns around and passes through the origin at 0.59 seconds. The particle then travels to the other side of the origin until it is 4.24 metres in the other direction, and then back to the origin at 1.37 seconds. It then oscillates backwards and forwards.
At maximum displacement the particle is at rest.

Consider the displacement of a particle in cm given by $x=\sqrt{3}\cos(2t)+\sin(2t)$, where t is time in seconds.

You can use the relationship $a\sin(\theta)+b\cos(\theta)=r\sin(\theta+\alpha)$ where $r=\sqrt{a^2+b^2}$ and $\tan(\alpha)=\frac{b}{a}$ to find the maximum distance from the equilibrium point.

Consider $x=\sqrt{3}\cos(2t)+\sin(2t)$

And

$$r=\sqrt{\left(\sqrt{3}\right)^2+1^2}$$
$$\therefore r=2$$

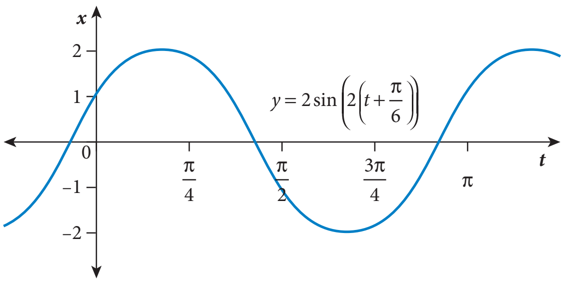
Using $\tan(\alpha)=\frac{b}{a}$

$$\tan(\alpha)=\frac{1}{\sqrt{3}}$$

$$\therefore \alpha=\frac{\pi}{6}$$

So $x=\sqrt{3}\cos(2t)+\sin(2t)=2\sin\left(2\left(t+\frac{\pi}{6}\right)\right)$

A sketch of the graph is shown on the right.



From the graph it can be seen that the maximum distance from the origin or centre is 2 metres.

Other periodic motion can be modelled by a sine or cosine curve, giving applications of trigonometric graphs in real life situations, such as wave height.

Example 24

The table shows the highest average monthly temperatures in Sydney.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
°C	26.1	26.1	25.1	22.8	19.8	17.4	16.8	18.1	20.1	22.2	23.9	25.6

Bureau of Meteorology

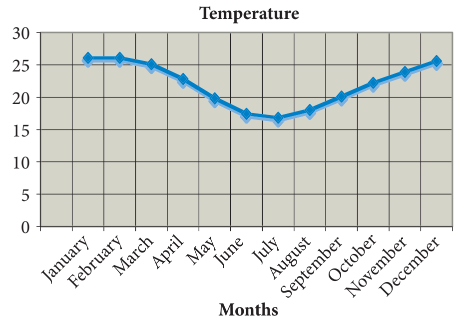
a Draw a graph of this data, by hand or on a calculator or computer.

b Is it periodic? Why would you expect it to be periodic?

c What is the period and amplitude?

Solution

a Draw the graph using any of the listed methods.



b Write the logical answer.

The graph looks like it is periodic, and we would expect it to be, since the temperature varies with the seasons. It goes up and down, and reaches a maximum in summer and a minimum in winter.

c This curve is approximately a cosine curve with one full period.
The maximum temperature is around 26° and the minimum is around 18°, so the centre of the graph is 22° with 4° on either side.

The period is 12 months.
The amplitude is 4.

EXERCISE 12.08 Modelling periodic motion

Reasoning and communication

1 Example 24 The graph shows the incidence of crimes committed over 24 years in Gotham City.

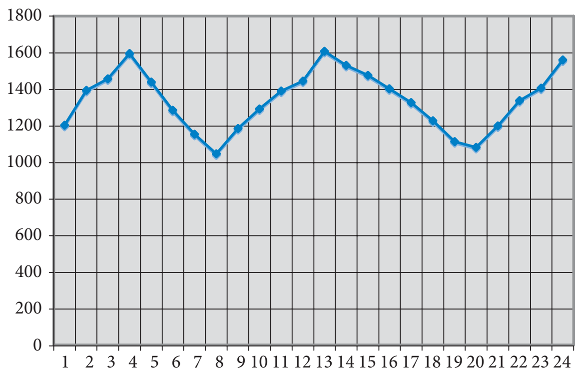
a Approximately how many crimes were committed in the 10th year?

b What was the

i highest and

ii lowest number of crimes?

c Find the amplitude and the period of the graph.



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2 Below is a table showing the average daylight hours over several months.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug
Daylight hours	15.3	14.7	13.2	13.1	12.7	12.2	12.5	13.8

a Draw a graph to show this data.

b Is it periodic? If so, what is the period?

c Find the amplitude.

3 The table below shows the high and low tides over a three-day period.

Day	**Friday**				**Saturday**				**Sunday**			
Time	6.20 a.m.	11.55 a.m.	6.15 p.m.	11.48 p.m.	6.20 a.m.	11.55 a.m.	6.15 p.m.	11.48 p.m.	6.20 a.m.	11.55 a.m.	6.15 p.m.	11.48 p.m.
Tide m	3.2	1.1	3.4	1.3	3.2	1.2	3.5	1.1	3.4	1.2	3.5	1.3

a Sketch a graph showing the tides.

b Find the period and amplitude.

c Estimate the height of the tide at around 8 a.m. on Friday.

Shutterstock.com/Michael Zysman

4 Examples 22, 23 A particle is moving in simple harmonic motion, with displacement at any time t seconds given by $x = 2\cos(t)$. Sketch the graph of its displacement.

5 A particle is moving in SHM such that its displacement at any time t seconds is given by $x = 5\sin(t)$. Sketch the graph of its displacement.

6 A particle is oscillating about a central point so that its displacement at any time t seconds is given by $x = 4\cos(2t)$.

a Sketch the graph of its displacement.

b Find the times when the particle will have maximum displacement, and find this maximum displacement.

7 A particle's displacement is given by $x = 2\cos\left(t + \frac{\pi}{4}\right)$ m at time t seconds.

a Find the times at which the particle will be at the origin.

b Write down the period of the motion.

c Find the maximum displacement.

12 CHAPTER SUMMARY
TRIGONOMETRIC FUNCTIONS AND GRAPHS

- Equations of the form $y = \sin(n\theta)$ and $y = \cos(n\theta)$ have the **period** $P = \frac{2\pi}{n}$.
- Equations of the form $y = \tan(n\theta)$ have the **period** $P = \frac{\pi}{n}$.
- Equations of the form $y = a\sin(\theta)$ and $y = a\cos(\theta)$ have the amplitude a.
- Horizontal translations of the graphs $y = \sin(\theta)$, $y = \cos(\theta)$ and $y = \tan(\theta)$ to $y = \sin(\theta + c)$, $y = \cos(\theta + c)$ and $y = \tan(\theta + c)$ move the graph $-c$ units horizontally: called a **phase shift**.
- **Summary of the features of sine and cosine functions**

  For the functions
  $y = a\sin(bx + c) + d$

  and $y = a\cos(bx + c) + d$
  - the amplitude is the magnitude of a
  - the period is $\frac{2\pi}{b}$
  - the value $\frac{c}{b}$ is called the **phase shift** and is the horizontal translation
  - the average (mean) value is d.
- Reciprocal trigonometric functions are found by reciprocating the trigonometric functions of sine, cosine and tangent.
- Reciprocal trigonometric relationships are $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$, $\sec(x) = \frac{1}{\cos(x)}$, $\cot(x) = \frac{1}{\tan(x)}$.
- Graph reciprocal trigonometric equations by doing one of the following.

  1 Sketch a sine, cosine, or tangent graph and reciprocate.

  2 Sketch the basic $y = \operatorname{cosec}(x)$, $y = \sec(x)$, $y = \cot(x)$ and transform.
- For the expression $a\sin(x) + b\cos(x) = r\sin(x + \alpha)$, where $r = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$
  - convert sums $a\cos(x) + b\sin(x)$ to $R\cos(x \pm a)$ or $R\sin(x \pm a)$ and apply these to sketch graphs
  - solve equations of the form $a\cos(x) + b\sin(x) = c$ and solve problems
  - model periodic motion using sine and cosine functions and understand the relevance of the period and amplitude of these functions in the model
- Applications of trigonometric functions to model other periodic phenomena.

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CHAPTER REVIEW 12

TRIGONOMETRIC FUNCTIONS AND GRAPHS

Multiple choice

1 Example 1 The period of the graph $y = -3\sin(4x)$ is:

A $-\frac{\pi}{2}$ B $\frac{\pi}{2}$ C π D $\frac{3\pi}{2}$ E 2π

2 Example 2 The amplitude of the graph $y = -3\sin(4x)$ is:

A -3 B 1 C 2 D 3 E 4

3 Example 3 The amplitude, period and phase shift of the graph $y = 3\sin\left(\theta - \frac{\pi}{4}\right)$ are respectively:

A $3, \pi, \frac{\pi}{4}$ horizontally to the left

B $-3, 2\pi, \frac{\pi}{4}$ horizontally to the right

C $\pi, 3, \frac{\pi}{4}$ horizontally to the left

D $3, 2\pi, \frac{\pi}{4}$ horizontally to the right

E $3, 2\pi, \frac{\pi}{4}$ horizontally to the left

4 Examples 4-6 The period and phase shift of the graph $y = \sin(3x + \pi)$ are respectively:

A $3\pi, \frac{\pi}{3}$ horizontally to the left

B $\frac{2\pi}{3}, \frac{\pi}{3}$ horizontally to the right

C $\frac{2\pi}{3}, \frac{\pi}{3}$ horizontally to the left

D $2\pi, \frac{\pi}{3}$ horizontally to the right

E $2\pi, \frac{\pi}{3}$ horizontally to the left

5 Examples 16-18 When $3\cos(x) + 3\sin(y)$ is written in the form $r\cos(x - \alpha)$,

A $r = 3$ and $\alpha = \frac{\pi}{4}$

B $r = 3\sqrt{2}$ and $\alpha = \frac{3\pi}{4}$

C $r = 3$ and $\alpha = -\frac{\pi}{4}$

D $r = 3\sqrt{2}$ and $\alpha = \frac{\pi}{4}$

E $r = \sqrt{6}$ and $\alpha = \frac{3\pi}{4}$

Short answer

6 Examples 4-6 Find the significant points of the cycle and hence sketch about one cycle of the graph of each of the following.

a $y = 3\cos\left(2x - \frac{\pi}{6}\right) + 4$ b $y = 3\cos\left(3x - \frac{\pi}{4}\right) - 2$ c $y = -2 - 2\sin\left(2x - \frac{\pi}{3}\right)$

7 Examples 4-6 CAS Draw each of the following.

a $y = 5\sin\left(2x + \frac{\pi}{4}\right) - 4$ b $y = 4\cos\left(3x + \frac{\pi}{6}\right) - 5$

c $y = 4\cos(x) + 2\cos(2x) + \cos(4x)$

8 Examples 8, 9 Solve $\sin(2x) = x$ for $0 \le x \le 2\pi$ by graphical means.

9 Example 8 Draw the graphs of $y = \sin(x)$ and $y = \cos(x)$ for $0 \le x \le 2\pi$ on the same set of axes. Use your graphs to solve the equation $\sin(x) = \cos(x)$ for $0 \le x \le 2\pi$

10 Example 10 CAS Solve the equation $-2\cos\left(2x+\frac{\pi}{6}\right)=1$ for $0 \le x \le 2\pi$.

11 Examples 11, 12 Solve the following for $\pi \le \alpha \le 2\pi$.

a $\sin(\alpha) = 0$
b $2\cos(\alpha) = 1$
c $\tan(\alpha) - 1 = 0$
d $1 + \tan(\alpha) = 1$
e $2\cos(\alpha) = \sqrt{3}$
f $\sqrt{3}\tan(\alpha) + 1 = 0$
g $\sqrt{2}\cos(\alpha) + 1 = 0$
h $\tan(\alpha) - \sqrt{3} = 0$
i $\cos(\alpha) = \cos\left(\frac{\pi}{8}\right)$
j $\sin(\alpha) = \sin\left(\frac{\pi}{6}\right)$

12 Examples 13-15 Sketch the following graphs from 0 to 2π.

a $y = \sec\left(x+\frac{\pi}{6}\right)$
b $y = \text{cosec}\left(2x+\frac{\pi}{4}\right)$

13 Examples 13-15 Use the definitions to find the following.

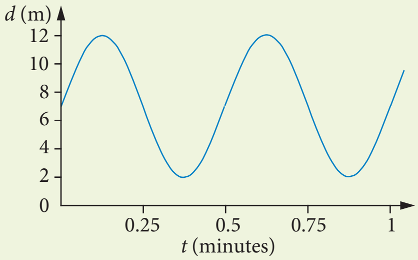
a $\cot\left(\frac{\pi}{3}\right)$
b $\text{cosec}\left(\frac{3\pi}{4}\right)$
c $\sec\left(-\frac{11\pi}{6}\right)$
d $\text{cosec}\left(\frac{4\pi}{3}\right)$

14 Examples 16-18 Write the expression $4\cos(x) + 3\sin(x)$ in the form $r\cos(x - \alpha)$.

Application

15 By expressing $6\cos(x) + \sin(x)$ in the form $r\cos(x - \alpha)$, solve the equation $6\cos(x) + \sin(x) = 1$ for $0 \le x \le 360°$.

16 Juanita is lying on a tropical beach, enjoying the sound of the waves. She has just finished her exams and so has plenty of time to observe the movement of the waves. She notices that the waves appear to roll up the beach at regular time intervals, and she is able to estimate the distance of the wave front from her toes over time. Idly, she scratches Cartesian axes in the sand and sketches the distance of the wave front from her toes against time in minutes. At this point she realises that the distance can be modelled by a sine curve
$d = a\sin(bt) + c$



where a, b and c are positive constants.

a State the maximum and minimum distances of the waves from her feet.
b How many waves wash up on the beach each hour?
c Find the values of a, b and c.

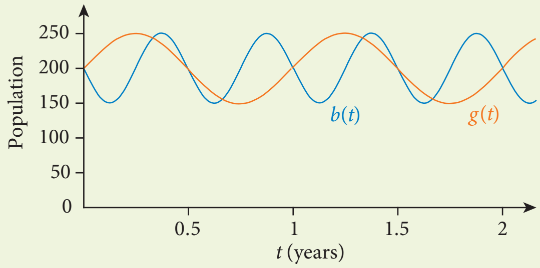
Her beach umbrella is stuck into the sand 4.5 m closer to the wave front than her toes and the waves are washing over its base.

d Calculate the percentage of time for which the base of the umbrella is in the water.

17 In a large butterfly enclosure the populations of two species are given by:

Orange: $g(t) = a + b\sin(mt)$

Blue: $b(t) = c - d\sin(nt)$



where t gives the number of years from when the observations began and a, b, c, d, m and n are positive constants.

Practice quiz

The graphs of $g(t)$ and $b(t)$ for the first 2 years are shown.

a State the values of a, b and m and hence write the rule for $g(t)$.

b State the values of c, d and n and hence write the rule for $b(t)$.

c Graphically determine the times when the populations are equal and state the populations at these times.

d Find the number of *days* after which the population of blues is first less than 180.

ANSWERS

1.01

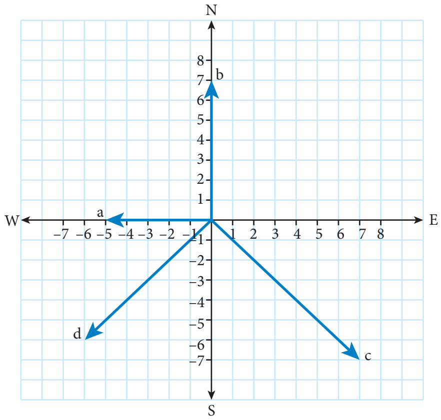
1 B

2 E

3 E

4 B

5



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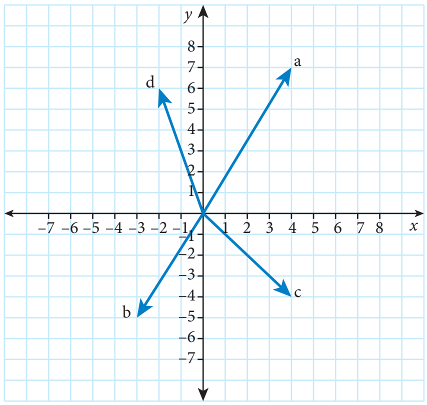

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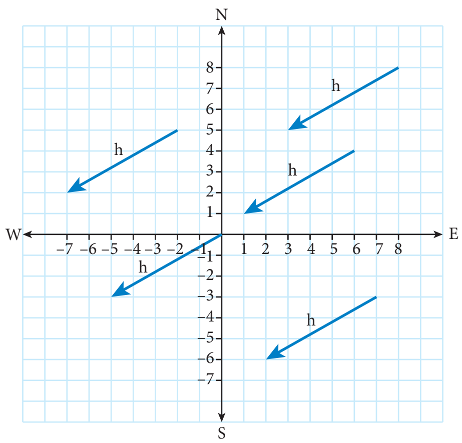


8 a 5 b 13 c 25 d $\sqrt{2}$

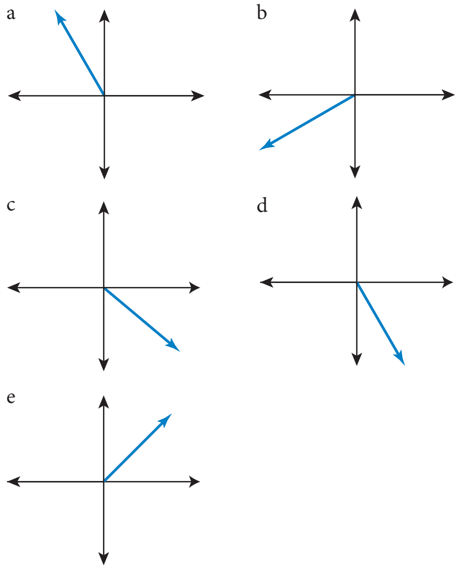
e $\sqrt{53}$ f $3\sqrt{5}$ g $\sqrt{a^2+16}$ h $\sqrt{36+b^2}$

9 a 53.1° b 112.6°

c 343.7° d 225°

e 105.9° f 296.6°

10 The vectors in your answer should be in the same direction but the scale could be different.



11 a (5, 306.9°) b (7.07, 135°)

c (12.21, 235.0°) d (12.65, 71.6°)

e (18.87, 122.0°)

12 a (17, 61.9°) b (13, 22.6°)

c (15, 233.1°) d (13.9, 239.7°)

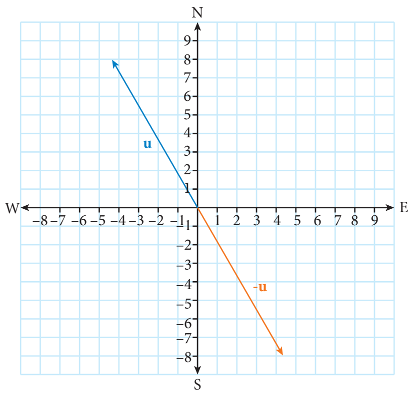
e (11.3, 315°)

13 a $AB \approx (7.3, 74.1°)$ b $RB \approx (5.8, 211.0°)$

c $MB \approx (10.8, 123.7°)$

14 a $-\mathbf{u}$ has a magnitude of 8 and is in the direction 60° south of east.

b



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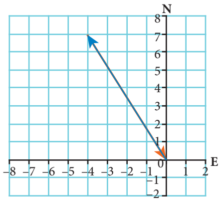
c



d The total displacement is 0.

1.02

1 D

2 B

3 a 14 east b 57 east
c 23 east d 7 west
e 73 east

4 a 60 N up b 9 N down
c 26 N up d 103 N down
e 110 N east

5 a (7.21, 146.1°) b (4.50, 12.6°)
c (10.93, 9.3°) d (10.82, 276.3°)
e (16.93, 77.7°)

6 a (13, 67.8°) b (6.4, 292.7°)
c (6.1, 274.7°) d (20.4, 220.3°)
e (24.1, 265.4°)

7 About 16.8 km on a bearing of 075.4° (14.6° north of east).

8 About 668 N at 14.9° to the vertical.

9 About 1433.2 m on a bearing of 347.6° (77.6° north of west).

10 About 9.6 km on a bearing of 089.8°.

11 About 37.1 km on a bearing of 062.8° (27.2° north of east).

12 22.4 ms^{-1} on a bearing of 167.2° (12.8° east of south).

1.03

1 a 74.1° b 83.7° c 63.4° d 58.0°
e 102.5° f 146.3° g 221.2° h 337.4°

2 a 5 b 13 c 6.32 d 11.70
e 11.40 f 15.26 g 13.89 h 13.15

3 a (5, 306.9°) b (7.1, 135°)
c (12.2, 235.0°) d (12.7, 71.6°)
e (18.9, 122.0°) f (17, 61.9°)
g (13, 22.6°) h (15, 233.1°)
i (13.9, 210.3°) j (11.3, 315°)

4 a (15, 53.13°) b (20.25, 57.09°)
c (17.49, 120.96°) d (22.20, 324.16°)
e (26.25, 220.37°) f (18.56, 27.95°)
g (14.30, 296.46°) h (12.26, 108.60°)

5 a (4.33, 2.5) b (5, −8.66)
c (0, 24) d (−11.31, 11.31)
e (−14, −24.25) f (0, −70)
g (35, 0) h (−22, 0)

6 a (3, 5.20) b (−6, 10.39)
c (10.47, 13.40) d (−22.32, 5.56)
e (−9.37, −10.40) f (4.39, −14.34)
g (−18.11, −15.75) h (41.12, −12.57)
i (−3.76, −1.37) j (4, −6.93)

7 a (1, 5) b (3, 9)
c (3, 8) d (7, 7)
e (−3, −10) f (6, −11)

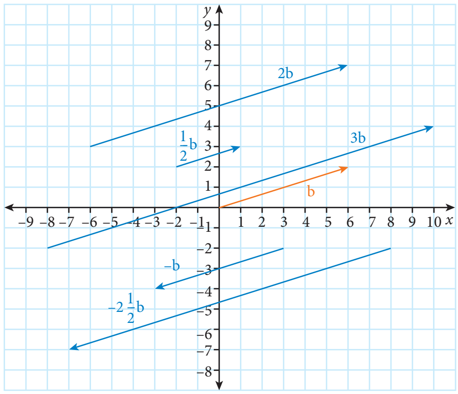
8 (10.63, 311.2°)

1.04

1 B

2 B

3



4 a $\begin{bmatrix} 54 \\ -36 \end{bmatrix}$ b $\begin{bmatrix} -36 \\ 24 \end{bmatrix}$
c $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$ d $\begin{bmatrix} 115.2 \\ -76.8 \end{bmatrix}$
e $\begin{bmatrix} 9 \\ -6 \end{bmatrix}$ f $\begin{bmatrix} 45 \\ -30 \end{bmatrix}$
g $\begin{bmatrix} 12 \\ -8 \end{bmatrix}$ h $\begin{bmatrix} -72 \\ 48 \end{bmatrix}$

5 a (18, 47°) b (24, 227°)
c (15, 47°) d (6, 227°)
e (30, 227°) f (60, 47°)
g (42, 227°) h (3, 47°)

6 a (8, 28) b (−9, 36)
c (−14, 31.5) d (19.4, 67.9)
e $(\frac{4}{3}, -3)$ f $(1\frac{1}{2}, -6)$
g $(-3, 6\frac{3}{4})$ h (−4, −14)

7 a (12, 195°) b (4, 124°)
c (7.5, 69°) d (16, 15°)
e (2.5, 249°) f $(\frac{2}{3}, 304°)$
g (20, 249°) h (20, 15°)

1.05

1 B

2 B

3 a $\left(\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right)$ b $\left(\frac{4\sqrt{41}}{41}, \frac{-5\sqrt{41}}{41}\right)$

c $\left(\frac{-\sqrt{10}}{10}, \frac{-3\sqrt{10}}{10}\right)$ d $(0.411, -0.912)$

e $\begin{bmatrix} \frac{-3\sqrt{10}}{10} \\ \frac{\sqrt{10}}{10} \end{bmatrix}$ f $\begin{bmatrix} \frac{-5}{13} \\ \frac{-12}{13} \end{bmatrix}$

g $\begin{bmatrix} \frac{24}{25} \\ \frac{-7}{25} \end{bmatrix}$ h $\begin{bmatrix} 0.869 \\ 0.496 \end{bmatrix}$

i $(1, 86°)$ j $(1, -165°)$

k $\left(1, \frac{2\pi}{3}\right)$ l $\left(1, \frac{7\pi}{5}\right)$

4 a $5\mathbf{i} - 3\mathbf{j}$ b $-6\mathbf{i} + 4\mathbf{j}$ c $-4\mathbf{i} - 7\mathbf{j}$
d $6\mathbf{i} + 5\mathbf{j}$ e $-3.2\mathbf{i} - 9.4\mathbf{j}$ f $\mathbf{i} - 4\mathbf{j}$
g $-2\mathbf{i} - 8\mathbf{j}$ h $-3\mathbf{i} + 5\mathbf{j}$ i $7.59\mathbf{i} + 3.68\mathbf{j}$
j $0.07\mathbf{i} + 0.19\mathbf{j}$ k $1.93\mathbf{i} - 6.73\mathbf{j}$ l $-5.16\mathbf{i} + 7.37\mathbf{j}$
m $9.53\mathbf{i} - 5.5\mathbf{j}$ n $-8\mathbf{i} - 13.86\mathbf{j}$ o $-2.33\mathbf{i} - 4.20\mathbf{j}$

5 $\hat{\mathbf{m}} = \left(-\frac{3}{5}, \frac{4}{5}\right)$

6 $\left(\frac{24\sqrt{65}}{65}, -\frac{42\sqrt{65}}{65}\right)$

1.06

1 a $(-13, 13)$ b $(18, -14)$ c $(-5, -1)$
d $(10, -11.8)$ e $(4.17, -4.22)$ f $\left(1\frac{3}{4}, 2\frac{13}{15}\right)$

2 a $\begin{bmatrix} 23 \\ -17 \end{bmatrix}$ b $\begin{bmatrix} -12 \\ -2 \end{bmatrix}$ c $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$ d $\begin{bmatrix} 3.3 \\ -13.3 \end{bmatrix}$

e $\begin{bmatrix} 1.1 \\ -6.44 \end{bmatrix}$ f $\begin{bmatrix} 6\frac{1}{4} \\ -5\frac{7}{9} \end{bmatrix}$

3 a $7\mathbf{i} - 13\mathbf{j}$ b $-17\mathbf{i} + 10\mathbf{j}$
c $-5\mathbf{i} + 5\mathbf{j}$ d $-3.4\mathbf{i} - 12\mathbf{j}$
e $-8.23\mathbf{i} + 5.15\mathbf{j}$ f $-10\frac{1}{10}\mathbf{i} - 9\frac{5}{8}\mathbf{j}$

4 a $(-4, 5)$ b $(0, 10)$
c $(5, 20)$ d $(12, -2)$
e $(-9, 14)$ f $(-30, 24)$
g $(5, -1)$ h $(0, 0)$ or $\mathbf{0}$
i $(-34, 69)$ j $(-4, -32)$

5 a $(14.33, 41.1°)$ b $(38.65, 104.6°)$
c $(9.32, 324.9°)$ d $(173.28, 271.7°)$
e $(6, 130°)$

6 a $\mathbf{AB} = (2, 7)$ b $\mathbf{RB} = (-5, -3)$
c $\mathbf{MB} = (-6, 9)$ d $\mathbf{QP} = (0, -18)$
e $\mathbf{BX} = (8, 14)$ f $\mathbf{FG} = (8, -11)$

7 $|c\mathbf{a}| = \sqrt{(cx_1)^2 + (cy_1)^2}$
$= \sqrt{c^2(x_1^2 + y_1^2)}$
$= \sqrt{c^2}\sqrt{x_1^2 + y_1^2}$
$= -c|\mathbf{a}|$ (to make it positive).

8 $c(\mathbf{a} + \mathbf{b}) = c(x_1 + x_2, y_1 + y_2)$
$= (cx_1 + cx_2, cy_1 + cy_2)$
$= (cx_1, cy_1 + cx_2, + cy_2)$
$= c(x_1, y_1) + c(x_2, y_2)$
$= c\mathbf{a} + c\mathbf{b}$

1.07

1 D
2 B
3 a $-\mathbf{m} = (5, -8)$ b $-\mathbf{d} = (11, 4)$
c $-\mathbf{r} = (0, 6)$ d $-\mathbf{q} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
e $-\mathbf{g} = \begin{bmatrix} -14 \\ -3 \end{bmatrix}$ f $-\mathbf{h} = \begin{bmatrix} 12 \\ -11 \end{bmatrix}$
g $-\mathbf{a} = -4\mathbf{i} + 9\mathbf{j}$ h $-\mathbf{p} = -\frac{2}{3}\mathbf{i} - \frac{4}{5}\mathbf{j}$
i $-\mathbf{n} = 0.24\mathbf{i} + 1.06\mathbf{j}$

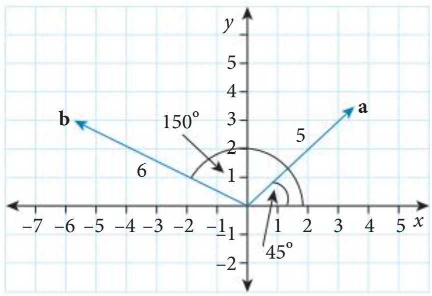
4–6 Demonstrations using procedure similar to Example **15**.
7–9 Proofs using procedure similar to Example **16**.

