

MATHEMATICS SPECIALIST

YEAR 11 ATAR COURSE – UNITS 1 & 2

REVISED EDITION



Greg Hine & Neil McNab



WACE Study Guide

MATHEMATICS SPECIALIST

YR 11 ATAR COURSE

Gregory Hine and Neil McNab

First published 2014
Second Edition published 2020

© ACADEMIC ASSOCIATES

Apart from any fair dealing for the purposes of private study, research, criticism or review, as permitted under the Copyright Act, no part of this publication may be reproduced by any process without written permission of the publisher.

National Library of Australia

ISBN 978-1-87691-886-6

Cover design by Charmaine Cave, Cave Design

Typeset by Midland Typesetters

Printed in Singapore

Acknowledgements

The W.A. School Curriculum and Standards Authority for extracts from the Mathematics Specialist syllabus.

My sincere gratitude is expressed to my wife, Cam, and my children, Mikayla and Jonathon. Your ongoing love and support is breathtaking.

CONTENTS

	Foreword	iv
1.	Counting Techniques	1
2.	Vectors I	14
3.	Vectors II	48
4.	Geometry	82
5.	Trigonometry	98
6.	Matrices	118
7.	Complex Numbers	138
8.	Mathematical Proof and Reasoning	148
	TRIAL TESTS	160
	ANSWERS	188
	SOLUTIONS TO TRIAL TESTS	202

FOREWORD

The purpose of this text is to assist Year 11 students with their preparation for tests and examinations in the new Specialist Mathematics course for Western Australia.

The *Syllabus Checklist* indicates to students which skills they must have acquired and the objectives they need to meet under each of the major headings of the course.

The *Worked Examples* are presented in a detailed manner, with brief notes and explanations being used to amplify the understanding for the particular question. Some of these worked examples could be used in the written notes that students are permitted to take into an examination.

The *Problems to Solve* section in each chapter provides students with a broad range of questions without the repetitive nature of problems usually associated with a course textbook.

The *Trial Tests* are an additional component to this text, and these allow students to familiarise themselves with examination questions. Suggested times are given for these tests, and students should be encouraged to adhere to these times to prepare properly for final examinations. Fully worked solutions are provided for students to receive immediate, accurate and useful feedback on their performance.

About the Units

In *Unit 1*, students examine three topics that complement the content of the course *Mathematical Methods*. The proficiency strand *Reasoning*, of the F-10 curriculum is continued explicitly in the topic *Geometry* through a discussion of developing mathematical arguments. This topic also provides the opportunity to summarise and extend students' studies in Euclidean Geometry, knowledge which is useful in the later study of topics such as vectors and complex numbers. The topic *Combinatorics* provides techniques that are very useful in many areas of mathematics, including probability and algebra. The topic *Vectors in the Plane* provides new perspectives on working with two-dimensional space, and serves as an introduction to techniques which can be extended to three-dimensional space in Unit 3. These three topics considerably broaden students' mathematical experience and therefore begin an awakening to the breadth and utility of the subject.

In *Unit 2*, students are introduced to three topics: *Trigonometry*, *Matrices*, and *Real and Complex Numbers*. The topic *Trigonometry* contains techniques that are used in other topics in both this unit and Units 3 and 4. *Matrices* provides new perspectives for working with two-dimensional space, while *Real and Complex Numbers* provides a continuation of the study of numbers. All of these topics develop students' ability to construct mathematical arguments. The technique of proof by the principle of mathematical induction is introduced in this unit.

Dr Gregory Hine, Ph.D.

Syllabus Checklist

By the end of this chapter, you should be able to:

Permutations (ordered arrangements)

- solve problems involving permutations
- use the multiplication and addition principles
- use factorial notation and ${}^n P_r$
- solve permutation problems involving restrictions, with or without repeated objects

The inclusion-exclusion principle for the union of two sets and three sets

- determine and use the formulas for finding the number of elements in the union of two and the union of three sets

The pigeon-hole principle

- solve problems and prove results using the pigeon-hole principle

Combinations (unordered selections)

- solve problems involving combinations
- use the notation $\binom{n}{r}$ or ${}^n C_r$
- derive and use associated simple identities associated with Pascal's triangle

FORMULAE AND DEFINITIONS

Combinations

A **combination** is a selection of objects. The **order** of the objects in a combination is not important.

The different combinations of three letters x, y, z taken two at a time are xy, xz, yz . This is denoted as ${}^3 C_2$, and ${}^3 C_2 = 3$.

The number of combinations of r objects chosen from a group of n different objects is given by:

$${}^n C_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Note: $n! = n(n-1)(n-2) \dots \times 3 \times 2 \times 1$

Permutations

Permutations are the different arrangements that can be made with a given number of things taking some or all of them at a time. **Order** is important with permutations.

Multiplication Principle

If there are a ways of performing the first operation, b ways of performing the second operation and so on, then the total number of ways of performing all operations is $a \times b \times \dots$

Addition Principle

If there are a ways of performing event A, b ways of performing event B and if A and B are mutually exclusive then A or B can be performed in $a + b$ ways.

The Binomial Theorem

The Binomial Theorem provides a method for expressing a binomial quantity raised to a power in the form of a series.

By expanding the parentheses in the examples below we obtain:

$$\begin{aligned}(a + b)^0 &= 1 \\(a + b)^1 &= 1a + 1b \\(a + b)^2 &= 1a^2 + 2ab + 1b^2 \\(a + b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3 \\(a + b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4\end{aligned}$$

Arranging the coefficients of the expanded expressions can give us Pascal's Triangle:

$$\begin{aligned}(a + b)^0 &= && 1 \\(a + b)^1 &= & 1 & 1 \\(a + b)^2 &= & 1 & 2 & 1 \\(a + b)^3 &= & 1 & 3 & 3 & 1 \\(a + b)^4 &= & 1 & 4 & 6 & 4 & 1\end{aligned}$$

Using Pascal's Triangle the coefficients of terms in the expansion of a binomial raised to a power can be written consecutively. Binomial coefficients are written in the form $\binom{n}{r}$, where n and r are positive integers ($n \geq r$), and where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

In this form, Pascal's Triangle can be rewritten as:

$$\begin{aligned}(a + b)^0 &= && \binom{0}{0} \\(a + b)^1 &= & \binom{1}{0} & \binom{1}{1} \\(a + b)^2 &= & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\(a + b)^3 &= & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\(a + b)^4 &= & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4}\end{aligned}$$

The pattern above can be generalised to the Binomial Expansion Formula:

$$(a + b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + \binom{n}{n} b^n$$

Pigeon-hole Principle

If more than n objects are to be placed into m boxes with $n > m$ boxes, then at least one box must contain more than one object.

For instance, if 10 pigeons need to roost in 9 pigeon-holes, then at least one pigeon-hole must contain 2 or more pigeons.

N.B. It might seem tempting to say (from the above example) that exactly one pigeon-hole will contain exactly 2 pigeons. This is not necessarily true, as we must consider *all* possible distributions of 10 pigeons into 9 pigeonholes (10 in 1 pigeon-hole, 0 in the other 9; 8 in 1, 2 in 1, 0 in the other 8 etc.). Keep in mind that the pigeon-hole principle states that for each and every distribution of pigeons, there must be at least one pigeon-hole with 2 or more pigeons in it.

Said another way, if each of the n objects is to be assigned one of k labels, then there must be at least $\frac{n}{k}$ objects with the same label.

Inclusion-Exclusion Principle

When using Venn Diagrams it is useful to know how many elements are in a particular region. If one region has unknown elements (and all other regions have a known number of elements) then the inclusion-exclusion principle can be used. The rules for 2-set and 3-set problems can be re-arranged according to whichever region requires solving. Note that the Universal Set ($n(U)$) and the 'non-participants' set ($n(A \cup B \cup C)$) do not feature in the following rules, but they can be appropriately included to solve problems.

For 2-set problems:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

For 3-set problems:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Worked Examples

- 1.1 How many different teams of 6 players can be chosen from 12 players?

$$\begin{array}{l} \text{total number} \\ \text{of players} \end{array} \rightarrow {}^{12}C_6 = \frac{12!}{(12-6)!6!}$$

$$\begin{array}{l} \text{number of} \\ \text{players chosen} \end{array} \rightarrow = 924 \text{ different teams}$$

- 1.2 In how many ways can a committee of 5 be selected from 7 men and 4 women if the committee must consist of 2 men and 3 women?

$$\begin{array}{l} \text{7 men in total} \rightarrow \binom{7}{2} \times \binom{4}{3} \leftarrow \text{4 women in total} \\ \text{2 men selected} \rightarrow \text{and} \leftarrow \text{3 women selected} \end{array} = 84 \text{ ways}$$

- 1.3 Including the empty set and the set itself, how many subsets can be made using the elements of set $X = \{x, y, z\}$?

Subset has no elements: 3C_0

Subset has 1 element: 3C_1

Subset has 2 elements: 3C_2

Subset has 3 elements: 3C_3

$$\text{Total} = {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3$$

$$= 1 + 3 + 3 + 1$$

$$= 8 \text{ subsets}$$

- 1.4 In how many ways can 5 members be selected to form a committee from 10 people if:

- there are no restrictions.
- two particular members must be included.
- two particular members, Jack and Jill, must not both be on the committee.

(a) $\binom{10}{5} = 252$ ways

(b) $\binom{2}{2} \binom{8}{3} = 56$ ways

2 particular members both included and *8 others 3 required*

$$\begin{aligned}
 & \text{(c)} \quad \begin{array}{ccc} \text{Jack} & \text{Jill not} & \\ \text{included} & \text{included} & \text{Others} \\ \binom{1}{1} & \binom{1}{0} & \binom{8}{4} \end{array} \quad \text{or} \\
 & \begin{array}{ccc} \text{Jack not} & \text{Jill} & \\ \text{included} & \text{included} & \text{Others} \\ \binom{1}{0} & \binom{1}{1} & \binom{8}{4} \end{array} \quad \text{or} \\
 & \begin{array}{ccc} \text{Jack not} & \text{Jill not} & \\ \text{included} & \text{included} & \text{Others} \\ \binom{1}{0} & \binom{1}{0} & \binom{8}{5} \end{array} \\
 & = 70 + 70 + 56 \\
 & = 196 \text{ ways}
 \end{aligned}$$

1.5 John is required to answer 10 of the 12 questions on the test paper. How many different choices does he have if:

- (a) there are no restrictions?
- (b) he must answer the first 3 questions?
- (c) he must answer at least 3 of the first 5 questions?

(a) $\binom{12}{10} = 66$ ways

(b) $\binom{3}{3}\binom{9}{7} = 36$ ways

$$\begin{aligned}
 & \text{(c)} \quad \begin{array}{cc} \text{3 of the} & \text{4 of the} \\ \text{1st 5} & \text{1st 5} \\ \binom{5}{3} & \binom{5}{4} \end{array} \begin{array}{c} \text{Others} \\ \binom{7}{7} \end{array} + \begin{array}{cc} \text{4 of the} & \text{5 of the} \\ \text{1st 5} & \text{1st 5} \\ \binom{5}{4} & \binom{5}{5} \end{array} \begin{array}{c} \text{Others} \\ \binom{7}{6} \end{array} + \begin{array}{cc} \text{5 of the} & \\ \text{1st 5} & \\ \binom{5}{5} & \binom{7}{5} \end{array} \\
 & = 10 + 35 + 21 \\
 & = 66 \text{ ways}
 \end{aligned}$$

1.6 How many 8 letter words can be formed using the letters of the word TRIANGLE if:

- (a) there are no restrictions?
- (b) the word must begin with a T and end in a G?
- (c) the vowels must be together?
- (d) the vowels must be separated?

(a) $\begin{array}{|c|c|c|c|c|c|c|c|} \hline 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline \end{array} = 8!$
 Eight objects can be permuted $8!$ ways = 40 320 ways

(b) $\begin{array}{|c|c|c|c|c|c|c|c|} \hline \text{T} & & & & & & & \text{G} \\ \hline 1 & 6 & 5 & 4 & 3 & 2 & 1 & 1 \\ \hline \end{array} = 6!$
 $= 720$ ways
1 T *6 remaining letters* *1 G*

(c) $\begin{array}{|c|c|c|c|c|c|} \hline \overbrace{3 \quad 2 \quad 1}^1 & & & & & \\ \hline 3 & 2 & 1 & & & & & & & \\ \hline \end{array} = 3! \times 6!$
 $= 4320$ ways
3 vowels together arranged 3! ways *6 objects are to be arranged 6! ways*

(d) If there are 4320 arrangements out of the 40 320 with the vowels together then the remaining arrangements $(40\,320 - 4320) = 36\,000$ must have the vowels **separated**.

- 1.7 How many odd numbers greater than 50 000 can be formed using the digits 2, 3, 4, 5 and 8 if each digit can only be used once?

For an odd number – the last digit can only be a 3 or a 5.

For a number > 50 000 – the first digit can be either a 5 or and 8

Possibilities

Ends in a 3 – begins in a 5 or 8

$$\begin{array}{|c|c|c|c|c|c|} \hline 5|8 & & & & & 3 \\ \hline 2 & 3 & 2 & 1 & 1 & \\ \hline \end{array} = 12 \text{ ways}$$

Ends in a 5 – Can only begin in an 8

$$\begin{array}{|c|c|c|c|c|c|} \hline 8 & & & & & 5 \\ \hline 1 & 3 & 2 & 1 & 1 & \\ \hline \end{array} = 6 \text{ ways}$$

$$\begin{aligned} \text{Total number} &= 12 + 6 \\ &= 18 \text{ ways} \end{aligned}$$

- 1.8 In how many ways can 4 people chosen from 10 people be arranged?

This is a combination → select 4, and then permutation → arrange

$$\text{Number of selections} = \binom{10}{4}$$

$$\text{Number of arrangement} = \binom{10}{4} \times 4!$$

$$= 5040$$

- 1.9 A shelf contains 7 encyclopedias, 5 magazines and 3 novels. How many ways can 2 encyclopedias, 3 magazines and 2 novels be selected and then arranged on a display shelf if the encyclopedias must be the end books?

$$\begin{aligned} \text{Number of selections:} & \quad \text{E} \quad \text{M} \quad \text{N} \\ & \binom{7}{2} \times \binom{5}{3} \times \binom{3}{2} \end{aligned}$$

$$\begin{aligned} \text{Number of arrangements:} & \binom{7}{2} \binom{5}{3} \binom{3}{2} \times \begin{array}{|c|c|c|c|c|c|c|} \hline \text{E} & & & & & & \text{E} \\ \hline 2 & 5 & 4 & 3 & 2 & 1 & 1 \\ \hline \end{array} \\ & = 151200 \end{aligned}$$

- 1.10 From a group of 1000 people, show that at least 3 people were born on the same day of the year.

Assign a label to each person for every day of a 365-day year (these days would act as ‘birth-days’). There are at least 365 different labels, and all 1000 people have these labels assigned.

Using the formula $\frac{n}{k}$ we find that $\frac{1000}{365} \approx 2.74$ people have the same label.

Therefore, at least 3 people share the same birthday.

- 1.11 Six integers are chosen from the set: {1, 2, 3, ..., 9, 10}. Prove that 2 of the chosen integers must have an odd sum.

Assign a label to each of the 10 integers as ‘even’ or ‘odd’. As there are at most 5 numbers with each label there must be 2 integers with an ‘opposite’ label. These 2 integers (one an odd, one an even) must sum to give an odd value.

- 1.12 60 people are enrolled in a Calculus class, and 40 people are enrolled in a Physics class. If 25 people are enrolled in both Calculus and Physics, how many people are enrolled in Calculus or Physics?

Looking at the situation we can see that there are two distinct sets of elements, i.e. Calculus students (C) and Physics students (P).

Substituting into the Inclusion-Exclusion rule we get:

$$\begin{aligned}n(C \cup P) &= n(C) + n(P) - n(C \cap P) \\ &= 60 + 40 - 25\end{aligned}$$

$$\therefore n(C \cup P) = 75 \text{ students}$$

- 1.13 Using the coefficients of Pascal's Triangle, determine a fully simplified expansion for the following:

(a) $(2.001)^3$

(b) $(x + 2y)^4$

(a) $(2.001)^3$
 $= (2 + 0.001)^3$
 $= \binom{3}{0}(2^3)0.001^0 + \binom{3}{1}(2^2)0.001^1 + \binom{3}{2}(2^1)0.001^2 + \binom{3}{3}(2^0)0.001^3$
 $= 8 + 0.012 + 0.000006 + 0.000000001$
 $= 8.012006001$

(b) $(x + 2y)^4$
 $= \binom{4}{0}(x^4)(2y)^0 + \binom{4}{1}(x^3)(2y)^1 + \binom{4}{2}(x^2)(2y)^2 + \binom{4}{3}(x^1)(2y)^3 + \binom{4}{4}(x^0)(2y)^4$
 $= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

PROBLEMS TO SOLVE

CHAPTER 1: COUNTING TECHNIQUES

1. How many 3 or 4 digit numbers can be formed using the digits 2, 3, 4, 7 and 8 without repetition?

2. A committee of 5 is to be chosen from 4 men and 7 women. In how many ways is this possible if there must be more men than women on the committee?

3. How many 4 digit numbers greater than 7000 can be made using the digits 3, 6, 7 and 8 without repetition:
 - (a) if there are no restrictions?

 - (b) if the number formed is even?

4. A committee of 4 is to be chosen from 10 athletes and 4 coaches. How many different committees can be formed if:
 - (a) 3 athletes must be selected?

 - (b) no coaches are selected?

 - (c) at least 2 coaches must be included on the committee?

5. In planning for a long trip a driver has only room to take 5 compact discs. If his main collection consists of 3 Rock and Roll, 2 Country and Western and 4 Pop Music compact discs, in how many ways can he choose:
 - (a) 1 Rock and Roll, 2 Country and Western and 2 Pop Music CDs?

 - (b) at least 2 Pop Music CDs?

 - (c) at least one CD of each type?

6. An exam consists of 2 sections A and B. Section A contains 6 questions and Section B 4 questions. A student must Answer 6 questions in total. In how many ways can the student complete the exam if he answers at least 3 questions from:

(a) Section A?

(b) Section B?

7. Two boys and three girls are chosen to speak at an assembly from five boys and seven girls. In how many different orders can they speak?

8. A bookshelf contains 3 novels, 4 magazines and 2 encyclopedias. In how many ways can they be arranged if:

(a) there are no restrictions?

(b) the encyclopedias are together?

(c) the encyclopedias are all to the left, magazines all in the middle and all the novels are to the right?

(d) each type must be together?

9. Words are formed using the letters of the word 'COMPUTERS'. How many different 5 letter words are possible if:

(a) there are no restrictions?

(b) 2 vowels must be used?

(c) at least 2 vowels must be used?

(d) the word begins and ends with a vowel?

(e) the word begins or ends with a vowel?

10. License plates consist of a digit followed by 3 letters and then 3 digits.

How many plates are possible if:

(a) there are no restrictions?

(b) repetition of letters and digits is not permitted?

(c) the letters and digits must be in ascending consecutive order?

11. A committee of 4 is to be chosen from 5 women and 7 men. How many committees are possible if:

(a) there are no restrictions?

(b) it must consist of 2 men and 2 women?

(c) it must have a least 1 man?

(d) two men refuse to be on the same committee?
(they are prepared to serve separately)

12. How many permutations using all the letters of the word 'FORCE' exist if:

(a) there are no restrictions?

(b) R immediately follows O?

(c) C and E are adjacent?

(d) C and E are not adjacent?

13. Five members of the student council are to be chosen to give a speech at an assembly. The student council consists of three students from Year 12, four from Year 11 and six from Years 8–10. How many different combinations of speakers can be chosen if:

(a) there are no restrictions?

(b) the number of students chosen in Year 11 and 12 must be greater than in Years 8–10?

If two Year 12 students are chosen together with one Year 11 and two Year 8–10 students, in how many ways can the order of speakers be arranged if:

(c) there are no restrictions?

(d) the two Year 12 students must start?

(e) they cannot start or finish with a Year 8–10 student?

14. The dots shown are to be used as vertices for a triangle.

A B C D

• • • •

E F G H I

• • • • •

How many different triangles are possible if:

(a) there is no restriction?

(b) vertex A must be included?

15. How many numbers greater than 50 can be made using the digits 2, 3, 4, 5 and 7 if repetition is not permitted?

16. The school relay team has to be selected from the top ten athletes. For the relay 6 athletes must be chosen.
- (a) In how many ways can the six athletes be chosen?
-
- (b) If two athletes, John and Ross must be included, how many groups of six are possible?
-
- (c) If Jack and Paul are in the top 10 athletes but cannot be selected together, in how many ways can the 6 students be selected?
-
- (d) The final six are chosen: John, Ross, Jack, Simon, Tzar and Braden. In how many ways can the relay be arranged if Ross must immediately follow John?
-
17. A committee of 5 students organising the school ball is selected from 6 male and 5 female students. How many different committees are possible if:
- (a) there are no restrictions?
-
- (b) the committee consists of all male students?
-
- (c) twins (Jason and Sarah) cannot be on the committee together?
-
-
18. There are 27 people in a room. Explain why there are at least 2 people whose first name starts with the same letter of the alphabet.
-
-
-
19. From the set of natural numbers, 8 positive numbers are chosen at random. Explain why 2 of these numbers must differ by a multiple of 7.
-
-
-

20. From a group of 2 000 people, prove that at least 6 people were born on the same day of the year.

21. Tom gets dressed every morning in the dark so as not to wake his wife. His sock drawer contains 8 pairs of blue socks and 10 pairs of black socks (i.e. a total of $16 \text{ blue} + 20 \text{ black} = 36$ socks). How many socks must Tom take out of the drawer to be sure of having a matching pair of socks?

22. In a Year 5 class all 22 students have a cat or dog (or both of these animals) as a pet. There are 12 students who have a cat and 15 students who own a dog. How many students have both animals as pets?

23. A group of 48 people are surveyed according to where they send their children to school. 25 people send their children to School A, 34 send their children to School B, and 4 people do not send their children to either School A or School B. How many people send their children to both School A and School B?

24. Seventy people were randomly chosen in a shopping centre and asked which of television, radio, or recorded music they played at home during a week. Twenty people used all three, 51 played radio, and 42 played music. 32 people played both radio and television, 31 played both radio and music, and 28 played both music and television. How many people played television?

25. One hundred and twenty people were surveyed regarding their relaxation activities. Forty-eight watched TV, fifty-eight exercised and sixty-two read a good book. Twenty-eight people read a book and watched TV, thirty-two read a book and exercised, while eight watched TV and exercised. Fifteen people surveyed did not participate in any of the activities. How many people participated in all three activities?

26. Using the coefficients of Pascal's Triangle, determine a fully simplified expansion for the following:

(a) $(1.05)^3$

(b) $(1.002)^4$

(c) $(0.97)^5$

(d) $(0.999)^6$

(e) $(m + 3n)^3$

(f) $(2a - 3b)^4$

(g) $(x + y)^4 - (x - y)^4$

(h) $(a + 2b)^7 - (a - 2b)^7$

2

VECTORS I

This section is an introduction to vector terminology, representation and methods based on coordinate geometry and trigonometry.

Syllabus Checklist

On the completion of this chapter you should be able to:

Representing vectors in the plane by directed line segments

- examine examples of vectors, including displacement and velocity
- define and use the magnitude and direction of a vector
- represent a scalar multiple of a vector
- use the triangle and parallelogram rules to find the sum and difference of two vectors

INTRODUCTION

A scalar quantity is fully specified by a single number with the appropriate units, e.g. the time taken was 5 minutes, the distance travelled is 12km, the egg has a mass (not weight!!) of 80g.

A vector quantity is multidimensional i.e. it needs two or more numbers to be fully specified, e.g. a force of 10N on a bearing of 120° , a velocity of 60 km/h on a bearing of 180° (i.e. south).

The types of vectors covered in this course are

- Displacement vectors
- Force vectors
- Velocity vectors
- Acceleration vectors

Vectors are interesting as they

- Are all illustrated by directed line segments, i.e. arrows
- All add and subtract in the same way via the triangle of vectors diagram
- Are versatile as they can model many tricky physical situations in elegant ways
- Are used to link up physical situations with the power of algebraic manipulations
- Lead onto huge sections of very powerful maths at the tertiary level.

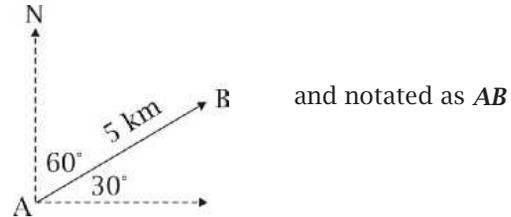
VECTOR NOTATION AND REPRESENTATION

- The way a vector is notated depends on what type of vector it is and where it is written. The following table summarises some of the common vector notations.

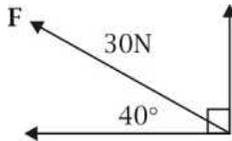
Type	In texts and on computers	By students on paper
Displacement	AB or b	\vec{AB} or \vec{a} or even \tilde{a}
Force	F	\vec{F}
Velocity and acceleration	v and a	\vec{v} and \vec{a}

- As mentioned earlier vectors are represented diagrammatically by directed line segments or in other words, as arrows. Some examples are

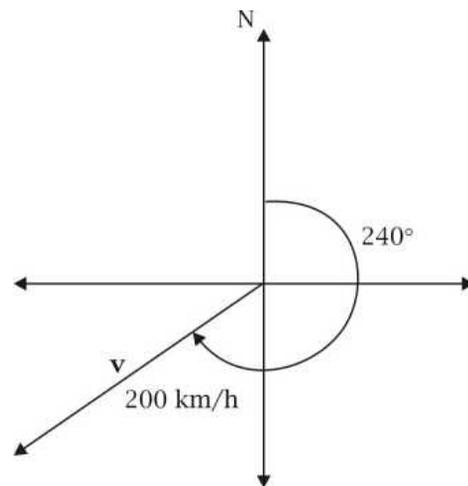
- (i) A 5 km journey from place A to place B on a bearing of 060° is drawn as



- (ii) A force F of 30N at 40° to the horizontal is acting on a block as shown



- (iii) A plane's velocity v is 200 km/h on a bearing of 240°



MAGNITUDE AND DIRECTION

The previous three diagrams showed a displacement vector, a force vector and a velocity vector which were all two dimensional vectors specified in magnitude/direction form.

In two dimensions a vector's **direction** can be specified

- By a diagram showing an angle
- As a bearing angle
- As a polar angle which is defined below

and

A vector's **magnitude** is its size and when represented diagrammatically the length of a vectors arrow diagram can be scaled according to its magnitude

The notation for a vector's magnitude is the same as the absolute value symbol. For example

- For the displacement vector AB , $|AB|$ is the magnitude of AB which is often called the **distance** between A and B .
- For the velocity vector v , the magnitude is $|v|$ which is often called its **speed**.
- For the force vector F , $|F|$ is its magnitude.

In three dimensions vectors can be specified by a magnitude and two angles. However the more practical method of specifying vectors (especially for more than two dimensions) is by its **components** which are defined now.

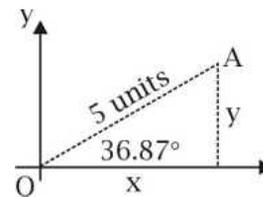
RECTANGULAR OR COMPONENT FORM OF A VECTOR

- All types of vectors can be written in component form but this section will consider **displacement** vectors. In two dimensions the x y Cartesian plane can show bearings by letting the positive y axis point North. Points on the plane can be shown by ordered pairs or as a distance from the origin O and a **polar angle**.
- For any point A on the Cartesian plane the vector OA will have a polar angle.

A vector's **polar angle** θ_p is the angle that the vector's arrow representation makes with the positive x axis. The positive direction for polar angles is anti clockwise and hence the negative direction is clockwise. The usual **domain** for a polar angle is

$$-180^\circ < \theta_p \leq 180^\circ$$

- Consider the point A which is 5 units from the origin and has a polar angle of 36.87° (2dp) shown to the right.



The x component of A is found by

$$\begin{aligned} \cos 36.87^\circ &= \frac{x}{5} \\ x &= 5 \cos 36.87^\circ \\ x &= 4 \text{ units} \end{aligned}$$

The y component of A is found by

$$\begin{aligned} \sin 36.87^\circ &= \frac{y}{5} \\ y &= 5 \sin 36.87^\circ \\ y &= 3 \text{ units} \end{aligned}$$

The vector OA can either be specified in magnitude/direction form or **polar form** as

$$\mathbf{OA} = \langle 5 \text{ units}, 36.87^\circ \rangle$$

Or now in component form or rectangular form as

$$\mathbf{OA} = \langle 4, 3 \rangle \text{ units}$$

The author prefers the triangular brackets to indicate a vector but the alternatives for the rectangular form are

$$\begin{aligned} \mathbf{OA} &= (4, 3) \text{ units} && \text{ordered pair form} \\ &= [4 \ 3] \text{ units} && \text{row matrix form} \\ &= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ units} && \text{column matrix form} \end{aligned}$$

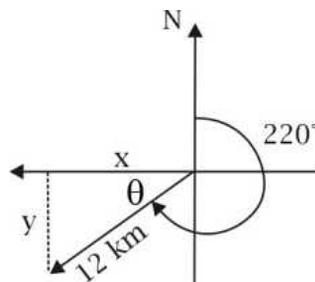
Conversion between rectangular and polar form will be done by most students using first principles, i.e. right angled trigonometry and Pythagoras. A quick sketch or diagram is essential to make sure that you know where the angles go and that you don't miss any negatives.

Sketches of vectors in the x y plane are very useful for giving feedback on whether calculated vectors look OK and hence are most likely correct or most definitely wrong!

Example 1. Change a 12 km journey on a bearing of 220° to rectangular form.

Step 1. Sketch

$$\begin{aligned}\theta &= 270^\circ - 220^\circ \\ &= 50^\circ\end{aligned}$$



Step 2. Forget signs and work out the distances for x and y by

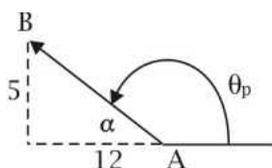
$$\begin{aligned}x &= 12 \cos 50^\circ & y &= 12 \sin 50^\circ \\ &= 7.71 \text{ km} \quad (2\text{dp}) & &= 9.19 \text{ km} \quad (2\text{dp})\end{aligned}$$

Step 3. Finish off, i.e. journey = $\langle -7.71, -9.19 \rangle$ km

Example 2. Change a journey from A(5, -3) km to B(-7, 2) km to magnitude / bearing form.

Step 1. Find $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$
 $= \langle -7, 2 \rangle - \langle 5, -3 \rangle$
 $= \langle -12, 5 \rangle$

Step 2. Sketch (forget negatives)



If you needed it, the Polar angle θ_p for \mathbf{AB} would be $180^\circ - 22.6^\circ = 157.4^\circ$

$$\begin{aligned}|\mathbf{AB}| &= \sqrt{5^2 + 12^2} \\ &= 13 \text{ km}\end{aligned}$$

$$\begin{aligned}\tan \alpha &= \frac{5}{12} \\ \alpha &= 22.6^\circ \quad (1\text{dp}) \\ \text{bearing} &= 270^\circ + 22.6^\circ \\ &= 292.6^\circ\end{aligned}$$

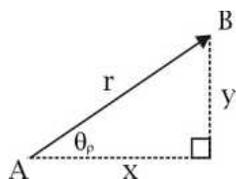
Step 3. Finish off. The journey from A to B was 13 km on a bearing of 292.6° (1dp.)

The conversion between rectangular and polar forms can be conveniently processed on all scientific calculators and most graphics calculators by using the R \rightarrow P and P \rightarrow R functions. For now the conversion formulae are shown below but don't ever forget that sketch!

In general

Let $\mathbf{AB} = \langle x, y \rangle$ R (Rectangular vector)

$= \langle r, \theta_p \rangle$ P (Polar vector)



Note: $r = |\mathbf{AB}|$

If $\mathbf{AB} = \langle x, y \rangle$ R is known then find r and θ_p

by $r = |\mathbf{AB}| = \sqrt{x^2 + y^2}$

$$\theta_p = \tan^{-1} \left(\frac{y}{x} \right)$$

If $\mathbf{AB} = \langle r, \theta_p \rangle$ P is known then find x and y by

$$x = r \cos \theta_p \quad y = r \sin \theta_p$$

POSITION AND DISPLACEMENT VECTORS

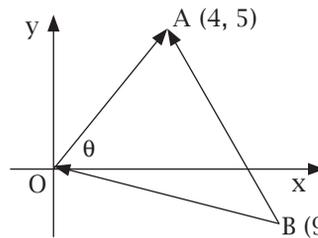
It is useful to consider a displacement, which is a change of position as a journey. When a journey starts at the origin and ends at place A, the journey is represented by a special type of displacement called a **position** vector and is notated by \mathbf{OA} .

Another journey which starts at A and ends at B is represented by the displacement \mathbf{AB} . When the **displacement** \mathbf{AB} is represented by an arrow diagram, A is said to be at the tail of the vector and B will be at the tip of the vector.

Sometimes it is convenient to let both position vectors and displacement vectors be notated by lower case letters. So we may let $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{AB}$ and $\mathbf{c} = \mathbf{OB}$ etc.

Example: The first leg of a yacht race went from the start O to the point A (4, 5)km. The boats then sailed directly to B (9, -7)km. Find the required bearings and the distances travelled to go around the course and back to O.

Step 1. A diagram is helpful



\mathbf{OA} is a position vector.

\mathbf{AB} and \mathbf{BO} are displacement vectors.

Each vector is specified in rectangular and then polar form.

Step 2. Rectangular to Polar

$$\begin{aligned} \mathbf{OA} &= \langle 4, 5 \rangle \quad |\mathbf{OA}| = \sqrt{4^2 + 5^2} \\ &= \sqrt{41} = 6.4 \text{ km (1dp)} \end{aligned}$$

$$\theta_p = \tan^{-1}\left(\frac{5}{4}\right)$$

$$= 51.34^\circ \text{ (2dp)}$$

$$\text{Bearing} = 90^\circ - 51.34^\circ$$

$$= 38.66^\circ$$

Step 3. $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$$\begin{aligned} &= \langle 9, -7 \rangle - \langle 4, 5 \rangle \\ &= \langle 5, -12 \rangle \end{aligned}$$

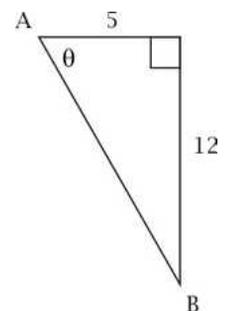
$$|\mathbf{AB}| = \sqrt{5^2 + 12^2} = 13 \text{ km}$$

$$\theta = \tan^{-1}\left(\frac{12}{5}\right)$$

$$\theta = 67.38^\circ \text{ (2dp)}$$

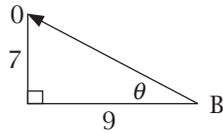
$$\text{Bearing} = 90^\circ + 67.38^\circ$$

$$= 157.38^\circ$$



$$\theta_p \text{ for } \mathbf{AB} \text{ is } -67.38^\circ$$

Step 4.



$$|\mathbf{BO}| = \sqrt{9^2 + 7^2}$$

$$= \sqrt{130} = 11.4 \text{ km (1dp)}$$

$$\theta = \tan^{-1}\left(\frac{7}{9}\right)$$

$$= 37.87^\circ$$

$$\theta_p \text{ for } \mathbf{BO} \text{ is } 180^\circ - 37.87^\circ$$

$$\text{Bearing} = 270^\circ + 37.87^\circ$$

$$= 142.13^\circ$$

$$= 307.87^\circ$$

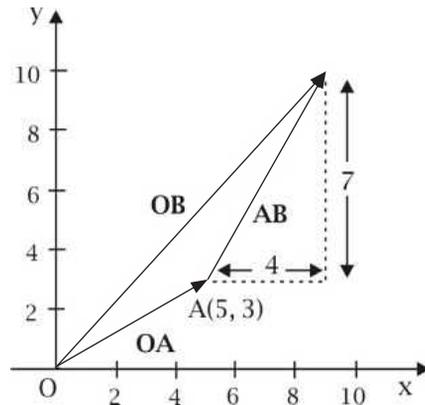
Step 5. Conclusion

The boats travel 6.4 km on a bearing of 38.7° , before travelling 13 km on a bearing of 157.4° and then 11.4 km on a bearing of 307.87° to get home.

This example shows that each of the vectors \mathbf{OA} , \mathbf{AB} and \mathbf{BO} can be specified in magnitude/direction form where the directions are quoted as either bearing or polar angles.

THE TRIANGLE RULE FOR VECTOR ADDITION

- Vectors in rectangular form are easily added.
Consider the vectors $\mathbf{OA} = \langle 5, 3 \rangle$, and $\mathbf{AB} = \langle 4, 7 \rangle$ and \mathbf{OB} as shown in the diagram below.



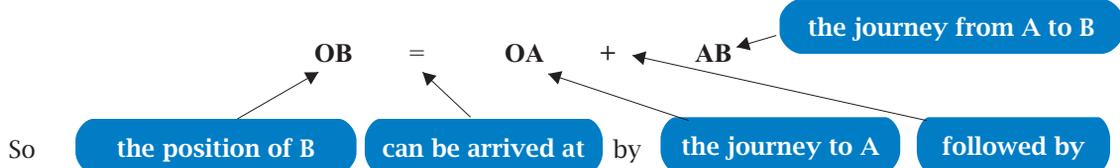
The horizontal component of B must be $x = 5+4$
 $= 9$

The vertical component of B must be $y = 3+7$
 $= 10$

So place B must have a position vector of $\mathbf{OB} = \langle 9, 10 \rangle$

The position of B can be arrived at by either;

- A direct journey from O to B. i.e. \mathbf{OB} or by
- A journey from O to A, i.e. \mathbf{OA} followed by a journey from A to B, i.e. \mathbf{AB} . The two alternative ways to get to B are modelled by the **vector equation**



For $\mathbf{OA} = \langle 5, 3 \rangle$ and $\mathbf{AB} = \langle 4, 7 \rangle$, \mathbf{OB} is found by

$$\begin{aligned}\mathbf{OB} &= \mathbf{OA} + \mathbf{AB} \\ &= \langle 5, 3 \rangle + \langle 4, 7 \rangle \\ &= \langle 5+4, 3+7 \rangle \\ &= \langle 9, 10 \rangle\end{aligned}$$

which means that B is at (9, 10)

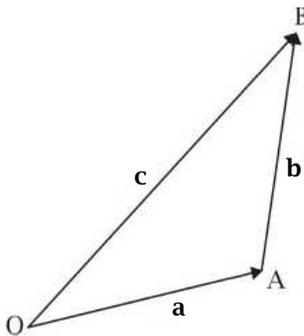
The vector equation $\mathbf{OB} = \mathbf{OA} + \mathbf{AB}$ is rearranged by following the usual rules of algebra. So,

$$\begin{aligned}\mathbf{OB} &= \mathbf{OA} + \mathbf{AB} \\ \text{or } \mathbf{OA} + \mathbf{AB} &= \mathbf{OB} \\ \text{and } \mathbf{AB} &= \mathbf{OB} - \mathbf{OA}\end{aligned}$$

Obviously then if $\mathbf{OA} = \langle 5, 3 \rangle$ and $\mathbf{OB} = \langle 9, 10 \rangle$ then the displacement \mathbf{AB} is found by

$$\begin{aligned}\mathbf{AB} &= \mathbf{OB} - \mathbf{OA} \\ &= \langle 9, 10 \rangle - \langle 5, 3 \rangle \\ &= \langle 9-5, 10-3 \rangle \\ &= \langle 4, 7 \rangle \text{ as expected}\end{aligned}$$

- In general then for displacements and in fact as we shall see later for all vectors, the **triangle law for vector addition and subtraction is**



$$\begin{aligned}\mathbf{c} &= \mathbf{a} + \mathbf{b} \quad \text{or } \mathbf{OB} = \mathbf{OA} + \mathbf{AB} \\ \text{and} \\ \mathbf{b} &= \mathbf{c} - \mathbf{a} \quad \text{or } \mathbf{AB} = \mathbf{OB} - \mathbf{OA}\end{aligned}$$

and the vector \mathbf{c} is the resultant vector of adding the vectors \mathbf{a} and \mathbf{b} by the **tip to tail** method as shown by the triangle diagram.

- The displacement \mathbf{AB} can be thought of as the change of position from \mathbf{A} to \mathbf{B} . i.e. new position \mathbf{B} take old position \mathbf{A} . This is a very important result which finds a displacement from two position vectors.

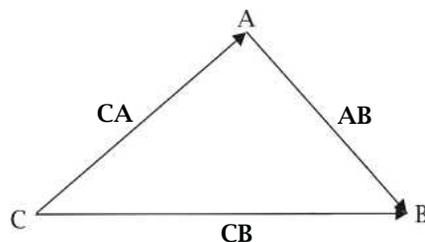
$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

The order of the letters A and B should be noted in the sense that on each side of the equation they are in opposite order. **You should be able to write this type of displacement equation without reference to a diagram.**

If the origin is not involved and point C is where \mathbf{CA} and \mathbf{CB} have their tails then

$$\mathbf{AB} = \mathbf{CB} - \mathbf{CA}$$

i.e.



SCALAR MULTIPLES AND OPPOSITE VECTORS

- When a vector quantity is multiplied by a scalar, s , (which may or may not have units) the vector's magnitude changes but not its direction. The vector is simply scaled up or down in size.

For example $\mathbf{OA} = \langle 4, 3 \rangle$

then $3\mathbf{OA} = 3\langle 4, 3 \rangle$
 $= \langle 12, 9 \rangle$

and $\frac{1}{2}\mathbf{OA} = \langle 4, 3 \rangle$
 $= \langle 2, 1.5 \rangle$

- If s is negative then the new vector is in the opposite direction,

i.e. $\mathbf{OA} = \langle 4, 3 \rangle$

$-2\mathbf{OA} = \langle -8, -6 \rangle$

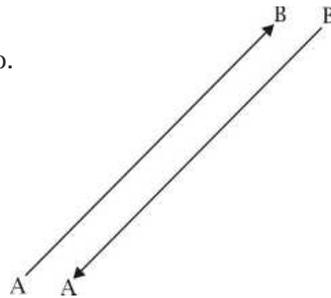
- When a journey of \mathbf{OA} is followed by the return journey of \mathbf{AO} then the overall displacement is zero.

i.e. $\mathbf{OA} + \mathbf{AO} = \mathbf{0}$

i.e. $\mathbf{OA} = -\mathbf{AO}$ Note the change in order

and $\mathbf{AO} = -\mathbf{OA}$ of the letters

\mathbf{AB} and \mathbf{BA} are called **opposite** vectors and they always sum to zero.



$$\mathbf{AB} + \mathbf{BA} = \mathbf{0}$$

$$\mathbf{AB} = -\mathbf{BA}$$

$$\mathbf{BA} = -\mathbf{AB}$$

- Often vectors connected by a scalar multiple will emerge from problems and proofs. The two statements below are major points to understand.

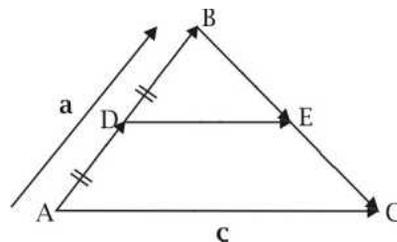
If two vectors \mathbf{a} and \mathbf{b} are related by a scalar multiple s i.e. $\mathbf{a} = s\mathbf{b}$ then the vectors are parallel and their magnitudes are related by s i.e. $|\mathbf{a}| = |s||\mathbf{b}|$.

If two vectors \mathbf{a} and \mathbf{b} are shown to be parallel then there is a scalar s such that $\mathbf{a} = s\mathbf{b}$ i.e. they are related by a scalar multiple.

Example of a vector proof

In $\triangle ABC$, D is the midpoint of AB and $DE \parallel AC$. Show that E is the midpoint of BC .

Step 1. Draw a good diagram showing vector arrows



Step 2. Let lower case letters be vectors $\mathbf{a} = \mathbf{AB}$ and $\mathbf{c} = \mathbf{AC}$

Let $\mathbf{DB} = \frac{1}{2}\mathbf{a}$ because D is the midpoint of AB

Let $\mathbf{DE} = h\mathbf{c}$ because $DE \parallel AC$

Let $\mathbf{BE} = k\mathbf{BC}$ because the value of k is needed to answer the question.

Step 3. Write vector equations from the triangles
 $\mathbf{BC} = \mathbf{c} - \mathbf{a}$ (1) $\mathbf{BE} = \mathbf{DE} - \mathbf{DB}$ (2)

Step 4. Substitute some equations from step 2 into (2)
 $\mathbf{BE} = \mathbf{DE} - \mathbf{DB}$

$$k\mathbf{BC} = h\mathbf{c} - \frac{1}{2}\mathbf{a} \quad \text{but } \mathbf{BC} = \mathbf{c} - \mathbf{a}$$

$$k(\mathbf{c} - \mathbf{a}) = h\mathbf{c} - \frac{1}{2}\mathbf{a}$$

$$k\mathbf{c} - k\mathbf{a} = h\mathbf{c} - \frac{1}{2}\mathbf{a}$$

Which means that $k = h$ (both scalar multiples of \mathbf{c})

$$\text{and } k = \frac{1}{2} \quad (\text{scalar multiples of } \mathbf{a})$$

$$\text{So } k = \frac{1}{2} \quad \text{and } h = \frac{1}{2}$$

Step 5. Finish off: $\mathbf{BE} = \frac{1}{2}\mathbf{BC}$

Which means that E is the midpoint of \mathbf{BC} as required,

Note that the proof also establishes that $\mathbf{DE} = \frac{1}{2}\mathbf{AC}$

UNIT VECTORS AND STANDARD UNIT VECTORS

- A unit vector is a vector which has a magnitude equal to one. So if $|\mathbf{b}|=1$ then \mathbf{b} is a unit vector.

The **standard unit** vectors are $\mathbf{i}=\langle 1,0\rangle$ and $\mathbf{j}=\langle 0,1\rangle$

and any vector in rectangular or component form is easily changed into $\mathbf{i} \mathbf{j}$ form by

$$\begin{aligned} \mathbf{r} &= \langle a, b \rangle \\ &= \langle a, 0 \rangle + \langle 0, b \rangle \\ &= a\langle 1, 0 \rangle + b\langle 0, 1 \rangle \\ &= a\mathbf{i} + b\mathbf{j} \end{aligned}$$

Examples are $\mathbf{AB} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{CD} = -4\mathbf{j}$,
 $\mathbf{a} = 3.5\mathbf{i} - 2.5\mathbf{j}$, $\mathbf{b} = 4\mathbf{j} - 5\mathbf{i}$ etc

As with the variations in notation for a vector in component form the $\mathbf{i} \mathbf{j}$ notation is popular but it is just as easy to write the above vectors as:

$$\mathbf{AB} = \langle -2, 3 \rangle, \mathbf{CD} = \langle 0, -4 \rangle, \mathbf{a} = \langle 3.5, -2.5 \rangle, \mathbf{b} = \langle -5, 4 \rangle$$

- Any non unit vector say, \mathbf{a} has a unit vector $\hat{\mathbf{a}}$ associated with it. To find $\hat{\mathbf{a}}$ from \mathbf{a} simply scale \mathbf{a} down by its magnitude i.e. $|\mathbf{a}|$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \quad \text{and} \quad \mathbf{a} = |\mathbf{a}| \hat{\mathbf{a}}$$

Obviously \mathbf{a} and $\hat{\mathbf{a}}$ have the same direction

Example 1. $\mathbf{a} = \langle 3, 4 \rangle$ then $|\mathbf{a}| = 5$

$$\text{and } \hat{\mathbf{a}} = \frac{\langle 3, 4 \rangle}{5} = \langle 0.6, 0.8 \rangle$$

Example 2. Find the vector that has the same direction as $\langle 4, 3 \rangle$ and the same magnitude as $\langle -5, -12 \rangle$

If $\mathbf{a} = \langle 4, -3 \rangle$ and $\mathbf{b} = \langle -5, -12 \rangle$

$|\mathbf{a}| = 5$ and $|\mathbf{b}| = 13$

$\hat{\mathbf{a}} = \frac{1}{5} \langle 4, -3 \rangle$

$= \langle 0.8, -0.6 \rangle$

\therefore Required vector $= 13 \langle 0.8, -0.6 \rangle$

$= \langle 10.4, -7.8 \rangle$

EQUIVALENT VECTORS

- Consider points A(-2, 3), B(-6, 8), C(3, 11) and D(7, 6)

then $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

and $\mathbf{DC} = \mathbf{OC} - \mathbf{OD}$

$= \langle -6, 8 \rangle - \langle -2, 3 \rangle$

$= \langle 3, 11 \rangle - \langle 7, 6 \rangle$

$= \langle -6 + 2, 8 - 3 \rangle$

$= \langle 3 - 7, 11 - 6 \rangle$

$= \langle -4, 5 \rangle$

$= \langle -4, 5 \rangle$

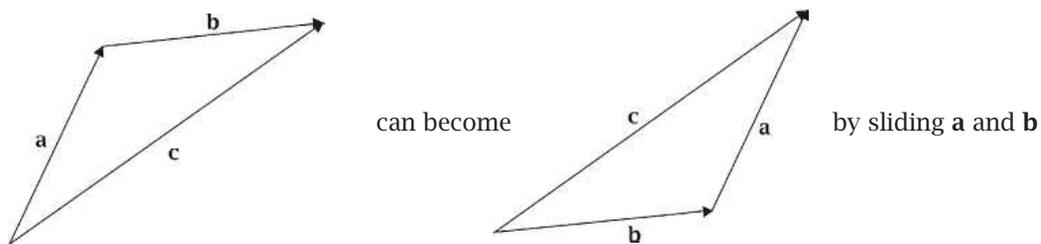
The displacements \mathbf{AB} and \mathbf{DC} are both $\langle -4, 5 \rangle$ and hence \mathbf{AB} and \mathbf{DC} are considered to be equivalent vectors even though they are at different places on the cartesian plane.

So

Equivalent vectors have the same magnitude and direction, i.e. when drawn as arrows they are the same length and are parallel. They also have the same rectangular components.

Also if $\mathbf{AB} = \mathbf{DC}$ then ABCD is a parallelogram and \mathbf{BC} will be equivalent to \mathbf{AD} .

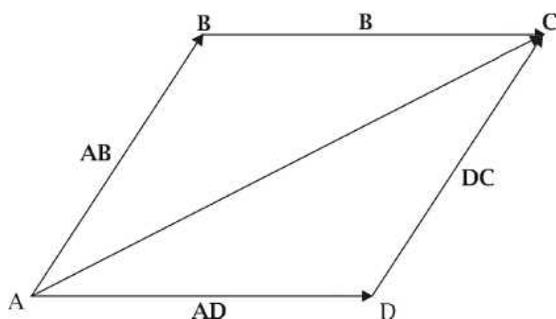
- This all means that when vectors are represented by arrows in diagrams they can be translated around to produce different looking diagrams which nevertheless yield the same equations. For example



and for both diagrams $\mathbf{a} + \mathbf{b} = \mathbf{c}$

THE PARALLELOGRAM RULE FOR VECTOR ADDITION

Consider the parallelogram ABCD shown below with its diagonal AC.



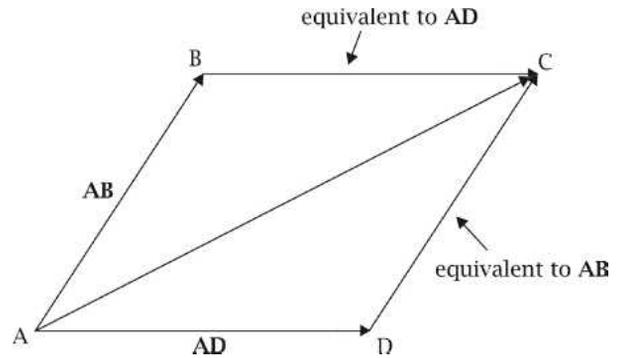
From the triangle law of vector addition

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$$

Now as \mathbf{AB} and \mathbf{DC} are equivalent vectors (because ABCD is a parallelogram, opposite sides are parallel and have the same length) the equation above becomes

$$\mathbf{AB} + \mathbf{AD} = \mathbf{AC}$$

This means that a way of adding two vectors which have their tails at the same point (in this case A) is to form a parallelogram using the two vectors to be added. The sum of the vectors is then the diagonal vector which also has its tail sharing the common point. The parallelogram consists of two triangles which include the equivalent vectors as shown opposite.

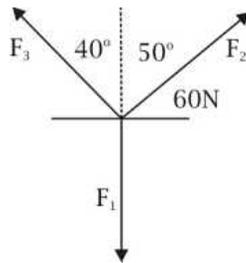


$$\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$$

So by sliding vectors around without changing their magnitude and direction, triangles can be formed. From the triangles, equations are written and solved as shown below with the three forces in equilibrium.

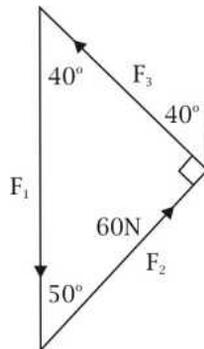
Example 1. Consider the system of forces in equilibrium shown below.

Find the values of $|\mathbf{F}_1|$ and $|\mathbf{F}_3|$



Step 1. Slide vectors to form a triangle of vectors which shows that

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0 \text{ due to the equilibrium condition.}$$



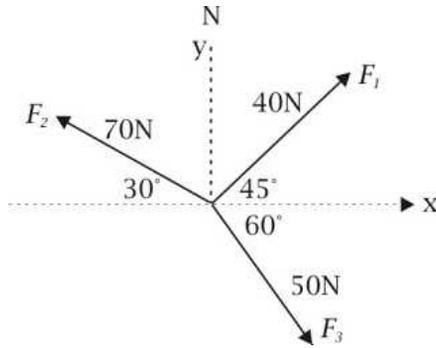
Step 2. Conveniently the triangle is right angled, so

$$\begin{aligned} \cos 50^\circ &= \frac{60}{|\mathbf{F}_1|} \\ |\mathbf{F}_1| &= \frac{60}{\cos 50^\circ} \\ &= 93.34 \text{ N (2dp)} \end{aligned}$$

$$\begin{aligned} \tan 50^\circ &= \frac{|\mathbf{F}_3|}{60} \\ |\mathbf{F}_3| &= 60 \tan 50^\circ \\ &= 71.51 \text{ N (2dp)} \end{aligned}$$

The next example has \mathbf{F}_1 , \mathbf{F}_2 and \mathbf{F}_3 fully defined and the resultant \mathbf{R} is easily found using the component method.

Example 2. For the three forces shown acting on the block find the resultant force and hence the new single force which will hold the block in equilibrium.



Method 1. No-one in their right mind will do these types using the sine and cosine rules. Instead each force is resolved into horizontal and vertical components and the sum of each type gives the resultant in component form.

$$\begin{aligned} \mathbf{R} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= (40\cos 45^\circ)\mathbf{i} + (40\sin 45^\circ)\mathbf{j} + (-70\cos 30^\circ)\mathbf{i} + (70\sin 30^\circ)\mathbf{j} + (50\cos 60^\circ)\mathbf{i} + (-50\sin 60^\circ)\mathbf{j} \\ &= (40\cos 45^\circ - 70\cos 30^\circ + 50\cos 60^\circ)\mathbf{i} + (40\sin 45^\circ + 70\sin 30^\circ - 50\sin 60^\circ)\mathbf{j} \\ &= -7.3375\mathbf{i} + 19.9830\mathbf{j} \end{aligned}$$

$$\begin{aligned} \therefore |\mathbf{R}| &= \sqrt{(-7.3375)^2 + (19.9830)^2} \\ &= 21.3\text{ N} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{19.9830}{7.3375}\right) \\ &= 69.8^\circ \end{aligned}$$



$$\begin{aligned} \text{polar angle of } \mathbf{R} &= 180^\circ - 69.8^\circ \\ &= 110.2^\circ \\ \text{bearing of } \mathbf{R} &= 270^\circ + 69.8^\circ \\ &= 339.8^\circ \end{aligned}$$

The resultant of the three forces is 21.3N on a bearing of 339.8°

The force required to keep the block in equilibrium is $-\mathbf{R}$ ie $7.3375\mathbf{i} - 19.983\mathbf{j}$ which has a magnitude of 21.3N and a bearing of $339.8^\circ - 180^\circ = 159.8^\circ$

Method 2. Here the $P \rightarrow R$ and $R \rightarrow P$ method will be shown but you will need to know how to use these functions on your calculator. Make sure you get the results as shown.