1.08

1 About 12 340 N at 11.6° to the passenger's side forward direction.
2 6.262 km at 153.2°
3 282.8 N
4 About 696 N at an angle of 46.6° to the shoreline.
5 About 26.9 m/s at 042.0°.
6 84.85 km h^{-1} at 045°.
7 About 181.8 knots at 066.5°.
8 About 10.7 m/s at 082.5°.
9 $v\sqrt{2[1 - \cos(\theta)]}$

CHAPTER 1 REVIEW

1 D
2 A
3 B
4 D
5 B
6 A
7 D
8 A
9 C
10



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11 a DE = –2i + 6j
b TS = 8i + 5j
c GJ = –4i + 4j
d RK = –2i + 4j
e YZ = 7i – 12j

12 a (9, –4) b (9, –4) c (0, 0) or **0**
d (3, 10) e (15, –1)

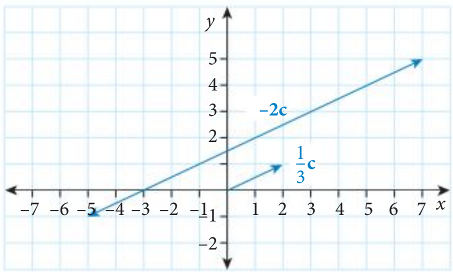
13 a (9.434, 58°) b (5, 126.9°)
c (11.662, 239°) d (9.487, 341.6°)
e (4, 180°)

14 a $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ b $(-4\sqrt{3}, 4)$
c $(2, -2\sqrt{3})$ d $\left(\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$
e (–5.785, –6.894)

15 a (5.831, 59°) b (6.325, 251.6°)
c (4.123, 284°) d (5, 90°)
e (8.246, 104°)

16 a 26 b 5 c 17
d $\sqrt{106} \approx 10.296$ e $\sqrt{244} \approx 15.621$

17



18 $\begin{bmatrix} -72 \\ 54 \end{bmatrix}$

19 $\left(\frac{7}{25}, -\frac{24}{25}\right)$

20 a $\mathbf{i} - 3\mathbf{j}$ b $4\mathbf{i} + 5\mathbf{j}$
c $-2\mathbf{i} + 5\mathbf{j}$ d $4\mathbf{i} - 3\mathbf{j}$
e $-2\mathbf{i} - 5\mathbf{j}$ f $3.36\mathbf{i} + 4.97\mathbf{j}$
g $-7.52\mathbf{i} - 2.74\mathbf{j}$ h $-3\mathbf{i}$
i $-7.88\mathbf{i} + 6.16\mathbf{j}$ j $-2.5\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}$

21 a $12\mathbf{i} - 5\mathbf{j}$ b (–6, –2) c $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

22 a (2, 9) b (–5, –8) c (–8, –1)
d (21, 34) e (0, 6)

23 a $\begin{bmatrix} -6 \\ 4 \end{bmatrix}$ b $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ c $\begin{bmatrix} 7 \\ 8 \end{bmatrix}$
d $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ e $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$

24 a (10.39, 121.1°) b (10.92, 176.1°)
c (4.99, 158.8°) d (9.85, 55.9°)
e (7.13, 275.8°)

25 a (9.43, 208°) b (16.86, 52.1°)
c (2.43, 166.6°) d (9.43, 92°)
e (7.36, 155.4°)

26 Demonstration.

27 Proof.

28 a $-\mathbf{a}$ b $-\mathbf{a} + \mathbf{c}$ c $\mathbf{a} + \mathbf{c}$
d $\frac{1}{2}(\mathbf{a} + \mathbf{c})$ e $\frac{1}{2}(-\mathbf{a} + \mathbf{c})$ f $\frac{1}{2}(-\mathbf{a} + \mathbf{c})$

29 Approximately 41 N at 10.7° to the horizontal.

30 1082 m on bearing 146.3°.

31 About 53 600 N in the direction N 14.5° W.

32 0.4013 m/s at 151.2° to the original direction.

2.01

1 a inductive b deductive
c deductive d by contradiction

2 *PQRS* has an interior angle sum of 360°.

3 Sea temperatures will always rise.

4 a necessary b necessary
c necessary d if and only if

5 a False, other plants may be prickly as well.
b False, the first premise is not correct.

6 Inductive, but the conclusion is wrong, Mary may have a girl.

7 This number is divisible by 4. ∴ It is composite.

8 Other shapes may have equal diagonals as well.
Corrected: This shape is a square.
∴ it has 2 equal diagonals.

9 It may be a coincidence that they have blond hair and scored well. Maybe all students in this class decided to dye their hair blond for a charity event.

10 Mandy may have more understanding than Dennis or may have done her homework through the term. Dennis may have had a bad day or lost his calculator.

2.02

1 a $x = -3$
b Not true for $x = 3$
c In 3D space, skew lines are not parallel and do not meet.
d octopus, some crustaceans

2 a false, rectangle b false, $y = -2$
c false, $a = -1$, $b = 2$ d false $n = 0$
e false – could be similar

3 a Yes, the statement is false for $x = 4$.
b No, the statement is about where fish live so it is not a counterexample.

4 False, try $a = 0$.

5 No, only if n is positive.

6 No, in 3D a vertical edge of a cube and an edge through one of the other vertical edges are neither parallel nor intersecting.

7 No, the intersections for an obtuse-angled triangle are outside.

8 No, just make a quadrilateral by choosing a fourth point that is not on the circumscribing circle of a triangle.

9 Yes, because it is necessarily convex.

10 No, draw a concave quadrilateral.

2.03

1 a If you surf at the beach, then you live in Queensland.
  b If you like Maths, then you are an interesting person.
  c If you went to school in Darwin, then you are the Premier.
  d If an animal has big teeth, then it is a lion.

2 a If you sweat, then you do a lot of exercise. False.
  b If your Maths improves, then you study hard. False.
  c If a boy has learnt to drive, then he has a P-plate. False.
  d If an animal lives in a tree, then it is a koala. False.

3 a If $3x = 15$, then $x = 5$. $x = 5$ iff $3x = 15$.
  b If a quadrilateral is a parallelogram then it has two pairs of opposite sides equal.
  A quadrilateral has two pairs of opposite sides equal iff it is a parallelogram.
  c If $a^2 > b^2$, then $a > b$. False. Try $a = -3$.
  d If you have big shoes, then you have big feet. False. (you may be a clown)

4 Converse: If students get a better job, then they study more maths. Bob has negated the statement.

5 In a way, they were both right. Female is the opposite gender to male, but the converse of male is either female or neuter. Bees are divided into queens (female), drones (male) and workers (sexless).

2.04

1 a If you don't live underground, then you don't live in Coober Pedy.
  b If you don't have a licence, then you cannot drive a car.
  c If you don't need hearing aids, then you are not old.
  d If $x^2 \neq 4$, then $x \neq 2$.
  e If an animal is not an amphibian, then it is not a cane toad.
  f If you are not mortal, then you are not a man.

2 a If it is sunny, then the sky is blue.
  b If a student passes his exams, then he studies.
  c If you live in the desert, then you live in Alice Springs.
  d If a number is a counting number, then it is positive.
  e If a vehicle has four wheels, then it is a car.
  f If you have money, then you have a job.

3 a If $x^2 \ngeq 9$, then $x \ngeq 3$.
  b If a quadrilateral does not have diagonals equal in length, then it is not a rectangle.
  c If people do not live in Australia, then they do not live in Tasmania.
  d If an animal is not a marsupial, then it is not a koala.

4 a If $x^2 \ngeq 9$, then $x \ngeq -3$. False.
  b If an animal can't fly, then it isn't a bird. False.
  c If you will not get a good job, then you have not been to university. False.
  d If a number is not real, then it is not rational. True.

5 a If they are not similar shapes, then two triangles do not have matching equal angles. True.
  b If a quadrilateral is not a rhombus, then it does not have diagonals that meet at right angles. False.
  c If an animal does not have a trunk, then it is not an elephant. True.

6 The contrapositive is: If a quadrilateral is not a square, then it does not have four equal sides. False.

7 D

2.05

1 a $a = 35$ b $x = 22\frac{1}{2}$ c $x = 10$
  d $x = 140$ e $w = 96$ f $y = 81$

2 a Proof b Proof

3 Proof

4 a AAS b SSS c SAS d RHS

5 Proof

6 a Proof, SAS b Proof

7 Proof

8 $x = 72$

2.06

1 Proof

2 a If it is a rhombus, then the diagonals of a parallelogram meet at right angles.
  b Proof
  c Proof
  d The diagonals of a rhombus bisect each other at right angles.

3 a $(a + b, a - b)$
  b $(a - b, -b - a)$
  c The vectors are the same length as they only differ in sign or direction.

4 Proof

5 Proof

6 Proof

7 a If a line is parallel to one side of a triangle and half its length, then it bisects the other two sides.
  b Proof
  c Proof

8, 9 Proof

10 a,b Proof
  c Both pairs of opposite sides parallel (or equal).

2.07

1 a $\forall\, x \in \boldsymbol{R}, x^2 \geq 0$
  b $\exists\, w \in \boldsymbol{Q}$ such that $2 < w \leq 7$
  c $\forall$ adult human beings $\exists$ a perfect match
  d $\forall\, a, b \in \boldsymbol{R}, \exists\, c \in \boldsymbol{R}$ such that $a < c < b$
  e $\forall\, x \in \boldsymbol{R}$ such that $x \geq 0$, $\exists\, y \in \boldsymbol{R}$ where $y \geq 0$ such that $y = \sqrt{x}$.

2 a Proof b Proof

3 Proof

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4 Proof
5 Proof
6 Proof

CHAPTER 2 REVIEW

1 D
2 B
3 D
4 A
5 C
6 a True b False, let $x = -3$.
c True d False, the number 2 is prime.
7 Elsie improves her playing.
8 There will be floods every 2 years in Queensland.
9 a If you tell the truth, then you are a politician.
b If the square of a number is positive, then the number is positive.
c If a student must resit an exam, then he/she has failed the exam.
d If an animal can swim, then it is a fish.
10 a If you do not tell the truth, then you are not a politician
b If the square of a number is not positive, then the number is not positive.
c If a student does not have to resit an exam, then he/she has not failed the exam.
d If an animal cannot swim, then it is not a fish.
11 a No
b A number is composite iff it has more than two factors.
c A quadrilateral is a rhombus iff it has diagonals that bisect each other at right angles.
d No
12 a Contrapositive: If a quadrilateral does not have two pairs of adjacent sides equal, then it is not a kite. Yes, the statement is true.
b Contrapositive: If $a^2 \not> b^2$, then $a \not> b$. No.
c Contrapositive: If a triangle does not have two equal sides, then it does not have two equal angles. Yes, the statement is true.
d Contrapositive: If a vehicle does not have wheels, then it is not a bike. Yes, the statement is true.
13 a Proof, let $2x + 2y = 180$
b Proof, AAS c Proof, AA
14 a $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$ b $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$
15 Proof
16 swimming $\Rightarrow$ wet, shower $\Rightarrow$ wet. There is more than one way of getting wet.
17 'if and only if', converse is also true.
18 Try $x = -\frac{1}{3}$.
19 If a triangle is isosceles, then a median of the triangle is perpendicular to the side it intersects. Converse is true.
20 If $\frac{1}{a} \not> \frac{1}{3}$, then $a \le 0$ or $a \ge 3$. True.

21 a use SSS test
b matching angles of congruent triangles equal, alternate angles
c matching angles of congruent triangles equal, alternate angles
d two pairs of opposite sides parallel
22 Proof

3.01

1 D
2 B
3 456 976
4 67 600
5 $26^5 \times 10^4$
6 260
7 $26^{10} \times 10^{15}$
8 1000
9 1 000 000
10 300
11 64
12 Yes
13 Yes
14 6
15 6840
16 360
17 7 880 400
18 210
19 271 252 800
20 $\frac{3}{10\,000}$
21 a 84 b $\frac{1}{84}$
22 a 10 000 000 b 1000
23 $\frac{1}{67\,600\,000}$
24 $\frac{1}{5184}$
25 a 9900 b $\frac{1}{9900}$
26 $\frac{1}{720}$
27 7
28 a 60 b 720 c 30 240
29 64
30 17 576 000. More number plates are possible with 3 letters.
31 72
32 36
33 a 25 b 20

3.02

1 36
2 7
3 9
4 22
5 960
6 a 84 b 504
7 a 840 b 480 c 480
d 480 e 240 f 720

8 a 1680 b 4096 c 720 d 720
e 120 f 1560 g 18 000

9 a 120 b 48

10 a 18 b 6 c 4
d 10 e 8 f 10

11 a 1680 b 6720 c 5400
d 840 e 4200

12 42

13 a 24 b 256

14 a 5 b 6 c 12

15 288

3.03

1 a 5 b 14

2 3

3 366 unless it is a leap year, when it is 367.

4 27

5 16

6 17

7 21

8 169

9 16

10 The 200 numbers can be divided into 100 pairs such that the second is double the first: 1\2, 2\4, 3\6, 4\8, 5\10, …, 100\200. By the principle, from 101 numbers between 1 and 200 there must be at least 2 that belong to the same pair, so there is at least one number that is double one of the others.

11 The 200 numbers can be divided into two separate groups: 66 pairs such that one is triple the other: 1\3, 2\6, 3\9, …, 66\198 and another 68 that are not triple any other number from 1 to 200. If 66 + 1 = 67 of the first group are placed into 66 boxes, then at least one box must contain a pair. The other 68 can be placed separately, making 67 + 68 = 135 numbers that must be chosen to ensure that at least one number is triple another.

12 If all the integers are the same value, then that value is the average and it is trivially true. Otherwise, assign the integers to two groups: those with values equal to the greatest value, and all others. By the pigeonhole principle, there must be an integer in each group, otherwise, revert to all values being the same. The average of the values lies between the greatest value and the least value, so it must be less than the greatest value. Hence, combining both parts, one of the integers must be greater than or equal to the average value.

13 Consider, say, Mary. There are 5 others, so by the principle she is either friends with 3 of them or there are 3 that are strangers to her. Of 3 friends, either 2 are friends with each other and Mary or the 3 are mutual strangers. Of 3 strangers, either 2 are strangers to each other and Mary, or the 3 are mutual friends. Either way, there are either 3 mutual strangers or 3 mutual friends.

14 Suppose there are n people. If they all have at least one friend, then they all have from 1 to $(n - 1)$ friends, so by the principle at least 2 must have the same number of friends. If someone has no friends, then there are $(n - 1)$ left. If one of them has no friends, then there are 2 people in the whole group with no friends. Otherwise, the remaining $(n - 1)$ people have between 1 and $(n - 2)$ friends, so by the principle there must be 2 of the $(n - 1)$ with the same number of friends.

3.04

1 B

2 C

3 C

4 E

5 a 120 b 362 880 c 3 628 800
d 24 e 479 001 600

6 5040

7 4.03×10^{26}

8 a 6 b 60 c 42
d 362 880 e 1680 f 120
g 110 h 30 240 i 592 620
j 240 240

9 a $n = 10$ b $n = 3$ c $n = 15$
d $n = 29$ e $n = 5$ f $n = 10$

10 24

11 5040

12 720

13 a 18 b 192

14 a 720 b 6840 c 24 360

15 a 24 b 720 c $\frac{1}{30}$

16 $\frac{1}{8}$

17 $(n + 1)! - n! = (n + 1) \times n! - n! = n!(n + 1 - 1)$
$= n \times (n - 1)! \times n = (n - 1)!\, n^2$

3.05

1 B

2 a 300 b 1080

3 a 96 b 261 c 93

4 a 720 b 4320 c 720

5 2880

6 a 3 628 800 b 362 880 c 28 800

7 a 6 b 720 c 5040
d 362 880 e 3 628 800

8 362 880

9 a 720 b 240 c 480 d 144

10 $\frac{2}{9}$

11 86 400

12 a 181 440 b 19 958 400
c 20 160 d 1 814 400
e 239 500 800

13 205 920

14 a 120 b 150

15 a $20!$ b $5!\,8!\,7!\,3!$
16 a 40 320 b 30 240 c 21 600
17 4680

3.06

1 a 60 480 b 2520 c 907 200
d 151 200 e 60 f 453 600
g 1 995 840 h 3360 i 83 160
j 145 297 152 000

2 a 60 b 48 c 36 d $\frac{1}{5}$
3 48
4 100
5 151 200
6 5040
7 About 4.22×10^{15}
8 12 600
9 a 6840 b 8000 c 7980
10 About 8.72×10^{10}, as you could start anywhere around the walls when viewing and in an exhibition, none of the paintings would be the same.
11 a 64 b 15 c $\frac{15}{64}$
12 a $\frac{3}{8}$ b $\frac{7}{64}$ c $\frac{5}{16}$
d $\frac{7}{64}$ e $\frac{184\,756}{1\,048\,576} \approx 0.176$

3.07

1 $\frac{1}{336}$
2 $\frac{1}{32\,760}$
3 a $\frac{1}{120}$ b $\frac{1}{210}$ c $\frac{1}{150}$ d $\frac{1}{7350}$
4 $\frac{1}{30}$
5 $\frac{1}{15\,400}$
6 $\frac{1}{140}$
7 $\frac{72}{325}$
8 $\frac{1}{56}$
9 $\frac{1}{3\,628\,800} \approx 0.000\,000\,28$

CHAPTER 3 REVIEW

1 E
2 E
3 B
4 D
5 D
6 456 976 000
7 20 160
8 $\frac{1}{128}$
9 32
10 14 400
11 6
12 9
13 9
14 362 880
15 30 240
16 720
17 256
18 576
19 420
20 $\frac{1}{4080}$
21 24
22 20 160
23 Separate the speakers by brand. Then if $12 \times 8 - 8 + 1 = 89$ speakers are placed in each group, by the strong pigeonhole principle (with $m_i = 12$) at least one group must contain at least 12 speakers.
24 64
25 1260

MIXED REVISION 1

Multiple choice

1 D
2 E
3 C
4 B
5 C
6 C
7 B
8 B
9 D

Short answer

1 (8.54, 290.6°)
2 Sample answers:
a Monkeys, people, dolphins or whales
b Any number between 0 and 1, e.g. $0.5^2 = 0.25$.
c A kite with angles of 90°, 110°, 50° and 110° is not a rectangle.
3 a 18 b $\frac{2}{19}$
4 (2, 4) or $2\mathbf{i} + 4\mathbf{j}$
5 Sample answers:
Inductive logic uses examples to reach a conclusion:
e.g. Holdens have four wheels
Falcons have four wheels
Toyotas have four wheels
Thus all cars have four wheels
Deductive logic uses logic to proceed from one (true) statement to another:
e.g. Reptiles lay eggs with soft shells
Goannas are reptiles
Thus goannas lay eggs with soft shells
6 2184

Application

1 (77.9, 81.5°), 77.9 km at the bearing 081.5°
2 About 39.3 knots at a bearing of 167.4°.
3 Use a counter example, of which 10 is the first: $10^2 + 10 + 11 = 121 = 11 \times 11$.
4 Proof
5 $\dfrac{1}{245\,157} \approx 0.000\,004$
6 $\dfrac{6}{77}$

4.01

1	a	12.12	b	5.5	c	28.28	d	−8.87
	e	0	f	30	g	−35.88	h	63.74
2	a	3	b	24.25	c	28.28	d	−45.55
	e	0	f	27	g	23.79	h	108.12
3	a	−240.99	b	−102.31	c	32.47	d	80.8
	e	4.89	f	59.06	g	−83.54	h	−61.52
4	a	−258.85	b	−141.38	c	233.41	d	−26.31
	e	−15.35	f	168.77	g	−39.6	h	−28.17
5	a	16	b	56	c	8		
	d	18	e	−29	f	−60		

4.02

1	a	10	b	12	c	5
	d	2	e	18		
2	a	44.50	b	46.36	c	57
3	a	19	b	23	c	28
	d	9	e	18	f	5
4	a	5	b	14	c	4
	d	20	e	2	f	−11
5	a	−14	b	0	c	−96
	d	−30	e	−85	f	73
6	a	27.59	b	11.31	c	11.19
	d	−8.12	e	31.89	f	−8.03
7	a	14.25°	b	8.13°	c	167.9°
	d	135°	e	147.53°	f	145.49°
8	a	49.4°	b	97.7°	c	170.84°
	d	99.73°	e	147.53°	f	31.33°

9 $\cos(\theta) = 0$, so $\theta = 90°$.
10 $\cos(A - B) = [\cos(A), \sin(A)] \cdot [\cos(B), \sin(B)]$
$= \cos(A)\cos(B) + \sin(A)\sin(B)$

4.03

1–4 Demonstration by substitution.
5–9 Proofs

4.04

1 B
2–3 For each pair of vectors **a** and **b**, **a** · **b** = 0.
4 For **a**, **c**, **f**, **a** · **b** = |**a**||**b**|, for **b**, **d**, **e**, **a** · **b** = −(|**a**||**b**|).
5 For **a**, **c**, **e**, **f**, **a** · **b** = |**a**||**b**|, for **b**, **d**, **a** · **b** = −(|**a**||**b**|).
6 a Neither
b Parallel (in opposite directions)
c Neither
d Parallel
e Perpendicular
f Neither
7 $a = 2\frac{1}{4}$
8 $b = 8$
9 Maximum value = 20 when **m** and **n** are parallel. Minimum value = −20 when **m** and **n** are parallel but in opposite directions.
10 49.4°

4.05

1 D
2 B
3 C
4 B
5 16.18
6 12.61
7 −2.21
8 29.67
9 0 (perpendicular)
10 19 (parallel)
11 $4\frac{3}{5}$
12 $-5\frac{9}{13}$
13 $\dfrac{31}{\sqrt{53}}$
14 $-\dfrac{9}{\sqrt{10}}$
15 $\dfrac{15}{\sqrt{5}}$

4.06

1 A
2 D

3	a	i	3.04**i** + 2.28**j**	ii	0.96**i** − 1.28**j**
	b	i	0.224**i** + 0.168**j**	ii	0.276**i** − 0.368**j**
	c	i	16**i** + 12**j**	ii	0
	d	i	−0.4**i** − 0.3**j**	ii	0
	e	i	−4.64**i** − 3.48**j**	ii	−0.36**i** + 0.48**j**
	f	i	−8**i** − 6**j**	ii	0
4	a	i	−6.92**i** − 4.62**j**	ii	6.92**i** − 10.38**j**
	b	i	2.88**i** − 0.58**j**	ii	−2.88**i** − 14.42**j**
	c	i	−15**j**	ii	0
	d	i	0	ii	−15**j**

5 About 1008 N at 13.4° towards the 500 N side of the 400 N force.
6 About 977 N each.
7 The component of the wind in the direction of the sprint is about 3.16 km/h.
8 About 1097 N parallel and 8933 N perpendicular to the slope.
9 About 1.25 N parallel and 5.87 N perpendicular, so the friction was about 0.55 N.
10 About 96 N
11 About 743 N

9780170250276

4.07

1 C
2 B
3 a 500 J b 400 J c 600 J
d 1200 J e 50 000 J f 900 J
4 a About 22.4 N b About 40.3 J
5 a 16 J b −10 J c 75 J
d −29 J e 0 f 0
6 7071 N
7 About 9888 J
8 72 J
9 a 76.53 N at 9° to the 50 N force.
b i 458.79 J ii 231.8 J iii 257.1 J
c 488.9 = 231.8 + 257.1, but because of the angles, the contributions of the 30 N and 50 N forces are almost the same.
10 a 8567 N perpendicular and 749.5 parallel to the slope, so about 250 N each.
b 89.94 kJ
11 7713 J
12 2.88 MJ
13 28.16 kJ
14 Proof
15 Proof

4.08

1 C
2 A
3 About 9.6 km at 089.8°.
4 520 m
5 About 181.8 knots at 066.5°
6 About 7.8 knots at 120.6°.
7 About 9.3 knots at 216.1°.
8 About 124.9° and 10.1 knots.
9 About 314.9° and 119.8 knots.
10 About 12:31:39 p.m.
11 About 9.8 knots.
12 9.434 knots at 32.01° upstream from the heading.
13 103.2 knots at 215.6°.
14 028.17°, to give a ground speed of 105 knots and an ETA of 1:35 p.m.

CHAPTER 4 REVIEW

1 A
2 C
3 B
4 C
5 B
6 A
7 E
8 D
9 B
10 a 20.54 b 14.65
11 a −32 b −26
12 63.4°
13 37.5
14 38
15 a −55 b −41 c 6
16 a 19.86 b −48.75
17 a 10.3° b 176.8°
18 Demonstration by substitution.
19 a, b Scalar product = 0
20 a $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$ b $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$
21 a Perpendicular b Neither
c Parallel but opposite in direction
22 17.10
23 7.23
24 a $1.10\mathbf{i} + 2.76\mathbf{j}$ b $-3.10\mathbf{i} + 1.24\mathbf{j}$
25 181.26 N (north) and 84.52 N (east)
26 a 300 J b 1904 J
27 Proof
28 123.69°
29 28.81 N at 10.7° to the right.
30 1476 N (parallel) and 8371 N (perpendicular).
31 675.3 N, 663.58 N
32 3236 J
33 About 149.5° and 7.7 knots
34 225° and 95 knots
35 Proof

5.01

1 a 15 b 1 c 25
d $\frac{n(n-1)(n-2)}{6}$ e $\frac{(n+1)n}{2}$
2 a 56 b 210 c 495
d 1287 e 1287
3 a 28 b 84 c 462
d 5005 e 38 760
4 a $n = 4$ b $n = 2$ c $n = 10$
d $n = 6$ e $n = 12$ f $n = 16$
5 2 598 960
6 a 45 b $\frac{n(n-1)}{2}$
7 10
8 252
9 15 504
10 210
11 296 010
12 792
13 3 838 380
14–15 Proofs

5.02

1 a 15 504 b 816
2 60
3 7260
4 a 120 b 112
5 935
6 a 45 b 120
7 a 100 947 b 462 c 924 d 36 300
e 26 334 f 74 613 g 27 225
8 $105

9 a 2 042 975 b 55 c 462 462 d 30 030

10 a 3003

b i 2450 ii 588 iii 56 iv 1176

11 a 1.58×10^{10} b 286

c 15 682 524 d 5 311 735

e 12 271 512

12 a 395 747 352 b 32 332 300

c 4 084 080 d 145 495 350

e 671 571 264

13 a 170 544 b 36 c 20 160

d 17 640 e 6300

14 a 7 b 27 132 c 13 860

d 20 790 e 27 720

15 a 5 b 360 c 126

16 a 715 b 495 c 330 d 1287

5.03

1–5 Proofs

6 $(x + y + z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 3xz^2 + 3y^2z + 3yz^2 + 6xyz$

7–8 Proofs

5.04

1 67

2 627

3 700

4 713

5 600

6 600

7 642

8 6600

9 188

10 440

11 978

12 43

5.05

1 a 44 b $\frac{4}{11} \approx 36\%$

2 $\frac{2}{3}$

3 $\frac{1}{26} \approx 0.03846$

4 a $\frac{1}{6}$ b $\frac{1}{3}$

5 a $\frac{1}{24} \approx 0.041\ 67$ b $\frac{3}{8} = 0.375$

6 a $\frac{1}{24} \approx 0.041\ 67$ b $\frac{1}{4} = 0.25$

c $\frac{3}{8} = 0.375$ d $\frac{1}{2} = 0.5$

5.06

1 500

2 a 22 b 84 c 270 d 690

3 1440

4 1422

5 4020

6 a $2! \times 19! \approx 2.433 \times 10^{17}$

b $3! \times 18! \approx 3.842 \times 10^{16}$

c $4! \times 17! \approx 8.537 \times 10^{15}$

d $20! - (6 \times \mathbf{a} - 4 \times \mathbf{b} + \mathbf{c})! \approx 1.118 \times 10^{18}$

7 ${}^{12}C_4 \times 7! \times 3! = 14\ 968\ 800$

5.07

1 $\frac{32}{663} \approx 0.048\ 27$

2 a $\frac{7}{102} \approx 0.068\ 63$ b $\frac{7}{306} \approx 0.022\ 88$

c $\frac{7}{17} \approx 0.4118$

3 $\frac{6}{4165}$

4 a $\frac{33}{16660}$ b $\frac{1}{899}$ c $\frac{7371}{3\ 235\ 501} \approx 0.022\ 278$

5 $\frac{1}{3}$

6 a 0.0976 b About 0.9785

c 28

7 a $\frac{1}{3}$ b $\frac{17}{24}$ c $\frac{9}{24}$

8 0.7

9 1

10 $\frac{1}{1200} \approx 0.000\ 83$

11 a $\frac{1}{253} \approx 0.003\ 95$ b $\frac{7}{253} \approx 0.0277$

12 a $\frac{1}{35} \approx 0.028\ 57$ b $\frac{1}{21} \approx 0.047\ 62$

c $\frac{10}{231} \approx 0.043\ 29$ d $\frac{1}{13} \approx 0.076\ 92$

CHAPTER 5 REVIEW

1 C

2 B

3 D

4 B

5 E

6 78

7 420

8 a 2600 b 4032

9 390

10 $\frac{23}{110} \approx 0.209$

11 150

12 440

13 $\frac{80}{221}$

9780170250276

14 $\frac{n}{r}\left({}^{n-1}C_{r-1}\right)$

$=\frac{n}{r}\times\frac{(n-1)!}{[(n-1)-(r-1)]!(r-1)!}$

$=\frac{n(n-1)!}{(n-1-r+1)\times!r(r-1)!}$

$=\frac{n!}{(n-r)!r!}$

$={}^nC_r$ **QED**

15 ${}^nC_r=\frac{n(n-1)\cdots(n-r+1)}{r!}$.