Step 1. Write the vector equation from the situation

$$\text{Resultant } \mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

Step 2. Write each known vector in polar form being careful to get the correct polar angles.

$$\mathbf{R} = \langle 40, 45^\circ \rangle + \langle 70, 150^\circ \rangle + \langle 50, -60^\circ \rangle P$$

Step 3. Change each polar vector to rectangular, i.e. use $P \rightarrow R$.

$$\mathbf{R} = \langle 28.2843, 28.2843 \rangle + \langle -60.6218, 35 \rangle + \langle 25, -43.3013 \rangle$$

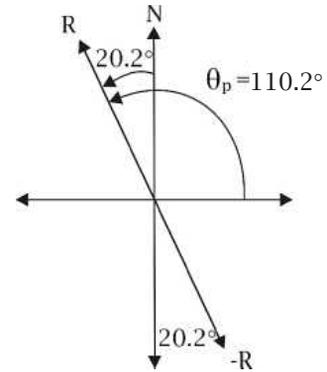
$$\mathbf{R} = \langle -7.3375, 19.983 \rangle$$

Step 4. Change back to polar

$$\mathbf{R} = \langle 21.3\text{N}, 110.2^\circ \rangle \text{P (1dp)}$$

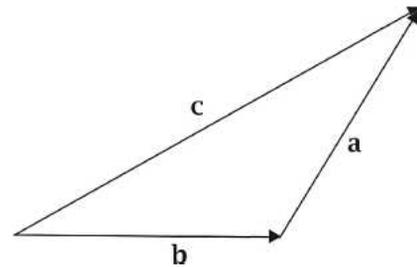
\therefore Resultant force is 21.3N
on a bearing of $360^\circ - 20.2^\circ = 339.8^\circ$

As before the force required to keep the block in equilibrium is $-\mathbf{R}$ which has a magnitude of 21.3N and a bearing of $180^\circ - 20.2^\circ = 159.8^\circ$.



THE TRIANGLE INEQUALITY

For $\mathbf{a} + \mathbf{b} = \mathbf{c}$
and its associated vector triangle

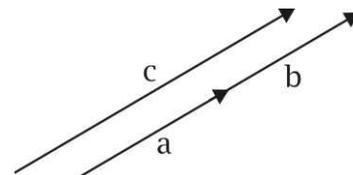


Consider the magnitudes $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{c}| = |\mathbf{a} + \mathbf{b}|$. It is intuitively obvious that

$$|\mathbf{c}| < |\mathbf{a}| + |\mathbf{b}|$$

i.e. $|\mathbf{a} + \mathbf{b}| < |\mathbf{a}| + |\mathbf{b}|$ (1)

When \mathbf{a} and \mathbf{b} are co-linear (in a straight line)
then there is no triangle i.e.



and $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$ (2)

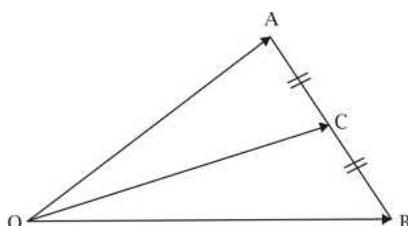
The combination of (1) and (2) above is called the triangular inequality i.e.

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

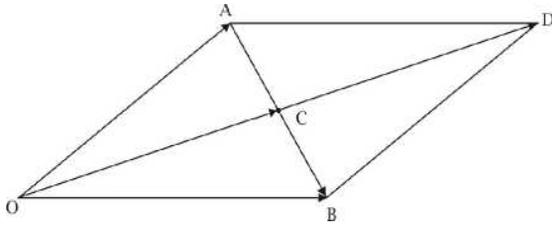
- The triangle inequality can be used to prove a variety of interesting algebraic results.

SUBDIVIDING A VECTOR

- The idea of a scalar multiplying a vector was introduced earlier and now versatility of scalar multiples is used in subdividing line segments. Consider the vector \mathbf{AB} and the point C which is the midpoint of \mathbf{AB} . The diagram with O the origin and the position vector \mathbf{OC} marked is:

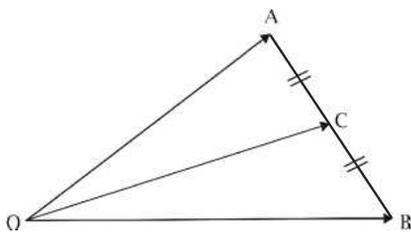


How is \vec{OC} connected to \vec{OA} and \vec{OB} ? The easiest way is to mark point D on the diagram above such that OADB is a parallelogram i.e.



Then $\vec{OD} = \vec{OA} + \vec{OB}$ by the parallelogram rule of vector addition. Now because $\vec{OC} = \frac{1}{2}\vec{OD}$ due to the property that the diagonals of a parallelogram bisect each other we have $\vec{OC} = \frac{1}{2}(\vec{OA} + \vec{OB})$ which suggests that the position vector of the midpoint is the average of the position vectors of the end points.

- The other method leads onto harder types

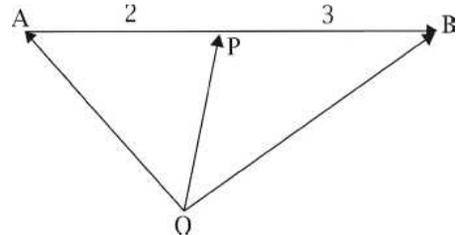


$$\begin{aligned} \vec{OC} &= \vec{OA} + \vec{AC} & \therefore \vec{OC} &= \vec{OA} + \frac{1}{2}(\vec{OB} - \vec{OA}) \\ \text{but } \vec{AC} &= \frac{1}{2}\vec{AB} & &= \vec{OA} + \frac{1}{2}\vec{OB} - \frac{1}{2}\vec{OA} \\ &= \frac{1}{2}(\vec{OB} - \vec{OA}) & &= \vec{OA} - \frac{1}{2}\vec{OA} + \frac{1}{2}\vec{OB} \\ & & &= \frac{1}{2}\vec{OA} + \frac{1}{2}\vec{OB} \\ \vec{OC} &= \frac{1}{2}(\vec{OA} + \vec{OB}) \text{ as before} \end{aligned}$$

Example 1. Find point P which divides AB in the ratio 2:3 if

A is (4, -6) and B is (14, -1)

Accurate diagrams on the x y plane are not needed for these types



$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= \langle 14, -1 \rangle - \langle 4, -6 \rangle$$

$$= \langle 10, 5 \rangle$$

$$\vec{AP} = \frac{2}{5}\vec{AB}$$

$$= \frac{2}{5}\langle 10, 5 \rangle$$

$$= \langle 4, 2 \rangle$$

$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\langle 4, -6 \rangle + \langle 4, 2 \rangle$$

$$= \langle 8, -4 \rangle$$

So P is (8, -4)

Note that this question could have been worded as

'Find point P if $\vec{AP} : \vec{PB}$ for A (4, -6) and B (14, -1).
= 2 : 3

The diagram is the same and it is obvious that P is between A and B.

The ratio $\vec{AP} : \vec{PB}$ can be written as $3\vec{AP} = 2\vec{PB}$
= 2 : 3

From which it follows that $\vec{AP} = \frac{2}{3}\vec{PB}$ and $\vec{PB} = \frac{3}{2}\vec{AP}$

This is an example of internal subdivision.

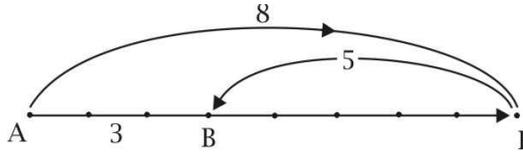
External subdivision types are easy enough when the technique over the page is used.

Example 2. Find P if $AP : PB = 8 : -5$ for A(-4, 5) and B(2, -4)

The ratio can be written as $AP : PB = 8 : -5$

And then changed to $AP : BP = 8 : 5$ which shows that point P is not between A and B.

The diagram can be found easily enough from the last ratio but most people simply draw A and P and put an 8 along the arced arrow and then another arced arrow from P back towards A (because of the negative) and put a 5 on the **PB** arc. Obviously **AB** now has a 3 along it as shown.



$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

$$= \langle 2, -4 \rangle - \langle -4, 5 \rangle$$

$$= \langle 6, -9 \rangle$$

$$\frac{1}{3}\mathbf{AB} = \frac{1}{3}\langle 6, -9 \rangle$$

$$= \langle 2, -3 \rangle$$

$$\mathbf{BP} = 5\langle 2, -3 \rangle$$

$$= \langle 10, -15 \rangle$$

$$\mathbf{OP} = \mathbf{OB} + \mathbf{BP}$$

$$= \langle 2, -4 \rangle + \langle 10, -15 \rangle$$

$$= \langle 12, -19 \rangle$$

$$\therefore P \text{ is } (12, -19)$$

RELATIVE DISPLACEMENT AND RELATIVE VELOCITY

- Relative displacement is a concept that seems to be completely covered by ordinary displacements as have been fully discussed already. However the notation is consistent with the hugely important topic of relative velocity so here goes.

For points A and B let $\mathbf{OA} = \mathbf{r}_A$ and $\mathbf{OB} = \mathbf{r}_B$. Then the displacement **AB** is

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} \quad \text{as usual}$$

$$= \mathbf{r}_B - \mathbf{r}_A$$

Now **AB** is given a new notation of ${}_B\mathbf{r}_A$ which reads as the position of B relative to that of A.

So ${}_B\mathbf{r}_A = \mathbf{r}_B - \mathbf{r}_A$ is the new notation (same order of B and A)

and $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$ is the original notation (opposite order of A and B)

One possible advantage of the new notation is that the order of subscripted letters is the same but you still need to know that ${}_B\mathbf{r}_A = \mathbf{AB}$.

- Relative velocity is a concept which has important applications to some very interesting problems.

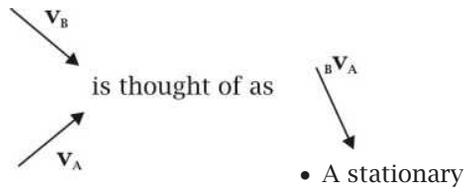
Consider two boats, A and B moving in straight lines on the x y plane with constant velocities \mathbf{v}_A and \mathbf{v}_B respectively. Imagine that you were on boat A observing the motion of boat B. Because you yourself are moving with velocity \mathbf{v}_A you will not be able to observe boat B's true velocity of \mathbf{v}_B but rather as the velocity of B relative to you with velocity \mathbf{v}_A .

The velocity of B as experienced by an observer travelling with velocity \mathbf{v}_A is notated as ${}_B\mathbf{v}_A$, and it can be demonstrated that:

$${}_B\mathbf{v}_A = \mathbf{v}_B - \mathbf{v}_A$$

In an ideal case where there was no friction, wind or wave action (i.e. you experience no acceleration and hence no force) you would not be able to tell that you were moving. In this case it is useful to pretend that you are actually stationary (even though you are actually moving with velocity \mathbf{v}_A) and in observing boat B's motion you will experience that velocity as the relative velocity ${}_B\mathbf{v}_A$. This means that two boats moving can be thought of as one boat moving with a relative velocity as observed by the other boat which is pretending to be stationary!

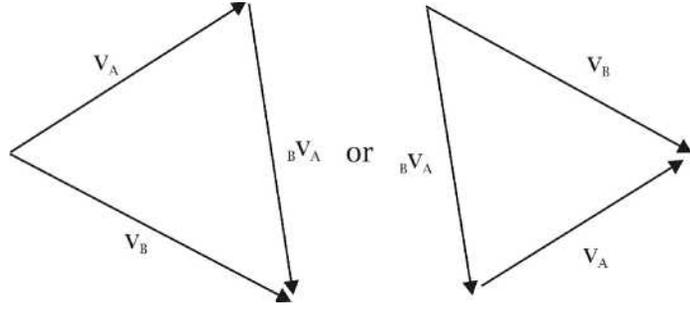
So



The three vectors \mathbf{v}_A , \mathbf{v}_B and $\mathbf{v}_B \mathbf{v}_A$ form a triangle of vectors according to the vector equation

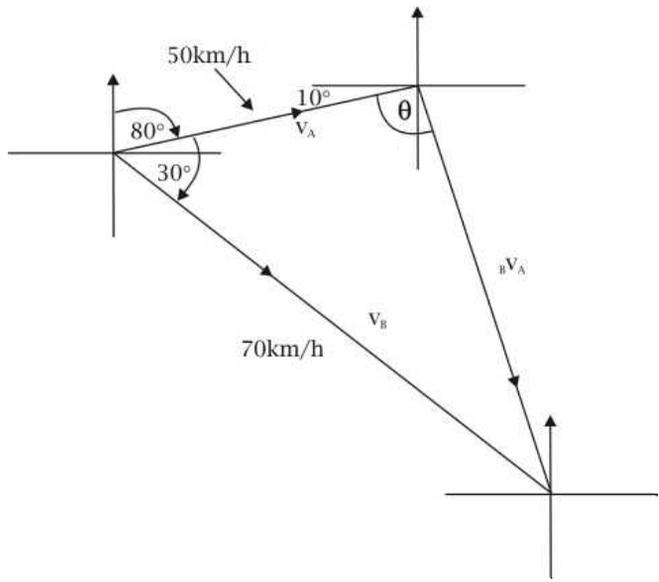
$$\mathbf{v}_B \mathbf{v}_A = \mathbf{v}_B - \mathbf{v}_A \quad \text{OR} \quad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_B \mathbf{v}_A$$

For example



Example 1. If boat B is travelling at 70 km/h on a bearing of 110°, and boat A is travelling at 50 km/h on a bearing of 80°, find the velocity of boat B relative to that of boat A.

Method 1. sine and cosine rule.



By the cosine rule

$$|\mathbf{v}_B \mathbf{v}_A|^2 = 50^2 + 70^2 - 2 \times 50 \times 70 \times \cos 30^\circ$$

$$\text{ie } |\mathbf{v}_B \mathbf{v}_A| = 36.576 \text{ km/h}$$

$$\cos \theta = \frac{50^2 + 36.576^2 - 70^2}{2 \times 50 \times 36.576}$$

$$\theta = 106.9^\circ \text{ (1 dp)}$$

The bearing of $\mathbf{v}_B \mathbf{v}_A$ will be $270^\circ - 10^\circ - 106.9^\circ = 153.1^\circ$

So boat B is seen travelling with a velocity of 36.6 km/h on a bearing of 153.1° by someone on boat A.

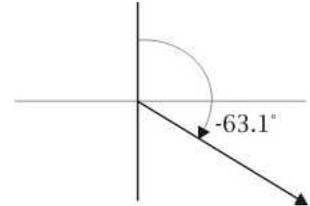
Method 2. Rectangular Polar conversion method

Firstly the polar angles of A and B are 10° and -20° respectively

$$\begin{aligned} {}_B \mathbf{V}_A &= \mathbf{V}_B - \mathbf{V}_A \\ &= \langle 70, -20^\circ \rangle - \langle 50, 10^\circ \rangle \text{ P} \\ &= \langle 65.7785, -23.9414 \rangle - \langle 49.2404, 8.6824 \rangle \text{ R} \\ &= \langle 16.5381, -32.6238 \rangle \text{ R} \\ &= \langle 36.576, -63.118^\circ \rangle \text{ P} \end{aligned}$$

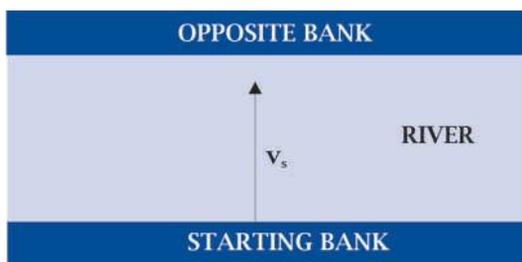
$$\begin{aligned} \text{Bearing} &= 90^\circ + 63.1^\circ \\ &= 153.1^\circ \end{aligned}$$

As before boat B is seen by boat A travelling at 36.6 km/h on a bearing of 153.1°



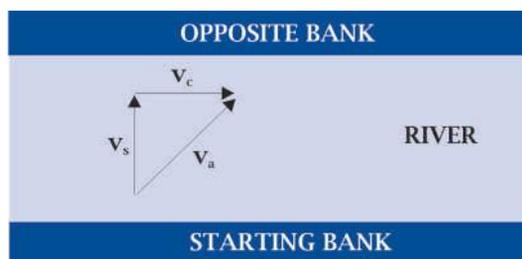
WIND AND CURRENT RELATIVE VELOCITY TYPES

- In the previous section vector theory modelled two objects moving independently each with their own velocities where they can view each other with a relative velocity. In this section a wind or current velocity influences the velocity of a plane or boat. Three velocities are still involved where one is a relative velocity but it is easier to rename the relative velocity as will be shown.
- Consider a river with no current flowing and a swimmer crossing the river with a velocity of \mathbf{v}_s via the shortest path. i.e. at right angles to each bank.



If the river is d metres wide and the journey takes t seconds then $|\mathbf{v}_s| = \frac{d}{t}$ as usual

- Now consider that the swimmer still heads directly towards the opposite bank in spite of the river flowing with a current \mathbf{v}_c parallel to the banks as shown.



The swimmer's actual velocity \mathbf{v}_a works out to be the vector addition of \mathbf{v}_s and \mathbf{v}_c as shown in

$$\mathbf{v}_a = \mathbf{v}_s + \mathbf{v}_c$$

\mathbf{v}_s is now called the steering velocity of the swimmer.

From $\mathbf{v}_a = \mathbf{v}_s + \mathbf{v}_c$ it follows that

$$\mathbf{v}_s + \mathbf{v}_c = \mathbf{v}_a$$

and $\mathbf{v}_s = \mathbf{v}_a - \mathbf{v}_c$

i.e. $\mathbf{v}_s = {}_a \mathbf{v}_c$

This shows that the steering velocity \mathbf{v}_s is the velocity of the swimmer as seen say by a duck which is allowing itself to be moved only by the current. In other words if we pretend that the duck is stationary, it will observe the swimmers velocity as \mathbf{v}_s .

The magnitude of \mathbf{v}_s i.e. $|\mathbf{v}_s|$ is called the still water speed in keeping with the idea that \mathbf{v}_s is the velocity as if there was no current flowing.

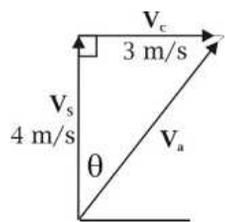
- A swimmer or the pilot of a boat or plane can choose the velocity \mathbf{v}_s . If there is no current obviously \mathbf{v}_s is the same as \mathbf{v}_a , the actual velocity.

If there is a current or wind the choice of \mathbf{v}_s is to usually allow for its influence in such a way to achieve the required destination. Two situations will be distinguished as follows:

Example 1. Betty chose to cross a 200 m wide river which had a 3 m/s current flowing by steering herself directly across the river. If Betty maintained a still water speed of 4 m/s find

- Her actual velocity
- The time for the swim
- Where she reached the opposite bank

Velocity diagram



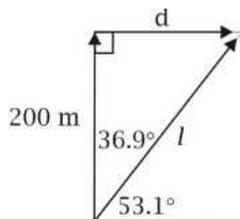
$$|\mathbf{v}_a| = \sqrt{3^2 + 4^2}$$

$$= 5 \text{ m/s}$$

$$\tan \theta = \frac{3}{4}$$

$$\theta = 36.9^\circ$$

Displacement diagram



$$\tan 36.9^\circ = \frac{d}{200}$$

$$d = 150 \text{ m}$$

$$l = \sqrt{200^2 + 150^2}$$

$$= 250 \text{ m}$$

The answers are:

- | | |
|--|--|
| $\text{Speed} = \frac{\text{dist}}{\text{time}}$ $\text{time} = \frac{\text{dist}}{\text{speed}}$ $= \frac{250}{5}$ $= 50 \text{ sec}$ | <ol style="list-style-type: none"> Betty's actual velocity is 5 m/s at 53.1° to the downstream bank as shown above. The swim takes 50 sec. Betty reached the opposite bank 150 m downstream from the point on the bank that she would have reached if there was no current. |
|--|--|

Note that the velocity triangle and displacement triangle above are similar, ie they are the same shape. This means that the time for the journey could have been found as:

$$\frac{\text{width of river}}{\text{still water speed}}$$

$$= \frac{200}{4}$$

$$= 50 \text{ sec as before}$$

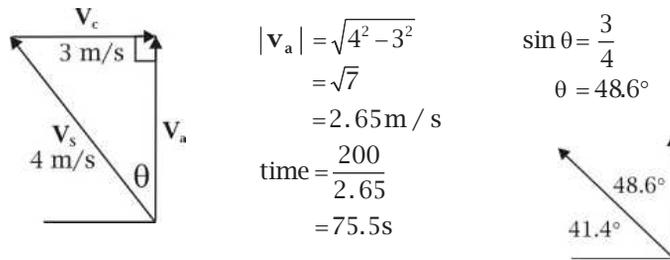
or even as $\frac{\text{distance swept downstream}}{\text{speed of current}}$

$$= \frac{150}{3}$$

$$= 50 \text{ sec as before}$$

Example 2. Billy wanted to cross the same river with the same current as Betty, but chose his v_s such that the direction of his actual velocity was directly across the river. If his still water speed is also 4m/s find:

- (a) The angle that Billy had to steer himself on
- (b) His actual speed
- (c) The time for the journey



The answers are

- (a) Billy has to steer himself at 41.4° to the upstream bank as shown
- (b) His actual speed was 2.65m/s
- (c) The journey took 75.5s

Note that in this case Billy swims a shorter distance and arrives on the opposite bank directly across from his starting position but that the swim takes much longer.

- Harder types of wind / current problems mostly involve use of the sine and cosine rules. There are two main types which are
 - (i) Basically the same as the Betty / Billy examples above but with non right angled triangles.
 - (ii) Called a b types which require a particular method which will be explained in detail

Firstly a few extremely important concepts need to be explained. These are

Fact (i) All wind / current types have to conform to the vector equation below which links the three velocities

- v_a the actual velocity
- v_s the steering velocity
- v_w or v_c the wind or current velocity

or

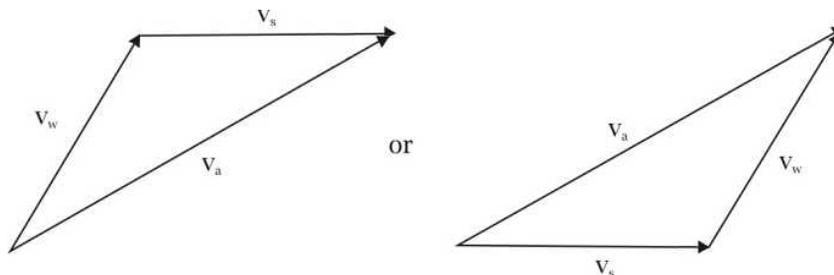
$$v_a = v_s + v_w$$

$$v_a = v_s + v_c$$

Where $|v_s|$ is called the still air or still water speed

Any vector diagram must agree with the above equations. If your diagram doesn't then you are on the wrong track.

The vector diagram (for a wind situation) could be



Note that either the tips of v_a and v_s meet or the tips of v_a and v_w meet

These facts can make drawing the diagram much easier.

Fact (ii) If an object travels from A to B, i.e. its displacement is \mathbf{AB} with a velocity \mathbf{v} and the journey takes t time then

$$\mathbf{AB} = t\mathbf{v}$$

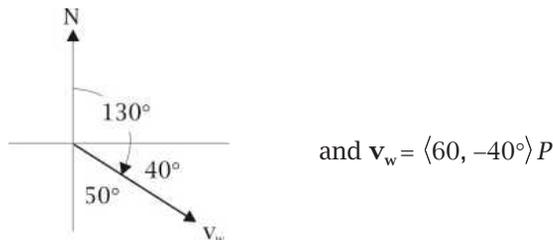
This is the vector equivalent of $\text{distance} = \text{time} \times \text{speed}$ and it is a very useful relationship.

Example. If $\mathbf{v} = \langle 2, 3 \rangle \text{m/s}$ find the displacement after 5 seconds of travel

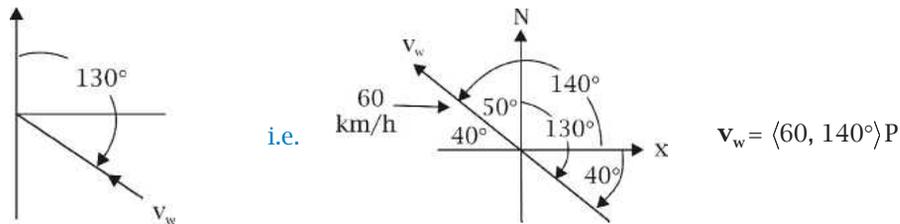
$$\begin{aligned} \mathbf{AB} &= t\mathbf{v} \\ &= 5\langle 2, 3 \rangle && \text{The time } t \text{ is acting as a scalar multiple.} \\ &= \langle 10, 15 \rangle \text{ m} \end{aligned}$$

Fact (iii) Directions in real life situations are usually given as a bearing and the associated wording needs to be carefully noted.

Example 1. A wind of 60km/h blowing on a bearing (or has a bearing) of 130° means this:



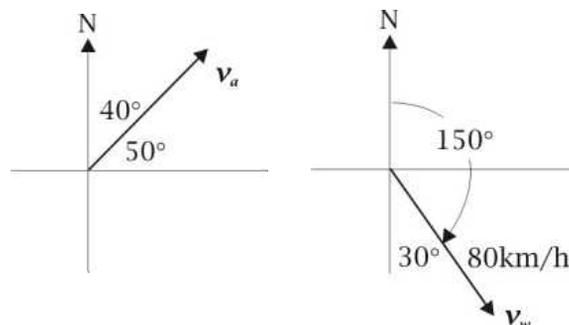
Example 2. A wind of 60 km/h blowing **from** a bearing of 130° means this:



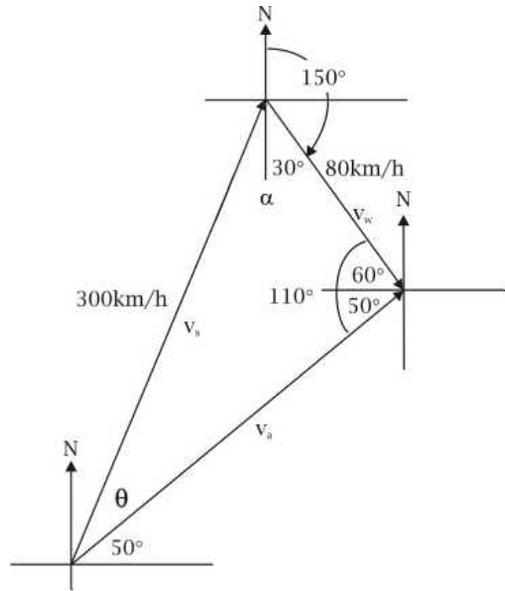
Big example 1. Vishnu's aircraft has a still air speed of 300 km/h. He wishes to land at an airfield which is 400 km away on a bearing of 040° . The wind is blowing at 80 km/h on a bearing of 150° .

- Find:
- (a) The bearing on which Vishnu must fly in order to reach the airfield.
 - (b) The actual velocity of the plane
 - (c) How long the trip takes

Step 1. Sketch the velocities that you know a bit about.



Step 2. Use the fact that the tips of \mathbf{v}_a and \mathbf{v}_w meet to draw the diagram and draw \mathbf{v}_a much longer than \mathbf{v}_w .



Step 3. Use sine rule twice

$$\frac{300}{\sin 110^\circ} = \frac{80}{\sin \theta}$$

$$\sin \theta = \frac{80 \sin 110^\circ}{300}$$

$$\theta = 14.512^\circ$$

$$\begin{aligned} \text{bearing} &= 90^\circ - 50^\circ - 14.5^\circ \\ &= 25.5^\circ \end{aligned}$$

$$\begin{aligned} \alpha &= 180^\circ - 110^\circ - 14.512^\circ \\ &= 55.488^\circ \end{aligned}$$

$$\frac{|\mathbf{v}_a|}{\sin 55.488^\circ} = \frac{300}{\sin 110^\circ}$$

$$\begin{aligned} |\mathbf{v}_a| &= \frac{300 \sin 55.488^\circ}{\sin 110^\circ} \\ &= 263.07 \text{ km/h} \end{aligned}$$

$$\text{speed} = \frac{\text{dist}}{\text{time}}$$

$$\text{time} = \frac{\text{dist}}{\text{speed}}$$

$$= \frac{400}{263.07}$$

$$= 1.52 \text{ h}$$

$$= 1 \text{ h } 31 \text{ min}$$

The answers are:

- The bearing that Vishnu must steer his plane on is 025.5°
- The actual velocity of the plane was 263.07 km/h on a bearing of 040°
- The time for the journey was approximately 1 h 31 min.

The main reason that the plane achieved the required destination is because Vishnu did exceptionally well with his vector studies!!

Note:

In general students find these types quite demanding. Important tips are to draw a big diagram with a \uparrow on each vertex of the triangle and to fully label the triangle with correct vector arrows and all missing angles marked in.

Big example 2. This is an a b type which will be broken down into easy to follow steps.

Karan, a long distance swimmer needed to swim from A(12, 19)km to B (-16, -2)km. Unfortunately the current was flowing at a steady $\langle -3.5, 2.5 \rangle$ km/h. If Karan's speed in still water was $\sqrt{30.5}$ km/h find the velocity \mathbf{v}_s that he needs to set himself in order to complete the journey and find the time taken.

Step 1. let $\mathbf{v}_s = \langle a, b \rangle$ km/h(1)

then $|\mathbf{v}_s| = \sqrt{a^2 + b^2} = \sqrt{30.5}$

i.e. $a^2 + b^2 = 30.5$

Step 2. Find $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

$= \langle -16, -2 \rangle - \langle 12, 19 \rangle$

$= \langle -28, -21 \rangle$ km

Step 3. Always $\mathbf{v}_a = \mathbf{v}_s + \mathbf{v}_c$

$= \langle a, b \rangle + \langle -3.5, 2.5 \rangle$

$= \langle a - 3.5, b + 2.5 \rangle$

Step 4. Required displacement $\mathbf{AB} = t \mathbf{v}_a$

So $\langle -28, -21 \rangle = t \langle a - 3.5, b + 2.5 \rangle$

$= \langle t(a - 3.5), t(b + 2.5) \rangle$

equate components $-28 = t(a - 3.5), -21 = t(b + 2.5)$

Step 5. Solve for a and b (this is called the Kubiak variation!)

$a = 3.5 - \frac{28}{t} \quad b = -2.5 - \frac{21}{t}$

Step 6. Substitute a and b into (1) above

$\left(3.5 - \frac{28}{t}\right)^2 + \left(-2.5 - \frac{21}{t}\right)^2 = 30.5$ (2)

Use solver on your CAS to solve for t or use algebra to transform (2) into $at^2 + bt + c = 0$ and use a quadratic solver in your CAS.

You should get $t = 7$ or -14.58 which is obviously rejected.

Step 7. Substitute $t = 7$ in to get a and b

$a = 3.5 - \frac{28}{7} \quad b = -2.5 - \frac{21}{7}$

$= -0.5 \quad = -5.5$

Step 8. Finish off.

So Karan needed to set his velocity at $\langle -0.5, -5.5 \rangle$ km/h in order to reach B(-16, -2) km in 7 hours

Note that for these particular types a diagram is not needed.

Worked Examples

Vectors I

2.1 Consider the points $A(-8, -3)$, $B(2, 5)$ and $C(7, -1)$. D is the mid-point of AB and E is the point which divides AC in the ratio of 1:2. Find

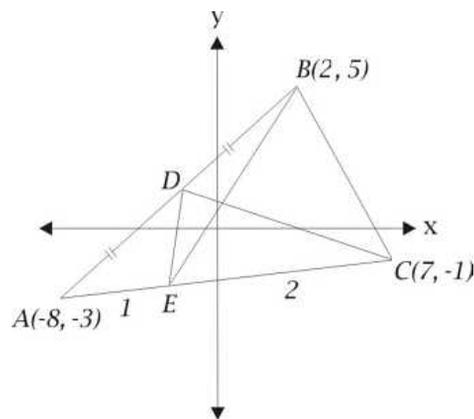
- (a) \mathbf{BC} (b) \mathbf{CD} (c) \mathbf{BE}

(d) Point F , such that $ABFC$ is a parallelogram

- (e) $|\mathbf{DE}|$

As usual a sketch diagram is useful for sorting out the necessary vector equations. If convenient, try to make it roughly to scale.

Each answer should be checked with the diagram for reasonableness.



(a) $\mathbf{BC} = \mathbf{OC} - \mathbf{OB}$
 $= \langle 7, -1 \rangle - \langle 2, 5 \rangle$
 $= \langle 5, -6 \rangle$ checked

(b) $\mathbf{CD} = \frac{1}{2}(\mathbf{CA} + \mathbf{CB})$
 $= \frac{1}{2}(\langle -15, -2 \rangle + \langle -5, 6 \rangle)$
 $= \frac{1}{2}\langle -20, 4 \rangle$
 $= \langle -10, 2 \rangle$ checked

(c) $\mathbf{BE} = \mathbf{BC} + \mathbf{CE}$
 $= \langle 5, -6 \rangle + \langle -10, -\frac{4}{3} \rangle$
 $= \langle -5, -7\frac{1}{3} \rangle$ checked

$$\mathbf{CA} = \mathbf{OA} - \mathbf{OC}$$

$$= \langle -8, -3 \rangle - \langle 7, -1 \rangle$$

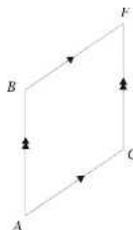
$$= \langle -15, -2 \rangle$$

$$\mathbf{CF} = \frac{2}{3}\mathbf{CA}$$

$$= \frac{2}{3}\langle -15, -2 \rangle$$

$$= \langle -10, -\frac{4}{3} \rangle$$

(d) point F ? let F be (x, y)



$$\mathbf{BF} = \mathbf{OF} - \mathbf{OB}$$

$$= \langle x, y \rangle - \langle 2, 5 \rangle$$

$$= \langle x - 2, y - 5 \rangle$$

$$\mathbf{BF} = \mathbf{AC}$$

$$\langle x - 2, y - 5 \rangle = \langle 15, 2 \rangle$$

$$x - 2 = 15 \quad y - 5 = 2$$

$$x = 17 \quad y = 7$$

$$\therefore F \text{ is } (17, 7)$$

(e) $|\mathbf{DE}|$? $\mathbf{BC} + \mathbf{CD} + \mathbf{DE} = \mathbf{BE}$

$$\therefore \mathbf{DE} = \mathbf{BE} - \mathbf{BC} - \mathbf{CD}$$

$$= \langle -5, -7\frac{1}{3} \rangle - \langle 5, -6 \rangle - \langle -10, 2 \rangle$$

$$= \langle 0, -3\frac{1}{3} \rangle$$

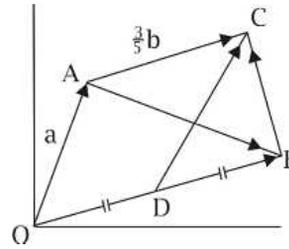
$$|\mathbf{DE}| = 3\frac{1}{3}$$

- 2.2 Consider the points A, B and C in the plane. If $OA = \mathbf{a}$ and $OB = \mathbf{b}$ and $AC = \frac{3}{5}\mathbf{b}$, find the vector for DC in terms of \mathbf{a} and \mathbf{b} where D is the midpoint of OB .

Solution

$$\begin{aligned} \mathbf{AB} &= \mathbf{OB} - \mathbf{OA} \\ &= \mathbf{b} - \mathbf{a} \end{aligned}$$

$$\begin{aligned} \mathbf{BC} &= \mathbf{AC} - \mathbf{AB} \\ &= \frac{3}{5}\mathbf{b} - (\mathbf{b} - \mathbf{a}) \\ &= \frac{3}{5}\mathbf{b} - \mathbf{b} + \mathbf{a} \\ &= \mathbf{a} - \frac{2}{5}\mathbf{b} \end{aligned}$$



$$\begin{aligned} \mathbf{DC} &= \mathbf{DB} + \mathbf{BC} \\ &= \frac{1}{2}\mathbf{b} + \mathbf{a} - \frac{2}{5}\mathbf{b} \\ &= \mathbf{a} + \frac{5}{10}\mathbf{b} - \frac{4}{10}\mathbf{b} \end{aligned}$$

$$\mathbf{DC} = \mathbf{a} + \frac{1}{10}\mathbf{b}$$

- 2.3 Find the vector in the same direction as $\langle 3, 4 \rangle$ which has three times the magnitude of $\langle 2, -5 \rangle$.

Solution

Let the required vector be \mathbf{a} . Let $\mathbf{b} = \langle 3, 4 \rangle$.

$$\begin{aligned} \text{then } |\mathbf{b}| &= |\langle 3, 4 \rangle| & |\langle 2, -5 \rangle| \\ \text{then } &= 5 & = \sqrt{2^2 + 5^2} \\ \text{so } \hat{\mathbf{b}} &= \frac{1}{5}\langle 3, 4 \rangle & = \frac{1}{\sqrt{29}} \end{aligned}$$

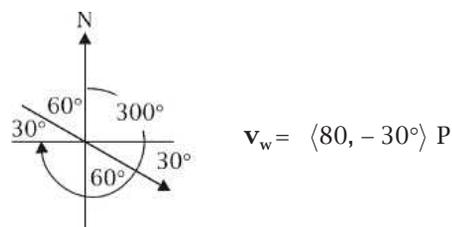
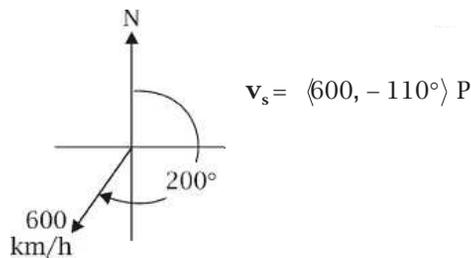
$$\begin{aligned} \mathbf{a} &= 3\sqrt{29} \times \frac{1}{5}\langle 3, 4 \rangle & \therefore |\mathbf{a}| = 3\sqrt{29} \\ &= \frac{3\sqrt{29}\langle 3, 4 \rangle}{5} \\ &= \left\langle \frac{9\sqrt{29}}{5}, \frac{12\sqrt{29}}{5} \right\rangle \text{ exact} \\ \mathbf{a} &= \langle 9.69, 12.92 \rangle \text{ (2dp)} \end{aligned}$$

- 2.4 A plane with an airspeed of 600km/h is being steered on a bearing of 200° . A steady wind of 80km/h is blowing from a bearing of 300° . Find the actual velocity of the plane.

Solution

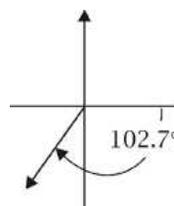
$$\mathbf{v}_a = \mathbf{v}_s + \mathbf{v}_w$$

$$\begin{aligned} \mathbf{v}_a &= \langle 600, -110^\circ \rangle + \langle 80, -30^\circ \rangle \text{ P} \\ &= \langle -205.21, -563.82 \rangle + \langle 69.28, -40 \rangle \text{ R} \\ &= \langle -135.93, -603.82 \rangle \text{ R} \\ &= \langle 618.93, -102.7^\circ \rangle \text{ P} \end{aligned}$$



$$\begin{aligned} \text{Bearing} &= 102.7^\circ + 90^\circ \\ &= 192.7^\circ \end{aligned}$$

The actual velocity of the plane is 619 km/h on a bearing of 193° .



2.5 Jan swims with a velocity of \mathbf{v}_s . The water that she is swimming in has a current of \mathbf{v}_c .

(a) What is Jan's actual velocity?

(b) A nearby duck in the water has a velocity such that $\mathbf{v}_{\text{duck}} - \mathbf{v}_{\text{current}} = 0$.
What is the duck doing?

(c) The duck observes Jan. Find $\mathbf{v}_{\text{Jan}}^{\text{duck}}$ and comment on its significance.

Solution

(a) $\mathbf{v}_{\text{actual}} = \mathbf{v}_a = \mathbf{v}_s + \mathbf{v}_c$

(b) $\mathbf{v}_{\text{duck}} - \mathbf{v}_{\text{current}} = 0 = \mathbf{v}_{\text{duck}} - \mathbf{v}_{\text{current}}$

$$\mathbf{v}_{\text{duck}} = \mathbf{v}_{\text{current}}$$

So the duck is not swimming at all but simply drifting with the current.

(c) $\mathbf{v}_{\text{Jan}}^{\text{duck}} = \mathbf{v}_{\text{Jan}} - \mathbf{v}_{\text{duck}}$ But $\mathbf{v}_{\text{Jan}} = \mathbf{v}_a$ and $\mathbf{v}_{\text{duck}} = \mathbf{v}_c$

$$\begin{aligned} &= \mathbf{v}_a - \mathbf{v}_c \\ &= \mathbf{v}_s + \mathbf{v}_c - \mathbf{v}_c \\ &= \mathbf{v}_s \end{aligned}$$

So by not swimming at all the duck observes Jan as having a velocity of \mathbf{v}_s which is the velocity that she has in still water.

PROBLEMS TO SOLVE

CHAPTER 2: VECTORS I

1. Find the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ which has a polar angle of $\frac{5\pi}{4}$ and a magnitude of 5.

2. Given $\mathbf{a} = -3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j}$, find:

- (a) a unit vector in the same direction as \mathbf{b}

- (b) a vector in the same direction as \mathbf{b} but equal in magnitude to \mathbf{a}

- (c) $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$

3. If $\mathbf{a} = \langle 3, 4 \rangle$, $\mathbf{b} = \langle 7, 24 \rangle$ and $\mathbf{c} = \langle 8, -15 \rangle$, find:

- (a) $2\mathbf{a} - \mathbf{c} - \mathbf{b}$

- (b) a vector that is parallel to \mathbf{a} and three times as long as \mathbf{a} .

4. Consider the points A (3, 4), B (-4, 5) and C (-2, -3). Use vector methods and notation to find:

- (a) the coordinates of D where D is the midpoint of the straight line joining A and B

- (b) the centroid E of $\triangle ABC$ where $\mathbf{OE} = \frac{1}{3}(\mathbf{OA} + \mathbf{OB} + \mathbf{OC})$

(c) the exact length of the straight line joining A and C

(d) a unit vector in the same direction as **BC**

(e) a vector which has the same direction as **BC** and three times the magnitude of **AC**.
(Give the components correct to 2 dp.)

5. Fran designs an orienteering course with 3 legs:
- O to A for a distance of 5 km on a bearing of 330°
 - A to B for a distance of 7 km on a bearing of 100°
 - B back to the start, O.

(a) Write the vector **BO** in terms of **OA** and **AB**.

(b) Find the vector **BO** in polar coordinates.

(c) Define the distance and bearing required for the third (i.e. final) leg of her course.

6. If $\mathbf{a} = \langle -2, 1 \rangle$ and $\mathbf{b} = \langle -1, -1 \rangle$, define the vectors below in terms of **a** and/or **b**.

(a) $\mathbf{c} = \langle 2, -1 \rangle$

(b) $\mathbf{d} = \langle 3, 3 \rangle$

(c) $\mathbf{e} = \langle 1, -2 \rangle$

(d) $\mathbf{f} = \langle 3, 0 \rangle$

(e) $\mathbf{g} = \langle 5, 4 \rangle$

7. Peter, flying with an air speed of 300 km/h wants to reach a town 140 km due west of his present location. The prevailing wind is $17\mathbf{i} + 33\mathbf{j}$ km/h.

(a) What course should Peter set?

(b) How long will the journey take?

8. Use a vector approach to determine whether A (2 , 4) , B (3 , 7) and C (-1 , -5) are collinear.

9. If $\mathbf{OA} = (3, -1)$ and $\mathbf{OB} = (5, 3)$ find the position of P on AB such that $2 \mathbf{AP} = 3 \mathbf{PB}$.

10. In the parallelogram OABC, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{c}$. The point D lies on \mathbf{AB} and $\mathbf{AD} : \mathbf{DB} = 1 : 2$. Express the following vectors in terms of \mathbf{a} and \mathbf{c} .

(a) \mathbf{BO}

(b) \mathbf{AC}

(c) \mathbf{AD}

(d) \mathbf{OD}

(e) \mathbf{DC}

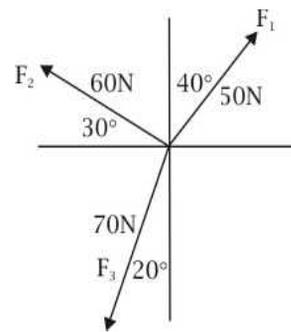
11. If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 7\mathbf{i} + 24\mathbf{j}$ find a vector \mathbf{v} which is parallel to \mathbf{a} and has the same magnitude as \mathbf{b} .

12. Ann's power boat is capable of a speed of 15 km/h. She sets off to cross a river flowing from East to West at 3 km/h.

(a) In what direction should she steer in order to cross the river from the South bank at right angles to the banks?

(b) How long will she take to cross the river, if it is 250 m wide?

13. Three forces act on a body as shown. Find the single force that has the same effect on the body as the three forces. Draw a diagram to show its magnitude and direction.



14. Leonie's aircraft can fly at 250 km/h in still air. It is to be flown from Hoptoun to Eclipse Island, 300 km from Hoptoun on a bearing of 310° . There is a wind of 40 km/h blowing from 020° . Find:

(a) the course Leonie must set to fly directly to Eclipse Island

(b) the time taken for the flight (to the nearest minute).

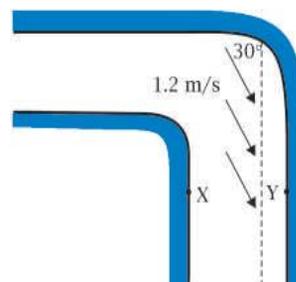
15. In an orienteering competition, competitors leave from the start, travel to Control 1, 1.35 km away on a bearing of 050° and then go to Control 2 which is 0.84 km from Control 1 and on a bearing of 110° . From Control 2 they head off on a bearing of 220° to Control 3 which is 1.06 km away. To complete the course they return to the start. Find the bearing and distance necessary for the runners to complete the last leg of the course.

16. ABCD is a quadrilateral. P and Q are the mid points of AD and DC respectively. Show that $\mathbf{PQ} = \mathbf{AP} + \mathbf{QC}$.

17. Three forces act on a body:
 50 N on a bearing of 40°
 49 N on a bearing of 180°
 75 N on a bearing of 340° .
 Find the magnitude and direction of the resultant force.

18. ABCD is a parallelogram and E is the point on CD such that $\mathbf{DE} : \mathbf{EC} = 3 : 1$. Let $\mathbf{u} = \mathbf{AD}$ and $\mathbf{v} = \mathbf{AB}$. Find \mathbf{BE} in terms of \mathbf{u} and \mathbf{v} .

19. Tahlia wishes to cross a river 150 m wide at a place where the current is flowing at an angle of 30° to the bank. She can swim 1.5 m/s in still water and the current is flowing at 1.2 m/s.



- (a) If Tahlia starts at point X and swims directly towards the bank at Y, where will she eventually come ashore?

(b) How far does she swim?

(c) What is her speed for the crossing?

20. ABCD is a quadrilateral where $\mathbf{a} = \mathbf{AB}$, $\mathbf{BD} = \mathbf{b}$ and $\mathbf{CA} = \mathbf{c}$. Find the following in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} :

(a) \mathbf{BC}

(b) \mathbf{AD}

(c) \mathbf{DC}

21. In a contest of strength, Alex has two ropes tied around his waist. One of the ropes is attached to Tim and the other to Paul. Tim and Paul attempt to move Alex by pulling on their ropes. If Tim's and Paul's force vectors are $6\mathbf{i} - 8\mathbf{j}$ N and $-\mathbf{i} - 3\mathbf{j}$ N respectively, find:

(a) the resultant force vector exerted on Alex due to the pull of the other two boys

(b) the force vector that Alex must apply to remain in his position

(c) the resultant force on Alex if Paul decides to pull in the same direction as Tim, assuming that the magnitudes of their forces don't change.

22. If A has position vector $\mathbf{OA} = \langle 3, 4 \rangle$ and B has position vector $\mathbf{OB} = \langle 6, -2 \rangle$, find

(a) C, such that $\mathbf{AC} : \mathbf{CB} = 1 : 2$

(b) D, such that $AD : BD = 5 : 3$

23. A yacht capable of sailing at 8 km/h is heading off on a bearing of 280° when it experiences a tide running at 3 km/h in a direction of 210° . Find the actual direction and speed of the yacht.

24. A plane wants to reach a destination 400 km away on a bearing of 260° . Unfortunately a wind of 50 km/h is blowing from a bearing of 160° . If the plane flies with an air speed of 300 km/h find:

(a) the actual velocity of the plane

(b) the bearing on which the plane should fly

(c) the time for the journey to the nearest 5 minutes.

25. A light aircraft has a maximum cruising speed in still air of 95 m/s. Jason needs to fly to his home field 100 km due East as quickly as possible, given that there is a wind of $6 \mathbf{i} + 15 \mathbf{j}$ m/s blowing. What course should he set, and how long will it take him to get home?

26. Triangle ABC has X a point on AC such that $AX = \frac{2}{3} AC$. If $BA = \mathbf{a}$ and $BX = \mathbf{b}$, show that $BC = \frac{1}{2}(3\mathbf{b} - \mathbf{a})$

27. A plane which had an air speed of 200 km/h set off at 8 am one morning on a bearing of 060° . The pilot knew that a wind of 40 km/h was blowing and at 11.12 am he found that he was 704 km away from his starting point. Find:

(a) the coordinates of his position at 11.12 am relative to his starting point

(b) the direction of the wind as a bearing.

28. Erica's plane usually averages 580 km/h on its trip from Sunbury to Wombat Flats, a distance of 480 km on a bearing of 025° . The wind speed last Monday was 70 km/h in a direction of 340° . On what course did Erica travel in order to get from Sunbury to Wombat Flats and how much faster was her trip?

29. If A (-5, 1) and B (5, 3) are two points find the position vector of P such that $AP : PB = 3 : -1$

30. In $\triangle ABC$, points D, E and F are the midpoints of sides BC, AC and AB respectively. Prove $\mathbf{AD} + \mathbf{BE} + \mathbf{CF} = \mathbf{0}$.

31. Using vector methods show that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half its length.

32. A parallelogram OABC has D the midpoint of AB, E the midpoint of BC and DC intersecting OE at F. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{c}$. Find the following in terms of \mathbf{a} and \mathbf{c} :

(a) \mathbf{OD}

(b) \mathbf{OE}

(c) \mathbf{DC}

(d) Now by letting $\mathbf{DF} = h \mathbf{DC}$ and $\mathbf{OF} = k \mathbf{OE}$, show that $h = \frac{3}{5}$ and $k = \frac{4}{5}$.

33. Prove that in a parallelogram OABC, if D is the midpoint of AB then OD cuts AC at the point of trisection nearest A.

(Hint: Let the point of intersection of OD and AC be M. Let $\mathbf{OM} = h \mathbf{OD}$ and $\mathbf{AM} = k \mathbf{AC}$.

34. In parallelogram ABCD, E is a point on AD such that it divides AD in the ratio 1 : 2. F is the point on the diagonal AC, such that it divides AC in the ratio 1 : 2. If G is on BC such that points E, F and G are collinear, show that point F also divides EG in the ratio 1 : 2.

This section expands vector methods to include vector representatives of lines and the dot product.

Syllabus Checklist

On the completion of this chapter you should be able to:

Algebra of vectors in the plane

- use ordered pair notation and column vector notation to represent a vector
- define unit vectors and the perpendicular unit vectors \mathbf{i} and \mathbf{j}
- express a vector in component form using the unit vectors \mathbf{i} and \mathbf{j}
- examine and use addition and subtraction of vectors in component form
- define and use multiplication of a vector by a scalar in component form
- define and use scalar (dot) product
- apply the scalar product to vectors expressed in component form
- examine properties of parallel and perpendicular vectors and determine if two vectors are parallel or perpendicular
- define and use projection of vectors
- solve problems involving displacement, force and velocity involving the above concepts

SCALAR OR DOT PRODUCT

Introduction. What does it mean?

When the dot product of two vectors \mathbf{a} and \mathbf{b} notated as $\mathbf{a} \cdot \mathbf{b}$ is computed the answer is always a scalar not a vector quantity. If this scalar answer is zero then \mathbf{a} and \mathbf{b} are perpendicular vectors. When it is not zero, the dot product is mostly used as a stepping stone to find out other useful quantities. So instead of asking what does the dot product mean it is more instructive to ask how it is used.

Development of the Dot Product

Consider the points A and B where $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$

$$\text{Let } \mathbf{a} = \langle a_1, a_2 \rangle \text{ and } \mathbf{b} = \langle b_1, b_2 \rangle$$

$$\text{then } |\mathbf{a}| = \sqrt{a_1^2 + a_2^2} \text{ and } |\mathbf{b}| = \sqrt{b_1^2 + b_2^2}$$

$$\text{and } |\mathbf{a}|^2 = a_1^2 + a_2^2, \quad |\mathbf{b}|^2 = b_1^2 + b_2^2$$

Also $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$

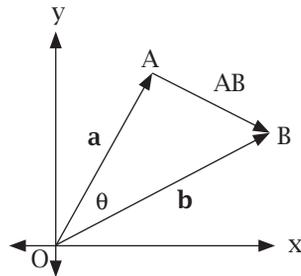
$$= \mathbf{b} - \mathbf{a}$$

$$= \langle b_1, b_2 \rangle - \langle a_1, a_2 \rangle$$

$$= \langle b_1 - a_1, b_2 - a_2 \rangle$$

$$|\mathbf{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$

and $|\mathbf{AB}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2$



Let θ be the angle between the vectors \mathbf{a} and \mathbf{b}

By the cosine rule

$$|\mathbf{AB}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\text{So } (b_1 - a_1)^2 + (b_2 - a_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

which when expanded out and simplified (you should confirm this) results in

$$a_1 b_1 + a_2 b_2 = |\mathbf{a}||\mathbf{b}|\cos\theta \quad \text{-----(1)}$$

This equation is used to find the angle θ between \mathbf{a} and \mathbf{b} when these vectors are specified in rectangular form. However the real power of mathematical notation is seen now when each side of equation (1) is called $\mathbf{a} \cdot \mathbf{b}$. i.e. the scalar product of \mathbf{a} and \mathbf{b} , which is a sort of vector multiplication.

$$\text{So now } \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 \quad \text{-----(2)}$$

$$\text{and } \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad \text{-----(3)}$$

The dot product notation $\mathbf{a} \cdot \mathbf{b}$ serves as a link between the two sides of equation (1) in keeping with it acting as a stepping stone quantity.

Also the equation (2) definition of the dot product can be thought of as the **rectangular version** as it involves the components of \mathbf{a} and \mathbf{b} whereas the equation (3) definition can be thought of as the **polar version** as it involves the magnitudes of \mathbf{a} and \mathbf{b} and the angle between them.

PROPERTIES OF THE DOT PRODUCT

- The angle θ between vectors \mathbf{a} and \mathbf{b} is given by rearranging equation (3)

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where $\mathbf{a} \cdot \mathbf{b}$ is computed from $a_1 b_1 + a_2 b_2$

- If $\theta = 90^\circ$, \mathbf{a} and \mathbf{b} are perpendicular and equation (3) becomes

$$\mathbf{a} \cdot \mathbf{b} = 0$$

- If $\mathbf{a} = \mathbf{b}$ then $\theta = 0^\circ$ and equation (3) becomes

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}| |\mathbf{a}|$$

$$\text{ie } \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

- For scalars r and s it can be shown that

$$r\mathbf{a} \cdot s\mathbf{b} = rs\mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$$

and the above algebraic properties of the dot product demonstrate the similarities with our usual rules of algebra. However there is at least one major exception which is

- If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ then it does not automatically mean that $\mathbf{b} = \mathbf{c}$!!

Example Find the angle between $\mathbf{a} = \langle 3, 2 \rangle$ and $\mathbf{b} = \langle -5, 4 \rangle$ correct to 2dp in degrees.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1 b_1 + a_2 b_2 & |\mathbf{a}| &= \sqrt{3^2 + 2^2} = \sqrt{13} \\ &= 3 \times -5 + 4 \times 2 & |\mathbf{b}| &= \sqrt{5^2 + 4^2} = \sqrt{41} \\ &= -7 \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{-7}{\sqrt{13} \sqrt{41}} \\ \therefore \theta &= 107.65^\circ \end{aligned}$$

Example If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ then determine the relationship between the vectors and give an example.

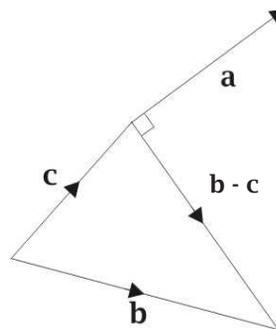
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$$

$$\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} = 0$$

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$$

So \mathbf{a} and $\mathbf{b} - \mathbf{c}$ are perpendicular

A possible diagram is



$$\begin{aligned} \text{If } \mathbf{a} &= \langle 6, 3 \rangle, \mathbf{b} = \langle 4, -1 \rangle \text{ and } \mathbf{c} = \langle 2, 3 \rangle \\ \text{then } \mathbf{a} \cdot \mathbf{b} &= \langle 6, 3 \rangle \cdot \langle 4, -1 \rangle \text{ and } \mathbf{a} \cdot \mathbf{c} = \langle 6, 3 \rangle \cdot \langle 2, 3 \rangle \\ &= 24 - 3 & &= 12 + 9 \\ &= 21 & &= 21 \end{aligned}$$

So $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 21$ and $\mathbf{b} \neq \mathbf{c}$

$$\begin{aligned} \text{Also } \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) &= \langle 6, 3 \rangle \cdot (\langle 4, -1 \rangle - \langle 2, 3 \rangle) \\ &= \langle 6, 3 \rangle \cdot \langle 2, -4 \rangle \\ &= 12 + -12 \\ &= 0 \text{ as expected showing that} \\ &\quad \mathbf{a} \text{ is perpendicular to } \mathbf{b} - \mathbf{c} \end{aligned}$$

Note 1. If $\mathbf{a} = \langle a_1, a_2 \rangle$ then the gradient of \mathbf{a} is $m_a = \frac{a_2}{a_1}$ i.e. $\frac{\text{rise}}{\text{run}}$.

Also if $\mathbf{b} = \langle b_1, b_2 \rangle$ then $m_b = \frac{b_2}{b_1}$

From your other studies you may have learnt that perpendicular lines have gradients which multiply to give -1 (i.e. they are negative reciprocals of each other).

$$\text{So } m_a \times m_b = -1$$

$$\frac{a_2}{a_1} \times \frac{b_2}{b_1} = -1$$

$$\frac{a_2 b_2}{a_1 b_1} = -1$$

$$a_2 b_2 = -a_1 b_1$$

$$a_1 b_1 + a_2 b_2 = 0$$

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle$$

$$\text{So } \mathbf{a} \cdot \mathbf{b} = 0$$

Which demonstrates that $m_a \times m_b = -1$ is equivalent to $\mathbf{a} \cdot \mathbf{b} = 0$.

Note 2. If $\mathbf{a} = \langle a_1, a_2 \rangle$ then the perpendicular vectors to \mathbf{a} are in the form of $\mathbf{b} = k \langle a_2, -a_1 \rangle$ as is shown by

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \langle a_1, a_2 \rangle \cdot k \langle a_2, -a_1 \rangle \\ &= k (a_1 a_2 + a_2 (-a_1)) \\ &= k (a_1 a_2 - a_1 a_2) \\ &= 0 \end{aligned}$$

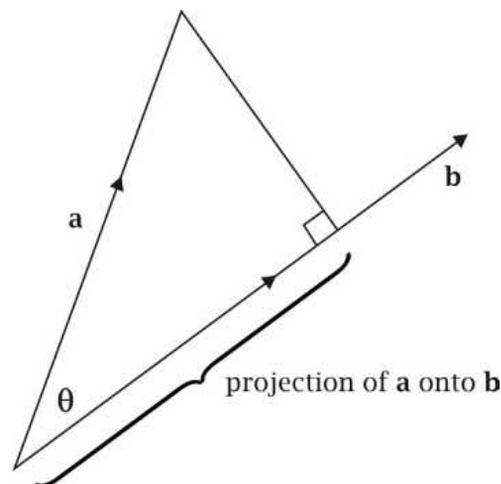
Example Find two vectors perpendicular to $\mathbf{a} = \langle 3, -4 \rangle$

$$\text{So } \mathbf{b}_1 = \langle -4, -3 \rangle \text{ and } \mathbf{b}_2 = \langle 4, 3 \rangle$$

The quick rule to find a perpendicular vector to \mathbf{a} is to swap the x and y components of \mathbf{a} and change **one** of the signs

PROJECTIONS

For two vectors \mathbf{a} and \mathbf{b} it is sometimes necessary to find the projection of one vector (say \mathbf{a}) onto the other vector \mathbf{b} . This projection is a new vector as shown in the diagram below.



The magnitude of the projection vector is $|a|\cos\theta$. The direction of the projection vector is represented by the unit vector for \mathbf{b} ie $\frac{\mathbf{b}}{|\mathbf{b}|}$. Combining these gives the required projection vector as

$$\begin{aligned} \text{Projection of } \mathbf{a} \text{ onto } \mathbf{b} &= \frac{\mathbf{b}}{|\mathbf{b}|} \times |a|\cos\theta \\ &= \frac{|a||b|\cos\theta}{|b|^2} \mathbf{b} \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} \end{aligned}$$

Example: Find the vector which is the projection of $\mathbf{a} = \langle 6, 8 \rangle$ onto $\mathbf{b} = \langle 12, -5 \rangle$.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \langle 6, 8 \rangle \cdot \langle 12, -5 \rangle & |\mathbf{b}| &= \sqrt{12^2 + 5^2} \\ &= 6 \times 12 + 8 \times -5 & &= 13 \\ &= 72 - 40 \\ &= 32 \end{aligned}$$

$$\text{Projection vector} = \frac{32}{13^2} \langle 12, -5 \rangle = \left\langle \frac{384}{169}, -\frac{160}{169} \right\rangle$$

FORCE DISPLACEMENT AND WORK

- If a body undergoes a displacement of \mathbf{s} through the action of a force \mathbf{F} then the work done by the force is $\mathbf{F} \cdot \mathbf{s}$. This is a case where the dot product gives a quantity which means something in its own right i.e. work.
- If after finding $\mathbf{F} \cdot \mathbf{s}$ the result is negative then it can be concluded that work is being done **against** \mathbf{F} and not by it.

Example:

A force of $\mathbf{F} = \langle 15, 12 \rangle$ Newtons acts on a body and moves it 4 m up a flat slope inclined at 30° to the horizontal. If the force due to its weight is $\langle 0, -40 \rangle$ Newtons and the force due to friction is 5 Newtons down the slope find the work done by (or against) each force.

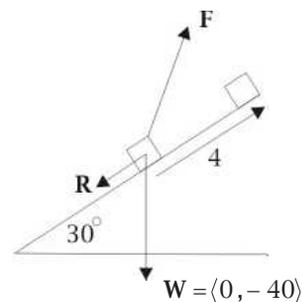
$$\begin{aligned} \text{Displacement } \mathbf{s} &= \langle 4, 30^\circ \rangle \text{ P} \\ &= \langle 3.464, 2 \rangle \text{ R} \end{aligned}$$

$$\begin{aligned} \text{Frictional Resistance } \mathbf{R} &= \langle 5, -150^\circ \rangle \text{ P} \\ &= \langle -4.33, -2.5 \rangle \text{ R} \end{aligned}$$

$$\begin{aligned} \text{Work done by } \mathbf{F} &= \mathbf{F} \cdot \mathbf{s} \\ &= \langle 15, 12 \rangle \cdot \langle 3.464, 2 \rangle \\ &= 75.96 \text{ joules} \end{aligned}$$

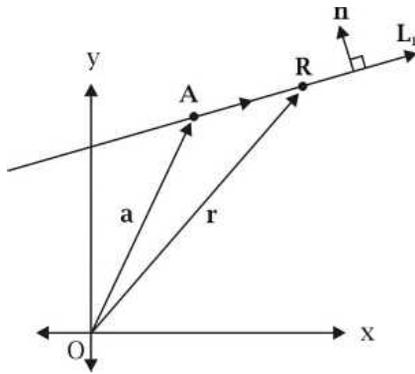
$$\begin{aligned} \text{Work done by } \mathbf{W} &= \mathbf{W} \cdot \mathbf{s} \\ &= \langle 0, -40 \rangle \cdot \langle 3.464, 2 \rangle \\ &= -80 \text{ joules so work done against } \mathbf{W} \text{ is } 80 \text{ joules} \end{aligned}$$

$$\begin{aligned} \text{Work done by } \mathbf{R} &= \mathbf{R} \cdot \mathbf{s} \\ &= \langle -4.33, -2.5 \rangle \cdot \langle 3.464, 2 \rangle \\ &= -20 \text{ joules so work done against } \mathbf{R} \text{ is } 20 \text{ joules} \end{aligned}$$



NORMAL FORM OF THE VECTOR EQUATION OF A STRAIGHT LINE

- The line L_1 passes through a known point A and is perpendicular (normal) to a known vector \mathbf{n} .
If R (x,y) is any point on the line then:



$$\begin{aligned} \mathbf{AR} &= \mathbf{r} - \mathbf{a} \\ \mathbf{AR} \cdot \mathbf{n} &= 0 \\ (\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} &= 0 \\ \mathbf{r} \cdot \mathbf{n} - \mathbf{a} \cdot \mathbf{n} &= 0 \\ \mathbf{r} \cdot \mathbf{n} &= \mathbf{a} \cdot \mathbf{n} \end{aligned}$$

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \quad \text{where} \quad \begin{aligned} \mathbf{r} &= \langle x, y \rangle && \text{is any point on the line} \\ \mathbf{a} &= \langle a_1, a_2 \rangle && \text{is a given point on the line} \\ \mathbf{n} &= \langle n_1, n_2 \rangle && \text{is a given normal vector to the line} \end{aligned}$$

is called the **normal form** or **scalar product** form of the vector equation of a straight line

The form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ reduces to the Cartesian form easily because after substitution

$$\langle x, y \rangle \cdot \langle n_1, n_2 \rangle = \langle a_1, a_2 \rangle \cdot \langle n_1, n_2 \rangle$$

$n_1 x + n_2 y = a_1 n_1 + a_2 n_2$ which is more usually written as $ax + by = c$

where $\mathbf{n} = \langle a, b \rangle$ and $c = a_1 n_1 + a_2 n_2$ is a scalar constant.

Example

If $\mathbf{n} = \langle 2, 3 \rangle$ and $\mathbf{a} = \langle -2, 4 \rangle$ find the normal form and Cartesian equations of the line.

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

$$\begin{aligned} \mathbf{r} \cdot \langle 2, 3 \rangle &= \langle -2, 4 \rangle \cdot \langle 2, 3 \rangle \\ &= -4 + 12 \end{aligned}$$

$$\mathbf{r} \cdot \langle 2, 3 \rangle = 8 \quad \text{this is normal form}$$

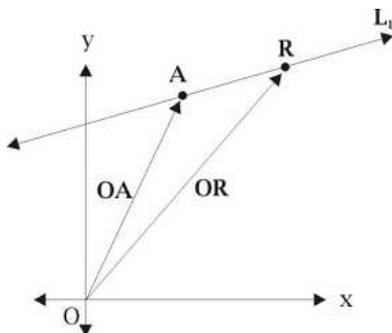
$$\langle x, y \rangle \cdot \langle 2, 3 \rangle = 8$$

$$2x + 3y = 8 \quad \text{this is the Cartesian equation}$$

VECTOR EQUATION OF A STRAIGHT LINE INVOLVING A PARAMETER

- The line L_1 is drawn on a set of x-y axes so it
 - passes through a given point A
 - is parallel to a given vector \mathbf{u} .

If R (x, y) is any point on the line then $\mathbf{OR} = \mathbf{OA} + \mathbf{AR}$.



\mathbf{AR} can be written as a scalar multiple of \mathbf{u}
and if $\mathbf{a} = \mathbf{OA}$
 $\mathbf{r} = \mathbf{OR}$
then $\mathbf{OR} = \mathbf{OA} + \mathbf{AR}$
becomes $\mathbf{r} = \mathbf{a} + t\mathbf{u}$

The general vector equation of a straight line is

$$\mathbf{r} = \mathbf{a} + t\mathbf{u}$$

where $\mathbf{r} = \mathbf{OR} = \langle x, y \rangle$
 $\mathbf{a} = \mathbf{OA} = \langle a_1, a_2 \rangle$
 $\mathbf{u} = \langle u_1, u_2 \rangle$
 $t = \text{scalar multiple}$

Example:

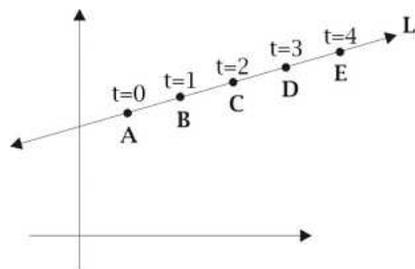
For $A(2, 3)$ and $\mathbf{u} = \langle 4, 5 \rangle$, then $\mathbf{r} = \langle 2, 3 \rangle + t \langle 4, 5 \rangle$

The value of t chosen determines the place on the line, so if $t = 3$,

$$\begin{aligned} \mathbf{r} &= \langle 2, 3 \rangle + 3 \langle 4, 5 \rangle \\ &= \langle 14, 18 \rangle \end{aligned}$$

A table of values illustrates how the line L_2 is “traced out” as t increases.

t	point	(x, y)
0	A	(2,3)
1	B	(6,8)
2	C	(10,13)
3	D	(14,18)
4	E	(18,23)
5	F	(22,28)



It is important to recognise that the vector between adjacent points on the line is the parallel vector i.e. \mathbf{u} in the general equation.

$$\begin{aligned} \mathbf{AB} &= \langle 4, 5 \rangle, \quad \mathbf{BC} = \langle 4, 5 \rangle, \quad \mathbf{CD} = \langle 4, 5 \rangle \\ \mathbf{DE} &= \langle 4, 5 \rangle, \quad \mathbf{EF} = \langle 4, 5 \rangle, \text{ etc} \end{aligned}$$

- For a line L_2 passing through two given points A and B the parallel vector \mathbf{u} can be

$$\begin{aligned} \mathbf{u} &= \mathbf{OB} - \mathbf{OA} \\ &= \mathbf{b} - \mathbf{a} \end{aligned}$$

Then $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ i.e. $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ is the equation of the straight line through A and B .

Note that the parallel vector could have been $\mathbf{a} - \mathbf{b}$ and the given point could have been B not A . so the line would now be written as

$$\mathbf{r} = \mathbf{b} + t(\mathbf{a} - \mathbf{b})$$

The point is that both equations above trace out the same line L_2 but the starting points (i.e. when $t = 0$) and their directions are different.

Example: Find a vector equation of the line through $(5, 6)$ and $(7, -2)$.

$$\begin{aligned} \mathbf{u} &= \langle 5, 6 \rangle - \langle 7, -2 \rangle \\ &= \langle -2, 8 \rangle \quad \therefore \mathbf{r} = \langle 5, 6 \rangle + t \langle -2, 8 \rangle \end{aligned}$$

Note: Alternative answers to this question are:

$$\begin{aligned} \mathbf{r} &= \langle 5, 6 \rangle + t \langle 2, -8 \rangle, \quad \mathbf{r} = \langle 7, -2 \rangle + t \langle -2, 8 \rangle \\ \mathbf{r} &= \langle 7, -2 \rangle + t \langle 2, -8 \rangle, \quad \mathbf{r} = \langle 5, 6 \rangle + t \langle 1, -4 \rangle \text{ etc} \end{aligned}$$

The line traced out by each equation is the same.

- The equation $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ can be written in **rectangular** or **component form** as

$$\begin{aligned}\mathbf{r} &= \langle a_1, a_2 \rangle + t \langle u_1, u_2 \rangle \\ &= \langle a_1, a_2 \rangle + \langle tu_1, tu_2 \rangle\end{aligned}$$

i.e. $\mathbf{r} = \langle a_1 + tu_1, a_2 + tu_2 \rangle$

This is an important form of the vector equation of a straight line and it leads on to:

$$\langle x, y \rangle = \langle a_1 + tu_1, a_2 + tu_2 \rangle$$

i.e. $x = a_1 + tu_1$ and $y = a_2 + tu_2$

These are called the **parametric equations** or the **scalar equations** of the line. If the **parameter t** is eliminated, the resulting equation becomes the **Cartesian equation** of the line.

Example: Find where the line through $(-8, -7)$ and parallel to $\langle 2, 3 \rangle$ intersects the line through $(-6, 14)$ and parallel to $\langle -4, 3 \rangle$.

$$\begin{aligned}\text{let } L_1 \text{ be } \mathbf{r}_1 &= \langle -8, -7 \rangle + t \langle 2, 3 \rangle \\ &= \langle 2t - 8, 3t - 7 \rangle\end{aligned}$$

$$\begin{aligned}\text{let } L_2 \text{ be } \mathbf{r}_2 &= \langle -6, 14 \rangle + \lambda \langle -4, 3 \rangle \\ &= \langle -4\lambda - 6, 3\lambda + 14 \rangle\end{aligned}$$

To find the intersection point, equate the x components of L_1 and L_2 and do the same with the y components.

$$\text{So } 2t - 8 = -4\lambda - 6 \quad \text{and} \quad 3t - 7 = 3\lambda + 14$$

$$\text{i.e. } 2t + 4\lambda = 2 \quad \quad \quad 3t - 3\lambda = 21$$

$$\text{or } t + 2\lambda = 1 \quad \quad \quad \text{or } t - \lambda = 7$$

Solving these equations simultaneously gives $t = 5$ and $\lambda = -2$. The point of intersection then is found from.

$$\begin{aligned}\mathbf{r}_1(5) &= \langle 2(5) - 8, 3(5) - 7 \rangle \\ &= \langle 2, 8 \rangle\end{aligned}$$

i.e. the required point is $(2, 8)$ and checked by finding

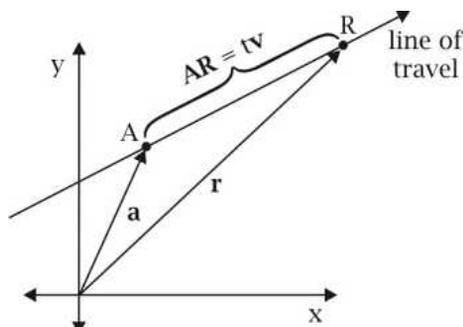
$$\begin{aligned}\mathbf{r}_2(-2) &= \langle -4(-2) - 6, 3(-2) + 14 \rangle \\ &= \langle 2, 8 \rangle\end{aligned}$$

i.e. $(2, 8)$ as before

USING LINES IN PARAMETER FORM TO REPRESENT MOTION

The vector equation of a straight line $\mathbf{r} = \mathbf{a} + t\mathbf{v}$ is useful for describing the position of an object travelling in a straight line with constant velocity \mathbf{v} as a function of time t . In the above equation the vector $t\mathbf{v}$ represents the displacement of the object from the $t = 0$ position of A.

A diagram is helpful here.



So when $t=0$ the object is at A which has a position vector of $\mathbf{OA} = \mathbf{a}$.

After t units of time the object has displaced itself by $\mathbf{AR} = t\mathbf{v}$ from A in the direction of its line of travel. At time t the object has a position vector of $\mathbf{r} = \mathbf{a} + t\mathbf{v}$.

Example: At 6 am a ship is at A $(-20, 8)$ km travelling with a speed of $\sqrt{89}$ km/h on a bearing of 122° . Represent the ship's path after 6 am as seen by an observer

(a) At the origin $(0, 0)$

(b) At B $(7, -10)$

(a) Change the velocity of the ship to rectangular form

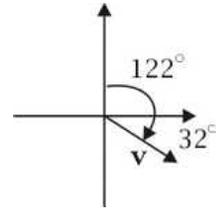
$$x = \sqrt{89} \cos 32^\circ = 8 \text{ km}$$

$$y = -\sqrt{89} \sin 32^\circ = -5 \text{ km}$$

$$\mathbf{OR} = \langle -20, 8 \rangle + t \langle 8, -5 \rangle = \langle 8t - 20, 8 - 5t \rangle \text{ km}$$

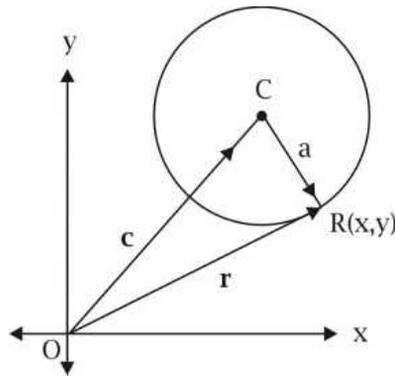
(b) $\mathbf{BR} = \mathbf{OR} - \mathbf{OB}$

$$= \langle 8t - 20, 8 - 5t \rangle - \langle 7, -10 \rangle = \langle 8t - 27, 18 - 5t \rangle \text{ km}$$



EQUATION OF CIRCLES

- A circle has centre $C(c_1, c_2)$ with radius a , and $R(x, y)$ is any point on the circle i.e.



$$\mathbf{OC} + \mathbf{CR} = \mathbf{OR}$$

$$\mathbf{OR} - \mathbf{OC} = \mathbf{CR}$$

$$\mathbf{r} - \mathbf{c} = \mathbf{CR}$$

$$|\mathbf{r} - \mathbf{c}| = |\mathbf{CR}|$$

$$|\mathbf{r} - \mathbf{c}| = a$$

$|\mathbf{r} - \mathbf{c}| = a$ is the vector equation of a circle where \mathbf{c} is the position vector of the centre and a = radius.

- The cartesian equation for a circle with centre $C(c_1, c_2)$ and radius a , follows from the vector equation as

$$|\mathbf{r} - \mathbf{c}| = a$$

$$|\langle x, y \rangle - \langle c_1, c_2 \rangle| = a$$

$$|\langle x - c_1, y - c_2 \rangle| = a$$

$$\sqrt{(x - c_1)^2 + (y - c_2)^2} = a$$

or $(x - c_1)^2 + (y - c_2)^2 = a^2$

Example (a)

If $\mathbf{c} = \langle 3, -4 \rangle$ and $a = \sqrt{34}$

then the vector equation of the circle is $|\mathbf{r} - \langle 3, -4 \rangle| = \sqrt{34}$

and $(x - 3)^2 + (y + 4)^2 = 34$ is the cartesian equation of the circle.

The cartesian equation can be written free of brackets as

$$x^2 - 6x + 9 + y^2 + 8y + 16 = 34$$

$$\text{or as } x^2 + y^2 - 6x + 8y - 9 = 0$$

Example (b)

Find where the above circle and the line through (13, 19) and parallel to $\langle 1, 4 \rangle$ intersect

$$\begin{aligned}\text{The equation of the line is } \mathbf{r} &= \langle 13, 19 \rangle + t \langle 1, 4 \rangle \\ &= \langle t + 13, 4t + 19 \rangle\end{aligned}$$

$$\text{i.e. } x = t + 13, y = 4t + 19$$

Sub into circle equation

$$(x - 3)^2 + (y + 4)^2 = 34$$

$$\text{i.e. } (t + 13 - 3)^2 + (4t + 19 + 4)^2 = 34$$

$$(t + 10)^2 + (4t + 23)^2 = 34$$

$$\text{or } t^2 + 20t + 100 + 16t^2 + 184t + 529 = 34$$

$$17t^2 + 204t + 595 = 0$$

$$t = -5, t = -7$$

$$\begin{aligned}\text{Sub back } \mathbf{r}(-5) &= \langle -5 + 13, 4 \times (-5) + 19 \rangle \\ &= \langle 8, -1 \rangle\end{aligned}$$

$$\begin{aligned}\text{and } \mathbf{r}(-7) &= \langle -7 + 13, 4(-7) + 19 \rangle \\ &= \langle 6, -9 \rangle\end{aligned}$$

So the line and the circle intersect at (8, -1) and (6, -9)

COLLISIONS

Two objects moving in space will collide if they are at the same place at the same time. In two dimensions there are four methods for determining whether two objects collide and if so, the time at which this happens and the place of impact.

Example 1: Shows the first method.

Consider an object A moving with velocity $\mathbf{v}_A = \langle 2, 3 \rangle$ which has a position at $t = 0$ of $A_0(-9, -8)$ and another object B moving with velocity $\mathbf{v}_B = \langle -1, 4 \rangle$ which has a position at time $t = 0$ of $B_0(9, -14)$. Find out whether the objects collide and if so when and where.

$$\begin{aligned}\mathbf{r}_A &= \langle -9, -8 \rangle + t \langle 2, 3 \rangle & \mathbf{r}_B &= \langle 9, -14 \rangle + t \langle -1, 4 \rangle \\ &= \langle -9 + 2t, -8 + 3t \rangle & &= \langle 9 - t, -14 + 4t \rangle\end{aligned}$$

Let the x values be the same

$$\begin{aligned}-9 + 2t &= 9 - t \\ \text{i.e. } t &= 6\end{aligned}$$

Let the y values be the same

$$\begin{aligned}-8 + 3t &= -14 + 4t \\ \text{i.e. } t &= 6\end{aligned}$$

$$\begin{aligned}\text{A and B have the same position } (x, y) \text{ when } t = 6 \text{ of } \mathbf{r}_A &= \langle -9 + 2(6), -8 + 3(6) \rangle \\ &= \langle 3, 10 \rangle\end{aligned}$$

So the objects collide at (3, 10) when $t = 6$

Example 2: Shows the second method.

Consider the motion of two boats A and B moving according to the details given below. Find whether or not they collide. If they do collide find when and where, and if they do not collide find the position vector of where their paths intersect.

A: $\mathbf{v}_A = \langle 15, -20 \rangle$ km/h
 $\mathbf{r}_A = \langle -18, 15 \rangle$ km at 6.48 am.

B: $\mathbf{v}_B = \langle 24, 16 \rangle$ km/h
 $\mathbf{r}_B = \langle -48, -221 \rangle$ km at 8.15 am.

This problem could be done by first finding where ship A was at 8:15am and then by using this place as the $t = 0$ position, then the first method above will soon find out if they collide. However if the ships do not collide more work will need to be done to find where the paths cross. The method below allows for both eventualities!

Let $\mathbf{r}_A = \langle -18, 15 \rangle + t \langle 15, -20 \rangle$
 $= \langle 15t - 18, 15 - 20t \rangle$
 and $\mathbf{r}_B = \langle -48, -221 \rangle + \lambda \langle 24, 16 \rangle$
 $= \langle 24\lambda - 48, 16\lambda - 221 \rangle$
 let $15t - 18 = 24\lambda - 48$ and $15 - 20t = 16\lambda - 221$
 i.e. $15t - 24\lambda = -30$ and $20t + 16\lambda = 236$

Solving these two equations simultaneously either algebraically or by using your graphics calculator gives $t = 7.2$ hours and $\lambda = 5.75$ hours.

Now $\mathbf{r}_A(7.2) = \langle 15(7.2) - 18, 15 - 20(7.2) \rangle$ and $\mathbf{r}_B(5.75) = \langle 24(5.75) - 48, 16(5.75) - 221 \rangle$
 $= \langle 90, -129 \rangle$ km $= \langle 90, -129 \rangle$ km

So the ships paths cross at (90, -129) but do they collide? Well the time that A gets to (90, -129) km is
 $6.48 + 7.2$ hrs
 $= 6.48 + 7$ hrs 12 min
 $= 2$ pm

And the time that B gets to (90, -129) km is
 8.15 am + 5.75 hrs
 $= 8.15$ am + 5hrs 45 mins
 $= 2$ pm

Because both ships get to (90, -129) at 2pm they must have collided!!

The third method to show that two object collide is shown in the next section.

RELATIVE VELOCITY AND COLLISIONS

If at $t = 0$ two objects A and B have positions of \mathbf{A}_0 and \mathbf{B}_0 and velocities of \mathbf{v}_A and \mathbf{v}_B then the equations of their paths are

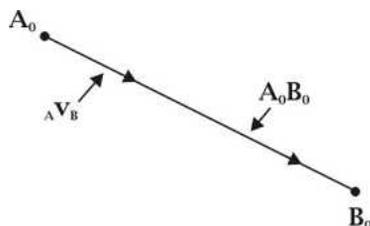
$\mathbf{OA} = \mathbf{OA}_0 + t \mathbf{v}_A$ where $\mathbf{OA} = \mathbf{r}_A$
 and $\mathbf{OB} = \mathbf{OB}_0 + t \mathbf{v}_B$ $\mathbf{OB} = \mathbf{r}_B$

The displacement $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$
 $= \mathbf{OB}_0 + t \mathbf{v}_B - (\mathbf{OA}_0 + t \mathbf{v}_A)$
 $= \mathbf{OB}_0 + t \mathbf{v}_B - \mathbf{OA}_0 - t \mathbf{v}_A$
 $= \mathbf{OB}_0 - \mathbf{OA}_0 + t(\mathbf{v}_B - \mathbf{v}_A)$
 $= \mathbf{A}_0 \mathbf{B}_0 + t_B \mathbf{v}_A$

The objects will collide if $\mathbf{AB} = 0$. For this to happen the equation for \mathbf{AB} becomes

$\mathbf{A}_0 \mathbf{B}_0 + t_B \mathbf{v}_A = 0$
 $\mathbf{A}_0 \mathbf{B}_0 = -t_B \mathbf{v}_A$
 $\mathbf{A}_0 \mathbf{B}_0 = t_A \mathbf{v}_B$

The last equation is the condition necessary for A and B to collide. $\mathbf{A}_0 \mathbf{B}_0$ is called the initial (because $t = 0$) displacement. ${}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}}$ is the velocity of A as seen by someone travelling on object B with a velocity of $\mathbf{v}_{\mathbf{B}}$. If we pretend that B is stationary observing A with the velocity ${}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}}$ then the diagram is:



So if the equation $\mathbf{A}_0 \mathbf{B}_0 = t {}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}}$ is true then the diagram shows that a collision will happen when B observes A coming straight at him! And then after t units of time the collision will happen.

In summary then

If it can be shown that either $\mathbf{A}_0 \mathbf{B}_0 = t {}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}}$ or $\mathbf{B}_0 \mathbf{A}_0 = t {}_{\mathbf{B}}\mathbf{v}_{\mathbf{A}}$ then objects A and B will collide after t units of time.

Example: Shows the third method for collisions.

Two objects A and B are at $\mathbf{A}_0 (-9, -8)$ and $\mathbf{B}_0 (9, -14)$ when $t = 0$ hours moving with velocities $\mathbf{v}_{\mathbf{A}} = \langle 2, 3 \rangle$ km/h and $\mathbf{v}_{\mathbf{B}} = \langle -1, 4 \rangle$ km/h. Show that the objects collide and find the place of the collision.

$$\begin{aligned} \mathbf{A}_0 \mathbf{B}_0 &= \mathbf{OB}_0 - \mathbf{OA}_0 & {}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}} &= \mathbf{v}_{\mathbf{A}} - \mathbf{v}_{\mathbf{B}} \\ &= \langle 9, -14 \rangle - \langle -9, -8 \rangle & &= \langle 2, 3 \rangle - \langle -1, 4 \rangle \\ &= \langle 18, -6 \rangle & &= \langle 3, -1 \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{A}_0 \mathbf{B}_0 &= \langle 18, -6 \rangle \\ &= 6 \langle 3, -1 \rangle \end{aligned}$$

$$\mathbf{A}_0 \mathbf{B}_0 = 6 {}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}} \quad \text{so A and B collide after 6 hrs.}$$

$$\begin{aligned} \mathbf{r}_{\mathbf{A}}(t) &= \langle -9, -8 \rangle + t \langle 2, 3 \rangle & \mathbf{r}_{\mathbf{B}}(t) &= \langle 9, -14 \rangle + t \langle -1, 4 \rangle \\ \mathbf{r}_{\mathbf{A}}(6) &= \langle -9 + 6 \times 2, -8 + 6 \times 3 \rangle & \mathbf{r}_{\mathbf{B}}(6) &= \langle 9 + 6(-1), -14 + 6 \times 4 \rangle \\ &= \langle 3, 10 \rangle & &= \langle 3, 10 \rangle \end{aligned}$$

So A and B collide at $(3, 10)$ km after 6 hrs.

The fourth method for collisions is shown in Worked Example 3.8.

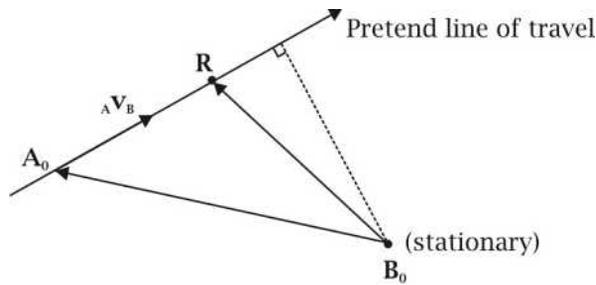
CLOSEST APPROACH USING THE SCALAR PRODUCT

If two objects don't collide then there may be a time after $t = 0$ when they are closest. There are two methods which both involve the dot product to find the closest distance between two moving objects and the time at which this happens. The Calculus method will not be covered in this guide.

The general situation will be two objects A and B having positions \mathbf{A}_0 and \mathbf{B}_0 at $t = 0$ and moving with velocities $\mathbf{v}_{\mathbf{A}}$ and $\mathbf{v}_{\mathbf{B}}$ respectively.

Method 1

Step 1 Decide on which object the observer is travelling. In this example I have chosen this to be B. Find the relative velocity ${}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}}$. Draw a diagram showing \mathbf{A}_0 and \mathbf{B}_0 with ${}_{\mathbf{A}}\mathbf{v}_{\mathbf{B}}$ coming off \mathbf{A}_0 and forming a "pretend line of travel". Let R be a point on this line.



Find the equation of the line of travel $\mathbf{OR} = \mathbf{OA}_0 + t \mathbf{A v}_B$

Step 2 Finding the vector $\mathbf{B}_0\mathbf{R}$ is central to this method ie.

$$\mathbf{B}_0\mathbf{R} = \mathbf{OR} - \mathbf{OB}_0 \text{ where } \mathbf{OR} \text{ is as above}$$

This equation now needs to be written in $\langle x(t), y(t) \rangle$ form.

Step 3 The closest distance is when $\mathbf{A v}_B$ is perpendicular to $\mathbf{B}_0\mathbf{R}$ and the time at which this happens is found by solving

$$\mathbf{B}_0\mathbf{R} \cdot \mathbf{A v}_B = 0 \text{ for } t$$

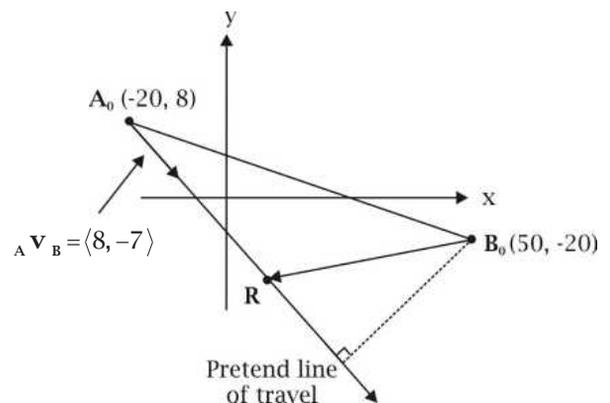
Step 4 Substitution of this value of t back into the vector $\mathbf{B}_0\mathbf{R}$ and finding $|\mathbf{B}_0\mathbf{R}|$ will give the required closest distance.

Example: A ship is at $\mathbf{A}_0(-20, 8)$ km at 10 am travelling with a velocity of $\langle 5, -3 \rangle$ km/h. At the same time, another ship is at $\mathbf{B}_0(50, -20)$ km travelling with a velocity of $\langle -3, 4 \rangle$ km/h.

Find the closest distance that they come to each other and the time that this happens.

Step 1 Let the observer be on ship B.

$$\begin{aligned} \text{Find } \mathbf{A v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= \langle 5, -3 \rangle - \langle -3, 4 \rangle \\ &= \langle 8, -7 \rangle \\ \mathbf{OR} &= \langle -20, 8 \rangle + t \langle 8, -7 \rangle \\ &= \langle 8t - 20, 8 - 7t \rangle \end{aligned}$$



Step 2

$$\begin{aligned} \mathbf{B}_0\mathbf{R} &= \mathbf{OR} - \mathbf{OB}_0 \\ &= \langle 8t - 20, 8 - 7t \rangle - \langle 50, -20 \rangle \\ &= \langle 8t - 70, 28 - 7t \rangle \end{aligned}$$

Step 3

$$\begin{aligned} \mathbf{B}_0\mathbf{R} \cdot \mathbf{A v}_B &= 0 \\ \langle 8t - 70, 28 - 7t \rangle \cdot \langle 8, -7 \rangle &= 0 \\ 64t - 560 - 196 + 49t &= 0 \\ 113t &= 756 \\ t &= \frac{756}{113} \\ t &= 6.69 \text{ hours} \\ t &= 6 \text{ hrs } 41 \text{ min} \end{aligned}$$

Step 4 Sub $t = 6.69$ into $|\mathbf{B}_0\mathbf{R}|$ gives

$$\begin{aligned} &|\langle 8 \times 6.69 - 70, 28 - 7 \times 6.69 \rangle| \\ &= | \langle -16.48, -18.83 \rangle | \\ &= 25.02 \text{ km} \end{aligned}$$

3.2 If $|\mathbf{b}| = 10$, $|\mathbf{c}| = 17$ and $\mathbf{b} \cdot \mathbf{c} = -72$, find the exact value of $|\mathbf{b} - \mathbf{c}|$.

Remembering that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

then $(\mathbf{b} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) = |(\mathbf{b} - \mathbf{c})|^2$

So $(\mathbf{b} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c})$

$$= \mathbf{b} \cdot \mathbf{b} - 2\mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}$$

$$= |\mathbf{b}|^2 - 2\mathbf{b} \cdot \mathbf{c} + |\mathbf{c}|^2$$

$$= 10^2 - 2 \times (-72) + 17^2$$

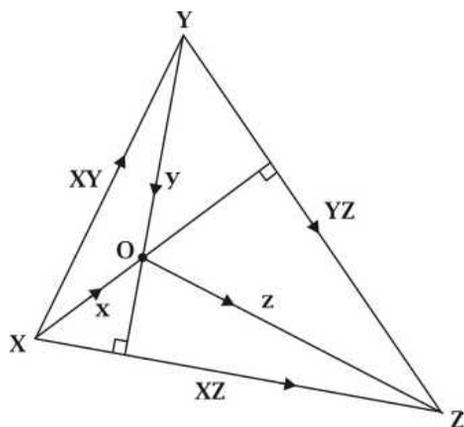
$$= 533$$

$$= |\mathbf{b} - \mathbf{c}|^2$$

$\therefore |\mathbf{b} - \mathbf{c}| = \sqrt{533}$ is the required exact value

3.3 In $\triangle XYZ$ let O be a point inside the triangle such that $YO \perp XZ$ and $XO \perp YZ$.

Prove that $OZ \perp XY$. Start by letting $XO = \mathbf{x}$, $YO = \mathbf{y}$ and $OZ = \mathbf{z}$.



$$\mathbf{XZ} = \mathbf{x} + \mathbf{z}$$

$$\mathbf{YZ} = \mathbf{y} + \mathbf{z}$$

$$\mathbf{YO} \perp \mathbf{XZ} \text{ means } \mathbf{y} \cdot (\mathbf{x} + \mathbf{z}) = 0$$

$$\mathbf{XO} \perp \mathbf{YZ} \text{ means } \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = 0$$

$$\mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{z} = 0$$

$$\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z} = 0$$

$$\mathbf{y} \cdot \mathbf{z} = -\mathbf{x} \cdot \mathbf{y} \quad (1)$$

$$\mathbf{x} \cdot \mathbf{z} = -\mathbf{x} \cdot \mathbf{y} \quad (2)$$

Consider $\mathbf{OZ} \cdot \mathbf{XY}$ $\mathbf{XY} = \mathbf{x} + -\mathbf{y}$

$$= \mathbf{z} \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{z} \cdot \mathbf{x} - \mathbf{z} \cdot \mathbf{y}$$

$$= \mathbf{z} \cdot \mathbf{x} - \mathbf{z} \cdot \mathbf{y}$$

$$= \mathbf{x} \cdot \mathbf{z} - \mathbf{y} \cdot \mathbf{z}$$

$$= -\mathbf{x} \cdot \mathbf{y} - (-\mathbf{x} \cdot \mathbf{y})$$

$$= -\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y}$$

$$= 0$$

Substitute using (1) and (2)

$\therefore \mathbf{OZ} \perp \mathbf{XY}$ as required

3.4 For the line which passes through $(-2, 3)$ and is parallel to $4\mathbf{i} - 5\mathbf{j}$ find:

- (a) a vector equation of the line
- (b) a pair of parametric equations
- (c) the Cartesian equation in $ax + by = c$ and $y = mx + b$ forms.

(a) $\mathbf{r} = \mathbf{a} + t\mathbf{u}$
 $\mathbf{r} = \langle -2, 3 \rangle + t\langle 4, -5 \rangle =$
 or $\mathbf{r} = \langle -2 + 4t, 3 - 5t \rangle$

(b) $x = -2 + 4t \quad y = 3 - 5t$

(c) $x + 2 = 4t \quad 5t = 3 - y$
 $t = \frac{x+2}{4} \quad t = \frac{3-y}{5}$
 So $\frac{x+2}{4} = \frac{3-y}{5}$

i.e. $5x + 10 = 12 - 4y$
 $5x + 4y = 2$ ----- $ax + by = c$ form

$4y = -5x + 2$
 $y = -\frac{5x}{4} + \frac{1}{2}$ ---- $y = mx + b$ form

3.5 If $(-6, 9)$ cm is the initial position ($t = 0$ s) of an ant moving with a velocity of $3\mathbf{i} - 2\mathbf{j}$ cm/s find:

- (a) the position of the ant when $t = 8$ sec
- (b) when the ant crosses the x-axis
- (c) the ant's speed
- (d) the ant's direction (as a positive polar angle in radians to 3dp).
- (e) when the ant's y coordinate is -10 cm
- (f) the ant's x coordinate when its y coordinate is -1.01 m.

(a) $\mathbf{r} = \langle -6, 9 \rangle + t\langle 3, -2 \rangle$
 $= \langle 3t - 6, 9 - 2t \rangle$
 $t = 8$ s $\mathbf{r} = \langle 3 \times 8 - 6, 9 - 2 \times 8 \rangle$
 So at $t = 8$ s the ant is at $(18, -7)$ cm.

(b) x axis, $y = 0$ $9 - 2t = 0$
 $9 = 2t$
 $t = 4.5$ s

(c) velocity $\mathbf{v} = \langle 3, -2 \rangle$ R
 $= \langle 3.6055\dots, -0.588\dots \rangle$ P
 $= \langle \sqrt{13}, -0.588 \rangle$
 Speed = $\sqrt{13}$ cm/s

(d) positive polar angle = $2\pi - 0.588$
 $= 5.695$

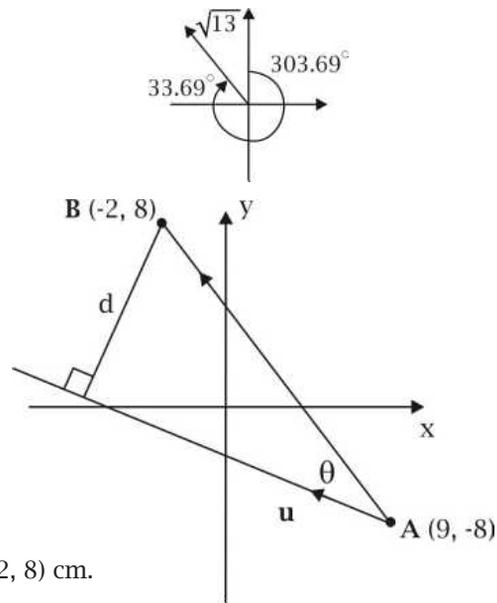
(e) $y = -10$ i.e. $-10 = 9 - 2t$
 $2t = 19$
 $t = 9.5$ s

(f) $-101 = 9 - 2t \quad x = 3t - 6$
 $2t = 9 + 101 \quad = 3 \times 55 - 6$
 $t = 55$ s $= 159$ cm
 $x = 1.59$ m

- 3.6 (a) An ant walked in a straight line with a speed of $\sqrt{13}$ cm/s on a bearing of 303.69° . It walked over the point $(9, -8)$ cm. Find the closest distance that the ant comes to the point $(-2, 8)$ cm.

$$\begin{aligned} \mathbf{u} &= \langle -\sqrt{13} \cos 33.69^\circ, \sqrt{13} \sin 33.69^\circ \rangle \\ &= \langle -3, 2 \rangle \text{ cm/s} \\ \mathbf{AB} &= \mathbf{OB} - \mathbf{OA} \\ &= \langle -2, 8 \rangle - \langle 9, -8 \rangle \\ &= \langle -11, 16 \rangle \\ \cos \theta &= \frac{\langle -3, 2 \rangle \cdot \langle -11, 16 \rangle}{\sqrt{3^2 + 2^2} \times \sqrt{11^2 + 16^2}} \\ &= \frac{65}{\sqrt{13} \sqrt{377}} \quad d = \sqrt{377} \sin 21.8^\circ \\ &= 7.21 \text{ cm} \\ \theta &= 21.8^\circ \end{aligned}$$

So the ant comes to within 7.21 cm of the point $(-2, 8)$ cm.



- (b) While on his walk, the ant walked through a circle with a vector equation of $|\mathbf{r} + 2\mathbf{i} + 3\mathbf{j}| = 6$. Find the points (correct to 2dp) where the ant crossed the circle.

$$\begin{aligned} |\mathbf{r} + 2\mathbf{i} + 3\mathbf{j}| &= 6 \\ |\langle x, y \rangle + \langle 2, 3 \rangle| &= 6 \\ |\langle x + 2, y + 3 \rangle| &= 6 \\ (x + 2)^2 + (y + 3)^2 &= 36 \text{ cartesian equation of circle} \\ \mathbf{r} &= \langle 9, -8 \rangle + t \langle -3, 2 \rangle \\ \langle x, y \rangle &= \langle 9 - 3t, 2t - 8 \rangle \\ x = 9 - 3t \quad y = 2t - 8 & \text{ parametric equations of ant's journey} \end{aligned}$$

Substitute into circle equation and solve for t

$$(9 - 3t + 2)^2 + (2t - 8 + 3)^2 = 36$$

$$(11 - 3t)^2 + (2t - 5)^2 = 36$$

$$121 - 66t + 9t^2 + 4t^2 - 20t + 25 = 36$$

$$13t^2 - 86t + 110 = 0$$

Quadratic formula or graphic calculator gives $t = 4.882$ s or 1.733 s.

Substitute these into $x = 9 - 3t$ $y = 2t - 8$

Gives $(-5.65, 1.76)$ cm and $(3.80, -4.53)$ cm as the points where the ant crosses the circle.

- 3.7 A boat was travelling in a straight line at a speed of $\sqrt{34}$ km/h and at 10 am Friday its position was (163.5, -136.5) km. At this time a yacht was at (6, 3.5) km travelling at $\langle 4, -5 \rangle$ km/h. Less than a day later the two vessels collided. Find:
- the velocity of the boat
 - the time of the collision
 - the position of the collision.

Using B for boat and Y for yacht and $\mathbf{Y}_0\mathbf{B}_0 = t \mathbf{Y}\mathbf{v}_B$ because it is a collision situation.

$$\begin{aligned}\mathbf{Y}_0\mathbf{B}_0 &= \mathbf{OB}_0 - \mathbf{OY}_0 \\ &= \langle 163.5, -136.5 \rangle - \langle 6, 3.5 \rangle \\ &= \langle 157.5, -140 \rangle \text{ km}\end{aligned}$$

$$\text{If } \mathbf{v}_B = \langle a, b \rangle \quad \mathbf{v}_Y = \langle 4, -5 \rangle$$

$$\begin{aligned}\text{Then } \mathbf{Y}\mathbf{v}_B &= \mathbf{v}_Y - \mathbf{v}_B \\ &= \langle 4, -5 \rangle - \langle a, b \rangle \\ &= \langle 4-a, -5-b \rangle\end{aligned}$$

$$\begin{aligned}\mathbf{Y}_0\mathbf{B}_0 &= t \mathbf{Y}\mathbf{v}_B \\ \langle 157.5, -140 \rangle &= t \langle 4-a, -5-b \rangle\end{aligned}$$

$$\text{also } |\mathbf{v}_B| = \sqrt{34} = \sqrt{a^2 + b^2} \text{ ie } a^2 + b^2 = 34$$

$$\begin{aligned}\text{So } 157.5 &= t(4-a) & -140 &= t(-5-b) \\ & & 140 &= t(5+b)\end{aligned}$$

$$a = 4 - \frac{157.5}{t} \quad b = \frac{140}{t} - 5 \quad \text{but } a^2 + b^2 = 34$$

$$\therefore \left(4 - \frac{157.5}{t}\right)^2 + \left(\frac{140}{t} - 5\right)^2 = 34$$

Solver gives $t = 17.5$ hours or $t = 362.5$ hours (reject)

$$\begin{aligned}a &= 4 - \frac{157.5}{17.5} & b &= \frac{140}{17.5} - 5 \\ &= -5 & &= 3\end{aligned}$$

Taking $t = 17.5$ h, then the position vector of the yacht would be

$$\begin{aligned}\langle 6, 3.5 \rangle + 17.5 \langle 4, -5 \rangle \\ = \langle 76, -84 \rangle\end{aligned}$$

- Answers**
- The velocity of the boat was $\langle -5, 3 \rangle$ km/h.
 - The time of the collision was 17.5 h after 10 am i.e. 3.30 am the next day.
 - The position of the collision was (76, -84) km.

When a collision situation is given where all of the vectors are in magnitude/direction form as opposed to rectangular form, the **t method** shown below is recommended. This is the fourth method mentioned on page 58 of the text.

- 3.8 At 10 am Ed was running at 7 km/h on a bearing of 200° and Neil was 12 km away from Ed and on a bearing of 150° running at 6 km/h. The two people met some time later. Find the time(s) for the meeting and the possible bearing(s) that Neil had to run on for the meeting to happen.

Let the couple meet after t hours.

So Ed will travel $7t$ km and Neil will travel $6t$ km in order to meet after t hours

$$\frac{6t}{\sin 50^\circ} = \frac{7t}{\sin \theta}$$

t can be cancelled

$$\sin \theta = \frac{7 \sin 50^\circ}{6}$$

$$\theta_1 = 63.34^\circ \quad \text{or}$$

$$\theta_2 = 180^\circ - 63.34^\circ$$

$$= 116.66^\circ$$

$$\alpha_1 = 180^\circ - 50^\circ - 63.34^\circ$$

$$\alpha_2 = 180^\circ - 50^\circ - 116.66^\circ$$

$$= 66.66^\circ$$

$$= 13.34^\circ$$

$$\frac{12}{\sin 66.66^\circ} = \frac{6t}{\sin 50^\circ}$$

$$\frac{12}{\sin 13.34^\circ} = \frac{6t}{\sin 50^\circ}$$

$$t = \frac{12 \sin 50^\circ}{6 \sin 66.66^\circ}$$

$$t = \frac{12 \sin 50^\circ}{6 \sin 13.34^\circ}$$

$$t = 1.67 \text{ hrs}$$

$$t = 6.64 \text{ hrs}$$

$$\text{Bearing} = 360^\circ - 30^\circ - 63.34^\circ$$

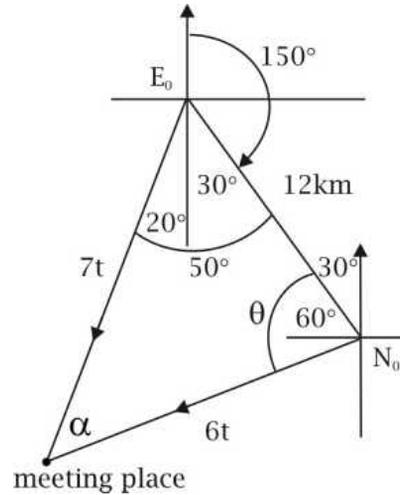
$$= 266.66^\circ$$

$$\text{Bearing} = 360^\circ - 30^\circ - 116.66^\circ$$

$$= 213.34^\circ$$

So either Neil will need to run for 1 h 40 min on a bearing of 266.66° and they meet at 11.40 am or Neil runs for 6 h 38 min on a bearing of 213.34° and they meet at 4.38 pm.

The next page shows an alternative method.



3.8 (continued)

Instead of using the sine rule as shown above where t can be cancelled out, the cosine rule could have been used. So from the diagram:

$$(6t)^2 = (7t)^2 + 12^2 - 2 \times 7t \times 12 \times \cos 50^\circ \quad (1)$$

$$36t^2 = 49t^2 + 144 - 168t \cos 50^\circ$$

$$49t^2 - 36t^2 - 168t \cos 50^\circ + 144 = 0$$

$$13t^2 - 168t \cos 50^\circ + 144 = 0$$

$$t = 1.67 \quad \text{or } 6.64 \text{ h as before}$$

The solver facility in your graphics calculator could have been used to solve equation (1) but you should be aware of the possibility of two solutions and hence enter another estimate in order to find the possible second case.

Now that the t values are known, substitute these back into the diagram i.e. find the the lengths $6t$ and $7t$ and use.

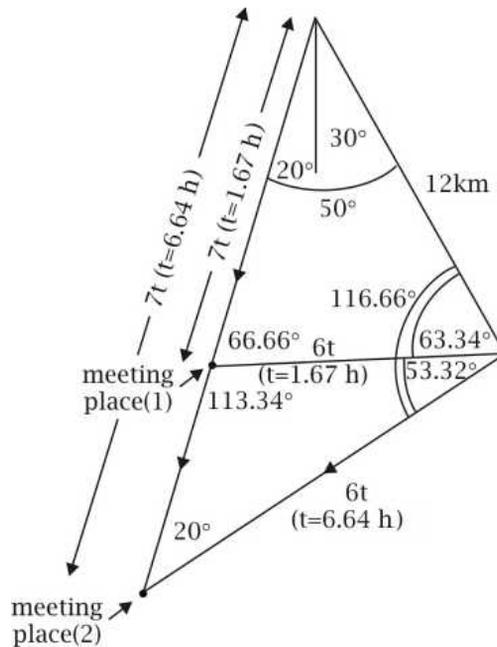
$$\cos \theta = \frac{(6t)^2 + 12^2 - (7t)^2}{2 \times 6t \times 12}$$

gives $\theta = 63.34^\circ$ for $t = 1.67$ h

and $\theta = 116.66^\circ$ for $t = 6.64$ h as before.

If you used solver on a Casio Graphics Calculator to find t , then in run mode the value of t is stored in the **X,θ,T** button and so θ can be found directly from $\cos \theta$ above which is quite handy.

The two possible meeting places can be shown on the same diagram below (not drawn to scale) for the two possible times of $t = 1.67$ h for the short run and $t = 6.64$ h for the long run.



PROBLEMS TO SOLVE

CHAPTER 3: VECTORS II

1. Find λ if vectors $2\mathbf{i} - 4\mathbf{j}$ and $-3\mathbf{i} - \lambda\mathbf{j}$ are:

(a) parallel

(b) perpendicular.

2. Three vectors \mathbf{a} , \mathbf{b} and \mathbf{c} are defined by $3\mathbf{i} + p\mathbf{j}$, $4\mathbf{i} + 5\mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$ respectively. Find the value of p if $\mathbf{a} + \mathbf{b}$ is perpendicular to \mathbf{c} .