For n prime and $0<r<n$, there is a factor of n in the numerator, and no factor in the denominator divides n, as it is prime. Thus, there is a factor of n in the number ${}^nC_r$. **QED**

16 a 2898 b 21 252 c 33 649

17 568

6.01

1 a $x = 68°$ b $y = 97°$
c $\beta = 40°$ d $x = y = 39°$

2 a $x = 112°, y = 56°, z = 34°$ b $x = 49°$
c $x = 166°, y = 7°$ d $x = 62°, \beta = 31°$

3 a $x = 5°$ b $y = 102°$

4 $x = 30°$ ($\angle$ at centre is double the $\angle$ at the circumference)
$y = 75°$ ($\angle$ sum of isosceles $\triangle$)

5 $360° - x = 2 \times 110°$ ($\angle$ at centre is double the $\angle$ at the circumference)
$x = 140°$
$y = 70°$ (similarly)

6 $\angle OAC = 30°$ (Base $\angle$s of isosceles $\triangle$)
$\angle BAO = 25°$ (similarly)
$\therefore\ \angle CAB = 30° + 25°$
$= 55°$
$x = 2 \times \angle CAB$ ($\angle$ at centre is double the $\angle$ at the circumference)
$= 2 \times 55°$
$= 110°$

7 Obtuse $\angle BOD = 2\theta$ ($\angle$ at centre double $\angle$ at circumference)
Reflex $\angle BOD = 360 - 2\theta$ (angle of revolution)
$\angle BCD = \frac{1}{2}\angle BOD$ ($\angle$ at centre double $\angle$ at circumference)
$= \frac{1}{2}(360 - 2\theta)$
$= 180 - \theta$
So $\angle BCD$ and $\angle DAB$ are supplementary (add to 180°)

6.02

1 a $x = 18°$ b $\alpha = 83°, \beta = 42°$
c $x = 97°$ d $m = 136°$

2 a $a = 65°$ ($\angle$ at centre = twice $\angle$ at circumference standing on arc ED)
$b = 65°$ ($\angle$s at circumference standing on arc ED are equal)
b $\angle PRQ = 41°$ ($\angle$s at circumference standing on arc PQ are equal)
$w = 33°$ ($\angle$sum of $\triangle QRT$)
c $\angle NKM = 28°$ ($\angle$s at circumference standing on arc NM are equal)
$a = 56°$ (exterior $\angle$ of $\triangle NPK$ = sum of 2 interior opposite $\angle$s)
d $\angle DEC = 141°$($\angle$sum of $\triangle DEC$)
$m = 141°$ (vertically opposite $\angle$s)
$\angle ACB = 31°$ ($\angle$s at circumference standing on arc AB are equal)
$x = 110°$ (exterior $\angle$ of $\triangle BEC$ = sum of 2 interior opposite $\angle$s)

3 a $\angle DCE = \angle ACB$ (vertically opposite $\angle$s)
$\angle EDC = \angle BAC$ ($\angle$s in the same segment)
$\angle DEC = \angle ABC$ (similarly)
$\therefore$ Since all pairs of $\angle$s are equal,
$\triangle ABC\ |||\ \triangle DEC$.
b $x = 5.5$ cm

4 $\angle STV = \angle WUV$ ($\angle$s in the same segment)
$\angle TSV = \angle UVW$ (similarly)
$\angle TVS = \angle UVW$ (vertically opposite $\angle$s)
$\therefore$ Since all pairs of angles are equal,
$\triangle STV\ |||\ \triangle WUV$.
$x = 2.4$ cm

5 $\angle ECB = 33°$ ($\angle$s in the same segment)
$\angle EBC = 180° - 114° + 33°$ ($\angle$sum of $\triangle EBC$)
$= 33°$
$\therefore\ \angle ECB = \angle ADE$
These are equal alternate angles.
$\therefore\ AD \parallel BC$

6 Proof

6.03

1 a $\theta = 29°$
b $x = 10$ cm
c $x = 15°, y = 150°, z = 75°$
d $x = 47°, y = 43°, z = 94°$

2 a $x = 55°, y = 43°$
b $x = y = 32°, z = 58°, v = 32°, w = 17°$
c $x = 57°\ 30', y = 32°\ 30'$
d $x = 75°, y = 77°, z = 13°$

3 $\angle ABC = 90°$ ($\angle$ in semicircle)
$\therefore\ \angle BAC = 90°$ ($\angle$sum of $\triangle ABC$)
$= 61°$
$\therefore\ x = 61°$ ($\angle$s in the same segment)

4 Proof.
Prove $RS = SP$. $\therefore$ S is centre of circle.

5 Proof. Use isosceles $\triangle OAC$ and $\triangle OBC$.

6 $\angle B = 90°$ ($\angle$in semicircle)
$AC^2 = AB^2 + BC^2$
$= 6^2 + 3^2$
$= 36 + 9$
$= 45$

$\angle AC = \sqrt{45}$
$= 3\sqrt{5}$
Radius $= \frac{1}{2}AC$
$= \frac{3\sqrt{5}}{2}$ cm

7 a $x = 52°, y = 76°$
b $AC = BD$ (equal diameters)
Diagonals are equal so $ABCD$ is a rectangle.
$\therefore AD = BC$ (opposite sides of a rectangle)

8 a $\angle AOB = 90°$ (given)
$\angle ABC = 90°$ (angle in semi-circle)
$\therefore \angle AOB = \angle ABC$
$\therefore \angle A$ is common
$\therefore \triangle AOB \mathbin{|||} \triangle ABC$ (AAA)
(Note 2 pairs of angles equal means 3 pairs will be equal by angle sum of triangle.)
b $AO = BO$ (equal radii)
$AB = \sqrt{r^2 + r^2}$
$= \sqrt{2r^2}$
$= \sqrt{2} \times \sqrt{r^2}$
$= \sqrt{2}r$
By similar triangles
$\frac{AO}{AB} = \frac{BO}{BC}$
But $AO = BO$ so $AB = BC$
So $BC = \sqrt{2}\,r$

6.04

1 a $\theta = 32°$ b $x = 8$ cm c $\theta = \alpha = 68°\ 30'$
d $\theta = 31°$ e $x = 9$ mm f $\theta = 45°$
2 a $x = 5$ cm b $y = 15$ cm c $x = 42°$
d $z = 90°$ e $x = 6$ m, $y = 3$ m
f $m \approx 13.4$ cm
3 $\sqrt{41}$ cm
4 144 mm
5 25.6 cm
6 $CE = \sqrt{11.5^2 - 6.9^2}$
$= 9.2$
$CD = 2 \times 9.2$ (perpendicular from O bisects chord)
$= 18.4$
$= AB$
7 $OB = 8.3$ cm
8 Proof
9 Proof

6.05

1 a $x = 2.4$ m b $x \approx 10.3$ m
c $y \approx 5$ cm d $a = 14$
2 a $x = 4.7$ m, $y = 1.8$ m
b $x \approx 4.4$ m, $\alpha = 78°$, $\beta = 38°$, $\theta = 64°$
3 a $x = 2$ b $y = 4$ c $z = 6$
4 a $\angle ECD = \angle ACB$ (vertically opposite angle)
$\angle A = \angle E$ (angle in same segment)
$\therefore \triangle ABC \mathbin{|||} \triangle CDE$ (AAA)
b By similar triangles
$\frac{AC}{CE} = \frac{BC}{CD}$
$AC \times CD = BC \times CE$

6.06

1 a $\theta = 47°$ b $x = 5$ m
c $y = 11.3$ cm d $x = y = 26°$
e $a = 64°, b = 32°$ f $\theta = 57°$
g $p = \sqrt{145} \approx 12$ cm h $y = 10$ mm
i $x \approx 5.79$ cm j $x = 33°, y = 33°$
2 a $x = 67°$ b $y \approx 7.5$ cm
c $x = 63°, y = 126°$ d $x = 8.9$ m, $y \approx 6.2$ m
e $x = 98°$ f $x = 57°, y = 57°$
g $x = 72°, y = 15°$ h $x = 61°, y = 70°, z = 52°$
3 a $x = 26°, y = 74°, z = 48°$
b $x = 68°, y = 44°, z = 68°$
c $x = y = z = 45°$
d $x = 70°, y = 31°$
e $x = 20°, y = 57°, z = 103°$
f $x \approx 5.4$ cm
g $x \approx 7.1$ cm
h $x = 77°, y = 13°$
i $x \approx 1.2$ cm, $y \approx 2.1$ cm
j $x = 55°, z = 57°$
4 $AB \approx 13$ m
5-6 proofs

6.07

1 a $x = 107°, y = 94°$
b $\theta = 134°, \gamma = 90°$
c $x = 112°, y = 112°, z = 68°$
d $x = 92°, y = 114°$
e $\beta = 73°, \alpha = 107°, \gamma = 107°$
f $x = 141°, y = 63°$
g $x = 65°, y = 43°$
h $w = 89°, x = 86°, y = 54°, z = 35°$
i $w = 69°, x = 111°, y = 82°, z = 98°$
j $x = 118°$
2 a $x = 62°, y = 31°$ b $x = 75°, y = 105°$
c $x = 88°, y = 65°$ d $x = 62°, y = 82°, z = 36°$
e $x = 90°, y = 113°$ f $x = 38°, y = 71°$
g $x = 85°, y = 95°$ h $x = 48°, y = 78°$
i $x = 107°, y = 73°$
j $a = 81°, b = 55°, c = 83°, d = 16°, e = 28°$
3 a $\angle A = 180° - 58°$ ($\angle A$ and $\angle B$ co-interior angles, $AD \parallel BC$)
$\angle D = 180° - 58°$ ($\angle C$ and $\angle D$ co-interior angles, $AD \parallel BC$)
So $\angle A = 180° - \angle C$ and $\angle D = 180° - \angle B$.
Since opposite angles are supplementary, $ABCD$ is a cyclic quadrilateral.
b $\angle B = \angle D = 90°$ (given)
$\therefore \angle B = 180° - \angle D$.
Let $\angle A = x$

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$\angle C = 360° - 90° + 90° + x$ ($\angle$sum of quadrilateral)
$= 360° - 180° - x$
$= 180° - x$
$= 180° - \angle A$
Since opposite angles are supplementary, $ABCD$ is a cyclic quadrilateral.

c $\angle CDA = 180° - \theta$ (straight angle)
$\therefore \angle B = 180° - \angle CDA$
Let $\angle A = x$
$\angle C = 360° - 90° + 90° + x$ ($\angle$sum of quadrilateral)
$= 360° - 180° - x$
$= 180° - x$
$= 180° - \angle A$
Since opposite angles are supplementary, $ABCD$ is a cyclic quadrilateral.

d $\angle DAB = 90°$ (tangent $\perp$ radius)
$\angle DCB = 90°$ (tangent $\perp$ radius)
Now $\angle DAB + \angle DCB = 90° + 90°$
$= 180°$
Since opposite angles are supplementary, $ABCD$ is a cyclic quadrilateral.

4 Proof

6.08

1 a $x = 58°, y = 116°, z = 58°$ b $x = 5$ mm

2 a $x = 10$ cm b $x = 64°, y = 26°$
c $x = 13$ cm d $x = 27°, y = 54°$
e $y = 5$ cm f $x = 32°, y = 7°$
g $x = 72°, y = 42°$ h $x = 35°, y = 90°$
i $m = 23°, n = 67°, p = 67°, q = 23°$
j $x = 71°, y = 62°$

3 $\therefore \angle OAB = 90°$ (tangent $\perp$radius)
$z = 90° - 48°$ ($\angle$sum of $\triangle ABC$)
$= 42°$
$OA = OC$ (equal radii)
$\therefore \angle OAC = \angle OCA = y$ (base $\angle$s of isosceles $\triangle$)
$y = (180° - 48°) \div 2$ ($\angle$ sum of $\triangle OAC$)
$= 66°$
$\angle ACD = 180° - \angle AED$ (opposite $\angle$s of cyclic quad.)
$\therefore y + u = 180° - 62°$
$66° + u = 118°$
$u = 52°$
$\angle BAC = \angle OAB - \angle OAC$
$\therefore x = 90° - 66°$
$= 24°$
$v = \frac{1}{2}\angle AOC$ ($\angle$ at centre twice $\angle$ at circumference)
$= \frac{1}{2} \times 48°$
$= 24°$

4 21 cm

5 $AC^2 + BC^2 = 3.9^2 + 5.2^2$
$= 42.25$
$AB^2 = 6.52$
$= 42.25$
$\therefore AB^2 = AC^2 + BC^2$
$\therefore \angle ACB = 90°$ (by Pythagoras' theorem)
$\therefore A$ lies on a diameter of the circle. (tangent $\perp$radius)

6 a $x = 72°, y = 121°$ b $x = 63°, y = 63°$

7 $\angle DAC = 63°$ ($\angle$sum of $\triangle DAE = 180°$)
$\therefore \angle DAC = \angle DBC = 63°$
$\therefore A, B, C, D$ are concyclic (equal $\angle$s at circumference standing on chord CD)

8 $OA = r$
$AC = \frac{x}{2}$ (perpendicular from the centre bisects a chord)
$OC = \sqrt{r - \left(\frac{x}{2}\right)^2}$ (Pythagoras' theorem)
$= \sqrt{r^2 - \frac{x^2}{4}}$
$= \sqrt{\frac{4r^2}{4} - \frac{x^2}{4}}$
$= \sqrt{\frac{4r^2 - x^2}{4}}$
$= \frac{\sqrt{4r^2 - x^2}}{2}$
$CD = r + \frac{\sqrt{4r^2 - x^2}}{2}$
$= \frac{2r + \sqrt{4r^2 - x2}}{2}$

9 $\angle BDE = \angle ABD + \angle BAD$
$\therefore 2\alpha = \angle ABD + \alpha$
$\alpha = \angle ABD$
$\therefore \triangle BAD$ is isosceles with $AD = BD$.
$\angle CDE = \angle ACD + \angle CAD$ (ext. $\angle$ of $\triangle BAD$)
$\therefore 2\beta = \angle ACD + \beta$
$\beta = \angle ACD$
$\therefore \angle CAD$ is isosceles with $AD = CD$.
$\therefore AD = BD = CD$
So a circle can be drawn through A, B and C with centre D.

10 Let $\angle ODC = x$ and $\angle OAB = y$.
Then you can find all these angles (giving reasons).
$\angle AOC + \angle COB + \angle BOD + \angle AOD = 360°$
($\angle$of revolution)
$90° - y + x + \angle COB + y + 90° - x + \angle AOD = 360°$
$\angle COB + \angle AOD + 180° = 360°$
$\therefore \angle COB + \angle AOD = 180°$

11 Let $ABCD$ be a quadrilateral with opposite angles supplementary.
i.e. $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$.
Assume the points are not concyclic. Draw a circle through A, B and C, cutting CD at E.
Now $ABCE$ is a cyclic quadrilateral, so $\angle AEC + \angle B = 180°$. (opposite $\angle$s are supplementary)
Also, $\angle D + \angle B = 180°$ (given)
$\angle D = \angle AEC$
These are equal corresponding angles, so $DA \parallel EA$ (this is impossible!)
$\therefore A, B, C$ and D must be concyclic.
$\therefore ABCD$ is a cyclic quadrilateral.

CHAPTER 6 REVIEW

1 C
2 E
3 B
4 D
5 E
6 C
7 D
8 $\theta = 56°$
9 $y = 2.3$ mm
10 $x = 7.2$ m
11 $x = y = 12$ cm
12 $z = 19°$ ($\angle$s in same segment)
$y = 180° - (131° + 19°)$ ($\angle$sum of Δ)
$= 30°$
$x = 30°$ ($\angle$s in same segment)
13 $x = 10$ cm
14 $\alpha = 3°$, $\beta = 44°$, $\gamma = 136°$
15 $a = \frac{1}{2} \times 100°$ ($\angle$ at centre twice $\angle$ at circumference)
$= 50°$
$\angle OCA = 90°$ (tangent perpendicular to radius)
$\therefore\ b = 90° - 83°$
$= 7°$
$OC = OE$ (equal radii)
$\therefore\ \triangle OCE$ is isosceles.
$\therefore\ \angle OCE = \angle OEC = c$
$2c + 100° = 180°$ ($\angle$ sum of Δ)
$c = 40°$
Reflex $\angle COE = 360° - 100°$ ($\angle$ of revolution)
$= 260°$
$d = 360° - (260° + 50° + 7°)$ ($\angle$ sum of quadrilateral)
$= 43°$
16 17 cm
17 5.3 m
18 $a = 101°$, $b = 98°$
19 $\alpha = 61°$, $\beta = 29°$
20 14.9 cm
21 $x = 4.9$ m
22 18 cm
23 $\alpha = 127°$, $\beta = 53°$
24 $\angle D = 180° - (80° + 53°)$ ($\angle$sum of Δ)
$= 47°$
$\therefore\ y = 47°$ ($\angle$s in same segment)
$x = 47°$ ($\angle$s in alternate segment)
25 $x = 55°$, $y = 56°$, $z = 54°$
26 $\angle C$ is common.
$\angle A = \angle CBD$ ($\angle$s in alternate segment)
$\therefore\ \triangle BCD \;|||\; \triangle ABC$ (AAA)
27 a $\angle OCB = \angle OCA = 90°$ (given)
$OA = OB$ (equal radii)
OC is common.
$\therefore\ \triangle OAC \equiv \triangle OBC$ (RHS)
b $AC = BC$ (corresponding sides in $\equiv \triangle$s)
$\therefore\ OC$ bisects AB.
28 Proof
29 Proof
30 Proof

MIXED REVISION 2

Multiple choice

1 D
2 A
3 B
4 A
5 A
6 E
7 E
8 E
9 B

Short answer

1 a 14 b 27.58
2 20 have all three.
3 $\angle LMN = 90°$ (angle in a semicircle)
$x = 41$ (complementary angles in $\triangle LMN$)
4 3.13
5 $\frac{496}{14\,259} \approx 3.479\%$
6 $IK \times KG = HK \times KF$ (product of intervals on chord IG = product of intervals on chord HF)
$y \times 12 = 18 \times 8$
$12y = 144$
$y = 12$

Application

1 About 74.1°
2 1.839 MJ
3 $\frac{264}{1885}$
4 1287
5 $3p = 360$ (revolution)
$p = 120$
$\therefore\ \angle BAD + \angle BCD = p + 60$
$= 120 + 60$
$= 180$
$\therefore\ ABCD$ is cyclic (opposite angles of quadrilateral supplementary).
6 Perimeter $= 2x + 2y + 40 = 2(16) + 40 = 72$ cm

7.01

1 D
2 a $2a + 1 + 2b + 1 = 2(a + b + 1)$
b $2a(2b + 1)$ or $2(ab + a)$
3 E
4 12, 18
5 a $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + 2048 + 4096 + 8191 + 16\,382 + 32\,764 + 65\,528 + 131\,056 + 262\,112 + 524\,224 + 1\,048\,448 + 2\,096\,896 + 4\,193\,792 + 8\,387\,584 + 16\,775\,168 = 33\,550\,336$
b $2^{12}(2^{13} - 1)$

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6 $1 + 2 + 4 + 8 + 16 + 32 + 37 + 74 + 148 + 296 + 592 = 1210$, $1 + 2 + 5 + 10 + 11 + 22 + 55 + 110 + 121 + 242 + 605 = 1184$

7 No, as 2 is prime.

8 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97.

9 a No b Yes

10 Yes

11 6

12 a 2^{16} b $2^{23} - 1$ is not prime

INVESTIGATION: PLAYING WITH INTEGERS

1 a i 65 ii 17 iii 31

b i 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97

ii 7919 iii 107 iv No

2 a i 5 ii 4 iii Yes

b i show that ii $2^4 \times 31$, $2^6 \times 127$

c $2^{p-1} \times$ prime, use $S_n = \dfrac{a(r^n - 1)}{r - 1}$, 2^{p-1} will have p factors, $1, 2, 2^2, \ldots, 2^{p-1}$, $S_p = \dfrac{1(2^p - 1)}{2 - 1} = 2^p - 1$, there will be p remaining terms which are multiples of the prime number, $b, 2b, \ldots, 2^{p-1} \times b$, $S_p^* = (2^p - 1)b$, $S_p + S_p^* = (2^p - 1) + (2^p - 1)b = 2 \times 2^{p-1} \times b = 2^p b$, $2^p - 1 - b = 0$, $b = 2^p - 1$

d

p	Is $2^p - 1$ prime?	Is $2^{p-1}(2^p - 1)$ a Perfect number?
2	Yes	Yes
3	Yes	Yes
5	Yes	Yes
7	Yes	Yes
11	No	No
13	Yes	Yes
17	Yes	Yes
19	Yes	Yes
23	No	No
29	No	No
31	Yes	Yes

e i No, the only proper divisor of a prime is 1, so there is no prime equal to the sum of its proper divisors.

ii If a Mersenne prime exists, a corresponding perfect number exists.

f 47 (the same as the number of Mersenne primes)

7.02

1 a $\dfrac{1}{3}$

b $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1$
$= 2(2ab + a + b) + 1$

2 a i Yes ii No iii No iv No

b i Closed ii $-1 - (-2) = 1$

iii $-1 \times (-2) = 2$

iv $\dfrac{-1}{-2} = 0.5$

3 No, $1 \div 2$

4 a No b Yes

5 a 2 b 0

6 $2k + 1 + 2k + 3 = 4k + 4 = 4(k + 1)$

7 $(k + 1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1$

8 Let $q = p + 2$, then $pq + 1 = p(p + 2) + 1$
$= p^2 + 2p + 1 = (p + 1)^2$, which is divisible by $p + 1$.
$p + 1 = \dfrac{2p + 2}{2} = \dfrac{p + p + 2}{2} = \dfrac{p + q}{2}$, which is the average of p and q.

9 a $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

b Show that if n^2 is even, n is even (9a) and if n is even, n^2 is even (Example 5a).

10 a $2a + 2b + 1 = 2(a + b) + 1$

b Show **10a** and $2a + 1 + 2b + 1 = 2(a + b + 1)$, $2a + 2b = 2(a + b)$

11 $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1$
$= 2(2ab + a + b) + 1$

12 $2a(2b + 1) = 2(2ab + a)$, $2a \times 2b = 2(2ab)$

13 Proof

14 Proof

15 Proof

16 Proof

7.03

1 a i $\dfrac{649}{200}$ ii $-\dfrac{260\,539\,287}{500\,000}$

b i -707.375 ii 62.258

2 C

3 a $51.\overline{27}$ b $18.\overline{384615}$ c $3.1\overline{42857}$

4 a $\dfrac{1}{9}$ b $\dfrac{7}{12}$ c $\dfrac{3097}{990}$

d $\dfrac{1\,714\,284}{999\,999}$ e $\dfrac{3}{2}$

5 a $0.\overline{9}$ b $\dfrac{1}{1}$

6 a 6, 16, 18

b $\dfrac{1}{p}$ where p is a prime has $p - 1$ recurring digits

c $\dfrac{1}{11} = 0.\overline{09}$

7 a $0.\overline{142857}$, $0.\overline{285714}$, $0.\overline{428571}$, $0.\overline{571428}$, $0.\overline{714285}$, $0.\overline{857142}$

b The recurring digits are rotating.

8 a $0.\overline{076923}$, $76 + 923 = 999$

b $\frac{1}{11} = 0.\overline{09}$, $0 + 9 = 9$,

$\frac{1}{17} = 0.\overline{058\ 823\ 529\ 411\ 764\ 705}$,

$05882352 + 94117647 = 99\ 999\ 999$

c $\frac{1}{77} = 0.\overline{012987}$, $12 + 987 = 999$

7.04

1 a $\frac{-2}{3}$ or $\frac{2}{-3}$ b $\frac{375}{1000} = \frac{3}{8}$

c $\frac{5670}{1}$ d $\frac{82}{1000} = \frac{41}{500}$

e $\frac{0}{1}$ f $\frac{11}{4}$

2 D

3 a $\frac{29}{21}$ b $\frac{-9}{10} = \frac{9}{-10}$ c $\frac{10}{21}$

d $\frac{17}{2}$ e $\frac{14}{15}$

4 $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{e}{f}$

5 $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} = \frac{e}{f}$

6 a $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$ b Yes

7 a i $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ ii $\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

iii $\frac{\frac{1}{2}}{\frac{1}{4}} = 2$

b $\frac{1}{2a} \times \frac{1}{2b} = \frac{1}{2(2ab)}$

8 $\left(\frac{a}{b}\right)^2 - 2\left(\frac{a}{b}\right) + 5$ is rational

9 Proof

10 Proof

11 Many possibilities, like $x^4 - 8x^2 + 15 = 0$, which factorises to $(x^2 - 3)(x^2 - 5) = 0$. It has x-intercepts $\pm\sqrt{3}, \pm\sqrt{5}$, y-intercept 15.
Its roots are $\pm\sqrt{3}, \pm\sqrt{5}$ which are not rational.

INVESTIGATION: INFINITE CONTINUED FRACTIONS FOR IRRATIONAL NUMBERS

a 1.5, 1.4, $1.41\overline{6}$, 1.41379… converging to $\sqrt{2} \approx 1.414\ 132\ 562\ldots$

b i $0 + \cfrac{1}{99 + \cfrac{1}{1}} = 0.01$, $0 + \cfrac{1}{99 + \cfrac{1}{1 + \cfrac{1}{9}}} = 0.01001$

ii $0 + \cfrac{1}{99 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{111 + \cfrac{1}{9 + \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{8}}}}}}}} = 0.010010001$

c $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ldots}}}}$, [1;1,2,1,1,4,1,1,6,1,1,8,…],

$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$, [3;7,15,1,292,1,1,1,2,1,3,1,…]

d Square roots eventually yield repetitive sequences, but cube roots and higher powers yield random sequences.

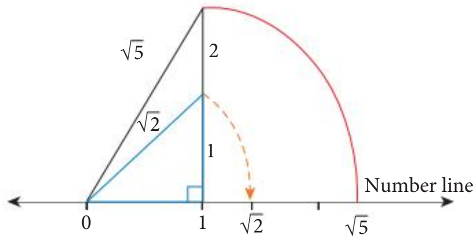
7.05

1 E

2 a $-\sqrt{2} + \sqrt{2} = 0$ b $\sqrt{2} - \sqrt{2} = 0$

c $\sqrt{2} \times \sqrt{8} = 4$ d $\frac{\sqrt{2}}{\sqrt{2}} = 1$

3



4 No, π cannot be written as a fraction.

5 e^π, e^2. Other answers are possible.

6 $\sqrt{5} = \frac{a}{b}$, $a^2 = 5b^2$

7 $\log_2(3) = \frac{a}{b}$, $a, b \in \mathbb{Z}^+$, $2^{\frac{a}{b}} = 3$, $2^a = 3^b$

8 Solve $a + b + c = 2$, $4a + 2b + c = 5$, $9a + 3b + c = 9$

9 $\frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{11}} + \cdots + \frac{1}{10^{\frac{1}{2}n^2 + \frac{5}{2}n - 1}} + \cdots$

10 $\frac{1}{10^2} + \frac{2}{10^5} + \frac{4}{10^9} + \frac{8}{10^{14}} + \cdots + \frac{2^{n-1}}{10^{\frac{1}{2}n^2 + \frac{3}{2}n}} + \cdots$

11 $b^2 - 4ac > 0$ and $b^2 - 4ac \neq \frac{m^2}{n^2}$, $n \neq 0$, $m, n \in \mathbb{Z}$

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12 $x^3 - 2 = 0$

13 a $\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}$

b i $\frac{1+\sqrt{5}}{2}$

ii $\frac{m}{s} = \frac{s}{m-s}$, $\frac{m}{s}$ in simplest form, $\frac{s}{m-s}$ simpler

7.06

1 a i 4 ii $2\sqrt{3}$ iii 1 iv $7+4\sqrt{3}$

b i $\boldsymbol{Z}^+$ or $\boldsymbol{N}$

ii $\boldsymbol{Q}'$

iii $\boldsymbol{Z}^+$

iv $\boldsymbol{Q}'$

2 a Yes b Yes c Yes d Yes

3 a $\frac{1}{9}$ b $\frac{1}{2}$ c $-\frac{1}{3}$ d $\frac{1}{6}$

4 Proof

5 Proof

6 Proof

7 a Proof b $a = 3, b = 4, c = 5$

8 $c = -\frac{a^2}{b^2} - \frac{a}{b}$, $b \neq 0$, $a, b \in \boldsymbol{Z}$ and $c \leq \frac{1}{4}$

9 $e = \sqrt{2}, f = \sqrt{3}, g = \sqrt{6}$

7.07

1–6 Proofs

CHAPTER 7 REVIEW

1 C

2 D

3 B

4 C

5 D

6 a $2k + 1 + 2k + 1 + 2k + 1 = 2(3k + 1) + 1$

b $(2k + 1)^3 = 2(4k^3 + 6k^2 + 3k) + 1$

7 $117 \neq 100$

8 a $2^9 - 1$, 9 is not prime b divisible by 7

9 a $\frac{2}{4} = \frac{1}{2}$ b $12 - 9 = 3$

10 Proof

11 Proof

12 a $\frac{-1\,015\,779}{15\,625}$ b 0.3376

13 a $934.7\overline{14285}$ b $0.\overline{012345679}$

14 $\frac{9463}{2475}$

15 a $\frac{-5}{6}$ b $\frac{7}{8}$ c $\frac{723}{10\,000}$

d $\frac{2400}{1}$ e $\frac{29}{5}$

16 a $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ b $\frac{7}{0}$ undefined

17 Proof

18 Proof

19 Proof

20 Proof

21 $\frac{1}{10^2} + \frac{1}{10^7} + \frac{1}{10^{15}} + \cdots + \frac{1}{10^{\frac{3n^2}{2}+\frac{n}{2}}} + \cdots$

22 Proof

23 a $2^{\frac{2}{3}}$ b $\sqrt{3} \times \sqrt{3} = 3$

24 $c \leq 0$

25 Proof

26 Proof

27 a i 1, 2, 4, 8, 16, 31, 62, 124, 248, 496

ii 31, prime

iii 62, 2×31

iv 124, 248, 496, $2^n \times 31$

v A perfect number

b i $2^4(2^5 - 1)$ ii 16 iii 31

c $1 + 2 + 4 + \cdots + 2^{p-1} = \frac{1(2^p - 1)}{2 - 1} = 2^p - 1$

d i $(x - 1)(1 + x)$, $(x - 1)(1 + x + x^2)$, $(x - 1)(1 + x + x^2 + x^3)$, $(x - 1)(1 + x + x^2 + x^3 + x^4)$, $(x - 1)(1 + x + x^2 + x^3 + \cdots + x^{n-1})$,

ii If p is not prime then $2^p - 1$ is not prime,

$\left(2^m\right)^n - 1$, $m, n \in \boldsymbol{Z}^+\backslash\{1\}$, $x = 2^m \geq 4$, $x^n - 1 = (x - 1)(1 + x + x^2 + \cdots + x^{n-1})$,

$x - 1 > 1$, and $(1 + x + x^2 + \cdots + x^{n-1}) > 1$

iii For $p = 11$, $2^{11} - 1 = 23 \times 89$ so it is not prime.

8.01

1 **a, e, f**

2 a $x_{12} = 2$ b $x_{21} = 1$ c $x_{34} = -2$

d $x_{22} = -5$ e $x_{32} = 3$

3 a $r_{31} = r_{25}$ b $2r_{14} = r_{45}$

c $r_{32} = 2r_{15}$ d $r_{42} = -3r_{44}$

e $r_{25} = -4r_{34}$ f $r_{35} = -5r_{34}$

4 a 2×2 square matrix with all elements 1

b 1×3 row matrix with all elements 2

c 2×1 column matrix

d 3×3 diagonal matrix with diagonal elements of 3

e 3×4 matrix with elements of 1 and 2

5 a $\mathbf{M} = \begin{bmatrix} 3 & 2 & 4 & 1 & 5 & 0 & 3 & 6 & 1 \\ 1 & 8 & 1 & 7 & 1 & 8 & 0 & 5 & 8 \\ 4 & 0 & 5 & 6 & 4 & 5 & 9 & 1 & 7 \\ 7 & 9 & 2 & 0 & 1 & 1 & 10 & 9 & 1 \end{bmatrix}$

b 0

c 1

d 0

e 4×9

6 a USD \$1.05664

b $\begin{bmatrix} 1 & 1.05664 & 0.65720 & 94.3892 \\ 0.94642 & 1 & 0.62192 & 89.3362 \\ 1.52148 & 1.60769 & 1 & 143.625 \\ 0.01060 & 0.01120 & 0.00696 & 1 \end{bmatrix}$

c 89.3362 d 1.60769 e $e_{ij} = \frac{1}{e_{ji}}$

7 **a** 7×6 **b** 50.7 **c** 1948

d maximum temperature range = 72.7

8 **a** $\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 6 & 4 & 0 \end{bmatrix}$

b Mosquito larvae do not eat the other animals.

c Water beetles do not eat each other.

9 $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix}$

8.02

1 **a** $\begin{bmatrix} 6 & 0 \\ -3 & -6 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{6}{5} & \frac{12}{5} \\ \frac{18}{5} & -\frac{6}{5} \end{bmatrix}$ **c** $\begin{bmatrix} 15 & 30 \\ 45 & -15 \end{bmatrix}$

d $\begin{bmatrix} -8 & 0 \\ 4 & 8 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

2 **a** $\begin{bmatrix} -8 & -6 & -2 \\ 0 & -6 & -12 \\ -10 & 0 & 2 \\ 0 & -10 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{1}{2} & -2 & -\frac{3}{2} \\ 1 & -\frac{3}{2} & \frac{1}{2} \\ 2 & 1 & 2 \\ -2 & -\frac{3}{2} & -1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 4 & 3 \\ -2 & 3 & -1 \\ -4 & -2 & -4 \\ 4 & 3 & 2 \end{bmatrix}$ **d** $\begin{bmatrix} 12 & 9 & 3 \\ 0 & 9 & 18 \\ 15 & 0 & -3 \\ 0 & 15 & -6 \end{bmatrix}$

3 **a** $\begin{bmatrix} 15 & 0 & -5 & 10 \\ 25 & -20 & 15 & 5 \\ -5 & 10 & 0 & 20 \end{bmatrix}$

b $\begin{bmatrix} -3 & -12 & -9 & 15 \\ 9 & -6 & 3 & 18 \\ 0 & -15 & 6 & -9 \end{bmatrix}$

c $\begin{bmatrix} 12 & 0 & -4 & 8 \\ 20 & -16 & 12 & 4 \\ -4 & 8 & 0 & 16 \end{bmatrix}$

d $\begin{bmatrix} -5 & -20 & -15 & 25 \\ 15 & -10 & 5 & 30 \\ 0 & -25 & 10 & -15 \end{bmatrix}$

4 **a** $\begin{bmatrix} 4.08 & 8.33 & 14.28 \\ 7.14 & 13.43 & 20.06 \\ 6.46 & 14.28 & 18.70 \\ 10.03 & 20.06 & 24.48 \end{bmatrix}$

b $\begin{bmatrix} 4.49 & 9.16 & 15.71 \\ 7.85 & 14.77 & 22.07 \\ 7.11 & 15.71 & 20.57 \\ 11.03 & 22.07 & 26.93 \end{bmatrix}$

5 **a** $\begin{bmatrix} 7000 \\ 5600 \\ 8750 \\ 8050 \\ 6650 \end{bmatrix}$ **b** $\begin{bmatrix} 240 \\ 60 \\ 420 \\ 120 \\ 180 \end{bmatrix}$ **c** Total sales = \$37 070

8.03

1 **a** $\begin{bmatrix} -4 & 4 \\ 7 & -12 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & -1 \\ 6 & -2 \end{bmatrix}$ **c** $\begin{bmatrix} -3 & 10 \\ 17 & -13 \end{bmatrix}$

d $\begin{bmatrix} 13 & 6 \\ 4 & -13 \end{bmatrix}$ **e** $\begin{bmatrix} 19 & 4 \\ 21 & -7 \end{bmatrix}$

2 **a** $\begin{bmatrix} 5 & 8 & 9 \\ 0 & 5 & 10 \\ 5 & 6 & 1 \\ 7 & 5 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & 8 & 9 \\ 0 & 5 & 10 \\ 5 & 6 & 1 \\ 7 & 5 & -2 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 & 5 \\ 2 & -1 & 5 \\ 4 & 8 & 6 \\ 3 & -3 & -2 \end{bmatrix}$

d $\begin{bmatrix} 8 & 23 & 33 \\ 0 & 11 & 22 \\ 5 & 24 & 7 \\ 28 & 5 & -2 \end{bmatrix}$ **e** $\begin{bmatrix} 12 & 11 & 13 \\ 4 & 7 & 28 \\ 23 & 16 & 9 \\ 6 & 9 & -10 \end{bmatrix}$

3 **a** $\begin{bmatrix} 4 & 4 & 2 & -3 \\ 2 & -2 & 2 & -5 \\ -1 & 7 & -2 & 7 \end{bmatrix}$

b $\begin{bmatrix} 7 & -1 & 7 & -8 \\ -4 & 10 & -10 & -4 \\ 4 & 12 & -4 & 8 \end{bmatrix}$

c $\begin{bmatrix} 30 & -15 & 8 & -1 \\ 17 & 8 & -15 & 10 \\ 8 & 29 & -6 & 31 \end{bmatrix}$

d $\begin{bmatrix} 37 & -21 & 21 & -16 \\ 2 & 34 & -40 & 6 \\ 18 & 44 & -12 & 38 \end{bmatrix}$

e $\begin{bmatrix} -3 & -39 & -1 & 11 \\ -10 & 32 & -36 & 37 \\ 17 & -19 & 6 & -23 \end{bmatrix}$

4 **a** $\begin{bmatrix} -8 & 4 \\ 9 & -8 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 13 \\ 12 & -4 \end{bmatrix}$ **c** $\begin{bmatrix} -9 & -2 \\ -1 & -7 \end{bmatrix}$

d $\begin{bmatrix} 7 & -6 \\ -14 & -7 \end{bmatrix}$ **e** $\begin{bmatrix} -1 & 32 \\ 33 & -11 \end{bmatrix}$

5 **a** $\begin{bmatrix} -3 & 2 & 7 \\ 0 & -1 & -2 \\ -5 & 6 & 3 \\ 7 & -5 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & -2 & -7 \\ 0 & 1 & 2 \\ 5 & -6 & -3 \\ -7 & 5 & -2 \end{bmatrix}$

c $\begin{bmatrix} -2 & -9 & -11 \\ 2 & -5 & -3 \\ 4 & -4 & 2 \\ -11 & -3 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 17 & 31 \\ 0 & 5 & 10 \\ -5 & 24 & 9 \\ 28 & -5 & 2 \end{bmatrix}$

e $\begin{bmatrix} -12 & -7 & 7 \\ 4 & -11 & -8 \\ -7 & 16 & 15 \\ 6 & -21 & 2 \end{bmatrix}$

6 **a** $\begin{bmatrix} 2 & -4 & -4 & 7 \\ 8 & -6 & 4 & 7 \\ -1 & -3 & 2 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & -9 & 1 & 2 \\ 2 & 6 & -8 & 8 \\ 4 & 2 & 0 & 2 \end{bmatrix}$

c $\begin{bmatrix} -6 & 15 & -16 & 17 \\ 23 & -40 & 39 & -2 \\ -16 & -13 & 6 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 5 & 25 & 9 & -13 \\ -4 & -6 & 10 & -30 \\ -6 & 22 & -8 & 18 \end{bmatrix}$

e $\begin{bmatrix} -3 & -39 & -1 & 11 \\ -10 & -8 & -38 & 37 \\ 17 & -19 & 6 & -23 \end{bmatrix}$

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7 a $\begin{bmatrix} 300 & 400 & 200 & 150 \\ 400 & 220 & 300 & 200 \\ 250 & 150 & 400 & 150 \end{bmatrix}$

b $\begin{bmatrix} 30 & 70 & 20 & 20 \\ 20 & 30 & 40 & 30 \\ 40 & 10 & 50 & 10 \end{bmatrix}$

c $\begin{bmatrix} 50 & 100 & 40 & 15 \\ 15 & 50 & 30 & 10 \\ 30 & 25 & 20 & 40 \end{bmatrix}$

d $\begin{bmatrix} 320 & 430 & 220 & 145 \\ 395 & 240 & 290 & 180 \\ 240 & 165 & 370 & 180 \end{bmatrix}$

8 a $\mathbf{F} + \mathbf{M} + \mathbf{L}$ b $\begin{bmatrix} 10 & 1 & 4 \\ 6 & 0 & 9 \\ 4 & 3 & 8 \\ 8 & 2 & 5 \\ 4 & 2 & 9 \\ 9 & 0 & 6 \end{bmatrix}$

c Team 1 (1st team)

8.04

1 a $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ b $\begin{bmatrix} -7 & 4 \end{bmatrix}$ c $\begin{bmatrix} 6 & -2 & 4 \\ -9 & 3 & -6 \\ 12 & -4 & 8 \end{bmatrix}$

d $\begin{bmatrix} -2 & 1 \\ 6 & 3 \end{bmatrix}$ e $\begin{bmatrix} -16 \\ 6 \end{bmatrix}$

2 a $\begin{bmatrix} 4 & 8 & -11 \end{bmatrix}$ b $\begin{bmatrix} -7 \\ -2 \\ -10 \end{bmatrix}$

c $\begin{bmatrix} 1 & 13 \\ -2 & 14 \end{bmatrix}$ d $\begin{bmatrix} -1 & -13 \\ 2 & -14 \end{bmatrix}$

e $\begin{bmatrix} -11 & 12 & 12 \\ 18 & -6 & 16 \\ 7 & -9 & 6 \end{bmatrix}$

3 Both $\begin{bmatrix} 33 & 21 \\ 35 & 40 \end{bmatrix}$, not usually expected

4 Both $\begin{bmatrix} -11 & 7 & -11 \\ 7 & 13 & 7 \\ -11 & 7 & -11 \end{bmatrix}$, not usually expected

5 Proof

6 Proof

7 a $\begin{bmatrix} 12 & 10 & 8 \\ 15 & 8 & 4 \\ 10 & 10 & 10 \\ 12 & 12 & 4 \end{bmatrix}$

b $\begin{bmatrix} 117.80 \\ 155.00 \\ 190.00 \end{bmatrix}$

c $\begin{bmatrix} 4483.60 \\ 3767.00 \\ 4628.00 \\ 4033.60 \end{bmatrix}$

8 a Diagonal zeros mean that there is no trucking from a city to the same city.

b $\mathbf{H} = \begin{bmatrix} 0 & 4 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 & 4 \\ 4 & 4 & 0 & 4 & 4 \\ 4 & 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 4 & 0 \end{bmatrix}$

$\mathbf{T} = \begin{bmatrix} 0 & 14.80 & 13.80 & 10.40 & 9.00 \\ 14.80 & 0 & 11.20 & 20.40 & 19.40 \\ 13.80 & 11.20 & 0 & 20.20 & 18.40 \\ 10.40 & 20.40 & 20.20 & 0 & 15.00 \\ 9.00 & 19.40 & 18.40 & 15.00 & 0 \end{bmatrix}$

c Delivery for medium parcel = $1.5\mathbf{C} + \mathbf{H}$

d Delivery for large parcel = $3\mathbf{C} + 2\mathbf{H}$

9 a $\mathbf{P} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

b $\mathbf{FP} = \begin{bmatrix} 12 \\ 9 \\ 4 \\ 7 \\ 3 \\ 9 \end{bmatrix}$, $\mathbf{MP} = \begin{bmatrix} 10 \\ 3 \\ 5 \\ 9 \\ 7 \\ 9 \end{bmatrix}$, $\mathbf{LP} = \begin{bmatrix} 9 \\ 6 \\ 6 \\ 10 \\ 4 \\ 9 \end{bmatrix}$

c Both give $\begin{bmatrix} 31 \\ 18 \\ 15 \\ 26 \\ 14 \\ 27 \end{bmatrix}$. Team 1 wins with 31 points, also it shows that matrix multiplication is distributive over addition.

8.05

1 a $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ b $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$ c $\begin{bmatrix} 0 & -1 \\ -\frac{1}{3} & -1 \end{bmatrix}$

2 a $\begin{bmatrix} \frac{1}{2} & -\frac{2}{3} \\ -\frac{1}{2} & 1 \end{bmatrix}$ b $\begin{bmatrix} \frac{1}{15} & \frac{4}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

c Does not exist.

3 a nilpotent b no c nilpotent

4 a idempotent b no c idempotent

5 a 34 b 58 c −12

d −42 e 21

6 a 53 b −175 c 165

d 112 e 43

7 a $\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ b $\begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$ c $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

d $\begin{bmatrix} 2 & 1 \\ -1\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ e Does not exist.

8 $\mathbf{B}^{-1} = \begin{bmatrix} 15 & 4 & -5 \\ -12 & -3 & 4 \\ -4 & -1 & 1 \end{bmatrix}$

9 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

10–12 Proofs

13 a $\mathbf{P} = \begin{bmatrix} 25 & 20 & 50 & 25 & 30 & 50 & 50 \\ 35 & 35 & 80 & 35 & 40 & 70 & 75 \\ 45 & 40 & 100 & 40 & 40 & 90 & 85 \end{bmatrix}$

b $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

c $\mathbf{PA} = \begin{bmatrix} 25 & 20 & 50 & 25 & 30 & 50 & 50 \\ 35 & 35 & 80 & 35 & 40 & 70 & 75 \\ 45 & 40 & 100 & 40 & 40 & 90 & 85 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 250 \\ 370 \\ 440 \end{bmatrix}$

d $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

e $\mathbf{BP} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times$

$\begin{bmatrix} 25 & 20 & 50 & 25 & 30 & 50 & 50 \\ 35 & 35 & 80 & 35 & 40 & 70 & 75 \\ 45 & 40 & 100 & 40 & 40 & 90 & 85 \end{bmatrix}$

$\mathbf{BP} = \begin{bmatrix} 105 & 95 & 230 & 100 & 110 & 210 & 210 \end{bmatrix}$

f $\mathbf{C} = \begin{bmatrix} 7.50 & 11.00 & 15.00 \end{bmatrix}$

g Income = $\mathbf{C} \times \mathbf{PA}$

$= \begin{bmatrix} 7.50 & 11.00 & 15.00 \end{bmatrix} \times \begin{bmatrix} 250 \\ 370 \\ 440 \end{bmatrix}$

= \$12 545

8.06

1 a $(\mathbf{M} + 4\mathbf{I})\mathbf{M}$ b $(7\mathbf{G} - \mathbf{I})\mathbf{X}$ c $\mathbf{A}(\mathbf{B} - 3\mathbf{I})$

d $\mathbf{H}(\mathbf{HG} + 4\mathbf{G})\mathbf{G}$ e no factors

2 a $\begin{bmatrix} 19 & 37 \\ -13 & -26 \end{bmatrix}$ b $\begin{bmatrix} \frac{1}{2} & -1\frac{1}{2} \\ 0 & -1 \end{bmatrix}$ c $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

d $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ e $\begin{bmatrix} -\frac{13}{25} & \frac{14}{25} \\ \frac{54}{25} & -\frac{37}{25} \end{bmatrix}$

3 a $\begin{bmatrix} 52 & 76 \\ -22 & -32 \end{bmatrix}$ b $\begin{bmatrix} 3 & 4 \\ -5 & -2 \end{bmatrix}$ c $\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$

d $\begin{bmatrix} 4\frac{4}{5} & -4\frac{3}{5} \\ 3\frac{2}{5} & -3\frac{4}{5} \end{bmatrix}$ e $\begin{bmatrix} 0 & -1\frac{1}{5} & 4\frac{1}{5} \\ 1 & -4 & 13 \end{bmatrix}$

4 a $\begin{bmatrix} -8 & 0 \\ 22 & 3 \end{bmatrix}$ b $\begin{bmatrix} -12 & 9 \\ 16 & -13 \end{bmatrix}$ c $\begin{bmatrix} 1.6 & -1.8 \\ -8.6 & -1.2 \end{bmatrix}$

d $\begin{bmatrix} -2 & -2\frac{1}{2} \\ -1 & -3 \end{bmatrix}$ e $\begin{bmatrix} -0.65 & -4.7 & 7.9 \\ 0.15 & 7.7 & -9.9 \\ -0.65 & 12.3 & -17.1 \end{bmatrix}$

CHAPTER 8 REVIEW

1 B

2 E

3 A

4 C

5 B

6 $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

7 $\begin{bmatrix} 4 & 5 & 4 & 5 \\ -8 & -7 & -8 & -7 \end{bmatrix}$

8 $\begin{bmatrix} -1 & 1 \\ -2\frac{1}{2} & 3 \end{bmatrix}$

9 a $\begin{vmatrix} 3 & 3 \\ 8 & 9 \end{vmatrix} = 3, \begin{bmatrix} 3 & 3 \\ 8 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -2\frac{2}{3} & 1 \end{bmatrix}$

b $\begin{vmatrix} -4 & 10 \\ -2 & 5 \end{vmatrix} = 0$, so no inverse exists.

10 $\mathbf{P}(\mathbf{P} + \mathbf{Q})\mathbf{Q}$

11 $\begin{bmatrix} \frac{1}{2} & -3 \\ -\frac{1}{4} & 7 \end{bmatrix}$

12 $\begin{bmatrix} 10 & -4\frac{1}{2} \\ -21\frac{1}{2} & 7\frac{1}{4} \end{bmatrix}$

13 Proof

14 a $\begin{bmatrix} 12.5 & 13.2 & 15.0 & 16.4 & 17.5 & 18.0 \end{bmatrix}$

b $\begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 5 \\ 2 & 3 & 0 \\ 3 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

c George = \$147.40, Frank = \$201.30 and Maria = \$168.40

9.01

1 A

2 D

3 D

4 D

5 C

6 a -1 b $-\sqrt{2}$ c $-\frac{2}{\sqrt{3}}$ d 2

e $\sqrt{3}$ f $-\sqrt{2}$ g $\sqrt{2}$ h $\sqrt{3}$

7 a $\sec(\theta)$ b $-\sec(\theta)$ c $-\operatorname{cosec}(\theta)$

d $\cot(\theta)$ e $-\operatorname{cosec}(\theta)$ f $-\cot(\theta)$

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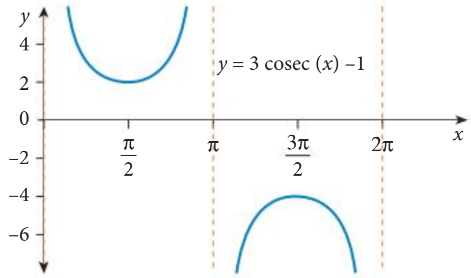
8



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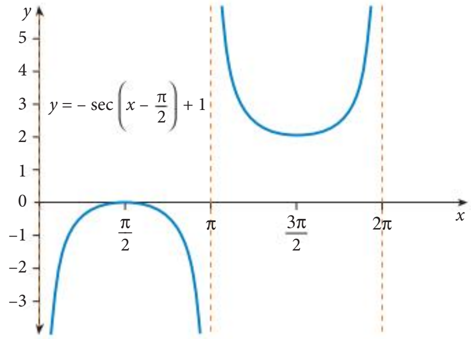

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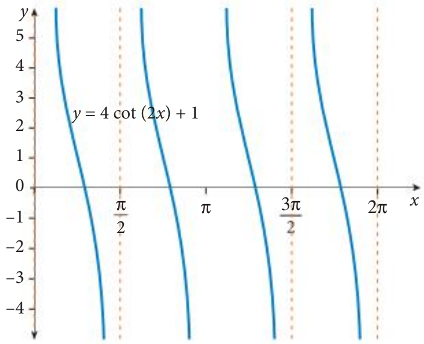

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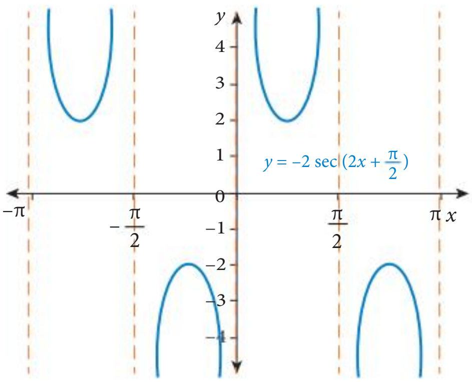


12 $\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$ and $\operatorname{cosec}\left(\frac{\pi}{4}\right) = \sqrt{2}$. These expressions are equal. Note that $\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$.

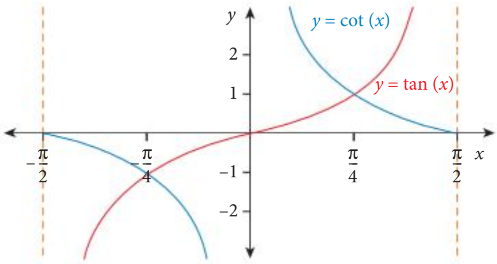
13 $\sec\left(\frac{\pi}{3}\right) = 2$ and $\operatorname{cosec}\left(\frac{\pi}{6}\right) = 2$. These expressions are equal. Note that $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$.

14 a $\frac{2\sqrt{3}}{3}$ b 1 c $-\frac{2\sqrt{3}}{3}$ d $\frac{2\sqrt{3}}{3}$

e 1 f not defined.

g $-\frac{\sqrt{3}}{3}$ h $\sqrt{2}$ i −1 j 2

15



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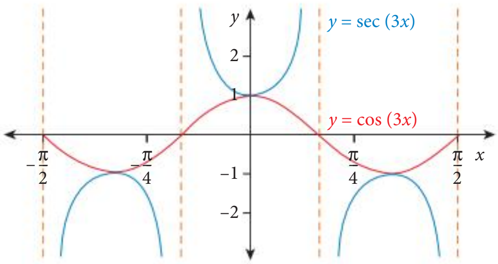

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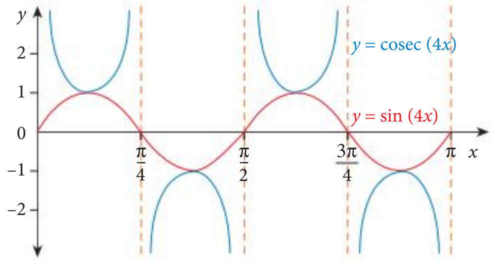

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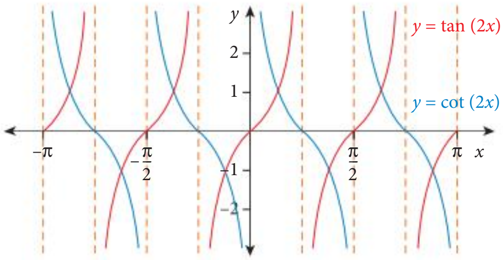

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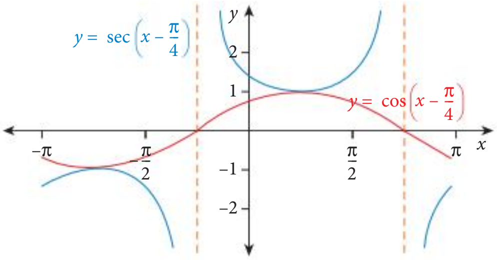

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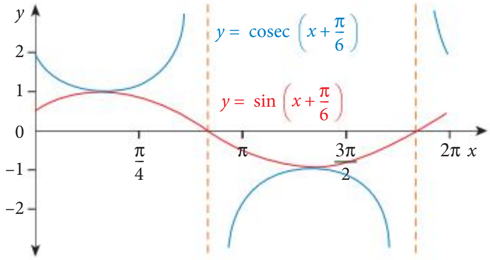

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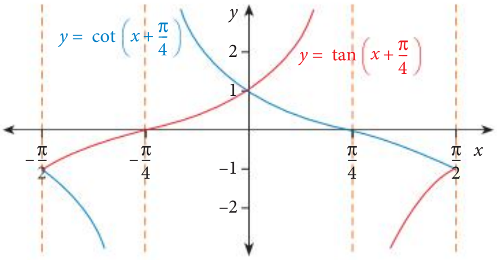

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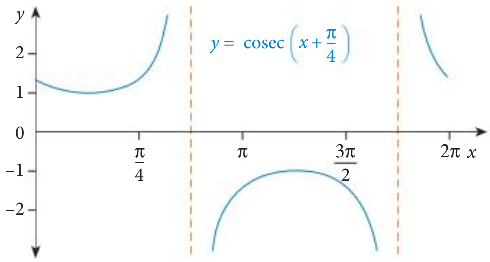

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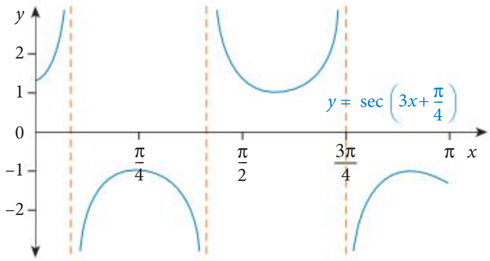

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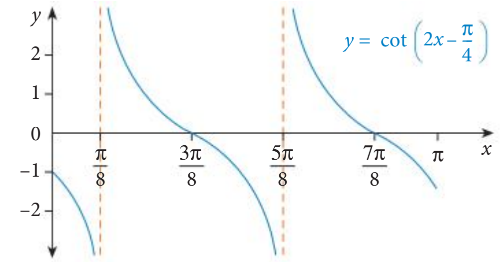

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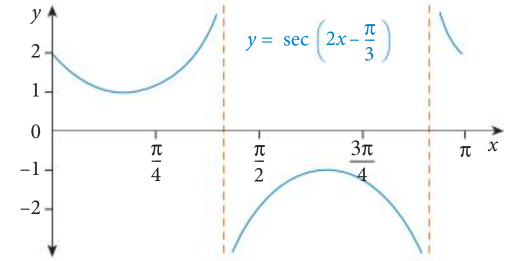

9.02

1–12 Proofs

9.03

1 **a** $\sin(2x)\cos(y) + \cos(2x)\sin(y)$
b $\cos(4x)\cos(3y) + \sin(4x)\sin(3y)$
c $\sin(3p)\cos(5q) - \cos(3p)\sin(5q)$
d $\dfrac{\tan(2e) - \tan(f)}{1 + \tan(2e)\tan(f)}$
e $\cos(3\alpha)\cos(2\beta) - \sin(3\alpha)\sin(2\beta)$
f $\sin(4m)\cos(n) + \cos(4m)\sin(n)$
g $\dfrac{\tan(w) + \tan(3x)}{1 - \tan(w)\tan(3x)}$
h $\sin(2D)\cos(5E) - \cos(2D)\sin(5E)$
i $\cos(P)\cos(4Q) + \sin(P)\sin(4Q)$

2 **a** $\sin(x)$ **b** $-\cos(x)$ **c** $\cot(x)$
d $\sec(x)$ **e** $\tan(x)$ **f** $-\operatorname{cosec}(x)$
g $-\sin(x)$ **h** $-\sin(x)$ **i** $\tan(x)$

3–12 Proofs

9.04

1–5 Proofs

6 **a** $\sin(4x) = 2\sin(2x)\cos(2x)$
b $\cos(6A) = 2\cos^2(3A) - 1$
c $\tan(4y) = \dfrac{2\tan(2y)}{1 - \tan^2(2y)}$
d $\cos(y) = \cos^2\left(\dfrac{y}{2}\right) - \sin^2\left(\dfrac{y}{2}\right)$
e $\tan(4x) = \dfrac{2\tan(2x)}{1 - \tan^2(2x)}$

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7 a $4\cos^3(x)\sin(x) - 4\cos(x)\sin^3(x)$

b $\cos^6(A) - \sin^6(A) - 15\cos^4(A)\sin^2(A) + 15\cos^2(A)\sin^4(A)$

c $\dfrac{4\tan(y) - 4\tan^3(y)}{\tan^4(y) - 6\tan^2(y) + 1}$

(Other answers are possible.)

8–14 Proofs

9.05

1 a $2-\sqrt{3}$ b $\dfrac{\sqrt{2-\sqrt{3}}}{2} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ c $-(2+\sqrt{3})$

d $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ e $-(2+\sqrt{3})$ f $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

g $-\dfrac{\sqrt{6}+\sqrt{2}}{4}$ h $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ i $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

2 $2+\sqrt{3}$

3 a $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ b $\dfrac{\sqrt{\sqrt{2}+2}}{2}$ c $\sqrt{3}+2$

d $\dfrac{\sqrt{2-\sqrt{2}}}{2}$ e $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ f $\sqrt{2}-1$

9.06

1 a 0.633 b 0.133 c 0.112 d 0.754

2 a $\sin(12x) - \sin(2x)$

b $4\cos(6x) - 4\cos(12x)$

c $3\cos(10x) + 3\cos(2x)$

d $2\sin(11x) + 2\sin(x)$

e $\frac{1}{2}\sin(16x) - \frac{1}{2}\sin(4x)$

f $1\frac{1}{2}\cos(24x) + 1\frac{1}{2}\cos(4x)$

g $2\frac{1}{2}\cos(x) - 2\frac{1}{2}\cos(5x)$

h $3\frac{1}{2}\cos(20x) + 3\frac{1}{2}\cos(2x)$

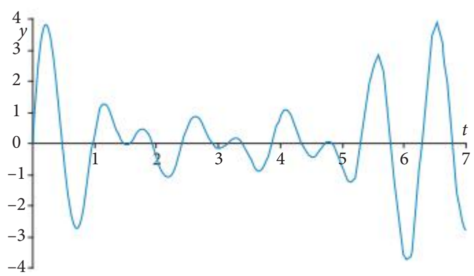
i $\frac{1}{2}\sin(9x) - \frac{1}{2}\sin(7x)$

3 a $\dfrac{\sqrt{3}+1}{4}$ b $\dfrac{\sqrt{3}+1}{4}$

c $\dfrac{\sqrt{3}-1}{4}$ d $\dfrac{\sqrt{3}-1}{2}$

4–9 Proofs

10 Write as $4\cos(x)\sin(6\frac{1}{2}x)\cos\left(\frac{1}{2}x\right)$. Graph shows the sizes clearly.



9.07

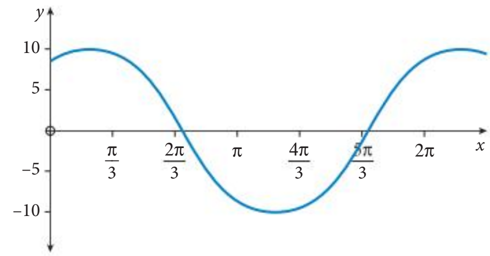
1 a i $10\sin(x+\theta)$, where $\theta = \arctan\left(\frac{4}{3}\right)$

ii $10\cos(x+\theta)$, where $\theta = -\arctan\left(\frac{3}{4}\right)$ or $2\pi - \arctan\left(\frac{3}{4}\right)$

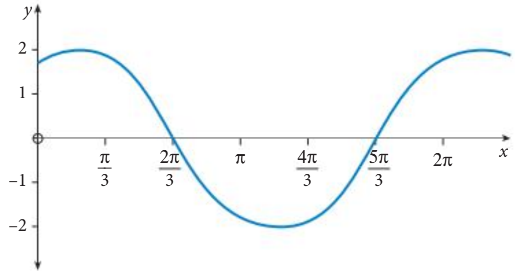
iii $10\sin(x-\theta)$, where $\theta = -\arctan\left(\frac{4}{3}\right)$ or $2\pi - \arctan\left(\frac{4}{3}\right)$

iv $10\cos(x-\theta)$, where $\theta = \arctan\left(\frac{3}{4}\right)$

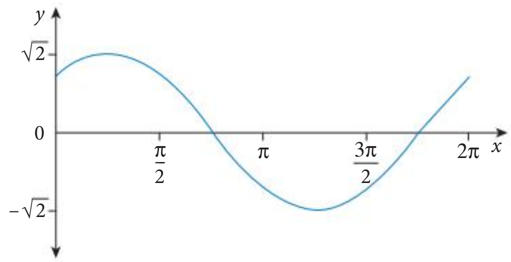
b The graphs are identical as:



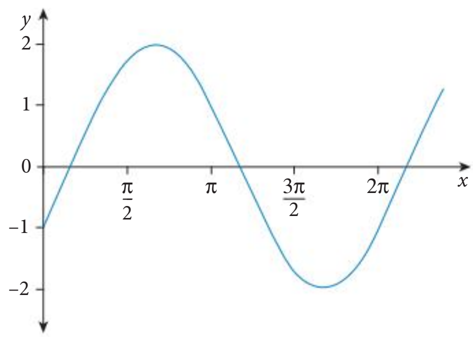
2 $2\sin\left(x+\frac{\pi}{3}\right)$



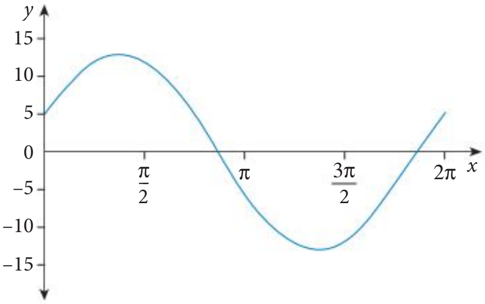
3 $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$