3. F and G are the Cartesian points $(3, -2)$ and $(-5, -1)$ respectively. Find:

(a) a vector equation of the line through F and G

(b) the vector equation of the line perpendicular to FG which passes through H $(12, 5)$

(c) the Cartesian coordinates of the point of intersection of the two lines

(d) the closest distance that H is to the line through F and G.

4. Given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 11$, find:

(a) $\mathbf{a} \cdot \mathbf{a}$

(b) $\mathbf{b} \cdot \mathbf{b}$

(c) the acute angle between \mathbf{a} and \mathbf{b} correct to 1 dp in degrees

(d) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

(e) $|\mathbf{a} - \mathbf{b}|$

5. Find the size of the acute angle between $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{b} = -5\mathbf{i} - 12\mathbf{j}$.

6. Find a vector equation of the line which passes through A (4, 2) and is perpendicular to $\mathbf{n} = 3\mathbf{i} - 5\mathbf{j}$.

7. If $\mathbf{a} = 24\mathbf{i} - 7\mathbf{j}$ and $\mathbf{b} = 8\mathbf{i} + 15\mathbf{j}$ find:

(a) a unit vector parallel to \mathbf{b}

(b) a vector parallel to \mathbf{b} and four times as long

(c) the size of the acute angle between \mathbf{a} and \mathbf{b} .

8. If $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and θ is the angle between \mathbf{a} and \mathbf{b} , find the exact value of $\cos \theta$.

9. If $\triangle BOA$ has a right angle at O , let $\mathbf{OB} = \mathbf{b}$, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{AB} = \mathbf{c}$.
If $|\mathbf{a}| = |\mathbf{b}| = 2$ units find the value of $\mathbf{b} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$.

10. Suppose that for any number θ , $\mathbf{a}_\theta = \langle \cos \theta, \sin \theta \rangle$.
(a) Show that \mathbf{a}_θ is a unit vector.

- (b) What is the angle between \mathbf{a}_θ and the positive x-axis?

- (c) Prove that $\mathbf{a}_\theta \cdot \mathbf{a}_\phi = \cos(\theta - \phi)$ where ϕ is another number.

11. If $|\mathbf{u}| = 4$, $|\mathbf{v}| = 5$ and $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}$ find $|\mathbf{u} - \mathbf{v}|$.

12. The unit vector $\mathbf{u} = a\mathbf{i} - b\mathbf{j}$ is perpendicular to $4\mathbf{i} + 3\mathbf{j}$. If $a > 0$, find a and b .

13. XOY is the diameter of a circle centre O , and Z is on the circle's circumference. If $\mathbf{OX} = \mathbf{x}$, and $\mathbf{OZ} = \mathbf{z}$, use vector methods to show that $\angle XZY$ is a right angle.

14. Change $3x^2 + 3y^2 - 6x + 4y - 4 = 0$ into vector form and then find the scalar product form of the vector equation of the tangent to the circle at $A(0, -2)$.

15. Find using the dot product the closest distance between $(2, 3)$ and the line specified by $\mathbf{r} = \langle 7, -8 \rangle + t \langle -4, 3 \rangle$.

16. A straight slope is defined by the line $\mathbf{r} \cdot \langle -2.5, 4 \rangle = 0$. An object is moved from the point $(16, 10)$ m on the slope, 5 m down the slope by the force $\mathbf{F} = \langle -6, -2 \rangle$ N. If the frictional force \mathbf{R} is 3 N up the slope and the force due to its mass (weight) is $\mathbf{W} = \langle 0, -25 \rangle$ N find the work done by or against each of the forces.

(a) **F**

(b) **W**

(c) **R**

17. At 7 am one day Colin was at $C_0(50, 30)$ km running with a velocity of $\langle -1, 6 \rangle$ km/h. At the same time Lyn was at $L_0(-20, -10)$ running at $\langle 5, 8 \rangle$ km/h. Find the time when Colin and Lyn are closest and the closest distance by using the scalar product method.

18. Repeat the above question for:

$\mathbf{A}_0(-28, 17)$ km at 9 am , $\mathbf{v}_A = \langle 5, -2 \rangle$ km/h

$\mathbf{B}_0(16, -30)$ km at 9 am , $\mathbf{v}_B = \langle -8, 3 \rangle$ km/h.

19. For two vehicles A and B with initial positions \mathbf{A}_0 and \mathbf{B}_0 and velocities \mathbf{v}_A and \mathbf{v}_B respectively, prove that the closest distance that they are apart occurs when

$$t = \frac{\mathbf{A}_0 \cdot \mathbf{B}_0 \cdot \mathbf{v}_B}{|\mathbf{v}_B|^2}$$

20. Two marathon swimmers, Shelley and Taylor set off from $(-15, -10)$ km and $(x, -20)$ km with velocities of $\langle 3, 4 \rangle$ km/hr and $\langle -2, 3 \rangle$ km/h respectively. Their closest distance to each other occurred 4 hs, 25 min and 23 seconds later. Find the value of x and hence the closest distance.

21. (a) Find the vector equation of the line passing through the point A with position vector $2\mathbf{i} + 3\mathbf{j}$ and parallel to the vector $4\mathbf{i} - 6\mathbf{j}$.

(b) Determine whether or not each of the following points lie on the line.

- B with position vector $10\mathbf{i} - 9\mathbf{j}$.

- C with position vector $8\mathbf{i} - 5\mathbf{j}$.

22. (a) Give the cartesian equation of a circle with centre $(0, -4)$ and radius $2\sqrt{2}$.

- (b) State the vector equation of the same circle.

23. If A has position vector $-2\mathbf{i} + \mathbf{j}$ and B has position vector $4\mathbf{i} - 3\mathbf{j}$, find:

- (a) a vector equation of the line containing A and B

- (b) the parametric equations of the line

- (c) the cartesian equation of the line in $ax + by = c$ form.

24. The vector equations

$$\mathbf{r}_1 = (4 - \lambda)\mathbf{i} + (2\lambda - 9)\mathbf{j}$$

$$\mathbf{r}_2 = (-3 - \lambda)\mathbf{i} + (5 + 2\lambda)\mathbf{j}$$

represent the same line. Explain how these two different looking equations can represent the same line.

25. (a) Find a vector equation of the line through $(2, 5)$ and $(5, 11)$ in component form.

- (b) Write down the parametric equations of this vector equation.

- (c) Use your answer to part b) to decide whether or not $(3, 7)$ is on the line.

26. Change $2x - y = 3$ into the vector form $\mathbf{r} = \mathbf{a} + t \mathbf{u}$.

27. Use vector methods to find the x-intercept of the line through $(-3, 2)$ and $(2, -7)$.

28. To the pilot of a bomber aircraft travelling with velocity $(150 \mathbf{i} - 200 \mathbf{j})$ km/h, a fighter aircraft appears to have a velocity of $(200 \mathbf{i} + 440 \mathbf{j})$ km/h. Find the speed and bearing of the fighter aircraft.

29. The circle $|\mathbf{r} - \langle 6, 3 \rangle| = 5$ is the outside ring of a shooting target which is mounted on a big board. If a person hits at point $(1, 5)$, show the working necessary to decide whether or not the target was hit.

30. Find the
(a) vector equation

(b) parametric equations

(c) Cartesian equation

of the line through the point with position vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and parallel to $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

31. (a) Determine the centre and radius of the circle with equation $x^2 + y^2 + 6x - 4y - 87 = 0$

- (b) Write the vector equation of this circle.

- (c) Show that the line $\mathbf{r} = \langle -1, -12 \rangle + t \langle 3, 4 \rangle$ is tangent to the above circle.

- (d) Find the point of contact.

32. A park ranger travels on a bearing of 130° with a speed of 40 km/h. A bird smuggler travels on a bearing of 220° with a speed of 60 km/h. Find the following in magnitude/bearing form:

- (a) the velocity of the ranger relative to the smuggler

- (b) the velocity of the smuggler relative to the ranger.

33. A circle has its centre at (1, -5) with (-11, 0) on its circumference. Determine whether the line $\mathbf{r} = 6\mathbf{i} + 7\mathbf{j} + \lambda(-12\mathbf{i} + 5\mathbf{j})$ intersects the circle in zero, one or two places.

If it does intersect the circle, find the coordinates of intersection.

34. Two boats are moving along straight paths and their position vectors at noon are

$$\mathbf{r}_1 = (7 - 4t)\mathbf{i} + (-5 + t)\mathbf{j}$$

$$\mathbf{r}_2 = (12 - 3t)\mathbf{i} + (13 - t)\mathbf{j} \quad \text{respectively.}$$

(a) Where was the first boat initially?

(b) Where was the second boat relative to the first boat initially?

(c) What is the velocity vector \mathbf{v}_2 of the second boat?

(d) Find whether or not the boats will collide.

(e) Find the coordinates of the point of intersection of their paths.

(f) Find the shortest distance between the ships in their motion after noon.

(g) Find the acute angle between the paths of the boats.

35. (a) Find the vector equation of the circle whose centre has the position vector $2\mathbf{i} + \mathbf{j}$ and which has a radius of 5 units.

(b) Find the Cartesian equation of this circle in the form $x^2 + y^2 + ax + by + c = 0$.

36. Find the centre and radius of the circle $x^2 + y^2 + 7x - 10y = 5$.

37. Elsie travelling on a bearing of 070° and at a speed of 70 km/h “sees” Vedran travelling on a bearing of 210° at a speed of 90 km/h. Find the speed and direction of Vedran’s car.

38. At 9.00 am SS Watson has position vector $8 \mathbf{i} + 27 \mathbf{j}$ km travelling with a velocity of $3 \mathbf{i} - 12 \mathbf{j}$ km/h. The liner, Holmes at 9.00 am is at position $47 \mathbf{i} + 21 \mathbf{j}$ km travelling with a velocity of $-4 \mathbf{i} - 9 \mathbf{j}$ km/h.

(a) At what time will the ships be 10 km apart? (Give your answer to the nearest minute.)

(b) If Watson and Holmes continue on their present courses, will they collide?

(c) If the ships do collide, find the position vector of the collision.

39. (a) Give the equation of the circle in Cartesian form which has E (-3, 7) and F (1, 3) as the end points of its diameter.

(b) Show algebraically that the line $y = -x$ is tangent to the circle.

(c) Is the point $(-1, 2)$ on, inside or outside the circle?

(d) Find the exact area of the circle.

40. Ships Anita and Britanis have position vectors $\mathbf{r}_A = 5\mathbf{i} + 2\mathbf{j}$ km and $\mathbf{r}_B = 7\mathbf{i} + 7\mathbf{j}$ km at midday. Anita has a velocity of $15\mathbf{i} + 10\mathbf{j}$ km/h and Britanis has a velocity of $9\mathbf{i} - 5\mathbf{j}$ km/h.

(a) Show that if they continue with these velocities, the ships will collide.

(b) Find the time, and the position vector of the collision.

(c) When the ships are 5 km apart, it becomes obvious that a collision might be about to happen and evasive action should be taken. At what time are the ships 5 km apart? (Give your answer to the nearest 1/2 minute.)

41. A bird is flying parallel to the ground at a speed of $\sqrt{41}$ m/s on a bearing of 231.34° .

(a) Find the vector equation of the bird's path given that it started at a place which had coordinates of $(-5, 6)$ m.

(b) Where is the bird after 8 seconds and how far had it travelled in that time?

(c) How close does the bird come to a hunter situated at (20, - 30) m?

42. At noon, two ships Alabama and Baltimore have the position and velocity vectors shown. The speeds are measured in kilometres per hour and the distances in kilometres.

	Position Vector	Velocity Vector
Alabama	$10 \mathbf{i} + 5 \mathbf{j}$	$-2 \mathbf{i} + 4 \mathbf{j}$
Baltimore	$2 \mathbf{i} - \mathbf{j}$	$2 \mathbf{i} + 8 \mathbf{j}$

If they continue on their respective courses find the closest distance between the ships to the nearest 10 m and the time at which this occurs.

43. Write the radii and Cartesian coordinates of the centres of each of the following circles.

(a) $36x^2 + 36y^2 = 9$

(b) $|\mathbf{r} - 4\mathbf{i} - 3\mathbf{j}| = 5$

(c) $|(x+3)\mathbf{i} + (y-2)\mathbf{j}| = 3$

44. Find the position vectors of the points where the straight line $\mathbf{r} = -6\mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j})$ intersects the circle $|\mathbf{r} - 2\mathbf{i} - 3\mathbf{j}| = 5\sqrt{2}$.

45. A pirate ship and an oil tanker, the Queen Juliana, have position and velocity vectors as given, at 4 pm.

$$\mathbf{r}_p = 3\mathbf{i} + 8\mathbf{j} \text{ km} \quad \mathbf{v}_p = 12\mathbf{i} - 10\mathbf{j} \text{ km/h}$$

$$\mathbf{r}_q = -24\mathbf{i} - 17\mathbf{j} \text{ km} \quad \mathbf{v}_q = a\mathbf{i} + b\mathbf{j} \text{ km/h}$$

The Queen Juliana has a speed of $\sqrt{327.76}$ km/h. Find the values of a and b if the two ships are to meet, and give the time of the meeting.

46. The position and velocity vectors (\mathbf{r} and \mathbf{v} respectively) of the two ships Anselm and Buderim are as follows:

$$\mathbf{r}_A = (6\mathbf{i} + 35\mathbf{j}) \text{ km} \quad \mathbf{v}_A = (4\mathbf{i} - 16\mathbf{j}) \text{ km/h at 11.30 am}$$

$$\mathbf{r}_B = (-2\mathbf{i} - 3\mathbf{j}) \text{ km} \quad \mathbf{v}_B = (12\mathbf{i} + 8\mathbf{j}) \text{ km/h at 12.00 noon}$$

- (a) Prove that the ships will collide and find when and where the collision will occur.

At the moment of the collision between the Anselm and the Buderim, a third ship the Charles is at a point with position vector $9\mathbf{i} + 3\mathbf{j}$ km and it hears the distress call over the radio.

The Charles immediately steams towards the scene of the collision, maintaining a steady speed of $10\sqrt{2}$ km/h.

- (b) Find the velocity of the Charles, in the form $a\mathbf{i} + b\mathbf{j}$ and calculate its time of arrival at the scene of the collision.

47. The Collie and the Durham are two ships 12 km apart when the Durham is on a bearing of 344° from the Collie. The Collie is moving west at 15 km/h and the Durham is moving on a bearing of 203° at 11 km/h. Find the closest distance between the ships and the time at which this occurs.

Syllabus Checklist

By the end of this chapter, you should be able to:

The nature of proof

- use implication, converse, equivalence, negation, inverse, contrapositive
- use proof by contradiction
- use the symbols for implication (\Rightarrow), equivalence (\Leftrightarrow)
- use the quantifiers 'for all' \forall and 'there exists' \exists
- use examples and counter-examples

Circle properties, including proof and use

- an angle in a semicircle is a right angle
- the size of the angle at the centre subtended by an arc of a circle is twice the size of the angle at the circumference subtended by the same arc
- angles at the circumference of a circle subtended by the same arc are equal
- the opposite angles of a cyclic quadrilateral are supplementary
- chords of equal length subtend equal angles at the centre, and conversely, chords subtending equal angles at the centre of a circle have the same length
- the angle in the alternate segment theorem
- when two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord
- when a secant (meeting the circle at A and B) and a tangent (meeting the circle at T) are drawn to a circle from an external point M , the square of the length of the tangent equals the product of the lengths to the circle on the secant ($AM \times BM = TM^2$)
- suitable converses of some of the above results
- solve problems determining unknown angles and lengths and prove further results using the results listed above

Geometric vectors in the plane, including proof and use

- the diagonals of a parallelogram intersect at right angles if, and only if, it is a rhombus
- the midpoints of the sides of a quadrilateral join to form a parallelogram
- the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides

FORMULAE AND DEFINITIONS

Conjectures

A conjecture is the expression of an opinion without sufficient evidence.

Conjectures may be true or false. Only one counter-example is necessary to prove that a conjecture is false.

Algebra can also be used to prove if a conjecture is true or false.

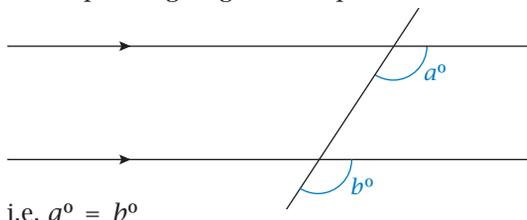
Deductive geometry – proofs

Axiom

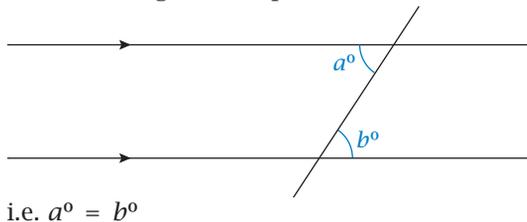
An **axiom** is a statement that is true.

The following axioms are commonly used in deductive geometry.

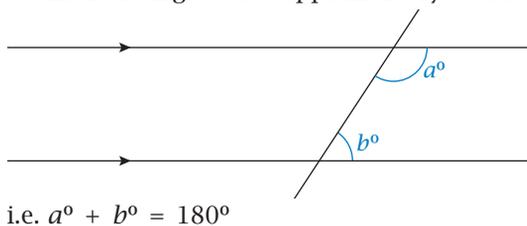
- 1) Corresponding Angles are equal



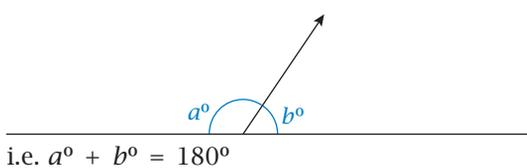
- 2) Alternate Angles are equal



- 3) Co-interior Angles are supplementary - add to 180°



- 4) Straight Angles are supplementary

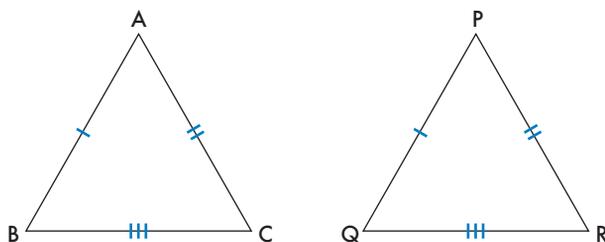


Congruent triangles

Congruent triangles have the same shape and same size.

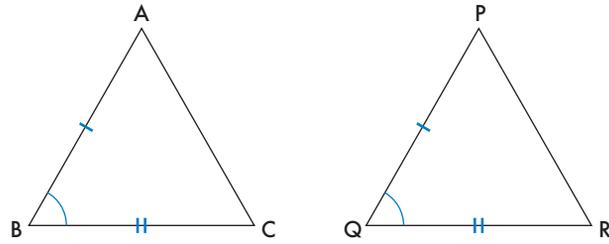
The following tests for congruence are commonly used in deductive geometry.

- 1) Corresponding sides are equal (S.S.S)



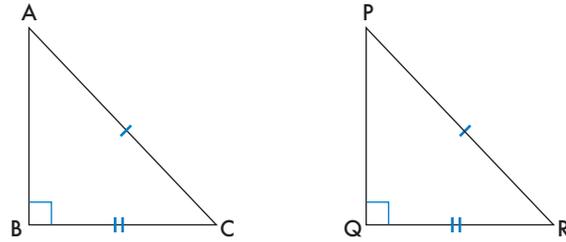
$$\triangle ABC \cong \triangle PQR \text{ (S.S.S)}$$

- 2) Two sides and the included angle are equal (S.A.S)



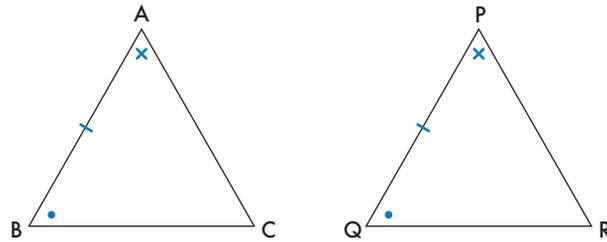
$$\triangle ABC \cong \triangle PQR \text{ (S.A.S)}$$

- 3) The hypotenuse and one pair of sides are equal for a right angled triangle (R.H.S)



$$\triangle ABC \cong \triangle PQR \text{ (R.H.S)}$$

- 4) Two angles and a pair of corresponding sides are equal (A.A.S)



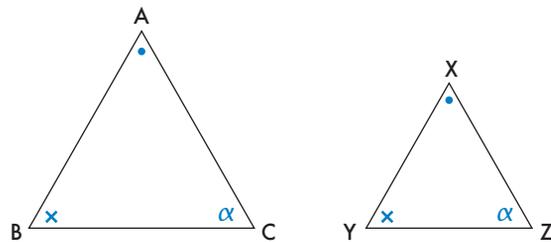
$$\triangle ABC \sim \triangle PQR \text{ (A.A.S)}$$

Similar triangles

Similar triangles have the same shape but not necessarily the same size.

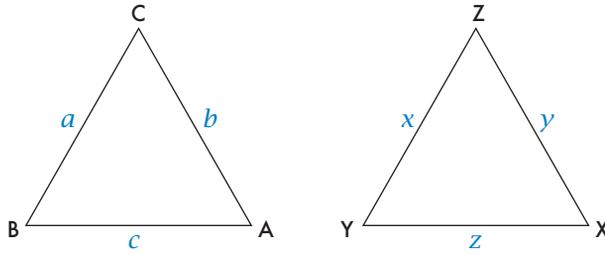
The following are tests for similarity.

- 1) Three pairs of corresponding angles are equal (A.A) (Third angle must be equal)



$$\triangle ABC \sim \triangle XYZ \text{ (A.A)}$$

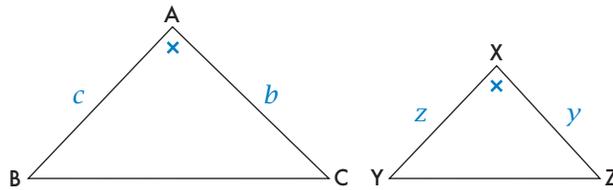
- 2) All three pairs of corresponding sides are in the same proportion (S.S.S)



$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$$

$\triangle ABC \sim \triangle XYZ$ (S.S.S)

- 3) Two pairs of sides are in the same proportion and included angles are equal. (S.A.S)



$$\frac{c}{z} = \frac{b}{y}; \angle A = \angle X$$

$\triangle ABC \sim \triangle XYZ$ (S.A.S)

Theorems and method of proof

A statement that can be proved by a series of logical steps is call a **theorem**.

A statement is not true unless it can be proved or deduced from the axioms.

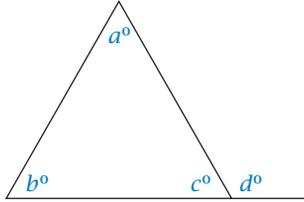
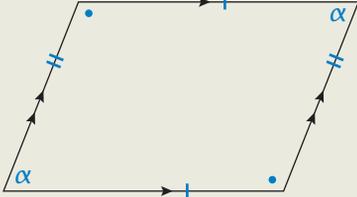
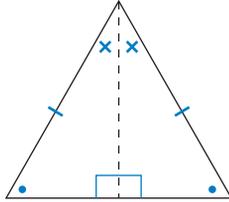
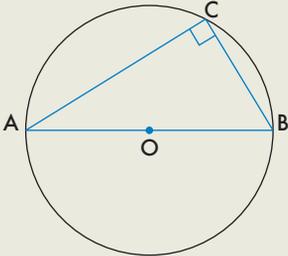
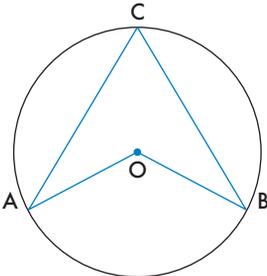
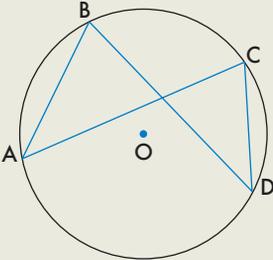
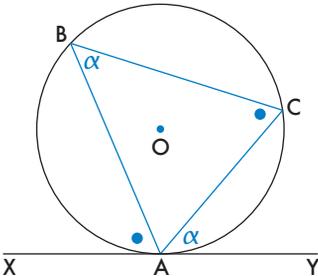
Method

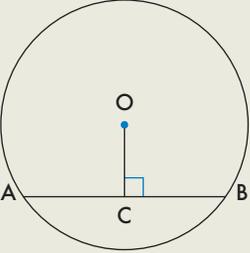
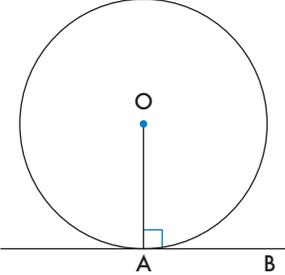
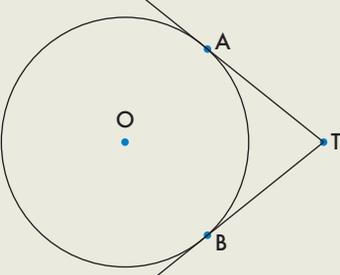
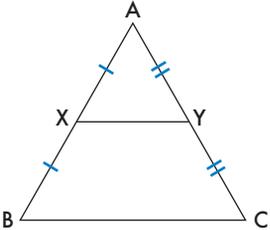
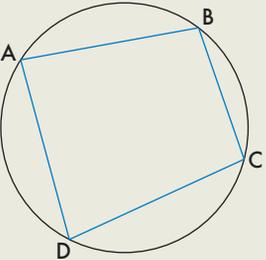
- 1) State 'What is given'
- 2) State 'What is to be proved'
- 3) Include a diagram. Show appropriate markings and any extra constructions needed.
- 4) Write related statements and give reasons (axioms used) for each. Statements should relate to parts of the diagram.
- 5) The final statement is the one to be proved.

If a theorem has been proved, it along with the axioms, can be used to prove further theorems.

Common theorems

Name	Diagram	Explanation
Vertically Opposite Angles		Vertically opposite angles are equal. i.e. $a^\circ = b^\circ$
Sum of the Angles in a Triangle		The interior angles in a triangle add to 180° . i.e. $a^\circ + b^\circ + c^\circ = 180^\circ$

Name	Diagram	Explanation
Exterior Angle of Triangle		<p>The exterior angle of a triangle is equal to the sum of the interior opposite angles.</p> <p>i.e. $d^\circ = a^\circ + b^\circ$</p>
Parallelogram		<p>Opposite sides and diagonally angles are equal in a parallelogram</p>
Isosceles Triangle		<p>Base angles are equal The line drawn from the vertex to the midpoint of the base is perpendicular to the base.</p>
Angle in a Semi Circle Theorem		<p>$\angle ACB = 90^\circ$</p>
Central Angle Theorem		<p>Size of the angle at the centre is twice the size of the angle at the circumference, if angles are subtended by the same arc.</p> <p>$\angle AOB = 2\angle ACB$</p>
Angles in the same Arc Theorem		<p>Angles at the circumference, subtended by the same arc, are equal.</p> <p>i.e. $\angle ABD = \angle ACD$</p>
Alternate Segment Theorem		<p>The angle between the chord and a tangent to the circle at the point of contact is equal to any angle in the alternate segment.</p> <p>i.e. $\angle XAB = \angle ACB$ and</p> <p>$\angle YAC = \angle ABC$</p>

Name	Diagram	Explanation
Chords		The perpendicular from the centre of the circle to the chord bisects the chord. i.e. $AC = CB$
Tangent and Radius		The angle formed from the radius to the tangent is 90° . i.e. $\angle BAO = 90^\circ$
Tangent to Circle		Tangents from an external point to the circle are equal in length. i.e. $TA = TB$
Midpoint Theorem		XY is parallel to BC $XY = \frac{1}{2} BC$
Cyclic Quadrilateral		Diagonally opposite angles of a cyclic quadrilateral are supplementary. i.e. $\angle A + \angle C = 180^\circ$ $\angle B + \angle D = 180^\circ$

Proof by Contradiction

Proving by contradiction is a commonly used method of indirect proof. The steps to follow in using this method are:

- i) Commence with claim P
- ii) Assume that P is an incorrect claim and develop an oppositional claim $\sim P$ (not P)
- iii) Work logically to determine an inconsistency between P and $\sim P$, and prove that $\sim P$ is false
- iv) Conclude that if claim $\sim P$ is false then by contradiction the premise P is true

There are some worked examples in this chapter and in Chapter 8 to guide you through these four steps.

The Language of Proof

Mathematical statements can be written succinctly with symbols. Some common 'Families of Numbers' and 'Operators' are summarised below.

Some symbols of proof ...

Families of Numbers		Operators	
Integers	\mathbb{Z}	Is an element of	\in
Positive Integers	\mathbb{Z}^+	For every	\forall
Negative Integers	\mathbb{Z}^-	There exists	\exists
Natural (Counting) Numbers	\mathbb{N}	Such that	i.e. / s.t.
Real Numbers	\mathbb{R}	That implies that	\Rightarrow
Irrational	\mathbb{P}	If and only if	\Leftrightarrow / iff
Rational	\mathbb{Q}	Therefore	\therefore

Other important terms help mathematicians work efficiently, and some of these are hereby explained and supported with an example.

Negation

The negation of statement p is 'not p ', which is symbolised by ' $\sim p$ '. The negation of any statement can be formed by inserting the clause 'It is not the case that' at the beginning of the statement. For instance, consider statement p and the negation of statement p :

p : All secondary school students have jobs.

$\sim p$: It is not the case that all secondary school students have jobs.

Contrapositive

To form the contrapositive of a conditional statement, switch both the hypothesis and the conclusion and negate both. The contrapositive of $p \Rightarrow q$ ('if p , then q ' or 'the truth of p implies the truth of q ') is $\sim q \Rightarrow \sim p$. To illustrate, the contrapositive of 'If it is raining then the ground is wet' is 'If the ground is not wet then it is not raining'. A conditional statement and its contrapositive are logically equivalent.

Converse

The converse of a statement can be formed by switching the 'if' and 'then' parts of that statement. For example, the converse of 'If two numbers are both odd, then their sum is even' would be 'If the sum of two numbers is even, then they are both odd numbers'. Note that this illustrative statement is true, and its converse is false. To summarise, the converse of $p \Rightarrow q$ is $q \Rightarrow p$.

Inverse

To form the inverse of a statement, negate both the hypothesis and the conclusion of a conditional statement. For instance, the inverse of 'If it is raining then the ground is wet' would be 'If it is not raining then the ground is not wet'. To summarise, the inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$. Note that a conditional statement may be true but its inverse may be false.

Worked Examples

4.1 **Conjecture:** The sum of five consecutive odd numbers is a multiple of five.

Let $n, n + 2, n + 4, n + 6, n + 8$ represent five consecutive odd numbers (n is an odd integer).

$$n + (n + 2) + (n + 4) + (n + 6) + (n + 8)$$

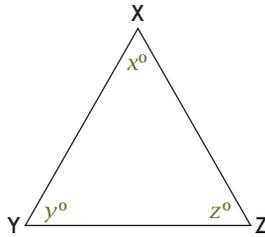
$$= 5n + 20$$

$$= 5(n + 4)$$

$5(n + 4)$ is divisible by 5 and hence is a multiple of 5.

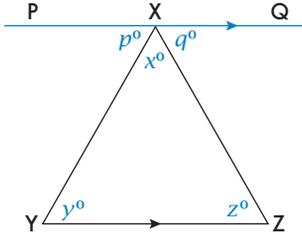
Conjecture is true.

4.2 Prove that the sum of the angles in a triangle is 180°



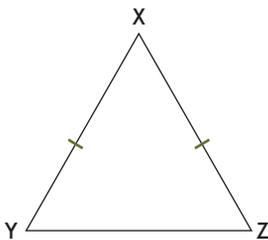
Method:

1. **Given:** $\triangle XYZ$
2. **To Prove:** $x^\circ + y^\circ + z^\circ = 180^\circ$
3. **Construct:** Line PQ through X parallel to YZ



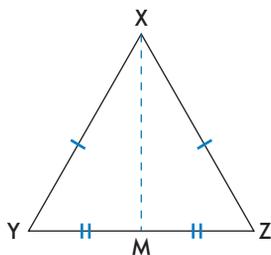
4. **Proof:**
 - $p^\circ = y^\circ$ (Alternate angles) *Statement*
 - $q^\circ = z^\circ$ (Alternate angles) *Reason*
 - $p^\circ + x^\circ + q^\circ = 180^\circ$ (Straight angle)
 - ∴ $y^\circ + x^\circ + z^\circ = 180^\circ$

4.3 Prove that the base angles of an isosceles triangle are equal.



Method:

1. **Given:** Isosceles $\triangle XYZ$ with $XY = XZ$
2. **To Prove:** $\angle XYZ = \angle XZY$
3. **Construct:** Draw a line from vertex X to the midpoint of YZ



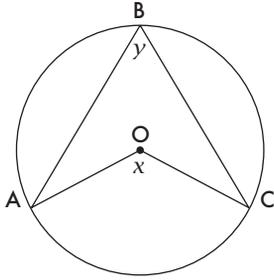
4. **Proof:**
 - $XM = XM$ (Common to both triangles)
 - $XY = XZ$ (Given)
 - $YM = MZ$ (From construction)
 - ∴ $\triangle XYM \cong \triangle XZM$ (S.S.S)
 - ∴ $\angle XYZ = \angle XZY$

- 4.4 The diagram below shows $\angle ABC$ and $\angle AOC$ on the circle and at the centre of the circle, respectively, subtended by the same arc.

$$\angle ABC = y$$

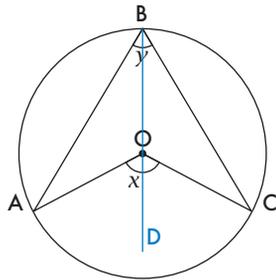
$$\angle AOC = x$$

Prove that $x = 2y$



Method:

1. **Given:** $\angle ABC = y$ and $\angle AOC = x$ subtended by the same arc.
2. **To Prove:** $x = 2y$
3. **Construct:** Draw line from B through O to D



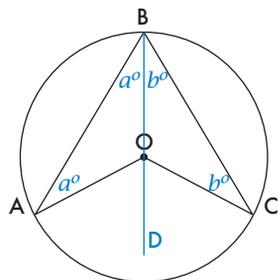
4. **Proof:** Let $\angle OAB = a^\circ$

$$\text{Let } \angle OCB = b^\circ$$

$$\overline{OA} = \overline{OB} = \overline{OC} \text{ (Radii of circle)}$$

$$\therefore \angle OBA = a^\circ$$

$$\angle OBC = b^\circ$$



$$\therefore y = a^\circ + b^\circ$$

$$\cdot \angle COD = 2b^\circ \text{ (Exterior angle of a } \Delta)$$

$$\cdot \angle AOD = 2a^\circ \text{ (Exterior angle of a } \Delta)$$

$$\therefore x = 2a^\circ + 2b^\circ$$

$$x = 2(a^\circ + b^\circ)$$

$$\therefore x = 2y$$

- 4.5 Prove by contradiction that the difference of any rational number and any irrational number is irrational.

Suppose we have a rational number x and an irrational number y such that their difference $(x - y)$ is **rational**. By definition of a rational number, we have

$$x = \frac{a}{b} \quad \text{for some integers } a \text{ and } b \text{ with } b \neq 0.$$

and $x - y = \frac{c}{d}$ for some integers c and d with $d \neq 0$.

Starting with $x - y = \frac{c}{d}$ *(We need to prove this result will give a rational number)*

$$\frac{a}{b} - y = \frac{c}{d} \quad \text{Substitute for } x$$

$$y = \frac{a}{b} - \frac{c}{d} = \frac{(ad - bc)}{bd} \quad \text{Add fractions and simplify}$$

Now the result $(ad - bc)$ is an integer (because a, b, c, d are all integers and products and differences of integers are integers), and $bd \neq 0$ (by zero product property).

Therefore, by definition of a rational number, y is rational (**on the RHS**). However, this finding contradicts our original supposition that y is irrational (**and on the LHS**). Hence, the supposition is false and the theorem is true.

- 4.6 What would happen if we investigated a more generalised statement:

If n is an integer, then $n^2 + 2$ is not divisible by 4

This cannot be proved by exhaustion since it involves infinitely many objects (integers).

Statement: If n is an integer then $n^2 + 2$ is not divisible by 4

The negation of the statement is: n is an integer and $n^2 + 2$ is divisible by 4

This is the claim we need to investigate. As a handy remark, recall that since n is an integer, n may be either even or odd.

Let's consider both cases and make some conclusions.

Evens

1. Assume first that n is even.

Then $n = 2m$, for some integer m

2. Thus, $n^2 + 2 = (2m)^2 + 2 = 4m^2 + 2$

3. Since $n^2 + 2$ is divisible by 4, we have that

$$4m^2 + 2 = 4k, \text{ for some integer } k.$$

4. By dividing both sides by 2 we get

$$2m^2 + 1 = 2k, \text{ with } k \text{ and } m^2 \text{ integers.}$$

5. So, there is an odd number that is equal to an even number (**which is a contradiction**)

& Odds

1. Assume now that n is odd.

Then $n = 2m + 1$, for some integer m

2. Thus, $n^2 + 2 = (2m + 1)^2 + 2 = 4m^2 + 4m + 3$

3. Since $n^2 + 2$ is divisible by 4, we have that

$$4m^2 + 4m + 3 = 4k, \text{ for some integer } k.$$

4. By dividing both sides by 2 we get

$$2m^2 + 2m + 1.5 = 2(m^2 + m) + 1.5 = 2k.$$

5. So again, there is a decimal number that is equal to an even number (another contradiction)

Because these two cases contradict the negation we can conclude that:

If n is an integer, then $n^2 + 2$ is not divisible by 4

PROBLEMS TO SOLVE

CHAPTER 4: GEOMETRY

A conjecture is true **only** if it is always true. If it is false, give a counter example, otherwise give one example of when it is true.

- The number 1 more than the square of an even integer is always a prime number.

- Every number has an even number of factors.

- Every factor of an odd number is odd.

- Every factor of an even number is even.

- The product of two consecutive counting numbers is a multiple of 4.

Prove the following conjectures.

- The sum of three consecutive even integers is always a multiple of three.

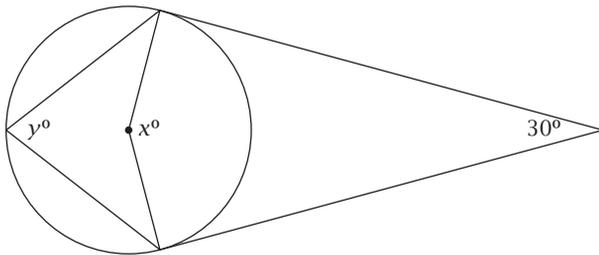
7. The sum of three counting numbers in an arithmetic progression is a multiple of three.

8. The product of three consecutive even numbers is always a multiple of eight.

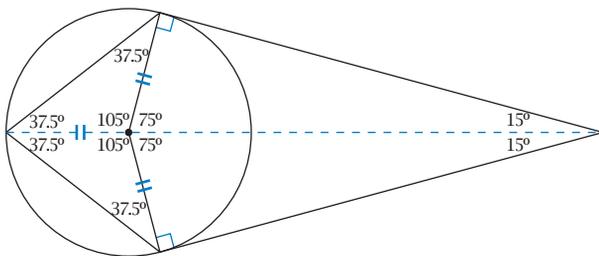
9. If you multiply two odd numbers the result is always an odd number.

10. If x^2 is an odd number then x is an odd number.

11. Mark was given the following diagram and asked to prove that $x^\circ = 2y^\circ$.



His proof is shown on the diagram below:



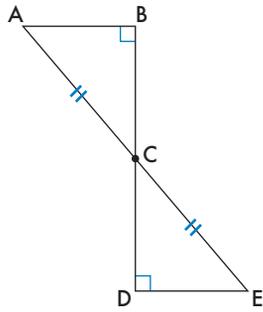
Using the diagram Mark found

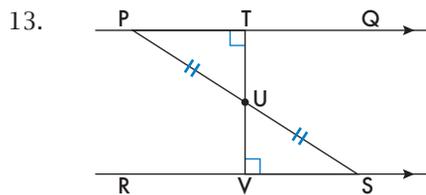
$$x^\circ = 150^\circ$$

$$y^\circ = 75^\circ$$

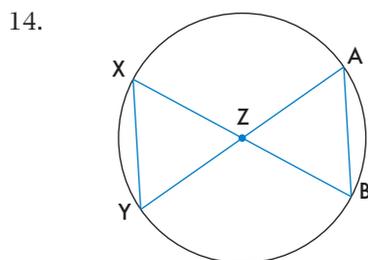
He concluded that $x^\circ = 2y^\circ$. Is his proof valid?

12. Given the diagram below prove that $\overline{BC} \cong \overline{DC}$.





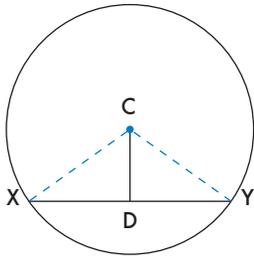
Lines PQ and RS are parallel with U being the midpoint of \overline{PS} . Prove that U is half way between lines PQ and RS.



\overline{XB} and \overline{AY} are diameters of a circle, centre Z. Prove $\overline{XY} \parallel \overline{AB}$.

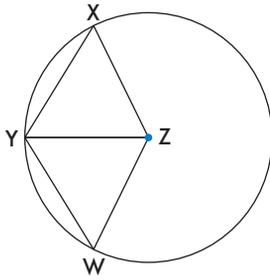
15. Two circles with centres P and Q, intersect at X and Y. Prove that $\angle PXQ = \angle PYQ$.

16.



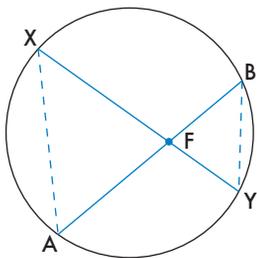
A circle centre C with chord \overline{XY} which has a midpoint at D. Prove $\overline{CD} \perp \overline{XY}$.

17.



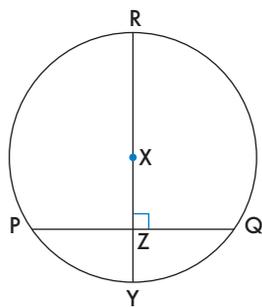
A circle centre Z has two points located on the circle, X and W, not in the same diameter. Prove reflex $\angle XZW = 2 \times \angle XYW$.

18.



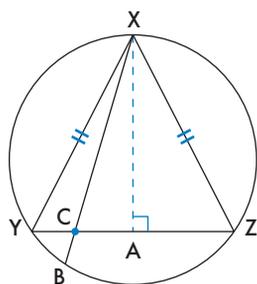
\overline{XY} and \overline{AB} are two chords of a circle that meet at point F. Prove $\overline{FX} \cdot \overline{FY} = \overline{FA} \cdot \overline{FB}$

19. A circle, centre X of radius 'r' is drawn below.



Show $\overline{PZ} \cdot \overline{ZQ} = r^2 - \overline{XZ}^2$

20.



Prove $XZ^2 - XC^2 = ZC \cdot YC$

21. Rewrite the following statements using plain English (and complete) sentences:

- (a) $\forall \text{IR } x, \frac{1}{x} \neq x$
- (b) $\exists \text{IR } x \text{ s.t. } x^2 < -x$
- (c) $f(x) = \sqrt{x} \Rightarrow D_f : \{x : x \in \text{IR}, x \geq 0\}$

22. Rewrite the following sentences as mathematical statements:

- (a) If a function $g(x)$ is defined as the quotient of 2 and $(x - 1)$, then the Domain of this function $g(x)$ is such that it is a set of values x such that x is an element of the Real numbers where x cannot equal zero.
- (b) For all positive integers m , the sum of m and the quotient of 1 and m is greater than or equal to two.
- (c) There exists a real number p such that the difference between zero and p do not equal p .

23. Consider the following statement:

If the Eagles are winning the game, then they are playing the Dockers.

Rewrite this statement as $a(n)$

- (a) inverse _____
- (b) converse _____
- (c) negation _____
- (d) contrapositive _____

24. Consider the statement $\forall x, x < 6 \Rightarrow |x| < 6$.

- (a) Write the converse of the statement.
- (b) Which is true: the statement, its converse, or both?

25. Rewrite the following mathematical statements (using mathematical notation) and according to the type of statement required:

- (a) If $x + y < y + 5$, then $x < y$ (inverse)
- (b) \forall even $x \in \mathbb{Z}$, x^2 is even (negation)
- (c) $\forall n \in \mathbb{Z}, n | 3 \Rightarrow n^2 | 3$ (contrapositive) (N.B. the symbol ' $|$ ' means 'divisible by').

Investigate the following ...

26. If a^2 is a multiple of p , then a is a multiple of p (suppose that $a \in \mathbb{Z}$ and p is prime).

27. If the square of an integer is odd, then that integer is odd.

28. For all integers n , prove that if $n^2 + 5$ is odd then n is even.

29. If $a^2 - 2a + 7$ is even, then a is odd (suppose that $a \in \mathbb{Z}$).

30. Prove for $\triangle ABC$ that if $\angle A$ is a right angle, then $\angle B$ cannot be an obtuse angle.

31. Prove that for every real number $x \in \left[0, \frac{\pi}{2}\right]$, we have $\sin(x) + \cos(x) \geq 1$.

32. Prove that no integers a and b exist for which $18a + 6b = 1$.

33. Prove that there are infinitely many prime numbers.

34. There is no greatest even integer.

In $\triangle OPR$, $OR = \cos B$ and $PR = \sin B$

In $\triangle ORS$, $OS = \cos A \cos B$ and $RS = \sin A \cos B$

In $\triangle TPR$, $PT = \cos A \sin B$ and $TR = \sin A \sin B$

$$\therefore QS = \sin A \sin B$$

Because

$$PQ = PT + TQ = PT + RS = RS + PT$$

$$\sin C = \sin A \cos B + \cos A \sin B \quad \text{and } C = A + B$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B \quad \dots\dots\dots(1)$$

$$OQ = OS - QS$$

$$\cos C = \cos A \cos B - \sin A \sin B \quad \text{and } C = A + B$$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B \quad \dots\dots\dots(2)$$

- If now B is replaced by (-B) in both of the above compound formulae then:

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\text{and } \cos(A + (-B)) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

It is now not hard to show that:

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

- When $A = B$ the double angle formulae are:

$$\sin(A + A) = \sin A \cos A + \cos A \sin A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos(A + A) = \cos A \cos A - \sin A \sin A$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad \text{- version (1) of } \cos 2A$$

If we let $\sin^2 A = 1 - \cos^2 A$ then:

$$\cos 2A = 2\cos^2 A - 1 \quad \text{- version (2) of } \cos 2A$$

If we let $\cos^2 A = 1 - \sin^2 A$ then:

$$\cos 2A = 1 - 2\sin^2 A$$

- version (3) of $\cos 2A$

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

SOLVING TRIGONOMETRIC EQUATIONS – THE BASIC TYPE

If a trig equation can be reduced to any of these -

$$\sin \theta = c \quad \text{where } -1 \leq c \leq 1$$

$$\cos \theta = c \quad \text{where } -1 \leq c \leq 1$$

$$\tan \theta = c \quad \text{where } c \text{ is any real number}$$

then it is considered to be of the basic type. You can be asked to solve any of these over a given domain.

Example:

Solve $\sin \theta = -0.3$ over $-180^\circ \leq \theta \leq 360^\circ$

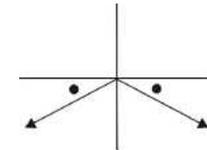
The negative sign in -0.3 means that the rays in the unit circle have to be drawn in the third and fourth quadrants according to

S	A
T	C

The reference angles (abbreviated RA hereunder) marked with dots are found by ignoring the negative sign in -0.3 and by finding inverse sine.

$$\begin{aligned} \text{i.e.} \quad \text{RA} &= \text{inv sin } 0.3 \\ &= 17.46^\circ \end{aligned}$$

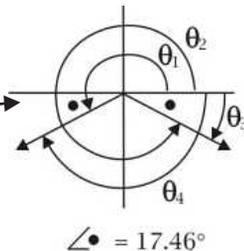
The possible angles for θ are drawn from the positive x axis in directions indicated by the domain. In this case both directions. The final diagram looks like this



The sizes of $\theta_1, \theta_2, \theta_3$ and θ_4 follow from the diagram.

$$\begin{aligned} \theta_1 &= 180^\circ + 17.46^\circ & \theta_3 &= -17.46^\circ \\ &= 197.46^\circ \end{aligned}$$

$$\begin{aligned} \theta_2 &= 360^\circ - 17.46^\circ & \theta_4 &= -(180^\circ - 17.46^\circ) \\ &= 342.54^\circ & &= -162.54^\circ \end{aligned}$$



The above steps are summarised as

(i) Draw the rays on $\frac{S}{T} \mid \frac{A}{C}$ according to the sign of c .

(ii) Do inv $\begin{cases} \sin \\ \cos \\ \tan \end{cases}$ of $|c|$ to get RA

(iii) Draw in all possible arcs for θ by noting the direction and the end points of the domain.

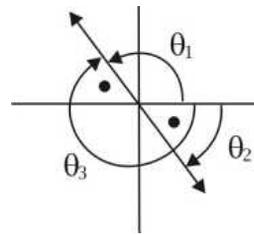
(iv) Find the angles shown on your diagram.

Example:

Solve $\tan \theta = -5$ to 4 dp over $-\frac{3\pi}{2} < \theta < \pi$

In Rad Mode

$$\begin{aligned}
 \text{RA} &= \text{inv tan } 5 \\
 &= 1.3734 \quad \leftarrow \text{store in memory} \\
 \theta_1 &= \pi - 1.3734 & \theta_3 &= -(\pi + 1.3734) \\
 &= 1.7682 & &= -4.5150 \\
 \theta_2 &= -1.3734
 \end{aligned}$$



SOLVING HARDER TYPES OF TRIGONOMETRIC EQUATIONS

- When an equation can be reduced to any one of $\sin(f(x)) = k$, $\cos(f(x)) = k$, $\tan(f(x)) = k$ then the easiest method to solve for x is to proceed as before by
 - drawing the diagram from the sign of k then
 - finding the reference angle (RA) by $\text{RA} = \text{inv trig function } |k|$

Now the "inside" (i.e. $f(x)$) is equated in turn to the angles found from the diagram above and solved for x . This process is repeated for the negative and positive directions until x goes outside the required domain. An example shows this best:

Example: Solve

$$\begin{aligned}
 4 \cos 3x &= -3 & -90^\circ \leq x \leq 180^\circ \\
 \cos 3x &= -\frac{3}{4} \\
 &= -0.75
 \end{aligned}$$

$$\begin{aligned}
 \text{RA} &= \text{inv cos}(0.75) \\
 &= 41.41^\circ
 \end{aligned}$$

$$3x = 180^\circ - 41.41^\circ$$

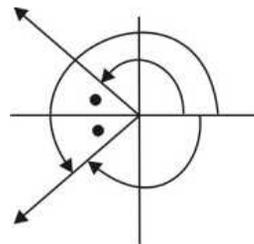
$$x_1 = 46.20^\circ \quad (2\text{dp})$$

$$3x = 180^\circ + 41.41^\circ$$

$$x_2 = 73.80^\circ \quad (2\text{dp})$$

$$3x = 540^\circ - 41.41^\circ$$

$$x_3 = 166.20^\circ \quad (2\text{dp})$$



$$3x = -(180^\circ - 41.41^\circ)$$

$$x_4 = -46.20^\circ \quad (2\text{dp})$$

$$3x = -(180^\circ + 41.41^\circ)$$

$$x_5 = -73.80^\circ \quad (2\text{dp})$$

Note:

This method can be used for nearly any inside function $f(x)$. If you are not sure whether another value of x exists within the domain, try the next one and reject it if you've gone too far!!

BASIC TECHNIQUES NEEDED TO PROVE IDENTITIES

- When proving an identity you must start with one side only and process it until you arrive at the other side. If this doesn't seem to be happening you can stop and process the other side of the identity with the hope of arriving at the same place.
- Start with the more complicated side, as it is easier to simplify an expression than to make it more complicated.
- Always keep an eye on the other side for clues.

USEFUL ALGEBRAIC TECHNIQUES

- You must be able to add and subtract fractions written in terms of $\sin \theta$ and $\cos \theta$. The basic algebraic process is

$$\frac{\frac{a}{b} + \frac{c}{d}}{= \frac{ad + bc}{bd}}$$

Be familiar with both forms.

Example: Consider

$$\begin{aligned} & \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \\ &= \frac{1}{\cos \theta \sin \theta} \\ &= \frac{2}{\sin 2\theta} \end{aligned}$$

- The process of **factorisation** is very important. Look out for
 - common factors
 - difference of two squares
 - trinomials

Don't expand parentheses unless you can see that it will simplify to something useful, otherwise you may miss the chance to cancel down.

- Multiplying top and bottom of a fraction by $\frac{a}{a}$ doesn't change the value of the fraction but it will change its appearance. The value of a is often chosen with the difference of two squares idea in mind.

Example:

Consider $\frac{\cos \theta}{1 - \sin \theta}$ (the "a" from above in this case is $1 + \sin \theta$)

$$\begin{aligned} &= \frac{\cos \theta}{1 - \sin \theta} \times \frac{1 + \sin \theta}{1 + \sin \theta} \\ &= \frac{\cos \theta(1 + \sin \theta)}{1 - \sin^2 \theta} \quad \text{don't expand the numerator} \\ &= \frac{\cos \theta(1 + \sin \theta)}{\cos^2 \theta} \\ &= \frac{1 + \sin \theta}{\cos \theta} \end{aligned}$$

cancel $\cos \theta$

COMBINATION TYPES i.e. $a \cos \theta + b \sin \theta \rightarrow R \sin(\theta + \alpha)$

The function $y = a \sin \theta + b \cos \theta$ can be written as a single sine function or as a single cosine function. The reason is as follows.

Consider
$$\begin{aligned} y &= R \sin(\theta + \alpha) \\ &= R(\sin \theta \cos \alpha + \cos \theta \sin \alpha) \\ &= R \sin \theta \cos \alpha + R \cos \theta \sin \alpha \end{aligned}$$

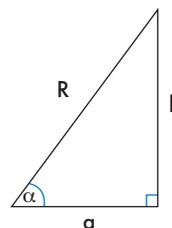
which is the same function as

if
$$\begin{aligned} y &= a \sin \theta + b \cos \theta \\ a &= R \cos \alpha \quad \text{and} \quad b = R \sin \alpha \end{aligned}$$

The values of a , b , R and α are linked by the right angled triangle

i.e.
$$\left. \begin{aligned} \cos \alpha &= \frac{a}{R} \\ \sin \alpha &= \frac{b}{R} \end{aligned} \right\} \text{ as above and } \tan \alpha = \frac{b}{a}$$

$$a^2 + b^2 = R^2$$



So
$$y = a \sin \theta + b \cos \theta$$

$$R \sin(x + \alpha) \quad \text{where} \quad R = \pm \sqrt{a^2 + b^2}$$

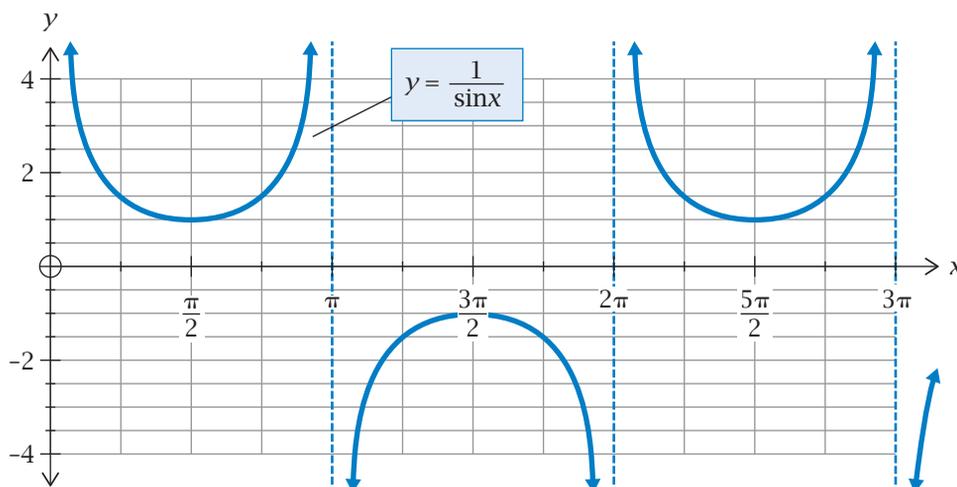
$$\tan \alpha = \frac{b}{a}$$

This is also given in your tables book. In practice it is strongly advised that you work from first principles because it is very easy to get mixed up.