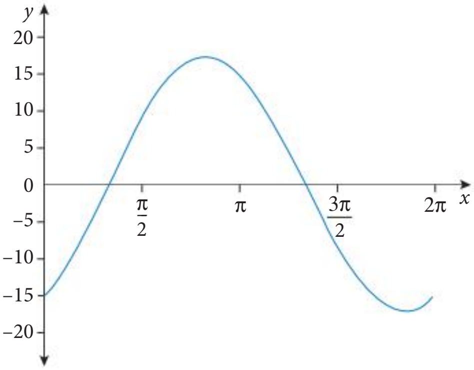
4 $2\cos\left(x+\frac{4\pi}{3}\right)$



5 $13 \sin(x + \theta)$, where $\theta = \arctan\left(\frac{5}{12}\right)$



6 $17 \sin(x - \theta)$, where $\theta = \arctan\left(\frac{15}{8}\right)$



7 a $2 \sin(5x) \cos(2x)$

b $2 \cos(5x) \sin(3x)$

c $2 \cos(7x) \cos(2x)$

d $-2 \sin(9x) \sin(2x)$

e $2 \sin\left(6\frac{1}{2}x\right) \cos\left(2\frac{1}{2}x\right)$

f $-2 \cos\left(3\frac{1}{2}x\right) \sin\left(1\frac{1}{2}x\right)$

g $2 \cos(3x) \cos(2x)$

h $2 \sin(7x) \sin(4x)$

i $2 \sin\left(1\frac{1}{2}x\right) \cos\left(\frac{1}{2}x\right)$

8 0

9 $\frac{\sqrt{6}}{2}$

10 $\sqrt{2}$

11–14 Proofs

15 a $2 \cos\left(x + \frac{1}{2}h\right) \sin\left(\frac{1}{2}h\right)$

b $\cos\left(x + \frac{1}{2}h\right) \frac{\sin\left(\frac{1}{2}h\right)}{\frac{1}{2}h}$

c $\cos(x)$, with $z = \frac{1}{2}h$

9.08

1–4 Proofs

5 $-\frac{4}{3}\sin^3(x) - 2\sin^2(x) + \sin(x) + 1$

6 $4\cos^4(x) - 6\cos^2(x) + 2$

7–10 Proofs

CHAPTER 9 REVIEW

1 C

2 A

3 E

4 C

5 A

6 E

7 E

8 B

9 a $\sec(\theta)$ b $\operatorname{cosec}(\theta)$

c $-\cot(\theta)$ d $-\sec(\theta)$

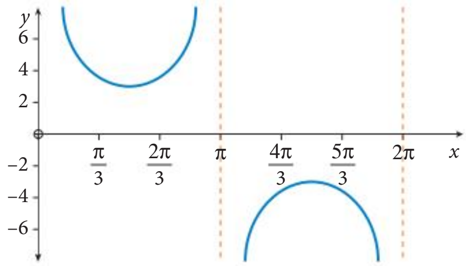
e $\cot(\theta)$ f $-\operatorname{cosec}(\theta)$

10 a $\operatorname{cosec}(\theta)$ b $-\sec(\theta)$

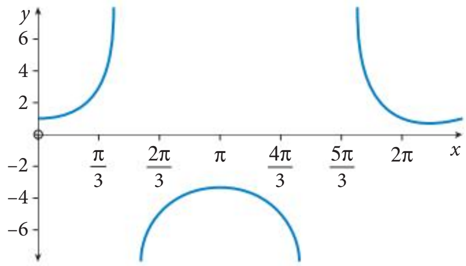
c $\tan(\theta)$ d $\operatorname{cosec}(\theta)$

e $\tan(\theta)$ f $\sec(\theta)$

11 $y = 3\operatorname{cosec}(x)$ for $0 \le x \le 2\pi$



12 $y = 2\sec(x) - 1$ for $0 \le x \le 2\pi$



13 Proof

14 $\frac{\sqrt{6}+\sqrt{2}}{4}$

15 $2-\sqrt{3}$

16 $\frac{\sqrt{6}+\sqrt{2}}{4}$

17 $4\sin(x)\cos^3(x) - 4\sin^3(x)\cos(x)$, or equivalent

18–21 Proofs

22 $-6\cos^4(x) + 6\cos^2(x) + 2\cos^3(x) - \cos(x)$

MIXED REVISION 3

Multiple choice

1 A

2 C

3 D

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4 D
5 C
6 C
7 B
8 E
9 A

Short answer

1 a LHS is even, but the RHS is odd.
  b (5, 5) or (10, 2)

2 $\begin{vmatrix} 12 & 28 \\ 15 & 35 \end{vmatrix} = 0$, so $\begin{bmatrix} 12 & 28 \\ 15 & 35 \end{bmatrix}$ has no inverse.

3 $\sin(x)\sin(2x)$
$= \sin(x) \times 2\sin(x)\cos(x)$
$= 2\sin^2(x)\cos(x)$
$= 2[1 - \cos^2(x)]\cos(x)$
$= 2\cos(x) - 2\cos^3(x)$

4 $\frac{1}{2\times10} + \frac{1}{2\times10^4} + \frac{1}{2\times10^8} + \dots + \frac{1}{2\times10^{\frac{1}{2}n^2+\frac{3}{2}n-1}} + \dots,$
$n \in \boldsymbol{N}$

5 $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

6 a $\cos(x + 2x)$
$= \cos(x)\cos(2x) - \sin(x)\sin(2x)$
$= \cos(x)[2\cos^2(x) - 1] - \sin(x) \times 2\sin(x)\cos(x)$
$= 2\cos^3(x) - \cos(x) - 2\sin^2(x)\cos(x)$
(Alternate answers are possible)
  b $2\cos^3(x) - \cos(x) - 2\sin^2(x)\cos(x)$
$= \cos(x)[2\cos^2(x) - 1 - 2\sin^2(x)]$
$= \cos(x)[2\{1 - 2\sin^2(x)\} - 1 - 2\sin^2(x)]$
$= \cos(x)[2 - 2\sin^2(x) - 1 - 2\sin^2(x)]$
$= \cos(x)[1 - 4\sin^2(x)]$

Application

1 Proof
2 a $2n + 7 = 11, n = 2$
  b $2(2k) + 7 = 4k + 7, 2(2k + 1) + 7 = 4k + 9$
  c Proof
  d $n = 1, n = 7, n = 49$
3 $\begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$
4 $\boldsymbol{X} = \begin{bmatrix} -8 & 0 \\ 22 & 3 \end{bmatrix}$
5 Proof
6 Proof

10.01

1 a $5i$ b $6i$ c $2i\sqrt{5}$ d $\frac{2i\sqrt{2}}{3}$
2 a $\pm i$ b $\pm 3i$ c $\pm 7i$ d $\pm\frac{i}{2}$
3 a -1 b $-i$ c 1 d i
4 a 1 b -1 c $-i$ d -45
5 a 2 b $-6 + 15i$
6 $1 + i$
7 $(x - 4i)(x + 4i)$
8 a 0 b $1 + i$
9 a $-i$ b i c 1
  d $-i$ e -1
10 a $(x - 2i)(x + 2i)$ b $(x - 9i)(x + 9i)$

10.02

1 a complex b purely imaginary
  c real d real
  e purely imaginary f real
  g complex h real
2 a $\text{Re}(z) = -2$, $\text{Im}(z) = -4$
  b $\text{Re}(z) = \frac{7}{4}$, $\text{Im}(z) = \frac{3}{4}$
  c $\text{Re}(z) = -6$, $\text{Im}(z) = \sqrt{2}$
  d $\text{Re}(z) = \frac{x^2}{x^2+y^2}$, $\text{Im}(z) = \frac{-y^2}{x^2+y^2}$
3 a $w = 5 - 4i$ b $w = \sqrt{2} + i\sqrt{7}$
  c $w = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ d $w = -2\sqrt{2} - 5i\sqrt{2}$
  e $w = \frac{y + xi}{x^2 + y^2}$
4 a $x = \pm 7i$ b $z = \pm i$ c $w = \pm 5i$
  d $x = -2 \pm i$ e $z = \frac{1 \pm i\sqrt{7}}{2}$ f $w = 1 \pm i$
  g $x = 1 \pm i\sqrt{3}$ h $z = 3 \pm i$ i $w = 2 \pm 2i$
5 a $\text{Re}(z) = x + 3$, $\text{Im}(z) = y - 2$
  b $\text{Re}(z) = x - 3y$, $\text{Im}(z) = 2x + y$
  c $\text{Re}(z) = -x + 4$, $\text{Im}(z) = 5 - y$
  d $\text{Re}(z) = 4x - y$, $\text{Im}(z) = -(3x - 2y)$
  e $\text{Re}(z) = 1$, $\text{Im}(z) = \frac{2xy}{x^2 + y^2}$
6 a $z = \frac{-3 \pm i\sqrt{23}}{4}$ b $z = \frac{\left(-3 \pm \sqrt{3}\right)i}{3}$
  c $z = a \pm bi$

10.03

1 a $\bar{z} = -1 + i$ b $\bar{z} = \frac{3 - i}{2}$
  c $\bar{z} = -i\sqrt{3} - 1$ d $x - y - (2x - 3y)i$
  e $\frac{x^2 - iy^2}{x^2 + y^2}$
2 a 5 b $7\frac{1}{4}$ c 14
3 $\left(x + 1 - i\sqrt{2}\right)\left(x + 1 + i\sqrt{2}\right)$
4 a $(x + 2 - i)(x + 2 + i)$ b $(x - 3 + 2i)(x - 3 - 2i)$
  c $(x - 1 - i)(x - 1 + i)$
5 a $x = \frac{-1 \pm i\sqrt{15}}{2}$ b $x = \frac{3 \pm i\sqrt{3}}{2}$
  c $x = -3 \pm i$ d $x = \frac{-1 \pm i\sqrt{31}}{2}$
6 a $x^2 + 2x + 2 = 0$ b $x^2 - 2x + 10 = 0$
  c $x^2 - 2\sqrt{3}x + 4 = 0$
7 a $x = \frac{-i \pm i\sqrt{5}}{2}$ b $x = i$ c $x = i, -2i$

8 a ... *if complex, must be complex conjugate pairs*
  b ...*are not complex conjugate pairs*

9 a Proof b Proof

10.04

1 a $a = 7, b = -4$ b $a = 9, b = 6$
  c $a = 1, b = -1$

2 a $3 - 2i$ b $3 + i$
  c $21 + 25i$ d $2 - 24i$

3 a $13 + 10i$ b $-3 + 5i$
  c $-5 + 12i$ d $16 + 11i$
  e $16 - 11i$ f $48 + 14i$
  g $21 + 20i$ h $4 + 4i$

4 a $\frac{1}{2} - \frac{1}{2}i$ b $\frac{3}{5} - \frac{4}{5}i$ c $-\frac{6}{13} + \frac{17}{13}i$
  d $\frac{1}{17} - \frac{4}{17}i$ e $-\frac{1}{5} - \frac{2}{5}i$

5 a $-\frac{1}{5} - \frac{2}{5}i$ b $-\frac{8}{13} + \frac{i}{13}$ c $-\frac{9}{25} + \frac{12}{25}i$
  d $\frac{3}{13} - \frac{2}{13}i$ e -1

6 a $18 - 4i$ b $25 + 28i$
  c $\frac{7}{5} + \frac{4}{5}i$ d $-\frac{9}{25} + \frac{13}{25}i$

7 $X = \frac{8}{25}, Y = -\frac{31}{25}$

8 a $\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$ b $\frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}i$
  c $\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}i$
  d $-\frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}i$
  e $\frac{x^2+y^2-1}{x^2+(y+1)^2} - \frac{2x}{x^2+(y+1)^2}i$

9 Substitute $z = 1 + i$ into $z^2 - 2z + 2 = 0$.

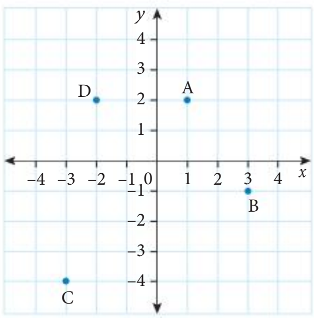
10 Substitute each solution in the LHS.

11 Substitute each solution in the LHS.

12 a, b Expand the LHS.

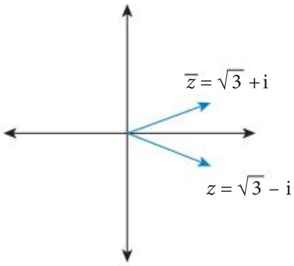
10.05

1 i a $A(1, 2)$ b $B(3, -1)$
  c $C(-3, -4)$ d $D(-2, 2)$

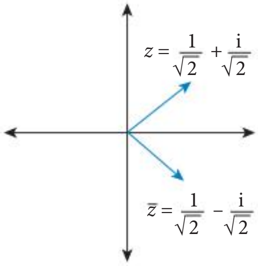


2 $z = 3 + i, u = -3 + 3i, v = -4 - 4i, w = 2 - 4i$

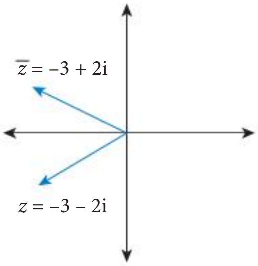
3 a



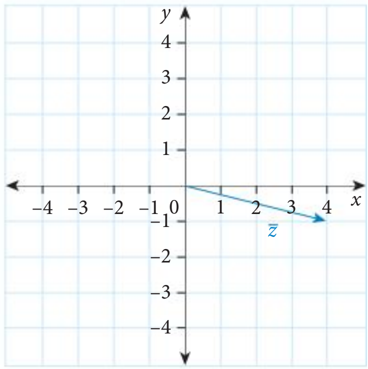
b

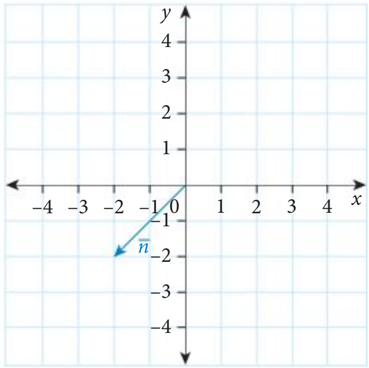


c



4 a $z = 4 + i, n = -2 + 2i, k = -5 - 2i, m = 2 - 3i$
  b $\bar{z} = 4 - i, \bar{n} = -2 - 2i, \bar{k} = -5 + 2i, \bar{m} = 2 + 3i$
  c





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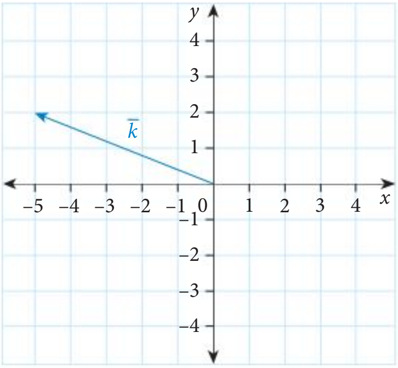
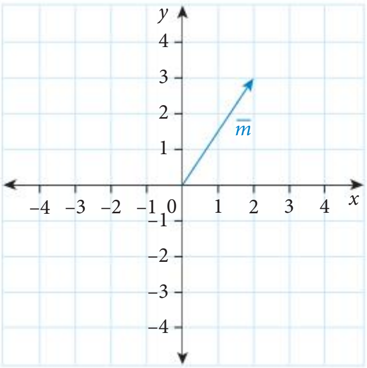
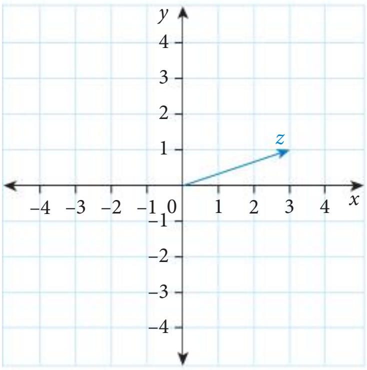
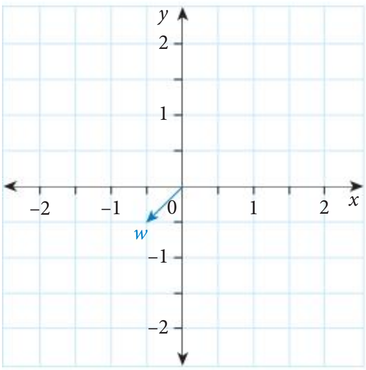
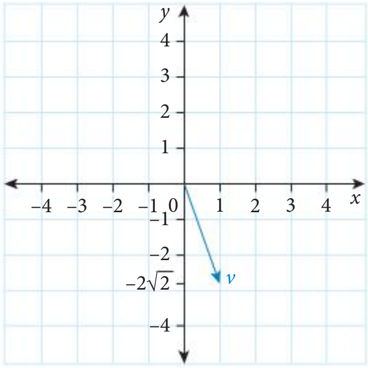




5 **a** $z = 3 + i$ b $w = -\frac{1}{2} - \frac{i}{2}$ c $v = 1 - 2i\sqrt{2}$



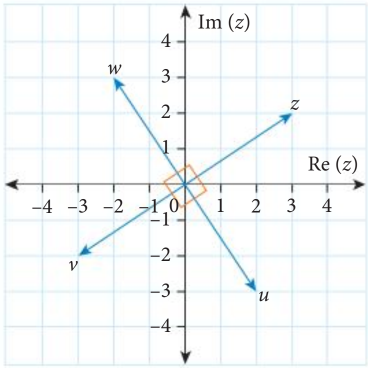




6 **a** **i** $w = -2 + 3i$ **ii** $v = -3 - 2i$

iii $u = 2 - 3i$

b



c Multiplying a vector by i rotates the vector 90° anticlockwise.

7 $z + \bar{z} = 2\text{Re}(z) \therefore \text{Re}(z) = 0$, i.e. z is purely imaginary.

8 $z + \bar{z} = 2\text{Re}(z)$ so $z + \bar{z}$ is real if $z + \bar{z} \neq 0$.

9 z must lie on the real axis.

10.06

1 **a** $\sqrt{34}$ **b** $\sqrt{5}$ **c** 2

d $\sqrt{3}$ **e** 1 **f** 1

2 **a** $\sqrt{13}$ **b** $\frac{1}{\sqrt{13}}$ **c** $\frac{1}{\sqrt{13}}$ **d** 13

3 **a** 2 **b** 3 **c** $\frac{1}{10}$ **d** 1

4 **a** $2\sqrt{17}$ **b** $\sqrt{221}$ **c** $2\sqrt{15}$

d $\frac{1}{2}$ **e** 1 **f** 3

5 **a** Proof **b** Proof

c Proof **d** Proof

6 **a** $\sqrt{x^2 + y^2}$ **b** $\frac{\sqrt{x^2 + y^2}}{x^2 + y^2}$ **c** 1

7 $|z| = |\bar{z}|$ because the modulus is the length of the vector. The vectors z and $\bar{z}$ are reflections so have the same length.

8 Show LHS = RHS.

9 Show LHS = RHS.

10 **a** Proof **b** Proof

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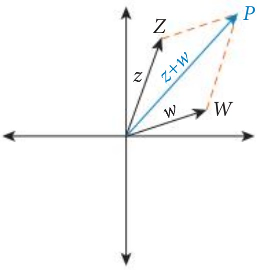
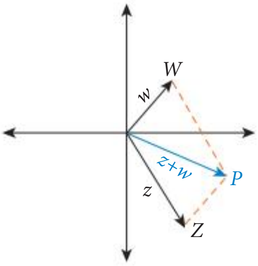
10.07

1 a



b



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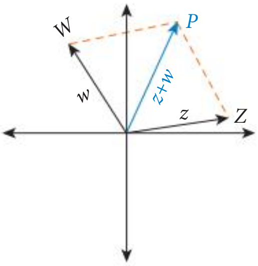
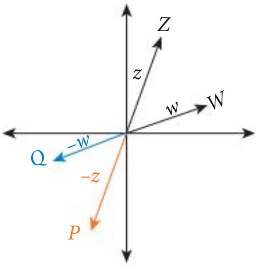
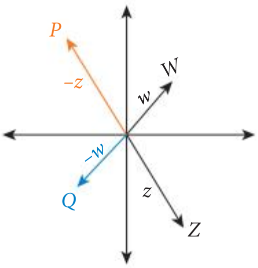


2 a



b



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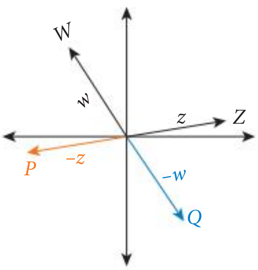
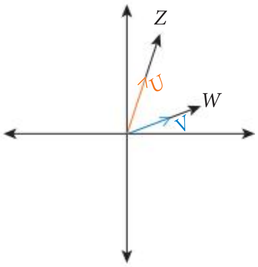
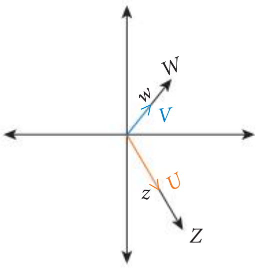


3 a



b



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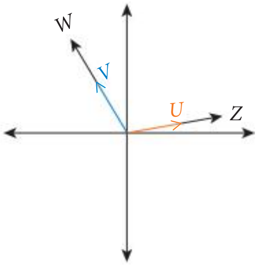
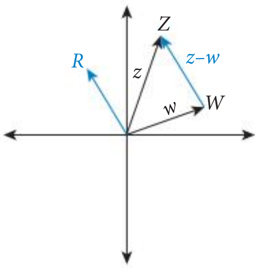


4 a



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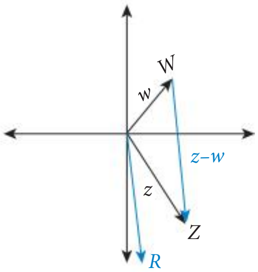

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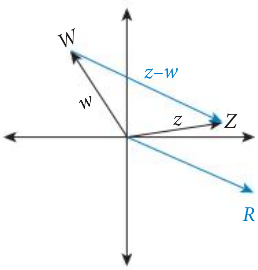

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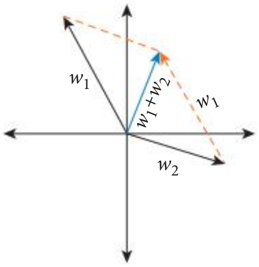
5 a $\mathbf{u}+\mathbf{v}$ b $\mathbf{v}-\mathbf{u}$ c $-\mathbf{u}-\mathbf{v}$

d $\mathbf{u}-\mathbf{v}$ e $\frac{1}{2}(\mathbf{u}+\mathbf{v})$

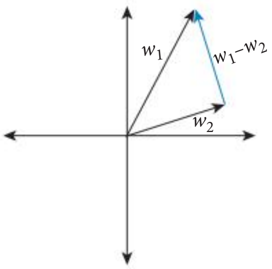
6 $1+3i$

7 a $\mathbf{z}_3=\mathbf{z}_1-\mathbf{z}_2$ b $\mathbf{z}_1=\mathbf{z}_2+\mathbf{z}_3$ c $\mathbf{z}_2=\mathbf{z}_1-\mathbf{z}_3$

8 a



b



The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

9 $\mathbf{z}_2=\mathbf{z}_1+\mathbf{w}$, etc.

10.08

1 Proofs

2 a $-z=-x-yi$ b $z^{-1}=\frac{x-yi}{x^2+y^2}$

c $0=0+0i$ d $1=1+0i$

3–7 Proofs

8 Yes, since $|z|^2=|z^2|$ if $z\neq 0$.

10.09

1 a $z=-1\pm i\sqrt{3}$ b $z=1\pm\sqrt{5}$

c $z=-2\pm 2i$ d $z=\frac{-1\pm i\sqrt{23}}{3}$

2 $x=-1\pm 3i$

3 $x=\frac{-7\pm i\sqrt{5}}{3}$

4 substitute, $\overline{\beta}=-5+i$

5 $m=-4$, other root $2+i$

6 a i $(x-1-2i)(x-1+2i)=0$

ii $x^2-2x+5=0$

b i $\left(x-\sqrt{3}-i\right)\left(x-\sqrt{3}+i\right)=0$

ii $x^2-2\sqrt{3}x+4=0$

c i $\left(x+4-i\sqrt{2}\right)\left(x+4+i\sqrt{2}\right)=0$

ii $x^2+8x+18=0$

d i $\left(x-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)\left(x-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=0$

ii $x^2-\sqrt{2}x+1=0$

e i $(x+5-6i)(x+5+6i)=0$

ii $x^2+10x+61=0$

f i $\left(x-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\left(x-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=0$

ii $x^2-\sqrt{3}x+1=0$

g i $\left(x+\frac{3}{5}-\frac{2i}{5}\right)\left(x+\frac{3}{5}+\frac{2i}{5}\right)=0$

ii $x^2+\frac{6}{5}x+\frac{13}{25}=0$

h i $\left(x+\frac{1}{\sqrt{3}}-\frac{i\sqrt{2}}{\sqrt{3}}\right)\left(x+\frac{1}{\sqrt{3}}+\frac{i\sqrt{2}}{\sqrt{3}}\right)=0$

ii $x^2+\frac{2}{\sqrt{3}}x+1=0$

7 a $k=1, n=-4, p=5$ b $k=1, n=-2, p=4$

c $k=2, n=1, p=2$ d $k=13, n=-10, p=13$

8 a $(z-1-i)(z-1+i)$

b $\left(z+1-i\sqrt{5}\right)\left(z+1+i\sqrt{5}\right)$

c $(z+2-i)(z+2+i)$

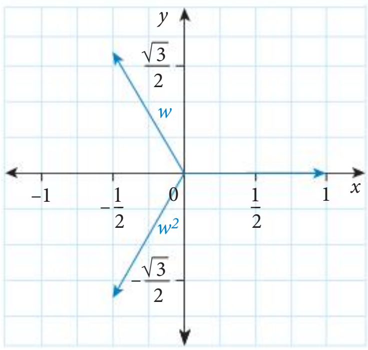
d $\left(z+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\left(z+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$

9 a $z=1,-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

10 a If $\omega=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ then $\omega^2=-\frac{1}{2}-\frac{i\sqrt{3}}{2}=\overline{\omega}$

b Proof. $|\omega|=\sqrt{\frac{1}{4}+\frac{3}{4}}$

11



$1,\omega,\omega^2$ are symmetrically placed around O.

CHAPTER 10 REVIEW

1 C

2 A

3 B

4 D

5 C

6 a $9i$ b $3i\sqrt{2}$ c $4i\sqrt{3}$ d $6i\sqrt{6}$

7 a $z=\pm 6i$ b $z=\pm 3i$ c $z=\pm\frac{i}{2}$

8 a $-i$ b 1 c i

d -1 e $-i$

9 a $\text{Re}(z) = 5$, $\text{Im}(z) = -\sqrt{5}$

b $\text{Re}(z) = \frac{1}{2}$, $\text{Im}(z) = \frac{\sqrt{3}}{2}$

c $\text{Re}(z) = x - 2y$, $\text{Im}(z) = 3y + x$

10 a $x = -2 \pm i\sqrt{2}$ b $x = \frac{5 \pm i\sqrt{11}}{3}$ c $x = \frac{-1 \pm i\sqrt{7}}{2}$

11 a $-4 - 7i$ b $2 - 5i$ c $2x + y + yi - 3xi$

12 a $(x + 2 - i)(x + 2 + i)$

b $(z - 5 - 2i)(z - 5 + 2i)$

c $\left(w + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(w + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

13 a $x^2 - 6x + 10 = 0$

b $x^2 - 2x + 7 = 0$

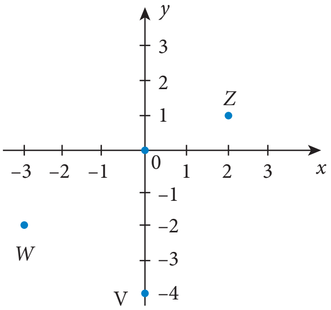
c $4x^2 - 8x + 7 = 0$

14 $x = 2.4$, $y = -0.8$

15 a $-2 + 2i$ b $31 + 5i$ c $5 - 12i$

16 a $\frac{1 + 3i}{5}$ b $\frac{\sqrt{2} - i}{3}$ c $-2i$

17



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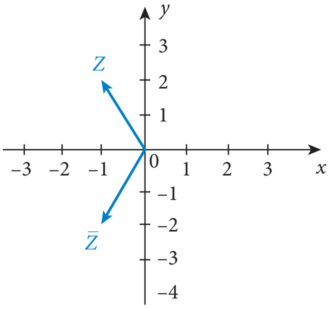

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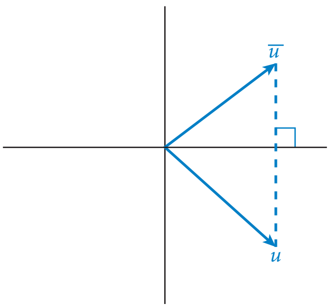
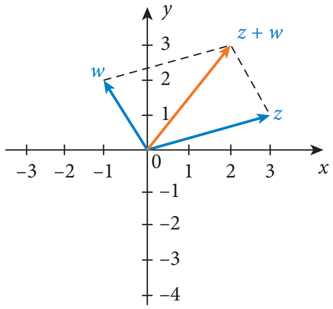


20 a $\sqrt{13}$ b $\frac{1}{2}$ c $\frac{1}{\sqrt{7}}$

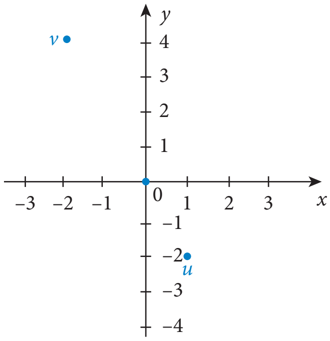
21 $\sqrt{5x^2 + 2xy + 2y^2}$

22

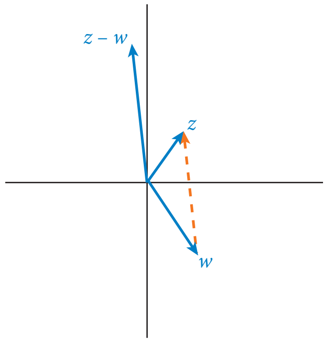


23 a $U(1, -2)$, $V(-2, 4)$

b



24



25 a $\frac{1 + i\sqrt{2}}{3}$ b $-\frac{1}{2}i$ c $3 - 4i$

26 a Proof b Proof

27 a $x = \pm 2i$, imaginary b $z = \frac{1}{3}, 3$ real

c $w = 1 \pm i$ complex

28 $4 + i$

29 $\left(z + 3 - i\sqrt{2}\right)\left(z + 3 + i\sqrt{2}\right) = 0$

30 a $44 + 23i$ b 10

c $\frac{1 + 2i\sqrt{2}}{3}$ d $3 + 4i$

31 a $\sqrt{3} - 1 - i\left(1 + \sqrt{3}\right)$ b $\sqrt{3} - 1$

c $-1 - \sqrt{3}$ d $2\sqrt{2}$

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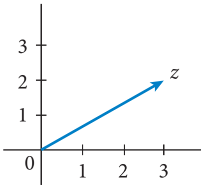
32 **a**



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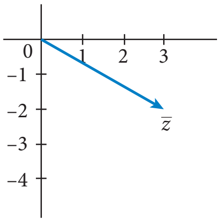
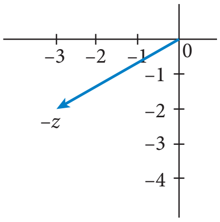
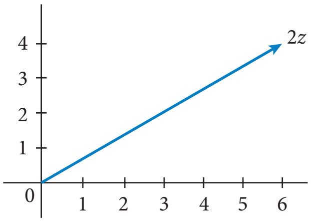


c



d



33 **a** $z = \pm 5i$ **b** $z = 5 \pm 3i$

c $w = 0$ or $w = -3$ **d** $u = -2 \pm \dfrac{\sqrt{2}}{2}i$

e $z = \dfrac{-7 \pm i\sqrt{11}}{6}$ **f** $w = \pm 3, \pm 2i$

34 $a(z^2 + 4z + 8)$

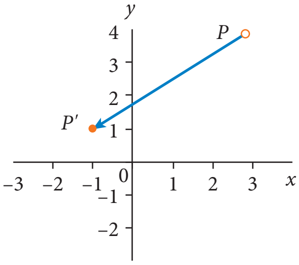
35 $x = \dfrac{3 \pm i\sqrt{11}}{2}$

36 $z = 2, -1 \pm i\sqrt{3}$

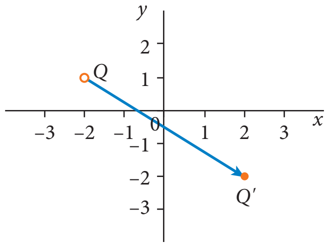
37 $X(2 + 4i), M\left(\dfrac{3}{2} + \dfrac{5}{2}i\right)$

11.01

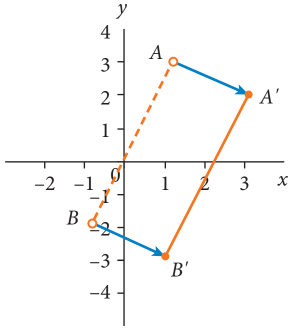
1 **a**



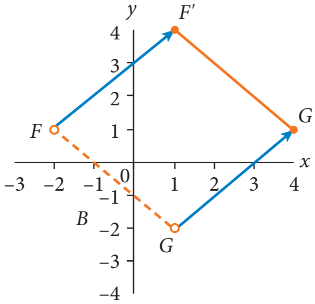
b



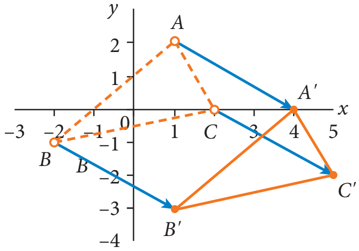
c



d



e



2 **a** $A'(-5, -2)$ **b** $B'(-2, 6)$ **c** $C'(-4, 3)$

d $D'(-3, 0)$ **e** $E'(-7, 9)$

3 **a** $A'(2, 2)$ **b** $B'(-6, -6)$ **c** $C'(3, -5)$

d $D'(5, -5)$ **e** $E'(-5, 6)$

4 **a** $A'(1, 1)$ **b** $B'(8, -9)$ **c** $C'(-3, -4)$

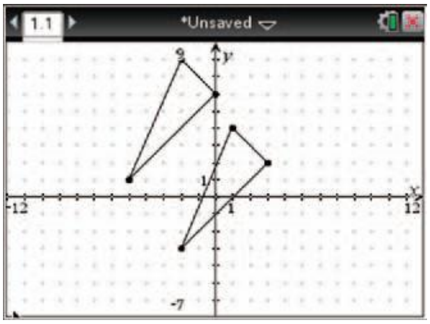
d $D'(-2, -11)$ **e** $E'(3, -10)$

5 $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$

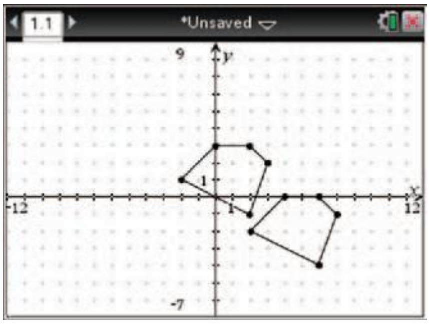
6 $\begin{bmatrix} 7 \\ -4 \end{bmatrix}$, 7 right and 4 down

7 $\begin{bmatrix} 4 \\ -6 \end{bmatrix}$, 4 across and 6 down

8



9



10 a $P'(2, 4)$ b $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ c $\begin{bmatrix} a+c \\ b+d \end{bmatrix}$

11 a $\mathbf{I} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ b $\mathbf{T}_i = \begin{bmatrix} -a \\ -b \end{bmatrix}$

12 For $\mathbf{S} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{T} = \begin{bmatrix} c \\ d \end{bmatrix}$, $\mathrm{S} \circ \mathrm{T} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} c+a \\ d+b \end{bmatrix}$

$= \mathrm{T} \circ \mathrm{S}$ QED

11.02

1 a Proof b $A'(6, -5)$, $B'(2, -5)$, $C'(2, 3)$
c A right-angled triangle

2 a Proof
b $A'(17, -13)$, $B'(6, 0)$, $C'(-10, -11)$, $D'(1, -2)$
c A parallelogram

3 a Proof
b $A'(15, -20)$, $B'(10, -10)$, $C'(30, 0)$, $D'(35, -10)$
c A parallelogram

4 a $\begin{bmatrix} 2 & -4 \\ 4 & 3 \end{bmatrix}$ b $A'(-14, 5)$, $B'(-16, 23)$, $C'(18, 3)$

5 a $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ b $A'(3, -3)$, $B'(8, -8)$, $C'(1, -1)$

6 a $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ b $A'(-7, 1)$, $B'(8, 10)$, $C'(-1, 1)$

7 a $\begin{bmatrix} -2 & 3 \\ 4 & 3 \end{bmatrix}$
b $A'(-5, -17)$, $B'(-1, -7)$, $C'(12, -6)$, $D'(8, -16)$

8 a $\begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$
b $A'(-5, 23)$, $B'(-16, 6)$, $C'(33, -1)$, $D'(23, 45)$

9 a $A'(2, 5)$, $B'(4, -5)$, $C'(5, -19)$, $D'(3, -9)$ and $A'(3, 8)$, $B'(11, -4)$, $C'(18, -22)$, $D'(10, -10)$
b Both parallelograms.

10 a $A'(2a + 3b, 2c + 3d)$, $B'(3a + 6b, 3c + 6d)$, $C'(5a + 4b, 5c + 4d)$, $D'(4a + b, 4c + d)$
b Proof

11.03

1 a $(x, y) \to (4x, 3y)$ b $(x, y) \to (1.2x, 1.2y)$
c $(x, y) \to (3x, 0.5y)$ d $(x, y) \to (0.25x, 3y)$

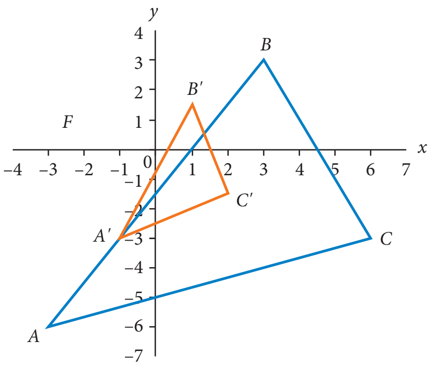
2 a $\begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}$
b $A'(4.5, 8)$, $B'(6, 16)$, $C'(10.5, 12)$
c A triangle

3 a $\begin{bmatrix} 0.4 & 0 \\ 0 & 0.8 \end{bmatrix}$
b $A'(-4, -6.4)$, $B'(-2, 3.2)$, $C'(7.6, -4.8)$, $D'(5.6, -14.4)$
c A parallelogram

4 a $\begin{bmatrix} 0.7 & 0 \\ 0 & 0.9 \end{bmatrix}$
b $A'(-7, -7.2)$, $B'(-2.8, 1.8)$, $C'(4.2, -3.6)$, $D'(0, -12.6)$
c A parallelogram

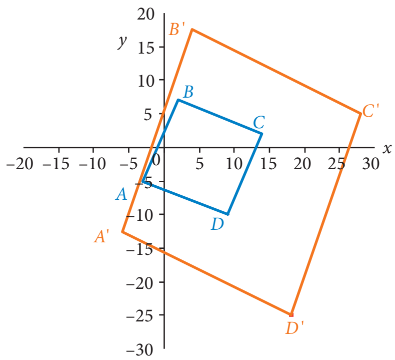
5 a $\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$
b $A'(-6, 2)$, $B'(0, 8)$, $C'(9, 8)$, $D'(9, -2)$, $E'(3, -4)$
c A pentagon

6 a $A'(-1, -3)$, $B'(1, 1.5)$, $C'(2, -1.5)$
b



c They are triangles with different angles.

7 a $A'(-6, -12.5)$, $B'(4, 17.5)$, $C'(28, 5)$, $D'(18, -25)$
b



c The square has changed to a parallelogram.

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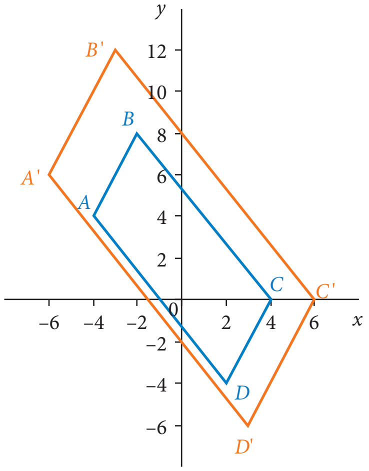
8 a $A'(-6, 6), B'(-3, 12), C'(6, 0), D'(3, -6)$

b



c They are similar.

11.04

1 $P'(3-3\sqrt{3}, -(3+3\sqrt{3}))$

2 $P'(0, -4)$

3 $A'(-0.83, -2.70), B'(-1.60, -7.65), C'(-5.77, -1.93), D'(0.83, 2.7)$

4 $A'(0.64, -7.59), B'(-3.25, -13.05), C'(-5.71, -4.17), D'(-1.46, 4.88)$

5 a $\begin{bmatrix} 0.80 & 0.60 \\ -0.60 & 0.80 \end{bmatrix}$

b $A'(6.2, 1.59), B'(9.61, 2.78), C'(4.99, -5.01)$

6 a $\begin{bmatrix} -0.73 & -0.68 \\ 0.68 & -0.73 \end{bmatrix}$

b $A'(1.56, -4.19), B'(-2.63, -5.75), C'(-8.38, -3.12), D'(-7.75, -0.98), E'(-5.55, -3.02)$

7 a $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

b $A'(2\sqrt{2}, 0), B'(-2\sqrt{2}, -\sqrt{2}), C'(-6\sqrt{2}, 3\sqrt{2}), D'(-2\sqrt{2}, 4\sqrt{2})$

8 a $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ b $-60°$ or $300°$

c $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

9 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ b $-\frac{7\pi}{4}$ or $\frac{\pi}{4}$

c $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

10 a $-\frac{\pi}{6}, \frac{11\pi}{6}, -30°$ or $330°$ b $\mathbf{R}_{-\theta} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

11.05

1 a $P'(-11, 2)$ b $P'(7, 3)$

c $P'(9, -3)$ d $P'(15, 20)$

2 $A'(2, -1), B'(3, 1), C'(-0.4, -2.8)$

3 $A'(1, -7), B'(-4, 3), C'(-4.4, -4.2)$

4 $A'(12.4, 6.8), B'(3.4, -6.2), C'(-9, -13), D'(-11, -2)$

5 $A'(-6.2, 1.6), B'(-11, 3), C'(-12.4, -1.8), D'(-6.2, -3.4), E'(-4, -3)$

6 $A'(6.33, 0.96), B'(10.79, 0.70), C'(7.56, -0.90)$

7 $A'(6.36, 0.71), B'(7.07, 14.14), C'(14.14, 14.14)$

8 $A'(-4.23, -3.33), B'(-6.56, 2.63), C'(-16.49, 1.44), D'(-15.89, -3.53)$

9 $A'(-1.12, -4.97), B'(1.82, -10.66), C'(11.57, -8.43), D'(10.46, -3.56), E'(6.59, -7.31)$

10 Proof

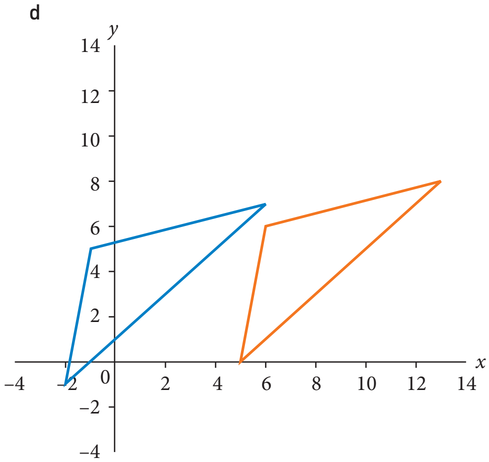
11.06

1 a $\mathbf{T}_1 \circ \mathbf{T}_2: (x, y) \rightarrow (1 + x, -3 + y)$

b $\mathbf{T}_3 \circ \mathbf{T}_2 = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$

c $A'(5, 0), B'(6, 6), C'(13, 8)$

d

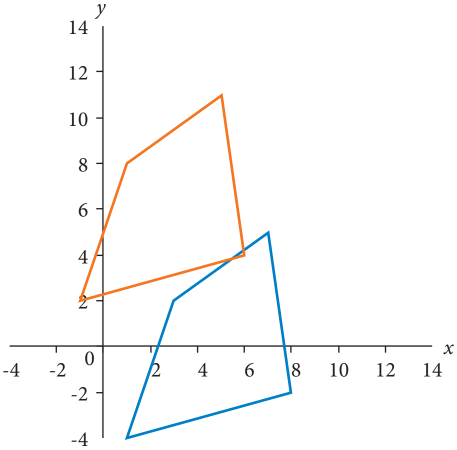


2 a $\mathbf{T}_1 \circ \mathbf{T}_2$: $(x, y) \rightarrow (1 + x, -2 + y)$

b $\mathbf{T}_3 \circ \mathbf{T}_2 = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$

c $A'(-1, 2)$, $B'(1, 8)$, $C'(5, 11)$, $D'(6, 4)$

d



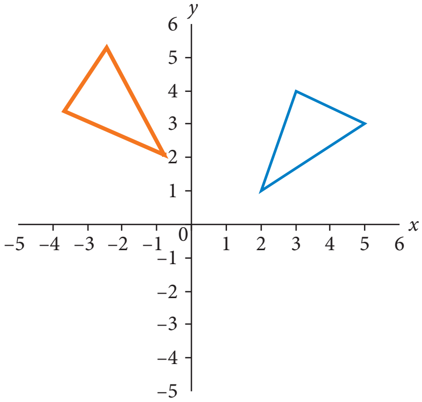
3 a $(x, y) \rightarrow (0.1045x - 0.9945y, 0.9945x + 0.1045y)$

b $(x, y) \rightarrow (0.1045x - 0.9945y, 0.9945x + 0.1045y)$

c Yes

d $A'(-3.66, 3.4)$, $B'(-2.46, 5.29)$, $C'(-0.79, 2.09)$

e

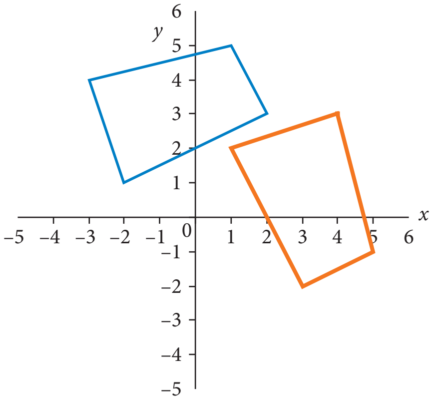


4 a $(x, y) \rightarrow (y, -x)$ b $(x, y) \rightarrow (-y, x)$

c No

d $A'(5, -1)$, $B'(3, -2)$, $C'(1, 2)$, $D'(4, 3)$

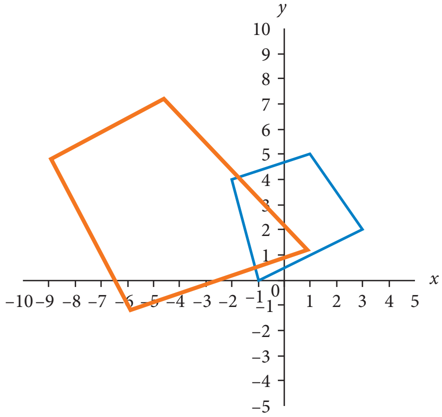
e



5 a $\begin{bmatrix} -0.9 & -1.2 \\ -1.6 & 1.2 \end{bmatrix}$ b $\begin{bmatrix} -0.9 & -1.6 \\ -1.2 & 1.2 \end{bmatrix}$ c No

d $A'(-4.6, 7.2)$, $B'(-8.9, 4.8)$, $C'(-5.9, -1.2)$, $D'(0.9, 1.2)$

e

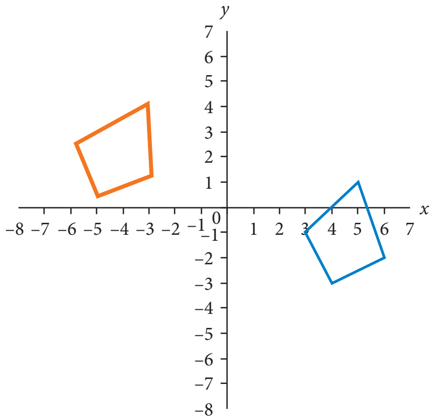


6 a $\begin{bmatrix} -0.7431 & 0.6691 \\ 0.6691 & 0.7431 \end{bmatrix}$ b $\begin{bmatrix} 0.9272 & -0.3746 \\ -0.3746 & -0.9272 \end{bmatrix}$

c No

d $A'(-2.9, 1.26)$, $B'(-3.05, 4.09)$, $C'(-5.8, 2.53)$, $D'(-4.98, 0.45)$

e



7 a $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ b $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ c Yes

8–10 Proofs

11.07

1 a $\mathbf{T} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\mathbf{T}^{-1} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$, $\mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{T}^{-1} \circ \mathbf{T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b $\mathbf{T} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$, $\mathbf{T}^{-1} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{T}^{-1} \circ \mathbf{T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

c $\mathbf{T} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$, $\mathbf{T}^{-1} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, $\mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{T}^{-1} \circ \mathbf{T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

d $\mathbf{T} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$, $\mathbf{T}^{-1} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$, $\mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{T}^{-1} \circ \mathbf{T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

9780170250276

2 $\mathbf{T}=\begin{bmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{2}\end{bmatrix}$, $\mathbf{T}^{-1}=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$, $\mathbf{T}\circ\mathbf{T}^{-1}=\mathbf{T}^{-1}\circ\mathbf{T}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

3 $\mathbf{T}=\begin{bmatrix}2\frac{1}{2} & 0\\ 0 & 2\frac{1}{2}\end{bmatrix}$, $\mathbf{T}^{-1}=\begin{bmatrix}\frac{2}{5} & 0\\ 0 & \frac{2}{5}\end{bmatrix}$,

$\mathbf{T}\circ\mathbf{T}^{-1}=\mathbf{T}^{-1}\circ\mathbf{T}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

4 $\mathbf{T}=\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$, $\mathbf{T}^{-1}=\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$,

$\mathbf{T}\circ\mathbf{T}^{-1}=\mathbf{T}^{-1}\circ\mathbf{T}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

5 $\mathbf{T}=\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$, $\mathbf{T}^{-1}=\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$,

$\mathbf{T}\circ\mathbf{T}^{-1}=\mathbf{T}^{-1}\circ\mathbf{T}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

6 $\mathbf{T}=\begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$, $\mathbf{T}^{-1}=\begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$,

$\mathbf{T}\circ\mathbf{T}^{-1}=\mathbf{T}^{-1}\circ\mathbf{T}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

7 $\mathbf{T}=\begin{bmatrix}\frac{4}{5} & -\frac{3}{5}\\ -\frac{3}{5} & -\frac{4}{5}\end{bmatrix}$, $\mathbf{T}^{-1}=\begin{bmatrix}\frac{4}{5} & -\frac{3}{5}\\ -\frac{3}{5} & -\frac{4}{5}\end{bmatrix}$,

$\mathbf{T}\circ\mathbf{T}^{-1}=\mathbf{T}^{-1}\circ\mathbf{T}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

8 a $\mathbf{T}_1\circ\mathbf{T}_2=\begin{bmatrix}\frac{\sqrt{6}-\sqrt{2}}{4} & \frac{-\sqrt{6}-\sqrt{2}}{4}\\ \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4}\end{bmatrix}$,

$(\mathbf{T}_1\circ\mathbf{T}_2)^{-1}=\begin{bmatrix}\frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{4}\\ \frac{-\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4}\end{bmatrix}$

$=\mathbf{T}_2^{-1}\circ\mathbf{T}_1^{-1}=\mathbf{T}_1^{-1}\circ\mathbf{T}_2^{-1}$

The composition of the inverses is the same in either order.

b $\mathbf{T}_1\circ\mathbf{T}_2=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$,

$(\mathbf{T}_1\circ\mathbf{T}_2)^{-1}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\mathbf{T}_2^{-1}\circ\mathbf{T}_1^{-1}$,

$\mathbf{T}_1^{-1}\circ\mathbf{T}_2^{-1}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$.

The composition of the inverses is different when the order is reversed.

9 a Both $\mathbf{I}$ b Both $\mathbf{I}$ c $\mathbf{T}_2$
d $\mathbf{T}_2^{-1}$ e Proof

10 Proof

11.08

1 a 182 b 360 c 300

2 A reflection in the line $y = -x$.

3 A rotation of $-45°$.

4 A dilation with factors 8 and 6 in the x and y directions.

5 A reflection in the line $y = -x$.

6 a 4 b $\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}$

c $A'(3, -8)$, $B'(-2, 10)$, $C'(0, 14)$

d 28 e 7

f The area of the image is det $\mathbf{T}$ (= 7 in this case) times the area of the object.

7 a 26 b $\begin{bmatrix}-2 & 1\\ 2 & -2\end{bmatrix}$ c 52

8 a 140 b 100 c 1080

9 a $\begin{bmatrix}\cos(50°) & -\sin(50°)\\ \sin(50°) & \cos(50°)\end{bmatrix}$

b A rotation through an angle of 50°.

10 a $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ b A rotation through an angle of 90°.

11–12 Proofs

CHAPTER 11 REVIEW

1 C

2 E

3 B

4 A

5 B

6 A

7 $A'(4, 2)$, $B'(8, 4)$, $C'(11, 0)$, $D'(15, -5)$

8 a Proof

b $A'(-3, 4)$, $B'(18, 18)$, $C'(3, 0)$, $D'(-18, -14)$

c A parallelogram

9 a $(x, y) \to (3x, 1.5y)$ b $\begin{bmatrix}3 & 0\\ 0 & 1.5\end{bmatrix}$

c $A'(-9, -3)$, $B'(9, 9)$, $C'(-3, 13.5)$

d A non-right-angled scalene triangle

10 a $\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

b $A\left(\frac{2+\sqrt{3}}{2}, \frac{2\sqrt{3}-1}{2}\right)$, $B\left(1-2\sqrt{3}, 2+\sqrt{3}\right)$, $C\left(\frac{-(5+4\sqrt{3})}{2}, \frac{4-5\sqrt{3}}{2}\right)$, $D\left(\frac{\sqrt{3}-5}{2}, \frac{-(1+5\sqrt{3})}{2}\right)$

c They are congruent.

11 $A'(-4.95, -0.71)$, $B'(-6.36, -4.95)$, $C'(-3.54, -6.36)$, $D'(0.71, -4.95)$

12 $A'(5, 6)$, $B'(14, 4)$, $C'(12, 3)$

13 a $A'(1.6, 6.6)$, $B'(-4.4, -6.9)$, $C'(2.4, -5.1)$ b No

14 a A translation 3 right and 4 down

b A dilation with x factor 2 and y factor $\frac{2}{3}$

c A rotation of $-\frac{4\pi}{3}$.

d A reflection in the line through the origin with inclination 25°.

15 225

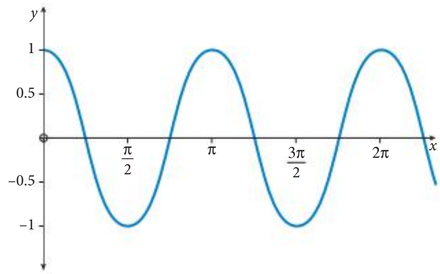
16 Proof

17 a 5.5 b Proof

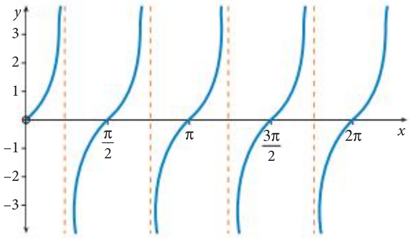
18 Proof

12.01

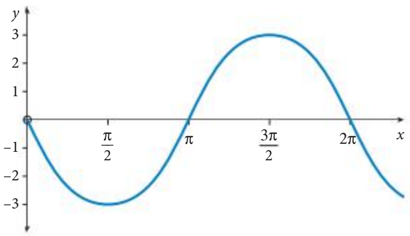
1 a



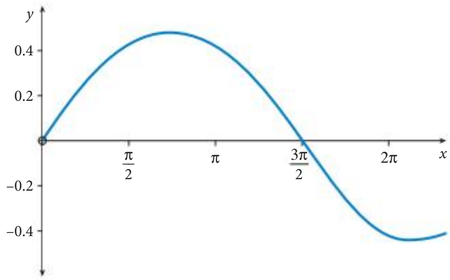
b



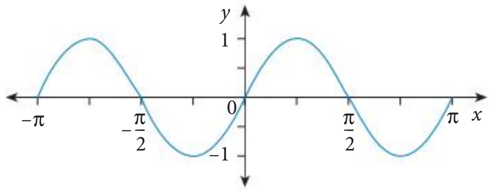
c



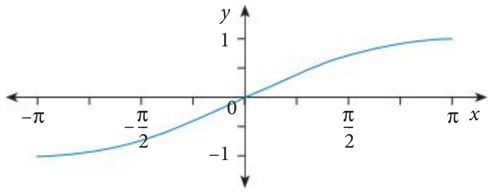
d



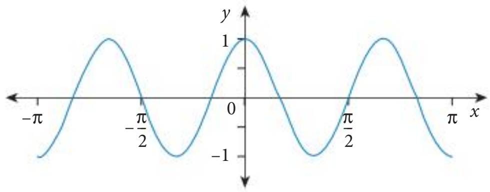
2 a $y = \sin(2x)$



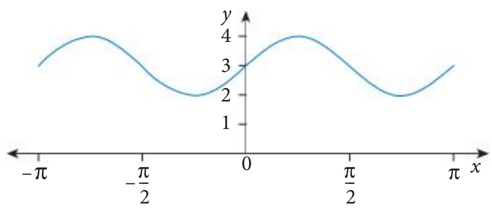
b $y = \sin\left(\frac{1}{2}x\right)$



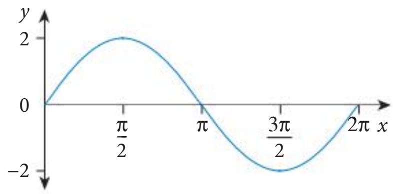
c $y = \cos(3x)$



d $y = \sin(2x) + 3$



3 a $y = 2\sin(x)$



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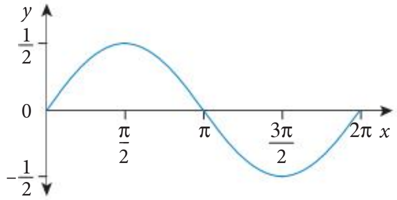
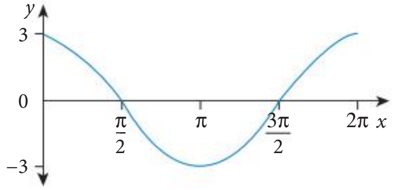
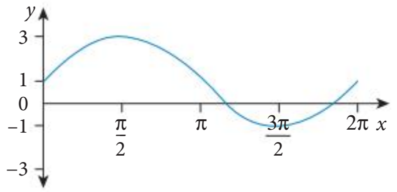
b $y = \frac{1}{2}\sin(x)$



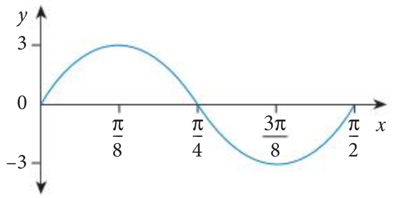
c $y = 3\cos(x)$



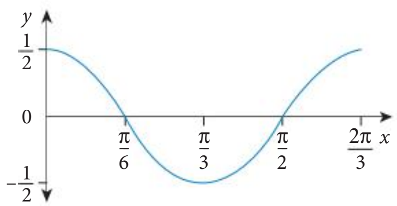
d $y = 2\sin(x) + 1$



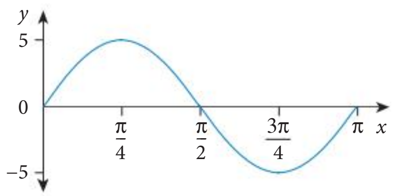
4 a $y = 3\sin(4x)$



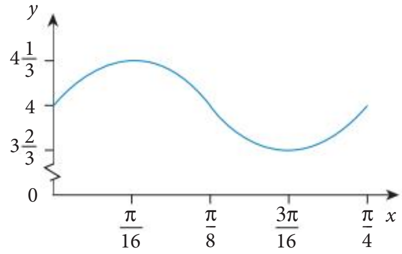
b $y = \frac{1}{2}\cos(3x)$



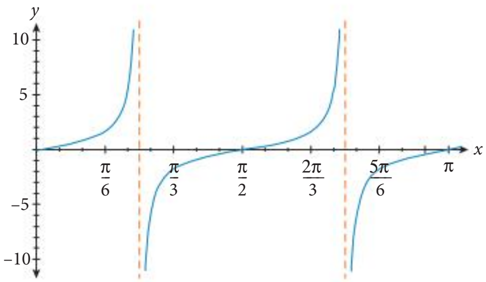
c $y = 5\sin(2x)$



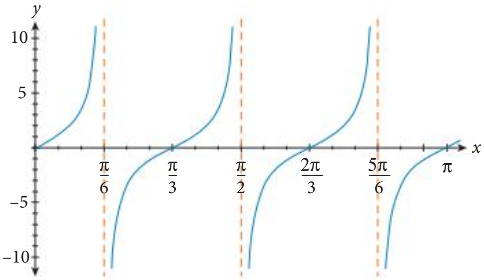
d $y = \frac{1}{3}\sin(8x) + 4$



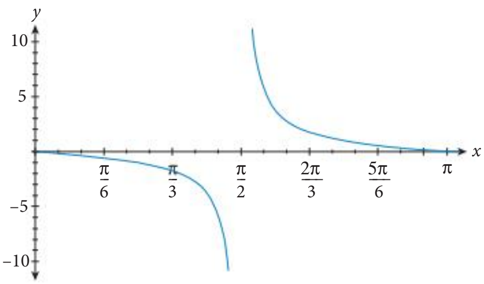
5 a $x = \frac{\pi}{4}, x = \frac{3\pi}{4}$