THE RECIPROCAL TRIGONOMETRIC FUNCTIONS

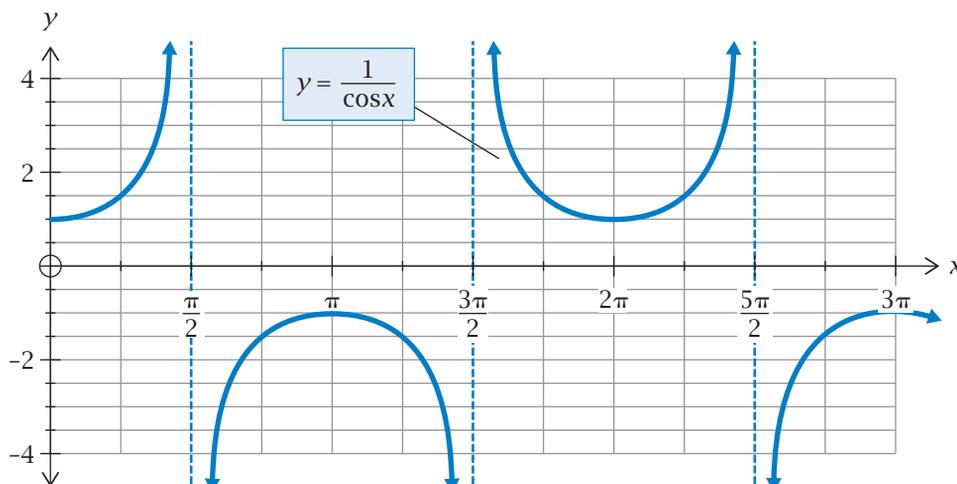
Cosecant

The reciprocal function of $y = \sin x$ is $y = \frac{1}{\sin x}$, and this is commonly expressed as $y = \operatorname{cosec} x$. The graph of $y = \operatorname{cosec} x$ is included below over the domain $0 \leq x \leq 3\pi$.



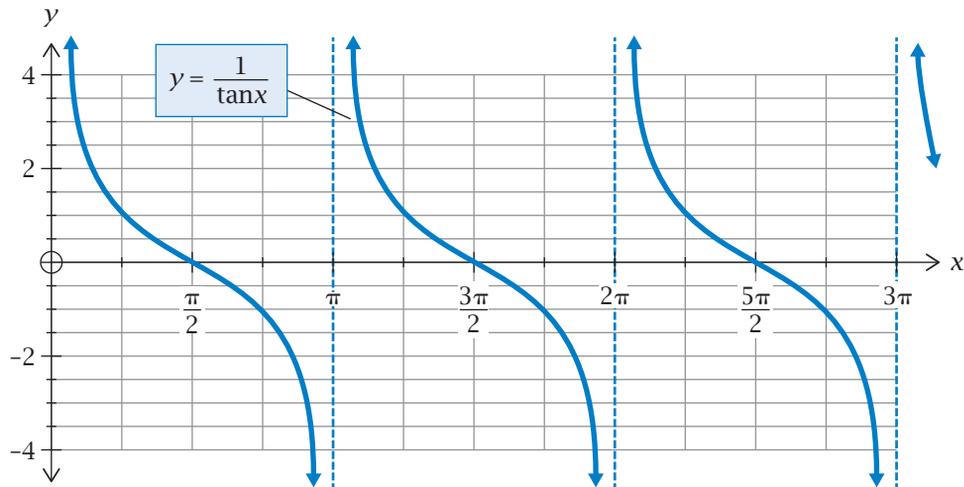
Secant

The reciprocal function of $y = \cos x$ is $y = \frac{1}{\cos x}$, and this is commonly expressed as $y = \operatorname{sec} x$. The graph of $y = \operatorname{sec} x$ is included below over the domain $0 \leq x \leq 3\pi$.



Cotangent

The reciprocal function of $y = \tan x$ is $y = \frac{1}{\tan x}$, and this is commonly expressed as $y = \cot x$. The graph of $y = \cot x$ is included below over the domain $0 \leq x \leq 3\pi$.



Worked Examples

Trigonometry

- 5.1 (a) If $\cos 2A = 0.28$, show that the exact value of $\tan A$ is ± 0.75 without finding $\text{inv } \cos 2A$ on your calculator.
- (b) Explain why $\tan A$ is either $+ 0.75$ or $- 0.75$.

Solution

$$\begin{aligned}
 \text{(a)} \quad \cos 2A &= 2 \cos^2 A - 1 & \cos 2A &= 1 - 2 \sin^2 A \\
 \therefore \cos^2 A &= \frac{1 + \cos 2A}{2} & \therefore \sin^2 A &= \frac{1 - \cos 2A}{2} \\
 \frac{\sin^2 A}{\cos^2 A} &= \frac{1 - \cos 2A}{2} \div \frac{1 + \cos 2A}{2} \\
 &= \frac{1 - \cos 2A}{1 + \cos 2A} \\
 \tan^2 A &= \frac{1 - 0.28}{1 + 0.28} \\
 \tan A &= \pm \sqrt{\frac{0.72}{1.28}} \\
 \tan A &= \pm \sqrt{0.5625} \\
 \tan A &= \pm 0.75
 \end{aligned}$$

- (b) When A and $2A$ are in the first quadrant $\tan A = 0.75$. When A and $2A$ are negative angles in the fourth quadrant $\cos 2A$ will still be 0.28 but $\tan A$ will be -0.75 due

$$\text{to } \begin{array}{c|c} \text{S} & \text{A} \\ \hline \text{T} & \text{C} \end{array} .$$

5.2 Find the amplitude, period, horizontal and vertical shifts (i.e. translations) of

$$y = -5 \sin\left(\frac{x}{3} + \frac{\pi}{12}\right) - 2$$

Solution

Rewrite as $y = -5 \sin\frac{1}{3}\left(x + \frac{\pi}{4}\right) - 2$

$$\therefore \text{amplitude } |-5| = 5$$

$$\text{period} = \frac{2\pi}{\frac{1}{3}} = 6\pi$$

horizontal shift $\frac{\pi}{4}$ to the left, vertical shift 2 units down.

5.3 Solve $4 \cos^2 x + 15 \sin x - 13 = 0$ over $0 \leq x \leq 2\pi$ correct to 2 dp.

Solution

	Replace $\cos^2 x$ by $1 - \sin^2 x$	
So	$4 \cos^2 x + 15 \sin x - 13 = 0$	
becomes	$4(1 - \sin^2 x) + 15 \sin x - 13 = 0$	
	$4 - 4 \sin^2 x + 15 \sin x - 13 = 0$	
	$-4 \sin^2 x + 15 \sin x - 9 = 0$	
	$4 \sin^2 x - 15 \sin x + 9 = 0$	
	$(4 \sin x - 3)(\sin x - 3) = 0$	

$$\therefore 4 \sin x - 3 = 0 \quad \text{or} \quad \sin x - 3 = 0$$

$$\therefore \sin x = \frac{3}{4} \quad \text{or} \quad \sin x = 3$$

(no solution)

$$\therefore x = \sin^{-1}\left(\frac{3}{4}\right)$$

$$\therefore x = 0.85 \text{ radians}$$

$$\text{or } x = 2.29 \text{ radians} \quad (2\text{dp})$$

5.4 Prove $\frac{1 - \cos A}{\sin A} = \tan \frac{A}{2}$

Solution

$$\begin{aligned} \text{Take LHS} &= \frac{1 - \cos A}{\sin A} \\ &= \frac{1 - \left(1 - 2\sin^2 \frac{A}{2}\right)}{2 \sin \frac{A}{2} \cos \frac{A}{2}} \\ &= \frac{1 - 1 + 2 \sin^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} \\ &= \frac{2 \sin^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} \\ &= \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} \\ &= \tan \frac{A}{2} \\ &= \text{R.H.S.} \end{aligned}$$

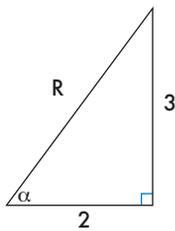
5.5 Prove $\frac{1}{\sin A} = \frac{\sin A}{1 - \cos A} - \frac{\cos A}{\sin A}$

Solution

$$\begin{aligned} \text{Take RHS} &= \frac{\sin A}{1 - \cos A} - \frac{\cos A}{\sin A} \\ &= \frac{\sin^2 A - \cos A(1 - \cos A)}{\sin A(1 - \cos A)} \\ &= \frac{\sin^2 A - \cos A + \cos^2 A}{\sin A(1 - \cos A)} \\ &= \frac{\sin^2 A + \cos^2 A - \cos A}{\sin A(1 - \cos A)} \\ &= \frac{(1 - \cos A)}{\sin A(1 - \cos A)} \\ &= \frac{1}{\sin A} \\ &= \text{L.H.S.} \end{aligned}$$

5.6 Write $2 \cos \theta - 3 \sin \theta$ in the form of $R \cos(\theta + \alpha)$

- (a) Expand $R \cos(\theta + \alpha)$
 $= R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$
- (b) Compare to given function $= 2 \cos \theta - 3 \sin \theta$
- (c) Equate corresponding 'pieces' i.e. $R \cos \alpha = 2$ $R \sin \alpha = 3$
- (d) Draw triangle



$$\begin{aligned} R &= \sqrt{2^2 + 3^2} = \sqrt{13} \\ \tan \alpha &= \frac{3}{2} \\ \alpha &= 56.31^\circ \end{aligned}$$

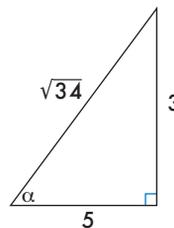
- (e) Write answer $2 \cos \theta - 3 \sin \theta = \sqrt{13} \cos(\theta + 56.31^\circ)$

5.7 Write $3 \cos \theta - 5 \sin \theta$ in the form $R \sin(\theta - \alpha)$

- (a) $R \sin(\theta - \alpha)$
 $= R \sin \theta \cos \alpha - R \cos \theta \sin \alpha$
 - (b) $= 3 \cos \theta - 5 \sin \theta$
 $= -5 \sin \theta + 3 \cos \theta$
- Compare—wrong way around so change order*

- (c) $R \cos \alpha = -5$ $-R \sin \alpha = 3$
- (d) If $0 < \alpha < 90^\circ$ R must be negative

$$\begin{aligned} R &= -\sqrt{3^2 + 5^2} \\ R &= -\sqrt{34} \\ \text{So } -\sqrt{34} \cos \alpha &= -5 & \sqrt{34} \sin \alpha &= 3 \\ \cos \alpha &= \frac{5}{\sqrt{34}} & \sin \alpha &= \frac{3}{\sqrt{34}} \end{aligned}$$



$$\alpha = 30.96^\circ$$

- (e) $3 \cos \theta - 5 \sin \theta = -\sqrt{34} \sin(\theta - 30.96^\circ)$

PROBLEMS TO SOLVE

CHAPTER 5: TRIGONOMETRY

1. In a right triangle $\tan A = \frac{5}{3}$

(a) Find the exact value of $\cos 2A$.

(b) Show that $\cos 2A + \sin 2A = \frac{7}{17}$

2. Solve:

(a) $\sin 3x = \frac{1}{\sqrt{2}} \quad 0 < x < \pi$

(b) $\tan^2 A = 3 \tan A - 2 \quad -\pi < A < \pi$

(c) $\sin 2\theta = \cos \theta \quad 0 < \theta < 2\pi$

(d) $\sin\left(2x - \frac{\pi}{6}\right) = \frac{1}{2} \quad -\pi \leq x \leq \pi$

3. State the maximum value of $3 - 2 \cos x$ and specify one angle in the range $0^\circ \leq x \leq 360^\circ$ for which this maximum occurs.

4. If $\sin A = \frac{5}{6}$ find $\cos 2A$ exactly.

5. (a) Find the exact value of $\tan(-15^\circ)$. (Hint: $\tan(-15^\circ) = \tan(30^\circ - 45^\circ)$)

- (b) Simplify $\frac{\sin(\theta - 90^\circ) \tan(180^\circ - \theta) \cos(\theta + 180^\circ)}{\sin(270^\circ - \theta) \sin^2(\theta - 180^\circ)}$

6. Show that the exact value of $\sin 15^\circ$ is $\frac{\sqrt{6} - \sqrt{2}}{4}$

7. Solve $\tan^2 2x = 3$ for $-\pi \leq x \leq \pi$.

8. Solve $\cos 2x = 1 + \sin x$ for the smallest positive value of $x > 180^\circ$.

9. $\triangle PQR$ is an isosceles triangle with $PR = QR$ and $\angle PQR = x^\circ$.

(a) Show that $\frac{\sin x^\circ}{p} = \frac{\sin 2x^\circ}{r}$

(b) State the value of x° when $p = r$.

(c) Using $p = r$ solve the equation in (a) above to justify your answer to (b).

10. $\triangle ABD$ has a right angle at D and $AD = 5$ cm. Point C is on BD where $BC = x$ cm and $CD = 2$ cm. If $\angle BAC = \angle CAD = \alpha$, find the exact value of x .

11. Find the exact value of $\cos \frac{\pi}{8}$

12. Find the possible answers for x as exact values if $\sqrt{2} \cos 2x - 1 = 0$ and $0 \leq x \leq 2\pi$.

13. $\triangle ABC$ is isosceles with a right angle at C . If $AC = 2$ cm and D the midpoint of BC then let $\angle BAD = \theta$.

(a) Write down the size of $\angle BAC$.

(b) Write down the exact value of $\tan \angle DAC$.

(c) Find exactly the size of $\tan \theta$.

14. Without using a calculator find the exact value of $\tan 75^\circ$.

15. If $\sin A = \frac{2}{3}$ find the exact value of $\tan 2A$ without using your calculator to find the value of A .

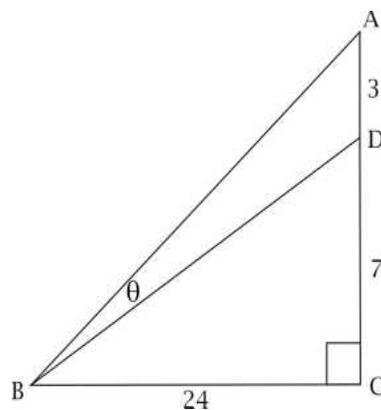
16. (a) What is the minimum value which the function $f(x) = 3 - 2 \sin(x - 60^\circ)$ can have?

(b) What is the smallest positive value of x which gives this minimum value of $f(x)$?

17. Find the exact value of the following expression. Rationalise the denominator if necessary.

$$\sin \frac{\pi}{3} \cdot \tan \frac{5\pi}{4} \cdot \tan^2 \left(-\frac{11\pi}{6} \right)$$

18. Consider this diagram which hasn't been drawn to scale.



Find the exact value of $\sin \theta$. (Hint: Let $\angle DBC = \mu$ and $\angle ABC = \gamma$.)

19. Find the exact value of $\frac{7 \sin 3x}{\sin x} - \frac{7 \cos 3x}{\cos x}$

20. Solve the following equations by giving your answers as exact values.

(a) $\sqrt{2} \sin \theta = -1$ $-2\pi \leq \theta \leq 2\pi$

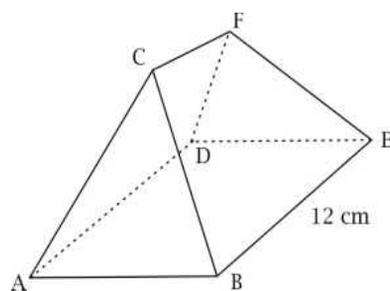
(b) $\cos^2 \theta \sin \theta - \sin^3 \theta = 0$ $0^\circ \leq \theta \leq 360^\circ$

21. The ends of the 3 dimensional figure ABCDEF shown in the diagram are equilateral triangles of side 8 cm which slant inwards at the base making an angle of 60° with the rectangle ABED. Find:

(a) the exact distance from C to the edge AB

(b) the exact distance from C to the base ABED

(c) the exact difference between the lengths BE and CF.



22. Solve the following equations.

(a) $3 \sin 2x = 1$ $0^\circ \leq \theta \leq 360^\circ$

(b) $\sin\left(3x + \frac{\pi}{6}\right) = \frac{1}{2}$ $0 \leq x \leq 2\pi$ as exact values.

23. Solve $\tan\left(\frac{x}{4} + \frac{\pi}{4}\right) = -0.5$ over $-2\pi \leq x \leq 2\pi$ correct to 4 dp.

24. Find all solutions to the following equations, giving your answers as exact values.

(a) $2 \cos x + 1 = 0$ $-\pi \leq x \leq \pi$

(b) $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ $0 \leq \theta \leq 2\pi$

25. In the cyclic quadrilateral ABCD, $AB = AD = 2$, $BC = 4$ and $DC = 3$.
Find the length of BD.

Prove these identities

26.
$$\frac{\tan(180^\circ + A) \cdot \cos(360^\circ - A)}{\sin^2(270^\circ - A)} = \frac{\sin A}{\cos^2 A}$$

27. $1 - 3 \sin^2 B = 3 \cos^2 B - 2$

28. $\cos(A + B) \cdot \cos(A - B) = \cos^2 A - \sin^2 B$

29.
$$\frac{\sin A}{1 - \cos A} - \frac{\sin A}{1 + \cos A} = \frac{2}{\tan A}$$

30. $\tan^2 A - \sin^2 A = \tan^2 A \cdot \sin^2 A$

31.
$$\frac{\cos^2 A}{1 + \sin A} + \sin A = 1$$

32.
$$\frac{1}{\sin B \cos A} + \frac{1}{\sin A \cos B} = \frac{4 \sin(A + B)}{\sin 2A \sin 2B}$$

33. $(1 + \tan^2 A) \cdot \sin^2\left(\frac{\pi}{2} - A\right) = 1$

34. $\cos^4 \frac{A}{2} - \sin^4 \frac{A}{2} = \cos A$

35. $\frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A$

36. $\frac{\cos A}{1 - \sin A} - \frac{\cos A}{1 + \sin A} = 2 \tan A$

37. $\frac{\sin 2x - \cos x}{\cos 2x - 2 \sin^2 x} = -\frac{\cos x}{1 + 2 \sin x}$

38. $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

39. $\frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \frac{A}{2}$

40. $\sin^4 A - \cos^4 A = 1 - 2 \cos^2 A$

41. $\frac{1}{1 + \sin A} + \frac{1}{1 - \sin A} = \frac{2}{\cos^2 A}$

42. $\frac{\cos A}{1 + \sin A} + \frac{1 + \sin A}{\cos A} = \frac{2}{\cos A}$

43. $\frac{\sin A + \tan A}{1 + \cos A} = \tan A$

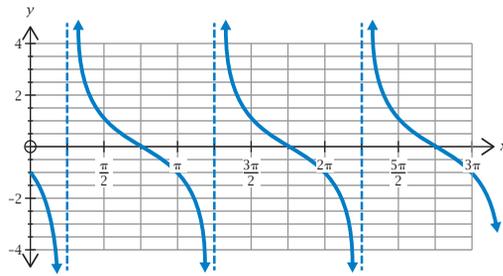
44. $\frac{\cos A}{\sin B} + \frac{\sin A}{\cos B} = \frac{2 \cos(A + B)}{\sin 2B}$

45. $\frac{\sin 2A - \sin A}{\cos 2A - \cos A + 1} = \tan A$

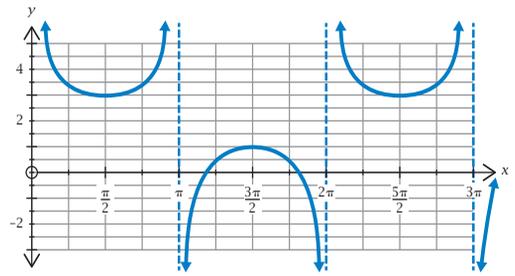
46. $\sin A \cos A \cos 2A \cos 4A = \frac{1}{8} \sin 8A$

47. From the graphs below, determine the equation of the reciprocal trigonometric function.

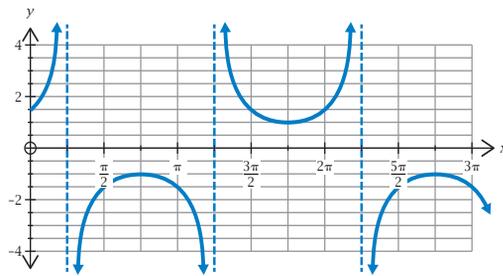
(a)



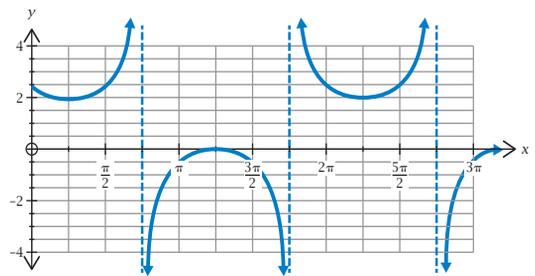
(b)



(c)



(d)



48. (a) Determine correct to 3 decimal places, the positive numbers p and q such that $3 \sin \theta + 4 \cos \theta = p \sin (\theta + q)$.

- (b) Hence find the smallest positive value of θ which satisfies the equation $6 \sin \theta + 8 \cos \theta = 10$.

49. (a) Show that $2 \cos (x + 30)^\circ - \sin x^\circ$ can be written as $\sqrt{3} \cos x^\circ - 2 \sin x^\circ$.

- (b) Express $\sqrt{3} \cos x^\circ - 2 \sin x^\circ$ in the form of $k \cos (x + \alpha)^\circ$ where $k > 0$ and $0^\circ \leq \alpha \leq 360^\circ$.

- (c) Solve $2 \cos (x + 30)^\circ = \sin x^\circ + 1$, $0^\circ \leq x \leq 360^\circ$.

50. (a) Express $2 \sin \theta + \sqrt{3} \cos \theta$ in the form $R \sin (\theta + \alpha)$ where $R > 0$.

- (b) Solve the equation $2 \sin \theta + \sqrt{3} \cos \theta = 1.36$ over the domain $0^\circ \leq \theta \leq 360^\circ$.

51. (a) Write $8 \cos x + 15 \sin x$ in the form $r \cos (x - A)$ with $0^\circ \leq x \leq 90^\circ$.

- (b) Solve $8 \cos x + 15 \sin x = 10$ over $0^\circ \leq x \leq 360^\circ$.

6

MATRICES

In this section, matrices are studied in their own right and the basic properties of matrix algebra are examined. The usefulness of matrices and matrix algebra is illustrated by the study of linear transformations in the plane, the solution of linear equations and in the solving of practical problems by the use of transition matrices.

Syllabus Checklist

By the end of this chapter, you should be able to:

Matrix arithmetic

- apply matrix definition and notation
- define and use addition and subtraction of matrices, scalar multiplication, matrix multiplication, multiplicative identity, and inverse
- calculate the determinant and inverse of 2×2 matrices and solve matrix equations of the form $AX = B$, where A is a 2×2 matrix and X and B are column vectors

Transformations in the plane

- examine translations and their representation as column vectors
- define and use basic linear transformations: dilations of the form $(x,y) \rightarrow (\lambda_1 x, \lambda_2 y)$, rotations about the origin and reflection in a line that passes through the origin and the representations of these transformations by 2×2 matrices
- apply these transformations to points in the plane and geometric objects
- define and use composition of linear transformations and the corresponding matrix products
- define and use inverses of linear transformations and the relationship with the matrix inverse
- examine the relationship between the determinant and the effect of a linear transformation on area
- establish geometric results by matrix multiplications; for example: show that the combined effect of 2 reflections is a rotation

Systems of linear equations

- interpret the matrix form of a system of linear equations in two variables and use matrix algebra to solve a system of linear equations

INTRODUCTION AND BASIC PROPERTIES

- A matrix is a rectangular array of numbers called elements which represent information in a convenient and concise way. Usually the elements of each row of a matrix are related by a common description and the elements of each column are related by another common description. This means that each element is fully described by two descriptions as illustrated below.

Three service stations S_1 , S_2 and S_3 owned by the same company sell regular premium and high octane petrol at the following prices:

$$C = \begin{matrix} & & \text{R} & \text{P} & \text{HO} \\ \begin{matrix} S_1 \\ S_2 \\ S_3 \end{matrix} & \left[\begin{array}{ccc} 1.15 & 1.20 & 1.30 \\ 1.18 & 1.25 & 1.37 \\ 1.13 & 1.19 & 1.28 \end{array} \right] & \text{\$/L} \end{matrix}$$

When a worded problem is represented in matrix form it is strongly advised that each row has a label and each column has a heading which combine to fully describe each element of the matrix. The nature of the row and column descriptions makes it much easier to see how the matrix combines with other matrices to produce new information.

• You should know that:

- (i) A matrix with m rows and n columns is said to have an order of $m \times n$.
- (ii) Matrices are represented by capital letters and elements by lower case letters.
- (iii) If a_{ij} is an element of A then it is positioned at the intersection of the i^{th} row and the j^{th} column. So a_{23} is in the second row and third column of a matrix.
- (iv) Only matrices of the same order can be added and subtracted.
- (v) When a matrix A is multiplied by a scalar multiple k , each element of the old matrix is multiplied by k to give the new matrix. For example, if petrol prices were increased by 5% then for the previous matrix C:

$$\begin{aligned}
 D = 1.05 C &= 1.05 \begin{bmatrix} 1.15 & 1.20 & 1.30 \\ 1.18 & 1.25 & 1.37 \\ 1.13 & 1.19 & 1.28 \end{bmatrix} \\
 &= \begin{bmatrix} 1.21 & 1.26 & 1.37 \\ 1.24 & 1.31 & 1.44 \\ 1.19 & 1.25 & 1.34 \end{bmatrix} \text{ \$/L 2dp}
 \end{aligned}$$

• Some special matrices are:

- (i) The zero matrix where all the elements are zero.
- (ii) A column matrix has any number of rows but just one column.
- (iii) A row matrix has any number of columns but just one row.
- (iv) A square matrix has the same number of rows as columns, i.e. a matrix with order $n \times n$ will be square.
- (v) The identity matrix I for multiplication applies only to square matrices and each element on the leading diagonal has a value of 1.

$$\text{So } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices and hence $AI = IA = A$.

MATRIX MULTIPLICATION

For matrices A and B, the product AB can only be performed when the number of columns of the first matrix A is equal to the number of rows of the second matrix B.

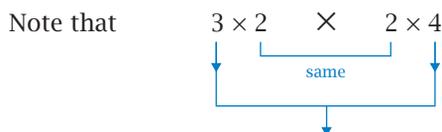
If A has an order of $m \times n$ and B has an order of $p \times q$, then AB is only possible when $n = p$.

If $AB = C$ and an element of C is c_{ij} , then c_{ij} results from multiplying the i^{th} row of A by the j^{th} row of B like this:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \text{ and } B = \begin{bmatrix} p & q & r & s \\ t & u & v & w \end{bmatrix}$$

order 3×2 order 2×4

$$\text{then } AB = \begin{bmatrix} ap + bt & aq + bu & ar + bv & as + bw \\ cp + dt & cq + du & cr + dv & cs + dw \\ ep + ft & eq + fu & er + fv & es + fw \end{bmatrix}$$



gives the order of $AB = 3 \times 4$

THE RULES OF MATRIX ALGEBRA

- (i) $A + B = B + A$ Because matrix addition is commutative.
- (ii) $AB \neq BA$ In most cases because matrix multiplication is not in general commutative.
- (iii) $(A + B) + C = A + (B + C)$ Matrix addition is associative.
- (iv) $(AB)C = A(BC)$ Matrix multiplication is associative.
- (v) $AI = IA = A$ I is the identity matrix of the same order as the square matrix A .
- (vi) $A(B + C) = AB + AC$ The distributive law for matrices where you will need to remember (ii) above (i.e. changing the order is not allowed).
- (vii) The transpose of a matrix A , denoted by A_T , is sometimes necessary when a matrix is written from a worded context and the rows and columns need to be interchanged before a matrix multiplication can be performed. An example shows this best.

$$\text{If } A = \begin{matrix} & \text{I} & \text{II} & \text{III} \\ a & \begin{bmatrix} 2 & 5 & 8 \\ 3 & 1 & 4 \end{bmatrix} & & \end{matrix} \text{ then } A_T = \begin{matrix} & \text{I} & \text{II} & \text{III} \\ a & \begin{bmatrix} 2 & 3 \\ 5 & 1 \\ 8 & 4 \end{bmatrix} & & \end{matrix}$$

Note You should make sure that for small order matrices you can perform all of the above operations both manually and on your calculator.

WORDED MATRIX PROBLEMS

- When a worded problem confronts you with a large amount of information you have to set up one or more matrices which should definitely have row labels and column headings to help you sort out the information.

Often, two of the matrices that you have set up need to be multiplied. Before this happens, say for AB , make sure that the orders are compatible and that the column headings of A match up with the row labels of B . Also make sure that multiplying each row of A by each column of B actually gives a meaningful result!

The example below is long but it does highlight many of the skills needed to process worded matrix problems. By carefully working through the stages you should get a fair idea of how different matrices link up to find new information.

Example

A company makes a range of paving products. Its range consists of pavers (P), bullnose edgings (BE), fancy edgings (FE) and wall cladding (WC). Each product can be purchased in three different finishes: Quarry, Reef and Classic. The products are all made from mixes of sand/cement, crushed limestone and granite, and ochre for colouring purposes.

Quarry is made from 50% sand/cement (SC) mixture, 20% crushed limestone (CL), 28% granite (G) and 2% ochre (O). Reef ingredients are 45% sand/cement, 42% crushed limestone, 12% granite and 1% ochre; Classic has in order, 60%, 20%, 19% and 1%.

Three months production for pavers is 650 tonnes of Quarry finish, 850 of Reef and 500 of Classic. For the same period the production for bullnose edging is 8 tonnes of Quarry finish, 12 tonnes of Reef and 6 tonnes of Classic. The orders for fancy edging are 3, 4 and 2 tonnes respectively and wall cladding can only be ordered in Reef and 80 tonnes are made.

- (a) (i) Summarise the above information into two matrices.
- (ii) Perform a suitable operation on these matrices which enables the company to find the quantity of raw materials required for each product. (calculators may be used).
- (iii) Find a new matrix, and show how to use it with the above answer, to find the total amount of raw materials that need to be ordered.
- (b) The paving product company buys from a supply company which itself buys from a quarry company at wholesale prices (W) and sells the materials to the paving product company at its retail prices (R) according to the matrix below in dollars per tonne.

Use matrix A and the matrix answer of part (a) (iii) to find a 1×2 matrix and explain what it means.

$$A = \begin{matrix} & & W & R \\ S/C & \begin{bmatrix} 80 & 110 \\ 130 & 170 \\ 115 & 135 \\ 200 & 250 \end{bmatrix} & & \\ CL & & & \\ G & & & \\ O & & & \end{matrix} \quad \$/\text{tonne}$$

- (c) Find and use the matrix necessary in order to calculate the total profit made by the supply company.

Answers

- (a) (i) The percentage matrix is

$$P = \begin{matrix} & & Q & R & C \\ S/C & \begin{bmatrix} 50 & 45 & 60 \\ 20 & 42 & 20 \\ 28 & 12 & 19 \\ 2 & 1 & 1 \end{bmatrix} & & & \\ CL & & & & \\ G & & & & \\ O & & & & \end{matrix}$$

order 4×3

The tonnes matrix is

$$T = \begin{matrix} & & P & BE & FE & WC \\ Q & \begin{bmatrix} 650 & 8 & 3 & 0 \\ 850 & 12 & 4 & 80 \\ 500 & 6 & 2 & 0 \end{bmatrix} & & & \\ R & & & & \\ C & & & & \end{matrix}$$

order 3×4

where Q stands for Quarry, R stands for Reef and C stands for Classic.

- (ii) let $M = 0.01 P T$

The 0.01 scalar multiple is necessary to change the percentages to decimals.

Note here that the column headings of P match the row labels of T. This is required for the matrix multiplication and results in a 4×4 matrix which has the same row labels of P and the same column headings of T. The headings Q, R and C disappear as part of the multiplication process. Using a calculator the matrix M is

$$M = \begin{matrix} & & \text{P} & \text{BE} & \text{FE} & \text{WC} \\ \text{S/C} & \left[\begin{array}{cccc} 1007.5 & 13 & 4.5 & 36 \\ 587 & 7.84 & 2.68 & 33.6 \\ 379 & 4.82 & 1.7 & 9.6 \\ 26.5 & 0.34 & 0.12 & 0.8 \end{array} \right] & & & & \text{tonnes} \end{matrix}$$

order 4×4

Note that multiplying P and T with the 0.01 makes sense because say, for m_{11} , we have:

$$m_{11} = \frac{50}{100} \times 650 + \frac{45}{100} \times 850 + \frac{60}{100} \times 500 = 1007.5 \text{ tonnes}$$

①
②
③
④
⑤
⑥

- ① fraction of each tonne of quarry pavers that is sand/cement
- ② tonnes of pavers with a quarry finish needed
- ③ fraction of each tonne of reef pavers that is sand/cement
- ④ tonnes of pavers with a reef finish needed
- ⑤ fraction of each tonne of classic pavers that is sand/cement
- ⑥ tonnes of classic pavers needed.

The 1007.5 answer is the amount of tonnes of sand/cement needed to accommodate the order. The same applies for the other elements of M.

(iii) When matrix M above is post multiplied by the 4×1 matrix, which has all of its elements equal to 1, then the quantities of each row get added together to give matrix S below. This shows how much sand/cement, crushed limestone, granite and ochre need to be ordered.

$$S = M \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{matrix} & & & \text{Tonnes} \\ & \text{S/C} & \text{CL} & \text{G} & \text{O} \\ \left[\begin{array}{cccc} 1061 & 631.12 & 395.12 & 27.76 \end{array} \right] & & & & \end{matrix}$$

(b) It is first necessary to transpose matrix S.

$$S_T = \text{Tonnes} \begin{bmatrix} \text{S/C} & \text{CL} & \text{G} & \text{O} \\ 1061 & 631.12 & 395.12 & 27.76 \end{bmatrix}$$

Then perform the multiplication with A as follows:

$$S_T A = \text{Tonnes} \begin{bmatrix} \text{S/C} & \text{CL} & \text{G} & \text{O} \\ 1061 & 631.12 & 395.12 & 27.76 \end{bmatrix} \begin{bmatrix} \text{W} & \text{R} \\ 80 & 110 \\ 130 & 170 \\ 115 & 135 \\ 200 & 250 \end{bmatrix} \begin{matrix} \text{S/C} \\ \text{CL} \\ \text{G} \\ \text{O} \end{matrix}$$

$$= \$ \begin{bmatrix} \text{W} & \text{R} \\ 217916.4 & 284281.6 \end{bmatrix}$$

This matrix shows the wholesale and retail prices for supplying the materials required by the paving product company.

(c) Profit is retail price take wholesale price. If the answer matrix of (b)

is post multiplied by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

$$\text{then Profit} = \$ \begin{bmatrix} 217916.40 & 284281.60 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1×2 2×1

$$= 217916.40 \times (-1) + 284281.60 \times (1)$$

$$= 284281.60 - 217916.40$$

$$= \$66365.20$$

THE INVERSE OF A MATRIX

- For the square matrix A, the inverse of A is written as A^{-1} and it is the matrix such that

$$A A^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

This means that A and A^{-1} are commutative that is, changing the order does not change the result, but, as explained previously this is not usual for matrices. Knowing the inverse of a matrix is needed for solving systems of linear equations. Essential to this process is the concept of pre-multiplying both sides of an equation by a new matrix and sometimes post multiplying by a new matrix.

For example, if $AB = C$ then premultiply both sides by A^{-1} in order to solve for B,

$$\begin{aligned} \text{i.e. } A^{-1} AB &= A^{-1}C \\ IB &= A^{-1}C \\ B &= A^{-1}C \end{aligned}$$

To solve $AB = C$ for A, then both sides need to be post multiplied by B^{-1} ,

$$\begin{aligned} \text{i.e. } AB &= C \\ ABB^{-1} &= CB^{-1} \\ AI &= CB^{-1} \\ A &= CB^{-1} \end{aligned}$$

So obviously 'pre' means in front and 'post' means behind.

- Not all square matrices have an inverse matrix. A matrix that does not have an inverse is said to be **singular**.

You will know that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse A^{-1} is $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The value of $ad - bc$ is called the **determinant** of the matrix A and is denoted by $|A|$.

$$\text{i.e. } \det A = |A| = ad - bc$$

When $\det A = 0$, A^{-1} does not exist and A is singular.

- Consider the matrix equation $AX = B$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e \\ f \end{bmatrix}. \quad \text{Then } AX = B$$

$$\text{becomes} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\text{or} \quad ax + by = e$$

$$cx + dy = f$$

When A is a known square matrix, B is a known column matrix and X is an unknown column matrix, $AX = B$ represents a **system of equations**, the solution of which is $X = A^{-1}B$.

LINEAR TRANSFORMATIONS IN THE PLANE

- If $\begin{bmatrix} x \\ y \end{bmatrix}$ is any point on the Cartesian plane called the **object** point then $\begin{bmatrix} x' \\ y' \end{bmatrix}$ will be the **image** point when $\begin{bmatrix} x \\ y \end{bmatrix}$ is transformed by the 2×2 transformation matrix A according to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This leads to the **transformation equations**

$$x' = ax + by$$

$$y' = cx + dy$$

- The two object points $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ have special significance because

$$\begin{array}{l} \text{for } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ = \begin{bmatrix} a \\ c \end{bmatrix} \text{ is the first column of A} \end{array} \qquad \begin{array}{l} \text{and for } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} b \\ d \end{bmatrix} \text{ is the second column of A.} \end{array}$$

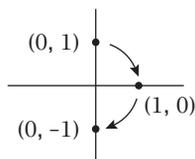
So for any transformation matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the first column is the image of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the second column is the image of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This feature enables us to quickly find A when the images of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are known.

Example

(a) Find the matrix that will rotate all points 90° clockwise.

(b) Find the equation of $y = -\frac{3}{4}x + 6$ after it is rotated 90° clockwise back again!!

(a) A diagram helps



The image of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and the image of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So the required matrix is $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) If $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ then $x' = y$ and $y' = -x$ or $x = -y'$

then by substitution, $y = -\frac{3}{4}x + 6$ becomes $x' = -\frac{3}{4}(-y') + 6$

$$\text{or } 4x' = 3y' + 24$$

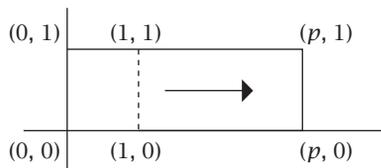
$$3y' = 4x' - 24$$

$$y' = \frac{4}{3}x' - 8$$

The required equation is $y = \frac{4}{3}x - 8$.

- The types of linear transformations that you need to be familiar with are:
 - reflections about a given line which is often the x axis or the y axis or one of the linear functions $y = \pm x$. Using the above method the necessary matrices are easy to find.
 - Dilations in the x direction, the y direction and in both directions using the origin as the centre of the dilations.

Consider the unit square shown being dilated by the scale factor p in the x direction to give the rectangle shown.



gives

To dilate points in the x direction by scale factor p the transformation matrix is

$$A = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly, if the unit square is dilated by the scale factor q in the y direction, the $(0, 0)$ and $(1, 0)$ stay where they are; $(0, 1)$ goes to $(0, q)$ and $(1, 1)$ goes to $(1, q)$.

To dilate points in the y direction by scale factor q the transformation matrix is

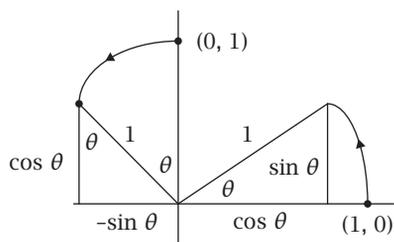
$$A_y = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$$

The above dilations can be performed simultaneously.

To dilate points in the x direction by scale factor p and in the y direction by scale factor q the transformation matrix is

$$A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

- rotations about the origin in both clockwise and anticlockwise directions by the polar angle θ .



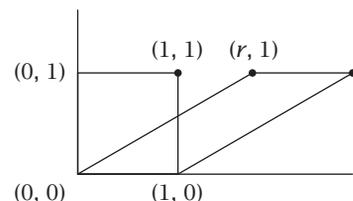
So $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$

In general to rotate points about the origin by θ the general transformation matrix is

$$A_r = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

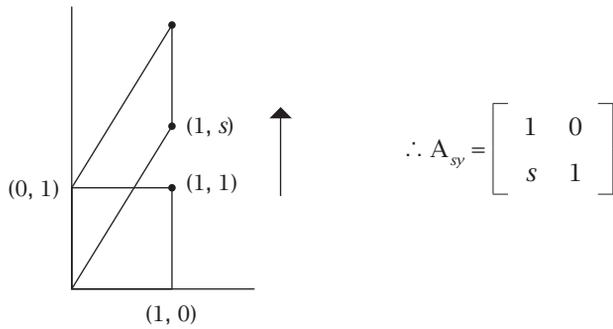
- shears in the x and y directions. Consider the unit square shown being sheared in the x direction by a shear factor of r .



So

$$A_{sx} = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$$

Similarly, a shear in the y direction with shear factor s is:



- When an object point $\begin{bmatrix} x \\ y \end{bmatrix}$ is transformed by A_1 and then the image point transformed by A_2 , the combined transformation A_c is $A_c = A_2 A_1$.

So $\begin{bmatrix} x' \\ y' \end{bmatrix} = A_2 A_1 \begin{bmatrix} x \\ y \end{bmatrix}$ or $X' = A_2 A_1 X$.

The combined transformation which reverses the above process (i.e. A_c^{-1}) is found as follows:

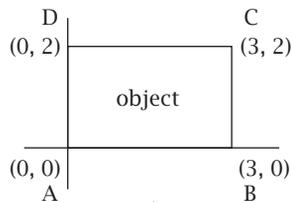
$$\begin{aligned} X' &= A_2 A_1 X \\ A_2^{-1} X' &= A_2^{-1} A_2 A_1 X \\ A_2^{-1} X' &= A_1 X \\ A_2^{-1} X' &= A_1 X \\ A_1^{-1} A_2^{-1} X' &= A_1^{-1} A_1 X \\ A_1^{-1} A_2^{-1} X' &= IX \\ \therefore X &= A_1^{-1} A_2^{-1} X' \\ \text{or } X &= A_c^{-1} X' \end{aligned}$$

Therefore, $(A_2 A_1)^{-1} = A_1^{-1} A_2^{-1}$

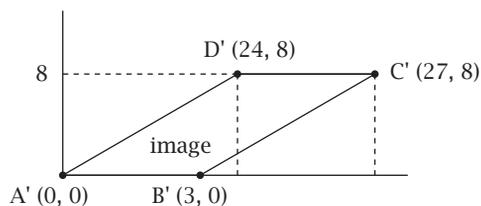
TRANSFORMATIONS, DETERMINANTS AND AREAS

When an object shape is transformed by a transformation matrix A , then the image shape will have an area that is the area of the object scaled by the determinant of the matrix A .

Example The rectangle as shown below is transformed by $A = \begin{bmatrix} 1 & 12 \\ 0 & 4 \end{bmatrix}$.



$$\begin{aligned} \text{Image} &= \begin{bmatrix} 1 & 12 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} A & B & C & D \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} A' & B' & C' & D' \\ 0 & 3 & 27 & 24 \\ 0 & 0 & 8 & 8 \end{bmatrix} \end{aligned}$$



$$\begin{aligned}\text{Object area} &= 3 \times 2 \\ &= 6\end{aligned}$$

$$\begin{aligned}\text{Image area} &= 3 \times 8 \\ &= 24\end{aligned}$$

$$\begin{aligned}|\text{Det } A| &= |1 \times 4 - 0 \times 12| \\ &= 4\end{aligned}$$

$$\therefore \text{Image area} = |\text{Det } A| \times \text{Object area}$$

$$24 = 4 \times 6 \text{ is confirmed}$$

Worked Examples

6.1 If $A = \begin{bmatrix} 2 & 2 \\ 0 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & a-1 \\ 0 & -3 \end{bmatrix}$

find the value(s) of a for which A and B are commutative.

$$\begin{aligned}AB &= \begin{bmatrix} 2 & 2 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} 2 & a-1 \\ 0 & -3 \end{bmatrix} & BA &= \begin{bmatrix} 2 & a-1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & -8 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2(a-1) + 2(-3) \\ 0 & (-8)(-3) \end{bmatrix} & &= \begin{bmatrix} 4 & 2(2) + -8(a-1) \\ 0 & (-3)(-8) \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2a-8 \\ 0 & 24 \end{bmatrix} & &= \begin{bmatrix} 4 & 12-8a \\ 0 & 24 \end{bmatrix}\end{aligned}$$

So for $AB = BA$ (making A and B commutative)

$$2a - 8 = 12 - 8a$$

$$10a = 20$$

$$a = 2 \text{ is the required value.}$$

6.2 Find any real value(s) of x for which the matrix below has an inverse.

$$A = \begin{bmatrix} x-3 & 3 \\ 2 & x+2 \end{bmatrix}$$

$$\text{Det } A: (x-3)(x+2) - 3(2)$$

$$= x^2 - x - 6 - 6$$

$$= x^2 - x - 12$$

$$= (x-4)(x+3)$$

$$\text{So } x = 4 \text{ or } x = -3$$

gives $\text{Det } A = 0$ and hence no inverse.

$\therefore A$ will have an inverse for all $x \neq 4, x \neq -3$.

6.3 If $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$$\text{find } AB \text{ and hence solve } y + 2z = 4$$

$$x + 2y = 5$$

$$x + z = 2.$$

$$AB = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5I \text{ from the calculator}$$

The system of equations in matrix form is

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$$\text{i.e. } B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1} \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$$\text{but } AB = 5I$$

$$\text{i.e. } \frac{1}{5} AB = I$$

$$\text{So } B^{-1} = \frac{1}{5} A$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5} A \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & -2 & 4 \\ 2 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \therefore x = 1, y = 2 \text{ and } z = 1 \text{ is the required solution.}$$

6.4 Consider the sum of square series

$$1^2 + 2^2 + 3^2 + 4^2 + \dots n^2.$$

(a) If S_n means the sum of first n terms, find the first 6 sums and complete the table.

n	1	2	3	4	5	6
S_n	S_1	S_2	S_3	S_4	S_5	S_6

Confirm that the rule for S_n is in the form of

$$S_n = an^3 + bn^2 + cn + d.$$

(b) Use matrix methods to find a , b , c and d and hence S_n .

(c) Write the rule for S_n in factorised form.

(a)

n	1	2	3	4	5	6
S_n	1	5	14	30	55	91

The third differences are constant

FD	4	9	16	25	36
SD		5	7	9	11
TD			2	2	2

$\therefore S_n$ has a cubic rule i.e.
 $S_n = an^3 + bn^2 + cn + d$

(b) letting $n = 1, 2, 3$ and 4 in the rule for S_n

gives $S_1 = 1 = a + b + c + d$

$$S_2 = 5 = 8a + 4b + 2c + d$$

$$S_3 = 14 = 27a + 9b + 3c + d$$

$$S_4 = 30 = 64a + 16b + 4c + d$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 5 \\ 14 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{6} \\ 0 \end{bmatrix} \text{ by calculator}$$

$\therefore S_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ is the required rule.

(c) $Sn = \frac{2}{6}n^3 + \frac{3}{6}n^2 + \frac{1}{6}n$

$$= \frac{1}{6}n(2n^2 + 3n + 1)$$

$$= \frac{1}{6}n(2n + 1)(n + 1) \text{ is the rule in factorised form.}$$

6.5 Find the combined transformation matrix which will first reflect any point in line $y = x$ and then rotate the new image 45° anticlockwise about the origin.

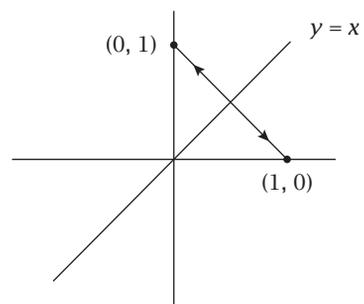
From the diagram $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

From the notes $A_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ where $\theta = 45^\circ$.

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore A_3 = A_2 A_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is the required combined transformation matrix.}$$



PROBLEMS TO SOLVE

CHAPTER 6: MATRICES

1. Find the matrix A without your calculator such that

$$\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 0 & 4 \\ 8 & 0 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix}$ find the following without your calculator.

(a) $3A$

(b) $AB - BA$

(c) A^{-1}

(d) B^{-1}

(e) $A^{-1}A$

3. A square ABCD with vertices A (2, 1), B (2, 3), C (4, 3), D (4, 1) is multiplied by matrix R, a rotation of 90° anti-clockwise, then the image is multiplied by matrix M, a reflection in the y axis.

- (a) Find the image of each vertex after these two transformations have been carried out.

- (b) What single matrix would have given the same result?

4. Two investments totalling \$8800 are made. In a particular year these investments yield \$663 in simple interest. Part of the \$8800 is invested at 7% and part at 8%.

(a) Write a pair of simultaneous equations for x and y , the amounts invested.

(b) Use matrix methods to find the amount invested at each rate.

5. Consider the transformation matrix: $S = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$

(a) If the point $A(2, -2)$ is transformed under the matrix S to a new point A' , find the coordinates of A' .

(b) A' is then transformed by the matrix $T = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}$ to the point A'' . Find the coordinates of A'' .

(c) Determine the single matrix which will transform A'' back to its original position $A(2, -2)$.

6. An airline is considering the purchase of two different types of plane: the Eagle, which can carry 40 first class passengers, 60 business class passengers and 120 economy class passengers, and the Falcon, which can carry 80 first class, 30 business class and 40 economy class passengers. The company flies two different routes, A and B. The costs per passenger on route A are 50c per km for economy class. The cost per passenger on route B are 45c per km for first class, 35c per km for business class and 25c per km for economy class.

(a) Write as a matrix the passenger numbers and types for each aircraft (matrix A).

(b) Write as a matrix the costs associated with each type of passenger over each route per km (matrix C).

(c) Find AC and describe the information obtained.

(d) Explain why it would or would not be useful to find the product CA .

7. A linear transformation is given by: $x' = 2y$
 $y' = -x$

(a) State the transformation matrix associated with the above linear transformation.

(b) Find the image, P' of the point P(3, 2) under this transformation.

(c) Find the matrix which will restore P' back to P.

8. A computer company is offering short courses in the software 'Word', 'Publisher' and 'Excel'.

Course A comprises 4 hours on Word, 2 hours on Publisher and 1 hour on Excel.
Course B comprises 2 hours on Word, 3 hours on Publisher and 2 hour on Excel.
Course C comprises 4 hours on Word, 1 hour on Publisher and 1 hour on Excel.

At the end of the week the computer company found that 138 hours had been totalled on Word, 83 hours had been totalled on Publisher, and 57 hours totalled on Excel. The cost of the courses A, B and C are \$65, \$60 and \$55 respectively. Find the number of people who took each course and the total revenue that the company received that week.

9. In each case find matrix A such that:

(a) $\begin{bmatrix} 3 & 6 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 3 & 6 \\ -2 & 1 \end{bmatrix}$

(b) $3A + \begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 0 & 4 \end{bmatrix}$

(c) $A \begin{bmatrix} 3 & 9 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 2 & 4 \end{bmatrix}$

10. Given $A = \begin{bmatrix} 0 & 6 & 3 \\ 3 & 3 & 3 \\ -3 & -6 & -6 \end{bmatrix}$

(a) Determine A^2 .

(b) By clearly using your result from part (a) solve the system of linear equations;

$$6y + 3z = -3$$

$$3x + 3y + 3z = 0$$

$$3x + 6y + 6z = -9.$$

11. A large sporting goods manufacturer specialising in the sale and supply of hockey sticks promotes three major brands: GREY, ATLAS and GRIFFON. The number of sales varies according to the seasons.

In Winter; 90 Grey, 40 Atlas and 70 Griffon sticks were sold.

In Spring; the numbers were respectively 100, 80 and 110.

In Summer; the sales were 30, 60 and 120 respectively.

(a) Display this information in a suitable matrix.

(b) If the takings in Winter, Spring and Summer were \$25760, \$37910 and \$28770 respectively, find the cost of each brand of hockey stick.

(c) If the number of hockey sticks sold is expected to increase by 10% in the following year, write down the matrix which shows the projected numbers of each brand.

(d) The manufacturer decided to increase the cost of each brand of hockey stick. If the new costs of the Grey, Atlas and Griffon are \$120, \$130 and \$150 respectively, carry out a suitable matrix operation to calculate the expected revenue for the following year.

12. The transformation C maps the point (x, y) onto (X, Y) where $X = 2x - y$ and $Y = x$.

The transformation D is a shear parallel to the y axis which maps the point with coordinates $(1, 1)$ onto the point with coordinates $(1, -1)$. Find:

(a) the matrix for the transformation C

(b) the matrix for the transformation D

(c) the matrix for the combined transformation D followed by C.

13. (a) If $A = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$, show that $A^2 = 4I$

where I is the 3×3 identity matrix.

(b) Show clearly how you can use the result of (a) to find the solution of the following simultaneous equations:

$$\begin{aligned} 4y + 2z &= 8 \\ 2x + 2y + 2z &= 10 \\ 2x + 4y + 4z &= 12 \end{aligned}$$

14. Square matrixes P, Q and R are such that $PQ = R$.

(a) Express matrix Q in terms of P and R.

(b) Hence, or otherwise, determine a 2×2 matrix Q if matrices P and R are respectively shown below. Do this without using your calculator.

$$\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 9 \\ -8 & 5 \end{bmatrix}$$

15. Each day Linda is either very busy or restful. A long term survey found that if she is very busy one day then the chance of her being very busy the next day is $\frac{4}{5}$. If Linda is restful one day then the chance of her being restful the next day is $\frac{1}{3}$. Find the long term probability of Linda being restful on any particular day.

16. If $A = \begin{bmatrix} \frac{2}{a-1} & a \\ 1 & 1 \end{bmatrix}$ find all value(s) of a such that A has no inverse.

17. A transformation matrix T maps point (x, y) onto (x', y') where $x' = y$ and $y' = 3y - x$.

(a) Find T .

(b) Find the transformation matrix for the combined transformation: a reflection in the y axis followed by T .

18. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -5 & -2 \\ 1 & -1 \end{bmatrix}$$

Find, where possible (stating why not possible if they are not), the following without using your calculator.

(a) $B - 2C$

(b) BC

(c) A^2

(d) B^2A

19. A group of students are hoping to go on a trip over East in the school holidays. Their parents decide to help fund raise by holding cake stalls twice weekly at the school. They decide to only make Butter Cakes and Tea Cakes because they are quick and easy and inexpensive to make.

In order to make a Butter Cake and Tea Cake, the following ingredients are required:

	Self-Raising Flour (Cupful)	Milk (Cupful)	Butter (g)	Vanilla (Teaspoon)	Sugar (Cupful)	Eggs
Butter Cake	2	$\frac{1}{2}$	120	1	$\frac{3}{4}$	2
Tea Cake	1	$\frac{1}{2}$	30	$\frac{1}{2}$	$\frac{1}{2}$	1

The cake stalls are held on Mondays and Wednesdays and they sell more cakes on Monday as there has been more time for baking on the weekend.

For Mondays, they make 28 Butter Cakes and 20 Tea Cakes, and on Wednesdays they make 22 Butter Cakes and 18 Tea Cakes.