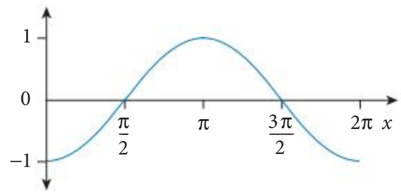
b $x = \frac{\pi}{6}, x = \frac{\pi}{2}$



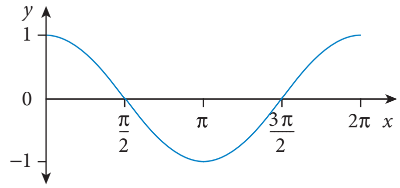
c $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$



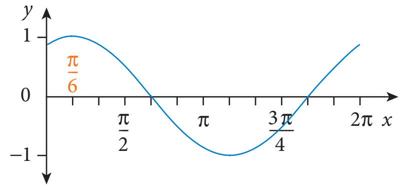
6 a $y = \sin\left(x - \frac{\pi}{2}\right)$



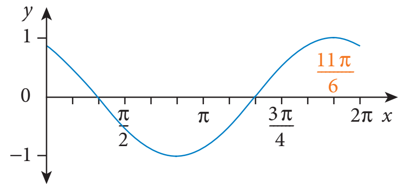
b $y = \sin\left(x + \frac{\pi}{2}\right)$



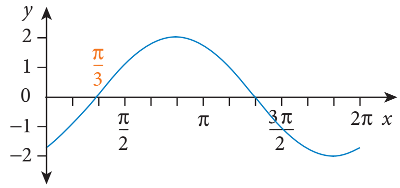
c $y = \cos\left(x - \frac{\pi}{6}\right)$



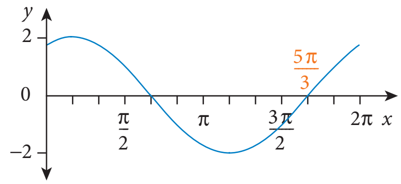
d $y = \cos\left(x + \frac{\pi}{6}\right)$



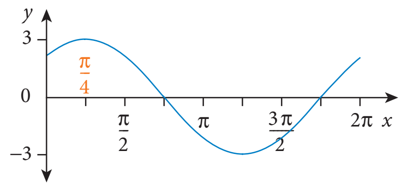
e $y = 2\sin\left(x - \frac{\pi}{3}\right)$



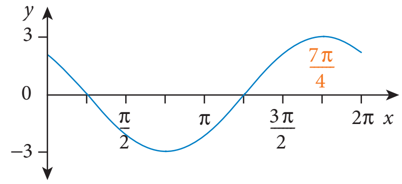
f $y = 2\sin\left(x + \frac{\pi}{3}\right)$



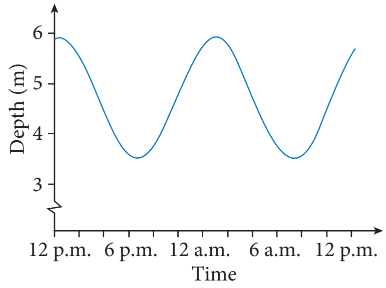
g $y = 3\cos\left(x - \frac{\pi}{4}\right)$



h $y = 3\cos\left(x + \frac{\pi}{4}\right)$



7 a $d = 4.7 + 1.2\cos(0.5t - 0.21)$



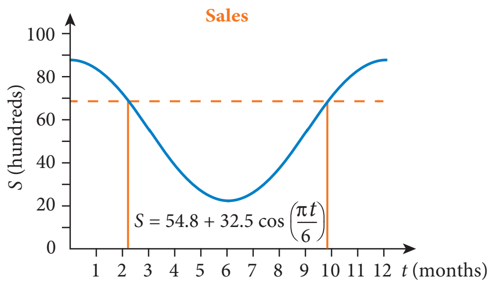
b About 6:42 p.m.

c About 5 h 16

d About 12 h 34 m

e About 25 minutes

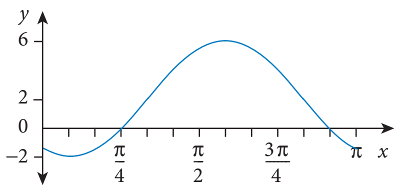
8 a



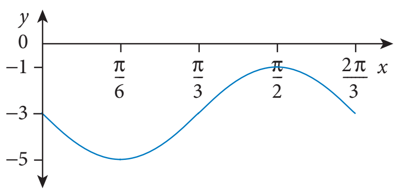
b Jan, Feb, Oct, Nov and Dec

12.02

1 a $y = 4\sin\left[2\left(x - \frac{\pi}{3}\right)\right] + 2$



b $y = 2\cos\left[3\left(x + \frac{\pi}{6}\right)\right] - 3$



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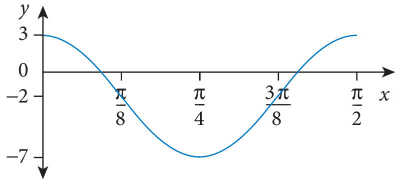
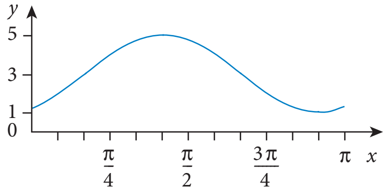
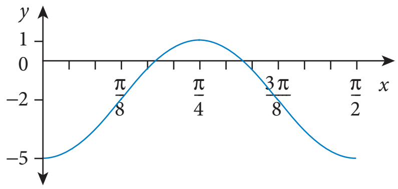
c $y = 5\sin\left[4\left(x + \frac{\pi}{8}\right)\right] - 2$



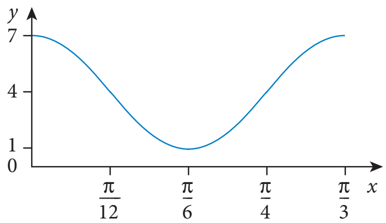
2 a $y = 2\sin\left(2x - \frac{\pi}{3}\right) + 3$



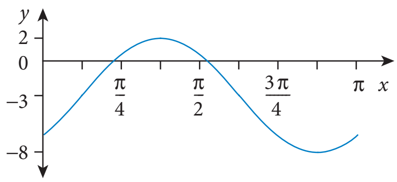
b $y = 3\cos(4x + \pi) - 2$



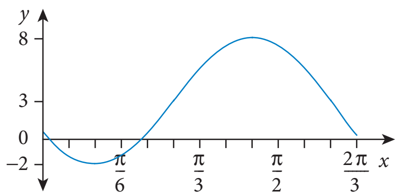
c $y = 4 - 3\sin\left(6x + \frac{3\pi}{2}\right)$



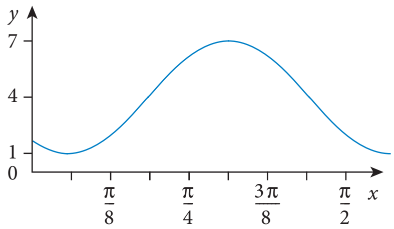
d $y = 5\cos\left(2x - \frac{3\pi}{4}\right) - 3$



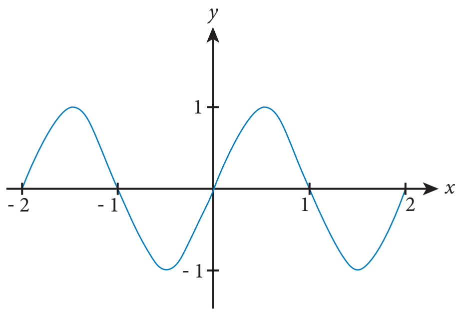
e $y = 3 - 5\cos\left(3x - \frac{\pi}{3}\right)$



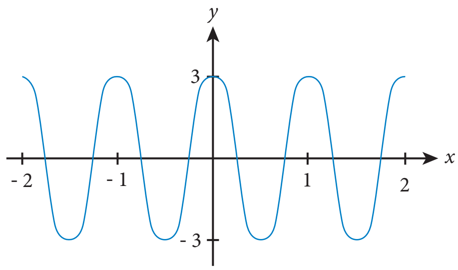
f $y = 4 - 3\sin\left(4x + \frac{\pi}{4}\right)$



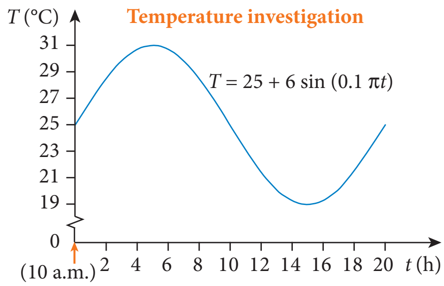
3 a



b



4 a



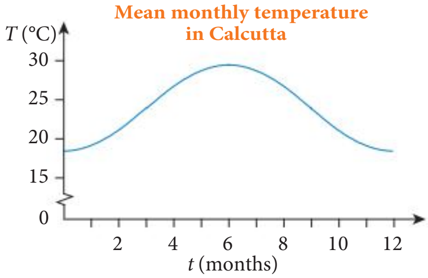
b 31°C, 19°C **c** 3 p.m., 1 a.m.

d **i** 11:05 a.m., 6:55 p.m., …

ii 11:08 p.m., 2:52 a.m., …

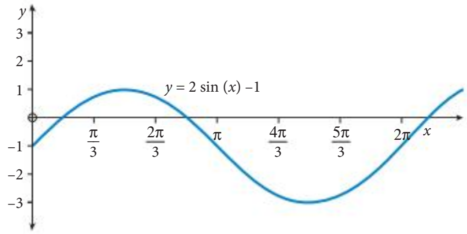
e Because the period was not 24 hours and it is unusual to have the maximum so early in the day.

5 $T = 23.875 + 5.575\cos\left(\dfrac{\pi t}{6}\right)$.



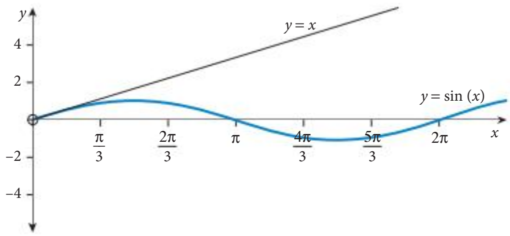
12.03

1

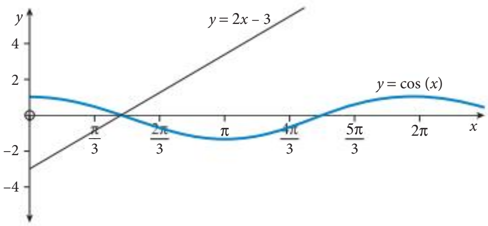


Solutions: $x = 0.52$, $x = 2.62$

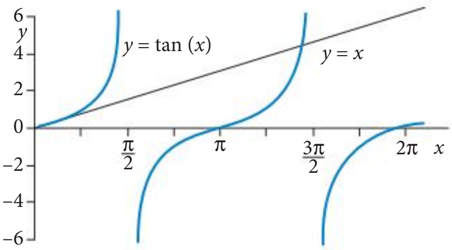
2 $x = 0$



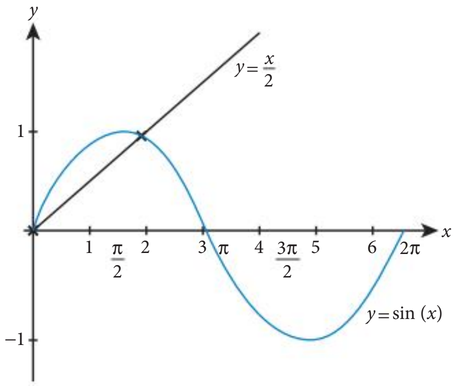
3 $x = 1.5$



4 $x = 0, 4.5$

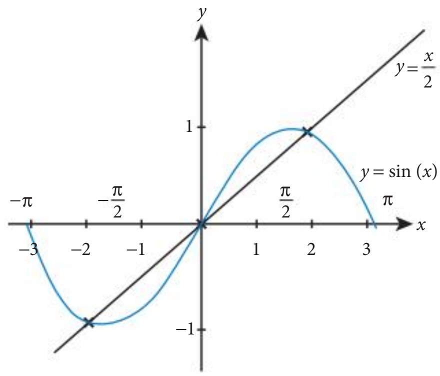


5 a



There are 2 points of intersection, so there are 2 solutions to the equation.

b



There are 3 points of intersection, so there are 3 solutions to the equation.

6 About 1.4 m

12.04

From **4** onwards, $n \in \boldsymbol{Z}$.

1 a $\frac{\pi}{6}, \frac{5\pi}{6}$ b $\frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}$

c $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$

2 a $x = -\pi, \pi$ b $x = \frac{\pi}{2}$ c $x = -\frac{7\pi}{4}, -\frac{3\pi}{4}$

d $x = 0, 2\pi$ e $x = -\pi, \pi$ f $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

g $x = -\frac{\pi}{4}, \frac{3\pi}{4}$ h $x = \frac{\pi}{6}, \frac{5\pi}{6}$ i $x = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

j $x = \frac{11\pi}{6}, \frac{13\pi}{6}$

3 a $\theta = 45°, 225°$ b $\theta = 30°, 150°$

c $\theta = 150°, 210°$ d $\theta = 225°, 315°$

e $\theta = 90°, 270°$ f $\theta = 60°, 240°$

g $\theta = 120°, 240°$ h $\theta = 150°, 330°$

i $\theta = 50°, 130°$ j $\theta = 115°, 245°$

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4 a $x = 2n\pi - \frac{\pi}{3}, 2n\pi + \frac{4\pi}{3}$

b $x = 2n\pi \pm \frac{\pi}{6}$

c $x = n\pi + \frac{\pi}{6}$

5 a $x = 0.3398, 2.8018$ b $x = 1.8235$

c $x = 0.3805$ d $x = 2.1518$

e $x = 0.4429, 2.6987$ f $x = 1.7197$

6 a $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ b $x = \frac{\pi}{2}, \frac{3\pi}{2}$

c $x = \frac{\pi}{4}, \frac{5\pi}{4}$ d $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

e $x = \frac{\pi}{4}, \frac{5\pi}{4}$ f $x = \frac{\pi}{3}, \frac{4\pi}{3}$

7 a $x = -\frac{\pi}{6} + n\pi$ or $\frac{2\pi}{3} + n\pi$

b $x = \frac{\pi}{18} + \frac{2n\pi}{3}$ or $\frac{5\pi}{18} + \frac{2n\pi}{3}$

c $x = \frac{5\pi}{24} + n\pi$ or $\frac{17\pi}{24} + n\pi$

d $x = \frac{7\pi}{6} + n\pi$ or $\frac{4\pi}{3} + n\pi$

e $x = n\pi$ or $\frac{3\pi}{4} + n\pi$

8 a −2.46, −0.16, 0.68, 2.98

b −2.40, −1.35, −0.30, 0.75, 1.79, 2.84

c −1.95, −1.59, −0.38, −0.02, 1.19, 1.56, 2.76, 3.13

d −3.12, −1.45, −1.02, 0.64, 1.07, 2.74

e −2.46, −2.11, −1.20, −0.86, 0.06, 0.40, 1.31, 1.66, 2.57, 2.91

9 $x \approx 0.47, -1.51$

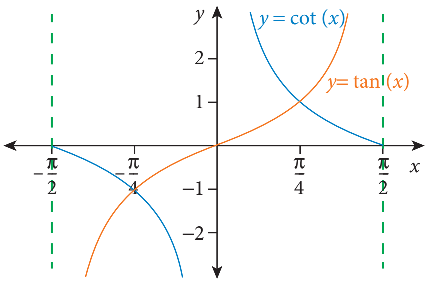
10 a 0.34 m

b 0.37 m

c 0.38 m when $t = \frac{\pi}{8}$

12.05

1



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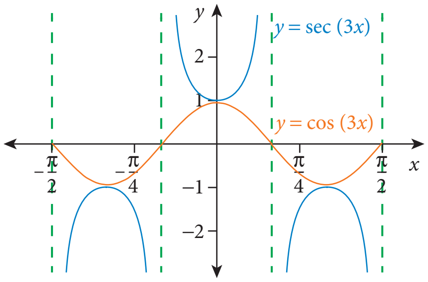

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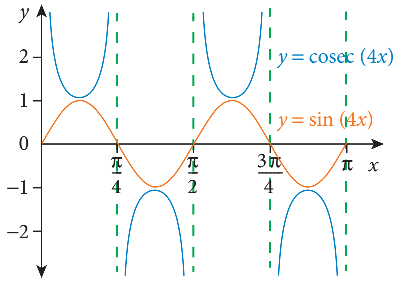

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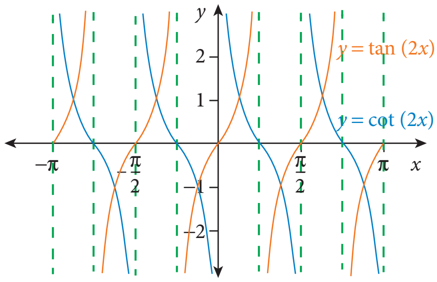

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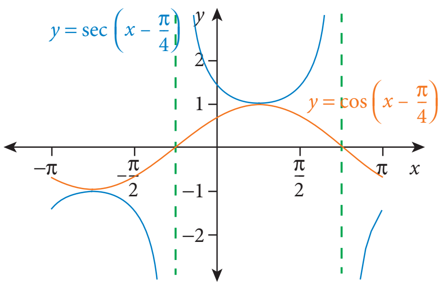

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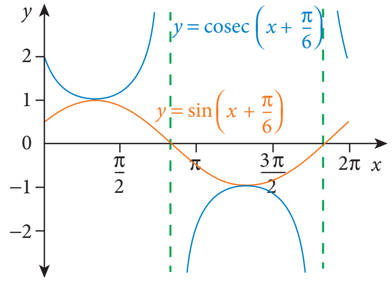

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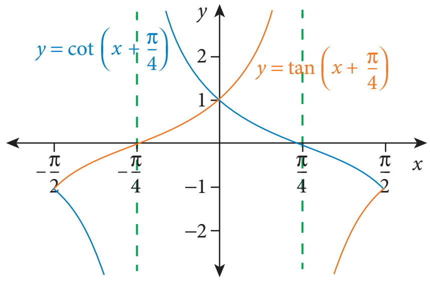

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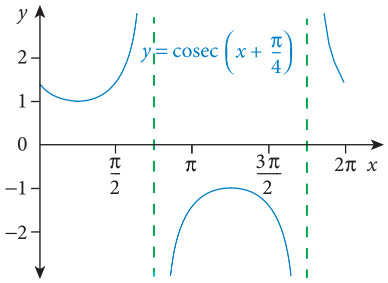

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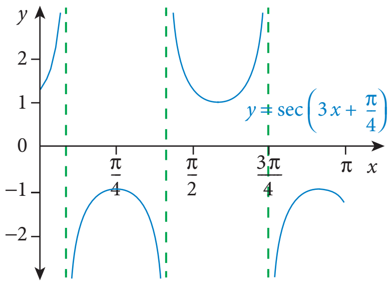

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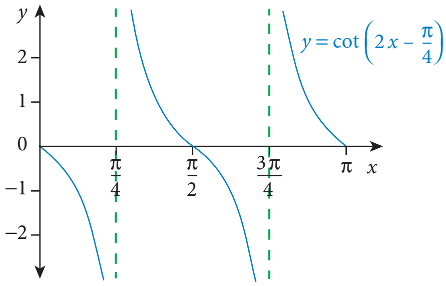
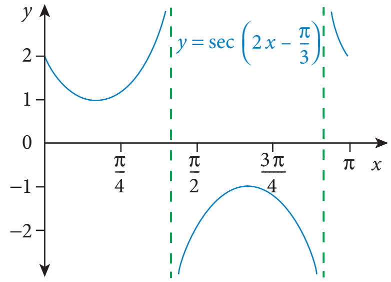


11



12.06

1 a $\sqrt{5}\sin(\theta + 26°34')$ b $2\sin(\theta + 60°)$
c $\sqrt{2}\sin(\theta + 45°)$ d $\sqrt{29}\sin(\theta + 21°48')$
e $\sqrt{17}\sin(\theta + 14°2')$ f $\sqrt{10}\sin(\theta + 18°26')$
g $\sqrt{13}\sin(\theta + 56°19')$ h $\sqrt{65}\sin(\theta + 60°15')$
i $\sqrt{41}\sin(\theta + 38°40')$ j $\sqrt{34}\sin(\theta + 59°2')$

2 a $\sqrt{2}\sin(\theta - 45°)$ b $\sqrt{5}\sin(\theta - 63°26')$
c $2\sin(\theta - 60°)$ d $2\sin(\theta - 30°)$
e $\sqrt{29}\sin(\theta - 21°48')$

3 $\sqrt{10}\cos(\theta - 18°26')$

4 $2\cos(\theta + 60°)$

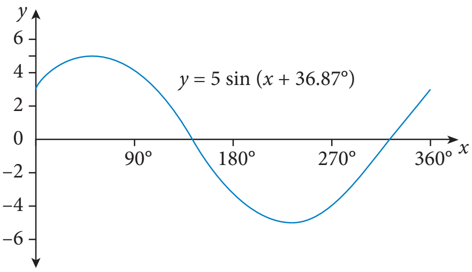
5 a $\sqrt{85}\sin(\theta + 12°28')$ b $\sqrt{85}\cos(\theta - 77°28')$

12.07

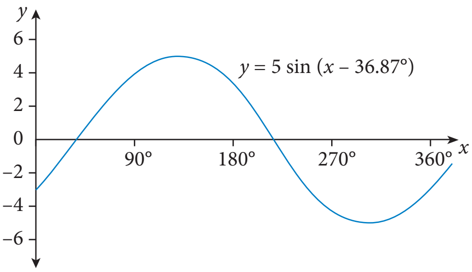
1 $r = 5$, $5\sin(x + \alpha) = 5[\sin(x)\cos(\alpha) + \cos(x)\sin(\alpha)] = 5\left[\sin(x) \times \frac{4}{5} + \cos(x) \times \frac{3}{5}\right]$

Where $\tan(\alpha) = \frac{3}{4}$, $\therefore\ \alpha = 36.87°$.

$y = 5\sin(x + 36.87°)$



2 $y = 5\sin(x - 36.87°)$



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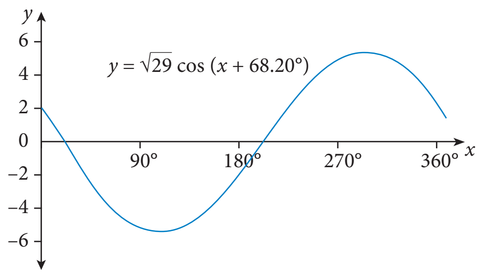
3 $r = \sqrt{29}$, $\sqrt{29}\cos(x + \alpha) = \sqrt{29}\,[\cos(x)\cos(\alpha) - \sin(x)\sin(\alpha)]$

$= \sqrt{29}\left[\cos(x) \times \dfrac{2}{\sqrt{29}} - \sin(x) \times \dfrac{5}{\sqrt{29}}\right]$

Where $\tan(\alpha) = \dfrac{5}{2}$, $\therefore\ \alpha = 68.20°$.

$y = \sqrt{29}\cos(x + 68.20°)$



4 **a** $\theta = 126°52', 306°52'$

b $\theta = 35°58', 189°16'$

c $\theta = 60°, 240°$

d $\theta = 180°, 270°$

e $\theta = 240°43', 327°21'$

f $\theta = 90°, 180°$

g $\theta = 90°, 340°32'$

h $\theta = 56°34', 176°34'$

i $\theta = 51°2', 190°5'$

j $\theta = 160°32', 270°$

5 $\theta = 2\pi k + 1.451,\ \theta = 2\pi k + 3.545,\ k \in \boldsymbol{Z}$

6 $\theta = 2\pi k,\ \theta = 2\pi k + \dfrac{2\pi}{3},\ k \in \boldsymbol{Z}$

7 **a** $h = 9\sin\left(\dfrac{\pi t}{4}\right) + \cos\left(\dfrac{\pi t}{4}\right)$

$\tan(\alpha) = \dfrac{1}{9}$, $\cos(\alpha) = \dfrac{9}{\sqrt{82}}$, $\sin(\alpha) = \dfrac{1}{\sqrt{82}}$, $r = \sqrt{82}$,

$\tan(\alpha) = \dfrac{1}{9}$, $\therefore\ \alpha = 0.1107^c$

Using $r\sin\left(\dfrac{\pi t}{4} + \alpha\right) =$

$r\left[\sin\left(\dfrac{\pi t}{4}\right)\cos(\alpha) + \cos\left(\dfrac{\pi t}{4}\right)\sin(\alpha)\right]$

gives

$\sqrt{82}\sin\left(\dfrac{\pi t}{4} + 0.1107\right) = \sqrt{82}\sin\left(\dfrac{\pi t}{4}\right) \times \dfrac{9}{\sqrt{82}}$

$+ \cos\left(\dfrac{\pi t}{4}\right) \times \dfrac{1}{\sqrt{82}}$

This gives $h = 9\sin\left(\dfrac{\pi t}{4}\right) + \cos\left(\dfrac{\pi t}{4}\right)$

$= \sqrt{82}\sin\left(\dfrac{\pi t}{4} + 0.1107\right)$

b 1 metre

c $h = \sqrt{82}\sin\left(\dfrac{\pi t}{4} + 0.1107\right)$, so at $t = 2$, $h = 9$ metres.

d Solutions $t = 0$, $t = 3.718$, $t = 8$, $t = 11.718$, $t = 16$, $t = 19.718$, $t = 24$; therefore: returns at 12:43 p.m., 5 p.m., 8:43 p.m., 1 a.m. (next day), 4:43 a.m. (next day), 9 a.m. (next day).

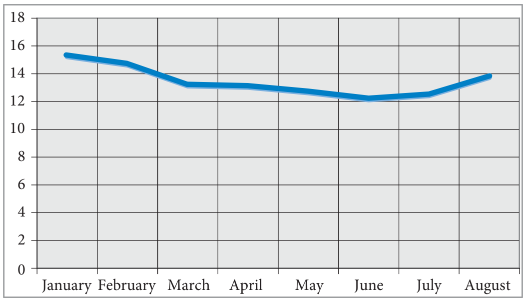
12.08

1 **a** 1300

b **i** 1600 **ii** 1010

c Amplitude 300, period 10 years

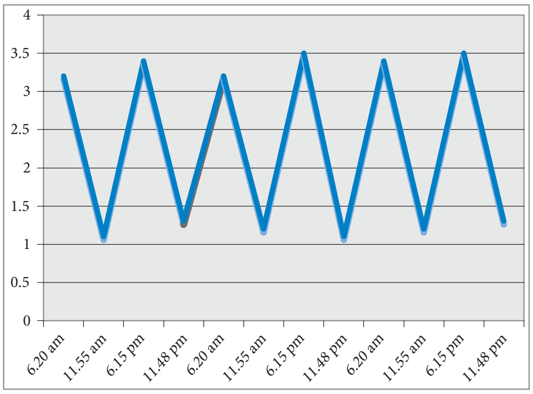
2 **a**



b It may be periodic – hard to tell from this data. Period would be about 10 months.

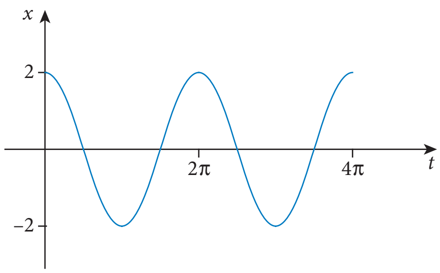
c Amplitude is 1.5 if it is periodic.

3 **a**

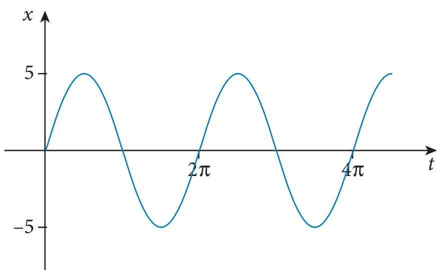


b Period 12 hours, amplitude 1.25. **c** 2.5 m

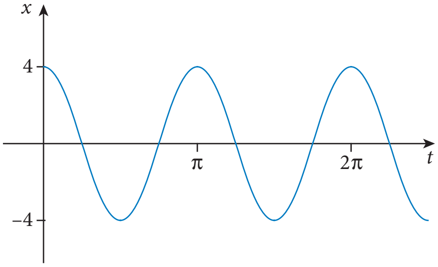
4



5



6 a $x = 4\cos(2t)$



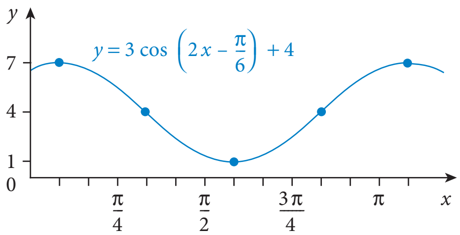
b $t = 0, \pi, 2\pi, \ldots, x = 4$

7 a $t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$ b 2π c $x = 2$

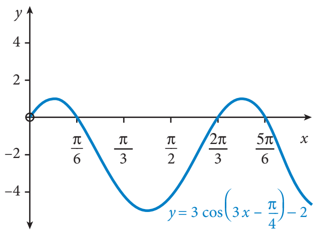
CHAPTER 12 REVIEW

1 B
2 D
3 D
4 C
5 D

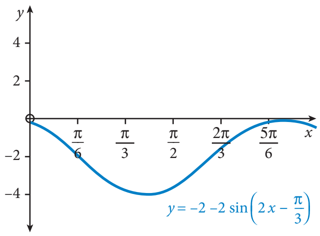
6 a $\left(\frac{\pi}{12}, 7\right), \left(\frac{\pi}{3}, 4\right), \left(\frac{7\pi}{12}, 1\right), \left(\frac{5\pi}{6}, 4\right), \left(\frac{13\pi}{12}, 7\right)$



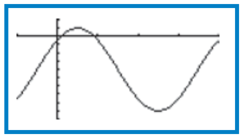
b $\left(\frac{\pi}{12}, 1\right), \left(\frac{5\pi}{12}, -5\right), \left(\frac{3\pi}{4}, 1\right), \left(\frac{5\pi}{12}, -2\right), \left(\frac{7\pi}{12}, -1\right)$



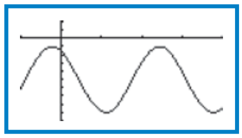
c $(0, -0.27), \left(\frac{5\pi}{12}, -4\right), \left(\frac{11\pi}{12}, 0\right), (\pi, -0.27)$



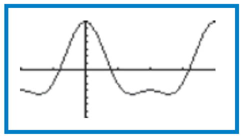
7 a $y = 5\sin\left(2x + \frac{\pi}{4}\right) - 4$



b $y = 4\cos\left(3x + \frac{\pi}{6}\right) - 5$



c $y = 4\cos(x) + 2\cos(2x) + \cos(4x)$



8 $x = 0, x = 0.948$

9 $x = 0.785, x = 3.927$

10 $\frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}$

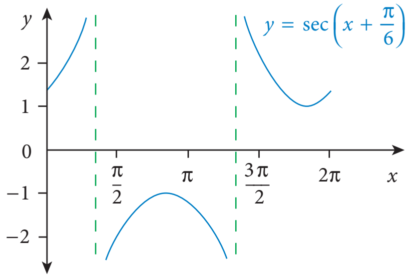
11 a $\alpha = \pi, 2\pi$ b $\alpha = \frac{5\pi}{3}$ c $\alpha = \frac{5\pi}{4}$

d $\alpha = \pi, 2\pi$ e $\alpha = \frac{11\pi}{6}$ f $\alpha = \frac{11\pi}{6}$

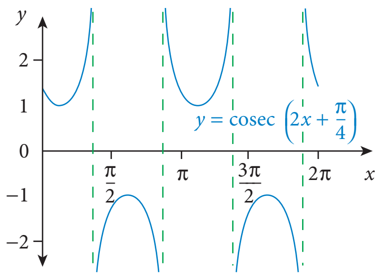
g $\alpha = \frac{5\pi}{4}$ h $\alpha = \frac{4\pi}{3}$ i $\alpha = \frac{15\pi}{8}$

j No solution in the domain

12 a



b



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13 a $\frac{\sqrt{3}}{3}$ b $\sqrt{2}$ c $\frac{2\sqrt{3}}{3}$ d $-\frac{2\sqrt{3}}{3}$

14 $4\cos(x) + 3\sin(x) = 5\cos(x - \alpha)$, where $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$.