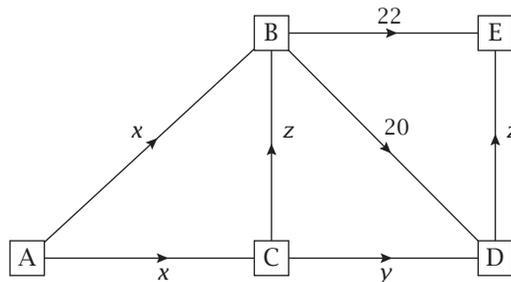
- (a) Express the information above as two matrices, and then use these matrices to find the quantities of each ingredient required on each day.

- (b) Self-raising flour costs 15c per cup, milk 30c per cup, butter \$3 per kg, vanilla 5c per spoonful, sugar 15c per cup and eggs 25c each. Express this as a matrix and use it to find the total cost of the ingredients (in dollars) for each day. Round your answer to the nearest dollar.

- (c) Assume they sell all the cakes each day. Find how much, in total, the cakes bring in for each day if the Butter Cakes sell for \$4 each and the Tea Cakes sell for \$3 each.

- (d) Consequently, what is the profit each day that is made by selling these cakes? Give your answer to the nearest dollar.

20. The network shows a distribution outlet in which computer components are analysed before being used. There are five work stations A, B, C, D and E. Computer components are directed according to the arrows.



The number of items leaving a work station each hour must be the same as the number coming in.

- (a) By considering work stations B, C and D, write down a set of three linear equations in x , y and z and express them in matrix form.

- (b) Solve for x , y and z .

- (c) State the total number of computer components analysed each hour.

21. A manufacturer of lawn fertiliser produces three different types of fertiliser, F_1 , F_2 and F_3 , each comprising different quantities of urea, phosphate and potash. The quantity of each ingredient in kg (kilograms) in each 20 kg bag of fertiliser is given in the following table.

Fertiliser			
	F_1	F_2	F_3
Urea	5	6	5
Phosphate	10	8	12
Potash	5	6	3

- (a) If the manufacturer had an order of 85 bags of F_1 , 110 bags of F_2 and 135 bags of F_3 , what quantity of each ingredient would be needed?
- _____
- _____
- (b) The manufacturer had a stockpile of 9930 kg of urea, 18900 kg of phosphate and 8370 kg of potash. Find out how many bags of each type of fertiliser could be manufactured to completely use the stockpile.
- _____
- _____
- (c) On the sale of the stockpile, the profit was \$18 on each bag of F_1 and \$15 on each bag of F_2 . Given that the overall profit was \$19.50 per bag, what was the profit on each bag of F_3 ?
- _____
- _____

22. If $A = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, find:

(a) A^2

(b) B^2

(c) $A + B$

(d) $A - B$.

Show that for A and B

(e) $A^2 - B^2 = (A + B)(A - B)$.

- (f) Will your result for (e) be true for all pairs of 2×2 matrices? Explain why, or why not.
- _____
- _____

Complex numbers have applications in many branches of science and engineering. The study of complex numbers enriches and unifies studies in algebra, geometry, trigonometry and calculus.

Syllabus Checklist

By the end of this chapter, you should be able to:

Complex numbers

- define the imaginary number i as a root of the equation $x^2 = -1$
- represent complex numbers in the rectangular form; $a + bi$ where a and b are the real and imaginary parts
- determine and use complex conjugates
- perform complex number arithmetic: addition, subtraction, multiplication and division

The complex plane

- consider complex numbers as points in a plane, with real and imaginary parts, as Cartesian coordinates
- examine addition of complex numbers as vector addition in the complex plane
- develop and use the concept of complex conjugates and their location in the complex plane

Roots of equations

- use the general solution of real quadratic equations
- determine complex conjugate solutions of real quadratic equations
- determine linear factors of real quadratic polynomials

INTRODUCTION

When attempting to find the exact solutions to the quadratic equation $x^2 - 4x + 13 = 0$ the quadratic formula can be used or as shown below by completing the square.

$$\begin{aligned}x^2 - 4x + 13 &= 0 \\x^2 - 4x + 4 - 4 + 13 &= 0 \\(x - 2)^2 + 9 &= 0 \\(x - 2)^2 &= -9 \\(x - 2) &= \sqrt[3]{-9} \\ \text{i.e. } x &= 2 \pm 3\sqrt{-1}\end{aligned}$$

As $\sqrt{-1}$ does not exist in the physical or real world there are no real solutions to $x^2 - 4x + 13 = 0$.

There are however two non-real or complex solutions which are

$$x_1 = 2 + 3\sqrt{-1} \text{ and } x_2 = 2 - 3\sqrt{-1}$$

For convenience we let $\sqrt{-1}$ be notated by the letter i (or j if you are an electrical engineer).

So the complex solutions to $x^2 - 4x + 13 = 0$

Are $x_1 = 2 + 3i$ and $x_2 = 2 - 3i$

$$\begin{aligned} \text{If } i = \sqrt{-1} \text{ then } i^2 &= -1, i^3 = i^2 i, i^4 = (i^2)^2 & i^5 &= i^4 i \\ &= -i & = 1 & = i \text{ etc} \end{aligned}$$

The interesting thing is that even though i doesn't exist in the real world, when complex numbers are used to represent alternating currents and voltages in electrical circuits the calculations are much simpler than the equivalent calculations in the real world!!

The other thing to note is that many of the familiar rules of algebra apply to complex numbers as will be seen.

THE REAL AND IMAGINARY PARTS

Complex numbers are usually denoted by letters like z and w .

- If $z = x + y i$ is a complex number the real part of $z = \text{Re } z = x$ and the Imaginary part of $z = \text{Im } z = y$ (not $y i$).
- Complex numbers in the form $x + y i$ are said to be in **rectangular** or **Cartesian form** and there are many parallels between complex numbers and vectors.

Often when manipulating complex number expressions the result is a mixture of real and imaginary parts.

Writing a mixture of real and imaginary parts in $z = x + y i$ form is a very common and often essential task before other tasks can be performed.

COMPLEX CONJUGATE PAIRS

The solutions $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$ of the quadratic $z^2 - 4z + 13 = 0$ are known as a complex conjugate pair.

$z = x + y i$ and the conjugate of z notated as $\bar{z} = x - y i$ have the same real parts whereas their imaginary parts always have opposite signs.

If $P(z)$ is a polynomial with real coefficients the complex solutions of $P(z) = 0$ always occur in complex conjugate pairs.

THE MAGNITUDE OF Z

If $z = x + y i$ then $|z|$ denotes the magnitude of z and $|z| = \sqrt{x^2 + y^2}$ which is simply Pythagoras on the real and imaginary parts of z . Don't ever make the mistake of squaring $y i$ and hence get $\sqrt{x^2 - y^2}$ which is quite wrong!

ADDING AND SUBTRACTING COMPLEX NUMBERS

If $z = x + yi$ and $w = a + bi$

$$\begin{aligned} \text{then } z + w &= x + yi + a + bi \\ &= x + a + (y + b)i \end{aligned}$$

$$\begin{aligned} z - w &= x + yi - (a + bi) \\ &= x + yi - a - bi \\ &= x - a + (y - b)i \end{aligned}$$

$$\begin{aligned} \text{also } z + \bar{z} &= x + yi + x - yi \\ &= 2x \\ &= 2 \operatorname{Re} z \end{aligned}$$

and

$$\begin{aligned} z - \bar{z} &= x + yi - (x - yi) \\ &= x + yi - x + yi \\ &= 2yi \\ &= 2i \operatorname{Im} z \end{aligned}$$

MULTIPLYING AND DIVIDING COMPLEX NUMBERS

For $z = x + yi$ and $w = a + bi$

$$\begin{aligned} zw &= (x + yi)(a + bi) \\ &= xa + xbi + yai + (yi)(bi) \\ &= xa + (bx + ay)i - yb \quad \text{because } i \times i = -1 \\ &= ax - by + (bx + ay)i \end{aligned}$$

$$\begin{aligned} \frac{z}{w} &= \frac{x + yi}{a + bi} \\ &= \frac{x + yi}{a + bi} \times \frac{a - bi}{a - bi} \quad \left. \begin{array}{l} \text{Multiply top and bottom by the} \\ \text{conjugate of the denominator} \end{array} \right\} \\ &= \frac{xa - xbi + yai + yb}{a^2 - abi + abi + b^2} \\ &= \frac{ax + by + (ay - bx)i}{a^2 + b^2} \\ &= \frac{ax + by}{a^2 + b^2} + \frac{ay - bx}{a^2 + b^2}i \end{aligned}$$

Note that the final denominator $a^2 + b^2$ results from the product of w and its conjugate \bar{w} .

In general for any complex number z
 $z\bar{z} = |z|^2$ is always real

Also note that factorising $a^2 - b^2 = (a - b)(a + b)$, the factors are real, whereas factorising $a^2 + b^2 = (a - bi)(a + bi)$ the factors are complex.

If $z = 2 + 3i$ and $\bar{z} = 2 - 3i$ are the solutions of a quadratic equation, then the two corresponding factors of the quadratic are

$$(z - (2 + 3i))(z - (2 - 3i)) = 0$$

$$\text{i.e. } (z - 2 - 3i)(z - 2 + 3i) = 0$$

$$(z - 2)^2 + 9 = 0$$

$$z - 4z + 4 + 9 = 0$$

$$z - 4z + 13 = 0$$

by using the above relationship $(a - bi)(a + bi) = a^2 + b^2$

which was the starting quadratic equation at the beginning of the chapter

The process demonstrated above highlights a very important result which you should be familiar with from your studies of real valued functions.

When the solution of $P(z) = 0$ are z_1, z_2, z_3, \dots then the factors of $P(z)$ are known and $P(z)$ factorises as

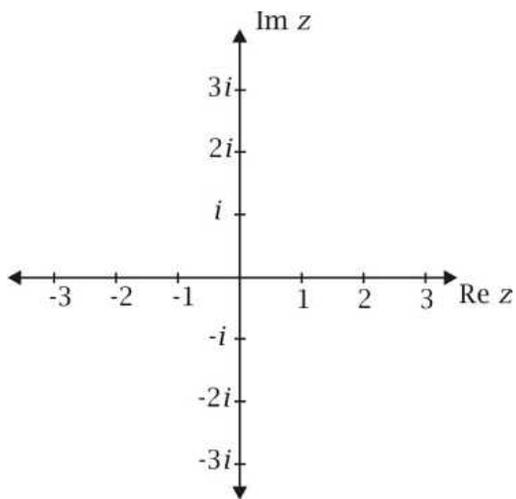
$$P(z) = (z - z_1)(z - z_2)(z - z_3) \dots$$

The reverse process is just as important i.e.

When the factors of $P(z)$ are $(z - z_1)(z - z_2)(z - z_3) \dots$ then the solutions of
 $P(z) = 0$ are $z = z_1, z = z_2, z = z_3, \text{ etc}$

DIAGRAMMATIC REPRESENTATION OF COMPLEX NUMBERS

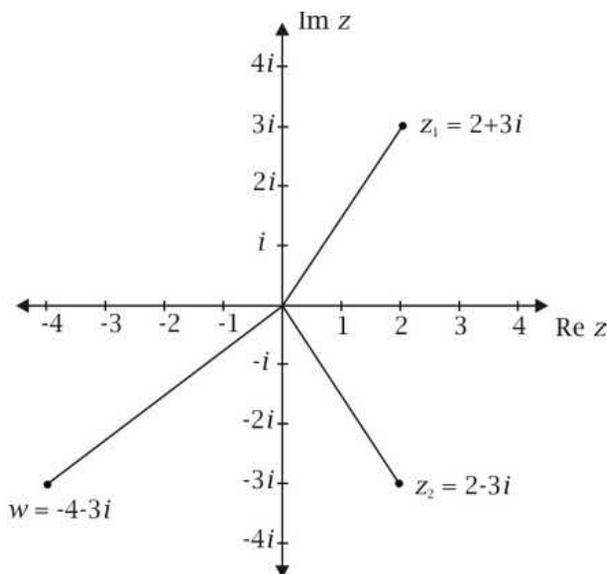
- The complex or Argand plane has a real axis and an imaginary axis as shown.



The real axis is the same as the x axis that we are used to in the Cartesian plane

The imaginary axis is marked out in terms of i as shown

- $z = 2 + 3i$, $\bar{z} = 2 - 3i$ and $w = -4 - 3i$ are plotted as shown.



- $|z - z_1|$ is the distance between say a variable point z and a fixed point z_1 .
Note that $|z_1 - z_2| = |z_2 - z_1|$.
- The equation $|z - z_1| = a$ represents the set of complex numbers z which are all the same distance of a from z_1 . When graphed on the complex plane these complex numbers z must lie on a circle of radius a and centre at z_1 .
Note the parallel between $|z - z_1| = a$ and $|r - c| = a$ which is the vector equation of a circle, centre at c with radius a .

The graph of the set of z 's bound by a constraint equation or inequality is called a locus

Worked Examples

Complex Numbers

- 7.1 Two of the solutions of $p(z) = 0$ where $p(z)$ is a polynomial of order 3 with real coefficients are $z = 3$ and $z = 1 - i$. Find the polynomial.

Answer. The third solution must be $z = 1 + i$ (as complex solutions travel in pairs)

$$\begin{aligned}
 \therefore \text{The polynomial is } P(z) &= (z-3)(z-(1-i))(z-(1+i)) \\
 &= (z-3)(z-1+i)(z-1-i) \\
 &= (z-3)\left((z-1)^2 + 1\right) \\
 &= (z-3)(z^2 - 2z + 2) \\
 &= z^3 - 2z^2 + 2z - 3z^2 + 6z - 6 \\
 &= z^3 - 5z^2 + 8z - 6
 \end{aligned}$$

- 7.2 If $z = -4 + 3i$ and $w = 2 - 5i$, Find

<p>(a) z^2</p>	<p>(b) $w - z$</p>
<p>(c) $3z + 4w$</p>	<p>(d) $\frac{w}{z}$</p>
<p>(a) $z^2 = (-4 + 3i)^2$ $= 16 - 24i - 9$ $= 7 - 24i$</p>	<p>(b) $w - z = 2 - 5i - (-4 + 3i)$ $= 2 - 5i + 4 - 3i$ $= 6 - 8i$ $= \sqrt{6^2 + 8^2}$ $= 10$</p>
<p>(c) $3z + 4w$ $= 3(-4 + 3i) + 4(2 - 5i)$ $= -12 + 9i + 8 - 20i$ $= -4 - 11i$</p>	<p>(d) $\frac{w}{z} = \frac{2 - 5i}{-4 + 3i} \times \frac{-4 - 3i}{-4 - 3i}$ $= \frac{-8 - 6i + 20i - 15}{16 + 9}$ $= -\frac{23 + 14i}{25}$ $= -\frac{23}{25} + \frac{14}{25}i$</p>

7.3 Find the real and imaginary parts of $\frac{1}{z}$ where $z = x + yi$.

$$\begin{aligned}\frac{1}{z} &= \frac{1}{x+yi} \times \frac{x-yi}{x-yi} \\ &= \frac{x-yi}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i \\ \therefore \operatorname{Re}\left(\frac{1}{z}\right) &= \frac{x}{x^2+y^2} \\ \text{and } \operatorname{Im}\left(\frac{1}{z}\right) &= -\frac{y}{x^2+y^2}\end{aligned}$$

7.4 Fully factorise $z^3 - 1$ Let $P(z) = z^3 - 1$
 $P(1) = 1^3 - 1$
 $= 0$
 $\therefore z - 1$ is a factor.

$$\begin{aligned}z^3 - 1 &= (z-1)(z^2 + bz + 1) \\ &= z^3 + bz^2 + z - z^2 - bz - 1 \\ &= z^3 + (b-1)z^2 + (1-b)z - 1 \\ \therefore b-1 &= 0, \quad 1-b = 0 \\ \text{so either way } b &= 1\end{aligned}$$

$$\begin{aligned}z^3 - 1 &= (z-1)(z^2 + z + 1) \\ &= (z-1)\left(z^2 + z + \frac{1}{4} - \frac{1}{4} + 1\right) \\ &= (z-1)\left(z + \left(\frac{1}{2}\right)^2 + \frac{3}{4}\right) \\ &= (z-1)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{3}{2}i\right)\end{aligned}$$

The above means that $z^3 - 1 = 0$ or $z^3 = 1$ has three solutions which are

$$\begin{aligned}z_1 &= 1 \\ z_2 &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ z_3 &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i\end{aligned}$$

And these in turn become the three cube roots of 1!!

7.5 Solve $z - 2\bar{z} = 5 + 6i$

Let $z = x + yi$ i.e. $\bar{z} = x - yi$

$$\begin{aligned}z - 2\bar{z} &= 5 + 6i \\ \text{becomes } x + yi - 2(x - yi) &= 5 + 6i \\ x + yi - 2x + 2yi &= 5 + 6i \\ -x + 3yi &= 5 + 6i \\ \therefore -x = 5 \text{ and } 3y &= 6 \\ x = -5 \quad y &= 2 \\ \therefore z = -5 + 2i &\text{ is the required solution.}\end{aligned}$$

7.6 Solve $\frac{z+1}{z-1} = 2+i$

$$z+1 = (2+i)(z-1)$$

$$z+1 = 2z-2+zi-i$$

$$1+2+i = 2z-z+zi$$

$$3+i = z+zi$$

$$3+i = z(1+i)$$

$$z = \frac{3+i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{3-3i+i+1}{1+1}$$

$$= \frac{4-2i}{2}$$

$$\therefore z = 2-i$$

7.7 Find the Cartesian equation of the locus $|z-1-i| = |z+2+3i|$

Let $z = x+yi$

$$|x+yi-1-i| = |x+yi+2+3i|$$

$$|x-1+i(y-1)| = |x+2+i(y+3)|$$

$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x+2)^2 + (y+3)^2}$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 + 4x + 4 + y^2 + 6y + 9$$

$$2 - 9 - 4 = 4x + 2x + 6y + 2y$$

$$6x + 8y = -11$$

$$\text{or } y = -\frac{3}{4}x - \frac{11}{8}$$

PROBLEMS TO SOLVE

CHAPTER 7: COMPLEX NUMBERS

1. If $z = 3 - 2i$ and $w = -4 + i$ show all working to find

(a) $4z - 3w$

(b) $\frac{1}{2}zw$

(c) z^2

(d) $\frac{z+w}{w}$

(e) $|z||w|$

(f) $\bar{z}\bar{w}$

2. If $z = 2 + 3i$ and $w = x + yi$ find the formula for $|z - w|$ and give the meaning of $|z - w|$ on the Argand plane.

3. Solve for z by using the standard rules of algebra, (i.e. no need to substitute $z = x + yi$) and leave your answers in $z = x + yi$ form.

(a) $3z + 4i = 6 - 5i$

(b) $2iz + 3 = z - 4i$

(c) $\frac{z-i}{z+i} = 1 + i$

(d) $z^2 + 2z = 9(i)^2 - 1$

(e) $\frac{z-i}{z+i} = -1 - i$

4. Solve $z^2 - 10z + 29 = 0$ by either completing the square or by using the quadratic formula.

5. A quadratic equation has solutions of $z = -3 \pm 7i$. Find the equation in $az^2 + bz + c = 0$ form.

6. Substitute $z = x + yi$ and $\bar{z} = x - yi$ into $z(2-i) + 2\bar{z} = 3 + 4i$ in order to solve for z .

7. Solve $\frac{z}{\bar{z}} = i$ for z and give your answer in general form.

8. Solve $\frac{z}{z+2} = 2-i$ and write your answer as $z = x + yi$.

9. Let $z = x + yi$ and $\bar{z} = x - yi$ in $z\bar{z} = 6(z + \bar{z})$ in order to find the relationship between x and y .

10. If $z_1 = 2 - 3i$ and $z_2 = -1 + 4i$ are the roots of $az^2 + bz + c = 0$ find a , b and c .

11. Graph each equation below on separate Argand planes.

(a) $\operatorname{Re} z = -4$

(b) $\operatorname{Im} z = 3$

(c) $\operatorname{Re} z + \operatorname{Im} z = 5$

(d) $|z| = 4$

(e) $|z-1| = 3$

(f) $|z-2i| = 3$

(g) $2 \operatorname{Re} z - 3 \operatorname{Im} z = -6$

12. Substitute $z = x + y i$ into $|z - 2i + 3| = |z + 4 - i|$ and simplify your answer into the form $ax + by = c$.

13. Fully factorise $z^3 - 8$ and hence provide all solutions to $z^3 - 8 = 0$.

14. Write a polynomial function f of least degree that has real coefficients, a leading coefficient of 1, and 2 and $1 + i$ as zeros.

Syllabus Checklist

By the end of this chapter, you should be able to:

Proofs involving numbers

- prove simple results involving numbers

Rational and irrational numbers

- express rational numbers as terminating or eventually recurring decimals and vice versa
- prove irrationality by contradiction for numbers such as $\sqrt{2}$

An introduction to proof by mathematical induction

- develop the nature of inductive proof, including the 'initial statement' and inductive step
- prove results for sums, such as $1 + 4 + 9 \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for any positive integer n
- prove divisibility results, such as $3^{2n+4} - 3^{2n}$ is divisible by 5 for any positive integer n

INTRODUCTION TO NUMBER PATTERNS AND DEFINITIONS

You will be familiar with the following sets of numbers:

Natural Numbers $N = \{1, 2, 3, 4, 5, 6, \dots\}$

Whole Numbers $W = \{0, 1, 2, 3, 4, 5, \dots\}$

Integers $I = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$

Prime Numbers $P = \{2, 3, 5, 7, 11, 13, \dots\}$

Note here that the dots in a set (i.e. ...) normally mean 'and so on continuing with the same pattern', but in the case of the set P it means continue on listing the prime numbers as there is no known pattern.

A **conjecture** is the expression of an opinion without sufficient evidence. Conjectures may be true or false. Only one counter example is necessary to prove that a conjecture is false.

An **axiom** is a statement that is accepted as being true without proof.

A **theorem** is a statement that has been proven to be true.

A **proposition** is a statement that can only be either true or false.

A **definition** is a statement of what we agree about something. For example, one definition of an even number is:

An **even number** is a natural number which has 2 as a factor,

$\therefore E = \{2, 4, 6, 8, \dots\}$ is the set of even numbers.

An **odd number** could be defined as a natural number which does not have 2 as a factor,

$$\therefore O = \{1, 3, 5, 7, 9, \dots\} \text{ is the set of odd numbers.}$$

(For the purposes of this text the author has chosen not to include zero in the set of even numbers and also not to include the negative integers in the sets E and O .)

If n is a natural number (i.e. $n \in N$), then $2n$ is always even and $2n + 1$ or $2n - 1$ is always odd. The sequence of consecutive natural numbers starting with n is

$$n, n + 1, n + 2, n + 3, \dots$$

The sequence of consecutive even numbers starting with $2n$ for any $n \in N$ is

$$2n, 2(n + 1), 2(n + 2), 2(n + 3), \dots$$

The sequence of consecutive odd numbers starting with $2n - 1$ for any $n \in N$ is

$$2n - 1, 2n + 1, 2n + 3, 2n + 5, \dots$$

A **terminating decimal** is a decimal number that has digits that do not go on forever (e.g. 0.25 has two digits before terminating; 2.0451 has four digits before terminating).

A **recurring decimal** is a decimal number that has digits that recur forever in a recognisable pattern (e.g. $\frac{1}{3} = 0.\dot{3}$ where the 3s recur forever; $\frac{1}{11} = 0.\overline{09}$ where the '09' digits will recur forever).

CONVERTING RECURRING DECIMALS TO FRACTIONS

Worked Examples

8.1 Convert the following recurring decimals to fractions

(a) $0.\dot{4}$ (b) $0.\overline{08}$ (c) $0.58\dot{3}$

- (a) Step 1 Let $x = 0.\dot{4}$
Step 2 Multiply both sides of the equation in Step 1 by 10 (this is because there is one zero in 10 and one repeating digit)
 $10x = 4.\dot{4}$
Step 3 Subtract x from both sides of the equation and simplify
 $10x - 1x = 4.\dot{4} - 0.\dot{4}$
 $9x = 4$
 $x = \frac{4}{9}$
- (b) Step 1 Let $x = 0.\overline{08}$
Step 2 Multiply both sides of the equation in Step 1 by 100 (this is because there are two zeroes in 100 and two repeating digits)
 $100x = 8.\overline{08}$
Step 3 Subtract x from both sides of the equation and simplify
 $100x - 1x = 8.\overline{08} - 0.\overline{08}$
 $99x = 8$
 $x = \frac{8}{99}$

- (c) Step 1 Let $x = 0.5\overline{83}$
- Step 2 Multiply both sides of the equation in Step 1 by 10 (this is because there is one zero in 10 and one repeating digit)
- $$10x = 5.8\overline{3}$$
- Step 3 Subtract x from both sides of the equation and simplify
- $$10x - 1x = 5.8\overline{3} - 0.5\overline{83}$$
- $$9x = 5.25$$
- $$x = \frac{5.25}{9}$$
- Step 4 Multiply answer by $\frac{100}{100}$ to remove two decimal places, and simplify
- $$x = \frac{525}{900}$$
- $$\therefore x = \frac{7}{12}$$

CONVERTING RECURRING DECIMALS TO FRACTIONS

8.2 Convert the following fractions to decimals and indicate appropriately any recurring digits

(a) $\frac{8}{9}$ (b) $\frac{9}{22}$

- (a) Step 1 Prepare the fraction for short division $9\overline{)8}$
- Step 2 Begin by dividing 9 into 8 (which can't be done evenly). Place a zero above the 8, followed by a decimal point. Also place a decimal point after the 8 followed by a decimal point. $9\overline{)8.0}$
- Step 3 Now divide 9 into 80 (which has been created by the previous operations). Place the dividend above the zero, and write the remainder in front of another zero placed after the '80'. $9\overline{)8.0^80}$
- Step 4 Continue this process until the pattern of recurring decimals appears $9\overline{)8.0^80^80}$
- Step 5 Conclude by writing the decimal correctly $0.\overline{8}$
- (b) Step 1 Prepare the fraction for short division $22\overline{)9}$
- Step 2 Begin by dividing 22 into 9 (which can't be done evenly). Place a zero above the 9, followed by a decimal point. Also place a decimal point after the 9 followed by a zero. $22\overline{)9.0}$

Step 3 Now divide 22 into 90 (which has been created by the previous operations). Place the dividend above the zero, and write the remainder in front of another zero placed after the '90'.

$$22 \overline{) 90.0}$$

Step 4 Continue this process until the pattern of recurring decimals appears

$$22 \overline{) 90.00}$$

Step 5 Conclude by writing the decimal correctly

$$0.4\overline{09}$$

PROOFS USING ALGEBRA

Algebra can be used to prove if a conjecture is true or false.

8.3 Conjecture: The sum of five consecutive odd numbers is a multiple of five.

Let $2n + 1$, $2n + 3$, $2n + 5$, $2n + 7$, $2n + 9$ represent five consecutive odd numbers.

$$\begin{aligned} 2n + 1 + 2n + 3 + 2n + 5 + 2n + 7 + 2n + 9 &= 10n + 25 \\ &= 5(2n + 5) \end{aligned}$$

$\therefore 5(2n + 5)$ is divisible by 5 and hence is a multiple of 5. Therefore the conjecture is true.

PROOF BY CONTRADICTION

Contradiction is a form of proof that establishes the truth or validity of a proposition. It achieves this by showing that the proposition being false would imply a contradiction.

8.4 Conjecture $\frac{p}{q} + \frac{r}{s} = \frac{p+r}{q+s}$ where p , q , r and s are counting numbers.

Start with the left hand side

$$\begin{aligned} &\frac{p}{q} + \frac{r}{s} \\ &= \frac{ps + rq}{qs} \quad (\text{Lowest common denominator is } qs) \end{aligned}$$

If the conjecture is assumed to be true, then we have

$$\frac{ps + rq}{qs} = \frac{p+r}{q+s}$$

$$\text{i.e. } (ps + rq)(q + s) = (qs)(p + r)$$

$$\text{i.e. } psq + ps^2 + rq^2 + rqs = pqs + rqs$$

$$\text{i.e. } ps^2 + rq^2 = 0$$

But since p , q , r and s are counting numbers

$$\cdot ps^2 \text{ must be greater than } 0$$

$$\cdot rq^2 \text{ must be greater than } 0$$

$$\therefore ps^2 + rq^2 \neq 0$$

$$\therefore \frac{p}{q} + \frac{r}{s} \neq \frac{p+r}{q+s}$$

\therefore Conjecture is false.

PROOF BY INDUCTION

A significant amount of mathematics involves the examination of patterns. Many of these patterns are concerned with generalisations about sequences and series.

Mathematical induction is a method of proof that is based in recursion, and it is used for proving conjectures which claim that a certain statement is true for integer values of some variable.

Let's say we were interested in finding a generalisation to explain the sum of n odd numbers, starting at 1. If we tabulate our findings for the first 10 odd numbers and their partial sums, we have:

n	1	2	3	4	5	6	7	8	9	10
T_n	1	3	5	7	9	11	13	15	17	19
S_n	1	4	9	16	25	36	49	64	81	100
n^2	1^2	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	10^2

The interesting thing here is that the last row of the table shows all integers n^2 , $n \geq 1$. Thus, the sum of all n odd numbers appears to be the square of n . In stating this, we have arrived at a conjecture - which is the first step in creating a theorem - but we may not know precisely why this is true.

The following worked exercise provides a precise mathematical statement of the result we are trying to prove.

8.5 General series: Prove by mathematical induction that for all odd integers $n \geq 1$ that

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$$

- Initial step: We need to show that the hypothesis is true for a small value of n , e.g. $n = 1$. Substitute this value into the series.

$$1 = 1^2$$

- Base step: We let $n = k$, which will assume that the statement will hold true for all values of n . After substituting $n = k$ into the series, we assume that the hypothesis is true for $n = k$.

$$1 + 3 + 5 + \dots (2k - 1) = k^2 \quad \rightarrow (1)$$

Assume that base step is true, i.e. statement is true for $n = k$.

- Base induction step: We let $n = k + 1$, to show that the statement will remain true for all values of k and the very next value after k . Note that this is not simply a case of substituting $k + 1$ for n in the original statement; doing so would be making another assumption. Instead, we add the next term of the series to both sides of the statement created in the 'base step'. This next term is $2(k + 1) - 1$.

Looking back at (1), we can see that the series $1 + 3 + 5 \dots (2k - 1)$ exists in (2).

We thus substitute k^2 for $1 + 3 + 5 + \dots (2k - 1)$

$$\begin{aligned}
 1 + 3 + 5 + \dots (2k - 1) + [2(k + 1) - 1] &= (k + 1)^2 \\
 1 + 3 + 5 + \dots (2k - 1) + (2k + 1) &= (k + 1)^2 \quad \rightarrow (2) \\
 \text{So } \underline{1 + 3 + 5 + \dots (2k - 1)} + (2k + 1) &= (k + 1)^2 \\
 k^2 + (2k + 1) &= (k + 1)^2 \\
 k^2 + 2k + 1 &= k^2 + 2k + 1 \\
 &= \text{R.H.S.}
 \end{aligned}$$

- Conclusion: Because we have proven that the LHS of the statement equals the RHS, we can conclude that the statement is true for all values of $n = k + 1$. Similarly, because the statement is true for all values of $n = k + 1$, it must also be true for all values of $n = k$. Therefore, the conjecture has been proven.

\therefore true for $n = k + 1$

\therefore true for $n = k$

\therefore conjecture is true.

8.6 Divisibility: Using mathematical induction, prove that $3^{2^n}-1$ is divisible by 8.

1. Initial step: We need to show that the conjecture is true for a small value of n , e.g. $n = 1$. Substitute this value into the expression, and check to see that the result is divisible by 8.

$$3^{2(1)} - 1 = 3^2 - 1 = 9 - 1 = 8$$

and this result is clearly divisible by 8.

\therefore initial step is true for $n = 1$.

2. Base step: We let $n = k$, and we assume that the statement will hold true for all values of n . After substituting $n = k$ into the series, we assume that the hypothesis is true for $n = k$.

$$3^{2(k)} - 1 = 3^{2k} - 1 \quad (1)$$

We assume that the base step is true, i.e. the statement is divisible by 8 for $n = k$.

3. Base induction step: We let $n = k + 1$ to show that the statement will remain true for all values of k and the very next value after k . Then, we manipulate this new statement using substitution and algebraic rules until it becomes divisible by 8.

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 3^{2k} \times 3^2 - 1 \\ &= 3^{2k}(9) - 1 \\ &= 3^{2k}(8 + 1) - 1 \\ &= 8 \times 3^{2k} + 3^{2k} - 1 \end{aligned} \quad (2)$$

Now because from (1) we have assumed that 3^{2k-1} is divisible by 8, there are two terms which are divisible by 8 – one proven through clear algebra, and the other via an assumption from the base step. As such, the base induction step is divisible by 8 and we can conclude the statement is true for all values of $n = k + 1$. Similarly, because the statement is true for all values of $n = k + 1$, it must also be true for all values of $n = k$. Therefore, the conjecture has been proven.

\therefore statement is true for $n = k + 1$

\therefore statement is true for $n = k$

\therefore conjecture is true.

8.7 Generalised Principle: For induction problems requiring the Generalised Principle, we first need S_n , a statement concerning the positive integer n .

Suppose that:

Step 1 - S_j is true

Step 2 - For any positive integer k , $k \geq j$, if S_k implies S_{k+1}

Then S_n is true for all positive integers $n \geq j$

8.8 Generalised Principle: Inequality Statements: Let S_n represent the statement

$$S_n : 2^n > 2n + 1$$

Show that S_n is true for all values of $n \geq 3$.

Since the statement is claimed to be true for values of n beginning with 3 and not 1, the proof uses the generalised principle of mathematical induction.

1. Show that S_n is true when $n = 3$.

S_3 is the statement $2^3 > 2 \cdot 3 + 1$ which is true since $8 > 7$. Therefore, S_n is true when $n = 3$.

2. Show that S_k implies S_{k+1} for $k \geq 3$.

Assume $S_k : 2^k > 2k + 1$ is true.

Multiply each side by 2, giving $2 \cdot 2^k > 2(2k + 1)$

or $2^{k+1} > 4k + 2$

or, equivalently $2^{k+1} > 2(k + 1) + 2k$

Since $k \geq 3$, then $2k > 1$ and it follows that $2^{k+1} > 2(k + 1) + 2k > 2(k + 1) + 1$

or $2^{k+1} > 2(k + 1) + 1$ which is the statement S_{k+1} .

Thus S_k implies S_{k+1} and, by the generalised principle, S_n is true for all $n \geq 3$.

PROBLEMS TO SOLVE

CHAPTER 8: MATHEMATICAL PROOF AND REASONING

For Questions 1–4 use the process of Proof by Contradiction to prove each of the propositions.

1. If n is even, then n^2 is even.

2. If n is odd, then n^2 is odd.

3. If n^2 is even, then n is even.

4. If n^2 is odd, then n is odd.

5. Prove that $\sqrt{2}$ is irrational.

Prove the following conjectures algebraically.

6. Prove that the difference between the squares of consecutive integers is always odd.

7. Prove that, if two even counting numbers differ by four, then their squares differ by a multiple of sixteen.

8. Prove that if the sum of the digits of a four digit number is a multiple of 3, then the four digit number is divisible by 3.

9. Express the following recurring decimals as fully simplified fractions:

(a) $0.\dot{6}$

(b) $0.\dot{7}$

(c) $0.1\dot{6}$

(d) $0.\overline{09}$

(e) $0.41\dot{6}$

(f) $0.\overline{428571}$

10. Convert the following fractions to decimals, clearly indicating which (if any) digits are recurring:

(a) $\frac{5}{6}$

(b) $\frac{4}{9}$

(c) $\frac{1}{15}$

(d) $\frac{7}{30}$

(e) $\frac{9}{22}$

(f) $\frac{5}{13}$

11. Prove that $\sqrt{3}$ is irrational.

12. Prove that $\sqrt{5}$ is irrational.

13. Prove that $\log_2 5$ is irrational.

14. Prove that $\log_2 7$ is irrational.

15. Use mathematical induction to prove that for all positive integers n :

(a) $2 + 4 + 6 + \dots + 2n = n(n + 1)$

(b) $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n + 1)$

(c) $1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n - 1)}{2}$

(d) $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

(e) $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

(f) $\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \dots + \frac{1}{(3n - 2)(3n + 1)} = \frac{n}{3n + 1}$

16. Use mathematical induction to prove the following divisibility tests for all positive integers n :

(a) $9^n - 3$ is a multiple of 6

(b) $3^{4n} - 1$ is divisible by 80

(c) $5^n + 3$ is divisible by 4

(d) $n(n^2 + 2)$ is divisible by 3

(e) $5^{2n} - 1$ is a multiple of 24

(f) $n^3 - n$ is a multiple of 6, n is a positive integer $n \geq 2$

17. Using mathematical induction, prove that for all $n \geq 1$

$$\frac{x^{n+1} - 1}{x - 1} = 1 + x + x^2 + \dots + x^n \quad \text{where } x \neq 1.$$

18. Using mathematical induction, show that for any positive integer n ,

$$a + ar + ar^2 + \dots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} \quad \text{where } r \neq 1.$$

19. Use mathematical induction to prove that $n < 2^n$ for all positive integers n .

For Q20-22, use mathematical induction to prove the following statements for all natural numbers $n \geq 5$.

20. $2^n > n^2$

21. $4n < 2^n$

22. $1 \times 2 \times 3 \times \dots \times (n - 1) > 2^n$



TRIAL TEST 1: COUNTING TECHNIQUES

Calculators allowed

Time Allowed: 50 minutes

Total Marks: 45

1. The letters of the word BLEAT are arranged to form five-letter 'words'.

How many arrangements are possible if:

- (a) there are no restrictions.

- (b) the word must start and end with a vowel.

- (c) the vowels must be together.

One of the five-letter words is selected at random. Determine the probability:

- (d) the vowels are not together.

[5]

2. A student is to answer 6 questions from a paper containing 10 questions, with 5 questions in section 1 and 5 questions in section 2.

How many different selections are possible:

- (a) without restriction.

- (b) if the first two questions are compulsory.

- (c) if two questions must be from section 1 and the remainder from section 2.

[5]

3. If $P(X|Y) = 0.6$
 $P(Y|X) = 0.4$
 $P(X \cap Y) = 0.2$

find:

- (a) $P(Y)$

- (b) $P(X \cup Y)$

- (c) $P(\bar{X} \cap \bar{Y})$

- (d) Are the events X and Y independent? Justify your answer.

- (e) Are the events X and Y mutually exclusive? Justify your answer.

[10]

4. To log into her school computer, Emily needs to enter her five-letter password.
How many different passwords are possible if:

- (a) there are no restrictions.

- (b) no letter is repeated.

- (c) the password begins with an E and ends in a Y and there are no repeated letters.

- (d) she forgets her password but knows that the first and last letter of her name were two of the letters used and there were no repeated letters.

[7]

5. In order to visit the Great Barrier Reef visitors travel by either ferry or helicopter to a floating pontoon. All visitors must return to the mainland on the same day. On the way to the reef 60% take the ferry. On the return journey 35% who took the ferry went back by helicopter. 5.5% of all visitors travel across to the reef by helicopter and return by ferry.

What is the probability that a randomly chosen visitor:

- (a) travels to the pontoon by ferry, given they return by ferry?

- (b) travels across to the reef and back using the same mode of transport?

- (c) travels on the helicopter given they travelled at least once on the ferry?

[8]

6. The Student Council at a particular school is made up of 1 Captain, 2 Vice Captains, 3 Sports Captains and 4 Junior Councillors.

A group of 4 is to be selected to attend a conference on leadership. If the group is selected at random, determine the probability the selection includes:

- (a) no Junior Councillors

- (b) the Captain, given no Junior Councillors are selected

The Student Council is to be arranged in 2 rows of 5 students for the official photograph. If all arrangements are equally likely, determine the probability:

- (c) that the Captain is in the middle of one of the rows.

- (d) that the Sports Captains are in the same row but separated, given the Captain is not in the middle of one of the rows.

[10]



TRIAL TEST 2: VECTORS I

Calculators allowed

Time Allowed: 60 minutes

Total Marks: 40

1. If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = -4\mathbf{i} + \mathbf{j}$ find c and d such that, $c\mathbf{a} + d\mathbf{b} = -2\mathbf{i} + 11\mathbf{j}$.

[7]

2. If $A(1, 5)$, $B(-5, 7)$ and $C(6, -4)$ are three points in the plane, find

(a) AB

(b) $|BC|$

(c) AC in polar form

(d) OD where D is the midpoint of AB

(e) E if $ABEC$ is a parallelogram.

[9]

3. $OABC$ is a parallelogram with D on OC such that $OD = \frac{2}{3}OC$ and E is the midpoint of OA . Let $OA = \mathbf{a}$ and $OC = \mathbf{c}$. Find the following in terms of \mathbf{a} and \mathbf{c} .

(a) BE

(b) DA

(c) GF where G is the midpoint of DA and F is the midpoint of BE .

[10]

4. If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j}$ find the vector \mathbf{c} in rectangular form which has five times the magnitude of \mathbf{a} and the same direction as \mathbf{b} .

[3]



TRIAL TEST 3: VECTORS II

Calculators allowed

Time Allowed: 50 minutes

Total Marks: 50

1. If $\mathbf{u} = (-2, 3, 1)$ and $\mathbf{v} = (3, 1, -5)$ find:

(a) $4\mathbf{u} - 3\mathbf{v}$

(b) the size of the angle between \mathbf{u} and \mathbf{v}

(c) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

(d) the acute angle between \mathbf{v} and the x axis

(e) the acute angle between \mathbf{u} and the $x - y$ plane.



TRIAL TEST 4: TRIGONOMETRY

Calculators allowed

Time Allowed: 70 minutes

Total Marks: 65

1. Write each of these functions in simplified form.

(a) $y = 2\sin^2 x + 2\cos^2 x$

(b) $y = \cos(-x)$

(c) $y = \sin\left(x - \frac{\pi}{2}\right)$

(d) $y = \cos(x + \pi)$

(e) $y = \tan(-x)$

(f) $y = \tan(\pi - x)$

[9]

2. Find the exact values of

(a) $\sin \frac{3\pi}{4}$

(b) $\tan \frac{4\pi}{3}$

(c) $\cos \frac{5\pi}{6}$

[6]

3. Write the period, amplitude, vertical shift and horizontal shift of:

(a) $y = -3\cos\left(2x + \frac{\pi}{3}\right)$

(b) $y = 2\tan\left(\frac{\pi}{4} - x\right) + 1$

(c) $y = 4\sin\left(3x - \frac{\pi}{2}\right) - 2$

[14]

4. Solve these equations correct to 2dp

(a) $\sin x = -0.3 \quad -\pi \leq x \leq \pi$

(b) $\cos x = 0.5 \quad -2\pi \leq x \leq 0$

(c) $\tan x = 1.7 \quad \frac{-3\pi}{2} \leq x \leq \frac{\pi}{2}$

[9]

5. If $\tan \theta = -\frac{3}{4}$ and $\sin \theta$ is negative, find the exact value of $\sin 2\theta$.

[4]

6. If $\tan A = -\frac{12}{5}$, $\sin A > 0$, $\cos B = -\frac{3}{5}$ and $\tan B < 0$, find the exact value of $\tan(A+B) \tan(A-B)$.

[8]

7. Prove

(a) $\tan \theta + \frac{\cos \theta}{1 + \sin \theta} = \frac{1}{\cos \theta}$

(b) $\frac{\tan^2 x - 1}{\sin x + \cos x} = \frac{\sin x - \cos x}{\cos^2 x}$

[8]



TRIAL TEST 5: MATRICES

Calculators allowed

Time Allowed: 50 minutes

Total Marks: 50

1. Given the following matrices:

$$A = \begin{bmatrix} -2 & 3 \\ -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -4 \end{bmatrix}$$

Find the following where possible, giving the reason if the operation is not possible.

- (a) A^2

- (b) AB

- (c) $B(3A - I)$

- (d) The value of k in $A^2 - 2A + kI = 0$.

[8]

2. If $A^2 - 3A + I = 0$ show that $A^{-1} = 3I - A$.

[3]

3. The matrices A and B below can be combined and the result used to solve the system of equations below. Show how to do this.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -3 & 1 \\ 6 & -7 & 2 \\ -3 & 4 & -1 \end{bmatrix} \quad \begin{array}{l} 2z + y = 5 \\ y = -3x - 4z \\ x = y + z - 6 \end{array}$$

[5]

4. (a) Find the relationship between b and c in the matrix $B = \begin{bmatrix} 4 & b \\ c & -4 \end{bmatrix}$ such that $B^2 = 6I$.

(b) Solve $A^2 - 2A = 5I$ by adding I to both sides and then using your result of part (a) with $b = 5$. You should give two solutions for matrix A.

- (c) How many solutions to $A^2 - 2A = 5I$ are there, when you consider the choices for the value of b available?

[9]

5. A company has two outlets which are the Uptown shop and the Downtown shop. Each shop sells four types of computer designated as A, B, C and D. At the beginning of the month the numbers of each computer available at the stores and the cost price/selling price of each type are summarised as shown below.

$$P = \begin{matrix} & \text{UT} & \text{DT} \\ \text{A} & \begin{bmatrix} 5 & 2 \end{bmatrix} \\ \text{B} & \begin{bmatrix} 3 & 6 \end{bmatrix} \\ \text{C} & \begin{bmatrix} 4 & 9 \end{bmatrix} \\ \text{D} & \begin{bmatrix} 7 & 3 \end{bmatrix} \end{matrix} \quad Q = \begin{matrix} \text{Cost Price} & \text{Selling Price} \\ \begin{bmatrix} 500 & 800 \\ 1100 & 1600 \\ 900 & 1300 \\ 2000 & 3000 \end{bmatrix} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{matrix} \quad \text{dollars}$$

- (a) Use these matrices to find a new matrix which is useful and fully describe it. Call this matrix R.

- (b) Now the total cost price and selling price are needed after the figures from both shops are combined. Show how to use a matrix method to achieve this.

- (c) Find, by using matrix methods, the profit that each shop will achieve if all computers are sold.

- (d) The big boss needs the total profit from selling all of the computers from both stores. How can this be achieved using matrix methods and what is the total profit?

[9]

6. If Kate exercises today then there is an 80% chance that she will exercise tomorrow, but if she doesn't exercise today there is only a 40% of her exercising tomorrow.

- (a) Write the transition matrix for this situation in the form

$$R = \begin{bmatrix} & E & \bar{E} \\ a & b \\ c & d \end{bmatrix} \begin{matrix} E \\ \bar{E} \end{matrix} \quad \text{where } E = \text{exercise } \bar{E} = \text{no exercise.}$$

- (b) If Kate exercised on Monday find the chance of her exercising on Saturday.

- (c) Find the steady state vector \mathbf{s} for this situation.

- (d) Show how to solve the matrix equation $R\mathbf{s} = \mathbf{s}$ for \mathbf{s} .

[9]

7. The point A (3, 2) is one vertex of a rectangle which has its opposite vertex at the origin. The rectangle undergoes the following transformations in order.

shear parallel to the x axis scale factor 4
 reflection in the line $y = -x$
 dilation parallel to the y axis scale factor -3

- (a) Show how to find the image of point A. i.e. A'''

(b) Find the single matrix which will transform A'' back to A .

(c) Find the area of the image of the original rectangle after the three transformations above.

[7]

TRIAL TEST 6: COMPLEX NUMBERS



Calculators allowed

Time Allowed: 30 minutes

Total Marks: 30

1. If $z = -2 + 5i$ and $w = 6 - 3i$ find without your calculator

(a) zw

(b) $3w - 4z$

(c) $(\bar{w})^2$

(d) $|z|$

(e) $\frac{w}{z}$

[12]

2. Find the exact complex solutions of $z^2 + 10z + 41 = 0$.

[4]

3. A quadratic equation with real coefficients has one of its roots as $z = 7 - 2i$. Find the equation.

[4]

4. (a) If $z_1 = 2 + 3i$ find $z_2 = iz_1$, $z_3 = iz_2$, $z_4 = iz_3$ and $z_5 = iz_4$.

- (b) Graph on the same Argand diagram z_1 , z_2 , z_3 and z_4 .

- (c) Comment on what the geometrical effect is of multiplying a complex number by i .

[10]

TRIAL TEST 7: MATHEMATICAL PROOF AND REASONING



Calculators NOT allowed

Time Allowed: 50 minutes

Total Marks: 63

1. 'Every even integer greater than two can be written as the sum of two prime numbers'.
Prove using counter-examples whether this conjecture is true or false.

[2]

2. Prove the following identity

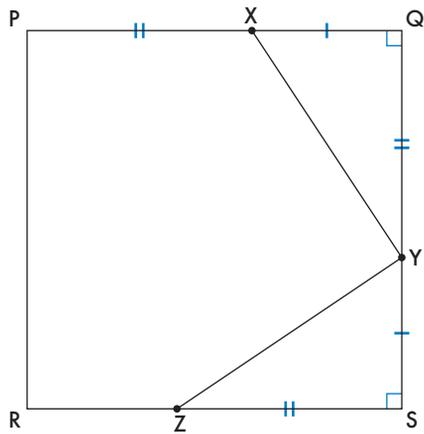
$$\frac{1 - \tan \theta}{1 + \tan \theta} = \frac{1 - \sin 2\theta}{\cos 2\theta}$$

[8]

Prove: $PQRS$ is a cyclic quadrilateral.

[6]

9.



Given: \cdot Square $PQRS$ where X lies on PQ , Y lies on QS and Z lies on RS .
 \cdot $PX = QY = SZ$

Prove: $XY = YZ$

[4]



ANSWERS

CHAPTER 1: Counting Techniques

$$1. \quad \begin{array}{|c|c|c|} \hline 5 & 4 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 5 & 4 & 3 & 2 \\ \hline \end{array}$$

$$= 60 + 120$$

$$= 180$$

$$2. \quad \binom{4}{3} \binom{7}{2} + \binom{4}{4} \binom{7}{1} = 91$$

$$3. \quad (a) \quad \begin{array}{|c|c|c|c|} \hline 2 & 3 & 2 & 1 \\ \hline \end{array}$$

$$= 12$$

$$(b) \quad \begin{array}{|c|c|c|c|} \hline 2 & 2 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 2 & 1 & 1 \\ \hline \end{array}$$

$$= 4 + 2$$

$$= 6$$

$$4. \quad (a) \quad \binom{10}{3} \binom{4}{1} = 480$$

$$(b) \quad \binom{10}{4} \binom{4}{0} = 210$$

$$(c) \quad \binom{10}{2} \binom{4}{2} + \binom{10}{1} \binom{4}{3} + \binom{10}{0} \binom{4}{4} = 311$$

$$5. \quad (a) \quad \binom{3}{1} \binom{2}{2} \binom{4}{2} = 18$$

$$(b) \quad \binom{4}{2} \binom{5}{3} + \binom{4}{3} \binom{5}{2} + \binom{4}{4} \binom{5}{1} = 105$$

$$(c) \quad \binom{3}{1} \binom{2}{2} \binom{4}{2} + \binom{3}{2} \binom{2}{1} \binom{4}{2} + \binom{3}{2} \binom{2}{2} \binom{4}{1} +$$

$$\binom{3}{3} \binom{2}{1} \binom{4}{1} + \binom{3}{1} \binom{2}{1} \binom{4}{3} = 98$$

$$6. \quad (a) \quad \binom{6}{3} \binom{4}{3} + \binom{6}{4} \binom{4}{2} + \binom{6}{5} \binom{4}{1} + \binom{6}{6} \binom{4}{0} = 195$$

$$(b) \quad \binom{6}{3} \binom{4}{3} + \binom{6}{2} \binom{4}{4} = 95$$

$$7. \quad \binom{5}{2} \binom{7}{3} \times 5! = 42000$$

$$8. \quad (a) \quad 9! = 362880$$

$$(b) \quad 2! \times 8! = 80640$$

$$(c) \quad 2! \times 4! \times 3! = 288$$

$$(d) \quad 3! \times 4! \times 2! \times 3! = 1728$$

$$9. \quad (a) \quad \begin{array}{|c|c|c|c|c|} \hline 9 & 8 & 7 & 6 & 5 \\ \hline \end{array}$$

$$= 15120$$

$$(b) \quad \binom{3}{2} \binom{6}{3} \times 5! = 7200$$

$$(c) \quad \binom{3}{2} \binom{6}{3} \times 5! + \binom{3}{3} \binom{6}{2} \times 5! = 9000$$

$$(d) \quad \begin{array}{|c|c|c|c|c|} \hline 3 & 7 & 6 & 5 & 2 \\ \hline \end{array}$$

$$= 1260$$

$$(e) \quad \begin{array}{|c|c|c|c|c|} \hline 3 & 8 & 7 & 6 & 5 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline 8 & 7 & 6 & 5 & 3 \\ \hline \end{array} - 1260$$

$$= 8820$$

$$10. \quad (a) \quad \begin{array}{|c|c|c|c|c|c|c|} \hline 10 & 26 & 26 & 26 & 10 & 10 & 10 \\ \hline \end{array}$$

$$= 175760000$$

$$(b) \quad \begin{array}{|c|c|c|c|c|c|} \hline 10 & 26 & 25 & 24 & 9 & 8 & 7 \\ \hline \end{array}$$

$$= 78624000$$

$$(c) \quad 24 \times 7 = 168$$

$$11. \quad (a) \quad \binom{12}{4} = 495$$

$$(b) \quad \binom{7}{2} \binom{5}{2} = 210$$

$$(c) \quad \binom{7}{1} \binom{5}{3} + \binom{7}{2} \binom{5}{2} + \binom{7}{3} \binom{5}{1} + \binom{7}{4} \binom{5}{0} = 490$$

$$(d) \quad \binom{1}{1} \binom{1}{0} \binom{10}{3} + \binom{1}{1} \binom{1}{0} \binom{10}{3} + \binom{2}{0} \binom{10}{4} = 450$$

$$12. \quad (a) \quad 5! = 120$$

$$(b) \quad 4! = 24$$

$$(c) \quad 2! \times 4! = 48$$

$$(d) \quad 120 - 48 = 72$$

$$13. \quad (a) \quad \binom{13}{5} = 1287$$

$$(b) \quad \binom{7}{3} \binom{6}{2} + \binom{7}{4} \binom{6}{1} + \binom{7}{5} \binom{6}{0} = 756$$

$$(c) \quad 5! = 120$$

$$(d) \quad 2! \times 3! = 12$$

$$(e) \quad \begin{array}{|c|c|c|c|c|} \hline 3 & 3 & 2 & 1 & 2 \\ \hline \end{array}$$

$$= 36$$

$$14. \quad (a) \quad \binom{4}{2} \binom{5}{1} + \binom{4}{1} \binom{5}{2} = 70$$

CHAPTER 2: Vectors I

(b) $\binom{1}{1}\binom{3}{1}\binom{5}{1} + \binom{1}{1}\binom{5}{2} = 25$

15. $\begin{array}{|c|c|} \hline 2 & 4 \\ \hline \end{array} = 8$
 $\begin{array}{|c|c|c|} \hline 5 & 4 & 3 \\ \hline \end{array} = 60$
 $\begin{array}{|c|c|c|c|} \hline 5 & 4 & 3 & 2 \\ \hline \end{array} = 120$
 $\begin{array}{|c|c|c|c|c|} \hline 5 & 4 & 3 & 2 & 1 \\ \hline \end{array} = \frac{120}{308}$

16. (a) $\binom{10}{6} = 210$

(b) $\binom{2}{2}\binom{8}{4} = 70$

(c) $\binom{10}{6} - \binom{2}{2}\binom{8}{4} = 140$

(d) $5! = 120$

17. (a) $\binom{11}{5} = 462$

(b) $\binom{6}{5}\binom{5}{0} = 6$

(c) $\binom{1}{0}\binom{1}{1}\binom{9}{4} \times 2 + \binom{2}{0}\binom{9}{5} = 378$

18. Proof

19. Proof

20. Proof

21. 3 socks - if Tom takes 2 socks he either has (i) matching blue socks (ii) matching black socks (iii) 1 each of black or blue. In the event that he has chosen 1 black sock + 1 blue sock, the sock selected next will either be black or blue. Thus, the third sock will guarantee a match to whatever has been chosen before.