15 $6\cos(x) + \sin(x) = \sqrt{37}\cos(x - \alpha)$, where $\alpha = \tan^{-1}\left(\frac{1}{6}\right)$. $x = 90°$, $x = 288.92°$.

16 a Maximum = 12 m, minimum = 2 m

b 120

c $a = 5$, $b = 4\pi$, $c = 7$

d $33\frac{1}{3}\%$

17 a $a = 200$, $b = 50$, $m = 2\pi$, $g(t) = 200 + 50\sin(2\pi t)$

b $c = 200$, $d = 50$, $n = 4\pi$, $b(t) = 200 - 50\sin(4\pi t)$

c

t (years)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1
Population	243 (240)	200	157 (160)	200
t (years)	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	2
Population	243 (240)	200	157 (160)	200

d 12 days

MIXED REVISION 4

Multiple choice

1 B
2 D
3 D
4 C
5 D
6 B
7 A
8 D
9 E

Short answer

1 $(x - 2 - i)(x - 2 + i)$

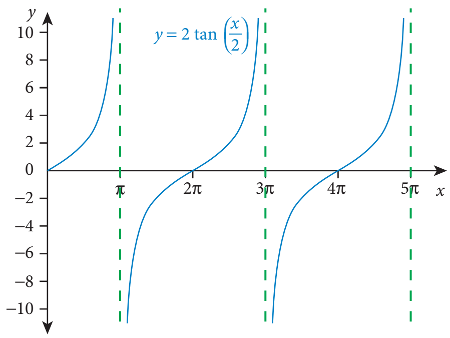
2 $A'(-6, 2)$, $B'(6, 8)$, $C'(21, 4)$, $D'(12, -6)$

3 $\sqrt{13}\cos(\theta - \alpha)$ where $\alpha = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.69$

4 a $3 - 7i$ b $\sqrt{37}$

5 $A'\left(-\frac{2+\sqrt{3}}{2}, \frac{1-2\sqrt{3}}{2}\right)$, $B'\left(\frac{3+2\sqrt{3}}{2}, \frac{3\sqrt{3}-2}{2}\right)$, $C'\left(\frac{4+7\sqrt{3}}{2}, \frac{4\sqrt{3}-7}{2}\right)$

6



Application

1 a $\overrightarrow{AD} = v - z$, $\overrightarrow{AB} = w - z$ b $u = v + w - z$

2 $u = -1 + 2i$

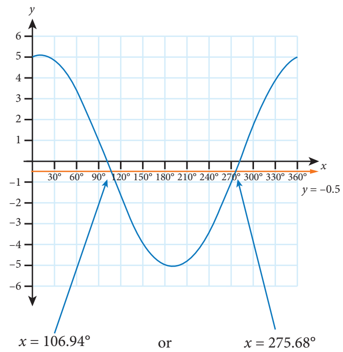
3 A reflection in the line through the origin with inclination 40°.

4 Proof

5 a $\sqrt{26}\sin(x + \alpha)$, where $\theta = \tan^{-1}(5) = 78.69°$, equals $\sqrt{26}\sin(x + 78.69°)$.

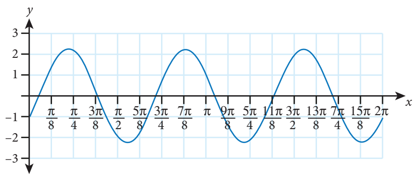
b $x = n \times 360° - 84.32°$ or $x = n \times 360° + 106.94°$ where $n \in \boldsymbol{Z}$ (other answers are acceptable).

c



Solutions: $x = 106.94°$, $x = 275.68°$

6 a



Graph oscillates in symmetry about the average position and the motion can be expressed in the form of SHM using

$a\sin(x) - b\cos(x) = r\sin(x - \alpha)$

b amplitude = $\sqrt{5}$, period = $\frac{2\pi}{3}$

c 2.236 metres either side of the origin.

GLOSSARY

Chapter 1

displacement A vector quantity representing the change in position of an object.

head-to-tail method A process for adding vectors whereby vectors are placed head-to-tail to calculate a resultant vector. The resultant vector is drawn from the head of the first vector to the tail of the last vector.

initial point See **position vector**.

linear combination A linear combination of vectors is a sum of scalar multiples of the vectors, so a linear combination of vectors **p** and **q** is $a\mathbf{p} + b\mathbf{q}$, where a and b are real numbers.

norm The magnitude of a vector is also called the norm of the vector. The norm of a vector **p** is written as $|\mathbf{p}|$ or p.

parallelogram rule If vectors **a** and **b** are the sides of a parallelogram $ABCD$, the diagonal AC is the resultant $\mathbf{a} + \mathbf{b}$.

polar form The vector **p** is represented in polar form by an ordered pair (p, θ). The first number is the magnitude of the vector and the second number gives the direction of the vector. In a Cartesian plane, the direction is given by the anticlockwise angle from the positive direction of the x-axis.

position vector The position vector of a point P is written as $\overrightarrow{OP}$ or $\overline{OP}$, where O is the origin or initial point and P is the terminal point of the vector.

resultant vector The result of adding vectors is known as the resultant vector.

scalar Quantities such as mass, distance, volume and speed, specified by a single magnitude are called scalar quantities.

terminal point See **position vector**.

triangle rule If two vectors **a** and **b** are placed head-to-tail to form two sides of a triangle, then $\mathbf{a} + \mathbf{b}$ is the third side of the triangle. The sum $\mathbf{a} + \mathbf{b}$ is also called the resultant of **a** and **b**.

unit vector A unit vector has magnitude 1. The unit vector in the same direction as a given vector **p** is symbolised by placing a circumflex (ˆ) over the vector symbol, giving $\hat{\mathbf{p}}$.

A unit vector may be derived from any vector **p** by multiplying by $\frac{1}{|\mathbf{p}|}$. Thus $\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$.

vector Quantities such as displacement, velocity and acceleration, need to have both magnitude and direction to complete their specification. These are known as vector quantities.

Chapter 2

axiom An assumption that begins a proof.

Contradiction (Proof by) A logical argument used to show that an assumption or statement is false.

contrapositive The contrapositive statement of *If P then Q* is *If not Q then not P*.

converse The converse of *P implies Q* is *Q implies P*.

counter example An example that shows that a statement is not true for all cases.

deductive (proof) Where a number of statements are made that lead logically to a conclusion.

equality Where two quantities are equal or LHS = RHS

equivalence If $P \Rightarrow Q$ and $Q \Rightarrow P$ then $P \Leftrightarrow Q$, if a statement and its converse are both true.

Euclidean (geometry) Geometry of the plane, or two dimensions, named after the classical Greek mathematician, Euclid.

hypothesis A general statement or conclusion for which tests are devised to prove or disprove.

if and only if $P \Leftrightarrow Q$, equivalence.

iff $P \Leftrightarrow Q$, abbreviation for *if and only if*.

implication P *implies* Q, written $P \Leftrightarrow Q$ or *if … then …* statement.

inductive Where a general conclusion is made from specific observations.

necessary A true statement that works only one way and is often written as 'If … then …'

negation *Not* P, written P', $\neg P$ or $\overline{P}$.

premise An assumption that begins a proof.

proof An argument that shows something to be true for all cases.

QED *Quod errat demonstrandum*, which can be translated from the Latin as *demonstrated as required*.

quantifier A mathematical symbol. $\forall$ for all. $\exists$ there exists.

reason A way of thinking attributed most often to the Classical Greek mathematicians and philosophers.

reasoning The reasons used to prove a statement.

RTP Required to prove.

sufficient A true statement that works only one way and is often written as 'If … then …'

Chapter 3

addition principle The total number of ways of making a choice made in mutually exclusive ways is the sum of the numbers of choices made in each way.

circular permutation A permutation of objects in a circle.

combinatorics Systematic counting of a finite collection, particularly selections and orders.

Dirichlet's box principle *See* pigeonhole principle.

factorial The factorial product of a number n is given by $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. $0! = 1! = 1$.

multiplication principle The number of ways of making a choice done in stages is the product of the numbers of choices in each stage. For two stages with a and b choices in each stage, there are $a \times b$ choices altogether.

mutually exclusive Mutually exclusive sets have no common elements.

permutation An ordered arrangement of some or all of a set of objects or symbols. The number of permutations of r objects from n objects is written as ${}^nP_r$.

pigeonhole principle If $m > n$ objects are placed in n boxes, then at least one box must contain more than one object. Also called Dirichlet's box principle.

recursion definition Defining (a sequence) by giving the first term and a method of obtaining each term from the previous term.

Chapter 4

airspeed In air navigation, the airspeed is the apparent speed of an aircraft relative to the air.

course made good The course made good is the true course of a boat or aircraft when the effects of the water current or wind are taken into account.

dot product For two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of θ between them, the dot product is given by: $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$

For two vectors $\mathbf{v}_1 = (a_1, a_2)$ and $\mathbf{v}_2 = (b_1, b_2)$, the dot product is given by:
$\mathbf{v}_1 \cdot \mathbf{v}_2 = (a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2$

ground speed Speed made good in air navigation.

heading In navigation, the direction in which a boat or aircraft is steered is called the heading.

orthogonal component A component that is perpendicular to a given vector or direction.

parallel component A component that is parallel to a given vector or direction.

projection For a single direction, the orthogonal component in that direction is called the projection of the vector in that direction.

resolution A process to express a given vector as a combination of simpler components – usually perpendicular to each other.

resolve The act of expressing a given vector as a combination of simpler components – usually perpendicular to each other.

scalar product For two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of θ between them, the scalar product is given by: $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| \times |\mathbf{v}_2| \cos(\theta)$

For two vectors $\mathbf{v}_1 = (a_1, a_2)$ and $\mathbf{v}_2 = (b_1, b_2)$, the scalar product is given by:
$\mathbf{v}_1 \cdot \mathbf{v}_2 = (a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2$

speed made good This is the true speed of a boat or aircraft when the effects of the water current or wind are taken into account.

speed through the water This is the apparent speed of a boat relative to the water.

true course The course of a boat or aircraft when the effects of the water current or wind are taken into account.

wind The air current.

work The energy transferred by a force when it causes an object to move.

When the force and the displacement are in the same direction, $W = F \times s$, where W = work, F (force) = $|\mathbf{F}|$ and s (displacement) = $|\mathbf{s}|$.

Otherwise, $W = \mathbf{F}_{\text{parallel}} \cdot \mathbf{s}$ where W = work, $\mathbf{F}_{\text{parallel}}$ = the component of the force parallel to motion and $\mathbf{s}$ = displacement.

Chapter 5

addition principle The total number of ways of making a choice made in mutually exclusive ways is the sum of the numbers of choices made in each way.

circular permutation An ordered arrangement of objects in a circle.

combination A selection of some or all of a set of objects or symbols without regard to order. The number of selections of r objects from n objects is written as ${}^nC_r$ or $\binom{n}{r}$.

combinatorics Systematic counting of a finite collection, particularly selections and orders.

inclusion-exclusion principle Rule for the number of elements in a union of sets.

multiplication principle The number of ways of making a choice done in stages is the product of the numbers of choices in each stage. For two stages with a and b choices in each stage, there are $a \times b$ choices altogether.

Pascal's triangle A triangular arrangement of numbers such that the number in position r of row n is ${}^nC_r$, where r and n both begin from 0.

permutation An ordered arrangement of some or all of a set of objects or symbols. The number of permutations of r objects from n objects is written as ${}^nP_r$.

Chapter 6

alternate segment In a circle, the segment other than the segment between a chord and a tangent.

arc Part of the circumference of a circle.

concyclic Four or more points that all lie on the same circle.

cyclic (quadrilateral) A quadrilateral whose four vertices lie on the circumference of a circle.

intercepts The intervals formed when two chords intersect.

proof An argument that shows something to be true for all cases.

secant A straight line that intersects a circle twice.

subtend Form an angle by two lines.

tangent A line that touches a circle at one point.

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Chapter 7

abundant numbers Numbers whose proper divisors sum to greater than the number; for example 12, is abundant as $1 + 2 \times 3 \times 4 \times 6 > 12$.

amicable numbers Pairs of numbers such that each one is the sum of the proper divisors of the other. The first pair of amicable numbers is 220 and 284.

composite numbers Positive integers that have more than two factors; for example 4 is a composite as its factors are 1, 2 and 4.

counter example An example which shows that a statement is not true.

deficient numbers Numbers whose proper divisors sum to less than the number; for example, 8 is deficient as $1 + 2 \ + 4 < 8$.

denominator The number below the vinculum in a fraction. If the fraction is $\frac{a}{b}$ where a and b are both integers, then b is the denominator.

direct proof A straightforward way of showing that a given statement is true by using established facts.

double or twin primes Double or twin primes differ by two; for example, three and five are twin primes

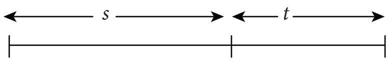
even numbers Integers which are divisible by 2. They are in the form $n = 2k$, where k is an integer.

The even numbers $= \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
$= \{n\!: n = 2k, k \in \boldsymbol{Z}\}$.

fraction A number of the form $\frac{a}{b}$, where a and b are both integers.

Golden Ratio A surd in the form $\frac{1+\sqrt{5}}{2}$. It is the ratio where

$$\frac{s+t}{s} = \frac{s}{t} = \frac{1+\sqrt{5}}{2} = 1.618... =$$ where $s > t$.



infinite continued fractions Fractions of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$
$\vdots$, where $a_0 \in \boldsymbol{Z}$ and $a_1, a_2, \ldots$ are positive integers.

integers The set of integers, $\boldsymbol{Z}$, contains the natural numbers, zero and the negative integers, $\boldsymbol{Z}^-$. So the set is $\boldsymbol{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

irrational number Real numbers which cannot be written in the form $\frac{a}{b}$ where $a, b \in \boldsymbol{Z}$, $b \neq 0$. Examples of such numbers are $\sqrt{2}$, $-3\sqrt{5}$, $\log_2(5)$, e and π.

law A mathematical law is a general statement that can be proven for all elements in the system. Closure law: A set S is closed under an operation $*$ if and only if, for all a and b in S, $a * b$ is also in S. In set notation: S is closed under $* \Leftrightarrow a * b \in S \ \forall a, b \in S$.

Mersenne primes Prime numbers that can be written in the form $2^p - 1$, where p is prime; for example, three is a Mersenne prime as $2^2 - 1 = 3$.

natural numbers These are the counting numbers or positive integers. They have the symbol $\boldsymbol{N}$. So the set is $\boldsymbol{N} = \{1, 2, 3, 4, 5, \ldots\}$.

numerator The number above the vinculum in a fraction. If the fraction is $\frac{a}{b}$, where a and b are both integers, then a is the numerator.

odd numbers Integers which are not divisible by 2. Odd numbers are of the form $n = 2k + 1$, where k is an integer. The odd numbers
$= \{\ldots -3, -1, 1, 3, \ldots\}$
$= \{ n\!: n = 2k + 1, k \in \boldsymbol{Z}\}$.

perfect numbers Positive integers that are equal to the sum of their positive proper divisors; for example, 6 is a perfect number as $1 + 2 + 3 = 6$.

prime numbers Positive integers which have exactly two factors, one and themselves. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, …

Proof by Contradiction This is a proof whereby to prove that a statement A is true, assume A is false or A' is true. Show that A' leads to B and B contradicts A'. Hence A' is false and A is true.

proof by contraposition This is a proof which is often used when a direct proof is too difficult. Rather than proving 'if p then q', 'if not q then not p' is proved.

proof by mathematical induction **The principle of mathematical induction.** Let there be associated with each positive integer n, a proposition $P(n)$.

If 1. $P(1)$ is true, and

2. for all k, $P(k)$ is true implies $P(k+1)$ is true,

then $P(n)$ is true for all positive integers n.

proper divisors These are the factors of the number excluding the number itself; for example, the proper divisors of 6 are 1, 2 and 3.

QED QED is put at the end of a proof. It is an abbreviation of the Latin words *quod erat demonstrandum* which means 'which had to be demonstrated'.

rational numbers The set of rational numbers, $\boldsymbol{Q}$ includes any number that can be written in the form $\frac{a}{b}$, where a and b are both integers. The set of integers $\boldsymbol{Z}$, terminating decimals and recurring decimals make up the set of rational numbers.

real numbers The set of real numbers, $\boldsymbol{R}$ is made up of the set of irrational numbers and the set of rational numbers. It is the universal set, where $\boldsymbol{N} \subset \boldsymbol{Z} \subset \boldsymbol{Q} \subset \boldsymbol{R}$ and {irrational numbers, $\boldsymbol{Q}'$} $\subset \boldsymbol{R}$.

recurring decimals Decimals which have digits with a repeated pattern after the decimal place; for example,

$$\tfrac{1}{7} = 0.142857142857142857\ldots = 0.\overline{142857}$$

$$= 0.\dot{1}4285\dot{7}.$$

sieve of Eratosthenes A method for finding prime numbers up to a particular number.

surd Irrational numbers that must be expressed in the form $\sqrt[a]{n}$, where $n \in \boldsymbol{N}$ and $a \in \boldsymbol{Q}$.

terminating decimals Decimals which have a finite set of digits after the decimal place; for example, 0.25.

transcendental number Irrational numbers which are not surds; for example, e and π.

vinculum A vinculum (bar) is the line above the denominator in a fraction.

Chapter 8

addition of matrices The addition of two matrices that have the same dimension, e.g. $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

column matrix A column matrix has only one column, e.g. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

conformable matrices Matrices **A** and **B** are said to be conformable if and only if **A** has the same number of columns as **B** has rows. The matrix product **AB** is only defined if the matrices are conformable.

determinant of a 2 × 2 matrix For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the quantity $ad - bc$ is a number called the determinant.

diagonal matrix A square matrix with all the non-diagonal elements zero.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

dimension (or size) The size (dimension or order) of a matrix is always given with the **number of rows × the number of columns,** e.g. 4×3

elements (entries) of a matrix Each number in a matrix is called an element of the matrix.

idempotent matrix An idempotent matrix **A** is such that $\mathbf{A}^2 = \mathbf{A}$.

identity matrix An identity matrix $\mathbf{I}_n$ is a diagonal matrix with 0s everywhere except the diagonal which has 1s.

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inverse matrix If it exists, the matrix $\mathbf{A}^{-1}$ whose product in either order with the square matrix $\mathbf{A}$ gives the identity: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

leading diagonal The elements in the diagonal of a square matrix from the top left corner to the bottom right corner.

matrix (matrices) A matrix (plural = matrices) is a rectangular array of elements shown in rows and columns enclosed by brackets. The size (or **dimension** or **order**) of a matrix is stated with the number of rows first. The values of a matrix **A** are indexed using the row and column numbers as subscripts on the lower case italic letter of the matrix name.

matrix equation An equation involving matrices, e.g. $\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ or $\mathbf{AX} = \mathbf{B}$.

matrix multiplication The product of two matrices **A** and **B** exists if and only if **A** has the same number of columns as **B** has rows. If **A** is an $n \times m$ matrix $[a_{ij}]$ and **B** is an $m \times p$ matrix $[b_{ij}]$, then the product matrix **C** is an $n \times p$ matrix $[c_{ij}]$.

multiplicative inverse For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse is the matrix $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

nilpotent matrix A nilpotent matrix **B** of order n is such that $\mathbf{B}^2 = 0_n$.

non-singular matrix A matrix for which an inverse exists. Also called an invertible matrix.

row matrix A row matrix has only one row, e.g. $\begin{bmatrix} -2 & 3 \end{bmatrix}$.

scalar multiplication Multiplication by a scalar quantity, e.g. $r[a_{ij}] = [ra_{ij}]$.

singular matrix A (square) matrix with no inverse, e.g. $\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$.

square matrix A square matrix that has the same number of rows and columns, e.g.
$\begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$.

zero (null) matrix A matrix with all elements zero. Sometimes called the null matrix, 0_n.

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Chapter 9

Angle sum and difference identities The following identities apply $\forall A$ and B.

$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

$\tan(A + B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$

$\tan(A - B) = \dfrac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$

complementary trig functions The following identities apply $\forall\theta$.

$\sin\left(\dfrac{\pi}{2} \pm \theta\right) = \cos(\theta)$

$\cos\left(\dfrac{\pi}{2} \pm \theta\right) = \pm\sin(\theta)$

$\tan\left(\dfrac{\pi}{2} \pm \theta\right) = \pm\cot(\theta)$

Also

$\sin\left(\dfrac{3\pi}{2} \pm \theta\right) = -\cos(\theta)$

$\cos\left(\dfrac{3\pi}{2} \pm \theta\right) = \pm\sin(\theta)$

$\tan\left(\dfrac{3\pi}{2} \pm \theta\right) = \pm\cot(\theta)$

double angle formulas The following identities apply $\forall A$.

$\sin(2A) = 2\sin(A)\cos(A)$

$$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A) \\ &= 2\cos^2(A) - 1 \\ &= 1 - 2\sin^2(A)\end{aligned}$$

$\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

exact values Values that relate to the angles 30°, 45°, 60° in degrees and $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ in radians.

products as sums and differences

$\cos(A)\cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

$\sin(A)\cos(B) = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos(A)\sin(B) = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

Pythagorean identities The Pythagorean identities are:

$\cos^2(A) + \sin^2(A) = 1$

$\tan^2(A) + 1 = \sec^2(A)$

$\cot^2(A) + 1 = \operatorname{cosec}^2(A)$

where A is any angle, measured in either degrees or radians.

reciprocal functions The secant of θ is the reciprocal of the cosine, so $\sec(\theta) = \frac{1}{x} = \frac{1}{\cos(\theta)}$.

The cosecant of θ is the reciprocal of the sine, so $\operatorname{cosec}(\theta) = \frac{1}{y} = \frac{1}{\sin(\theta)}$.

The cotangent of θ is the reciprocal of the tangent, so $\cot(\theta) = \frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)}$.

sums to products

$\sin(A) + \sin(B) = 2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$

$\sin(A) - \sin(B) = 2\cos\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$

$\cos(A) + \cos(B) = 2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$

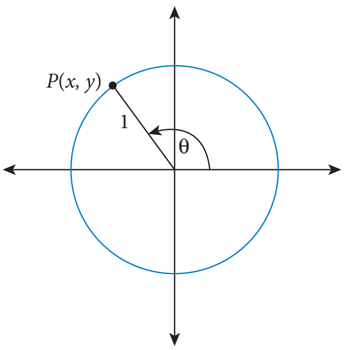
$\cos(A) - \cos(B) = -2\sin\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$

symmetry of the unit circle Angles in the 2nd, 3rd and 4th quadrants that match the angles in the 1st quadrant of the unit circle.

These angles are expressed as $\pi - \theta$, $\pi + \theta$, $2\pi - \theta$ for the 2nd, 3rd and 4th quadrants respectively.

trigonometric identities The triple bar symbol '≡' is often used in trigonometric identities, meaning that the statement is true for every value of the unknown. For example, you could write $\cos^2(\theta) + \sin^2(\theta) \equiv 1$.

trigonometric functions



Angles are measured anticlockwise from the positive x-axis. For a point $P(x, y)$ at an angle θ on the unit circle, we define each trigonometric ratio as below.

The sine of θ to be the y-coordinate of the point P, so $\sin(\theta) = y$.

The cosine of θ to be the x-coordinate of the point P, so $\cos(\theta) = x$.

The tangent of θ is the gradient of the line segment OP, so $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$.

Chapter 10

Argand diagram (or plane) A two-dimensional plane similar to the Cartesian plane, with a horizontal axis denoted by x or $\text{Re}(z)$ and vertical axis denoted by y or $\text{Im}(z)$.

complex (number) A number that can be written in the form $a + bi$, where a and b are real numbers.

complex conjugate This is denoted by $\bar{z}$, where $\bar{z} = a - ib$ for the complex number $z = a + ib$ (where a and b are real numbers).

imaginary (number) A number in the form bi, where b is real and $i = \sqrt{-1}$.

modulus The length of the vector $z = x + yi$, denoted by mod z or $|z|$ where $|z| = \sqrt{x^2 + y^2}$.

real (number) A number that has a position and can be plotted on the real number line.

realising the denominator When dividing two complex numbers, multiplying by the complex conjugate of the denominator.

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Chapter 11

angle of inclination The angle made by a line with the x-axis.

composition The combined transformation obtained by applying one transformation after another. $\mathbf{S} \circ \mathbf{T}$ is the result of applying the transformation **T** and then the transformation **S**.

dilation A dilation of the factor λ_1 in the x-direction and factor λ_2 in the y-direction changes coordinates so that points $P(x, y)$ change to $P'(\lambda_1 x, \lambda_2 y)$.

identity For a particular operation, the element that leaves all other elements unchanged. For transformations , the transformation I: $(x, y) \to (x, y)$.

image The result of applying a transformation to a point or set of points.

inverse If it exists, the inverse of a transformation **T** is the transformation $\mathbf{T}^{-1}$ such that $\mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{T}^{-1} \circ \mathbf{T} = \mathbf{I}$, where **I** is the identity transformation.

linear transformation A transformation of the plane such that points $P(x, y)$ change to $P'(ax + by, cx + dy)$, where a, b, c, and d are constants.

matrix A rectangular array of elements shown in rows and columns enclosed by brackets.

object A point or set of points to which a transformation is applied.

reflection A transformation that swaps all points to the other side of a line like a mirror.

rotation A transformation that turns all points through the same angle around a point, usually the origin.

slope The steepness of a line, given by $m = \dfrac{\Delta y}{\Delta x}$.

transformation For plane figures, operations that change the positions of points of the object, such as rotations and translations.

translation A transformation that moves all points the same distance in the same direction.

vector A quantity with magnitude and direction used to represent translation.

Chapter 12

amplitude The vertical dilation of a sine or cosine graph. The value a in $y = a \sin(x)$ or $y = a \cos(x)$. The value a in similar equations of other trigonometric functions is also sometimes called the amplitude.

cosec (x) $\operatorname{cosec}(x) = \dfrac{1}{\sin(x)}$

cot (x) $\cot(x) = \dfrac{1}{\tan(x)}$

equilibrium The central point of simple harmonic motion.

general solution The complete solution of a trigonometric equation for the real numbers, such as $n\pi + \frac{\pi}{6}$ or $n\pi - \frac{\pi}{6}$ for $n \in \mathbb{Z}$.

period The smallest interval of a trigonometric function before it repeats its values. The function $y = \sin(bx)$ has a period $\dfrac{2\pi}{b}$. For any value of x, $\sin\left(b\left(x + \dfrac{2\pi}{b}\right)\right) = \sin(bx)$.

periodic motion Motion that repeats itself in cycles.

phase shift The horizontal translation of a trigonometric graph. The value c in trigonometric functions such as $y = \sin(x + c)$. For $c > 0$ the graph moves to the left and for $c < 0$ to the right.

reciprocal functions The functions cosecant (cosec), secant (sec) and cotangent (cot) that are reciprocals of sine, cosine and tangent respectively, so $\sec(x) = \dfrac{1}{\cos(x)}$.

sec (x) $\sec(x) = \dfrac{1}{\cos(x)}$

simple harmonic motion (SHM) A particle in simple harmonic motion moves backwards and forwards (oscillates) in a regular way about a central point. Its equation of motion is a sinusoidal function.

sum and difference formulas The trigonometric expansions of $\sin(x \pm y)$, $\cos(x \pm y)$ and $\tan(x \pm y)$, like $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.

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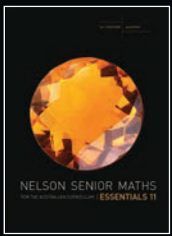

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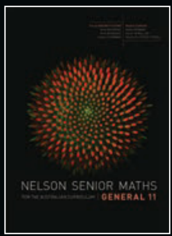

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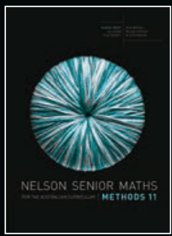

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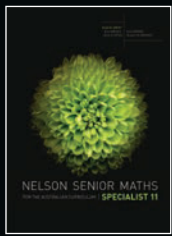

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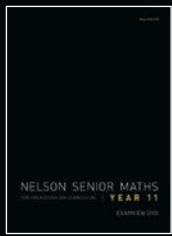

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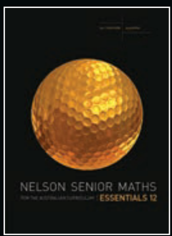

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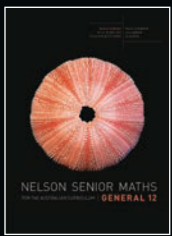

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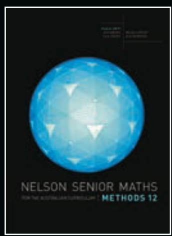

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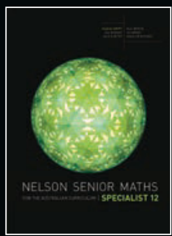

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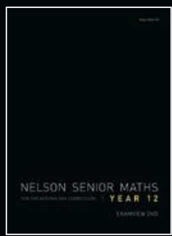

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