22. 5 students

23. 15

24. 48

25. 21

26. (a) 1.157625

(b) 1.008024032

(c) 0.0858734025

(d) 0.99401498

(e) $m^3 + 9m^2n + 27mn^2 + 27n^3$

(f) $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$

(g) $2x^4 + 12x^2y^2 + 2y^4$

(h) $28a^6b + 560a^4b^3 + 1344a^2b^5$

1. $\mathbf{v} = -3.54\mathbf{i} - 3.54\mathbf{j}$

2. (a) $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$

(b) $\langle 2\sqrt{5}, \sqrt{5} \rangle$

(c) $\sqrt{13}$

3. (a) $\langle -9, -1 \rangle$

(b) $\langle 9, 12 \rangle$

4. (a) $D(-0.5, 4.5)$

(b) $E(-1, 2)$

(c) $\sqrt{74}$

(d) $\left\langle \frac{2}{\sqrt{68}}, -\frac{8}{\sqrt{68}} \right\rangle$ or $\langle 0.24, -0.97 \rangle$

(e) $\langle 6.26, -25.04 \rangle$

5. (a) $\mathbf{BO} = -\mathbf{OA} - \mathbf{AB}$

(b) $\mathbf{BO} = \langle 5.386, -144.67^\circ \rangle_P$

(c) *third leg = 5.39 km on a bearing of 234.67°*

6. (a) $\mathbf{c} = -\mathbf{a}$

(b) $\mathbf{d} = -3\mathbf{b}$

(c) $\mathbf{e} = -\mathbf{a} + \mathbf{b}$

(d) $\mathbf{f} = -\mathbf{a} - \mathbf{b}$

(e) $\mathbf{g} = -\frac{1}{3}\mathbf{a} - 4\frac{1}{3}\mathbf{b}$

7. (a) *bearing = 263.69°*

(b) *time = 29 min 52 sec*

8. *Yes, they are collinear.*

9. $OP = \langle 4.2, 1.4 \rangle$

10. (a) $\mathbf{BO} = -\mathbf{a} - \mathbf{c}$

(b) $\mathbf{AC} = \mathbf{c} - \mathbf{a}$

(c) $\mathbf{AD} = \frac{1}{3}\mathbf{c}$

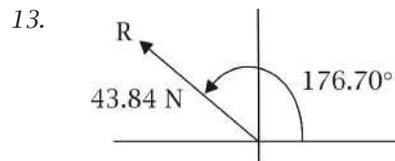
(d) $\mathbf{OD} = \mathbf{a} + \frac{1}{3}\mathbf{c}$

(e) $\mathbf{DC} = \frac{2}{3}\mathbf{c} - \mathbf{a}$

11. $\mathbf{v} = \langle 15, 20 \rangle$

12. (a) *The boat should be steered upstream at an angle of 78.46° to the bank.*

(b) *time = 1 min 1 sec*



14. (a) *bearing = 318.65°*

(b) *time = 1 hr 17 min*

15. *bearing = 281.46°
distance = 1.17 km*

17. *magnitude = 60.13 N
bearing = 006.19°*

18. $\mathbf{BE} = \mathbf{u} - \frac{1}{4}\mathbf{v}$

19. (a) Tahlia comes ashore 74.21 m down the bank from Y.
 (b) distance = 167.35 m
 (c) speed = 2.34 m/s
20. (a) $\mathbf{BC} = -\mathbf{a} - \mathbf{c}$
 (b) $\mathbf{AD} = \mathbf{a} + \mathbf{b}$
 (c) $\mathbf{DC} = -\mathbf{a} - \mathbf{b} - \mathbf{c}$
21. (a) Force on Alex = $\langle 5, -11 \rangle$ N
 (b) Force that Alex must apply = $\langle -5, 11 \rangle$ N
 (c) New force on Alex = $\langle 7.90, -10.53 \rangle$ N
22. (a) C (4, 2)
 (b) D (10.5, -11)
23. $|\mathbf{v}| = 9.46$ km/hr
 bearing = 262.66°
24. (a) $\mathbf{v}_a = 304.61$ km/hr on a bearing of 260°
 (b) bearing = 250.55°
 (c) time = 1 hr 20 min
25. bearing = 99.08° , time = 16.7 min
27. (a) position at 11.12 am = (659.30, 246.86) km or (543.43, 447.54) km
 (b) bearing of wind = 124.85° or 355.15°
28. bearing = 029.90°
 faster by 47.38 km/hr or by 3 min 45 sec
29. $\mathbf{OP} = \langle 10, 4 \rangle$
30. $\mathbf{AD} + \mathbf{BE} + \mathbf{CF}$
 $= \mathbf{AB} + \frac{1}{2}\mathbf{BC} + \mathbf{BC} + \frac{1}{2}\mathbf{CA} + \mathbf{CA} + \frac{1}{2}\mathbf{AB}$
 $= \frac{3}{2}\mathbf{AB} + \frac{3}{2}\mathbf{BC} + \frac{3}{2}\mathbf{CA}$
 $= \frac{3}{2}(\mathbf{AB} + \mathbf{BC} + \mathbf{CA})$
 $= \frac{3}{2} \times 0$
 $= 0$ Proved
31. In $\triangle ABC$ let D be the midpoint of AB and E be the midpoint of BC.
 $\mathbf{AD} + \mathbf{DE} + \mathbf{EC} = \mathbf{AC}$
 $\frac{1}{2}\mathbf{AB} + \mathbf{DE} + \frac{1}{2}\mathbf{BC} = \mathbf{AC}$
 $\frac{1}{2}(\mathbf{AB} + \mathbf{BC}) + \mathbf{DE} = \mathbf{AC}$
 $\frac{1}{2}\mathbf{AC} + \mathbf{DE} = \mathbf{AC}$
 $\mathbf{DE} = \mathbf{AC} - \frac{1}{2}\mathbf{AC}$
 $\mathbf{DE} = \frac{1}{2}\mathbf{AC}$ Proved
32. (a) $\mathbf{OD} = \mathbf{a} + \frac{1}{2}\mathbf{c}$
 (b) $\mathbf{OE} = \mathbf{c} + \frac{1}{2}\mathbf{a}$
 (c) $\mathbf{DC} = \frac{1}{2}\mathbf{c} - \mathbf{a}$

- (d) Substitute all relevant vectors into $\mathbf{OA} + \mathbf{AD} + \mathbf{DF} = \mathbf{OF}$ to get
 $(l - h)\mathbf{a} + (\frac{1}{2} + \frac{h}{2})\mathbf{c} = \frac{k}{2}\mathbf{a} + k\mathbf{c}$
 ie. $l - h = \frac{k}{2}$ and $\frac{1}{2} + \frac{h}{2} = k$
 Solve simultaneously to get
 $h = \frac{3}{5}, k = \frac{4}{5}$.

33. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{c}$. Use $\triangle OAM$ so
 $\mathbf{OA} + \mathbf{AM} = \mathbf{OM}$

i.e. $\mathbf{a} + k(\mathbf{c} - \mathbf{a}) = h(\mathbf{a} + \frac{1}{2}\mathbf{c})$

and $(l - k)\mathbf{a} + k\mathbf{c} = h\mathbf{a} + \frac{h}{2}\mathbf{c}$

$l - k = h$ $k = \frac{h}{2}$

giving $h = \frac{2}{3}$ and $k = \frac{1}{3}$

$\therefore \mathbf{AM} = \frac{1}{3}\mathbf{AC}$ proved

34. Let $\mathbf{AB} = \mathbf{b}$ and $\mathbf{AD} = \mathbf{d}$
 $\mathbf{AE} + \mathbf{EF} = \mathbf{AF}$

$\frac{1}{3}\mathbf{d} + \mathbf{EF} = \frac{1}{3}(\mathbf{b} + \mathbf{d})$

$\therefore \mathbf{EF} = \frac{1}{3}\mathbf{b}$ but now we need to show that

$\mathbf{b} = \mathbf{EG}$

Let $\mathbf{BG} = k\mathbf{BC}$ and $\mathbf{EG} = h\mathbf{EF}$
 $= k\mathbf{d}$ $= h \times \frac{1}{3}\mathbf{b}$
 $= \frac{h}{3}\mathbf{b}$

$\mathbf{AG} = \mathbf{AB} + \mathbf{BG} = \mathbf{AE} + \mathbf{EG}$

$\mathbf{b} + k\mathbf{d} = \frac{1}{3}\mathbf{d} + \frac{h}{3}\mathbf{b}$

$\therefore h = 3$ and $\mathbf{EG} = \frac{3}{3}\mathbf{b} = \mathbf{b}$ as required

From before $\mathbf{EF} = \frac{1}{3}\mathbf{b}$

$= \frac{1}{3}\mathbf{EG}$ ie F divides EG in

the ratio 1 : 2

CHAPTER 3: Vectors II

1. (a) $\lambda = -6$ (b) $\lambda = 1.5$
2. $p = -7\frac{1}{3}$
3. (a) one equation of the line is $\mathbf{r} = \langle 3, -2 \rangle + t\langle 8, -1 \rangle$
 (b) one equation of the line is $\mathbf{r} = \langle 12, 5 \rangle + \lambda\langle 1, 8 \rangle$
 (c) intersection point is (11, -3)
 (d) closest distance = $\sqrt{65}$ units
4. (a) $\mathbf{a} \cdot \mathbf{a} = 49$
 (b) $\mathbf{b} \cdot \mathbf{b} = 16$
 (c) $\theta = 66.9^\circ$
 (d) $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 43$
 (e) $|\mathbf{a} - \mathbf{b}| = \sqrt{43}$
5. $\theta = 59.49^\circ$

6. $\mathbf{r} = \langle 4 + 5t, 2 + 3t \rangle$ is one possible equation or $\mathbf{r} \cdot \langle 3, -5 \rangle = 2$ is another
7. (a) $\left\langle \frac{8}{17}, \frac{15}{17} \right\rangle$
 (b) $\langle 32, 60 \rangle$
 (c) $\theta = 78.19^\circ$
8. $\cos \theta = \frac{11}{5\sqrt{5}} = \frac{11\sqrt{5}}{25}$
9. $\mathbf{b} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 8$
10. (a) $|\mathbf{a}_0| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$
 (b) angle = θ
11. $|\mathbf{u} - \mathbf{v}| = 2\sqrt{10}$
12. $a = 0.6, b = 0.8$
14. vector form of circle is $\left| \mathbf{r} - \left\langle 1, \frac{2}{3} \right\rangle \right| = \frac{5}{3}$
 scalar product form of tangent line
 $\mathbf{r} \cdot \left\langle 1, \frac{4}{3} \right\rangle = -\frac{8}{3}$
15. closest distance = 5.8 units
16. (a) work done by $\mathbf{F} = 30.74$ joules
 (b) work done by $\mathbf{W} = 66.25$ joules
 (c) work done by $\mathbf{R} = 15$ joules
17. time = 7.30 pm for the closest distance of 15.81 km
18. time = 1.10 pm for the closest distance of 28.07 km
20. $x = 10$ km
 closest distance of 14.71 km
21. (a) $\mathbf{r} = \langle 2, 3 \rangle + t \langle 4, -6 \rangle$
 (b) B is on the line
 C is not on the line
22. (a) $x^2 + (y + 4)^2 = 8$
 (b) $|\mathbf{r} - \langle 0, -4 \rangle| = 2\sqrt{2}$
23. (a) One possible equation is
 $\mathbf{r} = \langle -2, 1 \rangle + t \langle 6, -4 \rangle$
 (b) $x = 6t - 2, y = 1 - 4t$
 (c) $2x + 3y = -1$
24. The equations can represent the same line because in $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ different points on the line can be used for \mathbf{a} and any vectors can be used for \mathbf{u} which are scalar multiples of each other.
25. (a) One possible equation is
 $\mathbf{r} = \langle 2 + 3t, 5 + 6t \rangle$
 (b) $x = 2 + 3t, y = 5 + 6t$
 (c) $(3, 7)$ is on the line
26. One possible equation is
 $\mathbf{r} = \langle 0, -3 \rangle + t \langle 1, 2 \rangle$
27. $\mathbf{x}_{\text{int}} = -1\frac{8}{9}$
28. magnitude = 424.38 km/hr
 bearing = 055.56°
29. The target was not hit.
30. (a) $\mathbf{r} = \langle 3, -4 \rangle + t \langle 2, 5 \rangle$
 (b) $x = 3 + 2t, y = -4 + 5t$
 (c) $5x - 2y = 23$
31. (a) centre $(-3, 2)$, radius = 10
 (b) $|\mathbf{r} - \langle -3, 2 \rangle| = 10$
 (d) point of contact = $(5, -4)$
32. (a) 72.11 km/hr on a bearing of 073.69°
 (b) 72.11 km/hr on a bearing of 253.69°
33. one intersection point at $(6, 7)$
34. (a) $(7, -5)$
 (b) $\langle 5, 18 \rangle$
 (c) $\mathbf{v}_2 = \langle -3, -1 \rangle$
 (d) no collision
 (e) intersection of paths is at $(-21, 2)$
 (f) shortest distance = 12.52 units
 (g) $\theta = 32.47^\circ$
35. (a) $|\mathbf{r} - \langle 2, 1 \rangle| = 5$
 (b) $x^2 + y^2 - 4x - 2y - 20 = 0$
36. centre $(-3.5, 5)$, radius = 6.5 units
37. speed of Vedran = 57.86 km/hr
 bearing of Vedran = 158.95°
38. (a) 10 km apart at 1.47 pm and 2.15 pm
 (b) No, the ships do not collide.
 (c) paths intersect at $(23, -33)$ km
39. (a) $(x + 1)^2 + (y - 5)^2 = 8$
 or $x^2 + y^2 + 2x - 10y + 18 = 0$
 (c) $(-1, 2)$ is outside the circle
 (d) area = 8π units²
40. (b) time of collision = 12.20 pm at
 position $(10, 5\frac{1}{3})$ km
 (c) 5 km apart at time 12.01:30 pm
41. (a) $\mathbf{r} = \langle -5, 6 \rangle + t \langle -5, -4 \rangle$ m
 (b) after 8 seconds the bird is at
 $(-45, -26)$ m and distance travelled
 in 8 sec is 51.22 m
 (c) closest distance = 43.73 m
42. The closest distance is 1.41 km which occurs at 1.45 pm.
43. (a) radius = $\frac{1}{2}$, centre = $(0, 0)$
 (b) radius = 5, centre = $(4, 3)$
 (c) radius = 3, centre = $(-3, 2)$
44. $\langle -5, 4 \rangle$ and $\langle -3, 8 \rangle$
45. $a = 17.4, b = -5$, time of collision is 9 pm

46. (a) ships collide at (13, 7) km at 1.15 pm
 (b) $\mathbf{v}_c = \langle 10, 10 \rangle \text{ km/hr}$
 time of arrival = 1.39 pm
47. shortest distance = 6.11 km
 time is after 42 min

CHAPTER 4: Geometry

1. False: Counter example -
 If the integer is 8 then $n^2 + 1 = 65$.
 This is not prime since divisible by 5.
2. False: Counter example -
 9 has factors 1, 3, 9. This is an odd number of factors.
3. True: Example -
 21 has factors 1, 3, 7, 21. All of the factors are odd.
4. False: Counter example -
 6 has factors 1, 2, 3, 6. Some of the factors are **not** even.
5. False: Counter example -
 The product of 5 and 6 is 30. 30 is **not** a multiple of 4.
6. Let $x, x + 2, x + 4$ be consecutive even integers.

$$\therefore x + x + 2 + x + 4 = 3x + 6$$

$$= 3(x + 2)$$
 So the sum has factors of 3 and $(x + 2)$
 \therefore The sum is a multiple of 3.
7. Let $a, a + d$ and $a + 2d$ be numbers in an arithmetic progression

$$\therefore a + a + d + a + 2d = 3a + 3d$$

$$= 3(a + d)$$
 \therefore The sum is a multiple of 3.
8. Let $2x, 2x + 2$ and $2x + 4$ be consecutive even numbers.

$$\therefore 2x(2x + 2)(2x + 4) = 2x \cdot 2(x + 1) \cdot 2(x + 2)$$

$$= 8x(x + 1)(x + 2)$$
 So the product is a multiple of 8.

9. Let x and y be the two odd numbers.
 Hence $x - 1$ and $y - 1$ are two even numbers.
 $\therefore x - 1 = 2a$
 $y - 1 = 2b$ where a and b are whole numbers
 $\therefore x = 2a + 1$
 $y = 2b + 1$
 $\therefore xy = (2a + 1)(2b + 1)$

$$= 4ab + 2a + 2b + 1$$
 $4ab, 2a$ and $2b$ are all even. Adding 1 will result in an odd number.

10. If x is an even number then $x = 2y$ for some x .
 Hence $x^2 = 4y^2$

$$= 2(2y^2)$$
 Which is also an even number.
 \therefore as x^2 is an odd number then x must be odd.

11. This is not a valid proof. This is a **specific** case when the angle given is 30° . This proof does not prove the general case.

12. **Given:**
- $\overline{AC} \cong \overline{EC}$
 - $\angle ABC = \angle EDC = 90^\circ$

To Prove: $\overline{BC} \cong \overline{DC}$

- Proof:**
- $\angle ABC \cong \angle EDC$ given
 - $\overline{AC} \cong \overline{EC}$ given
 - $\angle ACB \cong \angle ECD$ vertically opposite angles
- $\therefore \triangle ABC \cong \triangle EDC$ (A.A.S)
 $\therefore \overline{BC} \cong \overline{DC}$ corresponding sides

13. **Given:**
- $\overrightarrow{PQ} \parallel \overrightarrow{RS}$
 - $\overline{PU} \cong \overline{SU}$
 - $\angle PTU = \angle SVU = 90^\circ$

To Prove: $TU = VU$

- Proof:**
- $\angle PTU = \angle SVU = 90^\circ$ given
 - $\overline{PU} \cong \overline{SU}$ given
 - $\angle TPU \cong \angle VSU$ alternate angles
- $\therefore \triangle PTU \cong \triangle SVU$ (A.A.S)
 $\therefore \overline{TU} \cong \overline{VU}$
 $\therefore TU = VU$

14. **Given:** • \overline{XB} and \overline{AY} are diameters of a circle, centre Z.

To Prove: $\overline{XY} \parallel \overline{AB}$

Proof:

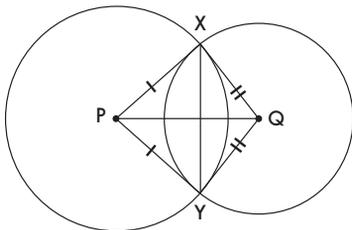
- $\angle XZY \cong \angle AZB$ vertically opposite angles
- $\overline{XZ} \cong \overline{ZB}$ radii
- $\overline{YZ} \cong \overline{ZA}$ radii

$\therefore \triangle XZY \cong \triangle AZB$ (S.A.S)

$\therefore \angle ZXY \cong \angle ZBA$ corresponding angles of congruent triangles

$\therefore \overline{XY} \parallel \overline{AB}$ alternate angles are congruent

15.



Given: • Circles centres P and Q intersecting at X and Y.

To Prove: $\angle PXQ = \angle PYQ$

Proof:

- $\overline{PX} \cong \overline{PY}$ same radii
- $\overline{QX} \cong \overline{QY}$ same radii
- $\overline{PQ} \cong \overline{PQ}$ common

$\therefore \triangle PXQ \cong \triangle PYQ$ (S.S.S)

$\therefore \angle PXQ \cong \angle PYQ$ corresponding angles

$\therefore \angle PXQ = \angle PYQ$

16. **Given:** • $\overline{XD} \cong \overline{YD}$
• Circle centre C with chord \overline{XY}

To Prove: $\overline{CD} \perp \overline{XY}$

Proof:

- $\overline{CX} \cong \overline{CY}$ radii
- $\overline{XD} \cong \overline{YD}$ given
- $\overline{CD} \cong \overline{CD}$ common

$\therefore \triangle CXD \cong \triangle CYD$ (S.S.S)

$\therefore \angle CDX \cong \angle CDY$ corresponding angles

• $\angle CDX$ and $\angle CDY$ ($\angle XDY$ is a straight angle) are supplementary

• $\angle CDX = \angle CDY = 90^\circ$

$\therefore \overline{CD} \perp \overline{XY}$

17. **Given:** • Circle centre Z
• Two points X and W located on the circle not in the same diameter
• $\angle XYW$ in the minor arc

To Prove: Reflex $\angle XZW = 2 \times \angle XYW$

Proof: Let $x^\circ = \angle XYZ$
 $y^\circ = \angle WYZ$

- $\overline{ZY} \cong \overline{ZW} \cong \overline{ZX}$ radii
- $\angle XYZ = \angle ZXY = x^\circ$ Isosceles \triangle
- $\angle WYZ = \angle ZWY = y^\circ$ Isosceles \triangle

$\therefore \angle XZY = 180^\circ - 2x^\circ$

$\therefore \angle WZY = 180^\circ - 2y^\circ$

- $\angle XZW = 360^\circ - 2x^\circ - 2y^\circ$
 $= 360^\circ - (2x^\circ + 2y^\circ)$

- reflex $\angle XZW = 2x^\circ + 2y^\circ$
 $= 2(x^\circ + y^\circ)$
 $= 2(\angle XYZ + \angle WYZ)$

\therefore reflex $\angle XZW = 2 \times \angle XYW$

18. **Given:** Circle with chords \overline{XY} and \overline{AB} which meet at F.

To Prove: $\overline{FX} \cdot \overline{FY} = \overline{FA} \cdot \overline{FB}$

Proof: $\angle AXF = \angle BFX$ angles in the same segment

$\angle BAX = \angle BYX$ angles in the same segment

$\angle XFA = \angle BFY$ vertically opposite

$\therefore \triangle AXF \sim \triangle BFX$ (A.A.)

$\therefore \frac{\overline{FX}}{\overline{FA}} = \frac{\overline{BF}}{\overline{FY}}$

ie. $\overline{FX} \cdot \overline{FY} = \overline{FA} \cdot \overline{FB}$

19. $\overline{PZ} \cdot \overline{ZQ} = \overline{YZ} \cdot \overline{ZR}$

$$\overline{RZ} = r + \overline{XZ}$$

$$\overline{YZ} = r - \overline{XZ}$$

$$\therefore \overline{PZ} \cdot \overline{ZQ} = (r - \overline{XZ})(r + \overline{XZ})$$

$$\therefore \overline{PZ} \cdot \overline{ZQ} = r^2 - \overline{XZ}^2$$

20. **To Prove:** $XZ^2 - XC^2 = ZC \cdot YC$

Proof: Using Pythagoras

$$XZ^2 = XA^2 + AZ^2$$

$$XC^2 = XA^2 + AC^2$$

$$\therefore XZ^2 - XC^2 = \cancel{XA^2} + AZ^2 - \cancel{XA^2} - AC^2$$

$$= AZ^2 - AC^2$$

$$= (ZC - AC)^2 - AC^2$$

$$= ZC^2 - 2ZC \cdot AC + \cancel{AC^2} - \cancel{AC^2}$$

$$= ZC^2 - 2ZC \cdot AC$$

$$= ZC(ZC - 2AC)$$

$$= ZC(ZA + AC - 2AC)$$

$$= ZC(ZA - AC)$$

$$= ZC(AY - AC)$$

$$= ZC \cdot YC$$

$$\therefore XZ^2 - XC^2 = ZC \cdot YC$$

21. (a) For all real values of x , the quotient of 1 and x cannot equal x
- (b) There exists a real value x such that the square of x is less than negative one multiplied by x
- (c) If a function of x is defined as the square root of x , then the domain of the function of x is such that it is a set of values x such that x is an element of the real numbers, and x is greater than or equal to zero.
22. (a) $g(x) = \frac{2}{x-1} \Rightarrow D_g : \{x : x \in \mathbb{R}, x \neq 1\}$
- (b) $\forall m \in \mathbb{Z}^+, m + \frac{1}{m} \geq 2$
- (c) $\exists p \in \mathbb{R}$ s.t. $0 - p \neq p$
23. (a) If the Eagles are not winning the game, then they are not playing the Dockers.
- (b) If the Eagles are playing the Dockers, then they are winning the game.
- (c) If the Eagles are winning the game, then they are not playing the Dockers.
- (d) If the Eagles are not playing the Dockers, then they are not winning the game.

24. (a) $\forall x, |x| < 6 \Rightarrow x < 6$

(b) Converse

25. (a) If $x + y > y + 5$, then $x > y$

(b) \forall even $x \in \mathbb{Z}$, x is odd

(c) $\forall n \in \mathbb{Z}, n^2 \not\equiv 3 \pmod{3} \Rightarrow n \not\equiv 3 \pmod{3}$ (or for every integer n , if n^2 is not divisible by 3 then n is not divisible by 3)

26. **Theorem:** If a^2 is a multiple of p , then a is a multiple of p (suppose that $a \in \mathbb{Z}$ and p is prime).

Proof: Assume the statement is false.

Then there is a prime p and positive integer a such that a^2 is a multiple of p but a is not a multiple of p . Without loss of generality we may assume a is the *smallest* such positive integer. By the division algorithm there are positive integers q and r such that

$$a = pq + r \text{ and } p > r > 0$$

hence r is not a multiple of p and $0 < r = a - pq < a$

$$\text{Thus } a^2 = (pq + r)^2 = p^2q^2 + 2pqr + r^2$$

$$a^2 - p^2q^2 = 2pqr + r^2$$

Since a^2 is a multiple of p , the LHS and hence r^2 is a multiple of p . That is, $0 < r < a$, r^2 is a multiple of p , and r is not a multiple of p —which contradicts the minimality of a .

27. **Theorem:** If the square of an integer is odd, then that integer is odd.

Proof: Use contraposition to develop an alternative statement:

i.e. evenness of integer \Rightarrow evenness of square of integer

Using contradiction this statement reads:

evenness of integer \Rightarrow oddness of square of integer

let $n = 2m$; n and $m \in \mathbb{Z}$

$$n^2 = (2m)^2 = 4m^2 = 2(2m^2)$$

which implies n^2 is even and contradicts the revised statement.

28. **Theorem:** For all integers n , prove that if $n^2 + 5$ is odd then n is even.

Contrapositive: oddness of $n \Rightarrow$ evenness of $n^2 + 5$

Contradiction: oddness of $n \Rightarrow$ oddness of $n^2 + 5$

Assuming $n^2 + 5$ is odd, then n^2 must be even and hence n is even.

29. **Theorem:** If $a^2 - 2a + 7$ is even, then a is odd (suppose that $a \in \mathbb{Z}$)

Contrapositive: evenness of $a \Rightarrow$ oddness of $a^2 - 2a + 7$

Contradiction: evenness of $a \Rightarrow$ evenness of $a^2 - 2a + 7$

Let $a = 2m$; a and $m \in \mathbb{Z}$

$$\begin{aligned} a^2 - 2a + 7 &= (2m)^2 - 2(2m) + 7 \\ &= 4m^2 - 4m + 7 \\ &= 2[2m^2 - 2m] + 7 \end{aligned}$$

which yields an odd and a contradiction.

30. **Theorem:** Prove for $\triangle ABC$ that if $\angle A$ is a right angle, then $\angle B$ cannot be an obtuse angle.

Proof: Assume that $\angle B$ is an obtuse angle.

If $\angle A + \angle B + \angle C = 180^\circ$, and $\angle A = 90^\circ$
then $90^\circ + \angle B + \angle C = 180^\circ$
 $\angle B + \angle C = 90^\circ$

If $\angle B$ is obtuse then $\angle C$ must be negative (which is impossible). Hence the claim that $\angle B$ is obtuse renders a contradiction.

31. **Theorem:** Prove that $\forall x: x \in \mathbb{R}, x \in \left[0, \frac{\pi}{2}\right]$ we have $\sin(x)$ and $\cos(x) \geq 1$

Proof: Assume that $\sin(x) + \cos(x) < 1$.

Now for $\left[0, \frac{\pi}{2}\right]$, $\sin(x) \geq 0$ and $\cos(x) \geq 0$

Thus $\sin(x) + \cos(x) \geq 0$

Squaring LHS gives $[\sin(x) + \cos(x)]^2$
 $= \sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x)$
 $= \sin^2(x) + \cos^2(x) + 2\sin(x)\cos(x)$
 $= 1 + 2\sin(x)\cos(x) \geq 1$

which creates a contradiction.

32. **Theorem:** Prove that no integers a and b exist for which $18a + 6b = 1$

Proof: Commence with statement
“ $\forall a, b \in \mathbb{Z}, 18a + 6b \neq 1$ ”

Negating this statement
“ $\exists a, b \in \mathbb{Z}, 18a + 6b = 1$ ”

Assume that integers a and b exist for which $18a + 6b = 1$

Dividing by 6 gives $3a + b = \frac{1}{6}$ which creates a contradiction as $(3a + b)$ will produce an integer but $\frac{1}{6}$ is not.

33. **Theorem:** Prove that there are infinitely many prime numbers.

Proof: Suppose that there are a finite number of primes. Then we can write them in a list: $\{2, 3, 5, 7, \dots, p_n\}$ where p_n is the last prime number. Let $p_1 = 2, p_2 = 3$, and so on, so this list can just be written as $\{p_1, p_2, p_3, \dots, p_{n-1}, p_n\}$. If we multiply all of the p_i together and add one, calling the result q , we have $q = p_1 \times p_2 \times p_3 \times \dots \times p_{n-1} \times p_n + 1$. Then q is a positive integer greater than 2, so it is either prime or composite. Clearly q is larger than any of the primes in $\{p_1, p_2, p_3, \dots, p_n\}$ so it cannot be located

anywhere on this list (and q itself is composite). Because q is composite, there is some prime number that divides evenly into q . Call this prime number P , and bear in mind that P must be somewhere on the list $\{p_1, p_2, p_3, \dots, p_n\}$, since it is a list of all of the prime numbers.

Since P is one of the p_p it divides evenly into $\{p_1, p_2, p_3, \dots, p_n\}$.

Then in the equation $q = p_1 \times p_2 \times p_3 \times \dots \times p_{n-1} \times p_n + 1$, P divides evenly into q and it divides evenly into $p_1 \times p_2 \times p_3 \times \dots \times p_{n-1} \times p_n$, so it must divide evenly into 1 as well. But that is impossible—since P is a prime number, it is greater than 1, so it cannot divide evenly into 1. Hence the supposition is false and the theorem correct.

34. **Theorem:** There is no greatest even integer.

Proof: Assume there is a greatest even integer N . Then for every even integer n , $N \geq n$. Now suppose $M = N + 2$, which makes M an even integer—but $M > N$! This contradicts the original assumption.

CHAPTER 5: Trigonometry

1. (a) $-\frac{8}{17}$

2. (a) $x_1 = \frac{\pi}{12}$ (b) $A_1 = \frac{\pi}{4}$
 $x_2 = \frac{\pi}{4}$ $A_2 = -\frac{3\pi}{4}$
 $x_3 = \frac{3\pi}{4}$ $A_3 = 1.11^R$
 $x_4 = \frac{11\pi}{12}$ $A_4 = -2.03^R$

(c) $\theta_1 = \frac{\pi}{2}$ (d) $x_1 = \frac{\pi}{6}$
 $\theta_2 = \frac{3\pi}{2}$ $x_2 = \frac{\pi}{2}$
 $\theta_3 = \frac{\pi}{6}$ $x_3 = -\frac{\pi}{2}$
 $\theta_4 = \frac{5\pi}{6}$ $x_4 = -\frac{5\pi}{6}$

3. $\max = 5$ at $x = 180^\circ$

4. $\cos 2A = -\frac{7}{18}$

5. (a) $\sqrt{3} - 2$
(b) $\frac{1}{\sin \theta}$

7. $\pm \frac{\pi}{6}, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{5\pi}{6}$

8. 210°

9. (b) $x = 60^\circ$
(c) $x = 60^\circ$ as before
10. $x = 2\frac{16}{21}$
11. $\cos \frac{\pi}{8} = \frac{\sqrt{\sqrt{2}+2}}{2}$
12. $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$
13. (a) $\angle BAC = 45^\circ$
(b) $\tan \angle DAC = \frac{1}{2}$
(c) $\tan \theta = \frac{1}{3}$
14. $\sqrt{3} + 2$
15. $4\sqrt{5}$
16. (a) $\min = 1$
(b) $x = 150^\circ$
17. $\frac{\sqrt{3}}{6}$
18. $\sin \theta = \frac{36}{325}$
19. 14
20. (a) $\theta_1 = \frac{5\pi}{4}, \theta_2 = \frac{7\pi}{4}$
 $\theta_3 = -\frac{\pi}{4}, \theta_4 = -\frac{3\pi}{4}$
(b) $\theta_1 = 0^\circ, \theta_2 = 180^\circ, \theta_3 = 360^\circ,$
 $\theta_4 = 45^\circ, \theta_5 = 135^\circ, \theta_6 = 225^\circ,$
 $\theta_7 = 315^\circ$
21. (a) $4\sqrt{3}$ cm
(b) 6 cm
(c) $4\sqrt{3}$ cm
22. (a) $x_1 = 9.74^\circ$
 $x_2 = 80.26^\circ$
 $x_3 = 189.74^\circ$
 $x_4 = 260.26^\circ$
(b) $x_1 = 0, x_2 = \frac{2\pi}{9}, x_3 = \frac{2\pi}{3}$
 $x_4 = \frac{8\pi}{9}, x_5 = \frac{4\pi}{3}, x_6 = \frac{14\pi}{9}$
 $x_7 = 2\pi$
23. $x = -4.9962$
24. (a) $x_1 = \frac{2\pi}{3}, x_2 = -\frac{2\pi}{3}$
(b) $\theta_1 = 0, \theta_2 = \frac{\pi}{2}, \theta_3 = \pi,$
 $\theta_4 = \frac{3\pi}{2}, \theta_5 = 2\pi$
25. $BD = 3.5$
26. $\tan(180^\circ + A) = \tan A$
 $\cos(360^\circ - A) = \cos A$
 $\sin(270^\circ - A) = -\cos A$
27. Use $\sin^2 B = 1 - \cos^2 B$ in the L.H.S. or
 $\cos^2 B = 1 - \sin^2 B$ in the R.H.S.
28. $\cos(A) + B = \cos A \cos B - \sin A \sin B$
 $\cos(A - B) = \cos A \cos B + \sin A \sin B$
Multiplying these together is best done by recognising the difference of two squares idea. Now use $\cos^2 B = 1 - \sin^2 B$ and $\sin^2 A = 1 - \cos^2 A$.
29. Use the lowest common denominator L.C.D. of $(1 - \cos A)(1 + \cos A)$ when adding the L.H.S. Simplifying the numerator gives $2 \sin A \cos A$. Denominator is $1 - \cos^2 A$ and then $\sin^2 A$.
30. $\tan A = \frac{\sin A}{\cos A}$ then subtract using the L.C.D. of $\cos^2 A$. Factorise the numerator and then $1 - \cos^2 A = \sin^2 A$. Put the denominator under one of the $\sin^2 A$ on the numerator and use $\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$
31. L.C.D. = $1 + \sin A$. Expand numerator and use $\cos^2 A + \sin^2 A = 1$.
32. L.C.D. = $\sin B \cos A \sin A \cos B$. Numerator then becomes $\sin(A + B)$. Use $\sin A \cos A = \frac{1}{2} \sin 2A$ etc in the denominator
33. $\sin\left(\frac{\pi}{2} - A\right) = \cos A$
 $\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$
L.C.D. = $\cos^2 A$ etc.
34. L.H.S. is the difference of two squares.
 $\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} = 1$
 $\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = \cos\left(2 \cdot \frac{A}{2}\right)$
35. In the numerator $\cos 2A = 1 - 2 \sin^2 A$. In the denominator $\cos 2A = 2 \cos^2 A - 1$. In both $\sin 2A = 2 \sin A \cos A$. Expand, simplify and factorise both numerator and denominator and cancel out.
36. Use L.C.D. = $(1 - \sin A)(1 + \sin A)$ to subtract. Simplify the numerator and cancel down.

37. $\sin 2x = 2 \sin x \cos x$
 $\cos 2x = 1 - 2 \sin^2 x$
38. $\sin 3\theta = \sin(\theta + 2\theta)$
 $= \sin \theta \cos 2\theta + \cos \theta \sin 2\theta$
 Use $\cos 2\theta = 1 - 2 \sin^2 \theta$
 and $\sin 2\theta = 2 \sin \theta \cos \theta$.
 Use $\cos^2 \theta = 1 - \sin^2 \theta$ expand and simplify.

39. $\sin 2A = 2 \sin A \cos A$ in both the numerator and denominator. Factorise both the numerator and the denominator. Cancelling down gives

$$\text{L.H.S.} = \frac{1 - \cos A}{1 + \cos A}$$

$$\text{Take R.H.S.} = \frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}$$

Now consider $\cos 2A = 2 \cos^2 A - 1$

$$\text{ie } \cos^2 A = \frac{\cos 2A + 1}{2}$$

$$\text{So now } \cos^2 \frac{A}{2} = \frac{\cos A + 1}{2}$$

Similarly $\cos 2A = 1 - 2 \sin^2 A$

$$\text{and } \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\text{ie } \sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}$$

Substitute and simplify gives

$$\text{R.H.S.} = \frac{1 - \cos A}{1 + \cos A}$$

40. L.H.S. is the difference of two squares.
41. L.C.D. is $(1 + \sin A)(1 - \sin A)$ which becomes $1 - \sin^2 A$ and then $\cos^2 A$.
42. L.C.D. is $(1 + \sin A) \cos A$
 Expand the numerator, simplify, use $\cos^2 A + \sin^2 A = 1$ and cancel down.
43. Replace $\tan A$ by $\frac{\sin A}{\cos A}$ in the L.H.S.
 Add the numerator and then factorise overall denominator becomes $\cos A(1 + \cos A)$ leads to cancelling down.
44. Add L.H.S. using L.C.D. = $\sin B \cos B$. Numerator becomes $\cos(A - B)$. Denominator becomes $\frac{1}{2} \sin 2B$ etc.
45. Replace $\sin 2A$ by $2 \sin A \cos A$ and $\cos 2A$ by $2 \cos^2 A - 1$ and simplify.
 Factorise the numerator and the denominator, and cancel down.
46. Group $\sin A \cos A$ and replace by $\frac{1}{2} \sin 2A$. Then group $\sin 2A \cos 2A$ and replace by $\frac{1}{2} \sin 4A$.
 Repeat and it's all over!

47. (a) $y = \cot(x - \frac{\pi}{4})$
 (b) $y = \sec x + 2$
 (c) $y = \sec(x + \frac{\pi}{2})$
 (d) $y = \operatorname{cosec}(x - \frac{\pi}{4}) + 1$

48. (a) $p = 5, q = 0.927$
 (b) $\theta = 0.64^R$

49. (a) $k = \sqrt{7}, \alpha = 49.11^\circ$
 (b) $x_1 = 18.68^\circ, x_2 = 243.10^\circ$

50. (a) $\sqrt{7} \sin(\theta + 40.89^\circ)$
 (b) $\theta_1 = 108.17^\circ, \theta_2 = 350.04^\circ$

51. (a) $17 \cos(x - 61.93^\circ)$
 (b) $x_1 = 7.96^\circ, x_2 = 115.90^\circ$

CHAPTER 6: Matrices

1. $\begin{bmatrix} 12 & 2 \\ 8 & 0 \end{bmatrix}$

2. (a) $\begin{bmatrix} 6 & -9 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 18 & 23 \\ 6 & -18 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$ (d) B^{-1} does not exist

(e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. (a) $A'' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $B'' = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$C'' = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad D'' = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

4. (a) $x + y = 8800, 0.07x + 0.08y = 663$

- (b) \$4100 is invested at 7%
 \$4700 is invested at 8%

5. (a) A' is $(-4, -2)$

(b) A'' is $(-4, 8)$

$$(c) \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

$$6. (a) A = \begin{array}{ccc} & \text{FC} & \text{BC} & \text{EC} \\ \begin{bmatrix} 40 & 60 & 120 \\ 80 & 30 & 40 \end{bmatrix} & & & \begin{matrix} \text{E} \\ \text{F} \end{matrix} \end{array}$$

$$(b) C = \begin{array}{cc} & \begin{matrix} \text{A} & \text{B} \end{matrix} \\ \begin{bmatrix} 0.5 & 0.45 \\ 0.4 & 0.35 \\ 0.35 & 0.25 \end{bmatrix} & \begin{matrix} \text{FC} \\ \text{BC} \\ \text{EC} \end{matrix} \end{array} \quad \$/\text{km}$$

$$(c) AC = \begin{array}{cc} & \begin{matrix} \text{A} & \text{B} \end{matrix} \\ \begin{bmatrix} 86 & 69 \\ 66 & 56.5 \end{bmatrix} & \begin{matrix} \text{E} \\ \text{F} \end{matrix} \end{array} \quad \text{dollars}$$

gives the cost of each route for each plane

(d) CA makes no sense as the row labels and the column headings don't match

$$7. (a) \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \quad (b) P'(4, -3)$$

$$(c) \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

8. 11 people took A
15 people took B
16 people took C
Total revenue is \$2495

$$9. (a) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 5 & -2 \\ 2 & 0 \end{bmatrix}$$

$$(c) \begin{bmatrix} \frac{1}{5} & \frac{9}{5} \\ \frac{8}{15} & -\frac{1}{5} \end{bmatrix}$$

$$10. (a) \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$(b) x = 3, y = 2, z = -5$$

$$11. (a) \begin{array}{ccc} & \text{GREY} & \text{ATLAS} & \text{GRIFFON} \\ \begin{bmatrix} 90 & 40 & 70 \\ 100 & 80 & 110 \\ 30 & 60 & 120 \end{bmatrix} & & & \begin{matrix} \text{Winter} \\ \text{Spring} \\ \text{Summer} \end{matrix} \end{array}$$

- (b) A GREY costs \$115
An ATLAS costs \$128
A GRIFFON costs \$147

$$(c) \begin{bmatrix} 99 & 44 & 77 \\ 110 & 88 & 121 \\ 33 & 66 & 132 \end{bmatrix} \quad \$$$

(d) Expected Revenue = \$104280

$$12. (a) C = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(b) D = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$(c) CD = \begin{bmatrix} 4 & -1 \\ 1 & 0 \end{bmatrix}$$

13. (a) you did it!!

$$(b) x = 4, y = 3, z = -2$$

14. (a) $Q = P^{-1}R$

$$(b) Q = \begin{bmatrix} -29 & 47 \\ 22 & -33 \end{bmatrix}$$

$$15. P(\text{restful}) = \frac{3}{13}$$

16. $a = 2$ or $a = -1$

$$17. (a) T = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

$$18. (a) \begin{bmatrix} 12 & 1 \\ -1 & 6 \end{bmatrix}$$

$$(b) \begin{bmatrix} -13 & -1 \\ -1 & -6 \end{bmatrix}$$

(c) Not possible, number of columns (3) \neq number of rows (2)

$$(d) \begin{bmatrix} -53 & 74 & -35 \\ 45 & -40 & 32 \end{bmatrix}$$

$$19. (a) \begin{bmatrix} F & M & B & V & S & E \\ 2 & \frac{1}{2} & 120 & 1 & \frac{3}{4} & 2 \\ 1 & \frac{1}{2} & 30 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{matrix} B \\ T \end{matrix}$$

$$\begin{bmatrix} B & T \\ 28 & 20 \\ 22 & 18 \end{bmatrix} \begin{matrix} M \\ W \end{matrix}$$

Quantities required =

$$\begin{bmatrix} F & M & B & V & S & E \\ 76 & 24 & 3960 & 38 & 31 & 76 \\ 62 & 20 & 3180 & 31 & 25\frac{1}{2} & 62 \end{bmatrix} \begin{matrix} M \\ W \end{matrix}$$

$$(b) \begin{matrix} F \\ M \\ B \\ V \\ S \\ E \end{matrix} \begin{bmatrix} \$ \\ 0.15 \\ 0.30 \\ 3.00 \\ 0.05 \\ 0.15 \\ 0.25 \end{bmatrix} \begin{matrix} \text{Monday } \$56 \\ \text{Wednesday } \$46 \end{matrix}$$

(c) Monday \$172 Wednesday \$142

(d) Profit Monday \$116,
Profit Wednesday \$96

$$20. (a) \begin{matrix} x + z = 42 \text{ at B} \\ x - y - z = 0 \text{ at C} \\ y - z = -10 \text{ at D} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 42 \\ 0 \\ -10 \end{bmatrix}$$

$$(b) x = 24\frac{2}{3} \quad y = 7\frac{1}{3} \quad z = 17\frac{1}{3}$$

(c) $49\frac{1}{3}$ on average per hour.

21. (a) 1760 kg of Urea
3350 kg of phosphate
1490 kg of potash

(b) Need 450 bags of F_1
630 bags of F_2
780 bags of F_3

(c) \$24

$$22. (a) \begin{bmatrix} -15 & -8 \\ 8 & -15 \end{bmatrix} \quad (b) \begin{bmatrix} -5 & 12 \\ -12 & -5 \end{bmatrix}$$

$$(c) \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \quad (d) \begin{bmatrix} -1 & -7 \\ 7 & -1 \end{bmatrix}$$

$$(e) \text{ Both} = \begin{bmatrix} -10 & -20 \\ 20 & -10 \end{bmatrix}$$

(f) No, only when $BA = AB$ which is not always the case.

CHAPTER 7: Complex Numbers

$$1. (a) 24 - 11i$$

$$(b) -5 + 5.5i$$

$$(c) 5 - 12i$$

$$(d) \frac{3}{17} + \frac{5}{17}i$$

$$(e) \sqrt{221}$$

$$(f) -10 - 11i$$

$$2. \sqrt{(2-x)^2 + (3-y)^2} \text{ or } \sqrt{(x-2)^2 + (y-3)^2}$$

For any point $w = x + yi$ the formula gives the distance between w and $z = 2 + 3i$.

$$3. (a) z = 2 - 3i$$

$$(b) z = -1 + 2i$$

$$(c) z = -2 - i$$

$$(d) z = -1 + 3i$$

$$(e) z = \frac{2}{5} - \frac{1}{5}i$$

$$4. z = 5 + 2i$$

$$5. z^2 + 6z + 58 = 0$$

$$6. z = -4 + 19i$$

$$7. z = xi + y \text{ for any } x, y \in \mathbb{R}$$

$$8. z = -3 - i$$

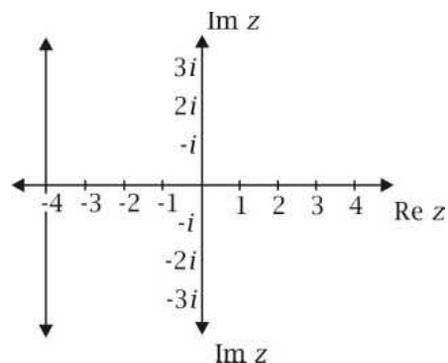
$$9. (x-6)^2 + y^2 = 6^2$$

$$10. a = 1$$

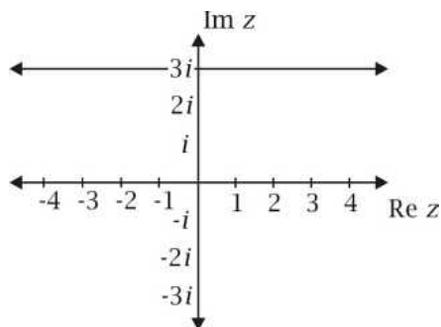
$$b = -1 - i$$

$$c = 10 + 11i$$

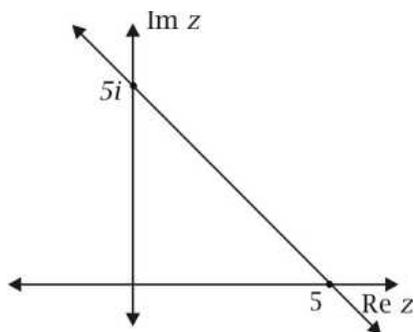
$$11. (a)$$



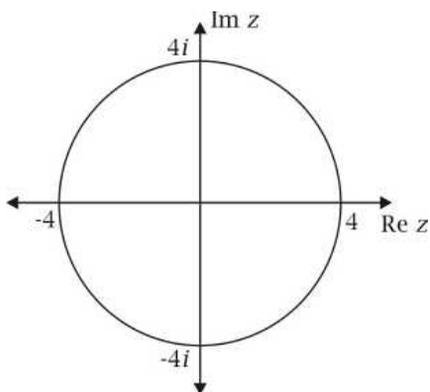
11. (b)
(cont)



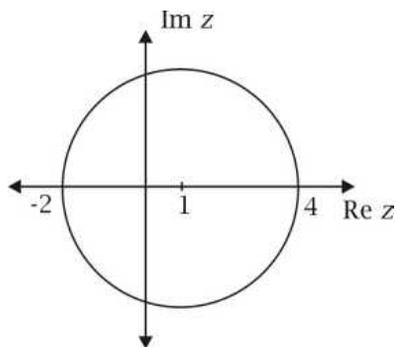
(c)



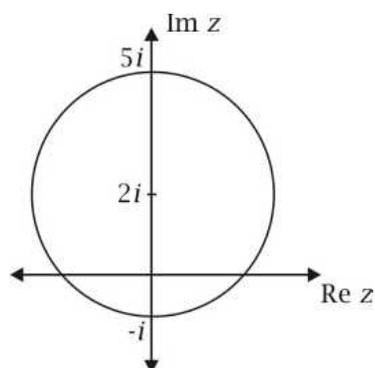
(d)



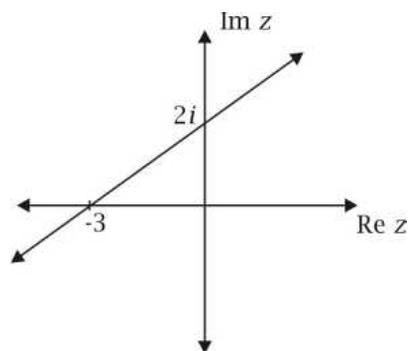
(e)



(f)



(g) $2 \operatorname{Re} z - 3 \operatorname{Im} z = -6$
 $2x - 3y = -6$
 $x = 0 \quad y = 2$
 $y = 0 \quad x = -3$



12. $x + y = -2$

13. $z_1 = 2, z_2 = -1 + \sqrt{3}i, z_3 = -1 - \sqrt{3}i$

14. $f(x) = x^3 - 4x^2 + 6x - 4$

CHAPTER 8: Mathematical Proof and Reasoning

1. So, our RTP statement becomes: that if n is even, then n^2 is odd. Letting $n = 2m \forall m \in \mathbb{Z}^+$ (through the definition of an even number) we substitute into n^2 and obtain $(2m)^2 = 4m^2 = 2(2m^2)$ which is even.

As such, we have a contradiction as we were expecting an odd result. Said another way, we have an odd equaling an even, which is impossible.

Therefore, we have proven that if n is even, then n^2 is even $\forall n \in \mathbb{Z}^+$

2. So, our RTP statement becomes: that if n is odd, then n^2 is even. Letting $n = 2m + 1 \forall m \in \mathbb{Z}^+$ (through the definition of an odd number) we substitute into n^2 and obtain $(2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ which is an even number plus an odd number which will always equal an odd number.

As such, we have a contradiction as we were expecting an even result. Said another way, we have an even equaling an odd, which is impossible.

Therefore, we have proven that if n is odd, then n^2 is odd $\forall n \in \mathbb{Z}^+$

3. For something different, let's reverse the current statement which reads:
 If n^2 is even $\rightarrow n$ is even $\forall n \in \mathbb{Z}^+$
 to: If n is odd $\rightarrow n^2$ is odd $\forall n \in \mathbb{Z}^+$
 This 'reversal' is known as the contrapositive. From here we can use any process to complete the proof.

Therefore, if n^2 is even $\rightarrow n$ is even $\forall n \in \mathbb{Z}^+$

4. Taking the original statement, we have:

$$\exists n \in \mathbb{Z} \text{ s.t. } n^2 \text{ odd} \Rightarrow n \text{ odd}$$

Negating this original statement (and supposing it is true) we obtain:

$$\exists n \in \mathbb{Z} \text{ s.t. } n^2 \text{ odd} \Rightarrow n \text{ even}$$

By definition of evenness, $n = 2k$ where $k \in \mathbb{Z}$. So, by substitution we have $n \cdot n = (2k)(2k) = 2(2k^2)$. Considering that products of integers are integers, we have obtained an even result for $n \cdot n$ (as 2 multiplied by anything is even). The conclusion is since n is even, n^2 is also even. This contradicts the supposition that n^2 is odd. Hence, the supposition is false and the proposition is true.

5. Let's start by asserting that the opposite is true, viz. $\sqrt{2}$ is a rational number. This means that $\sqrt{2}$ can be written as the ratio of two integers, a and b such that

$$\sqrt{2} = \frac{a}{b} \quad \text{Equation 1}$$

From this statement we can assume that a and b have no common factors (if there were any common factors, these could be cancelled in both numerator and denominator).

If we square both sides of Equation 1 we obtain

$$2 = \frac{a^2}{b^2} \quad \text{Equation 2}$$

And rearranging gives

$$2b^2 = a^2 \quad \text{Equation 3}$$

which implies that a^2 is even.

Furthermore, we can deduce that if a^2 is even, then a is even.

We can therefore write $a = 2w$ for some natural number w .

Substituting this value of a into an earlier equation ($2b^2 = a^2$), we can obtain

$$2b^2 = 4w^2$$

Dividing both sides of this equation by 2 gives us

$$b^2 = 2w^2$$

Now since the RHS of this equation is even, so must the LHS. Thus, b^2 is even. Furthermore, it follows from an earlier deduction that if b^2 is even, then b is even.

It has been shown that both a and b are multiples of 2 as they have the common factor of 2.

This contradicts the original assumption that $\frac{a}{b}$ was fully cancelled down (or in other words, the only natural number to divide both a and b is 1).

In summary, $\sqrt{2}$ cannot be written as a fraction and hence $\sqrt{2}$ is irrational.

9. (a) $\frac{2}{3}$

(b) $\frac{7}{9}$

(c) $\frac{1}{6}$

(d) $\frac{1}{11}$

(e) $\frac{5}{12}$

(f) $\frac{3}{7}$

10. (a) $0.8\bar{3}$

(b) $0.\bar{4}$

(c) $0.0\bar{6}$

(d) $0.2\bar{3}$

(e) $0.4\overline{09}$

(f) $0.\overline{384615}$



SOLUTIONS TO TRIAL TESTS

TRIAL TEST 1: Counting Techniques

1. (a) $5! = 120$ ✓

(b) $\boxed{2 \ 3 \ 2 \ 1 \ 1} = 12$ ✓

(c) $2! \times 4! = 48$ ✓

(d) $\frac{72}{120} = \frac{3}{5}$ ✓✓

2. (a) $\binom{10}{6} = 210$ ✓

(b) $\binom{2}{2} \binom{8}{4} = 70$ ✓✓

(c) $\binom{5}{2} \binom{5}{4} = 50$ ✓✓

3. (a) $P(X|Y) = \frac{P(X \cap Y)}{P(Y)}$

$$0.6 = \frac{0.2}{P(Y)} \quad \checkmark$$

$$P(Y) = \frac{1}{3} \quad \checkmark$$

(b) $P(X) = \frac{0.2}{0.4}$
 $= \frac{1}{2}$ ✓

$$P(X \cap Y) = P(X) + P(Y) - P(X \cup Y)$$

$$= \frac{1}{2} + \frac{1}{3} - 0.2 \quad \checkmark$$

$$= \frac{19}{30} \quad \checkmark$$

(c) $P(\bar{X} \cap \bar{Y}) = \frac{11}{30}$ ✓✓

(d) Independent events are when

$$P(X \cap Y) = P(X) \cdot P(Y)$$

$$\therefore 0.2 \neq \frac{1}{2} \times \frac{1}{3} \quad \checkmark$$

\therefore Events X and Y are not independent. ✓

(e) Events are mutually exclusive when $P(X \cap Y) = 0$

As $P(X \cap Y) = 0.2$ events X and Y are **not** mutually exclusive. ✓

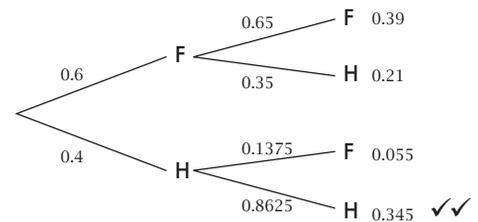
4. (a) $26^5 = 11881376$ ✓

(b) $\boxed{26 \ 25 \ 24 \ 23 \ 22} = 7893600$ ✓

(c) $\boxed{1 \ 24 \ 23 \ 22 \ 1} = 12144$ ✓✓

(d) $\binom{2}{2} \binom{24}{3} \times 5! = 242880$ ✓✓✓

5. (a)



$$P(F|F) \cong 0.8764 \quad \checkmark$$

(b) $P(FF) + P(HH) = (0.6)(0.65) + (0.4)(0.8625)$ ✓
 $= 0.735$ ✓

(c) $P(H | \text{more once on ferry}) = \frac{0.21 + 0.055}{1 - 0.345}$ ✓
 ≈ 0.4046 ✓

6. (a) $P(\text{no Junior}) = \frac{\binom{4}{0} \binom{6}{4}}{\binom{10}{4}}$ ✓
 $= \frac{1}{14}$ ✓

(b) $P(\text{Captain} | \text{No JC}) = \frac{\binom{1}{1} \binom{4}{0} \binom{5}{3}}{\binom{4}{0} \binom{6}{4}}$ ✓
 $= \frac{2}{3}$ ✓

(c) $P(\text{Captain in middle}) = \frac{2 \times 9!}{10!}$ ✓
 $= \frac{1}{5}$ ✓

(d) $P(\text{VC Separate} | \text{Captain not in middle}) = \frac{2 \times 3! \times 6 \times 6!}{10! - 2 \times 9!}$ ✓
 $= \frac{1}{56}$ ✓

TRIAL TEST 2: Vectors I

1.

$$c\mathbf{a} + d\mathbf{b} = -2\mathbf{i} + 11\mathbf{j}$$
$$c(2\mathbf{i} + 3\mathbf{j}) + d(-4\mathbf{i} + \mathbf{j}) = -2\mathbf{i} + 11\mathbf{j} \quad \checkmark$$
$$2c\mathbf{i} + 3c\mathbf{j} - 4d\mathbf{i} + d\mathbf{j} = -2\mathbf{i} + 11\mathbf{j} \quad \checkmark$$
$$(2c - 4d)\mathbf{i} + (3c + d)\mathbf{j} = -2\mathbf{i} + 11\mathbf{j} \quad \checkmark$$

$$2c - 4d = -2 \quad \text{or}$$

$$c - 2d = -1$$

$$c = 2d - 1 \quad 3c + d = 11$$

substitution gives

$$\begin{aligned} 3(2d - 1) + d &= 11 \\ 6d - 3 + d &= 11 \\ 7d &= 14 \\ d &= 2 \checkmark\checkmark \\ c &= 2d - 1 \\ c &= 3 \end{aligned}$$

$\therefore c = 3$ and $d = 2 \checkmark\checkmark$

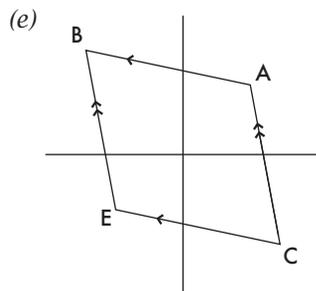
[7]

2. (a) $AB = OB - OA$
 $= \langle -5, 7 \rangle - \langle 1, 5 \rangle \checkmark$
 $= \langle -6, 2 \rangle = -6i + 2j$

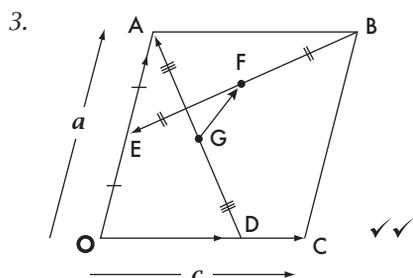
(b) $BC = OC - OB$
 $= |\langle 6, -4 \rangle - \langle -5, 7 \rangle|$
 $= |\langle 11, -11 \rangle| \checkmark$
 $= \sqrt{11^2 + 11^2}$
 $= 11\sqrt{2}$ units \checkmark

(c) $AC = OC - OA$
 $= \langle 6, -4 \rangle - \langle 1, 5 \rangle$
 $= \langle 5, -9 \rangle$ R \checkmark
 $= \langle 10.30, -60.95^\circ \rangle$ P \checkmark

(d) $OD = \frac{1}{2}(OA + OB)$
 $= \frac{1}{2}(\langle 1, 5 \rangle + \langle -5, 7 \rangle) \checkmark$
 $= \frac{1}{2}\langle -4, 12 \rangle$
 $= \langle -2, 6 \rangle \checkmark$



$CE = AB = \langle -6, 2 \rangle$
 $CE = OE - OC = \langle -6, 2 \rangle$
 $OE = \langle -6, 2 \rangle + \langle 6, -4 \rangle$
 $= \langle 0, -2 \rangle$
 $\therefore E$ is $(0, -2) \checkmark\checkmark$



(a) $BE = BA + AE \checkmark$
 $= -c + \frac{1}{2}(-a)$
 $= -c - \frac{1}{2}a \checkmark$

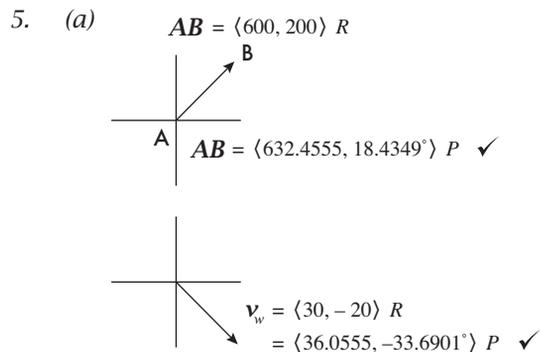
(b) $DA = DO + OA \checkmark$
 $= -\frac{2}{3}c + a$
 $= a - \frac{2}{3}c \checkmark$

(c) $GF + FB + BC + CD + DG = 0$
 $GF = -FB - BC - CD - DG$
 $= BF + CB + DC + GD$
 $= \frac{1}{2}BE + a + \frac{1}{3}c + \frac{1}{2}AD$
 $= \frac{1}{2}(-c - \frac{1}{2}a) + a + \frac{1}{3}c + \frac{1}{2}(-a + \frac{2}{3}c)$
 $= -\frac{1}{2}c - \frac{1}{4}a + a + \frac{1}{3}c - \frac{1}{2}a + \frac{1}{3}c$
 $= a(-\frac{1}{4} + 1 - \frac{1}{2}) + c(-\frac{1}{2} + \frac{1}{3} + \frac{1}{3})$
 $= \frac{1}{4}a + \frac{1}{6}c \checkmark\checkmark\checkmark\checkmark$

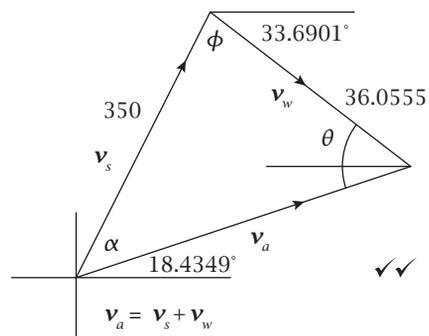
[10]

4. $a = 2i + 3j$
 $|a| = \sqrt{2^2 + 3^2} = \sqrt{13} \checkmark$
 $b = -3i + 4j$
 $= \langle 5, 126.8699^\circ \rangle$ P \checkmark
 $c = \langle 5\sqrt{13}, 126.8699^\circ \rangle$ P
 $= \langle -10.82, 14.42 \rangle \checkmark$

[3]



[9]



$\theta = 33.6901^\circ + 18.4349^\circ$
 $= 52.125^\circ \checkmark$
 $\frac{350}{\sin 52.125^\circ} = \frac{36.0555}{\sin \alpha}$

$$\sin \alpha = \frac{36.0555 \sin 52.125^\circ}{350}$$

$$\alpha = 4.6642^\circ \checkmark \checkmark$$

The plane must be set on a bearing of $90^\circ - 4.6642^\circ - 18.4349^\circ = 066.90^\circ \checkmark$

$$(b) \quad \phi = 180^\circ - 4.6642^\circ - 52.125^\circ \\ = 123.2108^\circ$$

$$\frac{|\mathbf{v}_a|}{\sin 123.2108^\circ} = \frac{350}{\sin 52.125^\circ}$$

$$|\mathbf{v}_a| = 370.9769 \checkmark \checkmark$$

The actual speed of the plane is 370.98 km/hr.

$$(c) \quad s = \frac{d}{t}$$

$$t = \frac{d}{s}$$

$$= \frac{632.4555}{370.9769}$$

$$= 1.7048 \text{ hrs}$$

Time for journey = 1 hr 42 min. \checkmark

[11]

$$6. \quad \mathbf{F}_1 = \langle -50 \cos 70^\circ, -50 \sin 70^\circ \rangle \checkmark$$

$$\mathbf{F}_2 = \langle 35 \cos 50^\circ, 35 \sin 50^\circ \rangle \checkmark$$

$$\mathbf{F}_3 = \langle 70 \cos 40^\circ, -70 \sin 40^\circ \rangle \checkmark$$

$$\mathbf{F}_4 = \langle -60 \cos 40^\circ, 60 \sin 40^\circ \rangle \checkmark$$

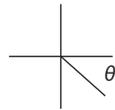
$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$$

$$= \langle 13.057, -26.601 \rangle \checkmark$$

$$\text{Mag } \mathbf{R} = \sqrt{13.057^2 + 26.601^2}$$

$$= 29.63 \text{ N } 2 \text{ dp } \checkmark$$

$$\tan \theta = \frac{26.601}{13.057}$$



$$\theta = 63.86^\circ 2 \text{ dp}$$

$$\text{Bearing} = 63.86^\circ + 90^\circ$$

$$= 153.86^\circ \checkmark$$

So the resultant force is 29.63 N on a bearing of $153.86^\circ \checkmark$

[8]

$$7. \quad \mathbf{AP} : \mathbf{BP} = 5 : -2$$

is the same as

$$\mathbf{AP} : \mathbf{PB} = 5 : 2 \checkmark$$



$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

$$= \langle 11, -13 \rangle - \langle -10, 15 \rangle$$

$$= \langle 21, -28 \rangle \checkmark$$

$$\mathbf{AP} = \frac{5}{7} \langle 21, -28 \rangle$$

$$= \langle 15, -20 \rangle \checkmark$$

$$\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$

$$= \langle -10, 15 \rangle + \langle 15, -20 \rangle$$

$$= \langle 5, -5 \rangle \checkmark$$

$$\therefore P \text{ is } (5, -5) \checkmark$$

[6]

$$8. \quad (a) \quad \mathbf{v}_c = \langle 15, -8 \rangle \quad \mathbf{wv}_c = \langle -12, 10 \rangle$$

$$\mathbf{wv}_c = \mathbf{v}_w - \mathbf{v}_c \checkmark$$

$$\therefore \mathbf{v}_w = \mathbf{wv}_c + \mathbf{v}_c \checkmark$$

$$= \langle -12, 10 \rangle + \langle 15, -8 \rangle$$

$$= \langle 3, 2 \rangle \text{ km/hr } \checkmark$$

The wind has a velocity of

$$3\mathbf{i} + 2\mathbf{j} \text{ km/hr } \checkmark$$

$$(b) \quad \mathbf{wv}_c = \mathbf{v}_w - \mathbf{v}_c$$

$$= \langle -8, 13 \rangle - \langle 15, -8 \rangle$$

$$= \langle -23, 21 \rangle \checkmark$$

The cyclist now experiences the wind

$$\text{velocity as } -23\mathbf{i} + 21\mathbf{j} \text{ km/hr. } \checkmark$$

[6]

TRIAL TEST 3: Vectors II

$$1. \quad (a) \quad 4\mathbf{u} - 3\mathbf{v}$$

$$= 4 \langle -2, 3, 1 \rangle - 3 \langle 3, 1, -5 \rangle$$

$$= \langle -8, 12, 4 \rangle - \langle 9, 3, -15 \rangle$$

$$= \langle -17, 9, 19 \rangle \checkmark \checkmark$$

$$(b) \quad \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}$$

$$= \frac{\langle -2, 3, 1 \rangle \cdot \langle 3, 1, -5 \rangle}{\sqrt{2^2 + 3^2 + 1} \sqrt{3^2 + 1 + 25}} \checkmark$$

$$= \frac{-6 + 3 - 5}{\sqrt{14} \sqrt{35}} \checkmark$$

$$= -0.3614 \dots$$

$$\theta = 111.19^\circ \checkmark$$

$$(c) \quad (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}$$

$$= |\mathbf{u}|^2 - |\mathbf{v}|^2 \checkmark$$

$$= 14 - 35$$

$$= -21 \checkmark$$

$$(d) \quad \mathbf{v} = \langle 3, 1, -5 \rangle \quad \mathbf{i} = \langle 1, 0, 0 \rangle$$

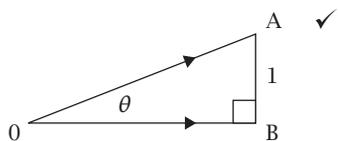
$$\cos \theta = \frac{3}{\sqrt{35}} \quad \theta = 59.53^\circ \checkmark$$

(e) The $x - y$ plane means $z = 0$

Let $\mathbf{u} = \mathbf{OA} = \langle -2, 3, 1 \rangle$ and

$$\mathbf{OB} = \langle -2, 3, 0 \rangle$$

Then $\triangle OBA$ is



$$\sin \theta = \frac{1}{|\mathbf{OA}|} = \frac{1}{\sqrt{14}}$$

$$\theta = 15.50^\circ \checkmark$$

[10]

2. $\langle 6, a, -4 \rangle \cdot \langle -1, 2a, a \rangle = 0$
 $-6 + 2a^2 - 4a = 0 \checkmark$
 $2a^2 - 4a - 6 = 0$
 $a^2 - 2a - 3 = 0$
 $(a - 3)(a + 1) = 0$
 $a = 3$ or $a = -1 \checkmark \checkmark$ [3]

3. (a) $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
 $\mathbf{r} \cdot \langle -1, 5, 3 \rangle = \langle 2, -3, 4 \rangle \cdot \langle -1, 5, 3 \rangle \checkmark$
 $\langle x, y, z \rangle \cdot \langle -1, 5, 3 \rangle = -2 - 15 + 12$
 $-x + 5y + 3z = -5$
 $x - 5y - 3z = 5 \checkmark$

(b) $\mathbf{r} = \langle 16, -17, -8 \rangle + t \langle -2, 3, 1 \rangle$
 $= \langle 16 - 2t, 3t - 17, t - 8 \rangle \checkmark$

(c) $x = 16 - 2t, y = 3t - 17, z = t - 8$

$$x - 5y - 3z = 5$$

$$16 - 2t - 5(3t - 17) - 3(t - 8) = 5 \checkmark$$

$$16 - 2t - 15t + 85 - 3t + 24 = 5$$

$$-20t = -120$$

$$t = 6 \checkmark$$

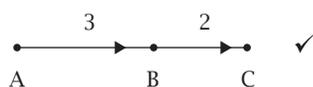
$$\therefore x = 16 - 2(6) = 4$$

$$y = 3(6) - 17 = 1$$

$$z = 6 - 8 = -2$$

The point of intersection is $(4, 1, -2) \checkmark$ [6]

4. $AB : AC = 3 : 5$



$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$$

$$= \langle -1, 15, 5 \rangle - \langle 2, 3, -1 \rangle$$

$$= \langle -3, 12, 6 \rangle \checkmark$$

$$\frac{1}{3} \mathbf{AB} = \langle -1, 4, 2 \rangle \checkmark$$

$$\mathbf{BC} = 2 \langle -1, 4, 2 \rangle$$

$$= \langle -2, 8, 4 \rangle \checkmark$$

$$\mathbf{OC} = \mathbf{OB} + \mathbf{BC}$$

$$= \langle -1, 15, 5 \rangle + \langle -2, 8, 4 \rangle \checkmark$$

$$= \langle -3, 23, 9 \rangle$$

$$\therefore C \text{ is } \langle -3, 23, 9 \rangle \checkmark$$

[6]

5. $\mathbf{AB} = \langle 1, -3, 10 \rangle - \langle 3, 2, 6 \rangle$
 $= \langle -2, -5, 4 \rangle \checkmark$

$$\mathbf{AC} = \langle 10, 0, 5 \rangle - \langle 3, 2, 6 \rangle$$

$$= \langle 7, -2, -1 \rangle \checkmark$$

$$\mathbf{AB} \cdot \mathbf{n} = 0, \quad \langle -2, -5, 4 \rangle \cdot \langle 1, p, q \rangle = 0$$

$$-2 - 5p + 4q = 0$$

$$-5p + 4q = 2 \quad (1) \checkmark$$

$$\mathbf{AC} \cdot \mathbf{n} = 0, \quad \langle 7, -2, -1 \rangle \cdot \langle 1, p, q \rangle = 0$$

$$7 - 2p - q = 0$$

$$2p + q = 7 \quad (2) \checkmark$$

Solving (1) and (2) gives

$$p = 2, \quad q = 3 \quad \checkmark$$

$$\therefore \mathbf{n} = \langle 1, 2, 3 \rangle$$

$$\mathbf{r} \cdot \langle 1, 2, 3 \rangle = \langle 3, 2, 6 \rangle \cdot \langle 1, 2, 3 \rangle \checkmark$$

$$\therefore x + 2y + 3z = 25 \quad \text{is the equation of the plane} \checkmark \quad [7]$$

6. (a) let J = jet and P = plane
 At 2pm let $t = 0$ and the initial displacement \mathbf{JP} is

$$\mathbf{JP} = \mathbf{OP} - \mathbf{OJ}$$

$$= \langle -238, 460, 4.52 \rangle - \langle 50, -20, 3.8 \rangle$$

$$= \langle -288, 480, 0.72 \rangle \checkmark$$

$$\text{also } {}_j\mathbf{v}_p = \mathbf{v}_j - \mathbf{v}_p$$

$$= \langle -200, 150, 0.5 \rangle - \langle -80, -50, 0.2 \rangle$$

$$= \langle -120, 200, 0.3 \rangle \checkmark$$

For collision $\mathbf{JP} = t {}_j\mathbf{v}_p$

$$\text{i.e. } \langle -288, 480, 0.72 \rangle = t \langle -120, 200, 0.3 \rangle \checkmark$$

$$-288 = t(-120) \quad 480 = 200t$$

$$t = \frac{-288}{-120} \quad t = \frac{480}{200}$$

$$t = 2.4 \quad t = 2.4$$

$$0.72 = t(0.3)$$

$$t = \frac{0.72}{0.3} \quad \checkmark \checkmark$$

$$t = 2.4$$

So $\mathbf{JP} = 2.4 {}_j\mathbf{v}_p$ which means that the aircraft do collide at 4.24pm. \checkmark

$$\mathbf{r}_j = \langle 50, -20, 3.8 \rangle + 2.4 \langle -200, 150, 0.5 \rangle$$

$$= \langle -430, 340, 5 \rangle$$

So the aircraft would collide at $(-430, 340, 5)$ km \checkmark

- (b) at $t = 2$

$$\mathbf{r}_j = \langle 50, -20, 3.8 \rangle + 2 \langle -200, 150, 0.5 \rangle$$

$$= \langle -350, 280, 4.8 \rangle \text{ km } \checkmark \quad \text{and}$$

$$\mathbf{r}_p = \langle -238, 460, 4.52 \rangle + 2 \langle -80, -50, 0.2 \rangle$$

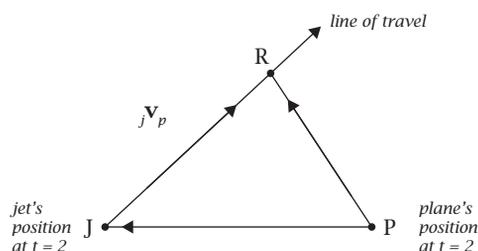
$$= \langle -398, 360, 4.92 \rangle \text{ km } \checkmark$$

Consider that the plane is at rest, so find

$${}_j\mathbf{v}_p = \mathbf{v}_j - \mathbf{v}_p$$

$$= \langle -220, 130, 0.8 \rangle - \langle -80, -50, 0.2 \rangle$$

$$= \langle -140, 180, 0.6 \rangle \checkmark$$



$$\mathbf{PJ} = \mathbf{OJ} - \mathbf{OP}$$

$$= \langle -350, 280, 4.8 \rangle - \langle -398, 360, 4.92 \rangle$$

$$= \langle 48, -80, -0.12 \rangle \checkmark$$

$$\begin{aligned} \mathbf{PR} &= \mathbf{PJ} + j\mathbf{V}_p \\ &= \langle 48, -80, -0.12 \rangle + t \langle -140, 180, 0.6 \rangle \checkmark \\ &= \langle 48 - 140t, 180t - 80, 0.6t - 0.12 \rangle \checkmark \end{aligned}$$

$$\text{Let } \mathbf{PR} \cdot j\mathbf{V}_p = 0$$

$$\langle 48 - 140t, 180t - 80, 0.6t - 0.12 \rangle \cdot \langle -140, 180, 0.6 \rangle = 0 \checkmark$$

$$\begin{aligned} -140(48 - 140t) + 180(180t - 80) \\ + 0.6(0.6t - 0.12) &= 0 \checkmark \end{aligned}$$

$$\begin{aligned} -6720 + 19600t + 32400t - 14400 \\ + 0.36t - 0.072 &= 0 \end{aligned}$$

$$52000.36t = 21120.072$$

$$\begin{aligned} t &= \frac{21120.072}{52000.36} \\ &= 0.4062 \checkmark \checkmark \end{aligned}$$

$$\text{at } t = 0.4062$$

$$\begin{aligned} |\mathbf{PR}| &= \sqrt{(-8.868)^2 + (-6.884)^2 + (0.1237)^2} \\ &= 11.23 \checkmark \end{aligned}$$

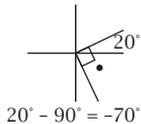
So the closest distance that the aircraft now come to each other is 11.23 km.

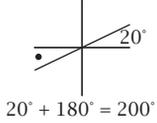
[18]

TRIAL TEST 4: Trigonometry

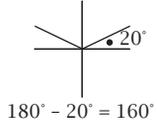
$$\begin{aligned} 1. (a) \quad y &= 2\sin^2 x + 2\cos^2 x \\ &= 2(\sin^2 x + \cos^2 x) \\ &= 2.1 \\ &= 2 \checkmark \end{aligned}$$

$$\begin{aligned} (b) \quad y &= \cos(-x) \\ &= \cos x \checkmark \end{aligned}$$

$$\begin{aligned} (c) \quad y &= \sin\left(x - \frac{\pi}{2}\right) \\ y &= -\cos x \checkmark \checkmark \end{aligned}$$


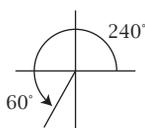
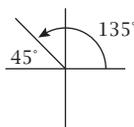
$$\begin{aligned} (d) \quad y &= \cos(x + \pi) \\ y &= -\cos x \checkmark \checkmark \end{aligned}$$


$$\begin{aligned} (e) \quad y &= \tan(-x) \\ y &= -\tan x \checkmark \end{aligned}$$

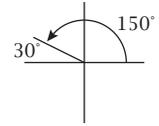
$$\begin{aligned} (f) \quad y &= \tan(\pi - x) \\ &= -\tan x \checkmark \checkmark \end{aligned}$$


$$\begin{aligned} 2. (a) \quad \sin \frac{3\pi}{4} \\ &= \sin 135^\circ \\ &= \sin 45^\circ \checkmark \\ &= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \checkmark \end{aligned}$$

$$\begin{aligned} (b) \quad \tan \frac{4\pi}{3} \\ &= \tan 240^\circ \checkmark \\ &= \tan 60^\circ \\ &= \sqrt{3} \checkmark \end{aligned}$$



$$\begin{aligned} (c) \quad \cos \frac{5\pi}{6} \\ &= \cos 150^\circ \checkmark \\ &= -\cos 30^\circ \\ &= -\frac{\sqrt{3}}{2} \checkmark \end{aligned}$$



[6]

$$3. (a) \quad y = -3\cos\left(2x + \frac{\pi}{3}\right)$$

$$y = -3\cos 2\left(x + \frac{\pi}{6}\right)$$

$$\text{Amplitude} = |-3| = 3 \checkmark$$

$$\text{Period} = \frac{2\pi}{2} = \pi \checkmark$$

$$\text{Vertical shift} = 0 \checkmark$$

$$\text{Horizontal shift} = \frac{\pi}{6} \text{ left. } \checkmark$$

$$(b) \quad y = 2\tan\left(\frac{\pi}{4} - x\right) + 1$$

$$y = 2\tan\left(-x + \frac{\pi}{4}\right) + 1$$

$$y = 2\tan\left(-\left(x - \frac{\pi}{4}\right)\right) + 1$$

$$y = -2\tan\left(x - \frac{\pi}{4}\right) + 1 \checkmark \checkmark$$

Amplitude is not defined but the basic tan graph is inverted. \checkmark

$$\text{Period} = \frac{\pi}{1} = \pi \checkmark$$

$$\text{Vertical shift} = 1 \text{ up } \checkmark$$

$$\text{Horizontal shift} = \frac{\pi}{4} \text{ right. } \checkmark$$

$$(c) \quad y = 4\sin\left(3x - \frac{\pi}{2}\right) - 2$$

$$y = 4\sin 3\left(x - \frac{\pi}{6}\right) - 2$$

$$\text{Amplitude} = 4 \checkmark$$

$$\text{Period} = \frac{2\pi}{3} \checkmark$$

$$\text{Vertical shift} = 2 \text{ down } \checkmark$$

$$\text{Horizontal shift} = \frac{\pi}{6} \text{ right. } \checkmark$$

[14]

$$[9] \quad 4. (a) \quad \sin x = -0.3 \quad -\pi \leq x \leq \pi$$

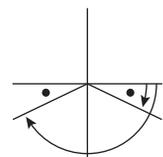
$$RA = \text{inv sin } 0.3 \checkmark$$

$$= 0.3047^R$$

$$x_1 = -0.30^R$$

$$x_2 = -(\pi - 0.3047) \checkmark$$

$$= -2.84^R \checkmark$$



(b) $\cos x = 0.5 \quad -2\pi \leq x \leq 0$

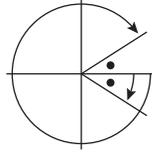
$RA = \text{inv} \cos 0.5$

$= \frac{\pi}{3} \checkmark$

$x_1 = -\frac{\pi}{3} (-1.0472) \checkmark$

$x_2 = -(2\pi - \frac{\pi}{3})$

$= -\frac{5\pi}{3} (-5.2360) \checkmark$



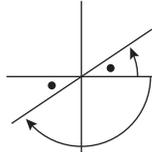
(c) $\tan x = 1.7 \quad -\frac{3\pi}{2} \leq x \leq \frac{\pi}{2}$

$RA = \text{inv} \tan 1.7 = 1.0391 \checkmark$

$x_1 = 1.04 \checkmark$

$x_2 = -(\pi - 1.0391)$

$= -2.10 \checkmark$



[9]

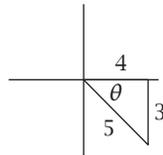
5. $\tan \theta = -\frac{3}{4}$

$\sin \theta < 0$

$\sin 2\theta = 2 \sin \theta \cos \theta$

$= 2 \cdot -\frac{3}{5} \cdot \frac{4}{5} \checkmark$

$\sin 2\theta = -\frac{24}{25} \checkmark$

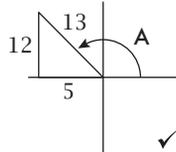


$\sin \theta = -\frac{3}{5} \checkmark$

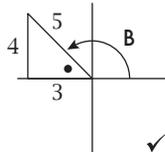
$\cos \theta = \frac{4}{5} \checkmark$

[4]

6. $\tan A = -\frac{12}{5}, \sin A > 0$



$\cos B = -\frac{3}{5}, \tan B < 0$



$\tan(A + B) \tan(A - B)$

$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \cdot \frac{\tan A - \tan B}{1 + \tan A \tan B} \checkmark$

$= \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B} \checkmark \quad \left(\tan B = -\frac{4}{3} \right)$

$= \frac{\left(-\frac{12}{5}\right)^2 - \left(-\frac{4}{3}\right)^2}{1 - \left(-\frac{12}{5}\right)^2 \times \left(-\frac{4}{3}\right)^2} \checkmark$

$= \frac{\frac{144}{25} - \frac{16}{9}}{1 - \frac{144}{25} \times \frac{16}{9}} \checkmark \quad \left(\times \text{top and bottom by } 9 \times 25 \right)$

$= \frac{144 \times 9 - 16 \times 25}{225 - 2304} \checkmark$

$= -\frac{896}{2079} \checkmark$

[8]

7. (a) $\tan \theta + \frac{\cos \theta}{1 + \sin \theta} = \frac{1}{\cos \theta}$

LHS = $\tan \theta + \frac{\cos \theta}{1 + \sin \theta}$

$= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} \checkmark$

$= \frac{\sin \theta (1 + \sin \theta) + \cos^2 \theta}{\cos \theta (1 + \sin \theta)} \checkmark$

$= \frac{\sin \theta + \sin^2 \theta + \cos^2 \theta}{\cos \theta (1 + \sin \theta)} \checkmark$

$= \frac{\sin \theta + 1}{\cos \theta (1 + \sin \theta)}$

$= \frac{1}{\cos \theta} \checkmark$

= RHS Proved

(b) $\frac{\tan^2 x - 1}{\sin x + \cos x} = \frac{\sin x - \cos x}{\cos^2 x}$

LHS = $\frac{\tan^2 x - 1}{\sin x + \cos x}$

$= \frac{\frac{\sin^2 x}{\cos^2 x} - 1}{\sin x + \cos x} \checkmark$

$= \frac{\sin^2 x - \cos^2 x}{\cos^2 x (\sin x + \cos x)} \checkmark$

$= \frac{(\sin x + \cos x)(\sin x - \cos x)}{\cos^2 x (\sin x + \cos x)} \div (\sin x + \cos x) \checkmark$

$= \frac{(\sin x - \cos x)(\sin x - \cos x)}{\cos^2 x} \times \frac{1}{(\sin x + \cos x)}$

$= \frac{\sin x - \cos x}{\cos^2 x} \checkmark$

= RHS Proved

[8]

8. (a) $2 \cos \theta + 3 \sin \theta$

$= R \sin(\theta + \alpha)$

$= R(\sin \theta \cos \alpha + \sin \alpha \cos \theta)$

$= R \sin \alpha \cos \theta + R \cos \alpha \sin \theta$

$\therefore R \sin \alpha = 2 \quad R \cos \alpha = 3$

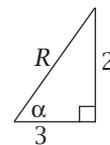
$\sin \alpha = \frac{2}{R} \quad \cos \alpha = \frac{3}{R}$

$\tan \alpha = \frac{2}{3}$

$\alpha = 33.69^\circ \checkmark \checkmark$

$R = \sqrt{2^2 + 3^2}$

$= \sqrt{13} \checkmark \checkmark$



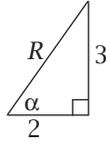
$\therefore 2 \cos \theta + 3 \sin \theta = \sqrt{13} \sin(\theta + 33.69^\circ)$

$$\begin{aligned}
 (b) \quad & 2 \cos \theta + 3 \sin \theta \\
 & = R \cos(\theta - \alpha) \\
 & = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha \\
 \therefore & R \cos \alpha = 2 \quad R \sin \alpha = 3
 \end{aligned}$$

$$\tan \alpha = \frac{3}{2}$$

$$\alpha = 56.31^\circ \checkmark \checkmark$$

$$\begin{aligned}
 R &= \sqrt{2^2 + 3^2} \\
 &= \sqrt{13} \quad \checkmark
 \end{aligned}$$



$$\therefore 2 \cos \theta + 3 \sin \theta = \sqrt{13} \cos(\theta - 56.31^\circ)$$

[7]

TRIAL TEST 5: Matrices

$$1. (a) \quad A^2 = \begin{bmatrix} -2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -1 & 4 \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} 1 & 6 \\ -2 & 13 \end{bmatrix} \checkmark$$

(b) Not possible because \checkmark

$$\begin{array}{ccc}
 A & \times & B \\
 2 \times 2 & & 3 \times 2 \quad \checkmark \\
 \uparrow & & \uparrow \\
 & \text{Not the same} &
 \end{array}$$

(c) $B(3A - I)$

$$= \begin{bmatrix} -17 & 29 \\ 9 & 17 \\ 1 & 13 \end{bmatrix} \checkmark \checkmark$$

(d) $A^2 - 2A = -kI$

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = -kI \checkmark$$

$$\therefore k = -5 \checkmark$$

[8]

$$2. \quad A^2 - 3A + I = 0$$

$$A^{-1}(A^2 - 3A + I) = A^{-1} \times 0 \checkmark$$

$$A - 3I + A^{-1} = 0 \checkmark$$

$$\therefore A^{-1} = 3I - A \checkmark \text{ as required [3]}$$

3. The system of equations suggests that it is in the form $AX = C$. So try BA which gives I i.e. $A^{-1} = B$. $\checkmark \checkmark$

$$\text{So } \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} \checkmark$$

$$AX = C$$

$$X = A^{-1}C$$

$$X = BC \checkmark$$

$$\therefore X = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \checkmark$$

$$\text{so } x = -3, y = 1 \text{ and } z = 2$$

[5]

4. (a) $B^2 = 6I$

$$\begin{bmatrix} 4 & b \\ c & -4 \end{bmatrix} \begin{bmatrix} 4 & b \\ c & -4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \checkmark$$

$$16 + bc = 6$$

$$bc = -10 \checkmark$$

(b) $b = 5, c = -2$

$$\therefore B = \begin{bmatrix} 4 & 5 \\ -2 & 4 \end{bmatrix} \checkmark$$

$$A^2 - 2A = 5I$$

$$A^2 - 2A + I = 6I \checkmark$$

$$(A - I)^2 = 6I$$

$$A - I = \pm B \checkmark$$

$$A = B + I \text{ or } A = I - B$$

$$\therefore A = \begin{bmatrix} 4 & 5 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -2 & -3 \end{bmatrix} \checkmark$$

$$\text{or } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix} \checkmark$$

(c) because $bc = -10$ has an infinite number of solutions for b and c , the equation $A^2 - 2A = 5I$ will also have infinite solutions $\checkmark \checkmark$

[9]

$$5. (a) P^T = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} UT \\ DT \end{matrix} & \begin{bmatrix} 5 & 3 & 4 & 7 \\ 2 & 6 & 9 & 3 \end{bmatrix} \end{matrix}$$

$$Q = \begin{matrix} & \begin{matrix} CP & SP \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 500 & 800 \\ 1100 & 1600 \\ 900 & 1300 \\ 2000 & 3000 \end{bmatrix} \end{matrix} \quad \checkmark$$

$$P^T Q = R = \begin{matrix} & \begin{matrix} CP & SP \end{matrix} \\ \begin{matrix} UT \\ DT \end{matrix} & \begin{bmatrix} 23400 & 35000 \\ 21700 & 31900 \end{bmatrix} \end{matrix} \quad \$ \checkmark$$

The matrix R shows the total cost price and total price of all the computers at each shop. \checkmark

$$(b) \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{matrix} \begin{matrix} CP & SP \end{matrix} \\ \begin{bmatrix} 23400 & 35000 \\ 21700 & 31900 \end{bmatrix} \end{matrix} \begin{matrix} UT \\ DT \end{matrix} \quad \checkmark$$

$$= \begin{bmatrix} 45100 & 66900 \end{bmatrix} \quad \$ \checkmark$$

$$(c) R \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{matrix} \begin{matrix} CP & SP \end{matrix} \\ \begin{bmatrix} 23400 & 35000 \\ 21700 & 31900 \end{bmatrix} \end{matrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \checkmark$$

$$= \begin{matrix} \text{Profit} \\ \begin{matrix} UT \\ DT \end{matrix} \end{matrix} \begin{bmatrix} 11600 \\ 10200 \end{bmatrix} \quad \$ \checkmark$$

$$(d) \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 11600 \\ 10200 \end{bmatrix} \quad \$ \checkmark$$

$$= \begin{bmatrix} 11600 + 10200 \end{bmatrix} \quad \$$$

= \$21800 is the total profit from selling all of the computers from both stores. \checkmark [9]

$$6. (a) R = \begin{matrix} & \begin{matrix} E & \bar{E} \end{matrix} \\ \begin{matrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{matrix} & \begin{bmatrix} E \\ \bar{E} \end{bmatrix} \end{matrix} \quad \checkmark$$

$$(b) \text{ For Saturday } R^5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \checkmark$$

$$= \begin{bmatrix} 0.67008 \\ 0.32992 \end{bmatrix} \quad \checkmark$$

$$(c) \lim_{n \rightarrow \infty} R^n = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \checkmark \quad \therefore \mathbf{s} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \begin{matrix} E \\ \bar{E} \end{matrix} \quad \checkmark$$

$$(d) R \mathbf{s} = \mathbf{s} \quad \text{let } \mathbf{s} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \checkmark$$

$$\begin{aligned} 0.8x + 0.4y &= x \\ 0.2x + 0.6y &= y \end{aligned}$$

$$\text{both give } 0.4y = 0.2x \quad \checkmark$$

$$\text{i.e. } x = 2y$$

$$x + y = 1$$

$$3y = 1$$

$$y = \frac{1}{3} \quad \checkmark$$

$$x = \frac{2}{3} \quad \checkmark$$

$$\therefore \mathbf{s} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \quad [9]$$

$$7. (a) A^3 = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \checkmark \checkmark$$

$$= \begin{bmatrix} 0 & -1 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 33 \end{bmatrix} \quad \checkmark$$

$$(b) T = \begin{bmatrix} 0 & -1 \\ 3 & 12 \end{bmatrix} \quad T^{-1} = \frac{1}{3} \begin{bmatrix} 12 & 1 \\ -3 & 0 \end{bmatrix} \quad \checkmark$$

$$= \begin{bmatrix} 4 & \frac{1}{3} \\ -1 & 0 \end{bmatrix} \quad \checkmark$$

$$(c) \text{ Area} = |\det T| \times \text{Area Rect} \quad \checkmark$$

$$= 3 \times (3 \times 2) = 18 \text{ square units} \quad \checkmark$$

[7]

TRIAL TEST 6: Complex Numbers

$$\begin{aligned}
 1. \quad (a) \quad zw &= (-2 + 5i)(6 - 3i) \\
 &= -12 + 6i + 30i + 15 \quad \checkmark \\
 &= 3 + 36i \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad 3w - 4z &= 3(6 - 3i) - 4(-2 + 5i) \\
 &= 18 - 9i + 8 - 20i \quad \checkmark \\
 &= 26 - 29i \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad (\bar{w})^2 &= (\overline{6 - 3i})^2 \\
 &= (6 + 3i)^2 \quad \checkmark \\
 &= 36 + 36i - 9 \quad \checkmark \\
 &= 27 + 36i \quad \checkmark
 \end{aligned}$$

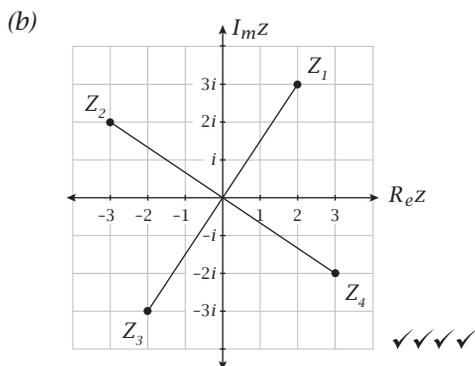
$$\begin{aligned}
 (d) \quad |z| &= |-2 + 5i| \\
 &= \sqrt{(-2)^2 + 5^2} \\
 &= \sqrt{29} \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 (e) \quad \frac{w}{z} &= \frac{6 - 3i}{-2 + 5i} \times \frac{-2 - 5i}{-2 - 5i} \quad \checkmark \\
 &= \frac{-12 - 30i + 6i - 15}{4 + 25} \quad \checkmark \\
 &= \frac{-27 - 24i}{29} \quad \checkmark \\
 &= \frac{-27}{29} - \frac{24i}{29} \quad \checkmark
 \end{aligned} \quad [12]$$

$$\begin{aligned}
 2. \quad z^2 + 10z + 41 &= 0 \\
 z^2 + 10z + 25 + 16 &= 0 \quad \checkmark \\
 (z + 5)^2 &= -16 \quad \checkmark \\
 z + 5 &= \pm 4i \quad \checkmark \\
 z &= -5 \pm 4i \quad \checkmark
 \end{aligned} \quad [4]$$

$$\begin{aligned}
 3. \quad z = 7 \pm 2i \text{ are the roots} \\
 (z - (7 + 2i))(z - (7 - 2i)) &= 0 \quad \checkmark \\
 (z - 7 - 2i)(z - 7 + 2i) &= 0 \quad \checkmark \\
 (z - 7)^2 + 4 &= 0 \quad \checkmark \\
 z^2 - 14z + 49 + 4 &= 0 \\
 z^2 - 14z + 53 &= 0 \quad \checkmark
 \end{aligned} \quad [4]$$

$$\begin{aligned}
 4. \quad (a) \quad Z_1 &= 2 + 3i & Z_4 &= i(-2 - 3i) \\
 Z_2 &= i(2 + 3i) & &= 3 - 2i \quad \checkmark \\
 &= -3 + 2i \quad \checkmark & Z_5 &= i(3 - 2i) \\
 Z_3 &= i(-3 + 2i) & &= 2 + 3i \quad \checkmark \\
 &= -2 - 3i \quad \checkmark & &
 \end{aligned}$$



(c) When a complex number is multiplied by i it rotates the complex number anticlockwise by 90° ✓
✓ [10]

TRIAL TEST 7: Mathematical Proof and Reasoning

1. True ✓✓ [2]

$$4 = 2 + 2$$

$$6 = 3 + 3$$

$$8 = 5 + 3$$

$$10 = 7 + 3$$

$$12 = 7 + 5$$

$$\begin{aligned}
 2. \quad \text{L.H.S.} &= \frac{1 - \tan \theta}{1 + \tan \theta} \\
 &= \frac{1 - \frac{\sin \theta}{\cos \theta} \times \cos \theta}{1 + \frac{\sin \theta}{\cos \theta} \times \cos \theta} \quad \checkmark \\
 &= \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \times \frac{\cos \theta - \sin \theta}{\cos \theta - \sin \theta} \quad \checkmark \\
 &= \frac{\cos^2 \theta - 2\sin \theta \cos \theta + \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} \quad \checkmark \\
 &= \frac{1 - 2\sin \theta \cos \theta}{\cos 2\theta} \quad \checkmark \\
 &= \frac{1 - \sin 2\theta}{\cos 2\theta} \quad \checkmark \\
 &= \text{R.H.S. Proved}
 \end{aligned} \quad [8]$$

3. Let the square number be n^2
Case 1. Let $n = 3p$ then $n^2 = (3p)^2 \checkmark$
 $= 9p^2$
which is a multiple of 3 so it has a remainder of 0 when divided by 3 ✓

Case 2. Let $n = 3p + 1$ then

$$\begin{aligned}
 n^2 &= (3p + 1)^2 \\
 &= 9p^2 + 6p + 1 \quad \checkmark \\
 &= 3(3p^2 + 2p) + 1
 \end{aligned}$$

so $\frac{n^2}{3} = 3p^2 + 2p$ remainder 1 ✓

Case 3. Let $n = 3p + 2$

then $n^2 = (3p + 2)^2 \checkmark$

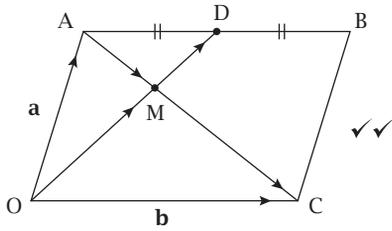
$$\begin{aligned}
 &= 9p^2 + 12p + 4 \\
 &= 9p^2 + 12p + 3 + 1 \\
 &= 3(3p^2 + 4p + 1) + 1 \quad \checkmark
 \end{aligned}$$

$\therefore \frac{n^2}{3} = 3p^2 + 4p + 1$ remainder 1 ✓

Because all whole numbers n can be written as either $3p$ or $3p + 1$ or $3p + 2$ the 3 cases above prove the statement by exhaustion. ✓

[8]

4.



Let $AM = hAC$ and $OM = kOD$ ✓

$$OA + AM = OM \quad \checkmark$$

$$a + hAC = kOD \quad \checkmark$$

$$a + h(b - a) = k\left(a + \frac{1}{2}b\right) \quad \checkmark$$

$$a + hb - ha = ka + \frac{k}{2}b \quad \checkmark$$

$$a(1 - h) + hb = ka + \frac{k}{2}b \quad \checkmark$$

$$1 - h = k \quad h = \frac{k}{2} \quad \checkmark$$

$$1 - \frac{k}{2} = k \quad \checkmark$$

$$1 = \frac{3k}{2} \quad \checkmark$$

$$k = \frac{2}{3} \quad \checkmark$$

$$h = \frac{1}{2}k \quad \checkmark$$

$$= \frac{1}{2} \times \frac{2}{3} \quad \checkmark$$

$$= \frac{1}{3} \quad \checkmark$$

$$\therefore AM = \frac{1}{3}AC \quad \checkmark$$

and hence OD cuts AC (at M) at the point of trisection of AC . [9]

5. Assume that $\log_{10}2$ can be written as $\frac{a}{b}$ where a and b are both counting numbers ✓

$$\text{Then } \log_{10}2 = \frac{a}{b}$$

$$10^{\frac{a}{b}} = 2 \quad \checkmark$$

$$10^a = 2^b$$

$$(2 \times 5)^a = 2^b$$

$$2^a \times 5^a = 2^b$$

$$5^a = 2^{b-a} \quad \checkmark$$

But L.H.S. always ends in a 5 i.e. is odd ✓ for any counting number a and R.H.S. is even for $b > a$ and fractional for $b < a$ ✓

∴ There are no counting numbers for which $5^a = 2^{b-a}$

∴ $\log_{10}2$ cannot be written as $\frac{a}{b}$ ✓

where a and b are counting numbers

and hence $\log_{10}2$ is irrational. [6]

6. $T_n = 3n - 2$

$$T_1 = 3(1) - 2$$

$$= 1 \text{ is true } \checkmark$$

Assume true for $n = k$

$$1 + 4 + 7 + 10 + \dots + (3k - 2) = \frac{k(3k - 1)}{2} \quad \checkmark$$

$$T_{k+1} = 3(k + 1) - 2$$

$$= 3k + 3 - 2$$

$$= 3k + 1 \quad \checkmark$$

add $3k + 1$ to both sides

$$1 + 4 + 7 + 10 + \dots + (3k - 2) + (3k + 1)$$

$$= \frac{k(3k - 1)}{2} + 3k + 1 \quad \checkmark$$

$$\text{R.H.S.} = \frac{k(3k - 1)}{2} + \frac{6k + 2}{2} \quad \checkmark$$

$$= \frac{3k^2 - k + 6k + 2}{2} \quad \checkmark$$

$$= \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{(3k + 2)(k + 1)}{2} \quad \checkmark$$

$$= \frac{(k + 1)(3(k + 1) - 1)}{2} \quad \checkmark$$

$$\text{which is } \frac{k(3k - 1)}{2} \quad \checkmark$$

with k replaced by $k + 1$ So because it is true for $n = 2$ and then 3 etc. for ever and so the proof is complete by induction ✓

[10]

7.

$$T_n = \frac{1}{n(n + 1)}$$

$$T_1 = \frac{1}{1(1 + 1)}$$

$$= \frac{1}{1.2} \text{ true } \checkmark$$

Assume true for $n = k$

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k + 1)} = \frac{k}{k + 1}$$

$$T_{k+1} = \frac{1}{(k + 1)(k + 1 + 1)}$$

$$= \frac{1}{(k + 1)(k + 2)} \quad \checkmark$$

Add $\frac{1}{(k + 1)(k + 2)}$ to both sides ✓

$$\text{R.H.S.} = \frac{k}{k + 1} + \frac{1}{(k + 1)(k + 2)} \quad \checkmark$$

$$= \frac{k(k + 2) + 1}{(k + 1)(k + 2)} \quad \checkmark$$

$$= \frac{k^2 + 2k + 1}{(k + 1)(k + 2)} \quad \checkmark$$

$$= \frac{(k + 1)^2}{(k + 1)(k + 2)} \quad \checkmark$$

$$= \frac{k + 1}{k + 2} \quad \checkmark$$

$$= \frac{k + 1}{k + 1 + 1} \text{ which is } \frac{k}{k + 1} \quad \checkmark$$

with k replaced by $k + 1$

∴ the statement is proved by mathematical induction ✓

[10]

8. **To prove:** PQRS is a cyclic quadrilateral 9.

Proof:

Let $\angle RQS = x^\circ$

Let $\angle QRP = y^\circ$ ✓

$\angle QSP = x^\circ$ (alternate angles)

$\angle QPR = x^\circ$ (alternate segment) ✓

$\angle RPS = y^\circ$ (alternate angles)

$\angle RSQ = y^\circ$ (alternate segment) ✓

$\therefore \angle RSP = (x + y)^\circ$

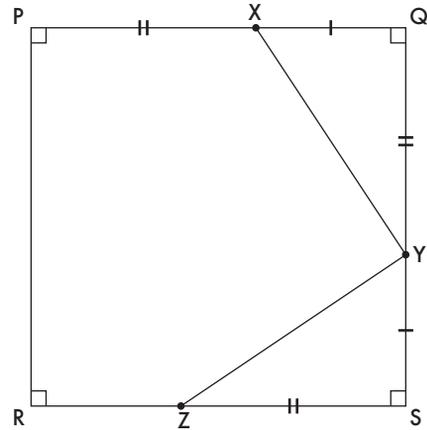
$\angle QPS = (x + y)^\circ$ ✓

$\angle QRS + \angle PSR = 180^\circ$ (cointerior angles)

$\therefore \angle QRS + \angle QPS = 180^\circ$ ✓

\therefore a cyclic quadrilateral. ✓

[6]



Given:
 • Square PQRS where X lies on PQ, Y lies on QS and Z lies on RS.

• $PX = QY = SZ$

To prove: $XY = YZ$

Proof: $XQ = YS$ (Since $PX = QY$ and $PQ = QS$) ✓

$QY = SZ$ (given)

$\angle XQY = \angle YSZ$ (90°) ✓

$\therefore \triangle XQY \cong \triangle YSZ$ (S.A.S) ✓

$\therefore XY = YZ$ ✓

[4]

The following titles are available from Academic Group Pty Ltd:

ACADEMIC ASSOCIATES STUDY GUIDES

Year 11

Accounting & Finance Year 11 ATAR Course Study Guide
Biology Year 11 ATAR Course Study Guide
Chemistry Year 11 ATAR Course Study Guide
Economics Year 11 ATAR Course Study Guide 2nd Edition
Human Biology Year 11 ATAR Course Study Guide
Mathematics Applications Year 11 ATAR Course Study Guide
Mathematics Methods Year 11 ATAR Course Study Guide
Mathematics Specialist Year 11 ATAR Course Study Guide
Physics Year 11 ATAR Course Study Guide
Psychology Year 11 ATAR Course Study Guide

Year 12

Accounting & Finance Year 12 ATAR Course Study Guide
Biology Year 12 ATAR Course Study Guide
Business Management and Enterprise Year 12 ATAR Course Study Guide
Chemistry Year 12 ATAR Course Study Guide
Drama Year 12 ATAR Course Study Guide
Economics Year 12 ATAR Course Study Guide
Human Biology Year 12 ATAR Course Study Guide
Mathematics Applications Year 12 ATAR Course Study Guide
Mathematics Methods Year 12 ATAR Course Study Guide
Mathematics Specialist Year 12 ATAR Course Study Guide
Physics Year 12 ATAR Course Study Guide
Psychology Year 12 ATAR Course Study Guide

ACADEMIC TASK FORCE REVISION SERIES

Year 11

Chemistry Year 11 ATAR Course Revision Series
Mathematics Applications Year 11 ATAR Course Revision Series
Mathematics Methods Year 11 ATAR Course Revision Series 2nd Edition
Mathematics Specialist Year 11 ATAR Course Revision Series 2nd Edition
Physics Year 11 ATAR Course Revision Series

Year 12

Chemistry Year 12 ATAR Course Revision Series
Human Biology Year 12 Revision Series
Mathematics Applications Year 12 ATAR Course Revision Series
Mathematics Methods Year 12 ATAR Course Revision Series
Mathematics Specialist Year 12 ATAR Course Revision Series
Physics Year 12 ATAR Course Revision Series

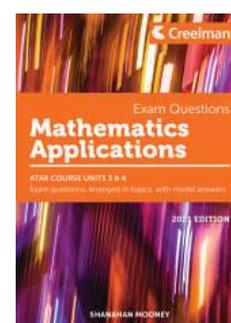
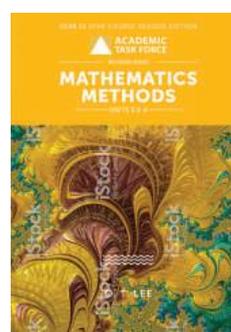
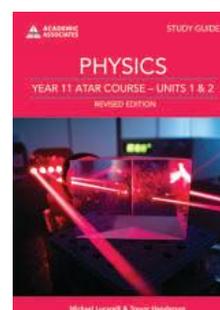
ATAR COURSE TEXTBOOKS

Mathematics Methods Year 11 2nd Edition
Mathematics Specialist Year 11 2nd Edition
Mathematics Methods Year 12
Mathematics Specialist Year 12

CREELMAN EXAM QUESTIONS

Year 12

Accounting & Finance Year 12 ATAR Course Units 3 & 4
Biology Year 12 ATAR Course Units 3 & 4
Business Management and Enterprise Year 12 ATAR Course Units 3 & 4
Chemistry Year 12 ATAR Course Units 3 & 4
Economics Year 12 ATAR Course Units 3 & 4
Geography Year 12 ATAR Course Units 3 & 4
Human Biology Year 12 ATAR Course Units 3 & 4
Mathematics Applications Year 12 ATAR Course Units 3 & 4
Mathematics Methods Year 12 ATAR Course Units 3 & 4
Mathematics Specialist Year 12 ATAR Course Units 3 & 4
Physical Education Studies Year 12 ATAR Course Units 3 & 4
Physics Year 12 ATAR Course Units 3 & 4
Politics and Law Year 12 ATAR Course Units 3 & 4
Psychology Year 12 ATAR Course Units 3 & 4



Achieve great ATAR results!

Found this study guide helpful? Academic Group offers a range of ATAR programs to help you achieve your academic goals.

About our ATAR Help Programs

Term Programs



Tuition Classes

Small group tutoring classes available throughout the year to provide on-going help as you need it. For years 7 - 12.



Master Classes

Weekly teaching classes by top ATAR teachers who will extend you and teach you how to gain maximum marks.



Exam Boost Seminars

Exam Boost Seminars provide targeted exam preparation answering practice exam questions. You will receive feedback from an exam marker and tips and strategies to improve your exam performance.

Holiday Programs



Head Start Program

Held in the January School Holidays. You will gain an overview of the syllabus for your ATAR courses, preview important concepts and get an academic boost for the year ahead.



ATAR Holiday Revision Program

Specially designed programs to help revise ATAR course content and teach you how to maximise your results. Held in the April, July and October school holidays.

Enrol in our programs at www.academicgroup.com.au

Get in touch



learn@academicgroup.com.au



(08) 9314 9500



872 Canning Highway Applecross 6153



MATHEMATICS SPECIALIST

YEAR 11 ATAR COURSE

NEW EDITION – NEW MATERIAL

This book follows the current Western Australian syllabus and is written by well-known West Australian teachers to meet WA educational needs and to prepare students for their ATAR examinations.

Featuring:

- ATAR syllabus checklist.
- Core theory clearly explained and illustrated.
- Wide range of revision questions for all topics with detailed answers.
- Trial tests and marking key – great preparation for tests and examinations.

Make success a reality with this essential student guide for test and exam preparation.

ACADEMIC ASSOCIATES STUDY GUIDES – YR 11 ATAR COURSE

